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ON SELFINJECTIVE ALGEBRAS OF FINITE
REPRESENTATION TYPE

by

© Luis Estrada Navas, B.Sc., Lic.Mat., Dipl.-Math.

A thesis submitted to the Faculty of
Graduate Studies and Research in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
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The undersigned recommend to the Faculty
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ABSTRACT

The stable Auslander-Reiten graph $\Gamma_s(A)$ of a basic, connected, finite dimensional, selfinjective algebra A of finite representation type over a field K is of the form $\mathbb{Z}\Delta/W$, where Δ is a Dynkin diagram and W is an admissible group of automorphisms of the universal stable translation graph $\mathbb{Z}\Delta$. The invariant Δ is called the class of A . The whole Auslander-Reiten graph $\Gamma(A)$ is itself determined by the set C_0 of the vertices of $\Gamma_s(A)$ representing the radicals of the projective-injective, indecomposable A -modules. Let $f: \mathbb{Z}\Delta \rightarrow \mathbb{Z}\Delta/W$ be the canonical projection, $C = f^{-1}(C_0)$. The modulation on $\Gamma_s(A)$ provided by the bimodules of irreducible maps can be extended to $\Gamma(A)$ and lifted to a modulation m of $\mathbb{Z}\Delta$ in such a way that C becomes a modulation of the mesh category $\text{mesh}(\mathbb{Z}\Delta, m)$ of the translation modulated graph $(\mathbb{Z}\Delta, m)$. Thus, the knowledge of all possible configurations C and all possible admissible groups of $(\mathbb{Z}\Delta, m)$ stabilizing C yields all possible stable Auslander-Reiten graphs $\Gamma_s(A)$.

We determine all the configurations and all the admissible groups of $(\mathbb{Z}\Delta, m)$ for the cases $\Delta = B_n, C_m, F_4, G_2$ ($n \geq 2, m \geq 3$). The configurations of $(\mathbb{Z}\Delta, m)$ are classified establishing a bijection with certain valued graphs which we introduce and call

(11)

Brauer graphs of type Δ . In the cases $\Delta = B_n$, and $\Delta = C_n$ an alternative classification is also given in terms of symmetric Brauer pairs of order $\omega = 2n - 1$.

Let C be a configuration of $(\mathbb{Z}\Delta, m)$, D its associated Brauer graph. From D we construct a valued infinite graph $\tilde{D}$ and from m a modulation $\tilde{m}$ of $\tilde{D}$ and a K -species $\tilde{\Sigma}$ with a suitable chosen relation ideal $\tilde{J}$ in such a way that, for every admissible group W of automorphisms of $\mathbb{Z}\Delta$ stabilizing C , a basic, connected, finite dimensional, selfinjective K -algebra $A = A(m, C, W)$ of finite representation type is given such that $\Gamma_s(A) \cong (\mathbb{Z}\Delta, m) / W$. The algebra A itself is described as a factor algebra $A \cong T(\Sigma) / J$ of the tensor algebra $T(\Sigma)$ of a K -species Σ by a relation ideal J , where Σ and J are determined by $\tilde{\Sigma}, \tilde{J}$ and $\tilde{W}$. The category $\text{mod } A$ is described as the category $R(\Sigma, J)$ of those representations of the category $R(\Sigma)$ bound by J , and its indecomposable representations are also completely described.

To my wife

Pilar

and to our children

Adriana

Fernando

Eduardo

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INTRODUCTION

(0.1) Let A be a finite dimensional, connected, selfinjective algebra over a field K , and let $\Gamma(A)$ be the Auslander-Reiten graph of the category $\text{mod } A$ of the finitely generated right modules over A . Let P be a projective-injective indecomposable A -module. Then the modules $\text{rad } P$ and $P/\text{soc } P$ are also both indecomposable and the Auslander-Reiten sequence with P in its middle term is of the form $0 \rightarrow \text{rad } P \rightarrow P \oplus \text{rad } P/\text{soc } P \rightarrow P/\text{soc } P \rightarrow 0$. Assume further that A is also an algebra of finite representation type, that is, there is only a finite number of non-isomorphic indecomposable A -modules. From the structure of the Auslander-Reiten sequences as the one above, it follows that the full subgraph $\Gamma_s(A)$ of $\Gamma(A)$, consisting of the indecomposable types x such that $\tau^t x \neq 0$ for every $t \in \mathbb{Z}$, is τ -connected, and every type is τ -periodic. Applying the results of Happel-Preiser-Ringel on the structure of components of $\Gamma_s(A)$ containing periodic types ([21]), it follows that $\Gamma_s(A) \cong \mathbb{Z}\Delta/W$, where Δ is a uniquely determined Dynkin graph, that is, one of the valued graphs $A_n: o \text{---} o \text{---} \dots o \text{---} o$ ($n \geq 1$), $B_n: o \xrightarrow{(1,2)} o \text{---} \dots o \text{---} o$ ($n \geq 2$), $C_n: o \xrightarrow{(2,1)} o \text{---} \dots o \text{---} o$ ($n \geq 3$), $D_n: \begin{smallmatrix} o \\ \diagdown \end{smallmatrix} o \text{---} o \text{---} \dots o \text{---} o$ ($n \geq 4$), each one of them

with n vertices, E_6 : , E_7 : ,
 E_8 : , F_4 : or G_2 : ,

and W is an admissible group of automorphisms of the stable valued graph Z_Δ , determined up to conjugation. The Dynkin graph Δ is thus an invariant of A , called the Dynkin class of A . This result suggests immediately the following.

Problem. Classify all finite dimensional, connected, self-injective algebras over a field K , which are of finite representation type and have a given Dynkin class.

Several investigations have dealt with this problem and in the case of an algebraically closed field K , that is, in the cases of Dynkin class A_n , D_n , E_6 , E_7 and E_8 , Riedtmann has obtained a complete classification ([24], [25]). In this work we deal with the remaining cases B_n , C_n , F_4 and G_2 .

Suppose we are given $\Gamma_s(A)$ and we know the positions within $\Gamma_s(A)$ of the set C_0 consisting of those vertices $x = [\text{rad } P]$ representing the radicals of the projective-injective indecomposable A -modules. Then we can recover the whole Auslander-Reiten graph $\Gamma(A)$ simply by adding new vertices $p(x)$ and new arrows $x \rightarrow p(x) \rightarrow \tau^{-1}x$ for every $x \in C_0$. Let $f: Z_\Delta \rightarrow Z_\Delta/W \cong \Gamma_s(A)$

be the canonical projection and let $C = f^{-1}(C_0)$. The modulation of $\Gamma_s(A)$ provided by the bimodules of irreducible maps can be lifted to a modulation m of $Z\Delta$ in such a way that C satisfies the following conditions:

- (C1) For each $x \in (Z\Delta)_0$ there exists $c \in C$ such that $\text{msh}(Z\Delta, m)(x, c) \neq 0$
- (C2) $\text{msh}(Z\Delta, m)(c, c') = 0$, for every $c, c' \in C$, $c \neq c'$
 $\text{msh}(Z\Delta, m)(c, c) = K_c$, for every $c \in C$.

Here, $\text{msh}(Z\Delta, m)$ is the mesh category of the translation modulated graph $(Z\Delta, m)$ defined as the quotient category of the tensor category $\otimes(Z\Delta, m)$ by its mesh ideal, which is constructed using the canonical elements of the bimodules of the modulation.

A set $C \subseteq Z\Delta$ satisfying (C1) and (C2), is called a configuration of $(Z\Delta, m)$, hence the knowledge of all configurations C together with all admissible groups of W stabilizing C yields all possible stable Auslander-Reiten graphs $\Gamma_s(A)$. We introduce a special kind of valued graphs ([12]), which we call Brauer graphs of type Δ , for each of the cases $\Delta = B_n, C_n, F_4$ and G_2 , to establish a fundamental bijection between the configurations of $Z\Delta$ and the Brauer graphs of type Δ .

Let D be a Brauer graph of type Δ . From D we construct a valued infinite graph $\tilde{D}$, which we call its universal covering, in such a way that from each modulation m of $\mathbb{Z}\Delta$ we obtain a modulation $\tilde{m}$ of $\tilde{D}$ and a species $\tilde{\Sigma}$, which we endow with a suitably chosen relation ideal $\tilde{J}$. The category $R(\tilde{\Sigma}, \tilde{J})$ of all finite dimensional representations of $R(\tilde{\Sigma})$ bound by $\tilde{J}$ comes equipped, by our construction, with a canonical selfequivalence μ , so we can consider the categories $R^{\mu^r}(\tilde{\Sigma}, \tilde{J})$ of μ^r -invariant representations, categories that can be interpreted as categories of modules $\text{mod } A_r$, for every $r \geq 1$, and we prove that the algebras A_r are finite dimensional, basic, connected, selfinjective algebras of finite representation type of class Δ . The algebras A_r are described as factor algebras $A_r = T(\tilde{\Sigma}_r)/J_r$ of the tensor algebra $T(\tilde{\Sigma}_r)$, with $\tilde{\Sigma}_r = \tilde{\Sigma}/\mu^r$, $J_r = \tilde{J}/\mu^r$, and the indecomposable representations of $R^{\mu^r}(\tilde{\Sigma}, \tilde{J}) \cong R(\tilde{\Sigma}_r, J_r) \cong \text{mod } A_r$ are also completely described.

(0.2) This thesis is divided in five chapters. In Chapter I, of preliminary nature, we recall those aspects of the Representation Theory of Associative Algebras over a field that will be used in the course of our investigation. Section 1 reviews the basic concepts of module and representation theory, Section 2 some of the results of Auslander and Reiten on Auslander-Reiten sequences and related

notions, and Section 3 some particular classes of algebras that appear frequently in our work. Chapter II deals with modulated graphs. In Section 1 we introduce the concept of modulated graph, a valued graph endowed with a modulation, giving a general definition to include the modulations of the universal coverings of the Brauer graphs of types B_n , C_n , F_4 or G_2 ; these latter are also defined here, to be used later to construct selfinjective algebras of finite representation type. Section 2 is dedicated to discuss modulations of translation graphs and Section 3 to the notion of species of a modulated graph and of a translation modulated graph. After discussing in Section 4 a general notion of relation ideal of the tensor algebra of a species, in Section 5 we introduce the OPE-relations as relations arising from one-point extension algebras. In Section 6 some functor categories, the mesh category of a translation modulated graph and the modulations of the stable translation graphs $\mathbb{Z}\Delta$, find their place, and Section 7 is dedicated to locally finite dimensional modulated graphs. Chapter III studies general properties of selfinjective algebras. After recalling in Section 1 basic results, in Section 2 we introduce the fundamental concept of configuration, and its main general properties are discussed. Section 3 deals with the modulated stable translation graphs $\mathbb{Z}\Delta$ and their mesh categories

mesh $\mathbb{Z}\Delta$. In Chapter IV we construct finite dimensional, basic, connected, selfinjective algebras of finite representation type of Dynkin class B_n or C_n for a given modulation of $\mathbb{Z}B_n$ or $\mathbb{Z}C_n$, respectively. In Section 1, of combinatorial nature, we give a complete classification of all possible configurations of $\mathbb{Z}B_n$ in terms of certain symmetric Brauer pairs canonically associated to a given configuration. Section 2 includes our main theorems on the construction of selfinjective algebras of Dynkin class B_n , their module categories being described as categories of representations $R(\Sigma, J)$, with species Σ obtained from the universal coverings $\tilde{D}$ of Brauer graphs of type B_n , and the algebras as factor algebras $T(\Sigma)/J$. In Section 3 the shape of the indecomposable representations of these categories is described, and an algorithm is provided to determine the position in the Auslander-Reiten graph of the category $R(\tilde{\Sigma}, \tilde{J})$, $\tilde{\Sigma}$ the species associated to $\tilde{D}$, of any indecomposable representation covering a given indecomposable representation of $R(\Sigma, J)$. Similar results are obtained for the case C_n . Chapter V deals with selfinjective algebras of finite representation type and of Dynkin class F_4 or G_2 . Being finite in number, a complete list of the configurations of $\mathbb{Z}F_4$ is given in Section 1, Section 2 discusses the algebras of Dynkin class F_4 and Section 3

their indecomposable representations. Finally, in Section 4 complete descriptions of the only configuration of $\mathbb{Z}\mathbb{C}_2$, the selfinjective algebras and the indecomposable representations, are given.

(0.3) The main results obtained in this investigation are as follows. Let Δ be one of the Dynkin graphs B_n , C_n , F_4 or G_2 . First, we consider the case $\Delta = B_n$. Let C be a configuration of $\mathbb{Z}B_n$ for a fixed modulation of this stable translation graph. We call a hill H_a of $\mathbb{Z}B_n$ the set of points $\{(a,b) \mid 1 \leq b \leq n\} \cup \{(r,s) \mid r+s=a+n\}$, for every $a \in \mathbb{Z}$. Then, on every hill lies exactly one configuration point, but every configuration point belongs to exactly two hills, except those lying on the upper border of $\mathbb{Z}B_n$. We introduce two functions $\alpha, \beta: \mathbb{Z} \rightarrow \mathbb{Z}$, defined in terms of the position of a configuration point within the hills to which it belongs, functions which we prove are both bijections of $\mathbb{Z}$ that satisfy the following fundamental properties:

$$(a) \quad \alpha(x+\omega) = \alpha x + \omega$$

$$\beta(x+\omega) = \beta x + \omega$$

$$(b) \quad \alpha\beta x = x + 2n$$

$$(c) \quad \beta(\beta x - n) = x + n,$$

for every $x \in \mathbb{Z}$, where $\omega = 2n - 1$. The properties (a) and (b)

state the fact ([19]) that (α, β) is a Brauer pair arising from a Brauer relation of order ω , whereas (c) establishes the fact that this relation is symmetric ([24]). Conversely, we prove that every symmetric Brauer pair gives rise to a configuration of $\mathbb{Z}B_n$, thus establishing:

Theorem (IV, (1.20)). The correspondence $C \mapsto (\alpha, \beta)$ establishes a bijection between the configurations C of $\mathbb{Z}B_n$ and the symmetric Brauer pairs of order $\omega = 2n - 1$.

Brauer pairs of order ω are well known to be related to self-injective algebras of finite representation type of Dynkin class A_ω ([24]). To relate them to selfinjective algebras of finite representation type of Dynkin class B_n , we introduce the notion of Brauer graph of type B_n : let Q be a Brauer quiver ([19]). We call transit vertices of Q those belonging to loops, nodes the other vertices. Consider the oriented graph D obtained from Q by erasing all possible loops. Select one transit vertex q_0 and endow D with the valuation w defined by $w(\delta) = (2, 1)$, $w(\epsilon) = (1, 2)$, $w(\gamma) = (1, 1)$ for every other arrow γ , $\gamma \neq \delta, \epsilon$, where δ and ϵ are the arrows of Q ending at q_0 and starting at q_0 respectively. If Q has n vertices, the resulting valued graph (D, w) together with the selected vertex q_0 is called a Brauer graph of type B_n .

We prove that this concept is the right one:

Theorem (IV, (1.4.1)). There exists a bijection between the configurations of $\mathbb{Z} B_n$ and the Brauer graphs of type B_n .

We stress the fact that the proof of the bijection established in this theorem is constructive, providing an effective algorithm to construct a configuration from a given Brauer graph and vice-versa.

In the course of arriving at the theorems above we have done in fact a fairly complete discussion of the combinatorial properties of the pairs $(\mathbb{Z} B_n, C)$, C a configuration of $\mathbb{Z} B_n$.

Consider next the case C_n . The corresponding main results for the configurations of $\mathbb{Z} C_n$, in complete analogy to the previous case B_n , are:

Theorem (IV, (1.43)). There is a bijection between the configurations of $\mathbb{Z} C_n$ and the symmetric Brauer pairs of order $w = 2n - 1$.

Theorem (IV, (1.44)). There is a bijection between the configurations of $\mathbb{Z} C_n$ and the Brauer graphs of type C_n .

A Brauer graph of type C_n is defined exactly as in the case of a Brauer graph of type B_n , except that the valuation w is

taken to be $\omega(\delta) = (1,2)$, $\omega(\epsilon) = (2,1)$. Thus, clearly every Brauer graph of type C_n gives rise canonically to a Brauer graph of type B_n , and conversely, in such a way that the associated configurations of the underlying graphs UZB_n and UZC_n are the same.

Next, we come to the selfinjective algebras. We consider again the case B_n first. Let (D, w, q_0) be a Brauer graph of type B_n . We can assume $D_0 = \mathbb{Z}/n\mathbb{Z}$, $q_0 = \bar{0}$. The valued graph $(\tilde{D}, \tilde{w}, \{i | i \equiv 0 \pmod{n}\})$ with $\mathbb{Z}$ is a set of vertices, arrows $i \rightarrow j$ whenever there is an arrow $\bar{i} \rightarrow \bar{j}$ in D , and valuation $\tilde{w}(\delta_i) = (2,1)$, $\tilde{w}(\epsilon_i) = (1,2)$, if δ_i and ϵ_i are the arrows of $\tilde{D}$ stopping at i , and starting at i , respectively, for every $i \in \mathbb{Z}$, and $w(\gamma) = (1,1)$ otherwise, is called the universal covering of (D, w, q_0) . We attach a K -species $\tilde{\Sigma}$ to this covering in the following way: $\tilde{\Sigma} = (K_i, M_{ij})_{i,j \in \mathbb{Z}}$ is the K -species with $K_i = G$ if $i \equiv 0$, $K_i = F$ otherwise, and bimodules $M_{ij} = {}_G^F$, ${}_F^G$ or ${}_F^F$ according to whether $i \equiv 0$, $j \equiv 0$ or neither of these two cases, respectively, for every arrow $j \rightarrow i$ of $\tilde{D}$. Let $\tilde{J}$ be the relation ideal of $\tilde{\Sigma}$ defined by the following minimal relations: (μ) $K_h \xrightarrow{\mu} K_i \rightarrow K_j$, $i \equiv 0$, $j \rightarrow i \rightarrow h$ path in $\tilde{D}$; (π) $K_{i+n} \xrightarrow{\pi} K_{i_1} \rightarrow \dots \rightarrow K_{i_r} \rightarrow K_i$, $i \equiv 0$, $\bar{i} \rightarrow \bar{i}_r \rightarrow \dots \rightarrow \bar{i}_1 \rightarrow \bar{i}$ the cycle through $\bar{0}$;

and

$\bar{i}$ a node, $\bar{i} \rightarrow \bar{a}_p \rightarrow \dots \rightarrow \bar{a}_1 \rightarrow \bar{i}$ and $\bar{i} \rightarrow \bar{b}_q \rightarrow \dots \rightarrow \bar{b}_1 \rightarrow \bar{i}$

the cycles through $\bar{i}$, and moreover $s \equiv 0$ in (c, μ) ; and

(0) $K_h \xrightarrow{0} K_i \rightarrow K_j$, $j \rightarrow i$ and $i \rightarrow h$ belonging to different

cycles. (All congruences to be taken modulo n .) Here, F/G is

a quadratic field extension with $K = G$ a perfect field. Let

$\mu: \mathbb{Z} \rightarrow \mathbb{Z}$, $i \mapsto i+n$; then μ induces a selfequivalence of the

category $R(\tilde{\Sigma}, \tilde{J})$ of the finite dimensional representations of $\tilde{\Sigma}$

bound by $\tilde{J}$, and we can consider the species $\Sigma_r := \tilde{\Sigma} / \mu^r$ and

the relation ideals $J_r := \tilde{J} / \mu^r$, for every $r \geq 1$. Set

further $B_r := T(\Sigma_r) / J_r$, $\tilde{B} := \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{R(\tilde{\Sigma}, \tilde{J})}(\underline{P}(i), \underline{P}(j))$,

and let finally C be the configuration of $(\mathbb{Z} B_n, m)$ associated

to D , m the given F/G -modulation of $\mathbb{Z} B_n$. With all these

constructions, we prove our main results:

Theorem (IV, (2.8)). Let D be a Brauer graph of type B_n , C its associated configuration of $\mathbb{Z} B_n$, $B_{r,D} = B_r$. Then

- (1) $B_{r,D}$ is a basic, connected, finite dimensional, self-injective K -algebra of finite representation type and of Dynkin class B_n .

$$(11) \quad \Gamma(B_{r,D}) \cong (\mathbb{Z} B_n)_C / W_r,$$

for every $r \geq 1$.

Here, $(\mathbb{Z} B_n)_C$ is the modulated translation graph obtained from $\mathbb{Z} B_n$ by adding new vertices $p(c)$, new arrows $c \rightarrow p(c) \rightarrow \tau^{-1}c$, with bimodules $M = {}_{K_c} (K)_{K_c}$, for every configuration point $c \in C$, and $K_{p(c)} = K_c$. Moreover, $W_r = \langle \tau^{\omega r} \rangle$, $\omega = 2n - 1$.

Theorem (IV, (2.11)). The algebras $B_{1,D}$, D running through a set of representatives of the isomorphisms classes of Brauer graphs of type B_n , are trivial extension algebras of finite representation type and of Dynkin class B_n .

We consider next the case of an F/G -modulation of $\mathbb{Z} C_n$. Let D be a Brauer graph of type C_n , $\tilde{D}$ its universal covering (constructed in a similar way to the case B_n). We attach to $\tilde{D}$ the K -species $\tilde{\Sigma} = (K_i, {}^M_{ij})_{i,j \in \mathbb{Z}}$ defined by $K_i = F$ if $i \equiv 0$, $K_i = G$, otherwise; ${}^M_{ij} = {}^F_{FG}$, ${}^F_{GF}$ or ${}^G_{GG}$, according to whether

$i \equiv 0$, $j \equiv 0$ or neither of these two cases, respectively, for every arrow $j \rightarrow i$ of $\tilde{D}$. Let $\tilde{J}$ be the ideal of $\tilde{\Sigma}$ defined by the following minimal relations: (π) $K_h \xrightarrow{\pi} K_i \rightarrow K_j$, $i \equiv 0$, $j \rightarrow i \rightarrow h$ path in $\tilde{D}$; (μ) $K_{i+n} \xrightarrow{\mu} K_{i_1} \rightarrow \dots \rightarrow K_{i_r} \rightarrow K_i$, $i \equiv 0$, $\bar{i} \rightarrow \bar{i}_r \rightarrow \dots \rightarrow \bar{i}_1 \rightarrow \bar{i}$ the cycle through $\bar{0}$;

(c)

(c, π)

$\bar{i}$ a node, $\bar{i} \rightarrow \bar{a}_p \rightarrow \dots \rightarrow \bar{a}_1 \rightarrow \bar{i}$ and $\bar{i} \rightarrow \bar{b}_q \rightarrow \dots \rightarrow \bar{b}_1 \rightarrow \bar{i}$ the cycles through $\bar{i}$, and moreover $s \equiv 0$ in (c, π) ; and

(0) $K_h \xrightarrow{0} K_i \rightarrow K_j$, $\bar{j} \rightarrow \bar{i}$ and $\bar{i} \rightarrow \bar{h}$ belonging to different

cycles. (All congruences are taken modulo n). The map

$\mu: \mathbb{Z} \rightarrow \mathbb{Z}$, $i \mapsto i+n$, induces a selfequivalence of the category

$R(\tilde{\Sigma}, \tilde{J})$ of the finite dimensional representations of $\tilde{\Sigma}$ bound by

$\tilde{J}$, and we can consider the species $\Sigma_r = \tilde{\Sigma} / \mu^r$ and the relation

ideals $J_r = \tilde{J} / \mu^r$, and the algebras $C_r = T(\Sigma_r) / J_r$,

$\tilde{C} = \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{R(\tilde{C}, \tilde{J})}(\underline{P}(i), \underline{P}(j))$, for every $r \geq 1$. Then, we prove

Theorem (IV, (2.21)). Let D be a Brauer graph of type C_n , C its associated configuration of $\mathbb{Z} C_n$, $C_{r,D} = C_r$. Then

(i) $C_{r,D}$ is a basic, connected, finite dimensional, self-injective K -algebra of finite representation type and of Dynkin class C_n

(ii) $\Gamma(C_{r,D}) \cong (\mathbb{Z} C_n)_C / W_r$

for every $r \geq 1$.

Here, again $W_r = \langle \tau^{\omega r} \rangle$, $\omega = 2n - 1$, and $(\mathbb{Z} C_n)_C$ is the modulated translation graph obtained from $\mathbb{Z} C_n$ by adding new vertices $p(c)$, new arrows $c \rightarrow p(c) \xrightarrow{\tau^{-1}} c$, with bimodules $M = K_c$, for every configuration point $c \in C$.

Theorem (IV, (2.23)). The algebras $C_{1,D}$, D running through a set of representatives of the isomorphism classes of Brauer graphs of type C_n , are trivial extension algebras of finite representation type and of Dynkin class C_n .

Finally, Chapter V contains the results on the configurations of the modulated translation graphs $\mathbb{Z}\mathbb{F}_4$ and $\mathbb{Z}\mathbb{C}_2$, the construction of selfinjective algebras of finite representation type and of Dynkin class $\mathbb{F}_4$ or $\mathbb{C}_2$, and their indecomposable representations. We refer to that chapter for the explicit description of the results.

CHAPTER I. PRELIMINARIES

In this chapter, we recall some basic concepts and fundamental results from the Representation Theory of Associative Algebras over a Field that will be used in the course of our investigation. In Section 1 we review the basic concepts of module and representation theory, Section 2 is devoted to Auslander-Reiten sequences, and in Section 3 some classes of algebras that will appear frequently in the sequel are discussed briefly.

Field will always mean commutative field; non-necessarily commutative fields will be called division rings. An algebra over a field will always be an associative algebra, and it will be assumed that has a unit element if it is of finite dimension over the field.

I.1 Basic concepts

Let A be an algebra over a field K . In this section we review the basic concepts of the module theory of A .

(1.1) The category of the finitely generated right modules over A will be denoted by $\text{mod } A$, its objects will be called A -modules, its morphisms will be called A -homomorphisms, or

simply morphisms or maps, and will usually be written on the opposite side of the scalars, thus a composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ of A -homomorphisms will be denoted by $g \circ f$ or simply gf . The left modules over A will be considered as A^{op} -modules, A^{op} the opposite algebra of A . By M_A resp. ${}_A M$ we stress the fact that M is an A -module resp. an A^{op} -module. The functor $\text{mod } A \rightarrow \text{mod } A^{\text{op}}, M_A \mapsto \text{Hom}_K(M_A, K)$, $f \mapsto \text{Hom}_K(f, K)$ establishes a duality between $\text{mod } A$ and $\text{mod } A^{\text{op}}$ which will be called the standard duality of A and will be denoted by D . The composition $D \circ \text{Hom}_A(-, A_A): \text{mod } A \rightarrow \text{mod } A$ will be called the Nakayama functor of A ; it sends projective A -modules to injective ones.

(1.2) Given an A -module M , its type $[M]$ is the isomorphism class of all A -modules isomorphic to M . The algebra A is said to be of finite representation type, if there is only a finite number of different types of indecomposable A -modules, a module M being indecomposable if it does not admit non-trivial direct sum decompositions $M = M' \oplus M''$.

By $\text{ind } A$ we denote the full subcategory of $\text{mod } A$ consisting of chosen representatives of the types of indecomposable A -modules, and by $\text{proj ind } A$ the full subcategory of $\text{mod } A$ consisting of chosen representatives of the types of the projective indecomposable A -modules.

(1.3) For a module M , its radical, the intersection of its maximal submodules, is denoted by $\text{rad } M$, and the module $\text{top } M := M/\text{rad } M$ is called the top of M . It is the largest semisimple factor module of M , a module being semisimple if its radical is trivial. On the other hand, the submodule $\text{soc } M$ generated by its simple submodules is the largest semisimple submodule of M and it is called the socle of M .

Any finitely generated A -module M has a finite composition series $M = M_m \supseteq M_{m-1} \supseteq \dots \supseteq M_1 \supseteq M_0 = 0$, the simple factors M_i/M_{i-1} are called its composition factors and the number m is called the composition length, or simply the length, of M , and will be denoted by $|M|$ or $\ell(M)$. By the theorem of Jordan-Hölder the types $[M_i/M_{i-1}]$ ($1 \leq i \leq m$) and $\ell(M)$ are invariants of M , so, if we choose a fixed ordered set of representatives $S(1), \dots, S(n)$ of the simple A -modules, the number $(\dim M)_i$ of composition factors of M isomorphic to $S(i)$ is also an invariant of M , and the map $\dim: \text{mod } A \rightarrow \mathbb{Q}^n$, $M \mapsto \dim M := ((\dim M)_i)_{1 \leq i \leq n}$ is well defined and called the dimension type.

(1.4) For a module M , an essential epimorphism $P \rightarrow M \rightarrow 0$ with P a projective module is called a projective cover of M and it is uniquely determined up to isomorphism. A fixed one will be called the projective cover of M , and also P will be so called,

a fact denoted by $P = P(M)$. Dually, essential monomorphisms $0 \rightarrow M \rightarrow I$ with injective I are called injective envelopes, and $I = I(M)$ will denote a fixed one. In particular, we can choose complete sets of representatives $P(1), \dots, P(n)$ and $I(1), \dots, I(n)$ of the indecomposable projective and injective modules, respectively, in such a way that $P(i)$ resp. $I(i)$ is the projective cover, resp. injective envelope of the simple $S(i)$ previously chosen ($1 \leq i \leq n$). In particular, $\text{top } P(i) = S(i) = \text{soc } I(i)$ and

$$\begin{aligned} (\dim M)_i &= \ell(\text{Hom}_A(P(i), M)_{\text{End } P(i)}) \\ &= \ell(\text{Hom}_A(M, I(i))_{\text{End } I(i)}) \end{aligned}$$

for every module M and for every $i = 1, \dots, n$.

Moreover, $A = \bigoplus_{i=1}^n d_i P(i)$, $d_i \in \mathbb{N}$, ($1 \leq i \leq n$). If $d_i = 1$ for every $i = 1, \dots, n$, A is said to be a basic algebra. Equivalently, A is basic if and only if $A/\text{rad } A$ decomposes as a product of division rings, $\text{rad } A$ denoting the (Jacobson) radical of A .

(1.5) Often we will find convenient to interpret $\text{mod } A$ as a functor category. Assuming that A is basic, as we will usually do (since, given any A we can consider its basic algebra

$\bar{A} = \bigoplus_{i,j=1}^n \text{Hom}_A(P(i), P(j))$, and A and $\bar{A}$ are Morita-equivalent algebras, i.e. $\text{mod } A \cong \text{mod } \bar{A}$, so we lose nothing from the point

of view of Representation Theory, which aims at describing the indecomposable A -modules), we have the well-known result

Lemma 1. $\text{mod } A \simeq ((\text{proj ind } A)^{\text{op}}, \text{Mod } K).$

Given a K -category C , we denote by $(C^{\text{op}}, \text{Mod } K)$ the full subcategory of the category of all K -linear contravariant functors from C to $\text{Mod } K$, consisting of those functors T such that $T(X)$ is a finite dimensional K -vector space for every object X of C . A K -category C is a category such that the morphism groups $C(X,Y) := \text{Hom}_C(X,Y)$, for every pair of objects X,Y in C , are endowed of a K -vector space structure in such a way that the composition of morphisms in C is K -bilinear. Clearly, $\text{proj ind } A$ is a K -category.

For the proof of the lemma, we only remark that $M \mapsto \text{Hom}_A(-, M) | \text{proj ind } A$ yields the equivalence.

I.2 Auslander-Reiten Theory

Let A be a finite dimensional algebra over a field K , and let

$$0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0 \quad (1)$$

be a short exact sequence in $\text{mod } A$.

(2.1) Definition. The sequence $\gamma(1)$ is an Auslander-Reiten sequence if the following conditions are satisfied:

- (i) X and Z are indecomposable
- (ii) The sequence does not split
- (iii) Every nonisomorphism $h: Z' \rightarrow Z$ with Z' indecomposable can be lifted to Y : there exists $u: Z' \rightarrow Y$ such that $h = gu$
- (iv) Every nonisomorphism $k: X \rightarrow X'$ with X' indecomposable can be extended to Y : there exists $v: Y \rightarrow X'$ such that $k = vf$.

Auslander and Reiten, in their series of papers on the Representation Theory of Artin Algebras ([3], [4], [5], [6]) and other related papers, introduced these sequences under the name of almost split sequences. From there we extract some of their concepts and results, of fundamental importance for our work. First, one has the basic existence and unicity theorem given by

(2.2) Theorem. Let A be an Artin algebra.

- (i) For every indecomposable non-projective A -module Z , there exists an Auslander Reiten sequence (1)
- (ii) For every indecomposable non-injective A -module X , there exists an Auslander Reiten sequence (1)
- (iii) An Auslander-Reiten sequence (1) is uniquely determined; up to isomorphism, either by X or by Z .

The unicity statement (iii) allows to speak of the Auslander-Reiten sequence starting at X , ending at Z , with kernel X , or with cokernel Z , to refer to an Auslander-Reiten sequence of the form (1). In particular, X and Z determine each other mutually, and there is an explicit way to do it: starting from Z , let

$$Q \xrightarrow{h} P \rightarrow Z \rightarrow 0$$

be the start of a minimal projective resolution of Z . Applying the functor $(-)^t = \text{Hom}_A(-, A)$ and setting $\text{Tr } Z := \text{Coker}(h, A)$ one obtains the exact sequence

$$P^t \rightarrow Q^t \rightarrow \text{Tr } Z \rightarrow 0,$$

and then $X \cong D\text{Tr } Z$. Similarly, $Z \cong \text{Tr } D X$. We denote

$\tau := D \operatorname{Tr}$ and call it the Auslander-Reiten translation; similarly,
 $\tau^{-1} := \operatorname{Tr} D$; $\operatorname{Tr} Z$ and τZ are called the transpose and the
 Auslander-Reiten translate of Z , respectively.

It is useful to relate Auslander-Reiten sequences to other concepts, in particular those of minimal left almost split, right almost split and irreducible morphisms, notions which we recall. A morphism $g: Y \rightarrow Z$ with Z indecomposable is right minimal, if for any $h: Y \rightarrow Y$ such that $gh = g$, then h is an isomorphism; and g is right almost split if it is not a retraction and for any $h: M \rightarrow Z$ with M indecomposable which is not an isomorphism, there exists $u: M \rightarrow Y$ such that $gu = h$. A minimal right almost split morphism is one which is both right minimal and right almost split. Similarly, a morphism $f: X \rightarrow Y$ with X indecomposable is left minimal if for any $h: Y \rightarrow Y$ such that $hf = f$, then h is an isomorphism, is left almost split if it is not a section and for any $h: X \rightarrow M$ with M indecomposable which is not an isomorphism, there exists $v: Y \rightarrow M$ such that $vf = h$, and is minimal left almost split if it is both left minimal and left almost split. A morphism $h: M \rightarrow N$ is irreducible if it is neither a section nor a retraction and in any factorization $h = fg$ then g is a section or f is a retraction. Clearly, an irreducible morphism is either a monomorphism or an epimorphism.

Some of the connections among these concepts and Auslander-Reiten sequences are given in the following propositions.

(2.3) Proposition.

(i) Let Z be an indecomposable non-projective A -module. Then a morphism $g: Y \rightarrow Z$ is minimal right almost split if and only if the sequence $0 \rightarrow \text{Ker } g \rightarrow Y \xrightarrow{g} Z \rightarrow 0$ is Auslander-Reiten.

(ii) Let X be an indecomposable non-injective A -module. Then a morphism $f: X \rightarrow Y$ is minimal left almost split if and only if $0 \rightarrow X \xrightarrow{f} Y \rightarrow \text{Coker } f \rightarrow 0$ is Auslander-Reiten.

(2.4) Proposition. Let $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ be an exact sequence in $\text{mod } A$. The following conditions are equivalent:

- (i) The sequence is Auslander-Reiten
- (ii) f is minimal left almost split and g is minimal right almost split
- (iii) f and g are irreducible morphisms.

(2.5) Proposition.

(i) Let Z be an indecomposable A -module and $g: Y \rightarrow Z$ a non-zero morphism. Then g is irreducible if and only if there exists a morphism $g': Y' \rightarrow Z$ such that $(g, g'): Y \oplus Y' \rightarrow Z$ is minimal right almost split.

(ii) Let X be an indecomposable A -module and $f: X \rightarrow Y$ a non-zero morphism. Then f is irreducible if and only if there exists a morphism $f': X \rightarrow Y'$ such that $(f, f')^t: X \rightarrow Y \oplus Y'$ is minimal left almost split.

(2.6) Remarks.

(i) Let P be an indecomposable projective A -module. Then a morphism $g: Y \rightarrow P$ is minimal right almost split if and only if g is isomorphic to the inclusion $\text{rad } P \hookrightarrow P$. Thus, by (2.5), (i), the irreducible morphisms $h: M \rightarrow P$ with indecomposable domain are those isomorphic to the inclusion $M' \hookrightarrow P$ where M' is an indecomposable direct summand of $\text{rad } P$.

(ii) Let I be an indecomposable injective A -module. Then a morphism $f: I \rightarrow Y$ is minimal left almost split if and only if f is isomorphic to the canonical projection $I \xrightarrow{\pi} I/\text{soc } I$ of I onto its socle factor module. Thus, by (2.5), (ii), the irreducible morphisms $h: I \rightarrow M$ with indecomposable codomain are those isomorphic to $\pi': I \rightarrow M'$, where M' is an indecomposable direct summand of $I/\text{soc } I$ and $\pi': I/\text{soc } I \rightarrow M'$ is the canonical projection.

The next concept we want to recall is that of the bimodules of irreducible maps. First, we recall the notion of the (Jacobson) radical of the category $\text{mod } A$. Given indecomposable A -modules X and Y , $\text{rad}(X, Y)$ denotes the group of all non-isomorphisms from X to Y . If X and Y are not necessarily indecomposable modules, we set $\text{rad}(X, Y) := \bigoplus_{i,j} \text{rad}(X_i, Y_j)$, where $X = \bigoplus X_i$, $Y = \bigoplus Y_j$ are decompositions of X and Y in indecomposable modules X_i, Y_j , using the canonical isomorphisms $\text{Hom}_A(X, Y) \cong \bigoplus_{i,j} \text{Hom}_A(X_i, Y_j)$. The ideal of $\text{mod } A$ generated by these groups $\text{rad}(X, Y)$ is called the radical of $\text{mod } A$. Similarly, defining, for every $n \in \mathbb{N}$, $\text{rad}^n(X, Y)$ as the subgroup of $\text{Hom}_A(X, Y)$ of all those morphisms $f: X \rightarrow Y$ which can be expressed as $f = \sum f_i$ with each f_i a composition of at least n morphisms in the radical of $\text{mod } A$, we obtain a family of ideals of $\text{mod } A$ such that $\text{rad}(X, Y) \supseteq \text{rad}^2(X, Y) \supseteq \dots \supseteq \text{rad}^n(X, Y) \supseteq \dots$ for all X and Y . Clearly, $\text{rad}(X, X) = \text{rad } \text{End}(X_A)$ and $\text{rad}(X, Y) = \text{Hom}_A(X, Y)$ if X and Y are nonisomorphic indecomposable modules; moreover, $\text{rad}(X, Y) \setminus \text{rad}^2(X, Y)$ is the set of irreducible morphisms from X to Y . Thus ([27]), the quotient group $\text{Irr}(X, Y) := \text{rad}(X, Y) / \text{rad}^2(X, Y)$ is called the bimodule of irreducible maps, for clearly it is a $K_Y - K_X$ -bimodule, where $K_X := \text{End}(X_A) / \text{rad } \text{End}(X_A)$, for every indecomposable module X .

(2.7) Proposition. Let $0 \rightarrow X \xrightarrow{f} \bigoplus_{i=1}^t d_i Y_i \xrightarrow{g} Z \rightarrow 0$ be an

Auslander-Reiten sequence with $Y_1, \dots, Y_t$ pairwise non-isomorphic indecomposable modules. Then

$$d_i = \dim_{K_{Y_i}} [\text{Irr}(X, Y_i)] = \dim_{K_{Y_i}} [\text{Irr}(Y_i, Z)]$$

$(1 \leq i \leq t)$.

(2.8) Recalling (2.4) that f is minimal left almost split and g minimal right almost split, (2.7) allows to consider the Auslander-Reiten graph $\Gamma(A)$ of $\text{mod } A$ as a valued graph (II, (2.3)): its vertices are the types of indecomposable A -modules, its arrows $[X] \rightarrow [Y]$, given if $\text{Irr}(X, Y) \neq 0$, are valued with $[X] \xrightarrow{(d, d')} [Y]$, where $d = d_{X, Y}$ is the number of indecomposable direct summands of M isomorphic to Y in a minimal left almost split morphism $f: X \rightarrow M$ and $d' = d'_{X, Y}$ is the number of indecomposable direct summands of N isomorphic to X in a minimal right almost split morphism $g: N \rightarrow Y$. This graph is a translation graph (II, (2.1)) with the translations $\tau[X] := [\tau X]$, $\tau^{-1}[Y] := [\tau^{-1} Y]$, for every non-projective resp. non-injective type $[X]$ resp. $[Y]$, and else not defined, which has no loops (there are no irreducible endomorphisms), is locally finite (the sets $[X]^+$ of all the types $[Y]$ with $\text{Irr}(X, Y) \neq 0$, and $[Y]^-$ of all types $[X]$ with $\text{Irr}(X, Y) \neq 0$ are finite) and $[X]^+ = [\tau^{-1} X]^-$ for every non-injective type $[X]$.

