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CONVOLUTION SUMS INVOLVING DIVISOR FUNCTIONS

by

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B.A.Sc., University of British Columbia

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Master of Science

School of Mathematics and Statistics
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submitted by
Nathalie Cheng, B.A.Sc.
in partial fulfilment of the requirements
for the degree of Master of Science

Dr. Kenneth S. Williams, Thesis Supervisor

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March, 2003

Abstract

Ramanujan, in his classic paper [31], [33, pp. 136-162] on arithmetical functions published in 1916, studied the relationships between the series

$$\begin{aligned} L(q) &= 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n, & |q| < 1, \\ M(q) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, & |q| < 1, \\ N(q) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, & |q| < 1, \end{aligned}$$

where

$$\sigma_f(n) = \sum_{d|n} d^f, \quad n, f \in \mathbb{N}.$$

From these relationships, he deduced nine arithmetic identities of the form

$$\sum_{m < n} \sigma_e(m) \sigma_f(n - m) = \sum_{k=1}^{(e+f+2)/2} (A_{2k-1} + B_{2k-1}n) \sigma_{2k-1}(n),$$

where e and f are positive odd integers and $A_1, \dots, A_{e+f+1}, B_1, \dots, B_{e+f+1}$ are rational numbers.

In 1998, Melfi [27] using the theory of modular forms proved seven more arithmetic identities similar to those of Ramanujan. In 2000, Huard, Ou,

Spearman and Williams [14] proved in an elementary manner thirty-seven arithmetic identities by using their generalization of an identity of Liouville. These identities included ten of Ramanujan, thirteen of Lahiri, seven of Melfi and seven new ones.

In one of his notebooks [32], Ramanujan gave without proof the evaluation of $L(q)$, $M(q)$ and $N(q)$ in terms of values of hypergeometric functions. These evaluations are proved in Berndt [3]. Taking these evaluations as our starting point, we develop an idea of Melfi [27] and carry out a systematic investigation of the relationships between $L(q)$, $M(q)$, $N(q)$, $L(q^2)$, $M(q^2)$, $N(q^2)$, $L(q^4)$, $M(q^4)$, $N(q^4)$, and some related functions. From these relationships we deduce a number of arithmetic identities including some new ones.

Acknowledgments

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Introduction

Let $\mathbb{N}$ denote the set of natural numbers. For $f, n \in \mathbb{N}$, let $\sigma_f(n)$ denote the sum of the f th powers of the positive divisors of n ,

$$(0.0.1) \quad \sigma_f(n) = \sum_{d|n} d^f.$$

We set $\sigma_1(n) = \sigma(n)$. If $n \notin \mathbb{N}$, we set $\sigma_f(n) = 0$.

In Chapter 1, we prove the nine arithmetic identities proved by Ramanujan in [31], [33, pp. 136-162], which are convolution sums of the form

$$\sum_{m < n} \sigma_e(m) \sigma_f(n - m) = \sum_{k=1}^{(e+f+2)/2} (A_{2k-1} + B_{2k-1}n) \sigma_{2k-1}(n),$$

where e and f are positive odd integers and $A_1, \dots, A_{e+f+1}, B_1, \dots, B_{e+f+1}$ are rational numbers.

For $|q| < 1$, we define

$$(0.0.2) \quad L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n,$$

$$(0.0.3) \quad M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$(0.0.4) \quad N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

$$(0.0.5) \quad P(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$(0.0.6) \quad Q(q) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n,$$

$$(0.0.7) \quad R(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n,$$

$$(0.0.8) \quad S(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n.$$

We start with the following evaluations of $L(q)$, $M(q)$, ..., $S(q)$, for $0 < q < 1$, which can be deduced from formulae in Ramanujan's Notebooks [3] and Lahiri's paper [18]:

$$(0.0.9) \quad L(q) = (1 - 5x)w^2 + 12x(1 - x)w \frac{dw}{dx},$$

$$(0.0.10) \quad M(q) = (1 + 14x + x^2)w^4,$$

$$(0.0.11) \quad N(q) = (1 + x)(1 - 34x + x^2)w^6,$$

$$(0.0.12) \quad P(q) = (1 + 14x + x^2)^2 w^8,$$

$$(0.0.13) \quad Q(q) = (1 + x)(1 + 14x + x^2)(1 - 34x + x^2)w^{10},$$

$$(0.0.14) \quad R(q) = \frac{1}{691} (691 + 2022x + 516381x^2 + 1792148x^3 \\ + 516381x^4 + 2022x^5 + 691x^6) w^{12},$$

$$(0.0.15) \quad S(q) = (1 + x)(1 + 14x + x^2)^2 (1 - 34x + x^2) w^{14}.$$

Here x ($0 < x < 1$) is given uniquely by

$$\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} = -\frac{1}{\pi} \log q$$

and

$$w = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x),$$

where ${}_2F_1(a, b; c; x)$ is the Gaussian hypergeometric function.

From (0.0.9)-(0.0.15), we obtain relationships between $L(q)$, $M(q)$, ..., $S(q)$. One such relationship is $L^2(q) = M(q) + 12q \frac{dL}{dq}$. We derive the nine identities proved by Ramanujan [31], [33, pp. 136-162] by substituting (0.0.2)-(0.0.8) into these relationships. For example, from $L^2(q) = M(q) + 12q \frac{dL}{dq}$, we obtain

$$\sum_{n=1}^{\infty} \sum_{m < n} \sigma(m) \sigma(n-m) q^n = \frac{1}{12} \sum_{n=1}^{\infty} \{ \sigma_3(n) + \sigma(n) - 6n\sigma(n) \} q^n.$$

By equating the coefficients of q^n , we obtain the arithmetic identity

$$\sum_{m < n} \sigma(m) \sigma(n-m) = \frac{1}{12} \{ \sigma_3(n) + (1-6n)\sigma(n) \},$$

given by Ramanujan [31], [33, pp. 136-162], but whose origins go back to Liouville [5].

In Chapter 2, we define new functions $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$, for $i = 0, 1$, as the even and odd parts of the sums in $L(q)$, $M(q)$, ..., $S(q)$. We have

$$\begin{aligned} L_{0,2}(q) &= \sum_{n=1}^{\infty} \sigma(2n) q^{2n}, \\ L_{1,2}(q) &= \sum_{n=1}^{\infty} \sigma(2n-1) q^{2n-1}, \end{aligned}$$

$$\begin{aligned}
M_{0,2}(q) &= \sum_{n=1}^{\infty} \sigma_3(2n)q^{2n}, \\
M_{1,2}(q) &= \sum_{n=1}^{\infty} \sigma_3(2n-1)q^{2n-1},
\end{aligned}$$

and so on. We obtain evaluations of $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$, for $i = 0, 1$ in terms of x and find relationships between $L(q)$, $M(q)$, ..., $S(q)$, $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$, for $i = 0, 1$. As in Chapter 1, we derive arithmetic identities from the relationships. In this chapter, we obtain twenty-two arithmetic identities. Some of these identities appear in the papers of Huard, Ou, Spearman and Williams [14] and Melfi [27].

In Chapter 3, we define new functions $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$, for $i = 0, 1, 2, 3$, by

$$\begin{aligned}
L_{0,4}(q) &= \sum_{n=1}^{\infty} \sigma(4n)q^{4n}, \\
L_{1,4}(q) &= \sum_{n=1}^{\infty} \sigma(4n-3)q^{4n-3}, \\
L_{2,4}(q) &= \sum_{n=1}^{\infty} \sigma(4n-2)q^{4n-2}, \\
L_{3,4}(q) &= \sum_{n=1}^{\infty} \sigma(4n-1)q^{4n-1}, \\
M_{0,4}(q) &= \sum_{n=1}^{\infty} \sigma_3(4n)q^{4n}, \\
M_{1,4}(q) &= \sum_{n=1}^{\infty} \sigma_3(4n-3)q^{4n-3}, \\
M_{2,4}(q) &= \sum_{n=1}^{\infty} \sigma_3(4n-2)q^{4n-2},
\end{aligned}$$

$$M_{3,4}(q) = \sum_{n=1}^{\infty} \sigma_3(4n-1)q^{4n-1},$$

and so on. We obtain representations of $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$, for $i = 0, 1, 2, 3$ in terms of x and find relationships between $L(q)$, $M(q)$, ..., $S(q)$, $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$, for $i = 0, 1, 2, 3$. From these relationships, we obtain seventeen arithmetic identities, four of which can be found in Huard, Ou, Spearman and Williams' paper [14].

In Chapter 4, we describe a computer search to look for possible new arithmetic identities of the form

$$\sum_{m < \frac{n}{k}} \sigma_e(m) \sigma_f(n - km) = \sum_{l=1}^{(e+f+2)/2} (A_{2l-1} + B_{2l-1}n) \sigma_{2l-1}(n),$$

which hold for all n in certain congruence classes coprime with the positive integer k , where e and f are positive odd integers and $A_1, \dots, A_{e+f+1}$, $B_1, \dots, B_{e+f+1}$ are rational numbers. Except for the seven identities Melfi [27] found in 1998, we did not find any other identities of this form.

We remark that in all the arithmetic identities the subscripts are odd. In 1959 Lehmer [20, p. 107], [21, p. 679] wrote "There is no theory for $\sigma_{2k}(n)$ and no observed phenomena." This remains as true today as it was in 1959.

Chapter 1

Basic Properties of Ramanujan's Functions $L(q)$, $M(q)$ and $N(q)$.

1.1 Ramanujan's functions $L(q)$, $M(q)$ and $N(q)$.

In 1916, Ramanujan [31], [33, pp. 136-162] introduced the three functions $L(q)$, $M(q)$, $N(q)$ defined by

$$(1.1.1) \quad L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n,$$

$$(1.1.2) \quad M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,$$

$$(1.1.3) \quad N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.$$

Ramanujan actually used P , Q , R in place of L , M , N but we prefer to follow Berndt [2, p. 318] and use L , M , N . Berndt [2, p. 318] shows that L , M and N are essentially the Eisenstein series of weights 2, 4 and 6, respectively, on the full modular group. However we make no use of this information in this thesis. The Eisenstein series $L(q)$, $M(q)$, $N(q)$ are algebraically independent [35, p. 69].

For all $k, n \in \mathbb{N}$, we have

$$0 < \sigma_k(n) = \sum_{d|n} d^k \leq n^k \sum_{d|n} 1 \leq n^{k+1},$$

so that

$$\left| \sum_{n=1}^{\infty} \sigma_k(n) q^n \right| \leq \sum_{n=1}^{\infty} n^{k+1} |q|^n.$$

The radius of convergence R of the infinite series

$$\sum_{n=1}^{\infty} n^{k+1} q^n$$

is given by

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{n^{k+1}} = \left(\limsup_{n \rightarrow \infty} n^{\frac{1}{n}} \right)^{k+1} = 1^{k+1} = 1$$

so that $R = 1$. Hence the infinite series

$$\sum_{n=1}^{\infty} \sigma_k(n) q^n \quad (k \in \mathbb{N})$$

converges absolutely for $|q| < 1$. Thus, in particular, $L(q)$, $M(q)$ and $N(q)$ are defined for $|q| < 1$.

It is convenient for our purposes to introduce the series

$$(1.1.4) \quad P(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n,$$

$$(1.1.5) \quad Q(q) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n,$$

$$(1.1.6) \quad R(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n,$$

$$(1.1.7) \quad S(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n,$$

which are also defined for $|q| < 1$. These functions are related to Ramanujan's functions $L(q)$, $M(q)$ and $N(q)$. Various relationships between $L(q)$, $M(q)$, $N(q)$, $P(q)$, $Q(q)$, $R(q)$ and $S(q)$ are exhibited in Section 1.5.

The general Eisenstein series is

$$E_k(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n, \quad k \in \mathbb{N},$$

where

$$\begin{aligned} B_0 &= 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \\ B_5 &= 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}, \quad B_9 = 0, \\ B_{10} &= \frac{5}{66}, \quad B_{11} = 0, \quad B_{12} = -\frac{691}{2730}, \quad B_{13} = 0, \quad B_{14} = \frac{7}{6}, \\ &\dots \end{aligned}$$

are the Bernoulli numbers given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi,$$

see for example [1, eqn (23.1.1), p. 804], so that

$$L(q) = E_1(q), \quad M(q) = E_2(q), \quad N(q) = E_3(q),$$

$$P(q) = E_4(q), \quad Q(q) = E_5(q), \quad R(q) = E_6(q), \quad S(q) = E_7(q).$$

1.2 Representations of $L(q)$, $M(q)$, $N(q)$, $P(q)$, $Q(q)$, $R(q)$, $S(q)$ in terms of x .

Let a , b , c be positive real numbers. The hypergeometric series is the series

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!},$$

where

$$(a)_n = a(a+1) \cdots (a+n-1), \quad n \in \mathbb{N}$$

$$(a)_0 = 1,$$

see for example [29, p. 45], so that

$$(1)_n = n!, \quad \left(\frac{1}{2}\right)_n = \frac{2n!}{2^{2n} n!}.$$

The hypergeometric series converges absolutely for $|x| < 1$. At $x = 1$ the series converges if and only if $c > a + b$. The sum of the hypergeometric series is the

hypergeometric function ${}_2F_1(a, b; c; x)$. Thus

$${}_2F_1(a, b; c; 0) = 1$$

and

$${}_2F_1(a, b; c; 1) = +\infty, \quad \text{if } a + b \geq c.$$

The function ${}_2F_1(a, b; c; x)$ is a solution of the hypergeometric differential equation [29, p. 54]

$$x(1-x)w'' + (c - (a+b+1)x)w' - abw = 0.$$

The gamma function $\Gamma(x)$ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt,$$

see for example [29, p. 9]. It is well known that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$ [29, p. 12]. Ramanujan stated [32, first notebook, entry 30] that if $\alpha + \beta + 1 = \gamma + \delta$ then

$$\frac{d}{dx} \left(\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)\Gamma(\delta)} \frac{{}_2F_1(\alpha, \beta; \delta; 1-x)}{{}_2F_1(\alpha, \beta; \gamma; x)} \right) = - \frac{x^{-\gamma}(1-x)^{-\delta}}{\{{}_2F_1(\alpha, \beta; \gamma; x)\}^2}.$$

A proof has been given by Berndt [2, p. 87].

From this point on, we take q to be a real number satisfying

$$(1.2.1) \quad 0 < q < 1.$$

Then

$$(1.2.2) \quad 0 < -\log q < \infty.$$

From the above formula of Ramanujan with $\alpha = \beta = \frac{1}{2}$ and $\gamma = \delta = 1$, as $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [29, p. 21], we see that the derivative y' of the function

$$(1.2.3) \quad y = y(x) = \frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}$$

is given by

$$(1.2.4) \quad y' = -\frac{x^{-1}(1-x)^{-1}}{\{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)\}^2}.$$

For $0 < x < 1$, we see that

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \binom{2n}{n}^2 x^n > 0$$

so that

$$(1.2.5) \quad y' < 0.$$

Hence y is a decreasing function of x for $0 < x < 1$. As

$$y(0) = \frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 1)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 0)} = \frac{\infty}{1} = \infty$$

and

$$y(1) = \frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 0)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1)} = \frac{1}{\infty} = 0,$$

the function y decreases from ∞ to 0 as x increases from 0 to 1. Hence there exists a unique value of x between 0 and 1 such that

$$(1.2.6) \quad y = \frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} = -\log q.$$

Therefore

$$(1.2.7) \quad q = e^{-y} = e^{-\frac{\pi_2 F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}}.$$

We also set

$$(1.2.8) \quad w = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right).$$

This thesis starts with the following representations of Ramanujan's functions $L(q)$, $M(q)$ and $N(q)$ in terms of x given by Ramanujan in his notebooks [32], and which are proved in [3, pp. 127, 129].

Theorem 1.2.1

$$(1.2.9) \quad L(q) = (1 - 5x)w^2 + 12x(1 - x)w \frac{dw}{dx},$$

$$(1.2.10) \quad M(q) = (1 + 14x + x^2)w^4,$$

$$(1.2.11) \quad N(q) = (1 + x)(1 - 34x + x^2)w^6.$$

Ramanujan [31, Table I], [33, Table I, p. 141] gives

$$(1.2.12) \quad M^2(q) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n,$$

$$(1.2.13) \quad M(q)N(q) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n,$$

$$(1.2.14) \quad \frac{441}{691}M^3(q) + \frac{250}{691}N^2(q) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n,$$

$$(1.2.15) \quad M^2(q)N(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n)q^n.$$

Since the right hand sides of (1.2.12), (1.2.13), (1.2.14) and (1.2.15) are P , Q , R and S respectively, we see that

$$(1.2.16) \quad P(q) = M^2(q),$$

$$(1.2.17) \quad Q(q) = M(q)N(q),$$

$$(1.2.18) \quad R(q) = \frac{441}{691}M^3(q) + \frac{250}{691}N^2(q),$$

$$(1.2.19) \quad S(q) = M^2(q)N(q) = P(q)N(q).$$

Appealing to (1.2.12)-(1.2.19), we obtain the representations of P , Q , R and S in terms of x .

Theorem 1.2.2

$$(1.2.20) \quad P(q) = (1 + 14x + x^2)^2 w^8,$$

$$(1.2.21) \quad Q(q) = (1 + x)(1 + 14x + x^2)(1 - 34x + x^2)w^{10},$$

$$(1.2.22) \quad R(q) = \frac{1}{691}(691 + 2022x + 516381x^2 + 1792148x^3 \\ + 516381x^4 + 2022x^5 + 691x^6)w^{12},$$

$$(1.2.23) \quad S(q) = (1 + x)(1 + 14x + x^2)^2(1 - 34x + x^2)w^{14}.$$

Ramanujan [31, Table I], [33, Table I, p. 141] also gives the representations of $E_8(q)$, ..., $E_{16}(q)$ in terms of $M(q)$ and $N(q)$, and proves by induction that in general

$$\frac{1}{2}\zeta(1 - 2k) + \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n = \sum_{\substack{r, s = 0 \\ 2r + 3s = k}}^{\infty} K(r, s)M(q)^r N(q)^s,$$

for rational numbers $K(r, s)$ [31], [33, eqn (26), p. 141], where $\zeta(s)$ is the Riemann zeta function. The Riemann zeta function $\zeta(s)$ is an analytic function of the complex variable s everywhere except for a simple pole at $s = 1$ with

residue 1. Moreover for real $s > 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Since

$$\zeta(1-2k) = -\frac{B_{2k}}{2k}, \quad k = 1, 2, \dots,$$

see for example [1, eqn (23.2.15), p. 807], we deduce that

$$E_k(q) = -\frac{4k}{B_{2k}} \sum_{\substack{r,s=0 \\ 2r+3s=k}}^{\infty} K(r,s) M(q)^r N(q)^s, \quad k = 2, 3, \dots$$

Thus, for example, Ramanujan [31], [33, Table I, eqn. 16., p. 141] gives

$$\begin{aligned} E_{16}(q) = & \frac{764412173217}{7709321041217} M(q)^8 \\ & + \frac{5323905468000}{7709321041217} M(q)^5 N(q)^2 \\ & + \frac{1621003400000}{7709321041217} M(q)^2 N(q)^4. \end{aligned}$$

In this thesis we do not consider $E_k(q)$ for $k \geq 8$. This will be explained in Section 1.5.

Skoruppa [35, pp. 68-69] has given recurrence relations which allow one to write $\left(q \frac{d}{dq}\right)^m E_n(q)$ inductively as a polynomial in $L(q)$, $M(q)$ and $N(q)$.

1.3 The quantities $\frac{dy}{dx}$, $\frac{dq}{dx}$, $\frac{d^2w}{dx^2}$ in terms of x .

In this section we express $\frac{dy}{dx}$, $\frac{dq}{dx}$ and $\frac{d^2w}{dx^2}$ in terms of x . These expressions will be used to determine the derivatives of $L(q)$, $M(q)$, $N(q)$, $P(q)$, $Q(q)$, $R(q)$ and $S(q)$ with respect to q .

Lemma 1.3.1

$$\frac{dy}{dx} = -\frac{1}{x(1-x)w^2}.$$

Proof This result follows immediately from (1.2.4) and (1.2.8). $\square$

Lemma 1.3.2

$$\frac{dq}{dx} = \frac{q}{x(1-x)w^2}.$$

Proof From (1.2.7) and Lemma 1.3.1 we obtain

$$\frac{dq}{dx} = -e^{-y} \frac{dy}{dx} = -q \frac{dy}{dx} = \frac{q}{x(1-x)w^2},$$

as asserted. $\square$

Lemma 1.3.3

$$x(1-x) \frac{d^2w}{dx^2} = \frac{1}{4}w - (1-2x) \frac{dw}{dx}.$$

Proof We have already mentioned that the hypergeometric function

$$F = {}_2F_1(a, b; c; x)$$

satisfies the hypergeometric differential equation

$$x(1-x) \frac{d^2F}{dx^2} + (c - (1+a+b)x) \frac{dF}{dx} - abF = 0.$$

Hence

$$w = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$$

satisfies

$$x(1-x)\frac{d^2w}{dx^2} + (1-2x)\frac{dw}{dx} - \frac{1}{4}w = 0.$$

Thus

$$x(1-x)\frac{d^2w}{dx^2} = \frac{1}{4}w - (1-2x)\frac{dw}{dx},$$

as claimed. $\square$

1.4 Evaluation of $\frac{dL}{dq}$, $\frac{dM}{dq}$, $\frac{dN}{dq}$, $\frac{dP}{dq}$, $\frac{dQ}{dq}$, $\frac{dR}{dq}$, $\frac{dS}{dq}$ in terms of x .

In this section we use Theorems 1.2.1 and 1.2.2 to determine $\frac{dL}{dq}$, ..., $\frac{dS}{dq}$ in terms of x . These determinations are used in Section 1.5 to find relationships between L , M , ..., S .

Theorem 1.4.1

$$(1.4.1) \quad q \frac{dL}{dq} = -2x(1-x)w^4 + 2x(1-x)(1-5x)w^3 \frac{dw}{dx} + 12x^2(1-x)^2w^2 \left(\frac{dw}{dx} \right)^2,$$

$$(1.4.2) \quad q \frac{dM}{dq} = 4x(1-x)(1+14x+x^2)w^5 \frac{dw}{dx} + 2x(1-x)(7+x)w^6,$$

$$(1.4.3) \quad q \frac{dN}{dq} = 6x(1-x)(1+x)(1-34x+x^2)w^7 \frac{dw}{dx} - 3x(1-x)(11+22x-x^2)w^8,$$

$$(1.4.4) \quad q \frac{dP}{dq} = 8x(1-x)(1+14x+x^2)^2w^9 \frac{dw}{dx}$$

$$\begin{aligned}
& +4x(1-x)(1+14x+x^2)(7+x)w^{10}, \\
(1.4.5) \quad q \frac{dQ}{dq} &= 10x(1-x)(1+x)(1+14x+x^2)(1-34x+x^2)w^{11} \frac{dw}{dx} \\
& -x(1-x)(19+988x+1482x^2+76x^3-5x^4)w^{12}, \\
(1.4.6) \quad q \frac{dR}{dq} &= \frac{12}{691}x(1-x)(691+2022x+516381x^2+1792148x^3 \\
& +516381x^4+2022x^5+691x^6)w^{13} \frac{dw}{dx} \\
& +\frac{1}{691}x(1-x)(2022+1032762x+5376444x^2 \\
& +2065524x^3+10110x^4+4146x^5)w^{14}, \\
(1.4.7) \quad q \frac{dS}{dq} &= 14x(1-x)(1+x)(1+14x+x^2)^2(1-34x+x^2)w^{15} \frac{dw}{dx} \\
& -x(1-x)(5+1910x+9435x^2+14012x^3 \\
& +1283x^4-18x^5-3x^6)w^{16}.
\end{aligned}$$

Proof First we prove (1.4.1). By (1.2.9), we have

$$\begin{aligned}
\frac{dL}{dx} &= \frac{d}{dx} \left((1-5x)w^2 + 12x(1-x)w \frac{dw}{dx} \right) \\
&= -5w^2 + (14-34x)w \frac{dw}{dx} + 12x(1-x) \left(\frac{dw}{dx} \right)^2 + 12x(1-x)w \frac{d^2w}{dx^2}.
\end{aligned}$$

Appealing to Lemma 1.3.3, we obtain

$$\frac{dL}{dx} = -2w^2 + (2-10x)w \frac{dw}{dx} + 12x(1-x) \left(\frac{dw}{dx} \right)^2.$$

Finally, by Lemma 1.3.2, we have

$$\begin{aligned}
q \frac{dL}{dq} &= q \frac{\frac{dL}{dx}}{\frac{dx}{dq}} = x(1-x)w^2 \frac{dL}{dx} \\
&= -2x(1-x)w^4 + 2x(1-x)(1-5x)w^3 \frac{dw}{dx} + 12x^2(1-x)^2w^2 \left(\frac{dw}{dx} \right)^2,
\end{aligned}$$

as asserted.

Next we prove (1.4.2). By (1.2.10), we have

$$\frac{dM}{dx} = \frac{d}{dx} ((1 + 14x + x^2)w^4) = 4(1 + 14x + x^2)w^3 \frac{dw}{dx} + 2(7 + x)w^4.$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dM}{dq} &= q \frac{\frac{dM}{dx}}{\frac{dx}{dq}} = x(1 - x)w^2 \frac{dM}{dx} \\ &= 4x(1 - x)(1 + 14x + x^2)w^5 \frac{dw}{dx} + 2x(1 - x)(7 + x)w^6, \end{aligned}$$

as asserted.

Next we prove (1.4.3). By (1.2.11), we have

$$\begin{aligned} \frac{dN}{dx} &= \frac{d}{dx} ((1 + x)(1 - 34x + x^2)w^6) \\ &= 6(1 + x)(1 - 34x + x^2)w^5 \frac{dw}{dx} - 3(11 + 22x - x^2)w^6. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dN}{dq} &= q \frac{\frac{dN}{dx}}{\frac{dx}{dq}} = x(1 - x)w^2 \frac{dN}{dx} \\ &= 6x(1 - x)(1 + x)(1 - 34x + x^2)w^7 \frac{dw}{dx} - 3x(1 - x)(11 + 22x - x^2)w^8, \end{aligned}$$

as asserted.

Next we prove (1.4.4). By (1.2.20), we have

$$\begin{aligned} \frac{dP}{dx} &= \frac{d}{dx} ((1 + 14x + x^2)^2 w^8) \\ &= 8(1 + 14x + x^2)^2 w^7 \frac{dw}{dx} + 4(8 + x)(1 + 14x + x^2)w^8. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dP}{dq} &= q \frac{\frac{dP}{dx}}{\frac{dq}{dx}} = x(1-x)w^2 \frac{dP}{dx} \\ &= 8x(1-x)(1+14x+x^2)^2 w^9 \frac{dw}{dx} + 4x(1-x)(7+x)(1+14x+x^2)w^{10}, \end{aligned}$$

as asserted.

Next we prove (1.4.5). By (1.2.21), we have

$$\begin{aligned} \frac{dQ}{dx} &= \frac{d}{dx} ((1+x)(1+14x+x^2)(1-34x+x^2)w^{10}) \\ &= 10(1+x)(1+14x+x^2)(1-34x+x^2)w^9 \frac{dw}{dx} \\ &\quad - (19+988x+1482x^2+76x^3-5x^4)w^{10}. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dQ}{dq} &= q \frac{\frac{dQ}{dx}}{\frac{dq}{dx}} = x(1-x)w^2 \frac{dQ}{dx} \\ &= 10x(1-x)(1+x)(1+14x+x^2)(1-34x+x^2)w^{11} \frac{dw}{dx} \\ &\quad - x(1-x)(19+988x+1482x^2+76x^3-5x^4)w^{12}, \end{aligned}$$

as asserted.

Next we prove (1.4.6). By (1.2.22), we have

$$\begin{aligned} \frac{dR}{dx} &= \frac{1}{691} \frac{d}{dx} (691 + 2022x + 516381x^2 + 1792148x^3 \\ &\quad + 516381x^4 + 2022x^5 + 691x^6)w^{12} \end{aligned}$$

$$\begin{aligned}
&= \frac{12}{691} (691 + 2022x + 516381x^2 + 1792148x^3 \\
&\quad + 516381x^4 + 2022x^5 + 691x^6) w^{11} \frac{dw}{dx} \\
&\quad + \frac{1}{691} (2022 + 1032762x + 5376444x^2 + 2065524x^3 \\
&\quad + 10110x^4 + 4146x^5) w^{12}.
\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}
q \frac{dR}{dq} &= q \frac{\frac{dR}{dx}}{\frac{dq}{dx}} = x(1-x) w^2 \frac{dR}{dx} \\
&= \frac{12}{691} x(1-x) (691 + 2022x + 516381x^2 + 1792148x^3 \\
&\quad + 516381x^4 + 2022x^5 + 691x^6) w^{13} \frac{dw}{dx} \\
&\quad + \frac{1}{691} x(1-x) (2022 + 1032762x + 5376444x^2 \\
&\quad + 2065524x^3 + 10110x^4 + 4146x^5) w^{14},
\end{aligned}$$

as asserted.

Lastly, we prove (1.4.7). By (1.2.23), we have

$$\begin{aligned}
\frac{dS}{dx} &= \frac{d}{dx} ((1+x)(1+14x+x^2)^2(1-34x+x^2)w^{14}) \\
&= 14(1+x)(1+14x+x^2)^2(1-34x+x^2)w^{13} \frac{dw}{dx} \\
&\quad - (5 + 1910x + 9435x^2 + 14012x^3 + 1283x^4 - 18x^5 - 3x^6)w^{14}.
\end{aligned}$$

By Lemma 1.3.2, we have

$$q \frac{dS}{dq} = q \frac{\frac{dS}{dx}}{\frac{dq}{dx}} = x(1-x) w^2 \frac{dS}{dx}$$

$$\begin{aligned}
&= 14x(1-x)(1+x)(1+14x+x^2)^2(1-34x+x^2)w^{15}\frac{dw}{dx} \\
&\quad -x(1-x)(5+1910x+9435x^2+14012x^3+1283x^4 \\
&\quad \quad -18x^5-3x^6)w^{16},
\end{aligned}$$

as asserted. □

1.5 Relationships between $L(q)$, $M(q)$, $N(q)$, $P(q)$, $Q(q)$, $R(q)$, $S(q)$.

In this section, we use Theorems 1.2.1, 1.2.2 and 1.4.1 to obtain relationships between $L(q)$, $M(q)$, $N(q)$, $P(q)$, $Q(q)$, $R(q)$ and $S(q)$. Formulae (1.5.6), (1.5.7), (1.5.8) and (1.5.9) are not surprising results as we have just recovered what was used to prove Theorem 1.2.2.

Theorem 1.5.1

$$(1.5.1) \quad L^2(q) = M(q) + 12q \frac{dL}{dq},$$

$$(1.5.2) \quad L(q)M(q) = N(q) + 3q \frac{dM}{dq},$$

$$(1.5.3) \quad L(q)N(q) = P(q) + 2q \frac{dN}{dq},$$

$$(1.5.4) \quad L(q)P(q) = Q(q) + \frac{3}{2}q \frac{dP}{dq},$$

$$(1.5.5) \quad L(q)R(q) = S(q) + q \frac{dR}{dq},$$

$$(1.5.6) \quad M^2(q) = P(q),$$

$$(1.5.7) \quad M(q)N(q) = Q(q),$$

$$(1.5.8) \quad M(q)Q(q) = S(q),$$

$$(1.5.9) \quad N(q)P(q) = S(q).$$

Proof First we prove (1.5.1). By (1.2.9), (1.2.10) and (1.4.1), we have

$$\begin{aligned} L^2(q) &= \left\{ (1-5x)w^2 + 12x(1-x)w\frac{dw}{dx} \right\}^2 \\ &= (1-10x+25x^2)w^4 + 24x(1-x)(1-5x)w^3\frac{dw}{dx} \\ &\quad + 144x^2(1-x)^2w^2\left(\frac{dw}{dx}\right)^2 \\ &= (1+14x+x^2)w^4 - 24x(1-x)w^4 \\ &\quad + 24x(1-x)(1-5x)w^3\frac{dw}{dx} + 144x^2(1-x)^2w^2\left(\frac{dw}{dx}\right)^2 \\ &= M(q) + 12q\frac{dL}{dq}. \end{aligned}$$

Next we prove (1.5.2). By (1.2.9), (1.2.10), (1.2.11) and (1.4.2), we have

$$\begin{aligned} L(q)M(q) &= (1-5x)(1+14x+x^2)w^6 + 12x(1-x)(1+14x+x^2)w^5\frac{dw}{dx} \\ &= (1+x)(1-34x+x^2)w^6 + 6x(1-x)(7+x)w^6 \\ &\quad + 12x(1-x)(1+14x+x^2)w^5\frac{dw}{dx} \\ &= N(q) + 3q\frac{dM}{dq}. \end{aligned}$$

Next we prove (1.5.3). By (1.2.9), (1.2.11), (1.2.20) and (1.4.3), we have

$$\begin{aligned} L(q)N(q) &= (1-5x)(1+x)(1-34x+x^2)w^8 \\ &\quad + 12x(1-x)(1+x)(1-34x+x^2)w^7\frac{dw}{dx} \end{aligned}$$

$$\begin{aligned}
&= (1 + 14x + x^2)^2 w^8 - 6x(1 - x)(11 + 22x - x^2)w^8 \\
&\quad + 12x(1 - x)(1 + x)(1 - 34x + x^2)w^7 \frac{dw}{dx} \\
&= P(q) + 2q \frac{dN}{dq}.
\end{aligned}$$

Next we prove (1.5.4). By (1.2.9), (1.2.20), (1.2.21) and (1.4.4), we have

$$\begin{aligned}
L(q)P(q) &= (1 - 5x)(1 + 14x + x^2)^2 w^{10} + 12x(1 - x)(1 + 14x + x^2)^2 w^9 \frac{dw}{dx} \\
&= (1 + x)(1 + 14x + x^2)(1 - 34x + x^2)w^{10} \\
&\quad + 6x(1 - x)(7 + x)(1 + 14x + x^2)w^{10} \\
&\quad + 12x(1 - x)(1 + 14x + x^2)^2 w^9 \frac{dw}{dx} \\
&= Q(q) + \frac{3}{2}q \frac{dP}{dq}.
\end{aligned}$$

Next we prove (1.5.5). By (1.2.9), (1.2.22), (1.2.23) and (1.4.6), we have

$$\begin{aligned}
L(q)R(q) &= \frac{1}{691}(1 - 5x)(691 + 2022x + 516381x^2 + 1792148x^3 \\
&\quad + 516381x^4 + 2022x^5 + 691x^6)w^{14} \\
&\quad + \frac{12}{691}x(1 - x)(691 + 2022x + 516381x^2 + 1792148x^3 \\
&\quad + 516381x^4 + 2022x^5 + 691x^6)w^{13} \frac{dw}{dx} \\
&= (1 + x)(1 + 14x + x^2)^2(1 - 34x + x^2)w^{14} \\
&\quad + \frac{1}{691}x(1 - x)(2022 + 1032762x + 5376444x^2 \\
&\quad + 2065524x^3 + 10110x^4 + 4146x^5)w^{14}, \\
&\quad + \frac{12}{691}x(1 - x)(691 + 2022x + 516381x^2 + 1792148x^3 \\
&\quad + 516381x^4 + 2022x^5 + 691x^6)w^{13} \frac{dw}{dx}
\end{aligned}$$

$$= S(q) + q \frac{dR}{dq}.$$

Next we prove (1.5.6). By (1.2.10) and (1.2.20), we have

$$M^2(q) = (1 + 14x + x^2)^2 w^8 = P(q).$$

Next we prove (1.5.7). By (1.2.10), (1.2.11) and (1.2.21), we have

$$M(q)N(q) = (1 + x)(1 + 14x + x^2)(1 - 34x + x^2)w^{10} = Q(q).$$

Lastly, we prove (1.5.8) and (1.5.9). By (1.2.10), (1.2.11), (1.2.20) and (1.2.21), we have

$$\begin{aligned} M(q)Q(q) = P(q)N(q) &= (1 + x)(1 + 14x + x^2)^2(1 - 34x + x^2)w^{14} \\ &= S(q). \end{aligned}$$

This completes the proof of Theorem 1.5.1. □

We note that $N^2(q)$ does not appear in Theorem 1.5.1. This is because its evaluation requires the Ramanujan tau function $\tau(n)$. Lahiri [19, Table A(2), eqn. 3., p. 33] has shown that

$$N(q)^2 = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n - \frac{762048}{691} \sum_{n=1}^{\infty} \tau(n)q^n.$$

It is unlikely that products not in the list (1.5.1)-(1.5.9) can be expressed in terms of Eisenstein series and their derivatives only. It is for this reason we restrict our attention only to $L(q) = E_1(q)$, ..., $S(q) = E_7(q)$.

We use the formulae in Theorem 1.5.1 to determine $\frac{d^2 L}{dq^2}$, ..., $\frac{d^2 S}{dq^2}$ in terms of x . These determinations are used in the next theorem to find relationships between L , ..., S and $\frac{dL}{dq}$, ..., $\frac{dS}{dq}$.

Lemma 1.5.2

$$(1.5.10) \quad q \frac{d^2 L}{dq^2} = -\frac{3}{2}x(1-x)^2 w^6 - 12x^2(1-x)^2 w^5 \frac{dw}{dx} \\ 6x^2(1-5x)(1-x)^2 w^4 \left(\frac{dw}{dx}\right)^2 \\ + 24x^3(1-x)^3 w^3 \left(\frac{dw}{dx}\right)^3,$$

$$(1.5.11) \quad q \frac{d^2 M}{dq^2} = 20x^2(1-x)^2(1+14x+x^2)w^6 \left(\frac{dw}{dx}\right)^2 \\ 20x^2(7+x)(1-x)^2 w^7 \frac{dw}{dx} + 5x(3+x)(1-x)^2 w^8,$$

$$(1.5.12) \quad q \frac{d^2 N}{dq^2} = 42x^2(1+x)(1-x)^2(1-34x+x^2)w^8 \left(\frac{dw}{dx}\right)^2 \\ - 42x^2(1-x)^2(11+22x-x^2)w^9 \frac{dw}{dx} \\ - \frac{21}{2}x(1-x)^2(3+14x-x^2)w^{10},$$

$$(1.5.13) \quad q \frac{d^2 P}{dq^2} = 72x^2(1-x)^2(1+14x+x^2)^2 w^{10} \left(\frac{dw}{dx}\right)^2 \\ + 72x^2(7+x)(1-x)^2(1+14x+x^2)w^{11} \frac{dw}{dx} \\ 6x(1-x)^2(5+137x+47w^2+3x^3)w^{12},$$

$$(1.5.14) \quad q \frac{d^2 Q}{dq^2} = 110x^2(1+x)(1-x)^2(1+14x+x^2) \\ (1-34x+x^2)w^{12} \left(\frac{dw}{dx}\right)^2 \\ - 22x^2(1-x)^2(19+988x+1482x^2$$

$$\begin{aligned}
& +76x^3 - 5x^4)w^{13}\frac{dw}{dx} \\
& -\frac{11}{2}x(1-x)^2(3+364x+858x^2 \\
& +60x^3-5x^4)w^{14}, \\
(1.5.15) \quad q\frac{d^2R}{dq^2} &= \frac{156}{691}x^2(1-x)^2(691+2022x+516381x^2+1792148x^3 \\
& +516381x^4+2022x^5+691x^6)w^{14}\left(\frac{dw}{dx}\right)^2 \\
& +\frac{156}{691}x^2(1-x)^2(337+172127x+896074x^2 \\
& +344254x^3+1685x^4+691x^5)w^{15}\frac{dw}{dx} \\
& \frac{39}{691}x(1-x)^2(105+53119x+426920x^2 \\
& +225246x^3+1453x^4+691x^5)w^{16}, \\
(1.5.16) \quad q\frac{d^2S}{dq^2} &= 210x^2(1+x)(1-x)^2(1-34x+x^2) \\
& (1+14x+x^2)^2w^{16}\left(\frac{dw}{dx}\right)^2 \\
& -30x^2(1-x)^2(1+14x+x^2) \\
& (5+1448x+2010x^2+128x^3-7x^4)w^{17}\frac{dw}{dx} \\
& -\frac{3}{2}x(1-x)^2(1+2030x+45339x^2+82484x^3 \\
& +13415x^4+126x^5-35x^6)w^{18}.
\end{aligned}$$

Proof First we prove (1.5.10). Differentiating (1.5.1) and multiplying it by q , we obtain

$$q\frac{dL}{dq}L(q) = \frac{1}{2}q\frac{dM}{dq} + 6q^2\frac{d^2L}{dq^2}$$

$$q^2 \frac{d^2 L}{dq^2} = \frac{1}{6} \left(\frac{1}{2} q \frac{dM}{dq} - q \frac{dL}{dq} L(q) \right).$$

Substituting (1.2.9), (1.4.1) and (1.4.2) into the equation, we obtain (1.5.10).

Next we prove (1.5.11). Differentiating (1.5.2) and multiplying it by q , we obtain

$$\begin{aligned} q \frac{dL}{dq} M(q) + q \frac{dM}{dq} L(q) &= q \frac{dN}{dq} + 3q^2 \frac{d^2 M}{dq^2} \\ q^2 \frac{d^2 M}{dq^2} &= \frac{1}{3} \left(q \frac{dL}{dq} M(q) + q \frac{dM}{dq} L(q) - q \frac{dN}{dq} \right). \end{aligned}$$

Substituting (1.2.9), (1.2.10), (1.4.1), (1.4.2) and (1.4.3) into the equation, we obtain (1.5.11).

Next we prove (1.5.12). Differentiating (1.5.3) and multiplying it by q , we obtain

$$\begin{aligned} q \frac{dL}{dq} N(q) + q \frac{dN}{dq} L(q) &= q \frac{dP}{dq} + 2q^2 \frac{d^2 N}{dq^2} \\ q^2 \frac{d^2 N}{dq^2} &= \frac{1}{2} \left(q \frac{dL}{dq} N(q) + q \frac{dN}{dq} L(q) - q \frac{dP}{dq} \right). \end{aligned}$$

Substituting (1.2.9), (1.2.11), (1.4.1), (1.4.3) and (1.4.4) into the equation, we obtain (1.5.12).

Next we prove (1.5.13). Differentiating (1.5.6) twice and multiplying it by q each time, we obtain

$$\begin{aligned} q \frac{dP}{dq} &= 2qM(q) \frac{dM}{dq} \\ q^2 \frac{d^2 P}{dq^2} &= 2q^2 \left(\frac{dM}{dq} \right)^2 + 2q^2 M(q) \frac{d^2 M}{dq^2}. \end{aligned}$$

Substituting (1.2.10), (1.4.2), and (1.5.11) into the equation, we obtain (1.5.13).

Next we prove (1.5.14). Differentiating (1.5.7) twice and multiplying it by q each time, we obtain

$$\begin{aligned} q \frac{dQ}{dq} &= qM(q) \frac{dN}{dq} + qN(q) \frac{dM}{dq} \\ q^2 \frac{d^2Q}{dq^2} &= q^2M(q) \frac{d^2N}{dq^2} + 2q^2 \frac{dM}{dq} \frac{dN}{dq} + q^2N(q) \frac{d^2M}{dq^2}. \end{aligned}$$

Substituting (1.2.10), (1.2.11), (1.4.2), (1.4.3), (1.5.11) and (1.5.12) into the equation, we obtain (1.5.14).

Next we prove (1.5.15). Differentiating (1.5.5) and multiplying it by q , we obtain

$$\begin{aligned} q \frac{dL}{dq} R(q) + q \frac{dR}{dq} L(q) &= q \frac{dS}{dq} + q^2 \frac{d^2R}{dq^2} \\ q^2 \frac{d^2R}{dq^2} &= q \frac{dL}{dq} R(q) + q \frac{dR}{dq} L(q) - q \frac{dS}{dq}. \end{aligned}$$

Substituting (1.2.9), (1.2.22), (1.4.1), (1.4.6) and (1.4.7) into the equation, we obtain (1.5.15).

Lastly we prove (1.5.16). Differentiating (1.5.8) twice and multiplying it by q each time, we obtain

$$\begin{aligned} q \frac{dS}{dq} &= qP(q) \frac{dN}{dq} + qN(q) \frac{dP}{dq} \\ q^2 \frac{d^2S}{dq^2} &= q^2P(q) \frac{d^2N}{dq^2} + 2q^2 \frac{dP}{dq} \frac{dN}{dq} + q^2N(q) \frac{d^2P}{dq^2}. \end{aligned}$$

Substituting (1.2.11), (1.2.20), (1.4.3), (1.4.4), (1.5.12) and (1.5.13) into the equation, we obtain (1.5.16). $\square$

We use Theorems 1.2.1, 1.2.2 and 1.4.1 and Lemma 1.5.2 to obtain relationships between $L(q)$, ..., $S(q)$ and $\frac{dL}{dq}$, ..., $\frac{dS}{dq}$.

Theorem 1.5.3

$$(1.5.17) \quad q \frac{dL}{dq} L(q) = \frac{1}{2} q \frac{dL}{dq} + 6q^2 \frac{d^2 L}{dq^2},$$

$$(1.5.18) \quad q \frac{dM}{dq} L(q) = \frac{2}{3} q \frac{dN}{dq} + \frac{12}{5} q^2 \frac{d^2 M}{dq^2},$$

$$(1.5.19) \quad q \frac{dN}{dq} L(q) = \frac{3}{4} q \frac{dP}{dq} + \frac{12}{7} q^2 \frac{d^2 N}{dq^2},$$

$$(1.5.20) \quad q \frac{dL}{dq} M(q) = \frac{1}{3} q \frac{dN}{dq} + \frac{3}{5} q^2 \frac{d^2 M}{dq^2},$$

$$(1.5.21) \quad q \frac{dM}{dq} M(q) = \frac{1}{2} q \frac{dP}{dq},$$

$$(1.5.22) \quad q \frac{dL}{dq} N(q) = \frac{1}{4} q \frac{dP}{dq} + \frac{2}{7} q^2 \frac{d^2 N}{dq^2}.$$

Proof First we prove (1.5.17). By (1.2.9), (1.4.1), (1.4.2) and (1.5.10), we obtain

$$\begin{aligned} q \frac{dL}{dq} L(q) &= 144x^3(1-x)^3w^3 \left(\frac{dw}{dx} \right)^3 \\ &\quad - 36x^2(1-5x)(1-x)^2w^4 \left(\frac{dw}{dx} \right)^2 \\ &\quad + 2x(1-x)(1-22x+37x^2)w^5 \frac{dw}{dx} \\ &\quad - 2x(1-x)(1-5x)w^6 \\ &= \frac{1}{2} q \frac{dM}{dq} + 6q^2 \frac{d^2 L}{dq^2}. \end{aligned}$$

Next we prove (1.5.18). By (1.2.9), (1.4.2), (1.4.3) and (1.5.11), we have

$$qL(q) \frac{dM}{dq} = 48x^2(1-x)^2(1+14x+x^2)w^6 \left(\frac{dw}{dx} \right)^2$$

$$\begin{aligned}
& +4x(1-x)(1+51x-105x^2-11x^3)w^7\frac{dw}{dx} \\
& +2x(1-x)(7+x)(1-5x)w^8 \\
& = \frac{2}{3}q\frac{dN}{dq} + \frac{12}{5}q^2\frac{d^2M}{dq^2}.
\end{aligned}$$

Next we prove (1.5.19). By (1.2.9), (1.4.3), (1.4.4) and (1.5.12), we have

$$\begin{aligned}
qL(q)\frac{dN}{dq} &= 72x^2(1-x)^2(1+x)(1-34x+x^2)w^8\left(\frac{dw}{dx}\right)^2 \\
&+ 6x(1-x)(1-104x+66x^2+304x^3-11x^4)w^9\frac{dw}{dx} \\
&+ 3x(1-x)(1-5x)(-11-22x+x^2)w^{10} \\
&= \frac{3}{4}q\frac{dP}{dq} + \frac{12}{7}q^2\frac{d^2N}{dq^2}.
\end{aligned}$$

Next we prove (1.5.20). By (1.2.10), (1.4.1), (1.4.3) and (1.5.11), we have

$$\begin{aligned}
qM(q)\frac{dL}{dq} &= 12x^2(1-x)^2(1+14x+x^2)w^6\left(\frac{dw}{dx}\right)^2 \\
&+ x(1-x)(2-10x)(1+14x+x^2)w^7\frac{dw}{dx} \\
&- 2x(1-x)(1+14x+x^2)w^8 \\
&= \frac{1}{3}q\frac{dN}{dq} + \frac{3}{5}q^2\frac{d^2M}{dq^2}.
\end{aligned}$$

Next we prove (1.5.21). By (1.2.10), (1.4.2) and (1.4.4), we have

$$\begin{aligned}
qM(q)\frac{dM}{dq} &= 4x(1-x)(1+14x+x^2)^2w^9\frac{dw}{dx} \\
&+ x(1-x)(14+2x)(1+14x+x^2)w^{10} \\
&= \frac{1}{2}q\frac{dP}{dq}.
\end{aligned}$$

Lastly we prove (1.5.22). By (1.2.11), (1.4.1), (1.4.4) and (1.5.12), we have

$$\begin{aligned}
 qN(q)\frac{dL}{dq} &= 12x^2(1+x)(1-x)^2(1-34x+x^2)w^8\left(\frac{dw}{dx}\right)^2 \\
 &\quad + x(1-x)(1+x)(2-10x)(1-34x+x^2)w^9\frac{dw}{dx} \\
 &\quad - 2x(1-x)(1+x)(1-34x+x^2)w^{10} \\
 &= \frac{1}{4}q\frac{dP}{dq} + \frac{2}{7}q^2\frac{d^2N}{dq^2}.
 \end{aligned}$$

This concludes the proof of Theorem 1.5.3. $\square$

We use Theorems 1.5.1 and 1.5.3 to obtain more relationships between $L(q)$, ..., $S(q)$.

Theorem 1.5.4

$$(1.5.23) \quad L^3(q) = N(q) + 9q\frac{dM}{dq} + 72q^2\frac{d^2L}{dq^2},$$

$$(1.5.24) \quad L^2(q)M(q) = P(q) + 4q\frac{dN}{dq} + \frac{36}{5}q^2\frac{d^2M}{dq^2},$$

$$(1.5.25) \quad L(q)M^2(q) = Q(q) + \frac{3}{2}q\frac{dP}{dq},$$

$$(1.5.26) \quad L^2(q)N(q) = Q(q) + 3q\frac{dP}{dq} + \frac{24}{7}q^2\frac{d^2N}{dq^2},$$

$$(1.5.27) \quad M^2(q)N(q) = S(q),$$

$$(1.5.28) \quad L^4(q) = P(q) + 8q\frac{dN}{dq} + \frac{216}{5}q^2\frac{d^2M}{dq^2} + 288q^3\frac{d^3L}{dq^3},$$

$$(1.5.29) \quad L^3(q)M(q) = Q(q) + \frac{9}{2}q\frac{dP}{dq} + \frac{72}{7}q^2\frac{d^2N}{dq^2} + \frac{72}{5}q^3\frac{d^3M}{dq^3},$$

$$\begin{aligned}
 (1.5.30) \quad L^5(q) &= Q(q) + \frac{15}{2}q\frac{dP}{dq} + \frac{240}{7}q^2\frac{d^2N}{dq^2} \\
 &\quad + 144q^3\frac{d^3M}{dq^3} + 864q^4\frac{d^4L}{dq^4}.
 \end{aligned}$$

Proof of (1.5.23)-(1.5.27) First we prove (1.5.23). By (1.5.1), (1.5.2) and (1.5.17), we obtain

$$\begin{aligned} L^3(q) &= L^2(q)L(q) = (M(q) + 12q\frac{dL}{dq})L(q) \\ &= M(q)L(q) + 12qL(q)\frac{dL}{dq} \\ &= N(q) + 9q\frac{dM}{dq} + 72q^2\frac{d^2L}{dq^2}. \end{aligned}$$

Next we prove (1.5.24). By (1.5.1), (1.5.6) and (1.5.20), we obtain

$$\begin{aligned} L^2(q)M(q) &= (M(q) + 12q\frac{dL}{dq})M(q) \\ &= M^2(q) + 12qM(q)\frac{dL}{dq} \\ &= P(q) + 4q\frac{dN}{dq} + \frac{36}{5}q^2\frac{d^2M}{dq^2}. \end{aligned}$$

Next we prove (1.5.25). By (1.5.6) and (1.5.4), we obtain

$$L(q)M^2(q) = L(q)P(q) = Q(q) + \frac{3}{2}q\frac{dP}{dq}.$$

Next we prove (1.5.26). By (1.5.1), (1.5.7) and (1.5.22), we obtain

$$\begin{aligned} L^2(q)N(q) &= (M(q) + 12q\frac{dL}{dq})N(q) \\ &= M(q)N(q) + 12qN(q)\frac{dL}{dq} \\ &= Q(q) + 3q\frac{dP}{dq} + \frac{24}{7}q^2\frac{d^2N}{dq^2}. \end{aligned}$$

Finally we prove (1.5.27). By (1.5.6) and (1.5.9), we obtain

$$M^2(q)N(q) = P(q)N(q) = S(q).$$

This concludes the proof of equations (1.5.23)-(1.5.27). □

Before proving equations (1.5.28)-(1.5.30), we first give the following lemma. The relationships in this lemma are used to prove (1.5.28)-(1.5.30). They are obtained from (1.5.23)-(1.5.26) and previous theorems.

Lemma 1.5.5

$$(1.5.31) \quad \left(q \frac{dL}{dq} \right)^2 = \frac{1}{5} q^2 \frac{d^2 M}{dq^2} + 2q^3 \frac{d^3 L}{dq^3},$$

$$(1.5.32) \quad q^2 \frac{d^2 L}{dq^2} L(q) = \frac{3}{10} q^2 \frac{d^2 M}{dq^2} + 4q^3 \frac{d^3 L}{dq^3},$$

$$(1.5.33) \quad q^2 \frac{dL}{dq} \frac{dM}{dq} = \frac{1}{105} (20q^2 \frac{d^2 N}{dq^2} + 42q^3 \frac{d^3 M}{dq^3}),$$

$$(1.5.34) \quad q^2 \frac{d^2 L}{dq^2} M(q) = \frac{1}{105} (15q^2 \frac{d^2 N}{dq^2} + 21q^3 \frac{d^3 M}{dq^3}),$$

$$(1.5.35) \quad q^2 \frac{d^2 M}{dq^2} L(q) = \frac{10}{21} q^2 \frac{d^2 N}{dq^2} + 2q^3 \frac{d^3 M}{dq^3},$$

$$(1.5.36) \quad q^3 \frac{d^2 L}{dq^2} \frac{dL}{dq} = \frac{1}{10} q^3 \frac{d^3 M}{dq^3} + q^4 \frac{d^4 L}{dq^4},$$

$$(1.5.37) \quad q^3 \frac{d^3 L}{dq^3} L(q) = \frac{1}{5} q^3 \frac{d^3 M}{dq^3} + 3q^4 \frac{d^4 L}{dq^4}.$$

Proof First we prove (1.5.31) and (1.5.32). By (1.5.1), (1.5.23), (1.5.3), (1.5.18), (1.5.6) and (1.5.20), we obtain

$$(1.5.38) \quad \begin{aligned} L^3(q)L(q) &= L^2(q)L^2(q) \\ \left(N(q) + 9q \frac{dM}{dq} + 72q^2 \frac{d^2 L}{dq^2} \right) L(q) &= \left(M(q) + 12q \frac{dL}{dq} \right)^2 \\ 72q^2 L(q) \frac{d^2 L}{dq^2} &= 144 \left(q \frac{dL}{dq} \right)^2 - \frac{36}{5} q^2 \frac{d^2 M}{dq^2}. \end{aligned}$$

Differentiating (1.5.17), we obtain

$$(1.5.39) \quad q^2 L(q) \frac{d^2 L}{dq^2} + \left(q \frac{dL}{dq} \right)^2 = \frac{1}{2} q^2 \frac{d^2 M}{dq^2} + 6q^3 \frac{d^3 L}{dq^3}.$$

By (1.5.38) and (1.5.39), we obtain (1.5.31) and (1.5.32).

Next we prove (1.5.33), (1.5.34) and (1.5.35). By (1.5.1), (1.5.23), (1.5.2), (1.5.21) and (1.5.22), we obtain

$$\begin{aligned}
 L^3(q)M(q) &= L^2(q)L(q)M(q) \\
 \left(N(q) + 9q\frac{dM}{dq} + 72q^2\frac{d^2L}{dq^2}\right)M(q) &= \left(M(q) + 12q\frac{dL}{dq}\right)\left(N(q) + 3q\frac{dM}{dq}\right) \\
 (1.5.40) \quad 72q^2M(q)\frac{d^2L}{dq^2} &= 36q^2\frac{dL}{dq}\frac{dM}{dq} + \frac{24}{7}q^2\frac{d^2N}{dq^2}.
 \end{aligned}$$

Differentiating (1.5.20), we obtain

$$(1.5.41) \quad q^2M(q)\frac{d^2L}{dq^2} + q^2\frac{dL}{dq}\frac{dM}{dq} = \frac{1}{3}q^2\frac{d^2N}{dq^2} + \frac{3}{5}q^3\frac{d^3M}{dq^3}.$$

Differentiating (1.5.18), we obtain

$$(1.5.42) \quad q^2L(q)\frac{d^2M}{dq^2} + q^2\frac{dL}{dq}\frac{dM}{dq} = \frac{2}{3}q^2\frac{d^2N}{dq^2} + \frac{12}{5}q^3\frac{d^3M}{dq^3}.$$

By (1.5.40) and (1.5.41), we obtain (1.5.33) and (1.5.34). By (1.5.40) and (1.5.42), we obtain (1.5.35).

