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Boundary Element Analysis in Contact Fracture Mechanics

by

Shubin Liu

*A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy*

**Ottawa-Carleton Institute of Mechanical and Aerospace Engineering
Department of Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario, Canada K1S 5B6**

March 1994

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**BOUNDARY ELEMENT ANALYSIS IN
CONTACT FRACTURE MECHANICS**

SHUBIN LIU

1994

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submitted by

Shubin Liu, B.Eng.Sc., M.Eng.Sc.

in partial fulfilment of the requirements
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Chair, Department of Mechanical and Aerospace Engineering

Thesis Supervisor

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Abstract

This thesis is concerned with the analysis of two-dimensional (2-D) and three-dimensional (3-D) elastic contact problems involving cracks using the boundary element method (BEM). The quadratic isoparametric formulation is used throughout, and for treating crack problems, the near-tip fields for both open and closed crack tips are properly modelled using special crack tip elements. These elements also provide a quick and direct means of obtaining the stress intensity factors. The classical contact algorithm using a load incremental-iterative approach is employed to treat frictional contact problems.

The BEM for 2-D contact mechanics is first applied to a number of angled crack problems involving crack surfaces in contact. Stress intensity factor solutions are presented for a range of geometries and coefficients of friction in each of the problems treated. The problem of a pin-loaded, straight attachment lug with a single edge crack at the pin hole, with or without a bushing, is then analysed. Stress intensity factor solutions are obtained for several cracked lug geometries under a wide range of frictional contact conditions between the pin and the lug. Finally, the BEM for 3-D frictional contact mechanics analysis is used to analyse four crack problems. Three of these problems involve crack closure under bending loads, and the fourth is that of a pin-loaded rectangular plate with corner cracks at the pin-hole.

Where possible, the BEM results are compared with those available in the literature and good agreement between them is obtained. The present study shows that friction between the crack surfaces in contact can have a significant effect indeed on the magnitude of the stress intensity factors for all the problems considered. It also shows the importance of carrying out a proper contact mechanics analysis for a number of crack problems.

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List of Principal Symbols

The notation used in the main text and Appendix are briefly described in alphabetical order in the following list. Where there are two or more definitions for the same symbol, the relevant chapter or section is indicated in parentheses. When no such indication is provided, the definition for the symbol applies everywhere other than for the alternative definitions.

The tensor components used in the main text and Appendix are defined by the indices h, i, j, k, l . A repeated index implies summation over its range (1, 2 in two-dimensional analysis and 1, 2, 3 in three-dimensional analysis) and a comma-index is used to indicate differentiation with respect to the coordinate in the direction specified by the index.

a	— Dimension of crack size
A	— A coefficient matrix of the unknowns in the interface and the contact zone
b	— Vector consisting of products of known values with their matrix coefficients (Section 2.5)
b	— A known vector in the equation $Ax = b$ (Section 4.3)
b	— Radius of a cylinder (Section 5.3)
b	— Thickness of a lug (Section 6.2)
b_i	— Element of the vector b
c	— Radius of the crack front of a circumferential edge crack
C	— Coefficient matrix of the unknown displacements and tractions (Section 2.4)
C	— A coefficient matrix of the unknowns in the contact zone
$C_{,j}$	— Geometrical coefficients of the boundary integral equation
d	— Distance between two opposite points coming into contact
$D = Eh^3/[12(1 - \nu^2)]$	— Flexural rigidity of a plate (Section 5.3)

D_{kij}	— Third-order displacement tensor for stresses
E, E_1, E_2	— Young's modulus
f	— Coefficient of friction
f_i	— Components of the body force vector per unit volume (Section 2.2)
f_2, f_3	— Pseudo-coefficients of friction (Section 3.6)
F	— External force
g_i	— Components of gap between the two bodies in contact (Chapter 3)
G	— Coefficient matrix of tractions (Chapter 2)
G	— Shear modulus
h	— Thickness of a plate
h	— Sub-matrix of displacements
H	— Coefficient matrix of displacements
J	— Jacobian of transformation from global coordinates to the local coordinates
K_I	— Mode I stress intensity factor
K_{II}	— Mode II stress intensity factor
K_{III}	— Mode III stress intensity factor
l	— Length of the crack tip element (Chapter 2)
L	— General linear second order differential operator
L	— Lower triangular matrix (Section 4.3)
L_{ij}	— Differential operators
M, M_o	— Bending moment
n_c	— Number of the unknowns in the contact zone
n_{ij}	— Unit vector of local coordinates η_i in the global coordinates x_j
n_s	— Number of the unknowns in the separated zone
N_c	— Number of elements on the surface of a domain

