**A Combination of Finite Element Method and
Complex Frequency Hopping Technique for
Solution of Electromagnetic Problems**

by

Meera Srinivasan

A thesis submitted to the Faculty of Graduate Studies and Research

in partial fulfillment of the requirements for the degree of

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**A Combination of Finite Element Method and Complex Frequency
Hopping Technique for Solution of Electromagnetic Problems**

submitted by Meera Srinivasan
in partial fulfillment of the requirements for
the degree of Master of Engineering

M. S. Nakhla, Thesis Supervisor
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Abstract

This thesis investigates the use of the Complex Frequency Hopping (CFH) technique to solve ElectroMagnetic (EM) problems. One of the main drawbacks of traditional EM simulators is the long computing time and numerical instability when obtaining both frequency-domain and time-domain responses. CFH, on the other hand has been used extensively in the circuit simulation area to obtain solutions for a large set of ordinary differential equations with a speed-up of two to three orders of magnitude. In this thesis, a new technique which combines CFH with the Finite Element Method (FEM) while maintaining the advantage of CFH, has been developed. Two approaches: (i) Netlist Generation and (ii) Direct Generation of Moments, to represent user supplied subnetwork information have been developed and implemented using a general CFH software. Several EM problems have been solved using the FEM & CFH technique to obtain a fairly accurate response with minimal CPU time.

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List of Symbols

E	Electric field intensity
D	Magnetic field density
H	Magnetic field intensity
B	Magnetic flux density (unless otherwise mentioned)
J	Electric current density
J_s	Surface current density
A	Magnetic vector potential (unless otherwise mentioned)
F	Electric vector potential
ρ	Electric charge density
ρ_s	Surface charge density
t	Time
ω	Angular frequency (unless otherwise mentioned)
j	Complex value indicator ($=\sqrt{-1}$)
ϵ_r	Relative permittivity
ϵ_0	Absolute permittivity ($= 8.854 \times 10^{-12} F/m$)
ϵ	Permittivity ($=\epsilon_r \epsilon_0$)
μ_r	Relative permeability
μ_0	Absolute permeability ($= 4\pi \times 10^{-7} H/m$)

μ	Permeability ($= \mu_r \mu_0$)
σ	Conductivity
ϕ	Potential (a scalar quantity)
c	Speed of light ($\approx 3 \times 10^8 m/s$)
E_z	Electric field intensity in the z direction
J_z	z component of electric current density
R	Any closed region
$\tilde{\Phi}$	Approximation of Φ
w_m	weighting functions
$[Y]$	matrix Y
R_e	residue
u_i	Basis functions
$\mathcal{F}(\tilde{\phi})$	Functional of $\tilde{\phi}$
h	Discretization step
L	Differential or Integral or Integro-differential operator
q	Charge
V	Potential
$G()$	Green's function
Ψ_j	Eigenfunctions
Ψ_j^*	Conjugate of Ψ_j
W_e	Energy associated with element e
V_e	Potential of element e
$[V_e]$	Matrix V_e
$[V_e]^t$	Matrix V_e transposed
$[C^{(e)}]$	Element coefficient matrix

$[C_{ij}^{(e)}]$	Element coefficient matrix with coupling between nodes i and j
α_i	Linear interpolation function
$\mathbf{X}(s)$	Response vector
$\mathbf{R}(s)$	Excitation vector
s	Laplace operator($= j\omega$)
N	Total number of nodes
$\mathcal{L}$	Laplace operator
λ	Eigenvalues
$\mathbf{Y}$	Matrix $\mathbf{Y}$ (unless otherwise stated)
$\mathbf{Y}^T$	Matrix $\mathbf{Y}$ transposed
$\mathbf{Y}^{(r)}$	r -th derivative of matrix $\mathbf{Y}$
$ \mathbf{Y} $	Determinant value of matrix $\mathbf{Y}$
$\mathbf{1}$	Unity matrix
$u(t)$	Unit step function
$\mathbf{M}_n$	Vector of n -th moments
m_n	n -th moment of an output
$h(t)$	Impulse response of an output
$H(s)$	Transfer function of an output
k_j	j -th system residue
p_j	p -th system pole
$\delta(t)$	Unit dirac delta function
$\mathbf{m}_A$	Moment matrix
$\mathbf{m}_v$	Padé moment vector
$\mathbf{b}_v$	Vector of unknown denominator coefficients
$\mathbf{a}_L$	Vector of unknown numerator coefficients

$\mathbf{G}_\phi$	MNA matrix $\mathbf{G}$ matrix for linear sub-network ϕ
$\mathbf{C}_\phi$	MNA matrix $\mathbf{C}$ matrix for linear sub-network ϕ
$\mathbf{V}_\phi(s)$	Vector of unknown variables in MNA formulation of linear sub-network ϕ
$\mathcal{R}^{N_\phi}$	The set of all real vectors of size N_ϕ
$\mathcal{R}^{N_\phi \times N_\phi}$	The set of all real matrices of size $N_\phi \times N_\phi$
$\mathbf{L}$	Lower triangular matrix
$\mathbf{U}$	Upper triangular matrix
f	Frequency
f_r	Resonant frequency
k_0	$= \sqrt{\omega^2 \mu_0 \epsilon_0}$

List of Acronyms and Abbreviations

AWE	Asymptotic Waveform Evaluation
CAD	Computer-Aided Design
CCCS	Current Controlled Current Source
CFH	Complex Frequency Hopping
COFFEE	COmplex Frequency hopping Electrical Evaluator
CPU	Central Processing Unit
DC	Direct Current
EM	ElectroMagnetic
EMP	ElectroMagnetic Pulse
FDM	Finite Difference Method
FEM	Finite Element Method
FEMFD	Finite Element Method Frequency Domain
FFT	Fast Fourier Transform
GHz	GigaHertz
GND	Ground
GTEM	General Transverse ElectroMagnetic
IE	Integral Equation
IFFT	Inverse Fast Fourier Transform

LDFEM	Laplace Domain Finite Element Method
MIC	Microwave Integrated Circuits
MNA	Modified Nodal Admittance
MOM	Method of Moments
NETGEN	Netlist Generator
PDE	Partial Differential Equation
TE	Transverse Electric
TM	Transverse Magnetic
VCCS	Voltage Controlled Current Source
TTEM	Triple Transverse ElectroMagnetic

Chapter 1

Introduction

1.1 Motivation

The high-tech industry today has seen a tremendous growth in the area of research and development of high frequency Electromagnetic (EM) based models. Analysis of these high frequency EM based problems have gained lot of importance in the past few years, because for these high frequencies (and short wavelengths), standard circuit theory cannot be used directly to solve certain microwave network problems. This is due to the fact that, in general, the lumped circuit element approximations of circuit theory are not valid at high frequencies. Microwave components usually are distributed elements, where the phase of a voltage or current changes significantly over the physical length of the device because the device dimensions are of the order of the microwave wavelength. Advanced techniques and computer-aided design (CAD) tools are being developed to solve these problems.

In the field of electrical engineering, EM problems are formulated in either space/frequency domain or space/time domain. In the space/frequency type, time-domain results are difficult to derive, while in the space/time domain, time-domain solutions suffer from instability and long computing time. Time-stepping schemes and the

Fast Fourier Transform (FFT) have provided solution using the space/time domain approach. In order to achieve a satisfactory resolution (stability with proper time-step) using FFT, longer computing time is needed. This is the main drawback of the time-stepping schemes. Also, with these conventional approaches obtaining both time-domain and frequency-domain solutions simultaneously has been rather difficult.

Asymptotic Waveform Evaluation (AWE) [1] and other moment-matching techniques have recently proven useful in the analysis of interconnect structures [2], [3] and various networks containing large linear structures with nonlinear terminations. Moment-matching, used in linear systems analysis as a method of model reduction, describes a method to extract a small set of dominant poles from a large network. The information is obtained from the Taylor series coefficients (moments) of the system. At the CPU cost of a little more than one DC analysis, AWE has proven to be two to three orders of magnitude faster than the full simulation techniques that require a nonlinear simulator similar to the one used in SPICE. The complicating issue of step size in numerical integration is not a problem for moment-matching in linear networks, because the entire waveform is generated simultaneously. The moment-matching technique has also shown the same speed-up over the Fast Fourier Transform (FFT) analysis needed for transmission line problems [4]. In order to improve the accuracy of Padé approximations, Complex Frequency Hopping (CFH) - a multi-point moment-matching algorithm was developed.

One of the well known techniques for numerically solving an EM problem is the Finite Element Method (FEM). Finite element analysis arose essentially as a discipline for solving problems in structural engineering, and its role in that field is still of fundamental importance today. It was soon clear that the method had implications far beyond those originally considered and that in fact the method presented a very

general and powerful technique for numerical solution of differential equations. FEM method used alone to solve EM problems has longer computing time. Hence solution of EM based models for designing practical high speed packages and boards still remains a challenging task.

1.2 Thesis Objective

The main objective of this thesis is to combine the advantages of both CFH technique and FEM approach to efficiently solve several EM problems. Based on this combined approach, a novel technique has been proposed to solve several types of EM problems. The desired features in the proposed technique are as follows:

- Considerable speed-up compared to the other conventional techniques
- Simultaneously obtaining both the frequency-domain and time-domain solutions
- Efficiently computing the dominant resonant frequencies

1.3 Outline of the Thesis

This thesis is organized as follows:

Chapter 2 This chapter starts with the basic Maxwell's equations and proceeds to obtain its second order differential equation. A brief overview of different analytical techniques is presented followed by the numerical techniques: variational technique and finite element method, used for the analysis of EM problems.

Chapter 3 describes two moment-matching techniques - AWE and CFH. Since this thesis is based on the CFH technique, more emphasis is placed on this technique.

The differences between the two approaches in CFH - Poles Approach and the Transfer Function Approach are also highlighted.

Chapter 4 (along with the following chapter) forms the core of this thesis. Here the ideas of FEM and CFH have been combined to obtain a new technique that will help solve EM problems more efficiently. The formulation used has been explained in detail and several examples dealing with diffusion and resonance problems have also been described. With these examples, an improvement in the CPU time has been demonstrated.

Chapter 5 deals with inhomogeneous cylindrical dielectric resonators. Due to the inhomogeneity in the structure, a more rigorous technique (as compared to the earlier chapter) was adopted to solve the problem. The chapter starts with the problem description and its formulation. The resonating frequency for a few of the resonators discussed in the previously published papers have been compared and CPU gain is discussed.

Chapter 6 concludes the thesis and discusses the scope for further research work.

Chapter 2

Numerical Techniques for Electromagnetic (EM) Problems

The problem of electromagnetic analysis is actually a problem of solving Maxwell's equation subject to given boundary conditions. Maxwell's equations are a set of fundamental equations governing all macroscopic electromagnetic phenomena. The equations can be written in both differential and integral form. The most common numerical techniques for solving differential equations are the Finite Difference Method (FDM), the variational technique or the Finite Element Method (FEM). Integral equations are mostly solved using the Method Of Moments (MOM).

2.1 General Methodology

Maxwell's Equations in General Differential Form

Analysis of EM problems starts with the basic Maxwell's equation which can be written in differential form [5] as follows:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (\text{Faraday's law}) \quad (2.1)$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad (\text{Maxwell - Ampere law}) \quad (2.2)$$

$$\nabla \cdot \mathbf{D} = \rho \quad (\text{Gauss's law}) \quad (2.3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss's law - magnetic}) \quad (2.4)$$

where $\mathbf{E}$ = electric field intensity (volts/meter)

$\mathbf{D}$ = electric flux density (coulombs/meter²)

$\mathbf{H}$ = magnetic field intensity (amperes/meter)

$\mathbf{B}$ = magnetic flux density (webers/meter²)

$\mathbf{J}$ = electric current density (amperes/meter²)

ρ = electric charge density (coulombs/meter³)

Another fundamental equation, which is known as the *equation of continuity*, can be written as

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad (2.5)$$

which represents the conservation of charge.

For the solution of any electromagnetic problem, only the first three equations (2.1) - (2.3) or equations (2.1), (2.2), (2.4) are sufficient. These are therefore called independent equations and the other two are called the auxiliary or dependent equations.

Electro- and Magnetostatic Fields

A static field is generated when there is no variation of the field quantity with respect to time. The equations (2.1), (2.2) and (2.5) can be written as

$$\nabla \times \mathbf{E} = 0 \quad (2.6)$$

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (2.7)$$

$$\nabla \cdot \mathbf{J} = 0 \quad (2.8)$$

$$(2.9)$$

whereas (2.3), (2.4) remain the same. It is evident that in this case there is no interaction between electric and magnetic fields, and therefore either an electrostatic case described by (2.3) and (2.6) or a magnetostatic case described by (2.4) and (2.7), with (2.8) being a result of (2.7), can be obtained separately.

Time-Harmonic Fields

When the field quantities in Maxwell's equations are harmonically oscillating functions with a single frequency, a time-harmonic field is generated. Using complex phasor notation, equations (2.1), (2.2), (2.5) can be written in a simplified form as

$$\nabla \times \mathbf{E} = -j\omega\mathbf{B} \quad (2.10)$$

$$\nabla \times \mathbf{H} = j\omega\mathbf{D} + \mathbf{J} \quad (2.11)$$

$$\nabla \cdot \mathbf{J} = -j\omega\rho \quad (2.12)$$

where the time convention $e^{j\omega t}$ is used and suppressed, and ω is angular frequency. It is obvious that the electric and magnetic fields must exist simultaneously and they interact with each other, and also static case is the limiting case of the harmonic fields as the frequency ω , approaches zero.

Constitutive relationships that are necessary to fully define Maxwell's equations are:

$$\mathbf{D} = \epsilon \mathbf{E} \quad (2.13)$$

$$\mathbf{B} = \mu \mathbf{H} \quad (2.14)$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (2.15)$$

where ϵ , μ and σ are the permittivity, permeability and conductivity of the medium under consideration.

Scalar and Vector Potentials

To solve Maxwell's equations, the first order differential equations involving two field quantities are converted into second order differential equations involving only one field quantity. The scalar electrostatic field is governed by (2.3) and (2.6) and can be satisfied by representing electric field intensity $\mathbf{E}$ as

$$\mathbf{E} = -\nabla \phi \quad (2.16)$$

where ϕ is the scalar potential. Substituting (2.16) into (2.3) with the aid of (2.13), one obtains

$$-\nabla \cdot (\epsilon \nabla \phi) = \rho \quad (2.17)$$

which is the second order differential equation governing ϕ . Equation (2.17) is the well-known Poisson equation. When $\rho = 0$, (2.17) becomes Laplace equation $-\nabla \cdot (\epsilon \nabla \phi) = 0$ or for constant ϵ , it is $\epsilon \nabla^2 \phi = 0$.

The magnetostatic field is governed by (2.4) and (2.7), where (2.4) can be satisfied by representing the magnetic flux density $\mathbf{B}$ as

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (2.18)$$

where $\mathbf{A}$ is called the magnetic vector potential. Substituting (2.18) into (2.7) with the aid of (2.14), yields the second order differential equation

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} \quad (2.19)$$

Since $\mathbf{A}$ here does not have a unique solution, one must impose a condition on its divergence

$$\nabla \cdot \mathbf{A} = 0 \quad (2.20)$$

Wave Equations

As mentioned earlier, Maxwell's equations are coupled first-order differential equations which are difficult to apply when solving boundary value problems [6]. The difficulty is overcome by decoupling the first-order equations, thereby obtaining the wave equation, a second-order differential equation for a linear, isotropic, homogeneous, source-free medium ($\rho = 0, J = 0$) from (2.1) - (2.4). The curl of (2.1) gives:

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) \quad (2.21)$$

and (2.2) yields:

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

since $J = 0$.

Equation (2.21) can be rewritten as:

$$\nabla \times \nabla \times \mathbf{E} = -\mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} \quad (2.22)$$

Applying the vector identity

$$\nabla \times \nabla \times \mathbf{F} = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$$

to (2.22),

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Since $\rho = 0$, $\nabla \cdot \mathbf{E} = 0$ from (2.3) and (2.13). The above equation simplifies to

$$\nabla^2 \mathbf{E} - \mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (2.23)$$

which is the time-dependent vector Helmholtz equation or simply wave equation.

Similarly the wave equation for $\mathbf{H}$ can be obtained as

$$\nabla^2 \mathbf{H} - \mu \epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad (2.24)$$

Equations (2.23) and (2.24) are the equations of motion of EM waves in the medium under consideration.

Boundary Conditions

A description of an electromagnetic problem is incomplete without its associ-

Figure 2.1: A closed region

ated boundary conditions. In addition to finding the unknown function ϕ of a partial differential equation, ϕ should satisfy certain boundary conditions on S , the boundary of the closed region, R . Usually, these boundary conditions are of the Dirichlet and Neumann types. When a boundary has both, a mixed boundary condition is said to exist.

Dirichlet boundary condition:

$$\phi(r) = 0, \quad r \text{ on } S. \quad (2.25)$$

Neumann boundary condition:

$$\frac{\partial \phi(r)}{\partial n} = 0, \quad r \text{ on } S. \quad (2.26)$$

i.e., the normal derivative of ϕ vanishes on S .

Mixed boundary condition:

$$\frac{\partial \phi(r)}{\partial n} + h(r) \phi(r) = 0, \quad r \text{ on } S \quad (2.27)$$

where $h(r)$ is a known function and $\partial \phi / \partial n$ is the directional derivative of ϕ along the outward normal to the boundary S , i.e., $\partial \phi / \partial n = \nabla \phi \cdot \hat{n}$ where $\hat{n}$ is a unit normal directed out of R as shown in figure 2.1.

The Neumann boundary condition is a special case of the mixed condition with $h(r) = 0$. The conditions in (2.25) to (2.27) are called homogeneous boundary conditions.

The more general ones are the inhomogeneous:

$$\begin{aligned} \text{Dirichlet} &: \phi(r) = p(r), \quad r \text{ on } S \\ \text{Neumann} &: \frac{\partial \phi(r)}{\partial n} = q(r), \quad r \text{ on } S \\ \text{Mixed} &: \frac{\partial \phi(r)}{\partial n} + h(r) \phi(r) = w(r), \quad r \text{ on } S \end{aligned}$$

where $p(r)$, $q(r)$ and $w(r)$ are explicitly known functions on the boundary S . The term *homogeneous* has been used to mean different things. The solution region could be homogeneous meaning that σ , ϵ , and μ are constant within R ; the partial differential equations could be homogeneous if $g = 0$ in

$$\nabla^2 \phi + k^2 \phi = g$$

and the boundary conditions are homogeneous when

$$p(r) = q(r) = w(r) = 0$$

At the interface between two media, say medium 1 and medium 2, the boundary conditions can be expressed mathematically [5] as:

$$\hat{n} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \quad (2.28)$$

$$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0 \quad (2.29)$$

for electric fields and similarly,

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = 0 \quad (2.30)$$

$$\hat{n} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \quad (2.31)$$

for magnetic fields, where $\hat{n}$ is the unit vector normal to the interface, pointing from medium 2 into medium 1.

If surface currents or surface charges exist at the interface, then the equations have to be denoted as

$$\hat{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (2.32)$$

$$\hat{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (2.33)$$

where $\mathbf{J}_s$ is the surface electric current density and ρ_s is the surface charge density.

