Single and Double Change Covering Designs

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by

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Abstract

A single change covering design (SCCD) is a sequence of b k-sets, called blocks, of a V-set in which exactly one element differs between consecutive blocks and every s-set of V is in some block.

We will review the literature and discuss several recursive constructions which completely solve the existence of SCCD for $k = 3, 4$ and partially for $k = 5$. We will examine optimizations on exhaustive search techniques. We determined that there are 313 unique circular SCCD(12,4,2,22). We determine a recursion for $s = 3$ and general k using expansion sets.

A double change covering design (DCCD) is similarly defined but consecutive blocks differ by two elements. We will completely solve the existence of tight DCCD $k = 3, s = 2$. We give several constructions using recursion and algebraic difference methods. This provides us with constructions for circular DCCD$(4k-2, 2k, 2, 2k-1)$, circular DCCD$(4k-1, 2k+1, 2, 2k-1)$ and circular DCCD$(4k-5k, 2, 4k-2)$ exists. We also find some other circular DCCD with given v and k.

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Contents

List of Tables

List of Figures

Chapter 1

Preliminaries

1.1 Introduction

A single change covering design (SCCD) is a sequence of b k-sets, called blocks, of a V-set in which exactly one element differs between consecutive blocks and every s-set of V is in some block. We notate this SCCD(v, k, s, b).

In this chapter we begin by reviewing relevant background information in design theory, graph theory and algebraic difference that will be needed in this theory. Continuing, we review all previous research on single change covering designs. Significant contributions were made to single change covering design theory during a twelve year period between 1990 and 2001 which will be discussed in Chapter 1.3. We conclude Chapter 1 by summarizing the known designs with $k = 2, ..., 5$.

In Chapter 2 we will discuss progress made to SCCD theory during the course of this thesis. We have a new recursive construction for strength 2 SCCD and CSCCD. We examine and combine previous exhaustive search techniques and implement an exhaustive search for circular SCCD. Using this we find all 313 possible non-equivalent CSCCD$(12, 4, 2, 21)$. We conclude this chapter by exploring preliminary results for strength 3 CSCCD with the goal of extending this search for higher strength designs.

We generalize the size of a change in Chapter 3 and focus double change covering designs. We will completely solve the existence question for tight DCCD and tight CDCCD when $k = 2, s = 2$. We prove some recursive constructions analogous to those of SCCD. We then find CDCCD by using algebraic difference methods. We conclude the chapter by combining some recursions and stating some sporadic results.

Finally, we will review the progress made in this thesis and consider future research paths that may be taken. Consequently we define some abstracted change designs for consideration.

1.2 Background

A **graph** is an ordered pair $G = (V, E)$ of vertices, V, and edges, E, such that each edge consists of two end vertices. [16]

A **walk** in a graph G is a sequence $W = v_0 e_1 v_1 ... v_{l-1} e_l v_l$ where v and e are alternating vertices and edges of G such that v_{i-1} and v_i are the ends of edge e_i, $1 \leq i \leq l$.

A **path** in a graph G is a walk with distinct vertices.

A **cycle** in a graph G is a path in G where the first and last vertex are joined by an edge.

A **Hamilton path (Hamilton cycle)** in a graph G is a path (cycle) visiting all the vertices V of G only once.

A **perfect matching or 1-factor** of a graph G is a subset of edges M such that every vertex $v \in V$ is adjacent to exactly one edge $e \in M$. A partition of the edges of a graph into perfect matchings is a **1-factorization**. Each color in Figure 1.3 is a 1-factor of the 1-factorization of K_6.

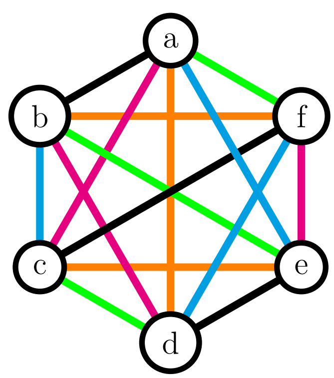

Figure 1.1: 1-factor of K_6

A **Steiner system** $\mathrm{S}(t, k, v)$, $2 \leq t \leq k \leq v$, is a v-set V with a family $\mathcal{B}$ of k-subsets of V (blocks), such that every t-subset of V is contained in exactly one block, $t, k, v \in \mathbb{Z}^+$. [4]

A **Steiner Triple System**, STS(v), is a S(2,3,v). A STS(7) is shown in Figure 1.2 and it's blocks are listed in Table 1.1. A **Steiner Quadruple System**, SQS(v), is a S(3,4,v).

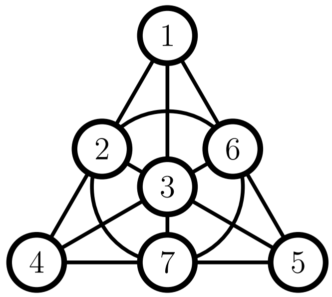

Figure 1.2: Fano Plane

B_1	B_2	B_3	B_4	B_5	B_6	B_7
1	2	3	4	5	6	7
2	3	4	5	6	7	1
4	5	6	7	1	2	3

Table 1.1: STS(7) constructed from the Fano Plane [13]

Fisher studied the applications of combinatorial designs to experimental design in his 1935 book "The Design of Experiments" [5]. In "The Design of Experiments" Fisher examines agricultural experiments where a scheme is used to plant wheat in fields and analyse the crop yield to improve farming techniques. We think of each field as a block with the space to grow k varieties of wheat. Today, designs are used for many diverse applications [2, 17, 4].

The designs Fisher implemented in his experiment were **balanced incomplete block designs (**(v, k, λ)**-BIBD)**, which is a set of blocks from a v-set X where each pair from X occurs in exactly λ blocks. An STS(v) is a $(v, 3, 1)$-BIBD.

Let $(G, +, 0)$ be a finite group of order v. For $D \subseteq G$ let $\partial D = \{d_1 - d_2 : d_1 \neq d_2 \in D\}$.

A (v, k, λ)**-difference family** is a collection $\{D_1, ..., D_t\}$ of k-subsets of G, $|G| = v$, where $\partial D_1 \cup ... \cup \partial D_t = \lambda G \backslash \{0\}$. A difference family with $t = 1$ is a $(v, k, \lambda)-$difference set.

Example 1.1. $D = \{1, 3, 4\}$ is a (7,3,1)-difference set in $\mathbb{Z}_7$.

Example 1.2. $\{\{0, 1, 4\}, \{0, 2, 8\}\}$ is a (13,3,1)-difference family over $\mathbb{Z}_{13}$.

Suppose an electronics company has seven versions of a component that it wants to test interacting over a network. They can only test three at a time. The STS(7) offers a solution that tests the interactions between all pairs with seven tests. Table 1.2 gives a set of tests derived this way.

	Test 1	Test 2	Test 3	Test 4	Test 5	Test 6	Test 7
Component 1:	1	1	1	2	2	3	3
Component 2:	2	4	6	4	5	4	5
Component 3:	3	5	7	6	7	7	6

Table 1.2: Using a STS(7) to test pairwise interactions. [13]

As the company increases the number of components they will need many more blocks to test them. For example, twelve tests are required if there are nine components, while twenty-six tests are required if there are thirteen components. These BIBD are shown in Table 1.3 and Table 1.4 respectively.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
1	4	7	1	2	3	1	2	3	1	2	3
2	5	8	4	5	6	5	6	4	6	4	5
3	6	9	7	8	9	9	7	8	8	9	7

Table 1.3: $(9, 3, 1)$-BIBD [13]

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}
1	2	3	4	5	6	7	8	9	10	11	12	13
3	4	5	6	7	8	9	10	11	12	13	1	2
9	10	11	12	13	1	2	3	4	5	6	7	8
B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}
2	3	4	5	6	7	8	9	10	11	12	13	1
6	7	8	9	10	11	12	13	1	2	3	4	5
5	6	7	8	9	10	11	12	13	1	2	3	4

Table 1.4: (13,3,1)-BIBD

Theorem 1.3. *If $\{D_1, ..., D_t\}$ is a (v, k, λ)-difference family in a group G then there exists a (v, k, λ)-BIBD.*

Proof. Let $\{D_1, ..., D_t\}$, be a (v, k, λ)-difference family in a group G. Let $B_{ij} = D_1 + j = \{d + j : d \in D_i\}$. We show that $(G, \{B_{ij} : 1 \leq i \leq t, j \in G\})$ is a (v, k, λ)-BIBD. Suppose $x, y \in G$ and let $d = x - y$. By definition of the (v, k, λ), every non-identity elements is covered by a difference of elements in some D_i exactly λ times. For $1 \leq l \leq \lambda$ let $d_{l_1}, d_{l_2} \in D_{i_l}$ with $d = d_{l_1} - d_{l_2}$. Then we have $x, y \in B_{i_l,(x-d_{l_1})}$. □

Let $(X, \mathcal{B})$ be a (v, k, λ)-BIBD and $0 \leq i \leq k$. The i-block intersection graph of $(X, \mathcal{B})$ is a graph G such that $\mathrm{V}(G) = \mathcal{B}$ and an edge $e \in E$ joins two vertices (blocks) if they have exactly i elements $x \in X$ in common. The block intersection graph is the union of i-blocks for $i > 0$[16].

The 1-intersection graph of the Fano Plane is given in Table 1.1. Each pair of block shares exactly one element in common so the 1-intersection graph of the Fano Plane is K_7.

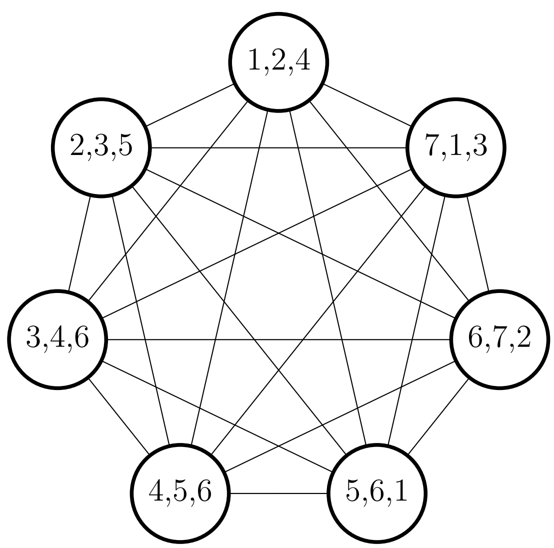

Figure 1.3: The block intersection graph of STS(7)

1.3 Literature Review

Single change covering designs differ from BIBDs and Steiner Systems. A **single change covering design** (SCCD(v, k, s, b)) $(X, \mathcal{L})$ is a v-set X and an ordered list of blocks $\mathcal{L} = (B_1, B_2, ..., B_b)$ of size k where every s-set, S, must occur in at least one block. Every block except the last differs from the next block by **removing** one element and **introducing** another. That is $|B_i \cap B_{i+1}| = (k-1)$ for all $1 \leq i < b$ and the point $x = B_i \backslash B_{i+1}$ is **removed** and $y = B_{i+1} \backslash B_i$ is **introduced** in B_{i+1}. If $|B_1 \cap B_b| = k-1$ then the SCCD is **circular**, denoted CSCCD. If the SCCD is circular one point is introduced in each block. If the SCCD is non-circular every point in B_1 is introduced and one point is introduced in each subsequent block. A s-set, S, is **covered** on block B_i if $S \subseteq B_i$ and one element of S was introduced in B_i.

An example SCCD(7,3,2,10) is found in Table 1.5. If no SCCD(v, k, s, b') exists for all $b' < b$ then a SCCD(v, k, s, b) is called **minimal**. We emphasize the introduced element in each block with an asterisk, *. Note that 1,2, and 3 are all introduced in the first block so the pairs $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are covered in B_1. In B_2 only 4 is introduced so only the pairs involving 4 are covered, namely $\{1, 4\}, \{2, 4\}$. Table 1.6 shows an example of a CSCCD(6,3,2,8) where the pair $\{1, 4\}$ is covered two times.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
1*	1	1	6*	7*	7	4*	4	4	1*
2*	2	2	2	2	3*	3	3	7*	7
3*	4*	5*	5	5	5	5	6*	6	6

Table 1.5: SCCD(7,3,2,10)

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
6*	6	6	6	4*	4	2*	2
4	3*	3	3	3	5*	5	4*
2	2	5*$_\wedge$	1*	1	1 $_\wedge$	1	1 $_\wedge$

Table 1.6: CSCCD(6,3,2,8). The pair {1,4} is covered in B_5 and B_8. [8]

We say that the set of elements that remain the same between B_i and B_{i+1} of a SCCD is the i^{th} **unchanged subset** between these blocks, $U_i = B_i \cap B_{i+1}$ [12]. If the SCCD is not circular, we also say that any $(k-1)$-subset of B_1 or B_b is an unchanged subset U_0 or U_b respectively. Consider the SCCD(7,3,2,10) in Table 1.5; the unchanged subsets are U_1=$\{1, 2\}$, U_2=$\{1, 2\}$, U_3=$\{2, 5\}$, U_4=$\{2, 5\}$, U_5=$\{7, 5\}$, U_6=$\{3, 5\}$, U_7=$\{4, 3\}$, $U_8 = \{4,$

6}, U_9={7, 6}. Additionally U_0 could be $\{1,2\}$, $\{1,3\}$, or $\{2,3\}$, and U_{10} could be $\{1,7\}$, $\{6,7\}$, or $\{1,6\}$.

If there exists $\{i_j, 1 \leq j \leq \frac{v}{k-1}\}$ such that

$$X = \dot{\bigcup}_{j=1}^{\frac{v}{k-1}} U_{i_j}$$

then we say that $\mathcal{E} = \{U_{i_1}, U_{i_2}, ..., U_{i_{\frac{v}{(k-1)}}}\}$ is an **expansion set**. If the SCCD is not circular and $\mathcal{E}$ contains u_0, u_b or both, then $\mathcal{E}$ is an **outer expansion set**. If $\mathcal{E}$ contains neither u_0 or u_b then we say $\mathcal{E}$ is an **inner expansion set**. We will denote expansion set locations in our tables with carets, $\wedge$. Consider Table 1.6, the carets between blocks 3 and 4, blocks 6 and 7, and after block 8 denote the expansion set where $U_{i_1} = U_3 = \{3,6\}$, $U_{i_2} = U_6 = \{1,5\}$, and $U_{i_3} = U_8 = \{2,4\}$. Now $X = U_{i_1} \dot{\cup}\, U_{i_2} \dot{\cup}\, U_{i_3}$ so this is a valid expansion set.

The earliest discussion of SCCDs was in 1968 and focused on applications to efficient computing [9]. Suppose we want to calculate AA^T for some matrix. We notice that the rows of A and columns of A^T are the same; so we must process each pair of rows in A. In the 1960's RAM (Random Access Memory) consisted of magnetic-core memory or magnetic-rope memory and held 4KB. This 4KB worth of memory cost \$320 in 1969 or about \$2,240 in 2021, and could be read at 1 MB/s. This meant we would not be able to store all the data we need to process in RAM through the whole calculation when the matrix is large. Thus, we supplement this fast memory with tape-drive memory. Tape-drive memory is much less expensive and held 46 MB per reel but was read at 15 KB/s. A naive program would move the relevant rows into the magnetic-core from the tape-drive and process the matrix by removing and adding rows in numerical order. However, reading information from the tape-drive to RAM would take a long time. Nelder sought to minimize reading from the tape-drive by methodically removing and adding the rows to RAM by using SCCD.

Gower and Preece continued the discussion in 1972 when they looked at successive incomplete blocks. The area was left alone after this until the 1990s when the group of Preece, Constable, Zhang, Yucas, Wallis, McSorley and Phillips re-started the discussion and a series of eleven papers substantially developed the theory. In 1992 [15] Yucas, Wallis and Zhang derived bounds on the number of blocks in a SCCD$(v, k, 2, b)$ and gave a construction for optimal SCCD with $k = 3$. In 1993 Rees [14] discussed upper bounds on SCCD$(v,k,2,b)$ for $k \geq 4$. In 1994 Zhang [18] introduced the idea of a missing pair (hole) from general design theory to single change covering designs. He also refined some new bounds on the number of blocks a SCCD can have. In 1995 [12] Preece, Constable, Zhang, Yucas, Wallis, McSorley, and Phillips essentially solved the existence of minimal SCCD with $k = 4$ using expansion sets. They also mentioned an extension to a double change covering design and gave a single example of a tight DCCD(9,3,2,12). In 1999 Phillips and Preece [11] found all 2554 SCCD(12,4,2,21) and show that there are 566 which are not minor variants of each other. Philips [10] closed this period in 2000 by finding two tight SCCD(20,5,2,46) using optimized code, a major breakthrough as the size of the search space grows exponentially as k increases.