Next we prove (1.5.36) and (1.5.37). By (1.5.1), (1.5.23), (1.5.28), (1.5.7), (1.5.22), (1.5.21), (1.5.33), (1.5.34), (1.5.4), (1.5.19) and (1.5.35), we obtain

$$\begin{aligned}
 L^3(q)L^2(q) &= L^4(q)L(q) \\
 \left(N(q) + 9q\frac{dM}{dq} + 72q^2\frac{d^2L}{dq^2}\right)\left(M(q) + 12q\frac{dL}{dq}\right) \\
 &= \left(P(q) + 8q\frac{dN}{dq} + \frac{216}{5}q^2\frac{d^2M}{dq^2} + 288q^3\frac{d^3L}{dq^3}\right)L(q) \\
 (1.5.43) \quad q^3L(q)\frac{d^3L}{dq^3} &= -\frac{1}{10}q^3\frac{d^3M}{dq^3} + 3q^3\frac{dL}{dq}\frac{d^2L}{dq^2}.
 \end{aligned}$$

Differentiating (1.5.32), we obtain

$$(1.5.44) \quad q^3 L(q) \frac{d^3 L}{dq^3} + q^3 \frac{dL}{dq} \frac{d^2 L}{dq^2} = \frac{3}{10} q^3 \frac{d^3 M}{dq^3} + 4q^4 \frac{d^4 L}{dq^4}.$$

By (1.5.43) and (1.5.44), we obtain (1.5.36) and (1.5.37).

This concludes the proof of Lemma 1.5.5. $\square$

We now return to the proof of the remaining three equations of Theorem 1.5.4.

Proof of (1.5.28)-(1.5.30) First we prove (1.5.28). By (1.5.1), (1.5.6), (1.5.20) and (1.5.31), we obtain

$$\begin{aligned} L^4(q) &= L^2(q)L^2(q) = (M(q) + 12q \frac{dL}{dq})^2 \\ &= M^2(q) + 24qM(q) \frac{dL}{dq} + 144 \left(q \frac{dL}{dq} \right)^2 \\ &= P(q) + 8q \frac{dN}{dq} + \frac{216}{5} q^2 \frac{d^2 M}{dq^2} + 288q^3 \frac{d^3 L}{dq^3}. \end{aligned}$$

Next we prove (1.5.29). By (1.5.23), (1.5.7), (1.5.21) and (1.5.34), we obtain

$$\begin{aligned} L^3(q)M(q) &= (N(q) + 9q \frac{dM}{dq} + 72q^2 \frac{d^2 L}{dq^2})M(q) \\ &= N(q)M(q) + 9qM(q) \frac{dM}{dq} + 72q^2 \frac{d^2 L}{dq^2} M(q) \\ &= Q(q) + \frac{9}{2} q \frac{dP}{dq} + \frac{72}{7} q^2 \frac{d^2 N}{dq^2} + \frac{72}{5} q^3 \frac{d^3 M}{dq^3}. \end{aligned}$$

Finally we prove (1.5.30). By (1.5.28), (1.5.4), (1.5.19), (1.5.35) and (1.5.37), we obtain

$$L^5(q) = L^4(q)L(q)$$

$$\begin{aligned}
&= \left(P(q) + 8q \frac{dN}{dq} + \frac{216}{5} q^2 \frac{d^2 M}{dq^2} + 288 q^3 \frac{d^3 L}{dq^3} \right) L(q) \\
&= P(q)L(q) + 8qL(q) \frac{dN}{dq} + \frac{216}{5} q^2 L(q) \frac{d^2 M}{dq^2} + 288 q^3 L(q) \frac{d^3 L}{dq^3} \\
&= Q(q) + \frac{15}{2} q \frac{dP}{dq} + \frac{240}{7} q^2 \frac{d^2 N}{dq^2} + 144 q^3 \frac{d^3 M}{dq^3} + 864 q^4 \frac{d^4 L}{dq^4}.
\end{aligned}$$

This concludes the proof of equations (1.5.28)-(1.5.30). $\square$

1.6 The arithmetic identities in Ramanujan's paper.

In this section, we use the relationships in Theorem 1.5.1 to deduce the nine arithmetic identities given in Ramanujan's paper [31, Table IV], [33, Table IV, p. 146]. Formula (1.6.1) is equivalent to formula (3.1) in Lahiri [18] and originally appeared in a letter from Besge to Liouville [5]. Rankin attributed it to Besge in [34, p. 115]. It also appears in the work of Glaisher [9], [10], [11], Lehmer [20, p. 106], [21, p. 678] and Skoruppa [35]. Formula (1.6.2) is a result attributed to Glaisher by MacMahon [24, p. 101], [25, p. 329]. It is also formula (5.1) of Lahiri [18]. Formula (1.6.3) is due to Ramanujan. It is also given in the work of MacMahon [24, p. 103], [25, p. 331]. Formula (1.6.4) is due to Ramanujan. Formula (1.6.6) is due to Glaisher [11, p. 35]. Formula (1.6.7) is due to Glaisher [11, p. 35]. Formulae (1.6.5), (1.6.8) and (1.6.9) are formulae (13.1), (13.2) and (13.4) of Lahiri [18]. Formulae (1.6.8) and (1.6.9)

are due to Glaisher [11, p. 35]. Formula (1.6.5) is due to Ramanujan.

Theorem 1.6.1 *For all positive integers n ,*

$$(1.6.1) \quad \sum_{m < n} \sigma(m)\sigma(n-m) = \frac{5}{12}\sigma_3(n) + \frac{1}{12}\sigma(n) - \frac{1}{2}n\sigma(n),$$

$$(1.6.2) \quad \sum_{m < n} \sigma(m)\sigma_3(n-m) = \frac{7}{80}\sigma_5(n) - \frac{1}{8}n\sigma_3(n) + \frac{1}{24}\sigma_3(n) - \frac{1}{240}\sigma(n),$$

$$(1.6.3) \quad \sum_{m < n} \sigma(m)\sigma_5(n-m) = \frac{5}{126}\sigma_7(n) - \frac{1}{12}n\sigma_5(n) + \frac{1}{24}\sigma_5(n) + \frac{1}{504}\sigma(n),$$

$$(1.6.4) \quad \sum_{m < n} \sigma(m)\sigma_7(n-m) = \frac{11}{480}\sigma_9(n) - \frac{1}{16}n\sigma_7(n) + \frac{1}{24}\sigma_7(n) - \frac{1}{480}\sigma(n),$$

$$(1.6.5) \quad \sum_{m < n} \sigma(m)\sigma_{11}(n-m) = \frac{691}{65520}\sigma_{13}(n) - \frac{1}{24}n\sigma_{11}(n) \\ + \frac{1}{24}\sigma_{11}(n) - \frac{691}{65520}\sigma(n),$$

$$(1.6.6) \quad \sum_{m < n} \sigma_3(m)\sigma_3(n-m) = \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n),$$

$$(1.6.7) \quad \sum_{m < n} \sigma_3(m)\sigma_5(n-m) = \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n),$$

$$(1.6.8) \quad \sum_{m < n} \sigma_3(m)\sigma_9(n-m) = \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n),$$

$$(1.6.9) \quad \sum_{m < n} \sigma_5(m)\sigma_7(n-m) = \frac{1}{10080}\sigma_{13}(n) + \frac{1}{504}\sigma_7(n) - \frac{1}{480}\sigma_5(n).$$

Proof First we prove (1.6.1). By (1.1.1), (1.1.2) and (1.5.1), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n} \sigma(m)\sigma(n-m)q^n \\ &= \left\{ \sum_{l=1}^{\infty} \sigma(l)q^l \right\}^2 = \left(\frac{1-L(q)}{24} \right)^2 \\ &= \frac{1-2L(q)+L^2(q)}{576} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{576} \left(1 - 2L(q) + M(q) + 12q \frac{dL}{dq} \right) \\
&= \frac{1}{576} - \frac{2}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) \\
&\quad + \frac{1}{576} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) + \frac{12}{576} \left(-24 \sum_{n=1}^{\infty} n \sigma(n) q^n \right) \\
&= \frac{1}{12} \sum_{n=1}^{\infty} (5\sigma_3(n) + \sigma(n) - 6n\sigma(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.1).

Next we prove (1.6.2). By (1.1.1), (1.1.2), (1.1.3) and (1.5.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n} \sigma(m) \sigma_3(n-m) q^n \\
&= \sum_{l=1}^{\infty} \sigma(l) q^l \sum_{k=1}^{\infty} \sigma_3(k) q^k \\
&= \left(\frac{1-L(q)}{24} \right) \left(\frac{M(q)-1}{240} \right) \\
&= \frac{-1-L(q)M(q)+M(q)+L(q)}{5760} \\
&= \frac{1}{5760} \left(-1-N(q)-3q \frac{dM}{dq} + M(q) + L(q) \right) \\
&= -\frac{1}{5760} - \frac{1}{5760} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\
&\quad - \frac{3}{5760} \left(240 \sum_{n=1}^{\infty} n \sigma_3(n) q^n \right) + \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \\
&\quad + \frac{1}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) \\
&= \frac{1}{240} \sum_{n=1}^{\infty} (21\sigma_5(n) - 30n\sigma_3(n) + 10\sigma_3(n) - \sigma(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.2).

Next we prove (1.6.3). By (1.1.1), (1.1.3), (1.1.4) and (1.5.3), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(m) \sigma_5(n-m) q^n \\
&= \sum_{l=1}^{\infty} \sigma(l) q^l \sum_{k=1}^{\infty} \sigma_5(k) q^k \\
&= \left(\frac{1-L(q)}{24} \right) \left(\frac{1-N(q)}{504} \right) \\
&= \frac{1+L(q)N(q)-N(q)-L(q)}{12096} \\
&= \frac{1}{12096} \left(1+P(q)+2q \frac{dN}{dq} - N(q) - L(q) \right) \\
&= \frac{1}{12096} + \frac{1}{12096} \left(1+480 \sum_{n=1}^{\infty} \sigma_7(n) q^n \right) \\
&\quad + \frac{2}{12096} \left(-504 \sum_{n=1}^{\infty} n \sigma_5(n) q^n \right) - \frac{1}{12096} \left(1-504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\
&\quad - \frac{1}{12096} \left(1-24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) \\
&= \frac{1}{1512} \sum_{n=1}^{\infty} (60\sigma_7(n) - 126n\sigma_5(n) + 63\sigma_5(n) + 3\sigma(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.3).

Next we prove (1.6.4). By (1.1.1), (1.1.4), (1.1.5) and (1.5.4), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(m) \sigma_7(n-m) q^n \\
&= \sum_{l=1}^{\infty} \sigma(l) q^l \sum_{k=1}^{\infty} \sigma_7(k) q^k \\
&= \left(\frac{1-L(q)}{24} \right) \left(\frac{P(q)-1}{480} \right) \\
&= \frac{-1-L(q)P(q)+P(q)+L(q)}{11520}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11520} \left(-1 - Q(q) - \frac{3}{2}q \frac{dP}{dq} + P(q) + L(q) \right) \\
&= -\frac{1}{11520} - \frac{1}{11520} \left(1 - 264 \sum_{n=1}^{\infty} \sigma_9(n)q^n \right) \\
&\quad - \frac{1}{7680} \left(480 \sum_{n=1}^{\infty} n\sigma_7(n)q^n \right) + \frac{1}{11520} \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n \right) \\
&\quad + \frac{1}{11520} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n \right) \\
&= \frac{1}{480} \sum_{n=1}^{\infty} (11\sigma_9(n) - 30n\sigma_7(n) + 20\sigma_7(n) - \sigma(n))q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.4).

Next we prove (1.6.5). By (1.1.1), (1.1.6), (1.1.7) and (1.5.5), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n} \sigma(m)\sigma_{11}(n-m)q^n \\
&= \sum_{l=1}^{\infty} \sigma(l)q^l \sum_{k=1}^{\infty} \sigma_{11}(k)q^k \\
&= \left(\frac{1-L(q)}{24} \right) \left(\frac{691R(q)-691}{65520} \right) \\
&= \frac{691(-1-L(q)R(q)+R(q)+L(q))}{1572480} \\
&= \frac{691}{1572480} \left(-1 - S(q) - q \frac{dR}{dq} + R(q) + L(q) \right) \\
&= -\frac{691}{1572480} - \frac{691}{1572480} \left(1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n)q^n \right) \\
&\quad - \frac{691}{1572480} \left(\frac{65520}{691} \sum_{n=1}^{\infty} n\sigma_{11}(n)q^n \right) + \frac{691}{1572480} \left(1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n \right) \\
&\quad + \frac{691}{1572480} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n \right)
\end{aligned}$$

$$= \frac{1}{65520} \sum_{n=1}^{\infty} (691\sigma_{13}(n) - 2730n\sigma_{11}(n) + 2730\sigma_{11}(n) - 691\sigma(n))q^n.$$

Equating the coefficients of q^n , we obtain the identity (1.6.5).

Next we prove (1.6.6). By (1.1.2), (1.1.4) and (1.5.6), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(m)\sigma_3(n-m)q^n \\ &= \left\{ \sum_{l=1}^{\infty} \sigma_3(l)q^l \right\}^2 = \left(\frac{M(q)-1}{240} \right)^2 \\ &= \frac{M^2(q) - 2M(q) + 1}{57600} \\ &= \frac{1}{57600} (1 + P(q) - 2M(q)) \\ &= \frac{1}{57600} + \frac{1}{57600} \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n \right) \\ &\quad - \frac{2}{57600} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right) \\ &= \frac{1}{120} \sum_{n=1}^{\infty} (\sigma_7(n) - \sigma_3(n))q^n. \end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.6).

Next we prove (1.6.7). By (1.1.2), (1.1.3), (1.1.5) and (1.5.7), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(m)\sigma_5(n-m)q^n \\ &= \sum_{l=1}^{\infty} \sigma_3(l)q^l \sum_{k=1}^{\infty} \sigma_5(k)q^k \\ &= \left(\frac{M(q)-1}{240} \right) \left(\frac{1-N(q)}{504} \right) \\ &= \frac{-M(q)N(q) + N(q) + M(q) - 1}{120960} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{120960}(-1 - Q(q) + N(q) + M(q)) \\
&= \frac{1}{120960} - \frac{1}{120960} \left(1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n \right) \\
&\quad + \frac{1}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) + \frac{1}{120960} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \\
&= \frac{1}{5040} \sum_{n=1}^{\infty} (11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.7).

Next we prove (1.6.8). By (1.1.2), (1.1.5), (1.1.7) and (1.5.8), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(m) \sigma_9(n-m) q^n \\
&= \sum_{l=1}^{\infty} \sigma_3(l) q^l \sum_{k=1}^{\infty} \sigma_9(k) q^k \\
&= \left(\frac{M(q) - 1}{240} \right) \left(\frac{1 - Q(q)}{264} \right) \\
&= \frac{-M(q)Q(q) + Q(q) + M(q) - 1}{63360} \\
&= \frac{1}{63360}(-1 - S(q) + Q(q) + M(q)) \\
&= -\frac{1}{63360} - \frac{1}{63360} \left(1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n \right) \\
&\quad + \frac{1}{63360} \left(1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n \right) + \frac{1}{63360} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \\
&= \frac{1}{2640} \sum_{n=1}^{\infty} (\sigma_{13}(n) - 11\sigma_9(n) + 10\sigma_3(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.8).

Lastly, we prove (1.6.9). By (1.1.3), (1.1.4), (1.1.7) and (1.5.9), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma_5(m) \sigma_7(n-m) q^n \\
&= \sum_{l=1}^{\infty} \sigma_5(l) q^l \sum_{k=1}^{\infty} \sigma_7(k) q^k \\
&= \left(\frac{1-N(q)}{504} \right) \left(\frac{P(q)-1}{480} \right) \\
&= \frac{-N(q)P(q) + P(q) + N(q) - 1}{241920} \\
&= \frac{1}{241920} (-1 - S(q) + P(q) + N(q)) \\
&= -\frac{1}{241920} - \frac{1}{241920} \left(1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n \right) \\
&\quad + \frac{1}{241920} \left(1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n \right) + \frac{1}{241920} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\
&= \frac{1}{10080} \sum_{n=1}^{\infty} (\sigma_{13}(n) + 20\sigma_7(n) - 21\sigma_5(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.6.8). $\square$

Formula (1.6.6) has been described recently by Elkies [7] as a “mysterious” identity. In fact the work of Huard, Ou, Spearman and Williams [14] and the work of this thesis show that (1.6.6) is in fact an elementary formula. Levitt [22] claims to have shown that the nine sums of Theorem 1.6.1 are the only sums of the form

$$\sum_{m < n} \sigma_e(m) \sigma_f(n-m)$$

for e, f positive odd integers which can be evaluated in terms of $\sigma_{e+f+1}(n)$, $\sigma_{e+f-1}(n)$, ..., $\sigma_3(n)$, $\sigma_1(n)$.

We note that the sum

$$\sum_{m < n} \sigma_5(n) \sigma_5(n - m)$$

does not appear in Theorem 1.6.1. This sum requires the “non-elementary” Ramanujan tau function $\tau(n)$ for its evaluation. The function $\tau(n)$ is defined by

$$q \prod_{k=1}^{\infty} (1 - q^k)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

and Ramanujan showed that

$$M(q)^3 - N(q)^2 = 1728 \sum_{n=1}^{\infty} \tau(n) q^n$$

and

$$\begin{aligned} \sum_{m < n} \sigma(m) \sigma_9(n - m) &= \frac{13650}{912120} \sigma_{11}(n) + \frac{1}{912120} (38005 - 45606n) \sigma_9(n) \\ &\quad + \frac{3455}{912120} \sigma(n) - \frac{9504}{912120} \tau(n), \\ \sum_{m < n} \sigma_3(m) \sigma_7(n - m) &= \frac{273}{331680} \sigma_{11}(n) - \frac{1392}{331680} \sigma_7(n) - \frac{691}{331680} \sigma_3(n) \\ &\quad + \frac{1800}{331680} \tau(n), \\ \sum_{m < n} \sigma_5(m) \sigma_5(n - m) &= \frac{65}{174132} \sigma_{11}(n) + \frac{691}{174132} \sigma_5(n) - \frac{756}{174132} \tau(n). \end{aligned}$$

The third of these follows from the evaluation of $N(q)^2$ mentioned in Section 1.5. It is not known what functions are needed to evaluate

$$\sum_{m < n} \sigma_e(n) \sigma_f(n - m)$$

for e, f positive odd integers with $e + f \geq 14$.

Recurrence relations for convolution sums involving divisor functions have been given by Grosjean [12], [13] and van der Pol [28].

1.7 Some arithmetic identities of Lahiri.

In this section, we use the relationships in Theorems 1.5.3 and 1.5.4 to deduce fourteen more arithmetic identities. These identities can be found in Lahiri's papers [18], [19]. Formula (1.7.1) is formula (3.2) of Lahiri [18]. This formula is due to Glaisher [11]. Formulae (1.7.2) and (1.7.4) are given in MacMahon [25, p. 331], [24, p. 103] and are also formulae (5.3) and (5.4) of Lahiri [18]. Formulae (1.7.3), (1.7.5) and (1.7.6) are formulae (7.6), (7.5) and (7.7) of Lahiri [18]. Formulae (1.7.7)-(1.7.14) are formulae (5.2), (7.3), (9.3), (9.4), (13.3), (7.4), (9.5) and (9.6) of Lahiri [18]. There is a slight typo in formula (9.6) of Lahiri. $-2^2.3^2.5^3(1, 4)$ should be $-2^2.3^2.5^2(1, 4)$.

Theorem 1.7.1 *For all positive integers n ,*

$$\begin{aligned}
 (1.7.1) \quad & \sum_{m < n} m\sigma(m)\sigma(n-m) \\
 &= \frac{5}{24}n\sigma_3(n) + \frac{1}{24}n\sigma(n) - \frac{1}{4}n^2\sigma(n), \\
 (1.7.2) \quad & \sum_{m < n} m\sigma_3(m)\sigma(n-m) \\
 &= \frac{7}{120}n\sigma_5(n) - \frac{1}{10}n^2\sigma_3(n) + \frac{1}{24}n\sigma_3(n),
 \end{aligned}$$

$$\begin{aligned}
(1.7.3) \quad & \sum_{m < n} m\sigma_5(m)\sigma(n-m) \\
&= \frac{5}{168}n\sigma_7(n) - \frac{1}{14}n^2\sigma_5(n) + \frac{1}{24}n\sigma_5(n),
\end{aligned}$$

$$\begin{aligned}
(1.7.4) \quad & \sum_{m < n} m\sigma(m)\sigma_3(n-m) \\
&= \frac{7}{240}n\sigma_5(n) - \frac{1}{40}n^2\sigma_3(n) - \frac{1}{240}n\sigma(n),
\end{aligned}$$

$$\begin{aligned}
(1.7.5) \quad & \sum_{m < n} m\sigma_3(m)\sigma_3(n-m) \\
&= \frac{1}{240}n\sigma_7(n) - \frac{1}{240}n\sigma_3(n),
\end{aligned}$$

$$\begin{aligned}
(1.7.6) \quad & \sum_{m < n} m\sigma(m)\sigma_5(n-m) \\
&= \frac{5}{504}n\sigma_7(n) - \frac{1}{84}n^2\sigma_5(n) + \frac{1}{504}n\sigma(n),
\end{aligned}$$

$$\begin{aligned}
(1.7.7) \quad & \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma(m_2)\sigma(m_3) \\
&= \frac{1}{192}((1-12n+24n^2)\sigma(n) + (10-30n)\sigma_3(n) + 7\sigma_5(n)),
\end{aligned}$$

$$\begin{aligned}
(1.7.8) \quad & \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma(m_2)\sigma_3(m_3) \\
&= \frac{1}{2880}((-1+6n)\sigma(n) + (-30n+36n^2)\sigma_3(n) \\
&\quad + (21-42n)\sigma_5(n) + 10\sigma_7(n)),
\end{aligned}$$

$$\begin{aligned}
(1.7.9) \quad & \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma_3(m_2)\sigma_3(m_3) \\
&= \frac{1}{240^2}(\sigma(n) + (-20+60n)\sigma_3(n) - 42\sigma_5(n) \\
&\quad + (20-30n)\sigma_7(n) + 11\sigma_9(n)),
\end{aligned}$$

$$(1.7.10) \quad \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma(m_2)\sigma_5(m_3)$$

$$= \frac{1}{12096}((2 - 12n)\sigma(n) + 10\sigma_3(n) \\ + (21 - 84n + 72n^2)\sigma_5(n) + (40 - 60n)\sigma_7(n) + 11\sigma_9(n)),$$

$$(1.7.11) \quad \sum_{m_1+m_2+m_3=n} \sigma_3(m_1)\sigma_3(m_2)\sigma_5(m_3) \\ = \frac{1}{1209600}(-20\sigma_3(n) + 21\sigma_5(n) \\ + 20\sigma_7(n) - 22\sigma_9(n) + \sigma_{13}(n)),$$

$$(1.7.12) \quad \sum_{m_1+m_2+m_3+m_4=n} \sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma(m_4) \\ = \frac{1}{3456}((1 - 18n + 72n^2 - 72n^3)\sigma(n) + (15 - 90n + 108n^2)\sigma_3(n) \\ + (21 - 42n)\sigma_5(n) + 5\sigma_7(n)),$$

$$(1.7.13) \quad \sum_{m_1+m_2+m_3+m_4=n} \sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma_3(m_4) \\ = \frac{1}{138240}((-3 + 36n - 72n^2)\sigma(n) \\ + (-20 + 216n^2 - 144n^3)\sigma_3(n) + (41 - 252n + 216n^2)\sigma_5(n) \\ + (60 - 90n)\sigma_7(n) + 11\sigma_9(n)),$$

$$(1.7.14) \quad \sum_{m_1+m_2+m_3+m_4+m_5=n} \sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma(m_4)\sigma(m_5) \\ = \frac{1}{24^4}((5 - 120n + 720n^2 - 1440n^3 + 864n^4)\sigma(n) \\ + (100 - 900n + 2160n^2 - 1440n^3)\sigma_3(n) \\ + (210 - 840n + 720n^2)\sigma_5(n) \\ + (100 - 150n)\sigma_7(n) + 11\sigma_9(n)).$$

Proof First we prove (1.7.1). By (1.1.1), (1.1.2) and (1.5.17), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} m \sigma(m) \sigma(n-m) q^n \\
&= \sum_{k=1}^{\infty} k \sigma(k) q^k \sum_{l=1}^{\infty} \sigma(l) q^l \\
&= \left(-\frac{1}{24} q \frac{dL}{dq} \right) \left(\frac{1-L(q)}{24} \right) \\
&= \frac{1}{576} \left(-q \frac{dL}{dq} + q \frac{dL}{dq} L(q) \right) \\
&= \frac{1}{576} \left(-q \frac{dL}{dq} + \frac{1}{2} q \frac{dM}{dq} + 6q^2 \frac{d^2 L}{dq^2} \right) \\
&= \frac{1}{576} \sum_{n=1}^{\infty} (24n\sigma(n) + 120n\sigma_3(n) - 144n^2\sigma(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.1).

Next we prove (1.7.2). By (1.1.1), (1.1.2), (1.1.3) and (1.5.18), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} m \sigma_3(m) \sigma(n-m) q^n \\
&= \sum_{l=1}^{\infty} l \sigma_3(l) q^l \sum_{k=1}^{\infty} \sigma(k) q^k \\
&= \left(\frac{1-L(q)}{24} \right) \left(\frac{1}{240} q \frac{dM}{dq} \right) \\
&= \frac{1}{5760} \left(q \frac{dM}{dq} - q \frac{dM}{dq} L(q) \right) \\
&= \frac{1}{5760} \left(q \frac{dM}{dq} - \frac{12}{5} q^2 \frac{d^2 M}{dq^2} - \frac{2}{3} q \frac{dN}{dq} \right) \\
&= \frac{1}{5760} \sum_{n=1}^{\infty} (336n\sigma_5(n) - 576n^2\sigma_3(n) + 240n\sigma_3(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.2).

Next we prove (1.7.3). By (1.1.1), (1.1.3), (1.1.4) and (1.5.19), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} m \sigma_5(m) \sigma(n-m) q^n \\
&= \sum_{l=1}^{\infty} l \sigma_5(l) q^l \sum_{k=1}^{\infty} \sigma(k) q^k \\
&= \left(\frac{1-L(q)}{24} \right) \left(-\frac{1}{504} q \frac{dN}{dq} \right) \\
&= \frac{1}{12096} \left(-q \frac{dN}{dq} + q \frac{dN}{dq} L(q) \right) \\
&= \frac{1}{12096} \left(-q \frac{dN}{dq} + \frac{12}{7} q^2 \frac{d^2 N}{dq^2} + \frac{3}{4} q \frac{dP}{dq} \right) \\
&= \frac{1}{12096} \sum_{n=1}^{\infty} (360n\sigma_7(n) - 864n^2\sigma_5(n) + 504n\sigma_5(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.3).

Next we prove (1.7.4). By (1.1.1), (1.1.2), (1.1.3) and (1.5.20), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} m \sigma(m) \sigma_3(n-m) q^n \\
&= \sum_{l=1}^{\infty} l \sigma(l) q^l \sum_{k=1}^{\infty} \sigma_3(k) q^k \\
&= \left(-\frac{1}{24} q \frac{dL}{dq} \right) \left(\frac{M(q)-1}{240} \right) \\
&= \frac{1}{5760} \left(-q \frac{dL}{dq} M(q) + q \frac{dL}{dq} \right) \\
&= \frac{1}{5760} \left(q \frac{dL}{dq} - \frac{3}{5} q^2 \frac{d^2 M}{dq^2} - \frac{1}{3} q \frac{dN}{dq} \right) \\
&= \frac{1}{5760} \sum_{n=1}^{\infty} (168n\sigma_5(n) - 144n^2\sigma_3(n) - 24n\sigma(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.4).

Next we prove (1.7.5). By (1.1.2), (1.1.4) and (1.5.21), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} m \sigma_3(m) \sigma_3(n-m) q^n \\
&= \sum_{l=1}^{\infty} l \sigma_3(l) q^l \sum_{k=1}^{\infty} \sigma_3(k) q^k \\
&= \left(\frac{1}{240} q \frac{dM}{dq} \right) \left(\frac{M(q) - 1}{240} \right) \\
&= \frac{1}{57600} \left(q \frac{dM}{dq} M(q) - q \frac{dM}{dq} \right) \\
&= \frac{1}{57600} \left(\frac{1}{2} q \frac{dP}{dq} - q \frac{dM}{dq} \right) \\
&= \frac{1}{57600} \sum_{n=1}^{\infty} (240n\sigma_7(n) - 240n\sigma_3(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.5).

Next we prove (1.7.6). By (1.1.1), (1.1.3), (1.1.4) and (1.5.22), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} m \sigma(m) \sigma_5(n-m) q^n \\
&= \sum_{l=1}^{\infty} l \sigma(l) q^l \sum_{k=1}^{\infty} \sigma_5(k) q^k \\
&= \left(-\frac{1}{24} q \frac{dL}{dq} \right) \left(\frac{1 - N(q)}{504} \right) \\
&= \frac{1}{12096} \left(q \frac{dL}{dq} N(q) - q \frac{dL}{dq} \right) \\
&= \frac{1}{12096} \left(-q \frac{dL}{dq} + \frac{2}{7} q^2 \frac{d^2 N}{dq^2} + \frac{1}{4} q \frac{dP}{dq} \right) \\
&= \frac{1}{12096} \sum_{n=1}^{\infty} (120n\sigma_7(n) - 144n^2\sigma_5(n) + 24n\sigma(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.6).

Next we prove (1.7.7). By (1.1.1), (1.1.2), (1.1.3), (1.5.1) and (1.5.23), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma(m_2)\sigma(m_3)q^n \\
&= \left(\frac{1-L(q)}{24} \right)^3 \\
&= \frac{1}{24^3} (1 - 3L(q) + 3L^2(q) - L^3(q)) \\
&= \frac{1}{24^3} \left(1 - 3L(q) + 3M(q) + 36q \frac{dL}{dq} - N(q) - 9q \frac{dM}{dq} - 72q^2 \frac{d^2L}{dq^2} \right) \\
&= \frac{1}{192} \sum_{n=1}^{\infty} ((1 - 12n + 24n^2)\sigma(n) + (10 - 30n)\sigma_3(n) + 7\sigma_5(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.7).

Next we prove (1.7.8). By (1.1.1), (1.1.2), (1.1.3), (1.1.4), (1.5.1), (1.5.2) and (1.5.24), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma(m_2)\sigma_3(m_3)q^n \\
&= \left(\frac{1-L(q)}{24} \right)^2 \left(\frac{M(q)-1}{240} \right) \\
&= \frac{1}{(240)24^2} (-1 + 2L(q) - L^2(q) + M(q) - 2L(q)M(q) + L^2(q)M(q)) \\
&= \frac{1}{(240)24^2} \left(-1 + 2L(q) - 12q \frac{dL}{dq} - 6q \frac{dM}{dq} + \frac{36}{5} q^2 \frac{d^2M}{dq^2} \right. \\
&\quad \left. - 2N(q) + 4q \frac{dN}{dq} + P(q) \right) \\
&= \frac{1}{2880} \sum_{n=1}^{\infty} ((-1 + 6n)\sigma(n) + (-30n + 36n^2)\sigma_3(n) \\
&\quad + (21 - 42n)\sigma_5(n) + 10\sigma_7(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.8).

Next we prove (1.7.9). By (1.1.1), (1.1.2), (1.1.3), (1.1.4), (1.1.5), (1.5.6), (1.5.2) and (1.5.25), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma_3(m_2)\sigma_3(m_3)q^n \\
&= \left(\frac{1-L(q)}{24} \right) \left(\frac{M(q)-1}{240} \right)^2 \\
&= \frac{1}{(24)240^2} (1 - 2M(q) + M^2(q) - L(q) + 2L(q)M(q) - L(q)M^2(q)) \\
&= \frac{1}{(24)240^2} \left(1 - L(q) - 2M(q) + 6q \frac{dM}{dq} + 2N(q) + P(q) - \frac{3}{2}q \frac{dP}{dq} - Q(q) \right) \\
&= \frac{1}{240^2} \sum_{n=1}^{\infty} (\sigma(n) + (-20 + 60n)\sigma_3(n) - 42\sigma_5(n) \\
&\quad + (20 - 30n)\sigma_7(n) + 11\sigma_9(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.9).

Next we prove (1.7.10). By (1.1.1), (1.1.2), (1.1.3), (1.1.4), (1.1.5), (1.5.1), (1.5.3) and (1.5.26), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3=n} \sigma(m_1)\sigma(m_2)\sigma_5(m_3)q^n \\
&= \left(\frac{1-L(q)}{24} \right)^2 \left(\frac{1-N(q)}{504} \right) \\
&= \frac{1}{(504)24^2} (1 - 2L(q) + L^2(q) - N(q) + 2L(q)N(q) - L^2(q)N(q)) \\
&= \frac{1}{(504)24^2} \left(1 - 2L(q) + 12q \frac{dL}{dq} + M(q) - N(q) + 4q \frac{dN}{dq} \right. \\
&\quad \left. - \frac{24}{7}q^2 \frac{d^2N}{dq^2} + 2P(q) - 3q \frac{dP}{dq} - Q(q) \right) \\
&= \frac{1}{12096} \sum_{n=1}^{\infty} ((2 - 12n)\sigma(n) + 10\sigma_3(n) + (21 - 84n + 72n^2)\sigma_5(n) \\
&\quad + (40 - 60n)\sigma_7(n) + 11\sigma_9(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.10).

Next we prove (1.7.11). By (1.1.2), (1.1.3), (1.1.4), (1.1.5), (1.1.7), (1.5.6), (1.5.7) and (1.5.27), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3=n} \sigma_3(m_1)\sigma_3(m_2)\sigma_5(m_3)q^n \\
&= \left(\frac{M(q)-1}{240} \right)^2 \left(\frac{1-N(q)}{504} \right) \\
&= \frac{1}{(504)240^2} (1 - 2M(q) + M^2(q) - N(q) + 2N(q)M(q) - N(q)M^2(q)) \\
&= \frac{1}{(504)240^2} (1 - 2M(q) - N(q) + P(q) - 2Q(q) - S(q)) \\
&= \frac{1}{1209600} \sum_{n=1}^{\infty} (-20\sigma_3(n) + 21\sigma_5(n) + 20\sigma_7(n) - 22\sigma_9(n) + \sigma_{13}(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.11).

Next we prove (1.7.12). By (1.1.1), (1.1.2), (1.1.3), (1.1.4), (1.5.1), (1.5.23) and (1.5.28), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3+m_4=n} \sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma(m_4)q^n \\
&= \left(\frac{1-L(q)}{24} \right)^4 \\
&= \frac{1}{24^4} (1 - 4L(q) + 6L^2(q) - 4L^3(q) + L^4(q)) \\
&= \frac{1}{24^4} \left(1 - 4L(q) + 72q \frac{dL}{dq} - 288q^2 \frac{d^2L}{dq^2} + 288q^3 \frac{d^3L}{dq^3} + 6M(q) \right. \\
&\quad \left. - 36q \frac{dM}{dq} + \frac{216}{5}q^2 \frac{d^2M}{dq^2} - 4N(q) + 8q \frac{dN}{dq} + P(q) \right) \\
&= \frac{1}{3456} \sum_{n=1}^{\infty} ((1 - 18n + 72n^2 - 72n^3)\sigma(n) + (15 - 90n + 108n^2)\sigma_3(n) \\
&\quad + (21 - 42n)\sigma_5(n) + 5\sigma_7(n)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.12).

Next we prove (1.7.13). By (1.1.1), (1.1.2), (1.1.3), (1.1.4), (1.1.5), (1.5.1), (1.5.23), (1.5.2), (1.5.24) and (1.5.29), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3+m_4=n} \sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma(m_4)q^n \\
&= \left(\frac{1-L(q)}{24} \right)^3 \left(\frac{M(q)-1}{240} \right) \\
&= \frac{1}{(240)24^3} (-1 + 3L(q) - 3L^2(q) + L^3(q) + M(q) - 3L(q)M(q) \\
&\quad + 3L^2(q)M(q) - L^3(q)M(q)) \\
&= \frac{1}{(240)24^3} \left(-1 + 3L(q) - 36q \frac{dL}{dq} + 72q^2 \frac{d^2L}{dq^2} - 2M(q) \right. \\
&\quad \left. + \frac{108}{5}q^2 \frac{d^2M}{dq^2} - \frac{72}{5}q^3 \frac{d^3M}{dq^3} - 2N(q) \right. \\
&\quad \left. + 12q \frac{dN}{dq} - \frac{72}{7}q^2 \frac{d^2N}{dq^2} + 3P(q) - \frac{9}{2}q \frac{dP}{dq} - Q(q) \right) \\
&= \frac{1}{138240} \sum_{n=1}^{\infty} \left((-3 + 36n - 72n^2)\sigma(n) + (-20 + 216n^2 - 144n^3)\sigma_2(n) \right. \\
&\quad \left. + (41 - 252n + 216n^2)\sigma_5(n) + (60 - 90n)\sigma_7(n) + 11\sigma_9(n) \right) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.13).

Next we prove (1.7.14). By (1.1.1), (1.1.2), (1.1.3), (1.1.4), (1.1.5), (1.5.1), (1.5.23), (1.5.28) and (1.5.30), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m_1+m_2+m_3+m_4+m_5=n} \sigma(m_1)\sigma(m_2)\sigma(m_3)\sigma(m_4)\sigma(m_5)q^n \\
&= \left(\frac{1-L(q)}{24} \right)^5 \\
&= \frac{1}{24^5} (1 - 5L(q) + 10L^2(q) - 10L^3(q) + 5L^4(q) - L^5(q))
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{24^5} \left(1 - 5L(q) + 120q \frac{dL}{dq} - 720q^2 \frac{d^2L}{dq^2} + 1440q^3 \frac{d^3L}{dq^3} - 864q^4 \frac{d^4L}{dq^4} \right. \\
&\quad + 10M(q) - 90q \frac{dM}{dq} + 216q^2 \frac{d^2M}{dq^2} - 144q^3 \frac{d^3M}{dq^3} - 10N(q) \\
&\quad \left. + 40q \frac{dN}{dq} - \frac{240}{7}q^2 \frac{d^2N}{dq^2} + 5P(q) - \frac{15}{2}q \frac{dP}{dq} - Q(q) \right) \\
&= \frac{1}{24^4} \sum_{n=1}^{\infty} \left((5 - 120n + 720n^2 - 1440n^3 + 864n^4)\sigma(n) \right. \\
&\quad + (100 - 900n + 2160n^2 - 1440n^3)\sigma_3(n) \\
&\quad + (210 - 840n + 720n^2)\sigma_5(n) \\
&\quad \left. + (100 - 150n)\sigma_7(n) + 11\sigma_9(n) \right) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (1.7.14). $\square$

We observe that formulae (1.7.1) and (1.7.5) can also be obtained from previously presented identities. Let

$$S = \sum_{m < n} m\sigma(m)\sigma(n-m).$$

Changing the summation variable from m to $n-m$, we obtain

$$S = \sum_{m < n} (n-m)\sigma(m)\sigma(n-m).$$

Adding both equations together, we have

$$2S = n \sum_{m < n} \sigma(m)\sigma(n-m).$$

By (1.6.1), we obtain (1.7.1).

Now let

$$T = \sum_{m < n} m\sigma_3(m)\sigma_3(n-m).$$

Changing m to $n - m$, we obtain

$$T = \sum_{m < n} (n - m) \sigma_3(m) \sigma_3(n - m).$$

Adding both equations together, we have

$$2T = n \sum_{m < n} \sigma_3(m) \sigma_3(n - m).$$

By (1.6.6), we obtain (1.7.5).

Chapter 2

The functions $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$.

2.1 Obtaining a formula by duplication.

Let x , q and w be as defined in Section 1.2. Suppose that

$$\Omega(x, q, w) = 0$$

is a relationship between x , q and w . Then it is shown in Berndt [3, p. 125] that

$$\Omega\left(\left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2, q^2, \frac{1}{2}w(1 + \sqrt{1-x})\right) = 0$$

also holds. The second formula is said to be obtained from the first formula by duplication since q is changed to q^2 .

2.2 Obtaining a formula by change of sign.

Berndt [3, p. 126] also describes the process of obtaining a new formula from a given formula by changing the sign of q , which is due to Jacobi. Suppose a relationship of the form

$$\Omega(x, q, w) = 0$$

holds, where x , q and w are as defined in Section 1.2. Then

$$\Omega\left(\frac{x}{x-1}, -q, w\sqrt{1-x}\right) = 0$$

holds.

2.3 Formulae for $L(q^2)$, $L(q^4)$, $M(q^2)$, $M(q^4)$, ..., $S(q^2)$, $S(q^4)$.

In this section we obtain, by the process of duplication, formulae for $L(q^2)$, $L(q^4)$, $M(q^2)$, $M(q^4)$, ..., $S(q^2)$, $S(q^4)$ in terms of x . Formulae (2.3.1)-(2.3.6) can be found in [3, pp. 126-129]

Theorem 2.3.1

$$(2.3.1) \quad L(q^2) = (1-2x)w^2 + 6x(1-x)w\frac{dw}{dx},$$

$$(2.3.2) \quad L(q^4) = \left(1 - \frac{5}{4}x\right)w^2 + 3x(1-x)w\frac{dw}{dx},$$

$$(2.3.3) \quad M(q^2) = (1-x+x^2)w^4,$$

$$(2.3.4) \quad M(q^4) = (1 - x + \frac{1}{16}x^2)w^4,$$

$$(2.3.5) \quad N(q^2) = (1 + x)(1 - \frac{1}{2}x)(1 - 2x)w^6,$$

$$(2.3.6) \quad N(q^4) = (1 - \frac{1}{2}x)(1 - x - \frac{1}{32}x^2)w^6,$$

$$(2.3.7) \quad P(q^2) = (1 - x + x^2)^2w^8,$$

$$(2.3.8) \quad P(q^4) = (1 - x + \frac{1}{16}x^2)^2w^8,$$

$$(2.3.9) \quad Q(q^2) = (1 + x)(1 - \frac{1}{2}x)(1 - 2x)(1 - x + x^2)w^{10},$$

$$(2.3.10) \quad Q(q^4) = (1 - \frac{1}{2}x)(1 - x - \frac{1}{32}x^2)(1 - x + \frac{1}{16}x^2)w^{10},$$

$$(2.3.11) \quad R(q^2) = \frac{441}{691}(1 - x + x^2)^3w^{12} \\ + \frac{250}{691}(1 + x)^2(1 - \frac{1}{2}x)^2(1 - 2x)^2w^{12},$$

$$(2.3.12) \quad R(q^4) = \frac{441}{691}(1 - x + \frac{1}{16}x^2)^3w^{12} \\ + \frac{250}{691}(1 - \frac{1}{2}x)^2(1 - x - \frac{1}{32}x^2)^2w^{12},$$

$$(2.3.13) \quad S(q^2) = (1 + x)(1 - \frac{1}{2}x)(1 - 2x)(1 - x + x^2)^2w^{14},$$

$$(2.3.14) \quad S(q^4) = (1 - \frac{1}{2}x)(1 - x - \frac{1}{32}x^2)(1 - x + \frac{1}{16}x^2)^2w^{14}.$$

Proof Under duplication we have

$$q \rightarrow q^2,$$

$$x \rightarrow x' = \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2,$$

$$w \rightarrow w' = \frac{1}{2}w(1 + \sqrt{1-x}).$$

Thus

$$\begin{aligned} \frac{dw}{dx} \rightarrow \frac{dw'}{dx'} &= \frac{\frac{dw'}{dx}}{\frac{dx'}{dx}} = \frac{\frac{1 + \sqrt{1-x}}{2} \frac{dw}{dx} - \frac{w}{4\sqrt{1-x}}}{\frac{2(1 - \sqrt{1-x})}{\sqrt{1-x}(1 + \sqrt{1-x})^3}} \\ &= \frac{(1 + \sqrt{1-x})^4(\sqrt{1-x})}{4(1 - \sqrt{1-x})} \frac{dw}{dx} - \frac{w(1 + \sqrt{1-x})^3}{8(1 - \sqrt{1-x})}. \end{aligned}$$

First we prove (2.3.1). By applying duplication to (1.2.9), we obtain

$$\begin{aligned} L(q^2) &= \left(1 - 5 \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2\right) \left(\frac{w^2}{4} (1 + \sqrt{1-x})^2\right) \\ &\quad + 6 \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2 \left(1 - \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2\right) (1 + \sqrt{1-x}) w \frac{dw'}{dx'} \\ &= \frac{1}{4} (1 + \sqrt{1-x})^2 w^2 - \frac{5}{4} (1 - \sqrt{1-x})^2 w^2 \\ &\quad + 24w \frac{(1 - \sqrt{1-x})^2 \sqrt{1-x}}{(1 + \sqrt{1-x})^3} \\ &\quad \left(\frac{(1 + \sqrt{1-x})^4(\sqrt{1-x})}{4(1 - \sqrt{1-x})} \frac{dw}{dx} - \frac{w(1 + \sqrt{1-x})^3}{8(1 - \sqrt{1-x})}\right) \\ &= (1 - 2x)w^2 + 6x(1 - x)w \frac{dw}{dx}. \end{aligned}$$

Next we prove (2.3.2). Applying duplication to (2.3.1), we obtain

$$\begin{aligned} L(q^4) &= \left(1 - 2 \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2\right) \left(\frac{w^2}{4} (1 + \sqrt{1-x})^2\right) \\ &\quad + 3 \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2 \left(1 - \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}}\right)^2\right) (1 + \sqrt{1-x}) w \frac{dw'}{dx'}, \\ &= \frac{1}{4} (1 + \sqrt{1-x})^2 w^2 - \frac{1}{2} (1 - \sqrt{1-x})^2 w^2 \\ &\quad + 12w \frac{(1 - \sqrt{1-x})^2 \sqrt{1-x}}{(1 + \sqrt{1-x})^3} \end{aligned}$$

$$\begin{aligned} & \left(\frac{(1 + \sqrt{1-x})^4(\sqrt{1-x})}{4(1 - \sqrt{1-x})} \frac{dw}{dx} - \frac{w(1 + \sqrt{1-x})^3}{8(1 - \sqrt{1-x})} \right) \\ &= \left(1 - \frac{5}{4}x \right) w^2 + 3x(1-x)w \frac{dw}{dx}. \end{aligned}$$

Next we prove (2.3.3). Applying duplication to (1.2.10), we obtain

$$\begin{aligned} M(q^2) &= \frac{1}{16} w^4 (1 + \sqrt{1-x})^4 \left(1 + 14 \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 + \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^4 \right) \\ &= (1 - x + x^2) w^4. \end{aligned}$$

Next we prove (2.3.4). Applying duplication to (2.3.3), we obtain

$$\begin{aligned} M(q^4) &= \frac{1}{16} w^4 (1 + \sqrt{1-x})^4 \left(1 - \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 + \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^4 \right) \\ &= \left(1 - x + \frac{1}{16} x^2 \right) w^4. \end{aligned}$$

Next we prove (2.3.5). Applying duplication to (1.2.11), we obtain

$$\begin{aligned} N(q^2) &= \frac{1}{64} w^6 (1 + \sqrt{1-x})^6 \left(1 + \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 \right) \\ &\quad \left(1 - 34 \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 + \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^4 \right) \\ &= (1+x) \left(1 - \frac{1}{2}x \right) (1-2x) w^6. \end{aligned}$$

Next we prove (2.3.6). Applying duplication to (2.3.5), we obtain

$$\begin{aligned} N(q^4) &= \frac{1}{64} w^6 (1 + \sqrt{1-x})^6 \left(1 + \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 \right) \\ &\quad \left(1 - \frac{1}{2} \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 \right) \left(1 - 2 \left(\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} \right)^2 \right) \\ &= \left(1 - \frac{1}{2}x \right) \left(1 - x - \frac{1}{32}x^2 \right) w^6. \end{aligned}$$

Next we prove (2.3.7). By (1.5.6) and (2.3.3), we have

$$P(q^2) = M^2(q^2) = (1 - x + x^2)^2 w^8.$$

Next we prove (2.3.8). By (1.5.6) and (2.3.4), we have

$$P(q^4) = M^2(q^4) = \left(1 - x + \frac{1}{16}x^2\right)^2 w^8.$$

Next we prove (2.3.9). By (1.5.7), (2.3.3) and (2.3.5), we have

$$Q(q^2) = M(q^2)N(q^2) = (1 + x) \left(1 - \frac{1}{2}x\right) (1 - 2x)(1 - x + x^2)w^{10}.$$

Next we prove (2.3.10). By (1.5.7), (2.3.4) and (2.3.6), we have

$$Q(q^4) = M(q^4)N(q^4) = \left(1 - \frac{1}{2}x\right) \left(1 - x - \frac{1}{32}x^2\right) \left(1 - x + \frac{1}{16}x^2\right) w^{10}.$$

Next we prove (2.3.11). By (1.2.18), (2.3.3), (2.3.5) and (2.3.7), we have

$$\begin{aligned} R(q^2) &= \frac{441}{691}M(q^2)P(q^2) + \frac{250}{691}N^2(q^2) \\ &= \frac{441}{691}(1 - x + x^2)^3 w^{12} + \frac{250}{691}(1 + x)^2 \left(1 - \frac{1}{2}x\right)^2 (1 - 2x)^2 w^{12}. \end{aligned}$$

Next we prove (2.3.12). By (1.2.18), (2.3.4), (2.3.6) and (2.3.8), we have

$$\begin{aligned} R(q^4) &= \frac{441}{691}M(q^4)P(q^4) + \frac{250}{691}N^2(q^4) \\ &= \frac{441}{691} \left(1 - x + \frac{1}{16}x^2\right)^3 w^{12} + \frac{250}{691} \left(1 - \frac{1}{2}x\right)^2 \left(1 - x - \frac{1}{32}x^2\right)^2 w^{12}. \end{aligned}$$

Next we prove (2.3.13). By (1.5.8), (2.3.3) and (2.3.9), we have

$$S(q^2) = M(q^2)Q(q^2) = (1 + x) \left(1 - \frac{1}{2}x\right) (1 - 2x)(1 - x + x^2)^2 w^{14}.$$

Lastly, we prove (2.3.14). By (1.5.8), (2.3.4) and (2.3.10), we have

$$S(q^4) = M(q^4)Q(q^4) = \left(1 - \frac{1}{2}x\right) \left(1 - x - \frac{1}{32}x^2\right) \left(1 - x + \frac{1}{16}x^2\right)^2 w^{14}.$$

This completes the proof of Theorem 2.3.1. $\square$

Relationships between $L(q)$, $M(q)$, $N(q)$, $L(q^2)$, $M(q^2)$, $N(q^2)$, $L(q^4)$, $M(q^4)$, $N(q^4)$ are given in Theorem 2.8.1, see (2.8.13)-(2.8.16).

2.4 Formulae for $L(-q)$, $M(-q)$, ..., $S(-q)$.

In this section we obtain formulae for $L(-q)$, $M(-q)$, ..., $S(-q)$ by applying Jacobi's change of sign formula to the formulae for $L(q)$, $M(q)$, ..., $S(q)$ given in Theorems 1.2.1 and 1.2.2.

Theorem 2.4.1

$$(2.4.1) \quad L(-q) = (1 - 2x)w^2 + 12x(1 - x)w \frac{dw}{dx},$$

$$(2.4.2) \quad M(-q) = (1 - 16x + 16x^2)w^4,$$

$$(2.4.3) \quad N(-q) = (1 - 2x)(1 + 32x - 32x^2)w^6,$$

$$(2.4.4) \quad P(-q) = (1 - 16x + 16x^2)^2 w^8,$$

$$(2.4.5) \quad Q(-q) = (1 - 2x)(1 - 16x + 16x^2)(1 + 32x - 32x^2)w^{10},$$

$$(2.4.6) \quad R(-q) = \frac{441}{691}(1 - 16x + 16x^2)^3 w^{12} \\ + \frac{250}{691}(1 - 2x)^2(1 + 32x - 32x^2)^2 w^{12},$$

$$(2.4.7) \quad S(-q) = (1 - 2x)(1 + 32x - 32x^2)(1 - 16x + 16x^2)^2 w^{14}.$$

Proof Under Jacobi's change of sign, we have

$$\begin{aligned} q &\rightarrow -q, \\ x &\rightarrow x' = \frac{x}{x-1}, \\ w &\rightarrow w' = w\sqrt{1-x}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{dw}{dx} \rightarrow \frac{dw'}{dx'} &= \frac{\frac{dw'}{dx}}{\frac{dx'}{dx}} = \frac{\sqrt{1-x} \frac{dw}{dx} - \frac{w}{2} \frac{1}{\sqrt{1-x}}}{\frac{-1}{(1-x)^2}} \\ &= \frac{1}{2} w (\sqrt{1-x})^3 - (\sqrt{1-x})^5 \frac{dw}{dx}. \end{aligned}$$

First we prove (2.4.1). Applying Jacobi's change of sign formula to (1.2.9), we obtain

$$\begin{aligned} L(-q) &= \left(1 - 5 \left(\frac{x}{x-1}\right) (1-x)w^2\right) - \frac{12xw}{(\sqrt{1-x})^3} \frac{dw'}{dx'} \\ &= (1+4x)w^2 - \frac{12xw}{(\sqrt{1-x})^3} \left(\frac{1}{2}w(\sqrt{1-x})^3 - (\sqrt{1-x})^5 \frac{dw}{dx}\right) \\ &= (1-2x)w^2 + 12x(1-x)w \frac{dw}{dx}. \end{aligned}$$

Next we prove (2.4.2). Applying Jacobi's change of sign formula to (1.2.10), we obtain

$$\begin{aligned} M(-q) &= w^4(1-x)^2 \left(1 + 14 \frac{x}{x-1} + \frac{x^2}{(1-x)^2}\right) \\ &= (1-16x+16x^2)w^4. \end{aligned}$$

Next we prove (2.4.3). Applying Jacobi's change of sign formula to (1.2.11), we obtain

$$\begin{aligned} N(-q) &= w^6(1-x)^3 \left(1 + \frac{x}{x-1}\right) \left(1 - 34\frac{x}{x-1} + \frac{x^2}{(1-x)^2}\right) \\ &= (1-2x)(1+32x-32x^2)w^6. \end{aligned}$$

Next we prove (2.4.4). By (1.5.6) and (2.4.2), we have

$$P(-q) = M^2(-q) = (1-16x+16x^2)^2w^8.$$

Next we prove (2.4.5). By (1.5.7), (2.4.2) and (2.4.3), we have

$$Q(-q) = M(-q)N(-q) = (1-2x)(1+32x-32x^2)(1-16x+16x^2)w^{10}.$$

Next we prove (2.4.6). By (1.2.18), (2.4.2), (2.4.3) and (2.4.4), we have

$$\begin{aligned} R(-q) &= \frac{441}{691}M(-q)P(-q) + \frac{250}{691}N^2(-q) \\ &= \frac{441}{691}(1-16x+16x^2)^3w^{12} \\ &\quad + \frac{250}{691}(1-2x)^2(1+32x-32x^2)^2w^{12}. \end{aligned}$$

Lastly, we prove (2.4.7). By (1.5.8), (2.4.2) and (2.4.5), we have

$$\begin{aligned} S(-q) &= M(-q)Q(-q) \\ &= (1-2x)(1+32x-32x^2)(1-16x+16x^2)^2w^{14}. \end{aligned}$$

This completes the proof of Theorem 2.4.1. □

2.5 The functions $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$.

In this section we define for $|q| < 1$ the functions

$$(2.5.1) \quad L_{0,2}(q) = \sum_{n=1}^{\infty} \sigma(2n)q^{2n},$$

$$(2.5.2) \quad L_{1,2}(q) = \sum_{n=1}^{\infty} \sigma(2n-1)q^{2n-1},$$

$$(2.5.3) \quad M_{0,2}(q) = \sum_{n=1}^{\infty} \sigma_3(2n)q^{2n},$$

$$(2.5.4) \quad M_{1,2}(q) = \sum_{n=1}^{\infty} \sigma_3(2n-1)q^{2n-1},$$

$$(2.5.5) \quad N_{0,2}(q) = \sum_{n=1}^{\infty} \sigma_5(2n)q^{2n},$$

$$(2.5.6) \quad N_{1,2}(q) = \sum_{n=1}^{\infty} \sigma_5(2n-1)q^{2n-1},$$

$$(2.5.7) \quad P_{0,2}(q) = \sum_{n=1}^{\infty} \sigma_7(2n)q^{2n},$$

$$(2.5.8) \quad P_{1,2}(q) = \sum_{n=1}^{\infty} \sigma_7(2n-1)q^{2n-1},$$

$$(2.5.9) \quad Q_{0,2}(q) = \sum_{n=1}^{\infty} \sigma_9(2n)q^{2n},$$

$$(2.5.10) \quad Q_{1,2}(q) = \sum_{n=1}^{\infty} \sigma_9(2n-1)q^{2n-1},$$

$$(2.5.11) \quad R_{0,2}(q) = \sum_{n=1}^{\infty} \sigma_{11}(2n)q^{2n},$$

$$(2.5.12) \quad R_{1,2}(q) = \sum_{n=1}^{\infty} \sigma_{11}(2n-1)q^{2n-1},$$

$$(2.5.13) \quad S_{0,2}(q) = \sum_{n=1}^{\infty} \sigma_{13}(2n)q^{2n},$$

$$(2.5.14) \quad S_{1,2}(q) = \sum_{n=1}^{\infty} \sigma_{13}(2n-1)q^{2n-1}.$$

These functions are the even and odd parts of the sums in $L(q)$, $M(q)$, ..., $S(q)$. We now obtain representations of $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$ in terms of x .