At a perfectly conducting surface, (2.28) becomes

$$\hat{n} \times \mathbf{E} = 0 \quad (2.34)$$

since a perfect conductor cannot sustain a field inside and (2.31) reduces to

$$\hat{n} \cdot \mathbf{B} = 0 \quad (2.35)$$

where $\mathbf{E}$ and $\mathbf{B}$ are the fields exterior to the conductor and $\hat{n}$ is the normal pointing away from the conductor.

2.2 Analytical Techniques

The most satisfactory solution of a field problem is an exact mathematical one, which is the analytical method. The most commonly used analytical methods [6] in solving EM-related problems include:

- Separation of variables
- Green's functions
- Conformal mapping
- Integral methods

The main disadvantage of this technique is that it can be applied only to restricted geometries. In some cases, the geometry is modified and the closest analytical solution is obtained. Although in many practical cases, such an analytical solution cannot be obtained and a numerical approximate solution is the fastest one, analytical solution is useful in checking solutions obtained from numerical methods.

2.3 Numerical Techniques

The numerical techniques commonly used are:

- Variational Method
- Finite Difference Method (FDM)
- Method of Moments (MOM)
- Finite Element Method (FEM)

2.3.1 Variational Method

In solving problems arising from mathematical physics and engineering, it is often possible to replace the problem of integrating a differential equation by the equivalent problem of seeking a function that gives a minimum value of some integral. Problems of this type are called the *variational methods* [7]. The variational methods form a

common base for both the method of moments (MOM) and the finite element method (FEM).

Variational methods [6] can be classified into two groups: direct and indirect methods. The direct method is the classical Rayleigh-Ritz method, while the indirect methods are collectively referred to as the method of weighted residuals: collocation (or point-matching), subdomain, Galerkin, and least square methods. The variational solution of a given Partial Differential Equation (PDE) using an indirect method usually involves two basic steps [8]:

1. cast the PDE into variational form, and
2. determine the approximate solution using one of the weighted residual methods.

Rayleigh-Ritz Method

The Rayleigh-Ritz method [6] is the direct variational method for minimizing a given functional. It is direct in that it yields a solution to the variational problem without recourse to the associated differential equation [9]. A general associated variational principle is given as:

$$I(\phi) = \int_S F(x, y, \phi, \phi_x, \phi_y) \, dS \quad (2.36)$$

The next objective is to minimize this integral. In the Rayleigh-Ritz method, a linearly independent set of functions called *expansion functions* (or *basis functions*) u_n is selected and an approximate solution to (2.36) is constructed satisfying some prescribed boundary conditions. The solution is in the form of a finite series

$$\tilde{\phi} \simeq \sum_{n=1}^N a_n u_n + u_o \quad (2.37)$$

where a_n are expansion coefficients to be determined and $\tilde{\phi}$ is an approximate solution to ϕ (the exact solution). Equation (2.37) is substituted into (2.36) and the integral

$I(\phi)$ is converted into a function of N coefficients $a_1, a_2, \dots, a_N$, i.e.,

$$I(\phi) = I(a_1, a_2, \dots, a_N) \quad (2.38)$$

The minimum of this function is obtained when its partial derivatives with respect to each coefficient is zero, i.e., :

$$\frac{\partial I}{\partial a_n} = 0, \quad n = 1, 2, \dots, N \quad (2.39)$$

A set of N simultaneous equations is obtained. The system of linear algebraic equations obtained is solved to get a_n , which are finally substituted into the approximate solution of (2.37). In the approximate solution of (2.37), if $\bar{\phi} \rightarrow \phi$ as $N \rightarrow \infty$, then the procedure is said to converge to the exact solution.

The Rayleigh-Ritz method has two major limitations: first, the variational principle in (2.36) may not exist in some problems such as in nonself-adjoint equations (odd order derivatives); second, it is difficult, if not impossible, to find the functions u_n satisfying the global boundary conditions for the domains with complicated geometries.

Weighted Residual Method

The Rayleigh-Ritz method is applicable only when a suitable functional exists. In cases where such a functional is difficult to find, one of the techniques collectively referred to as the *method of weighted residuals* [6] is applied. This method is more general and has wider application than the Rayleigh-Ritz method because it is not limited to a class of variational problems.

Consider the operator equation,

$$L \phi = g \quad (2.40)$$

In the weighted residual method, the solution to the above equation is approximated, in the same manner as in the Rayleigh-Ritz method using the expansion functions, u_n , i.e.,

$$\tilde{\phi} = \sum_{n=1}^N a_n u_n \quad (2.41)$$

where a_n is the expansion coefficient. If a set of *weighting functions* w_m (also known as *testing functions*) are chosen and the inner product of the above equation is taken for each w_m , then the summation results in:

$$\sum_{n=1}^N a_n \langle w_m, Lu_n \rangle = \langle w_m, g \rangle, \quad m = 1, 2, \dots, N \quad (2.42)$$

The system of linear equations (2.42) can be casted into matrix form as

$$[Y][X] = [R] \quad (2.43)$$

where $Y_{mn} = \langle w_m, Lu_n \rangle$, $R_m = \langle w_m, g \rangle$ - an *excitation vector*, $X_n = a_n$. Solving for $[X]$ in the above equation and substituting for a_n in (2.41) gives the approximate solution to (2.40). However, substitution of the approximate solution in the operator equation results in a *residual* R_e (an error in the equation), i.e.,

$$R_e = L(\tilde{\phi} - \phi) = L\tilde{\phi} - g \quad (2.44)$$

In the weighted residual method, the weighting functions w_m (which, in general are not the same as the expansion functions u_n) are chosen such that the integral of a weighted residual of the approximation is zero, i.e.,

$$\int w_m R_e dv = 0$$

or

$$\langle w_m, R_e \rangle = 0 \quad (2.45)$$

There are different ways [6] - [9], of choosing the weighting functions w_m leading to:

- Collocation (or point-matching method).
- Subdomain method,
- Galerkin method and
- Least square method.

Of these methods, the Galerkin method is of particular interest since this method has been adopted in this thesis for the analysis of inhomogeneous cylindrical dielectric resonators (Chapter 5). Hence, the Galerkin method is explained in detail in the following section.

Galerkin Method

In this method, the basis functions are selected as the weighting function, i.e., $w_m = u_n$. A special case of the Galerkin method is when the linear differential operator is of even order reducing the method to the Rayleigh-Ritz method. This is because the differentiation is transferred to the weighting function and the resulting coefficient matrix $[Y]$ in (2.43) will be symmetric. In order for the Galerkin method to be applicable, the operator must be of a certain specific type. Also, the expansion function u_n must span both the domain and the range of the operator.

The variational technique can be summarized as follows:

- There are systematic ways for finding the functional for any differential equation of interest.
- Once the functional is known, ϕ is expanded as a linear combination of basis function.

$$\phi \approx \tilde{\phi} = \sum_{i=1}^N \phi_i u_i$$

u_i are known *basis functions* and ϕ_i are *unknown coefficients*.

- $\mathcal{F}(\tilde{\phi})$ is then minimized by forcing

$$\frac{d\mathcal{F}(\tilde{\phi})}{d\phi_i} = 0$$

which produces an $N \times N$ matrix.

Other numerical techniques like FDM and MOM have been explained in detail in Appendix. The following section deals with FEM. FEM is a more powerful and versatile numerical technique for handling complex geometries and inhomogeneous media.

2.4 Finite Element Method (FEM)

The finite element analysis of any problem [6] involves basically four steps:

1. discretizing the solution region into a finite number of subregions or elements,
2. deriving the element matrix for a typical element,
3. assembling all the element matrices in a global matrix and
4. solving the system of equations obtained.

The above four steps can be applied to solve a Laplace's equation $\nabla^2 V = 0$ in domain S in figure 2.2, subject to particular boundary condition as follows: Discretization of the subregion involves dividing up the solution region into subdomains, called finite elements. These elements form a mesh over the solution domain. A convenient formulation is applied to these elements, [10] from which a system of equations is obtained. The solution to this system provides the solution to the problem. It is

Figure 2.2: (a) The domain of analysis; (b) its finite element discretization

clear that the smaller these elements are (or the denser the mesh), the closer the numerical solution will be to the exact, or continuous solution. To apply the finite element method to this problem, it is necessary to discretize the solution domain into small regions, figure 2.2. The finite elements chosen are triangular in shape. An important point in finite element discretization is that an element cannot contain two different materials. The boundary between two materials must also be a boundary between finite elements.

An approximation for the potential V_e within an element e is found and then inter-related to the potential distribution in various elements such that the potential is continuous across interelement boundaries. The approximate solution for the whole region is:

$$V(x, y) \approx \sum_{e=1}^n V_e(x, y) \quad (2.46)$$

where n is the number of triangular elements into which the solution region is divided. The most common form of approximation for V_e within an element is a polynomial approximation. The approximation for a triangular element is:

$$V_e(x, y) = a + bx + cy \quad (2.47)$$

and for a quadrilateral element is:

$$V_e(x, y) = a + bx + cy + dxy \quad (2.48)$$

The potential V_e in general is nonzero within element e but zero outside e . Linear variation of potential within a triangular element also implies that the electric field is uniform within the element, i.e.,

$$E_e = -\nabla V_e = -(ba_x + ca_y) \quad (2.49)$$

Governing Equations and Matrices

Figure 2.3: Typical triangular element with its local node numbering

A typical triangular element is shown in figure 2.3. The potential V_{e1} , V_{e2} , and

V_{e1} , at nodes 1, 2, and 3 respectively are obtained using (2.47), i.e.,

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad (2.50)$$

The coefficients a , b , and c are determined from (2.50) as:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \quad (2.51)$$

Substituting this into (2.47) gives

$$V_e = \begin{bmatrix} 1 & x & y \end{bmatrix} \frac{1}{2A} \begin{bmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \quad (2.52)$$

or

$$V_e = V_{ei} \sum_{i=1}^3 \alpha_i(x, y) \quad (2.53)$$

where

$$\begin{aligned} \alpha_1 &= \frac{1}{2A} [(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y] \\ \alpha_2 &= \frac{1}{2A} [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y] \\ \alpha_3 &= \frac{1}{2A} [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \end{aligned} \quad (2.54)$$

and A is the area of the element e , i.e.,

$$\begin{aligned} 2A &= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \\ &= (x_1y_2 - x_2y_1) + (x_3y_1 - x_1y_3) + (x_2y_3 - x_3y_2) \\ A &= \frac{1}{2} [(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] \end{aligned} \quad (2.55)$$

The potential at any point (x, y) within the element can be determined using (2.53) provided the potential at the vertices are known whereas in finite difference analysis,

the potential can be calculated only at the grid points. The function α_i are linear interpolation functions. They are called the *element shape functions* and they follow these properties [11]:

$$\alpha_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \quad (2.56)$$

$$\sum_{i=1}^3 \alpha_i(x, y) = 1 \quad (2.57)$$

The functional corresponding to Laplace's equation, $\nabla^2 V = 0$, is given by

$$W_e = \frac{1}{2} \int \epsilon |\mathbf{E}|^2 dS = \frac{1}{2} \int \epsilon |\nabla V_e|^2 dS \quad (2.58)$$

(Physically, the functional W_e is the energy per unit length associated with the element e .) Substituting (2.53) into the above equation gives

$$W_e = \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 \epsilon V_{e_i} \left[\int \nabla \alpha_i \cdot \nabla \alpha_j dS \right] V_{e_j}. \quad (2.59)$$

The term in the brackets can be defined as

$$C_{ij}^{(e)} = \int \nabla \alpha_i \cdot \nabla \alpha_j dS, \quad (2.60)$$

The functional W_e can be written in matrix form as

$$W_e = \frac{1}{2} \epsilon [V_e]^t [C^{(e)}] [V_e] \quad (2.61)$$

where

$$[V_e] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} \quad (2.62)$$

where superscript t denotes the transpose of the matrix and

$$[C^{(e)}] = \begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix} \quad (2.63)$$

The matrix $[C^e]$ is usually called the *element coefficient matrix* and the matrix element $C_{ij}^{(e)}$ denotes the coupling between the nodes i and j ; its value can be obtained from (2.60) and the corresponding relation from (2.54). For example,

$$\begin{aligned} C_{12}^{(e)} &= \int \nabla \alpha_1 \cdot \nabla \alpha_2 dS \\ &= \frac{1}{4A^2} [(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)] \int dS \\ &= \frac{1}{4A} [(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)] \end{aligned}$$

Similarly,

$$\begin{aligned} C_{13}^{(e)} &= \frac{1}{4A} [(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)] \\ C_{23}^{(e)} &= \frac{1}{4A} [(y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1)] \\ C_{11}^{(e)} &= \frac{1}{4A} [(y_2 - y_3)^2 + (x_3 - x_2)^2] \\ C_{22}^{(e)} &= \frac{1}{4A} [(y_3 - y_1)^2 + (x_1 - x_3)^2] \\ C_{33}^{(e)} &= \frac{1}{4A} [(y_1 - y_2)^2 + (x_2 - x_1)^2] \end{aligned}$$

Also,

$$C_{21}^{(e)} = C_{12}^{(e)}, \quad C_{31}^{(e)} = C_{13}^{(e)}, \quad C_{32}^{(e)} = C_{23}^{(e)}. \quad (2.64)$$

After having obtained the analysis for a single element, the next step in finite element analysis is to assemble all such elements in the solution region. The total energy W , associated with the assembly of elements is

$$W = \sum_{e=1}^n W_e = \frac{1}{2} \epsilon [V]^t [C] [V] \quad (2.65)$$

where

$$[V] = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_N \end{bmatrix} \quad (2.66)$$

N is the number of nodes, n is the number of elements, and $[C]$ is called the *global coefficient matrix*, which is the assembly of individual element coefficient matrices. In case an inhomogeneous solution region is considered, the region is so discretized that each finite element is homogeneous. In this case $\epsilon_r (= \epsilon/\epsilon_o)$ varies from element to element. Equation (2.60) can be rewritten taking ϵ_r into consideration as:

$$C_{ij}^{(e)} = \epsilon_r \int \nabla \alpha_i \cdot \nabla \alpha_j \, dS, \quad (2.67)$$

Figure 2.4: Assembly of three elements; $i-j-k$ corresponds to local numbering (1-2-3) of the element

The process by which the individual elements are assembled can be best illustrated with an example. Consider the finite element mesh consisting of three finite elements as shown in the figure 2.4. The numbering of nodes 1, 2, 3, 4, and 5 is called *global numbering*. The numbering $i-j-k$ is called *local numbering*, and it corresponds to 1-2-3 of an element in figure 2.3. For example, for element 3 in figure 2.4, the global numbering 3-5-4 corresponds with local numbering 1-2-3. The numbering has to be done either in clockwise or anticlockwise direction. Assuming an anticlockwise

numbering pattern as shown in figure 2.4, the global coefficient matrix has the form

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix} \quad (2.68)$$

which is a 5×5 matrix since five nodes ($N = 5$) are involved. Again, C_{ij} is the coupling between nodes i and j . Any C_{ij} can be obtained using the fact that the potential distribution must be continuous across interelement boundaries. The contribution to the i, j position in $[C]$ comes from all elements containing nodes i and j . For example, in figure 2.4, elements 1 and 2 have 1 in common; hence

$$C_{11} = C_{11}^{(1)} + C_{11}^{(2)}. \quad (2.69)$$

Node 2 belongs to element 1 only; hence

$$C_{33} = C_{33}^{(1)}. \quad (2.70)$$

Node 4 belongs to elements 1, 2, and 3; consequently

$$C_{44} = C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)}. \quad (2.71)$$

Nodes 1 and 4 belong simultaneously to elements 1 and 2; hence

$$C_{14} = C_{41} = C_{12}^{(1)} + C_{13}^{(2)}. \quad (2.72)$$

Since there is no coupling (or direct link) between nodes 2 and 3,

$$C_{23} = C_{32} = 0. \quad (2.73)$$

Similarly, all the elements in the global coefficient matrix can be obtained by inspec-

tion of figure 2.4 as:

$$\begin{bmatrix}
 C_{11}^{(1)} + C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & 0 \\
 C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} & 0 \\
 C_{21}^{(2)} & 0 & C_{22}^{(2)} + C_{11}^{(3)} & C_{23}^{(2)} + C_{13}^{(3)} & C_{12}^{(3)} \\
 C_{21}^{(1)} + C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)} + C_{31}^{(3)} & C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)} & C_{32}^{(3)} \\
 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} & C_{22}^{(3)}
 \end{bmatrix} \quad (2.74)$$

Summarizing, following are the properties of $[C]$ matrix:

- Matrix is symmetric ($C_{ij} = C_{ji}$)
- In general, the matrix is sparse. Matrix $[C]$ is also banded if the nodes are carefully numbered. Using all the relations from (2.64) results in:

$$\sum_{i=1}^3 C_{ij}^{(e)} = 0 = \sum_{j=1}^3 C_{ij}^{(e)}$$

- It is singular (Adding columns 2 and 3 of matrix (2.63) to column 1 yields zeros in column 1).

2.5 FEM as applied to Diffusion and Resonance Problems

For a homogeneous, isotropic and a linear medium, starting from Maxwell's equations the following scalar equation can be derived as:

$$\nabla^2 \Phi - \epsilon \mu \frac{\partial^2 \Phi}{\partial t^2} - \sigma \mu \frac{\partial \Phi}{\partial t} = R(x, y, z, t) \quad (2.75)$$

where $\Phi(x, y, z, t)$ represents either the electric or magnetic field components, and ϵ , μ , σ are the permittivity, permeability, conductivity of the medium respectively.