(2.9) To formulate the last result from the Auslander-Reiten theory that we recall here, let $((\text{mod } A)^{\text{op}}, \text{Ab})$ be the category of the additive contravariant functors from $\text{mod } A$ to the category Ab of abelian groups. The representable functors $(-, X) := \text{Hom}_A(-, X)$ with $X \in \text{ind } A$ form a complete set of representatives of the projective indecomposable objects, the functor $\text{rad}(-, X)$ such that $\text{rad}(-, X)(Y) := \text{rad}(Y, X)$ is the unique maximal subfunctor of $(-, X)$, for every $X \in \text{ind } A$, and the functors $S_X := (-, X) / \text{rad}(-, X)$, $X \in \text{ind } A$, form a complete set of representatives of the simple objects of this category.

Proposition. The following conditions are equivalent:

(i) The sequence $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is Auslander-Reiten

(ii) The sequence

$$0 \rightarrow (-, X) \xrightarrow{(-, f)} (-, Y) \xrightarrow{(-, g)} (-, Z) \xrightarrow{\pi_Z} S_Z \rightarrow 0$$

is a minimal projective resolution of S_Z in $((\text{mod } A)^{\text{op}}, \text{Ab})$,

where $\pi_Z: (-, Z) \rightarrow S_Z$ is the canonical projection.

of the Brauer graphs of type B_n or C_n . Throughout, F/G will be a quadratic field extension of a perfect field $K = G$.

(5.1) Let $B = \left(\begin{array}{c|c} A & 0 \\ \hline {}_H M_A & H \end{array} \right)$ be the one-point extension algebra of an algebra A by an A -module M_A with $H \subseteq \text{End}(M_A)$ a division ring (I, (3.5)). The category $\text{mod } B$ can be described as the category $R({}_H M_A)$ of representations of the bimodule ${}_H M_A$, whose objects are the triples (X_H, Y_A, φ) consisting of a right H -vector space X_H , an A -module Y_A and an A -linear map $\varphi: X_H \otimes {}_H M_A \rightarrow Y_A$, and whose morphisms are the pairs $(f, g): (X_H, Y_A, \varphi) \rightarrow (X'_H, Y'_A, \varphi')$ of H -linear maps $f: X_H \rightarrow X'_H$ and A -linear maps $g: Y_A \rightarrow Y'_A$ such that $g\varphi = \varphi'(f \otimes 1_M)$. We will write $B = E(H, M, A)$ to indicate that B is the one-point extension of A by the bimodule ${}_H M_A$, H a division subring of $\text{End}(M_A)$.

(5.2) Let $\Sigma = (K_i, {}_{i,j} M_{i,j})_{i,j \in I}$ be a K -species, $B = T(\Sigma)/J$, J a relation ideal of $T(\Sigma)$. We say that the relations ${}_{i,j} J$ are OPE-relations if $B \cong E(H, M, C)$ for some one-point extension algebra.

To introduce the OPE-relations that we need, we show first that the hereditary tensor algebras $T(\mathfrak{M}, \Omega)$, where $(\mathfrak{M}, \Omega)$ is an oriented modulation of the Dynkin graphs B_n or C_n , may be

(3.3) Let B be a finite dimensional algebra over a field K , and $Q = DB$ its minimal injective cogenerator. The matrix algebra ([22])

$$\hat{B} = \begin{pmatrix} M_{n-1} & B_{n-1} & & 0 \\ & M_n & B_n & \\ & 0 & M_{n+1} & B_{n+1} \\ & & & \ddots \end{pmatrix}$$

where $B_n = B$ and $M_n = Q$, for every $n \in \mathbb{Z}$, of those $\mathbb{Z} \times \mathbb{Z}$ -matrices with only finitely many entries different from zero and multiplication induced by the canonical morphisms $B \otimes_B Q \rightarrow Q$, $Q \otimes_B B \rightarrow Q$ and zero morphisms $Q \otimes_B Q \rightarrow 0$, can be used to describe the trivial extension algebra $B \ltimes Q$: the identity morphisms $B_n \rightarrow B_{n+1}$, $Q_n \rightarrow Q_{n+1}$ induce an automorphism h of $\hat{B}$, and the orbit space $\hat{B}/h$ has an induced structure of K -algebra such that $B \ltimes Q \cong \hat{B}/h$.

The algebras $\hat{B}$ are closely related to the universal coverings of selfinjective algebras of finite representation type, and are instances of the so called locally finite dimensional algebras (3.4).

(3.4) A K -algebra L is said to be locally finite dimensional, if there exists a set $\{u_i \mid i \in I\}$ of orthogonal idempotents u_i of L such that $L = \bigoplus_{i,j \in I} u_i L u_j$ with Lu_i and $u_i L$ finite dimensional for every $i \in I$. For an infinite index set I , clearly L does not possess a unit element, but the u_i can be chosen to be all primitive,

so the L -modules $P(i) = u_i L$ are projective indecomposable. L is said to be basic if $P(i) \sim P(j)$ only if $i = j$, and there exists, to any locally finite dimensional K -algebra L , another basic locally finite dimensional K -algebra L' which is Morita-equivalent to L , that is, their module categories are equivalent.

Let X_L be an L -module of finite dimension over K with $XL = X$. The algebra $L(X) := L/I_X$, where I_X is the ideal of L generated by all those idempotents u of L such that $Xu = 0$, is called the support algebra of X . Clearly, it is a finite dimensional K -algebra. Using these support algebras it is shown that mod L , the category of all those X_L of finite dimension over K such that $XL = X$, has Auslander-Reiten sequences, since the Auslander-Reiten translation of L coincides with the Auslander-Reiten translation of $L(X)$, for every X in mod L . The dimension type $\underline{\dim} X$ of X is defined in a similar way as in the finite dimensional case (1.3) as

$$(\underline{\dim} X)_i = \ell(\text{Hom}_L(P(i), X)_{\text{End } P(i)}),$$

for every $i \in I$.

(3.5) Let A be a finite dimensional K -algebra, M_A an A -module and F a division ring, $F \subseteq \text{End}(M_A)$. The algebra $B := \left(\begin{array}{c|c} A & 0 \\ \hline F^{M_A} & F \end{array} \right)$ is called ([20]) a one-point extension algebra, where, under special choices of M_A , is used to construct preprojective components of $\text{mod } B$ from preprojective components of $\text{mod } A$. We will use also suitable choices of A and B to describe algebras of the form $T(\Sigma)/J$, Σ a K -species, J a relation ideal of $T(\Sigma)$. (Cfr. II.5).

CHAPTER II. MODULATED GRAPHS

Brauer graphs of types B_n and C_n are defined in Section 1 as certain valued graphs with a selected vertex. Our definition of valued graph will allow us to endow the universal coverings of the Brauer graphs of suitable modulations which will be used in the next chapter to construct the categories of representations used to describe the module categories of selfinjective algebras of class B_n or C_n . Modulations of translation graphs are discussed in Section 2. Sections 3, 4 and 5 deal with species of modulated graphs and of translation modulated graphs, relation ideals of the tensor algebra of a species, and OPE-relations, respectively, the latter being relations arising from one-point extension algebras. Tensor categories and their quotient categories by relation ideals, in particular mesh categories, are studied in Section 6, and locally finite dimensional modulated graphs arising from coverings of mesh categories in Section 7.

II.1 Modulated graphs

In this section we introduce the fundamental concepts of valued graphs and modulations of valued graphs, to deal with the representation

theory of selfinjective algebras of finite representation type and Dynkin class B_n ($n \geq 2$), C_n ($n \geq 3$), F_4 and G_2 , and their universal coverings, thus extending in a natural way similar concepts introduced in [12] to include some infinite graphs.

(1.1) Definition. Let D be a non-empty set.

(i) A valuation v on D is a function $v: D \times D \rightarrow \mathbb{N}_0$, $(i,j) \mapsto v_{ij} := v(i,j)$, satisfying the following conditions:

(VG1) $v_{ii} = 0$, for all $i \in D$

(VG2) There exist $f_i \in \mathbb{N}$ such that $v_{ij} f_j = v_{ji} f_i$, for all $i, j \in D$.

(ii) A valued graph (D, v) is a non-empty set D endowed with a valuation v .

(1.2) Remarks.

(i) Let (D, v) be a valued graph. From (VG2) it follows that $v_{ij} \neq 0$ if and only if $v_{ji} \neq 0$, so (D, v) gives rise to a graph $U = U(D, v)$, its underlying graph, in the following way: U has D as set of vertices and edges those pairs $\{i, j\}$ such that $v_{ij} \neq 0$. If $\{i, j\}$ is an edge of $U(D, v)$, we shall write

$\underset{i}{o} \xrightarrow{(v_{ij}, v_{ji})} \underset{j}{o}$ to indicate this fact, and simply $\underset{i}{o} \xrightarrow{\quad} \underset{j}{o}$ if

$v_{ij} = v_{ji} = 1$, and say that $\overset{o}{i} \text{---} \overset{o}{j}$ is a trivially valued edge.

(ii) Let $T = (T_0, T_1)$ be a finite tree, T_0 its set of vertices, T_1 its set of edges. Let $v: T_0 \times T_0 \rightarrow \mathbb{N}_0$ be any function satisfying (VG1) and

$$(VG2') \quad v_{ij} \neq 0 \text{ iff } v_{ji} \neq 0, \text{ for all } i, j \in T_0.$$

Then (T_0, v) is a valued graph with underlying graph $U(T_0, v) = T$.

(1.3) Examples.

(i) The Dynkin graphs

$$A_n \quad \overset{o}{} \text{---} \overset{o}{} \text{---} \dots \text{---} \overset{o}{} \text{---} \overset{o}{} \quad (n \geq 1)$$

$$B_n \quad \overset{(1,2)}{\overset{o}{}} \text{---} \overset{o}{} \text{---} \dots \text{---} \overset{o}{} \text{---} \overset{o}{} \quad (n \geq 2)$$

$$C_n \quad \overset{(2,1)}{\overset{o}{}} \text{---} \overset{o}{} \text{---} \dots \text{---} \overset{o}{} \text{---} \overset{o}{} \quad (n \geq 3)$$

$$D_n \quad \overset{o}{} \text{---} \overset{o}{} \text{---} \dots \text{---} \overset{o}{} \begin{array}{l} \diagup \overset{o}{} \\ \diagdown \overset{o}{} \end{array} \quad (n \geq 4)$$

$$E_6 \quad \begin{array}{c} \overset{o}{} \\ | \\ \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \end{array}$$

$$E_7 \quad \begin{array}{c} \overset{o}{} \\ | \\ \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \end{array}$$

$$E_8 \quad \begin{array}{c} \overset{o}{} \\ | \\ \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{} \end{array}$$

$$F_4 \quad \overset{(1,2)}{\overset{o}{}} \text{---} \overset{o}{} \text{---} \overset{o}{} \text{---} \overset{o}{}$$

$$G_2 \quad \overset{(1,3)}{\overset{o}{}} \text{---} \overset{o}{}$$

where, in the first four families of graphs, there are n vertices in each case.

(ii) The Euclidean graphs

$$\tilde{A}_{11} \quad \begin{array}{c} (1,4) \\ \circ \text{---} \circ \end{array}$$

$$\tilde{A}_{12} \quad \begin{array}{c} (2,2) \\ \circ \text{---} \circ \end{array}$$

$$\tilde{A}_n \quad \begin{array}{c} \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad (n \geq 2)$$

$$\tilde{B}_n \quad \begin{array}{c} (1,2) \quad (2,1) \\ \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad (n \geq 2)$$

$$\tilde{C}_n \quad \begin{array}{c} (2,1) \quad (1,2) \\ \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad (n \geq 2)$$

$$\tilde{B}\tilde{C}_n \quad \begin{array}{c} (1,2) \quad (2,1) \\ \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad (n \geq 2)$$

$$\tilde{C}\tilde{D}_n \quad \begin{array}{c} (2,1) \\ \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad (n \geq 3)$$

$$\tilde{B}\tilde{D}_n \quad \begin{array}{c} (1,2) \\ \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad (n \geq 3)$$

$$\tilde{D}_n \quad \begin{array}{c} \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \end{array} \quad (n \geq 4)$$

$$\tilde{E}_6 \quad \begin{array}{c} \circ \\ | \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \end{array}$$

$$\tilde{E}_7 \quad \begin{array}{c} \circ \\ | \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \end{array}$$

where, in each case of a family, the graph has $n+1$ vertices.

(iii) The infinite graphs

(1.4) Definition. Let (D, v) be a valued graph.

(i) An orientation Ω of (D, v) is given by prescribing, for every edge of the valued graph, an ordering $\Omega \subseteq D \times D$

(11) An oriented valued graph (D, v, Ω) is a valued graph (D, v) endowed with an orientation Ω .

(1.5) Remarks. Let (D, v, Ω) be an oriented valued graph. We indicate the orientation of an edge $i \xrightarrow{(v_{ij}, v_{ji})} j$ by adding an arrow $i \xrightarrow{(v_{ij}, v_{ji})} j$, if $(i, j) \in \Omega$. Then we have the notions of sinks, sources, paths and length of paths: a vertex i is a source if $i \xrightarrow{(v_{ij}, v_{ji})} j$, for every j with $v_{ij} \neq 0$, a sink if $h \xrightarrow{(v_{hi}, v_{ih})} i$, for every h with $v_{ih} \neq 0$; a path is a sequence of vertices $i = i_1, i_2, \dots, i_t = j$ such that $i_r \xrightarrow{(v_{i_r, i_{r+1}}, v_{i_{r+1}, i_r})} i_{r+1}$ ($1 \leq r \leq t-1$), the length of the path is $t-1$, and $i = i_1, j = i_t$ are called the origin and end of the path, respectively. Paths of length one are called arrows, and we identify them with the corresponding valued edges. The graph is connected if for any two vertices i, j there exists a sequence $i = i_1, \dots, i_t = j$ such that $v_{i_s i_{s+1}} \neq 0$ ($1 \leq s \leq t-1$).

(1.6) Remarks.

(1) Let (D, v, Ω) be an oriented valued graph. Then there are neither "loops" $i \xrightarrow{v_{ii}} i$ nor "multiple" edges $i \xrightarrow{v_{ij}} j$ between two vertices.

(ii) In particular, the graphs

are not covered by our definition of oriented valued graphs. These graphs for the values $d = 1$, $d' = 2$ and $d = 1$, $d' = 3$ will appear naturally in the classification of selfinjective algebras of Dynkin class B_2 and of Dynkin class G_2 , respectively.

(Chapters IV and V). We can consider these graphs as oriented valued graphs extending our definition (1.1) to include them, or we can consider them as doubly oriented valued graphs $(D, v, v', \Omega, \Omega')$ with D a vertex set with two elements and valuations v, v' and orientations Ω, Ω' defined in the obvious way. A different approach to the concept of oriented valued graph, sometimes more convenient is discussed next.

(iii) Let $G = (V(G), A(G))$ be a directed graph, $V(G)$ its vertices, $A(G)$ its arrows. An arrow-valuation on G is a function $w: A(G) \rightarrow \mathbb{N} \times \mathbb{N}$, $\alpha \mapsto w(\alpha) = (w_\alpha, w'_\alpha)$; the pair (G, w) is called an (arrow-)valued directed graph. Sometimes valued directed graphs yield valued graphs, with an induced canonical orientation; for instance, if G is a directed finite tree, then any arrow-valuation w endows $V(G)$ of a structure of valued oriented graph $(V(G), v, \Omega)$ such that the underlying graph is G : simply set $v(i, j) = d$,

$v(j,i) := d'$, for every arrow α with origin $o(\alpha) = i$ and
 end $e(\alpha) = j$ with arrow-valuation $w(\alpha) = (d, d')$, and $(i, j) \in \Omega$
 in this case. The graphs of (ii) are thus valued directed graphs.
 By abuse of language, we will refer to them as oriented valued
 graphs.

The next example is of the uttermost importance for the clas-
 sification of selfinjective algebras of finite representation type
 of Dynkin classes B_n and C_n .

(1.7). Example. Brauer graphs. Let Q be a Brauer-quiver
 ([19]), that is, Q is a finite connected trivially valued oriented
 graph such that a) Q is the union of the cycles which are con-
 tained in Q , b) every vertex belongs to exactly two cycles and
 c) any two cycles meet in one vertex at most. We call transit
 vertices those belonging to loops, and nodes the other vertices.
 Consider the oriented graph D obtained from Q by erasing all
 possible loops; thus, only the nodes belong to two cycles. Select
 one transit vertex q_0 and endow D with the arrow-valuation w
 defined by $w(\delta) = (2, 1)$, $w(\epsilon) = (1, 2)$, where $e(\delta) = q_0 = o(\epsilon)$,
 and $w(\gamma) = (1, 1)$, for any other arrow $\gamma \neq \delta, \epsilon$. If Q has n
 vertices, we call the resulting valued graph (D, w, q_0) with the

selected transit vertex q_0 a Brauer graph of type B_n . Similarly, if we valueate (D, q_0) with the valuation w' given by $w'(\delta) = (1, 2)$, $w'(\epsilon) = (2, 1)$, $w'(\gamma) = (1, 1)$, for any other arrow $\gamma \neq \delta, \epsilon$, δ and ϵ as above, the valued graph (D, w', q_0) with the selected transit vertex q_0 is called a Brauer graph of type C_n .

In Figure 1 we give an example of a Brauer graph of type B_4 (Figure 1(a)) and of a Brauer graph of type C_5 (Figure 1(b)) (IV, (1.32)).

(Fig. 1(a))

(Fig. 1(b))

(Fig. 1)

The Brauer graphs of type B_n and C_n and their universal coverings (1.8) endowed with suitable modulations will allow us (Chapter IV) to describe the selfinjective algebras of class B_n and C_n , respectively. Similarly (Chapter V), the following valued graphs F_i ($1 \leq i \leq 6$) and G_1 will be used to describe the selfinjective algebras of class F_4 and C_2 , respectively, and will be called, accordingly, Brauer graphs of type F_4 , the F_i , and the Brauer graph of type C_2 , the valued graph G_1 .

F_1 F_2 F_3 F_4 F_5 F_6 G_1

(Fig. 2)

(1.8) Let (D, w, q_0) be a Brauer graph of type B_n . We can assume $D_0 = \mathbb{Z}/n\mathbb{Z}$, $q_0 = \bar{0}$. The valued graph $(\tilde{D}, \tilde{w}, \{i | i \equiv 0 \pmod{n}\})$ with $\mathbb{Z}$ as set of vertices, arrows $i \rightarrow j$ whenever there is an arrow $\bar{i} \rightarrow \bar{j}$ in D and valuation $\tilde{w}(\delta_1) = (2, 1)$, $\tilde{w}(\epsilon_1) = (1, 2)$ if $e(\delta_1) = 1 = o(\epsilon_1)$, for every $i \equiv 0 \pmod{n}$, $w(\gamma) = (1, 1)$ for every other arrow γ , is called the universal covering of the Brauer graph (D, w, q_0) . Similarly, if (D, w, q_0) is a Brauer graph of type C_n we construct its universal covering $(\tilde{D}, \tilde{w}, \{i | i \equiv 0 \pmod{n}\})$. By analogous procedures we define the universal coverings $\tilde{F}_i$ ($1 \leq i \leq b$) of type F_4 (V, (2.2)) and the universal covering $\tilde{G}_1$ of the Brauer graph G_1 (V, (4.5)).

Let (D, v) be again a valued graph. We introduce now the concept of modulation of (D, v) , following [10] and [12].

(1.9) Definition. Let (D, v) be a valued graph. A K -modulation $\mathfrak{M} = (K; F_i; {}^M_{ij})_{i,j \in D}$ of (D, v) is given by the following data:

- (i) K is a field
- (ii) F_i is a division ring of finite dimension over K and $K \subseteq Z(F_i)$, for every $i \in D$

(iii) ${}_i M_j$ is an $F_i - F_j$ -bimodule such that

(a) K acts centrally on ${}_i M_j$ ($\lambda m = m\lambda$, for every

$\lambda \in K$ and $m \in {}_i M_j$)

(b) $v_{ij} = \dim({}_i M_j)_{F_j}$

(c) $v_{ji} = \dim_{F_i}({}_i M_j)$

(d) ${}_j M_i \cong {}^* M_{ij} \cong {}_i M_j^* \cong D {}_i M_j$, for every $i, j \in D$.

Here, $Z(F)$ denotes the center of a division ring F , and

${}_i M_j^* := \text{Hom}_{F_i}({}_i M_j, F_i)$, ${}_i M_j^* := \text{Hom}_{F_j}({}_i M_j, F_j)$, $D {}_i M_j := \text{Hom}_K({}_i M_j, K)$ are the left dual, right dual and dual of ${}_i M_j$, respectively.

(1.10) Remark. Let Ω be an orientation of a valued graph (D, v) . Let K be a field and $\{F_i | i \in D\}$ a collection of division rings satisfying 1.9, (ii). Let further $\{{}_i M_j | (i, j) \in \Omega\}$ be a collection of $F_i - F_j$ -bimodules satisfying 1.9, (iii), (a), (b), (c). Define ${}_j M_i := {}^* M_{ij}$ for every $(i, j) \in \Omega$. Then $(K, F_i, {}_i M_j)_{i, j \in D}$ is a K -modulation of (D, v) . For, under these hypothesis on the ${}_i M_j$, $(i, j) \in \Omega$, it follows that (1.9), (i), (d) holds ([10]).

(1.11) Example. As a very simple example of a K -modulation of a valued graph, let $D = \{1, 2\}$, $v: D \times D \rightarrow \mathbb{N}_0$, $v_{11} = v_{22} = 0$, $v_{12} = 1$, $v_{21} = 2$, $F_1 = G$, $F_2 = F$, ${}_1 M_2 = {}_G F$, ${}_2 M_1 = {}_F G$, where F/G is a quadratic field extension. Then

- (i) $m = (G; G, F; {}_G^F F, F {}_G^F)$ is a G -modulation of (D, v)
- (ii) Let $\{(1, 2)\} = \Omega$ be an orientation of (D, v) . Let again $F_1 = G$, $F_2 = F$, ${}_1 M_2 = {}_G^F F$. Then $(G; G, F; {}_G^F F, {}_F^G G)$ is a G -modulation of (D, v) . In fact, it is m , for, ${}_F^G G \xrightarrow{\sim} {}_G^F F$.

(1.12) Remarks.

- (i) With respect to the previous example, (and any other of the same sort), it is convenient to use the following notation to describe the valued graph, orientation and modulation:

$$(D, v): \begin{array}{c} \textcircled{1, 2} \\ \text{---} \\ 1 \quad 2 \end{array}$$

$$\Omega: 1 \rightarrow 2$$

$$(m, \Omega): G \xrightarrow{{}_G^F F} F.$$

Note that in (m, Ω) we write only the bimodule ${}_1 M_2$. The bimodule ${}_2 M_1$ is tacitly described, according to (1.10).

- (ii) Let m, Ω be a modulation and an orientation, respectively, of a valued graph (D, v) . We simplify terminology and notation by calling $m, (m, \Omega)$ a modulated graph and an oriented modulated graph, respectively.

(1.13) Let (m, Ω) be an oriented modulated graph. The category $R(m, \Omega)$ of all (finite dimensional) representations is

the abelian category with objects the representations $\underline{X} = (X_i, {}_j\varphi_i)$ consisting of finite dimensional right F_i -vector spaces X_i for every $i \in D$ and F_j -linear maps ${}_j\varphi_i: X_i \otimes_{F_i} {}_iM_j \rightarrow X_j$ for every arrow $i \rightarrow j$, and morphisms $\underline{f} = (f_i): \underline{X} = (X_i, {}_j\varphi_i) \rightarrow \underline{Y} = (Y_i, {}_j\psi_i)$ families of F_i -linear maps $f_i: X_i \rightarrow Y_i$ such that $f_i {}_i\varphi_j = {}_i\psi_j (f_j \otimes 1_{{}_jM_i})$, for every arrow $j \rightarrow i$. It is well known ([10])

that $R(\mathfrak{M}, \Omega)$ is equivalent to the category $\text{mod}^* T(\mathfrak{M}, \Omega)$, where

$T(\mathfrak{M}, \Omega)$ is the tensor algebra of the oriented modulation:

$T(\mathfrak{M}, \Omega) := \bigoplus_{t \geq 0} M^{(t)}$ as abelian group, where $M^{(0)} := \prod_{i \in D_0} F_i$,

$M^{(1)} := \bigoplus_{i \rightarrow j} {}_iM_j$ and $M^{(t+1)} := M^{(t)} \otimes_{M^{(0)}} M^{(1)}$, with $M^{(1)}$ the

$M^{(0)}$ -module induced by the projections $M^{(0)} \rightarrow F_i$, and the

multiplication in $T(\mathfrak{M}, \Omega)$ is defined by the tensor products and distributivity.

II.2 Translation modulated graphs

We put a suitable modulation on translation graphs that allows to relate, in a convenient way, the Auslander-Reiten graphs $\Gamma(A)$, $\Gamma_S(A)$ and the categories $\text{ind } A$, $\underline{\text{ind}} A$ of an algebra A .

(2.1) Translation graphs. Introduced by Riedtmann ([23])

for trivially valued graphs, they soon became the Riedtmann-quivers,

with or without valuation ([21]). We follow this last approach (see (1.6, iii)). Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a directed graph, Γ_0 its set of vertices, Γ_1 its set of arrows; denote by x^+ (resp. x^-) the set of ends (resp. origins) of arrows starting (resp. ending) at x . Γ is said to be locally finite if x^+ and x^- are finite for every $x \in \Gamma_0$. A pair (Γ, τ) consisting of a locally finite directed graph and an injective function $\tau: \Gamma'_0 \rightarrow \Gamma_0$ where $\Gamma'_0 \subseteq \Gamma_0$, is called a translation graph if $(\tau x)^+ = x^-$ for every $x \in \Gamma'_0$. We call $P_0 := \Gamma_0 \setminus \Gamma'_0$ the set of projective vertices of (Γ, τ) , τ is called the translation, and (Γ, τ) is stable if $P_0 = \emptyset$. A non-projective vertex x is periodic if it has a finite closed orbit under τ , i.e., $\tau^t x = x$, for some $t \in \mathbb{N}$. Any arrow $x \xrightarrow{\alpha} y$ with non-projective end is paired with the unique arrow $\tau y \xrightarrow{\sigma\alpha} x$; sometimes, σ will be called the σ -translation of (Γ, τ) . If, moreover, w is an arrow-valuation of Γ , we say that (Γ, τ, w) is a valued translation graph if $w_{\sigma\alpha} = w'_\alpha$, $w'_\alpha = w_\alpha$, for every arrow α with non-projective end.

(2.2) Examples. Let T be a valued directed tree. The translation graph $\mathbb{Z}T = ((\mathbb{Z}T)_0, (\mathbb{Z}T)_1, \tau)$ is defined by $(\mathbb{Z}T)_0 := \mathbb{Z} \times T_0$, $(\mathbb{Z}T)_1 := \{(n, \alpha) \mid n \in \mathbb{Z}, \alpha \in T_1\}$, where (n, α) (resp. $\sigma(n, \alpha)$) is an arrow with origin (n, s) (resp. $(n-1, t)$)

and end (n,t) (resp. (n,s)), for every arrow $\alpha \in T_1$ with origin s and end t , and translation $\tau(n,t) := (n-1,t)$, for every $t \in T_0$. If, moreover, (T,w) is an arrow-valued directed tree, in particular a valued graph if T is finite (1.2, ii), then the valuation w can be extended to $\mathbb{Z} T$ by $w_{(n,\alpha)} := w_\alpha = : w'_{\sigma(n,\alpha)}$ and $w'_{(n,\alpha)} := \bar{w}'_\alpha = : w_{\sigma(n,\alpha)}$, for every $\alpha \in T_1$. Thus, $(\mathbb{Z} T, w)$ becomes a valued translation graph. In the sequel, we will always use the above labelling of $\mathbb{Z} T$, and denote again by $\mathbb{Z} T$ the graph $(\mathbb{Z} T, w)$. We are interested, in particular, in the valued translation graphs $\mathbb{Z} B_n$ ($n \geq 2$), $\mathbb{Z} C_n$ ($n \geq 3$), $\mathbb{Z} F_4$, $\mathbb{Z} G_2$.

(2.3) Example. The Auslander-Reiten graph of an algebra.

Let A be a finite dimensional algebra over a field K . The Auslander-Reiten graph $\Gamma(A)$ of the category $\text{mod } A$ is the valued translation graph with vertices the types of indecomposable A -modules, arrows $x \rightarrow y$ for every pair of types $x = [X]$, $y = [Y]$ such that $\text{Irr}(X,Y) \neq 0$, translation $\tau[X] := [\tau X]$, and valuation $w(\alpha) = (d,d')$ where $d(d')$ is the number of direct summands isomorphic to $Y(X)$ in a minimal left (right) almost split morphism with domain X (codomain Y), for every arrow α with origin x and end y . The full subgraph $\Gamma_s(A)$ of $\Gamma(A)$ with vertices the stable types is called the stable Auslander-Reiten graph of the category $\text{mod } A$.

$\mathbb{Z}B_n$ $\mathbb{Z}C_n$ $\mathbb{Z}F_4$ $\mathbb{Z}G_2$

Let L be a locally finite dimensional algebra over K . By (I, (3.4)), we can define the Auslander-Reiten graphs $\Gamma(L)$ and $\Gamma_s(L)$ exactly as above.

The graphs $\Gamma(A)$ and $\Gamma_s(A)$ are also called the Auslander-Reiten graph and the stable Auslander-Reiten graphs of the algebra A . Both have natural modulations (2.8).

(2.4) To endow a translation graph of a modulation, we recall first some concepts from [15] on dualization. Let ${}^M_F G$ be an F - G -bimodule with F and G division rings and let $\varphi: {}^M_F G \otimes_G X \rightarrow {}^Y_F$ be an F -linear map with ${}_G X$ and ${}_F Y$ vector spaces. The adjoint $\bar{\varphi}: {}_G X \rightarrow {}^M_F G \otimes_F {}^Y_F$ of φ is the G -linear map such that $\bar{\varphi}(x) = \sum_{i=1}^n \omega_i \otimes \varphi(e_i \otimes x)$, where $\{e_1, \dots, e_n\}$ is a basis of ${}_F M$ and $\{\omega_1, \dots, \omega_n\}$ is the dual basis of $({}^M_F G)_F$, for every $x \in X$.

In particular, we will consider also the adjoint $\bar{\chi}_M$ of the evaluation map $\chi_M: M \otimes X \rightarrow Y$, $f \otimes x \mapsto f(x)$, where M is an $\text{End } Y\text{-End } X$ -subbimodule of the bimodule $\text{Hom}(X, Y)$, for X and Y modules over an algebra.

Let ${}^M_F G$ and ${}_G N_F$ bimodules over division rings F and G such that ${}_F M$ and ${}_F N$ are finite dimensional vector spaces.

Any non-degenerate bilinear form $\epsilon: M \otimes_G N \rightarrow F$ establishes thus an isomorphism $\bar{\epsilon}: N \rightarrow {}^M_F$. Let $\{y_1, \dots, y_n\}$ be a basis of

N_F and $\{\omega_1, \dots, \omega_n\}$ the basis of $(^*M)_F$ such that $\omega_i = \bar{\epsilon}(y_i)$ ($1 \leq i \leq n$); let further $\{e_1, \dots, e_n\}$ be the dual basis of ${}_F M$ with respect to $\{\omega_1, \dots, \omega_n\}$. Then the element $c_\epsilon := \sum_{i=1}^n y_i \otimes e_i \in N \otimes_F M$ is called the canonical element of ϵ , and it is independent of the choice of the basis. In particular, the canonical element c_χ of the evaluation map $\chi: M \otimes_G {}^*M \rightarrow F$, $\chi(m \otimes \omega) := (m)\omega$, is called the canonical element of the bi-module ${}_F M_G$, and will be denoted by $c(M)$.

(2.5) Example: The canonical element $c({}_R C_C)$. Let $\{\omega_1, \omega_i\}$ be the basis of $(^*C)_R = \text{Hom}_R({}_R C, {}_R R)$ dual of the basis $\{1, i\}$ of ${}_R C$. Then, by definition, $c({}_R C_C) = \omega_1 \otimes 1 + \omega_i \otimes i \in {}^*C_C \otimes_R C_C$. We want to relate this element to the multiplication map

$$\mu: {}^C C_R \otimes_R {}^C C_C \rightarrow {}^C C_C, \quad x \otimes y \mapsto xy.$$

Lemma. Let $s := \frac{1}{2}(1 \otimes 1 - i \otimes i)$, $t := \frac{1}{2}(1 \otimes 1 + i \otimes i)$.

(i) μ is a $C-C$ -bimodule epimorphism with $\text{Ker } \mu = Ct$;

$$\text{moreover, } {}^C C_R \otimes_R {}^C C_C = Cs \oplus Ct$$

(ii) The map $f: {}^C C_R \otimes_R {}^C C_C \rightarrow {}^*C_C \otimes_R {}^C C_C$,

$$x \otimes y \mapsto f(x \otimes y) := \downarrow_x \otimes y, \quad \text{where}$$

$$\downarrow_x: {}^C C_C \rightarrow {}_R C, \quad \lambda \mapsto \downarrow_x(\lambda) = 2x_1 \lambda_1 - 2x_2 \lambda_2, \quad \text{if}$$

$$x = x_1 + x_2 i, \quad \lambda = \lambda_1 + \lambda_2 i, \quad x_i, \lambda_i \in R \quad (1 \leq i \leq 2),$$

is a $C-C$ -bimodule isomorphism; moreover, $f(s) = c({}_R C_C)$.

Corollary. $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) / \text{Ker } \mu \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}.$

Proof.

(i) Direct checking.

(ii) First, $\psi_x \in {}^*\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ for every $x \in \mathbb{C}$, for,

$$\begin{aligned} \psi_x(\lambda + \lambda') &= 2x_1(\lambda_1 + \lambda'_1) + 2x_2(\lambda_2 + \lambda'_2) = (2x_1\lambda_1 - 2x_2\lambda_2) + (2x_1\lambda'_1 - 2x_2\lambda'_2) = \\ &= \psi_x(\lambda) + \psi_x(\lambda') \quad \text{and} \quad \psi_x(r\lambda) = 2x_1(r\lambda_1) - 2x_2(r\lambda_2) = r(2x_1\lambda_1 - 2x_2\lambda_2) \\ &= r\psi_x(\lambda). \end{aligned}$$

Second, $\psi_{\alpha x} = \alpha \psi_x$, for, $\psi_{\alpha x}(\lambda) = 2(\alpha_1 x_1 - \alpha_2 x_2)\lambda_1 - 2(\alpha_1 x_2 + \alpha_2 x_1)\lambda_2$, and $(\alpha \psi_x)(\lambda) = \psi_x(\lambda \alpha) = 2x_1(\lambda_1 \alpha_1 - \lambda_2 \alpha_2) - 2x_2(\lambda_1 \alpha_2 + \lambda_2 \alpha_1)$, so $\psi_{\alpha x}(\lambda) = (\alpha \psi_x)(\lambda)$, for every λ . From here, $f(\alpha x \otimes y) = \psi_{\alpha x} \otimes y = (\alpha \psi_x) \otimes y = \alpha(\psi_x \otimes y)$. The linearity on the right is straightforward.

Finally, $f(s) = f(\frac{1}{2}(1 \otimes 1 - i \otimes i)) = \frac{1}{2}[\psi_1 \otimes 1 - \psi_i \otimes i] = \frac{1}{2}[2\omega_1 \otimes 1 + 2\omega_i \otimes i] = \omega_1 \otimes 1 + \omega_i \otimes i = c(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$, for, $\psi_1(1) = 2$, $\psi_1(i) = 0$, $\psi_i(1) = 0$, $\psi_i(i) = -2$, so $\psi_1 = 2\omega_1$, $\psi_i = -2\omega_i$. Hence, also the corollary.

(2.6) Example. The canonical element $c(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$. Let $\{\omega\}$ be the basis of ${}^*\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \text{Hom}_{\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$ dual of the basis $\{i\}$ of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. Then, by definition, $c(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) = \omega \otimes i \in {}^*\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. We want to relate this element to bimodule epimorphisms

$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$. Let $\pi: \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \mathbb{R} \otimes_{\mathbb{R}} \mathbb{R}$, $x \otimes y = (x_1 + ix_2) \otimes (y_1 + iy_2) \mapsto x_1 y_1 + x_2 y_2$.

Lemma.

(1) π is an R - R -bimodule epimorphism.

(11) The map $f: \frac{C}{R} \otimes_C \frac{C}{R} \rightarrow \frac{*C}{C} \otimes_C \frac{C}{R}$, $x \otimes y \mapsto \psi_x \otimes y$,

where $\psi_x: \frac{C}{R} \rightarrow C$, $\lambda \mapsto \psi_x(\lambda) = x\lambda$, is an R - R -bimodule isomorphism; moreover, $f(1 \otimes 1) = w \otimes 1$.

Corollary. $(\frac{C}{R} \otimes_C \frac{C}{R}) / \text{Ker } \pi \cong ((\frac{*C}{C}) \otimes_C \frac{C}{R}) / c(\frac{C}{R})$.

Proof. Similar to the previous one. Note in particular,

that $\text{Ker } \pi = \{\alpha \otimes \beta \mid \text{Im } (\alpha\beta) = 0\} = \{r(1 \otimes 1) \mid r \in R\}$.

(2.7) Translation modulated graphs. Let Γ be a translation graph, τ its translation. We denote by $o(\alpha)$, $e(\alpha)$, the origin, end of any $\alpha \in \Gamma_1$ and $\tau^{-1}x = y$ denotes that $\tau y = x$, if $y \in \Gamma'_0$; we say then that x is a non-injective vertex.

Definition. A translation modulated graph (Γ, m) is a translation graph Γ endowed with a modulation $m = (\frac{K}{x}, \frac{B}{x y x, y \in \Gamma_0})$ that satisfies the following conditions ([31]):

(TMG1) There exist isomorphisms $K_x \xrightarrow{\sim} K_{\tau x}$, for every $x \in \Gamma'_0$.

(TMG2) There exist non-degenerate bilinear forms

$$\epsilon_\alpha: B(o(\alpha)) \otimes_{K_{e(\alpha)}} B(\alpha) \rightarrow_{K_{o(\alpha)}} (K_{o(\alpha)})_{K_{o(\alpha)}}, \text{ for every } \alpha \in \Gamma_1$$

with $e(\alpha) \in \Gamma'_0$. Here, $B(\alpha); = e(\alpha)B_o(\alpha)$.

(2.8) Example. Let $\Gamma(A)$ be the Auslander-Reiten graph of an algebra A over a field K . This valued translation graph (2.3) will be endowed of the following canonical modulation: define

$K_x := \text{End } X_A / \text{rad } \text{End } X_A$, $B_{y,x} := \text{Irr}(X,Y)$ for every pair of vertices $x = [X]$, $y = [Y]$ of $\Gamma(A)$. Then $(\Gamma(A), m)$,

$m = (K_x, B_{y,x})_{x,y \in \Gamma(A)_0}$ is a translation modulated graph, for the properties of the Auslander-Reiten sequences ensure that (TMG1) and (TMG2) are satisfied. Since this m will be the only modulation that we need, we will denote again by $\Gamma(A)$ the modulated graph just defined.

(2.9) The structure of stable translation graphs, determined in [23], for trivially valued translation graphs, and established in full generality in [21], can be formulated as follows:

Theorem. Let Γ be a stable translation graph. Then, there exists a valued directed tree (T,w) and an admissible group of automorphisms W of the stable translation graph $\mathbb{Z}\mathbb{Z}T$, such that $\Gamma \cong \mathbb{Z}\mathbb{Z}T/W$. Moreover, the underlying tree $\bar{T}$ of T is uniquely determined, and W is determined up to conjugation in $\text{Aut } \mathbb{Z}\mathbb{Z}T$.

A group W of automorphisms of $\mathbb{Z}\mathbb{Z}T$ is said to be admissible if any orbit of W meets the subsets $\{x\} \cup x^+$ and $\{x\} \cup x^-$ in at most one point.

