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PRECISIONSM RESOLUTION TARGETS

OPTIMAL SEARCH IN DISCRETE AND CONTINUOUS SPACES

by

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**A thesis submitted to
the School of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Master in Computer Science***

**Carleton University
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Canada**

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ABSTRACT

We study the optimal search problem for stationary targets in discrete space and for moving targets in continuous space, and for both known and unknown target distributions. After a brief introduction we briefly review some known results for stationary targets. We then study the search problem for stationary targets for unknown target probability distributions and for non-regular detection functions. Error bounds and techniques for estimating the unknown distribution are also presented. Subsequently, after briefly reviewing some of the known results for the moving target problem, we study the latter when it is viewed as an optimal control problem.

We have designed algorithms to compute and simulate the optimal search plans for various scenarios. Based on these we have developed a complete SMALLTALK package which can easily compute and display the optimal search plan, the maximum detection probability, the error estimate, and the simulation result, in a variety of situations.

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results involving stationary targets. In Chapter 3 we present new results for the search problem for stationary targets when target probability distributions are either specified or unknown. Chapter 4 briefly reviews some known results for the moving target scenario. In Chapter 5 we study the search of moving targets modelled as an optimal control problem.

The main novel contributions of this thesis are the following:

1. We have studied the discrete search problem when the initial target distribution is unknown. In this case the searcher must guess the probability of the target's location before he can determine a search strategy. This situation gives rise to a variety of new problems such as those of guessing the unknown target distribution and estimating the error incurred when the strategy is compared to the strategies based on the *true* target distributions.
2. We have obtained upper and lower bounds for the detection probability for optimal search strategies. Based on these estimates we present selection criteria for determining an approximation for the target distribution, and for the error estimates.
3. We have analyzed the discrete search problem for non-regular detection functions. The results obtained are valid for discrete systems, but they can be easily extended to the continuous case.
4. We have modelled the moving target search problem as an optimal control problem. By utilizing the principle of optimality we have derived

Chapter 1

INTRODUCTION

1.1 Introduction

Search, in its primary sense, is an activity of looking for food, prey, shelter, mates, and detecting hostile enemies or natural dangers. It is performed by all animals capable of motion and is usually directed by their senses. Without this capability animals could not survive at all. During evolution the human being acquired (inherited) such instincts involving search in this primitive sense that - just as in breathing or walking - we habitually give it little or no conscious thought. With the advance of science and technology, researchers have endeavored to study search theory analytically, and to describe it mathematically as an operation having various structural patterns and obeying its own laws.

The scientific study of search before World War II was rather sporadic: methodology was applied to search for minerals or missing persons;

mass screening was made to find desirable physical or medicinal properties. Armies planned search on the basis of the sure detection of targets within unobstructed horizon distances, and the cross-over patrols for air search of surfaced targets were included in tactical handbooks. In 1924 a paper was read before the *Société Mathématique de France* on the search for aircraft at night by searchlights and the advantage of certain spiral searches were explained. Historically, these and a multitude of other cases could be cited, but each seemed to be so individual a nature so as to be united into a general theory.

During World War II, it became absolutely necessary to exploit all scientific methods to understand and improve the effectiveness of the instruments of search and the logic of their applications. Since then, the "Theory of Search" has emerged as a science, and is often perceived as a new branch of applied mathematics.

Optimal Search theory, as its name suggests, is about the theory of finding the "best" way for searching for a pre-claimed object (also referred to as the "target"). The three basic elements which are present in an optimal search problem are listed below.

1. Any optimal search problem involves the probability distribution for the target's location and motion. Information about the target's location at some initial time (for example, its last reported position) and its subsequent motion is quantified in terms of probability distributions. An example might be the last reported position of a submarine which is assumed to be traveling at a known speed, but its traveling direction

is unknown. The reported position has an error which is modelled by a circular normal distribution with standard deviation σ in any direction. This example was the prototype search problem used in the embryonic stages of Search Theory [107].

2. The second important notion in search theory is the concept of the detection function. The detection function relates the amount of effort utilized in searching an area to the probability of detecting the target given that it is located in that area.
3. The final concept that is typically needed is the concept of the constraint on the resources that are available. Typically, the searcher has a limited amount of resources (effort) available to conduct the search. For example, in the above situation, the searcher may have a limited time, T , as the constraint on resources. Often the constraint is not on the total resources available, but on the rate at which the resources are available.

The detection function and the probability distribution for the target's location can be used as follows. For each allocation of total search effort to the various regions of our search space, the probability of detecting the target with that allocation can be computed. The optimal search problem is then one of finding an allocation of resources (for example, search time) so as to maximize the probability of detection, or to minimize the expected consumption (cost) of resources for detecting the target.

We may categorize search problems based on the time used, the target motion and the constraints placed on our ability to allocate search effort. In other words, we categorize targets according to whether they are stationary or moving, whether they are operating in discrete or continuous time, and whether the search effort is infinitely divisible or discrete (in fixed length units). By infinitely divisible we mean that effort can be divided as finely as desired. As opposed to this, discrete effort scenarios correspond to the situations in which a searcher probes an area by making looks viewed as concentrated "peeps". Besides these, there are other constraints used to categorize search problem. For example, there are situations in which the searcher's motion is constrained, which is the case that occurs when one is searching for a lost person and the motion undertaken by the searcher is "walking". It is clear that as we impose more constraints on the search effort, our ability to find the optimal search strategy decreases.

1.2 Main Contributions

In this thesis we study the optimal search problem for stationary targets in discrete spaces, and moving targets in continuous spaces. We consider the scenarios in which the target distributions are both known or unknown. The thesis is organized as follows. In Chapter 2 we briefly review some known

results involving stationary targets. In Chapter 3 we present new results for the search problem for stationary targets when target probability distributions are either specified or unknown. Chapter 4 briefly reviews some known results for the moving target scenario. In Chapter 5 we study the search of moving targets modelled as an optimal control problem.

The main novel contributions of this thesis are the following:

1. We have studied the discrete search problem when the initial target distribution is unknown. In this case the searcher must guess the probability of the target's location before he can determine a search strategy. This situation gives rise to a variety of new problems such as those of guessing the unknown target distribution and estimating the error incurred when the strategy is compared to the strategies based on the *true* target distributions.
2. We have obtained upper and lower bounds for the detection probability for optimal search strategies. Based on these estimates we present selection criteria for determining an approximation for the target distribution, and for the error estimates.
3. We have analyzed the discrete search problem for non-regular detection functions. The results obtained are valid for discrete systems, but they can be easily extended to the continuous case.
4. We have modelled the moving target search problem as an optimal control problem. By utilizing the principle of optimality we have derived

the Hamilton-Jacobi-Bellman equation, and transferred the stochastic system to a deterministic semilinear system. We have thus presented a new tool for solving difficult moving target search problems.

5. We have designed algorithms to compute and simulate the optimal search plans for all the above scenarios. Using these we have developed a user friendly SMALLTALK environment to compute and display the optimal search strategy, the maximum detection probability, the error estimate, and the simulation result for a variety of situations.

Chapter 2

THE STATIONARY TARGET SEARCH PROBLEM : A REVIEW

2.1 Introduction

In this chapter we review some of the known results derived for the theory of search for a stationary target. Most of these results deal with the problem in discrete space. This problem has been studied by many authors from the earliest days when the theory of search was in its infancy, and even till today. In many cases optimal or uniformly optimal searching strategies have been devised. In this chapter we briefly list some of the interesting and powerful results available. We hasten to add that it is by no means a complete list of the work done in this area.

2.2 Koopman's Model for the Stationary Search Problem

Suppose an object is hidden in one of N boxes. More specifically, suppose that it is hidden in the k -th box with probability¹ $p(k)$. A search strategy is a finite or infinite sequence of box indices. The representation of this sequence is arbitrary, and so, for example, the list $f = (3, 1, 6, 8, 3, \dots)$ may be used to represent a search strategy in which box 3 is to be searched first, if the object is not found box 1 is to be searched, and so on.

Note that it is assumed that a search of the box containing the item may not lead to a successful detection. Consequently, a subsequent search of box 3 (the fifth search in the above sequence) might be necessary. Each search strategy has a positive probability of finding the target, and has a cost associated with each possible outcome. The constraints on the detection probability $P[f]$ and the associated cost $C[f]$ are specified below.

Let $E_{j,k}$ be the event that the j -th search of box k is successful. We denote

$$p_{j,k} = \text{Prob}\{E_{j,k}\},$$

and thus

$$P[f] = \sum p_{j,k},$$

where $p_{j,k}$ is included in the summation if and only if there is a j -th search of box k in the strategy f . Hence the probability of finding the object using

¹For the time being we shall assume that $p(k)$ is known to the algorithm computing the search strategy. We shall later study the scenario when $p(k)$ is unknown.

strategy f depends only on how many times each box is searched in f , and not on the order of the searches.

Let $C[f]$ be the upper bound for the cost associated with the strategy f . Suppose that the j -th search of box k costs $c_{j,k}$. Then the cost of the search strategy f is bounded by

$$C[f] \leq \sum c_{j,k},$$

where $c_{j,k}$ is included in the summation if and only if there is a j -th search of box k in the strategy f .

The study of the optimal distribution for the searching effort for a stationary target was pioneered by Koopman [107-111]. In these papers Koopman assumed an exponential form for the detection function, and considered the problem of maximizing the probability of detecting the target. He derived methods to allocate a given amount of continuous searching effort to the continuous target space.

Koopman observed that for the problem he studied, the distribution that maximized the detection probability with a given effort had the interesting property that it was the sum of the conditionally optimal distributions. More explicitly, he showed that the optimal distribution of $E1 + E2$ was the sum of the optimal distribution of $E1$ and the conditionally optimal distribution of $E2$, given that the target had not been found with the previous distribution of $E1$. Hence, he observed that no advantage could be gained from the additional information that the target was not found, if the distribution of the effort $E1$ was known. Koopman conjectured that the optimal

distribution had this property by virtue of the exponential law of search he assumed.

Dobbie [46] disproved Koopman's conjecture. He showed that the additive property was true even for more general cases.

If the optimal distribution has the additive property described above, the latter can be easily demonstrated after the solution has been found. On the other hand, if the additive property can be proved without finding the solution, the solution itself can be easily computed through the use of the property. In other words, whenever the property holds, we can find the optimal distribution in-the-large by conditionally optimizing in-the-small. This, in turn, leads to a simple sequential procedure in the discrete case, and to a procedure that is easy to describe in the continuous case.

This reverse procedure of finding optimal distributions was studied by Dobbie in [46]. His method yielded some of the known results (due to Koopman and others) for stationary targets, and for time-independent homogeneous detection functions [111], [43], [75].

Charness and Cooper [31] gave an algorithm for the numerical computation of the optimal distribution of searching effort as specified by Koopman [111].

Gluss [79] generalized Koopman's models to include penalty costs. He considered a complex system consisting of N modules containing $n(1), \dots, n(N)$ items or sub-circuits. A realistic example might be a radar-missile system, in which the modules are represented by the radar, the communication system, the host computer, the missile launcher and the missile, etc.

After the system breaks down, the procedure for detecting the faulty parts and repairing the system will determine the time taken to repair it. Gluss considered two different possible models. In the first model he assumed that over-all tests of each module may be performed, and individual item within modules can also be tested. In his second model he assumed that over-all module tests were not possible, and that penalty costs had to be paid whenever the search moved from one module to another. In both models, penalty costs (measured in time, for example) were associated with the testing of each item and module. For both models equations were developed for the optimum policy, i.e., the policy which minimizes the expected amount of time used or the penalties paid, and these equations were solved for the first model.

2.3 Continuous Versions of Koopman's Model

De Guenin [43] dealt with a generalized version of Koopman's model for regular detection functions. In his paper, De Guenin assumed that an object is located in a uni-dimensional space. Thus the position of the object can be defined by a single parameter X , which is a random variable with the probability density function $g(x)$. Note that $g(x)$ satisfies:

$$g(x) \geq 0,$$

$$g(x)dx = \text{Pr}\{x \leq X \leq x + dx\},$$

and

$$\int_{-\infty}^{\infty} g(x)dx = 1$$

The last condition implies that the object is necessarily in the given space. However, it can be assumed without complicating the problem that the object has only a probability $\alpha \leq 1$ of being in the given space. This covers the case in which we do not know whether there is actually an object in the space in which we are searching, as would be the case in oil exploration.

The search effort devoted to the interval between x and $x + dx$ is denoted by $\phi(x)dx$, where $\phi(x)$ is a non-negative and continuous function (effort allocation function) that is to be determined. If the total effort at our disposal is limited and equals to K , then we have:

$$\int_{-\infty}^{\infty} \phi(x)dx = K$$

The probability $b[\phi(x)]$ of detecting the target with an effort $\phi(x)$ when the target is indeed at x is called the detection probability or detection function. The notion of the detection probability comes from the experimental evidence that one may look directly at an object without actually seeing it, or that a plane can be in the range of a radar without any blip appearing on the screen for several revolutions of the antenna. However, if an observer continues to look towards an object, he will be almost certainly eventually locate it. These properties can be easily expressed as follows:

$$b(0) = 0; \quad b'(\phi) \geq 0; \quad \lim_{\phi \rightarrow \infty} b(\phi) = 1$$

Furthermore, De Guenin assumed that the detection function $b(\phi)$ is regular, that is, $b'(\cdot)$ is a decreasing function with $b'(0) > 0$ and $b'(\infty) = 0$. The probability of detecting an object between x and $x + dx$ is thus

$$g(x)b[\phi(x)]dx.$$

Hence the overall probability of detecting the target in the search process is given by:

$$P(\phi) = \int_{-\infty}^{\infty} g(x)b[\phi(x)]dx.$$

Thus the optimal search problem in this case is one of determining a function $\phi(x)$ that maximizes $P(\phi)$, subject to the conditions

$$\phi(x) \geq 0, \quad \int_{-\infty}^{\infty} \phi(x)dx = K.$$

The main result due to De Guenin [43] is given in Theorem 2.1 below.

Theorem 2.1. *A necessary condition for $\phi(x)$ to be optimum is that at any point x such that $\phi(x) > 0$,*

$$g(x)b'_{\phi}[\phi(x)] = C,$$

where C is a constant.

■

Based on this result De Guenin [43] gave a procedure of determining the optimal allocation $\phi(x)$ by successive approximations. In the interest of completeness the pseudo-code for his procedure is given below.

ALGORITHM De Guenin

Input: *The probability density function $g(x)$, detection function p .*

Output: *An optimal time allocation $\phi(x)$.*

Var λ, Δ : *Real numbers: These are chosen initially by the user.*

BEGIN

While $\Phi(\lambda) \neq \Phi$ **Do**

determine $E := \{x : g(x) > \lambda/p'_{(0)}\}$

$\phi(x) := f(\lambda/g(x))$ $\{f \text{ is the inverse of } p'\}$

$\Phi(\lambda) := \int_E \phi(x) dx$

$\Phi := \int_R \phi(x) dx$

If $\Phi(\lambda) > \Phi$ **Then**

$\lambda := \lambda + \Delta$

Else

If $\Phi(\lambda) < \Phi$ **Then**

$\lambda = \lambda - \Delta$

EndIf

EndIf

EndWhile

Return $\phi(x)$ $\{\text{found optimal plan}\}$

END

2.4 Minimum Expected Cost Model

Staroverov [170], Matula [125], and Kadane [95] investigated discrete versions of the model introduced above. We review Staroverov's result first.

There are two different considerations in optimal search. Sometimes one wishes to maximize the probability of a successful detection with total cost not exceeding the preset budget K . Alternatively one may wish to minimize the expected time spent in locating the target. Most of the results we reviewed earlier have used the first criteria. As opposed to these, Staroverov's paper is an example that uses the second criteria.

In his paper [170] Staroverov solved the following discrete optimal search problem. Assume that there are N cells numbered $1, 2, \dots, N$, one of which contains the object sought for. The probability that the object is in the cell k is $p_k > 0$, $k = 1, 2, \dots, N$, and $\sum p_k = 1$. During a unit of time only one cell j can be inspected. As a result of searching in a cell j the object may be found or not found. If the object is not in the cell the search will find the object with probability 0. However if a cell in which the object is located is searched, the object will be found with probability p (and thus will not be found with probability $1 - p$). In Staroverov's model, the result of searching for separate cells was independent of each other.

We define the search process as the sequence of numbers:

$$a = (a_1, a_2, \dots, a_j, \dots),$$

where all the a_j 's are integers and $1 \leq a_j \leq N$. The number a_j indicates that at time j the cell number a_j must be inspected if at the time $j - 1$ the target had not been found. A random variable τ_a is introduced with every searching process a , which is defined as the time of finding the object ($\tau_a = 1, 2, \dots$).

By deriving algebraic results concerning positive monotone decreasing series, Staroverov gave a method for obtaining a strategy a^* which satisfies

$$E\tau_{a^*} = \inf_a E\tau_a,$$

(where E denote the expectation operator). The process a^* is optimal under the second criteria.

The optimal distribution of searching effort which minimizes the expected time for detection was first studied by Gilbert [75]. In this paper he considered a two-box search model with an imposed loss time whenever the search region is switched. Gilbert's model was further generalized by Kisi [104].

Gluss [80] dealt with a special N -box problem of this type in which the boxes are arranged linearly. Onaga [139] investigated the N -box model for constant loss time. Lossner and Wegener [119] considered a discrete search model with switching loss, and obtained conditions for the optimal search minimizing the expected detection time.

2.5 Whereabouts Search

Generally speaking there are two main problems considered in search theory. The first is the question of choosing a searching strategy that maximizes the probability of finding the object when a budget ceiling is imposed. In such a case the total search cost is bounded by a pre-specified number K , and is called a **Detection Search**.

The second fundamental search problem is termed the **Whereabouts Search**. In a whereabouts search, the objective is to maximize the probability of correctly guessing which box contains the object (the target) after performing a search costing no more than K units, the budget ceiling. This can be accomplished by either finding the object during the search, or by guessing which box contains the object at the end of an unsuccessful search. Thus a strategy for a whereabouts search consists of a search strategy and a guessing strategy if the object is not found by the search strategy.

By elementary arguments Mela [127] showed that an optimal detection search strategy and an optimal whereabouts search strategy are different. We shall explain Mela's example below.

Suppose that an object is equally likely to be in one of two boxes, and that the probability of detection for one look in a box is $\frac{1}{2}$. Suppose that only two looks can be made. If we look twice in one box, the probability of detecting the object is $\frac{3}{4}$ when the object is in the box and 0 when the object

is not in the box. Hence the probability of detection is

$$\frac{1}{2} \times \frac{3}{4} + \frac{1}{2} \times 0 = \frac{3}{8}.$$

On the other hand, if we take a single look in each box, the probability of detection is

$$\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}.$$

To consider the whereabouts search problem we evaluate the amount of information obtained from the two possible search strategies. Suppose we looked twice in box *A*. If the object is not found, then the *a posteriori* probability that the object is still in the box *A* is 0.2 and the *a posteriori* probability that the object is in box *B* is 0.8. If the object is found, these probabilities are 1 and 0, respectively. From information theory we know that the amount of information contained in these two situations is given by the functions $I(0.2, 0.8)$ and $I(1, 0)$, where

$$I(p_1, p_2, \dots, p_n) = - \sum_{i=1}^n p_i \log(p_i).$$

Therefore the expected gain (increment) in information to be:

$$\Delta(2, 0) = I\left(\frac{1}{2}, \frac{1}{2}\right) - \left[\frac{3}{8}I(1, 0) + \frac{5}{8}I\left(\frac{1}{5}, \frac{1}{5}\right)\right] = 0.380.$$

For the case of one look in each box, a similar calculation yields the increment of information to be:

$$\Delta(1, 1) = I\left(\frac{1}{2}, \frac{1}{2}\right) - \left[\frac{1}{2}I(1, 0) + \frac{1}{2}I\left(\frac{1}{2}, \frac{1}{2}\right)\right] = 0.347.$$

Thus the strategy of using two looks in one box, (which yields a superior information gain), results in the lower detection probability. Consequently, the optimal strategies for detection search and whereabouts search are different.

Tognetti [193] generalized Mela's result but still stressed the distinction between optimal detection search and optimal whereabouts search strategies. In his paper Tognetti considered the case when the conditional probability is of arbitrary value p , the number of looks is equal to n , and the number of cells is equal to N . In this case the number of strategies is the number of partitions of n , i.e., each of these partitions represents a unique strategy for allocating the n looks to each of N boxes. If $N \geq n$, Tognetti specified a computation and proved that the optimal whereabouts search strategy was the partition that just missed one box.

Kadane [95] generalized Tognetti's results. In his paper he showed that an optimal whereabouts search strategy never searched the box to be guessed, and further showed that among all other boxes, the optimal whereabouts search strategy is an optimal detection search strategy. He also presented an algorithm for determining an optimal whereabouts search strategy.