$R(x, y, z, t)$ is related to the external excitation which could be a function of time, space or both. This expression can be applied to either a diffusion problem where the second term in the left-hand-side of (2.75) is dropped or to a wave equation where the third term is dropped. Applying the finite element method [12] to (2.75) where the interpolating functions are selected in exactly the same fashion as compared to the time independent problems except now the nodal values are taken to be functions of time rather than constants. Expanding the unknown function Φ in the triangular finite element domain Ω_e similar to (2.53) gives:

$$\Phi(x, y, z, t) = \sum_{n=1}^{N_{ep}} \phi_i(t) \alpha_i(x, y, z) \quad (2.76)$$

where N_{ep} is the total number of nodes in element Ω_{ep} , $\phi_i(t)$ denotes the local set of unknown time dependent expansion coefficients and $\alpha_i(x, y, z)$ are nodally based interpolation functions such as those regularly used in FEM triangular formulations [11]. It is of interest to note that in (2.76), the spatial variables x , y and z are discretized whereas the temporal variable t is not. An appropriate functional to be minimized for (2.75) is

$$F(\Phi) = \int_{\Omega} \left(\frac{1}{2} |\nabla \Phi|^2 + \epsilon \mu \Phi \Phi'' + \sigma \mu \Phi \Phi' + R \Phi \right) d\Omega \quad (2.77)$$

where Ω represents the finite element region. By minimizing the functional of the problem and applying associated Dirichlet and Neumann boundary conditions, a system of ordinary differential equations is obtained as

$$A \Psi + \sigma \mu \hat{B} \Psi' + \epsilon \mu \hat{C} \Psi'' = q \quad (2.78)$$

where Ψ , A , $\hat{B}$, $\hat{C}$, and q are defined as follows

$$\Psi = \{\phi_i\}; i = 1, \dots, N \quad (2.79)$$

$$\mathbf{A} = \sum_{e_p=1}^{N_e} [\mathbf{S}]_{e_p} \quad (2.80)$$

$$\hat{\mathbf{B}} = \hat{\mathbf{C}} = \sum_{e_p=1}^{N_e} [\mathbf{T}]_{e_p} \quad (2.81)$$

$$\mathbf{q} = \sum_{e_p=1}^{N_e} [\mathbf{G}]_{e_p} \quad (2.82)$$

where N_e denotes the total number of elements. $\mathbf{A}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{C}}$ are $N \times N$ symmetric, positive definite matrices assembled from $[\mathbf{S}]_{e_p}$ and $[\mathbf{T}]_{e_p}$ where N is the total number of nodes. $\mathbf{q}$ is a vector of dimension N , assembled from $[\mathbf{G}]_{e_p}$, containing the forced terms attributed to the time or space excitation. $[\mathbf{S}]_{e_p}$ and $[\mathbf{T}]_{e_p}$ are real symmetric square element matrices and $[\mathbf{G}]_{e_p}$ is a column vector and these matrices are:

$$[\mathbf{S}_{ij}]_{e_p} = \int_{\Omega_{e_p}} \nabla \alpha_i^p \cdot \nabla \alpha_j^p d\Omega_{e_p} \quad (2.83)$$

$$[\mathbf{T}_{ij}]_{e_p} = \int_{\Omega_{e_p}} \alpha_i^p \alpha_j^p d\Omega_{e_p} \quad (2.84)$$

$$[\mathbf{G}_i]_{e_p} = \int_{\Omega_{e_p}} \alpha_i^p f(t) d\Omega_{e_p} \quad (2.85)$$

subject to the initial conditions

$$\Psi(0) = \Psi_0$$

$$\Psi'(0) = \Psi'_0$$

where Ω_{e_p} denotes the domain of the finite elements. In (2.83) to (2.85), p refers to either 1, 2 or 3 corresponding to one, two or three dimensional finite element solution scheme while α^p is the corresponding shape function and e_p represents the finite elements. Taking the Laplace transform of (2.78) results in

$$(\mathbf{C}s^2 + \mathbf{B}s + \mathbf{A})\mathbf{X}(s) = \mathbf{R}(s) \quad (2.86)$$

and, in general can be written as:

$$\mathbf{Y}(s)\mathbf{X}(s) = \mathbf{R}(s) \quad (2.87)$$

where

$$\begin{aligned} \mathbf{B} &= \mu\sigma\hat{\mathbf{B}}, \\ \mathbf{C} &= \epsilon\mu\hat{\mathbf{C}}, \\ \mathbf{X}(s) &= \mathcal{L}[\Psi(t)], \\ \mathbf{R}(s) &= s\mathbf{C}(\Psi_0 + \Psi'_0) + \mathbf{B}\Psi_0 + \mathbf{Q}(s) \end{aligned} \quad (2.88)$$

where $\mathbf{Q}(s) = \mathcal{L}[\mathbf{q}(t)]$.

2.6 Solution to the System of Equations

Summarizing, any numerical technique converts a set of Partial Differential Equations (PDE) or Integral Equations (IE) to linear system of equations which can be defined by a matrix $[\mathbf{Y}]$, multiplied by a vector $\{\mathbf{X}\}$ to get a vector $\{\mathbf{R}\}$. Solution of this system of equations would be the final step in a finite element analysis. In general, the resultant system has one of the following two forms:

$$[\mathbf{Y}]\{\mathbf{X}\} = \{\mathbf{R}\} \quad (2.89)$$

or

$$[\mathbf{M}]\{\mathbf{X}\} = \lambda[\mathbf{N}]\mathbf{X} \quad (2.90)$$

Equation (2.89) is of the deterministic type, resulting from either an inhomogeneous differential equation or inhomogeneous boundary conditions or both. In electromagnetics, deterministic systems are usually associated with scattering, radiation,

and other deterministic problems where there exists a source or excitation. Equation (2.90) is of the eigenvalue type, resulting from a homogeneous governing differential equation and homogeneous boundary conditions. In electromagnetics, eigenvalue systems are usually associated with source free problems such as wave propagation in waveguides and resonances in cavities. In this case, the known vector $\{\mathbf{R}\}$ is zero and the matrix $[\mathbf{Y}]$ can be written as $[\mathbf{M}] - \lambda[\mathbf{N}]$, where λ denotes the unknown eigenvalues.

2.7 Computational Results

Let's illustrate the computational time by considering an example of rectangular waveguide described in more detail in Chapter 4 (FEM/CFH). The CPU time taken for the analysis varies super-linearly with the size of the matrix as shown in Table 2.1. For a matrix size of 1431 (the number of nodes chosen for finite element discretization being equal to 1431), the CPU time is given to be approximately 10 hours. The number of nodes, i.e., the matrix size was increased till the transverse resonant frequencies of the rectangular waveguide using FEMFD matched accurately with that of the analytical solution [Example 3 of Chapter 4].

Quite a number of approaches have been taken to solve this problem. The size of the equations to be solved is usually large and the conventional solution algorithms are restricted by computing time and stability conditions. For example, frequently used time-stepping schemes [13], [14], [15], [16], [17] require the satisfaction of the stability condition, that is, the ratio of spatial to temporal subdivision is to be greater than or equal to the speed of propagation [12]. This means the finer the mesh, the smaller the time step that must be chosen [18]. Implicit variable time step integration

Matrix Size	FEMFD (seconds)
25×25	5.8
255×255	532.35
289×289	753.26
357×357	921.39
925×925	9364.68
1431×1431	36538.36 (10.1 hours)

Table 2.1: CPU Time Comparison for a Rectangular Waveguide

algorithms [19] can be used. However, they require solution of a large set of algebraic equations at each time point. Frequently, both time-domain and frequency domain results are of interest [15] - [16]. In this case, the Fast Fourier Transform (FFT) and its inverse can be used to move from one solution space to the other. This can increase the computational time since in order to achieve satisfactory resolution, FFT has to span longer time and frequency intervals with smaller increments [20]. In an attempt to improve the efficiency of FEM solution techniques, a Laplace Domain FEM (LDFEM) has been proposed in [21]. This method is an extension of the technique used for solution of heat transfer problems [22], [23]. However, it is based on congruence transformation which involves the computationally expensive process of determining all the eigenvalues and eigenvectors of a large matrix.

From the conceptual point of view, the new technique proposed in this paper falls in the category of [21], [22], [23]. However, the main difference is that it requires the computation of the dominant natural modes only, thus eliminating the major computing cost.

Chapter 3

Moment-Matching Techniques

Moment-matching techniques such as Asymptotic Waveform Evaluation (AWE) [1] and Complex Frequency Hopping (CFH) [24] have been topics of intense research in the circuit simulation area in recent years. They have been successfully and efficiently applied for obtaining the solution of a large set of ordinary differential equations.

3.1 Asymptotic Waveform Evaluation

AWE [24] - a moment-matching technique in general, consists of two main parts: 1. Moment-generation, and 2. Moment-matching. AWE approximates the transient response of a linear circuit by first expanding the system equations using Taylor's series at $s = 0$ (Maclaurin series) in the frequency-domain. The first $2q$ moments (or coefficients) of the series (as well as the initial time conditions), are then matched to a low-order q -pole model. An efficient set of algorithms for this purpose is developed.

State Space Formulation

AWE can best be described in terms of the differential state equations for a lumped, linear, time-invariant circuit. The network [25] can be described as follows:

$$\begin{aligned}\frac{d}{dt}\mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) \\ \mathbf{y}(t) &= \mathbf{c}^T\mathbf{x}(t) + \mathbf{D}u(t)\end{aligned}\quad (3.1)$$

where $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}^T$, $\mathbf{D}$ are the matrices with $\mathbf{c}^T$ being the transpose of matrix $\mathbf{c}$, $\mathbf{x}(t)$ is the n -dimensional state vector, $u(t)$ the m -dimensional excitation vector, and $\mathbf{y}(t)$ the vector of required outputs.

Figure 3.1: Simple state space network.

Consider a simple example as shown in figure (3.1). The state space equations for the network are given as:

$$\begin{bmatrix} \frac{d}{dt}v_{C_1} \\ \frac{d}{dt}v_{C_2} \end{bmatrix} = \begin{bmatrix} -(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}) & \frac{1}{R_2C_1} \\ \frac{1}{R_2C_2} & -\frac{1}{R_2C_2} \end{bmatrix} \begin{bmatrix} v_{C_1} \\ v_{C_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} v_{in}$$

$$v_{out} = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} v_{C_1} \\ v_{C_2} \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} v_{in}$$

Taking the Laplace transform of (3.1) gives:

$$\begin{aligned} sX(s) - x(0^+) &= AX(s) + bU(s) \\ Y(s) &= c^T X(s) + DU(s) \end{aligned} \quad (3.2)$$

Solving for $Y(s)$,

$$Y(s) = c^T (sI - A)^{-1} (bU(s) + x(0^+)) + DU(s) \quad (3.3)$$

where I represents a unity matrix.

Superposition can be applied to the above linear system. $Y(s)$ may be decomposed into $DU(s)$, which is a direct coupling between the input and the output and needs no special calculation, $c^T (sI - A)^{-1} (bU(s))$, which is the response of the system to input $U(s)$, and $c^T (sI - A)^{-1} x(0^+)$, which is the response of the system to an impulse source equal to the initial conditions of the network, $x(0^+)$. The AWE calculation can be simplified by calculating an impulse response once, and then using the impulse response to calculate the network response of any other input waveform. However, the number of calculations required will depend on the vectors $U(s)$ and $x(0^+)$.

For a single input waveform $U(s) = U(s)$ (as in the example above, $U(s) = v_{in}$), and a set of zero initial conditions, the solution simplifies to:

$$Y(s) = c^T (sI - A)^{-1} bU(s) + DU(s) \quad (3.4)$$

The goal of AWE is to reduce the system transfer function, $c^T (sI - A)^{-1} b$, to an approximate transfer function, using an efficient model reduction method. Taking the single source $U(s) = 1.0$, a scalar representing an impulse $\delta(t)$ in the time domain,

the above equation becomes

$$\mathbf{Y}(s) = \mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + \mathbf{D} \quad (3.5)$$

Applying AWE, (3.5) is solved once to obtain an approximate transfer function. The response to any other input waveform, $u(t)$ may then be calculated by multiplying (in the frequency domain) or by convoluting (in the time domain) the new input waveform with the approximate impulse response generated by AWE. Any nonzero direct coupling term may be superimposed on this solution.

System Moments

Expansion of $\mathbf{Y}(s)$ in terms of a Maclaurin series results in

$$\begin{aligned} \mathbf{Y}(s) &= \mathbf{D} - \mathbf{c}^T \mathbf{A}^{-1} \mathbf{b} + s \mathbf{c}^T \mathbf{A}^{-1} \mathbf{b} + s^2 \mathbf{c}^T \mathbf{A}^{-2} \mathbf{b} + \dots \\ \text{or} \\ \mathbf{Y}(s) &= \mathbf{M}_0 + s \mathbf{M}_1 + s^2 \mathbf{M}_2 + \dots \end{aligned} \quad (3.6)$$

The moments, $\mathbf{M}_i$ for the expansion of $\mathbf{Y}(s)$ are given as:

$$\begin{aligned} \mathbf{M}_0 &= \mathbf{D} - \mathbf{c}^T \mathbf{A}^{-1} \mathbf{b} \\ \mathbf{M}_i &= \mathbf{c}^T \mathbf{A}^{-(i+1)} \mathbf{b} \quad (i > 0) \end{aligned} \quad (3.7)$$

The $\mathbf{M}_i$ represent the i -th moment vectors containing the moments of all outputs in $\mathbf{Y}(s)$. A particular output (row) of moment vectors is selected and scalar moments are labeled as m_i . These moments are the coefficients of the Maclaurin expansion of the selected output node, given an impulse at the input. In other words, the transfer function $H(s)$ of the selected output is given by:

$$H(s) = m_0 + s m_1 + s^2 m_2 + \dots \quad (3.8)$$

The m_i are identical to the time domain moments of the impulse response of the particular output, $h(t) = \mathcal{L}^{-1}(H(s))$,

$$m_i = \frac{(-1)^i}{i!} \int_0^\infty t^i h(t) dt \quad (i \geq 0) \quad (3.9)$$

This can be seen by expanding the definition of the Laplace transform in a Maclaurin series about s ,

$$H(s) = \int_0^\infty e^{-st} h(t) dt = \sum_{i=0}^\infty \frac{(-1)^i}{i!} s^i \int_0^\infty t^i h(t) dt = \sum_{i=0}^\infty s^i m_i \quad (3.10)$$

It has been established that the time moments provide a good measure of delays and rise times. AWE uses this fact to extract an approximate transfer function from the moments.

Generating an Approximate Response

The form of an actual one input, one output transfer function of an output can be described as,

$$H(s) = \frac{P(s)}{Q(s)} \quad (3.11)$$

where $P(s)$ and $Q(s)$ are polynomials in s where the degree of the numerator is less than or equal to the degree of the denominator. Applying a partial fraction decomposition on (3.11) yields,

$$H(s) = c + \sum_{j=1}^{N_p} \frac{k_j}{s - p_j} \quad (3.12)$$

N_p is the number of actual poles p_j , of the network and the number of residues k_j , of the selected output. While the poles of the network are the same at every output node, the residues will vary from node to node. The constant c also varies with the output and indicate direct coupling between the input and the output. $H(s)$ may

also be decomposed into N_z zeros as well as N_p poles,

$$H(s) = \frac{(s - z_1)(s - z_2) \cdots (s - z_{N_z})}{(s - p_1)(s - p_2) \cdots (s - p_{N_p})} \quad (3.13)$$

The form of (3.12) is more convenient for an inversion of Laplace transform into the time domain. The time domain impulse response of the output can be given in closed form in terms of the poles and residues as,

$$h(t) = c\delta(t) + \sum_{j=1}^{N_p} k_j e^{p_j t} \quad (3.14)$$

In large networks the number of poles N_p , and therefore the order of $Q(s)$, can be in the range of hundreds or thousands. Generating the N_p poles of even a small network is very CPU intensive. With large networks, it is completely impractical to attempt a direct extraction of $H(s)$ or the poles.

Using the moments generated, a low-order q -pole approximation to (3.12) can be constructed in the form,

$$H(s) \approx \hat{H}(s) = \frac{\hat{P}(s)}{\hat{Q}(s)} = \frac{(s - \hat{z}_1)(s - \hat{z}_2) \cdots (s - \hat{z}_q)}{(s - \hat{p}_1)(s - \hat{p}_2) \cdots (s - \hat{p}_q)} \quad q \ll N_p \quad (3.15)$$

where $\hat{z}_j$ are the approximate zeros and $\hat{p}_j$ are the approximate poles. Equation (3.15) can also be reformulated by partial fraction decomposition as,

$$\hat{H}(s) = \hat{c} + \sum_{j=1}^q \frac{\hat{k}_j}{s - \hat{p}_j} \quad (3.16)$$

In the time domain, an approximate impulse response can then be given by:

$$h(t) \approx \hat{h}(t) = \hat{c}\delta(t) + \sum_{j=1}^{N_p} \hat{k}_j e^{\hat{p}_j t} \quad (3.17)$$

Similar closed forms are also possible for an approximate step response, ramp response, or exponential response, by multiplying the input function by the impulse response in the frequency domain and decomposing the result into partial fractions. All other complex input waveforms may be decomposed into a linear combination of these forms.

Poles and Residues

In order to obtain the $\hat{p}_j$ and $\hat{k}_j$, (3.16) is expanded in a Maclaurin series,

$$\hat{H}(s) = \hat{c} - \sum_{n=0}^{\infty} s^n \left(\sum_{j=1}^q \frac{\hat{k}_j}{\hat{p}_j^{n+1}} \right) \quad (3.18)$$

The $\hat{c}$, in the case of a state space formulation, may be obtained directly from the $\mathbf{D}$. It is in the same position of the selected output of $\mathbf{Y}(s)$. It will therefore be made equal to c . When (3.18) is equated with (3.8) it yields the $2q$ equations needed to solve for the q approximate poles, $\hat{p}_j$, and the q approximate residues $\hat{k}_j$,

$$m_0 = c - \sum_{j=1}^q \frac{\hat{k}_j}{\hat{p}_j} \quad (3.19)$$

and

$$m_i = - \sum_{j=1}^q \frac{\hat{k}_j}{\hat{p}_j^{i+1}} \quad (0 < i < 2q) \quad (3.20)$$

This set of nonlinear equations may be solved directly. However, it is more convenient to transform it into a set of linear equations. This can be accomplished by solving instead for the q coefficients of

$$\hat{Q}(s) = b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + 1 \quad (3.21)$$

The result is a set of equations in the form,

$$\begin{bmatrix} m_0 & m_1 & \cdots & m_{q-1} \\ m_1 & m_2 & \cdots & m_q \\ \vdots & \vdots & \ddots & \vdots \\ m_{q-1} & m_q & \cdots & m_{2q-2} \end{bmatrix} \begin{bmatrix} b_q \\ b_{q-1} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_q \\ m_{q+1} \\ \vdots \\ m_{2q-1} \end{bmatrix}$$

or

$$\mathbf{m}_h \mathbf{b}_v = \mathbf{m}_v \quad (3.22)$$

The $\hat{p}_j$ can be obtained from (3.22) by applying a root solving algorithm on the resulting $\hat{Q}(s)$. In order to solve for the residues, the newly calculated poles are

substituted into (3.19) and (3.20), to give a set of linear equations for the $\hat{k}_j$. In order to improve the accuracy at $t = 0^+$, the equation for m_{q-1} is substituted for a time domain condition of:

$$\sum_{j=1}^q \hat{k}_j = h(0^+) \quad (3.23)$$

This condition is obtained from (3.14) at $t = 0^+$. The solution to all outputs of $y(t)$ at $t = 0^+$ is given by:

$$y(0^+) = c^T b = M_{-1} \quad (3.24)$$

M_{-1} is referred to as the negative first moment vector. Again, only the single output node is extracted from M_{-1} to give m_{-1} , yielding

$$\sum_{j=1}^q \hat{k}_j = m_{-1} \quad (3.25)$$

The equations are then given by:

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \hat{p}_1^{-1} & \hat{p}_2^{-1} & \dots & \hat{p}_q^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{p}_1^{-q+1} & \hat{p}_2^{-q+1} & \dots & \hat{p}_q^{-q+1} \end{bmatrix} \begin{bmatrix} \hat{k}_1 \\ \hat{k}_2 \\ \vdots \\ \hat{k}_q \end{bmatrix} = - \begin{bmatrix} m_{-1} \\ m_0 - c \\ \vdots \\ m_{q-2} \end{bmatrix} \quad (3.26)$$

Generating an approximate set of poles and residues from a set of moments is not costly in terms of CPU time. The matrices of (3.22) and (3.26), are relatively small for a small q and the only other expense is the root finding of $\hat{Q}(s)$, which is also relatively small for a small q . All these matrix inversions are of the order of q^3 . The accuracy of the approximation depends on the number of poles extracted.

3.2 Modified Nodal Admittance (MNA) Formulation

As shown above, it is not always necessary to reduce the network in the form of system matrix A , using this formulation. Assume that a linear subnetwork ϕ contains only

lumped elements, then the Modified Nodal Admittance (MNA) formulation of the subnetwork may be used.