Theorem 2.5.1

$$(2.5.15) \quad L_{0,2}(q) = \frac{1}{48} \left(2 - (2 - 7x)w^2 - 24x(1-x)w \frac{dw}{dx} \right),$$

$$(2.5.16) \quad L_{1,2}(q) = \frac{xw^2}{16},$$

$$(2.5.17) \quad M_{0,2}(q) = \frac{1}{480}(-2 + (2 - 2x + 17x^2)w^4),$$

$$(2.5.18) \quad M_{1,2}(q) = \frac{1}{32}(2 - x)xw^4,$$

$$(2.5.19) \quad N_{0,2}(q) = \frac{1}{1008}(2 + (-2 + 3x + 129x^2 - 65x^3)w^6),$$

$$(2.5.20) \quad N_{1,2}(q) = \frac{1}{16}(1 - x + x^2)xw^6,$$

$$(2.5.21) \quad P_{0,2}(q) = \frac{1}{960}(-2 + (2 - 4x + 486x^2 - 484x^3 + 257x^4)w^8),$$

$$(2.5.22) \quad P_{1,2}(q) = \frac{1}{192}(12 - 18x + 108x^2 - 51x^3)xw^8,$$

$$(2.5.23) \quad Q_{0,2}(q) = \frac{1}{528}(2 - (2 - 5x - 1054x^2 + 1586x^3 - 2579x^4 + 1025x^5)w^{10}),$$

$$(2.5.24) \quad Q_{1,2}(q) = \frac{1}{528}(33 - 66x + 2574x^2 - 2541x^3 + 1023x^4)xw^{10},$$

$$(2.5.25) \quad R_{0,2}(q) = \frac{1}{131040}(-1382 + (1382 - 4146x + 1053237x^2 - 2099564x^3 + 9538077x^4 - 8488986x^5 + 2831027x^6)w^{12}),$$

$$(2.5.26) \quad R_{1,2}(q) = \frac{1}{32}(2 - 5x + 1388x^2 - 2077x^3 + 2074x^4 - 691x^5)xw^{12},$$

$$(2.5.27) \quad S_{0,2}(q) = \frac{1}{48}(2 - (2 - 7x - 1527x^2 + 3835x^3 - 52229x^4 + 74505x^5 - 57349x^6 + 16385x^7)w^{14}),$$

$$(2.5.28) \quad S_{1,2}(q) = \frac{1}{16}(1 - 3x + 6231x^2 - 12457x^3 + 25341x^4 - 19113x^5 + 5461x^6)xw^{14}.$$

Proof First we prove (2.5.15) and (2.5.16). It is clear from (1.1.1) that

$$\begin{aligned} L_{0,2}(q) + L_{1,2}(q) &= \frac{1 - L(q)}{24}, \\ L_{0,2}(q) - L_{1,2}(q) &= \frac{1 - L(-q)}{24}. \end{aligned}$$

Solving for $L_{0,2}(q)$ and $L_{1,2}(q)$, we obtain

$$\begin{aligned} L_{0,2}(q) &= \frac{2 - L(q) - L(-q)}{48}, \\ L_{1,2}(q) &= \frac{L(-q) - L(q)}{48}. \end{aligned}$$

By (1.2.9) and (2.4.1), we obtain

$$\begin{aligned} L_{0,2}(q) &= \frac{1}{48} \left(2 - (2 - 7x)w^2 - 24x(1 - x)w \frac{dw}{dx} \right), \\ L_{1,2}(q) &= \frac{xw^2}{16}. \end{aligned}$$

Next we prove (2.5.17) and (2.5.18). From (1.1.2), we have

$$\begin{aligned} M_{0,2}(q) + M_{1,2}(q) &= \frac{M(q) - 1}{240}, \\ M_{0,2}(q) - M_{1,2}(q) &= \frac{M(-q) - 1}{240}. \end{aligned}$$

Solving for $M_{0,2}(q)$ and $M_{1,2}(q)$, we obtain

$$\begin{aligned} M_{0,2}(q) &= \frac{M(q) + M(-q) - 2}{480}, \\ M_{1,2}(q) &= \frac{M(q) - M(-q)}{480}. \end{aligned}$$

By (1.2.10) and (2.4.2), we obtain

$$\begin{aligned} M_{0,2}(q) &= \frac{1}{480}(-2 + (2 - 2x + 17x^2)w^4), \\ M_{1,2}(q) &= \frac{1}{32}(2 - x)xw^4. \end{aligned}$$

Next we prove (2.5.19) and (2.5.20). From (1.1.3), we have

$$\begin{aligned} N_{0,2}(q) + N_{1,2}(q) &= \frac{1 - N(q)}{504}, \\ N_{0,2}(q) - N_{1,2}(q) &= \frac{1 - N(-q)}{504}. \end{aligned}$$

Solving for $N_{0,2}(q)$ and $N_{1,2}(q)$, we obtain

$$\begin{aligned} N_{0,2}(q) &= \frac{2 - N(q) - N(-q)}{1008}, \\ N_{1,2}(q) &= \frac{N(-q) - N(q)}{1008}. \end{aligned}$$

By (1.2.11) and (2.4.3), we obtain

$$\begin{aligned} N_{0,2}(q) &= \frac{1}{1008}(2 + (-2 + 3x + 129x^2 - 65x^3)w^6), \\ N_{1,2}(q) &= \frac{1}{16}(1 - x + x^2)xw^6. \end{aligned}$$

Next we prove (2.5.21) and (2.5.22). From (1.1.4), we have

$$\begin{aligned} P_{0,2}(q) + P_{1,2}(q) &= \frac{P(q) - 1}{480}, \\ P_{0,2}(q) - P_{1,2}(q) &= \frac{P(-q) - 1}{480}. \end{aligned}$$

Solving for $P_{0,2}(q)$ and $P_{1,2}(q)$, we obtain

$$\begin{aligned} P_{0,2}(q) &= \frac{P(q) + P(-q) - 2}{960}, \\ P_{1,2}(q) &= \frac{P(q) - P(-q)}{960}. \end{aligned}$$

By (1.2.20) and (2.4.4), we obtain

$$\begin{aligned} P_{0,2}(q) &= \frac{1}{960}(-2 + (2 - 4x + 486x^2 - 484x^3 + 257x^4)w^8), \\ P_{1,2}(q) &= \frac{1}{192}(12 - 18x + 108x^2 - 51x^3)xw^8. \end{aligned}$$

Next we prove (2.5.23) and (2.5.24). From (1.1.5), we have

$$\begin{aligned} Q_{0,2}(q) + Q_{1,2}(q) &= \frac{1 - Q(q)}{264}, \\ Q_{0,2}(q) - Q_{1,2}(q) &= \frac{1 - Q(-q)}{264}. \end{aligned}$$

Solving for $Q_{0,2}(q)$ and $Q_{1,2}(q)$, we obtain

$$\begin{aligned} Q_{0,2}(q) &= \frac{2 - Q(q) - Q(-q)}{528}, \\ Q_{1,2}(q) &= \frac{Q(-q) - Q(q)}{528}. \end{aligned}$$

By (1.2.21) and (2.4.5), we obtain

$$\begin{aligned} Q_{0,2}(q) &= \frac{1}{528}(2 - (2 - 5x - 1054x^2 + 1586x^3 - 2579x^4 + 1025x^5)w^{10}), \\ Q_{1,2}(q) &= \frac{1}{528}(33 - 66x + 2574x^2 - 2541x^3 + 1023x^4)xw^{10}. \end{aligned}$$

Next we prove (2.5.25) and (2.5.26). From (1.1.6), we have

$$\begin{aligned} R_{0,2}(q) + R_{1,2}(q) &= \frac{691(R(q) - 1)}{65520}, \\ R_{0,2}(q) - R_{1,2}(q) &= \frac{691(R(-q) - 1)}{65520}. \end{aligned}$$

Solving for $R_{0,2}(q)$ and $R_{1,2}(q)$, we obtain

$$\begin{aligned} R_{0,2}(q) &= \frac{691(R(q) + R(-q) - 2)}{131040}, \\ R_{1,2}(q) &= \frac{691(R(q) - R(-q))}{131040}. \end{aligned}$$

By (1.2.22) and (2.4.6), we obtain

$$\begin{aligned} R_{0,2}(q) &= \frac{1}{131040}(-1382 + (1382 - 4146x + 1053237x^2 \\ &\quad - 2099564x^3 + 9538077x^4 - 8488986x^5 + 2831027x^6)w^{12}), \\ R_{1,2}(q) &= \frac{1}{32}(2 - 5x + 1388x^2 - 2077x^3 + 2074x^4 - 691x^5)xw^{12}. \end{aligned}$$

Lastly we prove (2.5.27) and (2.5.28). From (1.1.7), we have

$$\begin{aligned} S_{0,2}(q) + S_{1,2}(q) &= \frac{1 - S(q)}{24}, \\ S_{0,2}(q) - S_{1,2}(q) &= \frac{1 - S(-q)}{24}. \end{aligned}$$

Solving for $S_{0,2}(q)$ and $S_{1,2}(q)$, we obtain

$$\begin{aligned} S_{0,2}(q) &= \frac{2 - S(q) - S(-q)}{48}, \\ S_{1,2}(q) &= \frac{S(-q) - S(q)}{48}. \end{aligned}$$

By (1.2.23) and (2.4.7), we obtain

$$\begin{aligned} S_{0,2}(q) &= \frac{1}{48}(2 - (2 - 7x - 1527x^2 + 3835x^3 - 52229x^4 \\ &\quad + 74505x^5 - 57349x^6 + 16385x^7)w^{14}), \\ S_{1,2}(q) &= \frac{1}{16}(1 - 3x + 6231x^2 - 12457x^3 + 25341x^4 \\ &\quad - 19113x^5 + 5461x^6)xw^{14}. \end{aligned}$$

This completes the proof of Theorem 2.5.1. □

2.6 Evaluation of $\frac{dL(q^2)}{dq}$, $\frac{dL(q^4)}{dq}$, $\frac{dM(q^2)}{dq}$, $\frac{dM(q^4)}{dq}$, ..., $\frac{dS(q^2)}{dq}$, $\frac{dS(q^4)}{dq}$ in terms of x .

In this section, we use Theorem 2.3.1 to determine $\frac{dL(q^2)}{dq}$, $\frac{dL(q^4)}{dq}$, $\frac{dM(q^2)}{dq}$, $\frac{dM(q^4)}{dq}$, ..., $\frac{dS(q^2)}{dq}$, $\frac{dS(q^4)}{dq}$ in terms of x . These determinations are used in Section 2.8 to find relationships between $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$.

Theorem 2.6.1

$$(2.6.1) \quad q \frac{dL(q^2)}{dq} = -\frac{1}{2}x(1-x)w^4 + 2(1-x)(1-2x)xw^3 \frac{dw}{dx} \\ + 6(1-x)^2 x^2 w^2 \left(\frac{dw}{dx} \right)^2,$$

$$(2.6.2) \quad q \frac{dL(q^4)}{dq} = -\frac{1}{2}x(1-x)w^4 + (1-x) \left(2 - \frac{5}{2}x \right) xw^3 \frac{dw}{dx}$$

$$+3(1-x)^2x^2w^2\left(\frac{dw}{dx}\right)^2,$$

$$(2.6.3) \quad q\frac{dM(q^2)}{dq} = -x(1-x)(1-2x)w^6 + 4x(1-x)(1-x+x^2)w^5\frac{dw}{dx},$$

$$(2.6.4) \quad q\frac{dM(q^4)}{dq} = -\frac{1}{8}x(1-x)(8-x)w^6 \\ + \frac{1}{4}x(1-x)(16-16x+x^2)w^5\frac{dw}{dx},$$

$$(2.6.5) \quad q\frac{dN(q^2)}{dq} = 3x(1-x)(1+x)(2-x)(1-2x)w^7\frac{dw}{dx} \\ + \frac{3}{2}x(1-x)(-1-2x+2x^2)w^8,$$

$$(2.6.6) \quad q\frac{dN(q^4)}{dq} = \frac{3}{32}x(1-x)(2-x)(32-32x-x^2)w^7\frac{dw}{dx} \\ + \frac{3}{64}x(1-x)(-32+20x+x^2)w^8,$$

$$(2.6.7) \quad q\frac{dP(q^2)}{dq} = 8x(1-x)(1-x+x^2)^2w^9\frac{dw}{dx} \\ - 2x(1-x)(1-2x)(1-x+x^2)w^{10},$$

$$(2.6.8) \quad q\frac{dP(q^4)}{dq} = \frac{1}{32}x(1-x)(16-16x+x^2)^2w^9\frac{dw}{dx} \\ - \frac{1}{64}x(1-x)(8-x)(16-16x+x^2)w^{10},$$

$$(2.6.9) \quad q\frac{dQ(q^2)}{dq} = 5x(1-x)(2-x)(1-2x)(1+x)(1-x+x^2)w^{11}\frac{dw}{dx} \\ + \frac{1}{2}x(1-x)(-5+4x+6x^2-20x^3+10x^4)w^{12}$$

$$(2.6.10) \quad q\frac{dQ(q^4)}{dq} = \frac{5}{512}x(1-x)(2-x)(16-16x+x^2) \\ (32-32x-x^2)w^{11}\frac{dw}{dx} \\ + \frac{1}{1024}x(1-x) \\ (-2560+4160x-1680x^2+56x^3+5x^4)w^{12},$$

$$\begin{aligned}
(2.6.11) \quad q \frac{dR(q^2)}{dq} &= -\frac{6}{691}x(1-x)(1382 - 4146x + 4917x^2 \\
&\quad - 2924x^3 + 4917x^4 - 4146x^5 + 1382x^6)w^{13}\frac{dw}{dx} \\
&\quad - \frac{3}{691}x(1-x)(1-2x) \\
&\quad (691 - 257x + 948x^2 - 1382x^3 + 691x^4)w^{14},
\end{aligned}$$

$$\begin{aligned}
(2.6.12) \quad q \frac{dR(q^4)}{dq} &= \frac{3}{707584}x(1-x)(2830336 - 8491008x + 9021696x^2 \\
&\quad - 3891712x^3 + 536856x^4 - 6168x^5 + 691x^6)w^{13}\frac{dw}{dx} \\
&\quad + \frac{3}{1415168}x(1-x)(-1415168 + 3007232x \\
&\quad - 1945856x^2 + 357904x^3 - 5140x^4 + 691x^5)w^{14},
\end{aligned}$$

$$\begin{aligned}
(2.6.13) \quad q \frac{dS(q^2)}{dq} &= 7x(1-x)(1-x+x^2)^2(2-x) \\
&\quad (1-2x)(1+x)w^{15}\frac{dw}{dx} \\
&\quad + \frac{1}{2}x(1-x)(1-x+x^2) \\
&\quad (-7 + 11x + 3x^2 - 28x^3 + 14x^4)w^{16},
\end{aligned}$$

$$\begin{aligned}
(2.6.14) \quad q \frac{dS(q^4)}{dq} &= -\frac{7}{8192}x(1-x)(16 - 16x + x^2)^2 \\
&\quad (2-x)(-32 + 32x + x^2)w^{15}\frac{dw}{dx} \\
&\quad + \frac{1}{16384}x(1-x)(16 - 16x + x^2) \\
&\quad (-3584 + 5824x - 2352x^2 + 100x^3 + 7x^4)w^{16}.
\end{aligned}$$

Proof First we prove (2.6.1). By (2.3.1), we have

$$\frac{dL(q^2)}{dx} = \frac{d}{dx} \left((1-2x)w^2 + 6x(1-x)w\frac{dw}{dx} \right)$$

$$= -\frac{1}{2}w^2 + 8(1-2x)w\frac{dw}{dx} + 6x(1-x)\left(\frac{dw}{dx}\right)^2 + 6x(1-x)w\frac{d^2w}{dx^2}.$$

Appealing to Lemma 1.3.3, we obtain

$$\frac{dL(q^2)}{dx} = -\frac{1}{2}w^2 + 2(1-2x)w\frac{dw}{dx} + 6x(1-x)\left(\frac{dw}{dx}\right)^2.$$

Finally, by Lemma 1.3.2, we have

$$\begin{aligned} q\frac{dL(q^2)}{dq} &= q\frac{\frac{dL(q^2)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2\frac{dL(q^2)}{dx} \\ &= -\frac{1}{2}x(1-x)w^4 + 2(1-x)(1-2x)xw^3\frac{dw}{dx} \\ &\quad + 6(1-x)^2x^2w^2\left(\frac{dw}{dx}\right)^2, \end{aligned}$$

as asserted.

Next we prove (2.6.2). By (2.3.2), we have

$$\begin{aligned} \frac{dL(q^4)}{dx} &= \frac{d}{dx} \left(\left(1 - \frac{5}{4}x\right)w^2 + 3x(1-x)w\frac{dw}{dx} \right) \\ &= -\frac{5}{4}w^2 + \left(5 - \frac{17}{2}x\right)w\frac{dw}{dx} + 3x(1-x)\left(\frac{dw}{dx}\right)^2 + 3x(1-x)w\frac{d^2w}{dx^2}. \end{aligned}$$

Appealing to Lemma 1.3.3, we obtain

$$\frac{dL(q^4)}{dx} = -\frac{1}{2}w^2 + \left(2 - \frac{5}{2}x\right)w\frac{dw}{dx} + 3x(1-x)\left(\frac{dw}{dx}\right)^2.$$

Finally, by Lemma 1.3.2, we have

$$q\frac{dL(q^4)}{dq} = q\frac{\frac{dL(q^4)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2\frac{dL(q^4)}{dx}$$

$$\begin{aligned}
&= -\frac{1}{2}x(1-x)w^4 + (1-x) \left(2 - \frac{5}{2}x\right) xw^3 \frac{dw}{dx} \\
&\quad + 3(1-x)^2 x^2 w^2 \left(\frac{dw}{dx}\right)^2,
\end{aligned}$$

as asserted.

Next we prove (2.6.3). By (2.3.3), we have

$$\begin{aligned}
\frac{dM(q^2)}{dx} &= \frac{d}{dx}((1-x+x^2)w^4) \\
&\quad (-1+2x)w^4 + 4(1-x+x^2)w^3 \frac{dw}{dx}.
\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}
q \frac{dM(q^2)}{dq} &= q \frac{\frac{dM(q^2)}{dx}}{\frac{dq}{dx}} = x(1-x)w^2 \frac{dM(q^2)}{dx} \\
&= -x(1-x)(1-2x)w^6 + 4x(1-x)(1-x+x^2)w^5 \frac{dw}{dx},
\end{aligned}$$

as asserted. Next we prove (2.6.4). By (2.3.4), we have

$$\begin{aligned}
\frac{dM(q^4)}{dx} &= \frac{d}{dx} \left(\left(1-x + \frac{1}{16}x^2\right) w^4 \right) \\
&= \left(-1 + \frac{1}{8}x\right) w^4 + 4 \left(1-x + \frac{1}{16}x^2\right) w^3 \frac{dw}{dx}.
\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}
q \frac{dM(q^4)}{dq} &= q \frac{\frac{dM(q^4)}{dx}}{\frac{dq}{dx}} = x(1-x)w^2 \frac{dM(q^4)}{dx} \\
&= -\frac{1}{8}x(1-x)(8-x)w^6 + \frac{1}{4}x(1-x)(16-16x+x^2)w^5 \frac{dw}{dx},
\end{aligned}$$

as asserted.

Next we prove (2.6.5). By (2.3.5), we have

$$\begin{aligned} \frac{dN(q^2)}{dx} &= \frac{d}{dx} \left((1+x) \left(1 - \frac{1}{2}x \right) (1-2x)w^6 \right) \\ &\quad \left(-\frac{3}{2} - 3x + 3x^2 \right) w^6 + 6(1+x) \left(1 - \frac{1}{2}x \right) (1-2x)w^5 \frac{dw}{dx}. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dN(q^2)}{dq} &= q \frac{\frac{dN(q^2)}{dx}}{\frac{dq}{dx}} = x(1-x)w^2 \frac{dN(q^2)}{dx} \\ &= 3x(1-x)(1+x)(2-x)(1-2x)w^7 \frac{dw}{dx} \\ &\quad + \frac{3}{2}x(1-x)(-1-2x+2x^2)w^8, \end{aligned}$$

as asserted.

Next we prove (2.6.6). By (2.3.6), we have

$$\begin{aligned} \frac{dN(q^4)}{dx} &= \frac{d}{dx} \left(\left(1 - \frac{1}{2}x \right) \left(1 - x - \frac{1}{32}x^2 \right) w^6 \right) \\ &\quad \frac{3}{64}(-32 + 20x + x^2)w^6 + 6 \left(1 - \frac{1}{2}x \right) \left(1 - x - \frac{1}{32}x^2 \right) w^5 \frac{dw}{dx}. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dN(q^4)}{dq} &= q \frac{\frac{dN(q^4)}{dx}}{\frac{dq}{dx}} = x(1-x)w^2 \frac{dN(q^4)}{dx} \\ &= \frac{3}{32}x(1-x)(2-x)(32-32x-x^2)w^7 \frac{dw}{dx} \\ &\quad + \frac{3}{64}x(1-x)(-32+20x+x^2)w^8, \end{aligned}$$

as asserted.

Next we prove (2.6.7). By (2.3.7), we have

$$\begin{aligned}\frac{dP(q^2)}{dx} &= \frac{d}{dx}((1-x+x^2)^2w^8) \\ &= 2(-1+2x)(1-x+x^2)w^8 + 8(1-x+x^2)^2w^7\frac{dw}{dx}.\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}q\frac{dP(q^2)}{dq} &= q\frac{\frac{dP(q^2)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2\frac{dP(q^2)}{dx} \\ &= 8x(1-x)(1-x+x^2)^2w^9\frac{dw}{dx} \\ &\quad - 2x(1-x)(1-2x)(1-x+x^2)w^{10},\end{aligned}$$

as asserted.

Next we prove (2.6.8). By (2.3.8), we have

$$\begin{aligned}\frac{dP(q^4)}{dx} &= \frac{d}{dx}\left(\left(1-x+\frac{1}{16}x^2\right)^2w^8\right) \\ &= \left(-1+\frac{1}{8}x\right)\left(1-x+\frac{1}{16}x^2\right)w^8 + 8\left(1-x+\frac{1}{16}x^2\right)^2w^7\frac{dw}{dx}.\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}q\frac{dP(q^4)}{dq} &= q\frac{\frac{dP(q^4)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2\frac{dP(q^4)}{dx} \\ &= \frac{1}{32}x(1-x)(16-16x+x^2)^2w^9\frac{dw}{dx} \\ &\quad - \frac{1}{64}x(1-x)(8-x)(16-16x+x^2)w^{10},\end{aligned}$$

as asserted.

Next we prove (2.6.9). By (2.3.9), we have

$$\begin{aligned}\frac{dQ(q^2)}{dx} &= \frac{d}{dx} \left((1+x) \left(1 - \frac{1}{2}x \right) (1-2x)(1-x+x^2)w^{10} \right) \\ &= \left(-\frac{5}{2} + 2x + 3x^2 - 10x^3 + 5x^4 \right) w^{10} \\ &\quad + 10(1+x) \left(1 - \frac{1}{2}x \right) (1-2x)(1-x+x^2)w^9 \frac{dw}{dx}.\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}q \frac{dQ(q^2)}{dq} &= q \frac{\frac{dQ(q^2)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2 \frac{dQ(q^2)}{dx} \\ &= 5x(1-x)(2-x)(1-2x)(1+x)(1-x+x^2)w^{11} \frac{dw}{dx} \\ &\quad + \frac{1}{2}x(1-x)(-5+4x+6x^2-20x^3+10x^4)w^{12},\end{aligned}$$

as asserted.

Next we prove (2.6.10). By (2.3.10), we have

$$\begin{aligned}\frac{dQ(q^4)}{dx} &= \frac{d}{dx} \left(\left(1 - \frac{1}{2}x \right) \left(1 - x - \frac{1}{32}x^2 \right) \left(1 - x + \frac{1}{16}x^2 \right) w^{10} \right) \\ &= \left(-\frac{5}{2} + \frac{65}{16}x - \frac{105}{64}x^2 + \frac{7}{128}x^3 + \frac{5}{1024}x^4 \right) w^{10} \\ &\quad + 10 \left(1 - \frac{1}{2}x \right) \left(1 - x - \frac{1}{32}x^2 \right) \left(1 - x + \frac{1}{16}x^2 \right) w^9 \frac{dw}{dx}.\end{aligned}$$

By Lemma 1.3.2, we have

$$q \frac{dQ(q^4)}{dq} = q \frac{\frac{dQ(q^4)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2 \frac{dQ(q^4)}{dx}$$

$$\begin{aligned}
&= \frac{5}{512}x(1-x)(2-x)(16-16x+x^2)(32-32x-x^2)w^{11}\frac{dw}{dx} \\
&\quad + \frac{1}{1024}x(1-x)(-2560+4160x-1680x^2+56x^3+5x^4)w^{12},
\end{aligned}$$

as asserted.

Next we prove (2.6.11). By (2.3.11), we have

$$\begin{aligned}
\frac{dR(q^2)}{dx} &= \frac{d}{dx} \left(\frac{441}{691} (1-x+x^2)^3 w^{12} \right) \\
&\quad + \frac{d}{dx} \left(\frac{250}{691} (1+x)^2 \left(1 - \frac{1}{2}x \right)^2 (1-2x)^2 w^{12} \right) \\
&= \frac{1323}{691} (-1+2x)(1-x+x^2)^2 w^{12} \\
&\quad + \frac{5292}{691} (1-x+x^2)^3 w^{11} \frac{dw}{dx} \\
&\quad - \frac{375}{691} (2-x)(1-2x)(1+x)(1+2x-2x^2) w^{12} \\
&\quad + \frac{3000}{691} (1+x)^2 \left(1 - \frac{1}{2}x \right)^2 (1-2x)^2 w^{11} \frac{dw}{dx} \\
&= -\frac{3}{691} (1-2x)(691-257x+948x^2-1382x^3+691x^4) w^{12} \\
&\quad + \frac{6}{691} (1382-4146x+4917x^2-2924x^3+4917x^4 \\
&\quad -4146x^5+1382x^6) w^{11} \frac{dw}{dx}.
\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}
q \frac{dR(q^2)}{dq} &= q \frac{\frac{dR(q^2)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2 \frac{dR(q^2)}{dx} \\
&= -\frac{6}{691} x(1-x)(1382-4146x+4917x^2 \\
&\quad -2924x^3+4917x^4-4146x^5+1382x^6) w^{13} \frac{dw}{dx}
\end{aligned}$$

$$-\frac{3}{691}x(1-x)(1-2x) \\ (691 - 257x + 948x^2 - 1382x^3 + 691x^4)w^{14},$$

as asserted.

Next we prove (2.6.12). By (2.3.12), we have

$$\begin{aligned} \frac{dR(q^4)}{dx} &= \frac{d}{dx} \left(\frac{441}{691} \left(1 - x + \frac{1}{16}x^2 \right)^3 w^{12} \right) \\ &\quad + \frac{d}{dx} \left(\frac{250}{691} \left(1 - \frac{1}{2}x \right)^2 \left(1 - x - \frac{1}{32}x^2 \right)^2 w^{12} \right) \\ &= \frac{1323}{691} \left(-1 + \frac{1}{8}x \right) \left(1 - x + \frac{1}{16}x^2 \right)^2 w^{12} \\ &\quad + \frac{5292}{691} \left(1 - x + \frac{1}{16}x^2 \right)^3 w^{11} \frac{dw}{dx} \\ &\quad - \frac{375}{707584} (2-x)(32-32x-x^2)(32-20x-x^2)w^{12} \\ &\quad + \frac{3000}{691} \left(1 - \frac{1}{2}x \right)^2 \left(1 - x - \frac{1}{32}x^2 \right)^2 w^{11} \frac{dw}{dx}. \\ &= \frac{3}{1415168} (-1415168 + 3007232x - 1945856x^2 \\ &\quad + 357904x^3 - 5140x^4 + 691x^5)w^{12} \\ &\quad + \frac{3}{707584} (2830336 - 8491008x + 9021696x^2 \\ &\quad - 3891712x^3 + 536856x^4 - 6168x^5 + 691x^6)w^{11} \frac{dw}{dx}. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dR(q^4)}{dq} &= q \frac{\frac{dR(q^4)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2 \frac{dR(q^4)}{dx} \\ &= \frac{3}{707584} x(1-x)(2830336 - 8491008x + 9021696x^2 \end{aligned}$$

$$\begin{aligned}
& -3891712x^3 + 536856x^4 - 6168x^5 + 691x^6)w^{13}\frac{dw}{dx} \\
& + \frac{3}{1415168}x(1-x)(-1415168 + 3007232x - 1945856x^2 \\
& + 357904x^3 - 5140x^4 + 691x^5)w^{14},
\end{aligned}$$

as asserted.

Next we prove (2.6.13). By (2.3.13), we have

$$\begin{aligned}
\frac{dS(q^2)}{dx} &= \frac{d}{dx} \left((1+x) \left(1 - \frac{1}{2}x \right) (1-2x)(1-x+x^2)^2 w^{14} \right) \\
&= -\frac{1}{2}(1-x+x^2)(7-11x-3x^2+28x^3-14x^4)w^{14} \\
&\quad + 14(1+x) \left(1 - \frac{1}{2}x \right) (1-2x)(1-x+x^2)^2 w^{13} \frac{dw}{dx}.
\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}
q \frac{dS(q^2)}{dq} &= q \frac{\frac{dS(q^2)}{dx}}{\frac{dx}{dq}} = x(1-x)w^2 \frac{dS(q^2)}{dx} \\
&= 7x(1-x)(1-x+x^2)^2(2-x) \\
&\quad (1-2x)(1+x)w^{15} \frac{dw}{dx} \\
&\quad + \frac{1}{2}x(1-x)(1-x+x^2) \\
&\quad (-7+11x+3x^2-28x^3+14x^4)w^{16},
\end{aligned}$$

as asserted. Next we prove (2.6.14). By (2.3.14), we have

$$\begin{aligned}
\frac{dS(q^4)}{dx} &= \frac{d}{dx} \left(\left(1 - \frac{1}{2}x \right) \left(1 - x - \frac{1}{32}x^2 \right) \left(1 - x + \frac{1}{16}x^2 \right)^2 w^{14} \right) \\
&= \frac{1}{16384}(16-16x+x^2)(-3584+5824x-2352x^2+100x^3+7x^4)w^{14} \\
&\quad + 14 \left(1 - \frac{1}{2}x \right) \left(1 - x - \frac{1}{32}x^2 \right) \left(1 - x + \frac{1}{16}x^2 \right)^2 w^{13} \frac{dw}{dx}.
\end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned}
 q \frac{dS(q^4)}{dq} &= q \frac{\frac{dS(q^4)}{dx}}{\frac{dq}{dx}} = x(1-x)w^2 \frac{dS(q^4)}{dx} \\
 &= -\frac{7}{8192}x(1-x)(16-16x+x^2)^2 \\
 &\quad (2-x)(-32+32x+x^2)w^{15} \frac{dw}{dx} \\
 &\quad + \frac{1}{16384}x(1-x)(16-16x+x^2) \\
 &\quad (-3584+5824x-2352x^2+100x^3+7x^4)w^{16},
 \end{aligned}$$

as asserted. □

2.7 Evaluation of $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$ in terms of $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$.

In this section, we use Theorems 1.2.1, 1.2.2 and 2.3.1 to determine $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$ in terms of the corresponding functions, $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$. These determinations are used to prove the relationships in Section 2.8.

Theorem 2.7.1

$$(2.7.1) \quad L_{0,2}(q) = \frac{1}{24}(1 - 3L(q^2) + 2L(q^4)),$$

$$(2.7.2) \quad L_{1,2}(q) = \frac{1}{24}(-L(q) + 3L(q^2) - 2L(q^4)),$$

$$(2.7.3) \quad M_{0,2}(q) = \frac{1}{240}(-1 + 9M(q^2) - 8M(q^4)),$$

$$(2.7.4) \quad M_{1,2}(q) = \frac{1}{240}(M(q) - 9M(q^2) + 8M(q^4)),$$

$$(2.7.5) \quad N_{0,2}(q) = \frac{1}{504}(1 - 33N(q^2) + 32N(q^4)),$$

$$(2.7.6) \quad N_{1,2}(q) = \frac{1}{504}(-N(q) + 33N(q^2) - 32N(q^4)),$$

$$(2.7.7) \quad P_{0,2}(q) = \frac{1}{480}(-1 + 129P(q^2) - 128P(q^4)),$$

$$(2.7.8) \quad P_{1,2}(q) = \frac{1}{480}(P(q) - 129P(q^2) + 128P(q^4)),$$

$$(2.7.9) \quad Q_{0,2}(q) = \frac{1}{264}(1 - 513Q(q^2) + 512Q(q^4)),$$

$$(2.7.10) \quad Q_{1,2}(q) = \frac{1}{264}(-Q(q) + 513Q(q^2) - 512Q(q^4)),$$

$$(2.7.11) \quad R_{0,2}(q) = \frac{691}{65520}(-1 + 2049R(q^2) - 2048R(q^4)),$$

$$(2.7.12) \quad R_{1,2}(q) = \frac{691}{65520}(R(q) - 2049R(q^2) + 2048R(q^4)),$$

$$(2.7.13) \quad S_{0,2}(q) = \frac{1}{24}(1 - 8193S(q^2) + 8192S(q^4)),$$

$$(2.7.14) \quad S_{1,2}(q) = \frac{1}{24}(-S(q) + 8193S(q^2) - 8192S(q^4)).$$

Proof First we prove (2.7.1). By (2.5.15), (2.3.1) and (2.3.2), we have

$$\begin{aligned} L_{0,2}(q) &= \frac{1}{48} \left(2 - (2 - 7x)w^2 - 24x(1 - x)w \frac{dw}{dx} \right) \\ &= \frac{1}{24} - \frac{1}{8}L(q^2) + \frac{1}{12}L(q^4). \end{aligned}$$

Next we prove (2.7.2). By (2.5.16), (1.2.9), (2.3.1) and (2.3.2), we have

$$\begin{aligned} L_{1,2}(q) &= \frac{xw^2}{16} \\ &= -\frac{1}{24}L(q) + \frac{1}{8}L(q^2) - \frac{1}{12}L(q^4). \end{aligned}$$

Next we prove (2.7.3). By (2.5.17), (2.3.3) and (2.3.4), we have

$$\begin{aligned} M_{0,2}(q) &= \frac{1}{480}(-2 + (2 - 2x + 17x^2)w^4) \\ &= -\frac{1}{240} + \frac{3}{80}M(q^2) - \frac{1}{30}M(q^4). \end{aligned}$$

Next we prove (2.7.4). By (2.5.18), (1.2.10), (2.3.3) and (2.3.4), we have

$$\begin{aligned} M_{1,2}(q) &= \frac{1}{32}(2 - x)xw^4 \\ &= \frac{1}{240}M(q) - \frac{3}{80}M(q^2) + \frac{1}{30}M(q^4). \end{aligned}$$

Next we prove (2.7.5). By (2.5.19), (2.3.5) and (2.3.6), we have

$$\begin{aligned} N_{0,2}(q) &= \frac{1}{1008}(2 + (-2 + 3x + 129x^2 - 65x^3)w^6) \\ &= \frac{1}{504} - \frac{11}{168}N(q^2) + \frac{4}{63}N(q^4). \end{aligned}$$

Next we prove (2.7.6). By (2.5.20), (1.2.11), (2.3.5) and (2.3.6), we have

$$\begin{aligned} N_{1,2}(q) &= \frac{1}{16}(1 - x + x^2)xw^6 \\ &= -\frac{1}{504}N(q) + \frac{11}{168}N(q^2) - \frac{4}{63}N(q^4). \end{aligned}$$

Next we prove (2.7.7). By (2.5.21), (2.3.7) and (2.3.8), we have

$$\begin{aligned} P_{0,2}(q) &= \frac{1}{960}(-2 + (2 - 4x + 486x^2 - 484x^3 + 257x^4)w^8) \\ &= -\frac{1}{480} + \frac{43}{160}P(q^2) - \frac{4}{15}P(q^4). \end{aligned}$$

Next we prove (2.7.8). By (2.5.22), (1.2.20), (2.3.7) and (2.3.8), we have

$$\begin{aligned} P_{1,2}(q) &= \frac{1}{192}(12 - 18x + 108x^2 - 51x^3)xw^8 \\ &= \frac{1}{480}P(q) - \frac{43}{160}P(q^2) + \frac{4}{15}P(q^4). \end{aligned}$$

Next we prove (2.7.9). By (2.5.23), (2.3.9) and (2.3.10), we have

$$\begin{aligned} Q_{0,2}(q) &= \frac{1}{528}(2 - (2 - 5x - 1054x^2 + 1586x^3 \\ &\quad - 2579x^4 + 1025x^5)w^{10}) \\ &= \frac{1}{264} - \frac{171}{88}Q(q^2) + \frac{64}{33}Q(q^4). \end{aligned}$$

Next we prove (2.7.10). By (2.5.24), (1.2.21), (2.3.9) and (2.3.10), we have

$$\begin{aligned} Q_{1,2}(q) &= \frac{1}{528}(33 - 66x + 2574x^2 - 2541x^3 + 1023x^4)xw^{10} \\ &= -\frac{1}{264}Q(q) + \frac{171}{88}Q(q^2) - \frac{64}{33}Q(q^4). \end{aligned}$$

Next we prove (2.7.11). By (2.5.25), (2.3.11) and (2.3.12), we have

$$\begin{aligned} R_{0,2}(q) &= \frac{1}{131040}(-1382 + (1382 - 4146x + 1053237x^2 \\ &\quad - 2099564x^3 + 9538077x^4 - 8488986x^5 + 2831027x^6)w^{12}) \\ &= -\frac{691}{65520} + \frac{471953}{21840}R(q^2) - \frac{88448}{4095}R(q^4). \end{aligned}$$

Next we prove (2.7.12). By (2.5.26), (1.2.22), (2.3.11) and (2.3.12), we have

$$\begin{aligned} R_{1,2}(q) &= \frac{1}{32}(2 - 5x + 1388x^2 - 2077x^3 + 2074x^4 - 691x^5)xw^{12} \\ &= \frac{691}{65520}R(q) - \frac{471953}{21840}R(q^2) + \frac{88448}{4095}R(q^4). \end{aligned}$$

Next we prove (2.7.13). By (2.5.27), (2.3.13) and (2.3.14), we have

$$\begin{aligned} (2.7.15) \quad S_{0,2}(q) &= \frac{1}{48}(2 - (2 - 7x - 1527x^2 + 3835x^3 - 52229x^4 \\ &\quad + 74505x^5 - 57349x^6 + 16385x^7)w^{14}) \\ &= \frac{1}{24} - \frac{2731}{8}S(q^2) + \frac{1024}{3}S(q^4). \end{aligned}$$

Finally we prove (2.7.14). By (2.5.28), (1.2.23), (2.3.13) and (2.3.14), we have

$$\begin{aligned}
 (2.7.16) \quad S_{1,2}(q) &= \frac{1}{16}(1 - 3x + 6231x^2 - 12457x^3 + 25341x^4 \\
 &\quad - 19113x^5 + 5461x^6)w^{14} \\
 &= -\frac{1}{24}S(q) + \frac{2731}{8}S(q^2) - \frac{1024}{3}S(q^4).
 \end{aligned}$$

This completes the proof for Theorem 2.7.1. □

2.8 Relationships between $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$.

In this section we use Theorems 1.2.1, 1.2.2, 2.3.1, 2.5.1 and 2.6.1 to obtain relationships between the following functions: $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$. Using MAPLE, we take the product of two functions from the list above and check to see if the product can be expressed as a linear function of $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$ and their derivatives. Since there are thirty-five functions, in total, $35 + 34 + \dots + 2 + 1 = 630$ products were considered. Out of the 630, we found thirty-one which can be expressed as linear combinations of $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$ and their derivatives. Nine of these have been

presented in Section 1.5, so in this section we present the other twenty-two.

Theorem 2.8.1

$$(2.8.1) \quad L(q)L_{0,2}(q) = \frac{1}{24}L(q) - \frac{1}{240}(5M(q) + 21M(q^2) - 16M(q^4)) \\ - \frac{1}{24} \left(6q \frac{dL(q)}{dq} + 18q \frac{dL(q^2)}{dq} - 12q \frac{dL(q^4)}{dq} \right),$$

$$(2.8.2) \quad L(q)L_{1,2}(q) = -\frac{1}{240}(5M(q) - 21M(q^2) + 16M(q^4)) \\ - \frac{1}{24} \left(6q \frac{dL(q)}{dq} - 18q \frac{dL(q^2)}{dq} + 12q \frac{dL(q^4)}{dq} \right),$$

$$(2.8.3) \quad L(q^2)L_{0,2}(q) = \frac{1}{24}L(q^2) - \frac{1}{120}(13M(q^2) - 8M(q^4)) \\ - \frac{1}{8} \left(5q \frac{dL(q^2)}{dq} - 2q \frac{dL(q^4)}{dq} \right),$$

$$(2.8.4) \quad L(q^2)L_{1,2}(q) = -\frac{1}{120}(M(q) - 9M(q^2) + 8M(q^4)) \\ - \frac{1}{8} \left(q \frac{dL(q)}{dq} - 3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq} \right),$$

$$(2.8.5) \quad L(q^2)M_{0,2}(q) = -\frac{1}{240}L(q^2) + \frac{1}{5040}(181N(q^2) - 160N(q^4)) \\ + \frac{1}{160} \left(9q \frac{dM(q^2)}{dq} - 8q \frac{dM(q^4)}{dq} \right),$$

$$(2.8.6) \quad L(q^2)M_{1,2}(q) = \frac{1}{1008}(N(q) - 33N(q^2) + 32N(q^4)) \\ + \frac{1}{160} \left(q \frac{dM(q)}{dq} - 9q \frac{dM(q^2)}{dq} + 8q \frac{dM(q^4)}{dq} \right),$$

$$(2.8.7) \quad M(q^2)L_{0,2}(q) = \frac{1}{24}M(q^2) - \frac{1}{504}(53N(q^2) - 32N(q^4)) \\ - \frac{1}{8}q \frac{dM(q^2)}{dq},$$

$$(2.8.8) \quad M(q^2)L_{1,2}(q) = -\frac{1}{504}(N(q) - 33N(q^2) + 32N(q^4)),$$

$$(2.8.9) \quad L(q^4)L_{0,2}(q) = \frac{1}{24}L(q^4) - \frac{1}{120}(3M(q^2) + 2M(q^4))$$

$$\begin{aligned}
(2.8.10) \quad L(q^4)L_{1,2}(q) &= -\frac{1}{16} \left(3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq} \right), \\
(2.8.11) \quad L(q^4)M_{0,2}(q) &= -\frac{1}{480} (M(q) - 9M(q^2) + 8M(q^4)) \\
&\quad -\frac{1}{16} \left(q \frac{dL(q)}{dq} - 3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq} \right), \\
(2.8.12) \quad M(q^4)L_{0,2}(q) &= -\frac{1}{240} L(q^4) + \frac{1}{5040} (45N(q^2) - 24N(q^4)) \\
&\quad + \frac{1}{320} \left(9q \frac{dM(q^2)}{dq} - 8q \frac{dM(q^4)}{dq} \right), \\
(2.8.13) \quad L(q)L(q^2) &= \frac{1}{5} (M(q) + 4M(q^2)) \\
&\quad + 3 \left(q \frac{dL(q)}{dq} + 2q \frac{dL(q^2)}{dq} \right), \\
(2.8.14) \quad L(q)L(q^4) &= \frac{1}{20} (M(q) + 3M(q^2) + 16M(q^4)) \\
&\quad + \frac{1}{2} \left(3q \frac{dL(q)}{dq} + 12q \frac{dL(q^4)}{dq} \right), \\
(2.8.15) \quad L(q)M(q^2) &= \frac{1}{21} (N(q) + 20N(q^2)) + 3q \frac{dM(q^2)}{dq}, \\
(2.8.16) \quad M(q)L(q^2) &= \frac{1}{21} (5N(q) + 16N(q^2)) + \frac{3}{2} q \frac{dM(q)}{dq}, \\
(2.8.17) \quad L_{0,2}^2(q) &= \frac{1}{576} - \frac{1}{288} (3L(q^2) - 2L(q^4)) \\
&\quad + \frac{1}{2880} (33M(q^2) - 28M(q^4)) \\
&\quad + \frac{1}{48} \left(3q \frac{dL(q^2)}{dq} - 2q \frac{dL(q^4)}{dq} \right), \\
(2.8.18) \quad L_{0,2}(q)L_{1,2}(q) &= -\frac{1}{576} (L(q) - 3L(q^2) + 2L(q^4)) \\
&\quad + \frac{1}{1152} (M(q) - 9M(q^2) + 8M(q^4))
\end{aligned}$$

$$(2.8.19) \quad L_{1,2}^2(q) = \frac{1}{240}(M(q^2) - M(q^4)),$$

$$+ \frac{1}{96} \left(q \frac{dL(q)}{dq} - 3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq} \right),$$

$$(2.8.20) \quad L_{0,2}(q)M_{0,2}(q) = -\frac{1}{5760} + \frac{1}{5760}(3L(q^2) - 2L(q^4))$$

$$+ \frac{1}{5760}(9M(q^2) - 8M(q^4))$$

$$- \frac{1}{40320}(151N(q^2) - 144N(q^4))$$

$$- \frac{1}{1920} \left(9q \frac{dM(q^2)}{dq} - 8q \frac{dM(q^4)}{dq} \right),$$

$$(2.8.21) \quad L_{1,2}(q)M_{1,2}(q) = -\frac{1}{504}(N(q^2) - N(q^4)),$$

$$(2.8.22) \quad M_{0,2}(q)M_{1,2}(q) = -\frac{1}{57600}(M(q) - 9M(q^2) + 8M(q^4))$$

$$+ \frac{1}{115200}(P(q) - 129P(q^2) + 128P(q^4)).$$

Proof We use MAPLE (see Appendix A) to find out whether or not the 213 distinct products AB with A and B chosen from $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $L_{i,2}(q)$, $M_{i,2}(q)$, ..., $S_{i,2}(q)$ for $i = 0, 1$, can be expressed as linear combinations of $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$ and their derivatives. For example, we find such a relationship for

$$L(q)L_{1,2}(q) = \frac{1}{16}x(1-5x)w^4 + \frac{3}{4}x^2(1-x)w^3 \frac{dw}{dx},$$

see Theorems 1.2.1 and 2.5.1. The product $L(q)L_{1,2}(q)$ can only be related to $M(q)$, $M(q^2)$, $M(q^4)$, $q \frac{dL(q)}{dq}$, $q \frac{dL(q^2)}{dq}$ and $q \frac{dL(q^4)}{dq}$ as these are the only quantities whose evaluations contain at least one of the following terms: xw^4 ,

x^2w^4 , $x^2w^3\frac{dw}{dx}$, $x^3w^3\frac{dw}{dx}$. Thus we seek constants a, b, c, d, e and f such that

$$L(q)L_{1,2}(q) = aM(q) + bM(q^2) + cM(q^4) + dq\frac{dL(q)}{dq} + eq\frac{dL(q^2)}{dq} + fq\frac{dL(q^4)}{dq}.$$

Expanding both sides, we obtain

$$\begin{aligned} & \frac{xw^4}{16} - \frac{5x^2w^4}{16} + \frac{3x^2w^3}{4}\frac{dw}{dx} - \frac{3x^3w^3}{4}\frac{dw}{dx} \\ &= aw^4 + 14axw^4 + ax^2w^4 \\ & \quad + bw^4 - bxw^4 + bx^2w^4 \\ & \quad + cw^4 - cxw^4 + \frac{c}{16}x^2w^4 \\ & \quad - 2dxw^4 + 2dx^2w^4 + 2dxw^3\frac{dw}{dx} - 12dx^2w^3\frac{dw}{dx} + 10dx^3w^3\frac{dw}{dx} \\ & \quad + 12dx^2w^2\left(\frac{dw}{dx}\right)^2 - 24dx^3w^2\left(\frac{dw}{dx}\right)^2 + 12dx^4w^2\left(\frac{dw}{dx}\right)^2 \\ & \quad - \frac{e}{2}xw^4 + \frac{e}{2}x^2w^4 + 2exw^3\frac{dw}{dx} - 6ex^2w^3\frac{dw}{dx} + 4ex^3w^3\frac{dw}{dx} \\ & \quad + 6ex^2w^2\left(\frac{dw}{dx}\right)^2 - 12ex^3w^2\left(\frac{dw}{dx}\right)^2 + 6ex^4w^2\left(\frac{dw}{dx}\right)^2 \\ & \quad - \frac{f}{2}xw^4 + \frac{f}{2}x^2w^4 + 2fxw^3\frac{dw}{dx} - \frac{9f}{2}x^2w^3\frac{dw}{dx} + \frac{5f}{2}ex^3w^3\frac{dw}{dx} \\ & \quad + 3fx^2w^2\left(\frac{dw}{dx}\right)^2 - 6fx^3w^2\left(\frac{dw}{dx}\right)^2 + 3fx^4w^2\left(\frac{dw}{dx}\right)^2. \end{aligned}$$

Equating like terms, we obtain nine linear equations in the six unknowns $a, b,$

c, d, e and f .

$$\begin{aligned} a + b + c &= 0, \\ 14a - b - c - 2d - \frac{e}{2} - \frac{f}{2} &= \frac{1}{16}, \\ a + b + \frac{c}{16} + 2d + \frac{e}{2} + \frac{f}{2} &= -\frac{5}{16}, \end{aligned}$$

$$\begin{aligned}
2d + 2e + 2f &= 0, \\
-12d - 6e - \frac{9f}{2} &= \frac{3}{4}, \\
10d + 4e + \frac{5f}{2} &= -\frac{3}{4}, \\
12d + 6e + 3f &= 0, \\
-24d - 12e - 6f &= 0, \\
12d + 6e + 3f &= 0.
\end{aligned}$$

MAPLE was used to solve these linear equations for the unknowns a , b , c , d , e and f . In this case, MAPLE gives the solution $a = -\frac{1}{48}$, $b = \frac{7}{80}$, $c = -\frac{1}{15}$, $d = -\frac{1}{4}$, $e = \frac{3}{4}$, $f = -\frac{1}{2}$, which gives us (2.8.2).

We give another example for which a solution was not found. For

$$M(q)L_{1,2}(q) = \frac{1}{16}x(1 + 14x + x^2)w^6,$$

see Theorems 1.2.1 and 2.5.1, MAPLE did not return a result. The product $M(q)L_{1,2}(q)$ can only be related to $N(q)$, $N(q^2)$, $N(q^4)$, $q\frac{dM(q)}{dq}$, $q\frac{dM(q^2)}{dq}$ and $q\frac{dM(q^4)}{dq}$ as these are the only quantities whose evaluations contain at least one of the following terms: xw^6 , x^2w^6 and x^3w^6 . Thus we seek constants a , b , c , d , e and f such that

$$M(q)L_{1,2}(q) = aN(q) + bN(q^2) + cN(q^4) + dq\frac{dM(q)}{dq} + eq\frac{dM(q^2)}{dq} + fq\frac{dM(q^4)}{dq}.$$

Expanding both sides, we obtain

$$\frac{1}{16}xw^6 + \frac{14}{16}x^2w^6 + \frac{1}{16}x^3w^6$$

$$\begin{aligned}
&= aw^6 - 33axw^6 - 33ax^2w^6 + ax^3w^6 \\
&\quad + bw^6 - \frac{3}{2}bxw^6 - \frac{3}{2}bx^2w^6 + bx^3w^6 \\
&\quad + cw^6 - \frac{3}{2}cxw^6 + \frac{15}{32}cx^2w^6 + \frac{1}{64}cx^3w^6 \\
&\quad + 14dxw^6 - 12dx^2w^6 - 2dx^3w^6 \\
&\quad + 4dxw^5 \frac{dw}{dx} + 52dx^2w^5 \frac{dw}{dx} - 52dx^3w^5 \frac{dw}{dx} - 4dx^4w^5 \frac{dw}{dx} \\
&\quad - exw^6 + 3ex^2w^6 - 2ex^3w^6 \\
&\quad + 4exw^5 \frac{dw}{dx} - 8ex^2w^5 \frac{dw}{dx} + 8ex^3w^5 \frac{dw}{dx} - 4ex^4w^5 \frac{dw}{dx} \\
&\quad - fxw^6 + \frac{9}{8}fx^2w^6 - \frac{1}{8}fx^3w^6 \\
&\quad + 4fxw^5 \frac{dw}{dx} - 8fx^2w^5 \frac{dw}{dx} + \frac{17}{4}fx^3w^5 \frac{dw}{dx} - \frac{1}{4}fx^4w^5 \frac{dw}{dx}
\end{aligned}$$

Equating like terms, we obtain eight linear equations in the six unknowns a , b , c , d , e and f .

$$\begin{aligned}
a + b + c &= 0, \\
-33a - \frac{3b}{2} - \frac{3c}{2} + 14d - e - f &= \frac{1}{16}, \\
-33a - \frac{3b}{2} + \frac{15c}{32} - 12d + 3e + \frac{9f}{8} &= \frac{14}{16}, \\
a + b + \frac{c}{64} - 2d - 2e - \frac{f}{8} &= \frac{1}{16}, \\
4d + 4e + 4f &= 0, \\
52d - 8e - 8f &= 0, \\
-52d + 8e + \frac{17f}{4} &= 0, \\
-4d - 4e - \frac{f}{4} &= 0.
\end{aligned}$$

MAPLE was used to solve these linear equations for the unknowns a , b , c , d , e and f . In this case, MAPLE found the system of equations to be insolvable. This is clear in this case as it follows from the last three equations that

$$d = e = f = 0,$$

then the difference of the second and third equations gives

$$c = \frac{26}{63},$$

and finally the first and fourth equations give

$$\begin{aligned} a + b &= -\frac{26}{63}, \\ a + b &= \frac{113}{2016}, \end{aligned}$$

a contradiction.

Theorem 2.8.1 is the list of products for which our MAPLE program returns a solution.

We now prove the equalities in Theorem 2.8.1.

