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Bishop's Condition and Related Properties in Local Spectral Analysis of Operators

by

Moussa Boucekkine, D.E.S., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute of
Mathematics and Statistics

Carleton University

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submitted by
Moussa Boucekkine, D.E.S., M.Sc.
in partial fulfilment of the requirements
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Dr. V. Dlab, Chair

Department of Mathematics and Statistics

Dr. C.K. Fong, Thesis Supervisor

Dr. Peter Rosenthal, External Examiner

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Abstract

Bishop's condition (β) is a central concept in the theory of generalized spectral decomposition in the sense of Foias. In this thesis, we introduce a new local spectral property we denote by $(w\beta)$, a weak form of Bishop's condition (β) . This new property fits strictly between the single-valued extension property and property (β) . We show that a bounded linear operator T on a Banach space X has property $(w\beta)$ if and only if its Berberian-Quigley's extension has the single-valued extension property. We then apply this characterization to obtain one of the main results of this thesis, namely, the sum and product of two commuting Banach space operators with property (β) have property $(w\beta)$. We also show that $(w\beta)$ -property is stable under compact perturbation and asymptotic similarity.

Furthermore, we introduce and investigate a concept derived from property (β) for operators on Banach spaces, that we call Bishop spectrum. One of the main findings of this investigation is the establishment of the spectral mapping theorem for Bishop spectrum. Particular emphasis is given to the determination of the spectral residuum of bounded linear operators on Banach spaces, a notion introduced by B. Nagy in 1979.

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Introduction

This dissertation centres around Bishop's condition (β) (also called property (β) , as frequently used in this thesis) and related properties such as its dual property (β^*) , a weak form that we call property $(w\beta)$, and decomposability, etc.

Roughly speaking, an operator T has property (β) if a sequence of analytic vector-valued functions converges to zero in compact open topology provided that the same happens to this sequence applied by $z - T$; (for a precise definition, see p. 9).

Bishop's property (β) was introduced by E. Bishop [13] nearly forty-five years ago in connection with the duality theory for spectral decompositions. In [32], Foias introduced the notion of decomposability, a generalization of spectral operators of Dunford [21], and proved in [33] that every decomposable operator has property (β) .

Inspired by Foias' work, a number of simplifications and results have been obtained by many authors. Probably the most interesting earlier result concerning property (β) is the following: an operator $T \in B(X)$ is decomposable in the sense of Foias if and only if both T and its adjoint T^* have property (β) ; (for the proof in the case of reflexive spaces, see [45]).

It has been observed in [6] that an operator $T \in B(X)$ is decomposable if and only if T has both properties (β) and (β^*) . More significantly, it has recently been shown in [5] that the properties (β) and (β^*) are dual to each other in the sense that an operator $T \in B(X)$ satisfies (β) if and only if its adjoint T^* satisfies (β^*) ; (for the proof in the case of reflexive spaces, see [34]).

Furthermore, it has been shown in [5] that an operator $T \in B(X)$ satisfies (β) if and only if T is similar to the restriction of a decomposable operator to one of its invariant subspaces and that T satisfies (β^*) if and only if T is similar to a “quotient” of a decomposable operator; see [30], [28]. These characterizations have been very useful in recent work on the invariant subspace problem for operators on Banach spaces in the spirit of Scott Brown.

The plan of this thesis is as follows: in Chapter 1, we fix the notation, terminology and collect the relevant definitions and material from local spectral theory, as well as some auxiliary results. Most of the results are only stated and the reader is referred to the literature for their proofs.

In Chapter 2, we make an observation that allows us to restate property (β) in a form that has a clear technical advantage. This is then used to give a simple proof of an old result, namely, a decomposable operator has property (β) . Miscellaneous problems concerning the single-valued extension property, properties (β) and (β^*) are presented. We also discuss the asymptotic behaviour of property (β) and its dual (β^*) , as well as the stability of these properties with respect to the spectral equivalence relation.

In [74], F.H. Vasilescu introduced what he called residually decomposable operators. I. Baculu [10], [11] and others, studied the so-called S -decomposable operators, an equally important concept related to decomposability in the sense of Foias. These are operators that have a “good spectral behaviour” only outside certain parts of their spectra. Along this line, we introduce what we call Bishop’s spectrum or (β) -spectrum in Chapter 3; roughly speaking, we consider the smallest closed subset of the spectrum of an operator on which the operator in question does not have property (β) . We investigate some basic properties of (β) -spectrum and prove the spectral mapping theorem, one of our main results. In particular, we use (β) -spectrum to determine the so-called spectral residuum of a bounded linear operator, a notion introduced by B. Nagy [57]. We close Chapter 3 with some applications.

In Chapter 4, we introduce a natural weakening of property (β) , which we call weak Bishop's property (abbreviated; $(w\beta)$). Specifically, we impose the boundedness on the sequence of analytic functions in the definition of property (β) . Our motivation to study this weak form of condition (β) is partly due to the fact that property $(w\beta)$ behaves well with respect to compact perturbation and fits strictly between the two fundamental notions in local spectral theory, namely, single-valued extension property and Bishop's condition (β) . In our development, we rely heavily on the method of Berberian-Quigley's extension. This technique induces a functor which transforms the problems concerning $(w\beta)$ -property to those concerning single-valued extension property which have known answers, for we will show that an operator $T \in B(X)$ has property $(w\beta)$ if and only if its BQ-extension $\tilde{T}$ has the single-valued extension property. We then apply this characterization to prove one of our main results in this chapter, namely, the sum and the product of two commuting operators with property (β) have property $(w\beta)$ in support of the conjecture that commuting decomposable operators have decomposable sums and products. The rest of the chapter is devoted to compact perturbation and asymptotic behaviour of operators with property $(w\beta)$.

Finally, we notice that some of our results can be carried over to N -tuples of commuting, possibly unbounded, operators. For the sake of simplicity, we restrict ourselves to the case of single bounded linear operators, (inspite of the fact that the many interesting examples in the theory are unbounded operators, see [59]).

Chapter 1

Preliminaries

This chapter contains some facts concerning decomposable operators, spectral manifolds, and some closely related notions from local spectral theory. Included also in this chapter are some auxiliary results. Its main purpose is to fix the notation and terminology, and to state some results used in the sequel. Since these results are quite well-known, most of their proofs will not be given. Actually, for our purposes, not much prerequisite is needed, especially if one takes advantage of the simplified description of decomposable operators. In the first section we lay out those conventions of terminology and notation that we shall adhere to throughout this thesis.

1.1 Notation and Terminology

Throughout this thesis we adopt the following notation and terminology.

- All Banach spaces are complex.
- All operators are linear and bounded.
- All subspaces are closed.

X : a general complex Banach space.

$B(X)$: the Banach algebra of all bounded linear operators on X .

$B(X, Z)$: the space of all bounded linear transformations from a Banach space X to a Banach space Z .

X^* and T^* : the dual of the Banach space X and the adjoint of an operator $T \in B(X)$, respectively.

$\text{Lat}(T)$: the lattice of all closed subspaces of X invariant under T .

$T|Y$ and T^Y where $Y \in \text{Lat}(T)$: the restriction of T to Y and the operator induced by T on the quotient space X/Y , respectively.

$\sigma(T)$, $\rho(T)$ and $R(., T)$: the spectrum, the resolvent set, and the resolvent function respectively of $T \in B(X)$.

$\sigma_p(T)$ and $\sigma_a(T)$: the point spectrum and approximate point spectrum of T , respectively.

$SM(T)$: the set of all spectral maximal spaces of T ; $Y \in \text{Lat}(T)$ is in $SM(T)$, if Y contains every $Z \in \text{Lat}(T)$ with $\sigma(T|Z) \subset \sigma(T|Y)$.

$Y^\perp$: if $Y \subset X$, then $Y^\perp$ denotes the set of elements of X^* that vanish on Y .

$\mathbb{C}$: the field of complex numbers.

S^c : the complement of a set S .

F_r : for a closed set F in $\mathbb{C}$, and $r \geq 0$, let $F_r = \{z : \text{dist}(z, F) \leq r\}$.

$\sigma_s(T)$: the surjectivity spectrum; for $T \in B(X)$, $\sigma_s(T) = \{z \in \mathbb{C} : (z - T)X \neq X\}$, cf. [54].

SVEP: the single-valued extension property; see Definition 1.3.1.

(C): Dunford or closure condition (C); see p. 10 of this thesis.

(β): Bishop's property beta; see Definition 1.3.5.

($w\beta$): weak Bishop's property; see Definition 4.1.1.

$\rho_T(x)$ and $\sigma_T(x)$: the local resolvent and local spectrum of T at $x \in X$; see p. 7.

S_T : the analytic residuum; see p. 13.

1.2 Auxiliary Results

For an open set $G \subset \mathbb{C}$, $H(G, X)$ denotes the space of all locally analytic X -valued functions on G , endowed with the topology of uniform convergence on all compact subsets of G . The following well known lemma is an immediate consequence of analytic continuation.

Lemma 1.2.1 *Let G be a connected open subset of $\mathbb{C}$, and let Y be a closed subspace of X . If $f \in H(G, X)$ has the property that $f(D) \subset Y$ for a nonempty open set $D \subset G$, then $f(G) \subset Y$ and thus $f \in H(G, Y)$.*

Lemma 1.2.2 [1, Lemma 4] *Let $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$ be a short exact sequence of Banach spaces and continuous linear mappings. For every simply connected open set $G \subset \mathbb{C}$, the sequence*

$$0 \rightarrow H(G, Y) \rightarrow H(G, X) \rightarrow H(G, Z) \rightarrow 0$$

with the induced continuous mappings is also exact.

Corollary 1.2.3 [33, P. 1542-1543] *Let X_1, X_2 be closed subspaces of X with $X = X_1 + X_2$. Then for every $f \in H(G, X)$, there are functions $f_j \in H(G, X_j)$ ($j = 1, 2$) such that*

$$f(z) = f_1(z) + f_2(z), \quad \text{for all } z \in G.$$

The proof follows from the above lemma and the exactness of the sequence

$$0 \rightarrow X_1 \cap X_2 \rightarrow X_1 \oplus X_2 \rightarrow X = X_1 + X_2 \rightarrow 0$$

with the canonical continuous linear mappings. (In fact, only the surjectivity of $H(G, X_1 \oplus X_2) \rightarrow H(G, X)$ is used.)

Lemma 1.2.4 [8, Lemma 2.1] *Let $\{T_n\} \subset B(X)$ such that $\limsup \|T_n\|^{1/n} = r < \infty$ and $f \in H(K^c, X)$, where K is a compact subset of $\mathbb{C}$. Then for all compact subsets F of $\mathbb{C}$ with $F \cap K_r = \emptyset$, the series $\sum_{n=0}^{\infty} T_n f^{(n)}(z)/n!$ is uniformly convergent on F . Its sum is an analytic function in $\mathbb{C} \setminus K_r$.*

Proposition 1.2.5 [22, Proposition 1.14] *Given $T \in B(X)$ and $Y \in \text{Lat}(T)$, we have*

- (1) $\sigma(T) \subset \sigma(T^Y) \cup \sigma(T|Y)$.
- (2) $\sigma(T^Y) \subset \sigma(T) \cup \sigma(T|Y)$.
- (3) $\sigma(T|Y) \subset \sigma(T^Y) \cup \sigma(T)$.

1.3 Local Spectral Theory

We first recall some basic notions and results from local spectral theory; see the monographs [15], [75] and [23] for more information.

Definition 1.3.1 [15] An operator $T \in B(X)$ is said to have the single-valued extension property (abbrev. *SVEP*) if, for every analytic vector-valued function $f : G \rightarrow X$ (G open subset of $\mathbb{C}$), the condition $(z - T)f(z) = 0$ on G implies $f = 0$ on G .

By a *local resolvent* of T for $x \in X$, we mean an analytic vector-valued function $f : U \rightarrow X$ ($U \subseteq \mathbb{C}$) such that

$$(z - T)f(z) = x, \quad z \in U.$$

It is easily seen that, in case T has (*SVEP*), we can patch up any collection of its local resolvents for the same vector, say x , and hence there is a local resolvent for x (necessarily unique) with the largest domain denoted by $\rho_T(x)$. The complement of $\rho_T(x)$ in $\mathbb{C}$ is called the *local spectrum* of T at x and is denoted by $\sigma_T(x)$.

For $T \in B(X)$ and F a closed set in $\mathbb{C}$, we define $X_T(F)$ to be the set of vectors $x \in X$ such that $(z - T)f(z) = x$, $z \in F^c$, for some analytic function $f : F^c \rightarrow X$. $X_T(F)$ is called a *spectral manifold* of T . If F is arbitrary, we let

$$X_T(F) = \bigcup \{X_T(E) : E \subset F \text{ and } E \text{ is closed}\}.$$

If T has (SVEP), then the above definition is identical with Dunford and Schwartz's [21] original definition

$$X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$$

Elementary properties of spectral manifolds are given by the following proposition.

Proposition 1.3.2 *For an operator $T \in B(X)$ and a closed subset F in $\mathbb{C}$, we have*

- (1) $X_T(\emptyset) = \{0\}$, $X_T(F) = X_T(\sigma(T) \cap F)$, $X_T(\sigma(T)) = X$.
- (2) If $AT = SA$, for some $S \in B(Y)$ and some $A \in B(X, Y)$, then $AX_T(F) \subset Y_S(F)$; in particular, $X_T(F)$ is a hyperinvariant linear manifold of T .
- (3) If $x \in X_T(F)$ and f is its local resolvent, then $f(z) \in X_T(F)$ for $z \in F^c$ (see, [15, P.1]).
- (4) $X_T(F) = X$ implies that $z - T$ is surjective for all $z \in F^c$ and hence that $\sigma(T) \subseteq F$ whenever T has (SVEP).

Part (4) of the above proposition follows from the fact that surjective operators with (SVEP) have to be bijective [77], (see also [31]).

Lemma 1.3.3 [63] *Let $T \in B(X)$ and let f be an analytic function defined on a neighbourhood of $\sigma(T)$. Assume that f is nonconstant on every component of its domain, then*

$$X_{f(T)}(F) = X_T(f^{-1}(F)) \quad \text{for all closed sets } F.$$

It is well known that if T has (SVEP), the following statements are true.

- (1) For an arbitrary family of closed sets F_α of $\mathbb{C}$

$$X_T(\cap_{\alpha} F_{\alpha}) = \cap_{\alpha} X_T(F_{\alpha})$$

(2) If $X_T(F)$ is closed for a closed set F in $\mathbb{C}$, then

$$X_T(F) \in SM(T) \quad \text{and} \quad \sigma(T|_{X_T(F)}) \subseteq F \cap \sigma(T).$$

However, if T does not have (SVEP), then the above statements are false, cf. [70].

We shall need some closely related notions.

Definition 1.3.4 [44, Definition 1] Let $Y \in Lat(T)$. A subspace Y of X is said to be *strongly analytic* for T if whenever $f_n : G \rightarrow X$ is a sequence of analytic functions such that $\text{dist}((z - T)f_n(z), Y) \rightarrow 0$ uniformly on compact subsets of G it follows that $\text{dist}(f_n(z), Y) \rightarrow 0$ on compact subsets of G .

Strongly analytic spaces were introduced by [44], and Snader [67] characterized them as those Y for which T^Y satisfies Bishop's condition (β) ; see also [48].

Definition 1.3.5 [13] An operator $T \in B(X)$ is said to have *Bishop's property* (β) if, for every open subset U of $\mathbb{C}$ and for every sequence of analytic functions $f_n : U \rightarrow X$ for which $(z - T)f_n(z)$ converges uniformly to zero on each compact subset of U , it follows that also $f_n(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on each compact subset of U .

Definition 1.3.6 [34] An operator $T \in B(X)$ is said to have *property* (β^*) if, for every open covering $\{G_1, \dots, G_n\}$ of $\sigma(T)$,

$$X = X_T(\overline{G_1}) + \dots + X_T(\overline{G_n}).$$

In the above definition, $X_T(\overline{G_j})$ ($j = 1, 2, \dots, n$) are not necessarily closed. In [6], E. Albrecht *et al.* defined what they call property (δ) . The definition of this

property is the same as Definition 1.3.6 with $n = 2$. According to [5] the properties (β^*) and (δ) are equivalent. Indeed, if $T \in B(X)$ has property (δ) , then from [5] T is similar to a quotient of a decomposable operator, i.e., there exists a decomposable operator $R \in B(Z)$ on some Banach Z space and a continuous linear surjection Q such that $TQ = QR$, from which it follows that T has property (β^*) .

It is proved in [5] that the properties (β) and (β^*) are dual to each other in the sense that an operator $T \in B(X)$ satisfies (β) if and only if its adjoint T^* satisfies (β^*) , see also [30] (for the proof in the case of reflexive spaces see [34]). We frequently use this characterization, so we record it for reference.

Theorem 1.3.7 *An operator $T \in B(X)$ satisfies (β) if and only if its adjoint T^* on X^* satisfies (β^*) .*

These two properties will be one of our main topics in the subsequent chapters. Recall that an operator $T \in B(X)$ is said to satisfy *Dunford's condition (C)* [21], or *closure condition (C)*, if $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. We have the following lemma.

Lemma 1.3.8 $(\beta) \implies (C) \implies (SVEP)$.

The first implication is obvious, and the second follows from [63, Theorem 2.13]; (see also, [53]). It is a well-known intriguing question whether $(C) \implies (\beta)$ holds. The implication $(SVEP) \implies (C)$ is definitely false.

Since the spectral manifold $X_T(F)$ is not necessarily closed, it seems that there is no natural Banach space structure on it that allows us to take advantage of the theory of Banach spaces. To serve our purposes we make use of a variant of spectral manifold introduced by Fong [34]. The following construction with the proofs can be found in [34]. For an arbitrary closed set F in $\mathbb{C}$, let $X_T^b(F)$ be the set of vectors $x \in X$ such that there exists a *bounded* analytic function $f : F^c \longrightarrow X$ satisfying $(z - T)f(z) = x$, $z \in F^c$.

For $x \in X_T^b(F)$, we define

$$\|x\|_b^F = \inf_f \sup_{z \in F^c} \|f(z)\|$$

where the infimum is taken over all bounded X -valued analytic functions on F^c satisfying $(z - T)f(z) = x$.

It is easy to verify that $\|\cdot\|_b^F$ is a norm on $X_T^b(F)$. Now, fix $z_0 \in F^c$. Then, for $x \in X_T^b(F)$ and f a X -valued bounded analytic function satisfying $(z - T)f(z) = x$, we have

$$\|x\| = \|(z_0 - T)f(z_0)\| \leq \|z_0 - T\| \|f(z_0)\| \leq \|z_0 - T\| \sup_{z \in F^c} \|f(z)\|.$$

This shows that, there is a constant M independent of x such that

$$\|x\| \leq M \|x\|_b^F, \quad (\text{we may take } M = \|z_0 - T\|).$$

The proof of the completeness of $X_T^b(F)$ under the norm $\|\cdot\|_b^F$ is rather routine but we write it down for the sake of completeness. It suffices to show that, if $\{x_n\}$ is a sequence in $X_T^b(F)$ with $\sum_n \|x_n\|_b^F < \infty$, then the series $\sum_n x_n$ converges in $\|\cdot\|_b^F$ -norm. Let f_n be a bounded local resolvent on F^c for x_n such that

$$\sup_{z \in F^c} \|f_n(z)\| \leq \|x_n\|_b^F + 2^{-n}.$$

Then, the series $\sum_n f_n(z)$ converges uniformly to some vector-valued bounded analytic on F^c . Since

$$\sum_n \|x_n\| \leq M \sum_n \|x_n\|_b^F,$$

$\sum_n x_n$ converges in the usual norm to some $x \in X$. Clearly, $(z - T)f(z) = x$, for all $z \in F^c$. This shows that $x \in X_T^b(F)$. Now it is easy to see that $\sum_n x_n$ converges to x in $\|\cdot\|_b^F$ -norm.

Next we show that $X_T^b(F)$ is invariant under T and the restriction of T to it is bounded with respect to $\|\cdot\|_b^F$ -norm. To this end, let $x \in X_T^b(F)$ and choose some

X -valued bounded analytic function f on F^c such that $(z - T)f(z) = x$, $z \in F^c$. Then, $(z - T)g(z) = Tx$, $z \in F^c$, where $g(z) = Tf(z)$, $z \in F^c$ satisfies

$$\sup_{z \in F^c} \|g(z)\| \leq \|T\| \sup_{z \in F^c} \|f(z)\|.$$

(In fact, the same argument shows that $X_T^b(F)$ is hyperinvariant for T .) Therefore

$$\|Tx\|_b^F \leq \|T\| \|x\|_b^F.$$

Notice that, for a bounded analytic function $f : F^c \rightarrow X$ with $(z - T)f(z) = x$ and $\mu \in F^c$, the function

$$g(z) = \frac{f(z) - f(\mu)}{z - \mu}, \quad z \neq \mu,$$

has a removable singularity at μ and hence g is a bounded analytic function on F^c satisfying

$$(z - T)g(z) = f(\mu)$$

which shows that $f(\mu) \in X_T^b(F)$. In view of $(\mu - T)f(\mu) = x$, we have

$$(z - T)X_T^b(F) = X_T^b(F).$$

We have shown the following.

Lemma 1.3.9 (1) *With the norm $\|\cdot\|_b^F$ on $X_T^b(F)$, the inclusion map of $X_T^b(F)$ into X is bounded, i.e., there exists a positive constant M independent of x such that $\|x\| \leq M\|x\|_b^F$, for $x \in X_T^b(F)$.*

(2) *With the norm $\|\cdot\|_b^F$, $X_T^b(F)$ is a Banach space.*

(3) *The restriction T_F of T to $X_T^b(F)$ is bounded with respect to the norm $\|\cdot\|_b^F$ and $\|T_F\|_b^F \leq \|T\|$.*

(4) *For $z \in F^c$, $z - T_F$ is surjective.*

We end this section with a concept closely related to (SVEP), introduced in [74].

Definition 1.3.10 Let $T \in B(X)$. An open set $\Omega \subset \mathbb{C}$ is a set of *analytic uniqueness* for T if, for every open set $U \subset \Omega$ and each X -valued analytic function f on U , the identity $(z - T)f(z) = 0$ on U implies $f(z) = 0$ on U .

Denote by Ω_T the maximal open set of analytic uniqueness for T ; then $S_T = \mathbb{C} \setminus \Omega_T$ is called the *analytic residuum* of T .

Remark 1.3.11 $S_T = \emptyset$ if and only if T has (SVEP). Note also that $S_T \subset \sigma(T)$.