The whereabouts search problem is somewhat similar to the following problem proposed by Bellman [22]. Suppose we are given the information that a ball is in one of N boxes with the *a priori* probability p_k that it is in the k -th box. Let t_k be the time consumed in searching the k -th box, and let q_k be the probability that any particular searching of the k -th box will yield no information concerning its contents. When the latter occurs, we continue

the search operation with the information already available. Bellman showed that if we wish to obtain the ball, the optimal (i.e., minimum expected time) policy consists of searching the box for which $p_k(1 - q_k)/t_k$ is largest. On the other hand, if we merely wish to locate the box containing the ball in the minimum expected time, Bellman showed that the box for which the quantity $p_k(1 - q_k)/t_k$ is a maximum must be searched first, or not be searched at all.

The only difference between the whereabouts search problem and Bellman's problem is that in the latter, the searcher knows afterwards whether a look has been informative or not. An informative look fails to find the ball only if the ball is not in the box searched. In the detection and whereabouts search problem, however, a look can fail to find the target either because it is not in the box or because it was overlooked, and the searcher would not know which scenario has occurred.

2.6 Local vs. Global Optimization

Chew [34] considered the relation between locally optimal and globally optimal search strategies. In his paper he essentially presented the following optimal strategy: Let (j, k) denote the j -th search in the box k . In order to maximize the probability of finding the object in a fixed number N of

searches, one should choose those N searches (j, k) for which $p_k q_k^{j-1} (1 - q_k)$ is largest. Here q_k denotes the probability of searching box k and overlooking the target, given that the target is in box k . Such a strategy is also known as a **myopic policy**.

Chew's result was generalized by Kadane [93]. Kadane's primary result follows from the following well-known result, which is a variant of the Neyman-Pearson Lemma.

Theorem 2.2. (Variant of the Neyman-Pearson Lemma). *Let $\{p_i\}$ and $\{c_i\}$ be arbitrary nonnegative sequences such that*

$$\sum_{i=1}^{\infty} p_i < \infty.$$

Let X be the class of sequences x such that $0 \leq x_i \leq 1$ for all i . If $0 \leq C \leq \infty$, then the maximum of

$$\sum_{i=1}^{\infty} x_i p_i$$

subject to

$$\sum_{i=1}^{\infty} x_i c_i \leq C, \quad \{x_i\} \in X$$

occurs if and only if

$$x_i = \begin{cases} 1, & \text{if } p_i > r c_i \\ 0, & \text{if } p_i < r c_i \end{cases} \quad (2.1)$$

for some $r, 0 < r < \infty$, and

$$\sum_{i=1}^{\infty} x_i c_i = C.$$

The set of r 's satisfying (2.1) is the same for each optimal x and is a single point or a closed interval.

Proof. For the proof of the theorem see Kadane [93].

■

Based on this theorem, Kadane was able to prove the following result. Suppose X is the set of search strategies. For any $x \in X$, if x contains the search (j, k) it must also contain the search $(j - 1, k)$ if $j > 1$. Such a set is called **feasible**.

Let $q(j, k)$ be the probability that the j -th search of box k is successful. Suppose now that $q(j, k)$ is nonincreasing in j for each k , and that C is an integer. A feasible set of C searches maximizes the probability of finding the object in no more than C searches if and only if it includes all those searches for which for some r

$$q(j, k) > r,$$

and excludes all those searches for which

$$q(j, k) < r.$$

Kadane showed that such a feasible set always existed. Indeed, Chew's result (mentioned above) appears as a special case of Kadane's.

Stone [174-176] also investigated the relationship between incrementally optimal allocations and total optimal allocations. He considered the

problem of finding optimal solutions to a class of separably constrained extrema problems involving nonlinear functionals, and proved results for rather general situations. Suppose we want to optimally allocate a given amount of effort to detect a stationary target, hidden in Euclidean n -space R . The probability density function of the target's location is given by f . Suppose T amount of effort is available for the search. Then the search problem is to seek a function $q^* : R \rightarrow [0, \infty)$ such that

$$\int_R q^*(x) dx \leq T,$$

and

$$\int_R b(x, q^*(x)) f(x) dx = \max \left\{ \int_R b(x, q(x)) f(x) dx \mid q \geq 0, \int_R q(x) dx \leq T \right\}.$$

Stone called the function $b(x, \cdot)$ a **local effectiveness function** at x , because $b(x, y)$ gives the conditional probability of detecting the target given that it is located at x and the effort density at x is y . The integral on the left gives the probability of detecting the target when using allocation q^* , the optimal allocation. This problem can be reduced to the problem considered by Koopman, Dobbie and others when R is replaced by a finite number or a countable set of cells or boxes.

By using Lagrange Multipliers Stone proved that for any Borel measurable local effectiveness function, incrementally optimal allocations are totally optimal whenever the target's probability distribution is given by the density function f as above. In the case where the search space is countable and the local effectiveness function is concave, he showed that incremental optimal allocations are totally optimal. He also gave a counter example

demonstrating that this property will not hold if the local effectiveness function is not concave.

Stone also considered the uniformly optimal search problem described as follows: A search plan is called **uniformly optimal** if it maximizes the probability of detection *at each instant* during the search. Under some additional assumptions he was able to give a sufficient condition for the existence of uniformly optimal search plans.

In the case when $q(j, k)$ is a concave function, many authors have derived effective methods for determining optimal strategies. Among them we name Gilbert [75], Kisi [104], Onaga [139], Charnes and Cooper [31], and Stone [186]. While Gilbert and Kisi determined optimal strategies for some definite concave functions, $q(j, k)$, Onaga gave a solution of the problem under the assumption that all $q(j, k)$ are concave. Charnes and Cooper also solved this problem for some definite concave functions $q(j, k)$.

Stone improved all these results and obtained a general solution, which we shall briefly review presently. Indeed, all the other results mentioned above are special cases of Stone's [176].

2.7 The Lagrange Multiplier Method

Stone's strategy [186] explicitly utilized the well-known optimization technique involving Lagrange Multipliers. This technique has been widely used for centuries in many areas of mathematical science, such as the calculus of variations, optimization, numerical analysis, control theory, etc. In general, it can be applied to problems involving maximizing an arbitrary real valued objective function over any set of real numbers subject to the bounds on the values of any other finite collection of real valued functions defined on the same set. While the use of the Lagrange Multipliers does not guarantee that a solution will necessarily be found for all problems, it is guaranteed that any solution found by this method is indeed a solution. Since this method is so straightforward compared to other methods, it is often worth trying this first, and it works in a surprising number of cases. It is particularly well suited to the solution of problems of allocating limited resources to a set of independent activities, which is exactly what we encounter in search theory.

The main idea motivating the Lagrange Multipliers method is quite simple. We define a function ℓ as below:

$$\ell(k, \lambda, t) = p(k)q(j, k) - \lambda t, \quad 1 \leq k \leq N, \lambda > 0, t > 0.$$

This function is called the **Pointwise Lagrangian**. Stone has shown that a t -admissible allocation a_t is t -optimal if for some $\lambda \geq 0$, a_t satisfies

$$\ell(k, \lambda, a_t(k)) = \max\{\ell(k, \lambda, z) | z \geq 0\}, \quad k = 1, 2, \dots, N.$$

The problem of maximizing the Pointwise Lagrangian has been extensively studied in calculus. In the following we list some of its basic properties.

Theorem 2.3. *Let $q(\cdot, k)$ be a concave probability distribution function on $R_0^+ = \{t | t \geq 0\}$, with*

$$q_t^+(t, k) = \lim_{h \rightarrow 0^+} \frac{q(t+h, k) - q(t, k)}{h}$$

being well defined and non-negative for $t \geq 0$. $q_t^+(t, k)$ may not be finite. $q_t^+(\cdot, k)$ is decreasing and right-continuous. Similarly, we define

$$q_t^-(t, k) = \lim_{h \rightarrow 0^-} \frac{q(t+h, k) - q(t, k)}{h}.$$

$q_t^-(t, k)$ is well defined and non-negative for $t \geq 0$. $q_t^-(\cdot, k)$ is decreasing and left-continuous. Then we have

$$q_t^+(t, k) \leq q_t^-(t, k), \quad (\forall t, k),$$

and that a_t maximizes the Pointwise Lagrangian for some $\lambda \geq 0$ if

$$p(k)q_t^+(a_t(k), k) \leq \lambda, \quad p(k)q_t^-(a_t(k), k) \geq \lambda, \quad \text{if } a_t(k) > 0$$

and

$$p(k)q_t^+(a_t(k), k) \leq \lambda, \quad a_t(k) = 0.$$

■

By this result we can now choose for each $t \leq 0$ a definite t -optimal allocation as follows. Let

$$T^>(L, k) = \sup\{t \in R_0^+ | p(k)q_t^+(t, k) > L\}$$

and

$$T^{\geq}(L, k) = \sup\{t \in R_0^+ | p(k)q_t^+(t, k) \geq L\}$$

for $L \in R_0^+$ and $0 < k < N$.

Obviously, $T^>(\infty, k) = T^{\geq}(\infty, k) = 0$ and $T^{\geq}(0, k) = \infty$. For each $t \in R_0^+$ we may define an allocation a_t^* such that

1. a_t^* is t -admissible;
2. there is an $L(t) \in R^+$ and a box $j(t)$ so that $a_t^* = T^{\geq}(L(t), k)$ for $k < j(t)$, and $a_t^* = T^>(L(t), k)$ for $k > j(t)$;
3. $a_t^*(j(t)) \in [T^>(L(t), k), T^{\geq}(L(t), k)]$.

Thus by Theorem 2.3, a_t^* maximizes the Pointwise Lagrangian for $\lambda = L(t)$ and is therefore t -optimal. Consequently a_t^* is an optimal allocation for the search effort t .

The usefulness of the Lagrange Multipliers method in optimization with constraints is not limited by the differentiability requirement of functions. Many search problems involve discontinuous or non-differentiable functions, for example, integral valued functions. In these cases the Lagrange Multipliers method can still be used to solve optimization problems subject to certain constraints as shown by Everett [55] who viewed it from a different viewpoint. In general, the Lagrange Multiplier method provides a technique whose goal is the maximization of a functional with constraints, rather than

detecting the location of stationary objects. The domain of the function to be maximized can be quite general.

In general, suppose that there is an arbitrary set S , which is interpreted as the set of possible strategies or search plans. Defined on this strategy set is a real valued function H , called a *payoff function*. $H(x)$ is interpreted as the payoff which occurs from employing the strategy $x \in S$. In addition, there are n real valued functions c_k , ($k = 1, \dots, n$) defined on S , called *cost functions*. The meaning of these functions is that employment of the strategy $x \in S$ will consume $c_k(x)$ amount of resources. The problem here is to maximize the payoff subject to given constraints on resources, i.e.,

$$\max_{x \in S} \{H(x) | c_k(x) \leq C_k, k = 1, 2, \dots, n\},$$

for given constraints C_k , $k = 1, 2, \dots, n$.

Everett formulated the generalized Lagrange Multiplier method described below:

Theorem 2.4. *Let λ_k , $k = 1, 2, \dots, n$ be non-negative real numbers. If $x^* \in S$ maximizes the function*

$$H(x) - \sum_{k=1}^n \lambda_k c_k(x)$$

over all $x \in S$, then x^ maximizes $H(x)$ over all those $x \in S$ such that $c_k \leq c_k(x^*)$, for all $k = 1, 2, \dots, n$.*

A similar result was also obtained by Zahl [214]. In this paper he studied the general problem of maximizing

$$\int_c^d f(x, y(x)) dx$$

subject to

$$\int_c^d g(x, f(x)) dx \leq K,$$

where K is a constant, and $y(x)$ is a bounded measurable density function.

We denote $F[y(\cdot)]$ and $G[y(\cdot)]$ as follows:

$$F[y(\cdot)] = \int f(x, y(x)) dx,$$

$$G[y(\cdot)] = \int g(x, y(x)) dx.$$

Let M be the set of all measurable functions satisfying

$$a(x) \leq y(x) \leq b(x), \quad \forall y(x) \in M.$$

Zahl proved the following result:

Theorem 2.5. *A necessary and sufficient condition that a measurable function $y^*(\cdot)$ maximizes $F[y(\cdot)]$ over the set M is that there exists a constant λ such that for almost every x , $y^*(x)$ maximizes $f(x, y) - \lambda g(x, y)$ for all $y \in M$.*

■

Typically, optimal plans have a complicated form even when one is able to overcome the analytic difficulties involved in finding their explicit

expressions. This complicated form makes it difficult to provide the advice required for planning without the aid of computers. Even in the cases where the functional form of the optimal plan is simple, the plans typically require spreading small amounts of search over expanding areas, which is very difficult to achieve in practice.

As a method of overcoming some of the difficulties involved in using optimal plans, Stone [174] considered the approximation of optimal allocation by so called Δ plans. These plans are executed in a step-by-step fashion. At each step, one is required to calculate simple ratios and allocate Δ amount of effort to a single cell. These plans can be found easily by merely using a pocket calculator, and do not require small amounts of search effort to be spread over large areas. Furthermore, these plans approximate the optimal search plan in the sense that as $\Delta \rightarrow 0$, the mean time to find the target using a Δ plan approaches the mean time using the optimal search plan.

One major restriction on using Δ plans is that they apply only when the target location distribution is specified by a finite number of cells R_j , $1 \leq j \leq J$, such that the target is in cell R_j with probability p_j . For most search operations the target location distribution may be specified in this manner.

2.8 Sequential Search Problems

Kadane and Simon [97] has also obtained a similar result for discrete search problems in the case where only finitely many searches have a positive probability of success.

In this paper Kadane and Simon considered two important sequential problems which have been extensively studied by many other authors. The first sequential problem considers a single object hidden in one of the N boxes. The object is in the k -th box with probability p_k , ($\sum p_k = 1$). A search strategy is a permutation of a subset of the integers $\{1, 2, \dots, n\}$ which tells where to perform the next search if the object has not yet been found in the previous search. The cost for searching box k is c_k if the search is unsuccessful, and x_k if it is successful. The first sequential search problem is to determine a search strategy that includes each of the boxes and minimizes the expected cost of a detection search.

The second sequential problem can also be stated as a search problem. In this problem there are m objects ($0 \leq m \leq N$) hidden in N boxes. The event E_k that an object is hidden in the k -th of N boxes again has probability p_k . E_k now is independent of E_j if $k \neq j$, as opposed to the first sequential problem in which the set $\{E_k, k = 1, 2, \dots, n\}$ is disjoint. Again a search of box k will cost c_k if it is unsuccessful and x_k if it is successful. The search continues until the first object is found or until all boxes have been searched. We seek an optimal search strategy which has minimum expected cost.

By symbolically representing the problem the authors proposed a new class of sequential problems that included both the old problems.

Wegener [205] has generalized this method to non-concave detection functions and solved the discrete search problem where one may find the object only at the end of some definite searches. He investigated a search problem where the search space is discrete but where time is continuous, i.e., at any moment of time we may find the object. Without using the concavity assumption, he proved the existence of an optimal distribution of search effort and presented a procedure for its construction. His main result is the following:

Theorem 2.6. *Let $\bar{q}(\cdot, k)$ be the smallest concave function nowhere smaller than $q(\cdot, k)$. The optimal strategy a_1^* for the search problem where the detection function is $\bar{q}(\cdot, k)$, is also optimal for the problem where the detection function is replaced by $q(\cdot, k)$.*

Later, in an subsequent paper, Wegener [207] gave an algorithm for the construction of an optimal allocation, which is also a generalization of Kadane and Simon's algorithm. Wegener proved that it stops for each search problem and that it always computes an L -optimal allocation, where L is the given limit of time t .

Arkin [7] solved a search problem which may be called the continuous version of the search problem solved by Wegener. In this paper the hidden

object is a point of R^n . The *a priori* probability is given by a density function $f(x) : R^n \rightarrow R_0^+$. For each point $x \in R^n$, define $q(\cdot, x)$ as the probability distribution function for detecting an object at point x . A strategy s has to prescribe how the searcher distributes the search effort, i.e.,

$$s : R_0^+ \times R^n \rightarrow R_0^+.$$

$s(\cdot, x)$ is increasing and for all t , $s(\cdot, x)$ satisfying

$$\int_{R^n} s(t, x) dx = t.$$

The probability of detecting the object at time t is given by

$$Q(s, t) = \int_{R^n} f(x) q(s(t, x), x) dx.$$

An optimal search strategy is defined as the one which minimizes the expected search effort of a successful search. Arkin solved a dual search problem with $q(\cdot, x)$ being replaced by the smallest concave function that was nowhere smaller than $q(\cdot, x)$. This method is identical to the method used by Wegener [205].

Arkin [7] also tried to solve a more general and much more difficult search problem. He assumed that one has to pay some extra costs, called switch costs, if one changes the place of search.

2.9 Modelling Using a Moving Target Problem

Gal [71] treated the stationary optimal search problem as the special case of a moving target problem, and used a model that is quite different from previous models.

In his paper Gal considered search games of the following type: The search takes place in a bounded set Q called the Search Set, which is usually a two or more dimensional region, or in some cases is a network. The searcher start moving from a specified point O , the origin. The time unit can be always chosen so that the searcher is moving with maximum velocity of unity.

In the first part of the paper the author assumes that the hider is stationary, and that the searcher and the hider cannot see each other until the distance between them is less than or equal to a number r , known as the *Discovery Radius*, and at that moment capture occurs. In the case where Q is a network, the discovery radius r is taken as zero, while for cases when Q is a two or more dimensional region, r is assumed very small in comparison to the "magnitude" of the set Q .

This problem is modelled as a two-person zero-sum game with the loss of the searcher or the gain of the hider. The searcher's strategy, denoted by s , is a trajectory inside Q and starting from the origin point O . The searcher usually determines his strategies according to a probability distribution from a set of possible strategies. Let $V(s, h)$ be the expected capture time when

the searcher uses strategy s and the immobile hider is at point h . The value of the searcher's strategy s is defined as

$$V(s) = \sup_{h \in Q} V(s, h),$$

and the value of the searcher is defined as

$$\bar{V}(s) = \inf_s V(s).$$

The value of the hider's strategy h is defined as

$$V(h) = \inf_s v(s, h),$$

and the value of the hider is defined as

$$\underline{V}(s) = \sup_h V(h).$$

Usually $\bar{V} \leq \underline{V}$. If $\bar{V} = \underline{V} = V$, then V is called the value of the game. The main results of this paper are listed below:

1. The value of the search game with an immobile hider in an Eulerian network is $\mu/2$, where μ denotes the sum of the lengths of all arcs in the network.¹
2. Let Q be any network and S^* be a closed curve which passes through all the points of Q and which has minimum length. Let $L(s^*)$ denote the length of S^* . Then the value of the search game with the immobile hider satisfies

$$\frac{1}{2}\mu \leq v \leq \frac{1}{2}L(s^*).$$

¹An Eulerian network is a connected set of arcs which can be obtained by drawing a continuous closed curve S that goes through any of the arcs at most once.

3. In the case when Q is a tree, V , the value of the search game for an immobile hider satisfies:

$$V = \mu.$$

4. Let Q be a set of K (where K is odd) nonintersecting arcs of equal length which joins two points O and A . Then the value of the search game satisfies

$$V < \frac{1}{2}L(s^*).$$

These results can be extended to mobile target problems, and these problems are studied in greater detail in Chapter 4 and 5.

Despite the frequent use of search theory and the theoretical progress, the construction of explicit optimal search paths remains a computationally difficult problem. This empirical observation has recently been confirmed by Trummel and Weisinger [194], who proved that the problem of finding an optimal search path is NP-complete. Thus, in the construction of optimal search paths, it is often necessary to forge a compromise between the theoretically optimal and the computationally feasible.

Traditionally a searcher can use a two-stage approach to the planning. The first stage begins by assuming that the available search effort (often measured in swept area) is "infinitely divisible". This means that the effort can be allocated arbitrarily throughout the search region. Using this assumption, optimal search plans can be constructed for many search problems. The second stage in the search planning involves approximating the optimal infinitely divisible plan by a plan that can be executed by the

searcher, whose effort is not *really* infinitely divisible. Often this is done by approximating the optimal plan by rectangles of uniform coverage and then assigning each rectangle or group of rectangles to a searcher.

The key to this search-planning paradigm is the assumption of infinitely divisible effort in the first stage. This assumption led to a mathematically tractable, separable optimization problem. The solution to this optimization problem then serves as a guide for the construction of search paths in the second stage. However, the assumption of infinitely divisible effort limits the flexibility of this approach in modelling search sensors. In particular, this paradigm may not be appropriate for sensors which apply their effort in a series of discrete layers or glimpses. Optical sensors and radars are sensors of this type. Such sensors require search plans that assign an integral number of glimpses at each point — a property not guaranteed by optimal infinitely divisible search plans.