Describing a subnetwork ϕ in terms of the MNA formulation yields

$$\mathbf{C}_\phi \frac{d}{dt} \mathbf{v}_\phi(t) + \mathbf{G}_\phi \mathbf{v}_\phi(t) - \mathbf{e}_\phi(t) = 0 \quad (3.27)$$

where, $\mathbf{v}_\phi(t) \in \mathcal{R}^{N_\phi}$, is the vector of N_ϕ variables describing the subnetwork ϕ which include node voltage waveforms appended by independent voltage source current, linear inductor current, nonlinear capacitor charge and nonlinear inductor flux waveforms, $\mathbf{C}_\phi \in \mathcal{R}^{N_\phi \times N_\phi}$, $\mathbf{G}_\phi \in \mathcal{R}^{N_\phi \times N_\phi}$ are constant matrices describing the lumped memory and memoryless element of network ϕ respectively, and $\mathbf{e}_\phi(t) \in \mathcal{R}^{N_\phi}$ is the vector of source waveforms.

Taking the Laplace transform of (3.27) and assuming only an impulse source, $\mathbf{e}_\phi(t) = \mathbf{E}_\phi \delta(t)$ yields:

$$(\mathbf{G}_\phi + s\mathbf{C}_\phi)\mathbf{V}_\phi(s) = \mathbf{E}_\phi \quad (3.28)$$

Expression (3.28) can be used to derive the moments $\mathbf{V}_{\phi[i]}$ of $\mathbf{V}_\phi(s)$ in an efficient manner. Expansion for $\mathbf{V}_\phi(s)$ in a Maclaurin series is

$$(\mathbf{G}_\phi + s\mathbf{C}_\phi)(\mathbf{V}_{\phi[0]} + s\mathbf{V}_{\phi[1]} + s^2\mathbf{V}_{\phi[2]} + \dots) = \mathbf{E}_\phi \quad (3.29)$$

Equating like powers of s gives a recursive relationship for the moments:

$$\mathbf{G}_\phi \mathbf{V}_{\phi[0]} = \mathbf{E}_\phi \quad (3.30)$$

and

$$\mathbf{G}_\phi \mathbf{V}_{\phi[i]} = -\mathbf{C}_\phi \mathbf{V}_{\phi[i-1]} \quad (i > 0) \quad (3.31)$$

Because matrices $\mathbf{G}_\phi$ and $\mathbf{C}_\phi$ represent sparse matrices, the solution of (3.30) and (3.31) can be accomplished efficiently. Further, any invertible matrix $\mathbf{G}_\phi$ can be decomposed

(factorised) into two triangular matrices. The result is a lower triangular matrix L and an upper triangular matrix U where,

$$LU = G_\phi \quad (3.32)$$

After this, any system of equations,

$$G_\phi X = B \quad (3.33)$$

may be solved efficiently by first solving sequentially the two simpler equations,

$$\begin{aligned} LY &= B \\ \text{and} \\ UX &= Y \end{aligned} \quad (3.34)$$

The last two steps are solved by a forward and then a backward substitution of the unknown variables. The greatest CPU time is in the LU factorization of the matrix. Forward/backward substitution are less costly for each additional solution. Equations (3.30) and (3.31) can be solved recursively with one LU factorization of matrix G_ϕ . This is approximately equal in CPU time to the DC solution of circuit equations. In fact, (3.30) is equivalent to the DC solution of the network with constant DC sources E_ϕ . Each moment can then be obtained from the previous one by pre-multiplying by C_ϕ (a sparse matrix) and one forward/backward substitution using the LU decomposed matrix G_ϕ .

3.3 Padé Approximations

Equation (3.22) is a particular type of Padé rational approximation. Padé approximations can be applied to network transfer functions since these functions are in

a rational form (similar to a network transfer function), and the poles of the Padé approximant approach the poles of the system rather well. The residue strength of the typical electrical network poles is concentrated in a small region of the complex plane, making the approximation practical and accurate for these type of networks. Added to this property is the large CPU advantage of using this type of approximation compared to full network analysis. Padé approximations are useful when it is more efficient to obtain the moments of a function rather than to evaluate the function in closed form. This is typical to the case of linear electrical circuit network transfer functions.

The Rational Approximation

Padé approximations begin with a Maclaurin series expansion of a function,

$$F(s) = m_0 + sm_1 + s^2m_2 + s^3m_3 + \cdots = \sum_{n=0}^{\infty} s^n m_n \quad (3.35)$$

where the coefficients of the expansion are also known as moments. The series is then matched to a rational polynomial approximation where the Maclaurin expansion of the approximant matches the original Maclaurin series up to a set number of moments. The Maclaurin moments contain information of the function about the origin. In the frequency domain, they represent the low frequency behavior of a network transfer function.

Consider the response vector $\mathbf{X}(s)$ represented in (2.87) of Chapter 2. Taylor's series expansion of the output $\mathbf{X}(s)$ about a complex frequency point $s = s_o$, is given by

$$\mathbf{X}(s) = \sum_{n=0}^{\infty} \mathbf{M}_n (s - s_o)^n \quad (3.36)$$

where $\mathbf{M}_n$ is the n th moment vector of the Taylor's expansion about s_o and is given

by

$$M_n = \frac{\partial^n [Y^{-1}(s)R(s)]|_{s=s_0}}{\partial s^n} / n! \quad (3.37)$$

A recursive equation for the evaluation of the moments can be obtained in the form

$$[Y(s_0)]M_n = - \sum_{r=1}^n [Y]^r M_{n-r} / r! \quad (3.38)$$

where

$$[Y]^r = \frac{\partial^r}{\partial s^r} Y|_{s=s_0} \quad (3.39)$$

An L, M Padé approximant to $F(s)$ is denoted by,

$$[L/M] = \frac{\hat{P}_L(s)}{\hat{Q}_M(s)}$$

where

$$\hat{P}_L(s) = a_0 + sa_1 + s^2a_2 + \cdots + s^La_L$$

and

$$\hat{Q}_M(s) = b_0 + sb_1 + s^2b_2 + \cdots + s^Mb_M \quad (3.40)$$

The coefficients of $\hat{P}_L(s)$ and $\hat{Q}_M(s)$ are obtained uniquely from the coefficients of the Maclaurin series expansion when the normalizing condition of

$$\hat{Q}_M(0) = 1 \rightarrow b_0 = 1 \quad (3.41)$$

is imposed. Since there are $(L+M)$ unknowns, $(L+M)$ moments are to be matched.

Cross multiplication of (3.35) and (3.40) is:

$$(m_0 + sm_1 + s^2m_2 + s^3m_3 + \cdots + s^{L+M}m_{L+M}) \times \\ (b_0 + sb_1 + s^2b_2 + \cdots + s^Mb_M) = a_0 + sa_1 + s^2a_2 + \cdots + s^La_L \quad (3.42)$$

Equating the coefficients for powers $s^{L+1}, \dots, s^{L+M}$ and using $b_0 = 1$ gives,

$$\begin{bmatrix} m_{L-M+1} & m_{L-M+2} & \cdots & m_L \\ m_{L-M+2} & m_{L-M+3} & \cdots & m_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ m_L & m_{L+1} & \cdots & m_{L+M-1} \end{bmatrix} \begin{bmatrix} b_M \\ b_{M-1} \\ \vdots \\ b_1 \end{bmatrix} = - \begin{bmatrix} m_{L+1} \\ m_{L+2} \\ \vdots \\ m_{L+M} \end{bmatrix} \quad (3.43)$$

If $j < 0$, $c_j = 0$. Equation (3.22) is obtained when setting $L = M - 1$, i.e., when it is assumed that the degree of the numerator is one less than the degree of the denominator. The numerator coefficients can be found by equating powers $s^0, \dots, s^L$,

$$\begin{aligned} a_0 &= m_0 \\ a_1 &= m_1 + b_1 m_0 \\ a_2 &= m_2 + b_1 m_1 + b_2 m_0 \\ &\vdots \\ a_L &= m_L + \sum_{i=1}^{\min(L,M)} b_i m_{L-i} \end{aligned} \quad (3.44)$$

Various choices of L and M gives an infinite array of possible approximants. For example, an approximate expression for the exponential function using Maclaurin expansion is,

$$e^s = 1 + s + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!} + \cdots \quad (3.45)$$

Once the numerator and denominator coefficients are obtained, they may be decomposed into the form,

$$\frac{\hat{P}_L(s)}{\hat{Q}_M(s)} = \hat{c} + \sum_{i=1}^M \frac{\hat{k}_i}{s - \hat{p}_i} \quad (L \leq M) \quad (3.46)$$

An alternate approach to (3.44) is to factor the denominator in terms of its poles and then obtain the corresponding residues.

Padé approximations [4] attempt to extract pole/residue information from the moments. The detection of a particular pole will depend critically on the truncation

error that exists in that pole. The larger the truncation error, the more approximate will be the generated pole. The poles closest to the point of expansion are detected best. The error also depends on the order of the approximation. Generally, a Padé approximation is used to extract far fewer approximate poles than are actually in the network. Single Padé approximations are accurate near the point of expansion in the complex s -plane and decrease in accuracy with increased distance from the point of expansion.

AWE can extend the application of a single Padé approximation to an approximation containing moments from two points. The second point is conveniently chosen to be at $s = \infty$. This expansion is known as a *Laurent series* [26]. Expanding (3.5),

$$\begin{aligned} Y(s) &= \mathbf{c}^T (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{b} + \mathbf{D} \text{ at } s = \infty \text{ gives,} \\ Y(s) &= s^{-1} \mathbf{c}^T \mathbf{b} + s^{-2} \mathbf{c}^T \mathbf{A} \mathbf{b} + \cdots + s^{-n-1} \mathbf{c}^T \mathbf{A}^n \mathbf{b} + \cdots \end{aligned} \quad (3.47)$$

or

$$Y(s) = s^{-1} \mathbf{M}_{-1} + s^{-2} \mathbf{M}_{-2} + \cdots + s^{-n} \mathbf{M}_{-n} + \cdots \quad (3.48)$$

with

$$\mathbf{M}_{-i} = \mathbf{c}^T \mathbf{A}^{i-1} \mathbf{b} \quad (3.49)$$

This corresponds to expanding an output transfer function in (3.12) about $s = \infty$,

$$\begin{aligned} H(s) &= s^{-1} \left(\sum_{j=1}^{N_p} k_j \right) + s^{-2} \left(\sum_{j=1}^{N_p} k_j p_j \right) + \cdots + s^{-n} \left(\sum_{j=1}^{N_p} k_j p_j^{n-1} \right) + \cdots \\ &= s^{-1} m_{-1} + s^{-2} m_{-2} + \cdots + s^{-n} m_{-n} + \cdots \end{aligned} \quad (3.50)$$

These negative moments can also be incorporated in a Padé approximation. A Padé approximation using moments from the origin and infinity are known as a two-point Padé approximation.

Frequency Shifting

A Padé approximation faces problems when dealing with stiff systems since in these systems, a few poles are close to the origin while other poles are relatively farther away. A Padé approximation will only detect the few poles (sometimes only one) that are closest to the origin. In order to address this problem, a frequency shift is performed. In this process, a Taylor series expansion about a positive point s_0 away from the origin replaces the original Maclaurin expansion. The object of the shift is to move away from the closest poles blocking the view of the other poles farther away. The shift is chosen judiciously so as not to be too far into the right-half plane and therefore too far from the poles to detect them accurately. Generally, a fraction of the highest frequency f_{max} of interest is chosen as the shifting point.

When expanding about another point, it is sufficient to shift the system equations by that point and compute an expansion about the new shifted origin. Any poles obtained from the new moments can then be shifted back to the old origin. So for $s' = (s - s_0)$, (3.5) becomes:

$$\mathbf{Y}(s') = \mathbf{c}^T(s'\mathbf{1} + s_0\mathbf{1} - \mathbf{A})^{-1}\mathbf{b} + \mathbf{D} \quad (3.51)$$

giving positive and negative moments about $s' = 0$ in the form,

$$\begin{aligned} \mathbf{M}'_0 &= \mathbf{D} - \mathbf{c}^T(\mathbf{A} - s_0\mathbf{1})^{-1}\mathbf{b} \\ \mathbf{M}'_i &= -\mathbf{c}^T(\mathbf{A} - s_0\mathbf{1})^{-(i+1)}\mathbf{b} \quad (i > 0) \end{aligned} \quad (3.52)$$

If the network is given in the MNA formulation, then substituting $s' + s_0$ for s gives,

$$(\mathbf{G}_o + (s' + s_0)\mathbf{C}_o)\mathbf{V}_o(s) = \mathbf{E}_o \quad (3.53)$$

which in turn gives moments $\mathbf{M}'_i$ about $s = s_0$ in the form,

$$(\mathbf{G}_o + s_0\mathbf{C}_o)\mathbf{V}_{o[0]} = \mathbf{E}_o \quad (3.54)$$

and

$$(\mathbf{G}_\phi + s_0 \mathbf{C}_\phi) \mathbf{V}_{\phi[i]} = -\mathbf{C}_\phi \mathbf{V}_{\phi[i-1]} \quad (i > 0) \quad (3.55)$$

Another method, similar to shifting, is conformal mapping (transformation) of the s variable in a similar complex plane. The point of mapping is to bring the poles of the mapped function closer to the origin in order to improve, at times, a Padé approximation. A mapping may improve the detection of some poles and worsen the detection of others.

In the general case, it is possible to *hop* to any point in the complex s -plane and expand in a Taylor series there. However, expansions outside the real axis or $s = \infty$ result in complex moments and the calculations must be carried out with complex numbers which can be CPU intensive. The next section on complex frequency hopping solves problems which include information from multiple expansion points in a data set.

3.4 Complex Frequency Hopping (CFH)

Choosing any arbitrary complex frequency point $p = \alpha + j\beta$, one can perform a Taylor series approximation at the point of expansion in the same manner as a real frequency shift by the substitution $s' = s - p$. If the system consists of lumped element components given in the MNA formulation, analogous to (3.53) and (3.54), the shifted system moments can be derived by,

$$(\mathbf{G}_\phi + (\alpha + j\beta)\mathbf{C}_\phi) \mathbf{V}_{\phi[0]} = \mathbf{E}_\phi \quad (3.56)$$

and

$$(\mathbf{G}_\phi + (\alpha + j\beta)\mathbf{C}_\phi) \mathbf{V}_{\phi[i]} = -\mathbf{C}_\phi \mathbf{V}_{\phi[i-1]} \quad (i > 0) \quad (3.57)$$

These moments, in contrast to moments about $s = 0$, $s = \alpha$ and $s = \infty$, will be complex numbers. At each new expansion point (hop), a Padé approximation will be performed on the moments to determine the relevant poles. A complex frequency hop is used to detect poles in a region of the complex plane not visible from another expansion point. If one expansion point is near enough to another, many or some of the same poles will be detected thereby giving confirmation of the accuracy of pole/residue pairs. This is an error check not available by moment-matching methods using one Padé approximation, which may contain both accurate and inaccurate poles.

Complex frequency hopping overcomes this problem and is summarized in the following section. Complex frequency hopping is a method by which the frequency response of a system is expanded in multiple Taylor series expansions in the Laplace s -domain. CFH is so named because of the necessity to *hop* to and expand at multiple points. The expansion points are chosen on or near the imaginary axis because poles that dominate the transient and frequency response of a system are found there. The moments of the expansion are then matched to a rational Padé approximation. These Padé approximations have several useful properties - one of which is the convergence of the poles of the approximation to the actual poles of a system. There are two approaches for generating the response of a system:

- Poles Approach and
- Transfer Function Approach

They are described in detail in the following sections.

Advantages of CFH over AWE can be stated as follows: As the circuit operating frequency is increased and transmission line effects become prominent, more hops are needed to accurately describe the waveform and the CPU cost is proportional to this

complexity. CFH therefore provides an accuracy/CPU trade-off not available by one Padé approximation. Convergence to the actual dominant poles is always guaranteed within the frequency of interest using CFH.

3.5 Poles Approach

Padé approximation is very accurate near the point of expansion i.e., at $s = s_0$. However, the accuracy of Padé approximation decreases with increased distance from the point of expansion similar to the case of a Taylor's series expansion.

In order to check the accuracy of an approximation, two expansion points are necessary. The accuracies of these two expansion points can be verified by matching the poles generated at these two expansion points [4], [24]. Alternatively, the two expansion points can be verified for their accuracy by finding the value of the transfer function at a point intermediate to these two expansion points [27]. The number of expansions required to obtain a fairly accurate set of transfer functions over a specified frequency range is controlled by a binary search algorithm.

In the poles approach [4], [24], several expansion points are generated and the converged poles from each expansion are compared. If any two expansions find a common pole, then they are considered accurate within the radius of accuracy defined by the matching pole. If no poles are found in common, then an intermediate hop (expansion) is chosen. All poles within the radius of accuracy of each hop are then collected giving the actual system poles near the imaginary axis. The frequency and transient responses are then a closed-form function of these poles and their residues.

Pole Convergence

The strategy of CFH is to target and generate the actual dominant poles of the circuit rather than approximate poles. When trying to generate an approximating function to the original transfer function as in (3.16), a diagonal or near diagonal Padé approximation is generally used. However, the approximate poles generated from such an approximation will not necessarily reflect the actual system poles. To generate the real system poles accurately, one needs to use

$$\lim_{L \rightarrow \infty} [Q_{[L/M]}(s)] \rightarrow \prod_{j=1}^M \left(1 - \frac{s}{p_j}\right) \quad (3.58)$$

which allows convergence to the actual system poles. M is chosen large enough (usually 8-9) so that the sequence converges quickly and the number of moments needed is not excessive. Using this method, the poles generated about any hop are accurate with respect to the actual poles in the neighborhood of the expansion. The closer the point of expansion to a pole the more accurate the extracted pole. This is significant considering that the poles of distributed systems are transcendental and infinite in number and there is no general algorithm for generating them. Nevertheless, in practice, numerical truncation error affects the convergence of the above equation. As a result some poles far from the point of expansion do not converge well and a further series of steps are needed to ensure the accuracy of the poles.

Pole Selection

With two or more complex frequency hops, pole information from each is combined into an accurate set of poles and residues once convergence of each hop is achieved. The algorithm is based on properties 1, 2 and 3 of Padé approximations [4]. The accuracy of an approximation is good near the point of expansion but decreases with

increasing distance from the point.

If a pole generated in an approximation is confirmed to be accurate then all poles closer to the point of expansion are also tagged as accurate. The outermost pole known to be accurate consequently defines a radius of accuracy, R_{acc} , for an error tolerance ϵ_{err} . All poles within R_{acc} are marked as accurate. Poles are initially marked as accurate when they are confirmed to be detected by at least two different hops to within ϵ_{err} . If the same pole is detected in two hops, then that pole is confirmed accurate and its distance from the point of expansion defines R_{acc} .

The following steps are taken to extract a set of accurate poles:

Step 1: Poles from each hop are converged using (3.58)

Step 2: Residues from each hop are also extracted from the moments. This requires using the equation,

$$\begin{bmatrix} \hat{p}_1^{-L-2} & \hat{p}_2^{-L-2} & \dots & \hat{p}_q^{-L-2} \\ \hat{p}_1^{-L-3} & \hat{p}_2^{-L-3} & \dots & \hat{p}_q^{-L-3} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{p}_1^{-L-M-1} & \hat{p}_2^{-L-M-1} & \dots & \hat{p}_q^{-L-M-1} \end{bmatrix} \begin{bmatrix} \hat{k}_1 \\ \hat{k}_2 \\ \vdots \\ \hat{k}_q \end{bmatrix} = - \begin{bmatrix} m_{L+1} \\ m_{L+2} \\ \vdots \\ m_{L+M} \end{bmatrix} \quad (3.59)$$

The above equation derives the actual residues of the (converged) poles rather than approximate residues

Step 3: If the same poles (with the error tolerance ϵ_{err}) are detected in two different expansions, they are marked as accurate

Step 4: The distance between a hop and its farthest confirmed accurate pole defines a radius of accuracy R_{acc} . All poles within R_{acc} are also marked accurate

Step 5: Residues of marked accurate poles are also marked accurate

Step 6: Poles/residues not marked by steps above are rejected. There is no confirmation of their accuracy

The pole selection algorithm is illustrated in the figure (3.2).