The Cartan class of Γ is defined to be the Cartan matrix C of $T: C: T_0 \times T_0 \rightarrow \mathbb{Z}$, $(t, u) \mapsto C_{t, u}$ equals to 2, -d, -d', 0 if $t = u$, $0(\alpha) = t$ and $e(\alpha) = u$, $0(\alpha) = u$ and $e(\alpha) = t$, t and u unrelated, respectively, where $w(\alpha) = (d, d')$, $\alpha \in \Gamma_1$. If moreover, the underlying valued tree $\bar{T}$ is a Dynkin graph Δ , Γ is said to be of Dynkin class Δ .

II.3 Species

(3.1) Let K be a field. A K -species $\Sigma = (K_i, {}_iM_j)_{i, j \in I}$ is given by a finite set $\{K_i \mid i \in I\}$ of division rings such that $K \subseteq Z(K_i)$ and K_i is of finite dimension over K , for every $i \in I$, and a finite set $\{{}_iM_j \mid i, j \in I\}$ of K_i - K_j -bimodules of finite dimension over K and such that K acts centrally on every ${}_iM_j$ ($\lambda m = m\lambda$, for every $\lambda \in K$, $m \in {}_iM_j$) ([16]).

Given a species $\Sigma = (K_i, {}_iM_j)_{i, j \in I}$, its tensor algebra $T(\Sigma) := \bigoplus_{t \geq 0} M^{(t)}$ is the K -algebra obtained by imposing on the abelian group $\bigoplus_{t \geq 0} M^{(t)}$, $M^{(0)} := \prod_{i \in I} K_i$, $M^{(1)} := \bigoplus_{i, j \in I} {}_iM_j$, $M^{(t+1)} := M^{(t)} \otimes M^{(1)}$, where $M^{(1)}$ is considered as an $M^{(0)} \text{--} M^{(0)}$ -bimodule via the projections $M^{(0)} \rightarrow K_i$ ($i \in I$), the multiplication induced by the canonical isomorphisms $M^{(t)} \otimes M^{(u)} \xrightarrow{\sim} M^{(t+u)}$ and the distributivity of the tensor products.

(3.2) Let A be a finite dimensional algebra over a field K . The species of A , $\Sigma(A)$, is defined in the following way: let B be a basic algebra Morita equivalent to A . Then $B / \text{rad } B = \prod_{i \in I} K_i$, $\text{rad } B / (\text{rad } B)^2 = \bigoplus_{i,j \in I} M_{ij}$, where each M_{ij} is an K_i - K_j -bimodule, finite dimensional on both sides. Then, by definition, $\Sigma(A) := (K_i, M_{ij})_{i,j \in I}$.

(3.3) The notions of species of an algebra and of modulation $m = (F_i, M_{ij})_{i,j \in D_0}$ of a valued graph, although clearly related, are essentially different. It is of course possible to associate several species to m : let Ω be an orientation of D . Then $\Sigma(m, \Omega) := (F_i, M_{ij} \mid (i,j) \in \Omega)_{i,j \in D_0}$ is a species, and $(F_i, M_{ij} \mid (j,i) \in \Omega)_{i,j \in D_0}$ is a species, as well. Sometimes it is possible to relate these species.

(3.4) Let $\Sigma = (K_i, M_{ij})_{i,j \in I}$ be a K -species. The category $R(\Sigma)$ of the finite dimensional representations of Σ has as objects the families $\underline{V} = (V_i, j\varphi_i)$ of finite dimensional right K_i -vector spaces V_i , $i \in I$, and K_j -linear maps $j\varphi_i: V_i \otimes_{K_i} M_{ij} \rightarrow V_j$, for every $M_{ij} \neq 0$, and morphisms $\underline{f} = (f_i): \underline{V} \rightarrow \underline{W}$ between representations $\underline{V} = (V_i, j\varphi_i)$, $\underline{W} = (W_i, j\psi_i)$ the families of K_i -linear maps $f_i: V_i \rightarrow W_i$, $i \in I$, such that $f_j j\varphi_i = j\psi_i (f_i \otimes 1_{M_{ij}})$.

for every ${}_i M_j \neq 0$. The categories $R(\Sigma)$ and $\text{mod } T(\Sigma)$ are equivalent ([11]), an equivalence being given by $\underline{V} = (V_i, {}_j \varphi_i) \mapsto V := \bigoplus V_i$, as abelian group, endowed of the $T(\Sigma)$ -module structure obtained by setting $x_i \cdot {}_i m_j := {}_j \varphi_i(x_i \otimes {}_i m_j)$ and extending by distributivity, and inductively $x_i({}_i m_j \otimes \dots \otimes {}_r m_s \otimes {}_s m_t) := {}_t \varphi_s([x_i({}_i m_j \otimes \dots \otimes {}_r m_s)] \otimes {}_s m_t)$, and $\underline{f} \mapsto f: V \rightarrow W$, $f((v_i)_i) = (f_i v_i)_i$, for every morphism $\underline{f}: \underline{V} \rightarrow \underline{W}$.

(3.5) The category $T(\Sigma)\text{-mod}$ of finitely generated left modules over $T(\Sigma)$, can also be described as a category of representations of Σ : define $\mathcal{I}(\Sigma)$ as the category with objects the families $({}_j X, {}_j \lambda_i)$ of finite dimensional left K_j -vector spaces ${}_j X$, for every $j \in I$, and K_i -linear maps ${}_j \lambda_i: {}_i M_j \otimes {}_j X \rightarrow {}_i X$, for every ${}_i M_j \neq 0$, and morphisms $\underline{f} = ({}_j f): \underline{X} \rightarrow \underline{Y}$ between representations $\underline{X} = ({}_j X, {}_j \lambda_i)$, $\underline{Y} = ({}_j Y, {}_j \mu_i)$ the families ${}_j f: {}_j X \rightarrow {}_j Y$ of K_j -linear maps such that ${}_j \lambda_i \circ {}_i f = (1_{{}_i M_j} \otimes {}_j f) \circ {}_i \mu_i$, for every ${}_i M_j \neq 0$. Through the functor $\underline{X} = ({}_j X, {}_j \lambda_i) \mapsto X \oplus \bigoplus {}_j X$, where the abelian group X is endowed of a structure of a left $T(\Sigma)$ -module by setting ${}_i m_j \cdot {}_j x := ({}_i m_j \otimes {}_j x) \circ {}_j \lambda_i$, and extending inductively as usual, using the canonical isomorphisms of the tensor products and distributivity, one obtains, similarly to the equivalence $R(\Sigma) \simeq \text{mod } T(\Sigma)$ of (3.4), an equivalence $\mathcal{I}(\Sigma) \simeq T(\Sigma)\text{-mod}$. As a consequence $\mathcal{I}(\Sigma) \simeq \{R(\Sigma)\}^{\text{op}}$.

(3.6) The category $\mathcal{L}(\Sigma)$ introduced in (3.5) can also be described as a category of representations $\mathcal{R}(\Delta)$, with Δ another K -species. For, given $\Sigma = (K_i, {}^iM_j)_{i,j \in I}$, define $\Sigma^{\text{op}} := (K_j^{\text{op}}, {}^{K_j^{\text{op}}}({}^iM_j)_{K_i^{\text{op}}})_{i,j \in I}$. We claim that $\mathcal{L}(\Sigma) \simeq \mathcal{R}(\Sigma^{\text{op}})$. To see this, it is enough to prove the following

Lemma. $\mathcal{L}(\Sigma^{\text{op}}) \simeq \mathcal{R}(\Sigma)$.

For, clearly $(\Sigma^{\text{op}})^{\text{op}} = \Sigma$, hence $\mathcal{L}(\Sigma) = \mathcal{L}((\Sigma^{\text{op}})^{\text{op}}) \simeq \mathcal{R}(\Sigma^{\text{op}})$, by the lemma.

Proof of the Lemma. Let $\underline{V} = ({}^{K_j^{\text{op}}}V, {}_j\lambda_i)$ be a representation of $\mathcal{L}(\Sigma^{\text{op}})$. By definition, then, ${}_j\lambda_i: {}^{K_j^{\text{op}}}({}^iM_j)_{K_i^{\text{op}}} \otimes_{{}^{K_i^{\text{op}}}K_i^{\text{op}}} {}^{K_i^{\text{op}}}V \rightarrow {}^{K_i^{\text{op}}}V$ is a K_j^{op} -linear map. But ${}^{K_j^{\text{op}}}V = V_{K_j}$, so it suffices to show that the right K_j -modules ${}^{K_j^{\text{op}}}({}^iM_j)_{K_i^{\text{op}}} \otimes_{{}^{K_i^{\text{op}}}K_i^{\text{op}}} {}^{K_i^{\text{op}}}V$ and $V_{K_i} \otimes_{{}^{K_i}K_i} {}^iM_j$ are isomorphic, for, then, $\underline{V}$ can be seen as a representation $\underline{V}'$ of $\mathcal{R}(\Sigma)$, and the correspondence $\underline{V} \mapsto \underline{V}'$, $f \mapsto \underline{f}$, yields the desired equivalence. So, we finally need to prove the following claim:

Claim. Let X_R be a right R -module and R_M_S an R - S -bimodule over division rings R and S . Then, the right S -modules ${}^{S^{\text{op}}}({}^R_M_S)_{R^{\text{op}}} \otimes_{{}^{R^{\text{op}}}R^{\text{op}}} X_{R^{\text{op}}}$ and $X_R \otimes_{{}^R_R} {}^R_M_S$ are isomorphic.

But clearly $h: {}^{S^{\text{op}}}({}^R_M_S)_{R^{\text{op}}} \otimes_{{}^{R^{\text{op}}}R^{\text{op}}} X_{R^{\text{op}}} \rightarrow X_R \otimes_{{}^R_R} {}^R_M_S$, $(m, x) \mapsto x \otimes m$, is an R^{op} -balanced

map, thus induces a map $\text{Sop}_R(M) \otimes_{\text{Rop}} \text{Rop}^X \rightarrow X_R \otimes_R M_S$, which is easily seen to be S -linear, and provides the isomorphism of the claim.

(3.7) Let $\Sigma = (K_i, M_j)_{i,j \in I}$ be a K -species. The valued directed graph $D(\Sigma) = (D_0, D_1, w)$ defined by $D_0 = I$, $\alpha \in D_1$ if $o(\alpha)M_{e(\alpha)} \neq 0$, $w(\alpha) := (v_{ij}, v'_{ij})$, where $v_{ij} = \dim(M_j)_{F_j}$, $v'_{ij} = \dim(M_j)_{F_i}$, for every $\alpha \in D_1$ with $i = o(\alpha)$, $j = e(\alpha)$, is called the graph of Σ .

(3.8) Definition. Let $\Gamma = (\Gamma_0, \Gamma_1, \tau, m)$, $m = (K_x, B_{xy})_{x,y \in \Gamma_0}$ be a translation modulated graph. The K -species $\Sigma(\Gamma) := (K_y, B_{yx})_{x,y \in \Gamma_0}$, where $B(\alpha) = B_{yx}$ runs through all $\alpha \in \Gamma_1$ with $o(\alpha) = x$, $e(\alpha) = y$ will be called the species of Γ , and the tensor algebra $T(\Gamma) := T(\Sigma(\Gamma))$ will be called the tensor algebra of Γ .

II.4 Relations on species

Selfinjective algebras of finite representation type can be described (Chapters IV and V) as factor algebras $T(\Sigma)/J$ of the tensor algebra $T(\Sigma)$ of a species Σ by a suitable ideal J , and its modules can also be described as certain representations of the category $R(\Sigma)$ whose maps satisfy certain relations imposed by J .

We do not attempt here to develop a general theory of these "binding" ideals on a given species, and although some general concepts are discussed, we have to keep in mind the application to the description mentioned above, especially by our introduction of the OPE-relations (II.5).

(4.1) Let $\Sigma = (K_i, M_{ij})_{i,j \in I}$ be a K -species, $A = T(\Sigma)$ its tensor algebra, $D = D(\Sigma) = (D_0, D_1, w)$ its ~~graph~~ (3.7). Let $W(i,j)$, $i,j \in D_0$, be the set of all paths from i to j in D ; for every $p \in W(i,j)$ let ${}_i M_j(p) = {}_i M_{i_1 i_2} \otimes {}_{i_2} M_{i_2 i_3} \otimes \dots \otimes {}_{i_s} M_{i_s j}$, if $p = (i = i_1, i_2, \dots, i_s, i_{s+1} = j)$. We call ${}_i M_j(p)$ the bimodule of the path p , and we let ${}_i \tilde{M}_j = \bigoplus_p {}_i M_j(p)$, p running in $W(i,j)$. Clearly, ${}_i \tilde{M}_j = K_i A K_j$, for every pair of vertices i,j of D .

Let J be an ideal of A with $J \subseteq (\text{rad } A)^2$. The factor algebra $B = A/J$ is said to be given by Σ and the relation ideal J . The decomposition $A = \bigoplus_{i,j} K_i A K_j$ gives rise to a similar decomposition $J = \bigoplus_{i,j} K_i J K_j$ and each relation $K_i J K_j = : {}_i J_j$ is a subbimodule of ${}_i \tilde{M}_j$. Clearly, to determine J it is enough to give its components ${}_i J_j$, thus it suffices to give the canonical projections ${}_i \pi_j$ from ${}_i \tilde{M}_j$ to ${}_i \tilde{M}_j / {}_i J_j$, and even, to give those ${}_i J_j$ which are of minimal length, ${}_i J_j$ being of length t

if ${}_i J_j \subseteq (\text{rad } A)^t \setminus (\text{rad } A)^{t+1}$ and minimal if the length of any path from i to j is at least t .

Let $J = {}_i J_j$ be a relation. If $W' \subseteq W(i, j)$, J is said to be a zero relation if $J = \bigoplus_{p \in W'} {}_i M_j(p)$, a strict zero relation if $W' = W(i, j)$, a commutativity relation if $W(i, j) = \{p, q\}$, N and N' are isomorphic subbimodules of ${}_i M_j(p)$ and ${}_i M_j(q)$ respectively, and J is the diagonal subbimodule of $N \oplus N'$, and a strict commutativity relation if, moreover, $N = {}_i M_j(p)$, $N' = {}_i M_j(q)$.

(4.2) Let B be a finite dimensional basic algebra over a field K , and let $\Sigma = \Sigma(B)$ its K -species. If we assume that K is a perfect field, then $B \cong T(\Sigma)/J$ for some relation ideal of $T(\Sigma)$. For, under this assumption, one has a vector space direct sum decomposition $B = C \oplus \text{rad } B$, with $C \cong B/\text{rad } B$, and similarly $\text{rad } B = D \oplus (\text{rad } B)^2$, with $D \cong \text{rad } B/(\text{rad } B)^2$, by Wedderburn's theorem ([9]), so in particular $M^{(0)} \cong C$, $M^{(1)} \cong D$, and then by sending bases of $M^{(0)}$, $M^{(1)}$ to bases of C, D respectively, one obtains a surjective K -algebra homomorphism $T(\Sigma) \rightarrow B$ whose kernel, by the very construction, is a relation ideal. From now on, we will always assume that K is perfect.

(4.3) Let J be a relation ideal of $A = T(\Sigma)$. The canonical projection $A \rightarrow B := A/J$ gives rise to an embedding of $\text{mod } B$ in

mod A as those modules annihilated by J , and thus, using the equivalence $\text{mod } A \xrightarrow{\sim} \mathcal{R}(\Sigma)$, the B -modules may be realized as those representations $\underline{V} = (V_i, {}_j\varphi_i)$ of Σ such that ${}_j\tilde{\varphi}_i(X_i \otimes {}_iJ_j) = 0$, where ${}_j\tilde{\varphi}_i := \bigoplus_p {}_j\varphi_i(p)$, p running on $W(i,j)$ and

$${}_j\varphi_i(p) := {}_j\varphi_{i_s i_{s-1}} \otimes 1 \dots ({}_i\varphi_{i_2 i_3} \otimes 1 \otimes \dots \otimes 1) ({}_i\varphi_{i_1} \otimes 1 \otimes \dots \otimes 1),$$

for every $p \in W(i,j)$, $p = (i, i_2, \dots, i_s, i_{s+1} = j)$. Notice that ${}_j\varphi_i(p): X_i \otimes {}_iM_j(p) \rightarrow X_j$ and ${}_j\tilde{\varphi}_i: X_i \otimes {}_i\tilde{M}_j \rightarrow X_j$. The full subcategory of $\mathcal{R}(\Sigma)$ thus obtained will be denoted by $\mathcal{R}(\Sigma, J)$ and will be called the category of representations of Σ bound by J . We summarize in:

(4.4) Proposition. Let Σ be a K -species and J a relation ideal of the tensor algebra $T(\Sigma)$. Then $\text{mod } T(\Sigma)/J \xrightarrow{\sim} \mathcal{R}(\Sigma, J)$.

(4.5) Complete sets of representatives of the projective indecomposable, injective indecomposable and simple representations of the category $\mathcal{R}(\Sigma, J)$ can be given in the following way: first, the representations $\underline{e}(k) = (X_i, {}_j\varphi_i)$ with $X_k = K_k$, $X_i = 0$ for every $i \neq k$, and ${}_j\varphi_i = 0$ for every i, j , for every $k \in D_0$, are clearly simple, and pairwise non-isomorphic thus build up a complete set of representatives of the simple representations. The representations $\underline{p}(k) = (X_i, {}_j\varphi_i)$ with $X_j := \tilde{M}_j / {}_kJ_j$, for every

j , and ${}_j\varphi_i: X_i \otimes {}_iM_j \rightarrow X_j$ given by the canonical map
 $(\tilde{M}_i/J_i) \otimes_{{}_iK_1} {}_iM_j \rightarrow \tilde{M}_j/J_j$, are indecomposable and projective,
 for every $k \in D_0$, and pairwise non-isomorphic, thus furnish a
 complete set of representatives of the projective indecomposable
 representations of $R(\Sigma, J)$. Representatives $\underline{I}(k)$ of the in-
 decomposable injective representations can be described analogously,
 for every $k \in K_0$. In particular, we always have $\text{top } \underline{P}(k) \cong$
 $\underline{S}(k) \cong \text{soc } \underline{I}(k)$, for every $k \in D_0$, and $\text{Hom}(\underline{P}(k), \underline{X}) \cong X_k \cong$
 $D \text{Hom}(\underline{X}, \underline{I}(k))$, for every representation $\underline{X}$ of $R(\Sigma, J)$ and every
 $k \in D_0$.

(4.6) Let J be a relation ideal of a K -species Σ . The
 graph obtained from the graph $D(\Sigma)$ by adding dotted lines joining
 the vertices i and j together with the map ${}_i\pi_j$ defining every
 minimal relation ${}_iJ_j$ of J , sometimes will be denoted by
 $D(\Sigma, J)$ and called the graph of the algebra $T(\Sigma)/J$ or of the
 category $R(\Sigma, J)$.

(4.7) Example. Let $I = \{1, 2, 3\}$, F/G a quadratic field
 extension, and $\Sigma = (K_i, {}_iM_j)_{i,j \in I}$ the K -species, $K = G$,
 defined by $K_1 = F$, $K_2 = G$, $K_3 = F$, ${}_1M_2 = {}_F^F G$, ${}_2M_3 = {}_G^F F$.
 Let J be the relation ideal of the tensor algebra $A = T(\Sigma)$
 defined by $J = {}_1M_2 \otimes {}_2M_3$, and let $B = T(\Sigma)/J$. Then we have

$$D(\Sigma): 1 \xrightarrow{(2,1)} 2 \xrightarrow{(1,2)} 3$$

$$D(\Sigma, J): 1 \xrightarrow{(2,1)} 2 \xrightarrow{(1,2)} 3$$

0

It will be convenient to use the following notations:

$$\Sigma: F_1 \xrightarrow{F^F G} G_2 \xrightarrow{G^F F} F_3$$

$$(\Sigma, J): F_1 \xrightarrow{F^F G} G_2 \xrightarrow{G^F F} F_3$$

0

$$A = T(F_1 \xrightarrow{F^F G} G_2 \xrightarrow{G^F F} F_3)$$

$$B = T(F_1 \xrightarrow{F^F G} G_2 \xrightarrow{G^F F} F_3)$$

0

for this and other similar situations. Thus the notation F_i will denote that the field at the vertex i is F , and similarly for G_j . In a situation where $K_i = F = K_j$, $M_{ij} = F^F_F$, we usually omit the bimodule.

II.5 OPE-relations

In this section we describe by suitable one-point extension algebras, the relation ideals that will appear in the description of selfinjective algebras of finite representation type of Dynkin class B_n or C_n , and in the species of the universal coverings

of the Brauer graphs of type B_n or C_n . Throughout, F/G will be a quadratic field extension of a perfect field $K = G$.

(5.1) Let $B = \left(\begin{array}{c|c} A & 0 \\ \hline {}_H M_A & H \end{array} \right)$ be the one-point extension algebra of an algebra A by an A -module M_A with $H \subseteq \text{End}(M_A)$ a division ring (I, (3.5)). The category $\text{mod } B$ can be described as the category $R({}_H M_A)$ of representations of the bimodule ${}_H M_A$, whose objects are the triples (X_H, Y_A, φ) consisting of a right H -vector space X_H , an A -module Y_A and an A -linear map $\varphi: X_H \otimes_{{}_H M_A} \rightarrow Y_A$, and whose morphisms are the pairs $(f, g): (X_H, Y_A, \varphi) \rightarrow (X'_H, Y'_A, \varphi')$ of H -linear maps $f: X_H \rightarrow X'_H$ and A -linear maps $g: Y_A \rightarrow Y'_A$ such that $g\varphi = \varphi'(f \otimes 1_M)$. We will write $B = E(H, M, A)$ to indicate that B is the one-point extension of A by the bimodule ${}_H M_A$, H a division subring of $\text{End}(M_A)$.

(5.2) Let $\Sigma = (K_i, {}_i M_j)_{i,j \in I}$ be a K -species, $B = T(\Sigma)/J$, J a relation ideal of $T(\Sigma)$. We say that the relations ${}_i J_j$ are OPE-relations if $B \cong E(H, M, C)$ for some one-point extension algebra.

To introduce the OPE-relations that we need, we show first that the hereditary tensor algebras $T(\mathfrak{M}, \Omega)$, where $(\mathfrak{M}, \Omega)$ is an oriented modulation of the Dynkin graphs B_n or C_n , may be

obtained as one-point extension algebras of tensor algebras $T(\mathfrak{m}', \Omega')$ where $(\mathfrak{m}', \Omega')$ is an oriented ~~modulation~~ of the Dynkin graph A_{n-1} .

$$(5.3) \text{ Example. } T(G_n \xrightarrow{GF} F_{n-1} \rightarrow \dots \rightarrow F_1) = E(G, I(1), T(F_{n-1} \rightarrow F_{n-2} \rightarrow \dots \rightarrow F_1)), \quad (n \geq 2).$$

For,

$$\begin{pmatrix} F_1 & 0 & \dots & 0 \\ F & F_2 & & \vdots \\ \vdots & & \ddots & \\ F & \dots & F & F_{n-1} & 0 \\ F & \dots & F & F & G_n \end{pmatrix} = \left(\begin{array}{c|c} A & 0 \\ \hline I(1) & G \end{array} \right),$$

and the matrix algebra on the left hand side of the equality is

isomorphic to $T(G_n \xrightarrow{GF} F_{n-1} \rightarrow \dots \rightarrow F_1)$, and $A \cong T(F_{n-1} \rightarrow \dots \rightarrow F_1)$.

Note that $\text{End } I(1)_A = F \supseteq G$, $I(1)_A = (F_F, \dots, F_F)$.

$$(5.4) \text{ Example. } T(G_n \rightarrow \dots \rightarrow G_3 \rightarrow G_2 \xrightarrow{GF} F_1) = E(G, P(n-1), T(G_{n-1} \rightarrow \dots \rightarrow G_2 \xrightarrow{GF} F_1)), \quad (n \geq 2).$$

For,

$$\begin{pmatrix} F_1 & 0 & \dots & 0 \\ F & G_2 & & \vdots \\ F & G & & \\ \vdots & & \ddots & \\ F & G & & G_{n-1} & 0 \\ & & & G & G_n \end{pmatrix} = \left(\begin{array}{c|c} A & 0 \\ \hline P(n-1) & G \end{array} \right),$$

where the matrix algebra on the left hand side is isomorphic to

$$T(G_n \rightarrow \dots \rightarrow G_3 \rightarrow G_2 \xrightarrow{GF} F_1), \quad A \cong T(G_{n-1} \rightarrow \dots \rightarrow G_2 \xrightarrow{GF} F_1), \\ P(n-1) = (F_F, G_G, \dots, G_G) \text{ and } \text{End } P(n-1)_G \cong G.$$

(5.5) Similar procedures, with suitable modifications, show that $T(n, \Omega)$, Ω any orientation of the Dynkin graph B_n , can be expressed as a certain one-point extension algebra.

$$(5.6) \quad E(F, I(1), T(G_2 \xrightarrow{GF} F_1)) = T(F_3 \xrightarrow{FG} G_2 \xrightarrow{GF} F_1).$$

$$\text{For, } \begin{pmatrix} F_1 & 0 & 0 \\ F & G_2 & 0 \\ F & F & F_3 \end{pmatrix} = \left(\begin{array}{c|c} T(G_2 \rightarrow F_1) & 0 \\ \hline I(1) & F \end{array} \right),$$

$$\text{and } B := T(F_3 \xrightarrow{FG} G_2 \xrightarrow{GF} F_1) = T(F_3 \xrightarrow{FG} G_2 \xrightarrow{GF} F_1) / \text{Ker } \mu,$$

where $\mu: F_F^F \otimes G_G^F \rightarrow F_F^F$, $x \otimes y \mapsto xy$, is the multiplication map. Now, since $\underline{P}(1)_B = (F_F, 0, 0)$, $\underline{P}(2)_B = (F_F, G_G, 0)$,

$\underline{P}(3)_B = (F_F, (G_G^2)_G = F_G, F_F)$, at least the dimensions of the vector spaces X_1 defining the $\underline{P}(k)$ ($1 \leq k \leq 3$) are correct,

so our problem is to determine the maps $\phi_i: X_1 \otimes M_i \rightarrow X_j$. We

carry out this determination below for the case $G = R$, $F = \mathbb{C}$,

in order to use freely results from [12] and [13]. Thus, let

$$\Sigma: \mathbb{C}_3 \xrightarrow{\mathbb{C}R} R_2 \xrightarrow{R\mathbb{C}} \mathbb{C}_1, \quad A = T(\Sigma), \quad B = A/K, \quad K = \text{Ker } \mu. \quad \text{The}$$

Auslander-Reiten graph of B is

and the preprojective component P and preinjective component Q of $\Gamma(A)$ are

from where we see that the modules within the dotted lines are bound by K . Since $010 = \underline{\dim} S(2)$, it remains to describe the representation $\underline{X}$ with $\underline{\dim} \underline{X} = 121$, for the φ_i of the other representations are known from the general theory. Now, $\underline{X} \in \text{Reg}(\Sigma)$, the category of regular representations of Σ , in fact 121

generates the radical, so the problem is to determine that representation $\underline{X}$ such that $\underline{X} \cdot K = 0$ ($\underline{X} \mapsto \underline{X}$ the equivalence $R(\Sigma) \xrightarrow{\sim} \text{mod } T(\Sigma)$).

We have the full embedding $T: \text{Reg}(\mathbb{C} \xrightarrow{\mathbb{C}^{\mathbb{C}} \oplus \mathbb{C}^{\mathbb{C}}} \mathbb{C}) \rightarrow \text{Reg}(\Sigma)$ such that, in particular,

$$T(\mathbb{C} \xrightarrow{1} \mathbb{C}) = (\mathbb{C}, \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}_R, \mathbb{C}, \varphi_a, \psi_a),$$

with

$$\varphi_a: \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}_R \xrightarrow{1} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}_R$$

$$\psi_a: \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}_R \otimes_{R} \mathbb{C}_R \rightarrow \mathbb{C}$$

$$x \otimes s \mapsto x$$

$$x \otimes t \mapsto xa,$$

for every $a \in R$. Note that $\dim T(\mathbb{C} \xrightarrow{1} \mathbb{C}) = 121$, for every

$a \in R$.

Since

$$T(\Sigma) = \begin{pmatrix} \mathbb{C}_1 & 0 & 0 \\ \mathbb{C} & R_2 & 0 \\ P & \mathbb{C} & \mathbb{C}_3 \end{pmatrix},$$

with $P = \mathbb{C}_R \otimes_{\mathbb{C}} \mathbb{C}_R$, we have

$$(x_1, x_2, x_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{pmatrix} = x_3 \cdot k = {}_1\varphi_2[{}_2\varphi_3(x_3 \otimes k)],$$

in particular, for $\underline{X}_a = T(\mathbb{C} \xrightarrow{1} \mathbb{C})$, $x_3 \cdot k = \psi_a(x_3 \otimes k)$, and, for

$k \in K$, $k = \beta t$: $x_3 \cdot k = \psi_a(x_3 \otimes \beta t) = x_3 \beta a$, so $x_3 k = 0$ iff.

$a = 0$. Hence, $T(\mathbb{C} \xrightarrow{1} \mathbb{C}) \in R(\mathbb{C} \xrightarrow{\mu} \mathbb{C})$. As a consequence,

which we will use frequently in our description of the representations

of selfinjective algebras, we can assume that the representation $\underline{X}$

with $\dim X = 121$ is given by $X = (e_c, e_R, e_c, 2\varphi_3, 1\varphi_2)$ with

$$2\varphi_3 = 1, \quad 1\varphi_2 = 1.$$

$$(5.7) \quad T(F_n \xrightarrow{F^F G} G_{n-1} \rightarrow \dots \rightarrow G_2 \xrightarrow{G^F F} F_1) = \\ E(F, I(1), T(G_{n-1} \rightarrow \dots \rightarrow G_2 \xrightarrow{G^F F} F_1))$$

$$(5.8) \quad T(G_3 \xrightarrow{G^F F} F_2 \xrightarrow{F^F G} G_1) = E(G, I(1), T(F_2 \rightarrow G_1))$$

This algebra admits a similar description as the one in (5.6).

Now, we consider the species $\Sigma: G_3 \xrightarrow{G^F F} F_2 \xrightarrow{F^F G} G_1$ of Dynkin graph $\tilde{B}_2 \circ \underline{(1,2)} \circ \underline{(2,1)} \circ$. The algebra $B := T(\Sigma)/\text{Ker } \pi$,

$\pi: G^F F \otimes_F F^F G \rightarrow G^G$ a bimodule epimorphism, has as Auslander-Reiten graph

whereas the preprojective component P and the preinjective component Q of $\Gamma(A)$, $A = T(\Sigma)$ have the forms

and, again it remains to determine the B -modules $\underline{X}$ with $\dim \underline{X} = 010$ or 111 . The first one being the simple at the vertex 2, to determine $\underline{X} \in \text{Reg}(\Sigma)$ of dimension type 111 bound by $\text{Ker } \pi$, we use this time the full embedding $T: H(\frac{F}{G}) \rightarrow \text{Reg}(\Sigma)$,

$$(U_G, V_G, \varphi) \rightarrow (U_G, U_G \otimes_{G^F}^F V_G, U_G \otimes_{G^F}^F \frac{1}{G} U_G \otimes_{G^F}^F V_G, (U_G \otimes_{G^F}^F) \otimes_{F^F G}^F V_G).$$

In particular, since $H(\frac{F}{G}) = H(\frac{(G \oplus G)}{G})$, we can also describe the representations $H(\frac{F}{G})$ as $\underline{U} = (U_G, V_G, \varphi = (\varphi_1, \varphi_2): U_G \oplus U_G \rightarrow V_G)$,

and $T(\underline{U}) = (U_G, U_G \otimes_{G^F}^F V_G, \mathbb{1}, \varphi)$. The representations

$\underline{U}_a = (G \xrightarrow{\frac{1}{a}} G)$, $a \in G$, have $T(\underline{U}_a) = (G, G \otimes_{G^F}^F G, \mathbb{1}, \varphi_a)$ and dimension type 111 ; also $\underline{U}_\infty = (G \xrightarrow{0} G)$ has $\dim T(\underline{U}_\infty) = 111$, but,

computing φ_a for any bimodule epimorphism $\pi_a: \frac{F}{G} = G \cdot 1 \oplus G \cdot \alpha \rightarrow \frac{G}{G}$, $x + y\alpha \mapsto x$, $\alpha \in F \setminus G$, we obtain $\varphi_a: F \rightarrow G$, $x + y\alpha \mapsto x + ay$

and $\varphi_a(g \otimes \alpha) = ag$, so $\varphi_a(g \otimes \alpha) = 0$ iff $a = 0$. Hence,

$T(G \xrightarrow{\frac{1}{0}} G) = (G_G, F_F, G_G, G \otimes_{G^F}^F \frac{1}{F} F, F_F \otimes_{F^F G}^F (G_G \cdot 1 \oplus G\alpha) \rightarrow G_G, x + y\alpha \mapsto x)$ is the representation of Σ bound by $\text{Ker } \pi$. As a

consequence, we can assume that the representation $\underline{X}$ with $\dim \underline{X} = 111$

is given by $(G_G, F_F, G_G, {}_2\varphi_3, {}_1\varphi_2)$ with ${}_2\varphi_3 = \mathbb{1}$, ${}_1\varphi_2 = \pi$,

where π is a bimodule epimorphism $\pi: \frac{F}{G} \rightarrow \frac{G}{G}$.

$$(5.9) \quad T(G \xrightarrow{\frac{F}{G}} F_{n-1} \rightarrow \dots \rightarrow F_2 \xrightarrow{\frac{F}{G}} G_1) = E(G, I(1), T(F_{n-1} \rightarrow \dots \rightarrow F_2 \xrightarrow{\frac{F}{G}} G_1)).$$

(5.10)

$$\begin{aligned}
 & E\left(F, I(0), T\left(\begin{array}{c} F_P \rightarrow \dots \rightarrow F \xrightarrow{\mu} G \rightarrow F \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \\ F_{q'} \rightarrow \dots \quad \quad \quad \dots \rightarrow F_{1'} \rightarrow F_0 \end{array}\right)\right) \\
 &= T\left(\begin{array}{c} F \xrightarrow{F_P} \dots \rightarrow F \xrightarrow{\mu} G \rightarrow F \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \\ F \xrightarrow{F_{q'}} \dots \quad \quad \quad \dots \rightarrow F_{1'} \rightarrow F_0 \end{array} \quad (c, \mu) \right)
 \end{aligned}$$

(5.11)

$$\begin{aligned}
 & E\left(G, I(0), T\left(\begin{array}{c} G_P \rightarrow \dots \rightarrow G \xrightarrow{\pi} F \rightarrow G \rightarrow \dots \rightarrow G_1 \rightarrow G_0 \\ G_{q'} \rightarrow \dots \quad \quad \quad \dots \rightarrow G_{1'} \rightarrow G_0 \end{array}\right)\right) \\
 &= T\left(\begin{array}{c} G \xrightarrow{G_P} \dots \rightarrow G \xrightarrow{\pi} F \rightarrow G \rightarrow \dots \rightarrow G_1 \rightarrow G_0 \\ G \xrightarrow{G_{q'}} \dots \quad \quad \quad \dots \rightarrow G_{1'} \rightarrow G_0 \end{array} \quad (c, \pi) \right)
 \end{aligned}$$

II.6 Tensor categories

Let $\Sigma = (K_i, M_j)_{i,j \in I}$ be a K -species.

(6.1) Definition. The tensor category $\otimes \Sigma$ of Σ is the free category generated by $\otimes \Sigma(j,i) := {}_i M_j$ as morphisms, on the set of objects I .

(6.2) Proposition.

$$L: \otimes \Sigma \rightarrow \text{proj ind } T(\Sigma)$$

$$\begin{array}{ccc} j & \mapsto & L(j) := \underline{P}(j) \\ t \downarrow & & \downarrow L(t) := t \otimes _ \\ i & \mapsto & L(i) := \underline{P}(i) \end{array}$$

is an equivalence of categories.

Proof. $A := T(\Sigma)$ is a basic algebra with $\underline{P}(j) := K_j A$, $j \in I$ a complete set of representatives of the projective indecomposable A -modules. Let $i \rightarrow j$ in Σ ; then $L := L_{i,j}: M_{i,j} \rightarrow \text{Hom}_A(\underline{P}(j), \underline{P}(i))$, $t \mapsto L(t) := t \otimes _$; $L(t)(f_j, a) = t \otimes f_j, a$ is an isomorphism. Hence L is a fully faithful functor, so an equivalence, since it is dense by definition.

$$(6.3) \text{ Corollary. } T(\Sigma) \cong \bigoplus_{i,j \in I} (\otimes \Sigma)(j, i).$$

$$\text{For, } T(\Sigma) = \bigoplus_{i,j \in I} \text{Hom}_A(\underline{P}(j), \underline{P}(i)) \cong \bigoplus_{i,j \in I} (\otimes \Sigma)(j, i).$$

$$(6.4) \text{ Proposition. } (\otimes \Sigma)^{\text{op}}, \text{Ab}) \cong \text{mod } T(\Sigma).$$

Proof. The functor category $(\otimes \Sigma)^{\text{op}}, \text{Ab})$ is the category of the additive contravariant functors $T: \otimes \Sigma \rightarrow \text{Ab}$ from the tensor category $\otimes \Sigma$ in the category Ab of abelian groups such that $T(i)$ is a finite dimensional right K_i -vector space, for every

$i \in I$. Note that $T(i)$ is always a right K_1 -vector space, for any contravariant functor T from $\otimes \Sigma$ into Ab , under the product $xf := T(f)(x)$, for every $x \in T(i)$, $f \in K_1$; for, $x(fg) = T(fg)(x) = T(f \otimes g)(x) = T(fog)(x) = T(g) \circ T(f)(x) = (xf)g$.

Since $\text{mod } T(\Sigma) \xrightarrow{\sim} R(\Sigma)$, to prove the proposition we will construct functors $F: (\otimes \Sigma)^{op}, Ab \rightarrow R(\Sigma)$, $G: R(\Sigma) \rightarrow (\otimes \Sigma)^{op}, Ab$ inverse to each other. Let $T: \otimes \Sigma \rightarrow Ab$ be in $C := (\otimes \Sigma)^{op}, Ab$. Define $F(T) := (T(i), {}_j\varphi_i)_{i \in I}$, where ${}_j\varphi_i: T(i) \otimes {}_iM_j \rightarrow T(j)$, $x \otimes t \mapsto {}_j\varphi_i(x \otimes t) := T(t)(x)$. Conversely, given $\underline{X} = (X_i, {}_j\varphi_i) \in R(\Sigma)$, let $G(\underline{X}): \otimes \Sigma \rightarrow Ab$ be the functor defined by $G(\underline{X})(i) := X_i$, $G(\underline{X})(t): G(\underline{X})(i) = X_i \rightarrow G(\underline{X})(j) = X_j$, $x \mapsto G(\underline{X})(t)(x) := {}_j\varphi_i(x \otimes t)$, for every $t \in {}_iM_j = (\otimes \Sigma)(j, i)$. It is routine to check that G and F are inverse to each other.

(6.5) Remark. Propositions (6.2) and (6.4) can be resumed in the following commutative diagram:

$$\begin{array}{ccc}
 (\otimes \Sigma)^{op}, Ab & \xrightarrow{\sim} & \text{mod } T(\Sigma) \xrightarrow{\sim} R(\Sigma) \\
 \uparrow & & \uparrow \\
 \otimes \Sigma & \xrightarrow{\sim} & \text{proj ind } T(\Sigma)
 \end{array}$$

where $i \mapsto (\otimes \Sigma)(-, i)$ is the canonical embedding of $\otimes \Sigma$ in the functor category $(\otimes \Sigma)^{op}, Ab$. By (6.3), $T(\Sigma)$ can be recovered from $\otimes \Sigma$. In view of this situation, the category $\otimes \Sigma$, also

called the path category of the species Σ is sometimes identified with the algebra $T(\Sigma)$ (correctly, with the category $\text{proj ind } T(\Sigma)$). Notice that $\otimes \Sigma$ is a K -category (I, (1.5)).

(6.6) Let $J = \bigoplus_{i,j} K_i J K_j$ be a relation ideal of $T(\Sigma)$. We construct the category $\otimes \Sigma / J$ as the free category on the set of objects I with morphisms generated by $\otimes \Sigma / J(j, i) := (\otimes \Sigma)(j, i) / J_j$. In this way, J can be also regarded as an ideal of the tensor category $\otimes \Sigma$.

(6.7) Proposition.

$$(i) \quad ((\otimes \Sigma / J)^{\text{op}}, \text{Ab}) \sim \text{mod } T(\Sigma) / J \sim \mathcal{R}(\Sigma, J).$$

$$(ii) \quad \otimes \Sigma / J \sim \text{proj ind } T(\Sigma) / J.$$

Proof. Similar to those of (6.2) and (6.4) with the obvious modifications.