N_n	— Number of node on the surface of a domain
N^c	— Shape functions for the coordinates at c -th node
N_d^c	— Shape functions for the displacements
N_t^c	— Shape functions for the tractions
p	— Contact pressure
P	— A boundary point
$\mathbf{P}$	— Permutation matrix
P_o	— An interior point
$P^{d(b,c)}$	— c -th node of the b -th element
P^a	— a -th node
$\mathbf{p}$	— Surface force in and near the region of contact
p_N	— Tensile normal component of the force $\mathbf{p}$
p_T	— Shearing component of the force $\mathbf{p}$
Q_o	— An interior point
Q	— A boundary point
Q	— A force (Section 3.6)
r	— Distance between point P and Q
r	— Polar coordinate of radial distance
r	— Distance from the crack front
R, R_1, R_2, R_3, R'_3	— Radii
S_{kij}	— Third-order stress tensor for displacements
$S_p = F/(2Rt)$	— Nominal stress of pin-loading
S_t	— Tensile stress applied at the end of a lug
t	— Thickness of a plate
$\mathbf{t}$	— Vector of tractions
t_i	— Traction components
T_k	— Kelvin's point-load solution of tractions
$\mathbf{u}$	— Vector of displacements
u_i	— Displacement components

U_{ki}	— Kelvin's point-load solution of displacements
v	— Crack-opening displacement of a cracked plate
$\mathbf{v}$	— Slip of body I with respect to body II (Section 3.2)
w	— Transverse deflection of a plate
$\mathbf{x}$	— Vector of unknown displacements and tractions
x, y, z	— Rectangular coordinates
x_i	— Cartesian global coordinates
β	— Load incremental ratio
$\delta, \delta_1, \delta_2$	— Clearance of the fits
$\delta(P, Q)$	— Dirac delta function
Δ	— Increments
ϕ	— Angular span of the contact zone
ϕ_1	— Angular span of the contact zone between a pin and a bushing
ϕ_2	— Angular span of the contact zone between a bushing and the hole of a lug
Γ	— Boundary of a body
Γ_c	— Contact boundary
Γ_{ca}	— Sub-path of Γ_c corresponding to the friction state of adhesion
Γ_{cm}	— Sub-path of Γ_c corresponding to the friction state of adhesion and slip
Γ_{cp}	— Sub-path of Γ_c corresponding to the friction state of open
Γ_{cs}	— Sub-path of Γ_c corresponding to the friction state of slip
$\Gamma_\epsilon, \Gamma'_\epsilon$	— Boundaries of the domain Ω_ϵ
η, η_1, η_2	— Intrinsic local coordinates
φ, φ'	— Angular position of a circle
$\kappa = 3 - 4\nu$	— For plane strain
$\kappa = (3 - \nu)/(1 + \nu)$	— For plane stress

ν	— Poisson's ratio
θ	— Polar coordinate of angle (Section 2.4)
θ	— Angle from the crack plane
θ	— Angle measured from the plane of symmetry to the line of loading
θ_c	— Angular span of the crack-tip elements
σ_{br}	— Bearing stress in a lug
σ_{ij}	— Stress tensors
$\sigma_{r\varphi}$	— Shear stress on the surface of a lug
$\sigma_{\varphi\varphi}$	— Tangential stress along the hole of a lug
σ_o, σ^*	— Stresses
Σ	— Summation
Ω	— Domain of a body
Ω_ϵ	— Part of a cylinder or a sphere, of radius ϵ
ξ_1, ξ_2	— Intrinsic local coordinates
ψ	— Angular position measured in the crack plane
ζ_1, ζ_2	— Intrinsic local coordinates

Superscripts

A, B	— Domains
I, II, III	— Domains
n	— Step of incremental loading
$*$	— Normalised variable

Subscripts

s	— Interface boundary between the sub-domains
c	— Contact boundary

Abbreviations

BEM	— Boundary element method
BIE	— Boundary integral equation
FEM	— Finite element method
QP	— Quadratic programming
SIF	— Stress intensity factor
2D	— Two dimensions
2-D	— Two-dimensional
3D	— Three dimensions
3-D	— Three-dimensional

Chapter 1

Introduction

The study of contact mechanics and fracture mechanics have proved to be very useful in enhancing the reliability and utilization of engineering structures. However, despite the fact that, in practice, structural components are often in contact with each other during load transfer, contact effects are seldom taken into account in stress analysis. The reason is that the modelling of contact phenomena poses serious difficulties—conceptual, mathematical and computational—which are far more complex than those encountered in classical linear structural mechanics. Contact problems are inherently nonlinear. Prior to the application of loads to a body, the actual contact surface which the bodies meet is unknown. The boundary conditions on this unknown surface involve unknown stresses and displacements. As a result, mathematical models of contact involve systems of inequalities or nonlinear equations. Moreover, when friction is present, multiple solutions of the equations describing contact can exist[60], and the description of the motion of the bodies in contact becomes extremely complex. Yet, the contact area and the stresses in its neighbourhood have a very significant effect on the surface durability, and the rate at which cracks, once formed around there, will grow.

In the fracture mechanics analysis of crack problems in linear elasticity, the primary objective would be the determination of stress intensity factors (SIF) which characterise the asymptotic stress fields in the vicinity of the crack-tip. These stress intensity factors are then compared to the fracture toughness of the material to es-

establish the structural integrity, or otherwise, of the component.