Figure 3.2: Pole selection algorithm using two hops.

Search Strategy

Given a high enough uniform density of expansion points, the CFH method is guaranteed to find all the dominant network poles within a given search area. However, the CPU expense of such a strategy will be high. Any search strategy must therefore aim to extract accurate pole/residue information with a minimized number of hops. Several search strategies are possible with few constraints.

Constraint 1: The search should be confined to the upper left half complex plane.

Constraint 2: The search should be confined to or near the imaginary axis. Poles are critical to the transient and frequency responses if they are close to the imaginary axis. In the time domain the influence of poles relatively far into the left half plane is very transient. In the frequency domain poles far from the $j\omega$ axis do not have much influence on the response.

Constraint 3: The search should be confined to points below the highest frequency f_{max} . The expected operating frequency of the circuit, which depends on the input waveform and the type of circuit, limits the search up the frequency axis.

Figure 3.3: Binary search and CFH search area in the complex s plane.

The above search area uses a binary search strategy which drastically reduces the search space as shown in figure (3.3). The steps for the binary search strategy are:

- Step 1:** Expand at the origin ($s = 0$). Collect poles by convergence of (3.58).
- Step 2:** Determine the highest frequency of accuracy desired, $s = j\omega_{max} = j2\pi f_{max}$, and expand at that point. Collect poles by convergence.
- Step 3:** If common poles exist between $s = 0$ and $s = j\omega_{max}$, STOP. Define R_{acc} for each expansion and collect poles and residues between the two points. Else, expand at the midpoint $s = j\omega_{max}/2$ between the two end points.
- Step 4:** Continue the search in a similar binary fashion between $s = 0$ and $s = j\omega_{max}/2$ and between $s = j\omega_{max}/2$ and $s = j\omega_{max}$, defining and expanding at midpoints if no common poles are found.

Each hop must be near enough to the neighboring hop in order to confirm the same outer poles between the two expansions. CFH method guarantees an accurate waveform within the specified frequency range. The moments generated for any circuit must be accurate for this type of search. Any small truncation error in the moments will mean that poles far from a point of expansion will not be accurate, necessitating more hops and increasing the CPU cost.

CPU Requirements

Using the CFH method requires calculations using complex numbers in contrast to expansions about the real axis which require only real number processing. Consequently, the cost of an arbitrarily complex frequency expansion may, at first glance, seem more expensive than an expansion in real numbers about a point on the real axis. However, a dominant cost of a real expansion is the LU factorization of the MNA matrix. This is approximately equivalent to the CPU cost of one frequency point

analysis. Sparse matrix analysis, however, requires the cost of matrix reordering before performing the LU factorization. This reordering need not be done again for any additional frequency points [28]. The CPU cost at the origin is partly composed of a reordering and a (real) LU factorization. Every other expansion point consists of the following CPU cost: 1. Complex LU factorization (without reordering), 2. Padé table convergence, and 3. Complex moment generation (repeated forward-backward substitution). In general a sparse matrix reordering consumes more CPU time than one LU factorization.

The additional CPU expense of Padé table convergence and moment generation are more expensive using complex numbers compared to real numbers. The former however, is independent of circuit size and diminishes with the size of the circuit, while the latter is generally 3-4 times more expensive than real moment generation. However since moment generation and Padé analysis use a small fraction of the cost of LU factorization in a large circuit [1], [4], [29] the increase is minimal.

One important advantage to CFH is it can be readily given to parallel processing. This is not generally true of a differential equation solver such as SPICE. Parallelisation may be achieved by assigning one hop to each computed node and assuring that the hops tile the imaginary axis in a regular pattern. The information from each hop may then be gathered at the central compute node and collected in terms of poles and residues of a single transfer function. The result of such a process would be an accurate simulation at the cost of a little more than one complex LU factorization.

3.6 Transfer Function Approach

In the transfer function approach [27], in order to obtain the frequency response, a set of rational transfer functions are generated at a minimized set of expansion points. It is the value of these transfer functions that is compared at points intermediate to the expansion points rather than a search for same poles. If two transfer functions are found to give the same results at an intermediate frequency point between the two generating hops, then these transfer functions are considered accurate. If this is not the case, then another hop is chosen between the two expansion points and an expansion is performed there. It was found empirically, that this approach generally requires fewer hops compared to the poles approach. This means that the CPU time can be reduced. Further reduction was also noted due to the fact that no pole convergence was required at each expansion point. In addition, this approach avoids the complications arising from right-half plane poles (unstable) sometimes associated with Padé approximations. The time domain response is obtained using the generated frequency response and inverse Fast Fourier Transform (FFT).

Summarizing, CFH ensures accuracy of an approximation for a complete frequency range, using multiple expansion points and corresponding Padé approximations at the frequencies of interest. Additional hops are generated at an incremental CPU cost above the cost of the first. However, the number of hops typically needed ranges from 2-15, far less than full FFT analysis requiring hundreds or thousands of frequency point analyses.

Binary Search Algorithm for Transfer Function Approach

Figure 3.4: Generation of Transfer Function by CFH.

The steps involved in the binary search algorithm for the transfer function approach is summarized below:

- Step 1:** Set $f_L = 0$ and $f_H = f_{max}$ (figure (3.4)).
- Step 2:** Expand the system's response at frequency $f = f_L$. Determine the coefficients of the corresponding transfer function $H_L(s)$ using (3.43) and (3.44).
- Step 3:** Expand the system's response at frequency $f = f_H$. Determine the coefficients of the corresponding transfer function $H_H(s)$.
- Step 4:** Set $f = \frac{1}{2}(f_L + f_H)$. Calculate $H_L(j2\pi f)$ and $H_H(j2\pi f)$. If, $|H_H(j2\pi f) - H_L(j2\pi f)| < \epsilon$, where ϵ represents pre-specified threshold for relative error, GOTO step 5. Otherwise expand at $f_{mid} = \frac{1}{2}(f_L + f_H)$ and determine $H_{mid}(s)$.
- Step 5:** If no middle frequency f_{mid} is generated between any two other frequency

expansions, STOP, ELSE, repeat steps 2-4 between every two consecutive frequency points (e.g., between f_L & f_{mid} and f_{mid} & f_H).

At the completion of the binary search algorithm, a set of transfer functions is generated. The frequency response at a particular frequency point is computed by choosing a transfer function valid for that region. This is repeated for all other frequency points and the system's frequency response is computed. The time domain response is obtained as a closed form function of the generated poles and residues or alternatively by using Inverse Fast Fourier Transform (IFFT).

Having introduced the basic ideas of FEM and CFH techniques, a combination of these two approaches will be presented to efficiently solve several electromagnetic problems. Chapter 4 describes the formulation and analysis of homogeneous problems and Chapter 5 deals with inhomogeneous cylindrical dielectric resonators - their formulation and determination of its resonant frequency.

Chapter 4

A Combined Finite Element - Complex Frequency Hopping Method

4.1 Overview

One of the most common approaches used for the solution of electromagnetic field problems is the Finite Element Method (FEM) [30], [31], [32], [33], [11]. FEM formulations are either of the type space/frequency-domain or space/time-domain. To recapitulate, space/frequency formulation leads to a set of algebraic equations which have to be solved repeatedly at many frequency points, while space/time formulation leads to a set of ordinary differential equations which have to be solved in time-domain. The size of the equations to be solved is usually large and the conventional solution algorithms are restricted by computing time and stability conditions. For example, frequently used time-stepping schemes [13], [14], [15], [16], [17], require the satisfaction of the stability condition, i.e., the ratio of spatial to temporal subdivision is to be greater than or equal to the speed of propagation [12]. In order to obtain smaller time steps, the finite element mesh has to be denser [18], thus increasing

the size of the equations. In such cases, implicit variable time step integration algorithms [19] can be used. However, they require solution of large set of algebraic equations at each time point. More often than not, both time-domain and frequency-domain results are of interest [16], [17], in which case, Fast Fourier Transform (FFT) and its inverse can be used to move from one solution space to the other. This will increase computational time since in order to achieve satisfactory resolution, the FFT has to span longer time and frequency intervals with small increments [20].

In an attempt to improve the efficiency of FEM solution techniques, a Laplace-Domain FEM (LDFEM) has been proposed by Cai and Costache [21]. This method is an extension of the technique used for the solution of heat transfer problems [22], [23]. In these applications, the heat conduction equation, which describes the unsteady temperature distribution in a solid domain, together with proper initial/boundary conditions is transformed to Laplace-domain and then solved by a conventional node-based finite element method. This approach [22], [23] has been extended to three-dimensional problems but is restricted to diffusion-type partial differential equations, an important improvement being the adoption of congruence transformation of matrices. In [21], the same approach has been extended to wave equations. Important steps in LDFEM are as follows:

- The given problem in the form of damped wave equations associated with suitable initial and boundary conditions is transformed to Laplace-domain from time-domain and formulated with a conventional FEM scheme;
- The congruence transformation is used to deal with the matrices; and
- The numerical results derived in the Laplace-domain are inverted to time-domain in either an analytical or numerical way.

However, it is based on congruence transformation which involves computationally expensive process of determining all eigenvalues and eigenvectors of a large matrix.

From the conceptual point of view, the new technique proposed in this thesis falls in the category of [21], [22], [23]. However, the significant difference is that it only requires the computation of dominant natural modes resulting in considerable reduction in computational cost.

Complex Frequency Hopping (CFH) is particularly suitable for solving large sets of ordinary differential equations which makes it a logical approach for solving time-harmonic EM equations. The main steps involved can be summarized as follows:

- The given problem which is in the form of a damped wave equation is formulated using FEM and the resulting ordinary differential equation is transformed to the Laplace-domain;
- The Laplace-domain output is expanded using Padé approximation at selected frequency points;
- Information from each expansion point is used to generate the output frequency response or alternatively a unified set of dominant pole/residue pairs; and
- finally, the results are transformed to time-domain in either analytical or numerical form.

The main advantages of the proposed method can be summarized as follows:

- 10-1000 times faster than conventional FEM solution techniques.
- Solution algorithm does not suffer from instability problems associated with the time-stepping methods.
- Produces simultaneously both time- and frequency-domain results.

There are two ways of solving the ordinary differential equations generated from FEM formulation. In the first approach, an equivalent circuit is generated which is used as an input to CFH. CFH then generates moments to yield the frequency- and time-domain responses. This is a circuitous and an indirect approach. However, the second approach is a more efficient one, in which the moments are generated directly, thereby incorporating differential equations into the CFH algorithm at moments level. These two approaches are discussed in the following sections.

4.2 Approach 1 - Netlist Generation

Consider a second order differential equation in s ,

$$(s^2 \mathbf{C} + s \mathbf{B} + \mathbf{A}) \mathbf{X}_1(s) = \mathbf{R}(s) \quad (4.1)$$

where $\mathbf{C}, \mathbf{B}, \mathbf{A}$ are square matrices and $\mathbf{R}$ is the input excitation vector. This equation can be converted to time-domain using

$$\dot{X}_1 = X_2$$

$$\ddot{X}_1 = \dot{X}_2$$

and the matrix equation (4.1) can be rewritten as

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} J \\ 0 \end{bmatrix} \quad (4.2)$$

Most software packages employing the finite element method generate data from which the $\mathbf{C}, \mathbf{B}$ and the $\mathbf{A}$ matrices of (4.1) can be obtained. As part of this research project, a program (NETGEN) was developed to translate information from (4.2)

to an electrical netlist. This new netlist is then used as an input to an electrical simulator (COFFEE $\rightarrow$ complex frequency hopping electrical evaluator) to obtain the frequency and time-domain responses.

Figure 4.1: An equivalent circuit for netlist generation.

When X_1 and X_2 are scalars, there is no unique method to synthesize the matrices in (4.2). The method used in this thesis to synthesize these matrices and to obtain its electrical equivalent is explained in the following sections. The steps used in NETGEN are as follows and can be followed in figure 4.1:

1. Equation (4.2) has two unknowns, hence two nodes are needed to start with, in addition to a reference node (Ground node i.e., GND)
2. Consider diagonal entries in c and g matrices of (4.2): they are represented as capacitors (c_{11} , c_{22}) and resistors (g_{11} , g_{22}) connected between the corresponding row number (or column number) and the GND.

3. In the right hand side of (4.2), the term J is represented as a current source at the node corresponding to its row number.
4. A non-zero off-diagonal entry in g matrix represents a Voltage Controlled Current Source (VCCS). For example, a non-zero off-diagonal entry appears in the i th row and j th column, then the VCCS will have a value of $g_{ij}X_j$ and appears at the i th node with proper direction of current flow. In the example circuit shown in figure 4.1; VCCS are $g_{12}X_2$ at node 1 and $g_{21}X_1$ at node 2.
5. A non-zero off-diagonal entry in c matrix represents a Current Controlled Current Source (CCCS). For example, a non-zero off-diagonal entry appears in the i th row and j th column, then CCCS will have a value $\frac{c_{ij}}{c_{jj}}I_A$ in the i th node (w.r.t GND) where c_{jj} is the corresponding value at the diagonal entry and I_A is the value of the current in the capacitor at node j . The value of the current I_A is found by adding two very small resistors of equal but opposite polarities in series with the capacitor c_{jj} . This introduces two additional extra nodes 5 & 6 and two additional resistors R_3 & R_4 to the circuit and to the netlist. Since there are two non-zero off-diagonal entries in c matrix for the example considered, four additional nodes 3, 4, 5 & 6 and four new resistors R_1, R_2, R_3 & R_4 are added to the netlist.
6. One of the possibilities is that c_{jj} is zero while c_{ij} is a non-zero value. In this case, two new capacitors of equal and opposite polarities are added to the j th node and procedure (5) is repeated.

The netlist generated has additional nodes, resistors and capacitors, which increases the number of unknowns in its electrical equivalent. This netlist is then used as input

to the electrical evaluator (COFFEE) which generates the moments of the network to yield both frequency and time-domain responses.

4.3 Approach 2 - Direct Generation of Moments

In approach 2, instead of finding an equivalent circuit and generating the netlist, moments will be directly generated using the matrices from the FEM formulation. To derive an expression for moments M_n , rewrite (2.86) using (2.88) and (3.36) at any arbitrary complex frequency point $s = s_o$ and expand the right hand side of (2.86) using Taylor's series as

$$[(s - s_o)^2 \mathbf{C} + (s - s_o) \mathbf{B} + \mathbf{A} + (s - s_o) 2s_o \mathbf{C} + \mathbf{B}s_o + \mathbf{C}s_o^2] \sum_{n=0}^{\infty} \mathbf{M}_n (s - s_o)^n = [\mathbf{R}(s_o) + \frac{(s - s_o)}{1!} \mathbf{R}'(s_o) + \frac{(s - s_o)^2}{2!} \mathbf{R}''(s_o) + \dots + \frac{(s - s_o)^n}{n!} \mathbf{R}^n(s_o)] \quad (4.3)$$

where

$$\begin{aligned} \mathbf{R}(s_o) &= \mathbf{B}\Psi_o + \mathbf{Q}(s_o), \\ \mathbf{R}'(s_o) &= \mathbf{C}(\Psi_o + \Psi'_o) + \mathbf{Q}'(s_o), \\ \mathbf{R}''(s_o) &= \mathbf{Q}''(s_o) \\ &\vdots \\ \mathbf{R}^n(s_o) &= \mathbf{Q}^n(s_o) \end{aligned}$$

Equating the coefficients of powers of $(s - s_o)$ on both sides yields

$$[\mathbf{C}s_o^2 + \mathbf{B}s_o + \mathbf{A}]\mathbf{M}_o = \mathbf{R}(s_o) \quad (4.4)$$

$$[\mathbf{C}s_o^2 + \mathbf{B}s_o + \mathbf{A}]\mathbf{M}_1 = -[\mathbf{B} + 2s_o \mathbf{C}]\mathbf{M}_o + \mathbf{R}'(s_o) \quad (4.5)$$

Generalizing the above steps, the moments of higher order are

$$[Cs_o^2 + Bs_o + A]M_n = -BM_{n-1} - C[2s_oM_{n-1} + M_{n-2}] + \frac{R^n(s_o)}{n!} \quad (4.6)$$

for $n \geq 2$.

Equations (4.4), (4.5) and (4.6) are solved to obtain the moments M_n at all frequency point expansions. For the first frequency point expansion, sparse matrix re-ordering is followed by a LU decomposition to obtain all the moments M_0 - M_n . For the next expansion, only LU decomposition has to be performed while sparse matrix re-ordering is not essential. This yields considerable saving in terms of CPU time, thus resulting in an approach that is much faster than the conventional techniques. The magnitude of CPU time savings is illustrated in the examples of Section 4.4.

4.4 Examples

4.4.1 Diffusion Problems

Example 1:- The Magnetic Slot

Figure 4.2: The Magnetic Slot (Example 1).

For the purpose of evaluating the proposed approach, results from the problem

reported in [21] are compared. Consider a one dimensional diffusion problem shown in figure 4.2. The problem consists of a magnetic slot with bottom and side walls made of magnetic material with infinite permeability [34]. The time dependent magnetic field is illuminated on the top surface of the magnetic slot producing a time dependent current flowing through a metal-filled slot of infinite length. The excitation of the magnetic field $f(t)$ in (2.75) on the top surface of the magnetic slot is $f(t) = H(t) = e^{\alpha t} - e^{\beta t}$, which is an EMP type excitation and $\alpha = -4.0 \times 10^6$ and $\beta = -4.76 \times 10^8$. The other parameters are:

$$\begin{aligned} d &= 1.0 \times 10^{-4}(m), \\ \mu &= 4\pi \times 10^{-7}(H/m) \quad \text{and} \\ \sigma &= 5.76 \times 10^5(S/m). \end{aligned}$$

The boundary condition is: $H(t) = 0$ at $x = 0$.

Figure 4.3 shows the normalized time-domain magnetic field obtained by the proposed method at three different points ($x = \frac{d}{4}$, $x = \frac{d}{2}$, and $x = \frac{3d}{4}$). When results are compared with the response obtained by solving the ordinary differential equation (2.78) using conventional implicit numerical integration method [35], an accurate match is obtained (figure 4.3). The results are also in agreement with the analytical solution reported in [21].

Example 2:- The Rectangular Waveguide (Time Response)

The second problem used for evaluation of the proposed method is a two-dimensional metallic rectangular waveguide shown in figure 4.4. It is assumed that the waveguide is filled with a lossless dielectric material $\sigma = 0$ and its walls are perfectly conducting $\sigma_c = \infty$. For this problem it is also assumed that TM plane wave is propagated resulting in an electric field in z -direction which can be determined from scalar wave

Figure 4.3: Time response of magnetic field distribution (Example 1).