First we prove (2.8.1). By (1.2.9), (2.5.15), (1.2.10), (2.3.3), (2.3.4), (1.4.1), (2.6.1) and (2.6.2), we have

$$\begin{aligned} L(q)L_{0,2}(q) &= \frac{1}{24}(1-5x)w^2 - \frac{1}{48}(1-5x)(2-7x)w^4 + \frac{1}{2}x(1-x)w\frac{dw}{dx} \\ &\quad - x(1-x)\left(1 - \frac{17}{4}x\right)w^3\frac{dw}{dx} - 6x^2(1-x)^2w^2\left(\frac{dw}{dx}\right)^2 \\ &= \frac{1}{24}L(q) - \frac{1}{240}(5M(q) + 21M(q^2) - 16M(q^4)) \\ &\quad - \frac{1}{24}\left(6q\frac{dL(q)}{dq} + 18q\frac{dL(q^2)}{dq} - 12q\frac{dL(q^4)}{dq}\right). \end{aligned}$$

Next we prove (2.8.2). By (1.2.9), (2.5.16), (1.2.10), (2.3.3), (2.3.4), (1.4.1), (2.6.1) and (2.6.2), we have

$$\begin{aligned} L(q)L_{1,2}(q) &= \frac{1}{16}x(1-5x)w^4 + \frac{3}{4}x^2(1-x)w^3\frac{dw}{dx} \\ &= -\frac{1}{240}(5M(q) - 21M(q^2) + 16M(q^4)) \\ &\quad -\frac{1}{24}\left(6q\frac{dL(q)}{dq} - 18q\frac{dL(q^2)}{dq} + 12q\frac{dL(q^4)}{dq}\right). \end{aligned}$$

Next we prove (2.8.3). By (2.3.1), (2.5.15), (2.3.3), (2.3.4), (2.6.1) and (2.6.2), we have

$$\begin{aligned} L(q^2)L_{0,2}(q) &= \frac{1}{24}(1-2x)w^2 - \frac{1}{48}(1-2x)(2-7x)w^4 \\ &\quad + \frac{1}{4}x(1-x)w\frac{dw}{dx} - \frac{3}{4}x(1-x)\left(1-\frac{5}{2}x\right)w^3\frac{dw}{dx} \\ &\quad - 3x^2(1-x)^2w^2\left(\frac{dw}{dx}\right)^2 \\ &= \frac{1}{24}L(q^2) - \frac{1}{120}(13M(q^2) - 8M(q^4)) \\ &\quad - \frac{1}{8}\left(5q\frac{dL(q^2)}{dq} - 2q\frac{dL(q^4)}{dq}\right). \end{aligned}$$

Next we prove (2.8.4). By (2.3.1), (2.5.16), (1.2.10), (2.3.3), (2.3.4), (1.4.1), (2.6.1) and (2.6.2), we have

$$\begin{aligned} L(q^2)L_{1,2}(q) &= \frac{1}{16}x(1-2x)w^4 + \frac{3}{8}x^2(1-x)w^2\frac{dw}{dx} \\ &= -\frac{1}{120}(M(q) - 9M(q^2) + 8M(q^4)) \\ &\quad - \frac{1}{8}\left(q\frac{dL(q)}{dq} - 3q\frac{dL(q^2)}{dq} + 2q\frac{dL(q^4)}{dq}\right). \end{aligned}$$

Next we prove (2.8.5). By (2.3.1), (2.5.17), (2.3.5), (2.3.6), (2.6.3) and

(2.6.4), we have

$$\begin{aligned}
L(q^2)M_{0,2}(q) &= -\frac{1}{240}(1-2x)w^2 + \frac{1}{480}(1-2x)(2-2x+17x^2)w^6 \\
&\quad -\frac{1}{40}x(1-x)w\frac{dw}{dx} + \frac{1}{80}x(1-x)(2-2x+17x^2)w^5\frac{dw}{dx} \\
&= -\frac{1}{240}L(q^2) + \frac{1}{5040}(181N(q^2) - 160N(q^4)) \\
&\quad + \frac{1}{160}\left(9q\frac{dM(q^2)}{dq} - 8q\frac{dM(q^4)}{dq}\right).
\end{aligned}$$

Next we prove (2.8.6). By (2.3.1), (2.5.18), (1.2.11), (2.3.5), (2.3.6), (1.4.2), (2.6.3) and (2.6.4), we have

$$\begin{aligned}
L(q^2)M_{1,2}(q) &= \frac{1}{32}x(1-2x)(2-x)w^6 + \frac{3}{16}x^2(1-x)(2-x)w^5\frac{dw}{dx} \\
&= \frac{1}{1008}(N(q) - 33N(q^2) + 32N(q^4)) \\
&\quad + \frac{1}{160}\left(q\frac{dM(q)}{dq} - 9q\frac{dM(q^2)}{dq} + 8q\frac{dM(q^4)}{dq}\right).
\end{aligned}$$

Next we prove (2.8.7). By (2.3.3), (2.5.15), (2.3.5), (2.3.6) and (2.6.3), we have

$$\begin{aligned}
M(q^2)L_{0,2}(q) &= \frac{1}{24}(1-x+x^2)w^4 - \frac{1}{48}(2-7x)(1-x+x^2)w^6 \\
&\quad -\frac{1}{2}x(1-x)(1-x+x^2)w^5\frac{dw}{dx} \\
&= \frac{1}{24}M(q^2) - \frac{1}{504}(53N(q^2) - 32N(q^4)) \\
&\quad -\frac{1}{8}q\frac{dM(q^2)}{dq}.
\end{aligned}$$

Next we prove (2.8.8). By (2.3.3), (2.5.16), (1.2.11), (2.3.5) and (2.3.6), we have

$$M(q^2)L_{1,2}(q) = \frac{1}{16}x(1-x+x^2)w^6$$

$$= -\frac{1}{504}(N(q) - 33N(q^2) + 32N(q^4)).$$

Next we prove (2.8.9). By (2.3.2), (2.5.15), (2.3.3), (2.3.4), (2.6.1) and (2.6.2), we have

$$\begin{aligned} L(q^4)L_{0,2}(q) &= \frac{1}{24} \left(1 - \frac{5}{4}x\right) w^2 - \frac{1}{48} \left(1 - \frac{5}{4}x\right) (2 - 7x)w^4 \\ &\quad + \frac{1}{8}x(1-x)w \frac{dw}{dx} - \frac{5}{8}x(1-x) \left(1 - \frac{17}{10}x\right) w^3 \frac{dw}{dx} \\ &\quad - \frac{3}{2}x^2(1-x)^2 w^2 \left(\frac{dw}{dx}\right)^2 \\ &= \frac{1}{24}L(q^4) - \frac{1}{120}(3M(q^2) + 2M(q^4)) \\ &\quad - \frac{1}{16} \left(3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq}\right). \end{aligned}$$

Next we prove (2.8.10). By (2.3.2), (2.5.16), (1.2.10), (2.3.3), (2.3.4), (1.4.1), (2.6.1) and (2.6.2), we have

$$\begin{aligned} L(q^4)L_{1,2}(q) &= \frac{1}{16}x(1-5x)w^4 + \frac{3}{4}x^2(1-x)w^3 \frac{dw}{dx} \\ &= -\frac{1}{480}(M(q) - 9M(q^2) + 8M(q^4)) \\ &\quad - \frac{1}{16} \left(q \frac{dL(q)}{dq} - 3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq}\right). \end{aligned}$$

Next we prove (2.8.11). By (2.3.2), (2.5.17), (2.3.5), (2.3.6), (2.6.3) and (2.6.4), we have

$$\begin{aligned} L(q^4)M_{0,2}(q) &= -\frac{1}{240} \left(1 - \frac{5}{4}x\right) w^2 + \frac{1}{480} \left(1 - \frac{5}{4}x\right) (2 - 2x + 17x^2)w^6 \\ &\quad - \frac{1}{80}x(1-x)w \frac{dw}{dx} + \frac{1}{160}x(1-x) (2 - 2x + 17x^2) w^5 \frac{dw}{dx} \\ &= -\frac{1}{240}L(q^4) + \frac{1}{5040}(45N(q^2) - 24N(q^4)) \end{aligned}$$

$$+\frac{1}{320} \left(9q \frac{dM(q^2)}{dq} - 8q \frac{dM(q^4)}{dq} \right).$$

Next we prove (2.8.12). By (2.3.4), (2.5.15), (2.3.5), (2.3.6) and (2.6.4), we have

$$\begin{aligned} M(q^4)L_{0,2}(q) &= \frac{1}{24} \left(1 - x + \frac{1}{16}x^2 \right) w^4 - \frac{1}{48}(2 - 7x) \left(1 - x + \frac{1}{16}x^2 \right) w^6 \\ &\quad - \frac{1}{2}x(1 - x) \left(1 - x + \frac{1}{16}x^2 \right) w^5 \frac{dw}{dx} \\ &= \frac{1}{24}M(q^4) - \frac{1}{168}(N(q^2) + 6N(q^4)) \\ &\quad - \frac{1}{8}q \frac{dM(q^4)}{dq}. \end{aligned}$$

Next we prove (2.8.13). By (1.2.9), (2.3.1), (1.2.10), (2.3.3), (1.4.1) and (2.6.1), we have

$$\begin{aligned} L(q)L(q^2) &= (1 - 2x)(1 - 5x)w^4 + 18x(1 - x)(1 - 3x)w^3 \frac{dw}{dx} \\ &\quad + 72x^2(1 - x)^2w^2 \left(\frac{dw}{dx} \right)^2 \\ &= \frac{1}{5}(M(q) + 4M(q^2)) + 3 \left(q \frac{dL(q)}{dq} + 2q \frac{dL(q^2)}{dq} \right). \end{aligned}$$

Next we prove (2.8.14). By (1.2.9), (2.3.2), (1.2.10), (2.3.3), (2.3.4), (1.4.1) and (2.6.2), we have

$$\begin{aligned} L(q)L(q^4) &= (1 - 5x) \left(1 - \frac{5}{4}x \right) w^4 + 15x(1 - x)(1 - 2x)w^3 \frac{dw}{dx} \\ &\quad + 36x^2(1 - x)^2w^2 \left(\frac{dw}{dx} \right)^2 \\ &= \frac{1}{20}(M(q) + 3M(q^2) + 16M(q^4)) \\ &\quad + \frac{1}{2} \left(3q \frac{dL(q)}{dq} + 12q \frac{dL(q^4)}{dq} \right). \end{aligned}$$

Next we prove (2.8.15). By (1.2.9), (2.3.3), (1.2.11), (2.3.5) and (1.4.2), we have

$$\begin{aligned} L(q)M(q^2) &= (1-5x)(1-x+x^2)w^6 + 12x(1-x)(1-x+x^2)w^5 \frac{dw}{dx} \\ &= \frac{1}{21}(N(q) + 20N(q^2)) + 3q \frac{dM(q)}{dq}. \end{aligned}$$

Next we prove (2.8.16). By (2.3.1), (1.2.10), (1.2.11), (2.3.5) and (1.4.2), we have

$$\begin{aligned} M(q)L(q^2) &= (1-2x)(1+14x+x^2)w^6 + 6x(1-x)(1+14x+x^2)w^5 \frac{dw}{dx} \\ &= \frac{1}{21}(5N(q) + 16N(q^2)) + \frac{3}{2}q \frac{dM(q)}{dq}. \end{aligned}$$

Next we prove (2.8.17). By (2.5.15), (2.3.1), (2.3.2), (2.3.3), (2.3.4), (2.6.1) and (2.6.2), we have

$$\begin{aligned} L_{0,2}(q)L_{0,2}(q) &= \frac{1}{576} - \frac{1}{576}(2-7x)w^2 + \frac{1}{2304}(2-7x)^2w^4 - \frac{1}{24}x(1-x)w \frac{dw}{dx} \\ &\quad + \frac{1}{48}x(1-x)(2-7x)w^3 \frac{dw}{dx} + \frac{1}{4}x^2(1-x)^2w^2 \left(\frac{dw}{dx} \right)^2 \\ &= \frac{1}{576} - \frac{1}{288}(3L(q^2) - 2L(q^4)) \\ &\quad + \frac{1}{2880}(33M(q^2) - 28M(q^4)) \\ &\quad + \frac{1}{48} \left(3q \frac{dL(q^2)}{dq} - 2q \frac{dL(q^4)}{dq} \right). \end{aligned}$$

Next we prove (2.8.18). By (2.5.15), (2.5.16), (1.2.9), (2.3.1), (2.3.2), (1.2.10), (2.3.3), (2.3.4), (1.4.1), (2.6.1) and (2.6.2), we have

$$L_{0,2}(q)L_{1,2}(q) = \frac{1}{768}(2xw^2 - x(2-7x)w^4 - 24x^2(1-x)w^3 \frac{dw}{dx})$$

$$\begin{aligned}
&= -\frac{1}{576}(L(q) - 3L(q^2) + 2L(q^4)) \\
&\quad + \frac{1}{1152}(M(q) - 9M(q^2) + 8M(q^4)) \\
&\quad + \frac{1}{96}\left(q\frac{dL(q)}{dq} - 3q\frac{dL(q^2)}{dq} + 2q\frac{dL(q^4)}{dq}\right).
\end{aligned}$$

Next we prove (2.8.19). By (2.5.16), (2.3.3) and (2.3.4), we have

$$\begin{aligned}
L_{1,2}(q)L_{1,2}(q) &= \frac{1}{256}x^2w^4 \\
&= \frac{1}{240}(M(q^2) - M(q^4)).
\end{aligned}$$

Next we prove (2.8.20). By (2.5.15), (2.5.17), (2.3.1), (2.3.2), (2.3.3), (2.3.4), (2.3.5), (2.3.6), (2.6.1) and (2.6.2), we have

$$\begin{aligned}
L_{0,2}(q)M_{0,2}(q) &= -\frac{1}{5760} + \frac{1}{11520}(2 - 7x)w^2 + \frac{1}{480}x(1 - x)w\frac{dw}{dx} \\
&\quad + \frac{1}{11520}(2 - 2x + 17x^2)w^4 - \frac{1}{23040}(2 - 7x)(2 - 2x + 17x^2)w^6 \\
&\quad - \frac{1}{960}x(1 - x)(2 - 2x + 17x^2)w^5\frac{dw}{dx} \\
&= -\frac{1}{5760} + \frac{1}{5760}(3L(q^2) - 2L(q^4)) \\
&\quad + \frac{1}{5760}(9M(q^2) - 8M(q^4)) \\
&\quad - \frac{1}{40320}(151N(q^2) - 144N(q^4)) \\
&\quad - \frac{1}{1920}\left(9q\frac{dL(q^2)}{dq} - 8q\frac{dL(q^4)}{dq}\right).
\end{aligned}$$

Next we prove (2.8.21). By (2.5.16), (2.5.18), (2.3.5) and (2.3.6), we have

$$\begin{aligned}
L_{1,2}(q)M_{1,2}(q) &= \frac{1}{512}(2 - x)x^2w^6 \\
&= -\frac{1}{504}(N(q^2) - N(q^4)).
\end{aligned}$$

Finally we prove (2.8.22). By (2.5.17), (2.5.18), (1.2.10), (2.3.3), (2.3.4), (1.2.20), (2.3.7) and (2.3.8), we have

$$\begin{aligned} M_{0,2}(q)M_{1,2}(q) &= -\frac{1}{7680}x(2-x)w^4 + \frac{1}{15360}x(2-x)(2-2x+17x^2)w^8 \\ &= -\frac{1}{57600}(M(q) - 9M(q^2) + 8M(q^4)) \\ &\quad + \frac{1}{115200}(P(q) - 129P(q^2) + 128P(q^4)). \end{aligned}$$

This completes the proof for Theorem 2.8.1. □

2.9 Other arithmetic identities.

In this section, we use the relationships in Theorem 2.8.1 to derive twenty-two arithmetic identities. Formulae (2.9.1) and (2.9.2) are due to Huard, Ou, Spearman and Williams [14, Theorem 5 and Corollary 1, p. 249]. Formula (2.9.8) was stated by Ramanujan [31] [33, p. 146]. Although he never proved it, he mentioned that it can be proved by the theory of elliptic functions or by elementary methods. A result equivalent to (2.9.8) was first proved by Masser, see Berndt [2, p. 329], with later proofs by Atkin (see Berndt [2, p. 329]) and Ramamani [30]. None of these proofs is elementary. Huard, Ou, Spearman and Williams [14, Corollary 2, p.251] provided the first elementary proof of (2.9.8). Formula (2.9.21) was proved for odd n in Huard, Ou, Spearman and Williams' paper [14, Corollary 3, p. 252]. Formulae (2.9.13), (2.9.14), (2.9.15) and (2.9.16) are proved using the theory of modular forms in Melfi's paper

[27] for odd n . They are also proved in an elementary manner in Huard, Ou, Spearman and Williams' paper [14] for all n .

Theorem 2.9.1

$$\begin{aligned}
 (2.9.1) \quad & \sum_{m < n/2} \sigma(2m)\sigma(n-2m) \\
 &= \frac{1}{24} \{5\sigma_3(n) + 21\sigma_3(n/2) - 16\sigma_3(n/4) \\
 &\quad + (1-6n)(\sigma(n) + 3\sigma(n/2) - 2\sigma(n/4))\},
 \end{aligned}$$

$$\begin{aligned}
 (2.9.2) \quad & \sum_{m < \frac{n+1}{2}} \sigma(2m-1)\sigma(n-(2m-1)) \\
 &= \frac{1}{24} \{5\sigma_3(n) - 21\sigma_3(n/2) + 16\sigma_3(n/4) \\
 &\quad + (1-6n)(\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4))\},
 \end{aligned}$$

$$\begin{aligned}
 (2.9.3) \quad & \sum_{m < n} \sigma(2m)\sigma(n-m) \\
 &= \frac{1}{24} \{26\sigma_3(n) - 16\sigma_3(n/2) + 4\sigma(n) \\
 &\quad - 2\sigma(n/2) - 30n\sigma(n) + 12n\sigma(n/2)\},
 \end{aligned}$$

$$\begin{aligned}
 (2.9.4) \quad & \sum_{m < n} \sigma(2m-1)\sigma(n-m) \\
 &= \frac{1}{12} \{\sigma_3(2n-1) + (2-3n)\sigma(2n-1)\},
 \end{aligned}$$

$$\begin{aligned}
 (2.9.5) \quad & \sum_{m < n} \sigma_3(2m)\sigma(n-m) \\
 &= \frac{1}{240} \{181\sigma_5(n) - 160\sigma_5(n/2) \\
 &\quad + (10-30n)(9\sigma_3(n) - 8\sigma_3(n/2)) - \sigma(n)\},
 \end{aligned}$$

$$(2.9.6) \quad \sum_{m < n} \sigma_3(2m-1)\sigma(n-m)$$

$$= \frac{1}{48} \{ \sigma_5(2n-1) + (5-6n)\sigma_3(2n-1) \},$$

$$(2.9.7) \quad \sum_{m < n} \sigma(2m)\sigma_3(n-m) \\ = \frac{1}{240} \{ 53\sigma_5(n) - 32\sigma_5(n/2) + (10-60n)\sigma_3(n) \\ - 3\sigma(n) + 2\sigma(n/2) \},$$

$$(2.9.8) \quad \sum_{m < n} \sigma(2m-1)\sigma_3(n-m) \\ = \frac{1}{240} \{ \sigma_5(2n-1) - \sigma(2n-1) \},$$

$$(2.9.9) \quad \sum_{m < n} \sigma(2m)\sigma\left(\frac{n-m}{2}\right) \\ = \frac{1}{24} \{ 6\sigma_3(n) + 4\sigma_3(n/2) + 3\sigma(n) \\ - \sigma(n/2) - 9n\sigma(n) - 6n\sigma(n/2) \},$$

$$(2.9.10) \quad \sum_{m < n} \sigma(2m-1)\sigma\left(\frac{n-m}{2}\right) \\ = \frac{1}{48} \{ \sigma_3(2n-1) + (5-6n)\sigma(2n-1) \},$$

$$(2.9.11) \quad \sum_{m < n} \sigma_3(2m)\sigma\left(\frac{n-m}{2}\right) \\ = \frac{1}{240} \{ 45\sigma_5(n) - 24\sigma_5(n/2) \\ + (10-15n)(9\sigma_3(n) - 8\sigma_3(n/2)) - \sigma(n/2) \},$$

$$(2.9.12) \quad \sum_{m < n} \sigma(2m)\sigma_3\left(\frac{n-m}{2}\right) \\ = \frac{1}{240} \{ 3\sigma_5(n) + 18\sigma_5(n/2) \\ + (1-6n)(10\sigma_3(n/2)) - 3\sigma(n) + 2\sigma(n/2) \},$$

$$\begin{aligned}
(2.9.13) \quad & \sum_{m < n/2} \sigma(m)\sigma(n-2m) \\
&= \frac{1}{24} \{2\sigma_3(n) + 8\sigma_3(n/2) + \sigma(n) \\
&\quad + \sigma(n/2) - 3n\sigma(n) - 6n\sigma(n/2)\},
\end{aligned}$$

$$\begin{aligned}
(2.9.14) \quad & \sum_{m < n/4} \sigma(m)\sigma(n-4m) \\
&= \frac{1}{48} \{\sigma_3(n) + 3\sigma_3(n/2) + 16\sigma_3(n/4) + 2\sigma(n) \\
&\quad + 2\sigma(n/4) - 3n\sigma(n) - 12n\sigma(n/4)\},
\end{aligned}$$

$$\begin{aligned}
(2.9.15) \quad & \sum_{m < n/2} \sigma_3(m)\sigma(n-2m) \\
&= \frac{1}{240} \{\sigma_5(n) + 20\sigma_5(n/2) + (10 - 30n)\sigma_3(n/2) - \sigma(n)\},
\end{aligned}$$

$$\begin{aligned}
(2.9.16) \quad & \sum_{m < n/2} \sigma(m)\sigma_3(n-2m) \\
&= \frac{1}{240} \{5\sigma_5(n) + 16\sigma_5(n/2) + (10 - 15n)\sigma_3(n) - \sigma(n/2)\},
\end{aligned}$$

$$\begin{aligned}
(2.9.17) \quad & \sum_{m < n} \sigma(2m)\sigma(2n-2m) \\
&= \frac{1}{12} \{33\sigma_3(n) - 28\sigma_3(n/2) + (1 - 12n)(3\sigma(n) - 2\sigma(n/2))\},
\end{aligned}$$

$$\begin{aligned}
(2.9.18) \quad & \sum_{m < n} \sigma(2m-1)\sigma(2n-2m) \\
&= \frac{1}{24} 5\sigma_3(2n-1) + (7 - 12n)\sigma(2n-1)
\end{aligned}$$

$$\begin{aligned}
(2.9.19) \quad & \sum_{m < n+1} \sigma(2m-1)\sigma(2n-(2m-1)) \\
&= \sigma_3(n) - \sigma_3(n/2),
\end{aligned}$$

$$(2.9.20) \quad \sum_{m < n} \sigma(2m)\sigma_3(2n-2m)$$

$$= \frac{1}{240} \{ (1 - 6n)(9\sigma_3(n) - 8\sigma_3(n/2)) \\ + 453\sigma_5(n) - 432\sigma_5(n/2) - 3\sigma(n) + 2\sigma(n/2) \},$$

$$(2.9.21) \quad \sum_{m < n+1} \sigma(2m-1)\sigma_3(2n-(2m-1)) \\ = \sigma_5(n) - \sigma_5(n/2),$$

$$(2.9.22) \quad \sum_{m < \frac{n+1}{2}} \sigma_3(2m)\sigma_3(n-2m) \\ = \frac{1}{240} \{ \sigma_7(n) - \sigma_3(n) \}.$$

Proof First we prove (2.9.1). By (2.8.1), (1.1.1), (1.1.2), (2.5.1) and (2.7.1), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n/2} \sigma(2m)\sigma(n-2m)q^n \\ &= \sum_{k=1}^{\infty} \sigma(k)q^k \sum_{l=1}^{\infty} \sigma(2l)q^{2l} \\ &= \frac{L_{0,2}(q) - L(q)L_{0,2}(q)}{24} \\ &= \frac{1}{576} (1 - 3L(q^2) + 2L(q^4)) - \frac{1}{576} L(q) \\ &\quad + \frac{1}{5760} (5M(q) + 21M(q^2) - 16M(q^4)) \\ &\quad + \frac{1}{576} \left(6q \frac{dL(q)}{dq} + 18q \frac{dL(q^2)}{dq} - 12q \frac{dL(q^4)}{dq} \right) \\ &= \frac{1}{576} - \frac{3}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2)q^n \right) + \frac{2}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4)q^n \right) \\ &\quad - \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n \right) + \frac{5}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right) \\ &\quad + \frac{21}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2)q^n \right) - \frac{16}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4)q^n \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{6}{576} \left(-24 \sum_{n=1}^{\infty} n\sigma(n)q^n \right) + \frac{18}{576} \left(-24 \sum_{n=1}^{\infty} n\sigma(n/2)q^n \right) \\
& - \frac{12}{576} \left(-24 \sum_{n=1}^{\infty} n\sigma(n/4)q^n \right) \\
= & \frac{1}{24} \sum_{n=1}^{\infty} (\sigma(n)q^n + 3\sigma(n/2) - 2\sigma(n/4) \\
& + 5\sigma_3(n) + 21\sigma_3(n/2) - 16\sigma_3(n/4) \\
& - 6n\sigma(n) - 18n\sigma(n/2) + 12n\sigma(n/4))q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (2.9.1).

Next we prove (2.9.2). By (2.8.2), (1.1.1), (1.1.2), (2.5.2) and (2.7.2), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(2m-1)\sigma(n-(2m-1))q^n \\
= & \sum_{k=1}^{\infty} \sigma(k)q^k \sum_{l=1}^{\infty} \sigma(2l-1)q^{2l-1} \\
= & \frac{L_{1,2}(q) - L(q)L_{1,2}(q)}{24} \\
= & \frac{1}{576} (-L(q) + 3L(q^2) - 2L(q^4)) \\
& - \frac{1}{5760} (5M(q) - 21M(q^2) + 16M(q^4)) \\
& - \frac{1}{576} \left(6q \frac{dL(q)}{dq} - 18q \frac{dL(q^2)}{dq} + 12q \frac{dL(q^4)}{dq} \right) \\
= & -\frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n \right) + \frac{3}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2)q^n \right) \\
& - \frac{2}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4)q^n \right) - \frac{5}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{21}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) - \frac{16}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^n \right) \\
& - \frac{6}{576} \left(-24 \sum_{n=1}^{\infty} n \sigma(n) q^n \right) + \frac{18}{576} \left(-24 \sum_{n=1}^{\infty} n \sigma(n/2) q^n \right) \\
& - \frac{12}{576} \left(-24 \sum_{n=1}^{\infty} n \sigma(n/4) q^n \right) \\
& = \frac{1}{24} \sum_{n=1}^{\infty} (\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4) \\
& \quad + 5\sigma_3(n) - 21\sigma_3(n/2) + 16\sigma_3(n/4) \\
& \quad - 6n\sigma(n) + 18n\sigma(n/2) - 12n\sigma(n/4)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (2.9.2).

Next we prove (2.9.3). By (2.8.3), (1.1.1), (1.1.2), (2.5.1) and (2.7.1), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m) \sigma(n-m) q^{2n} \\
& = \sum_{k=1}^{\infty} \sigma(k) q^{2k} \sum_{l=1}^{\infty} \sigma(2l) q^{2l} \\
& = \frac{L_{0,2}(q) - L(q^2) L_{0,2}(q)}{24} \\
& = \frac{1}{576} (1 - 3L(q^2) + 2L(q^4)) - \frac{1}{576} L(q^2) \\
& \quad + \frac{1}{5760} (26M(q^2) - 16M(q^4)) \\
& \quad + \frac{1}{576} \left(15q \frac{dL(q^2)}{dq} - 6q \frac{dL(q^4)}{dq} \right) \\
& = \frac{1}{576} - \frac{4}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{2}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{26}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) - \frac{16}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) \\
& + \frac{15}{576} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n) q^{2n} \right) - \frac{6}{576} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/2) q^{2n} \right) \\
& = \frac{1}{24} \sum_{n=1}^{\infty} (4\sigma(n) - 2\sigma(n/2) + 26\sigma_3(n) - 16\sigma_3(n/2) \\
& \quad - 30n\sigma(n) + 12n\sigma(n/2)) q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.3).

Next we prove (2.9.4). By (2.8.4), (1.1.1), (1.1.2), (2.5.2) and (2.7.2), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(2m-1) \sigma\left(\frac{n-(2m-1)}{2}\right) q^n \\
& = \sum_{k=1}^{\infty} \sigma\left(\frac{k}{2}\right) q^k \sum_{l=1}^{\infty} \sigma(2l-1) q^{2l-1} \\
& = \frac{L_{1,2}(q) - L(q^2)L_{1,2}(q)}{24} \\
& = \frac{1}{576} (-L(q) + 3L(q^2) - 2L(q^4)) \\
& \quad + \frac{1}{2880} (M(q) - 9M(q^2) + 8M(q^4)) \\
& \quad + \frac{1}{576} \left(3q \frac{dL(q)}{dq} - 9q \frac{dL(q^2)}{dq} + 6q \frac{dL(q^4)}{dq} \right) \\
& = -\frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) + \frac{3}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^n \right) \\
& \quad - \frac{2}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^n \right) + \frac{1}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \\
& \quad - \frac{9}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) + \frac{8}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{3}{576} \left(-24 \sum_{n=1}^{\infty} n\sigma(n)q^n \right) - \frac{9}{576} \left(-24 \sum_{n=1}^{\infty} n\sigma(n/2)q^n \right) \\
& + \frac{6}{576} \left(-24 \sum_{n=1}^{\infty} n\sigma(n/4)q^n \right) \\
& = \frac{1}{24} \sum_{n=1}^{\infty} (\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4) \\
& \quad + 2\sigma_3(n) - 18\sigma_3(n/2) + 16\sigma_3(n/4) \\
& \quad - 3n\sigma(n) + 9n\sigma(n/2) - 6n\sigma(n/4))q^n.
\end{aligned}$$

Since $\sigma\left(\frac{n - (2m - 1)}{2}\right)$ is 0 if n is even, we only consider the case when n is odd. Changing $n \rightarrow 2n - 1$, we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m - 1)\sigma(n - m)q^{2n-1} \\
& = \frac{1}{24} \sum_{n=1}^{\infty} (\sigma(2n - 1) + 2\sigma_3(2n - 1) - (6n - 3)\sigma(2n - 1))q^{2n-1}.
\end{aligned}$$

Equating the coefficients of q^{2n-1} , we obtain the identity (2.9.4).

Next we prove (2.9.5). By (2.8.5), (1.1.1), (1.1.2), (1.1.3), (2.5.3) and (2.7.3), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(2m)\sigma(n - m)q^{2n} \\
& = \sum_{l=1}^{\infty} \sigma_3(2l)q^{2l} \sum_{k=1}^{\infty} \sigma(k)q^{2k} \\
& = \frac{M_{0,2}(q) - L(q^2)M_{0,2}(q)}{24} \\
& = \frac{1}{5760}(-1 + 9M(q^2) - 8M(q^4)) + \frac{1}{5760}L(q^2) \\
& \quad - \frac{1}{120960}(181N(q^2) - 160N(q^4))
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{3840} \left(9q \frac{dM(q^2)}{dq} - 8q \frac{dM(q^4)}{dq} \right) \\
= & -\frac{1}{5760} + \frac{9}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) \\
& -\frac{8}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) - \frac{181}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right) \\
& + \frac{160}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{2n} \right) - \frac{9}{3840} \left(240 \sum_{n=1}^{\infty} 2n \sigma_3(n) q^{2n} \right) \\
& + \frac{8}{3840} \left(240 \sum_{n=1}^{\infty} 2n \sigma_3(n/2) q^{2n} \right) + \frac{1}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) \\
= & \frac{1}{24} \sum_{n=1}^{\infty} \left(\frac{181}{10} \sigma_5(n) - 16 \sigma_5(n/2) + 9 \sigma_3(n) \right. \\
& \quad \left. - 8 \sigma_3(n/2) - 27n \sigma_3(n) + 24n \sigma_3(n/2) \right. \\
& \quad \left. - \frac{1}{10} \sigma(n) \right) q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.5).

Next we prove (2.9.6). By (2.8.6), (1.1.1), (1.1.2), (2.5.4) and (2.7.4), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma_3(2m-1) \sigma\left(\frac{n-(2m-1)}{2}\right) q^n \\
= & \sum_{l=1}^{\infty} \sigma_3(2l-1) q^{2l-1} \sum_{k=1}^{\infty} \sigma\left(\frac{k}{2}\right) q^k \\
= & \frac{M_{1,2}(q) - L(q^2) M_{1,2}(q)}{24} \\
= & \frac{1}{5760} (M(q) - 9M(q^2) + 8M(q^4)) \\
& - \frac{1}{24192} (N(q) - 33N(q^2) + 32N(q^4))
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{3840} \left(q \frac{dM(q)}{dq} - 9q \frac{dM(q^2)}{dq} + 8q \frac{dM(q^4)}{dq} \right) \\
& = \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) - \frac{9}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) \\
& \quad + \frac{8}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^n \right) - \frac{1}{24192} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\
& \quad - \frac{33}{24192} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^n \right) + \frac{32}{24192} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/4) q^n \right) \\
& \quad - \frac{1}{3840} \left(240 \sum_{n=1}^{\infty} n \sigma_3(n) q^n \right) + \frac{9}{3840} \left(240 \sum_{n=1}^{\infty} n \sigma_3(n/2) q^n \right) \\
& \quad - \frac{8}{3840} \left(240 \sum_{n=1}^{\infty} n \sigma_3(n/4) q^n \right) \\
& = \frac{1}{24} \sum_{n=1}^{\infty} \left(\frac{1}{2} \sigma_5(n) - \frac{33}{2} \sigma_5(n/2) + 16 \sigma_5(n/4) \right. \\
& \quad \left. + (1 - \frac{3}{2}n)(\sigma_3(n) - 9\sigma_3(n/2) + 8\sigma_3(n/4)) \right) q^n.
\end{aligned}$$

Since $\sigma\left(\frac{n-(2m-1)}{2}\right)$ is 0 if n is even, we only consider the case when n is odd. Changing $n \rightarrow 2n-1$, we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(2m-1) \sigma(n-m) q^{2n-1} \\
& = \frac{1}{24} \sum_{n=1}^{\infty} \left(\frac{1}{2} \sigma_5(2n-1) + \left(\frac{5}{2} - 3n \right) (\sigma_3(2n-1)) \right) q^{2n-1}.
\end{aligned}$$

Equating the coefficients of q^{2n-1} , we obtain the identity (2.9.6).

Next we prove (2.9.7). By (2.8.7), (1.1.1), (1.1.2), (1.1.3), (2.5.1) and (2.7.1), we have

$$\sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m) \sigma_3(n-m) q^{2n}$$

$$\begin{aligned}
&= \sum_{l=1}^{\infty} \sigma(2l) q^{2l} \sum_{k=1}^{\infty} \sigma_3(k) q^{2k} \\
&= \frac{M(q^2) L_{0,2}(q) - L_{0,2}(q)}{240} \\
&= \frac{1}{5760} M(q^2) - \frac{1}{120960} (53N(q^2) - 32N(q^4)) \\
&\quad - \frac{1}{1920} q \frac{dM(q^2)}{dq} - \frac{1}{5760} (1 - 3L(q^2) + 2L(q^4)) \\
&= -\frac{53}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right) + \frac{32}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{2n} \right) \\
&\quad + \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) - \frac{1}{1920} \left(240 \sum_{n=1}^{\infty} 2n \sigma_3(n) q^{2n} \right) \\
&\quad - \frac{1}{5760} + \frac{3}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) - \frac{2}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\
&= \frac{1}{240} \sum_{n=1}^{\infty} (53\sigma_5(n) - 32\sigma_5(n/2) + (10 - 60n)\sigma_3(n) \\
&\quad - 3\sigma(n) + 2\sigma(n/2)) q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.7).

Next we prove (2.9.8). By (2.8.8), (1.1.1), (1.1.2), (1.1.3), (2.5.2) and (2.7.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(2m-1) \sigma_3\left(\frac{n-(2m-1)}{2}\right) q^n \\
&= \sum_{l=1}^{\infty} \sigma(2l-1) q^{2l-1} \sum_{k=1}^{\infty} \sigma_3\left(\frac{k}{2}\right) q^k \\
&= \frac{M(q^2) L_{1,2}(q) - L_{1,2}(q)}{240} \\
&= -\frac{1}{120960} (N(q) - 33N(q^2) + 32N(q^4))
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{5760}(L(q) - 3L(q^2) + 2L(q^4)) \\
& = -\frac{1}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) + \frac{33}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^n \right) \\
& \quad - \frac{32}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/4) q^n \right) + \frac{1}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) \\
& \quad - \frac{3}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^n \right) + \frac{2}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^n \right) \\
& = \frac{1}{240} \sum_{n=1}^{\infty} (\sigma(n) - 33\sigma_5(n/2) + 32\sigma_5(n/4) \\
& \quad - \sigma(n) + 3\sigma(n/2) - 2\sigma(n/4)) q^n.
\end{aligned}$$

Since $\sigma_3\left(\frac{n - (2m - 1)}{2}\right)$ is 0 if n is even, we only consider the case when n is odd. Changing $n \rightarrow 2n - 1$, we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m - 1) \sigma_3(n - m) q^{2n-1} \\
& = \frac{1}{240} \sum_{n=1}^{\infty} (\sigma(2n - 1) - \sigma(2n - 1)) q^{2n-1}.
\end{aligned}$$

Equating the coefficients of q^{2n-1} , we obtain the identity (2.9.8).

Next we prove (2.9.9). By (2.8.9), (1.1.1), (1.1.2), (2.5.1) and (2.7.1), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m) \sigma\left(\frac{n - m}{2}\right) q^{2n} \\
& = \sum_{k=1}^{\infty} \sigma\left(\frac{k}{2}\right) q^{2k} \sum_{l=1}^{\infty} \sigma(2l) q^{2l} \\
& = \frac{L_{0,2}(q) - L(q^4) L_{0,2}(q)}{24} \\
& = \frac{1}{576} (1 - 3L(q^2) + 2L(q^4)) - \frac{1}{576} L(q^4)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2880}(3M(q^2) + 2M(q^4)) \\
& + \frac{1}{384} \left(3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq} \right) \\
= & \frac{1}{576} - \frac{3}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\
& + \frac{3}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) + \frac{2}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) \\
& + \frac{3}{384} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n) q^{2n} \right) + \frac{2}{384} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/2) q^{2n} \right) \\
= & \frac{1}{24} \sum_{n=1}^{\infty} (3\sigma(n) - \sigma(n/2) + 6\sigma_3(n) \\
& + 4\sigma_3(n/2) - 9n\sigma(n)q^n - 6n\sigma(n/2))q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.9).

Next we prove (2.9.10). By (2.8.10), (1.1.1), (1.1.2), (2.5.2) and (2.7.2), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(2m-1) \sigma\left(\frac{n-(2m-1)}{4}\right) q^n \\
= & \sum_{k=1}^{\infty} \sigma\left(\frac{k}{4}\right) q^k \sum_{l=1}^{\infty} \sigma(2l-1) q^{2l-1} \\
= & \frac{L_{1,2}(q) - L(q^4)L_{1,2}(q)}{24} \\
= & \frac{1}{576} (-L(q) + 3L(q^2) - 2L(q^4)) \\
& + \frac{1}{11520} (M(q) - 9M(q^2) + 8M(q^4)) \\
& + \frac{1}{384} \left(q \frac{dL(q)}{dq} - 3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{11520} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) - \frac{9}{11520} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) \\
&\quad + \frac{8}{11520} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^n \right) - \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) \\
&\quad + \frac{3}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^n \right) - \frac{2}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^n \right) \\
&\quad + \frac{1}{384} \left(-24 \sum_{n=1}^{\infty} n \sigma(n) q^n \right) - \frac{3}{384} \left(-24 \sum_{n=1}^{\infty} n \sigma(n/2) q^n \right) \\
&\quad + \frac{2}{384} \left(-24 \sum_{n=1}^{\infty} n \sigma(n/4) q^n \right) \\
&= \frac{1}{48} \sum_{n=1}^{\infty} (\sigma_3(n) - 9\sigma_3(n/2) + 8\sigma_3(n/4) \\
&\quad + 2\sigma(n) - 6\sigma(n/2) + 4\sigma(n/4) \\
&\quad - 3n\sigma(n) + 9n\sigma(n/2) - 6n\sigma(n/4)) q^n.
\end{aligned}$$

Since $\sigma\left(\frac{n - (2m - 1)}{4}\right)$ is 0 if n is even, we only consider the case when n is odd. Changing $n \rightarrow 2n - 1$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m - 1) \sigma\left(\frac{n - m}{2}\right) q^{2n-1} \\
&= \frac{1}{48} \sum_{n=1}^{\infty} (\sigma_3(2n - 1) + 2\sigma(2n - 1) - 3(2n - 1)\sigma(2n - 1)) q^{2n-1}.
\end{aligned}$$

Equating the coefficients of q^{2n-1} , we obtain the identity (2.9.10).

Next we prove (2.9.11). By (2.8.11), (1.1.1), (1.1.2), (1.1.3), (2.5.3) and (2.7.3), we have

$$\sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(2m) \sigma\left(\frac{n - m}{2}\right) q^{2n}$$

$$\begin{aligned}
&= \sum_{l=1}^{\infty} \sigma_3(2l) q^{2l} \sum_{k=1}^{\infty} \sigma\left(\frac{k}{2}\right) q^{2k} \\
&= \frac{M_{0,2}(q) - L(q^4)M_{0,2}(q)}{24} \\
&= \frac{1}{5760}(-1 + 9M(q^2) - 8M(q^4)) + \frac{1}{5760}L(q^4) \\
&\quad - \frac{1}{120960}(45N(q^2) - 24N(q^4)) \\
&\quad - \frac{1}{7680}\left(9q \frac{dM(q^2)}{dq} - 8q \frac{dM(q^4)}{dq}\right) \\
&= -\frac{1}{5760} + \frac{9}{5760}\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n}\right) \\
&\quad - \frac{8}{5760}\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n}\right) - \frac{45}{120960}\left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n}\right) \\
&\quad + \frac{24}{120960}\left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{2n}\right) - \frac{9}{7680}\left(240 \sum_{n=1}^{\infty} 2n\sigma_3(n) q^{2n}\right) \\
&\quad + \frac{8}{7680}\left(240 \sum_{n=1}^{\infty} 2n\sigma_3(n/2) q^{2n}\right) + \frac{1}{5760}\left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n}\right) \\
&= \frac{1}{24} \sum_{n=1}^{\infty} \left(\frac{9}{2}\sigma_5(n) - \frac{12}{5}\sigma_5(n/2) + 9\sigma_3(n) \right. \\
&\quad \left. - 8\sigma_3(n/2) - \frac{27}{2}n\sigma_3(n) + 12n\sigma_3(n/2) \right. \\
&\quad \left. - \frac{1}{10}\sigma(n/2)\right) q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.11).

Next we prove (2.9.12). By (2.8.12), (1.1.1), (1.1.2), (1.1.3), (2.5.1) and (2.7.1), we have

$$\sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m) \sigma_3\left(\frac{n-m}{2}\right) q^{2n}$$

$$\begin{aligned}
&= \sum_{l=1}^{\infty} \sigma(2l) q^{2l} \sum_{k=1}^{\infty} \sigma_3\left(\frac{k}{2}\right) q^{2k} \\
&= \frac{M(q^4)L_{0,2}(q) - L_{0,2}(q)}{240} \\
&= \frac{1}{5760} M(q^4) - \frac{1}{40320} (N(q^2) + 6N(q^4)) \\
&\quad - \frac{1}{1920} q \frac{dM(q^4)}{dq} - \frac{1}{5760} (1 - 3L(q^2) + 2L(q^4)) \\
&= -\frac{1}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right) - \frac{6}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{2n} \right) \\
&\quad + \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) - \frac{1}{1920} \left(240 \sum_{n=1}^{\infty} 2n \sigma_3(n/2) q^{2n} \right) \\
&\quad - \frac{1}{1920} + \frac{3}{1920} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) - \frac{2}{1920} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\
&= \frac{1}{240} \sum_{n=1}^{\infty} (3\sigma_5(n) + 18\sigma_5(n/2) + 10\sigma_3(n/2) \\
&\quad - 60n\sigma_3(n/2) - 3\sigma(n) + 2\sigma(n/2)) q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.12).

Next we prove (2.9.13). By (2.8.13), (1.1.1) and (1.1.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n/2} \sigma(m) \sigma(n-2m) q^n \\
&= \sum_{l=1}^{\infty} \sigma(l) q^l \sum_{k=1}^{\infty} \sigma(k) q^{2k} \\
&= \frac{1}{576} \{1 - L(q) - L(q^2) + L(q)L(q^2)\} \\
&= \frac{1}{576} (1 - L(q) - L(q^2)) + \frac{1}{2880} (M(q) + 4M(q^2)) \\
&\quad + \frac{1}{192} \left(q \frac{dL(q)}{dq} + 2q \frac{dL(q^2)}{dq} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{576} - \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) - \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^n \right) \\
&\quad + \frac{1}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) + \frac{4}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) \\
&\quad + \frac{1}{192} \left(-24 \sum_{n=1}^{\infty} n \sigma(n) q^n \right) + \frac{2}{192} \left(-24 \sum_{n=1}^{\infty} n \sigma(n/2) q^n \right) \\
&= \frac{1}{24} \sum_{n=1}^{\infty} (\sigma(n) + \sigma(n/2) + 2\sigma_3(n) + 8\sigma_3(n/2) \\
&\quad - 3n\sigma(n) - 6n\sigma(n/2)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (2.9.13).

Next we prove (2.9.14). By (2.8.14), (1.1.1) and (1.1.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n/4} \sigma(m) \sigma(n - 4m) q^n \\
&= \sum_{l=1}^{\infty} \sigma(l) q^l \sum_{k=1}^{\infty} \sigma(k) q^{4k} \\
&= \frac{1}{576} \{ 1 - L(q) - L(q^4) + L(q)L(q^4) \} \\
&= \frac{1}{576} - \frac{1}{576} L(q) - \frac{1}{576} L(q^4) + \frac{1}{11520} (M(q) + 3M(q^2) + 16M(q^4)) \\
&\quad + \frac{1}{384} q \frac{dL(q)}{dq} + 6q \frac{dL(q^4)}{dq} \} \\
&= \frac{1}{576} - \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) \\
&\quad - \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^n \right) + \frac{1}{11520} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \\
&\quad + \frac{3}{11520} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) + \frac{16}{11520} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{384} \left(-24 \sum_{n=1}^{\infty} n\sigma(n)q^n \right) + \frac{2}{384} \left(-24 \sum_{n=1}^{\infty} n\sigma(n/4)q^n \right) \\
& = \frac{1}{48} \sum_{n=1}^{\infty} (2\sigma(n) + 2\sigma(n/4) + \sigma_3(n) + 3\sigma_3(n/2) \\
& \quad + 16\sigma_3(n/4) - 3n\sigma(n) - 12n\sigma(n/4))q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (2.9.14).

Next we prove (2.9.15). By (2.8.15), (1.1.1), (1.1.2) and (1.1.3), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n/2} \sigma_3(m)\sigma(n-2m)q^n \\
& = \sum_{l=1}^{\infty} \sigma(l)q^l \sum_{k=1}^{\infty} \sigma_3(k)q^{2k} \\
& = \frac{1}{5760} \{-1 + L(q) + M(q^2) - L(q)M(q^2)\} \\
& = -\frac{1}{5760} + \frac{1}{5760}L(q) + \frac{1}{5760}M(q^2) \\
& \quad - \frac{1}{120960}(N(q) + 20N(q^2)) + \frac{1}{1920}q \frac{dM(q^2)}{dq} \\
& = -\frac{1}{5760} + \frac{1}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n \right) \\
& \quad + \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2)q^n \right) - \frac{1}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n \right) \\
& \quad - \frac{20}{120960} \left(1 + 504 \sum_{n=1}^{\infty} \sigma_5(n/2)q^n \right) + \frac{1}{8} \sum_{n=1}^{\infty} n\sigma_3(n/2)q^n \\
& = \frac{1}{240} \sum_{n=1}^{\infty} (\sigma_5(n) + 20\sigma_5(n/2) + (10 - 30n)\sigma_3(n/2) - \sigma(n))q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (2.9.15).

Next we prove (2.9.16). By (2.8.16), (1.1.1), (1.1.2) and (1.1.3), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n/2} \sigma(m) \sigma_3(n-2m) q^n \\
&= \sum_{l=1}^{\infty} \sigma(l) q^l \sum_{k=1}^{\infty} \sigma_3(k) q^{2k} \\
&= \frac{1}{5760} \{-1 + L(q^2) + M(q) - L(q^2)M(q)\} \\
&= -\frac{1}{5760} + \frac{1}{5760} L(q^2) + \frac{1}{5760} M(q) \\
&\quad - \frac{1}{120960} (5N(q) + 16N(q^2)) + \frac{1}{3840} q \frac{dM(q)}{dq} \\
&= -\frac{1}{5760} + \frac{1}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^n \right) + \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \\
&\quad - \frac{5}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) - \frac{16}{120960} \left(1 + 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^n \right) \\
&\quad - \frac{1}{16} \sum_{n=1}^{\infty} n \sigma_3(n) q^n \\
&= \frac{1}{240} \sum_{n=1}^{\infty} (5\sigma_5(n) + 16\sigma_5(n/2) + (10 - 15n)\sigma_3(n) - \sigma(n/2)) q^n.
\end{aligned}$$

Equating the coefficients of q^n , we obtain the identity (2.9.16).

Next we prove (2.9.17). By (2.8.17), (1.1.1) and (1.1.2), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m) \sigma(2n-2m) q^{2n} \\
&= \left(\sum_{l=1}^{\infty} \sigma(2l) q^{2l} \right)^2 = L_{0,2}^2(q) \\
&= \frac{1}{576} - \frac{1}{288} (3L(q^2) - 2L(q^4)) + \frac{1}{2880} (33M(q^2) - 28M(q^4)) \\
&\quad + \frac{1}{48} \left(3q \frac{dL(q^2)}{dq} - 2q \frac{dL(q^4)}{dq} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{576} - \frac{3}{288} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{2}{288} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\
&\quad + \frac{33}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) - \frac{283}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) \\
&\quad + \frac{3}{48} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n) q^{2n} \right) - \frac{2}{48} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/2) q^{2n} \right) \\
&= \frac{1}{12} \sum_{n=1}^{\infty} (33\sigma_3(n) - 28\sigma_3(n/2) + (1 - 12n)(3\sigma(n) - 2\sigma(n/2))) q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.17).

Next we prove (2.9.18). By (2.8.18), (1.1.1) and (1.1.2), we have for odd n

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(2m-1) \sigma(n - (2m-1)) q^n \\
&= \sum_{l=1}^{\infty} \sigma(2l) q^{2l} \sum_{k=1}^{\infty} \sigma(2k-1) q^{2k-1} \\
&= L_{0,2}(q) L_{1,2}(q) \\
&= -\frac{1}{576} (L(q) - 3L(q^2) + 2L(q^4)) \\
&\quad + \frac{1}{1152} (M(q) - 9M(q^2) + 8M(q^4)) \\
&\quad + \frac{1}{96} \left(q \frac{dL(q)}{dq} - 3q \frac{dL(q^2)}{dq} + 2q \frac{dL(q^4)}{dq} \right) \\
&= -\frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n \right) + \frac{3}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^n \right) \\
&\quad - \frac{2}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^n \right) + \frac{1}{1152} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \\
&\quad - \frac{9}{1152} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) + \frac{8}{1152} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^n \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{96} \left(-24 \sum_{n=1}^{\infty} n \sigma(n) q^n \right) - \frac{3}{96} \left(-24 \sum_{n=1}^{\infty} n \sigma(n/2) q^n \right) \\
& + \frac{2}{96} \left(-24 \sum_{n=1}^{\infty} n \sigma(n/4) q^n \right).
\end{aligned}$$

Since n is odd, we replace $n \rightarrow 2n - 1$ and we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m - 1) \sigma(2n - 2m) q^{2n-1} \\
& = \frac{5}{24} \sum_{n=1}^{\infty} (\sigma_3(2n - 1) + \frac{1}{24} (1 - 6(2n - 1)) \sigma(2n - 1)) q^{2n-1}.
\end{aligned}$$

Equating the coefficients of q^{2n-1} , we obtain the identity (2.9.18).

Next we prove (2.9.19). By (2.8.19) and (1.1.2), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n+1} \sigma(2m - 1) \sigma(2n - (2m - 1)) q^{2n} \\
& = \left(\sum_{l=1}^{\infty} \sigma(2l - 1) q^{2l-1} \right)^2 = L_{1,2}^2(q) \\
& = \frac{1}{240} (M(q^2) - M(q^4)) \\
& = \frac{1}{240} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) - \frac{1}{240} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) \\
& = \sum_{n=1}^{\infty} (\sigma_3(n) - \sigma_3(n/2)) q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.19).

Next we prove (2.9.20). By (2.8.20), (1.1.1), (1.1.2) and (1.1.3), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(2m) \sigma_3(2n - 2m) q^{2n} \\
& = \sum_{l=1}^{\infty} \sigma(2l) q^{2l} \sum_{k=1}^{\infty} \sigma_3(2k) q^{2k}
\end{aligned}$$

$$\begin{aligned}
&= L_{0,2}(q)M_{0,2}(q) \\
&= -\frac{1}{5760} + \frac{1}{5760}(3L(q^2) - 2L(q^4)) + \frac{1}{5760}(9M(q^2) - 8M(q^4)) \\
&\quad - \frac{1}{40320}(151N(q^2) - 144N(q^4)) - \frac{1}{1920} \left(9q \frac{dM(q^2)}{dq} - 8q \frac{dM(q^4)}{dq} \right) \\
&= -\frac{1}{5760} + \frac{3}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^{2n} \right) - \frac{2}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2)q^{2n} \right) \\
&\quad + \frac{9}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n} \right) - \frac{8}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2)q^{2n} \right) \\
&\quad - \frac{151}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n} \right) + \frac{144}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2)q^{2n} \right) \\
&\quad - \frac{9}{1920} \left(240 \sum_{n=1}^{\infty} 2n\sigma_3(n)q^{2n} \right) + \frac{8}{1920} \left(240 \sum_{n=1}^{\infty} 2n\sigma_3(n/2)q^{2n} \right) \\
&= \frac{1}{240} \sum_{n=1}^{\infty} ((1-6n)(9\sigma_3(n) - 8\sigma_3(n/2)) \\
&\quad + 453\sigma_5(n) - 432\sigma_5(n/2) - 3\sigma(n) + 2\sigma(n/2))q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.20).

Next we prove (2.9.21). By (2.8.19) and (1.1.2), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n+1} \sigma(2m-1)\sigma_3(2n-(2m-1))q^{2n} \\
&= \sum_{l=1}^{\infty} \sigma(2l-1)q^{2l-1} \sum_{k=1}^{\infty} \sigma_3(2k-1)q^{2k-1} \\
&= L_{1,2}(q)M_{1,2}(q) \\
&= -\frac{1}{504}(N(q^2) - N(q^4)) \\
&= -\frac{1}{504} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n} \right) + \frac{1}{504} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2)q^{2n} \right)
\end{aligned}$$

$$= \sum_{n=1}^{\infty} (\sigma_5(n) - \sigma_5(n/2)) q^{2n}.$$

Equating the coefficients of q^{2n} , we obtain the identity (2.9.21).

Next we prove (2.9.22). By (2.8.22), (1.1.2) and (1.1.4), we have for odd n

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma_3(2m) \sigma_3(n-2m) q^n \\ &= \sum_{l=1}^{\infty} \sigma_3(2l) q^{2l} \sum_{k=1}^{\infty} \sigma_3(2k-1) q^{2k-1} \\ &= M_{0,2}(q) M_{1,2}(q) \\ &= -\frac{1}{57600} (M(q) - 9M(q^2) + 8M(q^4)) \\ &\quad + \frac{1}{115200} (P(q) - 129P(q^2) + 128P(q^4)) \\ &= -\frac{1}{57600} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) + \frac{9}{57600} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^n \right) \\ &\quad - \frac{8}{57600} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^n \right) + \frac{1}{115200} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\ &\quad - \frac{129}{115200} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^n \right) + \frac{128}{115200} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/4) q^n \right). \end{aligned}$$

Since n is odd, we have

$$\sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma_3(2m) \sigma_3(n-2m) q^n = \frac{1}{240} \sum_{n=1}^{\infty} (\sigma_7(n) - \sigma_3(n)) q^n.$$

Equating the coefficients of q^n , we obtain the identity (2.9.22). $\square$

We did not find any identities among $L(q)$, $L(q^2)$, $L(q^4)$, $M(q)$, $M(q^2)$, $M(q^4)$, ... $S(q)$, $S(q^2)$, $S(q^4)$, involving products of three functions with at

least one $X(q^2)$ or $X(q^4)$ term. For example, no evaluation of $L^2(q)L(q^2)$ was found. Such an evaluation would lead to the determination of the arithmetic sum

$$\sum_{m_1+m_2+2m_3=n} \sigma(m_1)\sigma(m_2)\sigma(m_3).$$

We remark that there are interrelationships between the formulae of Theorem 2.9.1, as consequences of the simple identity

$$\sigma_k(2m) = (2^k + 1)\sigma_k(m) - 2^k\sigma_k(m/2), \quad k \in \mathbb{N}, \quad m \in \mathbb{N}.$$

For example we have

$$\begin{aligned} & \sum_{m < n} \sigma(2m)\sigma(n-m) \\ &= \sum_{m < n} (3\sigma(m) - 2\sigma(m/2))\sigma(n-m) \\ &= 3 \sum_{m < n} \sigma(m)\sigma(n-m) - 2 \sum_{m < n} \sigma(m/2)\sigma(n-m) \\ &= 3 \sum_{m < n} \sigma(m)\sigma(n-m) - 2 \sum_{m < n/2} \sigma(m)\sigma(n-2m) \end{aligned}$$

so that (2.9.3) follows from (1.6.1) and (2.9.13).

We conclude this chapter by noting that the sums (1.6.1) and (2.9.14) were used by Williams [36] to give an arithmetic proof of Jacobi's formula for the number of representations of $n \in \mathbb{N}$ as the sum of eight squares.

Chapter 3

The functions $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 1, 2, 3$.

3.1 Obtaining a formula by dimidiation.

The process of dimidiation is useful in obtaining formulae from previously derived formulae by changing q^2 to q . Berndt [3, p. 126] shows that given x , q and w defined in Section 1.2, if

$$\Omega(x, q, w) = 0,$$

then a second formula can be obtained by dimidiation using

$$\Omega\left(\frac{4\sqrt{x}}{(1+\sqrt{x})^2}, q^{\frac{1}{2}}, w(1+\sqrt{x})\right) = 0.$$

3.2 Obtaining a formula by rotation.

Theorem 3.2.1 *Suppose a formula is of the form*

$$\Omega(x, q, w) = 0,$$

where x , q and w are as defined in Section 1.2, then a second formula can be obtained by rotation using

$$\Omega\left(\frac{-8i(1-x)^{\frac{1}{4}}(1-\sqrt{1-x})}{(1-i(1-x)^{1/4})^4}, iq, i\frac{w}{2}(1-i(1-x)^{\frac{1}{4}})^2\right) = 0.$$

Sketch of a proof Suppose a formula is of the form

$$\Omega(x, q, w) = 0.$$

By duplication, we have

$$\Omega\left(\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right)^2, q^2, \frac{1}{2}w(1+\sqrt{1-x})\right) = 0.$$

By change of sign, we have

$$\Omega\left(\frac{(1-\sqrt{1-x})^2}{-4\sqrt{1-x}}, -q^2, w(1-x)^{\frac{1}{4}}\right) = 0.$$

By dimidiation, we have

$$\Omega\left(\frac{-8i(1-x)^{\frac{1}{4}}(1-\sqrt{1-x})}{(1-i(1-x)^{1/4})^4}, iq, i\frac{w}{2}(1-i(1-x)^{\frac{1}{4}})^2\right) = 0.$$

□

This is not a completely rigorous proof as dimidiation $q \rightarrow q^{\frac{1}{2}}$ only applies when $0 < q < 1$ and we are applying it to $-q^2$ which is between -1 and 0 . Care must therefore be taken to ensure that the correct squareroot i is taken. A rigorous proof is given in Chapter 5.