For $x \in X$, denote by $\delta_T(x)$ the open set of those points $z \in \mathbb{C}$ such that there exists an open neighbourhood V of z and a local resolvent for x on V . Set

$$\gamma_T(x) = \mathbb{C} \setminus \delta_T(x), \quad \rho_T(x) = \delta_T(x) \cap \Omega_T$$

and

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x) = \gamma_T(x) \cup S_T.$$

If T has (SVEP), then $\sigma_T(x) = \gamma_T(x)$ and $\sigma_T(x)$ has the usual meaning.

We need the following lemma, cf. [31, 53, 54, 77].

Lemma 1.3.12 For every $T \in B(X)$, $\sigma(T) = \bigcup_{x \in X} \sigma_T(x) = \sigma_s(T) \cup S_T$. Furthermore, if T has (SVEP), then $\sigma(T) = \sigma_s(T)$ and if T^* has (SVEP), then $\sigma(T) = \sigma_a(T)$. Finally, $\sigma_a(T^*) = \sigma_s(T)$ and $\sigma_a(T) = \sigma_s(T^*)$.

1.4 Equivalent Conditions for Decomposability

The notion of decomposable operators on a Banach space has been introduced by Foias in [32]. This class of operators is a major generalization of the class of spectral operators [15], but it still admits a rich spectral theory. An operator $T \in B(X)$ is said to be decomposable in the sense of Foias if, for every finite open covering $\{G_1, \dots, G_n\}$ of $\sigma(T)$, there exists a system $\{X_1, \dots, X_n\}$ in $SM(T)$ such that

$$X = X_1 + \dots + X_n, \quad \sigma(T|X_i) \subset G_i, \quad (1 \leq i \leq n).$$

There are numerous characterizations simplifying many technical aspects of the above definition. Among them, we mention the *SDP*: we say that T has the *spectral decomposition property* (*SDP*) if, in the above definition, $SM(T)$ is replaced by $Lat(T)$. It is known from [1], [46], [56], [62] that the *SDP* is equivalent to $2 - SDP$ (i.e., when $n = 2$). It can be shown without much difficulty that this is equivalent to the fact that T has properties (β) and (β^*) . In the following theorem, we summarize some of these known characterizations in the literature.

Theorem 1.4.1 *Let $T \in B(X)$. Then the following assertions are equivalent:*

- (1) T is decomposable,
- (2) T has the *SDP*,
- (3) T has properties (β) and (β^*) ,
- (4) T has condition (C) and property (β^*) , and
- (5) For every open set G in $\mathbb{C}$, there is a subspace $Y \in Lat(T)$ such that $\sigma(T|Y) \subset \overline{G}$ and $\sigma(T^Y) \subset G^c$.

The equivalence $(1) \Leftrightarrow (2)$ was proven in three different ways in [1], [46], [56], while $(1) \Leftrightarrow (5)$ was proven in [47, P. 403]. The equivalences $(1) \Leftrightarrow (3)$ and $(1) \Leftrightarrow (4)$ follow from [15, Proposition 1.3.8] and Lemma 1.3.8, see also [45].

In what follows, some definitions related to the class of decomposable operators are given. A decomposable operator T is said to be *strongly decomposable* [7] if $T|_{X_T(F)}$ is decomposable for every closed set F in $\mathbb{C}$. Unlike decomposable operators, the duality theorem does not hold for strongly decomposable operators, cf. [78]. We also point out that Eschmeier [27] has shown by an example that not every decomposable operator is strongly decomposable.

Definition 1.4.2 [43] Let $T \in B(X)$. We say that T is *weakly decomposable* if, for every finite open covering $\{G_1, \dots, G_n\}$ of $\sigma(T)$, there exists a system $\{X_1, \dots, X_n\}$ in $SM(T)$ such that

$$X = \overline{X_1 + \dots + X_n}, \quad \sigma(T|_{X_i}) \subset G_i, \quad (1 \leq i \leq n).$$

T is said to be *quasi-decomposable* if it is weakly decomposable and satisfies condition (C).

Note 1.4.3 Weakly decomposable operators have (SVEP) [43]. Albrecht [2] gave an example of a weakly decomposable operator which is not decomposable.

We close this chapter with the following definition:

Definition 1.4.4 [24] Let $T \in B(X)$. We say that T has the *2-weak spectral decomposition property* if, for every open cover $\{G_1, G_2\}$ of $\mathbb{C}$, there exist T -invariant subspaces X_1 and X_2 such that

$$X = \overline{X_1 + X_2}, \quad \sigma(T|X_i) \subset G_i, \quad (i = 1, 2).$$

A natural question is whether or not the 2-weak spectral decomposition property implies weak decomposability. To the best of our knowledge, this is still an open problem.

Chapter 2

On Bishop's Property (β) and its Dual (β^*)

Property (β) was introduced by E. Bishop [13] nearly forty-five years ago in connection with the duality theory for spectral decomposition; see Definition 1.3.4.

Probably the most interesting early result concerning property (β) is the observation in [5] that a continuous linear operator on a Banach space X is decomposable in the sense of C. Foias if and only if both T and T^* possess property (β) ; (for a proof in the case of reflexive spaces see [45]).

It has been observed in [6], [34] that an operator $T \in B(X)$ is decomposable if and only if it has both properties (β) and (β^*) . More significantly, Albrecht and Eschmeier [5] have shown that an operator T has property (β) if and only if it is similar to the restriction of a decomposable operator to one of its closed subspaces, and that T has property (β^*) if and only if it is similar to a quotient of a decomposable operator, cf. [28]. Furthermore, it has been shown in [5] that properties (β) and (β^*) are dual to each other, in the sense that an operator $T \in B(X)$ satisfies (β) if and only if its adjoint $T^* \in B(X^*)$ satisfies (β^*) and that the corresponding statement remains valid if both properties are interchanged (for a proof in the case of reflexive spaces see [34]).

These characterizations have been very useful in recent work on the invariant subspace problem for operators on Banach spaces, see for example [30]. In the case of multipliers on uniform algebras and on algebras with bounded approximate identity, property (β^*) is considerably more restrictive than property (β) . In fact, in this particular case (β^*) is equivalent to decomposability. Another example is that decomposable convolution operators induced by measures on a locally compact Abelian group G are characterized by property (β^*) , see [52].

The purpose of this chapter is to present a number of observations and remarks concerning these two properties and their asymptotic behaviour, along with some other related problems.

In Section 1, a simple observation about the locality of property (β) is obtained, and is then used to give a simple proof of an old result stating that a decomposable operator has property (β) . Miscellaneous problems concerning the single-valued extension property and properties (β) and (β^*) are presented in Section 2. The concluding Section 3 deals with spectral equivalence and spectral convergence of operators with various spectral properties; we also discuss spectral inclusions under some appropriate assumptions.

2.1 Property (β)

In this section a simple observation about property (β) is made, then it is used to prove that an operator T with the 2-SDP (and hence a decomposable operator) has property (β) . Our new proof is quite lengthy but the technique involved is rather unorthodox and may be useful for some other problems. Here we do not use the quotient space techniques as in [1]. Instead, we use elementary facts such as the locality of (β) and the maximum modulus principle. First, let us make an observation allowing us to restate property (β) in a form that has a clear technical

advantage.

Let $\{f_n\}$ be an arbitrary sequence of X -valued analytic functions defined on an open set U of the complex plane such that $\|(z - T)f_n(z)\| \rightarrow 0$ (as $n \rightarrow \infty$) uniformly on compact subsets of U . In order to show that T has property (β) , we have to verify that $\|f_n(z)\| \rightarrow 0$ uniformly on all compact subsets of U . However, by a compactness argument, it is easy to see that verifying local uniform convergence is sufficient, i.e., it is enough to show that every point in U is the center of a closed disk contained in U on which $\|f_n(z)\| \rightarrow 0$ uniformly. Furthermore, on a closed disk D in U , f_n can be uniformly approximated by polynomials and hence we can choose a sequence of polynomials $\{p_n\}$ such that $\|f_n(z) - p_n(z)\| \rightarrow 0$ uniformly on D . It is clear now that $\|(z - T)p_n(z)\| \rightarrow 0$ (as $n \rightarrow \infty$) uniformly on D . If we can show that $\|p_n(z)\| \rightarrow 0$ uniformly on a smaller concentric disk D_0 , then we also have $\|f_n(z)\| \rightarrow 0$ uniformly on D_0 . Thus, with no loss of generality, we may assume that f_n are polynomials. Furthermore, we may assume all f_n vanish at the center z_0 of D . In fact, we can replace $f_n(z)$ by $(z - z_0)f_n(z)$, which clearly vanishes at z_0 , so that $\|(z - T)(z - z_0)f_n(z)\| \rightarrow 0$ (as $n \rightarrow \infty$) (uniformly on D) still holds.

Once we derive $\|(z - z_0)f_n(z)\| \rightarrow 0$ uniformly on D_0 , we may apply the maximum modulus principle to conclude that $\|f_n(z)\| \rightarrow 0$ uniformly on D_0 . The above discussion is summarized as follows:

Lemma 2.1.1 *An operator $T \in B(X)$ has property (β) if, for every sequence $\{p_n\}$ of X -valued polynomials such that $\|(z - T)p_n(z)\| \rightarrow 0$ uniformly on $D(z_0, r)$ and such that $p_n(z_0) = 0$ for all n , we have $\|p_n(z)\| \rightarrow 0$ uniformly on $D(z_0, \delta r)$ for some δ with $0 < \delta < 1$. (Here, $D(z_0, r)$ denotes the closed disk centred at z_0 with radius r .)*

Next, we prove the key lemma:

Lemma 2.1.2 Assume $X = X_1 + \cdots + X_m$, where $X_i \in \text{Lat}(T)$, $1 \leq i \leq m$. If P is an X -valued polynomial, then there are X -valued polynomials $\{P_j\}_{j=1}^m$ such that $P = P_1 + \cdots + P_m$ and, for each $j = 1, 2, \dots, m$ and $z \in D(0, r/8)$, $P_j(z) \in X_j$ and $\|(z - T)P_j(z) - v_j\| \leq MC$ for some vector v_j in X_j not depending on z and $M > 0$ (depending only on the decomposition $X = X_1 + \cdots + X_m$ of the space X), where

$$C = \sup\{\|(z - T)P(z)\| : z \in D(0, r)\}$$

Proof. We first note that since $X = X_1 + \cdots + X_m$, by the open mapping theorem, there exists $M > 0$ such that each $x \in X$ can be written as $x = x_1 + \cdots + x_m$ where $x_j \in X_j$ and $\|x_j\| \leq M\|x\|$ for each j . Now let $P(z) = \sum_{k=0}^n a_k z^k$, where $a_k \in X$. Then

$$(z - T)P(z) = \sum_{k=0}^{n+1} b_k z^k$$

where $b_0 = -Ta_0$, $b_{n+1} = a_n$ and $b_k = a_{k-1} - Ta_k$ for $k = 1, 2, \dots, n$. Since $\|(z - T)p(z)\| \leq C$ provided $|z| = r$, we have $\|b_k\|r^k \leq C$ by Cauchy's inequality, for $k = 0, 1, \dots, n+1$. For each k , write $b_k = b_{k1} + \cdots + b_{km}$ with $b_{kj} \in X_j$ and $\|b_{kj}\|r^k \leq MC$. Define a_{kj} ($0 \leq k \leq n$, $1 \leq j \leq m$) inductively in a reverse way by $a_{nj} = b_{n+1,j}$ and $a_{k-1,j} = b_{kj} + Ta_{kj}$. Since $a_{nj} \in X_j$ and X_j is invariant for T , we have $a_{n-1,j} = b_{nj} + Ta_{nj} \in X_j$. Inductively, we can show that $a_{kj} \in X_j$ for all k, j . Let $P_j(z) = \sum_{k=0}^n a_{kj} z^k$, $1 \leq j \leq m$. Then $P_1 + \cdots + P_m = P$ and $P_j(z) \in X_j$. Now

$$(z - T)P_j(z) = v_j + \sum_{k=1}^{n+1} b_{kj} z^k,$$

where $v_j = -Ta_{0j}$, and hence, when $|z| \leq r/2$,

$$\left\| \sum_{k=1}^{n+1} b_{kj} z^k \right\| \leq \sum_{k=1}^{n+1} MC/2^k < MC.$$

The proof of the lemma is complete. ■

Now we are ready to give the proof of the following theorem we have promised.

Theorem 2.1.3 Every operator $T \in B(X)$ with the 2-SDP has property (β) .

Proof. Fix $r > 0$. Since T has the 2-SDP, there are invariant subspaces X_1, X_2 of T such that $\sigma(T|X_1) \subset D(0, r/4)$ and $\sigma(T|X_2) \cap D(0, r/6) = \emptyset$ and $X = X_1 + X_2$. In view of Lemma 2.1.1, let $\{p_n\}$ be a sequence of X -valued polynomials such that $p_n(z_0) = 0$ and $\|(z - T)p_n(z)\| \rightarrow 0$ uniformly on $D(z_0, r)$. With no loss of generality, we may assume $z_0 = 0$.

We apply Lemma 2.1.2 to each p_n in our previous discussion. We write $p_n = p_{n1} + p_{n2}$, where, for each j , p_{nj} is an X_j -valued polynomial and

$$\|(z - T)p_{nj}(z) - v_{nj}\| \rightarrow 0 \quad (2.1)$$

uniformly on $D(0, r/2)$ for some v_{nj} in X_j .

For convenience, we will simply write T_j for $T|X_j$, the restriction of T to X_j . Since $\sigma(T|X_1) \subset D(0, r/4)$, $\sigma(T_1)$ is contained in the interior of $D(0, r/2)$. Therefore, $z - T_1$ is invertible for $|z| = r/2$ and

$$\|p_{n1}(z) - (z - T)^{-1}v_{n1}\| \rightarrow 0 \quad (2.2)$$

uniformly on the circle $\{z : |z| = r/2\}$. Therefore

$$\|v_{n1}\| = \left\| \frac{1}{2\pi i} \int_{|z|=r/2} (p_{n1}(z) - (z - T)^{-1}v_{n1}) dz \right\| \rightarrow 0$$

as $n \rightarrow \infty$. Hence, by (2.2) and the maximum modulus principle $\|p_{n1}(z)\| \rightarrow 0$ uniformly on $D(0, r/2)$. Similarly, since $\sigma(T|X_2) \cap D(0, r/6) = \emptyset$, $z - T_2$ is invertible for $|z| = r/8$, so the same argument shows that $\|p_{n2}(z)\| \rightarrow 0$ uniformly on $D(0, r/8)$ and therefore $\|p_n(z)\| \rightarrow 0$ uniformly on $D(0, r/8)$. ■

The following theorem establishes the stability of properties (β) and (β^*) under functional calculus.

Theorem 2.1.4 *Let $T \in B(X)$, and let $f : G \rightarrow \mathbb{C}$ be an analytic function defined in an open neighbourhood G of $\sigma(T)$ which is nonconstant on every component of G . Then*

- (a) *if T has property (β^*) , then so does $f(T)$*
- (b) *if T has property (β) , then so does $f(T)$.*

Proof. Assume T has property (β^*) , and let $\{G_i\}_{i=1}^n$ be an open cover for $\sigma(f(T))$. Then $\sigma(T) \subset \cup_{i=1}^n f^{-1}(G_i)$. Hence

$$X = X_T(\overline{f^{-1}(G_1)}) + \cdots + X_T(\overline{f^{-1}(G_n)}),$$

since $\overline{f^{-1}(G_i)} \subset f^{-1}(\overline{G_i})$; this implies that

$$X = X_T(f^{-1}(\overline{G_1})) + \cdots + X_T(f^{-1}(\overline{G_n})).$$

By Lemma 1.3.3, we have

$$X = X_{f(T)}(\overline{G_1}) + \cdots + X_{f(T)}(\overline{G_n}).$$

Thus, $f(T)$ has property (β^*) . Statement (b) follows from (a) by a duality argument, for if T satisfies (β) , then T^* satisfies (β^*) by Theorem 1.3.7, but then $f(T^*) = f(T)^*$ satisfies (β^*) by (a). Therefore $f(T)^{**}$ on X^{**} has property (β) by Theorem 1.3.7, from which it follows that $f(T)$ on X has property (β) ($X \subset X^{**}$). ■

Remark 2.1.5 The statement (b) in the above theorem was proved by J. Eschmeier and M. Putinar in [28]. The above is an alternative proof of this result.

Recall that an invariant subspace Y of T is said to be *T -strongly analytic* if T^Y has property (β) , (see Definition 1.3.4). The following corollary slightly improves a result of [67].

Corollary 2.1.6 *Let $T \in B(X)$ and f as in the preceding theorem. If $Y \in \text{Lat}(T)$ is T -strongly analytic subspace, it is also strongly analytic subspace for $f(T)$.*

Proof. Since T has property (β) , by [37, Theorem 1] Y is analytically invariant for T , thus by [23, Theorems 2.3 and 2.5] $f(T)^Y = f(T^Y)$. By the above theorem $f(T)^Y$ has property (β) , i.e., Y is strongly analytic for $f(T)$. ■

2.2 The *SVEP*

It is well known that, if $T \in B(X)$ satisfies the closure condition (C), then $X_T(F) \in SM(T)$ and $\sigma(T|X_T(F)) \subset F \cap \sigma(T)$ for every closed set F in $\mathbb{C}$; this follows from Lemma 1.3.8 and [15, Proposition 1.3.8]. The following lemma establishes the converse.

Lemma 2.2.1 *Let $T \in B(X)$. If for every closed set F in $\mathbb{C}$, $\sigma(T|\overline{X_T(F)}) \subset F$, then T satisfies the closure condition (C). In particular T has the single-valued extension property.*

Proof. Let F be a closed set in $\mathbb{C}$ and suppose that $\sigma(T|\overline{X_T(F)}) \subset F$. Let $x \in \overline{X_T(F)}$ be given, then $f(z) = (z - T|\overline{X_T(F)})^{-1}x$ is analytic and $(z - T)f(z) = x$ on $\rho(T|\overline{X_T(F)}) \supset \mathbb{C} \setminus F$, i.e., $x \in X_T(F)$. Thus $X_T(F)$ is closed. The second statement follows from Lemma 1.3.8. ■

Remark 2.2.2 For an operator $T \in B(X)$ without the (*SVEP*) and a closed $X_T(F)$ for some closed subset F of $\mathbb{C}$, the inclusion $\sigma(T|X_T(F)) \subset F \cap \sigma(T)$ is not true in general. However, we have $\sigma(T|X_T(F)) \subset \hat{F} \cap \sigma(T)$, where $\hat{F}$ is the union of all bounded components in $\mathbb{C} \setminus F$, see [70].

Tanahashi and Yoshino, in their paper [73], asked whether or not a continuous linear operator with the 2-weak SDP has the single valued extension property. In the following we give a partial answer to this question. We need the following related notion: we say that $T \in B(X)$ has the 2-weak (β^*) property if $X_T(\overline{G}_1) + X_T(\overline{G}_2)$ is dense in X for every pair of open sets G_1 and G_2 covering $\sigma(T)$.

Proposition 2.2.3 *Let $T \in B(X)$. If T has the 2-weak (β^*) property, then its adjoint T^* has the single-valued extension property.*

Proof. Let $V \subset \mathbb{C}$ be an open set, and let $g : V \longrightarrow X^*$ be a holomorphic function satisfying

$$(z - T^*)g(z) = 0 \quad \text{on } V. \quad (2.3)$$

Without loss of generality, we may assume that V is connected. Let $\{G_1, G_2\}$ be an open cover of $\sigma(T)$ such that $\overline{G_1} \subset V$ and $V \setminus \overline{G_2} \neq \emptyset$. We have

$$X = \overline{X_T(\overline{G_1})} + \overline{X_T(\overline{G_2})}.$$

For $x_1 \in X_T(\overline{G_1})$ and $z \in V \setminus \overline{G_1}$, there exists an analytic function $f_1 : U \longrightarrow X$ on some neighbourhood U of z such that $(z - T)f_1(z) = x_1$ on U . Therefore

$$\langle x_1, g(z) \rangle = \langle (z - T)f_1(z), g(z) \rangle = \langle f_1(z), (z - T^*)g(z) \rangle = 0$$

on U by (2.3). Thus $\langle x_1, g(z) \rangle = 0$ on all V by analytic continuation. Similarly, one has $\langle x_2, g(z) \rangle = 0$ on V for all $x_2 \in X_T(\overline{G_2})$. It follows that $\langle x, g(z) \rangle = 0$ for all $x \in \overline{X_T(\overline{G_1})} + \overline{X_T(\overline{G_2})}$ and $z \in V$ and since the latter is dense, we conclude that $g(z) \equiv 0$ on V . So T^* has the (SVEP). ■

Corollary 2.2.4 *If an operator $T \in B(X)$ has the 2-weak SDP, then its adjoint T^* has the (SVEP).*

Corollary 2.2.5 *If an operator $T \in B(X)$ has the 2-weak (β^*) property, then $\sigma(T) = \sigma_a(T)$.*

Proof. By Lemma 1.3.12, it suffices to show that T^* has the single valued extension property, but this follows from Proposition 2.2.3. ■

In tackling the problem of perturbation we have the following assertion.

Proposition 2.2.6 *Let $T, S \in B(X)$ such that $TS = ST$. If S is decomposable, the planar Lebesgue measure of $\sigma(T)$ equals zero, and $\mathbb{C} \setminus \sigma(T)$ connected, then $S + T$ has the (SVEP).*

Proof. Let $f : U \longrightarrow X$ be an analytic function on an open set U in $\mathbb{C}$ such that

$$(S + T - z)f(z) = 0 \quad \text{on } U$$

Without loss of generality, we may assume that U is an open disk of radius r . Since the (planar) Lebesgue measure of $\sigma(T)$ is zero, then there exists $\delta > 0$ such that

$$G \equiv \{z \in \mathbb{C} : \text{dist}(z, \sigma(T)) < \delta\}$$

has Lebesgue measure less than πr^2 . Let $\{U_1, U_2, \dots, U_n\}$ be an open cover of $\sigma(S)$ such that the diameter of each U_j is less than $2r$.