Figure 4.4: The Rectangular Waveguide (Examples 2 & 3).

equations. The other components of the field can be obtained from z -components of the electric field. The application of the proposed method is demonstrated for a rectangular waveguide with dimensions ($a = 1.5m$ and $b = 1.0m$) subject to boundary conditions

$$\Phi(x, 1, t) = 0, \quad \Phi(x, 0, t) = 0, \quad \Phi(0, y, t) = 0, \quad \Phi(1.5, y, t) = 0 \quad (4.7)$$

with initial conditions

$$\Phi(x, y, 0) = 0, \quad \frac{\partial \Phi}{\partial t}(x, y, 0) = 0 \quad (4.8)$$

where the right-hand side of the wave equation (2.75) is a function of space only, given by

$$f(x, y) = \sin\left(\frac{2\pi x}{3}\right) \sin(\pi y) \quad (4.9)$$

on the domain of solution. The problem reduces to that of a wave equation given by

$$\nabla^2 \Phi - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} = f(x, y) \quad (4.10)$$

The analytical solution of (4.10) is obtained using the method of separation of variables as follows:

$$\Phi(x, y, t) = \frac{9}{13\pi^2} (\cos \omega t - 1) f(x, y) \quad (4.11)$$

where $\omega = \frac{\sqrt{13}\pi c}{3}$.

Figure 4.5 shows both time-domain response of the electric field at $(x = \frac{3a}{4}, y = \frac{b}{2})$ and analytical solution from (4.11) being indistinguishable.

4.4.2 Resonance Problems

Example 3:- Rectangular Waveguide

Figure 4.5: Time response of electric field distribution (Example 2).

In this example, transverse resonance frequencies of the rectangular waveguide shown in figure 4.4 (with the same dimensions) are calculated by the proposed method where the excitation is applied at $(x = \frac{a}{4}, y = \frac{b}{2})$ and output is taken at $(x = \frac{3a}{4}, y = \frac{b}{2})$. To verify the proposed technique the transverse resonant frequencies are also calculated by analytical method and the results are compared in Table 4.1 and a good agreement is observed. To illustrate the efficiency of CFH technique, the transverse resonance frequencies are also calculated using the FEMFD approach where the frequency response was obtained directly by solving (2.86). CPU time comparison is shown in Table 4.2, and it is evident that CPU speed-up ranging from 19 to 73 times were obtained.

Example 4:- EM Shield

The example chosen here is an application to a two-dimensional shielding problem reported in [21]. The third term in (2.75) is dropped due to the omission of con-

Mode	FEM & CFH GHz	Analytical Solution $f = \frac{c}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ GHz
TM_{11}	.18102	.180152
TM_{21}	.25158	.249827
TM_{31}	.33749	.335178
TM_{13}	.46328	.460658
TM_{23}	.49702	.492102
TM_{51}	.52770	.521654
TM_{33}	.54890	.540458
TM_{61}	.62588	.618038
TM_{53}	.68161	.672215
TM_{63}	.73713	.749481

Table 4.1: Comparison of Transverse Resonant Frequencies (Example 3).

Matrix Size	FEM & CFH (seconds)	FEMFD (seconds)	Speed-up Ratio	Number of hops
25 × 25	.3	5.8	19	2
255 × 255	11.75	532.35	45	15
289 × 289	18.66	753.26	41	19
357 × 357	32.6	921.39	29	21
925 × 925	212.8	9364.68	45	23
1431 × 1431	500.33 (8.3 minutes)	36538.36 (10.1 hours)	73	14

Table 4.2: CPU Time Comparison (Example 3).

Figure 4.6: Electromagnetic Shield (Example 4).

duction current and this corresponds to wave equations. The problem consists of two TM plane waves, independent of z -coordinate, oppositely impinging on a rectangular metallic cylinder serving as a electromagnetic shield in which four slots or apertures are symmetrically located as shown in figure 4.6. The boundary conditions are as follows:

$$\begin{aligned}\frac{\partial E_z(t)}{\partial y} &= 0 \quad \text{on } cb \text{ and } oa \\ \frac{\partial E_z(t)}{\partial x} &= 0 \quad \text{on } ba\end{aligned}$$

A pulse is used as an excitation on the boundary co . The physical structure and the dimensions of the problem are as follows - the two-dimensional shield with slots is made of perfect conductor; the excitation source has no spatial variation in z direction;

$$oa = 1.0m,$$

$$ga = 0.2m.$$

$$ha = 0.2m,$$

$$ba = 1.0m,$$

$$de = 0.05m$$

and the slot width $ef = 0.05m$.

The location of output is at $(x = 0.9571m, y = 0.05m)$. In [21] LDFEM is used to find the frequency response of the problem. Here, 841 nodes and 1568 elements were used in FEM formulation. Transverse resonant frequencies are compared in Table 4.3 and CPU time comparison is shown in Table 4.4 and it is evident that a speed-up ratio of 145 was obtained.

LDFEM	FEMFD	FEM & CFH
531.90	530.16	529.98
894.50	814.57	879.78
1132.00	1131.30	1131.15

Table 4.3: Comparison of Transverse Resonant Frequencies (MHz) (Example 4).

FEMFD (sec)	FEM & CFH (sec)	Speed-up Ratio
17228.53	118.92	145

Table 4.4: CPU Time Comparison (Example 4).

Example 5:- Cylindrical Waveguide

The problem chosen here is a cylindrical structure, problem 5.25 in [36]. The problem consists of a circular cavity with a conducting wedge as shown in figure 4.7. For this structure whose thickness is relatively much smaller than the radius a , the

Figure 4.7: Cylindrical Waveguide (Example 5).

transverse resonant frequency of the dominant mode is given by

$$f_r = \frac{w}{2\pi a \sqrt{\epsilon\mu}} \quad (4.12)$$

where w is the first root of Bessel's function $J_v(w) = 0$ and $v = \frac{\pi}{(2\pi - \phi_0)}$. For a specific case, w and f_r are calculated to be 3.28 and 154.5 MHz using (4.12) where $\phi_0 = 45^\circ$, $v = 0.5714$, $a = 1.0m$. For several modes, the resonant frequency f_r is calculated and is given in the first column of Table 4.5.

In this example, 396 nodes and 700 elements were used in the FEM formulation. The analytical results are compared with the results from FEMFD and FEM & CFH in Table 4.5 and CPU time comparison is shown in Table 4.6 and is evident that a speed-up ratio of 292 was obtained.

Example 6:- TTEM Cell

The TTEM (Triple Transverse ElectroMagnetic) cell considered is an example

Analytical Solution	FEMFD	FEM & CFH
$TM_{01} = 154.50$	156.85	156.25
$TM_{11} = 245.65$	229.34	229.49
$TM_{21} = 329.08$	312.94	313.48
$TM_{02} = 353.80$	351.68	351.56

Table 4.5: Comparison of Transverse Resonant Frequencies (MHz) (Example 5).

FEMFD (sec)	FEM & CFH (sec)	Speed-up Ratio
3326.82	11.38	292

Table 4.6: CPU Time Comparison (Example 5).

Figure 4.8: Triple TEM Cell.

reported in [37]. The TTEM cell has a tapered structure and has an additional electric field polarisation in the transverse plane compared to conventional TEM cells. It also has two separately perpendicular inner conductors (septums) distinguishing it from GTEM (General Transverse ElectroMagnetic) cell. In [37], LDFEM is used to find the frequency response of TTEM cell. The cross-sectional area of the TTEM cell is shown in figure 4.8. The problem domain is bounded by the following Dirichlet boundary conditions: septum A is set to 1V while septum B is set to 0V. The outer conductor is set to 0V. The assumed excitation is a line source of E_z field for TM modes. It is also assumed that the fields have no variations along z -coordinate. The physical dimensions of the cell are: $a = 1.0m$, $b = 1.0m$, $w = 0.5m$, $h = 0.875m$ and $g = 0.1875m$. The number of degrees of freedom chosen in finite element calculation was 676. The transverse resonant frequencies using three different approaches are shown in Table 4.7. Table 4.8 shows a speed-up of 710 being achieved.

LDFEM	FEMFD	FEM & CFH
242.52	241.61	241.82
381.21	382.40	382.47
388.89	383.35	383.15
484.72	481.61	481.59
545.28	535.84	535.94
555.00	547.65	547.39
606.04	609.39	609.59
640.66	615.83	615.75

Table 4.7: Comparison of Transverse Resonant Frequencies (MHz) (Example 6).

FEMFD (sec)	FEM & CFH (sec)	Speed-up Ratio
18393.43	25.92	710

Table 4.8: CPU Time Comparison (Example 6).

4.5 Discussion and CPU Analysis

Examples 1-6 indicate a very good speed-up achieved by the proposed technique compared to the conventional technique. The reason behind this efficiency is that it requires only one LU decomposition per frequency hop, whereas FEMFD requires one LU decomposition at each frequency point. Generally, the number of frequency hops required is far less than the number of frequency points to get an accurate solution. To illustrate this, consider the CPU cost for the two approaches,

$$(CPU)_{FEMFD} = (CPU)_s + N_p \times [(CPU)_{LU} + (CPU)_{FB}] \quad (4.13)$$

$$(CPU)_{CFH} = (CPU)_s + N_h \times [(CPU)_{LU} + N_m \times (CPU)_{FB}] \quad (4.14)$$

where

$(CPU)_{FEMFD}$ is the total cost for the FEMFD approach;

$(CPU)_{CFH}$ is the total cost for the FEM & CFH approach;

$(CPU)_s$ is the time for reordering the sparse matrix;

N_p is the number of frequency points considered;

N_h is the total number of hops; and

N_m is the total number of moments;

$(CPU)_{LU}$ and $(CPU)_{FB}$ are the cost for LU decomposition and Forward-Backward substitution respectively.

Table 4.9 indicates the actual and expected speed-up ratio obtained for different

examples included in this paper. The deviation from the expected value is due to approximations used in (4.13) and (4.14), as explained in the following paragraph.

Experiments	Expected Speed-up N_p/N_h	Actual Speed-up $(CPU)_{FEMFD}/(CPU)_{CFH}$
<i>Ex.3, matrix size = 925</i>	45	45
<i>Ex.3, matrix size = 1431</i>	74	72
<i>Ex.4, matrix size = 841</i>	164	145
<i>Ex.5, matrix size = 396</i>	456	292
<i>Ex.6, matrix size = 676</i>	820	710

Table 4.9: Comparison of Speed-up Ratio for Different Examples

For sparse matrices, $(CPU)_{LU}$ is proportional to N^α where N is the size of the matrix resulting from FEM formulation and α is problem dependent, ranging between 1.1 to 1.7 [19]. Assuming $(CPU)_{LU}$ to be the most dominant factor compared to other CPU costs yields: Speed-up ratio $\approx \frac{N_p}{N_h}$ and the CPU cost for the proposed FEM & CFH approach is

$$(CPU)_{CFH} = KN_h N^\alpha \quad (4.15)$$

where K is a constant.

Chapter 5

Inhomogeneous Cylindrical Dielectric Resonators

5.1 Introduction

The problem of finding the resonant frequency of resonators was dealt with in the examples of Chapter 4. An efficient and an inexpensive method was illustrated and proved for several homogeneous cases. Inhomogeneous dielectric resonators comes under a different class of problems as compared to the homogeneous one due to the presence of several dielectrics in the example. This chapter concentrates on the study and analysis of these problems and a way to efficiently find their resonant frequency.

Dielectric resonators have found wide applications in the field of Microwave Integrated Circuits (MIC). Dielectric resonators operating in high-Q cavity performance in MIC's are widely used in filters, oscillators, discriminators, etc., [38], [39] and [40]. Among such resonators, analysis of cylindrical cavity resonators with abrupt discontinuities in the permittivity along the cross-section is becoming increasingly important. The design of a dielectric resonator depends on its natural resonant frequencies.

Earlier Research Work

The resonant frequency of rectangular and cylindrical dielectric resonators have been traditionally analyzed using the magnetic wall method [41], [42], [43], [44], [45] and also a number of methods [46], [47], [48], [49] have been proposed for the accurate determination of the resonant frequency of the dielectric resonator. In references [46], [47], [48], the resonant frequency has been evaluated either in the presence of one or two conductor planes perpendicular to the dielectric resonator's axis. The open boundary surrounding the resonator introduces radiation losses. So, in practice the dielectric resonator is always housed in a conducting enclosure in order to avoid radiation losses which in turn changes the resonant frequency. Hence the effects of the enclosure and the substrate have to be considered in the analysis of the resonant frequency. In [50], analysis have been carried out based on field matching, which accounts for the enclosure and the supporting substrate. Kooi et al [51] presented the results applying the finite element method to evaluate the resonant frequency of a cylindrical dielectric resonator taking into account the effects of the conducting shield and the supporting substrate. The FEM technique used in [51] is based on the Galerkin-weighted residual approach. The same approach has been adapted in this research project. However, the significant difference is that it only requires the computation of dominant natural modes which results in considerable reduction in computational cost. Also the major difference between this Chapter and Chapter 4 (Examples of homogeneous resonators), is in the FEM formulation. This is explained in the following sections.

5.2 Cavity Resonator - Problem Description

The electromagnetic boundary value problem is formulated using vector potentials. Starting with Maxwell's equations [36],

$$\nabla \times \mathbf{E} = -j\omega\mu_0\mathbf{H} \quad (5.1)$$

$$\nabla \times \mathbf{H} = j\omega\epsilon_0\epsilon_r\mathbf{E} + \mathbf{J} \quad (5.2)$$

where $\mathbf{J}$ is the source or impressed current.

In homogeneous media, $\nabla \cdot \mathbf{H} = 0$. Any divergenceless vector is the curl of some other vector; so

$$\mathbf{H} = \nabla \times \mathbf{A} \quad (5.3)$$

where $\mathbf{A}$ is the magnetic vector potential.

Substituting (5.3) in (5.1) results in

$$\nabla \times (\mathbf{E} + j\omega\mu_0\mathbf{A}) = 0 \quad (5.4)$$

Any curl-free vector is the gradient of some scalar. Hence,

$$\mathbf{E} + j\omega\mu_0\mathbf{A} = -\nabla\phi \quad (5.5)$$

where ϕ is the electric scalar potential. Using equations (5.3) and (5.5) in (5.2) results in an equation in terms of $\mathbf{A}$ (the magnetic vector potential) given by,

$$\nabla \times \nabla \times \mathbf{A} - k_0^2\epsilon_r\mathbf{A} = \mathbf{J} - j\omega\epsilon_0\epsilon_r\nabla\phi \quad (5.6)$$

which by vector identity, becomes

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} - k_0^2\epsilon_r\mathbf{A} = \mathbf{J} - j\omega\epsilon_0\epsilon_r\nabla\phi \quad (5.7)$$

If

$$\nabla \cdot \mathbf{A} = -j\omega\epsilon_0\epsilon_r\phi \quad (5.8)$$

the equation for $\mathbf{A}$ simplifies to

$$\nabla^2 \mathbf{A} + k_0^2 \epsilon_r \mathbf{A} = -\mathbf{J} \quad (5.9)$$

where $k_0^2 = \omega^2 \mu_0 \epsilon_0$. This is the Helmholtz equation, or complex wave equation. Solutions to (5.9) are called wave potentials. In terms of magnetic wave potential, $\mathbf{A}$,

$$\mathbf{E} = -j\omega\mu_0 \mathbf{A} + \frac{1}{j\omega\epsilon_0\epsilon_r} \nabla(\nabla \cdot \mathbf{A}) \quad (5.10)$$

$$\mathbf{H} = \nabla \times \mathbf{A} \quad (5.11)$$

obtained from (5.3), (5.5) and (5.8).

Analogous to $\mathbf{A}$ is $\mathbf{F}$, the electric vector potential. Equation (5.11) gives the magnetic field in terms of magnetic vector potential when only electric sources exist. The dual of this when only magnetic sources exist is given by [36]:

$$\mathbf{E} = -\nabla \times \mathbf{F} \quad (5.12)$$

Similarly, the solution to Helmholtz equation (5.9) in terms of electric wave potential is:

$$\mathbf{H} = -j\omega\epsilon_0\epsilon_r \mathbf{F} + \frac{1}{j\omega\mu_0} \nabla(\nabla \cdot \mathbf{F}) \quad (5.13)$$

The potentials $\mathbf{A}$ and $\mathbf{F}$ in equations (5.10) and (5.13) satisfy the Helmholtz wave equation and in general can be written as

$$(\nabla^2 + k_0^2 \epsilon_r) \Psi = 0 \quad (5.14)$$

where Ψ can represent either $\mathbf{A}$ or $\mathbf{F}$ and taking $\mathbf{J}$ as zero.

The cross-section of the cylindrical dielectric resonator of length L , radius R and dielectric constant ϵ_r , located on a substrate of thickness t with dielectric constant ϵ_s ,

Figure 5.1: Cross-section of axisymmetric cylindrical cavity resonator

is shown in figure 5.1. Problems having boundaries which coincide with cylindrical coordinates are usually solved in cylindrical coordinate surfaces. The scalar Helmholtz equation (5.14) is written in cylindrical coordinates as follows:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{\partial^2 \Psi}{\partial z^2} + k_0^2 \epsilon_r \Psi = 0 \quad (5.15)$$

Commonly found dielectric resonators have axially symmetric geometries, hence the formulation is limited to these cases. In the axisymmetric case, the set of modes can be further divided into:

- Transverse Electric to z , denoted TE or H modes and
- Transverse Magnetic to z , denoted TM or E modes

Generally, dielectric resonators are operated in the axisymmetric modes which are basically the dominant $TE_{01\delta}$ -modes. The subscripts $(0, 1, \delta)$ refer to the cylindrical

coordinates φ , ρ , and z respectively. The electric vector potential with only a z component will generate field TE to z where $\hat{z}$ is the unit vector in z direction. Thus,

$$\mathbf{F} = \Psi(\rho, z) \hat{z} \quad (5.16)$$

which reduces (5.15) to the scalar wave equation

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \Psi}{\partial \rho} \right) - \frac{1}{\rho^2} \Psi + \frac{\partial^2 \Psi}{\partial z^2} + k_0^2 \epsilon_r \Psi = 0 \quad (5.17)$$

in cylindrical coordinates and corresponds to the φ -independent TE mode. By analogy the magnetic vector potential can be represented by

$$\mathbf{A} = \Phi(\rho, z) \hat{z} \quad (5.18)$$

which reduces (5.15) to the scalar equation given in (5.17) by replacing ψ by ϕ and in this case the φ -independent TM mode is described. In the axisymmetric case, it is customary to express the field components of the TE and TM modes in terms of E_ϕ and H_ϕ rather than ψ and ϕ . Hence in the TE case the field components are

$$H_\rho = \frac{1}{j\omega\mu_0} \frac{\partial}{\partial z} E_\phi \quad (5.19)$$

$$H_z = \frac{1}{j\omega\mu_0\rho} \frac{\partial}{\partial \rho} \rho E_\phi \quad (5.20)$$

and (5.17) reduces to

$$\left(\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial}{\partial \rho} \right) - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} + k_0^2 \epsilon_r \right) E_\phi = 0 \quad (5.21)$$

In the TM case the field components are

$$E_\rho = -\frac{1}{j\omega\epsilon_0\epsilon_r} \frac{\partial}{\partial z} H_\phi \quad (5.22)$$

$$E_z = \frac{1}{j\omega\epsilon_0\epsilon_r\rho} \frac{\partial}{\partial \rho} \rho H_\phi \quad (5.23)$$

and (5.17) reduces to (5.21) except E_c is replaced by H_c . Due to the similarity in the analysis of the TE and TM cases, the discussion in the following sections is carried out only for the TE case. Also included is the finite element method of solving the Helmholtz equation (5.21) and the evaluation of resonant frequencies of the cylindrical dielectric resonators.