(6.8) Let now Γ be a translation modulated graph, and $\Sigma = \Sigma(\Gamma)$ its species (3.8). The tensor category $\otimes \Gamma := \otimes \Sigma$ will be called the tensor category of Γ , and $T(\Gamma) := T(\Sigma)$ the tensor algebra of Γ . Let $\alpha \in \Gamma_1$, $B(\alpha) := {}_{e(\alpha)} B_{o(\alpha)} = {}_y B_x$, $x = o(\alpha)$, $y = e(\alpha)$. We put $c(\alpha) := c(x, y) := c(B(\alpha))$, the canonical element of the bimodule $B(\alpha)$. The relation ideal I_Γ of $\otimes \Gamma$ generated by the elements $c(x) := \sum_{o(\alpha)=x} c(\alpha)$, x running through the non-

injective vertices of Γ , is called the mesh ideal of Γ , and the quotient category $\text{msh } \Gamma := \oplus \Gamma / I_\Gamma$ is called the mesh category of Γ .

Proposition. Let Γ be a finite modulated translation graph. Then

$$T(\Gamma)/I_\Gamma \cong \bigoplus_{x,y \in \Gamma_0} (\text{msh } \Gamma)(x,y).$$

Proof. The map $T(\Gamma) \rightarrow \bigoplus_{x,y \in \Gamma_0} (\text{msh } \Gamma)(x,y)$,

$t_1 \otimes \dots \otimes t_r \mapsto t_1 \otimes \dots \otimes t_r + I_\Gamma$, clearly can be extended to a surjection which is an algebra-homomorphism with kernel I_Γ .

(6.9) Proposition. Let Γ be a finite modulated translation graph. Then

$$((\text{msh } \Gamma)^{\text{op}}, \text{Ab}) \cong \text{mod } T(\Gamma)/I_\Gamma.$$

Proof. Follows from (6.7) and (6.8).

(6.10) Examples. Let $\mathbb{Z}\Delta$, $\Delta = B_n, C_n, F_4$ or G_2 be the valued translation graphs of Example (2.2). Let F/G and H/G be a quadratic and cubic extension of a perfect field $K = G$. We modulate $\mathbb{Z}\Delta$ with the modulation $m = (K, B)_{x'xy, x,y \in (\mathbb{Z}\Delta)_0}$ defined in each case in the following ways (cfr. (1.10)):

(i) $\mathbb{Z} B_n$ ($n \geq 2$):

$$K_x: = \begin{cases} G, & x = (m, n), \quad m \in \mathbb{Z} \\ F, & \text{otherwise} \end{cases}$$

$${}_x B_y: = \begin{cases} F^F G, & x = (m, n-1), y = (m, n), \quad m \in \mathbb{Z} \\ G^F F, & x = (m, n), y = (m+1, n-1), \quad m \in \mathbb{Z} \\ F^F F, & \text{otherwise} \end{cases}$$

(ii) $\mathbb{Z} C_n$ ($n \geq 3$):

$$K_x: = \begin{cases} F, & x = (m, n), \quad m \in \mathbb{Z} \\ G, & \text{otherwise} \end{cases}$$

$${}_x B_y: = \begin{cases} G^F F, & x = (m, n-1), y = (m, n), \quad m \in \mathbb{Z} \\ F^F G, & x = (m, n), y = (m+1, n-1), \quad m \in \mathbb{Z} \\ G^G G, & \text{otherwise} \end{cases}$$

(iii) $\mathbb{Z} F_4$:

$$K_x: = \begin{cases} G, & x = (m, t) \quad t = 3, 4, \quad m \in \mathbb{Z} \\ F, & s = (m, t) \quad t = 1, 2, \quad m \in \mathbb{Z} \end{cases}$$

$$x^B_y = \begin{cases} G^G, & x = (m,3), y = (m,4) \\ & \text{or } x = (m,4), y = (m,3) \\ F^F, & x = (m,1), y = (m,2) \\ & \text{or } x = (m,2), y = (m+1,1) \\ F^G, & x = (m,2), y = (m,3) \\ G^F, & x = (m,3), y = (m+1,2) \end{cases}$$

for every $m \in \mathbb{Z}$

(iv) $\mathbb{Z} \mathbb{C}_2$:

$$K_x = \begin{cases} G, & x = (m,2) \\ H, & x = (m,1) \end{cases}$$

$$x^B_y = \begin{cases} H^G, & x = (m,1), y = (m,2) \\ G^H, & x = (m,2), y = (m+1,1) \end{cases}$$

for every $m \in \mathbb{Z}$.

The resulting mesh categories $(\text{msh } \mathbb{Z} \Delta, \mathbb{M})$ will be denoted by $(\text{msh } \mathbb{B}_n, F, G)$, $(\text{msh } \mathbb{C}_n, F, G)$, $(\text{msh } \mathbb{F}_4, F, G)$ and $(\text{msh } \mathbb{C}_2, H, G)$, respectively, or simply by $\text{msh } \Delta$, $\Delta = \mathbb{B}_n, \mathbb{C}_n, \mathbb{F}_4$ or $\mathbb{C}_2$ when the fields are fixed in a discussion.

II.7 Locally finite dimensional modulated graphs

(7.1) Let $\Gamma = (\Gamma_0, \Gamma_1, \tau, w, m)$ be a translation modulated graph, and let $(\Gamma_0, \Gamma_1, \tau)$ be the underlying (trivially valued) translation graph. Let $f: (\Delta_0, \Delta_1, \tau) \rightarrow (\Gamma_0, \Gamma_1, \tau)$ be a covering ([7]), that is, a morphism of translation graphs such that the induced maps $x^- \rightarrow (fx)^-$ and $x^+ \rightarrow (fx)^+$ are bijective, for every $x \in \Delta_0$, and such that τx and $\tau^{-1}y$ are defined whenever τfx and $\tau^{-1}fy$ are defined, respectively. Our purpose here is to lift the valuation and modulation given on Γ to a valuation w' and a modulation m' on $(\Delta_0, \Delta_1, \tau)$, to obtain in this way a translation modulated graph $\Delta = (\Delta_0, \Delta_1, \tau, w', m')$.

(7.2) We write $x'|x$ and $\alpha'|\alpha$ if $fx' = x$, $fx' = \alpha$, for $x' \in \Delta_0$, $\alpha' \in \Delta_1$. If we define $w'(\alpha') := w(\alpha)$, for every $\alpha'|\alpha$, we obtain a valuation on $(\Delta_0, \Delta_1, \tau)$ such that f respects the valuation. If $m = (K_x, B_{xy}, \epsilon_\alpha)$ is the given modulation on $(\Gamma_0, \Gamma_1, \tau)$, we define simply $K_{x'} := K_x$, $B(\alpha') := B(\alpha)$ and $\epsilon_{\alpha'} := \epsilon_\alpha$, for every $x'|x$ and $\alpha'|\alpha$ to obtain a modulation $m' := (K_{x'}, B_{xy}, \epsilon_{\alpha'})$ of $(\Delta_0, \Delta_1, \tau)$ such that $\Delta := (\Delta_0, \Delta_1, \tau, w', m')$ becomes a translation modulated graph, since, by construction, (TMG1) and (TMG2) are satisfied, which will be called a covering of Γ .

(7.3) In particular, if $(\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tau)$ is the universal covering ([7]) of $(\Gamma_0, \Gamma_1, \tau)$, the translation modulated graph $\tilde{\Gamma} = (\tilde{\Gamma}_0, \tilde{\Gamma}_1, \tau, \tilde{w}, \tilde{m})$ will be called the universal covering of Γ .

(7.4) Let $f: \Delta \rightarrow \Gamma$ be a covering of translation modulated graphs, that is, Δ is a covering of Γ and $f: (\Delta_0, \Delta_1, \tau) \rightarrow (\Gamma_0, \Gamma_1, \tau)$ is a covering of the underlying translation graphs. Then, f induces functors

$$(i) \text{ msh } f: \text{msh } \Delta \rightarrow \text{msh } \Gamma$$

$$(ii) f_*: ((\text{msh } \Delta)^{\text{op}}, \text{Ab}) \rightarrow ((\text{msh } \Gamma)^{\text{op}}, \text{Ab}).$$

An explicit description, for instance of (ii), follows from (6.9) in terms of the categories $R(\Sigma(\Delta), I_{\Sigma(\Delta)})$ and $R(\Sigma(\Gamma), I_{\Sigma(\Gamma)})$, as the following diagram indicates:

$$\begin{array}{ccc} ((\text{msh } \Delta)^{\text{op}}, \text{Ab}) & \xrightarrow{f_*} & ((\text{msh } \Gamma)^{\text{op}}, \text{Ab}) \\ \downarrow \wr & & \downarrow \wr \\ \text{mod } T(\Delta)/I_{\Delta} & \longrightarrow & \text{mod } T(\Gamma)/I_{\Gamma} \\ \downarrow \wr & & \downarrow \wr \\ R(\Sigma(\Delta), I_{\Sigma(\Delta)}) & \longrightarrow & R(\Sigma(\Gamma), I_{\Sigma(\Gamma)}) \\ \underline{V} = (V_x, y^{\varphi_x}) & \longmapsto & \underline{W} = (W_{fx}, fy^{\varphi_{fx}}), \end{array}$$

$$\text{where } W_{fx} := \bigoplus_{fz=fx} V_z, \quad fy^{\varphi_{fx}} := \bigoplus y^{\varphi_x}.$$

(7.5) Proposition. Let $f: \Delta \rightarrow \Gamma$ be a covering of translation modulated graphs. Then

$$(i) \quad (\text{msh } \Gamma)(fx, fy) \stackrel{\sim}{\sim} \bigoplus_{fz=fy} (\text{msh } \Delta)(x, z)$$

$$(ii) \quad (\text{msh } \Gamma)(fx, fy) \stackrel{\sim}{\sim} \bigoplus_{fu=fx} (\text{msh } \Delta)(u, y).$$

Proof. Clear.

(7.6) Let $\Sigma = (K_i, {}_iM_j)_{i,j \in I}$ be a K -species, J a relation ideal of $T(\Sigma)$. Following [30], any automorphism σ of I such that $K_{\sigma i} \stackrel{\sim}{\sim} K_i$ and ${}_{\sigma i}M_{\sigma j} \stackrel{\sim}{\sim} {}_iM_j$, for every $i, j \in I$, that moreover respects J , induces a selfequivalence, also denoted by σ , of the category $R(\Sigma, J)$ by $\underline{V} = (V_i, {}_j\varphi_i) \mapsto \sigma(\underline{V}) = (W_i, {}_j\psi_i)$, where $W_i := V_{\sigma^{-1}(i)}$ and ${}_j\psi_i := {}_{\sigma^{-1}(j)}\varphi_{\sigma^{-1}(i)}$, and $\sigma(\underline{f}) = \underline{g} = (g_i)$, $g_i := f_{\sigma^{-1}(i)}$, for every $i \in I$, for every morphism $\underline{f}: \underline{V} \rightarrow \underline{W}$. The subcategory of all representations $\underline{V}$ and all morphisms $\underline{f}$ which are σ -invariant, will be denoted by $R^\sigma(\Sigma, J)$. This category can be interpreted as a category of bound representations $R(\bar{\Sigma}, \bar{J})$ in the following canonical way: let I/σ be the set of σ -orbits of I ; let further $K_{\langle i \rangle} := K_i$, ${}_{\langle i \rangle}M_{\langle j \rangle} := {}_iM_j$, for every $i, j \in I$, where $\langle i \rangle, \langle j \rangle \in I/\sigma$, and $\bar{J}/\sigma$ the ideal generated by ${}_{\langle i \rangle}(\bar{J}/\sigma)_{\langle j \rangle} = {}_iJ_j$. Then, the functor that sends $(V_{\langle i \rangle}, {}_{\langle j \rangle}\varphi_{\langle i \rangle})$ to $(V_i, {}_j\varphi_i)$ with $V_i := V_{\langle i \rangle}$,

$j^{\varphi} i = \langle j \rangle^{\varphi} \langle i \rangle$, for every $i, j \in I$, clearly establishes an equivalence $R(\bar{\Sigma}, \bar{J}) \sim R^{\sigma}(\Sigma, J)$, with $\bar{\Sigma} = (K_{\langle i \rangle}, \langle i \rangle^M \langle j \rangle)$, $\bar{J} = J/\sigma$. In this situation, we say that $\Sigma((\Sigma, J))$ is a covering of $\bar{\Sigma}((\bar{\Sigma}, \bar{J}))$. Similar constructions are possible if one considers a group of automorphisms, instead of a single automorphism.

(7.7) Examples.

(i) Let $\tilde{B} = (\tilde{B}_0, \tilde{B}_1, \tilde{w})$ be the following valued graph:
 $\tilde{B}_0 = \mathbb{Z}$ as set of vertices, $\tilde{B}_1$ are the arrows $4k \rightarrow 4k+1$,
 $4k+1 \rightarrow 4k+3$, $4k+2 \rightarrow 4k+3$, $4k+3 \rightarrow 4(k+1)$, $4k+3 \rightarrow 4(k+1)+2$,
 $\tilde{w}$ is the valuation $4k \xrightarrow{(1,2)} 4k+1$, $4k+3 \xrightarrow{(2,1)} 4(k+1)$, otherwise
trivial, for every $k \in \mathbb{Z}$. Let $\tilde{\Sigma} = (K_{i,j}, i^M j)_{i,j \in \mathbb{Z}}$ be the
 $K = G$ species with $K_{4k} = G$, $K_i = F$ otherwise, F/G a quadratic
field extension, $4k^M 4k+1 = G^F F$, $4k+3^M 4(k+1) = F^F G$, $i^M j = F^F F$
otherwise. Finally, let $\tilde{J}$ be the relation ideal of $\tilde{\Sigma}$ generated
by the following minimal relations:

$$\begin{array}{ccc} \overset{\pi}{\curvearrowright} & & \overset{\mu}{\curvearrowright} \\ K_{4k} \rightarrow K_{4k+1} \rightarrow K_{4k+2} \rightarrow K_{4k+3} & , & K_{4k-1} \rightarrow K_{4k} \rightarrow K_{4k+1} \\ \overset{0}{\curvearrowright} & & \overset{0}{\curvearrowright} \\ K_{4k+1} \rightarrow K_{4k+3} \rightarrow K_{4k+6} & , & K_{4k+2} \rightarrow K_{4k+3} \rightarrow K_{4k+4} \end{array}$$

and

for every $k \in \mathbb{Z}$.

The map $\mu: k \mapsto k+4$, $k \in \mathbb{Z}$, clearly defines an automorphism of $\tilde{\Sigma}$ which respects $\tilde{J}$, by construction, so we can consider the species $\Sigma_r := \tilde{\Sigma}/\mu^r$ with relation ideal $J_r := J/\mu^r$, for every $r \in \mathbb{Z}$, $r \geq 1$. For instance, Σ_2 is given by

(Fig. 2)

Here, we omit to display the bimodules and the relations; the zero-relations are indicated by either $o \longrightarrow o \dots > o$ or $o \dots > o \longrightarrow o$. The division rings lying on the vertical dotted lines are to be identified, so Σ_r lies on a cylinder.

A similar description for any $r \geq 1$ shows that Σ_r lies on a cylinder, as well. In particular, for $r = 1$, $D(\Sigma_1)$ is simply the graph displayed on Figure 1a of (1.7). In fact, $D(\tilde{\Sigma})$ is the universal covering of this Brauer graph of type B_4 , and $\tilde{\Sigma}$ is the universal covering of Σ_r for any $r \geq 1$. The algebras $A_r := T(\Sigma_r)/J_r$ ($r \geq 1$) are all selfinjective algebras of finite representation type and Dynkin class B_4 , with Auslander-Reiten graph $\Gamma(A_r) \cong (\mathbb{Z} B_4)_C / \langle \tau^{7r} \rangle$, for a suitable configuration C of $\mathbb{Z} B_4$. Later we will see how to determine C from $D(\Sigma_1)$ and conversely, thus establishing the fundamental bijection between the configurations of $\mathbb{Z} B_n$ and the Brauer graphs of type B_n , for any $n \geq 2$.

(ii) A similar discussion as above produces a universal covering of the example displayed in Figure 1(b) of 1.7. Omitting easy to fill in details, we get

2

$\tilde{\Sigma}:$

In this case the algebras $C_r := T(\Sigma_r)/J_r$ ($r \geq 1$) are self-injective of finite representation type and of Dynkin class $\mathbb{C}_5$, with Auslander-Reiten graph $\Gamma(C_r) \cong (\mathbb{Z}\mathbb{C}_5)_C / \langle \tau^{9r} \rangle$, for a suitable configuration C of $\mathbb{Z}\mathbb{C}_5$ uniquely determined, up to isomorphism, by the Brauer graph of type $\mathbb{C}_5$ of Figure 1(b) of (1.7).

CHAPTER III. SELF-INJECTIVE ALGEBRAS

After discussing in Section 1 some general properties of selfinjective algebras, we center on Section 2 on the structure of the Auslander-Reiten graph of a selfinjective algebra A of finite representation type, to obtain it from the stable Auslander-Reiten graph $\Gamma_s(A)$ by adding suitable new points arising from configuration points. We extend this procedure to the whole modulation structures on the cases of algebras with Dynkin class $\Delta = B_n, C_n, F_4$ or G_2 . Section 3 is devoted to the modulated graphs Z_Δ and $(Z_\Delta)_C$; admissible groups, Nakayama permutations, ω_Δ . Throughout we assume a perfect ground field K .

III.1 General properties

Let A be a basic, connected, finite dimensional algebra over a field K .

(1.1) The algebra A is said to be selfinjective if the classes $\text{proj } A$, consisting of the finitely generated projective right A -modules, and $\text{inj } A$, consisting of the finitely generated injective right A -modules, coincide.

(1.2) Remarks.

(i) The condition $\text{proj } A = \text{inj } A$ of (1.1) is well known to be equivalent to the condition that the right A -module A_A is injective. There are, as well, many other characterizations of selfinjective algebras, also called Quasi-Frobenius algebras in the literature, but, from the point of view of our investigation, the condition stated in (1.1) is the one that suits best our purposes.

(ii) The vast class of selfinjective algebras includes, as well known examples, among others, semisimple algebras, group algebras, Frobenius algebras, in particular symmetric algebras and weakly symmetric algebras, and trivial extension algebras of Artin algebras.

(iii) If we drop the assumption of finite dimensionality on A , we still say that the algebra is selfinjective if $\text{proj } A = \text{inj } A$; this applies, in particular, to locally finite dimensional K -algebras.

(1.3) Let A be again a selfinjective algebra. Then the stable categories $\underline{\text{mod}} A$ and $\overline{\text{mod}} A$ clearly coincide. We recall that $\underline{\text{mod}} A$, the projectively stable category, has as objects the A -modules and morphisms $\underline{\text{Hom}}_A(X, Y) := \text{Hom}_A(X, Y) / P(X, Y)$, where $P(X, Y)$ is the subgroup of $\text{Hom}_A(X, Y)$ of those A -homomorphisms $f: X \rightarrow Y$ which factor through a projective A -module. Similarly, $\overline{\text{mod}} A$, the injectively stable category, has the objects of $\text{mod } A$.

as objects and morphisms $\overline{\text{Hom}}_A(X, Y) := \text{Hom}_A(X, Y) / I(X, Y)$, where now $I(X, Y)$ is the subgroup of $\text{Hom}_A(X, Y)$ of those A -homomorphisms that factor through an injective module. In the following, we denote by $\overline{\text{mod}} A$ the stable category of a selfinjective algebra A , $\bar{X}$, $\bar{f}$ its objects and morphisms. Clearly, an A -module P is the zero object of $\overline{\text{mod}} A$ if, and only if, P is a projective-injective A -module.

The category $\overline{\text{mod}} A$ has the canonical endofunctors Ω^1 and Σ^1 , the loop-functor of Heller, and the suspension-functor. These are defined, in general, by $\Omega^1(X) := \text{Ker } f$, where $f: P \rightarrow X$ is a projective cover of X , and by $\Sigma^1(Y) := \text{Coker } j$, where $j: Y \rightarrow I$ is an injective envelope of Y , for every $X, Y \in \text{mod } A$, and, for A selfinjective, are selfequivalences of $\overline{\text{mod}} A$, inverse to each other. In particular, $\Omega^1(S(1)) = R(1)$, $\Sigma^1(R(1)) = S(1)$, where $S(1) = \text{top } P(1)$, $R(1) = \text{rad } P(1)$, for every projective indecomposable A -module $P(i)$ ($1 \leq i \leq n$). Finally, $\text{DTr} \simeq (\Omega^1)^2$.

(1.4) The following two results of Auslander and Reiten will be used frequently in our work.

Proposition 1 ([4]). Let P be an indecomposable projective-injective A -module. Then the sequence

$$0 \rightarrow \text{rad } P \xrightarrow{\begin{pmatrix} 1 \\ q \end{pmatrix}} P \oplus \frac{\text{rad } P}{\text{soc } P} \xrightarrow{(p, -1)} \frac{P}{\text{soc } P} \rightarrow 0,$$

where $i: \text{rad } P \hookrightarrow P$, $j: \frac{\text{rad } P}{\text{soc } P} \hookrightarrow \frac{P}{\text{soc } P}$ are the inclusions

and $q: \text{rad } P \twoheadrightarrow \frac{\text{rad } P}{\text{soc } P}$, $p: P \twoheadrightarrow \frac{P}{\text{soc } P}$ are the canonical

projections, is an Auslander-Reiten sequence.

In general, $\text{rad } P / \text{soc } P$ is not an indecomposable module; let

$\text{rad } P / \text{soc } P = \bigoplus_{i=1}^r n_i X_i$ be an indecomposable decomposition of

$\text{rad } P / \text{soc } P$. Then, the Auslander-Reiten sequence of the proposition gives rise to the subgraph

of the Auslander-Reiten graph $\Gamma(A)$, which will be called the projective mesh of $[P]$.

The other result we will have occasions to use is the following:

Proposition 2 ([6]). Let A be a selfinjective algebra and M an indecomposable non-projective A -module. The following conditions are equivalent for X indecomposable:

(i) $X \overset{\sim}{\rightarrow} \text{Tr } D\Omega^1 M$

(ii) There exists a non-zero morphism $\bar{f}: \bar{X} \rightarrow \bar{M}$ such that

$\bar{f}\bar{h} = 0$ for every non-isomorphism $\bar{h}: \bar{Y} \rightarrow \bar{X}$ with $\bar{Y}$ indecomposable.

We will also need the dual version of the previous proposition:

Proposition 3. Let A be a selfinjective algebra and M an indecomposable non-injective A -module. The following conditions are equivalent for X indecomposable:

$$(i) \quad X \cong D \operatorname{Tr} \Sigma^1 M$$

(ii) There exists a non-zero morphism $\bar{f}: \bar{M} \rightarrow \bar{X}$ such that $\bar{h}\bar{f} = 0$ for every non-isomorphism $\bar{h}: \bar{X} \rightarrow \bar{Y}$, with $\bar{Y}$ indecomposable.

(1.5) In [22] Hughes and Waschbüsch investigated trivial extensions of finite representation type of Artin algebras B without oriented cycles on its ordinary graph. We quote from them the following results, in a form adapted to our needs.

Proposition 1. The following conditions are equivalent:

- (i) $B \ltimes DB$ is of finite representation type and Dynkin class Δ .
- (ii) There exists a tilted algebra T of Dynkin type Δ such that $B \ltimes DB = T \ltimes DT$.

Proposition 2. The algebra $\hat{B}$ associated to B has the following properties:

- (i) $\Gamma_s(\operatorname{mod} \hat{B}) \cong \mathbb{Z}\Delta$
- (ii) $\Gamma(\operatorname{mod} \hat{B}) \cong (\mathbb{Z}\Delta)_C$, for a suitable configuration C of $\mathbb{Z}\Delta$.

Proposition 3. Let $Z\Delta \cong \Gamma_s(\text{mod } \hat{B})$ and $\mathcal{F}$ a complete τ -slice through $Z\Delta$. Let $\text{Supp}(\mathcal{F})$ be the support of $\mathcal{F}$. Then

- (i) $\text{Supp } \mathcal{F} \subseteq [\tau^0 \mathcal{F}, \mathcal{F}]$
- (ii) $T := \bigoplus_{p,q} \text{msh}(Z\Delta)_C(p,q)$, p,q running through the support of $\mathcal{F}$, is a tilted algebra of type Δ such that $B \ltimes DB = T \ltimes DT$.

In the statement of the previous results, concepts and notations not discussed yet, will be introduced in the sequel.

III.2 Structure of the Auslander-Reiten graph

In this section, A will always be a basic, connected algebra over a field K .

(2.1) Let Δ be a component of the stable Auslander-Reiten graph $\Gamma_s(A)$. By (II, (2.9)) we know that $\Delta \cong ZT/W$, for some valued directed tree T and some admissible group of automorphisms W of ZT . The following theorem, proved in [23] for K algebraically closed and extended to arbitrary Artin algebras in [29] and [21], gives the structure of such a component:

Theorem. The Cartan class of any component of the stable Auslander-Reiten graph of an Artin algebra, containing periodic types, is either a Dynkin diagram or A_∞ .

(2.2) Let A be a selfinjective algebra of finite representation type. The only non-stable types are those corresponding to the projective-injective indecomposable A -modules. By (1.2), deletion of these non-stable types from the projective meshes does not break up the τ -connectedness, for $\tau[P/S] = [\text{rad } P]$, if $S \cong \text{soc } P$, P projective-injective indecomposable. So, $\Gamma_g(A)$ is connected, hence the only component of $\Gamma_g(A)$. Moreover, A being of finite representation type excludes A_∞ as a possible Cartan class, so the Cartan class of $\Gamma_g(A)$ is a Dynkin graph D , by (2.1). The algebra S is said to be of Dynkin class D if this is the case.

(2.3) By (2.2), we can recover the whole Auslander-Reiten graph $\Gamma(A)$ of a selfinjective algebra of finite representation type from the stable Auslander-Reiten graph $\Gamma_g(A)$, if we know which vertices of $\Gamma_g(A)$ represent the radicals of the projective-injective indecomposable A -modules, for then we have only to add new vertices $[P]$ corresponding to these projective-injectives and trivially valued arrows $[\text{rad } P] \rightarrow [P] \rightarrow [P/\text{soc } P]$: for, let $S := \text{soc } P$; the map $g \mapsto g|_S$ defines an epimorphism $\varphi: \text{End } P \rightarrow \text{End } S$, for P is injective, so any endomorphism of S can be lifted to an endomorphism of P , with $\ker \varphi = \text{rad } \text{End } P$. Hence, $\text{End } P / \text{rad } \text{End } P \cong \text{End } S \cong \overline{\text{End } S} \cong \overline{\text{End } \Sigma^1(S)} = \overline{\text{End } P/S} \cong \text{End } P/S \cong \text{End } (\text{rad } P)$; the last

isomorphism is always valid for the outer terms of any Auslander-Reiten sequence, whereas the isomorphisms $\text{End } S \cong \overline{\text{End } S}$ and $\overline{\text{End } P/S} \cong \text{End } P/S$ follow from the fact that S and P/S are indecomposable non-projective modules, in particular without projective summands. This discussion shows that it is of interest to know which conditions satisfies a given set $C = \{\text{rad } P \mid P \in \text{proj ind } A\}$ that is, a set of representatives of the radicals of the projective indecomposable A -modules, and then translate these conditions to the Auslander-Reiten graph of A . The following two conditions are of fundamental importance:

(1) For each non-projective X there exists R in C such that $\overline{\text{Hom}}_A(X, R) \neq 0$

(ii) $\overline{\text{Hom}}_A(R, R') = 0$ for every $R \neq R'$ in C ; $\overline{\text{Hom}}_A(R, R)$ is a division ring, for every R in C .

The validity of these conditions follows from the fact that under Σ^1 , C is mapped onto a set of representatives of the simple A -modules, and for this set (i) means that every non-projective X maps onto a simple and (ii) is simply Schur's Lemma.

(2.4) Definition. Let (Γ, m) be a stable translation module graph, $m = (K_x, B_y)_{x, y \in \Gamma_0}$. A configuration of (Γ, m) is a set of vertices C of Γ satisfying the following two conditions:

(C1) For each $x \in \Gamma_0$ there exists $c \in C$ such that $(\text{msh } \Gamma)(x, c) \neq 0$.

(C2) $(\text{msh } \Gamma)(c, c') = 0$, for every $c, c' \in C$, $c \neq c'$.

$(\text{msh } \Gamma)(c, c) = K_c$, for every $c \in C$.

Our aim is to show that the set $C = \{[\text{rad } P] \mid P \text{ projective-injective indecomposable}\}$ is a configuration of $\Gamma_s(A)$. To this end, we have the following proposition and theorem.

(2.5) Proposition. Let $f: (\Gamma', m') \rightarrow (\Gamma, m)$ be a covering of translation modulated graphs, and let $C' = f^{-1}(C)$ for $C \subseteq \Gamma_0$.

Then, the following conditions are equivalent:

- (i) C is a configuration of Γ .
- (ii) C' is a configuration of Γ' .

Proof. Immediate from (II, 7.5).

(2.6) Theorem. Let A be a selfinjective algebra of finite representation type of class Δ . Then, there exists a K -linear functor $F: \text{msh } \mathbb{Z}\Delta \rightarrow \overline{\text{mod}} A$ which is well-behaved, that is, satisfies the following two conditions:

- (i) $Fx \in \overline{\text{ind}} A$ and $[Fx] = \pi x$, for every $x \in (\mathbb{Z}\Delta)_0$
- (ii) $F(B(\alpha)) \cong \overline{\text{Irr}} (Fx, Fy)$ for every $\alpha \in (\mathbb{Z}\Delta)_1$ with $o(\alpha) = x$, $e(\alpha) = y$.

Proof. We follow [23] with the modifications that we have to introduce to take care of the non-trivial modulation we have in our general case. Let $\mathcal{F}$ be a complete τ -slice of $\mathbb{Z}\Delta$ such that every vertex of $\mathcal{F}$ is either a sink or a source. Clearly, $(\mathbb{Z}\Delta)_0$ is the disjoint union of the sets P , consisting of all predecessors of $\mathcal{F}$ together with the sources of $\mathcal{F}$, and S , consisting of all successors of $\mathcal{F}$ together with the sinks of $\mathcal{F}$. Now, the condition (i) determines F on the objects of $\text{msh } \mathbb{Z}\Delta$, so we have to construct F in such a way that condition (ii) is satisfied. To construct F on P , we use induction on $\ell(x) := 2m$, if $x = \tau^m t$, t source of $\mathcal{F}$, $\ell(x) := 2m-1$, if $x = \tau^m t$, t sink of $\mathcal{F}$, to define $F(B(\alpha))$ on the bimodules $B(\alpha)$ with $o(\alpha) = x$. First, if $\ell(x) = 0$, then $x \in \mathcal{F}$ is a source of $\mathcal{F}$, so we take any isomorphism $f: B(\alpha) \xrightarrow{\sim} \overline{\text{Irr}}(Fx, Fy)$, if $e(\alpha) = y$. This is possible under our assumption that the ground field K is perfect, for, then, we have $\text{rad}(X, Y) = M(X, Y) \oplus \text{rad}^2(X, Y)$, with $M(X, Y) \xrightarrow{\sim} \text{Irr}(\overline{X}, \overline{Y})$, and similarly $\overline{\text{rad}}(\overline{X}, \overline{Y}) = \overline{M}(\overline{X}, \overline{Y}) \oplus \overline{\text{rad}}^2(\overline{X}, \overline{Y})$ in $\overline{\text{mod}} A$, for every pair of indecomposable A -modules X, Y , and, by construction of $\text{msh } \Gamma_S(A)$, $B(\alpha) \xrightarrow{\sim} \overline{\text{Irr}}(Fx, Fy)$, so we can choose $B(\alpha) \xrightarrow{\sim} \overline{M}(Fx, Fy)$ as our starting isomorphisms. Next, let $\ell(x) > 0$, and let $x^+ = \{y_1, \dots, y_r\}$, $\{\alpha_1, \dots, \alpha_r\}$ all the arrows of $(\mathbb{Z}\Delta)_1$ such that $o(\alpha_i) = x$,

$e(\alpha_i) = y_i$ ($1 \leq i \leq r$); fix $i \in \{1, \dots, r\}$ and let $\beta_i = \sigma^{-1} \alpha_i$.

Since $\ell(y_i) = \ell(x) - 1$, by induction isomorphisms $B(\beta_i) \cong \overline{\text{Irr}}(Fy_i, F(\tau^{-1}x))$ have already been chosen, so we can take $f_{i1}, \dots, f_{ir_i} \in \overline{\text{Hom}}(Fy_i, F(\tau^{-1}x))$ such that their classes modulo $\overline{\text{rad}}^2(Fy_i, F(\tau^{-1}x))$ form a basis of

$\overline{\text{Irr}}(Fy_i, F(\tau^{-1}x))_{K_{Fy_i}}$. Consider now the Auslander-Reiten sequence

$$(*) \quad 0 \rightarrow Fx \xrightarrow{\begin{pmatrix} g_1 \\ \vdots \\ g_r \end{pmatrix}} \oplus_{i=1}^r (Fy_i) \xrightarrow{\begin{pmatrix} f_1 \\ \vdots \\ f_r \end{pmatrix}^t} F(\tau^{-1}x) \rightarrow 0, \text{ with}$$

$f_i = (f_{i1}, \dots, f_{ir_i})$. Since $\overline{M}(Fx, Fy_i) \cong {}^* \overline{M}(Fy_i, F(\tau^{-1}x))$ ([27]),

we can then choose $g_i = (g_{i1}, \dots, g_{ir_i})^t$ in such a way that they

are a basis of $[\overline{M}(Fx, Fy_i)]_{K_{Fy_i}}$, that is, their classes modulo

$\overline{\text{rad}}^2(Fx, Fy_i)$ form a basis of $\overline{\text{Irr}}(Fx, Fy_i)$. The construction of F

on S is completely similar.

(2.7) Theorem. Let $C = \{[\text{rad } P] \mid P \in \text{proj ind } A\}$, A a selfinjective algebra of finite representation type. Then, C is a configuration of $\Gamma_s(A)$.

Proof. Let $F: \text{msh } \mathbb{Z}\Delta \rightarrow \overline{\text{mod}} A$ be the functor of (2.6), and let $C_0 = \{[\text{rad } P] \mid P \in \text{proj ind } A\}$. Since C_0 satisfies the conditions (i) and (ii) of (2.3), $F^{-1}(C_0)$ satisfies the corresponding conditions, that is, it is a configuration of $\text{msh } \mathbb{Z}\Delta$. On the other hand, the canonical projection $\pi: \mathbb{Z}\Delta \rightarrow \Gamma_s(A)$ induces the covering

functor $\text{msh } \pi: \text{msh } \mathbb{Z}\Delta \rightarrow \text{msh } \Gamma_s(A)$, and $\pi^{-1}(C) = F^{-1}(C_0)$;

Hence C is a configuration of $\Gamma_s(A)$, by (2.5).

(2.8) Let A be selfinjective and of finite representation type, and let $C = \{[\text{rad } P] \mid P \in \text{proj ind } A\}$. By (2.3) we have that $\Gamma(A) = \Gamma_s(A) \cup C =: (\Gamma_s(A))_C$, where $(\Gamma_s(A))_C$ stresses the fact that we are working on the modulated graph, adding trivially valued arrows $[\text{rad } P] \rightarrow [P] \rightarrow [P/\text{soc } P]$, the division ring $E = \text{End } P/\text{rad End } P$ at $[P]$, and the bimodules ${}^E E_E$ at the arrows stopping and starting at $[P]$. On the other hand, let $\pi: \mathbb{Z}\Delta \rightarrow \mathbb{Z}\Delta/W \simeq \Gamma_s(A)$ be the covering of the underlying translation graph $\Gamma_s(A)$. If we consider now $(\Gamma_s(A))_C$, we can add to the modulated translation graph $\mathbb{Z}\Delta$ new vertices $p(c)$, one for each $c \in \pi^{-1}(C)$ and new arrows $c \rightarrow p(c) \rightarrow \tau^{-1}c$. If, moreover, we put $K_{p(c)} := K_c$, $B(\alpha) = B(\beta) := K_c (K_c)_{K_c}$ for α and β the arrows with $o(\alpha) = c$, $e(\alpha) = p(c) = o(\beta)$, $e(\beta) = \tau^{-1}c$, we obtain in this way a modulated valued graph $(\mathbb{Z}\Delta)_C$. Clearly, the action of W on $\mathbb{Z}\Delta$ extends to $(\mathbb{Z}\Delta)_C$, and, denoting again by $\pi: (\mathbb{Z}\Delta)_C \rightarrow (\mathbb{Z}\Delta)_C/W$ the canonical projection, we have $(\mathbb{Z}\Delta)_C/W \simeq (\Gamma_s(A))_C \simeq \Gamma(A)$.

$$\begin{array}{ccccc}
 (\mathbb{Z}\Delta)_C & \rightarrow & (\mathbb{Z}\Delta)_C/W & \simeq & (\Gamma_s(A))_C \simeq \Gamma(A) \\
 \uparrow & & & & \uparrow \\
 \mathbb{Z}\Delta & \rightarrow & \mathbb{Z}\Delta/W & \simeq & \Gamma_s(A)
 \end{array}$$

Since $\mathbb{Z}_\Delta$ is the universal covering of $\Gamma_\Delta(A)$, $(\mathbb{Z}_\Delta)_C$ will be called the universal covering of $\Gamma(A)$, and will be denoted by $\tilde{\Gamma}(A)$.

(2.9) An algebra A of finite representation type is said to be standard if it is basic and there is an isomorphism $\text{msr } \Gamma(A) \cong \text{ind } A$ ([7]). Such an algebra is uniquely determined, thus, up to isomorphism, by its Auslander-Reiten graph. The previous construction (2.8) then shows that a selfinjective standard algebra of finite representation type is uniquely determined, up to isomorphism, by the data $(\mathbb{Z}_\Delta, C, W)$, where C is a configuration of $\mathbb{Z}_\Delta$ and W is an admissible group of automorphisms of $\mathbb{Z}_\Delta$ stabilizing C , the group W itself uniquely determined up to conjugation. As a result, the determination of all possible selfinjective algebras of finite representation type and given Dynkin class Δ leads to determine all possible triples $(\mathbb{Z}_\Delta, C, W)$. In the next section we determine the admissible groups W for $\Delta = B_n, C_n, F_4, G_2$. We leave the discussion of the possible configurations of B_n and C_n for Chapter IV whereas the configurations of F_4 and G_2 are discussed in Chapter V.

(2.10) Let A be an algebra of finite representation type. We call $M_A := \bigoplus_{X \in \text{ind } A} X$ and $E_A := \text{End } M_A$ the Auslander module and the Auslander algebra of A , respectively. Both are uniquely determined.

up to isomorphism; moreover, E_A is a basic algebra. From [1], it is well known that A and E_A determine each other, up to Morita equivalence and isomorphism, respectively, hence up to isomorphism if A is also basic. Using the Auslander-Reiten graph $\Gamma(A)$, we can describe E_A and its modules in terms of $\Gamma(A)$:

Proposition. $\Sigma = (K_{x,y}, B_{x,y})_{x,y \in \Gamma(A)_0}$, where $K_x := \text{End } X / \text{rad } \text{End } X$, $x = [X]$, $B_{x,y} := K_x [\text{Irr}(Y, X)] K_y$, for every $X, Y \in \text{ind } A$, is the species of E_A .

Proof. Let $\text{mod } A \rightarrow \text{mod } E_A$, $X_A \mapsto X'_A := \text{Hom}_A(M_A, X_A)$, $f \mapsto f' := \text{Hom}_A(M_A, f)$, for every $f: X \rightarrow Y$ in $\text{mod } A$ be the representable functor $F := \text{Hom}_A(M_A, -)$ defined by an Auslander module M_A of A . Then F establishes an equivalence $\text{mod } A \simeq \text{proj } E_A$, and hence also $\text{ind } A \simeq \text{proj}_0 \text{ind } E_A$. Let $E_A \simeq T(\Delta)/J$ (II, (4.2)). Then (II, (6.7, fi)), $\text{proj ind } E_A \simeq \otimes \Delta/J$, and from $(\otimes \Delta/J)(x', y') \simeq \text{Hom}_{E_A}(X', Y') \simeq \text{Hom}_A(X, Y)$, and the fact that there is a bimodule ${}_y M_x \neq 0$ in Δ if and only if ${}_y M_x \simeq \text{rad}_{E_A}(X', Y') / \text{rad}_{E_A}^2(X', Y') \neq 0$, and this last bimodule is isomorphic to $\text{rad}_A(X, Y) / \text{rad}_A^2(X, Y) \simeq \text{Irr}(X, Y)$, together with the isomorphism $\text{End}_{E_A} X' / \text{rad } \text{End}_{E_A} X' \simeq \text{End}_A X / \text{rad } \text{End}_A X$, for every $X \in \text{ind } A$, we obtain that $\Delta \simeq \Sigma$, as claimed.