Among numerical methods, the finite element method (FEM) and the boundary element method (BEM) (also called as the boundary integral equation method—BIE method) have in recent years become the two most commonly used techniques for treating contact and crack problems. The superiority of FEM as a numerical tool lies in its general applicability and versatility for a variety of problems and structures. As far as FEM fracture mechanics analysis is concerned, special crack-tip elements, which incorporate the proper singularity of the asymptotic stress-strain-fields in the vicinity of the crack tip have been developed to obtain accurate stress intensity factor results. Nevertheless, a fairly refined mesh still needs to be used near the crack tip region. For FEM contact mechanics analysis, a refined mesh around the contact area is usually required to guarantee the accuracy and convergence of the analysis. If complex shapes of body are encountered, such as the helical gears recently treated by Vijaykar *et al* [142], the design of such meshes is an arduous task indeed.

In contrast to the FEM, the BEM is based on the discretization of the boundary integral equation of the problem. Thus it reduces the dimensions of the numerical problem by one. Preparing the boundary mesh becomes a relatively easier task. The effectiveness of the BEM to solve the problems with a rapidly stress-strain change rate, such as crack problems, has been well established for the past decade or so. But the BEM has the inherent disadvantage that its coefficient matrices are unsymmetric and dense. This presents some difficulties particularly with 3-D contact problems and when friction is encountered, since iterative methods are then required to solve a relatively large set of equations. Despite the advantages of the BEM and the demand for further developments to treat frictional contact fracture mechanics problems, relatively little has been reported in the literature. Reported works will be reviewed later but they all have several shortcomings and none in three dimensions has been presented, as far as the author is aware.

The present study is concerned with an analysis of 2-D and 3-D elastic contact problems involving the presence of cracks using the boundary element method. The

work primarily includes the following:

- (a) developing and implementing boundary element formulations of contact mechanics problems in 2-D and 3-D, which are capable of treating multi-body contact systems with or without cracks; and
- (b) investigation of the effects of the contact stresses on the stress intensity factors for cracks in pin-loaded lug joints. The motivation for this aspect of the study lies in the fact that pin-loaded lug joints provide some of the best practical examples of contact problems. They are, with respect to failure by fracture, also some of the most critical components in, for example, aircraft structures.

Chapter 2 starts with a review of the analytical formulation and numerical implementation of the BEM for two-dimensional (2-D) and three-dimensional (3-D) linear analysis of elastic crack problems using the multi-domain approach. The types of elements employed and the integration schemes are described in some detail. Also described are the special crack-tip elements used for fracture problems which properly model the near-tip fields and also provide a quick and direct means for obtaining the stress intensity factors.

Chapter 3 presents a BEM formulation for the analysis of 2-D and 3-D elastic contact problems with friction. This proceeds from a literature review of BEM analysis for elastic contact problems. The classical contact algorithm derived from using the load incremental theory and an iterative approach is employed to treat *a priori* unknown boundary conditions of the frictional contact problems. The solution of the system equations is discussed and several test problems are treated to demonstrate the veracity of the formulation and the convergence of the solutions.

A procedure for the analysis of 2-D contact problems involving cracks in the contact zone is presented in Chapter 4. This includes the derivation and treatment of the supplementary equations resulting from the use of the multi-domain technique, and the proper modelling for both the open and closed crack tips using special crack tip elements. Also, an efficient solver for the solution of the special system equations re-

sulting from the contact fracture mechanics problem is presented. All these represent new developments in the use of BEM in contact fracture mechanics. The BEM formulation is then applied to a number of angled crack problems involving crack surfaces in frictional contact. The extension of the iterative-load incremental approach of the BEM formulation to 3-D frictional contact mechanics analysis of fracture problems is discussed in Chapter 5. Four crack closure problems are then treated in 3D. The treatment of 3-D crack problems with contact mechanics analysis by the BEM, to the author's knowledge, has not been previously reported in the literature.

Finally, in Chapter 6, the multi-domain boundary element method developed for contact mechanics analysis, is applied to pin-loaded attachment lugs with either a single through-the-thickness crack or double symmetric corner cracks at the pin hole. For the single through-the-thickness cracks, the work consists of two parts. The first part deals with a crack in the lug without a bushing, the second part deals with a crack in the lug which has an interference-fit bushing. New stress intensity factor results are obtained for several cracked geometries and directions of pin-loading. The effects of friction, the types of fit, thickness of the bushing and the difference in material properties between the pin and the lug are investigated for the single through the thickness cracks. For the symmetric corner cracks, a neat-fit or full load assumption was made for the pin-lug assembly. Stress intensity factor results from the 3D BEM contact mechanics analysis are obtained for a few crack sizes. The effects of friction, difference in material properties between the pin and the lug are also investigated.

Chapter 2

Boundary Element Fracture Analysis in Elastostatics

2.1 Introduction

The main problem encountered in the static analysis of crack problems using the boundary element method (BEM) is the proper representation of the co-planar surfaces of the crack and the singular stress field at the crack-tip. In such an analysis, the primary aim is the determination of the crack-tip stress intensity factor which characterises the near-tip fields for the geometry and load conditions. Several BEM techniques have hitherto been developed specifically for treating crack problems. Three of the more established ones should perhaps be mentioned here. The first is the Green's function approach, see, e.g., Cruse[26]; the second is the displacement-discontinuity method, see, e.g. Guidera and Lardner[43], Crouch[23]; and the third is the multi-region method, see, e.g. Blandford, Ingraffea and Liggett[16]. While the Green's function method can, in theory, be applied to a wide range of crack shapes, the method has hitherto been limited to two dimensions. Its extension to three dimensions is hindered primarily by the development of a general Green's function for arbitrarily-shaped cracks. As for the displacement-discontinuity approach, it has two basic drawbacks. First, a complementary equation is needed to solve the system since the relative displacement and the total displacement are all unknowns. Second, when

tractions are only applied on the co-planar surfaces, there are no unique equation systems.