Take an arbitrary $x^* \in X^*$. It is enough to show that $\langle f(z), x^* \rangle = 0$. As S is decomposable, S^* has property (β^*) . Thus we can write

$$x^* = x_1^* + \dots + x_n^* \quad \text{with } x_j^* \in X_{S^*}^*(\overline{U_j}).$$

It suffices to show that $\langle f(z), x_j^* \rangle = 0$. Now, $X_{S^*}^*(\overline{U_j})$ is invariant for both S^* and T^* . Let S^j and T^j be the restrictions of S^* and T^* to $X_{S^*}^*(\overline{U_j})$, respectively. Then $\sigma(S^j) \subset \overline{U_j}$, which is of diameter less than $2r$ and $\sigma(T^j) \subset \sigma(T^*) = \sigma(T)$. Hence, $\sigma(T^j + S^j) \subset \sigma(T^*) + \sigma(S^j) = \sigma(S^j) + \sigma(T)$ which is a subset of a translate of G . Thus $V_j \equiv U \setminus \sigma(T^j + S^j)$ is nonempty. For $z \in V_j$, let $h^*(z) = (S^j + T^j - z)^{-1}x_j^*$. Then, for $z \in V_j$,

$$\langle f(z), x_j^* \rangle = \langle f(z), (S^j + T^j - z)h^*(z) \rangle = \langle (S + T - z)f(z), h^*(z) \rangle = 0.$$

Since $\langle f(z), x_j^* \rangle$ is analytic on U , it is identically zero. The proof is complete. ■

We end this section by showing another kind of permanence of property (β) . Let X, Y be two Banach spaces. For $T \in B(Y)$, define the left multiplication operator

$$L_T : B(X, Y) \longrightarrow B(X, Y)$$

by

$$L_TA = TA, \quad \text{for all } A \in B(X, Y).$$

We need the following lemma.

Lemma 2.2.7 *An operator $T \in B(X)$ has property (β) if, for every open set V of $\mathbb{C}$ and for every sequence of analytic functions $f_n : V \rightarrow X$ such that $(z-T)f_n(z) \rightarrow 0$ uniformly on all compact subsets of V , it follows that $\{f_n\}$ is bounded on compact subsets of V .*

Proof. Assume that T has the property prescribed in the lemma and $(z-T)f_n(z) \rightarrow 0$ uniformly on compact subsets of V . Let K be a compact subset of V , set:

$$\varepsilon_n = \sup_{z \in K} \|(z-T)f_n(z)\|$$

On the other hand, since $\varepsilon_n \rightarrow 0$, we also have $\sqrt{\varepsilon_n} = \sup_{z \in K} \|(z-T)\frac{f_n(z)}{\sqrt{\varepsilon_n}}\| \rightarrow 0$. It follows from the prescribed property for T that $\sup_{z \in K} \|\frac{f_n(z)}{\sqrt{\varepsilon_n}}\| \leq M$ for some $M > 0$. Therefore

$$\sup_{z \in K} \|f_n(z)\| \leq M\sqrt{\varepsilon_n} \rightarrow 0.$$

The proof of the lemma is complete. ■

Proposition 2.2.8 *For $T \in B(Y)$, L_T has property (β) if and only if T has the same property.*

Proof. Assume T has property (β) and let $F_n : V \rightarrow B(X, Y)$ be a sequence of analytic functions defined on an open $V \subset \mathbb{C}$ taking values in $B(X, Y)$ such that $(z - L_T)F_n(z) \rightarrow 0$ uniformly on every compact subset of V . Then, for a compact subset K of V , $(z - T)F_n(z)x \rightarrow 0$, for all $x \in X$ uniformly on K . Thus $F_n(z)x \rightarrow 0$ uniformly on K for all $x \in X$ and hence, by the uniform boundedness theorem, $F_n(z)$ is bounded on K , that is, $\sup_{n \in \mathbb{N}, z \in K} \|F_n(z)\| < \infty$. Consequently, L_T has property (β) by the above lemma.

The proof of the converse is the same as in [6, Corollary 1.2(a)]. ■

2.3 Spectral Equivalence and Spectral Convergence

In this section the asymptotic behaviour of property (β) under certain limits is obtained. We also discuss the stability of (β) and (β^*) with respect to spectral equivalence relation.

Let X, Y be Banach spaces, $T \in B(X), S \in B(Y)$. Define

$$C(S, T) : B(X, Y) \longrightarrow B(X, Y),$$

called the *commutator* of the pair (S, T) by

$$C(S, T)A = SA - AT, \quad \text{for all } A \in B(X, Y).$$

For every $n \geq 1$, we have

$$C^n(S, T)A = \sum_{k=0}^n \binom{n}{k} (-1)^k S^{n-k} AT^k = C^{n-1}(S, T)C(S, T)A$$

and

$$[C^n(S, T)A]^* = (-1)^n [C^n(T^*, S^*)A^*].$$

The operator T is called a *quasi-affine* transform of S if $C(S, T)A = 0$, for some quasi-affinity A . (An injective operator having a dense range is called a *quasi-affinity*.) The operators S and T are said to be *quasi-similar* if they are quasi-affine transforms of each other.

In case $X = Y$, denote by

$$d_{sp}(T, S) = \limsup_{n \rightarrow \infty} \|C^n(T, S)I\|^{1/n}$$

and

$$P(T, S) = \max\{d_{sp}(T, S), d_{sp}(S, T)\}, \quad \text{see [8].}$$

The map $T, S \longrightarrow P(T, S)$ is a semi-metric (also see [8]). If $TS = ST$, then $d_{sp}(T, S) = d_{sp}(S, T) = P(T, S)$. The non-Hausdorff topology determined by P is called the *perturbation topology* [8].

Definition 2.3.1 We say that $T, S \in B(X)$ are *spectral equivalent* (or *quasi-nilpotent equivalent*) if $P(T, S) = 0$, and we denote this relation by $T \stackrel{sp}{\sim} S$, (see [15], [18]). We say that a sequence $T_n \in B(X)$ converges *spectrally* to $T \in B(X)$ and denote it by $T_n \xrightarrow{sp} T$ if $P(T_n, T) \longrightarrow 0$, (see [8], [18]).

The proof of the following proposition is the same as the proof of (ii) $\implies$ (i) in Theorem 3.3 of [15].

Proposition 2.3.2 Assume $\lim \|C^n(S, T)A\|^{1/n} = 0$, where $T \in B(X)$, $S \in B(Y)$ and $A \in B(X, Y)$. Then $AX_T(F) \subset Y_S(F)$ for any closed set F of $\mathbb{C}$.

Proof. Let $x \in X_T(F)$. Then there exists an analytic $f : \mathbb{C} \setminus F \longrightarrow X$ such that

$$(z - T)f(z) = x. \quad (2.4)$$

Let $z_0 \in \mathbb{C} \setminus F$ be arbitrary and $0 < r_1 < r_2$ such that $D_2 \subset \mathbb{C} \setminus F$ (where $D_i = \{z : |z - z_0| < r_i\}$, $i = 1, 2$). Denote by $M = \max\{\|f(z)\| : |z - z_0| = r_2\}$. For $z \in D_1$, we then have

$$\left\| \frac{f^{(n)}(z)}{n!} \right\| = \|(2i\pi)^{-1} \int_{|\xi - z_0| = r_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi\| \leq \frac{Mr_2}{(r_2 - r_1)^{n+1}}, \quad \text{for all } n \geq 0. \quad (2.5)$$

Since $\lim_{n \rightarrow \infty} \|C^n(S, T)(A)\|^{1/n} = 0$, it follows that for every $\epsilon > 0$, there exists $M_\epsilon > 0$ such that

$$\|C^n(S, T)A\| \leq \epsilon^n M_\epsilon, \quad \text{for every } n \geq 0. \quad (2.6)$$

From (2.5) and (2.6) it results that

$$\|(C^n(S, T)A) \frac{f^{(n)}(z)}{n!}\| \leq \|C^n(S, T)A\| \left\| \frac{f^{(n)}(z)}{n!} \right\| < \frac{M_\epsilon M r_2 \epsilon^n}{(r_2 - r_1)^{n+1}}$$

from which by taking $\epsilon = (r_2 - r_1)/2$, we obtain

$$\|(C^n(S, T)A) \frac{f^{(n)}(z)}{n!}\| < \frac{M_0}{2^n}, \quad \text{for every } n \geq 0, \quad (2.7)$$

where M_0 does not depend on $z \in D_1$. This relation shows that the series

$$\sum_{n=0}^{\infty} (-1)^n (C^n(S, T)A) \frac{f^{(n)}(z)}{n!}$$

converges absolutely and uniformly in D_1 . Therefore, $D_1 \subset \mathbb{C} \setminus F$ being arbitrary, it converges absolutely and uniformly on every compact set $K \subset \mathbb{C} \setminus F$. It follows that the Y -valued function

$$h(z) = \sum_{n=0}^{\infty} (-1)^n (C^n(S, T)A) \frac{f^{(n)}(z)}{n!} \quad (2.8)$$

is analytic on $\mathbb{C} \setminus F$. Next, recall the following relation

$$(z - T)f^{(n)}(z) = -nf^{(n-1)}(z), \quad z \in \mathbb{C} \setminus F, \quad (2.9)$$

(see for example [23, P. 27]). Using relations (2.4), (2.8), and (2.9) for a brief computation, we obtain

$$(z - S)h(z) \equiv Ax$$

on $\mathbb{C} \setminus F$, (cf. [15, P. 51] for details.) Therefore, $Ax \in Y_S(F)$. Consequently

$$AX_T(F) \subset Y_S(F).$$

The proof is complete. ■

Corollary 2.3.3 *Let $R \in B(Y)$, $T \in B(X)$ and $A \in B(X, Y)$ satisfying*

$$\|C^n(R, T)A\|^{1/n} \rightarrow 0$$

as $n \rightarrow \infty$. If A has a finite dimensional null space, then $S_T \subset S_R$.

Proof. We follow the line of reasoning in the proof of Corollary 7.2 in [63]. Let D be an open disc in Ω_R and f an X -valued analytic function on D , such that $(z - T)f(z) = 0$. By Proposition 1.3.2 (3), $f(z) \in X_T(\mathbb{C} \setminus D)$ and, since $f(z)$ is an eigenvector of T corresponding to the eigenvalue z , $f(z) \in X_T(\{z\})$, ($z \in D$). Thus, by Proposition 2.3.2, $Af(z) \in Y_R(\{z\}) \cap Y_R(\mathbb{C} \setminus D) = \{0\}$ (since $D \subset \Omega_R$), and hence $Af(z) \equiv 0$. Let Z be the subspace generated by $f(z)$ for all $z \in D$. Since $AZ = 0$, Z is a finite dimensional invariant subspace of T . If $f(z)$ were nonzero, $T|Z$ would have infinitely many eigenvalues; a contradiction. Thus $f(z) \equiv 0$ on D . Hence $D \subset \Omega_T$. Consequently $\Omega_R \subset \Omega_T$. ■

The following corollary improves Corollary 3.2 of [53].

Corollary 2.3.4 *Let $T \in B(X)$, $R \in B(Y)$. Assume that there exist $V \in B(X, Y)$ surjective such that $\|C^n(R, T)V\|^{1/n} \rightarrow 0$ as $n \rightarrow \infty$ and $A \in B(Y, X)$ with finite dimensional null space, such that $\|C^n(T, R)A\|^{1/n} \rightarrow 0$ as $n \rightarrow \infty$. Then $\sigma(R) \subset \sigma(T)$.*

Proof. By the above corollary, $S_R \subset S_T$. According to Lemma 1.3.12 it is enough to show that $\sigma_s(R) \subset \sigma_s(T)$. Now by [54, Lemma 2] and Proposition 2.3.2, we have

$$\sigma_s(R) = \cup_{y \in Y} \gamma_R(y) = \cup_{x \in X} \gamma_R(Vx) \subset \cup_{x \in X} \gamma_T(x) = \sigma_s(T). \quad \blacksquare$$

Theorem 2.3.5 *Properties (β^*) and (β) are preserved under spectral equivalence.*

Proof. Let $S, T \in B(X)$. Assume T has property (β^*) and $T \stackrel{\mathfrak{sp}}{\sim} S$. We apply Proposition 2.3.2 twice with $X = Y$ to obtain $X_T(F) = X_S(F)$ for all closed sets of $\mathbb{C}$. We conclude that S also has property (β^*) . The proof for property (β) follows by the duality between these two properties (Theorem 1.3.7) and the fact that $T \stackrel{\mathfrak{sp}}{\sim} S$ if and only if $T^* \stackrel{\mathfrak{sp}}{\sim} S^*$. ■

We need the following lemma

Lemma 2.3.6 *Let $T, S \in B(X)$ and F be a compact set in $\mathbb{C}$. Then $X_T(F) \subset X_S(F_{P(T, S)})$.*

Proof. (Recall that $F_{P(T,S)}$ consists of those points (in the complex plane) whose distance to F is less than or equal to $P(T,S)$, see Section 1.)

Let $x \in X_T(F)$ and $f : \mathbb{C} \setminus F \rightarrow X$ be an analytic function such that $(z - T)f(z) = x$. By Lemma 1.2.4, the function g defined by the equation

$$g(z) = \sum_{n=0}^{\infty} (-1)^n \frac{(S - T)^{(n)}}{n!} f^{(n)}(z)$$

is analytic in $\mathbb{C} \setminus F_{P(T,S)}$. A routine check shows that $(z - S)g(z) = x$. Consequently, $x \in X_S(F_{P(T,S)})$. ■

Theorem 2.3.7 *Let $T_n, T \in B(X)$ such that $T_n \xrightarrow{sp} T$. Then*

- (a) *if T_n has property (β^*) for every $n \geq 1$, then so does T*
- (b) *if T_n has property (β) for every $n \geq 1$, then so does T .*

Proof. (a) Let $\{G_i\}_{i=1}^m$ be an open cover of $\sigma(T)$. Let F_i be compact sets such that $F_i \subset G_i$ and $\sigma(T) \subset \bigcup_{i=1}^m F_i^\circ$. By Proposition 2.2 in [8], we can find an integer N such that $\bigcup_{i=1}^m F_i^\circ \supset \sigma(T_n)$ for all $n \geq N$. Taking $n \geq N$ such that $F_{P(T,T_n)}^i = \{z \in \mathbb{C} : \text{dist}(z, F_i) \leq P(T, T_n)\} \subset G_i$, by Lemma 2.3.6, we have

$$X_{T_n}(F_i) \subset X_T(F_{P(T,T_n)}^i) \subset X_T(\overline{G_i}),$$

thus since T_n has property (β^*) , we conclude that

$$X = \sum_{i=1}^m X_T(\overline{G_i}),$$

from which it follows that T has property (β^*) . Part (b) is a consequence of (a), Theorem 1.3.7, and the fact that $P(T_n, T) = P(T_n^*, T^*)$. ■

A well-known result [8, Theorem 2.6] says that if T_n is a sequence of decomposable operators and if $T_n \xrightarrow{sp} T$, then T is also decomposable. This result is an immediate consequence of Theorem 2.3.7 and the fact that an operator is decomposable if and only if it has both properties (β) and (β^*) . The efficiency of our proof of this result

is another illustration of the technique based on the two facts:

- (1) (β) and $(\beta^*) \iff$ decomposability,
- (2) (β^*) is dual to (β) .

Definition 2.3.8 [53] An operator $A \in B(X, Y)$ is said to *asymptotically intertwine* $S \in B(X)$ and $T \in B(Y)$ if

$$\|C^n(S, T)A\|^{1/n} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

The operators T and S are said to be *asymptotically similar* if there exists a bijection $A \in B(X, Y)$ such that A asymptotically intertwines S and T and its inverse A^{-1} asymptotically intertwines T and S .

As noted in [53], asymptotic similarity generalizes slightly the notion of spectral equivalence (quasi-nilpotent equivalence), where, in the above definition $X = Y$ and $A = I$ the identity map on X , see Definition 2.3.1.

The following proposition is immediate from Corollary 2.3.4, see also [53].

Proposition 2.3.9 *If $T \in B(X)$ and $R \in B(Y)$ are asymptotically similar, then $\sigma(T) = \sigma(R)$.*

Definition 2.3.10 Let $T \in B(X)$ and $S \in B(Y)$. We say that T is *asymptotically quasi-affine transform* of S if $\|C^n(S, T)(A)\|^{1/n} \longrightarrow 0$, for some quasi-affinity A .

Note that the above definition covers the case that T is quasi-affine transform of S where it is required that $C(S, T)(A) = 0$

Proposition 2.3.11 *Suppose that T is quasi-affine transform of S . If T has property (β^*) and S has property (β) , then S is quasi-decomposable.*

Proof. Assume $\|C^n(S, T)(A)\|^{1/n} \longrightarrow 0$, for some quasi-affinity $A \in B(X, Y)$. By Proposition 2.3.3, $AX_T(F) \subset Y_S(F)$ for each closed set F in $\mathbb{C}$. Let $\{G_1, \dots, G_n\}$ be an open cover of $\mathbb{C}$. Since T has property (β^*) , we have

$$X = X_T(\overline{G_1}) + \dots + X_T(\overline{G_n}).$$

By the previous inclusion and the density of AX , we have

$$Y = \overline{Y_S(\overline{G_1}) + \cdots + Y_S(\overline{G_n})}.$$

Because S has property (β) , $Y_S(\overline{G_i})$ is closed by Lemma 1.3.8 and $\sigma(S|Y_S(\overline{G_i})) \subset \overline{G_i}$, for $i = 1, 2, \dots, n$. ■

Question 2.3.12 Proposition 2.3.11 makes the following question interesting. Suppose $T \in B(X)$ is a quasi-affine transform of $S \in B(Y)$, that is, $C(S, T)(A) = 0$ for some quasi-affinity A . Assume further that T has property (β^*) and S has property (β) . Then, is S decomposable?

Chapter 3

Bishop's Spectrum

Residually decomposable operators, introduced by F. H. Vasilescu [74], as well as bounded and unbounded S -decomposable operators studied by I. Baculu [10], [11] and others, cf. [57], [58], [72], [76], [79], are linear operators that show a "good spectral behaviour" only outside a certain part of their spectra. There are several related definitions of this type of good behaviour which are all connected with the concept of decomposability in the sense of C. Foias. As we have seen in Chapter 2, the class of operators with property (β) is a significant one; some of the recent advances in the work on invariant subspace problems illustrate this, see for instance [30]. It seems natural to investigate the role of property (β) in this context to gain a considerable new insight into operators with "good spectral behaviour" outside certain parts of their spectra. We do this by introducing the *Bishop's spectrum*. This concept will be made precise shortly. Loosely speaking, we consider the smallest closed subset of the spectrum on which the operator in question does not have property (β) .

The relevant preliminaries from local spectral theory and the theory of S -decomposable operators will be collected in Section 1. Section 2 contains the definitions and basic properties of Bishop's spectrum. In Section 3, we prove the spectral

mapping theorem for Bishop spectrum along with some properties pertinent to the restrictions and quotients. The central theme of Section 4 will be the discussion of the *spectral residuum*, defined in [57] as the minimal closed subset of the spectrum outside which the operator shows a good spectral behaviour (decomposability). By a known result due to [27], stating that decomposability is strictly weaker than strong decomposability, the spectral residuum is contained properly in the strong spectral residuum. In Section 5, we give some applications.

3.1 Preliminaries

We first recall some basic definitions and notions from spectral theory;

Definition 3.1.1 Let S be a closed subset of $\mathbb{C}$. An operator $T \in B(X)$ is said to possess *Bishop's property* (β) on $\mathbb{C} \setminus S$ if, for every open subset V of $\mathbb{C} \setminus S$ and every sequence of analytic functions $f_n : V \rightarrow X$ for which $(z - T)f_n(z)$ converges uniformly to zero on each compact subset of V , it follows that also $f_n(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on every compact subset of V . (See for instance [30]).

Given a closed set S in $\mathbb{C}$, a finite family of open sets $\{G_1, G_2, \dots, G_N, G_0\}$ is called an open S -covering of the closed set $F \subset \mathbb{C}$ if $G_0 \cup (\cup_{i=1}^N G_i) \supset F \cup S$ and $\overline{G_i} \cap S = \emptyset$ ($i = 1, 2, \dots, N$) [74].

The following definition is an extension of a definition of Fong [34] for the case $S = \emptyset$.

Definition 3.1.2 An operator $T \in B(X)$ is said to have *property* (β^*) on $\mathbb{C} \setminus S$ if

$$X = X_T(\overline{G_0}) + \sum_{i=1}^n X_T(\overline{G_i})$$

holds for every open S -covering $\{G_1, G_2, \dots, G_N; G_0\}$ of $\sigma(T)$.

The following lemma results immediately from Propositions 2.4 and 3.4 of [74].

Lemma 3.1.3 *If a closed set $F \subset \mathbb{C}$ contains S_T and if $X_T(F)$ is closed, then $X_T(F)$ is a spectral maximal subspace of T .*

Definition 3.1.4 Let $T \in B(X)$ and $S \subset \sigma(T)$ be a closed set. T is called *S-decomposable* if, for every open S -covering $\{G_1, G_2, \dots, G_n, G_0\}$ of $\sigma(T)$, there exists a system $\{X_1, X_2, \dots, X_n, X_0\}$ of spectral maximal spaces of T such that

- (1) $\sigma(T|X_i) \subset G_i$ for $i = 0, 1, 2, \dots, n$.
- (2) $X = X_0 + \sum_{i=1}^n X_i$.

We call T *strongly S-decomposable* if (2) is strengthened by

- (3) $Y = Y \cap X_0 + \sum_{i=1}^n Y \cap X_i$, for every spectral maximal space Y of T .

We call T *weakly S-decomposable* if (2) is weakened by

- (4) $X = \overline{X_0 + \sum_{i=1}^n X_i}$.

In the terminology of [74], $T \in B(X)$ is *S-decomposable* if and only if T is residually decomposable and has the so-called *property* (γ): for every open set with $G \supset S$, $X_T(\overline{G})$ is closed.

Several characterizations of *S-decomposable* operators are well known in the literature; see for example [74, Theorem 4.26], [72], [79]. Among them we mention the following so-called *S-decomposition property*: $T \in B(X)$ is *S-decomposable* if and only if, for every open covering $\{G, G_0\}$ of $\sigma(T)$ with $G_0 \supset S$ and $\overline{G} \cap S = \emptyset$, i.e., an *S-covering* consisting of two open sets, there exist subspaces Y and Y_0 that are invariant under T , such that $X = Y + Y_0$, $\sigma(T|Y) \subset G$ and $\sigma(T|Y_0) \subset G_0$.

We need the following lemma,

Lemma 3.1.5 *Let $T \in B(X)$ and $S \subset \sigma(T)$ be a closed set. If T has property (β) on $\mathbb{C} \setminus S$, then for every closed set $F \subset \mathbb{C}$ with $S \subset F$ or $S \cap F = \emptyset$ the spectral subspace $X_T(F)$ belongs to $SM(T)$ and satisfies $\sigma(T|X_T(F)) \subset F$.*

Proof. First we suppose $F \supset S$. Since T has property (β) on $S^c \supset F^c$, by the usual argument for $(\beta) \implies (C)$, we deduce that $X_T(F)$ is closed. Hence by Lemma 3.1.3, $X_T(F)$ has the required properties.