2.10 Layered Search Model

In a recent paper [209] Weisinger and Benkoski proposed a new approach for search planning. In this article they investigated the problem of planning an optimal layered search for a stationary target. A search is layered if it assigns each point in the search space an integer number of glimpses

or looks. In this alternate approach, the first stage of the search-planning paradigm becomes the solution of a layered search problem (defined below) rather than an infinitely divisible search problem. The second stage continues to be an approximation problem, although now layered search plans are to be approximated.

Let E^n denote the n -dimensional Euclidean space with the usual measure. Suppose that a stationary target resides in the search space E^n and that the target's location is described by the *a priori* probability density function g which is Borel measurable. Let Z be the set of non-negative integers. The set Φ of layered search allocations is the set of Borel measurable functions from E^n to Z . If $f \in \Phi$, then the corresponding search plan can be viewed as though it proceeds in a series of layers or steps. Here the I -th step of the search consists of applying one layer of effort to each point $x \in E^n$ such that $f(x) \geq I$.

Given that the target is located at a point $x \in E^n$, let p_i denote the probability of detecting the target with the i -th layer of search but not with earlier layers, where p_i is assumed to be independent of x . Furthermore, if $p_i = 1$ for some number i , we define $p_j = 1$ for all $j > i$. Hence, if effort k ($k \in Z$) is allocated to the point x and the target is located at x , the probability of detection is given by

$$b(k) = 1 - \prod_{i=1}^k (1 - p_i), \quad b(0) = 0.$$

The function b is called the detection function. It is clear that b is a non-decreasing function from $Z \rightarrow [0, 1]$. Conversely, given an non-decreasing

function $b : Z \rightarrow [0, 1]$, corresponding p_i can be constructed by

$$p_i = 1 - \frac{1 - b(I)}{1 - b(I - 1)}, \quad I \geq 1.$$

Hence any non-decreasing function from Z to $[0, 1]$ is a detection function.

The detection function b is **regular** if the distributed probabilities p_i satisfy

$$p_i > p_{I+1}(1 - p_i)$$

for all $I \geq 1$. This condition can be seen to be also equivalent to

$$b(I) - b(I - 1) > b(I + 1) - b(I)$$

for $I \geq 1$. Since $b(I) - b(I - 1)$ represents the incremental return from the I -th layer of search, the regularity condition implies a diminishing rate of return. The cost of applying the effort f at a point $x \in E^n$ is assumed to be $c(x)f$ for some positive Borel measurable function c . Suppose that $c(x)f(x)$ and $g(x)b(f(x))$ are integrable. Define the cost of the allocation f as

$$C[f] = \int_{E^n} c(x)f(x)dx,$$

and the probability of detection as

$$P[f] = \int_{E^n} g(x)b(f(x))dx.$$

The goal of the layered search problem is to find an allocation f^* which maximizes $P[f^*]$ subject to a cost constraint of $C[f^*] \leq K$ for some positive number K .

It is interesting to compare the layered search problem with its "non-layered" counterpart. In an infinitely divisible search problem, a regular

detection function is a function b from $[0, \infty)$ to $[0, 1]$ such that $b(0) = 0$ and the derivative of b is continuous, positive, and strictly decreasing. If b is a regular detection function in this "non-layered" context, then its restriction to the non-negative integers is a regular detection function for the layered search problem.

The Lagrange Multiplier method can be adopted for the layered search problem as is specified by the following result.

Theorem 2.7. For $x \in E^n$, λ a positive real number, and $k \in Z$, we define the Pointwise Lagrangian L by

$$L(x, \lambda, k) = g(x)b(k) - \lambda kc(x).$$

Suppose that $f \in \Phi$. Then a necessary and sufficient condition for f to be optimal within Φ for cost $C[f]$ is that there exists a $\lambda > 0$ such that

$$L(x, \lambda, f(x)) = \max\{L(x, \lambda, k) : k \in Z\}$$

for almost all x in E^n .

■

Hence the optimal layered search problem can be solved as in the "non-layered" case.

2.11 Algorithms for Computing Optimal Allocations

Below we give an algorithm from Stone [186] that can be used to find an allocation f^* that is optimal for cost K when the search space consists of a finite number of cells and the detection function is exponential.

Theorem 2.8. Suppose there are k cells and that for $j = 1, 2, \dots, k$, $p(j)$ is the probability that the target is located in cell j . Suppose the detection function is given by

$$b(j, z) = 1 - e^{-\alpha_j z}, \quad z \geq 0, \quad j = 1, 2, \dots, k,$$

where $\alpha_j \geq 0$. Assume that the cells are numbered so that

$$\alpha_1 p(1) \geq \alpha_2 p(2) \geq \dots \geq \alpha_k p(k).$$

Let

$$y_j = \ln[\alpha_j p(j)] - \ln[\alpha_{j+1} p(j+1)], \quad j = 1, \dots, k-1.$$

Then for a given number $K > 0$,

$$f^*(j) = \frac{1}{\alpha_j} (y_j + y_{j+1} + \dots + y_{l-1} + a), \quad (l = 1, 2, \dots, k.)$$

will yield an optimal allocation for the cost K .

Proof. We prove this theorem recursively. Let

$$y_1 = \ln[\alpha_1 p(1)] - \ln[\alpha_2 p(2)], \quad z_{11} = \frac{y_1}{\alpha_1}.$$

Denote

$$b'(j, z) = \frac{d}{dz} b(j, z).$$

We have

$$p(1)b'(1, z_{11}) = \alpha_1 p(1)e^{-\alpha_1 z_{11}} = \alpha_2 p(2) = p(2)b'(2, 0).$$

Let $\lambda = \alpha_2 p(2)$. Then f_λ^* defined by

$$f_\lambda^*(1) = z_{11}, \quad f_\lambda^*(j) = 0 \quad (j = 2, \dots, k)$$

satisfies

$$\ell(j, \lambda, f_\lambda^*(j)) = \max\{\ell(j, \lambda, z) : z \in Z\}.$$

That is, (λ, f_λ^*) maximizes the Pointwise Lagrangian. Hence f_λ^* is optimal for the cost $C[f_\lambda^*] = z_{11}$.

For any w , $(0 \leq w \leq z_{11})$, we may take $\lambda = p(1)b'(1, w)$ and verify that

$$f_\lambda^*(1) = w, \quad f_\lambda^*(j) = 0 \quad (j = 2, \dots, k)$$

is optimal for cost w . Thus all effort up to z_{11} should be placed entirely in cell 1. We show that the effort beyond z_{11} should be split between cell 1 and cell 2.

Let

$$y_2 = \ln[\alpha_2 p(2)] - \ln[\alpha_3 p(3)],$$

$$z_{22} = \frac{y_2}{\alpha_2}, \quad z_{12} = \frac{y_2}{\alpha_1},$$

and

$$\lambda = p(2)b'(2, z_{22}) = \alpha_3 p(3),$$

$$f_{\lambda}^*(1) = z_{11} + z_{12}, \quad f_{\lambda}^*(2) = z_{22}, \quad f_{\lambda}^*(j) = 0 \quad (j = 3, \dots, k)$$

It can be verified that f_{λ}^* is optimal for the cost $z_{11} + z_{22} + z_{22}$.

For any a , ($0 \leq a \leq 1$), consider

$$f_{\lambda}^*(1) = z_{11} + az_{12}, \quad f_{\lambda}^*(2) = az_{22}, \quad f_{\lambda}^*(j) = 0 \quad (j = 3, \dots, k).$$

Taking

$$\lambda = \alpha_2 p(2) e^{-az_{22}},$$

one may verify that f_{λ}^* is optimal for the cost $z_{11} + a(z_{12} + z_{22})$. Thus we have found the optimal allocation for any cost from 0 to $z_{11} + z_{12} + z_{22}$.

The above derivation gives the principle in developing an optimal search plan: The effort is applied only to cell 1 until z_{11} amount of effort has been placed there. Then additional effort is split between cell 1 and cell 2 so that the ratio of additional effort placed in cell 1 and cell 2 is equal to $\frac{\alpha_1}{\alpha_2}$. If there are only two cells, then this procedure gives a method of allocating any amount of effort.

For the general case of k cells, let

$$y_j = \ln[\alpha_j p(j)] - \ln[\alpha_{j+1} p(j+1)], \quad j = 1, \dots, k-1.$$

One may check that for any a , ($0 \leq a \leq 1$) and $1 \leq I \leq k-1$, f^* defined by

$$f^*(j) = \begin{cases} \frac{1}{\alpha_j}(y_j + y_{j+1} + \dots + y_{I-1} + ay_i), & \text{for } 1 \leq j \leq I \\ 0, & \text{for } I < j \leq k, \end{cases} \quad (2.2)$$

is optimal. Define

$$S(0) = 0, \quad S(I) = \sum_{j=1}^I \frac{1}{\alpha_j}(y_j + y_{j+1} + \dots + y_{I-1} + ay_i), \quad (I = 1, 2, \dots, k-1).$$

Then $S(I)$ is the amount of effort that is applied to cell 1, $\dots$, cell I . After that, any additional effort will be spread among cell 1, $\dots$, cell I , cell $I + 1$.

To find an optimal allocation for cost K , we must find the value of I for which

$$S(I - 1) \leq S(I)$$

Let

$$a = \frac{K - S(I - 1)}{S(I) - S(I - 1)},$$

then by (2.2) the allocation f^* is optimal for the cost K .

If $K > S(k - 1)$, taking

$$a = \frac{K - S(k - 1)}{\sum_{j=1}^k \frac{1}{\alpha_j}},$$

we observe that

$$f^*(j) = \frac{1}{\alpha_j}(y_j + y_{j+1} + \dots + y_{I-1} + a), \quad (I - 1, 2, \dots, k.)$$

will yield an optimal allocation for the cost K .

■

Now we present an algorithm based on the above theorem.

ALGORITHM Stone**Input:****integer** k , ;number of cells I, j , ;indices K ;upper limit of effort**Var** S : array of real numbers ;allocation of effort z : array of integers ;cost for searching y : array of integers ;intermediate variable**output:** $f^*[1...k]$ optimal allocation**BEGIN** **For** $j = 1$ to k **Do** $z_j := 0$. $S(0) = 0$. **EndFor** $i := 1$ **For** $i = 1$ to k **Do** $y[i] = \ln[\alpha_j p(i)] - \ln[\alpha_{i+1} p(i+1)]$. $S[i] = S[i-1] + y_i \sum_{j=1}^i \frac{1}{\alpha_j}$. **If** $S(i) < K$ **Then** $z_j = z_j + \frac{y_i}{\alpha_j}$ for $1 \leq j \leq i$,

$i = i + 1.$

If $i = k$ Then

$$a = [K - S(k - 1)] / [\sum_{j=1}^k \frac{1}{a_j}],$$

$$y_k = 1.$$

EndIf

Else

$$a = [K - S(i - 1)] / [S(i) - S(i - 1)].$$

EndIf

EndFor

For $j = 1$ to i Do

$$f^*(j) = z_j + a \frac{y_j}{a_j}.$$

EndFor

For $j = i + 1$ to k Do

$$f^*(j) = 0 \text{ for } i + 1 \leq j \leq k.$$

EndFor

END ALGORITHM Stone

An algorithm similar to this one has been used by the US Coast Guard to allocate search effort for some search and rescue actions at sea.

Below we describe another algorithm that is due to Wegener [207]. Suppose that one object is hidden in the k -th of n boxes, with probability $p(k)$. The boxes are to be searched sequentially. Associated with the j -th search of box k is a cost $c(j, k)$ and a conditional probability $q(j, k)$ that the first $j - 1$ searches of box k are unsuccessful while the j -th search is successful, given that the object is hidden in box k .

The goal is to maximize the probability of detecting the object, under the condition that the total cost will not exceed K . The algorithm below divides the original search problem to disjoint subsets of searches and computes the K_i -admissible allocation for sub-problem S_i , where $\{K_i\}$ is the set of decompositions of K . For the proof of this algorithm, see Wegener [207].

ALGORITHM Wegener

Input: *The search problem $S_0 = (n, p_0, c_0, q_0, K)$, where p_0, c_0, q_0 are the initial value of vectors p, c, q , as described above.*

Output: *An optimal time allocation $f^*(S)$.*

Var I : *Array of integers; {the set of indices of undivided search problems}*

BEGIN

If $K < 0$ **Then**

print("No optimal time allocation exist.")

Else

$i := 0, I[0] := 0;$

compute the admissible allocation a_i for S_0 ,

compute the upper bound $U - i$ and lower bound L_i for a_i ,

choose t in $I[i]$ such that $U_t = \max\{U_m | m \in I[i]\}$,

If $P(a_t) = U_t$ **Then**

a_t is the optimal allocation.

Else

If $j_t < \infty$ **Then**

divide S_t by inclusion/exclusion of (j_t, k_t)

to $S\{i + 1\}$ and $S\{i + 2\}$,

compute admissible allocations for $S\{i + 1\}$ and

$S\{i + 2\}$, increase i by 2,

set $i[i] = (i[i - 2] \cup \{i - 1, i\}) - \{t\}$.

Else

divide S_i by the R references inclusion/exclusion
to $S\{i + 1\}, \dots, S\{i + r\}$,

compute admissible allocations for

$S\{i + 1\}, \dots, S\{i + r\}$, increase i by r ,

set $i[i] = (i[i - r] \cup \{i - r + 1, \dots, i\}) - \{t\}$.

EndIf

EndIf

EndIf

END ALGORITHM Wegener

2.12 A Software Package for Computing Searching Strategies and Simulations

2.12.1 A Brief Descriptions of the Software Package

In this section we describe a software package written in Digitalk Smalltalk/V 2.0. This package includes a user friendly Graphical User Interface (GUI) that is both interactive and informative. It prompts the user for instructions and data input when necessary, performs a variety of searches or simulations, gives detailed description about the searching process, and the intermediate and final results.

In the discrete search problem, if the target distribution is known, it computes the optimal time allocation and the maximum detection probability. When the initial target distribution is unknown, it prompts the user and request him to guess the probability of target locations, and computes the best time allocation and detection probability for the user's input. It also allows the user to specify a time allocation and then compute the corresponding detection probability. When the guessed distribution differs from the *true target* distribution, the error estimates are obtained by comparing the non-optimal time allocation and detection probability with the optimal ones.

Upon starting the program a pop-up window (main window) appears. In the main window there are three smaller input/output windows. The first window labeled **System Input** gives details about the search process, in-

cluding the initial target distribution, the number of cells to be searched, and the total time allowed. The other two windows display inputs and results for Optimal and Non-optimal search strategies, including detection probabilities and guessed target distributions, respectively.

The main window also contains four boxes representing the cells to be searched. For every search plan the time allocated to each cell is displayed in the corresponding box.

Figure 2.1 shows an example of the main window.

Figure 1. The Main Window

Fig. 2.1 The main window

From Figure 2.1 we can see that in the main window there is a menu bar with three pull-down menus. The first one is named "Known", and

is used when the user wants to "search with a known target distribution". When this menu is activated, the GUI invokes the corresponding program to compute the optimal search strategy for the pre-specified (known) target distribution. Within the menu **Known** are two items. If user chooses the first item, the program will run once and compute the optimal time allocation and the optimal detection probability for given target distribution and time bound. If user chooses the second item and inputs a number N , the program will simulate the actual real-life search process N times, and report a statistical result of the total number of successful searches produced.

Fig. 2.2 The menu **Known** and two submenus

Figure 2.2 shows the appearance of the menu bar when the "Known" menu is activated, and the optimal search strategy for a sample search as well as the results of a simulation for a sample search executed 1000 times.

A typical use-case for the search under known target distributions is as follows.

- First, the program is initiated by executing "DemoSearch open". The "Main" window appears.
- Next, choose "Known" menu from the menu bar. A pull-down menu with two items is displayed.
- Next, choose "Discrete Search". A prompter window appears. The user must input the number of boxes that he is going to search.

Fig. 2.3 The prompter window requests the number of boxes to be searched

- Next, another prompter window appears requesting the user to input the time assigned for this search.

A screenshot of a Smalltalk/V Prompter window. The title bar is black with the text "Smalltalk/V Prompter" in white. The main area is white and contains the text "Total time allowed in search (in minute) ?" in bold. Below this text is a text input field containing the number "4". At the bottom of the window are two buttons: "OK" and "CANCEL".

Fig. 2.4 The prompter window requests the time bound for the search

- Next, the third prompter window appears. This time the user must input the initial target distribution.

A screenshot of a Smalltalk/V Prompter window. The title bar is black with the text "Smalltalk/V Prompter" in white. The main area is white and contains the text "What is the initial distribution probability?" in bold. Below this text is a text input field containing the text ".67 .33". At the bottom of the window are two buttons: "OK" and "CANCEL".

Fig. 2.5 The prompter window requests the target distribution

- Next, the data inputted is displayed in the "System Input" window, and a set of messages is sent to the intended receivers to compute the optimal time allocation and detection probabilities.
- Finally, the computation results are sent back to the main window. The contents of the main window are now updated.

Fig. 2.6 The main window that shows the optimal search strategy

The second menu in the menu bar is named "Unknown", representing the scenario when the user wants to "search with an unknown target distribution". When this menu is activated, the GUI invokes the corresponding program to compute the search strategy for the user inputted (guessed) target distribution. Within the menu "Unknown" there are also two items. If user chooses the first item, the program will prompt the user to guess a target distribution and compute the time allocation and detection probabilities based on this target distribution. If user chooses the second item, the program will prompt the user to input a time allocation and compute the detection probability for this case. The result is displayed in the windows named "Non-Optimal Case".

Fig. 2.7 The menu Unknown and two submenus

Figure 2.7 shows the appearance of the menu bar when "Unknown"

menu is activated, and the search strategy for a sample search based on a user-guessed distribution. A typical use-case for search under unknown target distribution is as follows.

- First, the program is initiated by executing "DemoSearch open". The main window appears.
- Next, choose "Unknown" menu from the menu bar. A pull-down menu with two items is displayed.
- Next, choose "Set Target Distribution". A prompter window appears. The user must guess an initial target distribution.

Fig. 2.8 The prompter window request the guessed target distribution

- Next, the guessed distribution is displayed in the "Non-Optimal Case" window, and a set of messages are sent to the intended receivers to compute the time allocation and detection probabilities.

- Finally, the computation results are sent back to the main window.
The contents of the main window are now updated.

Fig. 2.9 The main window that shows the search strategy based on guessed distribution

Figure 2.10 shows the result of a sample search when the time allocation is specified by the user.

Fig. 2.10 The main window that shows the search strategy based on user-assigned time allocation

2.12.2 The CRC Descriptions and the Class Hierarchy Diagram

The software package consists of 10 classes. A complete CRC Description is as follows.

DemoSearch Class

Responsibilities

Display and update the main window
Request and store user input information
Send input data to the intended receiver
Display the results

Collaborators

ViewManager
KnownDistribution
UnknownDistribution

KnownDistribution Class

Responsibilities

Analyze the input information
Invoke the corresponding subroutine
Collect the computation results
Analyze the validity of the results

Collaborators

DemoSearch
ExactSearch
Simulation

UnknownDistribution Class

Responsibilities

Collaborators

Analyze the input information

Invoke the corresponding subroutine

Collect the computation results

Analyze the validity of the results

DemoSearch

SetTargetDistribution

SetTimeAllocation

ExactSearch Class

Responsibilities

Compute the Optimal strategies

Compute the Optimal time allocation

Collaborators

KnownDistribution

Simulation

Simulation Class

Responsibilities

Invoke a random number generator

Decide if a search is successful

Repeat for pre-specified times

Collaborators

KnownDistribution

ExactSearch

SetTargetDistribution Class

Responsibilities

Get a guessed target distribution

Computer the detection probability

Computer the time allocation

Collaborators

UnknownDistribution

SetTimeAllocation Class

Responsibilities

Get user assigned time allocation
Computer the detection probability

Collaborators

UnknownDistribution

DemoSearchHelp Class**Responsibilities**

Display the **Help** window

Collaborators

SearchTheory
DemoProgram

SearchTheory Class**Responsibilities**

Store the Search Theory text file
Send text file for display

Collaborators

DemoSearchHelp

DemoProgram Class**Responsibilities**

Store the Demo Program help file
Send text file for display

Collaborators

DemoSearchHelp

Figure 2.11 gives the class hierarchy diagram

Class Hierarchy Diagram

Fig. 2.11 The class hierarchy diagram

2.13 Conclusion

In this chapter we have reviewed some of the known results derived for the theory of search for a stationary target. Section 2.2 is about the Koopman's model, a prototype of the search theory when it was in its infancy. In section 2.3 we reviewed De Guenin's model, which is a continuous version of Koopman's model. De Guenin's algorithm is formally included in section 2.3. In section 2.4 we discussed the minimum expected cost problem and the related work of Staroverov. In section 2.5 we reviewed Mela's result that showed that an optimal detection search strategy is different from an optimal whereabouts search strategy. In section 2.6 we compared the local and the global optimization issues and reviewed some results of Chew, Kadane, and Stone. Section 2.7 discussed the use of Lagrange Multipliers and some related results. Section 2.8 was devoted to the sequential searching problems. Section 2.9 discussed the method which treated the stationary target problem as a special case of studying a moving target problem, and presented the results due to Gal. In section 2.10 we discussed the layered search model, which was introduced by Weisinger and Benkoski.