In the multi-region approach, the cracked body is treated as being made up of two sub-regions coupled together along an artificial boundary which includes the co-planar surfaces of the crack. The boundary integral equation is written for each sub-region and the appropriate equilibrium and continuity conditions are applied at the sub-region interface boundary. The method is relatively simple to implement and ample evidence exists which show it giving good accuracy of the solutions obtained. It is therefore the approach chosen for crack modelling in the present study. A review of the boundary element method is now in order.

2.2 Boundary integral equations in elastostatics

2.2.1 Differential equations of equilibrium

The differential equations of equilibrium for an elastic domain Ω at point P_o under a Cartesian co-ordinate system x_i can be written as

$$\sigma_{ij,j} + f_i = 0; \quad i, j = 1, 2 \text{ in 2D}; \quad i, j = 1, 2, 3 \text{ in 3D}; \quad \forall (P_o) \in \Omega \quad (2.1)$$

where σ_{ij} are the stress components; and f_i are the components of the body force vector per unit volume. In terms of displacements, Eq. (2.1) can be written as

$$L_{ij}u_j + f_i = 0; \quad i, j = 1, 2 \text{ in 2D}; \quad i, j = 1, 2, 3 \text{ in 3D}; \quad \forall (P_o) \in \Omega \quad (2.2)$$

where u_j are the displacement components; and L_{ij} are the differential operators. For linear homogeneous materials in 3D,

$$L_{ij}(\quad) = E_{ijkl}(\quad)_{,kl} \quad (2.3)$$

where

$$E_{ijkl} = \frac{\nu E}{(1 - \nu)(1 - 2\nu)} \delta_{ij} \delta_{kl} + G(\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \quad (2.4)$$

and ν , E and G are Poisson's ratio, Young's modulus and the shear modulus respectively. Also δ_{ij} is the Kronecker delta which has the value of unity when $i = j$ and zero when $i \neq j$. In 2D, under plane strain conditions,

$$[L_{ij}] = 2G \begin{bmatrix} \frac{1-\nu}{1-2\nu} \frac{\partial^2}{\partial x_1^2} + \frac{1}{2} \frac{\partial^2}{\partial x_2^2} & \frac{1}{2(1-2\nu)} \frac{\partial^2}{\partial x_1 \partial x_2} \\ \frac{1}{2(1-2\nu)} \frac{\partial^2}{\partial x_1 \partial x_2} & \frac{1-\nu}{1-2\nu} \frac{\partial^2}{\partial x_2^2} + \frac{1}{2} \frac{\partial^2}{\partial x_1^2} \end{bmatrix}. \quad (2.5)$$

In the general form for 2D and 3D, L_{ij} can be written as

$$L_{ij}(\mathbf{x}) = A_{ijkl}(\mathbf{x}) \epsilon_{kl}. \quad (2.6)$$

2.2.2 Green's theorem and Kelvin's solutions

The general linear second order differential operator L can be written as, see, e.g. Zauderer[154]

$$Lv = a_{ij}v_{,ij} + b_i v_{,i} + cv \quad (2.7)$$

where the coefficients a_{ij} , b_i and c are the continuous second order differentiable functions of variables x_i . The adjoint operator M is then formulated as

$$Mv^* = (a_{ij}v^*)_{,ij} - (b_i v^*)_{,i} + cv^*. \quad (2.8)$$

If $L = M$, then L is self-adjoint. It can be verified that

$$v^*Lv - vMv^* = P_{i,i} \quad (2.9)$$

where

$$P_i = a_{ij}v^*v_{,j} - v(a_{ij}v^*)_{,j} + b_i v^*v. \quad (2.10)$$

Applying the divergence theorem on Eq. (2.7) yields

$$\int_{\Omega} (v^*Lv - vMv^*)d\Omega = \int_{\Gamma} P_i n_i d\Gamma \quad (2.11)$$

where n_i is the unit outward normal on the boundary Γ of the domain Ω . In Eq. (2.11), let L be replaced with L_{ij} , v with u_j and v^* with U_{ki} respectively. It is obvious then

that L_{ij} is self-adjoint. Thus the following equations may be obtained:

$$\int_{\Omega} U_{ki}(L_{ij}u_j + f_i)d\Omega = \int_{\Omega} u_j L_{ji}U_{ki}d\Omega + \int_{\Gamma} P_i n_i d\Gamma + \int_{\Omega} f_i U_{ki}d\Omega. \quad (2.12)$$