During this period in 1998 McSorley also explored circular single change covering designs [8]. He completely solved minimal CSCCD$(v,k,2,b)$ where $k = 2, 3$. He gave some lower bounds on the number of blocks in a circular design and some families of circular designs with a fixed k. Analysing properties of elements in a design, he explored substructures that must be present in circular designs to significantly reduce the work required in a computer search which lead to him constructing all the minimal CSCCD(9,4,2,12) and CSCCD(10,4,2,15). He ended this paper by discussing the non-existence of some tight SCCD.

I re-opened research on SCCD in my undergraduate honours thesis by formalizing a theorem given in [12] and expanding its utility to CSCCD and SCCD with some pairs covered more then once. I proved that all minimal circular SCCD$(v,4,2,b)$ and around half the possible circular and non-circular SCCD$(v,5,2,b)$ exist.

Let $g_1(v,k,2) = \frac{\binom{v}{2}-\binom{k}{2}}{k-1} + 1$ and $g_2(v,k,2) = \frac{\binom{v}{2}}{k-1}$. Wallis et al showed.

Lemma 1.4. *[15] In a SCCD$(v,k,2,b)$, $b \geq g_1(v,k,2)$. In a circular SCCD$(v,k,2,b)$, $b \geq g_2(v,k,2)$.*

We say a (circular) SCCD$(v,k,2,b)$ is **economical** if it has $\lceil g_1(v,k,2) \rceil$ $\big(\lceil g_2(v,k,2) \rceil\big)$ blocks. In other words, it has the fewest number of blocks possible for this v and k. A (circular) **tight** if it is economical and $g_1(v,k,2)$ $\big(g_2(v,k,2)\big)$ is an integer. In a tight SCCD every pairs is covered exactly once. With some v,k combinations it will be impossible to create tight SCCD, in these cases the economical design is our best option. A SCCD of strength two is economical if and only if no more than $k-1$ pairs are repeatedly covered, counting multiplicities.

It is worth noting that SCCD$(v, k, 2, b)$ do require more blocks to build than S$(2, k, v)$. However, when used to test components, SCCD offer savings if exchanging components between tests is more expensive than conducting each test. Recall the earlier scenario where a company is testing 7 variations of a component. Suppose the company is under a deadline and swapping components between tests costs the company 20 minutes, but running a test only requires 1 minute. A company using a (7,3,1,7)-BIBD would need to swap components out 15 times and run 7 tests, costing them 307 minutes. A company using a SCCD(7,3,2,7) would need to change components 12 times and run 10 tests, costing them 250 minutes. Helping to make statistical computing more efficient motivated Nelder, Gower and Preece in their early investigations. SCCD can help make statistical computing computations more efficient [6].

Wallis, Yucas and Zhang proved

Lemma 1.5. *[15] There is an economical SCCD$(v, k, 2, b)$ whenever $k = v, v-1$, or v is odd and $k = 2$.*

When $v = k$, $b = 1$. When $k = v - 1$, $b = 3$. When $k = 2$ and v is odd, $b = \binom{v}{2}$. For $k = 3$, they give a recursive construction.

Lemma 1.6. *[15] If there is a SCCD$(v, 3, 2, b)$ then there is a SCCD$(v + 4, 3, 2, b')$, $b' = 2v + 3 + b$.*

Using Lemma 1.6 they built an economical SCCD for any $v \geq 3$.

Theorem 1.7. *[15] There is an economical SCCD$(v, 3, 2, b)$ for all $v \geq 3$.*

They prove another lower bound on b by utilising the number of required introductions.

Theorem 1.8. *[15] If SCCD$(v, k, 2, b)$ exists then $b \geq g_3(v, k, 2) = 2(v - k) + 1$. Furthermore, if $k \geq 4$ and $v = 3k - 3$ then $b \geq 4k - 4$ blocks.*

They proved a number of recursive constructions.

Theorem 1.9. *[15] Suppose there exists a SCCD$(v, k, 2, b)$. There exists a SCCD$(v', k, 2, b')$, $v' = v + 2k - 2$, $b' = b + 2v + 3k - 6$.*

Lemma 1.10. *[15] The number of blocks in a SCCD$(v, k, 2, b)$ is at least $b = \left\lceil \frac{\binom{v}{2} - \binom{k-1}{2}}{k-1} \right\rceil$*

Theorem 1.11. *[15] Suppose a* $SCCD(v-j,k,2,b)$ *exists and* $j \leq k-1$*. Then a* $SCCD(v,k,2,b+v-k+j)$ *exists.*

They consider restrictions on block size by small v, a corollary of Theorem 1.8.

Theorem 1.12. *[15] For a SCCD*$(v,k,2,b)$*, if* $2k \leq v \leq 3k-3$ *then* $b \leq 3v-4k+1$.

Corollary 1.13. *[15] Given a SCCD*$(v,k,2,b)$*, if* $v \leq 2k$ *then* $b = 2(v-k)+1$

In 1993 Rees uses the frequency and introduction limits to show that :

Theorem 1.14. *[14] If* $k \geq 4$ *and a SCCD*$(3k-2,k,2,b)$ *exists then* $b \geq 4k-1$.

In 1994 Zhang shows the following in parallel to a theorem of [15].

Lemma 1.15. *[18] In a SCCD*$(v,k,2,b)$*,* $b \geq \left\lceil \frac{v(v-1)-(k-1)(k-2)-2}{2(k-1)} \right\rceil$ *with equality if* $k = 2,3$.

He then shows an upper bound when $v = n(k-1)+i$ and $k+1 \leq i \leq 2k-1$, $n \geq 0$, and $k \geq 4$ for $d(v,k) = 2(i-k) + n(i+k-3) + \frac{n(n-1)(k-1)}{2}$.

A **single change covering design with a hole**, SCCDH$(v,k,2,v)$ is a SCCD$(v,k,2,b)$ such that every pair of elements of V occur on at least one block except for one pair, $\{x,y\}$, called a **hole** such that one of x and y appear in B_1 or B_b. We use $f'(v,k,2)$ to denote the smallest b for which there exists a SCCDH$(v,k,2,b)$.

Theorem 1.16. *[18] It holds that* $f'(v,k,2) \leq d(v,k)$.

Corollary 1.17. *[18]*

1. $f'(v,4,2) = \left\lceil \frac{v^2-v-8}{6} \right\rceil$ *for all* $v \geq 5$.
2. $\left\lceil \frac{v^2-v-6}{6} \right\rceil \leq g_1(v,4,2) \leq \left\lceil \frac{v^2-v-2}{6} \right\rceil$ *for all* $v \geq 5$*. In particular,* $g_1(v,4,2) = \frac{v^2-v-2}{6}$ *if* $v \equiv 2 \pmod 3$

Theorem 1.18. *[18]* $g_1(v,k,2) \geq 3v-4k+1$ *with equality if* $k \geq 4$ *and* $2k+1 \leq v \leq 3k-1$.

Corollary 1.19. *[18]* $g_1(v,k,2) \geq 2(v-k)+1$ *with equality if and only if* $v \leq 2k$ *and* $(v,k) \neq (4,2)$.

Theorem 1.20. *[18]* $g_1(mv,mk,2) \leq \lceil f(v,k)-1 \rceil m+1$ *for any positive integer* m.

Corollary 1.21. *[18] If* $g_1(v,k,2) = 3v-4k+1$ *then* $g_1(mv,mk,2) = 3mv-4mk+1$ *for every positive integer* m.

Theorem 1.22. *[18] It holds that* $g_1(v+2,k+1,2) \leq g_1(v,k,2)+2$.

Corollary 1.23. *[18]*

1. $g_1(v+2m,k+m,2) \leq g_1(v,k+2m,2)$ *for all* $m \geq 1$
2. *If* $g_1(v,k,2) = 3v-4k+1$ *then* $g_1(v+2m,k+m,2) = 3(v+2m)-4(k+m)+1$ *for every positive integer* m.

In 1995 Preece et al. further considered tight single change covering designs [12]. They found a recursive construction for tight single change covering designs utilizing expansion sets. The proof is easily extended to economical SCCD.

Proposition 1.24. *If a (economical)* $SCCD(v,k,2,b)$ *with an expansion set exists, then a (economical)* $SCCD(v+1,k,2,b+\frac{v}{k-1})$ *exists.*

Theorem 1.25. *For* $k \geq 4$ *if a tight sccd*(v,k) *with* b *blocks and an outer expansion set exists then a economical sccd*$(v+2,k)$ *with* $\left(b+2\frac{v}{k-1}+1\right)$ *blocks exists if* $v+2$ *is an admissible value for an economical design* $(v+2,k)$.

They state this recursive construction informally. I formalized this and gave an explicit proof in my undergraduate thesis [3].

Theorem 1.26. *If there exists a tight SCCD*$(v,k,2,b)$ *and a tight SCCD*$(n(k-1),k,2,b')$ *with an outer expansion set, then there exists a tight SCCD*$(v+(n-1)(k-1),k,2,b^*)$*, with* $b^* = b+b'+\frac{v(v-k+1)}{k-1}$. *Furthermore, if the SCCD*$(v,k,2,b)$ *has an expansion set then the SCCD*$(v+(n-1)(k-1),k,2,b^*)$ *has an expansion set.*

I generalized Theorem 1.26 to construct economical SCCD. We say that **a block** B_i **in a SCCD is tight** if the pairs it covers are not covered in any other block of the SCCD. tight blocks allow us to use one economic SCCD and one tight SCCD to construct larger economic designs using the following lemma.

Lemma 1.27. *[3] Let* $(X, \mathcal{L} = (B_1, ..., B_b))$ *be a* $SCCD(v, k, 2, b)$ *and let* $d = (k-1)b + \binom{k-1}{2} - \binom{v}{2}$. *Suppose that* $(X', \mathcal{L}' = (B_1, ..., B_b, B'_{b+1}, ..., B'_{b'}))$ *is a* $SCCD(v', k, 2, b')$ *and* $X \subseteq X'$. *If* B'_i *is tight* $\forall\ i,\ b+1 \leq i \leq b'$, *then* $(X', \mathcal{L}')$ *is economical and has* $d = (k-1)b' + \binom{k-1}{2} - \binom{v'}{2}$.

In Lemma 1.27 d is a measure of tightness. Some SCCD$(v, k, 2, b)$ require that up to $k-2$ pairs be covered more then once in order cover all pairs. Consider Table 1.7 and Table 1.8, they make up the economic SCCD(8,3,2,14) in Table 1.9

1^*	4^*	4
2^*	2	1^*
$_\wedge 3^*$	3	$3_\wedge$

Table 1.7: economical SCCD(4,3,2,3)

4^*	7^*	7	7	7	7	7	6^*	6	6	6
8^*	8	8	8	8	8	4^*	4	1^*	2^*	3^*
3^*	3	6^*	1^*	2^*	5^*	5	5	5	5	5

Table 1.8: Tight single change blocks

1^*	4^*	4	4	7^*	7	7	7	7	7	6^*	6	6	6
2^*	2	1^*	8^*	8	8	8	8	8	4^*	4	1^*	2^*	3^*
$_\wedge 3^*$	3	$3_\wedge$	3	3	$6^*_\wedge$	1^*	2^*	5^*	5	5	5	5	$5_\wedge$

Table 1.9: economical SCCD(8,3,2,14)

Theorem 1.28. *[3] If there exists an economical* $SCCD(v, k, 2, b)$ *and a tight* $SCCD(n(k-1), k, 2, b')$ *with an outer expansion set, then an economical* $SCCD(v+(n-1)(k-1), k, 2, b^*)$, $b^* = b + b' + \frac{v(v-k+1)}{k-1}$ *exists. Furthermore, if the economical* $SCCD(v, k, 2, b)$ *has an expansion set then the economical* $SCCD(v+(n-1)(k-1), k, 2, b^*)$ *will have an expansion set.*

In 2000 Phillips used exhaustive search with the following three optimization Lemmas to find a SCCD(20,5,2,46). Consider the tight SCCD(12,4,2,21) from [10] in Table 1.10 for the following. First Phillips defines B_i, $B_{i+1} \in \mathcal{L}$ to have the **minor variant property** if x is introduced in B_i and removed in B_{i+1} and y is introduced in B_{i+1} and removed in B_{i+2}. For $i = b - 1$, we consider block $B_{b+1} = \{\}$, so all $x \in B_b$ are removed in B_{b+1}. For example in Table 1.10, 1 is introduced in B_{13} and removed in B_{14} and 3 is introduced in B_{14} and removed in B_{15}. There is a second example in B_{20} and B_{21} with elements 4 and 9.

Lemma 1.29. *[10] If a SCCD($v, k, 2, b$), $(X, \mathcal{L})$, has a minor variant in B_i then swapping B_i and B_{i+1} produces a SCCD($v, k, 2, b$).*

Phillips takes advantage of Lemma 1.29 in an exhaustive search by noticing we only need find one of each pair of minor variant SCCD. His search only advances on a particular SCCD with a minor variant $x \in B_i$, $y \in B_{i+1}$ if $x < y$. The earlier a minor variant occurs in the design the more significant the computational savings.

In a SCCD($v, k, 2, b$), $(X, \mathcal{L})$, let $Z \subseteq X$ and $x, y \in X \backslash Z$. If the pairs $\{z, x\}, \{z, y\}$ $\forall z \in Z$ are covered in $B_1, \dots, B_i$) we say that B_i has the **end permutation property**. Block B_8 with $x = 4$, $y = 5$ in Table 1.10, has the end permutation property. We can swap the points 4 and 5 in all blocks $B_8, ..., B_{21}$ and produce another SCCD.

Lemma 1.30. *[10] If B_i in a SCCD($v, k, 2, b$), $(X, \mathcal{L})$, has the end permutation property with elements x, y then we may swap (x, y) in $B_i, ..., B_b$ to produce a SCCD($v, k, 2, b$).*

Once again Phillips optimizes his search by checking for end permutations and only continuing the search if $x < y$ is introduced first in $B_i, ..., B_b$.

Finally, he considers the implications of removing an element from a block. When an element is removed, if it still needs to be reintroduced to cover a pair, the next introduction will cover $k - 1$ pairs. If there are at least one and fewer than $k - 1$ pairs left to cover with point x we know that x may not be removed as doing so will never produce a tight SCCD. For example, in B_{16} of Table 1.10 we cannot remove 7 as we have yet to cover $\{7, 5\}$, however this is the only pair involving 7 yet to be covered, so 7 must be in B_{16}. We call this checking the **removal conditions** of an element.

Lemma 1.31. *[10] In a tight SCCD($v, k, 2, b$), $(X, \mathcal{L})$ if $x \in B_i$ and at least $v - k + 1$ and less than $v - 1$ pairs involving x are covered in $B_1, ..., B_i$, x is not removed from B_i.*

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}
1^*	.	.	.	.	.	3^*	4^*	.	.	2^*	.	1^*	3^*	7^*	.	6^*	.	.	.	.
2^*	.	.	.	.	9^*	.	.	.	.	.	12^*	.	.	.	5^*	.	.	.	.	.
3^*	.	.	7^*	.	.	.	.	10^*	.	.	.	.	.	.	.	.	8^*	.	4^*	9^*
4^*	5^*	6^*	.	8^*	.	.	.	.	11^*	.	.	.	.	.	.	.	.	12^*	.	.

Table 1.10: SCCD(12,4,2,21) [10]

With these improvements to his search and using multiple CPUs, Phillips found two SCCD$(20, 5, 2, 26)$.

Using these SCCD$(20, 5, 2, 46)$, I was able to build a significant portion of tight circular and non-circular SCCD$(v, 5, 2, b)$ as seen in Theorem 1.40. We will discuss this methodology further in Section 2.1 when we discuss how we incorporated this in our recursive search.

In 1999 McSorley investigated circular SCCD. He proved the existence for all economic CSCCD$(v, k, 2, b)$ for $k = 2, 3$.

Theorem 1.32. *An economic CSCCD$(v, k, 2, b)$ exists for all v and $k = 2, 3$. These designs are tight if $k = 2$ or if $k = 3$ and $v \equiv 0$ or $1 \pmod 4$.*

He also finds some restrictions on the minimum number of blocks and constructions of families of CSCCD for small v. Let $B_0 = \{0, ..., k-1\} \in \mathbb{Z}_n$, $B_i = B_0 + i$ and $B_i' = B_i \cup \{\infty\}$.

Theorem 1.33. *[8]*

1. *For $k \geq 3$ and $k \leq v \leq 2k - 2$ the blocks $\mathcal{B} = (B_1', B_2', ..., B_b')$ form an economical circular $CSCCD(v, k, 2, b)$ over $\mathbb{Z}_{v-1} \cup \infty$. When $v = 2k - 2$ this CSCCD is tight.*

2. *For $k \geq 2$ $(\mathbb{Z}_{2k-1}, (B_i))$ is a tight circular $SCCD(2k - 1, k, 2, 2k - 1)$.*

3. *For $k \geq 3$ and $v = 2k$ let $C = B_{k-1} \backslash b_k \cup \{\infty\}$, $C' = B_{2k-2} \backslash 0 \cup \{\infty\}$, and $C'' = C' \backslash (2k - 1) \cup \{0\}$ the blocks $\mathcal{B} = (B_0, ..., B_{k-1}, C, B_k, ..., B_{2k-2}, C', C'')$ form an economical circular $SCCD(2k, k, 2, 2k + 2)$. If $k = 2$ then this design is tight.*

Using the concept of expansion sets with circular SCCD$(v, k, 2, b)$ he was able to construct circular SCCD with exactly one extra element.