Now suppose $S \cap F = \emptyset$. Let $Y = X_T(F \cup S)$. Then $Y \in SM(T)$ and $\sigma(T|Y) \subset F \cup S$ ($S \subset F \cup S$). Since F and S are disjoint closed sets, in light of Riesz decomposition theorem, there exists Y_1 and Y_2 in $\text{Lat}(T)$ such that $Y = Y_1 \oplus Y_2$ and $\sigma((T|Y)|Y_1) \subset F$, $\sigma((T|Y)|Y_2) \subset S$. We show first that $X_T(F) = Y_1$. Let $x \in Y_1$. Since $(T|Y)|Y_1 = T|Y_1$, we have $\sigma(T|Y_1) \subset F$. For $x \in Y_1$, $(z - T)(z - T|Y_1)^{-1}x = x$ on F^c and hence $x \in X_T(F)$. Conversely, let $x \in X_T(F)$. Then, there exists a holomorphic function $f : F^c \longrightarrow X$ such that $(z - T)f(z) = x$ on F^c . By [74, Proposition 2.2], $f(z) \in X_T(F)$ for all $z \in F^c$, hence x and $f(z)$ are in $X_T(F \cup S)$ and so $(z - T)f(z) = (z - T|Y)f(z) = x$ on F^c . We can write $x = x_1 + x_2$ where $x_i \in Y_i$, ($i=1,2$). Let D be a Cauchy domain with boundary Γ such that $S \subset D$ and $\overline{D} \cap F = \emptyset$. Then

$$x_2 = \frac{1}{2\pi i} \int_{\Gamma} (z - T|Y)^{-1} x dz = \frac{1}{2\pi i} \int_{\Gamma} f(z) dz = 0.$$

Therefore $x = x_1 \in Y_1$. We have shown that $X_T(F) = Y_1$, $X_T(F)$ is closed and $\sigma(T|X_T(F)) = \sigma(T|Y_1) \subset F$. Finally, let $Z \in \text{Lat}(T)$ with the property that $\sigma(T|Z) \subset F$. Then $Z \subset X_T(F \cup S)$ and we can proceed as in the above to show that one has actually the inclusion $Z \subset X_T(F)$. Namely, if $x \in Z$ is written as $x = y_1 + y_2$ with $y_1 \in Y_1$ and $y_2 \in Y_2$, then one shows that $y_2 = 0$. Therefore $X_T(F) \in SM(T)$. ■

3.2 Bishop's Spectra

Recall from Definition 3.1.1 that, given $T \in B(X)$ and an open set $G \subset \mathbb{C}$, T is said to have property (β) on G if, for every open $U \subset G$ and every sequence of

holomorphic functions $f_n : U \rightarrow X$ satisfying

$$(z - T)f_n(z) \rightarrow 0$$

uniformly on compact subsets of U , it follows that $f_n(z) \rightarrow 0$ uniformly on all compact subsets of U .

Proposition 3.2.1 *For every $T \in B(X)$, there exists a unique maximal open set $G_\beta(T)$ such that T has property (β) on $G_\beta(T)$.*

Proof. The family of open sets $\{G_\nu\}$ on which T has property (β) contains at least $\rho(T)$, the resolvent set of T . Let us put

$$G_\beta(T) = \cup_\nu G_\nu.$$

It is clear that $G_\beta(T)$ is open and it is straightforward to check that T has property (β) on $G_\beta(T)$. Indeed, if $U \subset G_\beta(T)$ is an open set and $f_n : U \rightarrow X$ is a sequence of holomorphic functions such that

$$(z - T)f_n(z) \rightarrow 0$$

uniformly on compact sets of U , then as each $z_0 \in U$ has an open neighbourhood V_{z_0} with a compact closure $\overline{V}_{z_0}$ completely contained in a set G_ν for some ν so that $f_n(z) \rightarrow 0$ uniformly on compact sets of $\overline{V}_{z_0}$, it follows that $f_n(z) \rightarrow 0$ uniformly on compact subsets of U . Obviously, $G_\beta(T)$ is the largest open set on which T has property (β) . ■

Definition 3.2.2 We set $\sigma_\beta(T) = \mathbb{C} \setminus G_\beta(T)$ (where $G_\beta(T)$ is given in the preceding Proposition) and call it the *Bishop spectrum* of $T \in B(X)$.

It follows from the above definition that $\sigma_\beta(T)$ is a closed subset of $\sigma(T)$. Notice that $T \in B(X)$ has property (β) if and only if $\sigma_\beta(T) = \emptyset$. Furthermore, by Lemma

1.3.8, which says that property (β) implies the $(SVEP)$, we have $S_T \subset \sigma_\beta(T)$. It is not hard to see that, just like S_T , $\sigma_\beta(T)$ has no isolated points.

Suggested by the duality between the properties (β) and (β^*) , we define the β^* -spectrum by setting

$$\sigma_{\beta^*}(T) = \sigma_\beta(T^*).$$

Example 3.2.3 Let T be the left weighted shift operator defined by

$$Te_n = w_{n-1}e_{n-1}, \quad \text{for } n \geq 1$$

$$Te_0 = 0$$

where $\{e_n\}$, $n \geq 1$ is the usual orthonormal basis for $X = \ell_2(\mathbb{N})$ and $\{w_n\}$ is a bounded sequence of positive real numbers.

For $z \neq 0$, put $\mu = z^{-1}$; then we have

$$\begin{aligned} (1 + \mu T + \mu^2 T^2 + \cdots)e_n &= e_n + \mu w_{n-1}e_{n-1} + \cdots + \mu^n w_{n-1}w_{n-2} \cdots w_0 \\ &= (\mu^n w_{n-1}w_{n-2} \cdots w_0, \dots, \mu w_{n-1}, 1, 0, \dots) \end{aligned}$$

and

$$(I - \mu T)(1 + \mu T + \mu^2 T^2 + \cdots)e_n = e_n,$$

or equivalently

$$(z - T)(z^{-1} + z^{-2}T + \cdots)e_n = e_n.$$

Let

$$\begin{aligned} f_n(z) &= (z^{-1} + z^{-2}T + z^{-3}T^2 + \cdots)e_n \\ &= (z^{-n-1}w_{n-1}w_{n-2} \cdots w_0, \dots, z^{-1}w_{n-1}, 1, 0, \dots). \end{aligned}$$

It is clear that

$$\|f_n(z)\| \geq |z|^{-n-1}w_{n-1}w_{n-2} \cdots w_0.$$

Suppose that

$$\lim_{n \rightarrow \infty} \sup(|z|^{-n} w_{n-1} w_{n-2} \dots w_0)^{1/n} > 1,$$

that is,

$$|z| < \lim_{n \rightarrow \infty} \sup(w_{n-1} w_{n-2} \dots w_0)^{1/n} = r(T).$$

In this case, $\lim_{n \rightarrow \infty} \sup \|f_n(z)\| = \infty$, for $0 < |z| < r(T)$ and $(z - T)f_n(z) = e_n$. Therefore, $U = \{z : 0 < |z| < r(T)\} \subset \sigma_\beta(T)$, and hence $\sigma(T) = \overline{U} \subset \sigma_\beta(T)$, in other words $\sigma_\beta(T) = \sigma(T) = \{z : |z| \leq r(T)\}$.

As a result of these definitions, decomposable as well as strongly decomposable operators can be characterized as follows. Let T be a bounded linear operator on a Banach space, then

- i) T is decomposable if and only if $\sigma_{\beta^*}(T) = \emptyset = \sigma_\beta(T)$.
- ii) T is strongly decomposable if and only if $\sigma_{\beta^*}(T|Y) = \emptyset = \sigma_\beta(T|Y)$, for every $Y \in SM(T)$.

In [44], R. Lange introduced the notion of *strongly analytic subspaces*, (see Definition 1.3.4) and investigated some of their properties. In [67], C. Snader characterized them as those $Y \in Lat(T)$ for which T^Y has property (β) . In our terminology, this is equivalent to : a subspace $Y \in Lat(T)$ is T -strongly analytic if and only if $\sigma_\beta(T^Y) = \emptyset$.

Lange, in the same paper, asked the following question: is every nontrivial spectral maximal space of a decomposable operator strongly analytic? We reformulate the question in the present context as follows:

Question 3.2.4 Let $T \in B(X)$ be decomposable and $Y \in SM(T)$. Does it follow that $\sigma_\beta(T^Y) = \emptyset$?

We notice here that such subspaces in the above question are *analytically invariant* by [37], that is, $S_{TY} = \emptyset$. ($S_{TY} \subset \sigma_\beta(T^Y)$).

The following is a generalization of Definition 1 in [44]. Given $T \in B(X)$ and an open set $G \subset \mathbb{C}$, an invariant subspace Y under T is said to be *G-strong analytically invariant* of T if, for every open $U \subset G$ and for any sequence of holomorphic functions $f_n : U \rightarrow X$, ($n = 1, 2, \dots$) such that

$$\text{dist}((z - T)f_n(z), Y) \rightarrow 0$$

uniformly on compact subsets of U , it follows that $\text{dist}(f_n(z), Y) \rightarrow 0$ uniformly on compact subsets of U . It is clear that there is a maximal open set G_Y , depending on Y such that Y is G_Y -strong analytically invariant under T .

The proof of the following Proposition is identical with that of [67, Theorem 2.1], hence it is omitted here.

Proposition 3.2.5 *Let $T \in B(X)$, an invariant subspace Y of T is G-strong analytically invariant of T if and only if the coinduced operator T^Y has property (β) on G .*

Remark 3.2.6 If $T = \bigoplus_{i=1}^n T_i$, $T_i \in B(X_i)$, then $\sigma_\beta(T) = \bigcup_{i=1}^n \sigma_\beta(T_i)$. However, unlike the analytic residuum the above equality does not hold for a countable direct sum. Indeed, let $T = \bigoplus_{n=1}^\infty T_n$ be as in Example 3.9 of [15] (see Example 4.2.4). For each positive integer n , T_n is nilpotent, hence $\sigma_\beta(T_n) = \emptyset$, ($n = 1, 2, \dots$). Thus we have $\overline{\bigcup_{n=1}^\infty \sigma_\beta(T_n)} = \emptyset$. On the other hand, T does not have property (β) , i.e., $\sigma_\beta(T) \neq \emptyset$.

3.3 Properties of Bishop's Spectrum

In this section, we state and prove one of the main results of this chapter, namely, the spectral mapping theorem for (β) -spectrum (Theorem 3.3.6). But before getting there, we will go over a few results concerning the restrictions and quotients.

Proposition 3.3.1 *Let $T \in B(X)$ and Y be an invariant subspace of T . Then we have*

$$(1) \sigma_\beta(T^Y) \subset \sigma_\beta(T) \cup \sigma(T|Y).$$

$$(2) \sigma_\beta(T) \subset \sigma_\beta(T^Y) \cup \sigma(T|Y).$$

$$(3) \sigma_\beta(T|Y) \subset \sigma_\beta(T) \cap \sigma(T|Y).$$

Proof. (1) Let $f_n : V \subset G_\beta(T) \cap \rho(T|Y) \rightarrow X/Y$ be a sequence of holomorphic functions such that

$$(z - T^Y)f_n(z) \rightarrow 0$$

uniformly on compact subsets of V . For each n , there exists a holomorphic function $g_n : U \subset V \rightarrow X$ such that $\pi g_n(z) = f_n(z)$ on U , (where π is the canonical mapping from X onto X/Y).

Let $z_0 \in U$, $0 < r_1 < r_2$ and $D_i = \{z \in \mathbb{C} : |z - z_0| < r_i\}$ ($i=1, 2$), such that $\overline{D}_2 \subset U$. Next, we have, $\pi((z - T)g_n(z)) \rightarrow 0$ uniformly on $\overline{D}_2$ for $n \rightarrow \infty$. By Lemma 1.2.2, $H(D_2, X/Y)$ is topologically isomorphic to the quotient space $H(D_2, X)/H(D_2, Y)$. Hence, there exist functions $h_n \in H(D_2, Y)$, $n = 1, 2, \dots$ such that

$$(z - T)g_n(z) + h_n(z) \rightarrow 0 \text{ uniformly on } \overline{D}_1. \quad (3.1)$$

Since $h_n \in H(D_2, Y)$, $n = 1, 2, \dots$ and $\overline{D}_1 \subset \rho(T|Y)$, we have

$$h_n(z) = (z - T)(z - T|Y)^{-1}h_n(z), \quad z \in \overline{D}_1,$$

and thus, by (3.1), we obtain

$$(z - T)(g_n(z) + (z - T|Y)^{-1}h_n(z)) \rightarrow 0 \text{ uniformly on } \overline{D}_1.$$

Hence

$$g_n(z) + (z - T|Y)^{-1}h_n(z) \rightarrow 0 \text{ uniformly on } \overline{D}_1$$

in view of $\overline{D}_1 \subset G_\beta(T)$. This implies

$$f_n(z) = \pi(g_n(z) + (z - T|Y)^{-1}h_n(z)) \rightarrow 0 \text{ uniformly on } \overline{D}_1.$$

To prove (2), let $U \subset G_\beta(T^Y) \cap \rho(T|Y)$ be an open set and $f_n : U \rightarrow X$ a sequence of holomorphic functions such that

$$(z - T)f_n(z) \rightarrow 0 \text{ uniformly on compact subsets of } U. \quad (3.2)$$

As in the first part of the proof, let $z_0 \in U$, $0 < r_0 < r_1 < r_2$ and $D_i = \{z \in \mathbb{C} : |z - z_0| < r_i\}$ ($i = 0, 1, 2$), such that $\overline{D}_2 \subset U$, (3.2) implies $(z - T^Y)\pi f_n(z) \rightarrow 0$ uniformly on $\overline{D}_2$. Since $\overline{D}_2 \subset G_\beta(T^Y)$, $\pi f_n(z) \rightarrow 0$ uniformly on $\overline{D}_2$. By Lemma 1.2.2, there exist functions $g_n \in H(D_2, Y)$ such that

$$f_n(z) + g_n(z) \rightarrow 0 \text{ uniformly on } \overline{D}_1. \quad (3.3)$$

Hence, $(z - T)(f_n(z) + g_n(z)) \rightarrow 0$ uniformly on $\overline{D}_1$ and we obtain

$$(z - T)g_n(z) = (z - T)(f_n(z) + g_n(z)) - (z - T)f_n(z) \rightarrow 0, \quad (3.4)$$

uniformly on $\overline{D}_1$. Because $D_0 \subset \rho(T|Y)$, we have $M = \sup_{z \in D_0} \|R(z, T|Y)\| < \infty$. We obtain from (3.4) that

$$\|g_n(z)\| = \|R(z, T|Y)(z - T)g_n(z)\| \leq M\|(z - T)g_n(z)\| \rightarrow 0$$

uniformly on D_0 . In view of (3.3), this implies

$$f_n(z) = (g_n(z) + f_n(z)) - g_n(z) \rightarrow 0$$

uniformly on D_0 , and (2) is proved. The inclusion (3) is obvious. ■

Corollary 3.3.2 *Let $T \in B(X)$ and $Y \in \text{Lat}(T)$. Then*

$$\sigma(T|Y) \cup \sigma_\beta(T^Y) = \sigma(T|Y) \cup \sigma_\beta(T).$$

Proposition 3.3.3 *Let $T \in B(X)$ and Y be an invariant subspace of T . Then we have*

$$\sigma_\beta(T) \subset \sigma_\beta(T|Y) \cup \sigma_\beta(T^Y).$$

Proof. Let $V \subset G_\beta(T|Y) \cap G_\beta(T^Y)$ be an open set and $f_n : V \rightarrow X$ a sequence of holomorphic functions such that

$$(z - T)f_n(z) \rightarrow 0, \quad (3.5)$$

uniformly on compact subsets of V . For $z_0 \in V$, let $0 < r_0 < r_1 < r_2$, and $D_i = \{z \in \mathbb{C} : |z - z_0| < r_i\}$, such that $\overline{D_2} \subset V$. It follows from (3.5) that $(z - T^Y)\pi f_n(z) \rightarrow 0$ uniformly on $\overline{D_2}$, and therefore $\pi f_n(z) \rightarrow 0$ uniformly on $\overline{D_2}$. By Lemma 1.2.2, $H(D_2, X/Y)$ is topologically isomorphic to $H(D_2, X)/H(D_2, Y)$, thus one can choose a sequence g_n in $H(D_2, Y)$ such that $f_n(z) + g_n(z) \rightarrow 0$ uniformly on $\overline{D_1}$, but then

$$(z - T|Y)g_n(z) = (z - T)(f_n(z) + g_n(z) - (z - T)f_n(z)) \rightarrow 0$$

uniformly on $\overline{D_1}$. This implies $g_n(z) \rightarrow 0$ uniformly on $\overline{D_0}$ and hence

$$f_n(z) = (f_n(z) + g_n(z)) - g_n(z) \rightarrow 0$$

uniformly on $\overline{D_0}$. The proposition is proved. ■

The following corollary, first proved in [27], is an immediate consequence of Proposition 3.3.3.

Corollary 3.3.4 *Let $T \in B(X)$. If $T|Y$ and T^Y have property (β) for some $Y \in \text{Lat}(T)$, then T also has property (β) .*

The following example shows that there exists a bounded linear operator T and $Y \in \text{Lat}(T)$ such that $\sigma_\beta(T^Y)$ is not in $\sigma_\beta(T|Y) \cup \sigma_\beta(T)$.

Example 3.3.5 Let $X = \ell_2(\mathbb{Z})$, T be the bilateral shift, and $Y = \overline{\text{span}}\{e_{-1}, e_{-2}, \dots\}$. Since T is unitary, we have $\sigma_\beta(T) = \emptyset = \sigma_\beta(T|Y)$. However, T^Y is isomorphic to the left shift on $\ell_2(\mathbb{N})$ and hence does not have property (β) , i.e., $\sigma_\beta(T^Y) \neq \emptyset$.

Next we prove the spectral mapping theorem for β -spectrum.

Theorem 3.3.6 *Let $T \in B(X)$. Let f be a scalar-valued analytic function defined on an open neighbourhood G of $\sigma(T)$. Assume f is non-constant in each component of its domain. Then*

$$\sigma_\beta(f(T)) = f(\sigma_\beta(T)).$$

Proof. First we show the inclusion $f(\sigma_\beta(T)) \subset \sigma_\beta(f(T))$. Let $\lambda_0 \in \sigma_\beta(T)$. Then, for any neighbourhood U of λ_0 , there exist an open set V containing λ_0 such that $V \subset U$ and functions $f_n \in H(V, X)$, $n = 1, 2, \dots$ such that f_n does not converge to 0 uniformly on every compact subset of V , but

$$(z - T)f_n(z) \longrightarrow 0 \quad (3.6)$$

uniformly on all compact subsets of V . Without loss of generality, we may assume that $f'(\lambda_0) \neq 0$. Let

$$g_\lambda(z) = \begin{cases} \frac{f(\lambda) - f(z)}{\lambda - z} & \text{for } z \neq \lambda \\ f'(\lambda) & \text{for } z = \lambda \end{cases}$$

The function $g_\lambda(T)$ is a well defined holomorphic function of λ in the domain of f , and

$$f(\lambda) - f(T) = g_\lambda(T)(\lambda - T).$$

Hence, by (3.6), we have

$$\|(f(\lambda) - f(T))f_n(\lambda)\| \longrightarrow 0 \quad (3.7)$$

uniformly on all compact subsets of V . Since $f'(\lambda_0) \neq 0$, there exists an open set D with $\lambda_0 \in D \subset V$ such that

$$f|D : D \longrightarrow f(D) = D'$$

is biholomorphic, see for instance [64, P. 281]. Therefore, (3.7) implies

$$\|(\nu - f(T))g_n(\nu)\| \longrightarrow 0$$

uniformly on all compact subsets of D' , where $g_n(\nu) = f_n(f^{-1}(\nu))$ for $\nu \in D'$. Since f_n does not converge to zero on all compact subsets of V , g_n does not converge to zero uniformly on all compact subsets of D' . Therefore $\nu_0 = f(\lambda_0) \in \sigma_\beta(f(T))$.

To prove the reverse inclusion, namely, $\sigma_\beta(f(T)) \subset f(\sigma_\beta(T))$, it is enough to show that if $\nu_0 \notin f(\sigma_\beta(T))$, then $\nu_0 \notin \sigma_\beta(f(T))$.

Let K be compact subset such that $\sigma(T) \subset \overset{\circ}{K} \subset K \subset G$. Notice that for every $\nu \in \mathbb{C}$, $\nu - f$ has only finitely many zeros in K , say $\nu_1, \nu_2, \dots, \nu_k$, counting multiplicities (both k and the zeros $\nu_1, \nu_2, \dots, \nu_k$, depend on ν).

Since $\nu_0 \notin f(\sigma_\beta(T))$, $\nu_0 - f$ has only finitely many zeros in K , say $\nu_1^0, \nu_2^0, \dots, \nu_m^0$, with $\nu_1^0, \nu_2^0, \dots, \nu_m^0 \notin \sigma_\beta(T)$. Let $F = \{z \in K : f'(z) = 0\}$ (consisting of finitely many points). For the time being, we make the extra assumption that $\nu_0 \notin f(F)$. It is clear that $\nu_1^0, \nu_2^0, \dots, \nu_m^0 \notin F$, i.e., $f'(\nu_j^0) \neq 0$ for $j = 1, 2, \dots, m$. In other words, $\nu_1^0, \nu_2^0, \dots, \nu_m^0$ are simple zeros of f .

Now, f behaves like a covering map on a neighbourhood of $\{\nu_1^0, \nu_2^0, \dots, \nu_m^0\}$, that is, there exists an open neighbourhood U of ν_0 such that $U \cap f(\sigma_\beta(T)) = \emptyset$, $f^{-1}(U)$ is a disjoint union of open sets $V_1, V_2, \dots, V_m$ with $\nu_j^0 \in V_j$, and $f_j \equiv f|_{V_j} : V_j \rightarrow U$ is biholomorphic. Clearly, $\sigma_\beta(T) \cap (V_1 \cup V_2 \cup \dots \cup V_m) = \emptyset$.

Consider

$$g(\nu, z) \equiv g_\nu(z) = \frac{\nu - f(z)}{(f_1^{-1}(\nu) - z)(f_2^{-1}(\nu) - z) \dots (f_m^{-1}(\nu) - z)}, \quad z \in K, \nu \in U.$$

Notice that $\nu_1 \equiv f^{-1}(\nu)$, $\nu_2 \equiv f^{-2}(\nu)$, $\dots$, $\nu_m \equiv f^{-m}(\nu)$ are removable singularities of $g_\nu(z)$ which, as a function of two variables ν and z , is holomorphic.