Summarizing, the major differences between the inhomogeneous and homogeneous problems can be stated as follows: as seen in (2.75), formulation for the homogeneous case yields

$$\nabla^2 \Phi - \epsilon \mu \frac{\partial^2 \Phi}{\partial t^2} - \sigma \mu \frac{\partial \Phi}{\partial t} = R(x, y, z, t) \quad (5.24)$$

where Φ represents either the electric or magnetic field components. Solution of this Maxwell's equation directly yields the electric and magnetic field. Formulation of the inhomogeneous problems starts with the basic Helmholtz equation (5.15) in cylindrical coordinates and the results are obtained in terms of A (the magnetic vector potential) or F (the electric vector potential). The fields H and E are related to A and F as shown in equations (5.11) and (5.12). Another important point to be noted here is the dependency of ϵ_r (5.28) in the matrices resulting from FEM formulation.

Various examples considered in the section on Examples proves the efficiency of the FEM & CFH technique for finding the resonant frequency of inhomogeneous cylindrical dielectric resonators. The speed-up and the number of hops required for the analysis is indicated.

5.3 FEM Formulation

Similar to the case of examples from Chapter 4, the solution domain of figure 5.1 is divided into three-noded triangular elements, and node numbering is done in such

a way that the resulting system matrix has a minimum bandwidth. More elements are taken in the dielectric resonator as most of the electromagnetic energy will be confined in it. Three-noded triangular elements are chosen to model the solution domain, and first order polynomials are selected to approximate the variation of E_z within each element. The next step in the analysis is the derivation of the matrix equations representing the properties of the individual elements.

Using the Galerkin's weighted residual method, the Helmholtz equation (5.21) is rewritten [51], [52] and equated to a small value R_e as shown below:

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{1}{\rho} \frac{\partial E_z}{\partial \rho} + (k_0^2 \epsilon_r - \frac{1}{\rho^2}) E_z = R_e \neq 0 \quad (5.25)$$

where R_e is the residue. The Galerkin weighted residual approach seeks to minimize the residue R_e . It is required to satisfy the following conditions,

$$\int_v R_e w \, dv = 0 \quad (5.26)$$

where w is a suitable weighting function and the integration is taken over the volume v of the domain. Since E_z is independent of φ , the elemental volume dv is given by $dv = 2\pi\rho \, da$, where da is the elemental area in the $\rho - z$ plane. Substituting this into (5.26) gives

$$\int_a \left\{ \rho w \frac{\partial^2 E_z}{\partial \rho^2} + \rho w \frac{\partial^2 E_z}{\partial z^2} + w \frac{\partial E_z}{\partial \rho} + (k_0^2 \epsilon_r \rho w - \frac{w}{\rho}) E_z \right\} da = 0 \quad (5.27)$$

By letting $E_z = \rho E$, the above equation results in

$$\int_a \left\{ \frac{\partial^2 \rho E}{\partial \rho^2} + \frac{\partial^2 \rho E}{\partial z^2} + \frac{1}{\rho} \frac{\partial \rho E}{\partial \rho} + (k_0^2 \epsilon_r - \frac{1}{\rho^2}) \rho E \right\} \rho w da = 0 \quad (5.28)$$

Applying the procedure similar to that described in [53], [54], the above equation can be reduced to a homogeneous algebraic equation

$$[Y][E] = [R] \quad (5.29)$$

where $[Y] = [A] + s^2[C]$ in the Laplace-domain where A and C are the square matrices, $[R]$ is the vector resulting from input excitation and boundary conditions. Finally, Approach 2 - Direct Generation of Moments of Chapter 4 is used to find the dominant resonant frequency. The input and the output points chosen for the CFH analysis are located inside the resonator. No change in the resonant frequency was observed with variations in the location of input and output.

5.4 Examples

5.4.1 Example 1

Problem Description

Figure 5.2: Cross-section of a resonator on a substrate

For purpose of comparison, the example chosen here is obtained from reference [51]. Figure 5.2 shows the cross-section of the cylindrical resonator of length L , radius R and dielectric constant ϵ_R supported by a substrate of thickness t and dielectric con-

start ϵ_S . The cylindrical conducting enclosure is of radius R_1 and height $(t + L + H)$, where H is the distance between the top conductor and the top of the resonator. For a structure with axial symmetry, it is convenient to carry out the analysis in the cylindrical coordinate system. The $TE_{01\delta}$ -mode has no azimuthal field variation $E_\phi = E_z = H_\phi = 0$, and the electric field component E_ρ satisfies the Helmholtz equation (5.25). Because of the symmetry, only half of the cross-section needs to be analyzed. In [51], finite element discretization has 576 elements and 325 nodes. As stated earlier, more elements are taken in the dielectric resonator as most of the electromagnetic energy will be confined in it.

Results and Discussion

The resonant frequency has been presented in [51] in a form independent of the absolute values by introducing the following normalizing factors:

$$t_n = t/R, L_n = L/R, H_n = H/R, R_n = R_1/R \text{ and } \epsilon_n = \epsilon_S/\epsilon_R,$$

with the resonator radial dimensions R and dielectric constant ϵ_R as reference. The normalized resonant frequency f_n is given by $f_n = 2Rf_0\sqrt{\mu_0\epsilon_0\epsilon_R}$. The dimensions for the resonator are:

$$L = 5 \text{ mm}$$

$$R = 5 \text{ mm}$$

$$\epsilon_R = 100$$

and for that of the substrate are:

$$R_1 = 15 \text{ mm}$$

$$t = 0.635 \text{ mm}$$

$$\epsilon_S = 9.6.$$

The normalized values are $\epsilon_n = 0.096$, $R_n = 1.5$, $t_n = 0.127$, $L_n = 1.0$.

In order to estimate the accuracy of FEM in computing the resonant frequency, results obtained in [51] have been compared with that obtained in [47] and [55]. In [47], the resonant frequencies have been computed based on a method which is a combination of both the magnetic wall and the dielectric waveguide models, while in [55], an analytical approach is used. Kooi et al [51] have used FEM for their analysis. In this thesis also, FEM approach has been used to obtain the matrices in the s -domain, and then these matrices are used as inputs to obtain the moments $M_0 - M_n$ as shown in the Approach 2 of last Chapter. This is proven to be two orders of magnitude faster than the other conventional approaches. The error in comparison with the other results lies within 5% which is well within the tolerance limits.

H_n	FEM & CFH $f_n(\text{GHz})$	FEMFD $f_n(\text{GHz})$
0.20	1.2427	1.2427
0.30	1.2199	1.2199
0.40	1.2069	1.2069
0.50	1.1988	1.1988
0.60	1.1939	1.1939
0.75	1.1890	1.1890
1.00	1.1874	1.1874
1.50	1.1857	1.1857
2.00	1.1857	1.1857
2.50	1.1857	1.1857

Table 5.1: Comparison of Resonant Frequencies for the Cylindrical Cavity Resonator on a substrate

In figure 5.3, f_n v/s H_n from Table 5.1 is plotted and this closely matches figure 7(a) in reference [51]. In [51], 325 nodes and 576 elements were used in the FEM analysis. Since [51] is used for comparison and validation of the results obtained.

Figure 5.3: Resonant frequencies for different distances H_n

Matrix Size	H_n	FEM & CFH (seconds)	FEMFD (seconds)	Speed-up Ratio	Number of hops
293 × 293	0.20	2.35	423.8	180	2
292 × 292	0.30	2.61	426.7	163	2
304 × 304	0.40	2.73	564.3	206	2
301 × 301	0.50	2.65	597.3	225	2
304 × 304	0.60	2.37	498.4	210	2
313 × 313	0.75	2.58	454.0	176	2
313 × 313	1.00	3.28	578.9	176	2
326 × 326	1.50	3.35	706.7	211	2
330 × 330	2.00	3.00	520.6	173	2
335 × 335	2.50	3.01	503.2	167	2

Table 5.2: CPU Time Comparison

approximately same number of nodes (node number corresponds to matrix size in column 1 of Table 5.2) is considered in FEM mesh generation. Table 5.2 also gives the CPU time comparison, speed-up ratio and number of hops required to simulate the resonant frequency.

5.4.2 Example 2

Figure 5.4: Cross-section of a resonator with inhomogeneous dielectrics

Problem Description

The second problem used for validation of the proposed method is the one reported in [49], where an iterative analytical method is presented for computing resonant frequencies of dielectric cylindrical resonators in inhomogeneous media. Figure 5.4 shows the configuration to be analyzed, which consists of a cylindrical high-dielectric

constant material (region 3) positioned between three layers of different dielectrics (regions 1, 2, and 4). The $TE_{10\epsilon}$ -mode field where J_0 , K_0 , and I_0 are the appropriate Bessel functions that describe the field distribution in the cylindrical region $r < R_1$ and the radially decaying fields for $r > R_1$. In [49], results are plotted for different values of h_4 (the distance between the top of the resonator and the boundary surface), and the resonant frequency, f_4 in GHz . Theoretical data and experimental results were computed in [49] using barium tetratitanate and assuming ϵ_r equal to 37.0 as the dielectric constant for the resonator. The dimensions of the example are:

$$h_1 = 0.0635 \text{ cm.}$$

$$\epsilon_1 = 10\epsilon_0,$$

$$\epsilon_3 = 37\epsilon_0,$$

$$\epsilon_5 = 1.4\epsilon_0,$$

$$h_3 = 0.48 \text{ cm.}$$

$$R_1 = 0.635 \text{ cm.}$$

$$R_2 = 1.4R_1.$$

The error in obtaining the resonant frequency f_0 lies within 5% and the computational speed-up achieved using FEM & CFH is very high compared to the regular analysis, FEMFD. In figure 5.5, $f_0(GHz)$ v/s h_4 from Table 5.3 is compared with the figure (7) of reference [49] for a specific $h_2 = 0.1 \text{ cm}$. Resonant frequencies for different values of h_4 are found. Table 5.4 gives the CPU time comparison, speed-up ratio and number of hops that are necessary for FEM & CFH analysis.

h_4	FEM & CFH f_0 (GHz)	FEMFD f_0 (GHz)
0.1	5.3516	5.3516
0.2	5.1563	5.1563
0.3	5.0488	5.0400
0.4	5.0195	5.0195
0.5	4.9902	4.9904
0.6	4.9902	4.9904
0.7	4.9805	4.9806
0.8	4.9805	4.9806
0.9	4.9805	4.9806
1.0	4.9805	4.9806

Table 5.3: Comparison of Resonant Frequencies for the Cylindrical Cavity Resonator with inhomogeneous dielectrics

Figure 5.5: Resonant frequencies for different distances h_4

Matrix Size	h_4	FEM & CFH (seconds)	FEMFD (seconds)	Speed-up Ratio	Number of hops
367×367	0.1	5.01	943.1	188	3
385×385	0.2	5.25	1075.2	204	3
405×405	0.3	5.07	1284.6	253	3
398×398	0.4	5.38	1099.0	204	3
397×397	0.5	5.05	1171.3	231	3
386×386	0.6	5.95	1216.0	204	3
398×398	0.7	5.70	1436.5	252	3
383×383	0.8	5.10	1220.5	239	3
404×404	0.9	5.70	1106.6	194	3
402×402	1.0	5.45	1314.2	241	3

Table 5.4: CPU Time Comparison

5.4.3 Example 3

This example is taken from [56] where the resonant frequencies of all the modes in cylindrical cavities axisymmetrically loaded with dielectrics are calculated. In this method, the radial variations of the field components in the resonator are expressed in terms of first-degree finite element polynomials while the axial variations of the field components are approximated by trigonometric functions. To calculate the resonant frequencies, an H -vector variational formulation is employed and minimized with respect to the coefficients of the expanded field components. Spurious solutions which are inherent in the finite element technique are effectively eliminated by means of a penalty term included in the variational expression, imposing a divergence-free magnetic field constraint. In this example, a cylindrical cavity loaded with a dielectric rod is considered. In [56], the eigenvalue equation has been solved to obtain the eigenvalues (resonant frequencies) and eigenvectors (field coefficients).

Problem Description

In its simplest form, the dielectric in the resonator extends fully along the cavity length as shown in figure 5.6. The dimensions of the problem are $a = 0.394in$, $b = 0.5in$ and $D = 2a$. The dielectric constant for the resonator is: $\epsilon_r = 37.6$. In the reference paper [56], the first four modes are compared with those in reference [57]. Resonant frequencies are found for different values of l .

Results and Discussion

In Table 5.5, resonant frequencies using different approaches have been tabulated. Results of FEM & CFH tally perfectly with FEMFD, while they are very close to the results of references [57] and [56]. CPU time comparison is shown in Table 5.6, where the speed-up ratio is one- to two-orders faster in magnitude than the FEMFD approach.

Figure 5.6: Cross-section of a cylindrical cavity loaded with a dielectric rod

$x = (\frac{D}{l})^2$	$l = \frac{D}{\sqrt{x}}$	FEM & CFH GHz	FEMFD GHz	Ref [57] GHz	Ref [56] GHz
2	1.4153	3.0176	3.0177	3.363	3.330
3	1.1556	3.2715	3.2716	3.582	3.613
4	1.0008	3.5059	3.5060	3.802	3.824
5	0.8951	3.7207	3.7208	4.013	4.020
6	0.8171	3.9250	3.9251	4.189	4.205
7	0.7565	4.1211	4.1212	4.403	4.381
8	0.7077	4.3066	4.3068	4.550	4.550
9	0.6672	4.4824	4.4826	4.710	4.713

Table 5.5: Comparison of Resonant Frequencies for a Cylindrical Cavity Resonator loaded with a dielectric rod

Matrix Size	$x = (\frac{D}{l})^2$	FEM & CFH (seconds)	FEMFD (seconds)	Speed-up Ratio	Number of hops
321 × 321	2	8.55	643.63	75	5
545 × 545	3	24.08	2647.93	110	4
478 × 478	4	6.53	1339.65	205	2
534 × 534	5	10.70	2489.52	232	2
484 × 484	6	8.15	1812.26	222	2
567 × 567	7	12.02	2685.67	223	2
434 × 434	8	5.71	1068.95	187	2
508 × 508	9	8.32	1964.17	236	2

Table 5.6: CPU Time Comparison

Chapter 6

Conclusions

6.1 Summary

In this thesis, Electromagnetic (EM) field problems are formulated using the Finite Element Method (FEM) and are solved using the Complex Frequency Hopping (CFH) technique. FEM & CFH has been used efficiently for obtaining the resonant frequency or both time-domain and frequency-domain solutions for these EM field problems. Problems consisting of Dirichlet, Neumann and combined boundary conditions can be solved using the proposed algorithm to obtain both time and frequency responses. EM problems can be broadly classified as:

- Homogeneous Problems
- Inhomogeneous Problems

As part of this research, several cases of diffusion and resonance problems were studied in the homogeneous category using the FEM & CFH technique and the speed-up advantage (one to three orders of magnitude) was compared to the conventional finite element technique. The numerical results obtained using the method proposed in this thesis closely agrees with previously published results.

In the inhomogeneous category, the weighted residual FEM approach was used to solve the problem of cylindrical dielectric resonators. The resonant frequencies obtained were close to the ones described in the previously published papers. The speed-up obtained for the inhomogeneous case was also of two to three orders of magnitude compared to conventional finite element technique.

To summarize, the advantages of the proposed technique are as follows:

Advantages

- Two to three orders of speed-up can be obtained with this technique as compared to other conventional techniques.
- Both time-domain and frequency-domain responses can be simultaneously obtained.
- It is capable of solving inhomogeneous media and inhomogeneous problems at the same time.

FEM & CFH technique is now in the hands of future researchers to further exploit to make it more universal such that it encompasses several types of EM problems.

6.2 Suggestions for Future Work

Further research can be conducted in the following areas:

- The examples considered in this thesis dealt with only structures that have axial symmetry. Hence only a cross-section of the structure was considered for the analysis. A study attempting to solve problems involving unsymmetrical structures using the proposed technique can be conducted.

- Only two-dimensional problems were considered in this thesis. This can be further extended to solve three-dimensional problems.
- The proposed technique can be made more general by solving several other types of EM problems like: Scattering problems, Antenna problems, Microstrip problems to name a few.

Bibliography

- [1] L. T. Pillage and R. A. Rohrer, "Asymptotic waveform evaluation for timing analysis," *IEEE Trans. Computer Aided-Design*, vol. 9, pp. 352–366, Apr. 1990.
- [2] M. Nakhla and Q. J. Zhang, editors. *Modeling and Simulation of High Speed VLSI Interconnects*, Kluwer, Boston, 1995.
- [3] Y. S. Tsuei, A. C. Cangellaris, and J. P. Prince, "Rigorous electromagnetic modeling of chip-to-package (first level) interconnections," *IEEE Trans. Components, Hybrids, and Manufacturing Tech.*, vol. 36, pp. 1775–1787, Dec. 1988.
- [4] E. Chiprout and M. Nakhla, *Asymptotic Waveform Evaluation and Moment Matching for Interconnect Analysis*, Kluwer, Boston, 1993.
- [5] J. Jin, *The Finite Element Method in Electromagnetics*, John Wiley and Sons, Inc., New York, NY, 1993.
- [6] M. N. O. Sadiku, *Numerical Techniques in Electromagnetics*, CRC Press, Inc., Florida, 1992.
- [7] S. G. Mikhlin, *Variational Methods in Mathematical Physics*, Macmillan, New York, 1964.
- [8] J. N. Reddy, *An Introduction to the Finite Element Method*, McGraw-Hill, New York, 1984.
- [9] R. F. Harrington, *Field Computation by Moment Methods*, Krieger, Malabar, Florida, 1968.

- [10] N. Ida and J. P. A. Bastos, *Electromagnetics and Calculation of Fields*, Springer - Verlag, Inc., New York, 1992.
- [11] P. P. Silvester and R. L. Ferrari, *Finite Elements for Electrical Engineers*, Cambridge University Press, New York, 1990.
- [12] D. S. Burnett, *Finite Element Analysis*, Addison - Wesley, MA, 1988.
- [13] J. Joseph, T. J. Sober, and K. J. Gohn, "Time domain analysis by the point-matched finite element method," *IEEE Trans. on Magnetism*, vol. 27, pp. 3852-3855, 1991.
- [14] N. K. Madsen and R. W. Ziolkowski, "Numerical solution of maxwell's equations in the time domain using irregular nonorthogonal grids," In *Wave Motion*, volume 10, pp. 583-596, 1988.
- [15] G. Mur, "The finite element modeling of three dimensional electromagnetic fields using edge and nodal elements," *IEEE Trans. on Antennas and Propagation*, vol. 41(7), pp. 948-953, July 1993.
- [16] S. Celozzi and M. Feliziani, "Time domain finite element simulation of conductive regions," *IEEE Trans. on Magnetism*, vol. MAG-29, pp. 1705-1710, 1993.
- [17] P. J. Leonard and D. Rodger, "Some aspects of two and three dimensional transient eddy current modelling using finite element and single step time matching algorithms," In *IEE Proc.*, volume 135 No. Part-H, pp. 159-166, 1988.
- [18] I. A. Tsukerman, A. Konrad, G. Bedrosian, and M. V. K. Chari, "A survey of numerical methods for transient eddy current problems," *IEEE Trans. on Magnetism*, vol. MAG-29, pp. 1711-1716, 1993.
- [19] J. Vlach and K. Singhal, *Computer Methods for Circuit Analysis and Design*, Van Nostrand Reinhold, New York, 1983.
- [20] H. J. Nussbaumer, *Fast Fourier Transform and Convolution Algorithms*, Springer-Verlag, 1982.