(2.11) Corollary. $E_A \cong T(\Gamma(A))/J$.

Proof. $\Sigma = \Sigma(\Gamma(A))$ (II, (6.8)).

We can be more specific about the relation ideal J of (2.11) in case the algebra A is standard:

(2.12) Corollary. Let A be standard. Then:

$$(i) \quad E_A \xrightarrow{\sim} T(\Gamma(A))/I_{\Gamma(A)}$$

$$(ii) \quad \text{mod } E_A \cong ((\text{msh } \Gamma(A))^{\text{op}}, \text{Ab}).$$

Proof.

$$(i) \quad E_A = \text{End } M_A \xrightarrow{\sim} \bigoplus_{X,Y} \text{Hom}_A(X,Y) = \bigoplus_{X,Y} (\text{ind } A)(X,Y) \xrightarrow{\sim} \bigoplus_{x,y} (\text{msh } \Gamma(A))(x,y) \\ \xrightarrow{\sim} T(\Gamma(A))/I_{\Gamma(A)}, \quad \text{by (II, (6.8))}.$$

$$(ii) \text{ mod } E_A \cong \text{mod } T(\Gamma(A))/I_{\Gamma(A)} \cong ((\text{msh } \Gamma(A))^{\text{op}}, \text{Ab}), \text{ by (i)}$$

and (II, (6.9)).

(2.13) The following diagram illustrates the situation of the previous propositions:

$$\begin{array}{ccccc}
 ((\text{msh } \Gamma(A)^{\text{op}}, \text{Ab}) & \simeq & \text{mod } E_A & & \\
 \uparrow & & \uparrow & & \\
 \text{msh } \Gamma(A) & \simeq & \text{proj } E_A & \simeq & \text{mod } A \simeq \mathcal{R}(\Sigma, J) \\
 & & \uparrow & & \uparrow \\
 & \simeq & \text{proj ind } E_A & \simeq & \text{ind } A
 \end{array}$$

(2.14) Let A be selfinjective and standard, and let $\mathbb{P}(A)_0$ be the set of projective vertices of $\Gamma(A)$, $B := \bigoplus_{p,q \in \mathbb{P}(A)_0} \text{msh } \Gamma(A)(p,q)$.

Proposition.

$$(i) \quad A \simeq B$$

$$(ii) \quad X_A \simeq \bigoplus_{p \in \mathbb{P}(A)_0} (\text{msh } \Gamma(A))(p, x), \text{ for every } X_A.$$

Proof.

$$(i) \quad A = \text{End } (A_A) \simeq \bigoplus_{P,Q} \text{Hom}_A(P,Q) = \bigoplus_{P,Q} (\text{ind } A)(P,Q) \simeq \bigoplus_{p,q} (\text{msh } \Gamma(A))(p,q).$$

Here, P and Q resp. p and q run through $\text{proj ind } A$ resp. $\mathbb{P}(A)_0$.

$$(ii) \quad X_A \simeq \text{Hom}_A(A, X_A) \simeq \bigoplus_P \text{Hom}_A(P, X) = \bigoplus_P (\text{ind } A)(P, X) \simeq \bigoplus_p (\text{msh } \Gamma(A))(p, x).$$

Here P resp. p runs through $\text{proj ind } A$ resp. $\mathbb{P}(A)_0$, $p = [P]$,

$\bar{x} = [X]$.

(2.15) Let $\mathbb{P}(A)$ be the full subcategory of $\text{msh } \Gamma(A)$ with $\mathbb{P}(A)_0$ as set of objects, and let $A \simeq T(\Sigma)/J$. Then

Proposition. $\mathbb{P}(A) \simeq \otimes \Sigma / J$.

Proof. The functor $\text{msh } \Gamma(A) \rightarrow (\mathbb{P}(A)^{\text{op}}, \text{Ab})$, $x \mapsto \text{msh } \Gamma(A)(_, x) | \mathbb{P}(A)$ yields, under our assumptions, an equivalence $\text{msh } \Gamma(A) \simeq \text{ind}(\mathbb{P}(A)^{\text{op}}, \text{Ab})$

([7]). Hence we have

$$\begin{array}{ccccc}
R(\Sigma/J) & \xrightarrow{\sim} & \text{mod } A & \xrightarrow{\sim} & (\mathbb{P}(A)^{\text{op}}, \text{Ab}) \\
& & \uparrow & & \uparrow \\
& & \text{ind } A & \xrightarrow{\sim} \text{msh } \Gamma(A) & \xrightarrow{\sim} \text{ind}(\mathbb{P}(A)^{\text{op}}, \text{Ab}) \\
& & \uparrow & & \uparrow \\
\otimes \Sigma/J & \xrightarrow{\sim} & \text{proj ind } A & \xrightarrow{\sim} & \text{proj ind}(\mathbb{P}(A)^{\text{op}}, \text{Ab}) \xrightarrow{\sim} \mathbb{P}(A),
\end{array}$$

thus, $\mathbb{P}(A) \xrightarrow{\sim} \otimes \Sigma/J$, as claimed.

(2.16) Remark. By the previous result, to describe the indecomposable A -modules all we need is to describe the functors $\text{msh } \Gamma(A)(_, x) \mid \mathbb{P}(A) \xrightarrow{\sim} \underline{X} = (\text{msh } \Gamma(A)(p, x), {}_q\varphi_p)$, p, q running through $\mathbb{P}(A)_0$. Although the actual description of course depends on Σ and J , we can, however, relate these functors to the corresponding functors $\text{msh } \tilde{\Gamma}(A)(_, \tilde{x}) \mid \tilde{\mathbb{P}}(A)$, thus reducing the description of the former to a description of the latter, in the following way: the canonical projection $\pi: \tilde{\Gamma}(A) \rightarrow \Gamma(A)$ induces functors $\text{msh } \pi: \text{msh } \tilde{\Gamma}(A) \rightarrow \text{msh } \Gamma(A)$ and $\pi_*: ((\text{msh } \tilde{\Gamma}(A)^{\text{op}}, \text{Ab}) \rightarrow (\text{msh } \Gamma(A)^{\text{op}}, \text{Ab})$, and from (II, (7.5)) we have

$$\text{msh } \Gamma(A)(p, x) \xrightarrow{\sim} \bigoplus_{\tilde{p} \mid p} \text{msh } \tilde{\Gamma}(A)(\tilde{p}, \tilde{x}),$$

for every $\tilde{x} \mid x$, so we are finally led to describe the representations

$$\tilde{\underline{X}} \xrightarrow{\sim} (\text{msh } \tilde{\Gamma}(A)(\tilde{p}, \tilde{x}), {}_{\tilde{q}}\varphi_{\tilde{p}}), \quad \tilde{p}, \tilde{q} \text{ running through } \tilde{\mathbb{P}}(A)_0, \text{ where}$$

$\tilde{\mathbb{P}}(A)$ is the full subcategory of $\text{msh } \tilde{\Gamma}(A)$ with objects $\tilde{\mathbb{P}}(A)_0$, the projective vertices of $\tilde{\Gamma}(A)$.

We notice that, in full analogy with the diagram in (2.13), we have the diagram

$$\begin{array}{ccccc}
 (\text{msh } \tilde{\Gamma}(A)^{\text{op}}, \text{Ab}) & \simeq & \text{mod } \tilde{E}_A & & \\
 \uparrow & & \uparrow & & \\
 \text{msh } \tilde{\Gamma}(A) & \xrightarrow{\sim} & \text{proj } \tilde{E}_A & \simeq & \text{mod } \tilde{A} \simeq R(\tilde{\Sigma}, \tilde{J}) \\
 & & \uparrow & & \uparrow \\
 & & \text{proj ind } \tilde{E}_A & \simeq & \text{ind } \tilde{A}
 \end{array}$$

and that we will describe (Σ, J) in terms of $(\tilde{\Sigma}, \tilde{J})$.

III.3 Universal stable modulated graphs

Let Δ be one of the Dynkin graphs B_n ($n \geq 2$), C_n ($n \geq 3$), F_4 or G_2 , and let m be a modulation of $\mathbb{Z}\Delta$ (II, (6.10)). For simplicity, we write $\mathbb{Z}\Delta$ to denote the translation modulated graph $(\mathbb{Z}\Delta, m)$.

Our first task is to determine the admissible groups of $\mathbb{Z}\Delta$:

(3.1) Proposition. The groups $\langle \tau^r \rangle$ are all the admissible groups of $\mathbb{Z}\Delta$, for every $r \geq 1$.

Proof. The map $\pi: x \mapsto \langle \tau \rangle x := \{ \tau^s x \mid s \in \mathbb{Z} \}$, that sends $x \in (\mathbb{Z}\Delta)_0$ to its orbit under τ , clearly establishes an isomorphism $\mathbb{Z}\Delta / \langle \tau \rangle \simeq \Delta$ of (unoriented) valued graphs, and gives rise to the exact sequence $1 \rightarrow \langle \tau \rangle \rightarrow \text{Aut } \mathbb{Z}\Delta \rightarrow \text{Aut } \Delta \rightarrow 1$. But in all

our cases $\text{Aut } \Delta = 1$, so $\langle \tau \rangle = \text{Aut } \mathbb{Z} \Delta$, thus every subgroup of $\text{Aut } \mathbb{Z} \Delta$ is of the form $\langle \tau^r \rangle$, $r \geq 1$, and it is clearly admissible.

(3.2) The Nakayama permutation of $\mathbb{Z} \Delta$ is the map

$v_\Delta: (\mathbb{Z} \Delta)_0 \rightarrow (\mathbb{Z} \Delta)_0$ defined, in each case, by the following formulas:

$$(i) \quad v_{B_n}(a, b) = (a + n - 1, b)$$

$$(ii) \quad v_{C_n}(a, b) = (a + n - 1, b)$$

$$(iii) \quad v_{F_4}(a, b) = (a + 5, b)$$

$$(iv) \quad v_{G_2}(a, b) = (a + 2, b)$$

for every $(a, b) \in (\mathbb{Z} \Delta)_0$.

(3.3) Proposition. Let $x \in (\mathbb{Z} \Delta)_0$ and let $d_x: (\mathbb{Z} \Delta)_0 \rightarrow \mathbb{N}_0$ be the function defined by

$$d_x(z) = \begin{cases} 1, & \text{if } z = x \\ \sum_{y \in z^-} d'_{yz} d_x(y) - d_x(\tau z), & \text{if } > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$(i) \quad \dim \text{msh } \mathbb{Z} \Delta(x, z)_{K_x} = d_x(z)$$

$$(ii) \quad \text{msh } \mathbb{Z} \Delta(x, z) \neq 0 \text{ iff } z \in \text{Supp } d_x$$

$$(iii) \quad |\text{Supp } d_x| < \infty.$$

If $\alpha \in (\mathbb{Z}\Delta)_1$ is an arrow with $o(\alpha) = r$, $e(\alpha) = s$, we set $(d_{rs}, d'_{rs}) = w(\alpha)$; $\text{Supp } d_x$ denotes the support of d_x , i.e. those z such that $d_x(z) \neq 0$.

Proof.

1. If $z = x$, $\dim \text{msh } \mathbb{Z}\Delta(x, x) = \dim(K_x)_{K_x} = 1$.

2. If $z \in x^+$, then $\text{msh } \mathbb{Z}\Delta(x, z) = {}_z B_x$, so

$\dim({}_z B_x)_{K_x} = d'_{xz} = \sum_{y \in z^-} d'_{yz} d_x(y) - d_x(\tau z) = d_x(z)$, for, $d_x(y) = 0$ for every $y \in z^-$, $y \neq x$.

3. Induction step.

We do induction on the distance from x to z , defined as the length of any path from x to z . Clearly, if there is no path from x to z , $d_x(z) = 0$, so we concentrate on positive distances.

Let $\{\dots \varphi'_1 \dots\}$ be the union of bases of the vector spaces

$K_{\tau z}({}_y B_{\tau z})$ and $\{\dots \varphi_1 \dots\}$ the union of the corresponding dual bases of $({}_z B_y)_{K_z} \cong {}^* B'_{y \tau z}$, for every $y \in (\tau z)^+$. Recall that $K_z \cong K_{\tau z}$, for every z , so $c(\tau z) = \sum_i \varphi'_i \otimes \varphi_i$.

Let $\pi: [x, z] \rightarrow (x, z) := [x, z] / I(x, z)$ be the canonical projection from $[x, z] := \otimes \mathbb{Z}\Delta(x, z)$ onto $(x, z) := \text{msh } \mathbb{Z}\Delta(x, z)$,

where $I(x, z) := I_{\mathbb{Z}\Delta}(x, z)$. Let us determine $I(x, z)$: let $\alpha: x \rightarrow z$ be in $I(x, z)$; then $\alpha = \sum_i \varphi_i \otimes \beta_i$, with $\beta_i: x \rightarrow y_i$ ($1 \leq i \leq s$), if $z^- = (y_1, \dots, y_s)$. From $0 = \pi\alpha = \sum (\pi\varphi_i)(\pi\beta_i)$ and $0 = \sum (\pi\varphi_i)(\pi\varphi'_i)$, it follows that $\pi\beta_i$ factors through τz , that is, $\pi((\beta_i)_i) = \pi((\varphi'_i)_i)(\pi\gamma)$, for some $\gamma: x \rightarrow \tau z$.

$$\begin{array}{ccccccc}
 0 & \rightarrow & \tau z & \xrightarrow{\pi((\varphi'_i)_i)} & \oplus y_i & \xrightarrow{\pi((\varphi_i)_i)} & z \rightarrow 0 \\
 & & & \nwarrow \pi\gamma & \uparrow \pi((\beta_i)_i) & & \\
 & & & & x & &
 \end{array}$$

So,

$$\alpha = \sum \varphi_i \otimes \beta_i = \sum \varphi_i \otimes (\beta_i - \varphi'_i \otimes \gamma) + \sum \varphi_i \otimes \varphi'_i \otimes \gamma \quad (1)$$

Note that $\beta_i - \varphi'_i \otimes \gamma \in I(x, y_i)$, for every $i = 1, \dots, s$, and $\sum \varphi_i \otimes \varphi'_i \otimes \gamma = (\sum \varphi_i \otimes \varphi'_i) \otimes \gamma \in I(x, z)$, with $\gamma \in [x, \tau z]$ determined modulo $I(x, \tau z)$. From (1) it follows that

$$\dim I(x, z) = \sum_{y \in z^-} d'_{yz} \dim I(x, y) + \dim(x, \tau z), \quad (2)$$

whence

$$\begin{aligned}
 \dim(x, z) &= \dim[x, z] - \dim I(x, z) \\
 &= \sum d'_{yz} \dim[x, y] - (\sum d'_{yz} \dim I(x, y) + \dim(x, \tau z)) \\
 &= \sum d'_{yz} (\dim[x, z] - \dim I(x, y)) - \dim(x, \tau z) \\
 &= \sum d'_{yz} d_x(y) - d_x(\tau z) \\
 &= d_x(z),
 \end{aligned}$$

by induction. All dimensions are right dimensions over K_x .

(ii) Trivial.

(iii) Let $\mathcal{T}_x$ be the complete τ -slice of $\mathbb{Z}\Delta$ with x as its unique source, and consider it as an oriented Dynkin graph with orientation the inherited orientation from $\mathbb{Z}\Delta$ but with valuation $x \xrightarrow{(d'_x, d_x)} y$. The assertion is now immediate once we observe that if $x_1, \dots, x_n$ is an admissible orientation of $\mathcal{T}_x$, the vector $(d_x(x_1), \dots, d_x(x_n))$ is simply the dimension vector of the indecomposable projective representation $\underline{P}(x)$ for any oriented modulation $\mathfrak{h}$ of $\mathcal{T}_x$, and $(d_x(\tau^{-1}(x_1)), \dots, d_x(\tau^{-1}(x_n))) = c^{-1}(d_x(x_1), \dots, d_x(x_n))$, for any $i \geq 1$, where c is the Coxeter transformation of $\mathcal{T}_x$, for, $c^{-m}(\dim \underline{P}(x)) = \dim(\phi^{-m} \underline{P}(x)) = 0$, if $\text{Card } \Delta = m$, with ϕ the Coxeter functor of $(\mathcal{T}_x, \mathfrak{h})$ ([12]).

(3.4) Remark. A case by case computation shows immediately that $\text{msh } \mathbb{Z}\Delta(x, \vee_{\Delta} x) \cong K_x(K_x)_{K_x}$, and that $\vee_{\Delta} x$ is the point z with maximal distance d_0 to x such that $\text{msh } \mathbb{Z}\Delta(x, z) \neq 0$. We define $\omega_{\Delta} = d_0 + 1$. Hence, $\omega_{B_n} = 2n - 1$, $\omega_{C_n} = 2n - 1$, $\omega_{F_4} = 11$, $\omega_{G_2} = 5$, since $d_0 = 2n - 2, 2n - 2, 10, 4$, respectively (3.2).

(3.5) Proposition. Let S be a simple $\tilde{A}$ -module, P its projective cover and I its injective envelope, respectively. Then

$$(1) \quad v^{-1}[S] = [P / \text{soc } P]$$

$$(11) \quad v[S] = [\text{rad } I] .$$

Proof. Since $\tilde{A}$, the universal covering algebra of A , is a locally finite dimensional selfinjective algebra, we can apply (1.4). Taking $M = S$ in Proposition 2, we have $\text{Tr } D\Omega^1 S \simeq \text{Tr } D \text{ rad } P \simeq P / \text{soc } P$ hence $v^{-1}[S] = [P / \text{soc } P]$, by (3.4). Similarly, taking again $M = S$ in Proposition 3, $D \text{Tr } \Sigma^1 S \simeq D \text{Tr } I / \text{soc } I \simeq \text{rad } I$, hence the result, again by (3.4).

(3.6) Let A be selfinjective of finite representation type with $\Gamma(A) \simeq (\mathbb{Z} \Delta)_C / W$. Let $\mathcal{F}$ be a complete τ -slice of $\mathbb{Z} \Delta$, that is, a connected subgraph of $\mathbb{Z} \Delta$ which contains exactly one representative of the τ -orbits of $\mathbb{Z} \Delta$ under τ . We write $\tau \mathcal{F} = \{\tau[x] \mid x \in \mathcal{F}\}$, similarly for τ^{-1} and $\tau^i \mathcal{F} = \tau(\tau^{i-1} \mathcal{F})$ for every $i \in \mathbb{N}$. We also write $[\tau^k \mathcal{F}, \mathcal{F}]$ for the set of all $x \in (\mathbb{Z} \Delta)_C$ such that there exist paths in $(\mathbb{Z} \Delta)_C$ $\tau^k y \rightarrow x$, $x \rightarrow z$, for some $y, z \in \mathcal{F}$.

Proposition. Let $\text{Card } (\Delta) = r$. Then, $\text{Card } (C \cap [\tau^{\omega-1} \mathcal{F}, \mathcal{F}]) = r$, for any complete τ -slice $\mathcal{F}$ of $\mathbb{Z} \Delta$.

Proof. This is contained in the proof of Proposition 3 of (1.5), since the support $\text{Supp}(\mathcal{F})$ of $\mathcal{F}$ is the set of all pro-

jective vertices p of $(\mathbb{Z} \Delta)_C$ such that there exists some $x \in \mathcal{F}$ with $\text{msh}(\mathbb{Z} \Delta)_C(p, x) \neq 0$. In the course of the proof of Proposition 3, it is established that $\text{Card}(\Delta) = \text{Card}(\text{Supp}(\mathcal{F}))$, hence the result.

CHAPTER IV. SELF-INJECTIVE ALGEBRAS OF DYNKIN

CLASSES B_n and C_n

In this chapter, we construct selfinjective algebras of finite representation type and of Dynkin class B_n or C_n . We give in Section 1 a complete description of the configurations of the stable modulated translation graphs ZZB_n and ZZC_n , establishing bijections with the universal coverings of the Brauer graphs of types B_n and C_n , respectively. The main theorems on the construction of the algebras are proved in Section 2. These are basic, connected, finite dimensional K -algebras of finite representation type and of Dynkin class B_n or C_n . We give descriptions of their modules categories as categories of representations $R(\Sigma, J)$ with Σ a K -species constructed from the K -species associated to the universal covering of the Brauer graphs. The indecomposable representations of these categories are described in Section 3.

Throughout K denotes a perfect field and F/G a quadratic field extension, $G = K$. We modulate ZZB_n and ZZC_n with the modulations introduced in (II, (6.10)) and use the coordinates of the valued translation graphs ZZB_n and ZZC_n defined in (II, (2.2)).

IV.1 Configurations of $\mathbb{ZB}_n$ and $\mathbb{ZC}_n$

In this section we determine all the configurations of the modulated translation graphs $\mathbb{ZB}_n$ ($n \geq 2$) and $\mathbb{ZC}_n$ ($n \geq 3$). Our main results establish a bijection between configurations of $\mathbb{ZB}_n$ and symmetric Brauer pairs (α, β) of order $\omega = 2n - 1$ (1.20) and between configurations of $\mathbb{ZB}_n$ and Brauer graphs of order n (1.40); similar results hold for the configurations of $\mathbb{ZC}_n$ (1.43 and 1.44).

Throughout the section, $\mathbb{ZB}_n$ will denote the modulated translation graph $(\mathbb{ZB}_n, m) \circ (\text{II}, (6.10))$ with a fixed modulation m ; we use the coordinates of $\mathbb{ZB}_n$ introduced in (II, (2.2)), and fix $\omega := \omega_{\mathbb{B}_n} = 2n - 1$, $v := v_{\mathbb{B}_n}$.

(Fig. 1)

(1.1) Definition. Let $x = (a, b) \in (\mathbb{ZB}_n)_0$. The butterfly $\text{Mar}(x)$ is the set of all points $y = (c, d) \in (\mathbb{ZB}_n)_0$ satisfying

either of the following conditions

$$(Mar\ 1) \quad a \leq c < a+b \leq c+d$$

$$(Mar\ 2) \quad a \leq c < a+n \leq r+s < a+b+n.$$

See Figure 2.

(Fig. 2)

(1.2) Remarks.

(i) From the definition (1.1), it is clear that the points $y \in Mar(x)$ are those lying within the shaded region of Figure 2. Since $vx = (a+n-1, b)$, $Mar(x)$ will sometimes be referred to as the butterfly starting at x and stopping at vx .

(ii) The butterfly $Mar((a,1))$ starting at $(a,1)$ will be called the hill starting at $(a,1)$, and will be denoted by H_a . Similarly, the butterfly stopping at $(t-1,1)$ will be called the hill stopping at $(t-1,1)$ and will be denoted by K_t . See Figure 3.

(Fig. 3.a)

(Fig. 3.b)

(Fig. 3)

It is clear that $H_a = K_{a+n}$, $K_t = H_{t-n}$, for every $a, t \in \mathbb{Z}$.

(iii) The sets $\Delta_a := \{(a, r) \mid 1 \leq r \leq n\}$ and $\nabla_t := \{(a, b) \mid a+b = t\}$ will be called the ascending diagonal Δ_a and the descending diagonal ∇_t , for every $a, t \in \mathbb{Z}$. Clearly,
 $H_a = \Delta_a \cup \nabla_{a+n}$, $K_t = \Delta_{t-n} \cup \nabla_t$.

(iv) The orbits $[0, n] = \{(k, n) \mid k \in \mathbb{Z}\}$ and $[0, 1] = \{(k, 1) \mid k \in \mathbb{Z}\}$ will be called the upper border and the lower border of $\mathbb{Z}B_n$ and will be denoted by u and l respectively.

The importance of these sets stems from the following two results:

(1.3) Lemma. Let $x \in (\mathbb{Z}B_n)_0$. The following conditions are equivalent for $y \in (\mathbb{Z}B_n)_0$:

- (i) $\text{msh}(\mathbb{Z}B_n)(x, y) \neq 0$
- (ii) $y \in \text{Mar}(x)$.

Proof. (III, (3.3)).

(1.4) Lemma. Let C be a configuration of $\mathbb{Z}B_n$. Then $|H_a \cap C| = 1$, for every $a \in \mathbb{Z}$.

Proof. Let $a \in \mathbb{Z}$ and consider $H_a = \text{Mar}((a,1))$. By condition (C1) (III, (2.4)), there exists $c \in C$ such that $\text{msh}(\mathbb{Z}B_n)((a,1),c) \neq 0$. By (1.3), $c \in H_a$, so $H_a \cap C \neq \emptyset$. Condition (C2) (III, (2.4)) then implies that c is the only point of C lying on H_a , by considering the butterflies starting at c and stopping at c respectively: $|\text{Mar}(c) \cap C| = 1$, by (C2), and all the successors of c on H_a lie on $\text{Mar}(c)$; similarly, $|\text{Mar}(v^{-1}c) \cap C| = 1$, again by (C2), and all the predecessors of c on H_a lie on $\text{Mar}(v^{-1}c)$.

(1.5) Let $x, y \in (\mathbb{Z}B_n)_0$. We say that x is a predecessor of y and y a predecessor of x , if there exists a path in $\mathbb{Z}B_n$ from x to y . If $x = (a,b)$ and $y = (c,d)$ and x is a predecessor of y , the number $d((a,b), (c,d)) := 2c + d - 2a - b$ will be called the distance from x to y . Clearly, the distance from x to y is the common length of all the paths in $\mathbb{Z}B_n$ from x to y .

(1.6) Let C be a configuration of $\mathbb{Z}B_n$ and let $a, t \in \mathbb{Z}$. The unique point of C lying on H_a will be denoted by h_a if $h_a \in \Delta_a$ and by h_a^* if $h_a^* \in \nabla_{a+n}$. Similarly, the unique point

of C lying on K_t will be denoted by k_t if $k_t \in \nabla_t$ and by k_t^* if $k_t^* \in \Delta_{t-n}$.

(1.7) Remarks.

(i) From the definitions of h_a , k_t , h_a^* and k_t^* , it is obvious that the following identities hold:

- (i) $h_a = k_{a+n}^*$
- (ii) $h_a^* = k_{a+n}$
- (iii) $k_t = h_{t-n}^*$
- (iv) $k_t^* = h_{t-n}$.

See Figure 4 for the cases (i) and (iv), for example.

(Fig. 4)

(ii) We stress the fact that although on every hill there is exactly one configuration point (1.4), every configuration point belongs to exactly two hills. See Figure 5.

(Fig. 5)

This is a fact that will lead us to seek for the relations between a and b , on the one side, and the relations between t and u , on the other, and eventually will allow us to describe the algebras arising from a given configuration. A first step in this direction is to measure the distances from a given configuration point to the starting and stopping points of the hills on which the point lies.

(1.8) Definition. Let C be a configuration of $\mathbb{Z}B_n$. Let $\beta = \beta_C: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by $\beta a := a + d((a, 1), c(a)) + 1$, where $c(a)$ is the point of C lying on the hill H_a .

(1.9) Remark. From the definition of β , it is clear that β is an increasing function satisfying $a < \beta a \leq a + \omega$, for every $a \in \mathbb{Z}$, that is,

$$1 \leq \beta a - a \leq \omega \quad (1)$$

Moreover, we have

$$\begin{aligned} \text{(i)} \quad c(a) = h_a & \quad \text{iff} \quad 1 \leq \beta a - a \leq n \\ \text{(ii)} \quad c(a) = h_a^* & \quad \text{iff} \quad n \leq \beta a - a \leq \omega, \end{aligned}$$

from where we obtain as coordinates of $c(a)$

$$c(a) = (a, \beta a - a), \quad \text{iff} \quad c(a) = h_a \quad (2)$$

$$c(a) = (\beta a - n, 2n - \beta a + a), \quad \text{iff} \quad c(a) = h_a^* \quad (3)$$

(1.10) Proposition. Let $\beta = \beta_C$, C a configuration of ZB_n . Then β is a bijection.

Proof. Let $\gamma: Z \rightarrow Z$ be the function defined by $\gamma t := t - d(e(t), (t-1, 1)) - 1$, if $e(t)$ is the configuration point lying on the hill K_t , $t \in Z$. We show that β and γ are both bijections on Z inverse to each other. We consider four cases:

(1⁰). $1 \leq \beta a - a \leq n$. Then $c(a) = h_a = k_{\beta a} = e(\beta a)$ (see Figure 1), for, $c(a) = (a, \beta a - a) \in \Delta_{\beta a} \subseteq K_{\beta a}$, hence $e(\beta a) = k_{\beta a}$.

(Fig. 1)

From the definitions of β and γ it follows that

$$\begin{aligned} \beta a - a &= d((a, 1), h_a) + 1 = d(k_{\beta a}, (\beta a - 1, 1)) + 1 \\ &= \beta a - \gamma \beta a, \end{aligned}$$

so $\gamma \beta a = a$.

(2⁰). $1 \leq t - \gamma t \leq n$. Then $e(t) = (\gamma t, t - \gamma t) = c(\gamma t)$

(see Figure 2), for, $(\gamma t, t - \gamma t) \in \Delta_{\gamma t} \subseteq H_{\gamma t}$, hence

$c(\gamma t) = (\gamma t, t - \gamma t)$. From here it follows that

(Fig. 2)

$$\begin{aligned}
 t - \gamma t &= d(e(\gamma), (t-1, 1)) + 1 \\
 &= d((\gamma t, 1), c(\gamma t)) + 1 \\
 &= \beta \gamma t - \gamma t,
 \end{aligned}$$

so $\beta \gamma t = t$.

(3⁰). $n \leq \beta a - a \leq \omega$. Then $c(a) = h_a^* = (\beta a - n, 2n - \beta a + a) = k_{\beta a}^* = (\beta a - n, 2n - (\beta a - \gamma \beta a))$. (See Figure 3).

(Fig. 3)

From here, $2n - \beta a + a = 2n - (\beta a - \gamma \beta a)$, so $\gamma \beta a = a$.

(4⁰). $n \leq t - \gamma t \leq \omega$. Then $e(t) = k_t^* = (t - n, 2n - (t - \gamma t)) = h_{\gamma t}^* = (\beta \gamma t - t, 2n - \beta \gamma t + t)$. (See Figure 4).

(Fig. 4)

From here, $t-n = \beta \gamma t - n$, hence $\beta \gamma t = t$.

The cases (1^o) to (4^o) together show that $\beta \gamma = \text{id}$, $\gamma \beta = \text{id}$, as claimed.

(1.11) Proposition. Let $\beta = \beta_C$, C a configuration of $\mathbb{Z}B_n$. Then

(i) β is a periodic function with ω as period:

$$\beta(a+\omega) = \beta a + \omega, \text{ for every } a \in \mathbb{Z}$$

$$(ii) \tau^\omega C = C.$$

Proof. The proposition follows from the following

Assertion. Let $a, b \in \mathbb{Z}$ with $a < b < a+\omega$. Then

$$\beta b \not\equiv \beta a \pmod{\omega}.$$

For, if the Assertion is true, $\beta a, \beta(a+1), \dots, \beta(a+\omega-1)$ are ω incongruent integers modulo ω , as well as $\beta(a+1), \beta(a+2), \dots, \beta(a+\omega-1), \beta(a+\omega)$. Thus, $\beta a \equiv \beta(a+\omega)$ and hence $\beta(a+\omega) = \beta a + \omega$,

since $a < \beta a \leq a + \omega < \beta(a + \omega) \leq (a + \omega) + \omega = a + 2\omega$, and (1) is proved.

On the other hand, let $c = (a, b)$ be a configuration point; then

$\tau^{-\omega} c = (a + \omega, b)$. We show that $(a + \omega, b) \in C$ by considering the two

cases $1 \leq \beta a - a \leq n$ and $n \leq \beta a - a \leq \omega$. In the first case,

$(a, b) = h_a = (a, \beta a - a)$, so $\beta(a + \omega) - (a + \omega) = \beta a + \omega - a - \omega = b$, and

hence $(a + \omega, b) = h_{a + \omega} \in C$. In the second case, $(a, b) = h_a^* =$

$(\beta a - n, 2n - \beta a + a)$, so $(a + \omega, b) = (\beta a - n + \omega, 2n - \beta a + a) = (\beta(a + \omega) - n, 2n - (\beta a + \omega) + (a + \omega))$

$= h_{a + \omega}^* \in C$. Thus, $\tau^{\omega} C = C$ and (ii) is proved, too. So, it remains

to prove the Assertion.

Proof of the Assertion. Suppose, on the contrary, that βb and βa are congruent modulo ω . Since we have the inequalities (1.8, (1)) valid for any $x \in \mathbb{Z}$, the hypothesis $a < b < a + \omega$ implies, together with (1.8; (1)) for $x = a$ and $x = b$, that $\beta a - \omega < \beta b < \beta a + 2\omega$. Thus, either $\beta b = \beta a$ or $\beta b = \beta a + \omega$, hence it must be $\beta b = \beta a + \omega$, since β is bijective. So, we are reduced to prove that $a < b < a + \omega$ and $\beta b = \beta a + \omega$ are incompatible. Let $c(a), c(b)$ be the configuration points lying on H_a and H_b respectively. There are four cases to consider, according to whether $c(x) = h_x$ or h_x^* , $x = a$ or b .

(1⁰). $c(a) = h_a$, $c(b) = h_b$. (See Figure 1).

(Fig. 1)

If $c(a) = h_a = (a, \beta a - a)$ and $c(b) = (b, \beta b - b)$, then

$c(b) \in \nabla_{\beta b} = \nabla_{\beta a + w} \subseteq K_{\beta a + w}$. Consider the point $x \in H_b \cap \nabla_{a+w}$;

its coordinates are $(b, a+w-b)$. Then, the butterfly stopping

at x — the shaded region in Figure 1 — does not contain any configuration point (compare with the butterfly starting at h_a and the butterfly stopping at h_b), in contradiction to condition (C1) of a configuration.

Hence, this case cannot occur.

(2°). $c(a) = h_a$, $c(b) = h_b^*$ (See Figure 2.)

(Fig. 2)

If $c(a) = h_a = (a, \beta a - a)$ and $c(b) = h_b^* = (\beta b - n, 2n - \beta b + b)$,

then $h_b^* \in \nabla_{b+n} \subseteq K_{b+n}$. Consider the point $x = (a+n-1, b-a+1)$;

then, since $(a+n-1) + (b-a+1) = b+n$, we have that $x \in K_{b+n}$,

$x \neq h_b^*$ so it is not a configuration point, and the butterfly

stopping at x — the shaded region of Figure 2 — does not contain

any configuration points, by comparing the overlappings of $\text{Mar}(\nabla^{-1}x)$

with $\text{Mar}(h_a)$ and $\text{Mar}(\nabla^{-1}h_b^*)$. This case, hence, does not occur.

(3⁰). $c(a) = h_a^*$, $c(b) = h_b^*$. (See Figure 3).

(Fig. 3)

If $c(a) = h_a^* = (\beta a - n, 2n - \beta a + a)$, $c(b) = h_b^* = (\beta b - n, 2n - \beta b + b)$, consider the point x on the intersection of ∇_{a+w} and ∇_b . Then $x = (b, w + a - b) \in H_b$, $x \neq h_b^*$ is not a configuration point and the butterfly stopping at x - the shaded region of Figure 3 - again does not contain any configuration points; compare with the butterflies $\text{Mar}(h_a^*)$ and $\text{Mar}(\nu^{-1} h_b^*)$.

(4⁰). $c(a) = h_a^*$, $c(b) = h_b$. Suppose $c(a) = h_a^*$; then $\beta a - a \geq n$, so $\beta b - b > \beta b - a - w = \beta a + w - a - w = \beta a - a \geq n$, so $c(b) = h_b^*$, and this last case does not occur at all, for then we are back in the impossible case (3⁰).

The proof of the Assertion is complete, and so is the Proposition.

(1.12) The function $\beta = \beta_C$ measures the distance of the configuration point $c(a)$ on a hill H_a to the starting point $(a, 1)$ of the hill. We now introduce another function $\alpha = \alpha_C$ that measures the distance from $c(a)$ to the ending point of the hill H_a .

Definition. Let C be a configuration of $\mathbb{Z}B_n$. Let $\alpha = \alpha_C: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by

$$\alpha t := t + d((t-n, 1), e(t)) + 1,$$

where $e(t)$ is the point of C lying on the hill K_t .

(1.13) Remark. From the definition of α , it is clear that α is an increasing function satisfying $t < \alpha t \leq t+w$, for every $t \in \mathbb{Z}$, that is

$$1 \leq \alpha t - t \leq w \quad (1)$$

Moreover, we have

$$(i) \quad e(t) = k_t \quad \text{iff} \quad n \leq \alpha t - t \leq w$$

$$(ii) \quad e(t) = k_t^* \quad \text{iff} \quad 1 \leq \alpha t - t \leq n,$$

from where we obtain as coordinates of $e(t)$

$$e(t) = (\alpha t - 2n, 2n - \alpha t + t), \quad \text{iff} \quad e(t) = k_t \quad (2)$$

$$e(t) = (t - n, \alpha t - t), \quad \text{iff} \quad e(t) = k_t^* \quad (3)$$

(1.14) Proposition. Let $\alpha = \alpha_C$, C a configuration of $\mathbb{Z}B_n$. Then α is a bijection.

Proof. Let $\delta: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by $\delta x = (x - d(e'(x-n), (x-n-1, 1)) - 1$, where $e'(x-n)$ denotes the configuration point lying on the hill K_{x-n} , for every $x \in \mathbb{Z}$. Similarly to the proof of (1.10), we show that α and δ are both bijections on $\mathbb{Z}$ inverse to each other, by considering the four possible cases $1 \leq \alpha t - t \leq n$, $n \leq \alpha t - t \leq \omega$, $1 \leq \delta x - x \leq n$ and $n \leq \delta x - x \leq \omega$.

(1⁰). $1 \leq \alpha t - t \leq n$. Then $e(t) = k_t^* = (t-n, \alpha t - t) = k_{\alpha t - n}$.

(See Figure 1).

(Fig. 1)

From here, we see that $d((t-n, 1), k_t^*) = d(k_{\alpha t - n}, (\alpha t - n - 1, 1))$,

hence

$$\begin{aligned} \delta(\alpha t) &= \alpha t - d(k_{\alpha t - n}, (\alpha t - n - 1, 1)) - 1 \\ &= t + d((t-n, 1), k_t^*) - d(k_{\alpha t - n}, (\alpha t - n - 1, 1)) - 1 \\ &= t. \end{aligned}$$

(2°). $n \leq \alpha t - t \leq \omega$. Then $e(t) = k_t = (\alpha t - 2n, 2n - \alpha t + t) = k_{\alpha t - n}^*$.

(See Figure 2.)

(Fig. 2)

From here, clearly $d((\alpha t - 2n, 1), k_t) = d(k_{\alpha t - n}^*, (t-1, 1))$,

hence $d((t-n, 1), k_t) = d(k_{\alpha t - n}^*, (\alpha t - n - 1, 1))$, and therefore

$$\begin{aligned} \delta(\alpha t) &= \alpha t - d(e'(\alpha t - n), (\alpha t - n - 1, 1)) - 1 \\ &= t + d((t-n, 1), e(t)) - d(e'(\alpha t - n), (\alpha t - n - 1, 1)) - 1 \\ &= t. \end{aligned}$$

The cases $1 \leq \delta x - x \leq n$ and $n \leq \delta x - x \leq \omega$ are quite similar.

In the following lemma we relate the functions α and β to obtain as a corollary that α is also a periodic function with ω as a period. This result could also be proved by methods similar to the ones used to prove the periodicity of β (1.11).

(1.15) Lemma. Let C be a configuration of $\mathbb{Z}B_n$, $\alpha = \alpha_C$ and $\beta = \beta_C$. Then $\alpha\beta a = a + 2n$, for every $a \in \mathbb{Z}$.

Proof. Let $c(a)$ be the configuration point on the hill H_a .

We have to consider two cases, according to the position of $c(a)$

on H_a : either $c(a) \in \Delta_a$ or $c(a) \in \nabla_{a+n}$.

(1⁰). $c(a) = h_a$. Then (see Figure 1),

(Fig. 1)

$h_a = (a, \beta a - a) \in \nabla_{\beta a} \subseteq K_{\beta a}$, so $h_a = k_{\beta a} = (\alpha(\beta a) - 2n, 2n - \alpha(\beta a) + \beta a)$

and hence $a = \alpha\beta a - 2n$, as claimed.

(2⁰). $c(a) = h_a^*$. Then (see Figure 2),

(Fig. 2)

$h_a^* = (\beta a - n, 2n - \beta a + a) = k_{\beta a}^* = (\beta a - n, \alpha(\beta a) - \beta a)$, hence $2n - \beta a + a = \alpha\beta a - \beta a$,

and $\alpha\beta a = a + 2n$, as claimed.

(1.16) Lemma.

(i) $\alpha(t+n) = \beta t + n$

(ii) $\alpha t = \beta(t-n) + n$, for every $t \in \mathbb{Z}$.

Proof.

(i) We consider two cases, according to the position of $c(t)$ on the hill H_t .