McSorley used the obvious extension of Proposition 1.24 to circular SCCD, specifically when moving from CSCCD(9,4,2,11) to CSCCD(10,4,2,14). This holds true, for all economical circular SCCD.

Proposition 1.34. *If an economical CSCCD*$(v, k, 2, b)$ *with an expansion set exists then a circular economical* $CSCCD(v+1, k, 2, b+\frac{v}{k-1})$ *exists. If the CSCCD*$(v, k, 2, b)$ *is tight then so is the* $CSCCD(v+1, k, 2, b+\frac{v}{k-1})$.

McSorley considered circular single change covering designs and developed SF-arrays and skeletons which significantly reduced the work in algorithmic searches for circular SCCD[8]. Let $(X, \mathcal{L})$ be a tight CSCCD$(v, k, 2, b)$. We denote the set of elements introduced i times in $\mathcal{L}$ by T_i and t_i is the cardinality of T_i. We denote the number of blocks containing the element x by f_x.

Note that for $x \in T_0$, x is never introduced but pairs containing x must be covered, so x must be in every block. If $x, y \in T_0$ then $\{x, y\}$ is never covered therefore $t_0 = 0$ or 1. McSorley showed that tight CSCCD$(v, k, 2, b)$ with $t_0 = 1$ must have $t_1 = v - 1$ and consequently $v = 2k - 2, b = 2k - 3$.

In a tight CSCCD we know that each time $x \in T_i$ is introduced we will cover $k - 1$ pairs that contain x. As there are $v - 1$ pairs in X that contain x we have for non-empty T_i that $i \le M = \left\lfloor \frac{v-1}{k-1} \right\rfloor$. We say M is the **maximum introduction limit**. McSorley proved the following.

Theorem 1.35. *In a tight CSCCD*$(v, k, 2, b)$*[8]*

1. $t_i = 0 \ \forall i > M$
2. $\sum_{i=1}^{A} t_i = v$
3. $\sum_{i=1}^{A} it_i = b$
4. $f_{\{x\}} = f_i = (v-1) - i(k-2) \ \forall x \in T_i$
5. *If* $f_i = i$ *then* $t_i = 0$ *or* 1
6. *If* $f_i = i + 1$ *then* $t_i = 0, 1,$ *or* 2.

Note that when we use a set notation in the subscript of $f_{\{x\}}$, we are talking about the frequency of element x. However if we use a number in the subscript of f_i, we are talking about the frequency of all elements introduced i times.

McSorley analyzed the introductions and removals of elements in T_1. For example consider the tight CSCCD(8,3,2,14) in Table 1.11. We see that $T_1 = \{0, 1\}$, $T_2 = \{2, 3, 4, 5, 6, 7\}$ and observe 0 and 1 are both in T_1 and the pair $\{0,1\}$ must be covered. Therefor 1 is introduced before 0 removed or visa versa.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}
0	0	0	5^*	6^*	6	6	4^*	4	5^*	5	5	5	2^*
1^*	1	1	1	1	1	7^*	7	7	7	6^*	6	3^*	3
2	7^*	4^*	4	4	3^*	3	3	2^*	2	2	0^*	0	0

Table 1.11: tight CSCCD(8,3,2,14)

For any given $x \in T_1$ we say that x **starts**, S, (is introduced), in block $B_{x,1}$ and **finishes**, F, in block B_{x,f_1}. We call the successive blocks $B_{x,i}$ containing x the **run** of x. We also know that in general for every pair to be covered, each other element in T_1 must either finish or start but not both during the run of x. For example, in Table 1.11, 0 is introduced in B_{12} and is removed in B_4 after 1 is introduced in B_1.

The **SF-array** of x is a list, A, of length f_1 with entries "-", "S", "F", or "SF" which describes how all elements in T_1 must interact in relation to a single element $x \in T_1$. If $y \in T_1$ is introduced in block $B_{x,i+j}$, A_{i+j} contains a "S". If $y \in T_1$ is present in $B_{x,i+j}$ but not $B_{x,i+j+1}$, A_{i+j} contains an "F". For the tight CSCCD(8,3,2,14) $f_1 = 6$ and in Table 1.11 a SF-arrays based on the element 0 and 1 are

	SF-array	wt(i)		SF-array	wt(i)
$R_{0,1}$	S	1	$R_{1,1}$	S	2
$R_{0,2}$	-	1	$R_{1,2}$	-	2
$R_{0,3}$	-	1	$R_{1,3}$	F	2
$R_{0,4}$	S	2	$R_{1,4}$	-	1
$R_{0,5}$	-	2	$R_{1,5}$	-	1
$R_{0,6}$	F	2	$R_{1,6}$	F	1
	$x = 0$			$x = 1$	

Table 1.12: SF-arrays for tight CSCCD(8,3,2,14)

We will write SS to indicate the event that B_i and B_{i+1}, $i < f_1$, both contain an S and similarly for FF. Let n_S be the number of "S" and n_F be the number of "F" in A. Let n_{SS} be the number of "SS" events and n_{FF} be the number of "FF" events. For $1 \leq i \leq f_1$, let the **weight of** i, wt(i) equal the number of "S" up to position j in A_i plus the number of "F" at or after $i+1$ less 1. Theorem 1.36 will explain the criteria for constructing a SF-array.

Theorem 1.36. *[8] Let A be a SF-array in a tight CSCCD($v,k,2,b$).*

1. *For any $1 \leq i \leq f_1$, A_i contains at most one "S" and at most one "F".*
2. *Let A_i contain "F" and the next "S" appears in A_j then $j \geq i+b-2f_1+1$.*
3. *Any three consecutive positions, A_i, A_{i+1}, A_{i+2}, may not all contain an S. Any three consecutive positions may not all contain an F.*
4. *There is an S in A_1, F in A_{f_1} and $n_S + n_F = t_1 + 1$.*
5. *For each $1 \leq i \leq f_1$, $wt(i) \leq k$.*
6. *$1 \leq t_1 \leq 2k-1$.*
7. *For any $1 \leq i \leq f_1 - 1$, if $wt(i) = k$ then $wt(i+1) \neq k$.*
8. *$n_{SS} \leq \left\lfloor \frac{k}{2} \right\rfloor$, $n_{FF} \leq \left\lfloor \frac{k}{2} \right\rfloor$ and $n_{SS} + n_{FF} \leq k-1$.*
9. *$t_1 + 1 - min\{t_1, k\} \leq n_S$, $n_F \leq min\{t_1, k\}$.*

Let $\mathcal{A}$ be all the possible SF-arrays for a tight SCCD($v,k,2,b$). An array A' of length b with t "S" and "F" is a **SF-skeleton** if each $S \in A'$ is the start of an SF-array $A \in \mathcal{A}$.

SF-skeletons allow us to use necessary conditions imposed by element introductions and exits to significantly reduce work while constructing tight circular SCCD. McSorley uses SF-skeletons to enumerate all possible tight CSCCD(9,4,2,12) and tight CSCCD(10,4,2,15).

For example, consider all eight possible SF-arrays for the tight circular SCCD($9,4,2,12$) given in Table 1.13. Each column is a SF-array based on the element $x \in T_1$.

$R_{x,1}$	xSF	xSF	xSF	xS	xSF	xSF	xSF	xS
$R_{x,2}$	xF	x-	x-	xSF	x-	x-	xF	xSF
$R_{x,3}$	x-	xSF	xS	x-	xSF	xSF	x-	xF
$R_{x,4}$	xSF	x-	xSF	xSF	x-	xF	xS	x-
$R_{x,5}$	x-	xS	x-	x-	xSF	x-	xSF	xS
$R_{x,6}$	xSF	xSF	xSF	xSF	xF	xSF	xF	xSF
	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)

Table 1.13: SF-arrays for the tight CSCCD(9,4,2,12)

For each SF-array we may turn this into a skeleton by appending empty rows to the array until we have the same number of rows as blocks in the design. Then for each S in the SF-array we place an F $f_1 - 1$ rows later, in this case $f_1 = 6$. Similarly, for each F in the SF-array we place an S $f_1 - 1$ rows earlier. Lets do this for column (a) from Table 1.13 in Table 1.14. Recall that the rows are circular, so if we reach the end we wrap around to continue.

$R_{x,1}$	xSF		R_1	SF	(a)
$R_{x,2}$	xF		R_2	F	
$R_{x,3}$	x-		R_3	-	
$R_{x,4}$	xSF	$\rightarrow$	R_4	SF	(b)
$R_{x,5}$	x-		R_5	-	
$R_{x,6}$	xSF		R_6	SF	(c)
			R_7	-	
			R_8	S	(d)
			R_9	SF	(e)
			R_{10}	-	
			R_{11}	SF	(f)
			R_{12}	-	

Table 1.14: skeleton for the SF-array (a) of a tight CSCCD(9,4,2,12)

Now for each S in the array we need to ensure that it is the beginning of a valid SF-array, if it is not we must discard the array as it will not produce a tight CSCCD. In our example, the first S corresponds to array (a), the second to array (b), and so on. As each S corresponds to the start of a valid we may use this skeleton to generate a partially completed design, found in Table 1.15, where the blocks are row instead of columns.

R_1	SF	1*	4	6	5
R_2	F	1		6	5
R_3	-	1		6	
R_4	SF	1	2*	6	
R_5	-	1	2		
R_6	SF	1	2	3*	
R_7	-		2	3	
R_8	S	4*	2	3	
R_9	SF	4	2	3	5*
R_{10}	-	4		3	5
R_{11}	SF	4	6*	3	5
R_{12}	-	4	6		5

Table 1.15: Partially filled in tight CSCCD(9,4,2,12)

From here we can complete the design much more efficiently. See Table for a completed tight CSCCD(9,4,2,12).

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
1*	1	1	1	1	1	7*	4*	4	4	4	4
4	7*	7	2*	2	2	2	2	2	8*	6*	6
6	6	6	6	9*	3*	3	3	3	3	3	9*
5	5	8*	8	8	9	9	7	5*	5	5	5

Table 1.16: Completed tight CSCCD(9,4,2,12)

We asked whether we could prove an analog of Proposition 1.24 for CSCCD. In addition to outer expansion sets we needed an additional property that allows us to avoid duplicate coverage. Suppose a SCCD$(v,k,2,b)$ has an outer expansion set that uses both U_0 and U_b. If $U_0 = U_1 = U_2 = \dots = U_{k-1}$ and $U_b = \bigcup_{i=1}^{k-1}(B_i \setminus B_{i+1})$, then the expansion set is **disjoint-capable**.

For example, consider a tight SCCD(10,3,2,22) in Table 1.17. Here we see that $U_0 = U_1 + U_2 = \{a,b\}$, and $U_b = \{c,d\} = B_1 \setminus B_2 \cup B_2 \setminus B_3$.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}
a*	a	a	a	a	4*	5*	6*	6	6	6
b*	b	b	2*	3*	3	3	3	c*	d*	d
$\wedge$c*	d*	1*	1	1	1	1	1 $\wedge$	1	1	2*

B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}	B_{19}	B_{20}	B_{21}	B_{22}
6	6	6	6	4*	4	4	4	3*	3	3
d	d	b*	a*	a	c*	2*	2	2	2	d*
4*	5*	5	5	5 $\wedge$	5	5	b*	b $\wedge$	c*	c $\wedge$

Table 1.17: Tight SCCD(10,3,2,22) [12] with a disjoint-capable expansion set.

To state our main theorem, we need a Lemma.

Lemma 1.37. *Let $(X, \mathcal{L} = (B_1, ..., B_b))$ be an economical $SCCD(v, k, 2, b)$. Suppose that $(X', \mathcal{M}' = (B_1, ..., B_b, B'_{b+1}, ..., B'_{b'}))$ is a circular $SCCD(v', k, 2, b')$ and $X \subseteq X'$. If B'_i is tight $\forall i$, $1 \leq i \leq b'$, and a tight SCCD$(v, k, 2, b)$ exists in $B'_{b+1}, ..., B'_{b'}$ then $(X', \mathcal{M}')$ is economical.*

With these tools we may now state a major theorem from my undergraduate honors thesis.

Theorem 1.38. *[3] If there exists a tight $SCCD(v, k, 2, b)$ with a disjoint-capable expansion set and a tight (economical) $SCCD(v', k, 2, b')$, then a tight (economical) circular $SCCD(v + v' - 2(k - 1), k, 2, b^*)$, $b^* = b + b' - (k - 1) + (\frac{v}{k-1})(v' - 2(k - 1))$, exists. Furthermore, if the $SCCD(v', k, 2, b')$ has an outer expansion set using both u'_0 and $u'_{b'}$ then the the circular $SCCD(v + v' - 2(k - 1), k, 2, b^*)$ has an expansion set.*

Proof. Suppose that $(X, \mathcal{L})$ is a *tight* $SCCD(v, k, 2, b)$ with a disjoint capable outer expansion set $\mathcal{E} = \{u_{i_j} : 1 \leq j \leq \frac{v'}{k-1}\}$ and $(X', \mathcal{L}')$ is an economic $SCCD(v', k, 2, b')$ with X' relabeled so $X' \cap X = U_0 \cup U_b$ and $B_1 \supseteq U_b$, $B_{b'} \supseteq U_0$. To build $(X^*, \mathcal{M}^*)$, the tight circular $SCCD(v + v' - 2(k - 1), k, 2, b^*)$, we delete the first $k - 1$ blocks from $\mathcal{L}$. Next, append the blocks $\mathcal{L}'$ to $\mathcal{L}$. Note, $\mathcal{E} \backslash \{U_0 \cup U_b\}$ partitions $X \backslash X'$. For all $x \in X' \backslash X$ and $2 \leq j \leq \frac{v'}{k-1} - 2$, we construct $B''_{i_j,x} = (B_{i_j} \cap B_{i_j+1}) \cup \{x\}$ and insert $B''_{i_j,x}$ between B_i and B_{i+1} in any order.

The pairs in $\{x, y\}$ from $X \backslash X'$ are covered only in $\mathcal{L}$ so are covered in $\mathcal{M}^*$. Similarly, the pairs $\{x, y\}$ from X' are covered in $\mathcal{L}'$ inside $\mathcal{M}^*$. Since

$(X', \mathcal{L}')$ is economical there are at most $k-1$ pairs covered more than once. $B''_{i_j,x}$ covers $\{x, y\}$ for all $x \in X' \backslash X$ and $y \in X \backslash X'$ and no other $B''_{i_j,x}$ covers this pair. Moreover, $U_k = U_0 \subset B'_{b'}$ so there is a single change between the last block of the design and the first block. Therefore $(X^*, \mathcal{M}^*)$ is a circular *economic SCCD*.

Suppose further that $(X', \mathcal{L}')$ has expansion set $\mathcal{E}' = \{U'_{i_j} : 1 \leq j \leq \frac{v'}{k-1}\}$. Then the economic CSCCD, $(X^*, \mathcal{M}^*)$, will have an expansion set

$$\mathcal{E}^* = \{(\mathcal{E} \cup \mathcal{E}') \backslash \{U'_{i_0}, U'_{i_{b'}}\}\}.$$

For the economical version of the proof, we use the same construction and apply Lemma 1.37. □

B_1	B_2	B_3	B_4	B_5	B_6	B_7
7*	7	a*	b*	b	b	b
d	d	d	d	c*	c	7*
c	8*	8	8	8	a*	a

Table 1.18: Tight SCCD(6,3,2,7)

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
a	a	a	4*	5*	6*	6	6	6	6	6	6	6	6	6	4*	4	4
b	2*	3*	3	3	3	7*	8*	c*	d*	d	d	d	b*	a*	a	7*	8*
1*	1	1	1	1	1	1	1	1	1	2*	4*	5*	5	5	5	5	5

B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}	B_{27}	B_{28}	B_{29}	B_{30}	B_{31}	B_{32}	B_{33}
4	4	4	3*	3	3	3	3	7*	7	a*	b*	b	b	b
c*	2*	2	2	2	2	2	*d**	d	d	d	d	c*	c	7*
5	5	b*	b	7*	8*	c*	c	c	8*	8	8	8	a*	a

Table 1.19: Tight CSCCD(12,3,2,33)

One might think that requiring a disjoint-capable outer expansion set is limiting for building circular SCCD. However, it is possible to build a design with this property as long as you have an outer expansion set that uses both U_0 and U_b.