3.3 Formulae for $L(q^8)$, $M(q^8)$, ..., $S(q^8)$.

In this section we obtain by the process of duplication formulae for $L(q^8)$, $M(q^8)$, ..., $S(q^8)$ in terms of x . For simplicity, we let $g = (1 - x)^{\frac{1}{4}}$ so that $x = 1 - g^4$.

Theorem 3.3.1

$$(3.3.1) \quad L(q^8) = \frac{1}{16}(-1 + 6g^2 + 11g^4)w^2 + \frac{3}{2}g^4(1 - g^4)w\frac{dw}{dx},$$

$$(3.3.2) \quad M(q^8) = \frac{1}{256}(1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^4,$$

$$(3.3.3) \quad N(q^8) = -\frac{1}{4096}(1 + 6g^2 + g^4) \\ (1 - 132g^2 - 250g^4 - 132g^6 + g^8)w^6,$$

$$(3.3.4) \quad P(q^8) = \frac{1}{65336}(1 + 60g^2 + 134g^4 + 60g^6 + g^8)^2w^8,$$

$$(3.3.5) \quad Q(q^8) = -\frac{1}{1048576}(1 + 6g^2 + g^4) \\ (1 - 132g^2 - 250g^4 - 132g^6 + g^8) \\ (1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^{10},$$

$$(3.3.6) \quad R(q^8) = \frac{441}{11593056256}(1 + 60g^2 + 134g^4 + 60g^6 + g^8)^3w^{12} \\ + \frac{250}{11593056256}(1 + 6g^2 + g^4)^2 \\ (1 - 132g^2 - 250g^4 - 132g^6 + g^8)^2w^{12},$$

$$(3.3.7) \quad S(q^8) = -\frac{1}{268435456}(1 + 6g^2 + g^4) \\ (1 - 132g^2 - 250g^4 - 132g^6 + g^8) \\ (1 + 60g^2 + 134g^4 + 60g^6 + g^8)^2w^{14}.$$

Proof Under duplication we have

$$\begin{aligned} q &\rightarrow q^2, \\ x &\rightarrow x' = \left(\frac{1-g^2}{1+g^2} \right)^2, \\ w &\rightarrow w' = \frac{1}{2}w(1+g^2). \end{aligned}$$

Thus

$$\begin{aligned} \frac{dw}{dx} \rightarrow \frac{dw'}{dx'} &= \frac{\frac{dw'}{dx}}{\frac{dx'}{dx}} = \frac{\frac{1+g^2}{2} \frac{dw}{dx} - \frac{w}{4g^2}}{\frac{2(1-g^2)}{dx} \frac{g^2(1+g^2)^3}{g^2(1+g^2)^3}} \\ &= \frac{g^2(1+g^2)^4}{4(1-g^2)} \frac{dw}{dx} - \frac{w(1+g^2)^3}{8(1-g^2)}. \end{aligned}$$

First we prove (3.3.1). By applying duplication to (2.3.2), we obtain

$$\begin{aligned} L(q^8) &= \left(1 - \frac{5}{4} \left(\frac{1-g^2}{1+g^2} \right)^2 \right) \left(\frac{w^2}{4} (1+g^2)^2 \right) \\ &\quad + \frac{3}{2} \left(\frac{1-g^2}{1+g^2} \right)^2 \left(1 - \left(\frac{1-g^2}{1+g^2} \right)^2 \right) (1+g^2) w \frac{dw'}{dx'} \\ &= \frac{1}{4} (1+g^2)^2 w^2 - \frac{5}{16} (1-g^2)^2 w^2 \\ &\quad + \frac{3}{2} g^4 (1-g^4) w \frac{dw}{dx} \\ &\quad - \frac{3}{4} (1-g^2) (g^2) w^2 \\ &= \frac{1}{16} (-1 + 6g^2 + 11g^4) w^2 + \frac{3}{2} g^4 (1-g^4) w \frac{dw}{dx}. \end{aligned}$$

Next we prove (3.3.2). Applying duplication to (2.3.4), we obtain

$$\begin{aligned} M(q^8) &= \frac{1}{16} w^4 (1+g^2)^4 \left(1 - \left(\frac{1-g^2}{1+g^2} \right)^2 + \frac{1}{16} \left(\frac{1-g^2}{1+g^2} \right)^4 \right) \\ &= \frac{1}{256} (1 + 60g^2 + 134g^4 + 60g^8 + g^8) w^4. \end{aligned}$$

Next we prove (3.3.3). Applying duplication to (2.3.6), we obtain

$$\begin{aligned} N(q^8) &= \frac{1}{64} w^6 (1+g^2)^6 \left(1 - \frac{1}{2} \left(\frac{1-g^2}{1+g^2} \right)^2 \right) \\ &\quad \left(1 - \left(\frac{1-g^2}{1+g^2} \right)^2 - \frac{1}{34} \left(\frac{1-g^2}{1+g^2} \right)^4 \right) \\ &= -\frac{1}{4096} (1+6g^2+g^4)(1-132g^2-250g^4-132g^6+g^8)w^6. \end{aligned}$$

Next we prove (3.3.4). By (1.5.6) and (3.3.2), we have

$$P(q^8) = M^2(q^8) = \frac{1}{65336} (1+60g^2+134g^4+60g^6+g^8)^2 w^8.$$

Next we prove (3.3.5). By (1.5.7), (3.3.2) and (3.3.3), we have

$$\begin{aligned} Q(q^8) &= M(q^8)N(q^8) \\ &= -\frac{1}{1048576} (1+6g^2+g^4)(1-132g^2-250g^4-132g^6+g^8) \\ &\quad (1+60g^2+134g^4+60g^6+g^8) w^{10}. \end{aligned}$$

Next we prove (3.3.6). By (1.2.18), (3.3.2), (3.3.3) and (3.3.4), we have

$$\begin{aligned} R(q^8) &= \frac{441}{691} M(q^8)P(q^8) + \frac{250}{691} N^2(q^8) \\ &= \frac{441}{11593056256} (1+60g^2+134g^4+60g^6+g^8)^3 w^{12} \\ &\quad + \frac{250}{11593056256} (1+6g^2+g^4)^2 \\ &\quad (1-132g^2-250g^4-132g^6+g^8)^2 w^{12}. \end{aligned}$$

Lastly, we prove (3.3.7). By (1.5.8), (3.3.2) and (3.3.5), we have

$$S(q^8) = M(q^8)Q(q^8)$$

$$= -\frac{1}{268435456} (1 + 6g^2 + g^4) (1 - 132g^2 - 250g^4 - 132g^6 + g^8) \\ (1 + 60g^2 + 134g^4 + 60g^6 + g^8)^2 w^{14}.$$

This completes the proof of Theorem 3.3.1. $\square$

3.4 Formulae for $L(iq)$, $L(-iq)$, ..., $S(iq)$, $S(-iq)$.

In this section we obtain formulae for $L(iq)$, $L(-iq)$, ..., $S(iq)$, $S(-iq)$ in terms of x . Let $g = (1 - x)^{\frac{1}{4}}$.

Theorem 3.4.1

$$(3.4.1) \quad L(iq) = -\frac{1}{4}(1 + 12ig + 18g^2 - 12ig^3 - 23g^4)w^2 \\ + 12g^4(1 - g^4)w \frac{dw}{dx},$$

$$(3.4.2) \quad L(-iq) = -\frac{1}{4}(1 - 12ig + 18g^2 + 12ig^3 - 23g^4)w^2 \\ + 12g^4(1 - g^4)w \frac{dw}{dx},$$

$$(3.4.3) \quad M(iq) = \frac{1}{16}(1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\ - 840ig^5 - 540g^6 + 120ig^7 + g^8)w^4,$$

$$(3.4.4) \quad M(-iq) = \frac{1}{16}(1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \\ + 840ig^5 - 540g^6 - 120ig^7 + g^8)w^4,$$

$$(3.4.5) \quad N(iq) = \frac{1}{64}(1 - 12ig - 6g^2 + 12ig^3 + g^4) \\ (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\ - 1848ig^5 - 996g^6 + 264ig^7 - g^8)w^6,$$

$$(3.4.6) \quad N(-iq) = \frac{1}{64}(1 + 12ig - 6g^2 - 12ig^3 + g^4) \\ (-1 + 264ig - 996g^2 - 1848ig^3 + 1978g^4 \\ + 1848ig^5 - 996g^6 - 264ig^7 - g^8)w^6,$$

$$(3.4.7) \quad P(iq) = \frac{1}{256}(1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\ - 840ig^5 - 540g^6 + 120ig^7 + g^8)^2w^8,$$

$$(3.4.8) \quad P(-iq) = \frac{1}{256}(1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \\ + 840ig^5 - 540g^6 - 120ig^7 + g^8)^2w^8,$$

$$(3.4.9) \quad Q(iq) = \frac{1}{1024}(1 - 12ig - 6g^2 + 12ig^3 + g^4) \\ (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\ - 1848ig^5 - 996g^6 + 264ig^7 - g^8) \\ (1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\ - 840ig^5 - 540g^6 + 120ig^7 + g^8)w^{10},$$

$$(3.4.10) \quad Q(-iq) = \frac{1}{1024}(1 + 12ig - 6g^2 - 12ig^3 + g^4) \\ (-1 + 264ig - 996g^2 - 1848ig^3 + 1978g^4 \\ + 1848ig^5 - 996g^6 - 264ig^7 - g^8) \\ (1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \\ + 840ig^5 - 540g^6 - 120ig^7 + g^8)w^{10},$$

$$(3.4.11) \quad R(iq) = \frac{441}{2830336}(1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\ - 840ig^5 - 540g^6 + 120ig^7 + g^8)^3w^{12}$$

$$\begin{aligned}
& + \frac{250}{2830336} (1 - 12ig - 6g^2 + 12ig^3 + g^4)^2 \\
& \quad (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\
& \quad - 1848ig^5 - 996g^6 + 264ig^7 - g^8)^2 w^{12}, \\
(3.4.12) \quad R(-iq) &= \frac{441}{2830336} (1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \\
& \quad + 840ig^5 - 540g^6 - 120ig^7 + g^8)^3 w^{12} \\
& + \frac{250}{2830336} (1 + 12ig - 6g^2 - 12ig^3 + g^4)^2 \\
& \quad (-1 + 264ig - 996g^2 - 1848ig^3 + 1978g^4 \\
& \quad + 1848ig^5 - 996g^6 - 264ig^7 - g^8)^2 w^{12},
\end{aligned}$$

$$\begin{aligned}
(3.4.13) \quad S(iq) &= \frac{1}{16384} (1 - 12ig - 6g^2 + 12ig^3 + g^4) \\
& \quad (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\
& \quad - 1848ig^5 - 996g^6 + 264ig^7 - g^8) \\
& \quad (1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\
& \quad - 840ig^5 - 540g^6 + 120ig^7 + g^8)^2 w^{14},
\end{aligned}$$

$$\begin{aligned}
(3.4.14) \quad S(-iq) &= \frac{1}{16384} (1 + 12ig - 6g^2 - 12ig^3 + g^4) \\
& \quad (-1 + 264ig - 996g^2 - 1848ig^3 + 1978g^4 \\
& \quad + 1848ig^5 - 996g^6 - 264ig^7 - g^8) \\
& \quad (1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \\
& \quad + 840ig^5 - 540g^6 - 120ig^7 + g^8)^2 w^{14}.
\end{aligned}$$

Proof Under rotation we have

$$\begin{aligned} q &\rightarrow iq, \\ x &\rightarrow x' = \frac{-8ig(1-g^2)}{(1-ig)^4}, \\ w &\rightarrow w' = i\frac{w}{2}(1-ig)^2. \end{aligned}$$

Thus

$$\begin{aligned} \frac{dw}{dx} \rightarrow \frac{dw'}{dx'} &= \frac{\frac{dw'}{dx}}{\frac{dx'}{dx}} = \frac{\frac{dw'}{dx}}{\frac{2i(1+3g^2+3g^4+g^6)}{g^3(2g+i(1-g^2))^2}} \\ &= \frac{g^3(1-ig)^7}{4(1+ig)^3} \frac{dw}{dx} + \frac{i(1-ig)^6 w}{8(1+ig)^3}. \end{aligned}$$

First we prove (3.4.1) and (3.4.2). By applying rotation to (1.2.9), we obtain

$$\begin{aligned} L(iq) &= \left(1 - 5\frac{-8ig(1-g^2)}{(1-ig)^4}\right) \left(-\frac{w^2}{4}(1-ig)^4\right) \\ &\quad + 12 \left(\frac{-8ig(1-g^2)}{(1-ig)^4}\right) \left(1 - \frac{-8ig(1-g^2)}{(1-ig)^4}\right) \left(i\frac{w}{2}(1-ig)^2\right) \\ &\quad \left(\frac{g^3(1-ig)^7}{4(1+ig)^3} \frac{dw}{dx} + \frac{i(1-ig)^6 w}{8(1+ig)^3}\right) \\ &= -\frac{1}{4}(1 + 12ig + 18g^2 - 12ig^3 - 23g^4)w^2 \\ &\quad + 12g^4(1-g^4)w \frac{dw}{dx}. \end{aligned}$$

Taking the complex conjugate of (3.4.1), we obtain (3.4.2).

Next we prove (3.4.3) and (3.4.4). By applying rotation to (1.2.10), we

obtain

$$\begin{aligned}
 M(iq) &= \left(1 + 14 \left(\frac{-8ig(1-g^2)}{(1-ig)^4} \right) + \left(\frac{-8ig(1-g^2)}{(1-ig)^4} \right)^2 \right) \left(\frac{w^4}{16} (1-ig)^8 \right) \\
 &= \frac{1}{16} (1 + 120ig - 540g^2 - 840ig^3 + 1094g^4 \\
 &\quad + 840ig^5 - 540g^6 - 120ig^7 + g^8) w^4.
 \end{aligned}$$

Taking the complex conjugate of (3.4.3), we obtain (3.4.4).

Next we prove (3.4.5) and (3.4.6). By applying rotation to (1.2.11), we obtain

$$\begin{aligned}
 N(iq) &= \left(-\frac{w^4}{64} (1-ig)^{12} \right) \left(1 + \left(\frac{-8ig(1-g^2)}{(1-ig)^4} \right) \right) \\
 &\quad \left(1 - 34 \left(\frac{-8ig(1-g^2)}{(1-ig)^4} \right) + \left(\frac{-8ig(1-g^2)}{(1-ig)^4} \right)^2 \right) \\
 &= \frac{1}{64} (1 - 12ig - 6g^2 + 12ig^3 + g^4) \\
 &\quad (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\
 &\quad - 1848ig^5 - 996g^6 + 264ig^7 - g^8) w^6.
 \end{aligned}$$

Taking the complex conjugate of (3.4.5), we obtain (3.4.6).

Next we prove (3.4.7) and (3.4.8). By (1.5.6) and (3.4.3), we have

$$\begin{aligned}
 P(iq) &= M^2(iq) \\
 &= \frac{1}{256} (1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\
 &\quad - 840ig^5 - 540g^6 + 120ig^7 + g^8)^2 w^8.
 \end{aligned}$$

Taking the complex conjugate of (3.4.7), we obtain (3.4.8).

Next we prove (3.4.9) and (3.4.10). By (1.5.7), (3.4.3) and (3.4.5), we have

$$\begin{aligned}
 Q(iq) &= M(iq)N(iq) \\
 &= \frac{1}{1024}(1 - 12ig - 6g^2 + 12ig^3 + g^4) \\
 &\quad (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\
 &\quad - 1848ig^5 - 996g^6 + 264ig^7 - g^8) \\
 &\quad (1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\
 &\quad - 840ig^5 - 540g^6 + 120ig^7 + g^8)w^{10}.
 \end{aligned}$$

Taking the complex conjugate of (3.4.9), we obtain (3.4.10).

Next we prove (3.4.11) and (3.4.12). By (1.2.18), (3.4.3), (3.4.5) and (3.4.7), we have

$$\begin{aligned}
 R(iq) &= \frac{441}{691}M(iq)P(iq) + \frac{250}{691}N^2(iq) \\
 &= \frac{441}{2830336}(1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\
 &\quad - 840ig^5 - 540g^6 + 120ig^7 + g^8)^3w^{12} \\
 &\quad + \frac{250}{2830336}(1 - 12ig - 6g^2 + 12ig^3 + g^4)^2 \\
 &\quad (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\
 &\quad - 1848ig^5 - 996g^6 + 264ig^7 - g^8)^2w^{12}.
 \end{aligned}$$

Taking the complex conjugate of (3.4.11), we obtain (3.4.12).

Lastly we prove (3.4.13) and (3.4.14). By (1.5.8), (3.4.3) and (3.4.9), we

have

$$\begin{aligned}
 S(iq) &= M(iq)Q(iq) \\
 &= \frac{1}{16384} (1 - 12ig - 6g^2 + 12ig^3 + g^4) \\
 &\quad (-1 - 264ig - 996g^2 + 1848ig^3 + 1978g^4 \\
 &\quad - 1848ig^5 - 996g^6 + 264ig^7 - g^8) \\
 &\quad (1 - 120ig - 540g^2 + 840ig^3 + 1094g^4 \\
 &\quad - 840ig^5 - 540g^6 + 120ig^7 + g^8)^2 w^{14}.
 \end{aligned}$$

Taking the complex conjugate of (3.4.13), we obtain (3.4.14). This completes the proof of Theorem 3.4.1. $\square$

3.5 The functions $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for

$$i = 0, 1, 2, 3.$$

In this section we define for $|q| < 1$ the functions

$$(3.5.1) \quad L_{0,4}(q) = \sum_{n=1}^{\infty} \sigma(4n)q^{4n},$$

$$(3.5.2) \quad L_{1,4}(q) = \sum_{n=1}^{\infty} \sigma(4n-3)q^{4n-3},$$

$$(3.5.3) \quad L_{2,4}(q) = \sum_{n=1}^{\infty} \sigma(4n-2)q^{4n-2},$$

$$(3.5.4) \quad L_{3,4}(q) = \sum_{n=1}^{\infty} \sigma(4n-1)q^{4n-1},$$

$$(3.5.5) \quad M_{0,4}(q) = \sum_{n=1}^{\infty} \sigma_3(4n)q^{4n},$$

$$(3.5.6) \quad M_{1,4}(q) = \sum_{n=1}^{\infty} \sigma_3(4n-3)q^{4n-3},$$

$$(3.5.7) \quad M_{2,4}(q) = \sum_{n=1}^{\infty} \sigma_3(4n-2)q^{4n-2},$$

$$(3.5.8) \quad M_{3,4}(q) = \sum_{n=1}^{\infty} \sigma_3(4n-1)q^{4n-1},$$

$$(3.5.9) \quad N_{0,4}(q) = \sum_{n=1}^{\infty} \sigma_5(4n)q^{4n},$$

$$(3.5.10) \quad N_{1,4}(q) = \sum_{n=1}^{\infty} \sigma_5(4n-3)q^{4n-3},$$

$$(3.5.11) \quad N_{2,4}(q) = \sum_{n=1}^{\infty} \sigma_5(4n-2)q^{4n-2},$$

$$(3.5.12) \quad N_{3,4}(q) = \sum_{n=1}^{\infty} \sigma_5(4n-1)q^{4n-1},$$

$$(3.5.13) \quad P_{0,4}(q) = \sum_{n=1}^{\infty} \sigma_7(4n)q^{4n},$$

$$(3.5.14) \quad P_{1,4}(q) = \sum_{n=1}^{\infty} \sigma_7(4n-3)q^{4n-3},$$

$$(3.5.15) \quad P_{2,4}(q) = \sum_{n=1}^{\infty} \sigma_7(4n-2)q^{4n-2},$$

$$(3.5.16) \quad P_{3,4}(q) = \sum_{n=1}^{\infty} \sigma_7(4n-1)q^{4n-1},$$

$$(3.5.17) \quad Q_{0,4}(q) = \sum_{n=1}^{\infty} \sigma_9(4n)q^{4n},$$

$$(3.5.18) \quad Q_{1,4}(q) = \sum_{n=1}^{\infty} \sigma_9(4n-3)q^{4n-3},$$

$$(3.5.19) \quad Q_{2,4}(q) = \sum_{n=1}^{\infty} \sigma_9(4n-2)q^{4n-2},$$

$$(3.5.20) \quad Q_{3,4}(q) = \sum_{n=1}^{\infty} \sigma_9(4n-1)q^{4n-1},$$

$$(3.5.21) \quad R_{0,4}(q) = \sum_{n=1}^{\infty} \sigma_{11}(4n)q^{4n},$$

$$(3.5.22) \quad R_{1,4}(q) = \sum_{n=1}^{\infty} \sigma_{11}(4n-3)q^{4n-3},$$

$$(3.5.23) \quad R_{2,4}(q) = \sum_{n=1}^{\infty} \sigma_{11}(4n-2)q^{4n-2},$$

$$(3.5.24) \quad R_{3,4}(q) = \sum_{n=1}^{\infty} \sigma_{11}(4n-1)q^{4n-1},$$

$$(3.5.25) \quad S_{0,4}(q) = \sum_{n=1}^{\infty} \sigma_{13}(4n)q^{4n},$$

$$(3.5.26) \quad S_{1,4}(q) = \sum_{n=1}^{\infty} \sigma_{13}(4n-3)q^{4n-3},$$

$$(3.5.27) \quad S_{2,4}(q) = \sum_{n=1}^{\infty} \sigma_{13}(4n-2)q^{4n-2},$$

$$(3.5.28) \quad S_{3,4}(q) = \sum_{n=1}^{\infty} \sigma_{13}(4n-1)q^{4n-1}.$$

We now obtain representations of $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 1, 2, 3$ in terms of x . Let $g = (1-x)^{\frac{1}{4}}$.

Theorem 3.5.1

$$(3.5.29) \quad L_{0,4}(q) = \frac{1}{192} \left(8 + (11 + 18g^2 - 37g^4)w^2 - 96g^4(1-g^4)w \frac{dw}{dx} \right),$$

$$(3.5.30) \quad L_{1,4}(q) = \frac{1}{32}(1-g)(1+g)^3w^2,$$

$$(3.5.31) \quad L_{2,4}(q) = \frac{3}{64}(1-g)^2(1+g)^2w^2,$$

$$(3.5.32) \quad L_{3,4}(q) = \frac{1}{32}(1+g)(1-g)^3w^2,$$

$$(3.5.33) \quad M_{0,4}(q) = \frac{1}{7680}(-32 + (137 - 540g^2 + 838g^4 - 540g^6 + 137g^8)w^4),$$

$$(3.5.34) \quad M_{1,4}(q) = \frac{1}{64}(1-g)(1-3g+6g^2-3g^3+g^4)(1+g)^3w^4,$$

$$(3.5.35) \quad M_{2,4}(q) = \frac{9}{512}(1+6g^2+g^4)(1-g)^2(1+g)^2w^4,$$

$$(3.5.36) \quad M_{3,4}(q) = \frac{1}{64}(1+g)(1-g)^3(1+3g+6g^2+3g^3+g^4)w^4,$$

$$(3.5.37) \quad N_{0,4}(q) = \frac{1}{64512}(128 + (1 + 6g^2 + g^4)(2081 - 8328g^2 + 12478g^4 - 8328g^6 + 2081g^8)w^6),$$

$$(3.5.38) \quad N_{1,4}(q) = \frac{1}{256}(1-g)(1+g)^3(8 - 15g + 30g^2 - 105g^3 + 172g^4 - 105g^5 + 30g^6 - 15g^7 + 8g^8)w^6,$$

$$(3.5.39) \quad N_{2,4}(q) = \frac{33}{1024}(1-g)^2(1+g)^2(1+2g+2g^2-2g^3+g^4)(1-2g+2g^2+2g^3+g^4)w^6,$$

$$(3.5.40) \quad N_{3,4}(q) = \frac{1}{256}(1+g)(1-g)^3(8 + 15g + 30g^2 + 105g^3 + 172g^4 + 105g^5 + 30g^6 + 15g^7 + 8g^8)w^6,$$

$$(3.5.41) \quad P_{0,4}(q) = \frac{1}{245760}(-512 + (32897 - 15480g^2 + 425756g^4 - 2089800g^6 + 3293766g^8 - 2089800g^{10} + 425756g^{12} - 15480g^{14} + 32897g^{16})w^8),$$

$$(3.5.42) \quad P_{1,4}(q) = \frac{1}{1024}(1-g)(1+g)^3(136 - 273g + 546g^2 - 273g^3$$

$$\begin{aligned}
& -120g^4 - 3822g^5 + 7644g^6 - 3822g^7 - 120g^8 \\
& -273g^9 + 546g^{10} - 273g^{11} + 136g^{12})w^8, \\
(3.5.43) \quad P_{2,4}(q) &= \frac{129}{16384}(1-g)^2(1+g)^2(1+6g^2+g^4) \\
& (17-60g^2+118g^4-60g^6+17g^8)w^8,
\end{aligned}$$

$$\begin{aligned}
(3.5.44) \quad P_{3,4}(q) &= \frac{1}{1024}(1+g)(1-g)^3(136+273g+546g^2+273g^3 \\
& -120g^4+3822g^5+7644g^6+3822g^7-120g^8 \\
& +273g^9+546g^{10}+273g^{11}+136g^{12})w^8,
\end{aligned}$$

$$\begin{aligned}
(3.5.45) \quad Q_{0,4}(q) &= \frac{1}{540672}(2048+(1+6g^2+g^4)(524801-3114948g^2 \\
& +12527492g^4-30472380g^6+41069814g^8 \\
& -30472380g^{10}+12527492g^{12}-3114948g^{14} \\
& +524801g^{16})w^{10}),
\end{aligned}$$

$$\begin{aligned}
(3.5.46) \quad Q_{1,4}(q) &= \frac{1}{4096}(1-g)(1+g)^3(3968-7935g+15870g^2 \\
& -28725g^3+35564g^4+74745g^5 \\
& -191070g^6-195525g^7+586344g^8 \\
& -195525g^9-191070g^{10}+74745g^{11} \\
& +35564g^{12}-28725g^{13}+15870g^{14} \\
& -7935g^{15}+3968g^{16})w^{10},
\end{aligned}$$

$$\begin{aligned}
(3.5.47) \quad Q_{2,4}(q) &= \frac{513}{16384}(1-g)^2(1+g)^2(31+60g^2+268g^4-1980g^6 \\
& +3498g^8-1980g^{10}+268g^{12}
\end{aligned}$$

$$\begin{aligned}
& +60g^{14} + 31g^{16})w^{10}, \\
(3.5.48) \quad Q_{3,4}(q) &= \frac{1}{4096}(1+g)(1-g)^3(3968 + 7935g + 15870g^2 \\
& +28725g^3 + 35564g^4 - 74745g^5 \\
& -191070g^6 + 195525g^7 + 586344g^8 \\
& +195525g^9 - 191070g^{10} - 74745g^{11} \\
& +35564g^{12} + 28725g^{13} + 15870g^{14} \\
& +7935g^{15} + 3968g^{16})w^{10}, \\
(3.5.49) \quad R_{0,4}(q) &= \frac{1}{65520}(-691 + \frac{1}{8192}(+5797943987 - 33562620g^2 \\
& -218152986g^4 - 371504640780g^6 \\
& +2176176141597g^8 - 5565555024120g^{10} \\
& +7510680250516g^{12} - 5565555024120g^{14} \\
& +2176176141597g^{16} - 371504640780g^{18} \\
& -218152986g^{20} - 33562620g^{22} \\
& +5797943987g^{24})w^{12}), \\
(3.5.50) \quad R_{1,4}(q) &= \frac{1}{16384}(1-g)(1+g)^3(176896 - 353793g \\
& +707586g^2 - 1017093g^3 + 973064g^4 \\
& -3936504g^5 + 6546408g^6 + 18615696g^7 \\
& -43600904g^8 - 18976914g^9 + 81731628g^{10} \\
& -18976914g^{11} - 43600904g^{12}
\end{aligned}$$

$$\begin{aligned}
& +18615696g^{13} + 6546408g^{14} - 3936504g^{15} \\
& +973064g^{16} - 1017093g^{17} + 707586g^{18} \\
& -353793g^{19} + 176896g^{20})w^{12}, \\
(3.5.51) \quad R_{2,4}(q) &= \frac{2049}{131072}(1-g)^2(1+g)^2(1+6g^2+g^4) \\
& (1-2g+2g^2+2g^3+g^4) \\
& (1+2g+2g^2-2g^3+g^4) \\
& (691-2760g^2+4154g^4-2760g^6+691g^8)w^{12}, \\
(3.5.52) \quad R_{3,4}(q) &= \frac{1}{16384}(1+g)(1-g)^3(+176896+353793g+707586g^2 \\
& +1017093g^3+973064g^4+3936504g^5 \\
& +6546408g^6-18615696g^7-43600904g^8 \\
& +18976914g^9+81731628g^{10}+18976914g^{11} \\
& -43600904g^{12}-18615696g^{13}+6546408g^{14} \\
& +3936504g^{15}+973064g^{16}+1017093g^{17} \\
& +707586g^{18}+353793g^{19}+176896g^{20})w^{12}, \\
(3.5.53) \quad S_{0,4}(q) &= \frac{1}{786432}(32768+(1+6g^2+g^4)(134225921 \\
& -805306368g^2+4127158266g^4 \\
& -19059441792g^6+59595890703g^8 \\
& -117582595968g^{10}+147180134380g^{12} \\
& -117582595968g^{14}+59595890703g^{16}
\end{aligned}$$

$$\begin{aligned}
& -19059441792g^{18} + 4127158266g^{20} \\
& -805306368g^{22} + 134225921g^{24})w^{14}), \\
(3.5.54) \quad S_{1,4}(q) &= \frac{1}{65536}(1-g)(1+g)^3(11184128 - 22368255g \\
& + 44736510g^2 - 67503345g^3 + 62308836g^4 \\
& + 18781035g^5 - 127832250g^6 - 1199866395g^7 \\
& + 2551508208g^8 + 4369924650g^9 \\
& - 11267414340g^{10} - 3303040650g^{11} \\
& + 17859165784g^{12} - 3303040650g^{13} \\
& - 11267414340g^{14} + 4369924650g^{15} \\
& + 2551508208g^{16} - 1199866395g^{17} \\
& - 127832250g^{18} + 18781035g^{19} \\
& + 62308836g^{20} - 67503345g^{21} \\
& + 44736510g^{22} - 22368255g^{23} \\
& + 11184128g^{24})w^{14}, \\
(3.5.55) \quad S_{2,4}(q) &= \frac{8193}{262144}(1-g)^2(1+g)^2(5461 + 10920g^2 + 1362g^4 \\
& - 207480g^6 + 1693371g^8 - 5394480g^{10} \\
& + 7785788g^{12} - 5394480g^{14} + 1693371g^{16} \\
& - 207480g^{18} + 1362g^{20} + 10920g^{22} \\
& + 5461g^{24})w^{14},
\end{aligned}$$

$$\begin{aligned}
(3.5.56) \quad S_{3,4}(q) = & \frac{1}{65536}(1+g)(1-g)^3(11184128 + 22368255g \\
& + 44736510g^2 + 67503345g^3 + 62308836g^4 \\
& - 18781035g^5 - 127832250g^6 + 1199866395g^7 \\
& + 2551508208g^8 - 4369924650g^9 \\
& - 11267414340g^{10} + 3303040650g^{11} \\
& + 17859165784g^{12} + 3303040650g^{13} \\
& - 11267414340g^{14} - 4369924650g^{15} \\
& + 2551508208g^{16} + 1199866395g^{17} \\
& - 127832250g^{18} - 18781035g^{19} \\
& + 62308836g^{20} + 67503345g^{21} \\
& + 44736510g^{22} + 22368255g^{23} \\
& + 11184128g^{24})w^{14}.
\end{aligned}$$

Proof First we prove (3.5.29), (3.5.30), (3.5.30) and (3.5.30). By the following equations,

$$\begin{aligned}
L_{0,4}(q) + L_{1,4}(q) + L_{2,4}(q) + L_{3,4}(q) &= \frac{1 - L(q)}{24}, \\
L_{0,4}(q) - L_{1,4}(q) + L_{2,4}(q) - L_{3,4}(q) &= \frac{1 - L(-q)}{24}, \\
L_{0,4}(q) + iL_{1,4}(q) - L_{2,4}(q) - iL_{3,4}(q) &= \frac{1 - L(iq)}{24}, \\
L_{0,4}(q) - iL_{1,4}(q) - L_{2,4}(q) + iL_{3,4}(q) &= \frac{1 - L(-iq)}{24},
\end{aligned}$$

we have

$$\begin{aligned} L_{0,4}(q) &= \frac{1}{96}(4 - L(q) - L(-q) - L(iq) - L(-iq)), \\ L_{1,4}(q) &= \frac{1}{96}(-L(q) + L(-q) + iL(iq) - iL(-iq)), \\ L_{2,4}(q) &= \frac{1}{96}(-L(q) - L(-q) + L(iq) + L(-iq)), \\ L_{3,4}(q) &= \frac{1}{96}(-L(q) + L(-q) - iL(iq) + iL(-iq)). \end{aligned}$$

Using MAPLE (see Appendix B) to substitute (1.2.9), (2.4.1), (3.4.1) and (3.4.2) into the above equations, we obtain (3.5.29), (3.5.30), (3.5.30) and (3.5.30).

Next we prove (3.5.33), (3.5.34), (3.5.34) and (3.5.34). By the following equations,

$$\begin{aligned} M_{0,4}(q) + M_{1,4}(q) + M_{2,4}(q) + M_{3,4}(q) &= \frac{M(q) - 1}{240}, \\ M_{0,4}(q) - M_{1,4}(q) + M_{2,4}(q) - M_{3,4}(q) &= \frac{M(-q) - 1}{240}, \\ M_{0,4}(q) + iM_{1,4}(q) - M_{2,4}(q) - iM_{3,4}(q) &= \frac{M(iq) - 1}{240}, \\ M_{0,4}(q) - iM_{1,4}(q) - M_{2,4}(q) + iM_{3,4}(q) &= \frac{M(-iq) - 1}{240}, \end{aligned}$$

we have

$$\begin{aligned} M_{0,4}(q) &= \frac{1}{960}(-4 + M(q) + M(-q) + M(iq) + M(-iq)) \\ M_{1,4}(q) &= \frac{1}{960}(M(q) - M(-q) - iM(iq) + iM(-iq)) \\ M_{2,4}(q) &= \frac{1}{960}(M(q) + M(-q) - M(iq) - M(-iq)) \\ M_{3,4}(q) &= \frac{1}{960}(M(q) - M(-q) + iM(iq) - iM(-iq)) \end{aligned}$$

Using MAPLE to substitute (1.2.10), (2.4.2), (3.4.3) and (3.4.4) into the above equations, we obtain (3.5.33), (3.5.34), (3.5.34) and (3.5.34).

Next we prove (3.5.37), (3.5.38), (3.5.38) and (3.5.38). By the following equations,

$$\begin{aligned} N_{0,4}(q) + N_{1,4}(q) + N_{2,4}(q) + N_{3,4}(q) &= \frac{1 - N(q)}{504}, \\ N_{0,4}(q) - N_{1,4}(q) + N_{2,4}(q) - N_{3,4}(q) &= \frac{1 - N(-q)}{504}, \\ N_{0,4}(q) + iN_{1,4}(q) - N_{2,4}(q) - iN_{3,4}(q) &= \frac{1 - N(iq)}{504}, \\ N_{0,4}(q) - iN_{1,4}(q) - N_{2,4}(q) + iN_{3,4}(q) &= \frac{1 - N(-iq)}{504}, \end{aligned}$$

we have

$$\begin{aligned} N_{0,4}(q) &= \frac{1}{2016}(4 - N(q) - N(-q) - N(iq) - N(-iq)), \\ N_{1,4}(q) &= \frac{1}{2016}(-N(q) + N(-q) + iN(iq) - iN(-iq)), \\ N_{2,4}(q) &= \frac{1}{2016}(-N(q) - N(-q) + N(iq) + N(-iq)), \\ N_{3,4}(q) &= \frac{1}{2016}(-N(q) + N(-q) - iN(iq) + iN(-iq)). \end{aligned}$$

Using MAPLE to substitute (1.2.11), (2.4.3), (3.4.5) and (3.4.6) into the above equations, we obtain (3.5.37), (3.5.38), (3.5.38) and (3.5.38).

Next we prove (3.5.41), (3.5.42), (3.5.42) and (3.5.42). By the following equations,

$$\begin{aligned} P_{0,4}(q) + P_{1,4}(q) + P_{2,4}(q) + P_{3,4}(q) &= \frac{P(q) - 1}{480}, \\ P_{0,4}(q) - P_{1,4}(q) + P_{2,4}(q) - P_{3,4}(q) &= \frac{P(-q) - 1}{480}, \end{aligned}$$

$$\begin{aligned} P_{0,4}(q) + iP_{1,4}(q) - P_{2,4}(q) - iP_{3,4}(q) &= \frac{P(iq) - 1}{480}, \\ P_{0,4}(q) - iP_{1,4}(q) - P_{2,4}(q) + iP_{3,4}(q) &= \frac{P(-iq) - 1}{480}, \end{aligned}$$

we have

$$\begin{aligned} P_{0,4}(q) &= \frac{1}{1920}(-4 + P(q) + P(-q) + P(iq) + P(-iq)) \\ P_{1,4}(q) &= \frac{1}{1920}(P(q) - P(-q) - iP(iq) + iP(-iq)) \\ P_{2,4}(q) &= \frac{1}{1920}(P(q) + P(-q) - P(iq) - P(-iq)) \\ P_{3,4}(q) &= \frac{1}{1920}(P(q) - P(-q) + iP(iq) - iP(-iq)) \end{aligned}$$

Using MAPLE to substitute (1.2.20), (2.4.4), (3.4.7) and (3.4.8) into the above equations, we obtain (3.5.41), (3.5.42), (3.5.42) and (3.5.42).

Next we prove (3.5.45), (3.5.46), (3.5.46) and (3.5.46). By the following equations,

$$\begin{aligned} Q_{0,4}(q) + Q_{1,4}(q) + Q_{2,4}(q) + Q_{3,4}(q) &= \frac{1 - Q(q)}{264}, \\ Q_{0,4}(q) - Q_{1,4}(q) + Q_{2,4}(q) - Q_{3,4}(q) &= \frac{1 - Q(-q)}{264}, \\ Q_{0,4}(q) + iQ_{1,4}(q) - Q_{2,4}(q) - iQ_{3,4}(q) &= \frac{1 - Q(iq)}{264}, \\ Q_{0,4}(q) - iQ_{1,4}(q) - Q_{2,4}(q) + iQ_{3,4}(q) &= \frac{1 - Q(-iq)}{264}, \end{aligned}$$

we have

$$\begin{aligned} Q_{0,4}(q) &= \frac{1}{1056}(4 - Q(q) - Q(-q) - Q(iq) - Q(-iq)), \\ Q_{1,4}(q) &= \frac{1}{1056}(-Q(q) + Q(-q) + iQ(iq) - iQ(-iq)), \end{aligned}$$

$$\begin{aligned} Q_{2,4}(q) &= \frac{1}{1056}(-Q(q) - Q(-q) + Q(iq) + Q(-iq)), \\ Q_{3,4}(q) &= \frac{1}{1056}(-Q(q) + Q(-q) - iQ(iq) + iQ(-iq)). \end{aligned}$$

Using MAPLE to substitute (1.2.21), (2.4.5), (3.4.9) and (3.4.10) into the above equations, we obtain (3.5.45), (3.5.46), (3.5.46) and (3.5.46).

Next we prove (3.5.49), (3.5.50), (3.5.50) and (3.5.50). By the following equations,

$$\begin{aligned} R_{0,4}(q) + R_{1,4}(q) + R_{2,4}(q) + R_{3,4}(q) &= \frac{691(R(q) - 1)}{65520}, \\ R_{0,4}(q) - R_{1,4}(q) + R_{2,4}(q) - R_{3,4}(q) &= \frac{691(R(-q) - 1)}{65520}, \\ R_{0,4}(q) + iR_{1,4}(q) - R_{2,4}(q) - iR_{3,4}(q) &= \frac{691(R(iq) - 1)}{65520}, \\ R_{0,4}(q) - iR_{1,4}(q) - R_{2,4}(q) + iR_{3,4}(q) &= \frac{691(R(-iq) - 1)}{65520}, \end{aligned}$$

we have

$$\begin{aligned} R_{0,4}(q) &= \frac{691}{262080}(-4 + R(q) + R(-q) + R(iq) + R(-iq)) \\ R_{1,4}(q) &= \frac{691}{262080}(R(q) - R(-q) - iR(iq) + iR(-iq)) \\ R_{2,4}(q) &= \frac{691}{262080}(R(q) + R(-q) - R(iq) - R(-iq)) \\ R_{3,4}(q) &= \frac{691}{262080}(R(q) - R(-q) + iR(iq) - iR(-iq)) \end{aligned}$$

Using MAPLE to substitute (1.2.22), (2.4.6), (3.4.11) and (3.4.12) into the above equations, we obtain (3.5.49), (3.5.50), (3.5.50) and (3.5.50).

Finally we prove (3.5.53), (3.5.54), (3.5.54) and (3.5.54). By the following

equations,

$$\begin{aligned} S_{0,4}(q) + S_{1,4}(q) + S_{2,4}(q) + S_{3,4}(q) &= \frac{1 - S(q)}{24}, \\ S_{0,4}(q) - S_{1,4}(q) + S_{2,4}(q) - S_{3,4}(q) &= \frac{1 - S(-q)}{24}, \\ S_{0,4}(q) + iS_{1,4}(q) - S_{2,4}(q) - iS_{3,4}(q) &= \frac{1 - S(iq)}{24}, \\ S_{0,4}(q) - iS_{1,4}(q) - S_{2,4}(q) + iS_{3,4}(q) &= \frac{1 - S(-iq)}{24}, \end{aligned}$$

we have

$$\begin{aligned} S_{0,4}(q) &= \frac{1}{96}(4 - S(q) - S(-q) - S(iq) - S(-iq)), \\ S_{1,4}(q) &= \frac{1}{96}(-S(q) + S(-q) + iS(iq) - iS(-iq)), \\ S_{2,4}(q) &= \frac{1}{96}(-S(q) - S(-q) + S(iq) + S(-iq)), \\ S_{3,4}(q) &= \frac{1}{96}(-S(q) + S(-q) - iS(iq) + iS(-iq)). \end{aligned}$$

Using MAPLE to substitute (1.2.23), (2.4.7), (3.4.13) and (3.4.14) into the above equations, we obtain (3.5.53), (3.5.54), (3.5.54) and (3.5.54). $\square$

3.6 Evaluation of $\frac{dL(q^8)}{dq}$, $\frac{dM(q^8)}{dq}$, ..., $\frac{dS(q^8)}{dq}$ in terms of x .

In this section, we use Theorem 3.3.1 to determine $\frac{dL(q^8)}{dq}$, $\frac{dM(q^8)}{dq}$, ..., $\frac{dS(q^8)}{dq}$ in terms of x . These determinations are used in Section 3.8 to find relationships between $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$, $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 1, 2, 3$. Let $g = (1 - x)^{\frac{1}{4}}$.

Theorem 3.6.1

$$\begin{aligned}
(3.6.1) \quad q \frac{dL(q^8)}{dq} &= \frac{1}{16}(1 - g^4)(3g^2 + 5g^4)w^4 \\
&\quad - \frac{1}{8}g^4(1 - g^4)(1 - 6g^2 - 11g^4)w^3 \frac{dw}{dx} \\
&\quad + \frac{3}{2}g^8(1 - g^4)^2w^2 \left(\frac{dw}{dx} \right)^2 \\
(3.6.2) \quad q \frac{dM(q^8)}{dq} &= \frac{1}{256}(1 - g^4)(-30g^2 - 134g^4 - 90g^6 - 2g^8)w^6 \\
&\quad + \frac{1}{64}g^4(1 - g^4)(1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^5 \frac{dw}{dx} \\
(3.6.3) \quad q \frac{dN(q^8)}{dq} &= \frac{3}{4096}g^2(1 - g^4)(-21 - 347g^2 - 882g^4 \\
&\quad - 694g^6 - 105g^8 + g^{10})w^8 \\
&\quad - \frac{3}{2048}g^4(1 - g^4)(1 + 6g^2 + g^4) \\
&\quad (1 + 6g^2 + 12g^3 + 12g + g^4) \\
&\quad (1 - 12g + 6g^2 - 12g^3 + g^4)w^7 \frac{dw}{dx} \\
(3.6.4) \quad q \frac{dP(q^8)}{dq} &= -\frac{1}{16384}g^2(1 - g^4)(15 + 67g^2 + 45g^4 + g^6) \\
&\quad (1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^{10} \\
&\quad + \frac{1}{8192}g^4(1 - g^4) \\
&\quad (1 + 60g^2 + 134g^4 + 60g^6 + g^8)^2w^9 \frac{dw}{dx} \\
(3.6.5) \quad q \frac{dQ(q^8)}{dq} &= \frac{1}{1048576}g^2(1 - g^4)(-33 - 8467g^2 \\
&\quad - 121572g^4 - 507868g^6 - 903870g^8 \\
&\quad - 761802g^{10} - 283668g^{12} - 33868g^{14} \\
&\quad - 297g^{16} + 5g^{18})w^{12}
\end{aligned}$$

$$\begin{aligned}
& -\frac{5}{524288}g^4(1-g^4)(1+6g^2+g^4) \\
& \quad (1-12g+6g^2-12g^3+g^4) \\
& \quad (1+12g+6g^2+12g^3+g^4) \\
& \quad (1+60g^2+134g^4+60g^6+g^8)w^{11}\frac{dw}{dx} \\
(3.6.6) \quad q\frac{dR(q^8)}{dq} = & -\frac{3}{5796528128}g^2(1-g^4)(1365+1398097g^2 \\
& +45327555g^4+351009887g^6 \\
& +1131762450g^8+1837552714g^{10} \\
& +1584467430g^{12}+702019774g^{14} \\
& +135982665g^{16}+6990485g^{18} \\
& +15015g^{20}+691g^{22})w^{14} \\
& +\frac{3}{2898264064}g^4(1-g^4)(691+16380g^2 \\
& +8388582g^4+181310220g^6 \\
& +1053029661g^8+2716229880g^{10} \\
& +3675105428g^{12}+2716229880g^{14} \\
& +1053029661g^{16}+181310220g^{18} \\
& +8388582g^{20}+16380g^{22} \\
& +691g^{24})w^{13}\frac{dw}{dx} \\
(3.6.7) \quad q\frac{dS(q^8)}{dq} = & \frac{1}{268435456}g^2(1-g^4)(-3-12113g^2 \\
& -169596g^4-711620g^6-1265418g^8
\end{aligned}$$

$$\begin{aligned}
& -1065918g^{10} - 397740g^{12} - 47156g^{14} \\
& -459g^{16} + 7g^{18})w^{16} \\
& -\frac{7}{134217728}g^4(1-g^4)(1+6g^2+g^4) \\
& (1+12g+6g^2+12g^3+g^4) \\
& (1-12g+6g^2-12g^3+g^4) \\
& (1+60g^2+134g^4+60g^6+g^8)^2w^{15}\frac{dw}{dx}
\end{aligned}$$

Proof First we prove (3.6.1). By (3.3.1), we have

$$\begin{aligned}
\frac{dL(q^8)}{dx} &= -\frac{1}{16} \left(\frac{3g+11g^3}{g^3} \right) w^2 - \frac{1}{8}(13-6g^2+35g^4)w\frac{dw}{dx} \\
&+ \frac{3}{2}g^4(1-g^4) \left(\frac{dw}{dx} \right)^2 + \frac{3}{2}g^4(1-g^4)w\frac{d^2w}{dx^2}.
\end{aligned}$$

Appealing to Lemma 1.3.3, we obtain

$$\begin{aligned}
\frac{dL(q^8)}{dx} &= -\frac{1}{16} \left(\frac{3g+5g^3}{g^3} \right) w^2 - \frac{1}{8}(1-6g^2-11g^4)w\frac{dw}{dx} \\
&+ \frac{3}{2}g^4(1-g^4) \left(\frac{dw}{dx} \right)^2.
\end{aligned}$$

Finally, by Lemma 1.3.2, we have

$$\begin{aligned}
q \frac{dL(q^8)}{dq} &= \frac{1}{16}(1-g^4)(3g^2+5g^4)w^4 \\
&- \frac{1}{8}g^4(1-g^4)(1-6g^2-11g^4)w^3\frac{dw}{dx} \\
&+ \frac{3}{2}g^8(1-g^4)^2w^2 \left(\frac{dw}{dx} \right)^2,
\end{aligned}$$

as asserted.

Next we prove (3.6.2). By (3.3.2), we have

$$\begin{aligned} \frac{dM(q^8)}{dx} &= -\frac{1}{1024g^3} (120g + 536g^3 + 360g^5 + 8g^7) w^4 \\ &\quad + \frac{1}{64} (1 + 60g^2 + 134g^4 + 60g^6 + g^8) w^3 \frac{dw}{dx}. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dM(q^8)}{dq} &= \frac{1}{256} (1 - g^4) (-30g^2 - 134g^4 - 90g^6 - 2g^8) w^6 \\ &\quad + \frac{1}{64} g^4 (1 - g^4) (1 + 60g^2 + 134g^4 + 60g^6 + g^8) w^5 \frac{dw}{dx}, \end{aligned}$$

as asserted.

Next we prove (3.6.3). By (3.3.3), we have

$$\begin{aligned} \frac{dN(q^8)}{dx} &= \frac{1}{16384g^3} (12g + 4g^3) \\ &\quad (1 - 132g^2 - 250g^4 - 132g^6 + g^8) w^6 \\ &\quad + \frac{1}{16384g^3} (1 + 6g^2 + g^4) \\ &\quad (-264g - 1000g^3 - 792g^5 + 8g^7) w^6 \\ &\quad - \frac{6}{4096} (1 + 6g^2 + g^4) \\ &\quad (1 - 132g^2 - 250g^4 - 132g^6 + g^8) w^5 \frac{dw}{dx}. \end{aligned}$$

By Lemma 1.3.2, we have

$$\begin{aligned} q \frac{dN(q^8)}{dq} &= \frac{3}{4096} g^2 (1 - g^4) (-21 - 347g^2 - 882g^4 \\ &\quad - 694g^6 - 105g^8 + g^{10}) w^8 \\ &\quad - \frac{3}{2048} g^4 (1 - g^4) (1 + 6g^2 + g^4) \end{aligned}$$

$$(1 + 6g^2 + 12g^3 + 12g + g^4)$$

$$(1 - 12g + 6g^2 - 12g^3 + g^4)w^7 \frac{dw}{dx},$$

as asserted.

Next we prove (3.6.4). By (1.5.6), (3.3.2) and (3.6.2), we have

$$\begin{aligned} q \frac{dP(q^8)}{dq} &= 2M(q^8)q \frac{dM(q^8)}{dq} \\ &= -\frac{1}{16384}g^2(1 - g^4)(15 + 67g^2 + 45g^4 + g^6) \\ &\quad (1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^{10} \\ &\quad + \frac{1}{8192}g^4(1 - g^4) \\ &\quad (1 + 60g^2 + 134g^4 + 60g^6 + g^8)^2w^9 \frac{dw}{dx}, \end{aligned}$$

as asserted.

Next we prove (3.6.5). By (1.5.7), (3.3.2), (3.3.3), (3.6.2) and (3.6.3), we have

$$\begin{aligned} q \frac{dQ(q^8)}{dq} &= N(q^8)q \frac{dM(q^8)}{dq} + M(q^8)q \frac{dN(q^8)}{dq} \\ &= \frac{1}{1048576}g^2(1 - g^4)(-33 - 8467g^2 \\ &\quad - 121572g^4 - 507868g^6 - 903870g^8 \\ &\quad - 761802g^{10} - 283668g^{12} - 33868g^{14} \\ &\quad - 297g^{16} + 5g^{18})w^{12} \\ &\quad - \frac{5}{524288}g^4(1 - g^4)(1 + 6g^2 + g^4) \\ &\quad (1 - 12g + 6g^2 - 12g^3 + g^4) \end{aligned}$$

$$(1 + 12g + 6g^2 + 12g^3 + g^4)$$

$$(1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^{11}\frac{dw}{dx},$$

as asserted.

Next we prove (3.6.6). By (1.2.18), (3.3.2), (3.3.3), (3.3.4), (3.6.2), (3.6.3) and (3.6.4) we have

$$\begin{aligned} q \frac{dR(q^8)}{dq} &= \frac{441}{691} \left(P(q^8)q \frac{dM(q^8)}{dq} + M(q^8)q \frac{dP(q^8)}{dq} \right) \\ &\quad + \frac{500}{691} \left(N(q^8)q \frac{dN(q^8)}{dq} \right) \\ &= -\frac{3}{5796528128} g^2(1 - g^4)(1365 + 1398097g^2 \\ &\quad + 45327555g^4 + 351009887g^6 \\ &\quad + 1131762450g^8 + 1837552714g^{10} \\ &\quad + 1584467430g^{12} + 702019774g^{14} \\ &\quad + 135982665g^{16} + 6990485g^{18} \\ &\quad + 15015g^{20} + 691g^{22})w^{14} \\ &\quad + \frac{3}{2898264064} g^4(1 - g^4)(691 + 16380g^2 \\ &\quad + 8388582g^4 + 181310220g^6 \\ &\quad + 1053029661g^8 + 2716229880g^{10} \\ &\quad + 3675105428g^{12} + 2716229880g^{14} \\ &\quad + 1053029661g^{16} + 181310220g^{18} \\ &\quad + 8388582g^{20} + 16380g^{22}) \end{aligned}$$

$$+691g^{24})w^{13}\frac{dw}{dx},$$

as asserted.

Next we prove (3.6.7). By (1.5.9), (3.3.3), (3.3.4), (3.6.3) and (3.6.4), we have

$$\begin{aligned} q\frac{dS(q^8)}{dq} &= N(q^8)q\frac{dP(q^8)}{dq} + P(q^8)q\frac{dN(q^8)}{dq} \\ &= \frac{1}{268435456}g^2(1-g^4)(-3-12113g^2 \\ &\quad -169596g^4-711620g^6-1265418g^8 \\ &\quad -1065918g^{10}-397740g^{12}-47156g^{14} \\ &\quad -459g^{16}+7g^{18})w^{16} \\ &\quad -\frac{7}{134217728}g^4(1-g^4)(1+6g^2+g^4) \\ &\quad (1+12g+6g^2+12g^3+g^4) \\ &\quad (1-12g+6g^2-12g^3+g^4) \\ &\quad (1+60g^2+134g^4+60g^6+g^8)^2w^{15}\frac{dw}{dx}, \end{aligned}$$

as asserted. □

3.7 Evaluation of $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 2$ in terms of $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$.

In this section, we use Theorems 2.3.1 and 3.3.1 to determine $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 1, 2, 3$ in terms of $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$. However, we were only able find relationships for $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 2$. These determinations are used to prove the relationships in Section 3.8.