From a computational perspective, we presented two algorithms for computing optimal search strategies. Finally, in section 2.12 we derived the design and implementation issues involved in developing a software package to compute and simulate the optimal search strategies for both known and unknown target distributions.

Chapter 3

SEARCH FOR STATIONARY TARGETS: SOME NEW RESULTS

3.1 Introduction

In this chapter we consider the optimal search problem for stationary targets. As we saw in Chapter 2, all the known results dealt with this problem using the assumption that the target distribution (the probabilities for target location) is known to the searcher before the search strategy is planned. In this chapter we consider the situations when the target distribution is unknown, which forces the searcher to guess the probability of the target locations before he can determine a good search strategy. This scenario gives rise to some new problems such as those involving how to guess the probability for the target locations, and the computation of the

error incurred by using these guesses as opposed to the strategies based on *true* target distributions. In section 2 we consider upper and lower bounds for the detection probability of optimal search strategies. In section 3 we consider some selection criteria for determining the target distribution, and the error estimates for the case of two cells. In section 4 we generate these results to the N -cell case. Finally, in section 5 we consider search problems with non-regular detection functions. Our results are mainly obtained for discrete systems, but we believe these results can be easily extended to the continuous case.

3.2 Searching When the Target Distribution is Unknown

Suppose an object is hidden in one of N boxes, but the probability distribution is unknown. The first issue to be considered in designing a search strategy is to choose a target distribution $q(k)$, ($k = 1, 2, \dots, N$). Thus the primary question is one of knowing how to choose a *good* target distribution that will maximize the detection probability. In the case where the chosen distribution is not the true (unknown) one, we should know how large the error can be in designing a search strategy based on this information rather than based on the *true* distribution. The study of these problems involves

theories of stochastic systems and optimal control, and we will try to answer these questions in this section and the next.

In this section we consider the problem of estimating error. First we consider a special case where the detection function is exponential (i.e., $d(t) = 1 - e^{-t}$). We derive below an estimate on the detection probability for optimal search strategies.

Theorem 3.1. (Upper and Lower Bounds of Optimal Strategies)

Suppose an object is located in one of N cells: $C_1, C_2, \dots, C_N$, with an arbitrary probability distribution, $p(i), (i = 1, 2, \dots, N)$. We assume that the detection function is exponential:

$$b(i, z) = 1 - e^{-z}, (i = 1, 2, \dots, N).$$

Then the detection probability $P[f^]$ of the optimal search strategy f^* satisfies the following:*

$$1 - e^{-K/N} \leq P[f^*] \leq 1 - C_N e^{-K/N}.$$

Where $C_N = N[p(1) \cdots p(N)]^{\frac{1}{N}}$ and $K > 0$ is the total cost constraint.

Proof: For a probability distribution function $p(i), (i = 1, 2, \dots, N)$, we have

$$p(i) \geq 0, \quad \sum_1^N p(i) = 1.$$

If $p(j) = 0$, we may discard the corresponding cell C_j and consider the remaining $N - 1$ cells. Thus, without loss of generality we may assume that

$$p(i) > 0, \quad \sum_1^N p(i) = 1.$$

Now consider the Lagrangian

$$\ell(i, \lambda, z) = p(i)b(i, z) - \lambda z = p(i)(1 - e^{-z}) - \lambda z.$$

We have

$$\frac{d\ell(i, \lambda, z)}{dz} = p(i)e^{-z} - \lambda.$$

We set $p(i)e^{-z} - \lambda = 0$ and solve the equation to yield the optimal search plan as:

$$z_i = f_\lambda^*(i) = \ln \frac{p(i)}{\lambda}, \quad (i = 1, 2, \dots, N).$$

In other words z_i is the optimal allocation of effort in cell C_i . Moreover, z_i satisfies the cost constraint condition:

$$\sum_1^N z_i = \sum_1^N \ln \frac{p(i)}{\lambda} \leq K$$

That is

$$\lambda \geq [p(1) \cdots p(N)]^{\frac{1}{N}} e^{-K/N}.$$

Consider now $P[f^*]$. It can be seen that

$$\begin{aligned} P[f^*] &= \sum_{i=1}^N p(i)b(i, f_\lambda^*(i)) \\ &= \sum_{i=1}^N p(i)(1 - e^{-z_i}) \\ &= \sum_{i=1}^N p(i)(1 - e^{-\ln \frac{p(i)}{\lambda}}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^N p(i) \left(1 - \frac{\lambda}{p(i)}\right) \\
&= \sum_{i=1}^N (p(i) - \lambda) \\
&= 1 - N\lambda
\end{aligned}$$

Therefore

$$P[f^*] \leq 1 - N[p(1) \cdots p(N)]^{\frac{1}{N}} e^{-K/N}.$$

In the boundary case where

$$\sum_1^N z_i = \sum_1^N \ln \frac{p(i)}{\lambda} = K,$$

we have

$$\lambda = [p(1) \cdots p(N)]^{\frac{1}{N}} e^{-K/N}$$

Using the inequality relating the geometric and arithmetic means we see:

$$[p(1) \cdots p(N)]^{\frac{1}{N}} \leq \frac{p(1) + \cdots + p(N)}{N}.$$

Therefore

$$\begin{aligned}
P[f^*] &= 1 - N[p(1) \cdots p(N)]^{\frac{1}{N}} e^{-K/N} \\
&\geq 1 - N \left[\frac{p(1) + \cdots + p(N)}{N} \right] e^{-K/N} \\
&= 1 - N \left[\frac{1}{N} \right] e^{-K/N} \\
&= 1 - e^{-K/N}
\end{aligned}$$

This completes the proof of the theorem.

We shall now generalize the above result for the case when the detection function is regular, that is, $b(i, z)$ is continuously differentiable and the derivative $b'(i, \cdot)$ is a decreasing function with $b'(i, 0) > 0$ and $b'(i, \infty) = 0$. We can show that the error bounds satisfy the following result.

Theorem 3.2. (Upper and Lower Bounds of Optimal Strategies)

Suppose an object is located in one of N cells: $C_1, C_2, \dots, C_N$, with an arbitrary probability distribution, $p(i), (i = 1, 2, \dots, N)$. We assume the detection function is regular with $b'(i, \cdot)$ decreasing and satisfying $b'(i, 0) > 0$ and $b'(i, \infty) = 0$. Then the detection probability $P[f^]$ of the optimal search strategy f^* satisfies the following:*

$$b(i, 0) + \lambda K \leq P[f^*] \leq b(i, 0) + b'(i, 0)K.$$

Proof: Consider the Lagrangian

$$\ell(i, \lambda, z) = p(i)b(i, z) - \lambda z,$$

and the derivative

$$\frac{d\ell(i, \lambda, z)}{dz} = p(i)b'(i, z) - \lambda.$$

Since $b'(i, z)$ is monotone and continuous, the inverse function $b'^{-1}(i, \cdot)$ is well defined. Solving $p(i)b'(i, z_i) - \lambda = 0$ we have

$$b'(i, z_i) = \frac{\lambda}{p(i)}, \quad z_i = b'^{-1}\left(i, \frac{\lambda}{p(i)}\right).$$

Since $b(i, z)$ is continuously differentiable, by the Lagrange formula of calculus we have

$$b(i, z) = b(i, 0) + b'(i, \theta z)z, \quad 0 < \theta < 1, \quad z > 0.$$

The detection probability of the optimal search plan is thus:

$$\begin{aligned} P[f^*] &= \sum_{i=1}^N p(i)b(i, z_i) \\ &= \sum_{i=1}^N p(i)[b(i, 0) + b'(i, \theta z_i)z_i] \\ &= b(i, 0) + \sum_{i=1}^N p(i)[b'(i, \theta z_i)z_i] \end{aligned}$$

Since $\theta z_i < z_i$ and $b'(i, z)$ is decreasing, we have $b'(i, \theta z_i) \geq b'(i, z_i)$.

Hence

$$\begin{aligned} P[f^*] &\geq b(i, 0) + \sum_{i=1}^N p(i)[b'(i, z_i)z_i] \\ &= b(i, 0) + \sum_{i=1}^N p(i)\left(\frac{\lambda}{p(i)}\right)z_i \\ &= b(i, 0) + \sum_{i=1}^N \lambda z_i \end{aligned}$$

In the boundary case we have

$$\sum_{i=1}^N z_i = K,$$

and hence

$$P[f^*] \geq b(i, 0) + \lambda K,$$

where λ is determined by

$$\sum_{i=1}^N b'^{-1}\left(i, \frac{\lambda}{p(i)}\right) = K.$$

On the other hand, since $b'(i, \theta z_i) \leq b'(i, 0)$, we have

$$\begin{aligned}
 P[f^*] &= \sum_{i=1}^N p(i) b(i, z_i) \\
 &= b(i, 0) + \sum_{i=1}^N p(i) [b'(i, \theta z_i) z_i] \\
 &\leq b(i, 0) + \sum_{i=1}^N p(i) [b'(i, 0) z_i] \\
 &= b(i, 0) + b'(i, 0) \sum_{i=1}^N p(i) z_i \\
 &\leq b(i, 0) + b'(i, 0) \sum_{i=1}^N z_i \\
 &\leq b(i, 0) + b'(i, 0) K.
 \end{aligned}$$

This completes the proof of the theorem. ■

Corollary 3.3. *With the assumptions of Theorem 3.2, if we further assume that $b(i, 0) = 0$, then we have*

$$\lambda K \leq P[f^*] \leq b'(i, 0) K,$$

where $K > 0$ is the total cost constraint. ■

3.3 Some Selection Criteria and Error Estimate

There are several aspects that have to be considered when we assume a probability distribution to the search problem with an unknown target distribution. To simplify our derivation, in this section we consider only a simple case of two cells and the exponential detection function. In the next section we will generalize these results to the case of N cells.

Suppose in a search problem the target distribution is $(a, 1 - a)$, but since this is unknown to the searcher, he assumes $(b, 1 - b)$ as the target distribution. Intuitively we believe that the optimal search plan f_a^* , which is based on the *true* target distribution, should have bigger probability of detection than that of the search plan f_b^* , based on the *assumed* target distribution. Our first result of this section establishes the correctness of this intuition.

Theorem 3.4. *Suppose that the true target distribution is $(a, 1 - a)$ for two-cell search problem. If the user assumes the target distribution to be $(b, 1 - b)$ where $b \neq a$, then*

$$P[f_b^*] \leq P[f_a^*].$$

Proof: For the assumed probability distribution function $p(1) = b, p(2) = 1 - b$, the Lagrangian is

$$\ell(i, \lambda, z) = p(i)b(i, z) - \lambda z = p(i)(1 - e^{-z}) - \lambda z.$$

Thus

$$\frac{d\ell(i, \lambda, z)}{dz} = p(i)e^{-z} - \lambda.$$

We set $p(i)e^{-z} - \lambda = 0$, and solve this equations to yield the optimal search plan for the assumed target distribution $(b, 1 - b)$ as follows:

$$z_{1,b} = \ln \frac{p(1)}{\lambda_b} = \ln \frac{b}{\lambda_b}, \quad z_{2,b} = \ln \frac{p(2)}{\lambda_b} = \ln \frac{1-b}{\lambda_b}.$$

We now consider the scenario that the total time to search the target is K .

Then

$$\sum_1^N z_{i,b} = \sum_1^N \ln \frac{p(i)}{\lambda_b} = K,$$

yielding

$$\lambda_b = [b(1-b)]^{\frac{1}{2}} e^{-K/2}.$$

Now since the real probability distribution is $(a, 1 - a)$, the detection probability for search plan $z_{1,b}$ and $z_{2,b}$ is given by

$$\begin{aligned} P[f_b^*] &= \sum_{i=1}^N p(i)(1 - e^{-z_{i,b}}) \\ &= a(1 - e^{-z_{1,b}}) + (1 - a)(1 - e^{-z_{2,b}}) \\ &= a(1 - \frac{\lambda_b}{b}) + (1 - a)(1 - \frac{\lambda_b}{1-b}) \\ &= 1 - \lambda_b \left[\frac{a}{b} + \frac{1-a}{1-b} \right] \\ &= 1 - \frac{a+b-2ab}{\sqrt{b(1-b)}} e^{-K/2}. \end{aligned}$$

As opposed to this the detection probability for the search plan based on *real* probability distribution $(a, 1 - a)$ is

$$P[f_a^*] = a(1 - e^{-z_{1,a}}) + (1 - a)(1 - e^{-z_{2,a}})$$

$$\begin{aligned}
&= a\left(1 - \frac{\lambda_a}{a}\right) + (1-a)\left(1 - \frac{\lambda_a}{1-a}\right) \\
&= 1 - 2\lambda_a \\
&= 1 - 2\sqrt{a(1-a)}e^{-K/2}.
\end{aligned}$$

Thus the error is given by

$$\begin{aligned}
P[f_a^*] - P[f_b^*] &= 1 - 2\sqrt{a(1-a)}e^{-K/2} - \left[1 - \frac{a+b-2ab}{\sqrt{b(1-b)}}e^{-K/2}\right] \\
&= \left[\frac{a+b-2ab}{\sqrt{b(1-b)}} - 2\sqrt{a(1-a)}\right]e^{-K/2} \\
&= \frac{a+b-2ab-2\sqrt{a(1-a)b(1-b)}}{\sqrt{b(1-b)}}e^{-K/2} \\
&= \frac{a+b-2\sqrt{ab}[\sqrt{ab} + \sqrt{(1-a)(1-b)}]}{\sqrt{b(1-b)}}e^{-K/2}
\end{aligned}$$

Now consider the factor in brackets in the numerator

$$\begin{aligned}
\sqrt{ab} + \sqrt{(1-a)(1-b)} &= \frac{a+b-(\sqrt{a}-\sqrt{b})^2}{2} \\
&\quad + \frac{(1-a)+(1-b)-(\sqrt{1-a}-\sqrt{1-b})^2}{2} \\
&= \frac{1}{2}[2 - (\sqrt{a}-\sqrt{b})^2 - (\sqrt{1-a}-\sqrt{1-b})^2] \\
&= 1 - \frac{1}{2}[(\sqrt{a}-\sqrt{b})^2 + (\sqrt{1-a}-\sqrt{1-b})^2] \\
&\leq 1.
\end{aligned}$$

Hence

$$\begin{aligned}
P[f_a^*] - P[f_b^*] &\geq \frac{a+b-2\sqrt{ab}}{\sqrt{b(1-b)}}e^{-K/2} \\
&= \frac{(\sqrt{a}-\sqrt{b})^2}{\sqrt{b(1-b)}}e^{-K/2} \\
&\geq 0.
\end{aligned}$$

This completes the proof of the theorem. ■

From the proof of this theorem we have the following corollaries:

Corollary 3.5. (First Criteria for Choosing Target Distributions)

In the case of two cells and the exponential detection function, one criteria for choosing the probability distribution $(b, 1 - b)$ is to maximize the following expression

$$P[f_b^*] = 1 - \frac{a + b - 2ab}{\sqrt{b(1 - b)}} e^{-K/2}$$

over a priority set, which is the set of those values of a that are the most likely ones. ■

Mathematically, we may define the priority set $\mathcal{A}$ to be the set of values which a can take with probability

$$P[a = a_0 \in \mathcal{A}] \geq p_0,$$

where p_0 is a fixed number. Usually $0.5 \leq p_0 \leq 1$.

Corollary 3.6. *In the case of two cells and exponential detection function, if the real target distribution is $(a, 1 - a)$ and chosen distribution is $(b, 1 - b)$, then the error in the detection probability of the optimal search plan is given by*

$$E(a, b) = \frac{a + b - 2\sqrt{ab}[\sqrt{ab} + \sqrt{(1-a)(1-b)}]}{\sqrt{b(1-b)}} e^{-K/2}.$$

■

This leads us to the second criterion for selecting a good "b".

Corollary 3.7. (Second Criterion for Choosing Target Distributions)

The target distribution should be chosen so as to minimize $E(a, b)$ over a priority set of a , where $E(a, b)$ is given by Corollary 3.6.

■

It is clear that the smaller the priority set is, the more accurate the assignment of target distribution can be. In other words, $E(a, b)$ is smaller when the priority set is smaller. Thus, in practice, the searcher should use all the sources of information to reduce the size of priority set, and to thus limit the range of possible target distributions.

In the following we show that if the priority set is an interval with length $\delta > 0$, then the error $E(a, b)$ tends to zero when δ tends to zero.

Corollary 3.8. (Asymptotical property of the error estimate)

In a search of two cells with exponential detection function, suppose that the target probability distribution is $(a, 1 - a)$, and a is a number which is evenly distributed in an interval of length $\delta > 0$. Let a_0 be the center of this interval. Then for any given number $\epsilon > 0$, we can choose

$\delta > 0$ such that if target distribution is located in the interval $[a_0 - \delta, a_0 + \delta]$, the error of the detection probability for the optimal search plan is less than ϵ .

Proof: Since a is evenly distributed and a_0 is the center of the interval, we have:

$$|a - a_0| < \delta$$

with probability 1. If we choose $b = a_0$, by Corollary 3.7 the error of the detection probability for the optimal search plan is

$$\begin{aligned} E(a, b) &= \frac{a + b - 2\sqrt{ab}[\sqrt{ab} + \sqrt{(1-a)(1-b)}]}{\sqrt{b(1-b)}} e^{-K/2} \\ &= \frac{a + b - 2\sqrt{ab}[1 - \frac{(\sqrt{a} - \sqrt{b})^2}{2} - \frac{(\sqrt{1-a} - \sqrt{1-b})^2}{2}]}{\sqrt{b(1-b)}} e^{-K/2} \\ &= \frac{2(\sqrt{a} - \sqrt{b})^2 + \sqrt{ab}[(\sqrt{a} - \sqrt{b})^2 + (\sqrt{1-a} - \sqrt{1-b})^2]}{2\sqrt{b(1-b)}} e^{-K/2} \\ &= \frac{(\sqrt{a} - \sqrt{a_0})^2(2 + \sqrt{aa_0}) + \sqrt{aa_0}(\sqrt{1-a} - \sqrt{1-a_0})^2}{2\sqrt{a_0(1-a_0)}} e^{-K/2}. \end{aligned}$$

Now since $|a - a_0| < \delta$, i.e., $a_0 - \delta < a < a_0 + \delta$, we obtain the following estimates:

$$\begin{aligned}
 |a - a_0| &= |\sqrt{a} - \sqrt{a_0}|(\sqrt{a} + \sqrt{a_0}) < \delta, \\
 |\sqrt{a} - \sqrt{a_0}| &< \frac{\delta}{\sqrt{a} + \sqrt{a_0}} < \frac{\delta}{\sqrt{a_0 - \delta} + \sqrt{a_0}} \\
 \sqrt{aa_0} &< \sqrt{(a_0 + \delta)a_0}, \\
 |a - a_0| &= |(1 - a) - (1 - a_0)| \\
 &= |\sqrt{1 - a} - \sqrt{1 - a_0}|(\sqrt{1 - a} + \sqrt{1 - a_0}) \\
 &< \delta, \\
 |\sqrt{1 - a} - \sqrt{1 - a_0}| &< \frac{\delta}{\sqrt{1 - a} + \sqrt{1 - a_0}} \\
 &< \frac{\delta}{\sqrt{1 - a_0 - \delta} + \sqrt{1 - a_0}}.
 \end{aligned}$$

Hence

$$E(a, a_0) < c_0 \delta^2$$

for some constant $c_0 > 0$. This completes the proof of the corollary. ■

Corollary 3.8 implies our claim that a smaller priority set corresponds to a better target distribution estimate.

If we know the mean value (mathematical expectation) of the target distribution a , we may treat a as a random variable that is evenly distributed in an interval centered at $E(a)$. As we have seen in Corollary 3.8, if we take $E(a)$ as the approximate of the target distribution, then the length

of the interval will affect the accuracy of the detection probability, $P[f^*]$.

We now consider another interesting question. Suppose that we know the mean value of the detection probability, $E[P(f^*)]$. From this, can we evaluate the corresponding mean value of the target distribution? In other words, if target distribution is $E(a)$, does the detection probability of the optimal search plan equal to $E[P(f^*)]$? In general this is not true, as the following example shows.

Example 3.9.