Theorem 1.39. *[3] Let $(X,\mathcal{L})$ be a tight $SCCD(v,k,2,b)$ with an outer expansion set that uses both U_0 and U_b. Then there exists a tight $SCCD(2v-k+1,k,2,2b+\frac{v}{k-1}(v-k+1))$ with a disjoint-capable expansion set.*

We can use the above lemmas, propositions, theorems, known SCCD, and SCCD to prove the following.

Theorem 1.40. *The following single-change covering designs exist.*

1. *There exists a tight $SCCD(v,2,2,b)$ for all v. [9]*
2. *An economical $SCCD(v,3,2,b)$ exists for all $v \geq 6$. These are tight if $v \equiv 2,3 \pmod 4$.[15]*
3. *An economical $SCCD(v,4,2,b)$ exists for all $v \geq 12$. [11] These are tight if $v \equiv 0,1 \pmod 3$. [12]*
4. *An economical $SCCD(v,5,2,b)$ exists for all $v \equiv 4,5,6 \pmod{16}$, $v \geq 20$. These are tight if $v \equiv 4,5 \pmod 6$[3]*
5. *A tight $CSCCD(v,2,2,b)$ exists for all $v \geq 3$. [8]*
6. *An economical CSCCD$(v,3,2,b)$ exists for all $v \geq 4$. These are tight if and only if $v \equiv 0,1 \pmod 4$. [8]*
7. *An economical CSCCD$(v,4,2,b)$ exists for all $v \geq 27$ [3]. These are tight if and only if $v \equiv 0,1 \pmod 3$.*
8. *An economical $CSCCD(v,5,2,b)$ exists for all $v \equiv 0,1,2 \pmod{16}$, $v \geq 480$. These are tight if $v \equiv 0,1 \pmod{16}$. [3]*

Proof. 1. (1) Nelder gives the construction. [9]

2. A tight SCCD(6,3,2,7) (Table 1.18) exists. An economic SCCD(8,3,2,14) exists. Using a tight SCCD(6,3,2,7) and Theorem 1.24 we form a tight SCCD(7,3,2,10). Using economic SCCD(8,3,2,14) and Theorem 1.24 we form an economic SCCD$(9,3,2,18)$. By Lemma 1.6 and tight SCCD(6, 3, 2, 7), tight SCCD$(7,3,2,10)$, economical SCCD$(8,3,2,14)$, economical SCCD$(9,3,2,18)$ we build tight SCCD$(10,3,2,24)$, tight SCCD$(11,3,2,27)$, economic SCCD$(12,3,2,33)$, and economical SCCD $(13,3,2,39)$ respectively. Suppose we form each tight SCCD$(6{+}4i,3,2,$ $7+\sum_{j=0}^{i}(2(4j+6)+3))$, tight SCCD$(7+4i,3,2,10+\sum_{j=0}^{i}(2(4j+$

$7)+3))$, economic SCCD$(8+4i,3,2,14+\sum_{j=0}^{i}(2(4j+8)+3))$, and economic SCCD$(9+4i,3,2,18+\sum_{j=0}^{i}(2(4j+9)+3))$ this way and this holds for each $i=n\geq 1$, when $i=n+1$ we have

$$\begin{aligned}
7+\sum_{j=0}^{n+1}(2(4j+6)+3) =& 7+\sum_{j=0}^{n}(2(4j+6)+3)\\
&+2(4(n+1)+6)+3\\
10+\sum_{j=0}^{n+1}(2(4j+7)+3) =& 10+\sum_{j=0}^{n}(2(4j+7)+3)\\
&+2(4(n+1)+7)+3\\
14+\sum_{j=0}^{n+1}(2(4j+8)+3) =& 14+\sum_{j=0}^{n}(2(4j+8)+3)\\
&+2(4(n+1)+8)+3\\
18+\sum_{j=0}^{n+1}(2(4j+9)+3) =& 18+\sum_{j=0}^{n}(2(4j+9)+3)\\
&+2(4(n+1)+9)+3
\end{aligned}$$

3. Similarly to (2), there exists a tight SCCD(12,4,2,21), a tight SCCD(15,4,2,34), and a tight SCCD(18,4,2,50) with expansion sets. Using the tight SCCD(12,4,2,21) and a tight SCCD(12,4,2,21), a tight SCCD(15,4,2,34), or a tight SCCD(18,4,2,50) respectively with Theorem 1.26 we may construct the tight SCCD(21,4,2,69), the tight SCCD(24,4,2,91), and the tight SCCD(27,4,2,116). We ultimately obtain all the tight SCCD$(v,4,2,b)$ with $v\equiv 0,3,6 \pmod 9$. So we have a construction for all the tight SCCD$(v,4,2,b)$ where $v\equiv 0 \pmod 3$. Using these designs which all have expansion sets and Theorem 1.24 or Theorem 1.25 we obtain the tight SCCD$(v,4,2,b)$ with $v\equiv 1 \pmod 3$ and $v\equiv 2 \pmod 3$ respectively, as desired.

4. Similarly to (2) and (3), a tight SCCD(20,5,2,46) exists with an expansion set. We recursively apply the results of Theorem 1.26 and a tight SCCD(20,5,2,46) to find all tight SCCD$(v,5,2,b)$ with $v\equiv 4 \pmod{16}$. From here we apply Proposition 1.24 to obtain the tight

SCCD$(v,5,2,b)$ with $v \equiv 5 \pmod{16}$. We use the tight SCCD$(v,5,2,b)$ with $v \equiv 4 \pmod{16}$ obtained in (4) and Theorem 1.25 to build the economic SCCD$(v,5,2,b)$ with $v \equiv 6 \pmod{16}$.

5. This follows from McSorley's Theorem 1.32.

6. This follows from McSorley's Theorem 1.32.

7. If a tight circular SCCD$(v,4,2,b)$ exists then $b = \frac{v(v-1)}{6} \in \mathbb{N}$, so $v \equiv 0,1 \pmod 3$. From (3) for all $v \geq 12, v \equiv 0,1 \pmod 3$ there exits a SCCD$(v,4,2,b)$. Using a tight SCCD$(21,4,2,69)$ with a disjoint-capable expansion set and Theorem 1.38 we can construct a circular tight SCCD$(v+15,4,2,b)$ therefore we can construct a circular tight SCCD$(v,4,2,b)$ for every $v \geq 27, v \equiv 0,1 \pmod 3$. From (3) we have that for all $v \geq 14, v \equiv 2 \pmod 3$ there exits an economic SCCD$(v,4,2,b)$. Using the tight SCCD$(21,4,2,69)$ with a disjoint-capable expansion set and Theorem 1.38 we can construct an economical circular SCCD$(v+15,4,2,b)$ therefore we can construct a tight circular SCCD$(v,4,2,b)$ for every $v \geq 29, v \equiv 2 \pmod 3$.

8. Similarly, we use two tight SCCD(20,5,2,46) to build a tight SCCD(36,5,2,156) with a disjoint capable outer expansion set and apply the same Theorems and Propositions to construct the tight CSCCD$(v,5,2,b)$ for all $v \equiv 0,1 \pmod{16}$. Similarly, we use the tight SCCD(36,5,2,156) from (9) and the economical SCCD$(v,5,2,b)$ from (5) to construct economical CSCCD$(v,5,2,b)$for all $v \equiv 2 \pmod{16}$, $v \geq 50$

□

Chapter 2

Single Change Covering Designs

We will begin by considering a new construction.

Theorem 2.1. *Let* $(X, \mathcal{L})$ *be a SCCD*$(v, 3, 2, b)$ *containing three consecutive blocks of the form*

B_i	B_{i+1}	B_{i+2}
2	3^*	3
1	1	4^*
0	0	0

Table 2.1: Consecutive blocks to replace

Then there exists a SCCD$(v+4, 3, 2, b+2v+3)$*,* $(X', \mathcal{L}')$*, containing three consecutive blocks of the form given in Table* 2.1*. Further, if* $(X, \mathcal{L})$ *is circular, economical, tight or contains an expansion set then so does* $(X', \mathcal{L}')$*.*

Proof. Let $(X, \mathcal{L})$ be a SCCD$(v, 3, 2, b)$ containing B_i, B_{i+1}, and B_{i+2} as proposed. Let $\{a, b, c, d\}$ be disjoint from X and $X' = X \cup \{a, b, c, d\}$. To construct $\mathcal{L}'$ we first replace blocks B_i, B_{i+1} and B_{i+2} from $\mathcal{L}$ with the following fourteen blocks, $\mathcal{N}$.

N_1	N_2	N_3	N_4	N_5	N_6	N_7	N_8	N_9	N_{10}	N_{11}	N_{12}	N_{13}	N_{14}
2	d^*	a^*	a	a	3^*	3	3	2^*	2	2	0^*	0	0
1	1	1	1	1	1	d^*	d	d	d	c^*	c	3^*	3
0	0	0	b^*	c^*	c	c	$a^*_\wedge$	a	b^*	$b_\wedge$	b	b	4^*

Table 2.2: $\mathcal{N}$

Note that $B_i = N_1$ and $B_{i+2} = N_{14}$ and the sequence $\mathcal{N}$ is single change so the new sequence of blocks remain a single change ordering. The sequence $\mathcal{N}$ covers exactly the same pairs from $\{0, 1, 2, 3\}$ as B_i, B_{i+1} and B_{i+2} did. Further, $\mathcal{N}$ covers all the pairs in $\{a, b, c, d\}$ as well as all the pairs between $\{0, 1, 2, 3\}$ and $\{a, b, c, d\}$. Thus, the only pairs not covered yet are those between $\{a, b, c, d\}$ and $X\backslash\{0, 1, 2, 3\}$. By using the expansion set on the points {a,b,c,d} in $\mathcal{N}$ at expansion locations u_8 and u_{11} we insert the blocks $\{a, d\}\cup\{x\}$ and $\{b, c\}\cup\{x\}$ for $x \in X\backslash\{0, 1, 2, 3\}$ respectively. Thus, $(X', \mathcal{L}')$ is a SCCD$(v+4, 3, 2, b+2v+3)$. Moreover, the blocks colored blue in Table 2.2 have the form given in Table 2.1 and no blocks are inserted between these, so $(X, \mathcal{L})$ has the form specified by Table 2.1.

As the first and last blocks of $(X', \mathcal{L}')$ remain unchanged from $(X, \mathcal{L})$ the design remains circular if it began as circular. As

$$\frac{\binom{v+4}{2} - \binom{v}{2}}{k-1} = \frac{\binom{v+4}{2} - \binom{v}{2}}{3-1} = 2v + 3,$$

if $(X, \mathcal{L})$ is tight or economical then $(X', \mathcal{L}')$ is tight or economical respectively. Finally, we note that the set of unchanged subsets of $(X', \mathcal{L}')$ is the union of the set of unchanged subsets from $(X, \mathcal{L})$ and the set of unchanged subsets from $\mathcal{N}$, which contains $\{a, d\}$ and $\{b, c\}$. Therefore, if $(X, \mathcal{L})$ contained an expansion set, then so does $(X', \mathcal{L}')$. $\square$

For example we consider the tight SCCD(6,3,2,7) in Table 2.3. Blocks B_3, B_4, B_5 have the required form. Table 2.4 shows the blocks after B_3, B_4, B_5 are replaced. Table 2.5 shows the tight SCCD(10,3,2,22).

B1	B2	B3	B4	B5	B6	B7
5^*	5	2^*	3^*	3	3	3
1^*	1	1	1	4^*	4	5^*
$_\wedge 4^*$	$0^*{}_\wedge$	0	0	0	$2^*{}_\wedge$	2

Table 2.3: Tight SCCD$(6, 3, 2, 7)$

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10
5^*	5	2^*	d^*	a^*	a	a	3^*	3	3
1^*	1	1	1	1	1	1	1	d^*	d
$_\wedge 4^*$	$0^*{}_\wedge$	0	0	0	b^*	c^*	c	c	$a^*_\wedge$
B11	B12	B13	B14	B15	B16	B17	B18		
2^*	2	2	0^*	0	0	2^*	2		
d	d	c^*	c	3^*	3	3	3		
a	b^*	$b_\wedge$	b	b	4^*	$4_\wedge$	5^*		

Table 2.4: The extended tight SCCD$(6, 3, 2, 7)$ with the 14 inserted blocks

B1	B2	B3	B4	B5	B6	B7	B8	B9	B10	B11
5^*	5	2^*	d^*	a^*	a	a	3^*	3	3	4^*
1^*	1	1	1	1	1	1	1	d^*	d	d
$_\wedge 4^*$	$0^*{}_\wedge$	0	0	0	b^*	c^*	c	c	$a^*_\wedge$	a
B12	B13	B14	B15	B16	B17	B18	B19	B20	B21	B22
5^*	2^*	2	2	4^*	5^*	0^*	0	0	2^*	2
d	d	d	c^*	c	c	c	3^*	3	3	3
a	a	b^*	$b_\wedge$	b	b	b	b	4^*	$4_\wedge$	5^*

Table 2.5: The completed tight SCCD$(10, 3, 2, 22)$ based on the tight SCCD(6,3,2,7)

The three consecutive blocks isomorphic to those in Table 2.1 are highlighted in blue. We can repeat this process on each newly constructed design to get all designs of the form $(6+4i, 3, 2, b+2(v+4(i-1))+3)$, $i \geq 0$. Further, each of these tight SCCD$(6+4i, 3, 2, b+2(v+4(i-1))+3)$ has an expansion set, therefore we construct *tight SCCD*$(7+4i, 3, 2, b+2(v+4(i-1))+3+\frac{v}{2})$ by Proposition 1.24. Thus, this construction provides to an alternate proof on the existence of SCCD$(v, 3, 2, b)$ and CSCCD$(v, 3, 2, b)$ in Theorem 3.

2.1 Efficient Computational Search Implementation

I adapted Phillips efficiencies in Lemmas 1.29, 1.30, and 1.31 to a search for tight CSCCD using SF-arrays.

After we generate a skeleton, we can partially fill in the CSCCD with all elements of T_1. We will choose the block that contains the largest number of elements from T_1 to be B_1. In a circular SCCD a cyclic shift can be used to make any block the first [8]. If we limit early choice, we reduce the exponential growth of a search tree. In the event that two blocks are partially filled equally, we use the block that initiates the longest run of initially filled blocks. We call this **front loading the design**.

We say that two CSCCD are **equivalent** if one is obtained from permuting the symbol set X, or cyclically shifting $\mathcal{L}$ or reversing $\mathcal{L}$ in our search ensures we fix $T_1 = \{0, 1, ..., t_1 - 1\}$, $T_2 = \{t_1, t_1 + 1, ..., t_1 + t_2 - 1\}, ..., T_M = \{\sum_{j=1}^{M-1} t_j, ..., v\}$. Further, we assume

Assumption 2.2. *For each $x_i, x_j \in T_n$, x_i must be introduced for the first time before x_j if $x_i < x_j$.*

Now we consider the structure of the SF-arrays in the skeleton.

Observation 2.3. Suppose there exists a partial construction of a circular SCCD$(v, k, 2, b)$ up until block B_i. If x's F is in block B_i in the skeleton, then all other elements of B_i are also in B_{i+1}.

Observation 2.4. Suppose there exists a partial construction of a circular SCCD$(v, k, 2, b)$ up until block B_i. If x's F is in block B_i in the skeleton, then all other element of B_i are in B_{i-1}.

Observation 2.5. If block B_i contains an "S" and "F" both observations apply.

In a tight CSCCD Phillip's notion of minor variants remains unchanged, and we use Lemma 1.29 exactly as he did to optimize our search.

However, when we consider Phillip's end permutations, we must now account for the circular property.

Lemma 2.6. *If there exists a CSCCD$(v, k, 2, b)$ and in blocks $B_1, ..., B_i$, $\{x, z\}, \{y, z\}, z \in Z$ are exactly the pairs counting x, y and either $x, y \in B_0$ or $x, y \notin B_0$. Then swapping x, y in blocks $B_{i+1}, ..., B_b$ produces a circular SCCD$(v, k, 2, b)$.*

Proof. Let $(X, \mathcal{L})$ be a CSCCD$(v, k, 2, b)$. Suppose elements $x, y \in X$ cover the pairs $\{x, z\}, \{y, z\}, \forall z \in Y \subseteq X$ in blocks $B_1, ..., B_i$. If $x, y \in B_0$ then x, y still need to cover the pairs $\{x, w\}, \{y, w\}$ $\forall w \in W \subset X \backslash Y$. The blocks

$B_{i+1}, ..., B_b$ would cover each of these pairs. So swapping x and y's introductions will produce a CSCCD$(v, k, 2, b)$ which will cover the pairs $\{x, w\}$ where $\{y, w\}$ was covered and $\{y, w\}$ where $\{x, w\}$ was covered. We have not effected the single change property either, so $(X, \mathcal{L})$ is still a CSCCD$(v, k, 2, b)$. We do similarly for $x, y \notin B_0$. If $x \in B_0$ and $y \notin B_0$ we may loose the single change between B_0 and B_b. □

In tight CSCCD$(v, k, 2, b)$ if $x \in B_i$ and there are $0 < m < (k-1)$ pairs containing x not covered in $B_0, ..., B_i$ then $x \in B_{i+1}$. We use this the same way as Phillips: if the conditions hold in a partially computed CSCCD we only continue the search if x next introduction after B_i is feasible.