From the definition of $g_\nu(z)$, we have

$$\nu - f(z) = (\nu_1 - z)(\nu_2 - z) \dots (\nu_m - z)g_\nu(z),$$

from which it follows that

$$\nu - f(T) = (\nu_1 - T)(\nu_2 - T) \dots (\nu_m - T)g_\nu(T), \quad (3.8)$$

where $g_\nu(T)$, $\nu_1, \nu_2, \dots, \nu_m$ are holomorphic functions of ν .

Now, we show that $f(T)$ has property (β) on U . To this end, let $\{h_n\}$ be a sequence of X -valued analytic functions on U such that

$$(\nu - f(T))h_n(\nu) \longrightarrow 0, \quad \text{on compact subsets of } U.$$

From (3.8), we have

$$(f_1^{-1}(\nu) - T)(f_2^{-1}(\nu) - T) \dots (f_m^{-1}(\nu) - T)g_\nu(T)h_n(\nu) \longrightarrow 0, \quad (3.9)$$

or, equivalently,

$$(\nu_1 - T)(f_2^{-1}(f(\nu_1)) - T) \dots (f_m^{-1}(f(\nu_1)) - T)g(f(\nu_1), T)h_n(f(\nu_1)) \longrightarrow 0.$$

Set $G_n(\nu_1) = (f_2^{-1}(f(\nu_1)) - T) \dots (f_m^{-1}(f(\nu_1)) - T)g(f(\nu_1), T)h_n(f(\nu_1))$. The function $G_n(\nu_1)$ is a vector-valued analytic function of ν_1 satisfying

$$(\nu_1 - T)G_n(\nu_1) \longrightarrow 0, \quad \text{uniformly on compact subsets of } V_1.$$

Because T has property (β) on V_1 (recall $\sigma_\beta(T) \cap V_1 = \emptyset$), we have $G_n(\nu_1) \longrightarrow 0$ on compact subsets of V_1 . Since f_1 is biholomorphic,

$$G_n(f_1^{-1}(\nu)) = (f_2^{-1}(\nu) - T) \dots (f_m^{-1}(\nu) - T)g_\nu(T)h_n(\nu) \longrightarrow 0,$$

uniformly on compact subsets of U . Applying the above argument successively, we obtain

$$g_\nu(T)h_n(\nu) \longrightarrow 0, \quad \text{uniformly on compact subsets of } U.$$

In view of the fact that g_ν has no zeros in $K \supset \sigma(T)$, $g_\nu(T)$ is invertible. Hence $h_n(\nu) \longrightarrow 0$ uniformly on compact subsets of U . Therefore, $f(T)$ has property (β) on U , and hence $\nu_0 \notin \sigma_\beta(f(T))$. We have shown that

$$\sigma_\beta(f(T)) \subset f(\sigma_\beta(T)) \cup f(F).$$

Since $f(F)$ is a finite set and the Bishop spectrum of an operator has no isolated points, we conclude that

$$\sigma_\beta(f(T)) \subset f(\sigma_\beta(T)). \quad \blacksquare$$

We close this section with a nontrivial example. Recall that a bounded linear operator T on a Hilbert space H is said to be hyponormal if $TT^* \leq T^*T$, or equivalently, $\|T^*x\| \leq \|Tx\|$ for all $x \in H$.

Example 3.3.7 Let $\{w_n\}$ be a bounded positive, eventually increasing sequence of real numbers, that is, there exists an integer $N > 0$ such that $w_N \leq w_{N+1} \leq \dots$. Assume furthermore that $0 < w_k < 1$ for $0 \leq k \leq N-1$ and $w_N \geq 1$. Consider the backward weighted shift operator T defined by

$$Te_n = w_{n-1}e_{n-1}, \quad \text{for } n \geq 1$$

$$Te_0 = 0$$

where $\{e_n\}$ is the standard orthonormal basis for $H = \ell^2(\mathbb{N})$. We will show that $\sigma_\beta(T) = \emptyset$, i.e., T has property (β) .

A straightforward verification shows that for every $n \geq 1$, we have

$$T^{*n}T^n(x_0, x_1, \dots) = (w_0^2w_1^2 \dots w_{n-1}^2x_0, w_1^2w_2^2 \dots w_n^2x_1, \dots) \quad (3.10)$$

and

$$T^nT^{*n}(x_0, x_1, \dots) = (0, 0, \dots, 0, w_0^2w_1^2 \dots w_{n-1}^2x_n, w_1^2w_2^2 \dots w_n^2x_{n+1}, \dots). \quad (3.11)$$

Now choose $m = 2N$. We are going to prove that $T^{*m}T^m \geq T^mT^{*m}$, that is, $(T^{*m}T^m x, x) \geq (T^mT^{*m} x, x)$ for all $x \in H$.

From (3.10) and (3.11), we have for $x \in H$

$$(T^{*m}T^m x, x) = \sum_{k=0}^{\infty} w_k^2 w_{k+1}^2 \dots w_{m+k-1}^2 |x_k|^2$$

and

$$(T^m T^{*m} x, x) = \sum_{k=0}^{\infty} w_k^2 w_{k+1}^2 \cdots w_{m+k-1}^2 |x_{m+k}|^2.$$

So it is enough to show

$$\frac{w_{m+k}^2 w_{m+k+1}^2 \cdots w_{2m+k-1}^2}{w_k^2 w_{k+1}^2 \cdots w_{m+k-1}^2} \geq 1, \quad k \geq 0. \quad (3.12)$$

Since $w_N \leq w_{N+1} \leq \cdots$, the inequality (3.12) holds for every $k \geq N$. For $0 \leq k \leq N-1$, it suffices to prove the inequality for $k=0$, or equivalently,

$$\frac{w_m^2 w_{m+1}^2 \cdots w_{2m-1}^2}{w_0^2 w_1^2 \cdots w_{m-1}^2} \geq 1 \quad (3.13)$$

(the proof for $0 < k \leq N-1$ is similar to the case $k=0$). Because $0 < w_0^2 w_1^2 \cdots w_{N-1}^2 < 1$, we have

$$w_0^2 w_1^2 \cdots w_{m-1}^2 < w_N^2 w_{N+1}^2 \cdots w_{m-1}^2.$$

On the other hand

$$w_N^2 w_{N+1}^2 \cdots w_{m-1}^2 \leq w_m^2 w_{m+1}^2 \cdots w_{m+N-1}^2 \leq w_{m+N}^2 w_{m+N+1}^2 \cdots w_{2m-1}^2.$$

It follows that

$$w_N^2 w_{N+1}^2 \cdots w_{m-1}^2 \leq w_m^2 w_{m+1}^2 \cdots w_{m+N-1}^2 w_{m+N}^2 w_{m+N+1}^2 \cdots w_{2m-1}^2.$$

Therefore

$$w_0^2 w_1^2 \cdots w_{m-1}^2 \leq w_m^2 w_{m+1}^2 \cdots w_{2m-1}^2,$$

and hence (3.13) follows. Next, since $T^{*m} T^m - T^m T^{*m} \geq 0$ if and only if (3.12) is satisfied, we conclude that T^m is hyponormal operator. Hence, by [61], T^m has property (β) . Consequently T has property (β) by Theorem 3.3.6, i.e., $\sigma_\beta(T) = \emptyset$.

3.4 S -Decomposable Operators

Our aim in this section is to determine the *spectral residuum* of a given bounded linear operator T on a Banach space. Let S be a closed set in $\mathbb{C}$. The following lemma is analogous to Theorem 2.1.3. The same argument can be used here; the only difference is that the disk $D(z_0, r)$ must be in $\mathbb{C} \setminus S$. So we omit its proof here. For a different proof see [75, Lemma 4.16].

Lemma 3.4.1 *Let $T \in B(X)$ be S -decomposable, and let G be an open set of $\mathbb{C} \setminus S$. If $\{f_n\}$ is a sequence of holomorphic functions on G , such that $(z - T)f_n(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact subsets of G , then $f_n(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on compact subsets of G .*

Corollary 3.4.2 *If T is S -decomposable, then $\sigma_\beta(T) \subset S$.*

If T is S -decomposable, by a result of Vasilescu [76], T^* is also S -decomposable. Thus by the above corollary $\sigma_\beta(T^*) \subset S$. We record this fact combined with the above corollary for reference.

Corollary 3.4.3 *If T is S -decomposable bounded linear operator on a Banach space, then*

$$\sigma_\beta(T) \cup \sigma_{\beta^*}(T) \subset S.$$

By Corollary 3.4.3, an equivalent way to say that an operator T on X is S -decomposable, is that T has properties (β) and (β^*) on $\mathbb{C} \setminus S$.

In the following we extend the results of [34] to the case $S \neq \emptyset$, specifically, we show that a bounded linear operator T has property (β) on $\mathbb{C} \setminus S$ if and only if its adjoint T^* has property (β^*) on $\mathbb{C} \setminus S$. The proofs are exactly the same as the corresponding proofs in [34] for the case $S = \emptyset$ but we include them for completeness.

Let T be a bounded linear operator on X with property (β) on $\mathbb{C} \setminus S$. It is well known that there is a Banach space Y with X as its subspace and an S -decomposable operator R on Y with $X \in \text{Lat}(R)$, such that $T = R|_X$, see for example [30]. Now $X^\perp \in \text{Lat}(R^*)$ and since X^* is canonically isomorphic to $Y^*/X^\perp$, T^* can be regarded as an operator on the quotient space $Y^*/X^\perp$ induced by R^* . Therefore, given an open S -covering $\{G_1, G_2, \dots, G_n; G_0\}$ of $\sigma(R^*)$, we have

$$Y^* = Y_{R^*}^*(\overline{G}_0) + \sum_{i=1}^n Y_{R^*}^*(\overline{G}_i).$$

Taking quotients on both sides, we obtain

$$X^* = X_{T^*}^*(\overline{G}_0) + \sum_{i=1}^n X_{T^*}^*(\overline{G}_i).$$

Thus, we have shown that

Proposition 3.4.4 *If T has property (β) on $\mathbb{C} \setminus S$, then T^* has property (β^*) on $\mathbb{C} \setminus S$.*

Notice that, without appealing to the extension idea, a direct proof of the above proposition is available using Bishop's techniques, see [35] for the case $S = \emptyset$. The converse of Proposition 3.4.4 is also true.

Proposition 3.4.5 *If the adjoint of an operator $T \in B(X)$ has property (β^*) on $\mathbb{C} \setminus S$, then T has property (β) on $\mathbb{C} \setminus S$.*

The proof follows from this variant:

Theorem 3.4.6 *If an operator T has property (β^*) on $\mathbb{C} \setminus S$, then T^* has property (β) on $\mathbb{C} \setminus S$.*

Indeed, suppose that T^* has property (β^*) on $\mathbb{C} \setminus S$. Then by the above theorem, T^{**} has property (β) on $\mathbb{C} \setminus S$. Hence, so does T which can be regarded as a restriction

of T^{**} . This shows the validity of Proposition 3.4.5, provided that the validity of Theorem 3.4.6 is accepted.

Proof. To show that T^* has (β) on $\mathbb{C} \setminus S$, let G be an open set in $\mathbb{C} \setminus S$ and $g_n \in H(G, X^*)$ such that

$$(z - T^*)g_n(z) \longrightarrow 0$$

uniformly on compact subsets of G . It is enough to show that each $z_0 \in G$ has a neighbourhood on which $g_n \longrightarrow 0$ uniformly (see Lemma 2.1.1). Pick four distinct closed disks D_j ($j=1,2,3,4$) in G all with z_0 as center and $D_j \subset D_{j+1}$. Since $\{D_3^o, D_2^c\}$ is an open S -cover of $\mathbb{C}$, by our assumption

$$X = X_T(D_3) + X_T(\mathbb{C} \setminus D_2^o).$$

For convenience, write $B_1 = X_T^b(D_4)$, $B_2 = X_T^b(\mathbb{C} \setminus D_1^o)$ (these are Banach spaces with $\|\cdot\|_b$ -norm by Lemma 1.3.9). Then $B_1 \supset X_T(D_3)$ and $B_2 \supset X_T(\mathbb{C} \setminus D_2^o)$. Hence

$$X = B_1 + B_2.$$

Consider the direct sum $B = B_1 \oplus B_2$ and the map Q from B onto X given by

$$Q(x_1 \oplus x_2) = x_1 + x_2.$$

The conjugate Q^* of Q can be identified with a bijection from X^* onto a closed subspace of $B_1^* \oplus B_2^*$.

Let R_1 and R_2 be the restriction of T to B_1 and B_2 respectively and $R = R_1 \oplus R_2$ an operator on B . Then R is bounded with

$$\|R\|_b = \max\{\|R_1\|_b, \|R_2\|_b\} \leq \|T\|.$$

Clearly, $TQ = QR$. Hence

$$Q^*T^* = R^*Q^*.$$

Put $Q^*g_n(z) = g_n^1 \oplus g_n^2$, $z \in G$. Then

$$(z - R^*)Q^*g_n(z) = (z - R_1^*)g_n^1 \oplus (z - R_2^*)g_n^2 = Q^*(z - T^*)g_n(z)$$

converges uniformly to zero on D_4 . For $z \in D_1^\circ$, we have by (4) of Lemma 1.3.9 $(z - R_2)B_2 = B_2$ and hence $(z - R_2)^*$ is bounded below; there exists $m_z > 0$ such that

$$\|(z - R_2^*)x_2^*\|_b \geq m_z \|x_2^*\|_b \text{ for all } x_2^* \in B_2^*.$$

Notice that, for $\mu \in D_1^\circ$ with $|\mu - z| \leq \epsilon < m_z$, we have

$$\|(z - R_2^*)x_2^*\|_b \geq (m_z - \epsilon) \|x_2^*\|_b.$$

Therefore, from

$$\lim_{n \rightarrow \infty} \sup_{z \in D_4} \|(z - R_2^*)x_2^*\| = 0,$$

we conclude that $g_n^2 \rightarrow 0$ uniformly on all compact subsets of D_1° . Similarly, we can show that $g_n^1 \rightarrow 0$ uniformly on all compact subsets of D_4° . In particular, $g_n^1 \rightarrow 0$ uniformly on the boundary of a closed disk D in G with $D_4 \subset D^\circ$. By the maximum modulus principle, $g_n^1 \rightarrow 0$ uniformly on D . Thus, we have shown that

$$Q^*g_n(z) = g_n^1 \oplus g_n^2 \rightarrow 0$$

uniformly on every subset of D_1° . Since Q^* is bounded below (because $QB = X$), $g_n \rightarrow 0$ uniformly on all compact subsets of D_1° . The proof is complete. ■

Let $S = \sigma_\beta(T) \cup \sigma_{\beta^*}(T)$. Then S is a closed subset of the spectrum $\sigma(T)$.

Proposition 3.4.7 *Given $T \in B(X)$ and $S = \sigma_\beta(T) \cup \sigma_{\beta^*}(T)$. Then T is S -decomposable.*

Proof. Let $\{G_1, G_2, \dots, G_n; G_0\}$ be an open S -covering of $\sigma(T)$. Then, there exists an open S -covering $\{D_1, D_2, \dots, D_n; D_0\}$ of $\sigma(T)$ such that $\overline{D_i} \subset G_i$ for $i = 0, 1, 2, \dots, n$. By our assumption T has property (β) on $\mathbb{C} \setminus S$ and hence T^* has property (β^*) on $\mathbb{C} \setminus S$ by Proposition 3.4.4. Thus

$$X^* = X_{T^*}^*(\overline{D_0}) + \sum_{i=1}^n X_{T^*}^*(\overline{D_i}).$$

By Lemma 3.1.5, $Z_i = X_{T^*}^*(\overline{D}_i)$ are spectral maximal spaces for T^* and $\sigma(T^*|Z_i) \subset \overline{D}_i \subset G_i$ (since T^* has property (β) on $\mathbb{C} \setminus S$). Therefore T^* is S -decomposable. Hence T is S -decomposable by [79, Theorem 2.7]. ■

In [57], B. Sz.Nagy introduced the notion of spectral residuum for a bounded linear operators on a Banach spaces and proved the existence of the spectral residuum for every bounded linear operator [57, Theorem 6].

Definition 3.4.8 [57, Definition 5] Let T be a bounded linear operator on a Banach space X , and let $\mathcal{Q} = \mathcal{Q}(T)$ denote the family of all closed sets S , such that $S_T \subset S \subset \sigma(T)$ and T is S -decomposable. If there exists $S^* \in \mathcal{Q}$ such that $S^* \subset S$ for every S in $\mathcal{Q}$, then S^* (clearly unique) is called the *spectral residuum* of T .

Now we are ready to prove the main theorem of this section.

Theorem 3.4.9 Let T be a bounded linear operator on a Banach space X . Then its spectral residuum is given by $\sigma_\beta(T) \cup \sigma_{\beta^*}(T)$.

Proof. Let S^* be the spectral residuum of T . Since T is S^* -decomposable, we have $\sigma_\beta(T) \cup \sigma_{\beta^*}(T) \subset S^*$ by Corollary 3.4.3. To show the other inclusion, let $S = \sigma_\beta(T) \cup \sigma_{\beta^*}(T)$. Then $S_T \subset \sigma_\beta(T) \subset S$. Moreover, T is S -decomposable by Proposition 3.4.7. On the other hand, since S^* is minimal, we have $S^* \subset S$. Therefore $S^* = \sigma_\beta(T) \cup \sigma_{\beta^*}(T)$. ■

Like decomposability, S -decomposability is preserved under asymptotic similarity.

Theorem 3.4.10 Let $T \in B(X)$ and $R \in B(Y)$ be asymptotically similar. Then

- (a) If T has property (β^*) on $\mathbb{C} \setminus S$, then R has property (β^*) on $\mathbb{C} \setminus S$.
- (b) If T has property (β) on $\mathbb{C} \setminus S$, then R has property (β) on $\mathbb{C} \setminus S$.
- (c) If T is S -decomposable, then R is S -decomposable.

Proof. Assume that T and R are asymptotically similar and choose a corresponding bijection $A \in B(X, Y)$ for the intertwining of (R, T) and (T, R) , that is, $\|C^n(R, T)A\|^{1/n} \rightarrow 0$ and $\|C^n(T, R)A^{-1}\|^{1/n} \rightarrow 0$. To prove (a), let $\{G_1, \dots, G_n; G_0\}$ be an open S -covering of $\sigma(R)$. By Proposition 2.3.9, $\sigma(R) = \sigma(T)$, and hence

$$X = X_T(\overline{G_0}) + \sum_{i=1}^n X_T(\overline{G_i}).$$

By assumption, $AX_T(F) = Y_R(F)$ for every closed set F in $\mathbb{C}$; see Proposition 2.3.2. Since A is onto, we have

$$\begin{aligned} Y = AX &= AX_T(\overline{G_0}) + \sum_{i=1}^n AX_T(\overline{G_i}) \\ &= Y_R(\overline{G_0}) + \sum_{i=1}^n Y_R(\overline{G_i}). \end{aligned}$$

Therefore, R has property (β^*) on $\mathbb{C} \setminus S$. Statement (b) follows from (a), Propositions 3.4.4-5, and the fact that T^* and R^* are also asymptotically similar. Part (c) is a consequence of (a), (b), and the fact T is S -decomposable if and only if T satisfies both (β) and (β^*) on $\mathbb{C} \setminus S$. ■

The preceding theorem (c) generalizes a classical result on quasi-nilpotent equivalence, see [10, Proposition 2.10].

Corollary 3.4.11 *The spectral residuum and the Bishop's spectrum of a given bounded linear operator on a Banach space are preserved under asymptotic similarity.*

3.5 Applications

In this section we prove some results in which Bishop's spectrum plays an important role in spectral decomposition.

We first recall that $T \in B(X)$ has the *weak 2-spectral decomposition property* (weak 2-SDP) if, for every open covering $\{G_1, G_2\}$ of the complex plane $\mathbb{C}$, there are

T -invariant linear subspaces Y_1 and Y_2 such that

$$X = \overline{Y_1 + Y_2} \quad (3.14)$$

$$\sigma(T|Y_i) \subset G_i \quad i = 1, 2.$$

Lemma 3.3.1 *Let $T \in B(X)$ and let Y_1, Y_2 as in (3.14). If $\sigma_\beta(T^{Y_1}) \subset \sigma(T|Y_2)$, then $\sigma(T^{Y_1}) \subset \sigma(T|Y_2)$.*

Proof. Let $z \in \rho(T|Y_2)$. There exists an open set V such that $z \in V \subset \rho(T|Y_2)$. For $x \in X$, $x = \lim_{n \rightarrow \infty} (y_n^1 + y_n^2)$ for some $y_n^i \in Y_i$, ($i=1,2$). Set

$$f_n(z) = (z - T|Y_2)^{-1} y_n^2, \quad \text{for } z \in V.$$

Then

$$y_n^2 = (z - T)f_n(z) \quad z \in V, \quad n = 1, 2, \dots$$

Let $\pi : X \longrightarrow X/Y_1$ be the canonical map. Then

$$\pi x = \lim_{n \rightarrow \infty} (z - T^{Y_1})\pi f_n(z) = \lim_{n \rightarrow \infty} \pi y_n^2, \quad z \in V. \quad (3.15)$$

Since $\pi y_n^2 = (z - T^{Y_1})\pi f_n(z)$, the above limit holds uniformly on V . Because $\rho(T|Y_2) \subset G_\beta(T^{Y_1})$, we deduce that $\{\pi f_n\}$ is a Cauchy sequence and hence converges to an analytic function, say g , uniformly on all compact sets of V . By (3.15), we have

$$\pi x = (z - T^{Y_1})g(z).$$

Hence $z - T^{Y_1}$ is surjective. Since $S_{T^{Y_1}} \subset \sigma_\beta(T^{Y_1}) \subset \sigma(T|Y_2)$, $z - T^{Y_1}$ must be injective, by Proposition 1.3.4(4). Thus $z \in \rho(T^{Y_1})$. ■

Theorem 3.5.2 *If $T \in B(X)$ has the weak 2-SDP with either $\sigma_\beta(T^{Y_1}) \subset \sigma(T|Y_2)$ or $\sigma_\beta(T^{Y_2}) \subset \sigma(T|Y_1)$, where Y_1, Y_2 satisfy (3.14), then T is decomposable.*

Proof. Let G be an open set in $\mathbb{C}$, and let U be open such that $U \subset \overline{U} \subset G$, $G \cup \overline{U}^c = \mathbb{C}$. Then

$$X = \overline{Y_1 + Y_2}$$

$$\sigma(T|Y_1) \subset G \quad \text{and} \quad \sigma(T|Y_2) \subset \overline{U}^c.$$

Suppose that $\sigma_\beta(T^{Y_1}) \subset \sigma(T|Y_2)$. Then, by the above lemma

$$\sigma(T^{Y_1}) \subset \sigma(T|Y_2) \subset \mathbb{C} \setminus \overline{U} \subset \mathbb{C} \setminus \overline{G}.$$

By Theorem 1.4.1 (5), T is decomposable. ■

Remark 3.5.3 The above theorem improves significantly Theorem 3.3 of [48], where it was assumed that either Y_1 or Y_2 is T -strongly analytic (i.e., $\sigma_\beta(T^{Y_1})$ or $\sigma_\beta(T^{Y_2})$ is empty).