In the case of two cells and the exponential detection function, assume that the target distribution is again a random variable with the distributions:

$$P[p_1 = a_i] = P[p_2 = 1 - a_i] = q_i, \quad (i = 1, 2, \dots, m).$$

Let $z_k (k = 1, 2)$ denote the time spent to search cell k in the optimal search plan, and $P[f^*]$ denote the detection probability of the optimal search plan. Suppose the cost is bounded by $K = 4$. Using the formula derived earlier we can tabulate the value of $P[f^*]$ as follows.

q_1	p_1	z_1	$P[f^*]$
0.1	0.50	2.00	0.86
0.15	0.60	2.20	0.87
0.20	0.70	2.42	0.88
0.35	0.80	2.69	0.89
0.15	0.90	3.10	0.92
0.05	1.00	4.00	0.99

From above data we see that $E[p_1] = 0.745$ but $E[P(f^*)] = 0.892$. The detection probability corresponding to $E[p_1]$ is only 0.8779, which is less than $E[P(f^*)]$. In fact, $P[f^*]$ is independent of q_1 , and thus the mean value of p_1 is not related to the mean value of $P[f^*]$.

■

3.4 Selection Criteria and Error Estimates for General Cases

In this section we generalize the results of section 3.2 to the case of N cells. Suppose that in a search problem the target distribution is $(a_1, a_2, \dots, a_N)$, but that this information is unknown to the searcher, and that the searcher chooses $(b_1, b_2, \dots, b_N)$ as the approximation of the target

distribution. Again, intuitively we believe that the optimal search plan f_a^* , which is based on the *true* target distribution, should have a superior detection probability than that of the search plan f_b^* , which is based on the assumed target distribution. We prove below that this is true.

Theorem 3.10. *In searching N cells with an exponential detection function, suppose that the true target distribution is $(a_1, a_2, \dots, a_N)$ and that the searcher assumes $(b_1, b_2, \dots, b_N)$ as the approximation of the target distribution. Let $P[f_a^*]$ denote the detection probability of the optimal search plan f_a^* , and $P[f_b^*]$ be the detection probability of the optimal search plan f_b^* . Then*

$$P[f_b^*] \leq P[f_a^*].$$

Proof: For the approximation of the target probability distribution function $(b_1, b_2, \dots, b_N)$, the corresponding Lagrangian is

$$\ell(i, \lambda, z) = p(i)b(i, z) - \lambda z = b_i(1 - e^{-z}) - \lambda z.$$

Thus,

$$\frac{d\ell(i, \lambda, z)}{dz} = b_i e^{-z} - \lambda.$$

We again set $p(i)e^{-z} - \lambda = 0$ and solve the equations to yield the optimal search plan for the assumed target distribution $(b_1, b_2, \dots, b_N)$ as follows:

$$z_{i,b} = \ln \frac{b_i}{\lambda_b} \quad (i = 1, 2, \dots, N).$$

If the total time allocated to search for the target is K , we have

$$\sum_1^N z_{i,b} = \sum_1^N \ln \frac{b_i}{\lambda_b} = K.$$

That is

$$\lambda_b = [(\prod_{i=1}^N b_i)^{\frac{1}{N}}]^{\frac{1}{N}} e^{-K/N}$$

Now since the real probability distribution is $(a_1, a_2, \dots, a_N)$, the detection probability for search plan $z_{i,b}$ is given by

$$\begin{aligned} P[f_b^*] &= \sum_{i=1}^N p(i)(1 - e^{-z_{i,b}}) \\ &= \sum_{i=1}^N a_i(1 - e^{-z_{i,b}}) \\ &= \sum_{i=1}^N a_i(1 - \frac{\lambda_b}{b_i}) \\ &= 1 - \lambda_b [\sum_{i=1}^N \frac{a_i}{b_i}] \\ &= 1 - e^{-K/N} [(\prod_{i=1}^N b_i)^{\frac{1}{N}}]^{\frac{1}{N}} \sum_{i=1}^N \frac{a_i}{b_i} \end{aligned}$$

Thus the detection probability for the search plan based on true probability distribution $(a_1, a_2, \dots, a_N)$ is

$$\begin{aligned} P[f_a^*] &= \sum_{i=1}^N p(i)(1 - e^{-z_{i,a}}) \\ &= \sum_{i=1}^N a_i(1 - \frac{\lambda_a}{a_i}) \\ &= 1 - N\lambda_a \\ &= 1 - Ne^{-K/N} [(\prod_{i=1}^N a_i)^{\frac{1}{N}}]^{\frac{1}{N}} \end{aligned}$$

Thus the error estimate is given by

$$P[f_a^*] - P[f_b^*] = 1 - N\lambda_a - [1 - \lambda_b [\sum_{i=1}^N \frac{a_i}{b_i}]]$$

$$\begin{aligned}
& \lambda_b \left| \sum_{i=1}^N \frac{a_i}{b_i} - N \lambda_a \right| \\
& \sum_{i=1}^N \frac{\lambda_b a_i - \lambda_a b_i}{b_i} \\
& = K/N \sum_{i=1}^N \frac{(\prod_{i=1}^N b_i)^{\frac{1}{N}} a_i - (\prod_{i=1}^N a_i)^{\frac{1}{N}} b_i}{b_i}.
\end{aligned}$$

Now consider the summation in the above expression

$$\begin{aligned}
& \sum_{i=1}^N \frac{(\prod_{i=1}^N b_i)^{\frac{1}{N}} a_i - (\prod_{i=1}^N a_i)^{\frac{1}{N}} b_i}{b_i} \\
& \frac{\sum_{i=1}^N [(\prod_{i=1}^N b_i)^{\frac{1}{N}} a_i \prod_{j \neq i} b_j - (\prod_{i=1}^N a_i)^{\frac{1}{N}} \prod_{i=1}^N b_i]}{\prod_{i=1}^N b_i} \\
& \frac{\sum_{i=1}^N [a_i \prod_{j \neq i} b_j - (\prod_{i=1}^N a_i)^{\frac{1}{N}} (\prod_{i=1}^N b_i)^{\frac{N-1}{N}}]}{(\prod_{i=1}^N b_i)^{\frac{N-1}{N}}} \\
& \frac{\sum_{i=1}^N a_i \prod_{j \neq i} b_j - N (\prod_{i=1}^N a_i)^{\frac{1}{N}} (\prod_{i=1}^N b_i)^{\frac{N-1}{N}}}{(\prod_{i=1}^N b_i)^{\frac{N-1}{N}}}.
\end{aligned}$$

It is easy to verify that

$$\left(\prod_{i=1}^N a_i \right) \left(\prod_{i=1}^N b_i \right)^{N-1} = \prod_{i=1}^N (a_i \prod_{j \neq i} b_j).$$

Hence

$$\begin{aligned}
& \sum_{i=1}^N \frac{(\prod_{i=1}^N b_i)^{\frac{1}{N}} a_i - (\prod_{i=1}^N a_i)^{\frac{1}{N}} b_i}{b_i} \\
& \frac{\sum_{i=1}^N a_i \prod_{j \neq i} b_j - (k+1) (\prod_{i=1}^N a_i \prod_{j \neq i} b_j)^{\frac{1}{N}}}{(\prod_{i=1}^N b_i)^{\frac{N-1}{N}}}.
\end{aligned}$$

We now need to show the above expression is non-negative for all $a_i \geq 0$, $b_i \geq 0$.

0. Let

$$x_i = (a_i \prod_{j \neq i} b_j)^{\frac{1}{N}}.$$

Hence

$$\begin{aligned} & \sum_{i=1}^N a_i \prod_{j \neq i} b_j = N \left(\prod_{i=1}^N a_i \prod_{j \neq i} b_j \right)^{\frac{1}{N}} \\ & = \sum_{i=1}^N x_i^N = (k+1) \prod_{i=1}^N x_i. \end{aligned}$$

We need to prove that $\sum_{i=1}^N x_i^N = N \left(\prod_{i=1}^N x_i \right) \geq 0$ for each $x_i \geq 0$. Indeed, this inequality is proved inductively.

For $N = 1$, $x_1 = x_1 \geq 0$, which proves a basis case.

For $N = 2$, $x_1^2 + x_2^2 - 2x_1x_2 \geq 0$, again proving a basis case.

Now suppose $\sum_{i=1}^{N-1} x_i^{N-1} = (N-1) \prod_{i=1}^{N-1} x_i \geq 0$ for any $x_i \geq 0$.

Assume $x_1 > 0$, then for arbitrary fixed $x_2, \dots, x_N$, ($N > 2$), we have

$$f(x_1) = x_1^N + \sum_{i=2}^N x_i^N = N \left(\prod_{i=2}^N x_i \right) x_1$$

Since

$$f'(x_1) = Nx_1^{N-1} = N \left(\prod_{i=2}^N x_i \right)$$

Solving $f'(x_1) = 0$, we have

$$x_1 = \left(\prod_{i=2}^N x_i \right)^{\frac{1}{N-1}}.$$

Since $f''(x_1) = N(N-1)x_1^{N-2} > 0$, we know that $f(x)$ attained its minimum at the point x_1 . That is

$$\begin{aligned} f(x_1) &= \left(\prod_{i=2}^N x_i \right)^{\frac{N}{N-1}} + \sum_{i=2}^N x_i^N = N \left(\prod_{i=2}^N x_i \right) \left(\prod_{i=2}^N x_i \right)^{\frac{1}{N-1}} \\ &= \left(\prod_{i=2}^N x_i \right)^{\frac{N}{N-1}} + \sum_{i=2}^N x_i^N = N \left(\prod_{i=2}^N x_i \right)^{\frac{N}{N-1}} \\ &= \sum_{i=2}^N x_i^N = (N-1) \left(\prod_{i=2}^N x_i \right)^{\frac{N}{N-1}} \end{aligned}$$

Let $y_i = x_{i+1}^{\frac{N}{N-1}}$, then the last line above becomes

$$\sum_{i=1}^{N-1} y_i^{N-1} - (N-1) \left(\prod_{i=1}^{N-1} y_i \right) \geq 0$$

by the inductive hypothesis. Thus, by induction we have proved that

$$\sum_{i=1}^N x_i^N - N \prod_{i=1}^N x_i \geq 0$$

for any $x_i \geq 0$. This implies that $P[f_a^*] - P[f_b^*] \geq 0$. ■

From Theorem 3.10 we have the following corollaries:

Corollary 3.11. (First Criterion for Choosing Target Distributions)

In the search of N cells with the exponential detection function, one criteria for choosing the probability distribution $(b_1, \dots, b_N)$ is to maximize the following expression

$$P[f_b^*] = 1 - e^{-K/N} \sum_{i=1}^N \left(\prod_{j=1}^N b_j \right)^{\frac{1}{N}} \frac{a_i}{b_i}$$

over a priority set, which is the set of most likely values for a . ■

Corollary 3.12.

In the case of N cells and an exponential detection function, if the real target distribution is $(a_1, \dots, a_N)$ and the chosen distribution is

$(b_1, \dots, b_N)$, then the error in the detection probability of the optimal search plan is given by

$$E(a, b) = \frac{1}{K} \sum_{i=1}^N \frac{(\prod_{t=1}^N b_t)^{\frac{1}{N}} a_i - (\prod_{t=1}^N a_t)^{\frac{1}{N}} b_i}{b_i}$$

■

The above results leads to the following criterion.

Corollary 3.13. (Second Criterion for Choosing Target Distributions)

The target distribution should be chosen so as to minimize $E(a, b)$ over a priority set of a where $E(a, b)$ is given by Corollary 3.12.

■

3.5 Optimal Search Problem with Non-Regular Detection Functions

In the book *Theory of Optimal Search* [186] Stone generalized the early results of optimal search [108] to a larger class of analytic functions that he called the set of "regular" detection functions. As per his definition, a function $b(i, z)$ is regular if $b(i, \cdot)$ is continuously derivable for each i and the derivative $b'(i, \cdot)$ is a monotone decreasing function with $b'(i, 0) = \infty$ and $b'(i, \infty) = 0$. It is easy to see that a regular function has an invertible

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derivative, a property which is essential to Stone's results. Unfortunately, this definition imposed strong limitations on the applications of search theory.

Consider the condition that a regular detection function must be monotone increasing. This implies that for each cell C_i , the longer the time spent to search for an object, the greater the likelihood of the search process finding it in that cell. This is not quite right in many situations. For example, suppose we are searching an airplane in the sky by eyes. After a few minutes if we still have not found the target, we are sure that the target does not exist or is not visible, and so any further searching in this direction only has the detection probability 0. Clearly, in this case the detection function $b(i, z)$ is not regular, since $b(i, z)$ increases at first and after a point it decreases to 0. Our hope is that a theory can be developed which considers "bell shaped" functions, such as the normal density function. Furthermore, since many real-life phenomena can be approximated by a normal distribution, such a choice can, indeed, be justified.

In this section, we shall revisit the search problem for the scenario when the detection function is non-regular, and is shaped like the Gaussian curve.

Theorem 3.14. (Optimal Search for a class of Non-Regular Detection Function)

Consider the discrete search of N cells. Suppose that at any moment only one cell can be reached, and that the probability of

detecting the target in cell C_i is given by the following detection functions

$$b(i, z) = \begin{cases} e^{-\alpha_i(z-z_i)^2}, & z > 0, \quad \alpha_i > 0, \quad (i = 1, 2, \dots, N) \\ 0, & z \leq 0. \end{cases}$$

Then the optimal search plan for any given cost constraint exists and is given by the solution of (3.1) below.

$$\begin{cases} \beta_i e^{\alpha_i \xi_i^2} - \xi_i = 0, & (i = 1, 2, \dots, N) \\ \sum_{i=1}^N (z_i - \xi_i) = K. \end{cases} \quad (3.1)$$

Proof. By invoking the form of the detection function we can express the Lagrangian as:

$$\ell(i, \lambda, z) = p(i) e^{-\alpha_i(z-z_i)^2} - \lambda z, \quad (z > 0),$$

where $p(i)$ is a probability distribution that has been chosen to approximate the target distribution. Utilizing the Lagrange Multiplier method, we want to maximizing $\ell(i, \lambda, z)$ with respect to z . Let

$$\frac{d\ell}{dz} = p(i) e^{-\alpha_i(z-z_i)^2} [-2\alpha_i(z-z_i)] - \lambda.$$

It is clear that

$$\frac{d\ell}{dz} < 0, \quad (z \geq z_i),$$

and hence

$$\ell(i, \lambda, z) \leq \ell(i, \lambda, z_i) \quad z \geq z_i.$$

This implies that the maximum of $\ell(i, \lambda, z)$ must be achieved in the interval $[0, z_i]$. If we assume that $z \in [0, z_i]$, we can compute the maximum by setting

$$\frac{d\ell}{dz} = 0$$

This yields

$$\alpha_i(z_i - z) = \frac{\lambda}{2p(i)} e^{\alpha_i(z_i - z)^2}$$

Let K be the cost constraint and $\xi_i = z_i - z$. Then by denoting $\beta_i = \frac{\lambda}{2p(i)\alpha_i}$, we obtain:

$$\begin{cases} \beta_i e^{\alpha_i \xi_i^2} - \xi_i = 0, & (i = 1, 2, \dots, N) \\ \sum_{i=1}^N (z_i - \xi_i) - K = 0 \end{cases} \quad (3.1)$$

If (3.1) does not have a solution, then the maximum of Lagrangian must occur at one of the end points of interval $[0, z_i]$. If (3.1) has a solution, we can solve this equation to obtain $(\xi_1, \xi_2, \dots, \xi_N, \lambda)$, and then we can find the optimal search plan $(z_1, z_2, \dots, z_N, K)$.

■

The following example illustrates the computation.

Example 3.15. Suppose we want to search two drawers for a key. The key is located in drawer 1 with probability $\frac{2}{3}$, and the probability for the key being in drawer 2 is $\frac{1}{3}$. Let the detection function be:

$$b(j, z) = e^{-(z-2.5)^2}, \quad (z > 0, j = 1, 2), \quad \text{with} \quad b(j, z) = 0, \quad (z = 0)$$

The total time allowed for the search is 4 minutes. Our goal is to find the optimal time allocation that maximizes the detection probability.

From Theorem 3.14 we have the following equations

$$\begin{cases} \frac{3}{4}\lambda e^{\xi_1^2} - \xi_1 = 0, \\ \frac{3}{2}\lambda e^{\xi_2^2} - \xi_2 = 0, \\ \xi_1 + \xi_2 - 1 = 0. \end{cases} \quad (3.2)$$

We can simplify equations (3.2) as follows:

$$\begin{cases} (1 - \xi_1)e^{2\xi_1 - 1} - 2\xi_1 = 0, \\ \xi_1 + \xi_2 - 1 = 0. \end{cases} \quad (3.3)$$

Solving this equation we have the optimal time allocation:

$$\begin{cases} z_1 = 2.28 \\ z_2 = 1.72. \end{cases}$$

If the detection function is the traditional *exponential* function, it is easy to verify that the time allocations would be $k_1 = 2.35$ and $k_2 = 1.65$. Observe that the time spent in the first drawer falls from 2.35 to 2.28 as a result of the non-regularity of the detection function $b(\cdot, \cdot)$. This is because it is futile to continue to search in the first drawer beyond a point.

3.6 Conclusion

In this chapter we have studied the scenario when the target distribution for searching a stationary target is unknown. In section 2 we obtained upper and lower bounds for the detection probability of the optimal search. Based on this result we presented several selection criteria for guessing a good estimate for the target distribution, and have derived formula for computing the error estimates when the guessed distribution is used to plan the search strategy. These results are found in section 3 and 4. The results have also been generalized for the N-cell case. Finally, in section 5 we considered search problems for non-regular detection functions having a bell-shaped form. An example illustrating this computation is included.

We would like to emphasize that all of the results presented in this chapter are novel to the area of optimal search.

Chapter 4

SEARCHING FOR MOVING TARGETS: A BRIEF REVIEW

4.1 One-sided Search

4.1.1 Introduction to moving targets

Prior to 1975, there were few results for optimally searching for a moving target. This is reflected in the first edition of Stone's book – Theory of Optimal search [186], which presents a well-developed theory of the search problem for a stationary target, but involves few moving-target solutions. Indeed, the solutions which were found were for a special types of moving target problems, such as the two-celled Markovian motion problem and the

conditionally deterministic target motion. For the case of target motions described by a continuous time and space Markov process and when the detection function is exponential, some necessary conditions for an optimal detection search were also found (see chapters 8 and 9 of Stone [186]). The results obtained during this period were often presented in a classical theorem-proof format with careful attention paid to the mathematical assumptions. An important part of the research effort spent during this period was devoted to understanding the mathematical nature of the problems. This resulted in the discovery of very general conditions necessary for yielding optimal search plans.

The period from 1975 to 1985, can be termed as "the Algorithmic Era". It was characterized by an intense effort to solve moving target problems motivated by the increasing availability of relatively cheap and powerful computers. These characteristics shifted the emphasis in search theory from mathematical and analytic solutions to algorithmic ones.

The literature on moving targets falls into two groups listed below.

- articles which address special types of target motion that are amenable to analysis, and
- articles which build on an observation made by Brown in 1977 and aim to develop general necessary and sufficient conditions for moving-target problems.

Two special types of target motions have been studied in the literature. The first is a conditionally deterministic motion with a factorable

Jacobian. A problem in this class can often be reduced to an equivalent problem involving a stationary target, and can consequently be solved with stationary-target techniques. Stone and Richardson [182], Stone [178], Pursiheimo [150], and Iida [89-91] have studied moving-target problems of this type.

The second special type of motion that has been studied is Markovian. In the discrete space, Pollock [148] Dobbie [47], and Schweitzer [166] studied the two-cell problem. Assuming perfect detection, Nakai [129] investigated the three-cell problem. Kan [100] considered N cells -- but for a limited class of transition matrices. In the continuous case, necessary and sufficient conditions for optimal search against a Markov Process were determined by Hellman [84] Saretsalo [165], and Hibey [87]. However these conditions have not been used to generate optimal plans.

In 1977, Brown investigated the search for a target with Markov motion in discrete time and space using an exponential detection function. He observed that if one selects a time t and conditions of failure for all time other than t , then the optimal plan maximizes the probability of detection for the stationary target problem that results from this conditioning. In particular, Brown provided a necessary and sufficient condition for an optimal search strategy and an efficient iterative algorithm for generating optimal plans. While Washburn [204] studied a discrete effort analogue of Brown's, Stone [179] derived the general necessary and sufficient conditions which extended Brown's results for an arbitrary stochastic process involving any mixture of discrete and continuous processes in both time and space.

Up to this point, the investigations primarily dealt with the probability of detection. Washburn [204] considered versions of Brown's results [30] which dealt with other payoffs, and presented an upper bound useful in implementing these approaches [203]. In turn Stromquist and Stone [187] extended this work to a wider class of constrained optimization problems. The resulting theory of search for a moving target has been termed "generalized search optimization". These techniques are discussed in Stone [180]. Some variations on the basic search problem such as the survivor search problems in Disenza and Stone [45] and Stone [177] can be viewed from within this general framework.