We may also consider the removal conditions of Phillips in Lemma 1.31 with a minor modification for circular. Namely, in block B_b we do not consider Lemma 1.31 as we would need to account for the elements in B_1. In our search we do not remove such an x from B_i, with the exception in block B_b.

Conversely, in tight CSCCD$(v, k, 2, b)$ for $x \in B_j$ suppose that $x \in T_i$ and has been introduced $i - p$ times and all but m pairs containing x are covered in $B_1, ..., B_j$. Then x will be in exactly $h = m - p(k-2)$ more blocks, which leads us to the following lemma. Let $\tilde{\mathcal{L}}$ be a partially completed design up to B_j, $B_j \in \tilde{\mathcal{L}}$. Let $\tilde{f}_x$ be the frequency of $x \in \tilde{\mathcal{L}}$. We say this is a **persistence condition**.

Lemma 2.7. *If $f_x < \tilde{f}_x + m - p(k-2)$ then x must be removed in B_i.*

Proof. If $f_x < \tilde{f}_x + m - p(k-2)$ then we will be forced to cover a pair a second time during an introduction, so x must leave. □

2.2 Algorithms

Using these Lemmas in conjunction we significantly speed up the search.

The algorithm starts by generating all valid SF arrays by a backtracking search using the conditions outlined in Theorem 1.36. Then we generate the skeletons by extending the SF-arrays. We place an "F" f_1 blocks after each "S" of the SF-array and an "S" f_1 blocks before each "F" of the SF-array. We need to validate the skeletons generated by confirming that each "S" in the skeleton is the beginning of a valid SF-array. We eliminate all cyclic shifts so all skeletons are non-isomorphic. For each skeleton we pick the cyclic first block such that B_1 will contain as many elements T_1 as possible.

Second, let $\mathcal{L}$ be a $k \times b$ array, each column is a block and rows are positions within a block. Now we fill in the blocks with the elements of T_1 starting with 0. To do so, we introduce the elements in increasing row order. Let j be the first block where a point from B_1 is removed. All "S" before B_j are filled in with the next available point from T_1 in the next available position. In block B_j or after, "S" must be replaced by an element in each possible position. Each of the resulting partially filled in skeletons is our potential block list, $\mathcal{L}$.

Third, now fill in $\mathcal{L}$ with a recursive search, Algorithm 2.2.2. The `Recursive Search` first checks if $\mathcal{L}$ is filled in. If so, we check that the design satisfies the end conditions. That is, we check if we are on B_b, if $|B_b| = k$, and if $|B_b \cap B_1| = k - 1$. Second, it checks if we are introducing an element on the first block. If we are, we call `Update First Block`. Now `Recursive Search` checks if the point introduced in this block, B_i, is forced by the skeleton. If it is, then we say that x is forced. Further, if x is not listed as an introduction, `Recursive Search` checks that x is valid by checking if the pairs $\{x, y\}$, $\forall y \in B_{i-1}$ not in the same position as x. After we determine x is valid in this situation we call `Update No New Points`. Finally, $x \in B_i$ is not an element from T_1, so we consider all possible points to introduce in increasing order. If we are not on the first block, suppose that $x \in T_j$ then we know that it will be introduced j times and appears in f_j blocks. We check that introducing x in B_i does not exceed j or f_j nor violates the assumptions of 2.2. If there exists a $y < x$ such that x and y satisfy the conditions of Lemma 2.6 we do not consider introducing x. If conditions allow we call `Update`. Otherwise, if we are not on B_1, `Recursive Search` checks that introducing x in B_i does not violate the assumptions of 2.2 and calls `Update First Block` if conditions allow.

`Update`, Algorithm 2.2.3, does the following for every empty row position in B_i. `Update` ensures we are using the smallest minor variant as in Lemma 1.29. Now we need to check for every y in B_{i-1} that does not share a row position with x such that no pairs $\{x, y\}$ were previously covered. This is U_{i-1}. The algorithm also checks that placing x and U_{i-1} in B_i will respect Observations 2.3, 2.4, and 2.5. Now we check that introducing x in B_i does not exceed j or f_j and that each $y \in U_{i-1}$, where $y \in T_w$ y, does not exceed f_w. `Update` checks if x must be in the next block as in Lemma 1.31 or if x must be removed as in Lemma 2.7 and that we can do these. When all these checks are passed, $\forall y \in U_{i-1}$ add y to B_i in the same row position and add x to B_i. Now increment the frequency count of each element in the block and the introduction count for x and add all the pairs covered. Call `Recursive Search(`$\mathcal{L}, i+1$`)`. When the search has returned, decrements frequency and introduction counts, remove covered pairs and remove x and the elements of U_{i-1} that were added from B_{i-1}.

`Update No New Points`, Algorithm 2.2.4, ensures we are using the smallest minor variant as in Lemma 1.29. Now we check that the introduced element x and elements of the unchanged subset that were previously forced forward by Observations 2.3, 2.4, and 2.5 are not covered more then once. Now we bring any elements from B_{i-1} not already in B_i to B_i. Now we add x to $\mathcal{L}$ and add the pairs covered and introduction and frequency counts to $\mathcal{L}$. Call `Recursive Search(`$\mathcal{L}, i+1$`)`. Now we remove the pairs covered, decrements the introductions of x, decrements the frequency count for every element in B_i. Remove x and the elements we brought froward from B_{i-1}.

`Update First Block`, Algorithm 2.2.5, does the following for each empty row position in B_1. Increment the frequency count and add the new pair that is covered to $\mathcal{L}$. If the block is now complete call `Recursive Search(`$\mathcal{L}$`, 2)` starting on the next block, otherwise call `Recursive Search(`$\mathcal{L}, 1$) on the first block. When `Recursive Search` returns we decrease the frequency count and remove the pair that was covered from $\mathcal{L}$.

`Update First Block No New Points`, Algorithm 2.2.6, adds x to $\mathcal{L}$ and increases the introduction and frequency count and save the new covered pairs. If the block is now complete it calls `Recursive Search(`$\mathcal{L}, 2$`)` starting on the next block, otherwise it calls `Recursive Search(`$\mathcal{L}, 1$`)` on the first block. When `Recursive Search` returns, decrease the introduction and frequency count and remove the pairs that were covered.

Algorithm 2.2.1: BUILD THE CSCCD(v, k)

$b \leftarrow \frac{\binom{v}{2}}{k-1}$
$M \leftarrow \lfloor \frac{v-1}{k-1} \rfloor$
for each t_1 that is in a valid solution $(t_0, t_1, ..., t_M)$ to Thm 1.35
do {
  generate all valid SF arrays (start of Section 2.2)
  generate and validate all skeletons
  for each valid distinct skeleton
  do {
    optimize initial block of the skeleton
    for each potential block list $\mathcal{L}$
      do {RECURSIVE SEARCH($\mathcal{L}, 0$)

Algorithm 2.2.2: RECURSIVE SEARCH($\mathcal{L}, i$)

```
if i = b then final check and save the design
if we are introducing an element in B_1
  then { UPDATE NO NEW POINT FIRST BLOCK(L)
 42 min
if If there exists a point in B_i such that x ∉ B_{i-1}
  then { if x is a valid introduction
         then { Add x to the introduction list
                UPDATE NO NEW POINTS(L, i)
  else { for x ∈ {t_1, ..., v}
         do { if i ≠ 1
                then { check that Assumption 2.2 holds
                       check that the intro lim is not exceeded
                       check that the freq lim is not exceeded
                       check that Lemma 2.6 holds
                       UPDATE(L, i)
                else { check Assumption 2.2 holds
                       UPDATE FIRST BLOCK(L)
```

Algorithm 2.2.3: UPDATE($\mathcal{L}, i$)

for each for each valid introduction position

do {
check that Lemma 1.29 holds
check that U_{i-1} does not cover pairs already covered
check that U_{i-1} and x sat Observations 2.3, 2.4, and 2.5
check that the intro limit of x is not exceeded
check that the frequency limit of U_{i-1} is not exceeded
check that Lemma 1.31 holds
check that Lemma 2.7 holds
add elem from U_{i-1} to B_i and future blocks when required
add pairs covered
increment number of introduction and frequency x
increment frequency count for $y \in U_{i-1}$
RECURSIVE SEARCH($\mathcal{L}, i+1$)
remove pairs covered
decrements number of introductions and frequency for x
decrements frequency count for $y \in U_{i-1}$
remove elements added to relevant blocks from U_{i-1}

Algorithm 2.2.4: UPDATE NO NEW POINTS($\mathcal{L}, i$)

check that Lemma 1.29 holds
check that elements forced by Observations 2.3, 2.4, and 2.5 hold
add elements from U_{i-1} to B_i and future blocks when required
increment number of introduction and frequency of x
increment number of introductions of unchanged subset
add pairs covered
RECURSIVE SEARCH($\mathcal{L}, i+1$)
remove the pairs covered
decrements number of introductions and frequency for x
decrements frequency for elements of U_{i-1}
removed elements added to relevant blocks from U_{i-1}

Algorithm 2.2.5: UPDATE FIRST BLOCK($\mathcal{L}$)

for each free position in block
do {
  increment frequency count for x
  add pair covered
  if $|B_1| = k$
    then {RECURSIVE SEARCH($\mathcal{L}, 2$)
    else {RECURSIVE SEARCH($\mathcal{L}, 1$)
  decrements frequency count for x
  remove pair covered

Algorithm 2.2.6: UPDATE FIRST BLOCK NO NEW POINTS($\mathcal{L}$)

add pairs involving x
increment x introduction and frequency count
if $|B_1| = k$
  then {RECURSIVE SEARCH($\mathcal{L}, 2$)
  else {RECURSIVE SEARCH($\mathcal{L}, 1$)
remove pairs involving x
decrements x introduction and frequency count

We found 313 tight CSCCD(12,4,2,21) in about 150045.29 seconds or just under 42 hours using the Carleton SageMath server which has 157GB of RAM and uses an Intel Xeon E5-2667 0 @ 2.90GHz CPU. You may find the algorithm here. *https://github.com/AmandaLynnC/CSCCD_Exhaustive_Search_Masters_Thesis*

Theorem 2.8. *There are 313 distinct CSCCD*$(12, 4, 2, 22)$

The 313 designs are available in the same repository.

2.2.1 SF-arrays

With the goal of using SF-arrays to construct circular SCCD$(v, k, 3, b)$ we begin an investigation into the number and restrictions of introductions in larger sets. Let (X,$\mathcal{M}$) be a tight CSCCD(v, k, s, b). We say that a **clout** is a set $C \subset X$ where $0 < c = |C| < s$. A clout is **introduced** in block B if $C \subseteq B$ and one element in the clout was introduced in this block. We say that $T_i^c \in X$ is the set of clouts of size c that are introduced in i blocks. Let $t_i^c = |T_i^c|$. Note that T_0^c is the set of clouts that are never introduced.

Lemma 2.9. *Each clout $C \in T_0^c$ in a tight CSCCD is in every block.*

Proof. Let $C \subseteq S$ $|S| = s$. Since C is never introduced but S must be covered, C must be in every block. This follows from the circularity. $\square$

Let f_c be the number of blocks that contain clout $C \subset X$. If $C \in T_i^c$, each time C is introduced $\binom{k-c}{s-c}$ s-sets containing C are covered. In each remaining $f_C - i$ blocks that contain C $\binom{k-c-1}{s-c-1}$ s-set containing C are covered.

Proposition 2.10. *The number of blocks containing any clout introduced i times is constant.*

Proof. We know $\binom{v-c}{s-c} = i\binom{k-c}{s-c} + \binom{k-c-1}{s-c-1}(f_C - i)$. In other words. $f_C = \binom{v-c}{s-c} - c - i(\binom{k-c}{s-c} - \binom{k-c-s}{s-c-1}))$, a constant. $\square$

Let $f_i^c = \binom{v-c}{s-c} - c - i(\binom{k-c}{s-c} - 1)$ be the number of blocks containing any fixed clout introduced i times.

When considering clouts of size 1, we may drop the super script. For $C \in T_1^c$, let B_i be the block where C is introduced and B_j be the last block containing C. We call the blocks $(B_i, ..., B_j)$ the **run of** C which **starts** at B_i and **finishes** at B_j. We will further explore properties of these clouts depending upon their size and strength of the design. When we need to clarify which clout the S or F of a SF-array corresponding to we will specify with S_C and F_C.

2.3 Higher Strength SCCD

Moving forward, we consider sets of **strength** $s > 2$. Let $g_1(v, k, s) = \frac{\binom{v}{s} - \binom{k}{s}}{\binom{k-1}{s-1}} + 1$. Let $g_2(v, k, s) = \frac{\binom{v}{s}}{\binom{k-1}{s-1}}$.

Proposition 2.11. *The number of blocks in a SCCD(v, k, s, b) is at least $b \geq \lceil g_1(v, k, s) \rceil$. The number of blocks in a CSCCD(v, k, s, b) is at least $b \geq \lceil g_2(v, k, s) \rceil$.*

Proof. Suppose we have a SCCD(v, k, s, b), $(X, \mathcal{L})$. The first block of $\mathcal{L}$ will cover $\binom{k}{s}$ s-sets and each subsequent block will cover at most $\binom{k-1}{s-1}$ new s-sets. There are $\binom{v}{s}$ s-sets which must be covered, therefore

$$\binom{v}{s} \leq \binom{k}{s} + (b-1)\binom{k-1}{s-1}$$
$$b \geq \left\lceil \frac{\binom{v}{s} - \binom{k}{s}}{\binom{k-1}{s-1}} + 1 \right\rceil.$$

Suppose we have a CSCCD(v, k, s, b), $(X, \mathcal{M})$. Each block of $\mathcal{M}$ will cover at most $\binom{k-1}{s-1}$ new s-sets. There are $\binom{v}{s}$ s-sets which must be covered. Therefore,

$$\binom{v}{s} \leq b\binom{k-1}{s-1}$$
$$b \geq \left\lceil \frac{\binom{v}{s}}{\binom{k-1}{s-1}} \right\rceil.$$

□

Corollary 2.12. *A CSCCD$(v, k, 3, b)$ will have at least $b = \left\lceil \frac{v(v-1)(v-2)}{3(k-1)(k-2)} \right\rceil$ blocks.*

We say a (circular) SCCD(v, k, s, b) is **economical** if it has $\lceil g_1(v, k, s) \rceil$ $\big(\lceil g_2(v, k, s) \rceil\big)$ blocks and **tight** if it is economical and $g_1(v, k, s)$ $\big(g_2(v, k, s)\big)$ is an integer. A SCCD(v, k, s, b) is **minimal** if there does not exist a SCCD(v, k, s, b'), $\forall b' < b$. We say that a **block B_i in a SCCD(b, k, s, b) is tight** if the pairs it covers are not covered in any other block of the SCCD.

The definition of **unchanged subset** of a SCCD(v, k, s) remains the same. Let $W \subseteq X$, $|W| = w$. A set $\mathcal{E} = \{U_{i_1}, ..., U_{i_l}\}$, $l = \frac{\binom{w}{s-1}}{\binom{k-1}{s-1}}$ of unchanged subsets which form the blocks of a $S(w, k-1, s-1)$ design is a **W-expansion set**. A V-expansion set is an **expansion set**. If $\mathcal{E}$ contains neither u_0 or u_b then it is an **inner expansion set**. If $\mathcal{E}$ contains u_0, u_b, or both, then it is an **outer expansion set**. We will be denoting expansion locations with a caret, $\wedge$. This generalized definition is key in some constructions in order to preserve tightness.