The left-hand-side of the above equation is equal to zero, and substitution of Eq. (2.10) into it finally results in the following equation:

$$\int_{\Gamma} t_i U_{ki}d\Gamma - \int_{\Gamma} u_i T_{ki}d\Gamma + \int_{\Omega} u_j L_{ji}U_{ki}d\Omega + \int_{\Omega} f_i U_{ki}d\Omega = 0 \quad (2.13)$$

where t_i are the traction components on the boundary Γ , i.e.

$$t_i = A_{ijkl}u_{j,k}n_l = \sigma_{ij}n_j \quad (2.14)$$

$$T_{ki} = A_{ijml}U_{kj,m}n_l. \quad (2.15)$$

If U_{ki} is a displacement solution for a homogeneous elastic body, then

$$L_{ji}U_{ki} + F_j = 0 \quad (2.16)$$

where F_j are the body force components. Substituting Eq. (2.16) into Eq. (2.13) gives

$$\int_{\Gamma} t_i U_{ki}d\Gamma + \int_{\Omega} f_i U_{ki}d\Omega = \int_{\Gamma} T_{ki}u_i d\Gamma + \int_{\Omega} F_i u_i d\Omega. \quad (2.17)$$

Eq. (2.17) is known as Betti-Rayleigh's reciprocal work theorem. If Kelvin's point-load solution U_{kj} , see, e.g. Fung[34], is chosen,

$$L_{ij}U_{kj}(P_o, Q_o) + \delta_{ki}\delta(P_o, Q_o) = 0 \quad (2.18)$$

where $\delta(P_o, Q_o)$ is a Dirac delta function, such that

$$\int_{\Omega} u_i(Q)\delta(P, Q)d\Omega(Q) = u_i(P) \quad (2.19)$$

Substitution of Eqs. (2.18) and (2.19) into Eq. (2.13) and carrying out the usual limiting process, see, e.g. Eshelby[128] will result in:

$$\begin{aligned} u_i(P_o) = & \int_{\Gamma} U_{ij}(P_o, Q)t_j(Q)d\Gamma(Q) - \int_{\Gamma} T_{ij}(P_o, Q)u_j(Q)d\Gamma(Q) + \\ & + \int_{\Omega} U_{ij}(P_o, Q_o)f_j(Q_o)d\Omega(Q_o) \end{aligned} \quad (2.20)$$

Eq. (2.20) is often known as Somigliana's identity for the displacements at an interior point P_0 . The quantities $U_{ij}(P, Q)$ and $T_{ij}(P, Q)$ are the fundamental solutions to the governing differential equations. They represent the displacement and traction in the x_j -direction at Q due to the application of a unit concentrated load in the x_i -direction at P in an infinite elastic domain. In 2D, under plane strain conditions,

$$U_{ij}(P, Q) = \frac{1}{8\pi G(1-\nu)} [(3-4\nu)\delta_{ij} \ln \frac{1}{r} + r_{,i}r_{,j}] \quad (2.21)$$

$$T_{ij}(P, Q) = \frac{-1}{4\pi(1-\nu)} \left\{ \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i) \right\} \quad (2.22)$$

where r is the distance between point P and point Q ; and $\partial r / \partial n = r_{,i}n_i$. Replacing ν in the above equations with $\nu/(1+\nu)$ results in the corresponding expressions for the plane stress condition. In 3D, the corresponding expressions for the fundamental solutions are:

$$U_{ij}(P, Q) = \frac{1}{16\pi Gr} [(3-4\nu)\delta_{ij} + r_{,i}r_{,j}] \quad (2.23)$$

$$T_{ij}(P, Q) = \frac{-1}{8\pi(1-\nu)r^2} \left\{ \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij} + 3r_{,i}r_{,j}] - (1-2\nu)(r_{,i}n_j - r_{,j}n_i) \right\}. \quad (2.24)$$

2.2.3 Boundary integral equations

Let point P be on the boundary Γ , and be excluded from domain Ω by a small domain Ω_ϵ , as shown in Figure 2.1. For convenience, the domain Ω_ϵ is chosen to be that part of a cylinder (2D) or a sphere (3D) of radius ϵ centered at P . Writing Eqs. (2.13) and (2.18) for the domain $(\Omega - \Omega_\epsilon)$ gives

$$\begin{aligned} \int_{\Gamma-\Gamma'_\epsilon} T_{ij}(P, Q)u_j(Q)d\Gamma(Q) - \int_{\Gamma-\Gamma'_\epsilon} U_{ij}(P, Q)t_j(Q)d\Gamma(Q) + \int_{\Gamma_\epsilon} T_{ij}(P, Q)u_j(Q)d\Gamma(Q) \\ - \int_{\Gamma_\epsilon} U_{ij}(P, Q)t_j(Q)d\Gamma(Q) - \int_{\Omega-\Omega_\epsilon} U_{ij}(P, Q_0)f_j(Q_0)d\Omega(Q_0) = 0 \end{aligned} \quad (2.25)$$

where Γ_ϵ and Γ'_ϵ are the boundaries of the domain Ω_ϵ , as shown in Figure 2 By

carrying out the usual limiting process, it can be verified that

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma - \Gamma'_\epsilon} T_{ij}(P, Q) u_j(Q) d\Gamma(Q) = \int_{\Gamma} T_{ij}(P, Q) u_j(Q) d\Gamma(Q) \quad (2.26)$$