(1⁰) $c(t) = h_t$. Then $1 \leq \beta t - t \leq n$ and $h_t = (t, \beta t - t)$
 $= k_{t+n}^* = ((t+n)-n, \alpha(t+n)-(t+n))$ (see Figure 1), hence $\beta t - t = \alpha(t+n)-(t+n)$,
 or $\alpha(t+n) = \beta t + n$, as claimed.

(Fig. 1)

(2⁰) $c(t) = h_t^*$. Then $n \leq \beta t - t \leq \omega$ and $h_t^* = (\beta t - n, 2n - \beta t + t)$
 $= k_{t+n} = (\alpha(t+n) - 2n, 2n - \alpha(t+n) - (t+n))$ (see Figure 2), hence

(Fig. 2)

$\beta t - n = \alpha(t+n) - 2n$, or $\alpha(t+n) = \beta t + n$, as claimed.

(ii) It follows from (i) by setting $t = s+n$: $\alpha t = \alpha(s+n)$
 $= \beta s + n = \beta(t-n) + n$.

(1.17) Corollary. Let $\alpha = \alpha_C$, C a configuration of $\mathbb{Z}B_n$. Then α is periodic with ω as period: $\alpha(t+\omega) = \alpha t + \omega$, for every $t \in \mathbb{Z}$.

Proof. $\alpha(t+\omega) = \alpha(t+n-1+n) = \beta(t+n-1)+n = \beta(x+2n-1)+n = \beta x + 2n - 1 + n = \beta(t-n)+3n-1 = \alpha t - n + 3n - 1 = \alpha t + \omega$, by setting $x = t-n$, using (1.16), (i) and (ii) and the periodicity of β (1.11).

(1.18) Let C be a configuration of $\mathbb{Z}B_n$, $\alpha = \alpha_C$ and $\beta = \beta_C$ the bijections of $\mathbb{Z}$ associated to C . Up to now, we have shown that both are periodic functions with $\omega = 2n-1$ as period (1.11), (1.17) and such that $\alpha\beta x = x+2n = x+\omega+1$ (1.15):

$$\alpha(x+\omega) = \alpha x + \omega \quad (1)$$

$$\beta(x+\omega) = \beta x + \omega \quad (2)$$

$$\alpha\beta x = x + \omega + 1 \quad (3)$$

Pairs of functions (α, β) satisfying (1), (2), (3) are well known to be related to the so called Brauer relations of order ω ([19]), and the associated concepts of Brauer-quivers, and Brauer trees. We call them here Brauer pairs of order ω .

Let (α, β) be a Brauer pair of order ω . Let $U(\omega)$ be the multiplicative group of complex ω -roots of unity, $\mathbb{Z}/\omega\mathbb{Z}$ the additive group of integers modulo ω . The functions $\bar{\alpha}, \bar{\beta}: \mathbb{Z}/\omega\mathbb{Z} \rightarrow \mathbb{Z}/\omega\mathbb{Z}$, defined by $\bar{\alpha}\bar{x} = \overline{\alpha x}$, $\bar{\beta}\bar{x} = \overline{\beta x}$, $\bar{x} = x + \omega\mathbb{Z}$, are well defined,

by (1) and (2) respectively, and satisfy $\overline{\alpha\beta x} = \overline{\gamma x}$, $\overline{\gamma x} = \overline{x+1}$,
 by (3). Through the isomorphism $\mathbb{Z}/\omega\mathbb{Z} \rightarrow U(\omega)$, $\overline{x} \mapsto u(x) := \exp(2\pi i x/\omega)$,
 we can represent each $\overline{x}$ by its image $u(x)$, and construct a trivially
 valued oriented graph Q with vertices $U(\omega)$ and arrows $u(x) \rightarrow u(\alpha x)$,
 $u(x) \rightarrow u(\beta x)$, which is called a Brauer quiver with ω vertices. We
 usually relax the notation, identifying $\overline{x}$ with $u(x)$ and write
 $\overline{x} \rightarrow \overline{\alpha x}$, $\overline{x} \rightarrow \overline{\beta x}$ for the arrows of Q . The set of orbits $\{\overline{\beta^r x} \mid r \geq 1\}$
 gives a partition of $U(\omega)$ such that the convex hulls of the equivalence
 classes, in the complex plane $\mathbb{C}$, do not intersect. Such a partition
 is called a Brauer relation of order ω . Alternatively, a Brauer
 quiver Q with ω vertices may be defined as a connected quiver which
 is the union of the cycles which are contained in Q , and such that
 every vertex belongs to exactly two cycles and any two cycles meet in
 one vertex at most. We call a vertex a transit point if one of the
 two cycles which contains the vertex is a loop, and call nodes the
 other vertices of Q . The cycles of Q may be divided into two
 camps, an α -camp and a β -camp, in such a way that neighbouring
 cycles belong to different camps. If we choose as vertex set Q_0
 precisely $\mathbb{Z}/\omega\mathbb{Z}$, the maps $\overline{x} \mapsto \overline{\alpha x} :=$ endpoint of the α -arrow
 starting at $\overline{x}$, $\overline{x} \mapsto \overline{\beta x} :=$ endpoint of the β -arrow starting at
 $\overline{x}$, we recover a Brauer pair (α, β) on $\mathbb{Z}$ that satisfies (1)-(2).

Thus, the notions of Brauer pair (α, β) of order ω , Brauer quiver Q with ω vertices and Brauer relation of order ω are essentially equivalent.

To each configuration C of $\mathbb{Z}B_n$ we have thus associated a Brauer pair of order ω , but the converse is not true: not every Brauer pair of order ω is associated to a configuration of $\mathbb{Z}B_n$. To characterize those Brauer pairs that correspond to configurations of $\mathbb{Z}B_n$, we need the notion of symmetric Brauer pair. A Brauer quiver is said to be symmetric ([24]) if it admits an automorphism exchanging the α -arrows with the β -arrows.

(1.19) Proposition. Let C be a configuration of $\mathbb{Z}B_n$, $\alpha = \alpha_C$, $\beta = \beta_C$ its associated functions on $\mathbb{Z}$. Then (α, β) is a symmetric Brauer pair of order ω .

Proof. The Brauer pair (α, β) is symmetric if and only if the β -arrows of the Brauer quiver Q associated with (α, β) are stable under the transformation $(\bar{x} \rightarrow \bar{y}) \mapsto (\overline{y-n} \rightarrow \overline{x-n+1})$ ([24]), that is, if and only if $\overline{\beta x - n} \rightarrow \overline{x - n + 1}$ is a β -arrow for every β -arrow $\bar{x} \rightarrow \bar{\beta x}$, and this is the case if and only if $\beta(\beta x - n) \equiv x - n + 1 \pmod{\omega}$, for every $x \in \mathbb{Z}$. Now, by (1.16, ii) and (1.15) we have $\beta(\beta x - n) = \alpha(\beta x) - n = (x + 2n) - n = x - n + 1 + 2n - 1 = x - n + 1 + \omega \equiv x - n + 1 \pmod{\omega}$, as claimed.

(1.20) Theorem. The assignment $C \mapsto (\alpha_C, \beta_C)$ establishes a bijection between the configurations of $\mathbb{Z}\mathbb{B}_n$ and the symmetric Brauer pairs of order w .

Proof. Let (α, β) be a symmetric Brauer pair of order w . Then $C := \{(a, \beta a - a) \in (\mathbb{Z}\mathbb{B}_n)_0 \mid 1 \leq \beta a - a \leq n\}$ clearly is a configuration of $\mathbb{Z}\mathbb{B}_n$. The converse has already been proven (1.19).

(1.21) Remark. The bijection of (1.20) has the interesting consequence that the configurations of $\mathbb{Z}\mathbb{B}_n$ are related to certain symmetric configurations of $\mathbb{Z}\mathbb{A}_w$, since there is a bijection between Brauer pairs of order w and configurations of $\mathbb{Z}\mathbb{A}_w$. As a result, the configurations of $\mathbb{Z}\mathbb{B}_n$ are related to selfinjective algebras of finite representation type of Dynkin class $\mathbb{A}_w$. Since we are interested, however, in the selfinjective algebras of finite representation type of Dynkin class $\mathbb{B}_n$, we will relate the configurations of $\mathbb{Z}\mathbb{B}_n$ to a valued graph with n vertices, a Brauer graph of type $\mathbb{B}_n$, out of which we will produce selfinjective algebras of type $\mathbb{B}_n$. The above bijection tells us that there will be a bijection, to be made precise in the sequel, between selfinjective algebras B of class $\mathbb{B}_n$, and selfinjective algebras A of class $\mathbb{A}_w$ with symmetric Auslander-Reiten graph, in such a way that $U\Gamma(B) \cong \Gamma(A)/\rho$, where ρ is the equivalence relation defined on $\mathbb{Z}\mathbb{A}_w$ by the reflection on the

central line of $\mathbb{Z}A_\omega$, $\rho(a,b) := (a+b-n, 2n-b)$ by setting $(a,b) \sim (a',b')$ if $(a',b') = \rho(a,b)$, and where $U\Gamma(B)$ denotes the underlying trivially valued translation graph of $\Gamma(B)$ (forget the valuation).

(1.22) We recall (II, (1.7)) that a Brauer graph of type B_n is the valued graph D obtained from a Brauer quiver Q with n vertices by selecting one transit point q_0 and endowing the arrows of Q with the valuation $w(\delta) = (2,1)$, $w(\epsilon) = (1,2)$, if $e(\delta) = q_0 = o(\epsilon)$ and $w(\eta) = (1,1)$ for any other arrow, and erasing all loops of Q . We usually put $D_0 = \mathbb{Z}/n\mathbb{Z}$, as above (1.18).

Our aim is to establish a bijection between Brauer graphs of type B_n and configurations of $\mathbb{Z}B_n$. This will be done via the bijection of (1.20) by establishing a bijection between the Brauer graphs of type B_n and the symmetric Brauer pairs of order ω , and, for this, it will be convenient to have some normalization procedures. First, although it is clear that we can always choose the selected transit point q_0 to be $\bar{0}$, we need to know how we can go from any other selected transit point to $\bar{0}$. The next proposition states the precise relationships:

(1.23) Proposition. Let D be a Brauer graph of type B_n , and let $(\bar{\alpha}, \bar{\beta})$ be its associated permutations. Then

(i) $\bar{\beta}' := \bar{\gamma} \bar{\beta} \bar{\gamma}^{-1}$ defines another Brauer graph D' of type B_n

(ii) $D' \cong D$.

Proof. Let $\bar{\beta} = \sigma_1 \dots \sigma_r$ be the cycle decomposition of $\bar{\beta} \in S_n$, the symmetric group of order n . Then, if $\sigma_i = (a_{i_1}, \dots, a_{i_{m_i}})$, then $\bar{\gamma} \sigma_i \bar{\gamma}^{-1} = (a_{i_1+1}, a_{i_2+1}, \dots, a_{i_{m_i}+1})$, for every $i = 1, \dots, r$, so $\bar{\beta}'$ yields a Brauer graph D' obtained from D by applying $\bar{\gamma}$ to the vertices of D : if $C_1 = \{\bar{k}_{11}, \dots, \bar{k}_{1m_1}\}, \dots, C_t = \{\bar{k}_{t1}, \dots, \bar{k}_{tm_t}\}$ are the equivalence classes of $D_0 \subseteq U(\mathbb{Z})$ under $\bar{\beta}$, then $\bar{\gamma} C_1 = \{\bar{\gamma} \bar{k}_{11}, \dots, \bar{\gamma} \bar{k}_{1m_1}\}, \dots, \bar{\gamma} C_t = \{\bar{\gamma} \bar{k}_{t1}, \dots, \bar{\gamma} \bar{k}_{tm_t}\}$, are the equivalence classes of D'_0 . Notice that $\bar{\gamma} \bar{k} = \overline{k+1}$, for every $k \in \mathbb{Z}$.

(1.24) Corollary. Let γD be the Brauer graph of type B_n defined by $\bar{\beta}' = \bar{\gamma} \bar{\beta} \bar{\gamma}^{-1}$ from the permutation $\bar{\beta}$ of a Brauer graph D of type B_n . Then the Brauer graphs $\gamma^t D$ ($0 \leq t \leq n-1$) are pairwise non-isomorphic and build up the isomorphism class $[D]$ of all Brauer graphs of type B_n isomorphic to D .

(1.25) Example.

The selected transit point is the encircled vertex, in each case.

(1.26) Proposition. Let C be a configuration of $\mathbb{Z}B_n$. Then $C \cap U \neq \emptyset$.

Proof. Let (α, β) be the symmetric Brauer pair associated to C , and let C' be the configuration of $\mathbb{Z}A_w$ associated to (α, β) . Then C' is a symmetric configuration of $\mathbb{Z}A_w$ that intersects the middle line $m = \{(k, n) \mid k \in \mathbb{Z}\}$ of $\mathbb{Z}A_w$, that is, there exist configuration points $(a, \beta a - a) = (a, n)$. But $(a, \beta a - a) \in (\mathbb{Z}B_n)_0$ is also a configuration point of $\mathbb{Z}B_n$, since $1 \leq \beta a - a \leq n$, hence C intersects the upper border U of $\mathbb{Z}B_n$.

(1.27) Proposition. Let C be a configuration of $\mathbb{Z}B_n$. Then $W_r = \langle \tau^{wr} \rangle$ ($r \geq 1$) are all the admissible groups of automorphisms of $\mathbb{Z}B_n$ that stabilize C .

Proof. Let $c = (a, n) \in C \cap U$ be a configuration point lying on the upper border of $\mathbb{Z}B_n$. (See Figure 1).

(Fig. 1)

Since C is stabilized by $\langle \tau^\omega \rangle$ (1.11), $(a+\omega, n)$ is also a configuration point. Consider the butterfly starting at (a, n) and the butterfly stopping at $(a+\omega, n)$. On these butterflies - the shaded regions of Figure 1 - (a, n) and $(a+\omega, n)$ are the only configuration points, in particular there are no configuration points between $(a+1, n)$ and $x = \nu(a, n) = (a+n-1, n)$, lying on U ; by the same reason, there are no configuration points between $(a+n, n) = \nu^{-1}(a+\omega, n)$ and $(a+\omega-1, n)$ lying on U . But $x = \tau y$, hence there are no configuration points at all lying on U between $(a+1, n)$ and $(a+\omega-1, n)$. As a consequence, no subgroup $\langle \tau^t \rangle$ of automorphisms of $\mathbb{Z}B_n$ with $t < \omega$ stabilizes C , thus if $\langle \tau^t \rangle$ stabilizes C , $t = r\omega$ for some $r \geq 1$.

(1.28) Remark. We can prove (1.26) in another way, if we know before hand that the given configuration C of $\mathbb{Z}B_n$ appears already as a configuration associated to a selfinjective algebra A of finite representation type and of Dynkin class B_n . Namely, if $\Gamma(A) \cong (\mathbb{Z}B_n)_C / W$, consider any simple A -module S with $\text{End } S_A = G$. Then (III, (2.3)) also $\text{End } \Omega^1 S = G$; but $\Omega^1 S \cong \text{rad } P$, $P \rightarrow S \rightarrow 0$ a projective cover of S . Hence $\pi^{-1}[\Omega^1 S]$ consists of configuration points all lying on U (III, (2.5)), where $\pi: (\mathbb{Z}B_n)_C \rightarrow \Gamma(A)$ is the canonical projection.

The last normalization condition that we shall use is a consequence of the following lemma, valid for any Brauer pair of any order.

(1.29) Lemma. Let (α, β) be a Brauer pair of order n . Then, there exists $x \in \mathbb{Z}$ such that $\beta x = x+1$.

Proof. Let $\bar{y}$ be a transit point of Q , the associated Brauer quiver of (α, β) . Without loss of generality we can assume that $\alpha \bar{y} = \bar{y}$, simply by taking the loop through $\bar{y}$ to be an α -loop. Then $\alpha y = y+n$, and $\alpha \beta(y-1) = (y-1)+n+1$ (1.18, (3)), hence $\beta(y-1) = y$, since α is bijective. Take then $x = y-1$.

(1.30) Corollary. Let D be a Brauer graph of type B_n , and let (α, β) be its associated Brauer pair. Then, we can always assume that $\bar{0}$ is the selected vertex of D and that $\beta(-1) = 0$, $\alpha 0 = n$.

Proof. By (1.23) we change, if necessary, D by an isomorphic Brauer graph D' with $\bar{0}$ as its selected vertex. By (1.29), then we take $\alpha 0 = n$, $\beta(-1) = 0$, as well.

(1.31) Proposition. Let D be a Brauer graph of type B_n , (α', β') its associated Brauer pair. Then, there exists a symmetric Brauer pair (α, β) of order $m = 2n-1$, and an embedding $UD \rightarrow Q$, where UD is the underlying Brauer quiver of D , and Q is the Brauer quiver associated to (α, β) .

Proof. Consider the interval $[0, \omega-1] = [0, n] \cup [n+1, \omega-1]$ of $\mathbb{Z}$. We will define functions $\alpha, \beta: \mathbb{Z} \rightarrow \mathbb{Z}$ in the following way. First, we define β on $[0, n]$ by setting

$$\beta a = \begin{cases} \beta' a, & \text{if } \beta' a < n \\ \beta' a + n - 1, & \text{if } \beta' a \geq n \\ \omega, & \text{if } a = n - 1, \end{cases} \quad (1)$$

that is

$$\beta a = \beta' a + v(a), \quad (2)$$

where $v(a) = 0$ if $\beta' a < n$ and $v(a) = n - 1$ if $\beta' a \geq n$.

We then extend β to the interval $[n, \omega-1]$ by

$$\beta(n+k) = \beta'^{-1}(k+1) + v(k+1) + 2n \quad (3)$$

($1 \leq k \leq n-2$), and then extend β to $\mathbb{Z}$ by periodicity

$$\beta(a+k\omega) = \beta a + k\omega, \quad (4)$$

for every $k \in \mathbb{Z}$, $a \in [0, \omega-1]$.

Let α be the corresponding permutation of $\mathbb{Z}$, if it exists, such that $\alpha\beta a = a + \omega + 1$, for all $a \in \mathbb{Z}$. We claim that α exists and that (α, β) is a symmetric Brauer pair of order ω . To this end, we define α on $[0, n-1]$ by

$$\alpha t = \begin{cases} \alpha' t, & \text{if } \alpha' t - t < n \\ \alpha' t + v(t), & \text{if } \alpha' t - t \geq n \end{cases} \quad (5)$$

Now, since (α', β') is a Brauer pair we have

$$\alpha' \beta' a = a + n + 1 \quad (6)$$

for every $a \in \mathbb{Z}$.

Consider next the functions φ, ψ defined on $[n, \omega-1]$ by

$$\varphi t = \beta(t-n) + n \quad (7)$$

$$\psi t = \alpha(t-n+1) + n - 1, \quad (8)$$

$$n \leq t < \omega.$$

Since (α, β) is a symmetric Brauer pair of order ω if it satisfies

$$\alpha \beta a = a + 2n \quad (9)$$

$$\alpha(a+n) = \beta a + n, \quad (10)$$

for every $a \in \mathbb{Z}$, these functions φ, ψ are the extensions of α and β to $[n, \omega-1]$, so we are led to show that $\varphi = \beta|_{[n, \omega-1]}$. In fact, suppose for a moment that α and β are really as claimed. Then, from (1), by setting $t = a+n$, we obtain $\alpha t = \beta(t-n) + n$, and setting $a = n+k$ we obtain $\beta(n+k) = \alpha(2n+k) - n = \alpha(2n-1+k+1) - n = \alpha(k+1) + 2n-1-n = \alpha(k+1) + n - 1$. From (6), on the other hand, setting $r = \beta' a$, we get $\alpha' r = \beta'^{-1} r + n + 1$, so together with the fact that $\beta' a = a + n$, we have $\alpha r = \alpha' r + v(r) = \beta'^{-1} r + n + 1 + v(r)$, and $\alpha(k+1) = \beta'^{-1}(k+1) + n + 1 + v(k+1)$, and finally

$$\begin{aligned}
 \beta(n+k) &= \alpha(k+1)+n-1 \\
 &= \beta'^{-1}(k+1)+n+1+v(k+1)+n-1 \\
 &= \beta'^{-1}(k+1)+v(k+1)+2n,
 \end{aligned}$$

which is (3), as claimed.

(1.32) Example. We illustrate the previous proposition with the following example of a Brauer graph D of type B_5 and the corresponding Brauer quiver Q with $9 = 2 \cdot 5 - 1$ vertices associated to the symmetric Brauer pair (α, β) of order 9 obtained from the Brauer pair (α', β') of D .

D

$$\bar{\beta}' = (04)(23)$$

$$\bar{\beta} = (04)(23)(568)$$

Here, the dotted arrows represent the α -arrows and the full arrows the β -arrows. The encircled vertex 0 of D is the selected vertex. The permutations $\bar{\alpha}$ and $\bar{\alpha}'$ are obtained from $\bar{\beta}$ and $\bar{\beta}'$ by means of the formulas $\bar{\alpha}\bar{\beta} = \bar{\gamma}$, $\bar{\alpha}'\bar{\beta}' = \bar{\gamma}'$, $\bar{\gamma} = (012345768)$, $\bar{\gamma}' = (01234)$. Below we display again D and Q in a way that makes transparent the symmetry property of the Brauer pair (α, β) associated to $(\bar{\alpha}, \bar{\beta})$:

In order to prove the converse of (1.31), we need ~~some~~ preliminary lemmas, of interest by themselves, and we use the bijection of Theorem (1.20) between configurations of $\mathbb{Z}B_n$ and symmetric Brauer pairs of order ω .

(1.33) Lemma. Let C be a configuration of $\mathbb{Z}B_n$, $\alpha = \alpha_C$.

Then $\alpha(\alpha y - n) = y + n$, for every $y \in \mathbb{Z}$.

Proof. We distinguish two cases: (1^o) $e(y) = k_y$, (2^o) $e(y) = k_y^*$, where $e(y)$ is the configuration point on the hill K_y .

(1^o) If $e(y) = k_y$, then (see Figure 1) $k_y = (\alpha y - 2n, 2n - \alpha y + y) = k_{\alpha y - n}^* = (\alpha y - n - n, \alpha(\alpha y - n) - (\alpha y - n))$, hence $2n - \alpha y + y = \alpha(\alpha y - n) - (\alpha y - n)$, or $\alpha(\alpha y - n) = n + y$.

(Fig. 1)

(2⁰) If $e(y) = k_y^*$, then (see Figure 2) $k_y^* = (y-n, \alpha y-y) =$
 $= k_{\alpha y-n} = (\alpha(\alpha y-n)-2n, 2n-\alpha(\alpha y-n)+(\alpha y-n))$, hence $y-n = \alpha(\alpha y-n)-2n$,
 or $\alpha(\alpha y-n) = n+y$, as claimed.

(Fig. 2)

(1.34) Definition and Lemma. Let (α, β) be a symmetric Brauer pair of order $\omega = 2n-1$. Define $x \sim y$ iff $y = \alpha x - n$. Then $x \sim y$ iff $y \sim x$.

Proof. If $x \sim y$, then $y = \alpha x - n$, so $\alpha y = \alpha(\alpha x - n) = x + n$, by (1.33), hence $x = \alpha y - n$ and $y \sim x$. If, conversely, $y \sim x$, then $x = \alpha y - n$, and $\alpha x = \alpha(\alpha y - n) = y + n$, as required.

(1.35) Remark. Let C be a configuration of $\mathbb{Z}B_n$. By (1.26) and (1.27) we can assume, without loss of generality that $C \cap U = [-n, n] = \{(-n+k\omega, n) \mid k \in \mathbb{Z}\}$. (See Figure 1).

(Fig. 1)

Under this normalization, which we will assume from now on, we have $(-n, n) = h_{-n} = (-n, \beta(-n) - (-n)) = k_0 = (\alpha 0 - 2n, 2n - \alpha 0 + 0)$, thus $\beta(-n) = 0$ and $\alpha 0 = n$, and the following lemma holds:

(1.36) Lemma. $x \sim x$ iff $x = kw$, $k \in \mathbb{Z}$.

Proof. If $x = kw$, then $\alpha x = \alpha(kw) = \alpha(0 + kw) = n + kw = n + x$, so $x \sim x$. Conversely, if $x \neq kw$, then $\alpha x - x \neq n$ (see Figure 1 in (1.35)), hence $x \not\sim x$.

(1.37) Lemma. Let $y \sim x$. Then $\alpha y \sim \beta x$.

Proof. If $y \sim x$, then $y = \alpha x - n$, so $\alpha y = \alpha(\alpha x - n) = x + n$, by (1.33). But $x + n = (\alpha \beta x - 2n) + n = \alpha \beta x - n$, by (1.15), so $\alpha y = \alpha \beta x - n$ and $\alpha y \sim \beta x$.

(1.38) Proposition. Let (α, β) be a symmetric Brauer pair of order $w = 2n - 1$. Then, there exists a Brauer graph D of type B_n , such that the symmetric Brauer pair associated to D (1.31) is (α, β) .

Proof. Let Q be the Brauer quiver of (α, β) . We construct $D = (D_0, D_1, w)$ in the following way: the elements of D_0 are the pairs $\langle \bar{x}, \bar{y} \rangle$ such that $x \sim y$. The elements of D_1 are the α -arrows $\langle \bar{x}, \bar{y} \rangle \xrightarrow{\langle \bar{\alpha}, \bar{\beta} \rangle} \langle \bar{\alpha} \bar{x}, \bar{\beta} \bar{y} \rangle$, and the β -arrows $\langle \bar{x}, \bar{y} \rangle \xrightarrow{\langle \bar{\beta}, \bar{\alpha} \rangle} \langle \bar{\beta} \bar{x}, \bar{\alpha} \bar{y} \rangle$, and the selected vertex of D_0

is the pair $\langle \bar{x}, \bar{x} \rangle$. By (1.34) - (1.37) this construction yields indeed a Brauer graph of type B_n , and by the construction of (1.31) it follows without difficulty that (α, β) is the symmetric Brauer pair associated to D .

(1.39) Example. Let (α, β) be the symmetric Brauer pair of order n defined by $\bar{\beta} = (04)(23)(568)$ (Example (1.32)). Then the Brauer graph D' of type B_5 associated to (α, β) is given by

and clearly D' is isomorphic to D given by

(1.40) Theorem. There exists a bijection between the configurations of $\mathbb{Z}B_n$ and the Brauer graphs of type B_n .

Proof. By (1.31) and (1.38) we have a bijection between Brauer graphs D of type B_n and symmetric Brauer pairs (α, β) of order $w = 2n - 1$. These, in turn, are paired bijectively to the configurations of $\mathbb{Z}B_n$, by (1.20).

(1.41) Remark. It is worth to note that the bijection of (1.40) is constructive, that is, using the constructions given in

the proofs of the various propositions involved we have in fact an algorithm to produce, given a Brauer graph of type B_n , its associated configuration, and viceversa.

We come next to the discussion of the configurations of $\mathbb{Z}\mathbb{C}_n$. Recall (II, (2.2)) that we use the coordinates on $\mathbb{Z}\mathbb{C}_n$ indicated on Figure 1.

We define the butterfly $\text{Mar}(a, b)$ starting at the point $(a, b) \in (\mathbb{Z}\mathbb{C}_n)_0$ exactly in the same way as in the case of $\mathbb{Z}B_n$ (1.1), and then, by (III, (3.3)) we have

(1.42) Lemma. Let $x \in (\mathbb{Z}\mathbb{C}_n)_0$. The following conditions are equivalent for $y \in (\mathbb{Z}\mathbb{C}_n)_0$:

(i) $\text{msh}(\mathbb{Z}\mathbb{C}_n)(x, y) \neq 0$

(ii) $y \in \text{Mar}(x)$.

As a consequence, all the previous results for the configurations of $\mathbb{ZB}_n$ apply, *mutatis mutandi*, to the configurations of $\mathbb{ZC}_n$.

We can thus state the main results:

(1.43) Theorem. There is a bijection between the configurations of $\mathbb{ZC}_n$ and the symmetric Brauer pairs of order $w = 2n - 1$.

(1.44) Theorem. There is a bijection between the configurations of $\mathbb{ZC}_n$ and the Brauer graphs of type C_n .

(1.45) Remark. Let $D = (D_0, D_1, w, q_0)$ be a Brauer graph of type C_n with q_0 as its selected vertex, $w(\delta) = (1, 2)$, $w(\epsilon) = (2, 1)$ for the arrows in D_1 with $e(\delta) = q_0 = o(\epsilon)$. Let $D' = (D_0, D_1, w', q_0)$ be the Brauer graph of type B_n with q_0 its selected vertex, $w'(\delta) = (2, 1)$, $w'(\epsilon) = (1, 2)$. Then clearly $D \mapsto D'$ establishes a bijection between the Brauer graphs of type C_n and the Brauer graphs of type B_n . As a consequence, there is also a bijection $C \mapsto C'$ between the configurations of $\mathbb{ZC}_n$ and the configurations of type $\mathbb{ZB}_n$, which is the identity map on the underlying configuration points.

(1.46) Examples. We list some small Brauer graphs D and the associated configurations C . By (1.45) these may be interpreted as configurations of $\mathbb{ZB}_n$ or as configurations of $\mathbb{ZC}_n$, according

to the valuation w on the arrows through the selected vertex q_0 of D . We use the normalization $q_0 = 0$, so $(+n-1, n)$ is the configuration point on the upper border of the figures representing the configuration C , of which only the part between the ascending diagonals Δ_0, Δ_{w-1} , $w = 2n-1$, is given. Thus the point furthest to the left is $(0,1)$ and the point furthest to right is $(w-1, n)$. The configuration points are the encircled ones.

Let $c(n)$ be the number of non-isomorphic configurations of $\mathbb{Z}\Delta$, $\Delta = B_n$ or C_n . Then $c(2) = 1$, $c(3) = 2$, $c(4) = 5$, $c(5) = 14$, $c(6) = 38$, $c(7) = 63$, The number $c(n)$, which is also the number of non-isomorphic Brauer graphs of type Δ , is related to the number $b(n)$ of non-isomorphic Brauer quivers with n vertices, or equivalently, of non-isomorphic Brauer trees. Thus, $c(n)$ measures the number of non-isomorphic Brauer quivers with n vertices with one distinguished transit point, or equivalently, of Brauer trees with one distinguished edge, up to the fact that different choices of transit points may give rise to isomorphic Brauer graphs.

Brauer graphs D and configurations C for small values of n .

(i) $n = 2$:

$D \quad 1 \rightleftharpoons 0$

(ii) $n = 3$:

(iii) $n = 4$

IV.2 Selfinjective algebras of classes B_n and C_n .

In this section we construct selfinjective algebras of Dynkin class B_n and of Dynkin class C_n such that their Auslander-Reiten graphs are of the form $(\mathbb{Z}B_n, m)_C / W$, for a fixed modulation m of the translation graph $(\mathbb{Z}B_n)_C$ with C a configuration of $(\mathbb{Z}B_n, m)$, for the algebras of Dynkin class B_n , and of the form $(\mathbb{Z}C_n, m)_C / W$ for algebras of Dynkin class C_n .

To simplify the notation, we denote in the sequel by $\mathbb{Z}B_n$ the modulated translation graph $(\mathbb{Z}B_n, m)$ with m a fixed modulation of $\mathbb{Z}B_n$ (II, (6.10)) and by $(\mathbb{Z}B_n)_C$ the modulated translation graph $((\mathbb{Z}B_n)_C, m)$, C a configuration of $\mathbb{Z}B_n$. Similarly, for the C_n case. We continue to use the coordinates on $\mathbb{Z}B_n$ and $\mathbb{Z}C_n$ introduced in (II, (2.2)).

Our main results are the construction of basic, connected, finite dimensional selfinjective K -algebras B of finite representation type of Dynkin class B_n or C_n , that is, with Auslander-Reiten graph $\Gamma(B) \cong (\mathbb{Z}B_n)_C / W$, for any configuration C of $\mathbb{Z}B_n$ and any admissible group of automorphisms W of $\mathbb{Z}B_n$ stabilizing C , in the B_n case, $\Gamma(B) \cong (\mathbb{Z}C_n)_C / W$, in the C_n case (Theorem (2.8) for B_n , Theorem (2.21) for C_n), in particular obtaining trivial extension algebras of finite representation type of Dynkin class B_n (2.11) or of Dynkin class C_n (2.23)). Examples are provided of the constructions required to prove the main results, as well as examples of selfinjective algebras of finite representation type of Dynkin class B_n or C_n .

In the first part of this section we consider the case of selfinjective algebras of class B_n ; the similar case of selfinjective algebras of class C_n is discussed afterwards.

(2.1) Let D be a Brauer graph of type B_n (II, (1.7)), $\tilde{D}$ its universal covering (II, (1.8)). As we know ((1.31)), we can suppose $D_0 = \mathbb{Z}/n\mathbb{Z}$ and $\bar{0}$ the selected transit vertex of D . Moreover, we can also assume $\alpha\bar{0} = n$; this simply means that the cycle through $\bar{0}$ is a β -cycle, the α -cycle of the corresponding Brauer quiver Q through $\bar{0}$ is a loop and therefore has been erased.

(2.2) We attach a K-species $\tilde{\Sigma} = \tilde{\Sigma}_D$ to $\tilde{D}$, in the following way: $\tilde{\Sigma} = (K_i, i_j^M)_{i,j \in \mathbb{Z}}$ is the K-species with

$$K_i = \begin{cases} G, & i \equiv 0 \\ F, & \text{otherwise} \end{cases}$$

$$i_j^M = \begin{cases} G^F_F, & i \equiv 0 \\ F^F_G, & j \equiv 0 \\ F^F_F, & \text{otherwise} \end{cases}$$

for every arrow $j \rightarrow i$ in $\tilde{D}$.

(2.3) Let $\tilde{J} = \tilde{J}_D$ be the relation ideal of $\tilde{\Sigma} = \tilde{\Sigma}_D$ defined by the following minimal relations:

$$(\mu) \quad K_h \xrightarrow{\mu} K_i \rightarrow K_j, \quad i \equiv 0$$

for every path $j \rightarrow i \rightarrow h$ in $\tilde{D}$

$$(\pi) \quad K_{i+n} \rightarrow K_{i_1} \rightarrow \dots \rightarrow K_{i_r} \rightarrow K_i, \quad i \equiv 0$$

where $\bar{i} \rightarrow \bar{i}_r \rightarrow \dots \bar{i}_1 \rightarrow \bar{i}$ is the cycle through $\bar{0}$.

where $\bar{i}$ is a node and $\bar{i} \rightarrow \bar{a}_p \rightarrow \dots \rightarrow \bar{a}_1 \rightarrow \bar{i}$, $\bar{i} \rightarrow \bar{b}_q \rightarrow \dots \rightarrow \bar{b}_1 \rightarrow \bar{i}$ are the cycles through $\bar{i}$, and $s \equiv 0$ in (c, μ)

(0) $K_h \rightarrow K_i \rightarrow K_j$, 0

whenever $\bar{j} \rightarrow \bar{i}$, $\bar{i} \rightarrow \bar{h}$ belong to different cycles.

All congruences above are taken modulo n .

(2.4) Definition and Lemma. Let $\mu: \mathbb{Z} \rightarrow \mathbb{Z}$, $i \mapsto i+n$.

Then μ induces a selfequivalence of the category $R(\tilde{\Sigma}, \tilde{J})$ of the finite dimensional representations of $\tilde{\Sigma}$ bound by $\tilde{J}$.

Proof. Clear from the constructions of $\tilde{\Sigma}$ and $\tilde{J}$ (II, (7.6)).

In the following we will denote also by μ the selfequivalence of $R(\tilde{\Sigma}, \tilde{J})$ induced by μ . The use of the same letter to denote a defining relation of $\tilde{J}$ is not likely to cause any confusion.

(2.5) Let $\tilde{\Sigma}$ and $\tilde{J}$ be as above. The equivalence μ of (2.4) induces equivalences μ^r of $R(\tilde{\Sigma}, \tilde{J})$, for every $r \geq 1$, so we can consider the species $\Sigma_r := \Sigma_{r,D} := \tilde{\Sigma} / \mu^r$ and the relation ideals $J_r := \tilde{J} / \mu^r$, for every $r \geq 1$. We set $B_r := B_{r,D} = T(\Sigma_r) / J_r$ ($r \geq 1$), and moreover we define $\tilde{B} := \tilde{B}_D = \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{R(\tilde{\Sigma}, \tilde{J})}(\underline{P}(i), \underline{P}(j))$.

(2.6) Proposition.

(i) $\text{mod } \tilde{B} \simeq R(\tilde{\Sigma}, \tilde{J})$.

(ii) $\tilde{B}$ is a basic, connected, selfinjective, locally finite dimensional K -algebra

(iii) $\text{mod } B_r \simeq R(\Sigma_r, J_r) \simeq R^{\mu^r}(\tilde{\Sigma}, \tilde{J})$, for every $r \geq 1$.

Proof. The equivalences $\text{mod } \tilde{B} \simeq R(\tilde{\Sigma}, \tilde{J})$ and $\text{mod } B_r \simeq R(\Sigma_r, J_r)$ being clear (II, (4.4)), as well as the equivalences $R(\Sigma_r, J_r) \simeq R^{\mu^r}(\tilde{\Sigma}, \tilde{J})$ (II, (7.6)), we have only to prove (ii).

Let $u_i := \text{id}_{\underline{P}(i)}$, for every $i \in \mathbb{Z}$. Clearly $\{u_i \mid i \in \mathbb{Z}\}$ form a set of orthogonal idempotents of $\tilde{B}$, $\tilde{B} = \bigoplus_{i,j \in \mathbb{Z}} u_i \tilde{B} u_j$,

and each $u_i \tilde{B} \simeq \underline{P}(i)$ is a finite dimensional K -vector space, from the very definition of $\tilde{J}$, with $\underline{P}(i) \simeq \underline{P}(j)$ if and only if

$i = j$. Thus, $\tilde{B}$ is indeed a basic, connected, locally finite dimensional K -algebra, $(I, (3.4))$ which is moreover selfinjective, since $\underline{P}(i+n) = \underline{I}(i)$, for every $i \in \mathbb{Z}$.

(2.7) Remark. The algebra $\tilde{B}$ admits a nice matrix representation: let

$$\hat{B} := \begin{pmatrix} K_{i+1} & 0 & 0 \\ {}_iE_{i+1} & K_i & 0 \\ {}_iE_j & {}_{i-1}E_i & K_{i-1} \end{pmatrix}$$

be the K -algebra of those $\mathbb{Z} \times \mathbb{Z}$ -matrices with only finitely many non-zero entries, with K_i , $i \in \mathbb{Z}$, the diagonal fields, ${}_iE_j := \tilde{M}_{ij} / \tilde{J}_{ij}$ for every $i, j \in \mathbb{Z}$, $i \geq j$, and multiplication induced by the relations $\tilde{J}_{ij}$. Then clearly $\tilde{B} \simeq \hat{B}$, since $\tilde{\Sigma}$ has no oriented cycles, so $\tilde{B}$ can be put in triangular form, as in $\hat{B}$. If we consider the matrix algebra

$$B_1 := \begin{pmatrix} K_n & 0 \\ {}_iE_j & K_1 \end{pmatrix},$$

with the entries ${}_iE_j$ as before, then $\hat{B} = B_1$ (I, (3.3)).

We are now ready to state and prove our main results.

(2.8) Theorem. Let D be a Brauer graph of type B_n , C its associated configuration of ZB_n . Then

(i) $B_{r,D}$ is a basic, connected, finite dimensional, self-injective algebra of finite representation type and of Dynkin class B_n

(ii) $\Gamma(B_{r,D}) \cong (ZB_n)_C / \langle \tau^{wr} \rangle$, for every $r \geq 1$.

Proof. From the definitions of $B_{r,D}$ and $\tilde{B}$, we have that $\Gamma(B_{r,D}) \cong \Gamma(\tilde{B})/W$, from where (i) will follow if we prove that $\Gamma(\tilde{B}) \cong (ZB_n)_C$, for, clearly, $B_{r,D}$ is a basic, connected and finite dimensional K -algebra. We prove below (2.12) this assertion about $\Gamma(\tilde{B})$, from where it will also follow (ii).

(2.9) Let $\tilde{\Sigma} = \tilde{\Sigma}_D$, $\tilde{J} = \tilde{J}_D$ be the K -species and its relation ideal associated to the universal covering $\tilde{D}$ of a Brauer graph D of type B_n , with respect to a modulation m . A full connected subspecies Σ of $\tilde{\Sigma}$ is called a complete μ -slice of $\tilde{\Sigma}$ if its set of vertices forms a complete set of representatives of the μ -orbits of Z under μ . We set $J := \tilde{J}|_{\Sigma}$, $B(\Sigma) := T(\Sigma)/J$, for a complete μ -slice Σ of $\tilde{\Sigma}$.