Theorem 3.7.1

$$(3.7.1) \quad L_{0,4}(q) = \frac{1}{24}(1 - 7L(q^4) + 6L(q^8)),$$

$$(3.7.2) \quad L_{2,4}(q) = \frac{1}{8}(-L(q^2) + 3L(q^4) - 2L(q^8)),$$

$$(3.7.3) \quad M_{0,4}(q) = \frac{1}{240}(-1 + 73M(q^4) - 72M(q^8)),$$

$$(3.7.4) \quad M_{2,4}(q) = \frac{1}{80}(3M(q^2) - 27M(q^4) + 24M(q^8)),$$

$$(3.7.5) \quad N_{0,4}(q) = \frac{1}{504}(1 - 1057N(q^4) + 1056N(q^8)),$$

$$(3.7.6) \quad N_{2,4}(q) = \frac{1}{168}(-11N(q^2) + 363N(q^4) - 352N(q^8)),$$

$$(3.7.7) \quad P_{0,4}(q) = \frac{1}{480}(-1 + 16513P(q^4) - 16512P(q^8)),$$

$$(3.7.8) \quad P_{2,4}(q) = \frac{1}{160}(43P(q^2) - 5547P(q^4) + 5504P(q^8)),$$

$$(3.7.9) \quad Q_{0,4}(q) = \frac{1}{264}(1 - 262657Q(q^4) + 262656Q(q^8)),$$

$$(3.7.10) \quad Q_{2,4}(q) = \frac{1}{88}(-171Q(q^2) + 87723Q(q^4) - 87552Q(q^8)),$$

$$(3.7.11) \quad R_{0,4}(q) = \frac{691}{65520}(-1 + 4196353R(q^4) - 4196352R(q^8)),$$

$$(3.7.12) \quad R_{2,4}(q) = \frac{691}{21840}(683R(q^2) - 1399467R(q^4) + 1398784R(q^8)),$$

$$(3.7.13) \quad S_{0,4}(q) = \frac{1}{24}(1 - 67117057S(q^4) + 67117056S(q^8)),$$

$$(3.7.14) \quad S_{2,4}(q) = \frac{1}{8}(-2731S(q^2) + 22375083S(q^4) - 22372352S(q^8)).$$

Proof First we prove (3.7.1). By (3.5.29), (2.3.2) and (3.3.1), we have

$$\begin{aligned} L_{0,4}(q) &= \frac{1}{192} \left(8 + (11 + 18g^2 - 37g^4)w^2 - 96g^4(1 - g^4)w \frac{dw}{dx} \right) \\ &= \frac{1}{24}(1 - 7L(q^4) + 6L(q^8)). \end{aligned}$$

Next we prove (3.7.2). By (3.5.31), (2.3.1), (2.3.2) and (3.3.1), we have

$$\begin{aligned} L_{2,4}(q) &= \frac{3}{64}(1 - g)^2(1 + g)^2w^2 \\ &= \frac{1}{8}(-L(q^2) + 3L(q^4) - 2L(q^8)). \end{aligned}$$

Next we prove (3.7.3). By (3.5.33), (2.3.4) and (3.3.2), we have

$$\begin{aligned} M_{0,4}(q) &= \frac{1}{7680}(-32 + (137 - 540g^2 + 838g^4 \\ &\quad - 540g^6 + 137g^8)w^4) \\ &= \frac{1}{240}(-1 + 73M(q^4) - 72M(q^8)). \end{aligned}$$

Next we prove (3.7.4). By (3.5.35), (2.3.3), (2.3.4) and (3.3.2), we have

$$\begin{aligned} M_{2,4}(q) &= \frac{9}{512}(1 + 6g^2 + g^4)(1 - g)^2(1 + g)^2w^4 \\ &= \frac{1}{80}(3M(q^2) - 27M(q^4) + 24M(q^8)). \end{aligned}$$

Next we prove (3.7.5). By (3.5.37), (2.3.6) and (3.3.3), we have

$$\begin{aligned} N_{0,4}(q) &= \frac{1}{64512} (128 + (1 + 6g^2 + g^4) \\ &\quad (2081 - 8328g^2 + 12478g^4 \\ &\quad - 8328g^6 + 2081g^8)w^6) \\ &= \frac{1}{504} (1 - 1057N(q^4) + 1056N(q^8)). \end{aligned}$$

Next we prove (3.7.6). By (3.5.39), (2.3.5), (2.3.6) and (3.3.3), we have

$$\begin{aligned} N_{2,4}(q) &= \frac{33}{1024} (1 - g)^2 (1 + g)^2 (1 + 2g + 2g^2 - 2g^3 + g^4) \\ &\quad (1 - 2g + 2g^2 + 2g^3 + g^4)w^6) \\ &= \frac{1}{168} (-11N(q^2) + 363N(q^4) - 352N(q^8)). \end{aligned}$$

Next we prove (3.7.7). By (3.5.41), (2.3.8) and (3.3.4), we have

$$\begin{aligned} P_{0,4}(q) &= \frac{1}{245760} (-512 + (32897 - 15480g^2 + 425756g^4 \\ &\quad - 2089800g^6 + 3293766g^8 - 2089800g^{10} \\ &\quad + 425756g^{12} - 15480g^{14} + 32897g^{16})w^8) \\ &= \frac{1}{480} (-1 + 16513P(q^4) - 16512P(q^8)). \end{aligned}$$

Next we prove (3.7.8). By (3.5.43), (2.3.7), (2.3.8) and (3.3.4), we have

$$\begin{aligned} P_{2,4}(q) &= \frac{129}{16384} (1 - g)^2 (1 + g)^2 (1 + 6g^2 + g^4) \\ &\quad (17 - 60g^2 + 118g^4 - 60g^6 + 17g^8)w^8) \\ &= \frac{1}{160} (43P(q^2) - 5547(q^4) + 5504P(q^8)). \end{aligned}$$

Next we prove (3.7.9). By (3.5.45), (2.3.10) and (3.3.5), we have

$$\begin{aligned} Q_{0,4}(q) &= \frac{1}{540672} (2048 + (1 + 6g^2 + g^4)(524801 - 3114948g^2 \\ &\quad + 12527492g^4 - 30472380g^6 + 41069814g^8 \\ &\quad - 30472380g^{10} + 12527492g^{12} - 3114948g^{14} \\ &\quad + 524801g^{16})w^{10}) \\ &= \frac{1}{264} (1 - 262657Q(q^4) + 262656Q(q^8)). \end{aligned}$$

Next we prove (3.7.10). By (3.5.47), (2.3.9), (2.3.10) and (3.3.5), we have

$$\begin{aligned} Q_{2,4}(q) &= \frac{513}{16384} (1 - g)^2 (1 + g)^2 (31 + 60g^2 + 268g^4 - 1980g^6 \\ &\quad + 3498g^8 - 1980g^{10} + 268g^{12} \\ &\quad + 60g^{14} + 31g^{16})w^{10} \\ &= \frac{1}{88} (-171Q(q^2) + 87723Q(q^4) - 87552Q(q^8)). \end{aligned}$$

Next we prove (3.7.11). By (3.5.49), (2.3.12) and (3.3.6), we have

$$\begin{aligned} R_{0,4}(q) &= \frac{1}{65520} (-691 + \frac{1}{8192} (+5797943987 - 33562620g^2 \\ &\quad - 218152986g^4 - 371504640780g^6 \\ &\quad + 2176176141597g^8 - 5565555024120g^{10} \\ &\quad + 7510680250516g^{12} - 5565555024120g^{14} \\ &\quad + 2176176141597g^{16} - 371504640780g^{18} \\ &\quad - 218152986g^{20} - 33562620g^{22} \end{aligned}$$

$$\begin{aligned}
& +5797943987g^{24})w^{12}) \\
& = \frac{691}{65520}(-1 + 4196353R(q^4) - 4196352R(q^8)).
\end{aligned}$$

Next we prove (3.7.12). By (3.5.51), (2.3.11), (2.3.12) and (3.3.6), we have

$$\begin{aligned}
R_{2,4}(q) &= \frac{2049}{131072}(1-g)^2(1+g)^2(1+6g^2+g^4) \\
&\quad (1-2g+2g^2+2g^3+g^4) \\
&\quad (1+2g+2g^2-2g^3+g^4) \\
&\quad (691-2760g^2+4154g^4-2760g^6+691g^8)w^{12} \\
&= \frac{691}{21840}(683R(q^2) - 1399467R(q^4) + 1398784R(q^8)).
\end{aligned}$$

Next we prove (3.7.13). By (3.5.53), (2.3.14) and (3.3.7), we have

$$\begin{aligned}
S_{0,4}(q) &= \frac{1}{786432}(32768 + (1+6g^2+g^4)(134225921 \\
&\quad -805306368g^2 + 4127158266g^4 \\
&\quad -19059441792g^6 + 59595890703g^8 \\
&\quad -117582595968g^{10} + 147180134380g^{12} \\
&\quad -117582595968g^{14} + 59595890703g^{16} \\
&\quad -19059441792g^{18} + 4127158266g^{20} \\
&\quad -805306368g^{22} + 134225921g^{24})w^{14}) \\
&= \frac{1}{24}(1 - 67117057S(q^4) + 67117056S(q^8)).
\end{aligned}$$

Lastly we prove (3.7.14). By (3.5.55), (2.3.13), (2.3.14) and (3.3.7), we

have

$$\begin{aligned}
 S_{2,4}(q) &= \frac{8193}{262144}(1-g)^2(1+g)^2(5461 + 10920g^2 + 1362g^4 \\
 &\quad - 207480g^6 + 1693371g^8 - 5394480g^{10} \\
 &\quad + 7785788g^{12} - 5394480g^{14} + 1693371g^{16} \\
 &\quad - 207480g^{18} + 1362g^{20} + 10920g^{22} \\
 &\quad + 5461g^{24})w^{14} \\
 &= \frac{1}{8}(-2731S(q^2) + 22375083S(q^4) - 22372352S(q^8)).
 \end{aligned}$$

This completes the proof for Theorem 3.7.1. □

3.8 Relationships between $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, $\dots$, $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$, $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 1, 2, 3$.

In this section we use the Theorems 1.2.1, 1.2.2, 2.3.1, 3.3.1, 3.5.1 2.6.1 and 3.6.1 to obtain relationships between the following functions: $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$, $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 1, 2, 3$. Using MAPLE, we take the product of two functions from the list above and check to see if the product can be expressed as a linear function of $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$ and their derivatives. Since there are fifty-six functions, in total, $56 + 55 + \dots + 2 + 1 = 1596$

products were considered. Out of the 1596, we found thirty which can be expressed as linear combinations of $L(q)$, $L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$ and their derivatives. Nine of these have been presented in Section 1.5 and four in Section 2.8, so in this section we present the other seventeen.

Theorem 3.8.1

$$(3.8.1) \quad L(q^2)L_{0,4}(q) = \frac{1}{24}L(q^2) - \frac{1}{240}(11M(q^2) + 47M(q^4) - 48M(q^8)) \\ - \frac{1}{8} \left(2q \frac{dL(q^2)}{dq} + 7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right),$$

$$(3.8.2) \quad L(q^2)L_{2,4}(q) = -\frac{1}{80}(5M(q^2) - 21M(q^4) + 16M(q^8)) \\ - \frac{1}{40} \left(15q \frac{dL(q^2)}{dq} - 36q \frac{dL(q^4)}{dq} + 30q \frac{dL(q^8)}{dq} \right),$$

$$(3.8.3) \quad L(q^4)L_{0,4}(q) = \frac{1}{24}L(q^4) - \frac{1}{120}(29M(q^4) - 24M(q^8)) \\ - \frac{1}{16} \left(11q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right),$$

$$(3.8.4) \quad L(q^4)M_{0,4}(q) = -\frac{1}{240}L(q^4) + \frac{1}{1680}(487N(q^4) - 480N(q^8)) \\ + \frac{1}{320} \left(73q \frac{dM(q^4)}{dq} - 72q \frac{dM(q^8)}{dq} \right),$$

$$(3.8.5) \quad M(q^4)L_{0,4}(q) = \frac{1}{24}M(q^4) - \frac{1}{168}(39N(q^4) - 32N(q^8)) \\ - \frac{1}{8}q \frac{dM(q^4)}{dq},$$

$$(3.8.6) \quad L(q^4)L_{2,4}(q) = -\frac{1}{840}(M(q^2) - 9M(q^4) + 8M(q^8)) \\ - \frac{1}{16} \left(3q \frac{dL(q^2)}{dq} - 9q \frac{dL(q^4)}{dq} + 6q \frac{dL(q^8)}{dq} \right),$$

$$(3.8.7) \quad L(q^4)M_{2,4}(q) = \frac{1}{112}(N(q^2) - 33N(q^4) + 32N(q^8))$$

$$\begin{aligned}
& + \frac{1}{320} \left(9q \frac{dM(q^2)}{dq} - 81q \frac{dM(q^4)}{dq} + 72q \frac{dM(q^8)}{dq} \right), \\
(3.8.8) \quad M(q^4)L_{2,4}(q) &= -\frac{1}{168} (N(q^2) - 33N(q^4) + 32N(q^8)) \\
(3.8.9) \quad L(q^8)L_{0,4}(q) &= \frac{1}{24}L(q^8) - \frac{1}{120}(7M(q^4) - 2M(q^8)) \\
& - \frac{1}{32} \left(7q \frac{dL(q^4)}{dq} - 2q \frac{dL(q^8)}{dq} \right), \\
(3.8.10) \quad L(q^8)M_{0,4}(q) &= -\frac{1}{240}L(q^8) + \frac{1}{5040}(365N(q^4) - 344N(q^8)) \\
& + \frac{1}{640} \left(73q \frac{dM(q^4)}{dq} - 72q \frac{dM(q^8)}{dq} \right), \\
(3.8.11) \quad M(q^8)L_{0,4}(q) &= \frac{1}{24}M(q^8) - \frac{1}{72}(N(q^4) - 2N(q^8)) \\
& - \frac{1}{8}q \frac{dM(q^8)}{dq}, \\
(3.8.12) \quad L(q^8)L_{2,4}(q) &= -\frac{1}{160}(M(q^2) - 9M(q^4) + 8M(q^8)) \\
& - \frac{1}{32} \left(3q \frac{dL(q^2)}{dq} - 9q \frac{dL(q^4)}{dq} + 6q \frac{dL(q^8)}{dq} \right), \\
(3.8.13) \quad L_{3,4}(q)L_{1,4}(q) &= \frac{1}{60}(M(q^4) - M(q^8)), \\
(3.8.14) \quad L_{2,4}(q)L_{0,4}(q) &= -\frac{1}{192}(L(q^2) - 3L(q^4) + 2L(q^8)) \\
& + \frac{1}{1920}(11M(q^2) - 99M(q^4) + 88M(q^8)) \\
& + \frac{1}{32} \left(q \frac{dL(q^2)}{dq} - 3q \frac{dL(q^4)}{dq} + 2q \frac{dL(q^8)}{dq} \right), \\
(3.8.15) \quad L_{2,4}^2(q) &= \frac{3}{80}(M(q^4) - M(q^8)), \\
(3.8.16) \quad L_{2,4}(q)M_{2,4}(q) &= -\frac{3}{56}(N(q^4) - N(q^8)), \\
(3.8.17) \quad L_{0,4}^2(q) &= \frac{1}{576} - \frac{1}{288}(7L(q^4) - 6L(q^8)) \\
& + \frac{1}{2880}(161M(q^4) - 156M(q^8))
\end{aligned}$$

$$+\frac{1}{48} \left(7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right).$$

Proof We use MAPLE (see Appendix C) to find out whether or not the 531 distinct products AB with A and B chosen from $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$, $L_{i,4}(q)$, $M_{i,4}(q)$, ..., $S_{i,4}(q)$ for $i = 0, 1, 2, 3$ can be expressed as linear combinations of $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$ and their derivatives. If a relationship exists, MAPLE returns the solution, otherwise, it returns NULL. The proof in Section 2.8 describes how the MAPLE program finds a solution. Theorem 3.8.1 is the list of products in which our MAPLE program returns a solution. We now prove the equalities in Theorem 3.8.1.

First we prove (3.8.1). By (2.3.1), (3.5.29), (2.3.3), (2.3.4), (3.3.2), (2.6.1), (2.6.2) and (3.6.1), we have

$$\begin{aligned} L(q^2)L_{0,4}(q) &= \frac{1}{24} \left((-1 + 2g^4)w^2 + 6g^4(1 - g^4)w \frac{dw}{dx} \right) \\ &\quad + \frac{1}{192}(-11 - 18g^2 + 54g^4 + 36g^6 - 64g^8)w^4 \\ &\quad + \frac{3}{32}g^4(1 - g^4)(9 + 6g^2 - 23g^4)w^3 \frac{dw}{dx} \\ &\quad - 3g^8(1 - g^4)^2w^2 \left(\frac{dw}{dx} \right)^2 \\ &= \frac{1}{24}L(q^2) - \frac{1}{240}(11M(q^2) + 47M(q^4) - 48M(q^8)) \\ &\quad - \frac{1}{8} \left(2q \frac{dL(q^2)}{dq} + 7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right) \end{aligned}$$

Next we prove (3.8.2). By (2.3.1), (3.5.31), (2.3.3), (2.3.4), (3.3.2), (2.6.1),

(2.6.2) and (3.6.1), we have

$$\begin{aligned}
 L(q^2)L_{2,4}(q) &= \frac{9}{32}g^4(1-g^2)^2(1-g^4)w^3\frac{dw}{dx} \\
 &\quad - \frac{3}{64}(1-g^2)^2(1-2g^4)w^4 \\
 &= -\frac{1}{80}(5M(q^2) - 21M(q^4) + 16M(q^8)) \\
 &\quad - \frac{1}{40}\left(15q\frac{dL(q^2)}{dq} - 36q\frac{dL(q^4)}{dq} + 30q\frac{dL(q^8)}{dq}\right).
 \end{aligned}$$

Next we prove (3.8.3). By (2.3.2), (3.5.29), (2.3.4), (3.3.2), (2.6.2) and (3.6.1), we have

$$\begin{aligned}
 L(q^4)L_{0,4}(q) &= \frac{1}{768}(1-5g^4)(-8-(11+18g^2-37g^4)w^2)w^2 \\
 &\quad - \frac{1}{64}g^4(1-g^4)(-8-(19+18g^2-77g^4)w^2)w\frac{dw}{dx} \\
 &\quad - \frac{3}{2}g^8(1-g^4)^2w^2\left(\frac{dw}{dx}\right)^2 \\
 &= \frac{1}{24}L(q^4) - \frac{1}{120}(29M(q^4) - 24M(q^8)) \\
 &\quad - \frac{1}{16}\left(11q\frac{dL(q^4)}{dq} - 6q\frac{dL(q^8)}{dq}\right).
 \end{aligned}$$

Next we prove (3.8.4). By (2.3.2), (3.5.33), (2.3.6), (3.3.3), (2.6.4) and (3.6.2), we have

$$\begin{aligned}
 L(q^4)M_{0,4}(q) &= -\frac{1}{30720}(1-5g^4) \\
 &\quad (-32 + (137 - 540g^2 + 838g^4 - 540g^6 + 137g^8)w^4)w^2 \\
 &\quad + \frac{1}{2560}g^4(1-g^4) \\
 &\quad (-32 + (137 - 540g^2 + 838g^4 - 540g^6 + 137g^8)w^4)w\frac{dw}{dx}
 \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{240}L(q^4) + \frac{1}{1680}(487N(q^4) - 480N(q^8)) \\
&\quad + \frac{1}{320} \left(73q \frac{dM(q^4)}{dq} - 72q \frac{dM(q^8)}{dq} \right).
\end{aligned}$$

Next we prove (3.8.5). By (2.3.4), (3.5.29), (2.3.6), (3.3.3) and (2.6.4), we have

$$\begin{aligned}
M(q^4)L_{0,4}(q) &= \frac{1}{3072}(8 + (11 + 18g^2 - 37g^4)w^2)(1 + 14g^4 + g^8)w^4 \\
&\quad - \frac{1}{32}g^4(1 - g^4)(1 + 14g^4 + g^8)w^5 \frac{dw}{dx} \\
&= \frac{1}{24}M(q^4) - \frac{1}{168}(39N(q^4) - 32N(q^8)) \\
&\quad - \frac{1}{8}q \frac{dM(q^4)}{dq}.
\end{aligned}$$

Next we prove (3.8.6). By (2.3.2), (3.5.31), (2.3.3), (2.3.4), (3.3.2), (2.6.1), (2.6.2) and (3.6.1), we have

$$\begin{aligned}
L(q^4)L_{2,4}(q) &= -\frac{3}{256}(1 - g^2)^2(1 - 5g^4)w^4 \\
&\quad + \frac{9}{64}g^4(1 - g^2)^2(1 - g^4)w^3 \frac{dw}{dx} \\
&= -\frac{1}{840}(1M(q^2) - 9M(q^4) + 8M(q^8)) \\
&\quad - \frac{1}{16} \left(3q \frac{dL(q^2)}{dq} - 9q \frac{dL(q^4)}{dq} + 6q \frac{dL(q^8)}{dq} \right).
\end{aligned}$$

Next we prove (3.8.7). By (2.3.2), (3.5.35), (2.3.5), (2.3.6), (3.3.3), (2.6.3), (2.6.4) and (3.6.2), we have

$$\begin{aligned}
L(q^4)M_{2,4}(q) &= -\frac{9}{2048}(1 - g^2)^2(1 - 5g^4)(1 + 6g^2 + g^4)w^6 \\
&\quad + \frac{27}{512}g^4(1 - g^2)^2(1 - g^4)(1 + 6g^2 + g^4)w^5 \frac{dw}{dx}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{112}(N(q^2) - 33N(q^4) + 32N(q^8)) \\
&\quad + \frac{1}{320} \left(9q \frac{dM(q^2)}{dq} - 81q \frac{dM(q^4)}{dq} + 72q \frac{dM(q^8)}{dq} \right).
\end{aligned}$$

Next we prove (3.8.8). By (2.3.4), (3.5.31), (2.3.5), (2.3.6) and (3.3.3), we have

$$\begin{aligned}
M(q^4)L_{2,4}(q) &= \frac{3}{1024}(1 - g^2)^2(1 + 14g^4 + g^8)w^6 \\
&= -\frac{1}{168}(N(q^2) - 33N(q^4) + 32N(q^8)).
\end{aligned}$$

Next we prove (3.8.9). By (3.3.1), (3.5.29), (2.3.4), (3.3.2), (2.6.2) and (3.6.1), we have

$$\begin{aligned}
L(q^8)L_{0,4}(q) &= -\frac{1}{3072}(1 - 6g^2 - 11g^4)(8 + (11 + 18g^2 - 37g^4)w^2)w^2 \\
&\quad + \frac{1}{128}g^4(1 - g^4)(8 + (15 - 6g^2 - 81g^4)w^2)w \frac{dw}{dx} \\
&\quad - \frac{3}{4}g^8(1 - g^4)^2w^2 \left(\frac{dw}{dx} \right)^2 \\
&= \frac{1}{24}L(q^8) - \frac{1}{120}(7M(q^4) - 2M(q^8)) \\
&\quad - \frac{1}{32} \left(7q \frac{dL(q^4)}{dq} - 2q \frac{dL(q^8)}{dq} \right).
\end{aligned}$$

Next we prove (3.8.10). By (3.3.1), (3.5.33), (2.3.6), (3.3.3), (2.6.4) and (3.6.2), we have

$$\begin{aligned}
L(q^8)M_{0,4}(q) &= \frac{1}{122880}(-1 + 6g^2 + 11g^4) \\
&\quad (-32 + (+137 - 540g^2 + 838g^4 - 540g^6 + 137g^8)w^4)w^2 \\
&\quad + \frac{1}{5120}g^4(1 - g^4)
\end{aligned}$$

$$\begin{aligned}
& (-32 + (137 - 540g^2 + 838g^4 - 540g^6 + 137g^8)w^4)w \frac{dw}{dx} \\
= & -\frac{1}{240}L(q^8) + \frac{1}{5040}(365N(q^4) - 344N(q^8)) \\
& + \frac{1}{640} \left(73q \frac{dM(q^4)}{dq} - 72q \frac{dM(q^8)}{dq} \right).
\end{aligned}$$

Next we prove (3.8.11). By (3.3.2), (3.5.29), (2.3.6), (3.3.3) and (3.6.2), we have

$$\begin{aligned}
M(q^8)L_{0,4}(q) &= \frac{1}{49152}(8 + (11 + 18g^2 - 37g^4)w^2) \\
& \quad (1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^4 \\
& \quad - \frac{1}{512}g^4(1 - g^4)(1 + 60g^2 + 134g^4 + 60g^6 + g^8)w^5 \frac{dw}{dx} \\
= & \frac{1}{24}M(q^8) - \frac{1}{72}(N(q^4) - 2N(q^8)) \\
& \quad - \frac{1}{8}q \frac{dM(q^8)}{dq}.
\end{aligned}$$

Next we prove (3.8.12). By (3.3.1), (3.5.31), (2.3.3), (2.3.4), (3.3.2), (2.6.1), (2.6.2) and (3.6.1), we have

$$\begin{aligned}
L(q^8)L_{2,4}(q) &= \frac{3}{1024}(-1 + 6g^2 + 11g^4)(1 - g^2)^2w^4 \\
& \quad + \frac{9}{128}g^4(1 - g^2)^2(1 - g^4)w^3 \frac{dw}{dx} \\
= & -\frac{1}{160}(M(q^2) - 9M(q^4) + 8M(q^8)) \\
& \quad - \frac{1}{32} \left(3q \frac{dL(q^4)}{dq} - 9q \frac{dL(q^4)}{dq} + 6q \frac{dL(q^8)}{dq} \right).
\end{aligned}$$

Next we prove (3.8.13). By (3.5.32), (3.5.30), (2.3.4) and (3.3.2), we have

$$\begin{aligned}
L_{3,4}(q)L_{1,4}(q) &= \frac{1}{1024}(1 + g)^4(1 - g)^4w^4 \\
&= \frac{1}{60}(M(q^4) - M(q^8)).
\end{aligned}$$

Next we prove (3.8.14). By (3.5.31), (3.5.29), (2.3.1), (2.3.2), (3.3.1) (2.3.3), (2.3.4), (3.3.2), (2.6.1), (2.6.2) and (3.6.1), we have

$$\begin{aligned} L_{2,4}(q)L_{0,4}(q) &= \frac{1}{4096}(1-g^2)^2(8+(11+18g^2-37g^4)w^2)w^2 \\ &\quad - \frac{3}{128}g^4(1-g^4)(1-g^2)^2w^3\frac{dw}{dx} \\ &= -\frac{1}{192}(L(q^2)-3L(q^4)+2L(q^8)) \\ &\quad + \frac{1}{1920}(11M(q^2)-99M(q^4)+88M(q^8)) \\ &\quad + \frac{1}{32}\left(q\frac{dL(q^2)}{dq}-3q\frac{dL(q^4)}{dq}+2q\frac{dL(q^8)}{dq}\right). \end{aligned}$$

Next we prove (3.8.15). By (3.5.31), (2.3.4) and (3.3.2), we have

$$\begin{aligned} L_{2,4}^2(q) &= \frac{9}{4096}(1-g^2)^4w^4 \\ &= \frac{3}{80}(M(q^4)-M(q^8)). \end{aligned}$$

Next we prove (3.8.16). By (3.5.31), (3.5.35), (2.3.6) and (3.3.3), we have

$$\begin{aligned} L_{2,4}(q)M_{2,4}(q) &= \frac{27}{32768}(1-g^2)^4(1+6g^2+g^4)w^6 \\ &= -\frac{3}{56}(N(q^4)-N(q^8)). \end{aligned}$$

Finally we prove (3.8.17). By (3.5.29), (2.3.2), (3.3.1) (2.3.4), (3.3.2), (2.6.2) and (3.6.1), we have

$$\begin{aligned} L_{0,4}^2(q) &= \frac{1}{24} + \frac{1}{192}(11+18g^2-37g^4)w^2 \\ &\quad - \frac{1}{192}g^4(1-g^4)(8+(11+18g^2-37g^4)w^2)w\frac{dw}{dx} \\ &\quad + \frac{1}{4}g^8(1-g^4)^2w^2\left(\frac{dw}{dx}\right)^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{576} - \frac{1}{288}(7L(q^4) - 6L(q^8)) \\
&\quad + \frac{1}{2880}(161M(q^4) - 156M(q^8)) \\
&\quad + \frac{1}{48} \left(7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right).
\end{aligned}$$

This completes the proof for Theorem 3.8.1 □

3.9 More arithmetic identities.

Let a and b be integers satisfying $b \geq 1$ and $0 \leq a \leq b - 1$. Set

$$S(a, b) = \sum_{\substack{m=1 \\ m \equiv a \pmod{b}}}^{n-1} \sigma(m)\sigma(n-m).$$

Huard, Ou, Spearman and Williams [14, Section 5] have given results for $S(a, b)$ for $b = 1, 2, 3$ and 4.

In this section, we use the relationships in Theorem 3.8.1 to derive seventeen arithmetic identities. Formula (3.9.13) gives the value of $S(1, 4)$ for $n \equiv 0 \pmod{4}$ [14, Theorem 9]. Formula (3.9.14) gives the value of $S(0, 4)$ for $n \equiv 2 \pmod{4}$ [14, Theorem 9]. Formula (3.9.15) gives the value of $S(2, 4)$ for $n \equiv 0 \pmod{4}$. Formula (3.9.17) gives the value of $S(0, 4)$ for $n \equiv 0 \pmod{4}$. The latter two formulae extend the result given in [14, Theorem 9].

Theorem 3.9.1

$$(3.9.1) \quad \sum_{m < n/2} \sigma(4m)\sigma(n-2m)$$

$$\begin{aligned}
&= \frac{1}{24} \{ 11\sigma_3(n) + 47\sigma_3(n/2) - 48\sigma_3(n/4) \\
&\quad + (\sigma(n) + 7\sigma(n/2) - 6\sigma(n/4)) \\
&\quad - 6n(2\sigma(n) + 7\sigma(n/2) - 6\sigma(n/4)) \}, \\
(3.9.2) \quad &\sum_{m < \frac{n+1}{2}} \sigma(4m-2)\sigma(n-(2m-1)) \\
&= \frac{1}{8} \{ 5\sigma_3(n) - 21\sigma_3(n/2) + 16\sigma_3(n/4) \\
&\quad + (1-6n)(\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4)) \}, \\
(3.9.3) \quad &\sum_{m < n} \sigma(4m)\sigma(n-m) \\
&= \frac{1}{12} \{ 29\sigma_3(n) - 24\sigma_3(n/2) + 4\sigma(n) \\
&\quad - 3\sigma(n/2) - 33n\sigma(n) + 18n\sigma(n/2) \}, \\
(3.9.4) \quad &\sum_{m < n} \sigma_3(4m)\sigma(n-m) \\
&= \frac{1}{240} \{ 1461\sigma_5(n) - 1440\sigma_5(n/2) \\
&\quad + (10-30n)(73\sigma_3(n) - 72\sigma_3(n/2)) - \sigma(n) \}, \\
(3.9.5) \quad &\sum_{m < n} \sigma(4m)\sigma_3(n-m) \\
&= \frac{1}{240} \{ 117\sigma_5(n) - 96\sigma_5(n/2) + (10-120n)\sigma_3(n) \\
&\quad - 7\sigma(n) + 6\sigma(n/2) \}, \\
(3.9.6) \quad &\sum_{m < n} \sigma(4m-2)\sigma(n-m) \\
&= \frac{1}{4} \{ \sigma_3(2n-1) + (2-3n)\sigma(2n-1) \}, \\
(3.9.7) \quad &\sum_{m < n} \sigma_3(4m-2)\sigma(n-m)
\end{aligned}$$

$$\begin{aligned}
&= \frac{3}{16} \{ \sigma_5(2n-1) + (5-6n)\sigma_3(2n-1) \}, \\
(3.9.8) \quad &\sum_{m < n} \sigma(4m-2)\sigma_3(n-m) \\
&= \frac{1}{80} \{ \sigma_5(2n-1) - \sigma(2n-1) \}, \\
(3.9.9) \quad &\sum_{m < n-1} \sigma(4m)\sigma\left(\frac{n-m}{2}\right) \\
&= \frac{1}{24} \{ 14\sigma_3(n) - 4\sigma_3(n/2) + 7\sigma(n) - 5\sigma(n/2) \\
&\quad - 21n\sigma(n) - 6n\sigma(n/2) \}, \\
(3.9.10) \quad &\sum_{m < n-1} \sigma_3(4m)\sigma\left(\frac{n-m}{2}\right) \\
&= \frac{1}{240} \{ 365\sigma_5(n) - 344\sigma_5(n/2) \\
&\quad + (10-15n)(73\sigma_3(n) - 72\sigma_3(n/2)) - \sigma(n/2) \}, \\
(3.9.11) \quad &\sum_{m < n-1} \sigma(4m)\sigma_3\left(\frac{n-m}{2}\right) \\
&= \frac{1}{240} \{ 7\sigma_5(n) + 14\sigma_5(n/2) \\
&\quad + (10-120n)(\sigma_3(n/2)) - 7\sigma(n) + 6\sigma(n/2) \}, \\
(3.9.12) \quad &\sum_{m < n-1} \sigma(4m-2)\sigma\left(\frac{n-m}{2}\right) \\
&= \frac{1}{16} \{ \sigma_3(2n-1) + (5-6n)\sigma(2n-1) \}, \\
(3.9.13) \quad &\sum_{m < n+1} \sigma(4m-3)\sigma(4n-(4m-3)) \\
&= 4\sigma_3(n) - 4\sigma_3(n/2), \\
(3.9.14) \quad &\sum_{m < n} \sigma(4m)\sigma(4n-2-4m) \\
&= \frac{1}{8} \{ 11\sigma_3(2n-1) + (3-24n)\sigma(n) \},
\end{aligned}$$

$$(3.9.15) \quad \sum_{m < n+1} \sigma(4m-2)\sigma(4n-(4m-2)) \\ = 9\sigma_3(n) - 9\sigma_3(n/2),$$

$$(3.9.16) \quad \sum_{m < n+1} \sigma_3(4m-2)\sigma(4n-(4m-2)) \\ = 27\sigma_5(n) - 27\sigma_5(n/2),$$

$$(3.9.17) \quad \sum_{m < n} \sigma(4m)\sigma(4n-4m) \\ = \frac{1}{12} \{161\sigma_3(n) - 156\sigma_3(n/2) \\ + (1-24n)(7\sigma(n) - 6\sigma(n/2))\}.$$

Proof First we prove (3.9.1). By (3.8.1), (1.1.1), (1.1.2), (3.5.1) and (3.7.1), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n/2} \sigma(4m)\sigma(n-2m)q^{2n} \\ &= \sum_{l=1}^{\infty} \sigma(4l)q^{4l} \sum_{k=1}^{\infty} \sigma(k)q^{2k} \\ &= \frac{L_{0,4}(q) - L^2(q)L_{0,4}(q)}{24} \\ &= \frac{1}{576}(1 - 7L(q^4) + 6L(q^8)) - \frac{1}{576}L(q^2) \\ & \quad + \frac{1}{5760}(11M(q^2) + 47M(q^4) - 48M(q^8)) \\ & \quad + \frac{1}{192} \left(2q \frac{dL(q^2)}{dq} + 7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right) \\ &= \frac{1}{576} - \frac{7}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2)q^{2n} \right) + \frac{6}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4)q^{2n} \right) \\ & \quad - \frac{1}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^{2n} \right) + \frac{11}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{47}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) - \frac{48}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^{2n} \right) \\
& + \frac{2}{192} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n) q^{2n} \right) + \frac{7}{192} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/2) q^{2n} \right) \\
& - \frac{6}{192} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/4) q^{2n} \right) \\
& = \frac{1}{24} \sum_{n=1}^{\infty} \{ 11\sigma_3(n) + 47\sigma_3(n/2) - 48\sigma_3(n/4) \\
& \quad + (\sigma(n) + 7\sigma(n/2) - 6\sigma(n/4)) \\
& \quad - 6n(2\sigma(n) + 7\sigma(n/2) - 6\sigma(n/4)) \} q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (3.9.1).

Next we prove (3.9.2). By (3.8.2), (1.1.1), (1.1.2), (3.5.3) and (3.7.2), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(4m-2) \sigma(n-(2m-1)) q^{2n} \\
& = \sum_{l=1}^{\infty} \sigma(4l-2) q^{4l-2} \sum_{k=1}^{\infty} \sigma(k) q^{2k} \\
& = \frac{L_{2,4}(q) - L^2(q) L_{2,4}(q)}{24} \\
& = \frac{1}{192} (-L(q^2) + 3L(q^4) - 2L(q^8)) \\
& \quad + \frac{1}{1920} (5M(q^2) - 21M(q^4) + 16M(q^8)) \\
& \quad + \frac{1}{192} \left(3q \frac{dL(q^2)}{dq} - 9q \frac{dL(q^4)}{dq} + 6q \frac{dL(q^8)}{dq} \right) \\
& = -\frac{1}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{3}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{2}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^{2n} \right) + \frac{5}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) \\
& -\frac{21}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) + \frac{16}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^{2n} \right) \\
& + \frac{3}{192} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n) q^{2n} \right) - \frac{9}{192} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/2) q^{2n} \right) \\
& + \frac{6}{192} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/4) q^{2n} \right) \\
& = \frac{1}{8} \sum_{n=1}^{\infty} \{ 5\sigma_3(n) - 21\sigma_3(n/2) + 16\sigma_3(n/4) \\
& \quad + (1 - 6n)(\sigma(n) - 3\sigma(n/2) + 2\sigma(n/4)) \} q^{2n}.
\end{aligned}$$

Equating the coefficients of q^{2n} , we obtain the identity (3.9.2).

Next we prove (3.9.3). By (3.8.3), (1.1.1), (1.1.2), (3.5.1) and (3.7.1), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(4m) \sigma(n-m) q^{4n} \\
& = \sum_{l=1}^{\infty} \sigma(4l) q^{4l} \sum_{k=1}^{\infty} \sigma(k) q^{4k} \\
& = \frac{L_{0,4}(q) - L^4(q) L_{0,4}(q)}{24} \\
& = \frac{1}{576} (1 - 7L(q^4) + 6L(q^8)) - \frac{1}{576} L(q^4) \\
& \quad + \frac{1}{2880} (29M(q^4) - 24M(q^8)) \\
& \quad + \frac{1}{384} \left(11q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right) \\
& = \frac{1}{576} - \frac{8}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{4n} \right) + \frac{6}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{4n} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{29}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n} \right) - \frac{24}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{4n} \right) \\
& + \frac{11}{384} \left(-24 \sum_{n=1}^{\infty} 4n\sigma(n) q^{4n} \right) - \frac{6}{384} \left(-24 \sum_{n=1}^{\infty} 4n\sigma(n/2) q^{4n} \right) \\
& = \frac{1}{12} \sum_{n=1}^{\infty} \{ 29\sigma_3(n) - 24\sigma_3(n/2) + 4\sigma(n) \\
& \quad - 3\sigma(n/2) - 33n\sigma(n) + 18n\sigma(n/2) \} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.3).

Next we prove (3.9.4). By (3.8.4), (1.1.1), (1.1.2), (3.5.5) and (3.7.3), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(4m) \sigma(n-m) q^{4n} \\
& = \sum_{l=1}^{\infty} \sigma_3(4l) q^{4l} \sum_{k=1}^{\infty} \sigma(k) q^{4k} \\
& = \frac{M_{0,4}(q) - L^4(q) M_{0,4}(q)}{24} \\
& = \frac{1}{5760} (-1 + 73M(q^4) - 72M(q^8)) + \frac{1}{5760} L(q^4) \\
& \quad - \frac{1}{40320} (487N(q^4) - 480N(q^8)) \\
& \quad - \frac{1}{7680} \left(73q \frac{dM(q^4)}{dq} - 72q \frac{dM(q^8)}{dq} \right) \\
& = -\frac{1}{5760} + \frac{73}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n} \right) \\
& \quad - \frac{72}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{4n} \right) + \frac{1}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{4n} \right) \\
& \quad - \frac{487}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{4n} \right) + \frac{480}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{4n} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{73}{7680} \left(240 \sum_{n=1}^{\infty} 4n\sigma_3(n)q^{4n} \right) + \frac{72}{7680} \left(240 \sum_{n=1}^{\infty} 4n\sigma_3(n/2)q^{4n} \right) \\
& = \frac{1}{240} \sum_{n=1}^{\infty} \{1461\sigma_5(n) - 1440\sigma_5(n/2) \\
& \quad + (10 - 30n)(73\sigma_3(n) - 72\sigma_3(n/2)) - \sigma(n)\} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.4).

Next we prove (3.9.5). By (3.8.5), (1.1.1), (1.1.2), (1.1.3), (3.5.1) and (3.7.1), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(4m)\sigma_3(n-m)q^{4n} \\
& = \sum_{l=1}^{\infty} \sigma(4l)q^{4l} \sum_{k=1}^{\infty} \sigma_3(k)q^{4k} \\
& = \frac{-L_{0,4}(q) + M(q^4)L_{0,4}(q)}{240} \\
& = -\frac{1}{5760}(1 - 7L(q^4) + 6L(q^8)) \\
& \quad + \frac{1}{5760}M(q^4) - \frac{1}{1920}q \frac{dM(q^4)}{dq} \\
& \quad - \frac{1}{40320}(39N(q^4) - 32N(q^8)) \\
& = -\frac{1}{5760} + \frac{7}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^{4n} \right) - \frac{6}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2)q^{4n} \right) \\
& \quad + \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{4n} \right) - \frac{1}{1920} \left(240 \sum_{n=1}^{\infty} 4n\sigma_3(n)q^{4n} \right) \\
& = \frac{1}{240} \sum_{n=1}^{\infty} \{117\sigma_5(n) - 96\sigma_5(n/2) + (10 - 120n)\sigma_3(n) \\
& \quad - 7\sigma(n) + 6\sigma(n/2)\} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.5).

Next we prove (3.9.6). By (3.8.6), (1.1.1), (1.1.2), (3.5.3) and (3.7.2), we have for odd n

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(4m-2) \sigma\left(\frac{n-(2m-1)}{2}\right) q^{2n} \\
&= \sum_{l=1}^{\infty} \sigma(4l-2) q^{4l-2} \sum_{k=1}^{\infty} \sigma(k) q^{4k} \\
&= \frac{L_{2,4}(q) - L(q^4) L_{2,4}(q)}{24} \\
&= \frac{1}{192} (-L(q^2) + 3L(q^4) - 2L(q^8)) \\
&\quad + \frac{1}{960} (M(q^2) - 9M(q^4) + 8M(q^8)) \\
&\quad + \frac{1}{384} \left(3q \frac{dL(q^2)}{dq} - 9q \frac{dL(q^4)}{dq} + 6q \frac{dL(q^8)}{dq} \right) \\
&= -\frac{1}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{3}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\
&\quad - \frac{2}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^{2n} \right) + \frac{1}{960} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) \\
&\quad - \frac{9}{960} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) + \frac{8}{960} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^{2n} \right) \\
&\quad + \frac{3}{384} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n) q^{2n} \right) - \frac{9}{384} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/2) q^{2n} \right) \\
&\quad + \frac{6}{384} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/4) q^{2n} \right).
\end{aligned}$$

Since n is odd, we replace $n \rightarrow 2n-1$ and we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(4m-2) \sigma(n-m) q^{4n-2} \\
&= \frac{1}{4} \sum_{n=1}^{\infty} \{ \sigma_3(2n-1) + (2-3n)\sigma(2n-1) \} q^{4n-2}.
\end{aligned}$$

Equating the coefficients of q^{4n-2} , we obtain the identity (3.9.6).

Next we prove (3.9.7). By (3.8.7), (1.1.1), (1.1.2), (1.1.3), (3.5.7) and (3.7.4), we have for odd n

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma_3(4m-2) \sigma\left(\frac{n-(2m-1)}{2}\right) q^{2n} \\
&= \sum_{l=1}^{\infty} \sigma_3(4l-2) q^{4l-2} \sum_{k=1}^{\infty} \sigma(k) q^{4k} \\
&= \frac{M_{2,4}(q) - L(q^4) M_{2,4}(q)}{24} \\
&= \frac{1}{1920} (3M(q^2) - 27M(q^4) + 24M(q^8)) \\
&\quad - \frac{1}{2688} (N(q^2) - 33N(q^4) + 32N(q^8)) \\
&\quad - \frac{1}{7680} \left(9q \frac{dM(q^2)}{dq} - 81q \frac{dM(q^4)}{dq} + 72q \frac{dM(q^8)}{dq} \right) \\
&= \frac{3}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) - \frac{27}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) \\
&\quad + \frac{24}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^{2n} \right) - \frac{1}{2688} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right) \\
&\quad + \frac{33}{2688} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{2n} \right) - \frac{32}{2688} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/4) q^{2n} \right) \\
&\quad - \frac{9}{7680} \left(240 \sum_{n=1}^{\infty} 2n \sigma_3(n) q^{2n} \right) + \frac{81}{7680} \left(240 \sum_{n=1}^{\infty} 2n \sigma_3(n/2) q^{2n} \right) \\
&\quad - \frac{72}{7680} \left(240 \sum_{n=1}^{\infty} 2n \sigma_3(n/4) q^{2n} \right).
\end{aligned}$$

Since n is odd, we replace $n \rightarrow 2n-1$ and we obtain

$$\sum_{n=1}^{\infty} \sum_{m < n} \sigma_3(4m-2) \sigma(n-m) q^{4n-2}$$

$$= \frac{3}{16} \sum_{n=1}^{\infty} \{ \sigma_5(2n-1) + (5-6n)\sigma_3(2n-1) \} q^{4n-2}.$$

Equating the coefficients of q^{4n-2} , we obtain the identity (3.9.7).

Next we prove (3.9.8). By (3.8.8), (1.1.1), (1.1.2), (1.1.3), (3.5.3) and (3.7.2), we have for odd n

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(4m-2) \sigma_3\left(\frac{n-(2m-1)}{2}\right) q^{2n} \\ &= \sum_{l=1}^{\infty} \sigma(4l-2) q^{4l-2} \sum_{k=1}^{\infty} \sigma_3(k) q^{4k} \\ &= \frac{L_{2,4}(q) - M(q^4) L_{2,4}(q)}{240} \\ &= -\frac{1}{1920} (-L(q^2) + 3L(q^4) - 2L(q^8)) \\ &\quad - \frac{1}{40320} (N(q^2) - 33N(q^4) + 32N(q^8)) \\ &= -\frac{1}{1920} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{3}{1920} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\ &\quad - \frac{2}{1920} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^{2n} \right) - \frac{1}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right) \\ &\quad + \frac{33}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{2n} \right) - \frac{32}{40320} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/4) q^{2n} \right). \end{aligned}$$

Since n is odd, we replace $n \rightarrow 2n-1$ and we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n} \sigma(4m-2) \sigma_3(n-m) q^{4n-2} \\ &= \frac{1}{80} \sum_{n=1}^{\infty} \{ \sigma_5(2n-1) - \sigma(2n-1) \} q^{4n-2}. \end{aligned}$$

Equating the coefficients of q^{4n-2} , we obtain the identity (3.9.8).

Next we prove (3.9.9). By (3.8.9), (1.1.1), (1.1.2), (3.5.1) and (3.7.1), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n-1} \sigma(4m) \sigma\left(\frac{n-m}{2}\right) q^{4n} \\
&= \sum_{l=1}^{\infty} \sigma(4l) q^{4l} \sum_{k=1}^{\infty} \sigma(k) q^{8k} \\
&= \frac{L_{0,4}(q) - L^8(q) L_{0,4}(q)}{24} \\
&= \frac{1}{576} (1 - 7L(q^4) + 6L(q^8)) - \frac{1}{576} L(q^8) \\
&\quad + \frac{1}{2880} (7M(q^4) - 2M(q^8)) \\
&\quad + \frac{1}{768} \left(7q \frac{dL(q^4)}{dq} - 2q \frac{dL(q^8)}{dq} \right) \\
&= \frac{1}{576} - \frac{7}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{4n} \right) + \frac{5}{576} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{4n} \right) \\
&\quad + \frac{7}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n} \right) - \frac{2}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{4n} \right) \\
&\quad + \frac{7}{768} \left(-24 \sum_{n=1}^{\infty} 4n\sigma(n) q^{4n} \right) - \frac{2}{38768} \left(-24 \sum_{n=1}^{\infty} 4n\sigma(n/2) q^{4n} \right) \\
&= \frac{1}{24} \sum_{n=1}^{\infty} \{ 14\sigma_3(n) - 4\sigma_3(n/2) + 7\sigma(n) - 5\sigma(n/2) \\
&\quad - 21n\sigma(n) - 6n\sigma(n/2) \} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.9).

Next we prove (3.9.10). By (3.8.10), (1.1.1), (1.1.2), (1.1.3), (3.5.5) and (3.7.3), we have

$$\sum_{n=1}^{\infty} \sum_{m < n-1} \sigma_3(4m) \sigma\left(\frac{n-m}{2}\right) q^{4n}$$

$$\begin{aligned}
&= \sum_{l=1}^{\infty} \sigma_3(4l) q^{4l} \sum_{k=1}^{\infty} \sigma(k) q^{8k} \\
&= \frac{M_{0,4}(q) - L^8(q) M_{0,4}(q)}{24} \\
&= \frac{1}{5760} (-1 + 73M(q^4) - 72M(q^8)) + \frac{1}{5760} L(q^8) \\
&\quad - \frac{1}{120960} (365N(q^4) - 344N(q^8)) \\
&\quad - \frac{1}{15360} \left(73q \frac{dM(q^4)}{dq} - 72q \frac{dM(q^8)}{dq} \right) \\
&= -\frac{1}{5760} + \frac{73}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n} \right) \\
&\quad - \frac{72}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{4n} \right) + \frac{1}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{4n} \right) \\
&\quad - \frac{365}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{4n} \right) + \frac{344}{120960} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{4n} \right) \\
&\quad - \frac{73}{15360} \left(240 \sum_{n=1}^{\infty} 4n \sigma_3(n) q^{4n} \right) + \frac{72}{15360} \left(240 \sum_{n=1}^{\infty} 4n \sigma_3(n/2) q^{4n} \right) \\
&= \frac{1}{240} \sum_{n=1}^{\infty} \{ 365 \sigma_5(n) - 344 \sigma_5(n/2) \\
&\quad + (10 - 15n)(73 \sigma_3(n) - 72 \sigma_3(n/2)) - \sigma(n/2) \} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.10).

Next we prove (3.9.11). By (3.8.11), (1.1.1), (1.1.2), (1.1.3), (3.5.1) and (3.7.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n-1} \sigma(4m) \sigma_3 \left(\frac{n-m}{2} \right) q^{4n} \\
&= \sum_{l=1}^{\infty} \sigma(4l) q^{4l} \sum_{k=1}^{\infty} \sigma_3(k) q^{8k}
\end{aligned}$$

$$\begin{aligned}
&= \frac{L_{0,4}(q) - M^8(q)L_{0,4}(q)}{240} \\
&= -\frac{1}{5760}(1 - 7L(q^4) + 6L(q^8)) + \frac{1}{5760}M(q^8) \\
&\quad - \frac{1}{17280}(N(q^4) + 2N(q^8)) - \frac{1}{1920}q \frac{dM(q^8)}{dq} \\
&= -\frac{1}{5760} + \frac{7}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^{4n} \right) \\
&\quad - \frac{6}{5760} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2)q^{4n} \right) + \frac{1}{5760} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2)q^{4n} \right) \\
&\quad - \frac{1}{17280} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^{4n} \right) - \frac{2}{17280} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2)q^{4n} \right) \\
&\quad - \frac{1}{1920} \left(240 \sum_{n=1}^{\infty} 4n\sigma_3(n/2)q^{4n} \right) \\
&= \frac{1}{240} \sum_{n=1}^{\infty} \{ 7\sigma_5(n) + 14\sigma_5(n/2) \\
&\quad + (10 - 120n)(\sigma_3(n/2)) - 7\sigma(n) + 6\sigma(n/2) \} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.11).

Next we prove (3.9.12). By (3.8.12), (1.1.1), (1.1.2), (3.5.3) and (3.7.2), we have for odd n

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(4m-2) \sigma\left(\frac{n-(2m-1)}{4}\right) q^{2n} \\
&= \sum_{l=1}^{\infty} \sigma(4l-2) q^{4l-2} \sum_{k=1}^{\infty} \sigma(k) q^{8k} \\
&= \frac{L_{2,4}(q) - L(q^8)L_{2,4}(q)}{24} \\
&= \frac{1}{192} (-L(q^2) + 3L(q^4) - 2L(q^8))
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{3840} (M(q^2) - 9M(q^4) + 8M(q^8)) \\
& + \frac{1}{768} \left(3q \frac{dL(q^2)}{dq} - 9q \frac{dL(q^4)}{dq} + 6q \frac{dL(q^8)}{dq} \right) \\
= & -\frac{1}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{3}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\
& - \frac{2}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^{2n} \right) + \frac{1}{3840} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) \\
& - \frac{9}{3840} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) + \frac{8}{3840} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^{2n} \right) \\
& + \frac{3}{768} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n) q^{2n} \right) - \frac{9}{768} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/2) q^{2n} \right) \\
& + \frac{6}{768} \left(-24 \sum_{n=1}^{\infty} 2n\sigma(n/4) q^{2n} \right).
\end{aligned}$$

Since n is odd, we replace $n \rightarrow 2n - 1$ and we obtain

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n} \sigma(4m - 2) \sigma\left(\frac{n - m}{2}\right) q^{4n-2} \\
& = \frac{1}{16} \sum_{n=1}^{\infty} \{ \sigma_3(2n - 1) + (5 - 6n) \sigma(2n - 1) \} q^{4n-2}.
\end{aligned}$$

Equating the coefficients of q^{4n-2} , we obtain the identity (3.9.12).

Next we prove (3.9.13). By (3.8.13), (1.1.2), (3.5.2) and (3.5.4), we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \sum_{m < n+1} \sigma(4m - 3) \sigma(4n - (4m - 3)) q^{4n} \\
& = \sum_{l=1}^{\infty} \sigma(4l - 3) q^{4l-3} \sum_{k=1}^{\infty} \sigma(4k - 1) q^{4k-1} \\
& = L_{1,4}(q) L_{3,4}(q) \\
& = \frac{1}{60} M(q^4) - \frac{1}{60} M(q^8)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{60} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n} \right) - \frac{1}{60} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{4n} \right) \\
&= 4 \sum_{n=1}^{\infty} \{ \sigma_3(n) - \sigma_3(n/2) \} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.13).

Next we prove (3.9.14). By (3.8.14), (1.1.1), (1.1.2), (3.5.1) and (3.5.3), we have for odd n

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < \frac{n+1}{2}} \sigma(4m) \sigma(2n - 4m) q^{2n} \\
&= \sum_{l=1}^{\infty} \sigma(4l) q^{4l} \sum_{k=1}^{\infty} \sigma(4k - 2) q^{4k-2} \\
&= L_{0,4}(q) L_{2,4}(q) \\
&= \frac{1}{192} (-L(q^2) + 3L(q^4) - 2L(q^8)) \\
&\quad + \frac{1}{1920} (11M(q^2) - 99M(q^4) + 88M(q^8)) \\
&\quad + \frac{1}{32} \left(q \frac{dL(q^2)}{dq} - 3q \frac{dL(q^4)}{dq} + 2q \frac{dL(q^8)}{dq} \right) \\
&= -\frac{1}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{2n} \right) + \frac{3}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{2n} \right) \\
&\quad - \frac{2}{192} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/4) q^{2n} \right) + \frac{11}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) \\
&\quad - \frac{99}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{2n} \right) + \frac{88}{1920} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/4) q^{2n} \right) \\
&\quad + \frac{1}{32} \left(-24 \sum_{n=1}^{\infty} 2n \sigma(n) q^{2n} \right) - \frac{3}{32} \left(-24 \sum_{n=1}^{\infty} 2n \sigma(n/2) q^{2n} \right) \\
&\quad + \frac{2}{32} \left(-24 \sum_{n=1}^{\infty} 2n \sigma(n/4) q^{2n} \right).
\end{aligned}$$

Since n is odd, we replace $n \rightarrow 2n - 1$ and we obtain

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n} \sigma(4m) \sigma(4n - 2 - 4m) q^{4n-2} \\ &= \frac{1}{8} \sum_{n=1}^{\infty} \{11\sigma_3(2n-1) + (13-24n)\sigma(2n-1)\} q^{4n-2}. \end{aligned}$$

Equating the coefficients of q^{4n-2} , we obtain the identity (3.9.14).

Next we prove (3.9.15). By (3.8.15), (1.1.2), and (3.5.3), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n+1} \sigma(4m-2) \sigma(4n - (4m-2)) q^{4n} \\ &= \sum_{l=1}^{\infty} \sigma(4l-2) q^{4l-2} \sum_{k=1}^{\infty} \sigma(4k-2) q^{4k-2} \\ &= L_{2,4}^2(q) \\ &= \frac{3}{80} M(q^4) - \frac{3}{80} M(q^8) \\ &= \frac{3}{80} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n} \right) - \frac{3}{80} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{4n} \right) \\ &= 9 \sum_{n=1}^{\infty} \{ \sigma_3(n) - \sigma_3(n/2) \} q^{4n}. \end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.15).

Next we prove (3.9.16). By (3.8.16), (1.1.3), (3.5.3) and (3.5.7), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \sum_{m < n+1} \sigma_3(4m-2) \sigma(4n - (4m-2)) q^{4n} \\ &= \sum_{l=1}^{\infty} \sigma_3(4l-2) q^{4l-2} \sum_{k=1}^{\infty} \sigma(4k-2) q^{4k-2} \\ &= L_{2,4}(q) M_{2,4}(q) \\ &= -\frac{3}{56} N(q^4) + \frac{3}{56} N(q^8) \end{aligned}$$

$$\begin{aligned}
&= -\frac{3}{56} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^{4n} \right) + \frac{3}{56} \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n/2) q^{4n} \right) \\
&= 27 \sum_{n=1}^{\infty} \{ \sigma_5(n) - \sigma_5(n/2) \} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.16).