Theorem 2.13. *Let $(X, \mathcal{L})$ be a SCCD(v, k, s, b) with an expansion set. Then a $(X', \mathcal{L}')$ SCCD$(v+1, k, s, b + \frac{\binom{v}{s-1}}{\binom{k-1}{s-1}})$ exists. Moreover, if $(X, \mathcal{L})$ is tight, economic, or circular then so is $(X', \mathcal{L}')$.*

Proof. Let $(X, \mathcal{L})$ be a SCCD(v, k, s, b) with an expansion set $\mathcal{E} = \{U_{i_1}, ..., U_{i_l}\}$, $l = \frac{\binom{v}{s-1}}{\binom{k-1}{s-1}}$. We build $(X', \mathcal{L}')$ by inserting a new block $B'_j = U_{i_j} \cup \{x\}, 1 \leq j \leq l$ at expansion set location i_j. The blocks we insert are single change so $(X', \mathcal{L}')$ is single change. The s-sets covered in $\mathcal{L}$ will be covered in $\mathcal{L}'$ in the same blocks. The s-sets of the form $\{x\} \cup S$, $|S| = s-1$, $S \subseteq U_i$, $1 \leq j \leq l$ will be covered in B'_j. As $|S| = s-1$, there exists a unique unchanged subset that contains S, since an expansion set is a design of strength $s-1$. So the s sets of the form $\{x\} \cup S$ are covered in B'_j. The new blocks only cover s-sets containing x and none are repeated, so if $(X, \mathcal{L})$ is tight then $(X', \mathcal{L}')$ is tight. The same is true for the economical case. As the unchanged subsets of the first and last blocks of $(X', \mathcal{L}')$ are the same as those in $(X, \mathcal{L})$, the new blocks do not alter circularity. $\square$

We proved the analogue of this construction for $s = 2$ in my undergraduate honours thesis [3].

Conjecture 2.14. *Let $(X, \mathcal{L} = (B_1, ..., B_b))$ be an economical SCCD(v, k, s, b). Suppose that $(X', \mathcal{L}' = (B_1, ..., B_b, B'_{b+1}, ..., B'_{b'}))$ is a SCCD(v', k, s, b') $\big($CSCCD$(v, k, s, b)\big)$ and $X \subseteq X'$. If B'_i is tight for all $1 \leq i \leq b'$, and a tight SCCD(v, k, s, b) $\big($tight CSCCD$(v, k, s, b)\big)$ exists in $(B'_{b+1}, ..., B'_{b'})$ then $(X', \mathcal{L}')$ is economical.*

For the rest of Chapter 2 we focus on $s = 3$.

Theorem 2.15. *Let $(X, \mathcal{L})$ be a SCCD$(v, k, 3, b)$ with an outer expansion set and $(X', \mathcal{L}')$ be a SCCD$(v', k, 3, b')$ with $X' \cap X = U_b \in B'_1$ and $(X' \backslash X)$-expansion set. Then there exists a SCCD$(v^*, k, 3, b^*)$, $(X^*, \mathcal{L}^*)$, with $v^* = (v + v' - (k-1))$ and $b^* = v + v' + \frac{v(v-1)}{(k-1)(k-2)}\big(v' - k + 1\big) + \frac{(v'-k+1)(v'-k)}{(k-1)(k-2)}\big(v - k + 1\big)$. Moreover,*

- *If $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ are tight then $(X^*, \mathcal{L}^*)$ is tight.*
- *If Conjecture 2.14 holds then if $(X, \mathcal{L})$ is tight and $(X', \mathcal{L}')$ is economical then $(X^*, \mathcal{L}^*)$ is economical.*

Proof. Let $(X, \mathcal{L})$ be a SCCD$(v, k, 3, b)$ with outer expansion set $\mathcal{E} = \{U_{i_j} : 1 \leq j \leq n\}$, $n = \frac{v(v-1)}{(k-1)(k-2)}$ and $U_b \in \mathcal{E}$. Let $(X', \mathcal{L}')$ be a SCCD$(v', k, 3, b')$ with $X' \backslash X$-expansion set, $\mathcal{E}' = \{U_{i_j} : 1 \leq j \leq n'\}$, $n' = \frac{(v'-k+1)(v'-k)}{(k-1)(k-2)}$

We construct $(X' \cup X, \mathcal{L}^*)$. We start by appending $\mathcal{L}'$ to $\mathcal{L}$. As $U_b \subset B_b$ and $U_b \subset B'_1$, this list of blocks is also single change. Next, insert $m' = v' - k + 1$ blocks, F_{i_f}, of the form $U_i \cup \{f\}$, for all $f \in X' \backslash X$ at each expansion location in $\mathcal{L}$. Finally, insert $m = v - k + 1$ blocks, G_{i_g}, of the form $U'_i \cup \{g\}$ for all $g \in X \backslash X'$ at each W-expansion location in $\mathcal{L}'$. With this we cover all triples:

1. The triples containing three elements of X are all covered in blocks $\mathcal{L}$.
2. The triples containing only elements of $X' \backslash X$ are covered in blocks $\mathcal{L}'$.
3. The triples containing two points from $X \backslash X'$ and one point from $X' \backslash X$ are covered in blocks F inserted at expansion locations in $\mathcal{L}$.
4. The triples containing two points from $X' \backslash X$ and one point from $X \backslash X'$ are covered in blocks G inserted at expansion locations in $\mathcal{L}'$.
5. The triples containing two points from $X \cap X'$ and one point from $X' \backslash X$ are covered in $\mathcal{L}'$.
6. The triples containing two points from $X' \backslash X$ and one point from $X \cap X$ are covered in $\mathcal{L}'$.
7. The triples containing one point each from $X \backslash X'$, $X' \backslash X$, and $X \cap X'$, are covered in the blocks F inserted at expansion locations in $\mathcal{L}$.

As $U_{i_j} \subset F_{i_f}$ and $U'_i \subset G_{i_g}$ for all F and G the unchanged subsets are not altered. If the blocks of $\mathcal{L}$ and $\mathcal{L}'$ were tight then $\mathcal{L}^*$ is tight as no blocks of F_{i_f} or G_{i_g} re-cover any pair. Therefor if $(X, \mathcal{L})$ and $(X', \mathcal{L}')$ were tight, so is $(X^*, \mathcal{L}^*)$. □

Chapter 3

Double Change Covering Designs

We now explore designs where more than one element changes between blocks. For $m < k < v$, a **multi-change covering design** (m-CCD(v, k, s, b)), $(X, \mathcal{L})$, is a v-set X and an ordered list of blocks $\mathcal{L} = (B_1, ..., B_b)$ of size k where every s-set, S, must occur in at least one block. Every block except the last differs from the next block by **removing** m elements and **introducing** m elements. So $|B_i \cap B_{i+1}| = (k - m)$ for all $1 \leq i < b$ and the points of $B_{i+1} \backslash B_i$ are **introduced** in B_{i+1} and $B_i \backslash B_{i+1}$ are **removed** in B_i. If $|B_1 \cap B_b| = k$-m then the SCCD is **circular** and m points are introduced in each block. In a non circular m-CCD every point in B_1 is introduced in B_1. A s-set, S, is **covered** on block B_i if $S \subseteq B_i$ and at least one element of S was introduced in B_i. If no DCCD(v, k, s, b') exists for all $b' < b$ then a DCCD(v, k, s, b) is called **minimal**.

We say that $U_i = B_i \cap B_{i+1}$ is the **unchanged subset** between these blocks. If the m-CCD is not circular we can take the unchanged subsets U_0 and U_b to be any $(k - m)$ subset of B_1 and B_b respectively. If $W \subseteq X$ and there exists a set of $l = \frac{\binom{|W|}{s-1}}{\binom{k-m}{s-1}}$ unchanged subsets, $\mathcal{E} = \{U_{i_j}\}$, $1 \leq j \leq l$ which form the blocks of a $S(s-1, k-m, v)$ we say that the m-CCD has an **W-expansion set**. If neither $U_0, U_b \in \mathcal{E}$ then $\mathcal{E}$ is an **inner expansion set**, otherwise $\mathcal{E}$ is an **outer expansion set**. We denote expansion locations with a caret, $\wedge$.

Similarly to single change covering designs we define lower bounds on the number of blocks. Let $g_1'(v,k,s,m) = \frac{\binom{v}{s}-\binom{k}{s}}{\sum_{i=0}^{s}\binom{m}{i}\binom{k-m}{s-i}} + 1$, and $g_2'(v,k,s,m) = \frac{\binom{v}{s}}{\sum_{i=0}^{s}\binom{m}{i}\binom{k-m}{s-i}}$.

Theorem 3.1. *A m-CCD(v,k,s,b) will have $b \geq \lceil g_1'(v,k,s,m) \rceil$ blocks. A circular m-CCD will have at least $b = \lceil g_2'(v,k,s,m) \rceil$ blocks.*

Proof. The first block of the m-CCD will cover $\binom{k}{s}$ s-sets. Every other block will have m elements introduced. The number of s-sets that contain i of the m introduced elements and $s-i$ of the remaining $k-m$ points is $\binom{m}{i}\binom{k-m}{s-i}$. Hence $\sum_{i=0}^{s}\binom{m}{i}\binom{k-m}{s-i}$ s-sets are introduced in each block after the first. There are $\binom{v}{s}$ s-sets to cover so $\binom{v}{s} \leq \binom{k}{s} + (b-1)\left(\sum_{i=0}^{s}\binom{m}{i}\binom{k-m}{s-i}\right)$

In the circular case, each block covers at most $\sum_{i=0}^{s}\binom{m}{i}\binom{k-m}{s-i}$ new s-sets and the calculation is similar. □

We say a (circular) m-CCD(v,k,s,b) is **economical** if it has $\lceil g_1'(v,k,s,m) \rceil$ $\left(\lceil g_2'(v,k,s,m) \rceil\right)$ blocks and **tight** if it is economical and $g_1'(v,k,s,m)$ $\left(g_2'(v,k,s,m)\right)$ is an integer. We say that a **block B_i in a DCCD is tight** if the s-sets the block covers are not covered in any other block of the DCCD.

We will be focusing on designs with $m = 2$ and will call these **double change covering designs** (DCCD(v,k,s,b)). When $m = 2$ we get $g_1'(v,k,s,2) = \frac{\binom{v}{s}-\binom{k}{s}}{2\binom{k-2}{s-1}+\binom{k-2}{s-2}} + 1$ and $g_2'(v,k,s,2) = \frac{\binom{v}{s}}{2\binom{k-2}{s-1}+\binom{k-2}{s-2}}$. When $m = s = 2$ we get $g_1'(v,k,2,2) = \frac{v(v-1)-k(k-1)}{4k-6} + 1$ and $g_2'(v,k,2,2) = \frac{v(v-1)}{4k-6}$.

An example of a CDCCD(7,3,2,7) is given in Table 3.1. The blocks are those of $STS(7)$. It is tight as either a DCCD or CDCCD. The unchanged subsets for the non-circular (circular) DCCD are $U_1 = \{0\}, U_2 = \{4\}, U_3 = \{3\}, U_4 = \{6\}, U_5 = \{6\}, U_6 = \{5\}$ where $U_0 = \{\{0\},\{1\},\{2\}\}$ and $U_7 = \{\{1\},\{3\},\{5\}\}$ $\big(U_1 = \{0\}, U_2 = \{4\}, U_3 = \{3\}, U_4 = \{6\}, U_5 = \{6\}, U_6 = \{5\}, U_0 = U_7 = \{1\}\big)$. Only the non-circular DCCD(7,3,2,7) has an expansion set.

B_1	B_2	B_3	B_4	B_5	B_6	B_7
0^*	0	2^*	0^*	1^*	5^*	5
1	4^*	4	6^*	6	6	3^*
$_\wedge 2^*{}_\wedge$	$5^*{}_\wedge$	$3^*{}_\wedge$	3	$4^*{}_\wedge$	$2^*{}_\wedge$	$1^*{}_\wedge$

Table 3.1: A tight CDCCD(7,3,2,7)

Theorem 3.2. *There exists a tight DCCD$(v, 3, 2, b)$ and tight CDCCD$(v, 3, 2, b)$ for all $v \equiv 1, 3 \pmod 6$.*

Proof. There exists a STS(v) for all $v = 1, 3 \pmod 6$[13]. Horak and Rosa proved the 1 block intersection graph of a STS(v) is Hamiltonian [7].

Let $C = \{B_1, B_2, ..., B_b\}$ be the Hamilton cycle in the 1-block intersection graph of a STS(v). As $|B_i \cap B_{i+1}| = 1$, the double change condition is met on C. As the blocks are those of a STS(v), every pair is covered and the number of blocks is $b = \frac{v(v-1)}{6}$ which matches both g_1' and g_2' □

Conjecture 3.3. *Let $(X, \mathcal{L} = (B_1, ..., B_b))$ be an economic DCCD$(v, k, 2, b)$ with $d = b\left(2\binom{k-2}{s-1} + \binom{k-2}{s-2}\right) + \binom{k}{s} - \binom{v}{s}$ blocks. Suppose that $(X', \mathcal{L}' = (B_1, ..., B_b,\ B'_{b+1}, ..., B'_{b'}))$ is a SCCD$(v', k, 2, b')$ with $X \subseteq X'$. If B'_i is tight $\forall\ i,\ b+1 \leq i \leq b'$, then $(X', \mathcal{L}')$ has $b'\left(2\binom{k-2}{s-1} + \binom{k-2}{s-2}\right) + \binom{k}{s} - \binom{v'}{s} = d$ blocks.*

Theorem 3.4. *Let $(X, \mathcal{L})$ be a DCCD$(v, k, 2, b)$ with v an odd multiple of $k-2$ and expansion set $\mathcal{E}$. Then a DCCD$(v+\frac{v}{k-2}+1, k, 2, b+\frac{v}{k-2}\frac{v}{2})$, $(X^*, \mathcal{L}^*)$, exists. Moreover, if $(E, \mathcal{L})$ is tight or circular then $(X^*, \mathcal{L}^*)$ is too. If Conjecture 3.3 holds then if $(X, \mathcal{L})$ is economical then $(X^*, \mathcal{L}^*)$ is economic.*

Proof. Let $(X, \mathcal{L})$ be a DCCD$(v, k, 2, b)$ with the expansion set $\mathcal{E} = \{U_{i_j} : 1 \leq j \leq t \leq \frac{v}{k-2}\}$. Let $\{F_j\}_1^t$ be a 1-factorization of K_{t+1} on set V[1].

For each edge $e \in F_j$ let $B_{j,e} = U_{i_j} \cup \{e\}$. Insert $B_{j,e}$ at each expansion location $U_{i_j} \in \mathcal{E}$. These insertions maintain the double change property. All the pairs that were covered in $(X, \mathcal{L})$ are still covered in $\mathcal{L}$. Since each pair of elements from V is in exactly one 1-factor all the pairs in V are covered. The blocks inserted at expansion set locations ensure each pair of elements $\{x, y\}$, $x \in X$, $y \in V$ is covered. So $(X^*, \mathcal{L}^*)$ is a DCCD$(v + \frac{v}{k-2} + 1, k, 2, b')$ with $b' = b + \frac{v^2}{2k-4}$.

Each B_{j_e} is tight, so if $(X, \mathcal{L})$ was tight so is $(X^*, \mathcal{L}^*)$. The block insertions do not affect the double change between the first and last block, so if $(X, \mathcal{L})$ was circular so is $(X^*, \mathcal{L}^*)$. □

We note that every new element x added to V in Theorem 3.4 is introduced in B_i and removed in B_{i+1} in $\mathcal{L}^*$. Consequently, x is never in an unchanged subset and any $\mathcal{L}^*$ constructed in this manner will never have an expansion set. Thus, we looked for recursions that will preserve expansion sets similar to when we use two SCCD to construct a larger SCCD as in Theorem 1.38.

Theorem 3.5. *Let $(X, \mathcal{L})$ and $(\tilde{X}, \tilde{\mathcal{L}})$ be a DCCD$(v, k, 2, b)$. Let $(X', \mathcal{L}')$ be a CDCCD$(v', k, 2, b')$ with an outer expansion set where $v' = \frac{v-k+2}{k-2} + 2k - 4$. Then a DCCD$(v^*, k, 2, b^*)$, $b^* = 2b + b' + (v - k + 2)^2$, $(X^*, \mathcal{L}^*)$, exists. Furthermore,*

- *If $(X, \mathcal{L})$ and $(\tilde{X}, \tilde{\mathcal{L}})$ are circular then $(X^*, \mathcal{L}^*)$ is circular.*
- *If $(X, \mathcal{L})$, $(\tilde{X}, \tilde{\mathcal{L}})$, and $(X', \mathcal{L}')$ are tight $\big($circular$\big)$ then $(X^*, \mathcal{L}^*)$ will only re-cover at most $k - 2$ $\big($ at most $3k - 6\big)$ pairs.*

Proof. Let $(X, \mathcal{L})$ and $(\tilde{X}, \tilde{\mathcal{L}})$ be two DCCD$(v, k, 2, b)$. Let $n = v - k + 2$ and $(X', \mathcal{L}')$ be a CDCCD$(v', k, 2, b')$ with an outer expansion set $\mathcal{E} = \{U_{i_j} : 1 \leq j \leq n\}$, and $v' = \frac{v-k+2}{k-2} + 2k - 4$ and $X' \cap X = U_0' = U_b' = X' \cap \tilde{X}$. Take $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}' \cup \tilde{\mathcal{L}}$ such that $|X \backslash X'| = |\tilde{X} \backslash X| = n$. Let $\{F_j\}_1^t$ be a 1-factorization of $K_{n,\tilde{n}}$ on set $\{X \backslash X', \tilde{X} \backslash X'\}$ [1].