(2.10) Theorem. Let Σ be a complete μ -slice of $\tilde{\Sigma}$. Then $B_1 \cong B(\Sigma) \ltimes DB(\Sigma)$.

Proof. It follows from (2.8) and the results in (III, (1.5)).

(2.11) Theorem. The algebras $B_{1,D}$, D running through a set of representatives of the isomorphism classes of Brauer graphs of type B_n , are trivial extension algebras of finite representation type and of Dynkin class B_n .

Proof. Immediate from the previous theorem.

(2.12) Theorem. Let D be a Brauer graph of type B_n , C its associated configuration of $\mathbb{Z}B_n$ and $\tilde{B}$ the algebra defined in (2.5). Then $\Gamma(\tilde{B}) \cong (\mathbb{Z}B_n)_C$.

Proof. Let (α, β) be the symmetric Brauer pair of order $\omega = 2n-1$ associated to D (1.19). By (1.38) we can assume, without loss of generality, that $D = (D_0, D_1, w, q_0)$ with D_0 the set of pairs $\langle \bar{y}, \bar{y}' \rangle$, $\bar{y}, \bar{y}'$ in $\mathbb{Z}/\omega\mathbb{Z}$, such that $y' = \alpha y - n$, D_1 the set of arrows $\langle \bar{y}, \bar{y}' \rangle \xrightarrow{\langle \bar{\alpha}, \bar{\beta} \rangle} \langle \bar{\alpha}\bar{y}, \bar{\beta}\bar{y}' \rangle$ and $\langle \bar{y}, \bar{y}' \rangle \xrightarrow{\langle \bar{\beta}, \bar{\alpha} \rangle} \langle \bar{\beta}\bar{y}, \bar{\alpha}\bar{y}' \rangle$, and $q_0 = \langle \bar{0}, \bar{0} \rangle$.

To (α, β) is also associated a configuration $C(Q)$ of $\mathbb{Z}A_\omega$, where Q is the Brauer quiver with ω vertices associated to (α, β) . To distinguish between both configurations $C(Q)$ of $\mathbb{Z}A_\omega$ and C of $\mathbb{Z}B_n$, we will denote this last one by $C(D)$ in the course of the proof. Let $\tilde{D}$ be the universal covering of D , $\tilde{\Sigma}$ the K -species constructed from $\tilde{D}$ and the given field extension F/G (2.2), $\tilde{J}$ the relation ideal of $\tilde{\Sigma}$ introduced in (2.3),

and $R(\tilde{\Sigma}, \tilde{J})$ the category of the finite dimensional representations of $\tilde{\Sigma}$ bound by $\tilde{J}$. Our aim is to relate $R(\tilde{\Sigma}, \tilde{J})$ to the category $R_G(\tilde{Q}, \tilde{I})$ of finite dimensional representations of the universal covering $\tilde{Q}$ over G bound by a suitable relation ideal $\tilde{I}$, and use the known results ([24]) on the classification of selfinjective algebras of class A_∞ . We state those results in our terminology. First, construct a K -species from $\tilde{Q}$ by setting $K_i = G$ for every $i \in \mathbb{Z}$, $M_j = {}_G G$ for every arrow $j \rightarrow i$ of $\tilde{Q}$, and a relation ideal $\tilde{I}$ on this species given by the following minimal relations:

for every $i \in \mathbb{Z}$, where $\bar{i} \rightarrow \bar{a}_p \rightarrow \dots \rightarrow \bar{a}_1 \rightarrow \bar{i}$ and $\bar{i} \rightarrow \bar{b}_q \rightarrow \dots \rightarrow \bar{b}_1 \rightarrow \bar{i}$ are the cycles through $\bar{i} \in \mathbb{Z} / n\mathbb{Z} = Q_0$,

$$(0) \quad K_h \xrightarrow{0} K_i \rightarrow K_j,$$

whenever $j \rightarrow i$ and $i \rightarrow h$ belong to different cycles of Q .

The category $R_G(\tilde{Q}, \tilde{I})$ is the category of finite dimensional representations of this K -species bound by the relation ideal $\tilde{I}$.

We set further $\tilde{A} := \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{\mathcal{R}(\tilde{Q}, \tilde{I})}(\underline{P}(i), \underline{P}(j))$. If we let Λ be the full subcategory of the mesh category $\text{msh}(\mathbb{Z}A_w)_{C(Q)}$ over G consisting of the projective vertices, then we have

$$\text{msh}(\mathbb{Z}A_w)_{C(Q)} \simeq \text{ind } \Lambda, \quad \Lambda \simeq \text{proj ind } \tilde{A} \quad \text{and} \quad \Gamma(\tilde{A}) \simeq (\mathbb{Z}A_w)_{C(Q)}.$$

Moreover, for any configuration $C(Q)$ of $\mathbb{Z}A_w$, in particular for the symmetric ones, we have $\Gamma(\tilde{A}) \simeq (\mathbb{Z}A_w)_{C(Q)}$.

Let $U(\mathbb{Z}B_n)$ denote the underlying translation graph of $\mathbb{Z}B_n$ obtained by forgetting the valuation, and similarly $U(\mathbb{Z}B_n)_{C(D)}$. From (1.40) we obtain then $U(\mathbb{Z}B_n)_{C(D)} \simeq (\mathbb{Z}A_w)_{C(Q)} / \rho$, where ρ is the reflection $(a,b) \mapsto (a+b-n, 2n-b)$ on the middle line $\{(k,n) \mid k \in \mathbb{Z}\}$ of $\mathbb{Z}A_w$, and $(\mathbb{Z}A_w)_{C(Q)} / \rho$ is the set of all pairs $\langle x, y \rangle$ with $y = \rho x$. Note that $y = \rho x$ if and only if $x = \rho y$, for $\rho^2 = \text{id}$, and that the isomorphism $U(\mathbb{Z}B_n)_{C(D)} \xrightarrow{\sim} (\mathbb{Z}A_w)_{C(Q)}$ is given by sending $x = (a,b)$ to the pair $\langle x, \rho x \rangle$. Under this isomorphism, moreover, we have $C(D) \simeq C(Q) / \rho$. From here we obtain our theorem, if we prove that $U\Gamma(\tilde{B}) \simeq U\Gamma(\tilde{A}) / \rho$.

From our choice of D , we have that $\tilde{D} = (\tilde{D}_0, \tilde{D}_1, \tilde{w}, \{q_t \mid t \in \mathbb{Z}\})$ can also be chosen as follows: the elements of $\tilde{D}_0$ are the pairs $\langle y, y' \rangle$ of integers with $y' = \alpha y - n$, the arrows are

$$\langle y, y' \rangle \xrightarrow{\langle \alpha, \beta \rangle} \langle \alpha y, \beta y' \rangle \quad \text{and} \quad \langle y, y' \rangle \xrightarrow{\langle \beta, \alpha \rangle} \langle \beta y, \alpha y' \rangle,$$

for every $y \in \mathbb{Z}$, the selected vertices are the one point pairs

$\langle k\omega, k\omega \rangle$, $k \in \mathbb{Z}$, and valuation $\tilde{w}$ defined by $\tilde{w}(\epsilon_{k\omega}) = (2,1)$,

$\tilde{w}(\delta_{k\omega}) = (1,2)$, where $\epsilon_{k\omega}$ and $\delta_{k\omega}$ are the arrows of $\tilde{D}_1$ such that $e(\delta_{k\omega}) = \langle k\omega, k\omega \rangle = o(\epsilon_{k\omega})$, for every $k \in \mathbb{Z}$, and

$\tilde{w}(\gamma) = (1,1)$ for any other arrow of $\tilde{D}_1$. Notice that since

$\langle \alpha k\omega, \beta k\omega \rangle = \langle \beta k\omega, \alpha k\omega \rangle$, the arrows $\langle k\omega, k\omega \rangle \xrightarrow{\langle \alpha, \beta \rangle} \langle \alpha k\omega, \beta k\omega \rangle$

and $\langle k\omega, k\omega \rangle \xrightarrow{\langle \beta, \alpha \rangle} \langle \beta k\omega, \alpha k\omega \rangle$ coincide, for every $k \in \mathbb{Z}$,

and similarly there is only one arrow with end point $\langle k\omega, k\omega \rangle$,

for every $k \in \mathbb{Z}$.

Let $\underline{X} = (X_{\langle y, y' \rangle}, \langle z, z' \rangle \varphi_{\langle y, y' \rangle})$ be a representation

of $R(\tilde{\Sigma}, \tilde{J})$. If $y = y'$, then $X_y := X_{\langle y, y \rangle}$ is a G -vector

space, whereas if $y \neq y'$, $X_{\langle y, y' \rangle}$ is an F -vector space and

$X_{\langle y, y' \rangle} \otimes_{F^G} = : X_y \oplus X_{y'}$, with X_y and $X_{y'}$ isomorphic G -vector

spaces. Let $j_y, j_{y'}$ be the canonical injections and $p_y, p_{y'}$ the

canonical projections. Let $H: R(\tilde{\Sigma}, \tilde{J}) \rightarrow R_G(\tilde{Q}, \tilde{I})$ be the functor defined

in the following way: $\underline{W} := H(\underline{X}) = (W_y, {}_Z\varphi_y)$ has $W_y := X_y$, for

every $y \in \mathbb{Z}$, and the maps ${}_Z\varphi_y: W_y \rightarrow W_z$ are defined according to

the following cases:

$$(I) \quad {}_y\varphi_{\alpha y} = \varphi j_{\alpha y}, \quad {}_y\varphi_{\beta y} = \varphi j_{\beta y}, \quad y \equiv 0 \pmod{\omega}, \text{ where}$$

$$\varphi: X_{\langle \alpha y, \beta y \rangle} \otimes_{F^G} \rightarrow X_{\langle y, y \rangle}$$

$$(II) \quad {}_{\alpha}^{-1}{}_y\varphi_y = p_{\alpha}^{-1}{}_y\bar{\varphi}, \quad {}_{\beta}^{-1}{}_y\varphi_y = p_{\beta}^{-1}{}_y\bar{\varphi}, \quad y \equiv 0 \pmod{\omega},$$

$$\text{where } \varphi: X_{\langle y, y \rangle} \otimes_{G^F} \rightarrow X_{\langle \alpha^{-1}y, \beta^{-1}y \rangle} \text{ and } \bar{\varphi}: X_{\langle y, y \rangle} \rightarrow X_{\langle \alpha^{-1}y, \beta^{-1}y \rangle} \otimes_{F^F}$$

is its adjoint map, and

$$(iii) \quad y^{\varphi}_x = p_y(\varphi \otimes 1)j_x, \quad y^{\varphi}_{x'} = p_{y'}(\varphi \otimes 1)j_{x'}, \quad \text{if}$$

$$\varphi: X_{\langle x, x' \rangle} \rightarrow X_{\langle y, y' \rangle}, \quad \text{for every } x \neq x', \quad y \neq y'.$$

Note that y^{φ}_x is defined, in each case, by the commutativity of the following diagrams:

(i)

$$\begin{array}{ccc} X_{\alpha y} & \xrightarrow{y^{\varphi}_{\alpha y}} & X_{\langle y, y \rangle} = X_y \\ j_{\alpha y} \searrow & & \uparrow j_{\beta y} \\ X_{\alpha y} \oplus X_{\beta y} = X_{\langle \alpha y, \beta y \rangle} \otimes_{FG} X_{\langle y, y \rangle} & \xrightarrow{\varphi} & X_{\langle y, y \rangle} = X_y \\ j_{\beta y} \nearrow & & \downarrow j_{\alpha y} \\ X_{\beta y} & \xrightarrow{y^{\varphi}_{\beta y}} & X_{\langle y, y \rangle} = X_y \end{array}$$

(ii)

$$\begin{array}{ccc} X_y = X_{\langle y, y \rangle} & \xrightarrow{\varphi} & X_{\langle \alpha^{-1}y, \beta^{-1}y \rangle} \otimes_{FG} X_{\langle y, y \rangle} = X_{\alpha^{-1}y} \oplus X_{\beta^{-1}y} \\ & \nearrow \alpha^{-1}y^{\varphi}_y & \downarrow p_{\alpha^{-1}y} \\ & & X_{\alpha^{-1}y} \\ & \searrow \beta^{-1}y^{\varphi}_y & \downarrow p_{\beta^{-1}y} \\ & & X_{\beta^{-1}y} \end{array}$$

(iii)

$$\begin{array}{ccc} X_x & \xrightarrow{y^{\varphi}_x} & X_y \\ j_x \searrow & & \uparrow j_{x'} \\ X_x \oplus X_{x'} = X_{\langle x, x' \rangle} \otimes_{FG} X_{\langle y, y' \rangle} & \xrightarrow{\varphi \otimes 1} & X_{\langle y, y' \rangle} \otimes_{FG} X_{\langle y, y' \rangle} = X_y \oplus X_{y'} \\ j_{x'} \nearrow & & \downarrow j_y \\ X_{x'} & \xrightarrow{y^{\varphi}_{x'}} & X_{y'} \end{array}$$

So far we have only defined H as a functor from $R(\tilde{\Sigma})$ to $R_G(\tilde{Q})$ and we still have to prove that (i) it sends the representations of $R(\tilde{\Sigma})$ bound by $\tilde{J}$ to representations of $R_G(\tilde{Q})$ bound by $\tilde{I}$ and that (ii) it satisfies the following property:

(*) If $\underline{X}$ is an indecomposable representation of $R(\tilde{\Sigma}, \tilde{J})$, then either

(a) $H(\underline{X})$ is an indecomposable representation of $R_G(\tilde{Q}, \tilde{I})$ such that $[\underline{X}] = \langle [H(\underline{X})], [H(\underline{X})] \rangle$, or

(b) $H(\underline{X}) = \underline{W} \oplus \underline{W}'$ with $\underline{W}$ and $\underline{W}'$ indecomposable representations of $R_G(\tilde{Q}, \tilde{I})$ such that $[\underline{X}] = \langle [\underline{W}], [\underline{W}'] \rangle$.

To prove finally (i) and (ii), we obviously have only to prove similar assertions for finite full connected subspecies Σ of $\tilde{\Sigma}$ with relation ideal $J: = \tilde{J} \mid \Sigma$ of the following types:

$$(1) \quad F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_{i+1} \xrightarrow{\mu} G_1 \rightarrow F_{1-1} \rightarrow \dots \rightarrow F_2 \rightarrow F_1$$

$$(2) \quad G_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_2 \xrightarrow{\pi} G_1$$

$$(3) \quad \begin{array}{ccccccc} & & F_{a_1} & \rightarrow & \dots & \rightarrow & F_r \xrightarrow{\mu} G_s \rightarrow F_t \rightarrow \dots \rightarrow F_{a_p} \\ & \nearrow & & & & & & \searrow \\ F_n & & & & & & & \\ & \searrow & & & & & & \nearrow \\ & & F_{b_1} & \rightarrow & \dots & \rightarrow & F_{b_q} \end{array}$$

(c, μ)

But, in each of these cases, the algebra $B = T(\Sigma)/J$ and the algebras $A = T(Q')/I'$ where Q' is the corresponding full connected

subgraph of $\tilde{Q}$, $I' := \tilde{I}/Q'$, are of finite representation type, have Auslander-Reiten graphs easy to compute and $U\Gamma(B) \cong \Gamma(A)/\rho$, where ρ is a reflection with respect to the middle line of $\Gamma(A)$. (See examples (2.14)-(2.16)). The proof of the theorem is complete.

(2.13) Example. Although this example does not fall under the cases (1)-(3) of (2.12), it does contain, in a very simple situation, the general principle underlying the whole proof of (2.12). Let $\Sigma_n: G \xrightarrow{GF} F \rightarrow \dots \rightarrow F$ be the K-species of the valued oriented graph $n \xrightarrow{(1,2)} n-1 \rightarrow \dots \rightarrow 2 \rightarrow 1$ of type B_n .

Let Q_{2n-1} be the quiver

of type A_{2n-1} . Let $B_n = T(\Sigma_n)$, $A_{2n-1} = G[Q_{2n-1}]$ the path algebra of Q_{2n-1} over G . Then, $U\Gamma(B_n) \cong \Gamma(A_{2n-1})/\rho$, ρ as above, for every $n \geq 2$.

We illustrate with the case $n = 3$:

$\Gamma(A_5)$:

The indecomposable representations are given by their dimension types. The dotted line indicates in $\Gamma(B_3)$ the τ -orbit u of indecomposable representations with endomorphism ring G , in $\Gamma(A_5)$ the middle line. We observe that under the functor $H: R(\Sigma_3) \rightarrow R_G(Q_5)$, the indecomposable representations $\underline{X}$ of $R(\Sigma_3)$ with $H(\underline{X})$ indecomposable are precisely those lying on u , the other indecomposable representations $\underline{X}$ satisfying $H(\underline{X}) = \underline{W} \oplus \underline{W}'$, $\underline{W}$ and $\underline{W}'$ indecomposable representations of $R_G(Q_5)$. For instance, $H(111) = \begin{pmatrix} 11 \\ 11 \end{pmatrix}$, $H(211) = \begin{pmatrix} 21 \\ 11 \end{pmatrix} = \begin{pmatrix} 11 \\ 00 \end{pmatrix} \oplus \begin{pmatrix} 10 \\ 11 \end{pmatrix}$. Clearly, $U\Gamma(B_3) = \Gamma(A_5)/\rho$.

(2.14) Example. Let $B_{n,1}$ be the algebra with graph

$D(\Sigma_{n,1}, J_{n,1})$

$$F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_{i+1} \xrightarrow{\mu} G_1 \xrightarrow{\nu} F_{i-1} \rightarrow \dots \rightarrow F_2 \rightarrow F_1$$

($n \geq 3$, $2 \leq i \leq n-1$), and let $A_{2n-1,i}$ be the algebra over G given by the quiver $Q_{2n-1,i}$

$$\begin{array}{c} n \rightarrow n-1 \rightarrow \dots \rightarrow i+1 \rightarrow 1 \rightarrow i-1 \rightarrow \dots \rightarrow 2 \rightarrow 1 \\ \uparrow \qquad \qquad \qquad \uparrow \\ n' \rightarrow (n-1)' \rightarrow \dots \rightarrow (i+1)' \rightarrow (i-1)' \rightarrow \dots \rightarrow 2' \rightarrow 1' \end{array}$$

with the zero relations $(i+1) \rightarrow i \rightarrow (i-1)'$ and $(i+1)' \rightarrow i \rightarrow (i-1)$.

Then, again, $\Gamma(B_{n,i}) \cong \Gamma(A_{2n-1,i})/\rho$. We illustrate with $B = B_{3,2}$,

$A = A_{5,2}$:

Under the functor $H: R(\Sigma_{3,2}) \rightarrow R_G(Q_{5,2})$ the representations $\underline{X}$ bound by $J_{3,2}$ have image $H(\underline{X})$ bound by I , the relation ideal generated by the given zero relations on $Q_{5,2}$. This is clear for all the indecomposable representations of B , except the projective-injective representation $\underline{P}$ with $\dim \underline{P} = (121)$. But, we have $\underline{P} = (F, G^2, F, \varphi: F \otimes_{F^G} F \rightarrow G^2, \psi: G^2 \otimes_{G^G} F \rightarrow F)$, where (II, (5.6)) we can take $\varphi = 1$, $\bar{\psi} = 1$, $\bar{\psi}: G^2 \rightarrow F_F \otimes_{F^G} F_G$ the adjoint map of ψ , and $\bar{\psi}\varphi = 1$. Moreover, $F_F \otimes_{F^G} F_G = G \oplus G'$, $G \simeq G'$ as G -vector spaces; let j, j' be the canonical injections, p, p' the canonical projections on $G \oplus G'$ then we have

$$H(\underline{P}) = \begin{pmatrix} G & & & \\ & f & & \\ & & G^2 & \\ & & & g \\ & & & & G \\ G & & & & & \\ & f' & & & & \\ & & g' & & & \\ & & & & & G \end{pmatrix}$$

$$f = \varphi j, \quad g = p\bar{\psi}, \quad f' = \varphi j', \quad g' = p'\bar{\psi},$$

$$\begin{array}{c} G \\ \searrow j \\ G \oplus G' = F \otimes_{F^G} F \xrightarrow{\varphi} G^2 \xrightarrow{\bar{\psi}} F \otimes_{F^G} F = G \oplus G' \begin{array}{l} \nearrow p \\ \searrow p' \\ G \end{array} \\ \nearrow j' \\ G' \end{array}$$

and

$$g'f = (p'\bar{\psi})(\varphi j) = p'(\bar{\psi}\varphi)j = p'.1.j = 0,$$

$$gf' = (p\bar{\psi})(\varphi j') = p(\bar{\psi}\varphi)j' = p.1.j' = 0,$$

so $H(\underline{P}) \in R(Q_{5,2}, I)$. Moreover, $gf = 1_G$, $g'f' = 1_{G'}$, and thus

$H(\underline{P}) = \underline{W} \oplus \underline{W}'$, with $\underline{W}$ and $\underline{W}'$ the projective-injective indecomposable

representations with dimension types $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ respectively.

Clearly, $\rho\left(\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, as required.

(2.15) Example. Let B_n be the algebra with graph

$D(\Sigma_n, J_n)$

$$G_n \xrightarrow{\pi} F_{n-1} \rightarrow \dots \rightarrow F_2 \rightarrow G_1$$

($n \geq 3$), and let A_{2n} be the algebra over G given by the quiver

Q_{2n-2}

with the commutative relation $(n \rightarrow n-1 \rightarrow \dots \rightarrow 2 \rightarrow 1) = (n \rightarrow (n-1)' \rightarrow \dots \rightarrow 2' \rightarrow 1)$.

We illustrate with $B = B_3$, $A = A_4$:

$\Gamma(B)$:

$\Gamma(A)$:

(2.16) Example. Let $B(p, q, i)$ be the algebra with graph $D(\Sigma(p, q, i), J(p, q, i))$

($p \geq 1, q \geq 1, 1 \leq i \leq p$), and let $A(2p, 2q, i)$ be the algebra over G given by the quiver $Q(2p, 2q, i)$

with the commutativity relations on both (non-oriented) cycles and zero-relations $i+1 \xrightarrow{0} a_1 \xrightarrow{0} a'_{i-1}$, $a'_{i+1} \xrightarrow{0} a_1 \xrightarrow{0} a_{i-1}$. We illustrate with $B = B(1, 1, 1)$ and $A = A(2, 2, 1)$:

$\Gamma(B)$:

(ii) If k is a node, then

$$P(k) = \left(\begin{array}{c} \begin{array}{c} \xrightarrow{1} F_{a_1} \xrightarrow{1} \dots \xrightarrow{1} F_{a_p} \xrightarrow{1} F_{k-n} \\ \xrightarrow{1} F_{b_1} \xrightarrow{1} \dots \xrightarrow{1} F_{b_q} \xrightarrow{1} F_{k-n} \end{array} \\ F_k \end{array} \right).$$

with $F_{a_t} = F = F_{b_s}$ for all t , $1 \leq t \leq p$, for all s ,
 $1 \leq s \leq q$, or

$$P(k) = \left(\begin{array}{c} \begin{array}{c} \xrightarrow{1} F_{a_1} \xrightarrow{1} \dots \xrightarrow{1} F_u \xrightarrow{1} G_v \xrightarrow{1} F_w \xrightarrow{1} \dots \xrightarrow{1} F_{a_p} \xrightarrow{1} F_{k-n} \\ \xrightarrow{1} F_{b_1} \xrightarrow{1} \dots \xrightarrow{1} F_{b_q} \xrightarrow{1} F_{k-n} \end{array} \\ F_k \end{array} \right).$$

with $F_{a_t} = F = F_{b_s}$ for all t , $1 \leq t \leq p$, $a_t \neq v$, for all s ,
 $1 \leq s \leq q$, where, in both cases, $\bar{k} \rightarrow \bar{a}_p \rightarrow \dots \rightarrow \bar{a}_1 \rightarrow \bar{k}$ and
 $\bar{k} \rightarrow \bar{b}_q \rightarrow \dots \rightarrow \bar{b}_1 \rightarrow \bar{k}$ are the two cycles through $\bar{k}$, with $\bar{0}$
 lying on the first cycle in the second case.

(3.4) Let $\underline{X} = (X_i, {}_j\varphi_i)$ be a stable indecomposable representation. Then, $(\dim \underline{X})_i \leq 2$, for every i . For, consider any complete τ -slice $\mathcal{F}$ of $\mathbb{Z}B_n$ that contains $[\underline{X}]$. Then, there exists a tilted algebra T of type B_n (III, (1.5)) such that $\mathcal{F}$ is also a complete τ -slice of the Auslander-Reiten graph $\Gamma(T)$, hence the assertion, since $(\dim M_T)_i \leq 2$, for every $i = 1, \dots, n$, for every indecomposable T -module M .

The relation ideal $\tilde{J} := \tilde{J}_D$ of $\tilde{\Gamma}$ is defined by the following minimal relations:

$$(\pi) \quad K_h \xrightarrow{\pi} K_i \rightarrow K_j, \quad i \equiv 0$$

for every path $j \rightarrow i \rightarrow h$ in $\tilde{D}$

$$(\mu) \quad K_{i+n} \xrightarrow{\mu} K_{i_1} \rightarrow \dots \rightarrow K_{i_r} \rightarrow K_i, \quad i \equiv 0$$

where $\bar{i} \rightarrow \bar{i}_r \rightarrow \dots \rightarrow \bar{i}_1 \rightarrow \bar{i}$ is the cycle through $\bar{0}$

where $\bar{i}$ is a node and $\bar{i} \rightarrow \bar{a}_p \rightarrow \dots \rightarrow \bar{a}_1 \rightarrow \bar{i}$, $\bar{i} \rightarrow \bar{b}_q \rightarrow \dots \rightarrow \bar{b}_1 \rightarrow \bar{i}$ are the cycles through $\bar{i}$, and $s \equiv 0$ in (c, μ)

$$(0) \quad K_h \xrightarrow{0} K_i \rightarrow K_j$$

whenever $\bar{j} \rightarrow \bar{i}$, $\bar{i} \rightarrow \bar{h}$ belong to different cycles.

All congruences above are taken modulo n .

(2.18) We set $\mu: \mathbb{Z} \rightarrow \mathbb{Z}$, $i \mapsto i+n$, and denote also by μ the selfequivalence of the category $R(\tilde{\Sigma}, \tilde{J})$ induced by μ (2.4). We construct the species $\Sigma_r := \Sigma_{r,D} := \tilde{\Sigma}/\mu^r$ and the relation ideals $J_r := \tilde{J}/\mu^r$ for every $r \geq 1$, and consider the algebras $C_r := C_{r,D} := T(\Sigma_r)/J_r$, $r \geq 1$, $\tilde{C} := \tilde{C}_D := \bigoplus_{i,j \in \mathbb{Z}} \text{Hom}_{R(\tilde{\Sigma}, \tilde{J})}(\mathcal{P}(i), \mathcal{P}(j))$.

(2.19) Proposition.

$$(i) \text{ mod } \tilde{C} \simeq R(\tilde{\Sigma}, \tilde{J})$$

(ii) $\tilde{C}$ is a basic, connected, selfinjective, locally finite dimensional K -algebra

$$(iii) \text{ mod } C_r \simeq R(\Sigma_r, J_r) \simeq R^{\mu^r}(\tilde{\Sigma}, \tilde{J}), \text{ for every } r \geq 1.$$

Proof. Similar to that of (2.6).

(2.20) The algebra $\tilde{C}$ has a matrix representation $\hat{C}$ in a way completely similar to the algebra $\tilde{B}$ in the B_n case. Since the main theorems for C_n stated below all have their similar counterparts for B_n , we state them without any further mention of similarity. Their proofs can be done, with the necessary modifications, as those of their counterparts.

(2.21) Theorem. Let D be a Brauer graph of type C_n , C its associated configuration of $\mathbb{Z} C_n$. Then

(i) $C_{r,D}$ is a basic, connected, finite dimensional K -algebra of finite representation type and of Dynkin class C_n

(ii) $\Gamma(C_{r,D}) \cong (ZC_n) / \langle \tau^{rw} \rangle$, for every $r \geq 1$.

(2.22) Theorem. Let Σ be a complete μ -slice of $\tilde{\Sigma}$, and let $C' := T(\Sigma)/J$, $J = \tilde{J}|_{\Sigma}$. Then $C_1 \cong C' \rtimes DC'$.

(2.23) Theorem. The algebras $C_{1,D}$, D running through a set of representatives of the isomorphism classes of Brauer graphs of type C_n , are trivial extension algebras of finite representation type and of Dynkin class C_n .

The following examples illustrate our theorems. We give, for a given Brauer graph D of type B_n or of type C_n , the objects $\tilde{D}$, $\tilde{\Sigma}$, $\tilde{J}$ that we have associated with it, as well as some algebras $B_{r,D}$ or $C_{r,D}$ which are basic, connected, finite dimensional, selfinjective algebras of finite representation type and of Dynkin class B_n or C_n .

(2.24) Example. Let $D = D_1$ (1.46, (ii)) be the Brauer graph of type B_3 :

Then

$\tilde{D}$: $\tilde{\Sigma}$: $\Gamma(\tilde{B})$: $B_{2,D}$: $J_{2,D}(\mu), (\pi), \text{rad}^4 = 0$ $\Gamma(B_{2,D})$:

$\tilde{J}: (\mu), (\pi), (c), (c, \mu), (0)$ [the zero relations are indicated by two consecutive arrows, one full and one broken, or viceversa].

$B_{1,D}: G_1 \xrightarrow{\sim} F_2 \xrightarrow{\sim} F_3 \xrightarrow{\sim} F_4$ with $J_{1,D}: (\mu), (\pi)$, all possible commutativity relations, $\text{rad}^3 = 0, (0)$.

$J_{2,D}: (\mu), (\pi), (0)$, all possible commutativity relations $((c), (c, \mu))$.

Notice that $D(B_{2,D}, J_{2,D})$ lies on a cylinder, the dotted lines indicating the identification points. It is clear that $D(B_{r,D}, J_{r,D})$ also lies on a cylinder, for every $r \geq 1$.

$\Gamma(B_{1,D})$:

(2.26) Remark. Let D'_1, D'_3 the Brauer graphs of types C_3 and C_4 given by

By (1.45) we obtain immediately $\tilde{D}', \tilde{\Sigma}', \tilde{J}', \tilde{C}', \tilde{C}'_{r,D'}$ from the corresponding objects of the Brauer graphs D_1, D_3 of Examples (2.24), (2.25). Thus, $\tilde{D}', \Gamma(\tilde{C}'), \Gamma(\tilde{C}'_{r,D'})$ have exactly the same shape as those of the B_n case. For example, the algebra C'_{2,D'_1} given by

with $J'_{2,D'_1}: (\pi), (\mu), \text{rad}^4 = 0$, or the algebra C'_{1,D'_3} given by

$$F_1 \xrightarrow{\alpha} G_2 \xrightarrow{\beta} G_3 \xrightarrow{\gamma} G_4$$

with $J'_{1,D_3} : (\pi), (\mu), (0), \text{rad}^3 = 0$ and all possible commutativity relations.

IV.3 Indecomposable representations

In this section, we use the theorems of Section 2 to describe the indecomposable representations of the categories $R(\Sigma_r, J_r) \bmod A_r$, for selfinjective algebras A_r of finite representation type and of Dynkin class B_n or C_n .

As in the previous section, we consider first the B_n case, discussing the similar C_n case afterwards. All the assumptions of that section remain in force.

(3.1) Let B be a basic, connected, finite dimensional, selfinjective algebra of finite representation type of Dynkin class B_n such that $B \cong T(\Sigma_r)/J_r$. By (2.6), $\text{mod } T(\Sigma_r)/J_r \cong R(\Sigma_r, J_r) \xrightarrow{\sim} R(\tilde{\Sigma}, \tilde{J})$, where $\tilde{\Sigma}$ is the K -species of the universal covering $\tilde{D}$ of the Brauer graph D such that $\Gamma(B) \cong (\mathbb{Z} B_n)_{n \in \mathbb{C}} / W_r$, C the configuration associated to D (1.40). Thus, to describe the indecomposable representations of $R(\Sigma_r, J_r)$ it suffices to describe the indecomposable representations of $R(\tilde{\Sigma}, \tilde{J})$. We will describe first the projective-injective indecompo-

sable representations, then the stable indecomposable representations and finally, using the functor $H: R(\tilde{C}, \tilde{J}) \rightarrow R_G(\tilde{Q}, \tilde{I})$ of (2.12), will give an algorithm to determine the position of each indecomposable representation on $(\mathbb{Z}B_n)_C$.

(3.2) Let $\underline{X} = (X_i, {}_j\varphi_i)$ be a representation of $R(\tilde{C}, \tilde{J})$.

To simplify the description of the X_i and of the maps ${}_j\varphi_i$, we will use the following notational conventions: we write $X_i = F_i^r$ if $X_i \cong F^r$, and $X_i = G_i^r$ if $X_i = G^r$, F^r and G^r the r -dimensional vector space over F and G respectively. We write $G_i \xrightarrow{1} F_j$ if ${}_j\varphi_i: G_i \otimes_{G^F} F \rightarrow (F_j)_F$ is isomorphic to the identity map $F \xrightarrow{1} F$, and $F_j \xrightarrow{1} G_i$ if ${}_j\varphi_i: F_i \otimes_{F^F} G \rightarrow (G_j)_G$ is the adjoint map of a map $F_i \rightarrow G_j \otimes_{G^F} F$ isomorphic to the identity map $F \xrightarrow{1} F$.

(3.3) The projective-injective indecomposable representations $\underline{P}(k)$, $k \in \mathbb{Z}$, are described according to whether k is transit vertex or a node of $\tilde{D}$:

(1) If k is a transit vertex, then $\underline{P}(k)$ is uniserial, with $\underline{P}(k) = (F_k \xrightarrow{1} F_{k_1} \xrightarrow{1} \dots \xrightarrow{1} F_{k_r} \xrightarrow{1} F_{k-n})$, $F_{k_t} = F$, for every t , $1 \leq t \leq r$, if $\bar{k} \rightarrow \bar{k}_r \rightarrow \dots \rightarrow \bar{k}_1 \rightarrow \bar{k}$ is the cycle through $\bar{k} \in D_0 = \mathbb{Z}/n\mathbb{Z}$, and $\bar{k} \neq \bar{0}$, and $\underline{P}(k) = (G_k \xrightarrow{1} F_{k_1} \xrightarrow{1} F_{k_2} \rightarrow \dots \xrightarrow{1} F_{k_r} \xrightarrow{1} G_{k-n})$, $F_{k_t} = F$, for all t , $1 \leq t \leq r$, if $\bar{k} = \bar{0}$, and $\bar{k} \rightarrow \bar{k}_r \rightarrow \dots \rightarrow \bar{k}_1 \rightarrow \bar{k}$ again is the cycle through $\bar{k}$.

(11) If k is a node, then

$$P(\underline{k}) = \left(\begin{array}{c} \begin{array}{c} \xrightarrow{1} F_{a_1} \xrightarrow{1} \dots \xrightarrow{1} F_{a_p} \xrightarrow{1} F_{k-n} \\ F_k \xrightarrow{1} F_{b_1} \xrightarrow{1} \dots \xrightarrow{1} F_{b_q} \xrightarrow{1} F_{k-n} \end{array} \end{array} \right).$$

with $F_{a_t} = F = F_{b_s}$ for all t , $1 \leq t \leq p$, for all s ,
 $1 \leq s \leq q$, or

$$P(\underline{k}) = \left(\begin{array}{c} \begin{array}{c} \xrightarrow{1} F_{a_1} \xrightarrow{1} \dots \xrightarrow{1} F_u \xrightarrow{1} G_v \xrightarrow{1} F_w \xrightarrow{1} \dots \xrightarrow{1} F_{a_p} \xrightarrow{1} F_{k-n} \\ F_k \xrightarrow{1} F_{b_1} \xrightarrow{1} \dots \xrightarrow{1} F_{b_q} \xrightarrow{1} F_{k-n} \end{array} \end{array} \right)$$

with $F_{a_t} = F = F_{b_s}$ for all t , $1 \leq t \leq p$, $a_t \neq v$, for all s ,
 $1 \leq s \leq q$, where, in both cases, $\bar{k} \rightarrow \bar{a}_p \rightarrow \dots \rightarrow \bar{a}_1 \rightarrow \bar{k}$ and
 $\bar{k} \rightarrow \bar{b}_q \rightarrow \dots \rightarrow \bar{b}_1 \rightarrow \bar{k}$ are the two cycles through $\bar{k}$, with $\bar{0}$
 lying on the first cycle in the second case.

(3.4) Let $\underline{X} = (X_i, {}_j\varphi_i)$ be a stable indecomposable representation. Then, $(\dim \underline{X})_i \leq 2$, for every i . For, consider any complete τ -slice $\mathcal{F}$ of $\mathbb{Z}B_n$ that contains $[\underline{X}]$. Then, there exists a tilted algebra T of type B_n (III, (1.5)) such that $\mathcal{F}$ is also a complete τ -slice of the Auslander-Reiten graph $\Gamma(T)$, hence the assertion, since $(\dim M_T)_i \leq 2$, for every $i = 1, \dots, n$, for every indecomposable T -module M .

(3.5) From (3.4), it follows that we can describe any indecomposable stable representation $\underline{X}$ by constructing the tilted algebra T of any complete τ -slice of ZB_n containing $[\underline{X}]$. For, $T = \text{End}(M_H)$, M_H a tilting module over a hereditary tensor algebra $H = T(\Omega, \Omega)$, where (Ω, Ω) is an oriented modulation of the Dynkin graph B_n . Thus, using appropriate tilting functors ([20]) we can describe the indecomposable T -modules in terms of the indecomposable representations of the category $R(\Omega, \Omega)$, which in turn have been completely described in [11].

(3.6) Let (a, b) be a point of ZB_n , and let $\underline{X}(a, b)$ be the indecomposable stable representation of $R(\tilde{E}, \tilde{J})$ standing at (a, b) . Our purpose is to describe $\underline{X}(a, b)$ purely in terms of the Brauer graph D such that $\Gamma(R(\tilde{E}, \tilde{J})) \simeq (ZB_n)_C$, C the configuration of ZB_n associated to D , or, equivalently (1.20), in terms of the symmetric Brauer pair (α, β) associated to D , by using the functor $H: R(\tilde{E}, \tilde{J}) \rightarrow R_G(\tilde{Q}, \tilde{I})$ of (2.12) that relates the indecomposable representations of $R(\tilde{E}, \tilde{J})$ with those of $R_G(\tilde{Q}, \tilde{I})$.

To this end, it will be convenient to use a set of coordinates for ZB_n different from the standard one (II, (2.2)) that we have used so far. This new set of coordinates is completely analogous to the coordinates used to describe the Brauer graph D taking pairs $\langle \bar{y}, \bar{y}' \rangle$ of integers y, y' such that $y' = \alpha y - n$

((1.38), (1.39)), and has the advantage of allowing to describe better the symmetric Brauer pair (α, β) .

Let $\rho: \mathbb{Z}A_w \rightarrow \mathbb{Z}A_w$ be the reflection through the middle line defined by $\rho(a, b) := (a+b-n, 2n-b)$. Let $[a, b]^* := \langle (a, b), \rho(a, b) \rangle$ for any $(a, b) \in (\mathbb{Z}/A_w)_0$, and let $\mathbb{Z}A_w/\rho$ be the set of all such pairs $[a, b]$. Given $[a, b]$ and $[c, d]$, define the arrow $[a, b] \rightarrow [c, d]$ if either there is an arrow $(a, b) \rightarrow (c, d)$ or an arrow $(a, b) \rightarrow \rho(c, d)$ in $\mathbb{Z}A_w$. Finally, define the valuation w on this set of arrows by $[a, n-1] \xrightarrow{(2,1)} [a, n]$, $[a, n] \xrightarrow{(1,2)} [a+1, n-1]$, for every $a \in \mathbb{Z}$, trivial on the remaining arrows. Then, clearly $\mathbb{Z}B_n \xrightarrow{\sim} (\mathbb{Z}A_w/\rho, (\mathbb{Z}A_w/\rho)_1, w)$, by sending $(a, b) \in (\mathbb{Z}B_n)_0$ to $[a, b]$. From now on we will identify $\mathbb{Z}B_n$ with $(\mathbb{Z}A_w/\rho, (\mathbb{Z}A_w/\rho)_1, w)$.