The following result appeared first in [10]. Using Theorem 3.3.6, we present a very easy short proof.

Proposition 3.5.4 *Let $T \in B(X)$ be S -decomposable and f as in Theorem 3.3.6. Then $f(T)$ is $f(S)$ -decomposable.*

Proof. Since T is S -decomposable then $\sigma_\beta(T) \cup \sigma_{\beta^*}(T) \subset S$, by Corollary 3.4.3. Therefore $\sigma_\beta(f(T)) \cup \sigma_{\beta^*}(f(T)) \subset f(S)$, by Theorem 3.3.6, i.e., $f(T)$ has properties (β) and (β^*) on $\mathbb{C} \setminus f(S)$. Thus $f(T)$ is $f(S)$ -decomposable. ■

It is well known that, in general, the restriction and quotient operators of a decomposable operator are not necessarily decomposable even on Hilbert spaces, see [27] and Example 3.3.5. However, we still have a similar behaviour (decomposability) outside the intersection of their spectra.

Proposition 3.5.5 *Let $T \in B(X)$ be a decomposable operator and Y an invariant subspace of T . Then T^Y is S -decomposable operator, where $S = \sigma(T|Y) \cap \sigma(T^Y)$.*

Proof. It is enough to show that $\sigma_\beta(T^Y) \cup \sigma_{\beta^\bullet}(T^Y) \subset S$. Since $\sigma_\beta(T^Y) \cup \sigma_{\beta^\bullet}(T^Y) \subset \sigma(T^Y)$ and $\sigma_{\beta^\bullet}(T^Y) = \sigma_\beta((T^Y)^*) = \sigma_\beta(T^*|Y^\perp) \subset \sigma_\beta(T^*) = \emptyset$ (T^* is decomposable), it remains to show that $\sigma_\beta(T^Y) \subset \sigma(T|Y)$. But this is immediate by Proposition 3.3.1(1) and the fact that $\sigma_\beta(T) = \emptyset$. The proof is complete. ■

Proposition 3.5.6 *Let $T \in B(X)$ be a decomposable operator and Y an invariant subspace of T . Then $T|Y$ is S -decomposable operator, where $S = \sigma(T|Y) \cap \sigma(T^Y)$.*

Proof. The proof is similar to the above one. Since T is decomposable we have $\sigma_\beta(T|Y) = \emptyset$, so it is sufficient to show that $\sigma_{\beta^\bullet}(T|Y) \subset \sigma(T^Y)$. By duality, we have

$$\sigma_{\beta^\bullet}(T|Y) = \sigma_\beta((T|Y)^*) = \sigma_\beta(T^*/Y^\perp) \subset \sigma(T^*|Y^\perp) = \sigma((T/Y)^*) = \sigma(T/Y).$$

The middle inclusion follows from part (1) of Proposition 3.3.1 and the decomposability of T^* . ■

Chapter 4

Weak Bishop's Property

In the present chapter, we introduce a new concept related to Bishop's property (β) that we call the *weak Bishop's property* (abbrev. $(w\beta)$). For a given operator $T \in B(X)$ on a complex Banach space X , if for every open subset G of $\mathbb{C}$ and for every *bounded* sequence of analytic functions $f_n : G \rightarrow X$, for which $(z - T)f_n(z)$ converges to zero on every compact subset of G , it follows that $f_n(z) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on all compact subsets of G , then we say that T has property $(w\beta)$.

One of the prime motivations for the study of property $(w\beta)$ is partly due to the fact that property $(w\beta)$ is well behaved under compact perturbation and fits strictly between the two fundamental notions in local spectral theory, namely, the single-valued extension property and Bishop's property (β) .

Most of the material presented in this chapter is based on the method of *Berberian-Quigley's* extension; this technique has already been used successfully for many problems in operator theory, see for instance [16], [35] and [38].

The plan of this chapter is as follows: in Section 1, we describe in detail how to construct *Berberian-Quigley's* extension, together with some auxiliary results needed for our purposes. The basic features of property $(w\beta)$ are contained in Section 2. In

Section 3, we first show that a Banach space operator T has property $(w\beta)$ if and only if its Berberian-Quigley's extension $\tilde{T}$ (abbreviated as BQ-extension) has the single-valued extension property, and then apply this characterization to obtain one of the main results of this chapter, namely, the sum and product of two commuting Banach space operators with property (β) have property $(w\beta)$. Section 4, is devoted to the compact perturbation, presenting our second main result. Also we discuss asymptotic behaviour of operators with property $(w\beta)$ in this Section.

4.1 Definitions and Preliminaries

Definition 4.1.1 We say that $T \in B(X)$ has property $(w\beta)$ if, for every bounded sequence of holomorphic functions $f_n : V \rightarrow X$ ($V \subset \mathbb{C}$ open) such that, as $n \rightarrow \infty$ $\|(z - T)f_n(z)\| \rightarrow 0$ uniformly on compact subsets of V , we have $\|f_n(z)\| \rightarrow 0$ uniformly on compact subsets of V .

It is clear that $(\beta) \implies (w\beta) \implies (SVEP)$. We will show by examples that both $(w\beta) \implies (\beta)$ and $(SVEP) \implies (w\beta)$ are false.

The following vector-valued version of *Vitali's* theorem will be used throughout this chapter, cf. [40, Theorem 3.14.1].

Theorem 4.1.2 *Let $f_n : V \rightarrow X$ be a bounded sequence of holomorphic functions. Assume that in V there is a set F on which $\{f_n\}$ is pointwisely convergent and that F has an accumulation point in V . Then $\{f_n\}$ converges uniformly on all compact subsets of V .*

The following proposition is a consequence of the above theorem.

Proposition 4.1.3 *Every operator with a thin spectrum in the sense that $\sigma(T)$ has no interior point has property $(w\beta)$.*

Proof. Let $f_n : G \longrightarrow X$ be a sequence of bounded holomorphic functions such that $\|(z - T)f_n(z)\| \longrightarrow 0$ as $n \rightarrow \infty$ uniformly on every compact subset of G . We notice first that $\|f_n(z)\| \longrightarrow 0$ uniformly on each compact subset of $G \cap \rho(T)$. For $z \in G \cap \sigma(T)$, there exists a sequence $\{z_k\}$ in $G \cap \rho(T)$ ($\overline{\rho(T)} = \mathbb{C}$) which converges to z . It follows that $\lim_{n \rightarrow \infty} \|f_n(z)\| = \lim_{n, k \rightarrow \infty} \|f_n(z_k)\| = 0$. Hence, for every $z \in G$, $\|f_n(z)\| \longrightarrow 0$. Thus by Vitali's theorem, T has property (w/β) . ■

Example 4.1.4 It is known that there is an operator T without property (β) such that $\sigma(T) = [0, 1]$; see [27]. Hence there are operators having property (w/β) but without property (β) .

As we have mentioned in the introduction, we rely heavily on the so-called *Berberian-Quigley's* extension, an adaptation of ideas of Berberian [12] and Quigley's [65, pp. 25-26]. We describe its construction here along with some auxiliary results which will be needed in the proofs.

Definition 4.1.5 Given a Banach space X , we define certain larger Banach spaces $\ell_\infty(X)$, $c_0(X)$ and $\tilde{X}$ as follows.

(1) Let $\ell_\infty(X)$ be the collection of all bounded sequences $(x_n)_{n=1}^\infty$ with the supremum norm

$$\|(x_n)_{n=1}^\infty\| = \sup_{n \in \mathbb{I}} \|x_n\|.$$

Let $J_X : X \longrightarrow \ell_\infty(X)$ be the isometric embedding of X in $\ell_\infty(X)$, which sends $x \in X$ to the constant sequence $(x_n)_{n=1}^\infty$ such that each $x_n = x$.

(2) Let $c_0(X)$ be the closed subspace of $\ell_\infty(X)$, consisting of those sequences in X which converge to zero in norm.

(3) Let $\tilde{X}$ denote the quotient space $\ell_\infty(X)/c_0(X)$, and let $J : X \longrightarrow \tilde{X}$ be the composition of the inclusion map J_X with the quotient map

$$Q : \ell_\infty(X) \longrightarrow \ell_\infty(X)/c_0(X).$$

Remark 4.1.6 The map $J : X \longrightarrow \hat{X}$ is an isometric embedding.

Note that if we set $\|(x_n)\| = \limsup_{n \rightarrow \infty} \|x_n\|$ for every sequence (x_n) , this defines a semi-norm on $\ell_\infty(X)$ which is zero exactly on the elements of $c_0(X)$; indeed, as pointed out in [16], it agrees with the usual norm on the quotient space $\ell_\infty(X)/c_0(X)$. In the case where X is a Hilbert space, Berberian [12] noted that the extension $\hat{X}$ is also a Hilbert space.

Definition 4.1.7 Given a Banach space X , let $\hat{X}$ and $J : X \longrightarrow \hat{X}$ as in Definition 4.1.5 above. Then, for each operator $T \in B(X)$, we define an extension $\hat{T} \in B(\hat{X})$ by the following:

$$\hat{T}Q\{x_n\}_n = Q\{Tx_n\}_n.$$

The map $T \longrightarrow \hat{T}$ defines an isometric embedding of $B(X)$ in $B(\hat{X})$, called the *Berberian-Quigley's extension*.

The correspondence $T \longrightarrow \hat{T}$ has the following fundamental property (see [16] and [17])

$$\hat{T} \text{ one to one} \iff T \text{ bounded below} \iff \hat{T} \text{ bounded below,}$$

which can be rephrased as

Proposition 4.1.8 [16, Proposition 1] *The correspondence $T \longrightarrow \hat{T}$ is an isometric homomorphism such that $\sigma_a(T) = \sigma_a(\hat{T}) = \sigma_p(\hat{T})$.*

This is the essence of the space $\hat{X}$: it is a space in which the "approximate eigenvectors" of an operator are represented as true eigenvectors. Moreover, it is noted in [14] that

Lemma 4.1.9 *An operator $T \in B(X)$ is invertible if and only if its BQ-extension $\hat{T}$ is invertible.*

We shall need the following lemma.

Lemma 4.1.10 *Let X be a Banach space and Y be a subspace of X . The sequence*

$$0 \rightarrow c_0(Y) \rightarrow c_0(X) \rightarrow c_0(X/Y) \rightarrow 0$$

is exact.

Proof. Let the maps $f : c_0(Y) \rightarrow c_0(X)$, and $g : c_0(X) \rightarrow c_0(X/Y)$, defined by $f((y_n)) = (y_n)$ for $(y_n) \in c_0(Y)$, and $g((x_n)) = (\pi x_n)$, for $(x_n) \in c_0(X)$ respectively. Note that f is one to one since it is an isometry. To show that g is onto, let $(\hat{x}_n) \in c_0(X/Y)$ (here $\hat{u} = \pi(u)$ where $\pi : X \rightarrow X/Y$ is the quotient map). Then $\|\hat{x}_n\| \rightarrow 0$. Put $\varepsilon_n = \sup\{\|\hat{x}_n\|, \|\hat{x}_{n+1}\|, \dots\}$, so that $\varepsilon_n \rightarrow 0$, and $\|\hat{x}_n\| \leq \varepsilon_n$. For each n , there exists $y_n \in Y$ such that $\|x_n - y_n\| < \varepsilon_n + 1/n$. Take $x'_n = x_n - y_n$. Then $\|x'_n\| \rightarrow 0$, and hence $(x'_n) \in c_0(X)$, and $\hat{x}'_n = \hat{x}_n$ for each n , which implies that $g((x'_n)) = (\hat{x}_n)$.

Clearly both maps f and g are continuous. It remains to show that $\text{Ker}(g) = \text{Im}(f)$. Let $(x_n) \in \text{Ker}(g) \subset c_0(X)$. Since $g((x_n)) = 0$, we have $x_n \in Y$ for each n , thus $(x_n) \in c_0(Y)$ and consequently $(x_n) \in \text{Im}(f)$. For the reverse inclusion, let $(x_n) \in \text{Im}(f)$. Then, there exists $(y_n) \in c_0(Y)$, such that $(x_n) = g((y_n)) = (y_n)$ and hence $x_n = y_n \in Y$, from which it follows that $g((x_n)) = 0$, that is, $(x_n) \in \text{Ker}(g)$. ■

Next we show that the Berberian-Quigley's extension of subspaces or quotient spaces of X can be naturally identified with subspaces or quotient spaces of $\tilde{X}$. What we will do is essentially showing that the functor $X \rightarrow \tilde{X}$ from the category of Banach spaces into itself is exact. In other words, the exact sequence

$$0 \rightarrow Y \xrightarrow{\alpha} X \xrightarrow{\eta} Z \rightarrow 0$$

(here α is an isometry and η is a quotient map in the sense that $\|\eta(x)\| = \text{dist}(x, \text{Ker}\eta)$ for all $x \in X$.) gives rise to the exact sequence

$$0 \rightarrow \tilde{Y} \xrightarrow{\tilde{\alpha}} \tilde{X} \xrightarrow{\tilde{\eta}} \tilde{Z} \rightarrow 0.$$

For convenience, we assume that Y is a subspace of X , α is the embedding, $Z = X/Y$, and η is the quotient map. Recall that an alternative description of the norm of an element, say $(x_n) + c_0(X)$ in $\hat{X}$, is $\|(x_n) + c_0(X)\| = \lim_{n \rightarrow \infty} \sup_{k \geq n} \|x_k\|$. The map $\hat{\alpha} : \hat{Y} = \ell_\infty(Y)/c_0(Y) \longrightarrow \hat{X} = \ell_\infty(X)/c_0(X)$ is an isometry because, for $(y_n) \in \ell_\infty(Y)$,

$$\begin{aligned} \|\hat{\alpha}((y_n) + c_0(Y))\| &= \|(\alpha y_n) + c_0(X)\| = \|(y_n) + c_0(X)\| \\ &= \lim_{n \rightarrow \infty} \sup \|y_n\| = \|(y_n) + c_0(Y)\|. \end{aligned}$$

Notice that the range of $\hat{\alpha}$ is

$$\{(y_n) + c_0(X) : (y_n) \in \ell_\infty(Y)\} = (\ell_\infty(Y) + c_0(X))/c_0(X)$$

and hence we may identify $\hat{Y}$ with the subspace $(\ell_\infty(Y) + c_0(X))/c_0(X)$ of $\hat{X}$. (Notice also that $\ell_\infty(Y) + c_0(X) = Q^{-1}(\text{image of } \hat{\alpha})$ which implies that $\ell_\infty(Y) + c_0(X)$ is closed). Thus $\hat{X}/\hat{Y}$ may be identified with

$$\frac{\ell_\infty(X)/c_0(X)}{(\ell_\infty(Y) + c_0(X))/c_0(X)} \approx \frac{\ell_\infty(X)}{\ell_\infty(Y) + c_0(X)}.$$

(Here we use the elementary fact that if E, F are subspaces of a Banach space G with $E \subset F \subset G$, then $(G/E)/(F/E)$ is isometrically isomorphic to G/F .)

Now, for $(x_n) \in \ell_\infty(X)$, we have

$$\begin{aligned} \|(x_n) + (\ell_\infty(Y) + c_0(X))\| &= \inf\{\|x_n + y_n + z_n\| : (y_n) \in \ell_\infty(Y), (z_n) \in c_0(X)\} \\ &= \inf\{\{\|x_n + y_n + z_n\| : (y_n) \in \ell_\infty(Y)\} : (z_n) \in c_0(X)\} \\ &= \inf\{\|x_n + z_n + Y\| : (z_n) \in c_0(X)\} \\ &= \limsup_{n \rightarrow \infty} \|x_n + Y\| \end{aligned}$$

which is the norm of $(x_n + Y)_n$ in $(X/Y)^\sim$.

Now we assume further that T is an operator on X and $Y \in \text{Lat}(T)$. Then we obtain operators T_Y and T^Y on the subspace Y and the quotient space X/Y respectively by means of restriction and "co-induction" (a term used in the literature

with a clear meaning.) By the above identification of $\hat{Y}$ as a subspace of $\hat{X}$ and $(X/Y)^\sim = \hat{X}/\hat{Y}$, we see that the BQ-extension of T_Y and T^Y can be identified with $(\tilde{T})_{\hat{Y}}$ and $(\tilde{T})^{\hat{Y}}$, respectively.

We summarize the above discussion in the following

Lemma 4.1.11 *Given $T \in B(X)$ and $Y \in \text{Lat}(T)$, $\hat{Y}$ can be identified as a subspace of $\hat{X}$ and $\hat{X}/\hat{Y}$ can be identified with $(X/Y)^\sim$. Under these identifications, $(T_Y)^\sim = (\tilde{T})_{\hat{Y}}$ and $(T^Y)^\sim = (\tilde{T})^{\hat{Y}}$.*

4.2 Basic Properties

In this section we present the basic properties of property $(w\beta)$. We start with the following simple but useful lemma which is of interest in its own right and serves as a test for operators that fail to have property $(w\beta)$.

Lemma 4.2.1 *Let $T \in B(X)$, with $\|T\| = 1$. Assume that, for every $\epsilon > 0$ and every positive integer N , there exists $n \geq N$ such that $\|T^n x\| = \|x\|$ and $\|T^{n+1} x\| < \epsilon \|x\|$ for some nonzero $x \in X$. Then T does not have property $(w\beta)$.*

Proof. Fix some $0 < r < 1$. Define a sequence $\{p_N\}$ of X -valued polynomials as follows. For each positive integer N , let $x \in X$ and $n \geq N$ (both depending on N) be such that $\|x\| = 1$, $\|T^n x\| = 1$ and $\|T^{n+1} x\| \leq r^{N+1}$. Put

$$p_N(z) = T^n x + zT^{n-1}x + z^2T^{n-2}x + \cdots + z^{n-1}T'x + z^n x.$$

A simple calculation shows

$$(z - T)p_N(z) = -T^{n+1}x + z^{n+1}x$$

and hence, for $|z| \leq r$,

$$\|(z - T)p_N(z)\| \leq 2r^{N+1}$$

$$\|p_N(z)\| \leq 1 + r + r^2 + \cdots + \cdots = \frac{1}{1-r}.$$

But $\|p_N(0)\| = \|T^N x\| = 1$. This shows that p_N is bounded on the disk D of radius r centred at 0, $(z - T)p_N(z)$ converges to zero uniformly on D , but $p_N(0)$ does not converge to zero. Hence T does not have property $(w\beta)$. ■

Remark 4.2.2 If the condition $\|T\| = 1$ is dropped in the previous lemma, then the same argument shows that T does not have property (β) .

Next, we give an example of operators having $(SVEP)$ but without property $(w\beta)$.

Example 4.2.3 Let T_n , ($n \geq 2$) be the $n \times n$ nilpotent Jordan blocks

$$T_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

considered as an operator defined on the Hilbert space $\mathbb{C}^n$. Let T be the direct sum $\bigoplus_{n=2}^{\infty} T_n$ defined on the direct sum $H = \bigoplus_{n=2}^{\infty} \mathbb{C}^n$ of Hilbert spaces. We have

$$\|T_n\| = \|T_n^{n-1}\| = 1 \quad (4.1)$$

and

$$(T_n)^n = 0 \quad (4.2)$$

hence

$$\|T\| = \sup\{\|T_n\| : n \geq 2\} = 1. \quad (4.3)$$

Let V be an open set in $\mathbb{C}$ and let $f = \bigoplus_{n=2}^{\infty} f_n : V \longrightarrow H$ be a holomorphic function (hence every component $f_n : V \longrightarrow \mathbb{C}^n$ is holomorphic) such that $(z - T)f(z) \equiv 0$, i.e.,

$$(z - T_n)f(z) \equiv 0, \quad \text{for every } n \geq 2. \quad (4.4)$$

From (4.2) and (4.4) it follows that $0 \equiv (T_n)^n f_n(z) = z^n f_n(z)$, from which it follows that $f_n(z) \equiv 0$ ($n = 2, 3, \dots$), hence $f(z) \equiv 0$. Thus T has the *(SVEP)*. But, according to the above lemma, T does not have property $(w\beta)$.

It is obvious that property $(w\beta)$ is preserved under similarity; however it is not preserved under quasisimilarity. In the following example, we exhibit quasisimilar operators T and S such that, S has property $(w\beta)$ but T does not. (However, at the end of Section 3, we will show that property $(w\beta)$ is preserved under asymptotic similarity.)

Example 4.2.4 Let T_n , ($n \geq 2$) and T as in Example 4.2.3. Let S_n ($n \geq 2$) be the $n \times n$ be given by the following matrix

$$S_n = \begin{pmatrix} 0 & 1/n & 0 & \cdots & 0 \\ 0 & 0 & 1/n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1/n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

which is considered as an operator defined on the Hilbert space $\mathbb{C}^n$. Then T_n and S_n are similar, in fact T_n is the Jordan canonical form for S_n .

If $S = \bigoplus_{n=2}^{\infty} S_n$, then it is easy to see that T is quasisimilar to S ; see [42, Theorem 2.5]. Now a routine check leads to

$$\|S_n^k\| = \|(T_n/n)^k\| = (1/n)^k \|T_n^k\|.$$

Therefore, by (2.1) and (2.2), we have

$$\|S_n^k\| = \begin{cases} 0 & \text{if } k \geq n \\ (1/n)^k & \text{if } k < n \end{cases}$$

Consequently

$$\|S_n^k\| = \sup_{n > k} (1/n)^k = \left(\frac{1}{k+1}\right)^k$$

hence

$$r(S) = \lim_{k \rightarrow \infty} \|S_n^k\|^{1/k} = 0$$

we conclude that S has property $(w\beta)$.

Given $T \in B(X)$, a subspace $Y \in Lat(T)$ gives two related linear operators: the restriction $T|Y$ acting on Y and the coinduced operator T^Y acting on the quotient space X/Y . Obviously, property $(w\beta)$ is inherited by the restriction to invariant subspaces, but not by T^Y .