4.1.2 Constrained Searcher Motion

The articles reviewed above place few restrictions on the searcher's motion. Now we move to the problem of constructing either an optimal searcher path or a sequential search plan that reflects travel costs. These problems are considerably harder than those discussed so far. In fact, Trummel and Weisinger [194] and Wegener [208] proved that this type of search problem is NP-hard.

The simplest version of the optimal searcher path problem is the linear search problem. Here the searcher and the (generally stationary) target

are restricted to one dimension. Gluss [80] studied a discrete version of this problem. Franck [63,64], Beck [17,18], Beck and Warren [19], and Balkhi [11] considered the search for a particle on the real line, and provided various existence theorems. Rousseau [159] used numerical methods to generate the optimal searcher paths against a target whose location is drawn from either a Normal, Student, Logistic, or a Laplace distribution. Beck *et. al.* [20] investigated the optimal searcher path for uniform, triangular, and normal distributions. McCabe [126] showed the existence of search paths against a uni-dimensional random walker. These paths have finite expected time to detection. Washburn [203] investigated a different type of one dimensional search -- the construction of patrol barriers.

The more general optimal searcher path problem involves higher dimensions and a moving target. In the discrete case, Eagle [51] described a dynamic programming technique, while Stewart [172] and Eagle and Yee [52] reported branch-and-bound approaches. In the continuous case, Lukka [120-122], Vereshchagin *et. al.* [196], Ohsumi [140-142], Mangel [123], and Sunahara *et. al.* [188,189] used optimal control theory to deal with the problem. As the complexity results mentioned above suggest, no general, effective algorithm for generating optimal searcher paths is known. We will discuss such a technique in more detail in the following chapter.

Closely related to the optimal searcher path problem is the problem of discrete sequential search with switch costs. Here a matrix of switch costs between cells is provided. The objective is to minimize the total expected cost which includes searching and switching. Gilbert [75] and Kisi

[104] considered this problem for two cells, and Onaga [139] and Lossner and Wegener [119] extend the analysis to N cells. Matula [125] considered connections involving ultimately periodic plans.

4.1.3 Some Extensions to Searching Non-Stationary Target

The basic search problem can be extended in, at least, four directions as listed below. In each case the basis for the extension is specified.

1. **Objective.** The objective function or rules for search can be changed. For example we can consider the strategies for whereabouts search and for "search and stop" criteria.
2. **Targets.** The assumption of a single target present during the entire search can be dropped. Examples for such extensions include multiple targets, false targets, and targets of limited life or unknown arrival time.
3. **Detection model.** The single, known sensor can be replaced to include, for example, multiple sensors with uncertain detection capabilities.
4. **Unknown initial target distribution.** In this case we would encounter problems which are extensions to the problems studied in chapter 3.

In this subsection we shall address some of these extensions.

A typical search objective is to maximize the probability of detection. We can call the resulting problem a "detection search". A related objective is to localize the target to within one of a finite number of cells. Here the searcher may succeed by detecting the target or, failing that, by guessing the target's location. This is a whereabouts search. A third type of objective arises in surveillance. In this case, the problem is to maximize the probability of detecting the target at a specified time or in a specified region. This is a surveillance search. Tognetti [193] and Kadane [95] studied whereabouts search for stationary targets (see Chapter 2), and Stone and Kadane [181] generalized these results to moving targets. In particular, they showed that solving the whereabouts search problem is equivalent to solving a finite number of optimal detection search problems. Assuming a Markov process for target motion, Tierney and Kadane [192] derived necessary conditions for optimality and an algorithm for constructing plans in the surveillance search problem.

Search and stop problems extend the cost structure of the basic problem by adding a reward for finding the target. Now the objective is to find a plan which specifies how to search and when to stop in order to maximize the total expected return (reward minus search costs). Consider first the discrete effort version of search and stop. Chew [34] studied this problem when all looks (searches) have the same cost, the reward is independent of target location, and the glimpse probability of detection is constant for each cell. These results were extended by Ross [158]. In particular, he showed the

existence of optimal plans. He derived conditions under which it is optimal to search until the target is found, and also obtained conditions when it is optimal not to search at all. Chew [35] determined conditions under which an optimal n -stage look-ahead policy is optimal. A counterexample to an appealing conjecture on the nature of the optimal policy space was given by Kan [100].

Next we move to the continuous effort version of search and stop. Motivated by Richardson and Belkin [185], Stone [186] showed that for a fixed stopping time, it is optimal to follow a plan that is uniformly optimal up to that time. This reduced the problem to an optimal stopping problem. Iida [89] and Iida and Hozaki [92] investigated this problem further for stationary and moving targets respectively. Kisi and Tatsuno [105] and Tatsuno and Kisi [191] added uncertain detection capability to the search and stop problem. Finally, Starr [171] considered a stopping problem in which search was modeled only through a capture rate.

Smith and Kimeldorf [167] considered the discrete search problem with an unknown number, M , of targets. In this case the objective was to minimize the expected cost to find at least a single target. When the number of cells is at least three and M has a Poisson distribution (conditioned on $M \geq 1$), they showed that a locally optimal plan is globally optimal. Further, in the two cell case they conjectured that this is true for all distributions on M .

Assaf and Zamir [9] found a counterexample to this conjecture, investigated the continuous effort analog, and demonstrated that the locally

optimal plan is optimal. Kimeldorf and Smith [103] and Assaf and Zamir [9] studied two further variations of this problem. The first changed the objective to finding all targets and the second included uncertainty to the parameters of the prior distribution for the target location. Assuming perfect detection, Kelly [102] discovered a connection between the single and multiple target problems.

A false target is a real object that can be mistaken for the target. A false alarm is an invalid detection generated by the sensor system which does not reflect a real object. Two false target models have been studied in the literature — the Poissonly distributed false targets and the independent and identically distributed false targets. The former was investigated by Stone and Stanshine [184,185], and Stone *et. al.* [136]; the latter by Dobbie [47]. In a related paper Iida [91] studied the problem of determining a stopping rule for terminating the investigation of a possibly false target contact. In all these cases, the objective is to minimize the mean time to find the target. However, these authors did not consider the problem of determining a plan to maximize the probability of detection before a given time. Engel [52] and Smith and Walsh [168] also discussed search problems involving false alarms.

Two types of search problems involve a target whose value or detectability varies with time. The first are the multi-state or survivor search problems (mentioned above in section 4.1). These have also been discussed in Discenza and Stone [45], Stone [177], Nakai [131], and Stromquist and Stone [187] in which there is a single target. The second type involves multiple targets which arrive randomly in time, which can occur simultaneously, and

which are of limited duration. These scenarios are typical of situations which may occur happen in a missile defense system. Blachman and Proschan [26], Blachman [25], Barnett and Mazzarino [13], Chelst [33], and Barnett [12] investigated versions of this problem.

One variation on the standard detection model involves introducing multiple, non-concurrent search modes. Posner [149] described a satellite search problem in which the radar beam widths can be varied. Persinger [147] considered a situation with two sensors - one having a deterministic sweep width and the other a stochastic sweep width. A model in which time must be allocated between search and improving detection rates was studied in Nakai [103].

Optimal search plans have been determined for situations in which the detection probabilities are random variables. In the discrete case, Hall [62, 63] determined extensions of the standard results on locally optimal plans. Kelly [81] investigated the asymptotic form for the optimal plans. For the case of continuous effort, Richardson and Belkin [114] considered search with an uncertain sweep width. They derived the optimal search plan for a stationary target with a bivariate normal prior location distribution. Belkin [34] studied the rate of expansion of this "gamma plan."

4.1.4 Modeling Search

With a sufficiently broad interpretation, a wide range of articles can be classified under the theme "search modeling". For example, many articles on stochastic processes may be seen as having some connection with modeling target motion. We have restricted our survey to articles which specifically model the search process.

Klein [106] described two Markov decision models for search. Cozzolino [39] dealt with a search model for an unknown number of targets of nonuniform size. One application for this model is the search for petroleum deposits. A conjugate family of distributions was postulated which permitted efficient Bayesian updating. Willman [210] formulated a class of unidimensional search problems in which the objective is to find a point near but not beyond a boundary of uncertain location. Problems of this type are encountered in mining operations.

Corwin [36,38] considered the target location distribution when the target motion is a diffusion process. Mangel [124] computed the probability of success in this situation by solving a backward diffusion equation. Corwin [37] discovered a representation of the minimax estimator for the target location when target motion is modeled as a Markov chain.

4.1.5 Computations

Most algorithms for computing optimal plans for moving targets apply only to one-sided search problems — problems in which the target does not react to the searcher. Problems in which the target reacts to the searcher are called two-sided problems (search games). In this section we only consider one-sided search problem; we shall discuss search games in section 4.3.

We can classify one-sided search problems into two different categories:

1. optimal search density problems, and
2. optimal search path problems.

When search effort can be divided as finely as desired, and when the application of search in one place and time does not constrain the effort at any other time, we have an optimal search density problem. When the location of search effort at one time constrains the location of effort at a later time, the problem involved is an optimal searcher path problem. Optimal searcher path problems arise when the searcher's speed is roughly the same as the target's, for example, when one submarine is searching for another. Optimal search density problems are reasonable approximations to situations where the searcher's speed is much faster than the target's, for example, in the case when a plane is looking for a ship lost at sea.

1. Optimal Search Density Problems

The first substantial progress in developing algorithms for moving target detection problems was obtained by Brown in 1977. For the case of an exponential detection function, he applied the Karush-Kuhn-Tucker conditions to find an algorithm for computing the optimal detection search allocation for a target moving in discrete space and time according to a Markov process. For this case, he was able to reduce the moving target problem to that of solving a sequence of stationary target problems (see Brown [30]). Washburn [201] gave an analogous algorithm for discrete search effort which finds plans that satisfy a necessary condition for optimality. Stone *et. al.* [186] developed algorithms for arbitrary discrete time target motions and exponential detection functions, and later Stone [179] found necessary and sufficient conditions for optimal detection problems involving regular detection functions involving an essentially arbitrary stochastic process modelling the target motion. Stromquist and Stone [187] extended these results to a wider class of constrained optimization problems which include problems not related to search as well as many search related ones. The necessary conditions derived in this paper are distinct from those developed for nonlinear programming (e.g., the Karush-Kuhn-Tucker conditions) and those developed for control theory (the Pontryagin's maximum principle). Thus none of these results imply or is a special case of those available for the other problems. Algorithms were developed by Discenza and Stone [45] for computing survivor search plans and by Stone and Kadance [181] for computing optimal whereabouts plans for the moving target. Washburn's [204] Forward

and Backward (FAB) algorithm generalized Brown's algorithm to apply to a wider class of payoff functions than that of maximizing the probability of detection. Tierney and Kadane [192] developed an algorithm for computing optimal plans for a special type of surveillance searches. Typically, these algorithms are based on the necessary and sufficient conditions mentioned above.

2. Optimal Search Path Algorithms

We divide this area of problems into discrete and continuous time-and-space ones. In the discrete-time-and-space problem the target motion takes place in discrete cells. The searcher moves among the cells according to constraints. For example, if the searcher is in cell j at time t , he is constrained to be in one of the set $S(j)$ of cells at time $t+1$. The probability of detecting the target at time t depends on the location of the target and the searcher. Detection at one time is independent of detection at all others. The problem is to find a search path that satisfies the constraints and yet maximizes the probability of detecting the target by time T . Trummel and Weisinger [194] have shown that the optimal searcher path problem is NP-complete.

A number of heuristic approaches have been used to produce algorithms for computing optimal searcher paths (for example, Stewart [172,173] and Eagle [52]). More recently Eagle and Yee [53] have developed an efficient algorithm for this problem based on the work of Stewart [172]. In the continuous time-and-space case, the constraints on the searcher's motion are given in terms of a differential equation that the searcher's path must

satisfy. The target's motion is modeled as a diffusion process, and the detection function is taken to be exponential. These assumptions on the target motion and the detecting function have been made so that the problem fits into the framework encountered in standard optimal stochastic control. One then can find a dynamic programming equation whose satisfaction provides a sufficient condition for optimality. This is the approach taken by Ohsumi [140,141]. Optimal control theory also has been applied to this problem by Lukka [121] and Mangel[123]. We shall discuss this approach in detail in the next section.

4.2 Optimal Control Theory and the Optimal Search Problem

4.2.1 Introduction to Optimal Control Theory

Calculus of variation has a long history, starting almost at the same time as the invention of calculus. With the calculus of variation one seeks a curve which maximizes (or minimizes) a certain function J associated with the curve. A straightforward example is the length of the curve, where

the minimum is obtained by the straight line segment joining the given end points.

The original motivation for the calculus of variation emerged from classical mechanics, optics, and geometry. Since 1950 many new applications have been found, some of them were to problems in the aerospace sciences, industrial process control, economics, biology, and management. To deal with these new applications the theory of calculus of variations has been extended, certain variables called controls were introduced into the system, and a more general theory, which now is called the theory of optimal control, was developed.

An optimal control problem includes the following information:

1. A control u belonging to some set $\mathcal{U}_{ad}$ (the set of admissible controls, usually a compact subset of a Polish space).
2. For a given control u , the state $x(u)$ of the system which is to be controlled is given by the solution of an equation (the **state equation**)

$$\mathcal{P}(x, u) = 0,$$

where $\mathcal{P}$ is a known operator that relates the state and the control. Under different situations, the state equation may be specified by a difference equation, an ordinary differential equation (as in a lumped system), a partial differential equation (as in a distributed system), an integro-differential equation, or a functional equation, which is either deterministic or stochastic.

3. The **observation** $y = f(x, u)$, which is an index of available information about the state given the input history. In general, it is not possible to determine the state x when the given observation y and input u . Such problems are known as partially observable control problems.
4. The **cost function** $J(u)$, (also called the **economic function** or **performance index**), defined in terms of a numerical function $z \rightarrow \Phi(z) \geq 0$ on the space of observations, and is given by

$$J(u) = \Phi(y) = \Phi(f(x, u)).$$

The optimal control problem is to find a control $u^0 \in \mathcal{U}_{ad}$ such that

$$\inf \{J(u) : u \in \mathcal{U}_{ad}\} = J(u^0).$$

4.2.2 Application of Optimal Control Theory to Search Problem

It is natural to consider the optimal search problem within the framework of optimal control theory. We shall now review the work of several authors in this field.

Lukka [121] presented necessary conditions for the optimal searching tracks for a moving target, where the motion of the target is determined

by a parameter vector having a known probability distribution. In this paper, the target is assumed to be moving in R^2 . The position of the target at time t will be given by the function $\eta(y, \xi, t)$ having values in R^2 , where $y \in R^2$ is the position of the target at time $t = 0$ and $\xi \in R^m$ is a parameter vector. Suppose that $\eta(y, \xi, t)$ and $d\eta/dt$ are continuous and uniformly bounded for all $y \in R^2, \xi \in R^m$ and $t \in [0, T]$. The initial position y and the parameter vector ξ are assumed to have a joint probability distribution function $F(y, \xi)$.

Let $\lambda(x(t), u(t), y)$ be the intensity of the search effort at y at time t , when the searcher is at $x(t)$ and has the velocity $u(t)$. The function λ and its first and second order partial derivatives with respect to x and u are assumed to be continuous and uniformly bounded for all x and u in R^2 . Now the search effort accumulated on the target during the time interval $[0, T]$ is

$$\phi(y, \xi) = \int_0^T \lambda(x(t), u(t), \eta(y, \xi, t)) dt$$

given that the track of the target is $\{\eta(y, \xi, t), 0 \leq t \leq T\}$ and the track of the searcher is $\{x(t), 0 \leq t \leq T\}$ with $\dot{x}(t) = u(t)$.

Let $q(\phi)$ be the probability that the target is not detected assuming that the effort ϕ has been accumulated on it. The functions q and q' are assumed to be continuous for all $\phi \in [0, \infty)$. Then the probability that the target is not detected during the time interval $[0, T]$ is

$$Q(x, u) = \int_{R^2} \int_{R^m} q\left(\int_0^T \lambda(x(t), u(t), \eta(y, \xi, t)) dt\right) dF(y, \xi)$$

given that the track of the searcher is $\{x(t), 0 \leq t \leq T\}$.

Assume that the function $x(\cdot)$ is in the space of absolutely continuous functions defined on $[0, T]$, and that the function $u(\cdot)$ is in the space $L_\infty(0, T)$. Moreover we assume that $x(0) = a^0$, $x(T) = a^T$ and $|u(t)| \leq v$ for all $t \in [0, T]$. Thus we have an optimal control problem defined as follows.

minimize $Q(x, u)$ under the following conditions:

$$\begin{aligned}\frac{dx}{dt} &= u(t), \\ x(0) &= a^0, \\ x(T) &= a^T, \\ |u(t)| &\leq v.\end{aligned}$$

Using optimal control theory Lukka [121] derived necessary conditions for the optimal track in the form of Pontryagin's maximum principle.

Mangel [124] calculated the joint density for target location and unsuccessful search. In this paper, he assumed that the target moves as a diffusion process. For simplicity, the target is assumed to move in the plane and the searcher in space. The descriptive search problem involves evaluating the joint density for the target location and an unsuccessful search. Let $(X(t), Y(t))$ be the position of the target at time t and let $S(t) = (S_x(t), S_y(t), S_z(t))$ be the position of the searcher at time t . The joint density $f(x, y, t, S)$ is defined by

$$f(x, y, t, S) dx dy = \text{Prob}[x \leq X(t) \leq x + dx, y \leq Y(t) \leq y + dy | \text{search along } S(\tau), (0 \leq \tau \leq t), \text{ was not successful}].$$

Once $f(x, y, t, S)$ is known, the conditional density, given unsuccessful search, is given by

$$\rho(x, y, t|S) = \frac{f(x, y, t, S)}{\int \int_D f(x, y, t, S) dx dy},$$

where D is the domain in which the target moves. The probability of detection by time t is

$$P_t = 1 - \int \int_D f(x, y, t, S) dx dy$$

The joint density function $f(x, y, t, S)$ satisfies the following equations (search equations):

$$\begin{aligned} \frac{df}{dt} &= \frac{1}{2} \left[\frac{d^2}{dx^2} (a_{11}(x, y, t)f) + 2 \frac{d^2}{dx dy} (a_{12}(x, y, t)f) + \frac{d^2}{dy^2} (a_{22}(x, y, t)f) \right] \\ &\quad - \frac{d}{dx} (b_1(x, y, t)f) - \frac{d}{dy} (b_2(x, y, t)f) - \psi(x, y, t, S)f, \\ f(x, y, 0, S) &= \rho_0(x, y). \end{aligned}$$

Search equations contain information about the target motion model and the detection model. The function $\rho_0(x, y)$ is the initial density for the location of the target and satisfies:

$$\rho_0(x, y) dx dy = Prob[x \leq X(0) \leq x + dx, y \leq Y(0) \leq y + dy].$$

In Mangel [124], the search for a moving target by a single searcher is studied. The density $f(x, y, t, S)$ can be approximately calculated by asymptotically solving the search equations that the density satisfies, since only ordinary differential equations are involved.

Ohsumi [142] considered optimal search problem for detecting a randomly moving target whose dynamics are described by a stochastic differential equation. The key notions involve formulating the problem as one of optimal control and establishing the searcher's strategy by finding the control signal minimizing the probability that the searcher fails to detect the target. Sufficient conditions are given for the existence of an optimal search control.

The search problem considered in his article [142] is roughly visualized as follows. A target moves randomly according to a diffusion process. At the beginning of the search its location is not known precisely, but is given by a probability density function (pdf). The search action of the searcher is characterized by a function which specifies the probability of detecting the target, given the positions of the searcher and the target. The search for the randomly moving target (called a Markovian target) is to be optimized by selecting the control variables of the searcher's dynamics so as to maximize the probability of detecting the target by a fixed time.

Let $x(t) \in R^m$ be the track of a target whose dynamics is given by the Itô stochastic differential equation

$$dx(t) = f[t, x(t)]dt + G[t, x(t)]dw(t), \quad x(0) = x_0,$$

where $w(t) \in R^d$ is the standard Wiener process; f and G are known vector and matrix functions (nonlinear), respectively. x_0 is the initial value which is unknown but its probability density function $p_0(x)$ is known to the searcher.

In the paper [142] the optimal search plan has been performed by minimizing the cost functional. Models of the search and the target are given

and the problem of the optimal search is formulated as an optimal control problem of minimizing the cost functional of exponential type which reflects the probability of detecting the target. A function called the search function is introduced and its time evolution is obtained. Sufficient conditions for the existence of optimal control for search are examined, and it is noted that in order to realize the optimal search a Cauchy problem must be solved. A successful approach to solve the Cauchy problem is proposed by applying the quasilinearization technique to the first order nonlinear partial differential equation. Based on this technique two types of simulations are given.