(3.7) Let $\underline{X} = (X_1, j^{\varphi_1}) = (X_{\langle y, y' \rangle}, \langle z, z' \rangle^{\varphi_{\langle y, y' \rangle}})$ be an indecomposable representation of $R(\tilde{C}, \tilde{J})$. We call support graph of $\underline{X}$ the full subgraph $D(\underline{X})$ of $\tilde{D}$ whose vertices are those $i = \langle y, y' \rangle$ such that $X_i \neq 0$. Clearly, $D(\underline{X})$ is of the form given in Figure 1, where the broken arrows are $\langle \alpha, \beta \rangle$ -arrows, the full arrows are $\langle \beta, \alpha \rangle$ -arrows, $0 \leq a_1 < a(y_1)$, $0 \leq b_r < b(y_r)$, $0 < b_1 < b(y_1)$, $0 < a_2 < a(y_2), \dots, 0 < a_r < a(y_r)$, with $r \geq 1$. Here, $a(y)$ and $b(y)$ are the integers defined

(Fig. 1)

3 3
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by $\alpha^{a(y)} y = y + \omega = \beta^{b(y)} y$, for every $y \in \mathbb{Z}$. In Figure 2, we give some support graphs of indecomposable representations of the category $R(\tilde{\Sigma}, \tilde{J})$ of Example (3.11). All subsequent examples will be taken also from (3.11) to facilitate the exposition.

(Fig. 2)

Notice that the set of vertices of any support graph has always at most n elements (3.4) and that it contains at most one G -vertex, that is one of the form $\langle k\omega, k\omega \rangle$, $k \in \mathbb{Z}$. We call G -representations those whose support graph contains a G -vertex, F -representations the others. For instance, (a), (e)

and (f) in Figure 2 are support graphs of F-representations, the other ones of G-representations.

(3.8) Let $\underline{X}$ be an F-representation. Then, clearly

$\underline{X} = (X_i, {}_j\varphi_i)$ with $X_i = F$, ${}_j\varphi_i = 1$, for every $i, j \in D(\underline{X})$

and $X_i = 0$, ${}_j\varphi_i = 0$, otherwise. By (2.12), $H(\underline{X}) = \underline{W} \oplus \underline{W}'$,

with $\underline{W}, \underline{W}'$ indecomposable representations of $R_G(\tilde{Q}, \tilde{I})$ such

that $\rho([W]) = [W']$ and with support graphs of the forms (a) and

(b) of Figure 3, one of them that of $\underline{W}$, the other that of $\underline{W}'$,

if $D(\underline{X})$ has the form given in Figure 1. Suppose that $D(\underline{W})$ is

of the form (a) and $D(\underline{W}')$ of the form (b). Then by [24] we

know that $\underline{W} = \underline{X}(t, \alpha s - t)$ and $\underline{W}' = \underline{X}(t', \alpha s' - t')$, where

$t = \beta^b y_r$, $s = \alpha^a y_1$ and $t' = \alpha^a y_r$, $s' = \beta^a y_1$. As a

consequence, $\underline{X} = \underline{X}[t, \alpha s - t] = \underline{X}[t', \alpha s' - t']$, for,

$[\underline{X}] = \langle [\underline{W}], [\underline{W}'] \rangle$, by (2.13), and $\rho(t, \alpha s - t) = (t', \alpha s' - t')$,

by (1.37) and (1.33): $s' = \beta^a y_1 \sim \alpha^a y_1 = s$, so $s' = \alpha s - n$;

similarly, $t' \sim t$, so $t' = \alpha t - n$. Thus, $\rho(t, \alpha s - t) = (t + (\alpha s - t) - n,$

$2n - (\alpha s - t)) = (\alpha s - n, 2n - \alpha s + t) = (s', \alpha t' - s')$.

For instance, the simple representation $\underline{S}(\langle 2, 5 \rangle)$ is given

by $\underline{X}[2, \alpha 2 - 2] = \underline{X}[2, 8] = \underline{X}[5, 2] = \underline{X}[5, \alpha 5 - 5]$. (Figure 2(a)).

Similarly, the representation $\underline{X}$ with support graph the one given

in Figure 2 (f) is given by $\underline{X}[13, \alpha(19) - 13] = \underline{X}[15, \alpha(16) - 15] = \underline{X}[15, 3]$.

(a)

(b)

(Fig. 3)

(3.9) Let finally $\underline{X}$ be a G -representation. Here, we have the additional complication that there are several indecomposable representations with the same support graph. For example, the representations:

$$\underline{X}_1 := (F_{\langle 11,14 \rangle} \xrightarrow{1} G_{\langle 9,9 \rangle}),$$

$$\underline{X}_2 := (F_{\langle 11,14 \rangle} \xrightarrow{1} G_{\langle 9,9 \rangle}^2)$$

both have the support graph (Figure 2(d)).

Similarly, the representations

$$\underline{Y}_1 := \begin{pmatrix} F_{\langle 12,8 \rangle} \xrightarrow{1} F_{\langle 4,7 \rangle} \\ G_{\langle 9,9 \rangle} \xrightarrow{1} F_{\langle 4,7 \rangle} \end{pmatrix}$$

$$\underline{Y}_2 := \begin{pmatrix} F_{\langle 12,18 \rangle} \xrightarrow{1} F_{\langle 4,7 \rangle} \\ G_{\langle 9,9 \rangle}^2 \xrightarrow{1} F_{\langle 4,7 \rangle} \end{pmatrix}$$

$$\underline{Y}_3 := \begin{pmatrix} F_{\langle 12,18 \rangle} \xrightarrow{2} F_{\langle 4,7 \rangle} \\ G_{\langle 9,9 \rangle}^2 \xrightarrow{1} F_{\langle 4,7 \rangle} \end{pmatrix}$$

have the support graph

(Fig. 2(g)).

(We describe the representations $\underline{X} = (\tilde{X}_1, j\varphi_1)$, in an obvious manner, displaying only the non-zero entries $X_1 \neq 0$.)

Consider first the case of a support graph of the form

Applying H to $\underline{X} = (F_{\langle y, y' \rangle} \rightarrow G_{\langle 1, 1 \rangle})$ we have

$$H(\underline{X}) = \left(\begin{array}{c} G_{y'} \\ \searrow \\ G_y \end{array} \rightarrow G_1 \right) = \underline{X} \left(\begin{array}{c} \nearrow x' \\ 1 \\ \searrow y \end{array} \right) =: \underline{W},$$

so $[\underline{X}] = \langle [\underline{W}], [\underline{W}] \rangle$ and $\underline{X} = \underline{X}[1 \begin{smallmatrix} \nearrow y' \\ \searrow y \end{smallmatrix}] = \underline{X}[y, \alpha y' - y]$, whereas,

applying H to $\underline{Y} = (F_{\langle y, y' \rangle} \rightarrow G_{\langle 1, 1 \rangle}^2)$ we obtain

$$\begin{aligned} H(\underline{Y}) &= \left(\begin{array}{c} G_{y'} \\ \searrow \\ G_y \end{array} \rightarrow G_1^2 \right) = \left(\begin{array}{c} G_{y'} \\ \searrow \\ 0 \end{array} \rightarrow G_1 \right) \oplus \left(\begin{array}{c} 0 \\ \searrow \\ G_y \end{array} \rightarrow G_1 \right) = \\ &= \underline{X}(1 \begin{smallmatrix} \nearrow y' \\ \searrow y \end{smallmatrix}) \oplus \underline{X}(\begin{smallmatrix} 1 \\ \searrow y \end{smallmatrix}) =: \underline{W} \oplus \underline{W}', \end{aligned}$$

so $[\underline{Y}] = \langle [\underline{W}], [\underline{W}'] \rangle$, and $\underline{Y} = \underline{X}[1 \begin{smallmatrix} \nearrow y' \\ \searrow y \end{smallmatrix}] = \underline{X}[1, \alpha y' - 1] = \underline{X}[\begin{smallmatrix} 1 \\ \searrow y \end{smallmatrix}] = \underline{X}[y, \alpha y - y]$

In particular,

$$\begin{aligned} \underline{X}_1 &= \underline{X}[9 \begin{smallmatrix} \nearrow 14 \\ \searrow \end{smallmatrix}] = \underline{X}[9, \alpha(14) - 9] = \underline{X}[9, 7] \\ &= \underline{X}[\begin{smallmatrix} 9 \\ \searrow 11 \end{smallmatrix}] = \underline{X}[11, \alpha 9 - 11] = \underline{X}[11, 3]. \end{aligned}$$

Consider now a support graph of the form

Applying H to the representations $\underline{X} = \begin{pmatrix} F_{\langle z, z' \rangle} & \rightarrow & F_{\langle y, y' \rangle} \\ G_{\langle i, i \rangle} & \rightarrow & \end{pmatrix}$,

$$\underline{Y} = \begin{pmatrix} F_{\langle z, z' \rangle} & \rightarrow & F_{\langle y, y' \rangle} \\ G_{\langle i, i \rangle}^2 & \rightarrow & \end{pmatrix} \quad \text{and} \quad \underline{Z} = \begin{pmatrix} F_{\langle z, z' \rangle} & \rightarrow & F_{\langle y, y' \rangle}^2 \\ G_{\langle i, i \rangle}^2 & \rightarrow & \end{pmatrix},$$

we obtain

$$H(\underline{X}) = \begin{pmatrix} G_{z'} \rightarrow G_{y'} \\ G_i \rightarrow G_y \\ G_z \rightarrow \end{pmatrix} = \underline{X} \begin{pmatrix} y \rightarrow z \\ y' \rightarrow i \\ y' \rightarrow z' \end{pmatrix} = : \underline{U},$$

$$H(\underline{Y}) = \begin{pmatrix} G_{z'} \rightarrow G_{y'} \\ G_i^2 \rightarrow G_y \\ G_z \rightarrow \end{pmatrix} = \begin{pmatrix} G_{z'} \rightarrow G_{y'} \\ G_i \rightarrow 0 \\ 0 \rightarrow 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \rightarrow 0 \\ G_i \rightarrow G_y \\ G_z \rightarrow \end{pmatrix}$$

$$= \underline{X} \begin{pmatrix} y' \rightarrow i \\ y' \rightarrow z' \end{pmatrix} \oplus \underline{X} \begin{pmatrix} y \rightarrow z \\ y' \rightarrow i \end{pmatrix} = : \underline{V} \oplus \underline{V}'$$

$$H(\underline{Z}) = \begin{pmatrix} G_{z'} \rightarrow G_{y'}^2 \\ G_i^2 \rightarrow G_y \\ G_z \rightarrow \end{pmatrix} = \begin{pmatrix} G_{z'} \rightarrow G_{y'} \\ G_i \rightarrow G_y \\ 0 \rightarrow \end{pmatrix} \oplus \begin{pmatrix} 0 \rightarrow G_{y'} \\ G_i \rightarrow G_y \\ G_z \rightarrow \end{pmatrix}$$

$$= \underline{X} \begin{pmatrix} y \rightarrow i \\ y' \rightarrow z' \end{pmatrix} \oplus \underline{X} \begin{pmatrix} y \rightarrow z \\ y' \rightarrow i \end{pmatrix} = : \underline{W} \oplus \underline{W}',$$

so, $[X] = \langle [U], [U] \rangle$, $[Y] = \langle [V], [V'] \rangle$, $[Z] = \langle [W], [W'] \rangle$,

and

$$\underline{X} = \underline{X} \left[\begin{array}{c} y \rightarrow z \\ \quad \searrow \\ \quad i \\ y' \rightarrow z' \\ \quad \searrow \\ \quad i \end{array} \right] = \underline{X}[z', \alpha z - z'] ,$$

$$\underline{Y} = \underline{X} \left[\begin{array}{c} i \\ y' \rightarrow z' \\ \quad \searrow \\ \quad i \end{array} \right] = \underline{X}[z', \alpha i - z']$$

$$= \underline{X} \left[\begin{array}{c} z \\ y \rightarrow i \\ \quad \searrow \\ \quad i \end{array} \right] = \underline{X}[i, \alpha z - i] ,$$

$$\underline{Z} = \underline{X} \left[\begin{array}{c} y \rightarrow i \\ \quad \searrow \\ \quad i \\ y' \rightarrow z' \\ \quad \searrow \\ \quad i \end{array} \right] = \underline{X}[x', \alpha y - z']$$

$$= \underline{X} \left[\begin{array}{c} z \\ y \rightarrow i \\ \quad \searrow \\ \quad i \\ y' \rightarrow i \end{array} \right] = \underline{X}[y', \alpha z - y'] .$$

In particular,

$$\underline{Y}_1 = \underline{X} \left[\begin{array}{c} 12 \\ 4 \rightarrow 12 \\ \quad \searrow \\ \quad 9 \\ 7 \rightarrow 9 \\ \quad \searrow \\ \quad 8 \end{array} \right] = \underline{X}[8, \alpha(12) - 8] = \underline{X}[8, 5]$$

$$\underline{Y}_2 = \underline{X} \left[\begin{array}{c} 9 \\ 7 \rightarrow 9 \\ \quad \searrow \\ \quad 8 \end{array} \right] = \underline{X}[8, \alpha 9 - 8] = \underline{X}[8, 6]$$

$$= \underline{X} \left[\begin{array}{c} 12 \\ 4 \rightarrow 9 \\ \quad \searrow \\ \quad 9 \end{array} \right] = \underline{X}[9, \alpha(12) - 9] = \underline{X}[9, 4] ,$$

$$\begin{aligned}
 \underline{Y}_3 &= \underline{X} \left[\begin{array}{c} 4 \rightarrow 9 \\ 7 \rightarrow 8 \end{array} \right] = \underline{X}[8, \alpha 4 - 8] = \underline{X}[8, 4] \\
 &= \underline{X} \left[\begin{array}{c} 4 \rightarrow 12 \\ 7 \rightarrow 9 \end{array} \right] = \underline{X}[7, \alpha(12) - 7] = \underline{X}[7, 6].
 \end{aligned}$$

From the examples above it is clear how to obtain any indecomposable G-representation with a given support graph.

(3.10) Remark. The description of the indecomposable representations of the category $R(\tilde{\mathcal{C}}, \tilde{\mathcal{J}})$ where $\tilde{\mathcal{C}}$ is the K-species of the universal covering $\tilde{D}$ of a Brauer graph D of type C_n is entirely analogous to the one just given in the B_n case, for, obviously, all our considerations are valid, mutatis mutandi, also in this case. We omit them.

(3.11) Example. Let D be the Brauer graph of type B_5 given by

We want to illustrate all the concepts associated to D which we have introduced in the course of our investigation. We refer thus to the previous pages for full explanations of these concepts and notations.

$$(ii) \quad \bar{\beta}' = (024)$$

$$\bar{\alpha}' = (12)(34)$$

$$(iii) \quad \beta = (024)(56)(78)$$

$$\alpha = (057)(12)(34)$$

(iv) Q:

(v) D':

(vi) $\tilde{D}$:

(vii) $D(\tilde{\Sigma})$: (viii) $\tilde{Q}$: (ix) $D(\tilde{\Sigma})$:

(x) $\Gamma(\tilde{B})$:

CHAPTER V. SELF-INJECTIVE ALGEBRAS OF DYNKIN

CLASS F_4 OR G_2

In this final chapter of our investigation, we construct self-injective algebras of finite representation type and of Dynkin class F_4 or G_2 . In each case, we give a complete description of all the configurations of the stable translation modulated graphs, the Brauer graphs and its universal coverings, the basic, connected, finite dimensional self-injective algebras of finite representation type of class F_4 or G_2 , and indicate how to describe its indecomposable modules.

As always, K will denote a perfect field. In the case of F_4 , F/G will be a quadratic field extension, and in the case of G_2 H/G will be a cubic field extension, with $G = K$ in both cases. We modulate $\mathbb{Z}\Delta$ with the modulations introduced in (II, (6.10)) and use the coordinates of the valued translation graphs $\mathbb{Z}\Delta$ defined in (II, (2.2)), $\Delta = F_4$ or G_2 .

Sections 1 to 3 deal with the case of F_4 and Section 4 is devoted to the case of G_2 .

V.1 Configurations of $\mathbb{Z}\mathbb{F}_4$

We put $\omega := \omega_{\mathbb{F}_4} = 11$, $W = \langle \tau^\omega \rangle$, $W_r := \langle \tau^{\omega r} \rangle$, for every $r \geq 1$. Further, for $x = (a, b) \in (\mathbb{Z}\mathbb{F}_4)_0$, we put $[a, b] := W(a, b) = \{ \tau^{kw}(a, b) \mid k \in \mathbb{Z} \}$ to denote the orbit of x under W . We recall (III, (3.1)) that all the groups W_r , $r \geq 1$, are admissible subgroups of automorphisms of $\mathbb{Z}\mathbb{F}_4$.

(1.1) Proposition. There are exactly six non-isomorphic configurations of $\mathbb{Z}\mathbb{F}_4$, represented by the following configurations:

$$\begin{aligned} C_1 &= [0, 1] \cup [1, 1] \cup [4, 4] \cup [5, 4], \\ C_2 &= [0, 1] \cup [5, 2] \cup [9, 4] \cup [10, 1], \\ C_3 &= [0, 1] \cup [1, 1] \cup [5, 3] \cup [10, 1], \\ C_4 &= [0, 1] \cup [4, 4] \cup [5, 2] \cup [10, 3], \\ C_5 &= [0, 1] \cup [3, 4] \cup [4, 1] \cup [7, 4], \\ C_6 &= [0, 1] \cup [1, 4] \cup [5, 4] \cup [7, 1]. \end{aligned}$$

See Figure 1.

Proof. Case by case checking.

(1.2) Remarks.

(1) In Figure 1 we have displayed only the configuration points contained in the rectangle starting at $(0, 1)$, the point farthest to the left, and stopping at $(10, 4)$, the point farthest to the right. since, by (1.1), each

(Fig. 1)

configuration has ω as period. We omit to put the direction of the arrows, always going from left to right.

(ii) Let C be one of the configurations listed in (1.1). Then $\tau^r C$ ($1 \leq r \leq 10$) are also configurations of $\mathbb{Z}F_4$; in fact, they are all the configurations isomorphic to C .

(1.3) Proposition. The groups W_r ($r \geq 1$) are all the admissible groups of $\mathbb{Z}F_4$ that stabilize configurations.

Proof. Let $\langle \tau^i \rangle$, $i \geq 1$, be an admissible group of automorphisms of $\mathbb{Z}F_4$. A glance at each of the configurations listed in (1.1) shows that none of them is stable under $\langle \tau^i \rangle$ for $i = 1, \dots, 10$, hence ω is the smallest period of the configurations, and $\langle \tau^i \rangle = \langle \tau^{r\omega} \rangle = W_r$.

V.2 Selfinjective algebras of class F_4 .

We describe the basic, connected, selfinjective algebras of finite representation type and of Dynkin class F_4 by means of suitable modulations of the universal coverings of Brauer graphs of type F_4 , which we now introduce.

(2.1) Definition. Let $I = \{0, 1, 2, 3\}$. We call Brauer graphs of type F_4 each of the following six valued oriented graphs:

(2.2) Definition. Let F_i ($1 \leq i \leq 6$) be a Brauer graph of type F_4 . We call universal covering of F_i , and denote by $\tilde{F}_i$ ($1 \leq i \leq 6$), each of the following valued oriented graphs with $(\tilde{F}_i)_0 = \mathbb{Z}$:

$\tilde{F}_1$:

$\tilde{F}_2$:

$\tilde{F}_3$:

$\tilde{F}_4$:

$\tilde{F}_5:$ $\tilde{F}_6:$ (2.3) Definition.

(a) Let $\tilde{\Sigma}_h$ the K -species of the universal covering $\tilde{F}_h$ of the Brauer graph F_h ($1 \leq h \leq 6$), defined, in each case, by the following $\tilde{\Sigma}_h = (K_i, {}_iM_j)_{i,j \in \mathbb{Z}}$

(i) $\tilde{\Sigma}_1:$

$$K_i = \begin{cases} G, & i \equiv 0, 2 \\ F, & i \equiv 0, 1 \end{cases}$$

$${}_iM_j = \begin{cases} G^F_F, & i \equiv 0, 2 \\ F^F_G, & i \equiv 1, 3 \\ F^F_F, & \text{otherwise} \end{cases}$$

(ii) $\tilde{\Sigma}_2$:

$$K_i = \begin{cases} G, & i \equiv 0,1 \\ F, & i \equiv 2,3 \end{cases}$$

$${}_i M_{i+1} = \begin{cases} G^G_G, & i \equiv 0 \\ G^F_F, & i \equiv 2 \\ F^F_G, & i \equiv 3 \end{cases}$$

$${}_i M_{i+2} = G^F_F, \quad i \equiv 1$$

$${}_i M_{i+3} = F^F_F, \quad i \equiv 3$$

(iii) $\tilde{\Sigma}_3$:

$$K_i = \begin{cases} G, & i \equiv 0,3 \\ F, & i \equiv 1,2 \end{cases}$$

$${}_i M_{i+1} = \begin{cases} G^F_F, & i \equiv 0 \\ F^F_F, & i \equiv 1 \\ G^G_G, & i \equiv 3 \end{cases}$$

$${}_i M_{i+2} = F^F_G, \quad i \equiv 2$$

$${}_i M_{i+3} = G^G_G, \quad i \equiv 3$$

(iv) $\tilde{\Sigma}_4$:

$$K_i = \begin{cases} G, & i \equiv 0, 3 \\ F, & i \equiv 2, 4 \end{cases}$$

$$i^{M_{i+1}} = F^F_F, \quad i \equiv 1$$

$$i^{M_{i+2}} = \begin{cases} G^F_F, & i \equiv 0 \\ F^F_G, & i \equiv 2 \end{cases}$$

$$i^{M_{i+3}} = \begin{cases} G^G_G, & i \equiv 0 \\ F^F_F, & i \equiv 3 \end{cases}$$

(v) $\tilde{\Sigma}_5$:

$$K_i = \begin{cases} G, & i \equiv 0, 1 \\ F, & i \equiv 2, 3 \end{cases}$$

$$i^{M_{i+1}} = \begin{cases} G^G_G, & i \equiv 0 \\ G^F_F, & i \equiv 1 \end{cases}$$

$$i^{M_{i+2}} = F^F_G, \quad i \equiv 2$$

$$i^{M_{i+3}} = G^F_F, \quad i \equiv 0$$

$$i^{M_{i+4}} = F^F_F, \quad i \equiv 2$$

(vi) $\tilde{F}_6$:

$$K_i = \begin{cases} G, & i \equiv 2, 3 \\ F, & i \equiv 0, 1 \end{cases}$$

$$iM_{i+1} = \begin{cases} F^F_F, & i \equiv 0 \\ F^F_G, & i \equiv 1 \end{cases}$$

$$iM_{i+2} = G^F_F, \quad i \equiv 2$$

$$iM_{i+3} = F^F_G, \quad i \equiv 0$$

$$i+1M_i = G^G_G, \quad i \equiv 2.$$

(b) Let further $\tilde{J}_h$ be the relation ideal of $\tilde{\Sigma}_h$ ($1 \leq h \leq 6$) defined by the following minimal relations:

$$(i) \quad \tilde{J}_1: (\mu): F_{i-1} \rightarrow G_i \rightarrow F_{i+1}, \quad i \equiv 0, 2$$

$$(\pi): G_i \rightarrow F_{i+1} \rightarrow F_{i+2} \rightarrow F_{i+3} \rightarrow G_{i+4}, \quad i \equiv 0, 2$$

$$(p): iM_j(p), \quad \text{for every path of length } 5.$$

$$(ii) \quad \tilde{J}_2: (\mu): F_i \rightarrow G_{i+1} \rightarrow G_{i+2} \rightarrow F_{i+3}, \quad i \equiv 3$$

$$(\pi): G_i \rightarrow F_{i+2} \rightarrow G_{i+3}, \quad i \equiv 1,$$

$$(0): G_i \rightarrow F_{i+2} \rightarrow F_{i+5}, \quad i \equiv 1$$

$$F_i \rightarrow F_{i+1} \rightarrow G_{i+2}, \quad i \equiv 2$$

(c):

$$(iii) \quad \mathfrak{J}_3: (\mu): F_i \rightarrow G_{i+2} \rightarrow F_{i+3}, \quad i \equiv 2$$

$$(\pi): G_i \rightarrow F_{i+1} \rightarrow F_{i+2} \rightarrow G_{i+4}, \quad i \equiv 0$$

$$(0): F_i \rightarrow G_{i+2} \rightarrow G_{i+5}, \quad i \equiv 2$$

$$G_i \rightarrow G_{i+1} \rightarrow F_{i+2}, \quad i \equiv 3$$

(c):

$$(iv) \quad (\mu): F_i \rightarrow G_{i+2} \rightarrow F_{i+4}, \quad i \equiv 2$$

$$(\pi): G_i \rightarrow F_{i+2} \rightarrow G_{i+4}, \quad i \equiv 0$$

$$(0): G_i \rightarrow F_{i+2} \rightarrow F_{i+5}, \quad i \equiv 0$$

$$F_i \rightarrow F_{i+1} \rightarrow G_{i+3}, \quad i \equiv 1$$

$$F_i \rightarrow G_{i+2} \rightarrow G_{i+5}, \quad i \equiv 2$$

$$G_i \rightarrow G_{i+1} \rightarrow F_{i+3}, \quad i \equiv 3$$

(c)

$$(v) \quad (\mu): F_i \rightarrow G_{i+2} \rightarrow F_{i+5}, \quad i \equiv 2$$

$$(\pi): G_i \rightarrow F_{i+1} \rightarrow G_{i+3}, \quad i \equiv 1$$

$$(0): G_i \rightarrow F_{i+1} \rightarrow G_{i+3} \rightarrow F_{i+6}, \quad i \equiv 1$$

$$F_i \rightarrow F_{i-1} \rightarrow G_{i+1} \rightarrow G_{i+2}, \quad i \equiv 3$$

(c):

$$(p): M_{ij}(p), \quad \text{for every path } p \text{ of length } 4$$

$$(vi) \quad (\mu): F_i \rightarrow G_{i+1} \rightarrow F_{i+3}, \quad i \equiv 1$$

$$(\pi): G_i \rightarrow F_{i+2} \rightarrow G_{i+5}, \quad i \equiv 2$$

$$(0): F_i \rightarrow G_{i+1} \rightarrow F_{i+3} \rightarrow G_{i+6}, \quad i \equiv 1$$

$$G_i \rightarrow G_{i-1} \rightarrow F_{i+1} \rightarrow F_{i+2}, \quad i \equiv 3$$

$$(p): M_{ij}(p), \quad \text{for every path of length } 4.$$

(c) Let $\mu: \mathbb{Z} \rightarrow \mathbb{Z}$, $i \mapsto i+4$. Let

$$\Sigma_{h,r} := \tilde{\Sigma}_h / \mu^r$$

$$J_{h,r} := \tilde{J}_r / \mu^r,$$

$(1 \leq h \leq 6, r \geq 1)$.

(d) Let $A_{h,r} := T(\Sigma_{h,r}) / J_{h,r}$ $(1 \leq h \leq 6, r \geq 1)$.

All congruences are modulo 4, in the definitions above.

The introduction of these algebras $A_{h,r}$ is justified by the following theorem, in which we construct selfinjective algebras of Dynkin class F_4 with prescribed modulated Auslander-Reiten graph:

(2.4) Theorem. Let $A_{h,r}$ $(1 \leq h \leq 6, r \geq 1)$ be the algebras defined in (2.3, (d)).

(i) Each algebra $A_{h,r}$ is a selfinjective, basic, connected, finite dimensional K -algebra, of Dynkin class F_4

$$(ii) \Gamma(A_{h,r}) \cong (\mathbb{Z}F_4)_{C_h} / \langle \tau^{\omega^r} \rangle.$$

(2.5) Let $\tilde{A}_h := \bigoplus_{p,q} msh (\mathbb{Z}F_4)_{C_h}(p,q)$, p,q running through the projective vertices of $(\mathbb{Z}F_4)_{C_h}$ $(1 \leq h \leq 6)$.

Let Σ_h be a full connected subspecies of $\tilde{\Sigma}_h$ whose vertices form a complete μ -slice, and let $J_h := \tilde{J}_h|_{\Sigma_h}$. Let further

$$A_h := T(\Sigma_h) / J_h \quad (1 \leq h \leq 6).$$

(2.6) Theorem.

$$(i) \quad \tilde{A}_h = \hat{A}_h$$

$$(ii) \quad A_{h,1} \cong A_h \ltimes D A_h$$

(iii) The algebras $A_{h,1}$ ($1 \leq h \leq 6$) are trivial extension algebras of finite representation type and of Dynkin class F_4 .

(2.7) The proofs of (2.4) and (2.6) are immediate consequences of the results of (III, (1.5)): one simply constructs, from the known tilted algebras of type F_4 , the corresponding universal modulated graphs $(ZF_4)_C$. The results of these constructions are summarized in (2.8).

(2.8) The following list contains all the tilted algebras of type F_4 arising from a given quadratic field extension F/G . The number h under the configuration indicates that if $A = T(\Sigma)/J$, then $A \ltimes DA = A_{h,1}$. The column (Σ, J) gives the K -species Σ and the relation ideal J such that $A = T(\Sigma)/J$ is a tilted algebra of type F_4 .

Tilted Algebras of Type F_4

No.	(Σ, J)	Configuration
(1) Hereditaries		
1	$G \rightarrow G \rightarrow F \rightarrow F$	1
1^0	$G \leftarrow G \leftarrow F \leftarrow F$	1
2	$G \rightarrow G \rightarrow F \leftarrow F$	2
2^0	$G \leftarrow G \leftarrow F \rightarrow F$	2

No.	(Σ, J)	Configuration
3	$G \rightarrow G \leftarrow F \leftarrow F$	3
3^0	$G \leftarrow G \leftarrow F \rightarrow F$	3
4	$G \rightarrow G \leftarrow F \rightarrow F$	4
4^0	$G \leftarrow G \rightarrow F \leftarrow F$	4

(ii) One zero-relation

5	$G \rightarrow G \rightarrow F \rightarrow F$	2
5^0	$G \leftarrow G \leftarrow F \leftarrow F$	2
6	$G \rightarrow G \rightarrow F \rightarrow F$	3
6^0	$G \leftarrow G \leftarrow F \leftarrow F$	3
7	$G \leftarrow G \rightarrow F \rightarrow F$	4
7^0	$G \rightarrow G \leftarrow F \leftarrow F$	4
8	$G \rightarrow G \rightarrow F \leftarrow F$	4
8^0	$G \leftarrow G \leftarrow F \rightarrow F$	4
9	$G \rightarrow G \rightarrow F \rightarrow F$	6
9^0	$G \leftarrow G \leftarrow F \leftarrow F$	5

(iii) One non-zero relation

10	$F \rightarrow G \rightarrow G \rightarrow F$	1
11	$F \rightarrow G \rightarrow F$	5

11^o

6

12

1

13

6

13^o

5

(iv) Mixed relations

14

2

14^o

2

15

3

15^o

3

16

5

16^o

6

17

6

17^o

5

In $(\mathbb{C}, J)$ we have not displayed the bimodules, which are as follows: ${}_i M_j = {}_F F_G$, if $K_i = F$, $K_j = G$, ${}_i M_j = {}_G F_F$, if $K_i = G$, $K_j = F$, ${}_i M_j = {}_G G_G$, if $K_i = G$, $K_j = G$, and ${}_i M_j = {}_F F_F$, if $K_i = F$, $K_j = F$. Notice that in each case the algebra number N^0 is the opposite algebra of the algebra number N ; the algebras 10 and 12 coincide with their opposite algebras.

V.3 Indecomposable representations

By (1.3), in order to describe the indecomposable representations, it is enough to describe those lying on $[\tau^w \mathcal{F}, \mathcal{F}]$, where $\mathcal{F}$ is a complete τ -slice through $(\mathbb{Z}F_4)_C$. In the cases C_h , $h = 1, 2, 3, 4$, the description is simple, due to the fact (2.6) that we can restrict ourselves to the hereditary algebras of type F_4 , whose representation theory is completely known ([11]): Let A be a hereditary algebra of type F_4 . Then $A \ltimes DA = A_{h,1}$, for some $h = 1, 2, 3, 4$. From the table in 2.8 we also have $A^{op} \ltimes DA^{op} = A_{h,1}$, so we have embeddings $\text{mod } A \rightarrow \text{mod } A_{h,1}$, $\text{mod } A^{op} \rightarrow \text{mod } A_{h,1}$, from which we can reproduce $[\tau^w \mathcal{F}, \mathcal{F}]$, since the corresponding embeddings $\Gamma(A) \rightarrow \Gamma(A_{h,1})$, $\Gamma(A^{op}) \rightarrow \Gamma(A_{h,1})$ are nicely put together, and $\Gamma(A_{h,1}) \rightarrow (\mathbb{Z}F_4)_{C_h} / \langle \tau^w \rangle$ is obtained from $[\tau^w \mathcal{F}, \mathcal{F}]$ identifying the borders $\tau^w \mathcal{F}$ and $\mathcal{F}$.

We illustrate this discussion with the configuration C_3 , the other cases being clearly similar. Start with the algebra $A = T(\Sigma)$, $\Gamma: G_1 \rightarrow G_2 \leftarrow F_3 \leftarrow F_4$. Its Auslander-Reiten graph has the shape

and similarly, the Auslander-Reiten graph $\Gamma(A^{\text{op}})$ of its opposite algebra has the shape

where $P(i)$, $Q(i)$ and $I(i)$, $J(i)$ represent the indecomposable projective and injective A -modules (A^{op} -modules) respectively.

The construction of $(\mathbb{Z}F_4)_{C_3}$ yields the subgraphs

and

from where we see that the shape of the whole graph $(\mathbb{Z}F_4)_{C_3}$ is the one shown in the following picture:

To discuss the indecomposable representations of the algebras with configuration C_5 or C_6 , we can proceed as follows: consider a complete τ -slice $\mathcal{F}$ of $(\mathbb{Z}F_4)_{C_h}$, $h = 5, 6$, say $h = 5$ to fix ideas. Notice that C_6 is the opposite configuration of C_5 , in a selfexplaining sense. Construct a tilted algebra A of type F_4 having $\mathcal{F}$ as a complete slice. Then, by (2.8), A is isomorphic to one of the tilted algebras $T(\Sigma)/J$, with

(Σ, J) one of the numbers $(9^0, 11, 13^0, 16 \text{ or } 17^0)$ of (2.8).

Then $A \simeq \text{End } (T_H)$, where T_H is a multiplicity-free tilting module over a hereditary algebra H , of type F_4 , whose representations are known, as above. Using an appropriate tilting functor we can then describe the indecomposable representations of A , as well.

V.4 Selfinjective algebras of class $\mathbb{C}_2$

In this final section of this investigation, we do a similar discussion to the one done in V.1 to V.3 for F_4 , to describe the objects associated with the only configuration, up to isomorphism, of $\mathbb{C}_2$.

Recall (III, (3.4)) that $w_{\mathbb{C}_2} = 5$. We put $W_r := \langle \tau^{5r} \rangle$, $r \geq 1$, $[a, b] := W_1(a, b) = \{ \tau^{5k}(a, b) \mid k \in \mathbb{Z} \}$ for $(a, b) \in \mathbb{Z} \mathbb{C}_2$. The groups W_r are admissible subgroups of automorphisms of $\mathbb{Z} \mathbb{C}_2$, for every $r \geq 1$ (III, (3.1)).

(4.1) Proposition. There is exactly one isomorphism class of configurations of $\mathbb{Z} \mathbb{C}_2$, represented by the configuration $C = [0, 1] \cup [2, 2]$.

See Figure 1.

(Fig. 1)

(4.2) Remarks.

(i) In Figure 1 we have displayed only the configuration points contained in the rectangle starting at $(0,1)$ and stopping at $(2,2)$, indicated by the encircled points farthest to the left and the right of the picture, respectively.

(ii) The remaining configurations of $\mathbb{Z}\mathfrak{C}_2$ are given by $\tau^r C$, $r = 1, 2, 3, 4$, which are obviously isomorphic to C .

(4.3) Proposition. The groups W_r ($r \geq 1$) are all the admissible groups of automorphisms of $\mathbb{Z}\mathfrak{C}_2$ that stabilize the configurations.

Proof. Clearly no group $\langle \tau^k \rangle$ with $k < 5$ stabilizes C .

(4.4) Definition.

(i) The valued oriented graph

$$G_1: \quad 1 \begin{array}{c} \xleftarrow{(3,1)} \\ \xrightarrow{(3,1)} \end{array} 0$$

will be called the Brauer graph of type $\mathfrak{C}_2$.

(ii) The valued oriented graph $\tilde{G}_1$ with $(\tilde{G}_1)_0 = \mathbb{Z}$

$$\tilde{G}_1: \quad \begin{array}{ccccccccc} & -2 & & 0 & & 2 & & 4 & & 6 \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\ & & -1 & & 1 & & 3 & & 5 & \end{array} \quad \dots$$

will be called the universal covering of G_1 .

(4.5) Definition.

(i) Let $\tilde{\Sigma} = (K_i, {}_iM_j)_{i,j \in \mathbb{Z}}$ be the K species defined by

$$K_i = \begin{cases} G, & i \equiv 0 \\ H, & i \equiv 1 \end{cases}$$

$${}_iM_{i+1} = \begin{cases} \begin{smallmatrix} H & \\ G & H \end{smallmatrix}, & i \equiv 0 \\ \begin{smallmatrix} H & \\ H & G \end{smallmatrix}, & i \equiv 1 \end{cases}$$

(ii) Let $\tilde{J}$ be the relation ideal of $\tilde{\Sigma}$ defined by the following minimal relations:

$$(\nu) \quad H_i \rightarrow G_{i+1} \rightarrow H_{i+2} \quad i \equiv 1$$

$$(\rho) \quad G_i \rightarrow H_{i+1} \rightarrow G_{i+2} \quad i \equiv 0$$

$$(\rho) \quad {}_iM_j(p), \quad \text{for every path of length } 3.$$

Here, the relations ν and ρ arise from the one point extension algebras $E(H, I(1), T(G \xrightarrow{G_H} H))$ and $E(G, I(1), T(H \xrightarrow{H_G} G))$ of the hereditary algebras $T(G \rightarrow H) \simeq \begin{pmatrix} H & 0 \\ G_H & G \end{pmatrix}$ and $T(H \rightarrow G) \simeq \begin{pmatrix} G & 0 \\ H_G & H \end{pmatrix}$, respectively.

(iii) Let $\mu: \mathbb{Z} \rightarrow \mathbb{Z}$, $i \mapsto i+2$. Let

$$\Sigma_r := \tilde{\Sigma} / \mu^r$$

$$J_r := \tilde{J} / \mu^r$$

($r \geq 1$)

(iv) Let $A_r := T(\Sigma_r) / J_r$ ($r \geq 1$).

All congruences are modulo 2, in the definitions above.

In the following theorem we construct selfinjective algebras of Dynkin class $\mathbb{C}_2$ with prescribed modulated Auslander-Reiten graph:

(4.6) Theorem. Let A_r ($r \geq 1$) be the algebras defined in (4.5, (iv)).

(i) Each algebra A_r is a selfinjective, basic, connected, finite dimensional K -algebra of Dynkin class $\mathbb{C}_2$

$$(ii) \Gamma(A_r) \cong (\mathbb{Z} \mathbb{C}_2)_C / \langle \tau^{5r} \rangle.$$

(4.7) Let $\tilde{A} := \bigoplus_{p,q} \text{msh} (\mathbb{Z} \mathbb{C}_2)_C(p,q)$, p, q running through the projective vertices of $(\mathbb{Z} \mathbb{C}_2)_C$.

Let Σ be a full connected subspecies of $\tilde{\Sigma}$ whose vertices form a complete μ -slice, and let $J := J|_{\Sigma}$. Let further $A := T(\Sigma) / J$.

(4.8) Theorem.

$$(i) \tilde{A} = \hat{A}$$

$$(ii) A_1 \cong A \ltimes DA$$

(iii) The algebra A_1 is a trivial extension algebra of finite representation type and Dynkin class $\mathbb{C}_2$.

(4.10) It is clear that $A \cong T(H \rightarrow G)$ or $A \cong T(G \rightarrow H)$, in (4.7), and these two algebras are opposite to each other. Thus, the trivial extension algebra A_1 of Dynkin class $\mathbb{C}_2$ is $A \ltimes DA$, $A = T(H \rightarrow G)$, for instance.

The proofs of (4.6) and (4.8) follow from the same sort of arguments as those used in (2.7).

(4.11) The indecomposable representations lying on $[\tau^5 \mathcal{F}, \mathcal{F}]$, $\mathcal{F}$ a complete τ -slice of $(\mathbb{Z} \mathbb{C}_2)_C$ are enough to describe all the indecomposable representations of a given selfinjective algebra of finite representation type of Dynkin class $\mathbb{C}_2$ (cfr. (V.3)).

Let $A = T(G_2 \rightarrow H_1)$. Then $A^{\text{op}} \cong T(H_2 \rightarrow G_1)$ and the Auslander-Reiten graphs of A and A^{op} are

$\Gamma(A)$:

$\Gamma(A^{\text{op}})$:

with $P(i)$ ($Q(i)$) and $I(i)$ ($J(i)$) the projective and injective indecomposable A -modules (A^{op} -modules), respectively. The construction of $(\mathbb{Z} \mathbb{C}_2)_C$ yields thus the graph

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END

3	0	H	1	0	H	8	5
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FIN