Next we prove (3.9.17). By (3.8.17), (1.1.1), (1.1.2) and (3.5.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sum_{m < n} \sigma(4m) \sigma(4n - 4m) q^{4n} \\
&= \sum_{l=1}^{\infty} \sigma(4l) q^{4l} \sum_{k=1}^{\infty} \sigma(4k) q^{4k} \\
&= L_{0,4}^2(q) \\
&= \frac{1}{288} (-7L(q^4) + 6L(q^8)) \\
&\quad + \frac{1}{2880} (161M(q^4) - 156M(q^8)) \\
&\quad + \frac{1}{48} \left(7q \frac{dL(q^4)}{dq} - 6q \frac{dL(q^8)}{dq} \right) \\
&= -\frac{7}{288} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^{4n} \right) + \frac{6}{288} \left(1 - 24 \sum_{n=1}^{\infty} \sigma(n/2) q^{4n} \right) \\
&\quad + \frac{161}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^{4n} \right) - \frac{156}{2880} \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n/2) q^{4n} \right) \\
&\quad + \frac{7}{48} \left(-24 \sum_{n=1}^{\infty} 4n \sigma(n) q^{4n} \right) - \frac{6}{48} \left(-24 \sum_{n=1}^{\infty} 4n \sigma(n/2) q^{4n} \right) \\
&= \frac{1}{12} \sum_{n=1}^{\infty} \{ 161 \sigma_3(n) - 156 \sigma_3(n/2) \\
&\quad + (1 - 24n)(7\sigma(n) - 6\sigma(n/2)) \} q^{4n}.
\end{aligned}$$

Equating the coefficients of q^{4n} , we obtain the identity (3.9.17). □

The sums

$$\begin{aligned}
& \sum_{m < n/4} \sigma(m) \sigma_3(n - 4m), \\
& \sum_{m < n/4} \sigma_3(m) \sigma(n - 4m), \\
& \sum_{m < n/2} \sigma(m) \sigma_5(n - 2m), \\
& \sum_{m < n/2} \sigma_3(m) \sigma_3(n - 2m), \\
& \sum_{m < n/2} \sigma_5(m) \sigma(n - 2m), \\
& \sum_{m < n/4} \sigma(m) \sigma_5(n - 4m), \\
& \sum_{m < n/4} \sigma_3(m) \sigma_3(n - 4m), \\
& \sum_{m < n/4} \sigma_5(m) \sigma(n - 4m)
\end{aligned}$$

do not appear in Theorems 2.9.1 and 3.9.1 and it is highly unlikely that they can be given only in terms of the values of $\sigma(t)$, $\sigma_3(t)$, $\sigma_5(t)$, $\sigma_7(t)$ with $t = n, n/2, n/4$. Huard and Williams [16] have used a linear combination of the first two sums which can be so evaluated to give elementary derivations of the formulae of Liouville [23] and Ewell [8] for the number of representations of $n \in \mathbb{N}$ as the sum of twelve squares. Similarly they have used properties of the last six sums to determine the number of representations of a positive integer as the sum of sixteen squares, the sum of sixteen triangular numbers and the sum of twenty-four triangular numbers [15], [17].

We did not find any identities among $L(q)$, $L(q^2)$, $L(q^4)$, $L(q^8)$, $M(q)$,

$M(q^2)$, $M(q^4)$, $M(q^8)$, ... $S(q)$, $S(q^2)$, $S(q^4)$, $S(q^8)$, involving products of three functions with at least one $X(q^2)$, $X(q^4)$ or $X(q^8)$ term. For example, we did not find an identity for $L^2(q)L(q^8)$. Such an identity would lead to the evaluation of the arithmetic sum

$$\sum_{m_1+m_2+8m_3=n} \sigma(m_1)\sigma(m_2)\sigma(m_3).$$

Chapter 4

Computer search for identities similar to those of Melfi.

Let n be a positive integer. In 1997/98, Melfi [26], [27] proved seven identities of the following form:

$$(4.0.1) \quad \sum_{m=0}^{\lfloor n/k \rfloor} \sigma_e(m) \sigma_f(n - km) = A \sigma_{e+f+1}(n) + B n \sigma_{e+f-1}(n),$$

which hold for all n in certain congruence classes modulo a multiple of k , $k \geq 2$, $e, f = 1$ or 3 , and for rationals A and B , where (here and only here)

$$\sigma_e(0) = \frac{1}{2} \zeta(-e)$$

and $\zeta(s)$ is the Riemann zeta function.

Using the theory of modular forms, Melfi [27] obtained the following iden-

ties. We obtained formulae (4.0.2)-(4.0.5) by our methods in Chapter 2.

$$(4.0.2) \quad \sum_{m < n/2} \sigma(m)\sigma(n-2m) = \frac{1}{24}\{2\sigma_3(n) + (1-3n)\sigma(n)\}, \quad n \equiv 1 \pmod{2},$$

$$(4.0.3) \quad \sum_{m < n/4} \sigma(m)\sigma(n-4m) = \frac{1}{48}\{\sigma_3(n) + (2-3n)\sigma(n)\}, \quad n \equiv 1 \pmod{2},$$

$$(4.0.4) \quad \sum_{m < n/2} \sigma_3(m)\sigma(n-2m) = \frac{1}{240}\{\sigma_5(n) - \sigma(n)\}, \quad n \equiv 1 \pmod{2},$$

$$(4.0.5) \quad \sum_{m < n/2} \sigma(m)\sigma_3(n-2m) = \frac{1}{240}\{5\sigma_5(n) + (10-15n)\sigma_3(n)\}, \quad n \equiv 1 \pmod{2}.$$

If $n \not\equiv 0 \pmod{3}$, then

$$(4.0.6) \quad \sum_{m < n/3} \sigma(m)\sigma(n-3m) = \frac{1}{24}\{\sigma_3(n) + (1-2n)\sigma(n)\}.$$

If $n \equiv 8 \pmod{16}$ and $n \not\equiv 0 \pmod{5}$, then

$$(4.0.7) \quad \sum_{m < n/5} \sigma(m)\sigma(n-5m) = \frac{5}{312}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n).$$

If n satisfies one of the following conditions:

(i) $n \equiv 2 \pmod{3}$;

(ii) $n \equiv 1 \pmod{3}$ and there exists a prime $p \equiv 2 \pmod{3}$ such that $p \mid n$ but $p^2 \nmid n$; then

$$(4.0.8) \quad \sum_{m < n/9} \sigma(m)\sigma(n-9m) = \frac{1}{216}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{36}n\right)\sigma(n).$$

By explicit computation, Melfi [26] has shown that for $e = f = 1$ and $2 \leq k < 100$, there are no other identities of the form (4.0.1). We extended this search by using MAPLE to seek possibly new identities of the form

$$(4.0.9) \quad \sum_{m < \frac{n}{k}} \sigma_e(m) \sigma_f(n - km) = \sum_{l=1}^{(e+f+2)/2} (A_{2l-1} + B_{2l-1}n) \sigma_{2l-1}(n),$$

$$n \equiv r \pmod{s}, \quad (n, k) = 1,$$

where e and f are positive odd integers and $A_1, \dots, A_{e+f+1}, B_1, \dots, B_{e+f+1}$ are rational numbers and where

$$\begin{aligned} k &= 1, 2, 3, \dots, 16, \\ e, f &= 1, 3, 5, \dots, 15, \\ r &= 0, 1, \dots, s-1, \\ s &= 1, 2, \dots, 50. \end{aligned}$$

For $2(e + f + 2)$ values of n coprime with k in each of the residue classes $r \pmod{s}$, the $2(e + f + 2)$ linear equations in the $e + f + 2$ unknowns $A_1, A_3, \dots, A_{e+f+1}, B_1, B_3, \dots, B_{e+f+1}$ resulting from (4.0.9) were solved. If these equations were solvable then these values gave a possible formula of the type (4.0.9). See Appendix D for the MAPLE program. This program took approximately 3 months to run on a 633MHz PC as there were 1,205,600 for loops, each of which took 4-6 seconds to run. However, no new identities were found.

Chapter 5

Proof of the formula for obtaining a formula by rotation.

As usual we let $\mathbb{C}$ denote the complex plane and $\mathbb{R}$ the real number line in $\mathbb{C}$.

We recall that the function $z^{\frac{1}{4}}$ is defined for all $z \in \mathbb{C}$ by

$$z^{\frac{1}{4}} = \begin{cases} |z|^{\frac{1}{4}} e^{\frac{1}{4}i \arg z}, & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

where the argument $\arg z$ of the complex number z is chosen to satisfy

$$-\pi < \arg z \leq \pi, \quad z \in \mathbb{C} \setminus \{0\}.$$

The function $z^{\frac{1}{4}}$ is regular [6, p. 36] in $\mathbb{C} \setminus (-\infty, 0]$. Let D denote the domain in the complex plane given by

$$(5.0.1) \quad D = \mathbb{C} \setminus [1, \infty).$$

Hence

(5.0.2) the function $(1 - z)^{\frac{1}{4}}$ is defined for all $z \in \mathbb{C}$ and is regular in D .

We define the function h by

$$(5.0.3) \quad h(z) = \left(1 - i(1 - z)^{\frac{1}{4}}\right)^4, \quad z \in \mathbb{C}.$$

We show that

$$(5.0.4) \quad h(z) \neq 0 \text{ for all } z \in \mathbb{C}.$$

To see this we suppose that $z \in \mathbb{C}$ is such that $h(z) = 0$. From (5.0.3) we see that

$$h(0) = (1 - i)^4 = -4 \neq 0$$

so that $z \neq 0$. Also from (5.0.3) we have

$$1 - i(1 - z)^{\frac{1}{4}} = 0$$

so that

$$(1 - z)^{\frac{1}{4}} = -i.$$

Taking the fourth power of both sides, we obtain

$$1 - z = 1,$$

contradicting $z \neq 0$.

In view of (5.0.2), (5.0.3) and (5.0.4) we can define

$$(5.0.5) \quad f(z) = \frac{-8i(1 - z)^{\frac{1}{4}} \left(1 - (1 - z)^{\frac{1}{2}}\right)}{\left(1 - i(1 - z)^{\frac{1}{4}}\right)^4}, \quad z \in \mathbb{C}$$

and

$$(5.0.6) \quad g(z) = \frac{1}{2}i \left(1 - i(1 - z)^{\frac{1}{4}} \right)^2, \quad z \in \mathbb{C}.$$

From (5.0.2), (5.0.3), (5.0.4), (5.0.5) and (5.0.6) we see that

(5.0.7) $f(z)$ and $g(z)$ are defined for all $z \in \mathbb{C}$ and are regular in D .

For $r \in \mathbb{R}^+$ we set $C(r) = \{z \in \mathbb{C} \mid |z| < r\}$. We note that

$$(5.0.8) \quad f(0) = 0,$$

$$(5.0.9) \quad g(0) = 1,$$

$$(5.0.10) \quad f'(0) = \lim_{z \rightarrow 0} \frac{f(z)}{z} = i,$$

$$(5.0.11) \quad g'(0) = \frac{1}{4}(i - 1),$$

$$(5.0.12) \quad f(z) \in C(\tfrac{1}{2}) \text{ for } z \in C(\tfrac{1}{16}),$$

$$(5.0.13) \quad f(x) \notin \mathbb{R} \text{ for } x \in (0, \tfrac{1}{16}).$$

We just prove (5.0.12) and (5.0.13) as the others are clear.

Proof of (5.0.12) and (5.0.13) First we prove (5.0.12). For $z \in C(\frac{1}{16})$

we have

$$|1 - z| \leq 1 + |z| < \frac{17}{16} < 16$$

so that

$$|1 - z|^{\frac{1}{4}} < 2.$$

By the generalized binomial theorem, we see that for $a = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and $z \in C(\frac{1}{16})$ we have

$$|(1-z)^a - 1| \leq \sum_{n=1}^{\infty} \left| \binom{a}{n} \right| |z|^n \leq \sum_{n=1}^{\infty} a|z|^n = \frac{a|z|}{1-|z|} < \frac{a}{15}.$$

Thus

$$\left| -8i(1-z)^{\frac{1}{4}} \left(1 - (1-z)^{\frac{1}{2}} \right) \right| < 8 \cdot 2 \cdot \frac{1}{30} = \frac{8}{15}$$

and

$$\begin{aligned} & \left| \left(1 - i(1-z)^{\frac{1}{4}} \right)^4 \right| \\ &= \left| 1 - 4i(1-z)^{\frac{1}{4}} - 6(1-z)^{\frac{1}{2}} + 4i(1-z)^{\frac{3}{4}} + (1-z) \right| \\ &= \left| -4 - z + 4i - 4i(1-z)^{\frac{1}{4}} + 6 - 6(1-z)^{\frac{1}{2}} - 4i + 4i(1-z)^{\frac{3}{4}} \right| \\ &\geq 4 - |z| - 4|1 - (1-z)^{\frac{1}{4}}| - 6|1 - (1-z)^{\frac{1}{2}}| - 4|1 - (1-z)^{\frac{3}{4}}| \\ &> 4 - \frac{1}{16} - \frac{1}{15} - \frac{3}{15} - \frac{3}{15} \\ &= \frac{833}{240}. \end{aligned}$$

Then, for $z \in C(\frac{1}{16})$, we have

$$|f(z)| = \frac{\left| -8i(1-z)^{\frac{1}{4}} \left(1 - (1-z)^{\frac{1}{2}} \right) \right|}{\left| \left(1 - i(1-z)^{\frac{1}{4}} \right)^4 \right|} < \frac{\frac{8}{15}}{\frac{833}{240}} = \frac{128}{833} < \frac{1}{2}.$$

Now we prove (5.0.13). Suppose that $f(x) \in \mathbb{R}$ for $x \in (0, \frac{1}{16})$. Then $f(x) = \overline{f(x)}$. As $1-x > 0$ we have $(1-x)^{\frac{1}{4}} \in \mathbb{R}$ so that by (5.0.5) we have

$$\frac{-i}{\left(1 - i(1-x)^{\frac{1}{4}} \right)^4} = \frac{i}{\left(1 + i(1-x)^{\frac{1}{4}} \right)^4}$$

and so

$$-\left(1 + i(1-x)^{\frac{1}{4}}\right)^4 = \left(1 - i(1-x)^{\frac{1}{4}}\right)^4.$$

Expanding the fourth powers, we obtain after cancelling the $(1-x)^{\frac{1}{4}}$ and $(1-x)^{\frac{3}{4}}$ terms

$$-1 + 6(1-x)^{\frac{1}{2}} - (1-x) = 1 - 6(1-x)^{\frac{1}{2}} + (1-x)$$

that is

$$-2 + 12(1-x)^{\frac{1}{2}} - 2(1-x) = 0.$$

Solving for $(1-x)^{\frac{1}{2}}$, we obtain

$$(1-x)^{\frac{1}{2}} = 3 + 2\sqrt{2} \text{ or } 3 - 2\sqrt{2}$$

so that

$$x = -16 - 12\sqrt{2} \ (\approx -33) \text{ or } -16 + 12\sqrt{2} \ (\approx 0.97).$$

This is a contradiction as neither of these values lies in the interval $(0, \frac{1}{16})$. $\square$

The integral

$$\int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-zt)^{-\frac{1}{2}} dt$$

is uniformly convergent in any closed domain of D and so represents an analytic function of z , which is regular in D [6, p. 249]. We set

$$w(z) := \frac{1}{\pi} \int_0^1 t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}(1-zt)^{-\frac{1}{2}} dt, \quad z \in D.$$

Hence

$$(5.0.14) \quad w(z) \text{ is defined and regular in } D.$$

For $z \in C(1)$ it is known that [6, p. 249]

$$\frac{1}{\pi} \int_0^1 t^{-\frac{1}{2}} (1-t)^{-\frac{1}{2}} (1-zt)^{-\frac{1}{2}} dt = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} \frac{z^n}{n!},$$

where for $a \in \mathbb{R}^+$ and $n \in \{0, 1, 2, \dots\}$ the Pochhammer symbol $(a)_n$ is defined by

$$(a)_n = a(a+1) \cdots (a+n-1), \quad n \in \mathbb{N}; \quad (a)_0 = 1.$$

Thus $w(z)$ is the analytic continuation of the hypergeometric function ${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)$ from $C(1)$ to D , so that

$$w(z) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right), \quad z \in C(1).$$

The function $w(z)$ is the unique solution of the hypergeometric differential equation

$$(5.0.15) \quad z(1-z)w'' + (1-2z)w' - \frac{1}{4}w = 0,$$

which is regular in D and satisfies

$$(5.0.16) \quad w(0) = 1, \quad w'(0) = \frac{1}{4},$$

see [6, pp. 246-248]. We now show that

$$(5.0.17) \quad w(z) \neq 0 \quad \text{for } z \in C\left(\frac{1}{2}\right).$$

As

$$\left(\frac{1}{2}\right)_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} = \frac{2n!}{2^{2n}n!}, \quad (1)_n = n!,$$

we have

$$\frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(1)_n} = \frac{(2n!)^2}{n!^3 2^{4n}}$$

so that

$$|w(z) - 1| = \left| \sum_{n=1}^{\infty} \frac{(2n!)^2}{n!^4 2^{4n}} z^n \right| \leq \sum_{n=1}^{\infty} \frac{(2n!)^2}{n!^4 2^{4n}} |z|^n, \quad |z| < 1.$$

The binomial coefficient

$$\binom{2n}{n} = \frac{2n!}{n!n!}$$

is one of the terms in the binomial expansion of $(1+1)^{2n}$ so that

$$\left(\frac{2n!}{n!^2} \right)^2 \leq 2^{4n}.$$

Hence

$$|w(z) - 1| \leq \sum_{n=1}^{\infty} |z|^n = \frac{|z|}{1 - |z|}.$$

If $z \in C(\frac{1}{2})$ then $\frac{|z|}{1 - |z|} < 1$, so that $|w(z) - 1| < 1$, proving (5.0.17). $\square$

For $x \in \mathbb{R}$ with $0 \leq x < 1$ we have

$$w(x) = 1 + \sum_{n=1}^{\infty} \frac{(2n!)^2}{n!^4 2^{4n}} x^n$$

so that

$$w(x) \geq 1, \quad 0 \leq x < 1,$$

and thus in particular

$$(5.0.18) \quad w(x) \neq 0, \quad 0 \leq x < 1.$$

It is convenient to define the subset D^* of D by

$$(5.0.19) \quad D^* = \{z \in D \mid f(z) \in D\}.$$

By (5.0.7), (5.0.14) and (5.0.19) we see that

$$w(f(z)) \text{ is defined and regular in } D^*$$

and

$$g(z)w(z) \text{ is defined and regular in } D.$$

Let $z \in C(\frac{1}{16})$. By (5.0.1) we have $z \in D$. By (5.0.12) we have $f(z) \in C(\frac{1}{2})$ so that $f(z) \in D$. Hence

$$(5.0.20) \quad C\left(\frac{1}{16}\right) \subseteq D^* \subseteq D.$$

Theorem 5.0.1 *For $z \in D^*$ we have*

$$w(f(z)) = g(z)w(z).$$

Proof We set

$$(5.0.21) \quad a(z) = w(f(z)), \quad z \in D^*,$$

$$(5.0.22) \quad b(z) = g(z)w(z), \quad z \in D^*.$$

Let $z \in C(\frac{1}{16}) \subseteq D^*$. We know that $a(z)$ and $b(z)$ are defined and regular in D^* . Differentiating (5.0.21) twice and appealing to (5.0.15), we find that $a(z)$ satisfies the differential equation

$$a'' + p(z)a' + q(z)a = 0,$$

where

$$(5.0.23) \quad p(z) = \frac{(1 - 2f(z))f'(z)}{f(z)(1 - f(z))} - \frac{f''(z)}{f'(z)},$$

$$(5.0.24) \quad q(z) = \frac{-f'(z)^2}{4f(z)(1 - f(z))}.$$

Similarly, differentiating (5.0.22) twice and appealing to (5.0.15), we find that $b(z)$ satisfies the differential equation

$$b'' + r(z)b' + s(z)b = 0,$$

where

$$(5.0.25) \quad r(z) = \frac{1 - 2z}{z(1 - z)} - \frac{2g'(z)}{g(z)},$$

$$(5.0.26) \quad s(z) = \frac{2g'(z)^2}{g(z)^2} - \frac{g''(z)}{g(z)} - \frac{1}{4z(1 - z)} - \frac{(1 - 2z)g'(z)}{z(1 - z)g(z)}.$$

A simple MAPLE calculation shows that

$$p(z) = r(z), \quad q(z) = s(z),$$

so that $a(z)$ and $b(z)$ satisfy the same second order homogeneous linear differential equation

$$(5.0.27) \quad y'' + r(z)y' + q(z)y = 0.$$

From (5.0.9), (5.0.11) and (5.0.25), we deduce that

$$(5.0.28) \quad \lim_{z \rightarrow 0} zr(z) = \lim_{z \rightarrow 0} \left(\frac{1 - 2z}{1 - z} - \frac{2zg'(z)}{g(z)} \right) = 1$$

and, from (5.0.8), (5.0.10) and (5.0.24), that

$$(5.0.29) \quad \lim_{z \rightarrow 0} z^2 q(z) = \lim_{z \rightarrow 0} \left(\frac{-z f'(z)^2}{4 \frac{f(z)}{z} (1 - f(z))} \right) = 0.$$

We deduce from (5.0.28) and (5.0.29) that $z = 0$ is a regular singular point of (5.0.27) [6, p. 237]. The indicial equation is $\alpha(\alpha - 1) + 1\alpha + 0 = 0$ [6, p. 238], which has the double root $\alpha = 0$. Thus the general solution of (5.0.27) in a neighbourhood of $z = 0$ is of the form

$$y = Ay_1(z) + By_2(z), \quad A, B \in \mathbb{C},$$

where $y_1(z)$ is a solution of (5.0.27), which is regular in the neighbourhood of $z = 0$, and $y_2(z)$ is a solution of (5.0.27) possessing a logarithmic branch point at $z = 0$ [6, p. 242]. Hence (5.0.27) has a unique solution $y(z)$ regular in a neighbourhood of $z = 0$ with a prescribed value for $y(0)$. Since (by (5.0.21), (5.0.16), (5.0.9) and (5.0.22))

$$a(0) = w(f(0)) = w(0) = g(0)w(0) = b(0),$$

this proves that

$$w(f(z)) = a(z) = b(z) = g(z)w(z) \quad \text{for } z \in C\left(\frac{1}{16}\right).$$

Thus as $w(f(z))$ and $g(z)w(z)$ are defined and regular on D^* , and equal on the subset $C\left(\frac{1}{16}\right)$ of D^* , we have

$$w(f(z)) = g(z)w(z) \quad \text{for } z \in D^*.$$

This completes the proof of Theorem 5.0.1. □

We define the subset D^{**} of D^* by

$$(5.0.30) \quad D^{**} = \{z \in D^* \mid w(z) \neq 0, w(f(z)) \neq 0, 1 - z \in D, 1 - f(z) \in D\}.$$

It is clear from (5.0.30) that

$$(5.0.31) \quad \frac{\pi w(1-z)}{w(z)} - \frac{\pi w(1-f(z))}{w(f(z))} \text{ is defined and regular in } D^{**}.$$

For $x \in (0, \frac{1}{16})$ we have

$$(5.0.32) \quad x \in D^*, \quad \text{by} \quad (5.0.20),$$

$$(5.0.33) \quad w(x) \neq 0, \quad \text{by} \quad (5.0.17),$$

$$(5.0.34) \quad w(f(x)) \neq 0, \quad \text{by} \quad (5.0.12) \text{ and } (5.0.17),$$

$$(5.0.35) \quad 1 - x \in D, \quad \text{as} \quad \frac{15}{16} < 1 - x < 1,$$

$$(5.0.36) \quad 1 - f(x) \in D, \quad \text{by} \quad (5.0.1) \text{ and } (5.0.13).$$

From (5.0.30) and (5.0.32)-(5.0.36) we see that

$$(5.0.37) \quad (0, \frac{1}{16}) \subseteq D^{**}.$$

We use Theorem 5.0.1 to prove the following important identity satisfied by $w(z)$.

Theorem 5.0.2 *For $z \in D^{**}$ we have*

$$e^{-\frac{\pi w(1-f(z))}{w(f(z))}} = ie^{-\frac{\pi w(1-z)}{w(z)}}.$$

Proof By (5.0.31) the function

$$\frac{\pi w(1-z)}{w(z)} - \frac{\pi w(1-f(z))}{w(f(z))}$$

is defined and regular in $D^{**} \supseteq (0, \frac{1}{16})$. For $x \in (0, \frac{1}{16})$ we have by [2, p. 87]

$$\frac{d}{dx} \left(\frac{\pi w(1-x)}{w(x)} \right) = \frac{-1}{x(1-x)w(x)^2}.$$

Thus

$$\frac{d}{dx} \left(\frac{\pi w(1-f(x))}{w(f(x))} \right) = \frac{-f'(x)}{f(x)(1-f(x))w(f(x))^2}.$$

By Theorem 5.0.1 we have $w(f(x)) = g(x)w(x)$ so that

$$\begin{aligned} & \frac{d}{dx} \left(\frac{\pi w(1-x)}{w(x)} - \frac{\pi w(1-f(x))}{w(f(x))} \right) \\ &= \left(\frac{f'(x)}{f(x)(1-f(x))g(x)^2} - \frac{1}{x(1-x)} \right) \frac{1}{w(x)^2}. \end{aligned}$$

A MAPLE calculation shows that

$$\frac{f'(x)}{f(x)(1-f(x))g(x)^2} = \frac{1}{x(1-x)}$$

so that there is a constant $K \in \mathbb{C}$ such that

$$\frac{\pi w(1-x)}{w(x)} - \frac{\pi w(1-f(x))}{w(f(x))} = K, \quad \text{for } 0 < x < \frac{1}{16}.$$

Thus

$$\begin{aligned} & \lim_{x \rightarrow 0^+} (g(x)w(1-x) - w(1-f(x))) \\ &= \lim_{x \rightarrow 0^+} \left(g(x)w(x) \left(\frac{w(1-x)}{w(x)} - \frac{w(1-f(x))}{w(f(x))} \right) \right) \\ &= g(0)w(0) \frac{K}{\pi} \\ &= \frac{K}{\pi}, \end{aligned}$$

by (5.0.9) and (5.0.16). On the other hand by [1, formula (15.3.6), p. 559], we have

$$w(1-x) = -\frac{1}{\pi}w(x)\log x + A(x), \quad |x| < 1,$$

where

$$A(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^2} \left(2\psi(n+1) - \psi\left(n+\frac{1}{2}\right) - \psi\left(n+\frac{1}{2}\right)\right) x^n,$$

and

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0,$$

is the digamma function. Hence

$$\begin{aligned} & \lim_{x \rightarrow 0^+} (g(x)w(1-x) - w(1-f(x))) \\ &= \lim_{x \rightarrow 0^+} \left(g(x)A(x) - A(f(x)) + \frac{1}{\pi}(\log f(x) - \log x)w(f(x)) \right) \\ &= g(0)A(0) - A(0) + \frac{1}{\pi} \lim_{x \rightarrow 0^+} \log \frac{f(x)}{x} \quad (\text{by (5.0.8) and (5.0.16)}) \\ &= \frac{1}{\pi} \log(i) \quad (\text{by (5.0.10)}) \\ &= \frac{1}{\pi} i \left(\frac{\pi}{2} \right) \\ &= \frac{i}{2}. \end{aligned}$$

Hence

$$\frac{1}{\pi}K = \frac{i}{2}$$

so that

$$K = \frac{\pi i}{2}.$$

Thus

$$(5.0.38) \quad \frac{\pi w(1-x)}{w(x)} - \frac{\pi w(1-f(x))}{w(f(x))} - \frac{\pi i}{2} = 0 \text{ in } (0, \frac{1}{16}).$$

As the lefthand side of (5.0.38) is defined and regular in D^{**} and $D^{**} \supseteq (0, \frac{1}{16})$, we have

$$\frac{\pi w(1-z)}{w(z)} - \frac{\pi w(1-f(z))}{w(f(z))} - \frac{\pi i}{2} = 0 \text{ in } D^{**},$$

which is Theorem 5.0.2. □

In Section 3.4, we apply Theorems 5.0.1 and 5.0.2 to $L(q)$, $M(q)$, ..., $S(q)$ to obtain $L(iq)$, $M(iq)$, ..., $S(iq)$. By Theorems 5.0.1 and 5.0.2 we have

$$\begin{aligned} x &\rightarrow f(x), \\ w(x) &\rightarrow w(f(x)) = g(x)w(x), \\ q = e^{-\pi \frac{w(1-x)}{w(x)}} &\rightarrow e^{-\pi \frac{w(1-f(x))}{w(f(x))}} = ie^{-\pi \frac{w(1-x)}{w(x)}} = iq. \end{aligned}$$

In other words, suppose a formula is of the form

$$\Omega(x, q, w) = 0,$$

then a second formula can be obtained using

$$\Omega \left(\frac{-8i(1-x)^{\frac{1}{4}} \left(1 - (1-x)^{\frac{1}{2}}\right)}{\left(1 - i(1-x)^{\frac{1}{4}}\right)^4}, iq, \frac{w}{2}i \left(1 - i(1-x)^{\frac{1}{4}}\right)^2 \right) = 0.$$

Conclusion

This thesis presents a simple way of finding and proving arithmetic identities similar to those found by Ramanujan. We find relationships between $L(q)$, $M(q)$, ..., $S(q)$ in terms of x and derive the arithmetic identities from these relationships. In this thesis, we only consider $L(q^e)$, $M(q^e)$, ..., $S(q^e)$, for $e = 2, 4, 8$, which were derived by applying duplication to $L(q)$, $M(q)$, ..., $S(q)$. It is likely that more identities could be found by including the derivatives of $L_{i,j}(q)$, $M_{i,j}(q)$, $N_{i,j}(q)$, ... when searching for relationships. Further, one can also derive $L(q^3)$, $M(q^3)$, ..., $S(q^3)$ by triplication [4] and obtain identities such as Melfi's identity:

If $n \not\equiv 0 \pmod{3}$, then

$$\sum_{m < n/3} \sigma(m)\sigma(n-3m) = \frac{1}{24} \{ \sigma_3(n) + (1-2n)\sigma(n) \}.$$

Further work is needed in this area to be able to derive identity (4.0.8). In addition the identities among $L(q^5)$, $M(q^5)$, ... remain to be discovered which give the identity (4.0.7).

Appendix A

MAPLE program to find

relationships between $L(q)$,

$L(q^2)$, $L(q^4)$, ..., $S(q)$, $S(q^2)$, $S(q^4)$,

$L_i(q)$, $M_i(q)$, ..., $S_i(q)$ for $i = 0, 1$.

```
#####
# This program finds the relationship between
# a product from L, M, ..., S, L_i, M_i, ...,
# S_i for i=0,1, say L*M_0 and L, M, ..., S.
# For example, L(q^2)*M_0(q)=L(q)+L(q^2)+M(q).
# To calculate specific products, change the
# values of Factor1 and Factor2 below.
#####

#####
# Initialize the arrays that hold the values
# for L(q), L(q^2), L(q^4), dL(q)/dq,
```

```

# dL(q^2)/dq, dL(q^4)/dq, L_0(q), L_1(q),...
# same for M, N, P, Q, R, S.
#####
restart;
Lq  := array(0..599, [seq(0, j=0..599)]):
Lq2 := array(0..599, [seq(0, j=0..599)]):
Lq4 := array(0..599, [seq(0, j=0..599)]):
dLq := array(0..599, [seq(0, j=0..599)]):
dLq2 := array(0..599, [seq(0, j=0..599)]):
dLq4 := array(0..599, [seq(0, j=0..599)]):
L0q := array(0..599, [seq(0, j=0..599)]):
L1q := array(0..599, [seq(0, j=0..599)]):
Mq  := array(0..599, [seq(0, j=0..599)]):
Mq2 := array(0..599, [seq(0, j=0..599)]):
Mq4 := array(0..599, [seq(0, j=0..599)]):
dMq := array(0..599, [seq(0, j=0..599)]):
dMq2 := array(0..599, [seq(0, j=0..599)]):
dMq4 := array(0..599, [seq(0, j=0..599)]):
M0q := array(0..599, [seq(0, j=0..599)]):
M1q := array(0..599, [seq(0, j=0..599)]):
Nq  := array(0..599, [seq(0, j=0..599)]):
Nq2 := array(0..599, [seq(0, j=0..599)]):
Nq4 := array(0..599, [seq(0, j=0..599)]):
dNq := array(0..599, [seq(0, j=0..599)]):
dNq2 := array(0..599, [seq(0, j=0..599)]):
dNq4 := array(0..599, [seq(0, j=0..599)]):
N0q := array(0..599, [seq(0, j=0..599)]):
N1q := array(0..599, [seq(0, j=0..599)]):
Pq  := array(0..599, [seq(0, j=0..599)]):
Pq2 := array(0..599, [seq(0, j=0..599)]):
Pq4 := array(0..599, [seq(0, j=0..599)]):
dPq := array(0..599, [seq(0, j=0..599)]):
dPq2 := array(0..599, [seq(0, j=0..599)]):
dPq4 := array(0..599, [seq(0, j=0..599)]):
P0q := array(0..599, [seq(0, j=0..599)]):
P1q := array(0..599, [seq(0, j=0..599)]):
Qq  := array(0..599, [seq(0, j=0..599)]):
Qq2 := array(0..599, [seq(0, j=0..599)]):
Qq4 := array(0..599, [seq(0, j=0..599)]):
dQq := array(0..599, [seq(0, j=0..599)]):

```

```

dQq2 := array(0..599, [seq(0, j=0..599)]):
dQq4 := array(0..599, [seq(0, j=0..599)]):
Q0q  := array(0..599, [seq(0, j=0..599)]):
Q1q  := array(0..599, [seq(0, j=0..599)]):
Rq   := array(0..599, [seq(0, j=0..599)]):
Rq2  := array(0..599, [seq(0, j=0..599)]):
Rq4  := array(0..599, [seq(0, j=0..599)]):
dRq  := array(0..599, [seq(0, j=0..599)]):
dRq2 := array(0..599, [seq(0, j=0..599)]):
dRq4 := array(0..599, [seq(0, j=0..599)]):
R0q  := array(0..599, [seq(0, j=0..599)]):
R1q  := array(0..599, [seq(0, j=0..599)]):
Sq   := array(0..599, [seq(0, j=0..599)]):
Sq2  := array(0..599, [seq(0, j=0..599)]):
Sq4  := array(0..599, [seq(0, j=0..599)]):
dSq  := array(0..599, [seq(0, j=0..599)]):
dSq2 := array(0..599, [seq(0, j=0..599)]):
dSq4 := array(0..599, [seq(0, j=0..599)]):
S0q  := array(0..599, [seq(0, j=0..599)]):
S1q  := array(0..599, [seq(0, j=0..599)]):
const:= array(0..599, [seq(0, j=0..599)]):

#####
# Assigning values to the arrays. Note that
# x, w, dw/dx are represented by different
# numbers.
# The coefficients of  $x^a$  are stored in 1*a.
# The coefficients of  $w^b$  are stored in 10*b.
# The coefficients of  $(dw/dx)^c$  are in 200*c.
# For example, the  $x^2w(dw/dx)$  term in  $L(q)$ 
# will be stored in the 2+10+200 spot in the
# array, Lq[212]=-12.
#####
# L(q)
Lq[20]:=1:      Lq[21]:=-5:      Lq[211]:=12:
Lq[212]:=-12:
# L(q^2)
Lq2[20]:=1:     Lq2[21]:=-2:     Lq2[211]:=6:
Lq2[212]:=-6:
# L(q^4)

```

```

Lq4[20]:=1:      Lq4[21]:=-5/4:      Lq4[211]:=3:
Lq4[212]:=-3:
# dL(q)/dq
dLq[41]:=-2:      dLq[42]:= 2:      dLq[231]:= 2:
dLq[232]:=-12:    dLq[233]:= 10:     dLq[422]:= 12:
dLq[423]:=-24:    dLq[424]:= 12:
# dL(q^2)/dq
dLq2[41]:=-1/2:   dLq2[42]:= 1/2:     dLq2[231]:= 2:
dLq2[232]:=-6:    dLq2[233]:= 4:      dLq2[422]:= 6:
dLq2[423]:=-12:   dLq2[424]:= 6:
# dL(q^4)/dq
dLq4[41]:=-1/2:   dLq4[42]:= 1/2:     dLq4[231]:= 2:
dLq4[232]:=-9/2:  dLq4[233]:= 5/2:    dLq4[422]:= 3:
dLq4[423]:=-6:    dLq4[424]:= 3:
# L0(q)
L0q[0]:=2/48:      L0q[20]:=-2/48:      L0q[21]:=7/48:
L0q[211]:=-1/2:    L0q[212]:=1/2:
# L1(q)
L1q[21]:=1/16:
# M(q)
Mq[40]:= 1:      Mq[41]:= 14:      Mq[42]:= 1:
# M(q^2)
Mq2[40]:= 1:      Mq2[41]:= -1:      Mq2[42]:= 1:
# M(q^4)
Mq4[40]:= 1:      Mq4[41]:= -1:      Mq4[42]:= 1/16:
# dM(q)/dq
dMq[61]:= 14:     dMq[62]:= -12:     dMq[63]:= -2:
dMq[251]:= 4:     dMq[252]:= 52:     dMq[253]:= -52:
dMq[254]:= -4:
# dM(q^2)/dq
dMq2[61]:= -1:    dMq2[62]:= 3:      dMq2[63]:= -2:
dMq2[251]:= 4:    dMq2[252]:= -8:     dMq2[253]:= 8:
dMq2[254]:= -4:
# dM(q^4)/dq
dMq4[61]:= -1:    dMq4[62]:= 9/8:     dMq4[63]:= -1/8:
dMq4[251]:= 4:    dMq4[252]:= -8:     dMq4[253]:= 17/4:
dMq4[254]:= -1/4:
# M0(q)
M0q[0]:=-2/480:    M0q[40]:=2/480:      M0q[41]:=-2/480:
M0q[42]:=17/480:

```



```

# M1(q)
M1q[41] := 1/16:      M1q[42] := -1/32:
# N(q)
Nq[60] := 1:          Nq[61] := -33:      Nq[62] := -33:
Nq[63] := 1:
# N(q^2)
Nq2[60] := 1:         Nq2[61] := -3/2:    Nq2[62] := -3/2:
Nq2[63] := 1:
# N(q^4)
Nq4[60] := 1:         Nq4[61] := -3/2:    Nq4[62] := 15/32:
Nq4[63] := 1/64:
# dN(q)/dq
dNq[81] := -33:       dNq[82] := -33:     dNq[83] := 69:
dNq[84] := -3:        dNq[271] := 6:      dNq[272] := -204:
dNq[274] := 204:      dNq[275] := -6:
# dN(q^2)/dq
dNq2[81] := -3/2:     dNq2[82] := -3/2:    dNq2[83] := 6:
dNq2[84] := -3:       dNq2[271] := 6:     dNq2[272] := -15:
dNq2[274] := 15:      dNq2[275] := -6:
# dN(q^4)/dq
dNq4[81] := -3/2:     dNq4[82] := 39/16:    dNq4[83] := -57/64:
dNq4[84] := -3/64:    dNq4[271] := 6:      dNq4[272] := -15:
dNq4[273] := 189/16:  dNq4[274] := -87/32: dNq4[275] := -3/32:
# N0(q)
N0q[0] := 2/1008:     N0q[60] := -2/1008:    N0q[61] := 3/1008:
N0q[62] := 129/1008:  N0q[63] := -65/1008:
# N1(q)
N1q[61] := 1/16:      N1q[62] := -1/16:     N1q[63] := 1/16:
# P(q)
Pq[80] := 1:          Pq[81] := 28:         Pq[82] := 198:
Pq[83] := 28:         Pq[84] := 1:
# P(q^2)
Pq2[80] := 1:         Pq2[81] := -2:        Pq2[82] := 3:
Pq2[83] := -2:        Pq2[84] := 1:
# P(q^4)
Pq4[80] := 1:         Pq4[81] := -2:        Pq4[82] := 9/8:
Pq4[83] := -1/8:      Pq4[84] := 1/256:
# dP(q)/dq
dPq[101] := 28:       dPq[102] := 368:     dPq[103] := -312:
dPq[104] := -80:      dPq[105] := -4:      dPq[291] := 8:

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dPq[292] := 216:      dPq[293] := 1360:      dPq[294] := -1360:
dPq[295] := -216:     dPq[296] := -8:
# dP(q^2)/dq
dPq2[101] := -2:      dPq2[102] := 8:      dPq2[103] := -12:
dPq2[104] := 10:      dPq2[105] := -4:     dPq2[291] := 8:
dPq2[292] := -24:     dPq2[293] := 40:     dPq2[294] := -40:
dPq2[295] := 24:      dPq2[296] := -8:
# dP(q^4)/dq
dPq4[101] := -2:      dPq4[102] := 17/4:   dPq4[103] := -21/8:
dPq4[104] := 25/64:   dPq4[105] := -1/64:  dPq4[291] := 8:
dPq4[292] := -24:     dPq4[293] := 25:     dPq4[294] := -10:
dPq4[295] := 33/32:   dPq4[296] := -1/32:
# P0(q)
P0q[0] := -2/960:     P0q[80] := 2/960:     P0q[81] := -4/960:
P0q[82] := 81/160:    P0q[83] := -121/240:  P0q[84] := 257/960:
# P1(q)
P1q[81] := 1/16:      P1q[82] := -3/32:     P1q[83] := 9/16:
P1q[84] := -17/64:
# Q(q)
Qq[100] := 1:         Qq[101] := -19:       Qq[102] := -494:
Qq[103] := -494:      Qq[104] := -19:       Qq[105] := 1:
# Q(q^2)
Qq2[100] := 1:        Qq2[101] := -5/2:     Qq2[102] := 1:
Qq2[103] := 1:        Qq2[104] := -5/2:     Qq2[105] := 1:
# Q(q4)
Qq4[100] := 1:        Qq4[101] := -5/2:     Qq4[102] := 65/32:
Qq4[103] := -35/64:   Qq4[104] := 7/512:    Qq4[105] := 1/1024:
# dQ(q)/dq
dQq[121] := -19:      dQq[122] := -969:     dQq[123] := -494:
dQq[124] := 1406:     dQq[125] := 81:       dQq[126] := -5:
dQq[311] := 10:       dQq[312] := -200:     dQq[313] := -4750:
dQq[314] := 0:        dQq[315] := 4750:     dQq[316] := 200:
dQq[317] := -10:
# dQ(q^2)/dq
dQq2[121] := -5/2:    dQq2[122] := 9/2:     dQq2[123] := 1:
dQq2[124] := -13:     dQq2[125] := 15:      dQq2[126] := -5:
dQq2[311] := 10:      dQq2[312] := -35:     dQq2[313] := 35:
dQq2[314] := 0:       dQq2[315] := -35:     dQq2[316] := 35:
dQq2[317] := -10:
# dQ(q^4)/dq

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dQq4[121] := -5/2:      dQq4[122] := 105/16:      dQq4[123] := -365/64:
dQq4[124] := 217/128:    dQq4[125] := -51/1024:    dQq4[126] := -5/1024:
dQq4[311] := 10:         dQq4[312] := -35:         dQq4[313] := 725/16:
dQq4[314] := -825/32:    dQq4[315] := 1435/256:    dQq4[316] := -65/512:
dQq4[317] := -5/512:
# Q0(q)
Q0q[0] := 1/264:          Q0q[100] := -1/264:
Q0q[101] := 5/528:         Q0q[102] := 527/264:
Q0q[103] := -793/264:      Q0q[104] := 2579/528:
Q0q[105] := -1025/528:
# Q1(q)
Q1q[101] := 1/16:         Q1q[102] := -1/8:
Q1q[103] := 39/8:         Q1q[104] := -77/16:
Q1q[105] := 31/16:
# R(q)
Rq[120] := 1:             Rq[121] := 2022/691:
Rq[122] := 516381/691:    Rq[123] := 1792148/691:
Rq[124] := 516381/691:    Rq[125] := 2022/691:
Rq[126] := 1:
# R(q^2)
Rq2[120] := 1:           Rq2[121] := -3:
Rq2[122] := 4917/1382:   Rq2[123] := -1462/691:
Rq2[124] := 4917/1382:   Rq2[125] := -3:
Rq2[126] := 1:
# R(q^4)
Rq4[120] := 1:           Rq4[121] := -3:
Rq4[122] := 51/16:       Rq4[123] := -11/8:
Rq4[124] := 67107/353792: Rq4[125] := -771/353792:
Rq4[126] := 1/4096:
# dR(q)/dq
dRq[141] := 2022/691:     dRq[142] := 1030740/691:
dRq[143] := 4343682/691: dRq[144] := -3310920/691:
dRq[145] := -2055414/691: dRq[146] := -5964/691:
dRq[147] := -6:          dRq[331] := 12:
dRq[332] := 15972/691:   dRq[333] := 6172308/691:
dRq[334] := 15309204/691: dRq[335] := -15309204/691:
dRq[336] := -6172308/691: dRq[337] := -15972/691:
dRq[338] := -12:
# dR(q^2)/dq
dRq2[141] := -3:         dRq2[142] := 6990/691:

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dRq2[143] := -9303/691:      dRq2[144] := 14220/691:
dRq2[145] := -20199/691:     dRq2[146] := 21:
dRq2[147] := -6:            dRq2[331] := 12:
dRq2[332] := -48:           dRq2[333] := 54378/691:
dRq2[334] := -47046/691:     dRq2[335] := 47046/691:
dRq2[336] := -54378/691:     dRq2[337] := 48:
dRq2[338] := -12:
# dR(q^4)/dq
dRq4[141] := -3:            dRq4[142] := 75/8:
dRq4[143] := -21/2:         dRq4[144] := 431955/88448:
dRq4[145] := -272283/353792: dRq4[146] := 17493/1415168:
dRq4[147] := -3/2048:      dRq4[331] := 12:
dRq4[332] := -48:          dRq4[333] := 297/4:
dRq4[334] := -219/4:        dRq4[335] := 1660713/88448:
dRq4[336] := -101817/44224: dRq4[337] := 20577/707584:
dRq4[338] := -3/1024:
# R0(q)
R0q[0] := -691/65520:       R0q[120] := 691/65520:
R0q[121] := -691/21840:     R0q[122] := 351079/43680:
R0q[123] := -524891/32760:  R0q[124] := 3179359/43680:
R0q[125] := -1414831/21840: R0q[126] := 2831027/131040:
# R1(q)
R1q[121] := 1/16:          R1q[122] := -5/32:
R1q[123] := 347/8:         R1q[124] := -2077/32:
R1q[125] := 1037/16:       R1q[126] := -691/32:
# S(q)
Sq[140] := 1:              Sq[141] := -5:
Sq[142] := -759:           Sq[143] := -7429:
Sq[144] := -7429:          Sq[145] := -759:
Sq[146] := -5:             Sq[147] := 1:
# S(q^2)
Sq2[140] := 1:             Sq2[141] := -7/2:
Sq2[142] := 9/2:           Sq2[143] := -5/2:
Sq2[144] := -5/2:          Sq2[145] := 9/2:
Sq2[146] := -7/2:          Sq2[147] := 1:
# S(q^4)
Sq4[140] := 1:             Sq4[141] := -7/2:
Sq4[142] := 147/32:         Sq4[143] := -175/64:
Sq4[144] := 11/16:         Sq4[145] := -3/64:
Sq4[146] := -1/8192:       Sq4[147] := 1/16384:

```

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# dS(q)/dq
dSq[161] := -5:
dSq[163] := -20769:
dSq[165] := 25921:
dSq[167] := 37:
dSq[351] := 14:
dSq[353] := -10556:
dSq[355] := 0:
dSq[357] := 10556:
dSq[359] := -14:
# dS(q^2)/dq
dSq2[161] := -7/2:
dSq2[163] := -33/2:
dSq2[165] := 65/2:
dSq2[167] := 28:
dSq2[351] := 14:
dSq2[353] := 112:
dSq2[355] := 0:
dSq2[357] := -112:
dSq2[359] := -14:
# dS(q^4)/dq
dSq4[161] := -7/2:
dSq4[163] := -1113/64:
dSq4[165] := -191/64:
dSq4[167] := 19/16384:
dSq4[351] := 14:
dSq4[353] := 1813/16:
dSq4[355] := 1533/32:
dSq4[357] := 2681/4096:
dSq4[359] := -7/8192:
# S0(q)
S0q[0] := 1/24:
S0q[141] := 7/48:
S0q[143] := -3835/48:
S0q[145] := -24835/16:
S0q[147] := -16385/48:
# S1(q)
S1q[141] := 1/16:
S1q[143] := 6231/16:
S1q[145] := 25341/16:
dSq[162] := -1513:
dSq[164] := -7429:
dSq[166] := 3765:
dSq[168] := -7:
dSq[352] := -84:
dSq[354] := -93380:
dSq[356] := 93380:
dSq[358] := 84:
dSq2[162] := 25/2:
dSq2[164] := -5/2:
dSq2[166] := -87/2:
dSq2[168] := -7:
dSq2[352] := -63:
dSq2[354] := -98:
dSq2[356] := 98:
dSq2[358] := 63:
dSq4[162] := 203/16:
dSq4[164] := 701/64:
dSq4[166] := 957/4096:
dSq4[168] := -7/16384:
dSq4[352] := -63:
dSq4[354] := -3283/32:
dSq4[356] := -329/32:
dSq4[358] := 21/8192:
S0q[140] := -1/24:
S0q[142] := 509/16:
S0q[144] := 52229/48:
S0q[146] := 57349/48:
S1q[142] := -3/16:
S1q[144] := -12457/16:
S1q[146] := -19113/16:

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S1q[147]:= 5461/16:          const[0] := 1:

#####
# Maximum number of arrays. Each product can
# only be related to a few other functions,
# i.e. the ones that contain the same terms
# as the product. Each product can only be
# related to at most 10 other functions.
#####
MAXNUMARR := 10:

#####
# Factor1: Stores the values for the first
#         factor of the product.
# Factor2: Stores the values for the second
#         factor of the product.
#####
Factor1 := array(0..599, [seq(0, j=0..599)]):
Factor2 := array(0..599, [seq(0, j=0..599)]):
Lq_arr:= array(0..6, [Lq, Mq, Nq, Pq, Qq, Rq, Sq]):
Lq2_arr:= array(0..6, [Lq2, Mq2, Nq2, Pq2, Qq2, Rq2, Sq2]):
Lq4_arr:= array(0..6, [Lq4, Mq4, Nq4, Pq4, Qq4, Rq4, Sq4]):
dLq_arr:= array(0..6, [dLq, dMq, dNq, dPq, dQq, dRq, dSq]):
dLq2_arr:= array(0..6, [dLq2, dMq2, dNq2, dPq2, dQq2, dRq2, dSq2]):
dLq4_arr:= array(0..6, [dLq4, dMq4, dNq4, dPq4, dQq4, dRq4, dSq4]):
L0q_arr:= array(0..6, [L0q, M0q, N0q, P0q, Q0q, R0q, S0q]):
L1q_arr:= array(0..6, [L1q, M1q, N1q, P1q, Q1q, R1q, S1q]):

#####
# Go through for loops to find relationships
# with products of the form Factor1*Factor2.
# The choices for Factor1 and Factor2 are:
# Lq_arr and L0q_arr,
# Lq_arr and L1q_arr,
# Lq2_arr and L0q_arr,
# Lq2_arr and L1q_arr,
# Lq4_arr and L0q_arr,
# Lq4_arr and L1q_arr,
# Lq_arr and Lq2_arr,
# Lq_arr and Lq4_arr,

```

[illegible]

```

                                dLq4_arr[first+second],
                                const]);

# Construct equations to find the unknowns.
for i from 0 to 599
do
    if (sum('abs(arr[j][i])', 'j'=0..MAXNUMARR-1)
        +abs(res[i])= 0)
    then
        # do nothing
    else
        eqn[numeqn] := sum('unk[j]*arr[j][i]',
                            'j'=0..MAXNUMARR-1) = res[i];
        numeqn := numeqn+1;
    fi;
od;
print("Solving unknowns for ", first, " and ", second);
if (not(solve({seq(eqn[j], j=0..numeqn-1)},
                {seq(unk[l], l=0..MAXNUMARR-1)})= NULL))
then
    print(solve({seq(eqn[j], j=0..numeqn-1)},
                {seq(unk[l], l=0..MAXNUMARR-1)}));
fi;
od;
od:

```


Appendix B

MAPLE program to calculate
 $L(q)$, ..., $L(q^8)$, $L(-q)$, $L(iq)$,
 $L(-iq)$, $L_0(q)$, ..., $L_3(q)$, $\frac{dL(q)}{dq}$, ...,
 $\frac{dL(q^8)}{dq}$, ..., etc.