Example 4.2.5 Here we give an example of an operator T and $Y \in Lat(T)$ such that T has property $(w\beta)$, but T^Y does not, cf. [19, Example 5.29]. Let T be the bilateral shift on $\ell_2(\mathbb{Z})$ and let $Y = \overline{\text{span}}\{e_{-1}, e_{-2}, \dots\}$, where $\{e_n\}_{n \in \mathbb{Z}}$ is the usual orthonormal basis for $\ell_2(\mathbb{Z})$. Since T is unitary, it has property $(w\beta)$; but T^Y is isomorphic to the left shift on $\ell_2(\mathbb{N})$, and hence does not have $(SVEP)$, (take for example $f(z) = \sum_{n=0}^{\infty} z^n e_n$, for $z \in V = \{z : |z| < 1\}$) and consequently does not have property $(w\beta)$ either.

It is well known that if both $T|Y$ and T^Y have property (β) , then so does T . Next we give a result parallel to this fact for property $(w\beta)$.

Proposition 4.2.6 *Given $T \in B(X)$, if for some $Y \in Lat(T)$, $T|Y$ and T^Y both have property $(w\beta)$, then T has the same property.*

Proof. Let $\{f_n\}$ be a sequence of bounded holomorphic functions on an open set G satisfying

$$(z - T)f_n(z) \longrightarrow 0 \quad (4.5)$$

uniformly on all compact subsets of G . Let $z_0 \in G$ and $r > 0$ such that $\overline{D} \subset G$ where $D = \{z : |z - z_0| < r\}$. Now expand f_n ($n = 1, 2, \dots$) as a power series in D as

$$f_n(z) = \sum_{j=0}^{\infty} a_{nj}(z - z_0)^j. \quad (4.6)$$

By (4.5) and the hypothesis on T^Y , we have $\pi f_n \longrightarrow 0$ uniformly in the quotient norm of X/Y on $\overline{D}$, where $\pi : X \longrightarrow X/Y$ is the quotient map. Furthermore, (4.6) implies that

$$\pi f_n(z) = \sum_{j=0}^{\infty} \pi a_{nj}(z - z_0)^j.$$

Let $\varepsilon_n = \sup\{\|\pi f_n(z)\| : |z - z_0| = r\}$. Then $\varepsilon_n \longrightarrow 0$. By Cauchy's inequality, we have

$$\|\pi a_{nj}\| \leq \varepsilon_n / r^j.$$

Next, for each n and each j , choose $b_{nj} \in Y$ such that

$$\|a_{nj} - b_{nj}\| \leq 3\varepsilon_n / r^j, \quad (4.7)$$

from which it follows that

$$\|b_{nj}\| \leq \|a_{nj}\| + 3\varepsilon_n / r^j. \quad (4.8)$$

Define

$$h_n(z) = \sum_{j=0}^{\infty} b_{nj}(z - z_0)^j.$$

By (4.8), $\{h_n\}$ is a Y -valued sequence of holomorphic functions on D . Moreover, (4.5) and (4.7) imply that

$$(z - T)h_n(z) = (z - T)(h_n(z) - f_n(z)) + (z - T)f_n(z) \longrightarrow 0$$

uniformly on all compact sets of D . Since $T|Y$ has property $(w\beta)$, we deduce that $h_n(z) \rightarrow 0$ uniformly on compact sets of D . By (4.7), $f_n(z) \rightarrow 0$ uniformly on compact subsets of D . The proof is complete. ■

Remark 4.2.7 Example 4.2.3 tells us that property $(w\beta)$ is not preserved under countable direct sum.

We have already mentioned earlier that property $(w\beta)$ is inherited on restrictions to invariant subspaces. The following shows that the converse is false.

Example 4.2.8 Let $X = \ell_2(\mathbb{N}) \oplus \ell_2(\mathbb{N})$, let $T_1, T_2 : \ell_2(\mathbb{N}) \rightarrow \ell_2(\mathbb{N})$ be the right, left shift operators, respectively [68]. If we let $T = T_1 \oplus T_2$, then T does not have property $(w\beta)$. However, if we take $Y = \{0\} \oplus \ell_2(\mathbb{N})$, then $Y \in Lat(T)$, and $T|Y$ is isomorphic to T_1 . Since T_1 is a restriction of the unitary bilateral operator shift on $\ell_2(\mathbb{Z})$, we conclude that $T|Y$ has property $(w\beta)$.

From the above discussion, a natural question comes to mind in the present context.

Question 4.2.9 Let $T \in B(X)$. If $X = X_1 + X_2$, $X_i \in Lat(T)$ ($i = 1, 2$) and $T|X_i$ has property $(w\beta)$ ($i = 1, 2$), does T have property $(w\beta)$?

We also remark that a similar question can be asked for $(SVEP)$ and property (β) .

As in the case of the $(SVEP)$, property $(w\beta)$ is not preserved under the adjoint operation neither the sum operation. For the first one, we just take the right shift operator on $\ell_2(\mathbb{N})$ whose adjoint is the left shift. For the sum operation, take any operator T without property $(w\beta)$. Then $T = (T + T^*)/2 + (T - T^*)/2$. Since $(T + T^*)/2$ and $(T - T^*)/2$ are normal operators, they do have property $(w\beta)$. Finally, it is clear that the class of operators with property $(w\beta)$ does not form an ideal in $B(X)$. We close this section with another kind of permanence; its proof is similar to that of Proposition 2.2.8.

Proposition 4.2.10 *Given $T \in B(X)$, then T has property $(w\beta)$ if and only if its left multiplication L_T has the same property.*

4.3 Characterization of Property $(w\beta)$

A well-known open problem in the local spectral theory is whether the sum of two commuting decomposable operators is also decomposable. One may also ask if the following stronger statement holds: the sum of two commuting operators with property (β) also has property (β) . In this section, we give a characterization of property $(w\beta)$ in terms of a weaker property; the $(SVEP)$, and prove also that the sum of two commuting operators with property (β) has property $(w\beta)$. Our method to do so is the technique of BQ-extension. For a Banach space X , let $\tilde{X}$, Q , J as defined in Section 1.

Theorem 4.3.1 *For $T \in B(X)$, let $\tilde{T} \in B(\tilde{X})$ be as in Definition 1.7. Then T has property $(w\beta)$ if and only if $\tilde{T}$ has the $(SVEP)$.*

Proof. "If" part. Assume $f_n : G \rightarrow X$ be a bounded sequence of holomorphic functions, and

$$\|(z - T)f_n(z)\| \rightarrow 0$$

uniformly on compact subsets of G . For $z \in G$, define $\tilde{f}(z) \in \tilde{X}$ by putting $\tilde{f}(z) = Q\{f_n(z)\}_n$. It is easy to check that $\tilde{f} : G \rightarrow \tilde{X}$ is holomorphic and $(z - \tilde{T})\tilde{f}(z) \equiv 0$. The $(SVEP)$ of $\tilde{T}$ implies that $\tilde{f}(z) \equiv 0$. Hence $\|f_n(z)\| \rightarrow 0$ for each $z \in G$. Now apply Vitali's theorem.

"Only if" part. Suppose $\tilde{f} : G \rightarrow \tilde{X}$ is holomorphic on an open connected set G and

$$(z - \tilde{T})\tilde{f}(z) \equiv 0.$$

Let D be a closed disk in G . Then, by Lemma 1.2.2, there exist an open set V with $D \subset V \subset G$ and a holomorphic function $F : V \rightarrow \ell_\infty(X)$ such that $Q(F(z)) = \tilde{f}(z)$. Let $F(z) = \{f_n(z)\}_n$, where $f_n : V \rightarrow X$ are holomorphic. Since

$$\sup\{\|f_n(z)\| : n \in \mathbb{N}, z \in D\} \leq \sup_{z \in D} \|F(z)\| < \infty$$

and $\|(z - T)f_n(z)\| \rightarrow 0$ for each $z \in D$, by Vitali's theorem $\|(z - T)f_n(z)\| \rightarrow 0$ uniformly on every disk having the same center as D with a smaller radius. Since T has property $(w\beta)$, $\|f_n(z)\| \rightarrow 0$ uniformly on a disk D_0 , and hence $\tilde{f}(z) \equiv 0$ on D_0 . Hence $\tilde{f}(z) \equiv 0$ on G by analytic continuation. ■

As an application of the foregoing theorem, we present a simple proof of the stability of property $(w\beta)$ under Reisz-Dunford functional calculus. But before that, we need an easy lemma.

Lemma 4.3.2 *Let $T \in B(X)$, and $\tilde{T}$, J as in Section 1, and let f be a scalar-valued analytic function defined on an open neighbourhood of $\sigma(T)$. Then $Jf(T) = f(\tilde{T})J$.*

Proof. Since $\sigma(\tilde{T}) = \sigma(T)$ by Lemma 4.1.9, $f(\tilde{T})$ makes sense. Moreover, $\sigma(f(\tilde{T})) = f(\sigma(\tilde{T})) = f(\sigma(T)) = \sigma(f(T))$. By construction, J intertwines T and $\tilde{T}$, that is, $JT = \tilde{T}J$. Obviously, the assertion holds for polynomials and hence also holds for rational functions. Therefore, by Runge's theorem, the assertion holds for an analytic function. ■

Corollary 4.3.3 *Let T be bounded linear operator on a complex Banach space X and let f be a scalar-valued analytic function defined in an open neighbourhood of $\sigma(T)$. Assume f is nonconstant on every component of its domain. Then T has property $(w\beta)$ if and only if $f(T)$ does so.*

Proof. Suppose T has property $(w\beta)$. Then, by Theorem 4.3.1, its BQ-extension $\tilde{T}$ has the $(SVEP)$, and so by Theorem 1.1.5 of [15], $f(\tilde{T})$ has the $(SVEP)$. Now, by Theorem 4.3.1 and the preceding lemma, we conclude that $f(T)$ has property $(w\beta)$.

Conversely, if $f(T)$ has property $(w\beta)$, then by Theorem 4.3.1 and Lemma 4.3.2, $f(\hat{T})$ has the $(SVEP)$. Once again, by Theorem 1.1.5 of [15], $\hat{T}$ has the $(SVEP)$. Another application of Theorem 4.3.1 completes the proof. ■

For another application of Theorem 4.3.1, we give an alternative proof of Proposition 4.2.6. Indeed, suppose that T_Y and T^Y both have property $(w\beta)$ for some $Y \in Lat(T)$. In view of Lemma 4.1.11 and Theorem 4.3.1, $(T_Y)^\sim = (\hat{T})_{\hat{Y}}$ and $(T^Y)^\sim = (\hat{T})^{\hat{Y}}$ both have the $(SVEP)$ from which it follows that $\hat{T}$ has the $(SVEP)$. Consequently, T has property $(w\beta)$ by Theorem 4.3.1.

Lemma 4.3.4 *Given $T \in B(X)$. Then T has property (β) if and only if $\hat{T}$ has the same property.*

Proof. The "If" part is obvious because $J : X \longrightarrow \hat{X}$ is an isometric injection and $JT = \hat{T}J$.

The "only if" part can be proved as follows. Suppose that T has property (β) . Let $\tilde{f}_n : G \longrightarrow \hat{X}$ be a sequence of holomorphic functions on an open G and

$$\|(\hat{T} - z)\tilde{f}_n(z)\| \longrightarrow 0 \quad (4.9)$$

uniformly on compact subsets of G . Then, for each $z_0 \in G$, there exists a sequence $\{F_n\}$ of $\ell_\infty(X)$ -valued holomorphic functions on a neighbourhood V of z_0 so that $V \subset G$ and $QF_n = \tilde{f}_n$, ($n = 1, 2, \dots$). Let $F_n(z) = (f_{nk}(z))_k \in \ell_\infty(X)$ and K a compact subset of V .

We will show that $\tilde{f}_n$ converges to zero uniformly on K . To this end, let us take another compact subset F of V such that K is contained in the interior $\overset{\circ}{F}$ of F . Notice that, since T has property (β) , there exists $M > 0$ such that

$$\sup_{z \in K} \|g(z)\| \leq M \sup_{z \in F} \|(z - T)g(z)\|, \quad (4.10)$$

for every analytic function g defined on an open neighbourhood of F , taking values in X . (Otherwise, we can find a sequence of analytic functions g_n from an open

neighbourhood of F into X such that $\sup_{z \in K} \|g_n(z)\| \geq 1$ but $\sup_{z \in F} \|(z - T)g_n(z)\| \rightarrow 0$. Clearly, $(z - T)g_n(z)$ converges uniformly on every compact subset of $\overset{\circ}{F}$. By property (β) of T , g_n should also converge on every compact subset of $\overset{\circ}{F}$, in particular on K . This contradicts the inequality $\sup_{z \in K} \|g_n(z)\| \geq 1$.

From (4.9), we have

$$\|(z - \hat{T})\hat{f}_n(z)\| < \varepsilon_n, \quad (z \in F)$$

where ε_n are some positive numbers converging to zero. This inequality can be rewritten as

$$\lim_{m \rightarrow \infty} \sup_{k \geq m} \|(z - T)f_{nk}(z)\| = \lim_{k \rightarrow \infty} \|(z - T)f_{nk}(z)\| < \varepsilon_n, \quad (z \in F).$$

Thus, for each z , there exists a positive integer $m = m(n, z)$ such that

$$\sup_{k \geq m} \|(z - T)f_{nk}(z)\| < \varepsilon_n.$$

Hence, by (4.10),

$$\|f_{nk}(z)\| < M\varepsilon_n, \quad \text{for } k \geq m(n, z).$$

Therefore, $\|\hat{f}_n(z)\| < M\varepsilon_n$ for all $z \in K$ and all n . The proof is complete. ■

Now we are in a position to prove the following important result.

Theorem 4.3.5 *Let T and S be in $B(X)$ such that $TS = ST$. If T and S have property (β) , then $T + S$ and ST have property $(w\beta)$.*

Proof. Suppose T and S have property (β) . Then by the preceding lemma, $\hat{T}$ and $\hat{S}$ have property (β) . Now by Sun's result [69], $\hat{T} + \hat{S}$ and $\hat{T}\hat{S}$ have the $(SVEP)$. By Theorem 4.3.1, we conclude that $T + S$ and ST have property $(w\beta)$. ■

Naturally one asks: If T and T^* have property $(w\beta)$ does it follow that T is decomposable? An affirmative answer would solve the problem raised at the beginning of this section. Unfortunately the answer is no, see the example below.

Example 4.3.6 Let T be the operator given in Example 4.1.4. Since $\sigma(T) = [0, 1] = \sigma(T^*)$, both T and T^* have property (w, β) by Proposition 4.1.3. However, T is not decomposable.

Proposition 4.3.7 *If an operator $T \in B(X)$ has property (β^*) , then $\hat{T}$ has the same property.*

Proof. Let $\{G_1, \dots, G_N\}$ be an open cover of $\sigma(\hat{T}) (= \sigma(T))$. Choose a second open cover $\{D_1, \dots, D_N\}$ of $\sigma(\hat{T})$ which satisfies $D_k \subset \overline{D_k} \subset G_k$, ($k = 1, 2, \dots, N$). Because T has property (β^*) , we have

$$X = \sum_{k=1}^N X_T(\overline{D_k}) = \sum_{k=1}^N X_T(\overline{G_k}) \quad (4.11)$$

According to the selection of $\{G_1, \dots, G_N\}$ and $\{D_1, \dots, D_N\}$, we have

$$X_T(\overline{D_k}) \subseteq X_T^b(\overline{G_k}) \subseteq X_T(\overline{G_k}), \quad i = 1, 2, \dots, N,$$

where $X_T^b(\overline{G_k})$ is the Banach space described in Section 3 of the first chapter with the norm denoted by $\|\cdot\|_b^{\overline{G_k}}$, which is abbreviated as $\|\cdot\|_b^j$. It follows from (4.11) that

$$X = X_T^b(\overline{G_1}) + \dots + X_T^b(\overline{G_N}). \quad (4.12)$$

Consider the direct sum

$$X^b = X_T^b(\overline{G_1}) \oplus \dots \oplus X_T^b(\overline{G_N}),$$

with the norm defined as $\|x\|_b = \|x_1\|_b^1 + \dots + \|x_N\|_b^N$, for $x = x_1 \oplus \dots \oplus x_N$, $x_k \in X_T^b(\overline{G_k})$, for $k = 1, \dots, N$, where $\|x_j\|_b^j = \|x_j\|_b^{\overline{G_j}}$, ($j = 1, 2, \dots, N$), (see Lemma 1.3.9).

Let $P : X^b \longrightarrow X$ defined as $Px = x_1 + \dots + x_N$ for $x = x_1 \oplus \dots \oplus x_N$, $x_k \in X_T^b(\overline{G_k})$. By the way the norm in X^b is defined, we see that P is a bounded linear

map. Indeed, by (1) of Lemma 1.3.9, there exist $M_j > 0$ such that $\|x_j\| \leq M_j \|x_j\|_b^j$ for $j = 1, 2, \dots, N$. Hence, we have

$$\begin{aligned}\|Px\| &= \|x_1 + \dots + x_N\| \leq \|x_1\| + \dots + \|x_N\| \\ &\leq M_1 \|x_1\|_b^1 + \dots + M_N \|x_N\|_b^N \leq M \|x\|_b,\end{aligned}$$

where $M = \max\{M_1, \dots, M_N\}$. It follows from (4.12) that P is surjective. So, by the open mapping theorem, there exists a constant $M_b > 0$ such that, for each $x \in X$, there are $x_k \in X_T^b(\overline{G}_k)$, $k = 1, 2, \dots, N$, with

$$x = x_1 + \dots + x_N \quad \text{and} \quad \|x_k\|_b^k \leq M_b \|x\|. \quad (4.13)$$

Next, let $\hat{x} \in \hat{X}$, that is, $\hat{x} = (x_m)_{m=1}^\infty + c_0(X)$, where $(x_m)_{m=1}^\infty \in \ell^\infty(X)$. According to (4.13), we can write

$$x_m = x_m^1 + \dots + x_m^N, \quad x_m^k \in X_T^b(\overline{G}_k), \quad (4.14)$$

with $\|x_m^k\|_b^k \leq M_b \|x_m\|$.

According to the norm $\|\cdot\|_b^F$ defined on p. 11, there exist bounded analytic functions $f_m^k : \mathbb{C} \setminus \overline{G}_k \longrightarrow X$ such that

$$(z - T)f_m^k(z) = x_m^k, \quad (k = 1, 2, \dots, N, \quad m = 1, 2, \dots), \quad (4.15)$$

with

$$\sup\{\|f_m^k(z)\| : z \in \overline{G}_k^c\} \leq \|x_m^k\|_b^k + \frac{1}{2^m}.$$

The inequality $\|x_m^k\|_b^k \leq M_b \|x_m\|$ tells us that, for every $k = 1, 2, \dots, N$, we have $(x_m^k)_{m=1}^\infty \in \ell^\infty(X)$. Furthermore, from (4.14), we obtain

$$\hat{x} = \hat{x}_1 + \dots + \hat{x}_N, \quad \text{where} \quad \hat{x}_k = (x_m^k) + c_0(X).$$

It remains only to show that $\hat{x}_k \in \hat{X}_{\hat{T}}(\overline{G}_k)$, $k = 1, 2, \dots, N$. Define the holomorphic functions, $\hat{f}_k : \mathbb{C} \setminus \overline{G}_k \longrightarrow \hat{X}$ by putting $\hat{f}_k(z)$ as the coset $(f_m^k(z))_{m=1}^\infty + c_0(X)$ for

each $z \in \overline{G}_k^c$. It follows from (4.15) that

$$(z - \hat{T})\hat{f}_k(z) = \hat{x}_k, \quad k = 1, 2, \dots, N$$

i.e., $\hat{x}_k \in \hat{X}_{\hat{T}}(\overline{G}_k)$, as claimed. Therefore

$$\hat{X} = \hat{X}_{\hat{T}}(\overline{G}_1) + \dots + \hat{X}_{\hat{T}}(\overline{G}_N).$$

The proof is complete. ■

The following corollary is immediate from Lemma 4.3.4 and Proposition 4.3.7.

Corollary 4.3.8 *If $T \in B(X)$ is decomposable, then $\hat{T}$ is also decomposable.*

The above corollary can also be proved directly using Lemma 4.1.11 and the open mapping theorem. Indeed, let $\{G_1, G_2\}$ be an open cover of $\mathbb{C}$. Because T is decomposable, there exist $X_1, X_2 \in Lat(T)$ such that $X = X_1 + X_2$ and $\sigma(T|X_i) \subset \overline{G}_i$, $i = 1, 2$. By Lemma 4.1.9 and 4.1.11, we have $\hat{X}_i = (\ell_\infty(X_i) + c_0(X))/c_0(X) \in Lat(\hat{T})$ and $\sigma(\hat{T}|\hat{X}_i) \subset \overline{G}_i$. Next, by the open mapping theorem, $\ell_\infty(X) = \ell_\infty(X_1) + \ell_\infty(X_2)$, thus $\hat{X} = \hat{X}_1 + \hat{X}_2$. Therefore $\hat{T}$ is decomposable.

In what follows we prove the converse of Proposition 4.3.7 in the case where the Banach space X is reflexive by a nice application of the *generalized Banach limit*. To do so, we first recall the notion of *Banach limit*, see for example [20, p.73]. There exists a linear functional $LIM : \ell^\infty \rightarrow \mathbb{C}$ such that $LIM\{a_1, a_2, \dots\} = LIM\{a_2, a_3, \dots\}$, $\|LIM\| = 1$, $LIM\{a, a, \dots\} = a$ and, if a sequence $\{x_n\}$ is convergent, then $LIM\{x_n\}$ coincides with the usual limit.

Assume that X is a reflexive Banach space. Given $\{x_n\} \in \ell^\infty(X)$, for each $x^* \in X^*$, $\langle x_n, x^* \rangle_n \in \ell^\infty$. Consider the linear functional

$$x^* \mapsto LIM \langle x_n, x^* \rangle.$$

Since X is reflexive, there exists a unique $x \in X$ such that $LIM\{\langle x_n, x^* \rangle\} = \langle x, x^* \rangle$ for every $x^* \in X^*$. Define $x = LIM\{x_n\}$; thus we have a bounded linear

map from $\ell^\infty(X)$ to X again called LIM . Note that if $\{x_n\} \in c_0(X)$, then $LIM\{\langle x_n, x^* \rangle\} = \lim_{n \rightarrow \infty} \langle x_n, x^* \rangle = 0$ and hence $LIM\{x_n\} = 0$. Accordingly, there exists a linear map $L : \tilde{X} \longrightarrow X$ so that the

$$\begin{array}{ccc} \ell^\infty(X) & \xrightarrow{LIM} & X \\ & \searrow Q \quad \nearrow L & \\ & \tilde{X} & \end{array}$$

commutes. It is not difficult to see that L is surjective and $L \circ J = I$, where I is the identity map. Now, let $\tilde{x} \in \tilde{X}$ which is represented by $\{x_n\} \in \ell^\infty(X)$. Then $\tilde{T}\tilde{x}$ is represented by $\{Tx_n\}$. We have

$$\begin{aligned} \langle L\tilde{T}\tilde{x}, x^* \rangle &= LIM \langle Tx_n, x^* \rangle = LIM \langle x_n, T^*x^* \rangle = \langle LIM x_n, T^*x^* \rangle \\ &= \langle L\hat{x}, T^*x^* \rangle = \langle TL\hat{x}, x^* \rangle. \end{aligned}$$

Therefore $L\tilde{T} = TL$.