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma - \Gamma'_\epsilon} U_{ij}(P, Q) t_j(Q) d\Gamma(Q) = \int_{\Gamma} U_{ij}(P, Q) t_j(Q) d\Gamma(Q) \quad (2.27)$$

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega - \Omega'_\epsilon} U_{ij}(P, Q_o) f_j(Q_o) d\Gamma(Q_o) = \int_{\Omega} U_{ij}(P, Q_o) f_j(Q_o) d\Gamma(Q_o) \quad (2.28)$$

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} U_{ij}(P, Q) t_j(Q) d\Gamma(Q) = 0 \quad (2.29)$$

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} T_{ij}(P, Q) u_j(Q) d\Gamma(Q) = C_{ij}(P) u_j(P) \quad (2.30)$$

where

$$C_{ij}(P) = \lim_{\epsilon \rightarrow 0} \int_{\Gamma_\epsilon} T_{ij}(P, Q) d\Gamma(Q) \quad (2.31)$$

and is equal to $\frac{1}{2}\delta_{ij}$ if P lies on a smooth plane. Substituting the above equations into Eq. (2.25) gives the conventional boundary integral equation, see e.g., Rizzo[108] and Cruse[24]:

$$\begin{aligned} C_{ij}(P) u_j(P) + \int_{\Gamma} T_{ij}(P, Q) u_j(Q) d\Gamma(Q) - \int_{\Gamma} U_{ij}(P, Q) t_j(Q) d\Gamma(Q) \\ - \int_{\Omega} U_{ij}(P, Q_o) f_j(Q_o) d\Omega(Q_o) = 0 \end{aligned} \quad (2.32)$$

2.2.4 Calculation of stresses inside the domain

After the displacements and the tractions for all the points on the boundary have been obtained, by differentiating Eq. (2.20) and invoking strain-displacement relations and Hooke's law, the resulting expression, known as Somigliana's identity for stress, is obtained as follows.

$$\begin{aligned} \sigma_{ij}(P_o) = \int_{\Gamma} D_{kij}(P_o, Q) t_k(Q) d\Gamma(Q) - \int_{\Gamma} S_{kij}(P_o, Q) u_k(Q) d\Gamma(Q) + \\ \int_{\Omega} D_{kij}(P_o, Q_o) f_k d\Omega(Q_o) \end{aligned} \quad (2.33)$$

where for 3D,

$$D_{kij} = \frac{1}{8\pi(1-\nu)r^2} \{ 3r_{,k} r_{,i} r_{,j} + (1-2\nu)(\delta_{ki} r_{,j} + \delta_{kj} r_{,i} - \delta_{ij} r_{,k}) \} \quad (2.34)$$

$$\begin{aligned}
S_{kij} = & \frac{G}{4\pi(1-\nu)r^3} \left\{ 3 \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{ki}r_{,j} + \delta_{kj}r_{,i}) - 5r_{,k}r_{,i}r_{,j}] + \right. \\
& 3\nu(n_i r_{,k} r_{,j} + n_j r_{,i} r_{,k}) + (1-2\nu)(3n_k r_{,i} r_{,j} + n_j \delta_{ki} + n_i \delta_{kj}) \\
& \left. - (1-4\nu)n_k \delta_{ij} \right\}
\end{aligned} \tag{2.35}$$

and for 2D,

$$D_{kij} = \frac{1}{4\pi(1-\nu)r} \{ (1-2\nu)(\delta_{ki}r_{,j} + \delta_{kj}r_{,i} - \delta_{ij}r_{,k}) + 2r_{,i}r_{,j}r_{,k} \} \tag{2.36}$$

$$\begin{aligned}
S_{kij} = & \frac{G}{2\pi(1-\nu)r^2} \left\{ 2 \frac{\partial r}{\partial n} [(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{ki}r_{,j} + \delta_{kj}r_{,i}) - 4r_{,k}r_{,i}r_{,j}] + \right. \\
& 2\nu(n_i r_{,k} r_{,j} + n_j r_{,i} r_{,k}) + (1-2\nu)(2n_k r_{,i} r_{,j} + n_j \delta_{ki} + n_i \delta_{kj}) \\
& \left. - (1-4\nu)n_k \delta_{ij} \right\}
\end{aligned} \tag{2.37}$$

2.3 Numerical Solution of the BIE in 2D and 3D

In the present study, body forces are neglected. In what follows therefore, all terms involving f_j are set to zero. To solve the BIE, Eq. (2.32), numerically, the boundary is discretized into an assemblage of, say, N_e elements and N_n nodes. Piece-wise smooth variations of the geometry, the displacements and the tractions is assumed on each element. Each of these elements is defined by M nodes and an intrinsic local co-ordinate system, namely (ξ) in 2D and (ξ_1, ξ_2) in 3D, is introduced.