- [21] X. D. Cai and G. I. Costache. "A Laplace Domain Finite Element Method (LD-FEM) applied to diffusion and propagation problems in electrical engineering." *Inter. J. of Numerical Modelling. Electronic Networks, Devices and Fields*, vol. 7, pp. 419-432, 1994.
- [22] H. T. Chen and C. K. Chen. "Hybrid laplace transform/finite element method for two dimensional transient heat conduction." *J. Thermophysics*, vol. 2, pp. 31-36, 1988.
- [23] C. K. Chen and T. M. Chen. "New hybrid laplace transform/finite element method for three dimensional transient heat conduction problem." *Int. J. Num. Meth. Eng.*, vol. 32, pp. 45-61, 1991.
- [24] E. Chiprout and M. Nakhla, "Analysis of interconnect networks using complex frequency hopping," *IEEE Trans. Computer Aided Design*, vol. 14, pp. 186-200, Jan. 1995.
- [25] L. O. Chua and P. Lin, *Computer-Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques*, Prentice-Hall, Englewood Cliffs, NJ, 1975.
- [26] X. Huang, V. Raghavan, and R. A. Rohrer, "AWESim: a program for the efficient analysis of linear(ized) circuits," In *Proc. IEEE International Conference on Computer-Aided Design (ICCAD)*, pp. 534-537, Nov. 1990.
- [27] R. Sanaie, E. Chiprout, M. Nakhla, and Q. J. Zhang, "A fast method for frequency and time domain simulation of high speed VLSI interconnects," *IEEE Trans. Microwave Theory Tech.*, vol. 42, pp. 2562-2571, 1994.
- [28] K. S. Kundert, *Circuit Analysis, Simulation and Design*, volume 3 of pt. 1, North-Holland, 1986.
- [29] T. Tang and M. S. Nakhla, "Analysis of high-speed VLSI interconnect using the Asymptotic Waveform Evaluation technique," *IEEE Trans. Computer Aided-Design*, vol. 11, Mar. 1992.

- [30] U. Ghoshai and L. N. Smith, "Finite element analysis of skin effect in copper interconnects at 77 K and 300 K," *IEEE MTT-S Int. Symp. Dig.*, vol. , pp. 773-776, 1988.
- [31] C. K. Tzuang and T. Itoh, "Finite element analysis of slow wave Schottky contact printed lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-34(2), pp. 1483-1489, 1986.
- [32] M. A. Kolbehdari, "Analysis of quasi-static characteristic of cylindrical coupled microstrip transmission line by finite element method," *IEEE AP-MTT Benjamin Franklin Symp. Dig.*, vol. , pp. 69-73, May 1994.
- [33] M. A. Kolbehdari and M. Sadiku, "Finite and infinite analysis of coupled cylindrical microstrip line in a nonhomogeneous dielectric media," In *Proceedings IEEE Southeastcon*, pp. 269-273, March 1995.
- [34] M. A. Kolbehdari, M. S. Nakhla, and Q. J. Zhang, "Solution of EM problems using finite element method and complex frequency hopping techniques," In *25th European Microwave Conf.*, volume 2, pp. 1079-1081, Sep. 1995.
- [35] MATLAB. *User's Guide*, The Math Works, Inc., Natick, MA, 1992.
- [36] R. Harrington, *Time-harmonic electromagnetic fields*, McGraw-Hill Book Company Inc., New York, 1961.
- [37] X. D. Cai and G. I. Costache, "Finite element analysis of a triple-TEM cell," *IEEE Trans. on Electromagnetic Compatibility*, vol. 36(4), pp. 398-404, Nov. 1994.
- [38] J. Cohen and M. Gilden, "Temperature stability of an MIC Gunn-effect oscillator," *IEEE Trans. Microwave Theory Tech.*, vol. 21, pp. 115-116, Feb. 1973.
- [39] H. Abe, Y. Takayama, A. Higashisaka, and H. Takamizawa, "A highly stabilized low-noise GaAs FET integrated oscillator with a dielectric resonator in the C-band," *IEEE Trans. Microwave Theory Tech.*, vol. 26, pp. 156-162, Mar. 1978.

- [40] W. R. Day, "Dielectric resonators as microstrip-circuit elements," *IEEE Trans. Microwave Theory Tech.*, vol. 18, pp. 1175-1176, Dec. 1970.
- [41] S. B. Cohn, "Microwave bandpass filters containing high-Q dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. 16, pp. 218-227, 1968.
- [42] Y. Yee, "Natural resonant frequencies of microwave dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. , pp. 256, Mar. 196.
- [43] Chow, "In the solution of field pattern of cylindrical dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. , pp. 439, Sep. 196.
- [44] Sethares and etc., "Design of microwave dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-14, pp. 2, Jan. 1966.
- [45] Okaya and Barash, "The dielectric microwave resonator," *Proc. I.R.E.*, vol. , Oct. 1962.
- [46] T. Itoh and R. Rudokas, "New method for computing the resonant frequency of dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. 25, pp. 52-54, Jan. 1977.
- [47] P. Guillon and Y. Garault, "Accurate resonant frequency of dielectric resonators," *IEEE Trans. Microwave Theory Tech.*, vol. 25, pp. 916-922, 1977.
- [48] M. Jaworski and M. W. Pospieszalski, "An accurate solution of the cylindrical dielectric resonator problem," *IEEE Trans. Microwave Theory Tech.*, vol. 27, pp. 639-642, Jul. 1979.
- [49] R. R. Bonetti and A. E. Atia, "Design of cylindrical dielectric resonators in inhomogeneous media," *IEEE Trans. Microwave Theory Tech.*, vol. 29(4), Apr. 1981.
- [50] S. Tatsuguchi and etc., "An integrated 18 GHz receiver front end using a dielectric resonator stabilized generator," In *ICC 79, Conf. Rec.*, volume 2, pp. 26.2.1-5, 1979.

- [51] P. S. Kooi, M. S. Leong, and A. L. SatyaPrakash, "Finite-element analysis of the shielded cylindrical dielectric resonator," In *IEE Proceedings*, volume 132 No. H-1, Feb. 1985.
- [52] M. A. Kolbehdari and M. S. Nakhla, "Design of inhomogeneous waveguide using FEM and CFH techniques," *ANTEM*, vol. 1, pp. 35-38, Aug. 1996.
- [53] M. A. Kolbehdari, M. Srinivasan, M. S. Nakhla, Q. J. Zhang, and R. Achar, "Simultaneous time and frequency domain solutions of EM problems using finite element and CFH techniques," *IEEE Trans. Microwave Theory Tech.*, vol. 44, pp. 1526-1534, Sep. 1996.
- [54] M. A. Kolbehdari, "VLSI Interconnections with a MMTL in a multilayered dielectric media," *Journal of Electromagnetic Waves and Applications*, vol. 10, pp. 1483-1504, 1996.
- [55] M. W. Pospieszalski, "Cylindrical dielectric resonators and their applications in TEM line microwave circuits," *IEEE Trans. Microwave Theory Tech.*, vol. 27, pp. 233-238, 1979.
- [56] M. M. Taheri and D. Mirshekar-Syahkal, "Accurate determination of modes in dielectric loaded cylindrical cavities using a one-dimensional finite element method," *IEEE Trans. Microwave Theory Tech.*, vol. 37 No. 10, pp. 1536-1541, Oct 1989.
- [57] K. A. Zaki and C. Chen, "Loss mechanism in dielectric-loaded resonators," *IEEE Trans. Microwave Theory Tech.*, vol. 33, pp. 1448-1452, Dec. 1985.

Appendix A

Other Numerical Techniques for Solving EM Problems

A.0.1 Finite Difference Method (FDM)

The idea of FDM is to transform the Partial Differential Equation (PDE) into a set of difference equations over a set of points of the studied region (*DISCRETIZATION*). Using finite differences formulas for the grid points, PDE is transformed into a system of equations $[Y][\phi] = [R]$ where the column vector $[\phi]$ are the unknowns. The solution of the system is the approximate solution of the problem. FDM can handle both

1. Deterministic problems: Eg. $-\nabla^2\phi = \rho/\epsilon$
2. Eigenvalue problems: Eg. $\nabla^2\phi + \lambda^2\phi = 0$

Finite Differences Formulas

They are used to approximate the derivatives:

$$x_{i+1} - x_i = h$$

Figure A.1: Discretization for FDM

$$x_i - x_{i-1} = h$$

where h is the discretization step.

The forward difference formula is given by

$$\left. \frac{d\phi}{dx} \right|_i \approx \frac{\phi_{i+1} - \phi_i}{h}$$

for $h \rightarrow 0$, this is the derivative.

The backward difference formula is given by

$$\left. \frac{d\phi}{dx} \right|_i \approx \frac{\phi_i - \phi_{i-1}}{h}$$

The central difference formula is given by

$$\left. \frac{d\phi}{dx} \right|_i \approx \frac{\phi_{i+1} - \phi_{i-1}}{2h}$$

A few important observations can be made for the FDM, they are:

- Forward and backward difference formulas are used to compute derivatives at extreme ends.

- The central difference formula should be used if data are available on both sides of the point being considered.
- The replacement of the derivative by an algebraic operation is the key in FDM.
- The error decreases quadratically for the central difference formula and linearly for others.

The algebraic expression for the second derivative is:

$$\begin{aligned}\frac{d\phi}{dx}\bigg|_{i+\frac{1}{2}} &= \frac{\phi_{i+1} - \phi_i}{h} \\ \frac{d\phi}{dx}\bigg|_{i-\frac{1}{2}} &= \frac{\phi_i - \phi_{i-1}}{h} \\ \frac{d}{dx}\left(\frac{d\phi}{dx}\right)\bigg|_i &= \frac{\frac{d\phi}{dx}\big|_{i+\frac{1}{2}} - \frac{d\phi}{dx}\big|_{i-\frac{1}{2}}}{h} \\ \frac{\partial^2 \phi}{\partial x^2} &= \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}\end{aligned}$$

Finite Differences Formulas For Laplace Operator

Figure A.2: Laplace operator formula for FDM

For simplicity, two dimensions are chosen for the analysis of FDM. The above formulas give:

$$\begin{aligned}\nabla^2\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} \\ \nabla^2\phi &\approx \frac{\phi_{i-1,j} - 2\phi_{i,j} + \phi_{i+1,j} + \phi_{i,j-1} - 2\phi_{i,j} + \phi_{i,j+1}}{h^2} \\ \nabla^2\phi &\approx \frac{\phi_{i,j+1} + \phi_{i-1,j} + \phi_{i,j-1} + \phi_{i+1,j} - 4\phi_{i,j}}{h^2}\end{aligned}$$

This is called the *five point finite difference operator* corresponding to equispaced node points. It can be established for unequal arms and different systems of co-ordinates.

The salient features of FDM can be summarized as:

- FDM is used only for differential equations not integral equations,
- It can be used for both deterministic and eigenvalue problems,
- It is simple to understand and program.

The main disadvantages are:

- Not well suited for non-uniform discretization,
- Discontinuities in the geometry cannot be taken care of.

A.0.2 Method of Moments (MOM)

$$L \phi = R \tag{A.1}$$

Most of the EM problems can be stated in terms of an inhomogeneous equation (A.1), where L is an operator (which may be differential), R is the known excitation or source function, and ϕ is the unknown function to be determined. In other methods, L has been differential. MOM is suited for problems where L is an integral or integro-differential operator.

MOM is a general procedure for solving (A.1). It is essentially the method of weighted residuals [section 2.3.1]. Therefore, the method is applicable for solving both differential and integral equations. The method is so called [9] because it involves the process of taking moments by multiplying with appropriate weighting functions and integrating.

The procedure for applying MOM to solve (A.1) usually involves four steps:

1. derivation of the appropriate Integral Equation (IE),
2. conversion (discretization) of the IE into a matrix equation using basis (or expansion) function and weighting (or testing) function,
3. evaluation of the matrix elements, and
4. solving the matrix equation and obtaining the parameters of interest.

The basic tools for the second step have been discussed in the variational technique. An integral equation is any equation involving an unknown function ϕ within the integral. Simple examples of integral equations are the Fourier, Laplace, and Hankel equations. Most ordinary differential equations can be expressed as integral equations, but the reverse is not true. While boundary conditions are imposed externally in differential equations, they are incorporated within an integral equation.

Green's Functions

A more systematic means for obtaining an IE from a PDE is by constructing an auxiliary function known as the *Green's function* [6] of that problem. The Green's function, also known as the *source function* or *influence function*, is the kernel function obtained from a linear boundary value problem and forms the essential link between

the differential and integral formulations. Green's function also provides a method of dealing with the source term R in a PDE. In other words, it provides an alternative approach to the series expansion method for solving inhomogeneous boundary-value problems by reducing the inhomogeneous problem to a homogeneous one. When the domain to be solved is bounded by one or more grounded planes, there are two ways to obtain the Green's function:

1. *The method of images:* This is a powerful technique for obtaining the field due to one or more sources with conducting boundary planes. If a point charge q is at some distance h from a grounded conducting plane, the boundary condition imposed by the plane on the resulting potential field may be satisfied by replacing the plane with an *image charge* $-q$ located at a position which is the mirror location of q .

This can be best explained with an example. Consider the region between the ground planes at $y = 0$ and $y = h$ as shown in figure (A.3). The Green's function $G(x, y; x', y')$ is the potential at any point (x, y) which results when a unit charge of 1 C/m is placed at the point (x', y') . If no ground planes were present, the potential at distance ρ from a unit line charge would be:

$$V(\rho) = \frac{1}{4\pi\epsilon} \ln \rho^2 \quad (\text{A.2})$$

In order to satisfy the boundary conditions on the ground planes, an infinite set of images is derived as shown in figure (A.3). The potential due to such a sequence of line charges (including the original) within the strip is the superpo-

Figure A.3: Method of images for a single charge placed between two conducting planes

sition of an infinite series of images as given below:

$$\begin{aligned}
 G(x, y; x', y') &= \frac{1}{4\pi\epsilon} (\ln[(x-x')^2 + (y-y')^2] - \ln[(x-x')^2 + (y+y')^2] \\
 &\quad + \sum_{n=1}^{\infty} (-1)^n [\ln[(x-x')^2 + (y+y'-2nh)^2] \\
 &\quad - \ln[(x-x')^2 + (y-y'-2nh)^2] \\
 &\quad + \ln[(x-x')^2 + (y+y'+2nh)^2] \\
 &\quad - \ln[(x-x')^2 + (y-y'+2nh)^2]] \\
 &= \frac{1}{4\pi\epsilon} \sum_{n=-\infty}^{\infty} \ln \left[\frac{(x-x')^2 + (y+y'-2nh)^2}{(x-x')^2 + (y-y'-2nh)^2} \right] \quad (A.3)
 \end{aligned}$$

This series converges slowly and is difficult for numerical computation. It can be summed to give

$$G(x, y; x', y') = \frac{1}{4\pi\epsilon} \ln \left[\frac{\sinh^2(\frac{\pi(x-x')}{2h}) + \sin^2(\frac{\pi(y+y')}{2h})}{\sinh^2(\frac{\pi(x-x')}{2h}) + \sin^2(\frac{\pi(y-y')}{2h})} \right] \quad (A.4)$$

This expression satisfies the appropriate boundary conditions along the ground plane, i.e., $G(x, y; x', y') = 0$ at $y = 0$ or $y = h$. $G(x, y; x', y')$ has one singularity point at $x = x', y = y'$ in the region $0 \leq y \leq h$.

In order to evaluate an integral involving $G(x, y; x', y')$ in (A.4), it is convenient to take out the singular portion of the unit source function. Equation (A.4) can be rewritten as

$$G(x, y; x', y') = -\frac{1}{4\pi\epsilon} \ln[(x-x')^2 + (y-y')^2] + g(x, y; x', y') \quad (A.5)$$

where

$$\begin{aligned}
 g(x, y; x', y') &= \\
 &\frac{1}{4\pi\epsilon} \ln \left[\frac{[(x-x')^2 + (y+y')^2] [\sinh^2(\frac{\pi(x-x')}{2h}) + \sin^2(\frac{\pi(y+y')}{2h})]}{\sinh^2(\frac{\pi(x-x')}{2h}) + \sin^2(\frac{\pi(y-y')}{2h})} \right] \quad (A.6)
 \end{aligned}$$

The term $g(x, y; x', y')$ is finite everywhere in $0 \leq y \leq h$. The integral involving g is evaluated numerically, while the one involving the singular logarithmic term is evaluated analytically with the aid of integral tables.

2. *The eigenfunction expansion:* This method is suitable for deriving the Green's function for differential equations whose homogeneous solution is known. The Green's function is represented in terms of a series of orthonormal functions that satisfy the boundary conditions associated with the differential equation.

Consider the Green's function for the wave equation, $\nabla^2 G + \lambda^2 G = \delta(r - r')$
[Modified steady state Helmholtz (or wave) equation]

$$\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \lambda^2 \Psi = 0 \quad (\text{A.7})$$

subject to

$$\frac{\partial \Psi}{\partial n} = 0 \quad \text{or} \quad \Psi = 0. \quad (\text{A.8})$$

Let the eigenfunctions and eigenvalues of (A.7) that satisfy (A.8) be Ψ_j and λ_j , respectively, i.e.,

$$\nabla^2 \Psi_j + \lambda_j^2 \Psi_j = 0 \quad (\text{A.9})$$

Since Ψ_j form an orthonormal set,

$$\int_S \Psi_j^* \Psi_i \, dx \, dy = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases} \quad (\text{A.10})$$

where the asterisk (*) denotes complex conjugation. Assuming that Ψ_j form a complete set of orthonormal functions, $G(x, y; x', y')$ can be expanded in terms of Ψ_j , i.e.,

$$G(x, y; x', y') = \sum_{j=1}^{\infty} a_j \Psi_j(x, y) \quad (\text{A.11})$$

Since the Green's function must satisfy

$$(\nabla^2 + \lambda^2) G(x, y; x', y') = \delta(x - x') \delta(y - y'), \quad (\text{A.12})$$

substituting (A.9) and (A.11) into (A.12) yields:

$$\sum_{j=1}^{\infty} a_j (\lambda^2 - \lambda_j^2) \Psi_j = \delta(x - x') \delta(y - y'). \quad (\text{A.13})$$

Multiplying both sides by Ψ_i^* and integrating over the region S gives

$$\sum_{j=1}^{\infty} a_j (\lambda^2 - \lambda_j^2) \int_S \Psi_j \Psi_i^* dx dy = \Psi_i^*(x', y'). \quad (\text{A.14})$$

Imposing the orthonormal property in (A.10) leads to

$$a_i (\lambda^2 - \lambda_i^2) = \Psi_i^*(x', y')$$

or

$$a_i = \frac{\Psi_i^*(x', y')}{(\lambda^2 - \lambda_i^2)} \quad (\text{A.15})$$

Thus,

$$G(x, y; x', y') = \sum_{j=1}^{\infty} \frac{\Psi_j(x', y') \Psi_j^*(x, y)}{(\lambda^2 - \lambda_j^2)} \quad (\text{A.16})$$

The approach is limited to a separable coordinate system [6] since the requisite eigenfunctions can be determined for only these cases.