4.3 Two-Sided Search — Search Games

This section considers articles which treat the two-sided search problem. All but a few of these articles formulate the search problem as a game-theory problem. In particular, each of these articles involve a searcher and a hider who knows that he is being pursued. The information that the hider has concerning the searcher varies from complete information of the searcher's strategy to a complete lack of information. Various payoff functions are considered, such as the probability of detection or capture and the time until detection. The hider may be mobile and moving to avoid capture, or he may be immobile and only able to confront the searcher by the choice

of his initial position.

4.3.1 Mobile Evader

Probably the most general form of the limited information problem is the princess and monster game posed by Isaacs [56]. A mobile hider (the princess) and a searcher (the monster) move along continuous paths within a bounded region. Neither has any information about the other's actions. The monster captures the princess if he gets closer than some fixed distance from her. The game ends upon capture and the payoff is the time until capture. The special case of this problem where the searcher and the evader move on the circumference of a circle is considered by Alpern *et. al.* [6] and Zelikin [215]. The solutions given by Alpern and Zelikin are obtained under the assumption that the searcher's and evader's initial positions are determined by a uniform distribution on the circle. Foreman solved the game when the initial positions are given by an arbitrary distribution on the circle. These results are generalized in Gal [71] to the case that the players move on k nonintersecting arcs, of equal length, connecting two points. Gal [71] also considered the case of the searcher and evader moving in an arbitrary bounded convex region. Results on the value of the game and ϵ -optimal strategies for the players are presented.

Fitzgerald [56] extended these results to the case where the playing space is the finite union of convex sets in R^n . Discrete versions of the game

on the circle are presented in Wilson [211] and Worsham [213]. In Foreman [61], a version of the princess and monster game is studied where the game ends at a specified time if the searcher has not captured the evader, and the payoff is the probability of capture. One case solved in Foreman [61] is that in which the playing space is an arbitrary region in the plane and the two players are initially uniformly distributed in an area far from the boundary of the playing space. Lalley and Robbins [114,115] modified the strategies given in Gal [71] and Fitzgerald [56] and presented asymptotically minimax strategies for the searcher which are suitable when the evader gets intermittent information on the searcher's position. Such cases of occasional information could serve as models for the situation of a surface ship with an active sonar attempting to locate a submarine.

In Alpern and Asic [5,6], and Gal [71], the searcher and evader move along a graph. Bounds on the value of the game are obtained in Alpern and Asic [5]. The searcher in Alpern and Asic [6] is allowed to loiter at a node. Gal [71] also considered the case of an immobile evader. A pursuit-evasion game on a graph with multiple searchers is studied in Parsons [143].

The games considered above assume that no information is available on the opposing player's strategy. An alternate formulation is to have complete information available. In this situation a typical payoff from the searcher to the evader is the closest distance of approach between the players. Flynn [58-60], Croft [40], Lewin [117], and Halpern [83] considered problems of this type. Flynn [58] solved a version of the "lion and man" problem, where the searcher (the lion) moves on a closed disc and the evader (the

man) is confined to move on the circumference of the disc. The searcher and the evader are continuously knowledgeable of each other's position. Both the searcher and the evader move in the disc in Flynn [59,60]. In Flynn [60], the max-min evasion part of the problem is considered. In particular, the searcher is required to select his trajectory before the game begins. The evader can continuously observe the searcher and adjust his path as a function of the searcher's moves. However, the evader must tell the searcher ahead of time how he will react to the searcher's actions. The searcher can use this information when selecting his trajectory. Flynn [59] concentrated on the min-max pursuit part of the problem and developed strategies for the searcher. Lewin [117] investigated the lion and man problem using methods of the theory of singular surfaces in differential games. Results when curvature constraints are placed upon the man's path are obtained in Croft [40].

Halpern [83] considered a problem where the searcher must select a pure strategy at the start of the game. This strategy is known to the evader. It is known that there exists a $\delta > 0$ and a pure strategy for the evader such that the evader stays at least a distance δ away from the searcher for any strategy selected by the searcher. Washburn [201] looked at a searcher and evader moving on a checkerboard, where the searcher and evader are aware of each other's previous moves. Asymptotic results are obtained for games with a large number of squares.

A discrete game is considered in Efron [54]. A solution to the game is found when the searcher is not permitted to revisit a cell after searching it

once. No movement restrictions are placed on the evader. Both a single-stage and a discrete sequential search game are studied in Sakaguchi [164]. Dobbie [48] presented a discrete problem where the evader moves from cell to cell in an effort to avoid the cell in which search is currently being conducted. Note that this problem is one sided in the sense that the rate at which the evader moves from cell to cell and the initial distribution of the evader are known to the searcher.

Stewart [173] studied a two-cell search-evasion game. The evader is initially in the first cell and has to pass, in order, through the first and second cells to reach a goal lying beyond the second cell. He must do this within some fixed time. The searcher's probability of detecting the evader is a function of the cell the evader is currently in, and whether or not the evader is moving. Attention is focused upon the situation where at each stage the searcher is only allowed to move resources from the first cell to the second. Both the divisible and indivisible search effort cases are considered. Similar problems are considered in Nakai [132] and Nakai [134]. In Nakai [132] the playing space consists of N ordered cells. The evader is only allowed to move in one direction through the cells. No restrictions are placed upon the searcher. After each stage, the searcher is informed of the evader's position. Results on the value of game and on the player's optimal strategies are obtained. Nakai [134] extends the playing space to an infinite number of ordered cells and allows the evader to move forward and backward through the cells.

4.3.2 Immobile Evader

Search for an immobile evader has been considered in game-theory versions of the linear search problem has been studied by Beck [17,18], Beck and Newman [21], Beck and Warren [20], and Beck and Beck [19]. Recall that in this problem a point on the real line is selected according to a distribution known to the searcher. A searcher strategy consists of starting in the positive or negative direction and proceeding until the point is found or until x_1 is reached. Upon reaching x_1 and not finding the point, the searcher turns around and searches the other half of the line until the point is found or until x_2 is reached. Upon reaching x_2 and not finding the point, the searcher turns around and searches until x_3 , and so on. The searcher seeks to minimize the expected distance traveled until detection. In the game-theory versions of this problem studied by Fristedt and Heath [66] and Fristedt [65], the evader chooses a distribution (unknown to the searcher) from which to select a hiding point. The payoff function is the time it takes the searcher to find the evader divided by the distance of the evader from the origin. Gal [69] extended the linear search game and considered, for instance, searching for the evader when the probability of detecting the evader at a point is less than unity. Further results are given in Gal [67] and Gal and Chazan [72].

In Gittins and Roberts [78] and Roberts and Gittins [157], the

search space is discrete and the evader selects a distribution on his initial position. It is shown that selecting that distribution which equalizes $p_i q_i$ is a near-optimal strategy for the evader, where p_i is the evader's probability of being in cell i and q_i is the searcher's probability of detecting the evader in cell i if the evader is there.

Roberts and Gittins [157] also solves a continuous-time version of the problem when the target is in one of two cells. Gittins [77] extends the results on the continuous-time problem to N cells. Techniques from the theory of optimal control are employed to obtain these results. Not only is the evader stationary in a discrete problem studied by Neuts [137], but the searcher is also restricted to stationary strategies. In other words, the searcher must select a probability distribution over the N cells at the start of the game, which will serve to determine which cell will be searched at each stage of the game. The results on this problem are used to study the case where the searcher is allowed nonstationary strategies. Nakai [136] determines the optimal evader distribution and searcher allocation scheme for a one-stage discrete search game, where the probability of detecting the evader in a cell is given by an exponential function. A discrete search game played on three paths connecting two points with an immobile evader is solved in Bostock [29].

4.3.3 Ambush Games

Ambush games make up another category of search games. In an ambush game, the evader wishes to move from one area of a given region to another area in the region without passing through areas occupied by the searcher. The searcher, meanwhile, wants to select areas to occupy so as to guarantee an ambush. Ruckle *et. al.* [162] considered an ambush game played on a lattice of points. Ruckle [161] and Ruckle and Reay [163] solved a continuous version of an ambush game, where the evader wants to cross a rectangle. A one-dimensional ambush game is presented in Baston and Bostock [14]. A discrete ambush game where both the evader and searcher are allowed to move is given in Lalley [112]. Several different ambush games are studied in Ruckle [160].

4.3.4 Tactical Games

Articles with a distinct operational flavor include Charnes and Schroeder [32], where a Anti-Submarine Warship pursuit game is studied. In Baston and Bostock [14], the searcher and evader move on a line. The searcher wants to destroy the evader. To accomplish this the searcher has

b bombs available, each with destructive radius R . A complicating factor is that after the searcher drops a bomb, there is a time lag before the bomb explodes. A complete solution to the game is obtained for $b = 1$. A similar problem is considered in Dubins [49]. Danskin [42] determined an optimal search strategy for a helicopter with a dipping sonar attempting to detect an evading submarine.

Another type of tactical game is one in which two searchers compete to find a target. Nakai [134] investigated such a situation. A stationary target is located in one of N cells. Two searchers compete in a non-zero-sum game, each trying to locate the target first. A general formulation of the problem is presented, where each searcher places a prior distribution on the target's location and the search effectiveness between searchers within a cell may differ. An unexpected result is that the searchers have the same equilibrium strategies if they have the same prior target distribution, even if their search effectiveness within cells are different.

Although not strictly formulated as a game, the results of Giammo [73] are also of some tactical interest. This article discusses the problem of two adversarial forces searching for one another within a fixed region.

4.3.5 Other Games

Suppose a point (the evader) is known to lie in an interval. The searcher tries to locate the evader by making a series of guesses as to the evader's position. After each guess the searcher is told whether it has guessed too high or too low. The searcher desires to construct a strategy which minimizes a given cost function. This problem, which might be termed search with directional information, has been considered in various forms. In Baston and Bostock [14], the problem is treated as game with the cost function as the payoff function. Alpern [4] shows that the game has a value. The strategy which minimizes the cost to find any evader (the minimax strategy) is also determined in Alpern [4]. The evader selects an integer between 1 and N in the discrete game investigated by Gal [68]. The payoff function is the number of guesses that the searcher needs to locate the selected number. The value of the game and the optimal strategies for the players are given.

Arising from a problem stated by Ulam, several articles considered the situation where wrong answers may be given to the searcher's queries. In Ulam [195], the following was posed: What is the minimum number of yes-no questions needed to find an integer, selected by the "evader", between one and one million, if the evader is allowed to lie once. Spencer [169] treats the case where the evader has selected a number between 1 and N and the searcher is allowed k questions. All questions are of the form: Is the selected number greater than a ? The evader is allowed to lie at most once. Bounds estimating the minimum number of questions necessary to locate the number are found.

For example, if $n \leq 2^{k-1}/(k+1)$, then the searcher wins. Moreover, it is shown that either 25 or 26 questions are needed in Ulam's problem.

In Rivest *et. al.* [156], the evader may lie up to E times. Continuous and discrete cases are considered. In the continuous case, where the evader selects a number in the interval $(0, 1)$, the minimum number of questions necessary in the worst case to narrow the location of the number to a specified size set is determined. Bounds on the minimum number of questions for the discrete case are also specified. The bounds, again, show that either 25 or 26 questions are necessary in Ulam's problem. Ravikumar and Lakshmanan [153] found the minimum number of questions necessary to reach a given level of uncertainty when the pattern of lies of the evader is known. For example, it might be known that the evader lies at most twice, or never lies twice in a row.

Pelc [145], among more general results, settled one version of Ulam's problem. If the searcher is able to ask questions of the form: "Is the selected number an element of T , where T is a subset of $(1, \dots, n)$ "; then a minimum of 25 questions is necessary to locate the number. Using the theory of error-correcting codes, Niven [138] also showed that a minimum of 25 questions is necessary. Czyzowicz and Pelc [41] showed that 29 questions are necessary to find a number between one and one million, if up to two lies are allowed. The problem of identifying an unknown subset of $(1, \dots, n)$ in m questions is studied in Pelc [144]. The patterns of lies are characterized for which it is feasible to do this. Gal [70] and Pelc [146] treated the case where there is some positive probability that a wrong answer is given to the

searcher each time he asks a question. The feasibility of constructing search strategies which localize the evader to within a given degree of accuracy is studied. Both continuous and discrete versions of the game are considered in Pelc [146]. Another version of directional information search is given in Berry and Mensch [24].

4.4 Conclusion

In this chapter we have briefly reviewed some known results for searching for a moving target. Section 4.1 covers some older results, most of which were obtained in 70's or early 80's. Section 4.2 presents some results which reflects the recent effort to perceive search theory in the framework of optimal control theory. In section 4.3 we reviewed some results of two-sided search games, which is the most complicated and interesting part of search theory.

The materials of section 4.1 and 4.3 are mainly adapted from two survey papers by L.D. Stone (What Happened in Search Theory Since the 1975 Lanchester Prize? *Operations Research*, Vol. 37, 501-506, 1989) and Benkoski *et. al.* (A Survey of the Search Theory Literature, *Naval Research Logistics*, Vol.38, 469-494, 1991).

Chapter 5

SEARCH FOR MOVING TARGETS: SOME NEW RESULTS

5.1 Introduction to Stochastic Optimal Control

Stochastic control theory is a relatively young branch of mathematics. Intensive research in this field began in the late 1950's and early 1960's. During that period an extensive body of literature appeared on optimal stochastic control using the quadratic performance criterion (see references in Wonham [212]). At the same time, Girsanov [76] and Howard [88] made the first steps in constructing a general theory, based on Bellman's technique of dynamic programming, developed by him somewhat earlier [22].

Two different families of engineering problems initiated the study

of two different sub-areas of stochastic optimal control theory. Problems of the first type are associated with multistep decision making in discrete time, and are treated in the theory of discrete stochastic dynamic programming. For more on this theory, in addition to the work of Howard and Bellman mentioned above, we refer the reader to the books by Derman [44], Mine and Osaki [128], and Dynkin and Yushkevich [50].

Another class of engineering problems that encouraged the development of the theory of stochastic control involves the time continuous control of a dynamic system in the presence of random noise. The case where the system is described by a differential equation and the noise is modeled as a time continuous random process is the core of the optimal control theory of diffusion processes. Our problem falls into this category.

As we stated in section 4.2.1, an optimal control problem includes the following information:

1. A control u belonging to some set $\mathcal{U}_{ad}$ (the set of admissible controls, usually a compact subset of a Polish space).
2. For a given control u , the state $x(u)$ of the system which is to be controlled is given by the solution of an equation (the state equation)

$$\mathcal{P}(x, u) = 0,$$

where $\mathcal{P}$ is a known operator that relates the state and the control. In other situations, the state equation may be specified by a differ-

ence equation, an ordinary differential equation (as in a lumped system), a partial differential equation (as in a distributed system), an integro-differential equation, or a functional equation, which is either deterministic or stochastic.

3. The **observation** $y = f(x, u)$, which is an index of available information about the state given the input history. In general, it is not possible to determine the state x when the given observation y and input u . Such problems are known as partially observable control problems.
4. The **cost function** $J(u)$, (also called the **economic function** or **performance index**), defined in terms of a numerical function $z \rightarrow \Phi(z) \geq 0$ on the space of observations, and is given by

$$J(u) = \Phi(y) = \Phi(f(x, u)).$$

The optimal control problem is to find a control $u^0 \in \mathcal{U}_{ad}$ such that

$$\inf \{J(u) : u \in \mathcal{U}_{ad}\} = J(u^0).$$

One of the most fascinating and challenging problems in optimal control theory today is the proof of existence of optimal controls, especially feedback controls, i.e., controls dependent on the state of the system. For systems governed by stochastic differential equations or integral equations the existence problem is far more complex (see Ahmed [1-3]). In this chapter we will mainly be concerned with the question of existence of optimal search strategies as optimal feedback controls, which use dynamic programming principle and Hamilton-Jacobi-Bellman (HJB) equations.

5.2 Principle of Optimality and HJB Equations

5.2.1 Modelling of Stochastic Optimal Control System

Assume $X = R^n$ is n dimensional real space. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space. That is, $(\Omega, \mathcal{F}, P)$ is a probability space and $\mathcal{F}_t$ is a set of sub- σ -algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_r$ if $s < r$. Suppose $\{W_t, t \geq 0\}$ is an $\mathcal{F}_t$ adapted X -valued Brownian motion (Wiener process). We consider the optimal control problem of stochastic evolution equations driven by W_t . More precisely, we consider the following Controlled Stochastic Differential Equation (CSDE):

$$(CSDE) \quad \begin{cases} dy(t) &= a^u(t, y(t))dt + \sigma^u(t, y(t))dW_t \\ y(0) &= y_0 \end{cases}$$

where $a^u(t, y(t)) \equiv a(t, y(t), u(t, y(t)))$, $\sigma^u(t, y(t)) \equiv \sigma(t, y(t), u(t, y(t)))$ is a linear operator in X .

Definition 5.1. (Fleming and Rishel [57, p.156])

A feedback control u is said to be **admissible** if u is a Borel measurable function from $[0, T] \times X \rightarrow X$, satisfying the following conditions:

(a) For each $y_0 \in X$, there exists a solution y of (CSDE), unique in probability law.

(b) For each $k > 0$

$$E \int_0^T |u(t, y(t))|^k dt < \infty.$$

We denote by $\mathcal{U}_{ad}$ the class of all admissible feedback controls.

Remark 5.2. There are several ways to define admissible controls. Some authors define it as a family of probability measures satisfying measurability condition (see Gihman and Skorohod [74, p.79]). Other authors define it as a U -valued stochastic process with measurable sample paths satisfying the non-anticipativity condition: for $t \geq s \geq r$, $W(t) - W(s)$ is independent of $u(z)$, $z \leq r$. (Borkar [28, p.5-6])

Definition 5.3. The performance index (cost function) of a feedback control system is given by

$$J(u) = E \left\{ \int_0^T \ell(s, y(s), u(s, y(s))) ds + \phi_0(y(T)) \right\},$$

where $y(t)$ is a solution of (CSDE), and ℓ, ϕ_0 are known functions.

The optimal feedback control problem considered here is to find a feedback control law u^0 which minimizes $J(u)$. That is,

$$J(u^0) = \inf \{ J(u) : u \in \mathcal{U}_{ad} \}.$$

5.2.2 Principle of Optimality

There are two different approaches for the optimal control problems described in section 5.2.1. One is probabilistic approach (see Ahmed [2], for example). This approach is based on selection theorems and the so-called "compactness-continuity" argument — every continuous (more generally, lower semi-continuous) function on a compact set attains its minimum. The distinguishing feature of the probabilistic method is that it works with the actual sample paths of the process, so that one does not lose sight of "the stochastic process within". The other strategy is the analytic approach. In this approach the optimal control problem of a stochastic system reduces to an equivalent problem of a deterministic distributed parameter system, governed by a partial differential equation. Compared to the stochastic method, this approach works with average quantities. It is more general and powerful, especially when one deals with analytic properties of the system, such as the regularity properties of solutions.

In this thesis we follow the analytic approach. Using this technique the stochastic optimal control problem can be reduced to an equivalent problem of a deterministic system governed by a partial differential equation, called the **Hamilton-Jacobi-Bellman (HJB) equation**. This reduction is done by means of the principle of optimality (also called the dynamic programming principle), introduced by R. Bellman [22] (see also Ahmed [3]).

The essence of the principle of optimality is the following: if u^* is the optimal control over the interval $I = [0, T]$, then its restriction to any

subinterval of I is also optimal. If we define the value function $\phi(t, x)$ as:

$$\phi(t, x) = \inf\{J(t, x, u) : u \in \mathcal{U}_{ad}\}$$

we can state the principle of optimality as in the lemma below.

Lemma 5.4. (Principle of Optimality)

For $0 \leq t < T$, $x \in X$ and $u \in \mathcal{U}_{ad}$, let

$$\begin{aligned} \ell^u(\theta, y_{t,x}(\theta)) &= \ell(\theta, y_{t,x}(\theta), u(\theta, y_{t,x}(\theta))), \\ \phi^u(t, x) &= E\left\{\int_t^T \ell^u(\theta, y_{t,x}(\theta))d\theta + \phi_0(y_{t,x}(T))\right\}, \end{aligned}$$

where $y_{t,x}$ is the solution of (CSDE) with initial condition $y(t) = x$. Let u be an admissible control restricted to the interval $[t, s]$, and $\phi(t, x) = \inf\{\phi^u(t, x) : u \in \mathcal{U}_{ad}\}$. Then for all $t < s < T$

$$\phi(t, x) = \inf_{u \in \mathcal{U}_{ad}} E\left\{\int_t^s \ell^u(\theta, y_{t,x}(\theta))d\theta + \phi(s, y_{t,x}(s))\right\}. \quad (5.1)$$

The principle of optimality says that the minimum cost of the control problem equals the minimum of the cost at a given stage plus the minimum cost to go thereafter. In other words, whatever may have been the control policy over the initial period of time $[t, s]$, the subsequent policy must be such that it minimizes the cost of operation for the remaining period $[s, T]$.