2.3.1 Boundary elements and shape functions

In the present 2D study, two- and three-node isoparametric elements, shown in Figure 2.2a, are used. The geometry x_i , displacements u_i and tractions t_i at any point over the element can be expressed as the products of their values at the nodes and the corresponding interpolation functions which are expressed in the local co-ordinate system. Thus,

$$x_i(\xi) = \sum_{c=1}^3 N^c(\xi) x_i^c \tag{2.38}$$

$$u_i(\xi) = \sum_{c=1}^3 N_d^c(\xi) u_i^c \quad (2.39)$$

$$t_i(\xi) = \sum_{c=1}^3 N_t^c(\xi) t_i^c \quad (2.40)$$

where $N^c = N_d^c = N_t^c$ except for the elements which are adjacent to the crack front in fracture mechanics analysis which will be discussed later. In Eqs.(2.38)-(2.40), the superscript c denotes the c -th node, and $N^c(\xi)$ are the shape functions defined as

$$N^2(\xi) = \begin{cases} 1 - \xi^2 & \text{if node 2 is included} \\ 0 & \text{if node 2 is excluded} \end{cases} \quad (2.41)$$

$$N^1(\xi) = \frac{1}{2}(1 - \xi) - \frac{1}{2}N^2(\xi) \quad (2.42)$$

$$N^3(\xi) = \frac{1}{2}(1 + \xi) - \frac{1}{2}N^2(\xi). \quad (2.43)$$

Substituting Eqs. (2.38)-(2.40) into Eq. (2.32) results in the following expression:

$$\begin{aligned} & C_{ij}(P^a) u_j(P^a) + \sum_{b=1}^{N_e} \sum_{c=1}^3 u_j(P^{d(b,c)}) \int_{\Gamma_b} T_{ij}(P^a, Q) N^c(\xi) J(\xi) d\xi \\ &= \sum_{b=1}^{N_e} \sum_{c=1}^3 t_j(P^{d(b,c)}) \int_{\Gamma_b} U_{ij}(P^a, Q) N^c(\xi) J(\xi) d\xi; \quad P^a = 1, N_n \end{aligned} \quad (2.44)$$

where $P^{d(b,c)}$ is the c -th node of the b -th element, $J(\xi)$ is the Jacobian of transformation from x_i to ξ , which along with the vector of unit outward normal $\vec{n}$, is obtained from the following:

$$\vec{n} = \frac{1}{J(\xi)}(g_1, g_2) \quad (2.45)$$

$$g_1 = \frac{dx_2}{d\xi}, \quad g_2 = -\frac{dx_1}{d\xi} \quad (2.46)$$

$$J(\xi) = \sqrt{g_1^2 + g_2^2}. \quad (2.47)$$

For fracture mechanics analysis, special shape functions to represent the $O(\sqrt{r})$ displacement variation, and the $O(\frac{1}{\sqrt{r}})$ traction variation near the crack tip, developed

by Jia, Rizzo and Shippy[50], are used. If the 1st node is at the crack tip, these shape functions are written as

$$\begin{aligned} N_d^1(\xi) &= 1 - \left(1 + \frac{\sqrt{2}}{2}\right)\sqrt{1+\xi} + \frac{\sqrt{2}}{2}(1+\xi), \\ N_d^2(\xi) &= (2 + \sqrt{2})\sqrt{1+\xi} - (1 + \sqrt{2})(1+\xi), \\ N_d^3(\xi) &= -\left(1 + \frac{\sqrt{2}}{2}\right)\left[\sqrt{1+\xi} - (1+\xi)\right] \end{aligned} \quad (2.48)$$

$$\begin{aligned} N_t^1(\xi) &= N_d^1(\xi)/\sqrt{1+\xi}, \\ N_t^2(\xi) &= N_d^2(\xi)/\sqrt{1+\xi}, \\ N_t^3(\xi) &= \sqrt{2}N_d^3(\xi)/\sqrt{1+\xi} \end{aligned} \quad (2.49)$$

where the subscripts "d" and "t" denote the shape functions for displacements and tractions, respectively. When the 3rd node is at the crack front, a similar set of shape functions can be obtained as

$$\begin{aligned} N_d^1(\xi) &= -\left(1 + \frac{\sqrt{2}}{2}\right)\left[\sqrt{1-\xi} - (1-\xi)\right], \\ N_d^2(\xi) &= (2 + \sqrt{2})\sqrt{1-\xi} - (1 + \sqrt{2})(1-\xi), \\ N_d^3(\xi) &= 1 - \left(1 + \frac{\sqrt{2}}{2}\right)\sqrt{1-\xi} + \frac{\sqrt{2}}{2}(1-\xi) \end{aligned} \quad (2.50)$$

$$\begin{aligned} N_t^1(\xi) &= \sqrt{2}N_d^1(\xi)/\sqrt{1-\xi}, \\ N_t^2(\xi) &= N_d^2(\xi)/\sqrt{1-\xi}, \\ N_t^3(\xi) &= N_d^3(\xi)/\sqrt{1-\xi}. \end{aligned} \quad (2.51)$$