For each edge $e \in F_j$ let $B_{j_e} = U_{i_j} \cup e$. Insert B_{j_e} at each expansion location $U_{i_j} \in \mathcal{E}$.

These insertions maintain the double change property. All the pairs that were covered in $(X, \mathcal{L})$ are covered in $\mathcal{L}$. All the pairs that were covered in $(X', \mathcal{L}')$ are covered in $\mathcal{L}'$. All the pairs that were covered in $(\tilde{X}, \tilde{\mathcal{L}})$ are covered in $\tilde{\mathcal{L}}$. The pairs $\{x, \tilde{x}\}$, $\{x', \tilde{x}\}$, $\{x, x'\}$ where $x \in X \backslash X', x' \in X' \backslash (X \cup \tilde{X})$, and $\tilde{x} \in \tilde{X} \backslash X'$ are covered in B_{j_e}'.

As The pairs $\{x, y\}$ where $x, y \in X \cap X' \cap \tilde{X}$ are covered in both $\mathcal{L}$ and $\mathcal{L}'$. These are the only pairs of elements that may repeat their coverings. So for $\mathcal{L}, \tilde{\mathcal{L}}$ non-circular there are at most $v - k + 2$ pairs that are re-covered. For $\mathcal{L}, \tilde{\mathcal{L}}$ circular we now have $\tilde{U_b} = U_0 = U_b = U_0' = U_b' = \tilde{U_0}$, so the pairs $\{x, y\}$ where $x, y \in X \cap X' \cap \tilde{X}$ are covered in $\mathcal{L}, \mathcal{L}'$ and $\tilde{\mathcal{L}}$ and are recovered at most $v - 3k + 6$ times. □

For example consider the non-circular DCCD(7,3,2,7) and expansion set in Table 3.1. We may insert four new blocks at the expansion set locations that consist of the unchanged subset and the vertices of an edge in a 1-factor of K_8 in Figure 3.1 to construct at DCCD(13,3,2,35) as seen in Table 3.2.

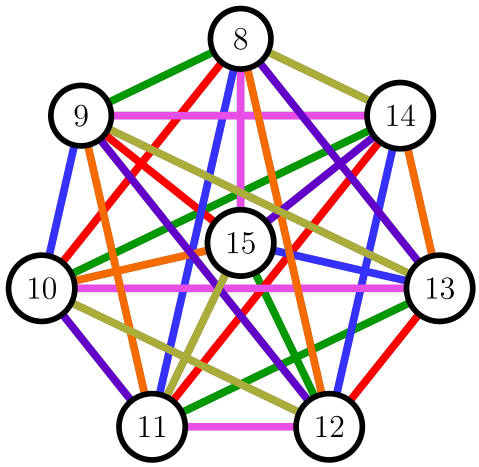

Figure 3.1: K_8

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}	B_{18}
8	10	11	12	0*	0	0	0	0	0	8	9	12	13	2*	8	9	10
9	14	13	15	1*	8	11	12	9	4*	4	4	4	4	4	15	14	13
2	2	2	2	2	10	14	13	15	5*	11	10	14	15	3*	3	3	3

B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}	B_{27}	B_{28}	B_{29}	B_{30}	B_{31}	B_{32}	B_{33}	B_{34}	B_{35}
11	0*	1*	8	13	9	10	5*	5	5	5	5	5	8	9	10	11
12	6*	6	6	6	6	6	6	8	9	10	14	3*	14	13	12	15
3	3	4*	12	14	11	15	2*	13	12	11	15	1*	1	1	1	1

Table 3.2: A DCCD(15,3,2,35) built from the DCCD(7,3,2,2) and K_8

3.1 Difference Methodology

We can construct some CDCCD$(v, k, 2, b)$ using a construction similar to difference sets and families. McSorley proved that $\{0, 1, ..., k-1\}$ is a base block of a CSCCD$(2k-1, k, 2, 2k-1)$ over $\mathbb{Z}_{2k-1}$ in [8].

Every element in the tight CSCCD($2k-1, k, 2, 2k-1$) is introduced exactly once. If we replace $x \in \mathbb{Z}_{2k-1}$ in B_i with $2x$ and $2x+1$ in $\mathbb{Z}_{4k-2}$ to produce a list of $2k-1$ blocks B_i' of size $2k$ from $\mathbb{Z}_{2k} \times \{0,1\}$ each element is still introduced once. The pairs $\{2x, 2x+1\}$ are covered when introduced and $\{y, z\} \in \mathbb{Z}_{4k-2}$ are covered in block where $\{\lfloor \frac{y}{2} \rfloor, \lfloor \frac{z}{2} \rfloor\}$ where covered in CSCCD($2k-1, k, 2, 2k-1$). Thus

Theorem 3.6. *A tight CDCCD*($4k-2, 2k, 2, 2k-1$) *exists for all* $k \geq 2$.

Proof. The only thing to check is the number of blocks and $b' = b = 2k+1 = g_2'(4k-2, 2k, 2, 2)$. □

For example, consider the tight CSCCD(3,2,2,3) and construct a tight CDCCD(6,4,2,3) in Table 3.3.

B_1	B_2	B_3		B_1'	B_2'	B_3'
0	2*	2		0	4*	4
1*	1	0*		1	5*	5
				2*	2	0*
				3*	3	1*

Table 3.3: A tight CSCCD(3,2,2,3) and tight CSCCD(6,4,2,3)

It is worth noting that $k-1$ does not divide $2k-1$ for $k > 2$ but when $k = 2$ the CSCCD(3,2,2,3) built this way has an expansion set, so the blown up CDCCD(6,4,2,3) has one as well. Using this in conjunction with the 1-factor of K_4 and Theorem 3.4 we build the CDCCD(10,4,2,9) seen in Table 3.4.

0	a*	c*	4*	4	4	4	a*	b*
1	b*	d*	5*	5	5	5	d*	c*
2*	2	2	2	a*	b*	0*	0	0
3*	3	3	3	c*	d*	1*	1	1

Table 3.4: A tight CDCCD(10,4,2,9)

Adding a single new point to every block yields

Theorem 3.7. *A tight CDCCD*($4k-1, 2k+1, 2, 2k-1$) *exists for all* $k \geq 2$.

Theorem 3.8. *A circular DCCD(*$4k-5, k, 2, 4k-5$*) exist for all* k.

Proof. Let $B_0 = \{0, 1, ..., k-2, 2k-3\} \subseteq \mathbb{Z}_{4k-5}$. We construct $B_i = B_0 + i \forall i \in \mathbb{Z}_{4k-5}$. Let $\mathcal{L} = (B_i)_0^{4k-6}$. Note that $B_j \cap B_{j+1} = \{i+1, ..., k+i-2\}$ so B_{i+1} covers pairs $\{k+i-1, z\}$ and $\{2k-2+i, z\}$ for $z \in B_i \cap B_{i+1}$ and $\{k+i-1, 2k+i-2\}$. These pairs have differences $\pm\{1, 2, ..., k-2, k-1, k, ..., 2k-3\} = \mathbb{Z}_{4k-5} \backslash \{0\}$. As i transfers $\mathbb{Z}_{4k-5}$ every pair is covered.

$$b = 4k - 5 = \frac{(4k-5)(4k-6)}{4k-6} = g_2'(4k-5, k, 2, 2)$$

□

We can think of these vertices in a circular manor. For example, consider a CDCCD(11,4,2,11), so $k = 4$, in Figure 3.2. The first block is $\{0, 1, 2, 5\}$ and the points 2 and 5 are introduced. In the figure the vertices only in the first block are highlighted in blue, the vertices only on the second block are highlighted in pink, and the vertices in the unchanged subset of B_1 is in black. Now we place a line between each introduced element and the other elements in the block, blue for the first block and pink for the second. Each subsequent block is obtained by rotating the previous block by one. As this is a circle we will eventually return to our starting position and when we do we will have placed an edge between every vertex exactly once. We also see pictorially that there will only ever be double change between consecutive blocks. Note that the blue blocks cover the difference $\{1, 2, 3, 4, 5\}$ and so every block will as well.

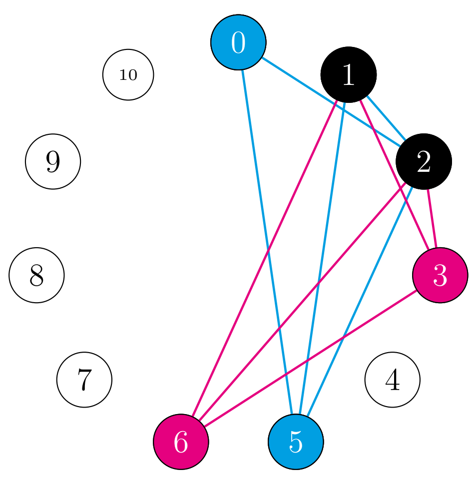

Figure 3.2: $c = 1$, $k = 4$ difference construction for DCCD(11,4,2,11)

Table 3.5 shows the table form the CDCCD(11,4,2,11).

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}
0	3*	3	3	6*	6	6	9*	9	9	1*
1	1	4*	4	4	7*	7	7	10*	10	10
2*	2	2	5*	5	5	8*	8	8	0*	0
5*	6*	7*	8*	9*	10*	0*	1*	2*	3*	4*

Table 3.5: A CDCCD(11,4,2,11)

When $k-2|4k-5$ these have expansion sets $\mathcal{E} = \bigcup_{i=0}^{\frac{4k-5}{k-2}} U_{i(k-2)}$

Corollary 3.9. *A tight CDCCD(15,3,2,35) exists. A tight CDCCD(21,5, 2,30) exists.*

A backtracking search yielded sets of base blocks for some parameters.

Theorem 3.10. *The following tight CDCCD exist.*

1. *A tight CDCCD*$(10c+1, 4, 2, 10c^2+c)$ *exists for all* $2 \leq c \leq 6$.
2. *A tight CDCCD*$(c(4k-6)+1, k, 2, c^2(4k-6)+c)$ *exists for all* $2 \leq c \leq 4$, $5 \leq k \leq 15$.
3. *A tight CDCCD*$(c(4k-6)+1, k, 2, c^2(4k-6)+c)$ *exists for all* $2 \leq c \leq 3$, $16 \leq k \leq 20$.

Proof. To build each of the tight CDCCD$(c(4k-6)+1, k, 2, c^2(4k-6)+c)$ we start the with the initial c blocks given in order. Each contains $\{0, 1, ..., k-3\}$ and $B_{0,c}$ contains $\{0, 1, ..., k-2\}$ thus $(B_{0,1}, B_{0,2}, ..., B_{0,c}, B_{1,1}, ..., B_{1,c}, ..., B_{v,c})$ is double change.

$B_{0,i} = \{0, 1, ..., k-3, x, y\}$ covers differences $\pm\{y-x, x-k+3, ..., x, y-k+3, ..., y\}$ and in each case x, y can be changed such that these cover all difference in $\mathbb{Z}_v$

Table 3.6: Initial Blocks for some $CDCCD(c(4k-6)+1, k, 2, c^2(4k-6)+c)$

k	c	**Initial Blocks**
4	2	$\{0, 1, 4, 12\}, \{0, 1, 2, 7\}$

Table 3.6: (Initial Blocks continued)

k	c	**Initial Blocks**
4	3	$\{0,1,4,24\},\{0,1,6,22\},\{0,1,2,14\}$
4	4	$\{0,1,4,25\},\{0,1,6,32\},\{0,1,8,19\},\{0,1,2,14\}$
4	5	$\{0,1,4,17\},\{0,1,6,37\},\{0,1,8,33\},\{0,1,10,40\},\{0,1,2,24\}$
4	6	$\{0,1,4,19\},\{0,1,6,22\},\{0,1,8,25\},\{0,1,10,48\},\{0,1,12,32\},$ $\{0,1,2,28\}$
5	2	$\{0,1,2,6,17\},\{0,1,2,3,10\}$
5	3	$\{0,1,2,6,33\},\{0,1,2,9,30\},\{0,1,2,3,20\}$
5	4	$\{0,1,2,6,26\},\{0,1,2,9,36\},\{0,1,2,12,40\},\{0,1,2,3,16\}$
6	2	$\{0,1,2,3,8,22\},\{0,1,2,3,4,13\}$
6	3	$\{0,1,2,3,8,17\},\{0,1,2,3,13,31\},\{0,1,2,3,4,23\}$
6	4	$\{0,1,2,3,8,25\},\{0,1,2,3,12,55\},\{0,1,2,3,16,47\},$ $\{0,1,2,3,4,36\}$
7	2	$\{0,1,2,3,4,10,27\},\{0,1,2,3,4,5,16\}$
7	3	$\{0,1,2,3,4,10,21\},\{0,1,2,3,4,16,38\},$ $\{0,1,2,3,4,5,28\}$
7	4	$\{0,1,2,3,4,10,26\},\{0,1,2,3,4,15,49\},$ $\{0,1,2,3,4,21,54\},\{0,1,2,3,4,5,32\}$
8	2	$\{0,1,2,3,4,5,12,32\},\{0,1,2,3,4,5,6,19\}$
8	3	$\{0,1,2,3,4,5,12,25\},\{0,1,2,3,4,5,19,45\},$ $\{0,1,2,3,4,5,6,33\}$
8	4	$\{0,1,2,3,4,5,12,25\},\{0,1,2,3,4,5,19,65\},$ $\{0,1,2,3,4,5,32,58\},\{0,1,2,3,4,5,6,39\}$
9	2	$\{0,1,2,3,4,5,6,14,37\},\{0,1,2,3,4,5,6,7,22\}$
9	3	$\{0,1,2,3,4,5,6,14,29\},\{0,1,2,3,4,5,6,22,52\},$ $\{0,1,2,3,4,5,6,7,38\}$
9	4	$\{0,1,2,3,4,5,6,14,36\},\{0,1,2,3,4,5,6,21,67\},$ $\{0,1,2,3,4,5,6,29,74\},\{0,1,2,3,4,5,6,7,44\}$
10	2	$\{0,1,2,3,4,5,6,7,16,42\},\{0,1,2,3,4,5,6,7,8,25\}$
10	3	$\{0,1,2,3,4,5,6,7,16,33\},\{0,1,2,3,4,5,6,7,25,59\},$ $\{0,1,2,3,4,5,6,7,8,43\}$
10	4	$\{0,1,2,3,4,5,6,7,16,50\},\{0,1,2,3,4,5,6,7,24,102\},$ $\{0,1,2,3,4,5,6,7,33,58\},\{0,1,2,3,4,5,6,7,8,68\}$
11	2	$\{0,1,2,3,4,5,6,7,8,18,47\},\{0,1,2,3,4,5,6,7,8,9,28\}$

Table 3.6: (Initial Blocks continued)

k	c	**Initial Blocks**
11	3	$\{0,1,2,3,4,5,6,7,8,18,37\}, \{0,1,2,3,4,5,6,7,8,28,66\},$ $\{0,1,2,3,4,5,6,7,8,9,48\}$
11	4	$\{0,1,2,3,4,5,6,7,8,18,46\}, \{0,1,2,3,4,5,6,7,8,27,85\},$ $\{0,1,2,3,4,5,6,7,8,37,94\}, \{0,1,2,3,4,5,6,7,8,9,56\}$
12	2	$\{0,1,2,3,4,5,6,7,8,9,20,52\}, \{0,1,2,3,4,5,6,7,8,9,10,31\}$
12	3	$\{0,1,2,3,4,5,6,7,8,9,20,41\}, \{0,1,2,3,4,5,6,7,8,9,31,73\},$ $\{0,1,2,3,4,5,6,7,8,9,10,53\}$
12	4	$\{0,1,2,3,4,5,6,7,8,9,20,51\}, \{0,1,2,3,4,5,6,7,8,9,30,94\},$ $\{0,1,2,3,4,5,6,7,8,9,41,104\}, \{0,1,2,3,4,5,6,7,8,9,10,62\}$
13	2	$\{0,1,2,3,4,5,6,7,8,9,10,22,57\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,34\}$
13	3	$\{0,1,2,3,4,5,6,7,8,9,10,22,103\},$ $\{0,1,2,3,4,5,6,7,8,9,10,57,80\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,35\}$
13	4	$\{0,1,2,3,4,5,6,7,8,9,10,22,45\},$ $\{0,1,2,3,4,5,6,7,8,9,10,34,115\},$ $\{0,1,2,3,4,5,6,7,8,9,10,57,103\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,69\}$
14	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,24,62\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,12,37\}$
14	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,24,87\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,37,62\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,12,50\}$
14	4	$\{0,1,2,3,4,5,6,7,8,9,10,11,24,49\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,37,125\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,62,112\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,12,75\}$
15	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,26,67\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,40\}$
15	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,26,53\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,12,40,94\},$ $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,68\}$

Table 3.6: (Initial Blocks continued)

k	c	**Initial Blocks**
15	4	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,26,53\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,40,135\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,67,121\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,81\}$
16	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,28,72\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,43\}$
16	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,28,57\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,43,101\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,73\}$
17	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,30,77\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,46\}$
17	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,30,61\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,46,108\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,78\}$
18	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,32,82\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,49\}$
18	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,32,65\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,49,115\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,83\}$
19	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,34,87\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,52\}$
19	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,34,69\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,52,122\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,88\}$
20	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,36,92\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,55\}$
20	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,36,73\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,55,129\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,93\}$
21	2	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,38,97\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,58\}$
21	3	$\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,38,77\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,58,136\}$, $\{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,98\}$

□

For example, consider the initial two blocks given for the tight CDCCD(21,4,2,42) in Theorem 3.10. We construct the following tight CDCCD(12,4,2,42) in Table 3.7. In this Table we highlight every $x \in X$ such that the pair $(0, x)$ is covered.