```
#####
# This program calculates L_i, M_i, ..., S_i
# for i=0,1,2,3, etc... It also uses the
# relationships between L, M, ..., S to
# calculate the functions P, Q, R, S.
# Note that g=(1-x)^{1/4}
#####
restart;
Lq:=(-4+5*g^4)*w^2+12*g^4*(1-g^4)*w*d:
Lq2:=(-1+2*g^4)*w^2+6*g^4*(1-g^4)*w*d:
Lq4:=(-1/4+5/4*g^4)*w^2+3*g^4*(1-g^4)*w*d:
Lq8:=-1/16*(1-6*g^2-11*g^4)*w^2+3/2*g^4*(1-g^4)*w*d:
Lmq:=(-1+2*g^4)*w^2+12*g^4*(1-g^4)*w*d:
Lmiq:=-1/4*(1+12*I*g+18*g^2-12*I*g^3-23*g^4)*w^2+12*g^4*(1-g^4)*w*d:
Liq:=-1/4*(1-12*I*g+18*g^2+12*I*g^3-23*g^4)*w^2+12*g^4*(1-g^4)*w*d:
```

```

L0q:=(1/96)*(4-Lq-Lmq-Liq-Lmiq):
L2q:=(1/96)*(-Lq-Lmq+Liq+Lmiq):
L1q:=(1/96)*(-Lq+Lmq-I*Liq+I*Lmiq):
L3q:=(1/96)*(-Lq+Lmq+I*Liq-I*Lmiq):
dLq:=(1-g^4)*(1-(1-g^4))*w^2*(-2*w^2+(2-10*(1-g^4))*w*d+12*(1-g^4)
    *(1-(1-g^4))*d^2):
dLq2:=g^4*(1-g^4)*w^2*(-(1/2)*w^2+(-2+4*g^4)*w*d+6*g^4
    *(1-g^4)*d^2):
dLq4:=g^4*(1-g^4)*w^2*(-(1/2)*w^2+(-1/2+5/2*g^4)*w*d+3*g^4
    *(1-g^4)*d^2):
dLq8:=-1/16*(1-g^4)*(3*g^2+5*g^4)*w^4-1/8*g^4*(1-g^4)
    *(1-6*g^2-11*g^4)*w^3*d+3/2*g^8*(1-g^4)^2*w^2*d^2:
Mq:=w^4*(1+14*(1-g^4)+(1-g^4)^2):
Mq2:= w^4*(1-(1-g^4)+(1-g^4)^2):
Mq4:= w^4*(1-(1-g^4)+(1/16)*(1-g^4)^2):
Mq8:= w^4/256*(1+60*g^2+134*g^4+60*g^6+g^8):
Mmq:=w^4*(1-16*(1-g^4)+16*(1-g^4)^2):
Mmiq:=w^4/16*(1-120*I*g-540*g^2+840*I*g^3+1094*g^4
    -840*I*g^5-540*g^6+120*I*g^7+g^8):
Miq:=w^4/16*(1+120*I*g-540*g^2-840*I*g^3+1094*g^4
    +840*I*g^5-540*g^6-120*I*g^7+g^8):
M0q:=(1/960)*(-4+Mq+Mmq+Miq+Mmiq):
M2q:=(1/960)*(Mq+Mmq-Miq-Mmiq):
M1q:=(1/960)*(Mq-Mmq+I*Miq-I*Mmiq):
M3q:=(1/960)*(Mq-Mmq-I*Miq+I*Mmiq):
dMq:=(4*w^3*d*(1+14*(1-g^4)+(1-g^4)^2)+(14+2*(1-g^4))*w^4)
    *(1-g^4)*(1-(1-g^4))*w^2:
dMq2:=(4*w^3*d*(1-(1-g^4)+(1-g^4)^2)+(-1+2*(1-g^4))*w^4)
    *(1-g^4)*(1-(1-g^4))*w^2:
dMq4:=(4*w^3*d*(1-(1-g^4)+(1/16)*(1-g^4)^2)+(-1+(1/8)
    *(1-g^4))*w^4)*(1-g^4)*(1-(1-g^4))*w^2:
dMq8:=1/256*(-30*g^2-134*g^4-90*g^6-2*g^8)*(1-g^4)*w^6
    +1/64*(1+60*g^2+134*g^4+60*g^6+g^8)*g^4*(1-g^4)*w^5*d:
Nq:=w^6*(1+(1-g^4))*(1-34*(1-g^4)+(1-g^4)^2):
Nq2:=w^6*(1+(1-g^4))*(1-1/2*(1-g^4))*(1-2*(1-g^4)):
Nq4:=w^6*(1-1/2*(1-g^4))*(1-(1-g^4)-1/32*(1-g^4)^2):
Nq8:=-1/4096*(1+6*g^2+g^4)*(1-132*g^2-250*g^4-132*g^6+g^8)*w^6:
Nm q:=w^6*(1-2*(1-g^4))*(1+32*(1-g^4)-32*(1-g^4)^2):
Nmiq:=-w^6/64*(((1-I*g)^4-8*I*g*(1-g^2))*((1-I*g)^8
    +272*I*g*(1-g^2)*(1-I*g)^4-64*g^2*(1-g^2)^2)):

```

```

Niq:=-w^6/64*(((1+I*g)^4+8*I*g*(1-g^2))*((1+I*g)^8
-272*I*g*(1-g^2)*(1+I*g)^4-64*g^2*(1-g^2)^2)):
N0q:=(1/2016)*(4-Nq-Nmq-Niq-Nmiq):
N2q:=(1/2016)*(-Nq-Nmq+Niq+Nmiq):
N1q:=(1/2016)*(-Nq+Nmq-I*Niq+I*Nmiq):
N3q:=(1/2016)*(-Nq+Nmq+I*Niq-I*Nmiq):
dNq:=(6*d*w^5*(1+(1-g^4))*(1-34*(1-g^4)+(1-g^4)^2)
+w^6*(1-34*(1-g^4)+(1-g^4)^2)+w^6*(1+(1-g^4))
*(-34+2*(1-g^4)))*(1-g^4)*(1-(1-g^4))*w^2:
dNq2:=(6*d*w^5*(1+(1-g^4))*(1-1/2*(1-g^4))*(1-2*(1-g^4))
+w^6*(1-1/2*(1-g^4))*(1-2*(1-g^4))+w^6*(1+(1-g^4))
*(-1/2)*(1-2*(1-g^4))+w^6*(1+(1-g^4))
*(1-1/2*(1-g^4))*(-2))*(1-g^4)*(1-(1-g^4))*w^2:
dNq4:=(6*d*w^5*(1-1/2*(1-g^4))*(1-(1-g^4))-1/32*(1-g^4)^2)
+w^6*(-1/2)*(1-(1-g^4))-1/32*(1-g^4)^2)+w^6*(1-1/2*(1-g^4))
*(-1-1/16*(1-g^4)))*(1-g^4)*(1-(1-g^4))*w^2:
dNq8:=(-1/4096*(-12*I*g/(4*I*g^3)-4*I*g^3/(4*I*g^3))
*(1-132*g^2-250*g^4-132*g^6+g^8)*w^6
-1/4096*(1+6*g^2+g^4)*(132*2*I*g/(4*I*g^3)+250*4*I*g^3/(4*I*g^3)
+132*6*I*g^5/(4*I*g^3)-8*I*g^7/(4*I*g^3))*w^6
-1/4096*(1+6*g^2+g^4)*(1-132*g^2-250*g^4
-132*g^6+g^8)*6*w^5*d)*g^4*(1-g^4))*w^2:
Pq:=Mq^2:
Pq2:=Mq2^2:
Pq4:=Mq4^2:
Pq8:=Mq8^2:
Pmq:=Mmq^2:
Pmiq:=Miq^2:
Piq:=Mmiq^2:
P0q:=(1/1920)*(-4+Pq+Pmq+Piq+Pmiq):
P2q:=(1/1920)*(Pq+Pmq-Piq-Pmiq):
P1q:=(1/1920)*(Pq-Pmq+I*Piq-I*Pmiq):
P3q:=(1/1920)*(Pq-Pmq-I*Piq+I*Pmiq):
dPq:=2*Mq*dMq:
dPq2:=2*Mq2*dMq2:
dPq4:=2*Mq4*dMq4:
dPq8:=2*Mq8*dMq8:
Qq:=Mq*Nq:
Qq2:=Mq2*Nq2:
Qq4:=Mq4*Nq4:

```

```

Qq8:=Mq8*Nq8:
Qmq:=Mmq*Nmq:
Qmiq:=Miq*Niq:
Qiq:=Mmiq*Nmiq:
Q0q:=(1/1056)*(4-Qq-Qmq-Qiq-Qmiq):
Q2q:=(1/1056)*(-Qq-Qmq+Qiq+Qmiq):
Q1q:=(1/1056)*(-Qq+Qmq-I*Qiq+I*Qmiq):
Q3q:=(1/1056)*(-Qq+Qmq+I*Qiq-I*Qmiq):
dQq:=dMq*Nq+Mq*dNq:
dQq2:=dMq2*Nq2+Mq2*dNq2:
dQq4:=dMq4*Nq4+Mq4*dNq4:
dQq8:=dMq8*Nq8+Mq8*dNq8:
Rq:=(441/691)*Mq*Pq+(250/691)*Nq*Nq:
Rq2:=(441/691)*Mq2*Pq2+(250/691)*Nq2*Nq2:
Rq4:=(441/691)*Mq4*Pq4+(250/691)*Nq4*Nq4:
Rq8:=(441/691)*Mq8*Pq8+(250/691)*Nq8*Nq8:
Rmq:=(441/691)*Mmq*Pmq+(250/691)*Nmq*Nmq:
Rmiq:=(441/691)*Miq*Piq+(250/691)*Niq*Niq:
Riq:=(441/691)*Mmiq*Pmiq+(250/691)*Nmiq*Nmiq:
R0q:=(691/262080)*(-4+Rq+Rmq+Riq+Rmiq):
R2q:=(691/262080)*(Rq+Rmq-Riq-Rmiq):
R1q:=(691/262080)*(Rq-Rmq+I*Riq-I*Rmiq):
R3q:=(691/262080)*(Rq-Rmq-I*Riq+I*Rmiq):
dRq:=(441/691)*(dMq*Pq+Mq*dPq)+(250/691)*(2*Nq*dNq):
dRq2:=(441/691)*(dMq2*Pq2+Mq2*dPq2)+(250/691)*(2*Nq2*dNq2):
dRq4:=(441/691)*(dMq4*Pq4+Mq4*dPq4)+(250/691)*(2*Nq4*dNq4):
dRq8:=(441/691)*(dMq8*Pq8+Mq8*dPq8)+(250/691)*(2*Nq8*dNq8):
Sq:=Mq*Qq:
Sq2:=Mq2*Qq2:
Sq4:=Mq4*Qq4:
Sq8:=Mq8*Qq8:
Smq:=Mmq*Qmq:
Smiq:=Miq*Qiq:
Siq:=Mmiq*Qmiq:
S0q:=(1/96)*(4-Sq-Smq-Siq-Smiq):
S2q:=(1/96)*(-Sq-Smq+Siq+Smiq):
S1q:=(1/96)*(-Sq+Smq-I*Siq+I*Smiq):
S3q:=(1/96)*(-Sq+Smq+I*Siq-I*Smiq):
dSq:=dMq*Qq+Mq*dQq:
dSq2:=dMq2*Qq2+Mq2*dQq2:

```

$$dS_{q4} = dM_{q4} \cdot Q_{q4} + M_{q4} \cdot dQ_{q4} :$$

$$dS_{q8} = dM_{q8} \cdot Q_{q8} + M_{q8} \cdot dQ_{q8} :$$

Appendix C

MAPLE program to find
relationships between $L(q)$, ...,
 $L(q^8)$, ..., $S(q)$, ..., $S(q^8)$, $L_i(q)$,
 $M_i(q)$, ..., $S_i(q)$ for $i = 0, 1, 2, 3$.

```
#####
# This program finds the relationship between
# a product from L, M, ..., S, L_i, M_i, ...,
# S_i for i=0,1,2,3, say L*M_0 and L, M, ..., S.
# For example,  $L(q^2)*M_0(q)=L(q)+L(q^2)+M(q)$ .
# To calculate specific products, change the
# values of Factor1 and Factor2 below.
#####
#####
# Initialize the arrays that hold the values
# for  $L(q)$ ,  $L(q^2)$ ,  $L(q^4)$ ,  $L(q^8)$ ,  $dL(q)/dq$ ,
#  $dL(q^2)/dq$ ,  $dL(q^4)/dq$ ,  $dL(q^8)/dq$ ,  $L_0(q)$ ,
#  $L_1(q)$ ,  $L_2(q)$ ,  $L_3(q)$ , ..., same for
# M, N, P, Q, R, S.
#####
restart;
Lq  := array(0..1699, [seq(0, j=0..1699)]):
Lq2 := array(0..1699, [seq(0, j=0..1699)]):
Lq4 := array(0..1699, [seq(0, j=0..1699)]):
Lq8 := array(0..1699, [seq(0, j=0..1699)]):
```

```

dLq  := array(0..1699, [seq(0, j=0..1699)]):
dLq2 := array(0..1699, [seq(0, j=0..1699)]):
dLq4 := array(0..1699, [seq(0, j=0..1699)]):
dLq8 := array(0..1699, [seq(0, j=0..1699)]):
L0q  := array(0..1699, [seq(0, j=0..1699)]):
L1q  := array(0..1699, [seq(0, j=0..1699)]):
L2q  := array(0..1699, [seq(0, j=0..1699)]):
L3q  := array(0..1699, [seq(0, j=0..1699)]):
Mq   := array(0..1699, [seq(0, j=0..1699)]):
Mq2  := array(0..1699, [seq(0, j=0..1699)]):
Mq4  := array(0..1699, [seq(0, j=0..1699)]):
Mq8  := array(0..1699, [seq(0, j=0..1699)]):
dMq  := array(0..1699, [seq(0, j=0..1699)]):
dMq2 := array(0..1699, [seq(0, j=0..1699)]):
dMq4 := array(0..1699, [seq(0, j=0..1699)]):
dMq8 := array(0..1699, [seq(0, j=0..1699)]):
M0q  := array(0..1699, [seq(0, j=0..1699)]):
M1q  := array(0..1699, [seq(0, j=0..1699)]):
M2q  := array(0..1699, [seq(0, j=0..1699)]):
M3q  := array(0..1699, [seq(0, j=0..1699)]):
Nq   := array(0..1699, [seq(0, j=0..1699)]):
Nq2  := array(0..1699, [seq(0, j=0..1699)]):
Nq4  := array(0..1699, [seq(0, j=0..1699)]):
Nq8  := array(0..1699, [seq(0, j=0..1699)]):
dNq  := array(0..1699, [seq(0, j=0..1699)]):
dNq2 := array(0..1699, [seq(0, j=0..1699)]):
dNq4 := array(0..1699, [seq(0, j=0..1699)]):
dNq8 := array(0..1699, [seq(0, j=0..1699)]):
N0q  := array(0..1699, [seq(0, j=0..1699)]):
N1q  := array(0..1699, [seq(0, j=0..1699)]):
N2q  := array(0..1699, [seq(0, j=0..1699)]):
N3q  := array(0..1699, [seq(0, j=0..1699)]):
Pq   := array(0..1699, [seq(0, j=0..1699)]):
Pq2  := array(0..1699, [seq(0, j=0..1699)]):
Pq4  := array(0..1699, [seq(0, j=0..1699)]):
Pq8  := array(0..1699, [seq(0, j=0..1699)]):
dPq  := array(0..1699, [seq(0, j=0..1699)]):
dPq2 := array(0..1699, [seq(0, j=0..1699)]):
dPq4 := array(0..1699, [seq(0, j=0..1699)]):
dPq8 := array(0..1699, [seq(0, j=0..1699)]):

```

```

P0q := array(0..1699, [seq(0, j=0..1699)]):
P1q := array(0..1699, [seq(0, j=0..1699)]):
P2q := array(0..1699, [seq(0, j=0..1699)]):
P3q := array(0..1699, [seq(0, j=0..1699)]):
Qq  := array(0..1699, [seq(0, j=0..1699)]):
Qq2 := array(0..1699, [seq(0, j=0..1699)]):
Qq4 := array(0..1699, [seq(0, j=0..1699)]):
Qq8 := array(0..1699, [seq(0, j=0..1699)]):
dQq  := array(0..1699, [seq(0, j=0..1699)]):
dQq2 := array(0..1699, [seq(0, j=0..1699)]):
dQq4 := array(0..1699, [seq(0, j=0..1699)]):
dQq8 := array(0..1699, [seq(0, j=0..1699)]):
Q0q  := array(0..1699, [seq(0, j=0..1699)]):
Q1q  := array(0..1699, [seq(0, j=0..1699)]):
Q2q  := array(0..1699, [seq(0, j=0..1699)]):
Q3q  := array(0..1699, [seq(0, j=0..1699)]):
Rq   := array(0..1699, [seq(0, j=0..1699)]):
Rq2  := array(0..1699, [seq(0, j=0..1699)]):
Rq4  := array(0..1699, [seq(0, j=0..1699)]):
Rq8  := array(0..1699, [seq(0, j=0..1699)]):
dRq  := array(0..1699, [seq(0, j=0..1699)]):
dRq2 := array(0..1699, [seq(0, j=0..1699)]):
dRq4 := array(0..1699, [seq(0, j=0..1699)]):
dRq8 := array(0..1699, [seq(0, j=0..1699)]):
R0q  := array(0..1699, [seq(0, j=0..1699)]):
R1q  := array(0..1699, [seq(0, j=0..1699)]):
R2q  := array(0..1699, [seq(0, j=0..1699)]):
R3q  := array(0..1699, [seq(0, j=0..1699)]):
Sq   := array(0..1699, [seq(0, j=0..1699)]):
Sq2  := array(0..1699, [seq(0, j=0..1699)]):
Sq4  := array(0..1699, [seq(0, j=0..1699)]):
Sq8  := array(0..1699, [seq(0, j=0..1699)]):
dSq  := array(0..1699, [seq(0, j=0..1699)]):
dSq2 := array(0..1699, [seq(0, j=0..1699)]):
dSq4 := array(0..1699, [seq(0, j=0..1699)]):
dSq8 := array(0..1699, [seq(0, j=0..1699)]):
S0q  := array(0..1699, [seq(0, j=0..1699)]):
S1q  := array(0..1699, [seq(0, j=0..1699)]):
S2q  := array(0..1699, [seq(0, j=0..1699)]):
S3q  := array(0..1699, [seq(0, j=0..1699)]):

```



```

const:= array(0..1699, [seq(0, j=0..1699)]):
#####
# Assigning values to the arrays. Note that
#  $g=(1-x)^{1/4}$ ,  $w$ ,  $dw/dx$  are represented by
# different numbers.
# The coefficients of  $g^a$  are stored in  $1*a$ .
# The coefficients of  $w^b$  are stored in  $40*b$ .
# The coefficients of  $(dw/dx)^c$  are in  $800*c$ .
# For example, the  $g^8w(dw/dx)$  term in  $L(q)$ 
# will be stored in the  $8+40+800$  spot in the
# array,  $Lq[848]=-12$ .
#####
#  $L(q)$ 
Lq[80]:=-4:          Lq[84]:=5:          Lq[844]:=12:
Lq[848]:=-12:
#  $L(q^2)$ 
Lq2[80]:=-1:         Lq2[84]:=2:         Lq2[844]:=6:
Lq2[848]:=-6:
#  $L(q^4)$ 
Lq4[80]:=-1/4:       Lq4[84]:=5/4:       Lq4[844]:=3:
Lq4[848]:=-3:
#  $L(q^8)$ 
Lq8[80]:=-1/16:      Lq8[82]:=3/8:       Lq8[84]:=11/16:
Lq8[844]:=3/2:       Lq8[848]:=-3/2:
#  $dL(q)/dq$ 
dLq[164]:= -2:       dLq[168]:= 2:       dLq[924]:= -8:
dLq[928]:= 18:       dLq[932]:= -10:      dLq[1688]:= 12:
dLq[1692]:= -24:     dLq[1696]:= 12:
#  $dL(q^2)/dq$ 
dLq2[164]:= -1/2:    dLq2[168]:= 1/2:    dLq2[924]:= -2:
dLq2[928]:= 6:       dLq2[932]:= -4:    dLq2[1688]:= 6:
dLq2[1692]:= -12:    dLq2[1696]:= 6:
#  $dL(q^4)/dq$ 
dLq4[164]:= -1/2:    dLq4[168]:= 1/2:    dLq4[924]:= -1/2:
dLq4[928]:= 3:       dLq4[932]:= -5/2:   dLq4[1688]:= 3:
dLq4[1692]:= -6:     dLq4[1696]:= 3:
#  $dL(q^8)/dq$ 
dLq8[162]:= -3/16:   dLq8[164]:= -5/16:   dLq8[166]:= 3/16:
dLq8[168]:= 5/16:   dLq8[924]:= -1/8:   dLq8[926]:= 3/4:
dLq8[928]:= 3/2:    dLq8[930]:= -3/4:   dLq8[932]:= -11/8:

```

dLq8[1688] := 3/2:	dLq8[1692] := -3:	dLq8[1696] := 3/2:
# L0(q)		
L0q[0] := 1/24:	L0q[80] := 11/192:	L0q[82] := 3/32:
L0q[84] := -37/192:	L0q[844] := -1/2:	L0q[848] := 1/2:
# L1(q)		
L1q[80] := 1/32:	L1q[81] := 1/16:	L1q[83] := -1/16:
L1q[84] := -1/32:		
# L2(q)		
L2q[80] := 3/64:	L2q[82] := -3/32:	L2q[84] := 3/64:
# L3(q)		
L3q[80] := 1/32:	L3q[81] := -1/16:	L3q[83] := 1/16:
L3q[84] := -1/32:		
# M(q)		
Mq[160] := 16:	Mq[164] := -16:	Mq[168] := 1:
# M(q^2)		
Mq2[160] := 1:	Mq2[164] := -1:	Mq2[168] := 1:
# M(q^4)		
Mq4[160] := 1/16:	Mq4[164] := 7/8:	Mq4[168] := 1/16:
# M(q^8)		
Mq8[160] := 1/256:	Mq8[162] := 15/64:	Mq8[164] := 67/128:
Mq8[166] := 15/64:	Mq8[168] := 1/256:	
# dM(q)/dq		
dMq[244] := 16:	dMq[248] := -18:	dMq[252] := 2:
dMq[1004] := 64:	dMq[1008] := -128:	dMq[1012] := 68:
dMq[1016] := -4:		
# dM(q^2)/dq		
dMq2[244] := 1:	dMq2[248] := -3:	dMq2[252] := 2:
dMq2[1004] := 4:	dMq2[1008] := -8:	dMq2[1012] := 8:
dMq2[1016] := -4:		
# dM(q^4)/dq		
dMq4[244] := -7/8:	dMq4[248] := 3/4:	dMq4[252] := 1/8:
dMq4[1004] := 1/4:	dMq4[1008] := 13/4:	dMq4[1012] := -13/4:
dMq4[1016] := -1/4:		
# dM(q^8)/dq		
dMq8[242] := -15/128:	dMq8[244] := -67/128:	dMq8[246] := -15/64:
dMq8[248] := 33/64:	dMq8[250] := 45/128:	dMq8[252] := 1/128:
dMq8[1004] := 1/64:	dMq8[1006] := 15/16:	dMq8[1008] := 133/64:
dMq8[1012] := -133/64:	dMq8[1014] := -15/16:	dMq8[1016] := -1/64:
# M0(q)		
M0q[0] := -1/240:	M0q[160] := 137/7680:	M0q[162] := -9/128:

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M0q[164] := 419/3840:    M0q[166] := -9/128:    M0q[168] := 137/7680:
# M1(q)
M1q[160] := 1/64:        M1q[161] := -1/64:    M1q[163] := 7/64:
M1q[165] := -7/64:       M1q[167] := 1/64:     M1q[168] := -1/64:
# M2(q)
M2q[160] := 9/512:       M2q[162] := 9/128:    M2q[164] := -45/256:
M2q[166] := 9/128:       M2q[168] := 9/512:
# M3(q)
M3q[160] := 1/64:        M3q[161] := 1/64:     M3q[163] := -7/64:
M3q[165] := 7/64:        M3q[167] := -1/64:    M3q[168] := -1/64:
# N(q)
Nq[240] := -64:          Nq[244] := 96:        Nq[248] := -30:
Nq[252] := -1:
# N(q^2)
Nq2[240] := -1:          Nq2[244] := 3/2:      Nq2[248] := 3/2:
Nq2[252] := -1:
# N(q^4)
Nq4[240] := -1/64:       Nq4[244] := 33/64:    Nq4[248] := 33/64:
Nq4[252] := -1/64:
# N(q^8)
Nq8[240] := -1/4096:     Nq8[242] := 63/2048:  Nq8[244] := 1041/4096:
Nq8[246] := 441/1024:    Nq8[248] := 1041/4096: Nq8[250] := 63/2048:
Nq8[252] := -1/4096:
# dN(q)/dq
dNq[324] := -96:         dNq[328] := 156:      dNq[332] := -57:
dNq[336] := -3:          dNq[1084] := -384:     dNq[1088] := 960:
dNq[1092] := -756:       dNq[1096] := 174:     dNq[1100] := 6:
# dN(q^2)/dq
dNq2[324] := -3/2:       dNq2[328] := -3/2:    dNq2[332] := 6:
dNq2[336] := -3:         dNq2[1084] := -6:     dNq2[1088] := 15:
dNq2[1096] := -15:       dNq2[1100] := 6:
# dN(q^4)/dq
dNq4[324] := -33/64:     dNq4[328] := -33/64:  dNq4[332] := 69/64:
dNq4[336] := -3/64:      dNq4[1084] := -3/32:  dNq4[1088] := 51/16:
dNq4[1096] := -51/16:    dNq4[1100] := 3/32:
# dN(q^8)/dq
dNq8[322] := -63/4096:   dNq8[324] := -1041/4096:
dNq8[326] := -2583/4096: dNq8[328] := -1041/4096:
dNq8[330] := 2331/4096:  dNq8[332] := 2085/4096:
dNq8[334] := 315/4096:   dNq8[336] := -3/4096:

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dNq8[1084] := -3/2048:	dNq8[1086] := 189/1024:
dNq8[1088] := 1563/1024:	dNq8[1090] := 2457/1024:
dNq8[1094] := -2457/1024:	dNq8[1096] := -1563/1024:
dNq8[1098] := -189/1024:	dNq8[1100] := 3/2048:
# N0(q)	
N0q[0] := 1/504:	N0q[240] := 2081/64512:
N0q[242] := 33/512:	N0q[244] := -11803/21504:
N0q[246] := 231/256:	N0q[248] := -11803/21504:
N0q[250] := 33/512:	N0q[252] := 2081/64512:
# N1(q)	
N1q[240] := 1/32:	N1q[241] := 1/256:
N1q[243] := -61/256:	N1q[244] := -1/16:
N1q[245] := 97/128:	N1q[247] := -97/128:
N1q[248] := 1/16:	N1q[249] := 61/256:
N1q[251] := -1/256:	N1q[252] := -1/32:
# N2(q)	
N2q[240] := 33/1024:	N2q[242] := -33/512:
N2q[244] := 495/1024:	N2q[246] := -231/256:
N2q[248] := 495/1024:	N2q[250] := -33/512:
N2q[252] := 33/1024:	
# N3(q)	
N3q[240] := 1/32:	N3q[241] := -1/256:
N3q[243] := 61/256:	N3q[244] := -1/16:
N3q[245] := -97/128:	N3q[247] := 97/128:
N3q[248] := 1/16:	N3q[249] := -61/256:
N3q[251] := 1/256:	N3q[252] := -1/32:
# P(q)	
Pq[320] := 256:	Pq[324] := -512:
Pq[328] := 288:	Pq[332] := -32:
Pq[336] := 1:	
# P(q^2)	
Pq2[320] := 1:	Pq2[324] := -2:
Pq2[328] := 3:	Pq2[332] := -2:
Pq2[336] := 1:	
# P(q^4)	
Pq4[320] := 1/256:	Pq4[324] := 7/64:
Pq4[328] := 99/128:	Pq4[332] := 7/64:
Pq4[336] := 1/256:	
# P(q^8)	
Pq8[320] := 1/65536:	Pq8[322] := 15/8192:

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Pq8[324] := 967/16384:
Pq8[328] := 12579/32768:
Pq8[332] := 967/16384:
Pq8[336] := 1/65536:
# dP(q)/dq
dPq[404] := 512:
dPq[412] := 672:
dPq[420] := 4:
dPq[1168] := -6144:
dPq[1176] := -2560:
dPq[1184] := -8:
# dP(q^2)/dq
dPq2[404] := 2:
dPq2[412] := 12:
dPq2[420] := 4:
dPq2[1168] := -24:
dPq2[1176] := -40:
dPq2[1184] := -8:
# dP(q^4)/dq
dPq4[404] := -7/64:
dPq4[412] := 39/32:
dPq4[420] := 1/64:
dPq4[1168] := 27/32:
dPq4[1176] := -85/16:
dPq4[1184] := -1/32:
# dP(q^8)/dq
dPq8[402] := -15/16384:
dPq8[406] := -1515/4096:
dPq8[410] := -2025/8192:
dPq8[414] := 2505/4096:
dPq8[418] := 105/16384:
dPq8[1164] := 1/8192:
dPq8[1168] := 3867/8192:
dPq8[1172] := 10645/4096:
dPq8[1176] := -10645/4096:
dPq8[1180] := -3867/8192:
dPq8[1184] := -1/8192:
# P0(q)
P0q[0] := -1/480:
P0q[322] := -129/2048:

Pq8[326] := 2025/8192:
Pq8[330] := 2025/8192:
Pq8[334] := 15/8192:

dPq[408] := -1088:
dPq[416] := -100:
dPq[1164] := 2048:
dPq[1172] := 6400:
dPq[1180] := 264:

dPq2[408] := -8:
dPq2[416] := -10:
dPq2[1164] := 8:
dPq2[1172] := 40:
dPq2[1180] := 24:

dPq4[408] := -23/16:
dPq4[416] := 5/16:
dPq4[1164] := 1/32:
dPq4[1172] := 85/16:
dPq4[1180] := -27/32:

dPq8[404] := -967/16384:
dPq8[408] := -2903/4096:
dPq8[412] := 4839/8192:
dPq8[416] := 725/4096:
dPq8[420] := 1/16384:
dPq8[1166] := 15/1024:
dPq8[1170] := 1005/512:
dPq8[1174] := 0:
dPq8[1178] := -1005/512:
dPq8[1182] := -15/1024:

P0q[320] := 32897/245760:
P0q[324] := 106439/61440:

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P0q[326] := -17415/2048:
P0q[330] := -17415/2048:
P0q[334] := -129/2048:
# P1(q)
P1q[320] := 17/128:
P1q[323] := 547/1024:
P1q[325] := -4881/1024:
P1q[328] := 0:
P1q[331] := 4881/1024:
P1q[333] := -547/1024:
P1q[336] := -17/128:
# P2(q)
P2q[320] := 2193/16384:
P2q[324] := -9417/4096:
P2q[328] := -104877/8192:
P2q[332] := -9417/4096:
P2q[336] := 2193/16384:
# P3(q)
P3q[320] := 17/128:
P3q[323] := -547/1024:
P3q[325] := 4881/1024:
P3q[328] := 0:
P3q[331] := -4881/1024:
P3q[333] := 547/1024:
P3q[336] := -17/128:
# Q(q)
Qq[400] := -1024:
Qq[408] := -2080:
Qq[416] := -14:
# Q(q^2)
Qq2[400] := -1:
Qq2[408] := -1:
Qq2[416] := 5/2:
# Q(q4)
Qq4[400] := -1/1024:
Qq4[408] := 247/512:
Qq4[416] := 19/1024:
# Q(q8)
Qq8[400] := -1/1048576:
Qq8[404] := 8467/1048576:

P0q[328] := 548961/40960:
P0q[332] := 106439/61440:
P0q[336] := 32897/245760:

P1q[321] := -1/1024:
P1q[324] := -1/4:
P1q[327] := 11979/1024:
P1q[329] := -11979/1024:
P1q[332] := 1/4:
P1q[335] := 1/1024:

P2q[322] := 129/2048:
P2q[326] := 17415/2048:
P2q[330] := 17415/2048:
P2q[334] := 129/2048:

P3q[321] := 1/1024:
P3q[324] := -1/4:
P3q[327] := -11979/1024:
P3q[329] := 11979/1024:
P3q[332] := 1/4:
P3q[335] := -1/1024:

Qq[404] := 2560:
Qq[412] := 560:
Qq[420] := -1:

Qq2[404] := 5/2:
Qq2[412] := -1:
Qq2[420] := -1:

Qq4[404] := 19/1024:
Qq4[412] := 247/512:
Qq4[420] := -1/1024:

Qq8[402] := 33/524288:
Qq8[406] := 10131/131072:

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Qq8[408] := 126967/524288:	Qq8[410] := 90387/262144:
Qq8[412] := 126967/524288:	Qq8[414] := 10131/131072:
Qq8[416] := 8467/1048576:	Qq8[418] := 33/524288:
Qq8[420] := -1/1048576:	
# dQ(q)/dq	
dQq[484] := -2560:	dQq[488] := 6720:
dQq[492] := -5840:	dQq[496] := 1736:
dQq[500] := -51:	dQq[504] := -5:
dQq[1244] := -10240:	dQq[1248] := 35840:
dQq[1252] := -46400:	dQq[1256] := 26400:
dQq[1260] := -5740:	dQq[1264] := 130:
dQq[1268] := 10:	
# dQ(q^2)/dq	
dQq2[484] := -5/2:	dQq2[488] := 9/2:
dQq2[492] := 1:	dQq2[496] := -13:
dQq2[500] := 15:	dQq2[504] := -5:
dQq2[1244] := -10:	dQq2[1248] := 35:
dQq2[1252] := -35:	dQq2[1256] := 0:
dQq2[1260] := 35:	dQq2[1264] := -35:
dQq2[1268] := 10:	
# dQ(q^4)/dq	
dQq4[484] := -19/1024:	dQq4[488] := -969/1024:
dQq4[492] := -247/512:	dQq4[496] := 703/512:
dQq4[500] := 81/1024:	dQq4[504] := -5/1024:
dQq4[1244] := -5/512:	dQq4[1248] := 25/128:
dQq4[1252] := 2375/512:	dQq4[1256] := 0:
dQq4[1260] := -2375/512:	dQq4[1264] := -25/128:
dQq4[1268] := 5/512:	
# dQ(q^8)/dq	
dQq8[482] := -33/1048576:	dQq8[484] := -8467/1048576:
dQq8[486] := -121539/1048576:	dQq8[488] := -499401/1048576:
dQq8[490] := -391149/524288:	dQq8[492] := -126967/524288:
dQq8[494] := 310101/524288:	dQq8[496] := 363967/524288:
dQq8[498] := 283371/1048576:	dQq8[500] := 33873/1048576:
dQq8[502] := 297/1048576:	dQq8[504] := -5/1048576:
dQq8[1244] := -5/524288:	dQq8[1246] := 165/262144:
dQq8[1248] := 10585/131072:	dQq8[1250] := 202455/262144:
dQq8[1252] := 1227335/524288:	dQq8[1254] := 350625/131072:
dQq8[1256] := 0:	dQq8[1258] := -350625/131072:
dQq8[1260] := -1227335/524288:	dQq8[1262] := -202455/262144:

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dQq8[1264] := -10585/131072:      dQq8[1266] := -165/262144:
dQq8[1268] := 5/524288:
# Q0(q)
Q0q[0] := 1/264:
Q0q[402] := 513/8192:
Q0q[406] := 157491/2048:
Q0q[410] := 1405107/4096:
Q0q[414] := 157491/2048:
Q0q[418] := 513/8192:
# Q1(q)
Q1q[400] := 31/32:
Q1q[403] := -4921/4096:
Q1q[405] := 30517/1024:
Q1q[408] := 5/2:
Q1q[411] := -642279/2048:
Q1q[413] := 155017/1024:
Q1q[416] := 39/16:
Q1q[419] := -1/4096:
# Q2(q)
Q2q[400] := 15903/16384:
Q2q[404] := 91827/16384:
Q2q[408] := 1981719/8192:
Q2q[412] := 1981719/8192:
Q2q[416] := 91827/16384:
Q2q[420] := 15903/16384:
# Q3(q)
Q3q[400] := 31/32:
Q3q[403] := 4921/4096:
Q3q[405] := -30517/1024:
Q3q[408] := 5/2:
Q3q[411] := 642279/2048:
Q3q[413] := -155017/1024:
Q3q[416] := 39/16:
Q3q[419] := 1/4096:
# R(q)
Rq[480] := 4096:
Rq[488] := 13056:
Rq[496] := 536856/691:
Rq[504] := 1:
# R(q^2)
Q0q[400] := 524801/540672:
Q0q[404] := -5637395/540672:
Q0q[408] := -64618487/270336:
Q0q[412] := -64618487/270336:
Q0q[416] := -5637395/540672:
Q0q[420] := 524801/540672:
Q1q[401] := 1/4096:
Q1q[404] := -39/16:
Q1q[407] := -155017/1024:
Q1q[409] := 642279/2048:
Q1q[412] := -5/2:
Q1q[415] := -30517/1024:
Q1q[417] := 4921/4096:
Q1q[420] := -31/32:
Q2q[402] := -513/8192:
Q2q[406] := -157491/2048:
Q2q[410] := -1405107/4096:
Q2q[414] := -157491/2048:
Q2q[418] := -513/8192:
Q3q[401] := -1/4096:
Q3q[404] := -39/16:
Q3q[407] := 155017/1024:
Q3q[409] := -642279/2048:
Q3q[412] := -5/2:
Q3q[415] := 30517/1024:
Q3q[417] := -4921/4096:
Q3q[420] := -31/32:
Rq[484] := -12288:
Rq[492] := -5632:
Rq[500] := -6168/691:

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Rq2[480] := 1:
Rq2[488] := 4917/1382:
Rq2[496] := 4917/1382:
Rq2[504] := 1:
# R(q^4)
Rq4[480] := 1/4096:
Rq4[488] := 516381/2830336:
Rq4[496] := 516381/2830336:
Rq4[504] := 1/4096:
# R(q^8)
Rq8[480] := 1/16777216:
Rq8[482] := 4095/2898264064:
Rq8[484] := 4194291/5796528128:
Rq8[486] := 45327555/2898264064:
Rq8[488] := 1053029661/11593056256:
Rq8[490] := 339528735/1449132032:
Rq8[492] := 918776357/2898264064:
Rq8[494] := 339528735/1449132032:
Rq8[496] := 1053029661/11593056256:
Rq8[498] := 45327555/2898264064:
Rq8[500] := 4194291/5796528128:
Rq8[502] := 4095/2898264064:
Rq8[504] := 1/16777216:
# dR(q)/dq
dRq[564] := 12288:
dRq[572] := 43008:
dRq[580] := 2178264/691:
dRq[588] := 6:
dRq[1328] := -196608:
dRq[1336] := -224256:
dRq[1344] := -6516288/691:
dRq[1352] := -12:
# dR(q^2)/dq
dRq2[564] := 3:
dRq2[572] := 9303/691:
dRq2[580] := 20199/691:
dRq2[588] := 6:
dRq2[1328] := -48:
dRq2[1336] := -47046/691:
dRq2[1344] := -54378/691:
Rq2[484] := -3:
Rq2[492] := -1462/691:
Rq2[500] := -3:
Rq4[484] := 1011/1415168:
Rq4[492] := 448037/707584:
Rq4[500] := 1011/1415168:
dRq[568] := -38400:
dRq[576] := -13822560/691:
dRq[584] := -34986/691:
dRq[1324] := 49152:
dRq[1332] := 304128:
dRq[1340] := 53142816/691:
dRq[1348] := 82308/691:
dRq2[568] := -6990/691:
dRq2[576] := -14220/691:
dRq2[584] := -21:
dRq2[1324] := 12:
dRq2[1332] := 54378/691:
dRq2[1340] := 47046/691:
dRq2[1348] := 48:

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dRq2[1352] := -12:
# dR(q^4)/dq
dRq4[564] := -1011/1415168:
dRq4[572] := -2171841/1415168:
dRq4[580] := 1027707/1415168:
dRq4[588] := 3/2048:
dRq4[1328] := 3993/707584:
dRq4[1336] := 3827301/707584:
dRq4[1344] := -1543077/707584:
dRq4[1352] := -3/1024:
# dR(q^8)/dq
dRq8[562] := -4095/5796528128:
dRq8[564] := -4194291/5796528128:
dRq8[566] := -67989285/2898264064:
dRq8[568] := -524417685/2898264064:
dRq8[570] := -3259304685/5796528128:
dRq8[572] := -4459628481/5796528128:
dRq8[574] := -339528735/1449132032:
dRq8[576] := 851649705/1449132032:
dRq8[578] := 4345454295/5796528128:
dRq8[580] := 2085087867/5796528128:
dRq8[582] := 203951475/2898264064:
dRq8[584] := 10484691/2898264064:
dRq8[586] := 45045/5796528128:
dRq8[588] := 3/8388608:
dRq8[1324] := 3/4194304:
dRq8[1326] := 12285/724566016:
dRq8[1328] := 25163673/2898264064:
dRq8[1330] := 33992595/181141504:
dRq8[1332] := 3133923237/2898264064:
dRq8[1334] := 1901189745/724566016:
dRq8[1336] := 7866227301/2898264064:
dRq8[1340] := -7866227301/2898264064:
dRq8[1342] := -1901189745/724566016:
dRq8[1344] := -3133923237/2898264064:
dRq8[1346] := -33992595/181141504:
dRq8[1348] := -25163673/2898264064:
dRq8[1350] := -12285/724566016:
dRq8[1352] := -3/4194304:
# R0(q)
dRq4[568] := -257685/707584:
dRq4[576] := 413865/353792:
dRq4[584] := 1491/707584:
dRq4[1324] := 3/1024:
dRq4[1332] := 1543077/707584:
dRq4[1340] := -3827301/707584:
dRq4[1348] := -3993/707584:

```

```

R0q[0] := -691/65520:
R0q[480] := 5797943987/536739840:
R0q[482] := -2049/32768:
R0q[484] := -36358831/89456640:
R0q[486] := -22680381/32768:
R0q[488] := 725392047199/178913280:
R0q[490] := -169888737/16384:
R0q[492] := 1877670062629/134184960:
R0q[494] := -169888737/16384:
R0q[496] := 725392047199/178913280:
R0q[498] := -22680381/32768:
R0q[500] := -36358831/89456640:
R0q[502] := -2049/32768:
R0q[504] := 5797943987/536739840:
# R1(q)
R1q[480] := 691/64:
R1q[481] := -1/16384:
R1q[483] := 44287/16384:
R1q[484] := -259/8:
R1q[485] := -3051755/16384:
R1q[487] := 30779477/16384:
R1q[488] := 259/8:
R1q[489] := -57667517/8192:
R1q[491] := 106536227/8192:
R1q[492] := 0:
R1q[493] := -106536227/8192:
R1q[495] := 57667517/8192:
R1q[496] := -259/8:
R1q[497] := -30779477/16384:
R1q[499] := 3051755/16384:
R1q[500] := 259/8:
R1q[501] := -44287/16384:
R1q[503] := 1/16384:
R1q[504] := -691/64:
# R2(q)
R2q[480] := 1415859/131072:
R2q[482] := 2049/32768:
R2q[484] := -4222989/65536:
R2q[486] := 22680381/32768:
R2q[488] := -521861859/131072:

```

```

R2q[490] := 169888737/16384:
R2q[492] := -460473819/32768:
R2q[494] := 169888737/16384:
R2q[496] := -521861859/131072:
R2q[498] := 22680381/32768:
R2q[500] := -4222989/65536:
R2q[502] := 2049/32768:
R2q[504] := 1415859/131072:
# R3(q)
R3q[480] := 691/64:
R3q[481] := 1/16384:
R3q[483] := -44287/16384:
R3q[484] := -259/8:
R3q[485] := 3051755/16384:
R3q[487] := -30779477/16384:
R3q[488] := 259/8:
R3q[489] := 57667517/8192:
R3q[491] := -106536227/8192:
R3q[492] := 0:
R3q[493] := 106536227/8192:
R3q[495] := -57667517/8192:
R3q[496] := -259/8:
R3q[497] := 30779477/16384:
R3q[499] := -3051755/16384:
R3q[500] := 259/8:
R3q[501] := 44287/16384:
R3q[503] := -1/16384:
R3q[504] := -691/64:
# S(q)
Sq[560] := -16384:
Sq[568] := -75264:
Sq[576] := -11264:
Sq[584] := 2:
# S(q^2)
Sq2[560] := -1:
Sq2[568] := -9/2:
Sq2[576] := 5/2:
Sq2[584] := 7/2:
# S(q^4)
Sq4[560] := -1/16384:
Sq[564] := 57344:
Sq[572] := 44800:
Sq[580] := 768:
Sq[588] := -1:
Sq2[564] := 7/2:
Sq2[572] := 5/2:
Sq2[580] := -9/2:
Sq2[588] := -1:
Sq4[564] := 5/16384:

```

```

Sq4[568] := 759/16384:
Sq4[576] := 7429/16384:
Sq4[584] := 5/16384:
# S(q^8)
Sq8[560] := -1/268435456:
Sq8[562] := 3/134217728:
Sq8[564] := 12293/268435456:
Sq8[566] := 149463/67108864:
Sq8[568] := 6255351/268435456:
Sq8[570] := 13483053/134217728:
Sq8[572] := 60845317/268435456:
Sq8[574] := 9885201/33554432:
Sq8[576] := 60845317/268435456:
Sq8[578] := 13483053/134217728:
Sq8[580] := 6255351/268435456:
Sq8[582] := 149463/67108864:
Sq8[584] := 12293/268435456:
Sq8[586] := 3/134217728:
Sq8[588] := -1/268435456:
# dS(q)/dq
dSq[644] := -57344:
dSq[652] := -284928:
dSq[660] := -48896:
dSq[668] := 19:
dSq[1404] := -229376:
dSq[1412] := -1856512:
dSq[1420] := -784896:
dSq[1428] := -10724:
dSq[1436] := 14:
# dS(q^2)/dq
dSq2[644] := -7/2:
dSq2[652] := -33/2:
dSq2[660] := 65/2:
dSq2[668] := 28:
dSq2[1404] := -14:
dSq2[1412] := -112:
dSq2[1420] := 0:
dSq2[1428] := 112:
dSq2[1436] := 14:
# dS(q^4)/dq
Sq4[572] := 7429/16384:
Sq4[580] := 759/16384:
Sq4[588] := -1/16384:
dSq[648] := 207872:
dSq[656] := 179456:
dSq[664] := 3828:
dSq[672] := -7:
dSq[1408] := 1032192:
dSq[1416] := 1680896:
dSq[1424] := 168448:
dSq[1432] := -42:
dSq2[648] := 25/2:
dSq2[656] := -5/2:
dSq2[664] := -87/2:
dSq2[672] := -7:
dSq2[1408] := 63:
dSq2[1416] := 98:
dSq2[1424] := -98:
dSq2[1432] := -63:

```

```

dSq4[644] := -5/16384:
dSq4[652] := -20769/16384:
dSq4[660] := 25921/16384:
dSq4[668] := 37/16384:
dSq4[1404] := -7/8192:
dSq4[1412] := 2639/4096:
dSq4[1424] := -23345/4096:
dSq4[1432] := -21/4096:
# dS(q^8)/dq
dSq8[642] := -3/268435456:
dSq8[644] := -12293/268435456:
dSq8[646] := -896775/268435456:
dSq8[648] := -12498409/268435456:
dSq8[650] := -66518487/268435456:
dSq8[652] := -170025249/268435456:
dSq8[654] := -209370363/268435456:
dSq8[656] := -60845317/268435456:
dSq8[658] := 155438151/268435456:
dSq8[660] := 212104513/268435456:
dSq8[662] := 118059291/268435456:
dSq8[664] := 31202997/268435456:
dSq8[666] := 3288147/268435456:
dSq8[668] := 73765/268435456:
dSq8[670] := 39/268435456:
dSq8[672] := -7/268435456:
dSq8[1404] := -7/134217728:
dSq8[1406] := 21/67108864:
dSq8[1408] := 43029/67108864:
dSq8[1410] := 2092461/67108864:
dSq8[1412] := 21850703/67108864:
dSq8[1414] := 92288889/67108864:
dSq8[1416] := 191064881/67108864:
dSq8[1418] := 182404257/67108864:
dSq8[1420] := 0:
dSq8[1422] := -182404257/67108864:
dSq8[1424] := -191064881/67108864:
dSq8[1426] := -92288889/67108864:
dSq8[1428] := -21850703/67108864:
dSq8[1430] := -2092461/67108864:
dSq8[1432] := -43029/67108864:
dSq4[648] := -1513/16384:
dSq4[656] := -7429/16384:
dSq4[664] := 3765/16384:
dSq4[672] := -7/16384:
dSq4[1408] := 21/4096:
dSq4[1416] := 23345/4096:
dSq4[1428] := -2639/4096:
dSq4[1436] := 7/8192:

```

```

dSq8[1434] := -21/67108864:
dSq8[1436] := 7/134217728:
# S0(q)
S0q[0] := 1/24:
S0q[560] := 134225921/786432:
S0q[562] := 8193/131072:
S0q[564] := -570454021/786432:
S0q[566] := 408183453/65536:
S0q[568] := -16877867261/262144:
S0q[570] := 36822217743/131072:
S0q[572] := -498719550725/786432:
S0q[574] := 26996483931/32768:
S0q[576] := -498719550725/786432:
S0q[578] := 36822217743/131072:
S0q[580] := -16877867261/262144:
S0q[582] := 408183453/65536:
S0q[584] := -570454021/786432:
S0q[586] := 8193/131072:
S0q[588] := 134225921/786432:
# S1(q)
S1q[560] := 5461/32:
S1q[561] := 1/65536:
S1q[563] := -398581/65536:
S1q[564] := -9557/16:
S1q[565] := 38146971/32768:
S1q[567] := -756322611/32768:
S1q[568] := 792:
S1q[569] := 9709824531/65536:
S1q[571] := -29741019351/65536:
S1q[572] := -584:
S1q[573] := 12645048737/16384:
S1q[575] := -12645048737/16384:
S1q[576] := 584:
S1q[577] := 29741019351/65536:
S1q[579] := -9709824531/65536:
S1q[580] := -792:
S1q[581] := 756322611/32768:
S1q[583] := -38146971/32768:
S1q[584] := 9557/16:
S1q[585] := 398581/65536:

```

```

S1q[587] := -1/65536:
S1q[588] := -5461/32:
# S2(q)
S2q[560] := 44741973/262144:
S2q[562] := -8193/131072:
S2q[564] := -123034281/262144:
S2q[566] := -408183453/65536:
S2q[568] := 17284714749/262144:
S2q[570] := -36822217743/131072:
S2q[572] := 166056698967/262144:
S2q[574] := -26996483931/32768:
S2q[576] := 166056698967/262144:
S2q[578] := -36822217743/131072:
S2q[580] := 17284714749/262144:
S2q[582] := -408183453/65536:
S2q[584] := -123034281/262144:
S2q[586] := -8193/131072:
S2q[588] := 44741973/262144:
# S3(q)
S3q[560] := 5461/32:
S3q[561] := -1/65536:
S3q[563] := 398581/65536:
S3q[564] := -9557/16:
S3q[565] := -38146971/32768:
S3q[567] := 756322611/32768:
S3q[568] := 792:
S3q[569] := -9709824531/65536:
S3q[571] := 29741019351/65536:
S3q[572] := -584:
S3q[573] := -12645048737/16384:
S3q[575] := 12645048737/16384:
S3q[576] := 584:
S3q[577] := -29741019351/65536:
S3q[579] := 9709824531/65536:
S3q[580] := -792:
S3q[581] := -756322611/32768:
S3q[583] := 38146971/32768:
S3q[584] := 9557/16:
S3q[585] := -398581/65536:
S3q[587] := 1/65536:

```



```

S3q[588]:= -5461/32:
const[0] := 1:
#####
# Maximum number of arrays. Each product can
# only be related to a few other functions,
# i.e. the ones that contain the same terms
# as the product. Each product can only be
# related to at most 13 other functions.
#####
MAXNUMARR := 13:
#####
# Factor1: Stores the values for the first
#         factor of the product.
# Factor2: Stores the values for the second
#         factor of the product.
#####
Factor1 := array(0..1699, [seq(0, j=0..1699)]):
Factor2 := array(0..1699, [seq(0, j=0..1699)]):
Lq_arr:= array(0..6, [Lq, Mq, Nq, Pq, Qq, Rq, Sq]):
Lq2_arr:= array(0..6, [Lq2, Mq2, Nq2, Pq2, Qq2, Rq2, Sq2]):
Lq4_arr:= array(0..6, [Lq4, Mq4, Nq4, Pq4, Qq4, Rq4, Sq4]):
Lq8_arr:= array(0..6, [Lq8, Mq8, Nq8, Pq8, Qq8, Rq8, Sq8]):
dLq_arr:= array(0..6, [dLq, dMq, dNq, dPq, dQq, dRq, dSq]):
dLq2_arr:= array(0..6, [dLq2, dMq2, dNq2, dPq2, dQq2, dRq2, dSq2]):
dLq4_arr:= array(0..6, [dLq4, dMq4, dNq4, dPq4, dQq4, dRq4, dSq4]):
dLq8_arr:= array(0..6, [dLq8, dMq8, dNq8, dPq8, dQq8, dRq8, dSq8]):
L0q_arr:= array(0..6, [L0q, M0q, N0q, P0q, Q0q, R0q, S0q]):
L1q_arr:= array(0..6, [L1q, M1q, N1q, P1q, Q1q, R1q, S1q]):
L2q_arr:= array(0..6, [L2q, M2q, N2q, P2q, Q2q, R2q, S2q]):
L3q_arr:= array(0..6, [L3q, M3q, N3q, P3q, Q3q, R3q, S3q]):
#####
# Go through for loops to find relationships
# with products of the form Factor1*Factor2.
# The choices for Factor1 and Factor2 are:
# Lq_arr and L0q_arr,
# Lq_arr and L1q_arr,
# Lq_arr and L2q_arr,
# Lq_arr and L3q_arr,
# Lq2_arr and L0q_arr,
# Lq2_arr and L1q_arr,

```

```

# Lq2_arr and L2q_arr,
# Lq2_arr and L3q_arr,
# Lq4_arr and L0q_arr,
# Lq4_arr and L1q_arr,
# Lq4_arr and L2q_arr,
# Lq4_arr and L3q_arr,
# Lq8_arr and L0q_arr,
# Lq8_arr and L1q_arr,
# Lq8_arr and L2q_arr,
# Lq8_arr and L3q_arr,
# L0q_arr and L0q_arr,
# L0q_arr and L1q_arr,
# L0q_arr and L2q_arr,
# L0q_arr and L3q_arr,
# L1q_arr and L1q_arr,
# L1q_arr and L2q_arr,
# L1q_arr and L3q_arr,
# L2q_arr and L2q_arr,
# L2q_arr and L3q_arr,
# L3q_arr and L3q_arr,
# Lq_arr and Lq8_arr.
#####

for first from 0 to 5
do
  # CHANGE HERE !!!
  Factor1 := Lq_arr[first]:

  for second from 0 to (6-first-1)
  do
    # CHANGE HERE !!!
    Factor2 := Lq8_arr[second];

    # Obtain desired product
    res := array (0..3399,[seq(0, j=0..3399)]):
    for i from 0 to 1699
    do
      if (not(Factor1[i]=0))
      then
        for j from 0 to 1699

```

```

        do
            res[i+j] := res[i+j]+(Factor1[i]*Factor2[j]):
        od:
    fi:
od:

# Unknown constants, equations
unk := array(0..MAXNUMARR-1):
eqn := array(0..1699):
numeqn := 0:
arr := array(0..MAXNUMARR-1, [ Lq_arr[first],
                                Lq2_arr[first],
                                Lq4_arr[first],
                                Lq8_arr[first],
                                Lq_arr[first+second+1],
                                Lq2_arr[first+second+1],
                                Lq4_arr[first+second+1],
                                Lq8_arr[first+second+1],
                                dLq_arr[first+second],
                                dLq2_arr[first+second],
                                dLq4_arr[first+second],
                                dLq8_arr[first+second],
                                const]):

for i from 0 to 1699
do
    if (sum('abs(arr[j][i])', 'j'=0..MAXNUMARR-1)
        +abs(res[i]) = 0)
    then
        # do nothing
    else
        eqn[numeqn] := sum('unk[j]*arr[j][i]',
                            'j'=0..MAXNUMARR-1) = res[i];
        numeqn := numeqn+1;
    fi;
od;

print("Solving unknowns for ", first, " and ", second);
if (not(solve({seq(eqn[j], j=0..numeqn-1)},
                {seq(unk[l], l=0..MAXNUMARR-1)})= NULL))
then

```

```
        print(solve({seq(eqn[j], j=0..numeqn-1)},
                      {seq(unk[l], l=0..MAXNUMARR-1)}));
    fi;
od:
od:
```

Appendix D

MAPLE program to find arithmetic identities of the same type as Melfi's identities.

```
#####
# This program calculates the unknowns A, B, C,... for the
# following convolution sum equation given
# e = 1, 3, 5, 7, 9, 11, 15
# f = 1, 3, 5, 7, 9, 11, 15
# k = 1, 2, 3,...16
# s = 1, 2, 3, ... 50
# r = 0, 1, 2, 3, 4, ... s-1
#
# There are (e+f+2) unknowns, A, B, C, D,...
# The unknowns are solved for by formulating (e+f+2)
# equations and solving for the unknowns. The solution,
# A, B, C,... should hold for all "n congruent (r+xs) mod s".
# Where x is a constant integer less than k and
# gcd (n,k) = 1.
#
# sum(sigma[e](m)*sigma[f](n-km),m = 1 .. [n/k])
#       = (A+Bn)*sigma[e+f+1](n)+(C+D*n)*sigma[e+f-1](n)
#       +(E+F*n)*sigma[e+f-3](n)+%?;
#
# and so on until sigma[1](n). Where
# sigma[x](n) := sum(divisors(n)[i]^x,i = 1 .. tau(n));
```

```
#####
restart;
with(numtheory):

#####
# Run the program from
# e = 1, 3, 5, 7, 9, 11, 13, 15
# f = 1, 3, 5, 7, 9, 11, 13, 15
# k = 1 to 16
# s = 1 to 50
# r = 1 to s-1
#
# Assuming that there are at most 100 sets of solutions
# {A,B,C,...} for each k, let goodR and goodS be arrays
# of 100 integers. goodR and goodS store the combinations
# of (r,s) that had a solution. Before doing a
# calculation for a (r,s) pair, check goodR and goodS
# first to make sure that (r,s) is not a multiple of a
# previously calculated pair. If it is a multiple, there
# is no need to calculate again. For example, if a
# solution is found for (r1,s1), there is no need to
# calculate (r2 = r1_x*s1, s2 = y*s1), where y is any
# positive integer and x = 0,1,2,... y-1.
#
# There are at most 15+15+2 =32 unknowns. Set an array
# for the unknowns.
#####
unk := array(1..32):
for e from 1 by 2 to 15
do
  for f from 1 by 2 to 15
  do
    # The following functions are used in the calculations.
    sigma := (power,n) -> sum(divisors(n)[i]^power, i=1..tau(n)):
    LHS := n -> sum(sigma(e,m)* sigma(f,n-k*m), m=1..n/k):

    for k from 1 by 1 to 16
    do
      print ("Calculating unknowns for e =",e,"f =",f,"k =",k);
      goodRSIndex := 0:
```

```

goodRSMaX    := 100:
goodR := array(1..goodRSMaX) :
goodS := array(1..goodRSMaX) :

for s from 1 by 1 to 50
do
  for r from 0 by 1 to (s-1)
  do
    nFirst := r;
    modVal := s;

    # Check if the (r,s) pair needs to be calculated.
    # The (r,s) doesn't need to be checked
    # If the current (r,s) pair can be derived from
    # a previously calculated (r,s), then
    # it doesn't need to be repeated.
    pleaseCheck := 1;
    for checkIndex from 1 by 1 to goodRSIndex
    do

      if (gcd(s,goodS[checkIndex]) = goodS[checkIndex])
      then
        for lambda from 0 to ((s/goodS[checkIndex])-1)
        do
          if (r = goodR[checkIndex]
              + lambda*goodS[checkIndex])
          then
            pleaseCheck := 0;
          fi;
        od;
      fi;
    od;

    # Count the number of n in {r+s, r+2s, ...
    # r+4(e+f+2)s} that satisfy gcd(n,k)=1
    goodN      := array(1..4*(e+f+2)):
    numGoodN   := 0:
    for counter from 1 by 1 to 4*(e+f+2)
    do
      if (gcd(r+counter*s, k) = 1)

```

```

    then
        numGoodN      := numGoodN + 1:
        goodN[numGoodN] := r+counter*s:
    fi:
od;

# If there are 2*(e+f+2) or more goodN's solve for
# the unknowns
if (numGoodN < 2*(e+f+2))
then
    pleaseCheck := 0:
fi:

if (pleaseCheck = 1)
then

    # Construct 2*(e+f+2) equations to solve for
    # e+f+2 unknowns.
    for numEqn from 1 by 1 to 2*(e+f+2)
    do
        eqn[numEqn] :=
            LHS(goodN[numEqn]) =
                sum((unk[j*2-1]+unk[j*2]*goodN[numEqn])
                    *sigma(j*2-1, goodN[numEqn]),
                    j=1..((e+f+2)/2));
    od;
    solution := solve ({seq(eqn[j], j=1..2*(e+f+2))},
                        {seq(unk[j], j=1..e+f+2)});

    if (not(solution = NULL))
    then
        print (k,s,r, solution);
        # While there's still room, store the (r,s)
        # pair that had a solution for the unknowns
        if (goodRSIndex < goodRSMax)
        then
            goodRSIndex := goodRSIndex+1:
            goodR[goodRSIndex] := r;
            goodS[goodRSIndex] := s;
        fi:
    end if
end if

```



```
                fi;  
            fi;  
        od;  
    od;  
od;  
od;  
od;
```

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