In the next section we derive the HJB equations for the optimal search problem.

5.3 The HJB Equation for the Optimal Search Problem

5.3.1 Derivation of HJB Equation for the Optimal Search Problem

The function $\phi(t, x)$ introduced in section 5.2.2 is called the **value function**. Loosely speaking, it yields the minimum cost of future operation of the system starting from (t, x) . Using the principle of optimality and Itô's formula we can derive the HJB equation for the optimal search problem.

Definition 5.5. Let I be an subset (index set) of the real line R . A random process $\xi = \{\xi(t), t \in I\}$ on a probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t \geq 0}, P)$ is said to be an $\mathcal{F}_t$ -martingale if

$$\begin{aligned} E|\xi(t)| &< \infty, & t \in I, \\ E\{\xi(t)|\mathcal{F}_t\} &= \xi(s) & s \leq t. \end{aligned}$$

A random process $\xi = \{\xi(t), t \in I\}$ is said to be a semimartingale if $\xi(t)$ admits the following decomposition

$$\xi(t) = \xi(0) + V(t) + M(t),$$

where $V(t)$ is an $\mathcal{F}_t$ -measurable process with bounded variation and $M(t)$ is an $\mathcal{F}_t$ -martingale.

In classical calculus, if $f = f(t, x)$ has first order derivatives ($f \in C^1$) in both t and x on $I \times R^n$, then for any $\xi(t) \in C^1(I, R^n)$, we have

$$\frac{d}{dt}f(t, \xi(t)) = f_t + (f_x, \frac{d\xi}{dt}),$$

or equivalently

$$df = f_t dt + (f_x, d\xi),$$

where $(\cdot, \cdot)$ denotes the inner product. In the case of stochastic process this formula is no longer true. The correct chain rule of stochastic differentials is given by Itô formula.

Lemma 5.6. (Itô formula) *Let $\xi(t)$ be a semimartingale and suppose that both $V(t)$ and $M(t)$ are continuous. Suppose that $f(t, x)$ has first order derivative in $t \in I$ and second order derivatives in $x \in R^n$, and that all these derivatives are continuous. Then the differential of $f(t, \xi(t))$ is given by*

$$\begin{aligned} df &= f_t dt + (f_x, d\xi) + \frac{1}{2} \langle f_{xx} d\xi, d\xi \rangle \\ &= f_t dt + (f_x, dV) + (f_x, dM) + \frac{1}{2} \langle f_{xx} dM, dM \rangle. \end{aligned}$$

In the above, $\langle \cdot, \cdot \rangle$ is defined as follows. Suppose

$$\xi(t) = \int_0^t f(\theta) dW(\theta), \quad \eta(t) = \int_0^t g(\theta) dW$$

then

$$\langle \xi(t), \eta(t) \rangle = \int_0^t |f(\theta)g(\theta)| d\theta.$$

In particular,

$$\langle \xi(t) \rangle = \langle \xi(t), \xi(t) \rangle = \int_0^t |f(\theta)|^2 d\theta$$

is called the quadratic variation process.

The proof of Itô formula is lengthy but straitforward. It can be found in any book on stochastic analysis, and so we omit the proof here.

■

Now we begin to define the optimal control model for the search problem. Suppose a target is moving randomly according to the following stochastic differential equation

$$dx(t) = a(t, x(t))dt + \sigma(t, x(t))dW(t), \quad x(0) = x_0 \quad (5.2)$$

where $x(t) \in R^n$ is the track of the moving target, $W(t) \in R^d$ is the standard Wiener process (Brownian motion), a and σ are known vector and matrix nonlinear functions, respectively. The initial value of the target x_0 is unknown but its distribution is known (or pre-assumed) to the searcher.

In the searcher's part, the searching path is described by

$$dz(t) = h(t, z(t), u(t, x))dt, \quad z(0) = z_0, \quad (0 \leq t \leq T) \quad (5.3)$$

where $z(t) \in R^n$ and $u(t, x) \in R^m$, ($m \leq n$) are the state and control functions, and h is a nonlinear function.

The controlled search problem may formulated as follows: find $u^*(t, x)$ among the admissible control set such that the time when the searcher meets with the target is minimized. In other words, we want to find $u^*(t, x) \in \mathcal{U}_{ad}$ such that $\{t \mid x(t) = z(t)\}$ is minimized.

Before we start to derive the HJB equation for the controlled search problem described above, it will be helpful to consider an example first.

Example 5.7. We consider here a simple case. Suppose a target is moving on a circle C of radius 1 and centered at 0, with constant speed 1. A searcher started at point $(-z_0, 0)$ and moving along a straight line L with constant speed v . Our goal is to choose a direction (the angle between L and the x -axis) such that the time when the searcher meets the target is minimum.

Fig. 5.1 Search for a target moving along a circle

Suppose that the target's initial position is $\omega_0 = 0$. Hence the motion of the target is governed by

$$x_1(t) = \cos t, \quad x_2(t) = \sin t.$$

The motion of the searcher is given by

$$z_1(t) = vt \cos u - z_0, \quad z_2(t) = vt \sin u.$$

Assume the searcher and target meet at time t , then we have

$$\cos t = vt \cos u - z_0, \quad \sin t = vt \sin u.$$

If we solve these equations we get a set of solutions $S = \{(t_i, u_i) | i = 1, 2, \dots\}$. Thus the optimal control is $u^* = \{u_i | t_i \leq t_j, \forall j\}$.

Now suppose the target's initial position is unknown, but its probability distribution $P[\omega_0 = x] = p(x)$ is known to the searcher. For each point x on the circle we have a set of solutions $S(x) = \{(t_i(x), u_i(x)) | i = 1, 2, \dots\}$. Taking expectation we have the set of mean values $S = \{(Et_i, Eu_i) | i = 1, 2, \dots\}$, where

$$Et_i = \int_C t_i(x) dP = \int_C t_i(x) p(x) dx, \quad Eu_i = \int_C u_i(x) dP = \int_C u_i(x) p(x) dx.$$

Thus the optimal control is $u^* = \{Eu_i | Et_i \leq Et_j, \forall j\}$.

■

In the general case, the principle of optimality provides a powerful tool for designing optimal feedback control laws. Using this principle one obtains a first-order hyperbolic partial differential equation (called the Hamilton-Jacobi-Bellman equation), and the solution of HJB equation gives the optimal cost and hence the optimal control law.

We formulate the controlled optimal search problem and derive the related HJB equation. For simplicity let $y^u(t) = x(t) - z(t)$, from (5.2) and (5.3) we have

$$dy^u(t) = [a(t, x) - h(t, z, u(t, x))]dt + \sigma(t, x)dW(t) = b^u(t, y) + \sigma(t, x)dW(t) \quad (5.4)$$

$$y_u(0) = y_0 = x_0 - z_0.$$

Now the optimal search problem is equivalent to find the smallest positive root of the equation $y^u(t) = y(t, u(t)) = 0$ for all $u \in \mathcal{U}_{ad}$. It is again equivalent to minimizing the following expression

$$E\left\{\int_0^T g^u(\theta, y^u(\theta))d\theta\right\}$$

for all $u \in \mathcal{U}_{ad}$, where

$$g^u(t, y) = |(y^u)^{-1}(t)\delta(t)|/T, \quad (0 \leq t \leq t_0)$$

$$g^u(t, y) = 0, \quad (t_0 < t \leq T).$$

Where $(y^u)^{-1}$ is the inverse of y^u in the interval $[0, t_0]$ where $t_0 = t_0(u)$ is the smallest root of the equation $y^u(t) = 0$, and $\delta(t)$ is the Kronecker Delta function:

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0. \end{cases}$$

Define the cost functional

$$J(u) = E\left\{\int_0^T g^u(\theta, y^u(\theta))d\theta + f(y^u(T))\right\}$$

For $0 \leq t < T$, $x \in R^n$ and $u \in \mathcal{U}_{ad}$, let

$$\phi^u(t, x) = E\left\{\int_t^T g^u(\theta, y_{t,x}^u(\theta))d\theta + f(y_{t,x}^u(T))\right\}.$$

The function ϕ^u determines the cost of the future control policy given the current state $y(t) = x$. Now the value function is defined by:

$$\phi(t, x) = \inf\{\phi^u(t, x) | u \in \mathcal{U}_{ad}\}. \quad (5.5)$$

Denote $G = [0, T] \times X$, and $C^{1,2}(G) = \{\phi(t, x) : \phi_t, \phi_x, \phi_{xx} \text{ exists and continuous}\}$.

Using these concepts we are now ready to derive the HJB equation for the optimal search problem.

Theorem 5.8. (Hamilton-Jacobi-Bellman Equation)

Suppose the value function ϕ in (5.5) belongs to $C^{1,2}(G)$. Then it satisfies the Hamilton-Jacobi-Bellman equation

$$(HJB) \quad \begin{cases} \phi_t(t, x) + \inf_{u \in \mathcal{U}_{ad}} \{A^u(t)\phi(t, x) + g^u(t, x)\} = 0 \\ \phi(T, x) = f(x). \end{cases}$$

where

$$A^u(t)\phi(t, x) = \frac{1}{2}Tr(\sigma\sigma'\phi_{xx}) + (b^u(t, x), \phi_x), \quad (5.6)$$

$Tr(\cdot)$ denotes the trace operator.

For definition and properties of trace operator, see Reed and Simon [154].

Proof: For simplicity we write y instead of $y_{t,x}^u$. By Itô's formula (Lemma 5.6) and (5.4), we have

$$\begin{aligned} d\phi(t, y(t)) &= \phi_t(t, y)dt + (\phi_x(t, y), dy) + \frac{1}{2}\langle \phi_{xx}(t, y)dy, dy \rangle \\ &= \phi_t(t, y)dt + (\phi_x(t, y), b^u(t, y))dt + \frac{1}{2}Tr(\sigma\sigma'\phi_{xx})dt + (\phi_x, \sigma dW) \\ &= L^u\phi dt + (\sigma'\phi_x, dW), \end{aligned}$$

where $L^u\phi = \phi_t + A^u(t)\phi$.

Integrating and then taking expectation, we have

$$E\phi(s, y(s)) = \phi(t, x) + E \int_t^s (L^u\phi)(\theta, y(\theta))d\theta.$$

It follows from (5.5) that for all $t < s < T$

$$\inf_{u \in \mathcal{U}_{ad}} E \left\{ \int_t^s \ell(\theta, y, u)d\theta + \int_t^s (L^u\phi)(\theta, y)d\theta \right\} = 0.$$

Dividing by $s - t$ and letting $s \rightarrow t^+$, since $y(t) = x$ we obtain

$$\begin{aligned} \inf_{u \in \mathcal{U}_{ad}} \{ L^u(t)\phi(t, x) + \ell^u(t, x) \} &= 0 \\ \phi(T, x) &= f(x). \end{aligned}$$

Or equivalently

$$(HJB) \quad \begin{cases} \frac{d}{dt}\phi(t, x) + \frac{1}{2}Tr(\sigma\sigma'\phi_{xx}) + \inf_{u \in \mathcal{U}_{ad}} \{ (b^u(t, x), \phi_x) + g^u(t, x) \} = 0, & \text{on } G, \\ \phi(T, x) = f(x), & \text{on } R^n. \end{cases}$$

The terminal condition follows from the definition of $\phi(t, x)$ in (5.5).

■

As we have stated before, the solution of HJB equation gives the optimal cost and hence the optimal control law. This result is contained in the following theorem.

Theorem 5.9. (Verification Theorem)

Let $\phi^ \in C^{1,2}(G)$ be a solution of the HJB equation and let the minimum in HJB equation be attained at some control $u_0 \in \mathcal{U}_{ad}$. In other words,*

$$A^{u_0}\phi + g(t, x, u_0) = \inf_{u \in \mathcal{U}_{ad}} \{A^u\phi + g(t, x, u)\}. \quad (5.7)$$

Then the solution of HJB equation coincides with the value function, i.e.,

$$\phi^* = \inf_{u \in \mathcal{U}_{ad}} \phi^u \equiv \phi_B.$$

Thus u_0 is the optimal control law.

Proof. Suppose ϕ^* is a solution of the HJB equation, then for any u

$$L^u\phi^* + g^u = \frac{d}{dt}\phi^* + A^u\phi^* + g^u \geq 0.$$

Let $y_{t,x}^u(\theta)$, $\theta \geq t$, be the solution of equation (5.4) corresponding to the control u . Then by Itô's formula

$$E\phi^*(s, y_{t,x}^u(s)) = \phi^*(t, x) + E \int_t^s (L^u\phi^*)(\theta, y_{t,x}^u(\theta)) d\theta$$

$$\geq \phi^*(t, x) - E \int_t^s g^u(\theta, y_{t,x}^u(\theta)) d\theta.$$

Letting $s \rightarrow T$, by using the terminal condition of the equation (HJB) we obtain

$$Ef(y_{t,x}^u(T)) + E \int_t^T g^u(\theta, y_{t,x}^u(\theta)) d\theta \geq \phi^*(t, x).$$

Since, by definition, the left hand is $\phi^u(t, x)$, we have $\phi^u(t, x) \geq \phi^*(t, x)$, $(t, x) \in G = [0, T] \times R^n$ for all $u \in \mathcal{U}_{ad}$. Hence

$$\phi_B(t, x) = \inf\{\phi^u(t, x)\} \geq \phi^*(t, x), \quad (t, x) \in G.$$

Next we show that $\phi_B(t, x) \leq \phi^*(t, x)$. Suppose that the infimum is attained at $u_0 \in \mathcal{U}$, and the corresponding solution is denoted by $y_{t,x}^0(\theta)$, $t \leq \theta \leq T$. Define

$$\phi_0(t, x) = E\{f(y_{t,x}^0(T)) + \int_t^T g^{u_0}(\theta, y_{t,x}^0(\theta)) d\theta\}.$$

Since ϕ^* is the solution of the HJB equation and the infimum is attained at u_0 , we have

$$0 = \phi_t^* + \inf\{A^u \phi^* + g^u\} = \phi_t^* + \{A^{u_0} \phi^* + g^{u_0}\}.$$

That is $\phi^{u_0} = \phi_0 = \phi^*$. Therefore

$$\phi_B = \inf \phi^u \leq \phi^{u_0} = \phi^*.$$

This implies that $\phi_B = \phi^*$, thus the solution of HJB equation coincides with the value function (Ahmed [3, p.403]).

■

The verification theorem reduces the stochastic optimal search problem to two deterministic problems. The first is to prove the existence of solutions for the semilinear second order partial differential equation (HJB equation), or more effectively, to solve it with the given terminal condition. The second is to find u_0 by performing the minimization indicated in (5.7). In many applications the second problem is easy, but the question of existence of a solution of HJB equation poses a major problem. For more discussions about this topic, the interested reader may refer to the author's recent work [216,217] and the references therein.

5.3.2 Computation of Optimal Search Plans

The method we discussed in section 5.3.1 is known as **Dynamic Programming**. This technique is very powerful since it deals with more general situations than the methods we reviewed in Chapter 4, and hence may have much wider applications. The virtue of the dynamic programming technique is that it converts the original stochastic problem to two equivalent deterministic problems of solving a semilinear second order PDE and minimizing a continuously differentiable function. The latter two problems have been well studied for hundreds of years and, in general, will be easier than the original stochastic problem. Furthermore, there are many well-known numerical algorithms which have been developed to solve these problems computation-

ally. Therefore, at least theoretically, we can solve any random moving target problem of the type (5.2) (5.3) by dynamic programming.

In the more general case, the computation of optimal search plans is not a simple task. It may involve a lot of matrix multiplications and iterations in either solving equations or determining the minimum (infimum) of the underlying function. But the good news is that in some applications we may easily obtain the optimal search plan without solving the HJB equation, as the following example shows.

Example 5.10. We consider a modified stochastic version of example 5.7. Suppose this time the target's motion is perturbed by a random process $W(t)$ which can be modelled as a Wiener process.

A searcher started at point $(-z_0, 0)$ and moving along a straight line L with constant speed v . Our goal is to choose a direction (the angle between L and the x -axis) such that the time when the searcher meets the target is minimum. (See Figure 5.1.)

Let $x(t) \in R^2$ be the track of the target. Suppose the dynamics of the target is given by the stochastic differential equation

$$d(x_1(t), x_2(t)) = (-\sin t, \cos t)dt + dW(t), \quad x(0) = (1, 0),$$

where $W(t) \in R^2$ is standard Wiener process.

Let $z(t) \in R^2$ be the search path which can be described by

$$d(z_1(t), z_2(t)) = (v \cos u, v \sin u)dt, \quad z(0) = (-z_0, 0), \quad (0 \leq t \leq T).$$

The state equation is

$$dy(t) = b^u(t) + dW(t).$$

where $u(t) \in R$ is the control variable.

In this example it is easy to solve the state equation and obtain

$$(y_1(t), y_2(t)) = (-\sin t - vt \cos u + z_0, \cos t - vt \sin u) + (W_1(t), W_2(t)).$$

However, on the other hand it is difficult to solve the equation $y(t) = 0$.

By dynamic programming (Theorem 5.8) we obtain the HJB equation

$$\begin{cases} \frac{d}{dt}\phi(t, x) + \frac{1}{2}\text{Tr}(\phi_{xx}) + \inf_{u \in U_{ad}} \{ (b^u(t, x), \phi_x) + g^u(t, x) \} = 0, \\ \phi(T, x) = f(x), \quad x \in R^2, \quad 0 \leq t \leq T \end{cases}$$

where

$$\begin{aligned} b^u(t, x) &= a(t, x) - h(t, x, u) = (-\sin t, \cos t) - (v \cos u, v \sin u), \\ g^u(t, y) &= [(y^u)^{-1}(t)\delta(t)]/T = [y^{u^{-1}}(0)]/T = \frac{t_0}{T}. \end{aligned}$$

Since $\phi(x)$ is independent of u , it is easy to see that the infimum in HJB equation is attained at $u = 0$. Thus, by Theorem 5.9, $u_0 = 0$ is the optimal strategy.

■

5.3.3 Conclusion

In this chapter we have considered the optimal search problem as a special case of the optimal control problem. After a brief introduction to optimal

control theory, we presented some basic results in this field. In particular, we presented the principle of optimality and Itô's formula. Based on these results we derived the HJB equation for the problem of optimal search for a randomly moving target, which can be described by a standard Brownian motion (Wiener process). Finally, in a verification theorem we proved that the solution of HJB equation is the optimal strategy.

Our result is derived for the general case, and hence may have wider applications than some traditional methods used by other authors. As a suggestion for further research, it will be interesting to apply this method to all the search scenarios of significant interest and to compare the results with those obtained by other methods.

Chapter 6

CONCLUSION

In this thesis we have studied some interesting problems in optimal search theory. The contents of the thesis can be divided into two parts. In the first part we have considered optimal search problem for a stationary target in discrete space. Although this topic has been well investigated by many authors since the 1950's, we considered it from a new angle. In particular, we have considered the situation in which the target distribution is *not* known.

In the second part of the thesis we have developed a method to solve the problem of optimal search for a randomly moving target, which can be described by a standard Brownian motion (Wiener process). This approach treats the optimal search problem as a special case of the optimal control problem. The results have been derived for the general case, and hence may have wider applications than many traditional methods used by other authors.

Chapters 2 and 4 were primarily of a review nature. In Chapter 2 we reviewed some significant results about the optimal search for stationary

targets. Similarly in Chapter 4 we surveyed a number of books and papers on the problem of optimal search for moving targets, as well as some methods used to study this rather difficult topic. The novel and pioneering results presented in this thesis are in Chapters 3 and 5 respectively. These results are listed below along with suggestions for future research possibilities.

- In Chapter 3 we have studied the scenario when the target distribution for searching a stationary target is unknown. We have proved the upper and lower bounds for the detection probability of the optimal search. Based on this result we have presented several selection criteria for guessing a good estimate for the target distribution, and have derived formula for computing the error estimates when the guessed distribution is used to plan the search strategy. We also have considered search problems for non-regular detection functions having a bell-shaped form. All of these results are novel. It will be very interesting to see how these results can be extended to either the deterministic or the stochastic moving target problems.
- In Chapter 5 we have considered the optimal search as a special case of the optimal control problem. In particular, we derived HJB equation for the problem of optimal search for a randomly moving target. In a verification theorem we proved that the solution of HJB equation is the optimal search strategy. It will be very interesting to apply this method to all the search scenarios of significant interest and to compare the results with those obtained by other methods.

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