$B_{0,1}$	$B_{0,2}$	$B_{1,1}$	$B_{1,2}$	$B_{2,1}$	$B_{2,2}$	$B_{3,1}$	$B_{3,2}$	$B_{4,1}$	$B_{4,2}$	$B_{5,1}$	$B_{5,2}$	$B_{6,1}$	$B_{6,2}$
0	0	5*	3*	3	3	3	3	8*	6*	6	6	6	6
1	1	1	1	6*	4*	4	4	4	4	9*	7*	7	7
4*	2*	2	2	2	2	7*	5*	5	5	5	5	10*	8*
12*	7*	13*	8*	14*	9*	15*	10*	16*	11*	17*	12*	18*	13*
$B_{7,1}$	$B_{7,2}$	$B_{8,1}$	$B_{8,2}$	$B_{9,1}$	$B_{9,2}$	$B_{10,1}$	$B_{10,2}$	$B_{11,1}$	$B_{11,2}$	$B_{12,1}$	$B_{12,2}$	$B_{13,1}$	$B_{13,2}$
11*	9*	9	9	9	9	14*	12*	12	12	12	12	17*	15*
7	7	12*	10*	10	10	10	10	15*	13*	13	13	13	13
8	8	8	8	13*	11*	11	11	11	11	16*	14*	14	14
19*	14*	20*	15*	0*	16*	1*	17*	2*	18*	3*	19*	4*	20*
$B_{14,1}$	$B_{14,2}$	$B_{15,1}$	$B_{15,2}$	$B_{16,1}$	$B_{16,2}$	$B_{17,1}$	$B_{17,2}$	$B_{18,1}$	$B_{18,2}$	$B_{19,1}$	$B_{19,2}$	$B_{20,1}$	$B_{20,2}$
15	15	15	15	20*	18*	18	18	18	18	2*	0*	0	0
18*	16*	16	16	16	16	0*	19*	19	19	19	19	3*	1*
14	14	19*	17*	17	17	17	17	1*	20*	20	20	20	20
5*	0*	6*	1*	7*	2*	8*	3*	9*	4*	10*	5*	11*	6*

Table 3.7: A Tight CDCCD(21,4,2,42)

We may also think of Theorem 3.10 in a circular manor as we did for Theorem 3.8. Consider the tight CDCCD(13,3,2,26). The first block is $\{0, 1, 4\}$ and the second block is $\{0, 2, 7\}$. These blocks are highlighted in blue and pink respectively in Figure 3.3 and the unchanged subset is highlighted black.

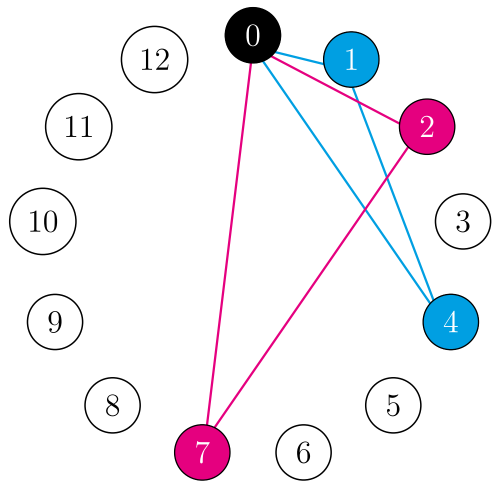

Figure 3.3: $c = 2$, $k = 3$ tight CDCCD(13,3,2,26)

We alternate rotating the colours by 1 so that the odd blocks are blue and the even pink. Note that taking every difference in B_1 and B_2 involving at least one introduced point produces the differences $1, 2, ..., 12$ (mod 13) and so these differences are in every pair of consecutive blocks. This produces the tight CDCCD(13,3,2,26) as seen in Table 3.8.

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}	B_{17}
0^*	0	1^*	1	2^*	2	3^*	3	4^*	4	5^*	5	6^*	6	7^*	7	8^*
1	2^*	2	3^*	3	4^*	4	5^*	5	6^*	6	7^*	7	8^*	8	9^*	9
4^*	7^*	5^*	8^*	6^*	9^*	7^*	10^*	8^*	11^*	9^*	12^*	10^*	0^*	11^*	1^*	12^*

B_{18}	B_{19}	B_{20}	B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}
8	9^*	9	10^*	10^*	11^*	11	12^*	12
10^*	10	11^*	11	12^*	12	0^*	0	1^*
2^*	0	3	1	4	2	5	3	6

Table 3.8: Tight CDCCD(13,3,2,26)

Chapter 4

Conclusion

In this paper we reviewed the literature on circular and non-circular SCCD. We develop a small recursion for all circular and non-circular SCCD$(v, 3, 2, b)$ which provides an alternative proof of existence. We integrated search techniques from exhaustive searches for tight SCCD and tight CSCCD to implement an efficient search for CSCCDs. With this search we generate all 313 distinct CSCCD(12,4,2,22). We began a preliminary examination for higher strength SCCD and CSCCD.

In Chapter 3 we explore DCCD and CDCCD. Our examination produced a theorem to increase v in CDCCD and DCCD analogous to the SCCD$(v+1, k, 2, b)$ construction. We also completely solved the existence problem for tight CDCCD$(v, 3, 2, b)$ and tight DCCD$(v, 3, 2, b)$ and produced some families of designs for some given k. We solve CDCCD$(4k-2, 2k, 2, 2k-1)$, CDCCD$(4k-1, 2k+1, 2, 2k-1)$, CDCCD$(4k-5, k, 2, 4k-5)$, and CDCCD$(c(4k-6)+1, k, 2, c^2(4k-6)+c)$ for specific values of c, k.

We believe that the most significant contributions to the field involve the examination of double change covering designs. This is because we completely solve the existence problem for tight $k = 3, s = 2$ as well as provide some insight into more possible single change constructions using the difference methods technique explored. We also think that the strength 3, $v+1$ recursion with the expansion set and the $v + v'$ recursions are significant.

4.1 Looking forward

We would like to continue the examination of SF-arrays for strength 3 CSCCD. Once we know the restrictions on the skeletons of singletons and pairs, we can use both to significantly reduce the work of a recursive search. This search will provide us with small designs we can use as ingredients for some recursive constructions on v.

We would like to re-visit the recursive search for CSCCD and parallelize it to find CSCCD with larger k. Furthermore, we would like to examine how we may construct SF-arrays for economical CSCCD with strength 2 in order to make this recursion more dynamic.

We want to consider the use of SCCD, CSCCD, DCCD, and CDCCD for use in cache-aware programming. Change designs suite situations where operations or tests are inexpensive compared to the cost of interchanging elements. Updating the cache is time expensive compared to reading as it requires reading RAM which is an order of magnitudes slower, so algorithms which utilized double and single change designs to manage cache content could offer significant time savings.

We believe that we can find the following family of circular DCCD.

Conjecture 4.1. *All circular DCCD*$(c(4k-6)+1, k, 2, c^2(4k-6)+c)$*,* $c \geq 1$ *exist.*

By Theorem 3.8 we know that this is true when $c = 1$. We have code that will run for small values of c and v which produces the base blocks for such designs. We am currently using the idea related to Skolem sequences to examine these cases when $k = 4$.

Furthermore, we ask whether we may use a set of initial blocks as in Conjecture 4.1 to produce initial blocks for families of CSCCD.

We would like to get a general construction for economic CDCCD$(v, 3, 2, b)$.

We would like to generalize Theorem 2.1 for values of $k > 3$. In particular we would like to start by examining potential constructions for SCCD$(v, 4, 2, b)$ by examining all possible SCCD(12,4,2,22).

Suppose $c : X \to \mathbb{R}$ is a cost function and $(X, \mathcal{L} = (B_i)_{i=1}^{b})$ is a m-CCD. The **cost** of $(X, \mathcal{L})$ is

$$\sum_{i=1}^{b} \sum_{x \in B_{i+1} \setminus B_i} c(x)$$

If $c(x) = c\ \forall x$ then a minimal m-CCD minimizes the cost. Suppose that $c : X \to \mathbb{R}$ is is not a constant function, can we develop methods to construct minimal SCCD(v, k, s, b)?

Can we use SCCD or DCCD to formulate a logic puzzle with a unique solution? If so what would the minimum amount of information required be to solve the puzzle?

Let $\binom{X}{s}$ denote the set of all s-subsets of X. A **disjoint single change covering design**, DSCCD(v, v', k, s, b), $(X, X', \mathcal{L})$, is a v-set X, a v'-set X' and a single change list of blocks $\mathcal{L} = (B_i)_{i=1}^{b}$ of size k where every s-set in $\binom{X}{s}\dot{\cup}\binom{X'}{s}$ must occur on at least one block. What is minimum b for v, v', k, s?

Let $H \subseteq X$ and $\mathcal{L} = (B_i)_{i=1}^{b}$, what is the minimum b required to cover all the pairs $\binom{X}{2}\setminus\binom{H}{2}$ [18].

We end with a speculative idea. Suppose we had a group of v people and restricted contact in effect in society. Suppose that kn of them meet simultaneously in n groups of k people. Suppose that only one person can leave and enter a meeting at a time. After leaving these meetings each of the n people must isolate for t days to monitor for possible infections. For example, consider Table 4.1.

	Group 1			
B_1^1	B_2^1	B_3^1	B_4^1	...
1*	4*	4	4	
2	2	5*	5	
3	3	3	6*	
	⋮			
	Group n			
B_1^n	B_2^n	B_3^n	B_4^n	...
v*	v-3*	v-3	v-3	
v-1	v-1	v-4*	1*	
v-2	v-2	v-2	v-2	

Table 4.1: Partial example of kn people in n groups of k, $t = 2$

Let a single change covering family, SCCF(v, k, n, t, s, b), be a set of n single change lists of blocks $(B_i^j)_{i=1}^{b}$ for $1 \leq j \leq n$ and if $x \in B_i^j$ then $x \notin B_l^j$, $\forall i < l \leq i + t$ What is the minimum b for a SCCF(v, k, n, t, s, b). Can minimal, economical or tight SCCF be constructed?

Suppose further the possibility that blocks vary in size. Let a single change covering family, SCCF$(v, k_1, ..., k_r, n, t, s, b)$ be a set of n single change lists of blocks with the same isolation time constraints. For what b do SCCF$(v, k_1, ..., k_r, n, t, s, b)$ exist? For what $v, k_1, ..., k_r, n, t, s$ can tight, economical or minimal SCCF be constructed?

Glossary

1-factor A perfect matching or 1-factor of a graph G is a subset of edges M such that every vertex $v \in V$ is adjacent to exactly one edge $e \in M$. 2

1-factorization A partition of the edges of a graph into perfect matchings is a 1-factorization. 2

balanced incomplete block designs A balanced incomplete block design ((v, k, λ)-BIBD), is a set of k blocks from a v-set X where each pair from X occurs in exactly λ blocks. 3

circular The property that the number of changes required between consecutive blocks holds between B_1 and B_b. 6, 39

clout A clout is a set of elements less then the strength of a design that we consider for the sake of SF-arrays of higher strength designs. 34

covered A s-set, S, is covered on B_i if at least one $x \in S$ is introduced in B_i. 6, 39

cycle A cycle in a graph G is a path in G where the first and last vertex are joined by an edge. 2

difference family A (v, k, λ)-difference family is a collection $\{D_1, ..., D_t\}$ of k-subsets of G, $|G| = v$, where $\partial D_1 \cup ... \cup \partial D_t = \lambda G \backslash \{0\}$. 3

difference set A difference family with $t = 1$ is a $(v, k, \lambda)-$difference set. 3

disjoint-capable An expansion set is disjoint-capable if it satisfies conditions that utilize the unchanged subsets. 19

double change covering design A double change covering design (DCCD(v, k, s, b)) is a v set and an ordered list of blocks of size k where every s-set must occur on at least one block. Every block differs from the next by exactly two elements. 40

economical A change design is economical if the lower bound on the minimum required blocks is met. 8, 40

end permutation An end permutation is an SCCD obtained by swapping two points in all blocks after the i^{th} block. 13

equivalent Two designs are equivalent if one may be obtained by permuting the elements or cyclically shifting the blocks of the other. 28

expansion set A partition of V using unchanged subsets. 7, 36

finishes An element finishes in a block of an SF-array if it is only introduced once and is removed in the next block. 16, 35

front loading Front loading the design is when we cycle the SF-skeleton so that the first block will contain as many elements introduced exactly once as possible. 28

graph A graph is an ordered pair $G = (V, E)$ of vertices, V, and edges, E, such that each edge consists of two end vertices. 2

Hamilton cycle A Hamilton cycle in a graph G is a cycle visiting all the vertices V of G only once. 2

Hamilton path A Hamilton path in a graph G is a path visiting all the vertices V of G only once. 2

hole A hole is the pair of elements missing from a single change covering design with a hole. 10

inner expansion set An expansion set that does not use U_0 or U_b. 7, 36, 39

introduced An element x is introduced in a block B_i if $x \notin B_{i-1}$. In a non-circular SCCD every element in B_1 is introduced. 6, 35, 39

maximum introduction limit The most number of introductions any element may be introduced in a single design. 15

minimal A change design is minimal if it uses the fewest blocks possible to complete the change design. 6, 39

minor variant A minor variant is an SCCD obtained by swapping two consecutive blocks where this preserves the single change property. 12

multi-change covering design A multi-change covering design (m-CCD(v, k, s, b)) is a v set and an ordered list of blocks of size k where every s-set is must occur on at least one block. Every block differs from the next by exactly m elements. 39

outer expansion set An expansion set that uses U_0, U_b or both. 7, 36, 39

path A path in a graph G is a walk with distinct vertices. 2

perfect matching See 1-factor. 2

persistence condition Conditions that dictate if an element must be in the unchanged subset based on the number of remaining pairs to be covered. 29

removal conditions Conditions that dictate if an element may be removed in the next block based on the number of remaining pairs to be covered. 13

removed An element x is removed in block B_i if $x \in B_{i-1}$ and $x \notin B_i$. 6, 39

run A group of successive blocks containing x. 16, 35

SCCD A single change covering design (SCCD(v, k, s, b)) is a sequence of b k-sets, called blocks, of a V-set in which exactly one element differs between consecutive blocks and every s-set of V is in some block. 1, 5

SF-array The SF-array of x in a design is a list describing the introductions and removals of all elements introduced exactly once over a run of x where x is introduced once. 16

SF-skeleton A SF-skeleton is an expanded SF-array that may be used to partially fill in the design. 17

single change covering design with a hole A single change covering design with a hole, SCCDH is a SCCD such that every pair of elements of V occur on at least one block except for one pair, {x,y}, the hole, where one of x or y appear in B_1 or B_b. 10

starts An elements starts in a block of an SF-array if it is only introduced once and is introduced on the current block. 16, 35

Steiner system A Steiner system $S(t, k, v)$, $2 \leq t \leq k \leq v$, is a v-set V with a family $\mathcal{B}$ of k-subsets of V (blocks), such that every t-subset of V is contained in exactly one block, $t, k, v \in \mathbb{Z}^+$. 2

Steiner Triple System A Steiner Triple System, STS(v), is a S(2,3,v). 3

strength The strength of a design is the size of the set S that we must cover in at least one block of the design. 35

tight A change design is tight if it is economical and the bound on minimum blocks is met with equality. 8, 40

tight block A block is tight if it does not cover any s-set covered in a different block. 11

unchanged subset The unchanged subset is $U_i = B_i \cap B_{i+1}$, the elements that do not change between consecutive blocks. 6, 36, 39

V-expansion set A V-expansion set is an expansion set. 36

W-expansion set A W-expansion set is a partition of $W \subset V$ using the expansion set. 36, 39

walk A walk in a graph G is a sequence $W = v_0e_1v_1...v_{l-1}e_lv_l$ where v and e are alternating vertices and edges of G such that v_{i-1} and v_i are the ends of edge e_i, $1 \leq i \leq l$. 2

weight The weight of a block is a value based on the number of ‘S’ and ‘F’ in the SF-array. 16

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