Now we are in a position to prove that T has property (β^*) whenever $\tilde{T}$ does. Let $\{G_1, G_2, \dots, G_n\}$ be an open cover of $\sigma(T) = \sigma(\tilde{T})$ and $x \in X$. Since L is surjective, there exists $\tilde{x} \in \tilde{X}$ such that $L\tilde{x} = x$. Because $\tilde{T}$ has property (β^*) ,

$$\tilde{x} = \tilde{x}_1 + \tilde{x}_2 + \dots + \tilde{x}_n, \quad \text{where } \tilde{x}_i \in \tilde{X}_{\tilde{T}}(\overline{G_i}), \quad i = 1, 2, \dots, n.$$

Let $x_i = L\tilde{x}_i$, $i = 1, 2, \dots, n$. Hence

$$x = x_1 + x_2 + \dots + x_n.$$

It remains to show that $x_i \in X_T(\overline{G_i})$ for $1 \leq i \leq n$. For each $i = 1, 2, \dots, n$, there exists a holomorphic function $\tilde{f}_i : \mathbb{C} \setminus \overline{G_i} \longrightarrow \tilde{X}$ such that $(z - \tilde{T})\tilde{f}_i(z) = \tilde{x}_i$, from

which it follows that $(z - T)f_i(z) = x_i$, where $f_i = L\tilde{f}_i$ is an X -valued holomorphic function. Hence $x_i \in X_T(\overline{G}_i)$, $(1 \leq i \leq n)$. We conclude that

$$X = X_T(\overline{G}_1) + X_T(\overline{G}_2) + \cdots + X_T(\overline{G}_n).$$

We have proved the following fact:

Theorem 4.3.9 *Suppose X is a reflexive Banach space. If $\hat{T}$ has property (β^*) , then T has the same property.*

Corollary 4.3.10 *Assume X is a reflexive Banach space. If $\hat{T}$ is decomposable, then T is decomposable.*

The rest of this section deals with the stability of property $(w\beta)$ under asymptotic similarity. Recall that the operators $T \in B(X)$ and $S \in B(Y)$ are said to be asymptotically similar if there exists a bijection $A \in B(X, Y)$ such that A asymptotically intertwines S and T and its inverse A^{-1} asymptotically intertwines T and S .

Theorem 4.3.11 *Property $(w\beta)$ is preserved under asymptotic similarity. In particular, property $(w\beta)$ is stable under quasi-nilpotent equivalence.*

Proof. Assume that $T \in B(X)$ and $S \in B(Y)$ are asymptotically similar and choose a corresponding bijection $A \in B(X, Y)$ for the asymptotic intertwining of the pairs (S, T) and (T, S) , i.e.,

$$\|C^n(S, T)A\|^{1/n} \rightarrow 0$$

and

$$\|C^n(T, S)A^{-1}\|^{1/n} \rightarrow 0.$$

Now, by Lemma 4.1.9, $\tilde{A}$ is also a bijection. Next, note that since the correspondence $T \rightarrow \tilde{T}$ is an isometric homomorphism, it follows that $\|C^n(\tilde{S}, \tilde{T})\tilde{A}\|^{1/n} \rightarrow 0$ and

$\|C^n(\tilde{T}, \tilde{S})\tilde{A}\|^{1/n} \rightarrow 0$, i.e., $\tilde{T}$ and $\tilde{S}$ are asymptotically similar. Now, if T has property $(w\beta)$, then by Theorem 4.3.1, $\tilde{T}$ has the $(SVEP)$ and hence $\tilde{S}$ has the $(SVEP)$ by [53, Theorem 3.5]. Another application of Theorem 4.3.1 shows that S has property $(w\beta)$.

4.4 Perturbation

In this section we will prove that, for every operator $T \in B(X)$ with property $(w\beta)$ and every compact operator $K \in B(X)$ such that $T + K$ has the $(SVEP)$, $T + K$ also has property $(w\beta)$. We also prove that property $(w\beta)$ is stable under certain limits. We also give some examples.

Theorem 4.4.1 *If T has property $(w\beta)$, K is a compact operator and $T + K$ has the $(SVEP)$, then $T + K$ also has property $(w\beta)$.*

Proof. Let $f_n : G \rightarrow X$ be a sequence of holomorphic functions such that

$$M = \sup\{\|f_n(z)\| : z \in G, n \in \mathbb{N}\} < \infty,$$

$$\sup_G \|(T + K - z)f_n(z)\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In view of the vector-valued version of *Vitali's* theorem (Theorem 4.1.2) and the usual sub-subsequence argument (a sequence converges to zero if every subsequence of it contains a subsequence converging to zero), it is enough to show that a subsequence of $\{f_n\}$ converges uniformly to zero on a closed disk $D = D(z_0, r)$ contained in G .

For each positive integer n , write $f_n(z) = \sum_{k=0}^{\infty} a_{nk}(z - z_0)^k$, $a_{nk} \in X$. By Cauchy's inequality, $\|a_{nk}\|r^k < M$ for each n and each k . Since $\{a_{nk}\}_n$ is bounded and K is compact, $\{Ka_{nk}\}_n$ has a convergent subsequence (for each k). By a diagonal process, we can pick $n_1 < n_2 < \dots$, such that $\{Ka_{n_j k}\}_j$ converges in norm, say, to c_k .

Since $\|K a_{n,k}\| r^k < M \|K\|$, we have $\|c_k\| r^k < M \|K\|$. This shows that the radius of convergence of $h(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k$ is at least r . Now it is easy to check that

$$K f_n(z) = \sum_{k=0}^{\infty} K a_{n,k} (z - z_0)^k$$

converges uniformly on $D(z_0, r/2)$ to $h(z)$ (by noticing that the “tails” of the power series expansions of $K f_n$, ($j \geq 1$) are uniformly small on $D(z_0, r/2)$). Thus, from $\|(T + K - z)f_n(z)\| \rightarrow 0$ uniformly on all compact subsets of G , we have

$$\sup\{\|(T - z)f_n(z) + h(z)\| : z \in D(z_0, r/2)\} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Since T has property $(w\beta)$, $\{f_n\}$ converges uniformly on $D(z_0, r/4)$, say, to f . Clearly, $K f(z) = h(z)$ and $(T - z)f(z) + h(z) = 0$ on $D(z_0, r/4)$, and hence $(T + K - z)f(z) = 0$ on G by analytic continuation. Since $T + K$ has the $(SVEP)$, $f(z) \equiv 0$. Thus $\{f_n\}$ converges uniformly to zero on $D(z_0, r/4)$, which completes the proof. ■

Remark 4.4.2 In the above theorem, the assumption that, $T + K$ has the $(SVEP)$ cannot be dropped. Indeed, let T be the bilateral shift ($T e_n = e_{n+1}$, where $\{e_n : n \in \mathbb{Z}\}$ is the standard orthonormal basis of $\ell_2(\mathbb{Z})$), and let K be the rank one operator (hence compact) defined by $K x = -(x, e_{-1}) e_0$, $x \in \ell_2(\mathbb{Z})$. Then $T + K = U^* \oplus U$, where U is the right shift and U^* its adjoint (the left shift), see [41, Problem 183]. Now T has property $(w\beta)$ since it is unitary; on the other hand $T + K$ does not have the $(SVEP)$.

Example 4.4.3 This is another example of an operator that enjoys property $(w\beta)$, but without condition (β) . (The previous one is given in Example 4.1.4.)

In [39], Herrero constructed a compact operator K such that $T = U + K$ (where U is the bilateral shift) has the following curious property: $\sigma(T)$ is the unit circle and, for every invariant subspace $Y \neq (0)$ of T , $\sigma(T|Y)$ is either the unit circle or the unit disk. Thus T is not decomposable and hence either T or T^* does not have

property (β) . This shows that property (β) cannot be preserved under compact perturbation. In other words, the analogue of Theorem 4.4.1 for property (β) fails. However, since $\sigma(T)$ is the unit circle, a closed set in the complex plane without interior points, Proposition 4.1.3 or Theorem 4.4.1 tells us that both T and T^* have property $(w\beta)$.

The rest of this section is devoted to the discussion of the asymptotic behaviour of property $(w\beta)$. Recall that $P(S, T)$ is the pseudo-distance of two operators S and T introduced in Section 3 of Chapter 2.

Theorem 4.4.4 *Let $T_n, T \in B(X)$, and assume that the operators T_n have property $(w\beta)$. If $P(T_n, T) \rightarrow 0$ as $n \rightarrow \infty$, then T also has property $(w\beta)$.*

Proof. Since the correspondence $T \rightarrow \hat{T}$ is an isometric homomorphism, it follows that $P(\hat{T}_n, \hat{T}) \rightarrow 0$. By Theorem 4.3.1, the operators $\hat{T}_n$ have the $(SVEP)$. Hence, by Theorem 2.6 of [8], $\hat{T}$ has the $(SVEP)$. Thus, by Theorem 4.3.1, T has property $(w\beta)$. ■

Corollary 4.4.5 *Given $T \in B(X)$, let $\{T_n\}$ be a sequence in $B(X)$ satisfying*

- (a) *Each T_n has property $(w\beta)$*
- (b) *$TT_n = T_nT$ for all $n \in \mathbb{N}$*
- (c) *The sequence $\{T_n\}$ converges to T in the uniform operator topology.*

Then T has property $(w\beta)$.

Proof. Suppose $TT_n = T_nT$ for all n , then we have $P(T_n, T) = d_{sp}(T_n, T)$, and thus T has property $(w\beta)$ by the above theorem. ■

Condition (b) in the above corollary is essential as the following example shows.

Example 4.4.6 In this example we exhibit a sequence of bounded linear operators $\{T_n\}$ on a Hilbert space satisfying conditions (i) and (iii) of Corollary 4.4.5, but its

uniform limit does not have property $(w\beta)$. For this, let H be a separable Hilbert space and $\{e_k\}$ the orthonormal basis of H . Consider $T, K_n \in B(H)$ defined as

$$Te_k = w_k e_{k+1}, \quad k \in \mathbb{Z}$$

where $w_0 = 0$ and $w_k = 1$ for $k \neq 0$

and

$$K_n e_k = \alpha_k e_{k+1}, \quad k \in \mathbb{Z}$$

where $\alpha_0 = 1/n$ and $\alpha_k = 0$ for $k \neq 0$.

It is clear that $T_n = T + K_n$ is similar to the bilateral shift on H . Thus, T_n has property $(w\beta)$. Moreover, since $\|K_n\| \rightarrow 0$, $T_n \rightarrow T$ in the uniform topology. However, T does not even have the $(SVEP)$, since $T = U \oplus U^*$, where U is the unilateral shift (on the subspace spanned by $\{e_k\}_{k>0}$) and U^* its adjoint (on the orthogonal complement), cf. [66, P. 56].

Question 4.4.7 It would be interesting to know if there is a duality theory for property $(w\beta)$ analogous to that of property (β) .

Bibliography

- [1] E. Albrecht, *On decomposable operators*, Integral Equations Operator Theory, **2**(1979), 1-10.
- [2] E. Albrecht, *An example of weakly decomposable operator which is not decomposable*, Rev. Roumaine Math. Pures Appl., **19** (1974), 131-139.
- [3] E. Albrecht, *Spectral decompositions for systems of commuting operators*, Proc. Roy. Irish Acad., 1981.
- [4] E. Albrecht, *Decomposable systems of operators in harmonic analysis*, in Toeplitz Centennial, I. Gohlborg (ed.) Birkhauser, Basel, (1982), pp. 19-35.
- [5] E. Albrecht and J. Eschemeier, *Analytic functional models and local spectral theory*, Preprint.
- [6] E. Albrecht, J. Eschemeier and M.M. Neumann, *Some topics in the theory of decomposable operators*, Oper. Theory: Adv. Appl., **17**(1986), 15-34.
- [7] C. Apostol, *Restrictions and quotients of decomposable operators in a Banach space*, Rev. Roumaine Math. Pures Appl., **13**(1968), 147-150.
- [8] C. Apostol, *Remarks on the perturbation and a topology for operators*, J. Funct. Anal., **2**(1968), 395-405.

- [9] C. Apostol, *Decomposable multiplication operators*, Rev. Roumaine Math. Pures Appl., **17**(1972), 323-333.
- [10] I. Bacalu, *S-decomposable operators in Banach spaces*, Rev. Roumaine Math. Pures Appl., **20**(1975), 1101-1107.
- [11] I. Bacalu, *Some properties of decomposable operators*, Rev. Roumaine Math. Pures Appl., **21**(1976), 177-194.
- [12] S. K. Berberian, *Approximate proper vectors*, Proc. Amer. Math. Soc., **13**(1962), 111- 114.
- [13] E. Bishop, *A duality theory for an arbitrary operator*, Pacific J. Math., **9**(1959), 379-397.
- [14] J.J. Buoni, R. Harte and T. Wickstead, *Upper and lower Fredholm spectra*, Proc. Amer. Soc., **66**(1977), 309-314.
- [15] I. Colojoara and C. Foias, *Theory of Generalized Spectral Operators*, Gordon and Breach, New York, 1968.
- [16] M.D. Choi and C. Davis, *The spectral mapping theorem for point approximate point spectrum*, Bull. Amer. Math. Soc., **80**(1974), 317- 321.
- [17] C. Davis and P. Rosenthal, *Solving linear operator equations*, Can. J. Math., **27**(1974), 1384-1389.
- [18] G. Dinescu, *Weak spectral equivalence and weak spectral convergence*, Czechlovak Math. J. **34**(1984), 203- 226.
- [19] H.R. Dowson, *Spectral Theory of Linear Operators*, London Math. Soc., Monograph Series **12**, Academic Press, London, 1978.

- [20] N. Dunford and J.T. Schwartz, *Linear Operators, Part I: General Theory*, Wiley-Interscience, New York, 1967.
- [21] N. Dunford and J.T. Schwartz, *Linear Operators, Part III: Spectral Theory*, Wiley-Interscience, New York, 1971.
- [22] I. Erdelyi and R. Lange, *Spectral Decomposition on Banach Spaces*, Lect. Notes Math., **623**, Berlin-Heidelberg-New York, Springer-Verlag, 1977.
- [23] I. Erdelyi and S. Wang, *A Local Spectral Theory for Closed Operators*, London Math. Soc. Lect. Series, **105**, Cambridge, Univ. Press, 1985.
- [24] J. Eschmeier, *Operator decomposability and weakly continuous representations of locally compact Abelian groups*, J. Operator Theory, **7**(1982), 201-208.
- [25] J. Eschmeier, *Equivalence of decomposability and 2-decomposability for several commuting operators*, Math. Ann., **262**(1983), 305-312.
- [26] J. Eschmeier, *Are commuting systems of decomposable operators decomposable?*, J. Operator Theory, **12**(1984), 213-219.
- [27] J. Eschmeier, *A decomposable Hilbert space operator which is not strongly decomposable*, J. Integral Equations and Operator Theory, **11**(1988), 161-171.
- [28] J. Eschmeier and M. Putinar, *Bishop's condition (β) and rich extensions of linear operators*, Indiana Univ. Math. J., **37**(1988), 325-348.
- [29] J. Eschmeier and M. Putinar, *Spectral theory and sheaf theory III*, J. Reine Angew. Math., **354**(1984), 150-163.
- [30] J. Eschmeier and B. Putinar, *Invariant subspaces for operators with Bishop's property (β) and thick spectrum*, J. Funct. Anal., **94**(1990), 196-222.

- [31] J.K. Finch, *The single-valued extension property on a Banach space*, Pacific J. Math., **58**(1975), 61-69.
- [32] C. Foias, *Spectral maximal spaces and decomposable operators in Banach spaces*, Arch. Math., **14**(1963), 311-349.
- [33] C. Foias, *On the spectral maximal spaces of decomposable operators*, Rev. Roumaine Math. Pures Appl., **15**(1970), 1599-1606.
- [34] C.K. Fong, *Decomposability into spectral manifolds and Bishop's property (β)*, Northeastern Math. J., **5**(1989), 391-394.
- [35] C.K. Fong, *On Bishop's duality theory of spectral subspaces and operators*, Preprint.
- [36] S. Frunza, *A duality theorem for decomposable operators*, Rev. Roumaine Math. Pures Appl., **16**(1971), 1055-1058.
- [37] S. Frunza, *The single-valued extension property for coinduced operators*, Rev. Roumaine Math. Pures Appl., **18**(1973), 1061-1065.
- [38] S. Frunza, *Spectral decomposition and duality*, Illinois J. Math., **20** (1976), 314-321.
- [39] D.A. Herrero, *Indecomposable compact perturbations of the bilateral shift*, Proc. Amer. Math. Soc., **62**(1977), 254-258.
- [40] E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, A.M.S., Providence, 1957.
- [41] P.R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, 2nd ed., 1982.
- [42] T.B. Hoover, *Quasi-similarity of operators*, Ill. J. Math., **16**(1972), 678- 686.

- [43] A.A. Jafarian, *Weak and quasi-decomposable operators*, Rev. Roumaine Math. Pures Appl., **22**(1977), 195-212.
- [44] R. Lange, *Strongly analytic subspaces*, in Operator Theory and Functional Analysis, edited by I. Erdelyi, Research Notes in Math., **38**, Pitman, San Francisco, 1979.
- [45] R. Lange, *A purely analytic criterion for a decomposable operator*, Glasgow Math. J., **21**(1980), 69-70.
- [46] R. Lange, *On generalization of decomposability*, Glasgow Math. J., **22**(1981), 77-81.
- [47] R. Lange, *Equivalent conditions for a decomposable operator*, Proc. Amer. Math. Soc., **82**(1981), 401-406.
- [48] R. Lange and S. Wang, *Strongly analytic spaces in spectral decomposition*, Glasgow Math. J., **30**(1988), 249-257.
- [49] R. Lange, *Duality and asymptotic spectral decompositions*, Pacific J. Math., **121**(1986), 93-108.
- [50] K.B. Laursen and M.M. Neumann, *Decomposable operators and automatic continuity*, J. Operator Theory, **15**(1986), 33-51.
- [51] K.B. Laursen and M.M. Neumann, *Decomposable operators and applications to harmonic analysis*, Studia Math., **101**(1992), 193-214.
- [52] K.B. Laursen and M.M. Neumann, *Local spectral properties of multipliers on Banach algebras*, Arch. Math., **58**(1992), 368-375.
- [53] K.B. Laursen and M.M. Neumann, *Asymptotic intertwining and spectral inclusions on Banach spaces*, Submitted.

- [54] K.B. Laursen and P. Vrbova, *Some remarks on the surjectivity spectrum of linear operators*, Czechoslovak Math. J., **39**(1989), 730-739.
- [55] H.P. Lotz, *Über das Spektrum positiver Operatoren*, Math. Z., **108**(1968), 15-32.
- [56] B. Nagy, *Operators with spectral decomposition property are decomposable*, Math. Hungaria, **13**(1978), 429-432.
- [57] B. Nagy, *On S-decomposable operators*, J. Operator Theory, **2**(1979), 277- 286.
- [58] B. Nagy, *Restrictions, quotients and S-decomposability of operators*, Rev. Roumaine Math. Pures Appl., **25**(1980), 1085-1090.
- [59] B. Nagy, *Differential operators and spectral decomposition*, Functions Series, Operators, Vol. I-II (Budapest 1980), 891-917. Colloq. Math. Soc. János Bolyai **35**, North Holland, Amsterdam 1983.
- [60] M.M. Neumann, *Commutative Banach algebras and decomposable operators*, Mh. Math., **113**(1992), 227-243.
- [61] M. Putinar, *Hyponormal operators are subscalar*, J. Operator Theory, **12** (1984), 385- 397.
- [62] M. Radjabalipour, *Equivalence of decomposable and 2-decomposable operators*, Pacific J. Math., **77**(1978), 243-247.
- [63] M. Radjabalipour, *Decomposable operators*, Bull. Iranian Math. Soc., **9**(1978), 1-49.
- [64] R. Remmert, *Theory of Complex Functions*, Springer-Verlag, New York, 1991.
- [65] C.E. Rickart, *General Theory of Banach Algebras*, Krieger, Huntington, 1974.

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- [66] A.L. Shields, *Weighted shift operators and analytic function theory*, Topics in Operator Theory, Math. Surveys Monographs, Vol. **13**, Amer. Math. Soc. 2nd ed. 1978.
- [67] J.C. Snader, *Strongly analytic subspaces and strongly decomposable operators*, Pacific J. Math., **115**(1984), 193-202.
- [68] J.C. Snader, *Bishop's condition (β)*, Glasgow Math. J., **26**(1985), 35-46.
- [69] S.L. Sun, *The sum and product of decomposable operators*, Northeastern Math. J., **5**(1989), 105-117.(In Chinese)
- [70] S.L. Sun, *The single-valued extension property and spectral manifolds*, Proc. Amer. Math. Soc., **118**(1993), 77-87.
- [71] K. Tanahashi, *A characterization of decomposable operators*, Tohoku Math. J., **34**(1982), 295-300.
- [72] K. Tanahashi, *Characterization of S -decomposable operators on a Banach space*, Tohoku Math. J., **35**(1983), 261-265.
- [73] K. Tanahashi and T. Yoshino, *Weak decomposable and spectral operators on a weakly complete Banach spaces*, Math. Japonica, **34**(1989), 831-835.
- [74] F.H. Vasilescu, *Residually decomposable operators in Banach spaces*, Tohoku Math. J., **21**(1969), 509-522.
- [75] F.H. Vasilescu, *Analytic Functional Calculus and Spectral Decomposition*, Reidel, Dordrecht, Boston and London, 1982.
- [76] F.H. Vasilescu, *On the residual decomposability in dual spaces*, Rev. Roumaine Math. Pure Appl., **16**(1971), 1573-1587.

- [77] P. Vrbova, *On local spectral properties of operators in Banach spaces*, Czechoslovak Math. J., **23**(1973), 483-492.
- [78] S. Wang, *A characterization of strongly decomposable operators and duality theorem*, Acta Math. Sinica, **29**(1986), 145-155. (In Chinese)
- [79] S. Wang and L. Guangyu, *On the duality theorem of bounded S-decomposable operators*, J. Math. Anal. Appl., **99**(1984), 150-163.

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