In the case of 3D, two types of elements are used. The first is the conventional four- and eight-node quadrilateral isoparametric element, as shown in Figure 2.2b. The shape functions for the quadrilateral elements are

$$\begin{aligned} N^1(\xi_1, \xi_2) &= \frac{1}{4}(1 - \xi_1)(1 - \xi_2) - \frac{1}{2}\left[N^5(\xi_1, \xi_2) + N^8(\xi_1, \xi_2)\right], \\ N^2(\xi_1, \xi_2) &= \frac{1}{4}(1 + \xi_1)(1 - \xi_2) - \frac{1}{2}\left[N^5(\xi_1, \xi_2) + N^6(\xi_1, \xi_2)\right], \end{aligned}$$

$$\begin{aligned}
N^3(\xi_1, \xi_2) &= \frac{1}{4}(1 + \xi_1)(1 + \xi_2) - \frac{1}{2} [N^6(\xi_1, \xi_2) + N^7(\xi_1, \xi_2)] , \\
N^4(\xi_1, \xi_2) &= \frac{1}{4}(1 - \xi_1)(1 + \xi_2) - \frac{1}{2} [N^7(\xi_1, \xi_2) + N^8(\xi_1, \xi_2)] .
\end{aligned} \tag{2.52}$$

$$N^5(\xi_1, \xi_2) = \begin{cases} \frac{1}{2}(1 - \xi_1^2)(1 - \xi_2) & \text{if node 5 is included} \\ 0 & \text{if node 5 is excluded} \end{cases} \tag{2.53}$$

$$N^6(\xi_1, \xi_2) = \begin{cases} \frac{1}{2}(1 - \xi_2^2)(1 + \xi_1) & \text{if node 6 is included} \\ 0 & \text{if node 6 is excluded} \end{cases} \tag{2.54}$$

$$N^7(\xi_1, \xi_2) = \begin{cases} \frac{1}{2}(1 - \xi_1^2)(1 + \xi_2) & \text{if node 7 is included} \\ 0 & \text{if node 7 is excluded} \end{cases} \tag{2.55}$$

$$N^8(\xi_1, \xi_2) = \begin{cases} \frac{1}{2}(1 - \xi_2^2)(1 - \xi_1) & \text{if node 8 is included} \\ 0 & \text{if node 8 is excluded} \end{cases} \tag{2.56}$$

The second type of element used is the conventional three- and six-node triangle element, as shown in Figure 2.2c. The shape functions for the triangle elements are given as

$$\begin{aligned}
N^1(\xi_1, \xi_2) &= 1 - \xi_1 - \xi_2 - \frac{1}{2} [N^4(\xi_1, \xi_2) + N^6(\xi_1, \xi_2)] , \\
N^2(\xi_1, \xi_2) &= \xi_1 - \frac{1}{2} [N^4(\xi_1, \xi_2) + N^5(\xi_1, \xi_2)] , \\
N^3(\xi_1, \xi_2) &= \xi_2 - \frac{1}{2} [N^5(\xi_1, \xi_2) + N^6(\xi_1, \xi_2)] .
\end{aligned} \tag{2.57}$$

$$N^4(\xi_1, \xi_2) = \begin{cases} 4\xi_1(1 - \xi_1 - \xi_2) & \text{if node 4 is included} \\ 0 & \text{if node 4 is excluded} \end{cases} \tag{2.58}$$

$$N^5(\xi_1, \xi_2) = \begin{cases} 4\xi_1\xi_2 & \text{if node 5 is included} \\ 0 & \text{if node 5 is excluded} \end{cases} \tag{2.59}$$

$$N^6(\xi_1, \xi_2) = \begin{cases} 4\xi_2(1 - \xi_1 - \xi_2) & \text{if node 6 is included} \\ 0 & \text{if node 6 is excluded} \end{cases} \tag{2.60}$$

Substitution of isoparametric representations above into the BIE, Eq. (2.32), in the absence of body forces, again results in the following discretized form of the BIE.

$$\begin{aligned} C_{ij}(P^a)u_j(P^a) + \sum_{b=1}^{N_e} \sum_{c=1}^{8 \text{ or } 6} u_j(P^{d(b,c)}) \int_{\Gamma_b} T_{ij}(P^a, Q) N^c(\xi_1, \xi_2) J(\xi_1, \xi_2) d\xi_1 d\xi_2 \\ = \sum_{b=1}^{N_e} \sum_{c=1}^{8 \text{ or } 6} t_j(P^{d(b,c)}) \int_{\Gamma_b} U_{ij}(P^a, Q) N^c(\xi_1, \xi_2) J(\xi_1, \xi_2) d\xi_1 d\xi_2; \quad P^a = 1, N_n \end{aligned} \quad (2.61)$$

with these elements, the vector of the unit outward normal $\vec{n}$ and the Jacobian of transformation, $J(\xi_1, \xi_2)$, from x_i to ξ_i can be obtained as follows:

$$\vec{n} = \frac{\partial \vec{r}}{\partial \xi_1} \times \frac{\partial \vec{r}}{\partial \xi_2} \bigg/ \left| \frac{\partial \vec{r}}{\partial \xi_1} \times \frac{\partial \vec{r}}{\partial \xi_2} \right| = \frac{1}{J(\xi_1, \xi_2)} (g_1, g_2, g_3) \quad (2.62)$$

$$g_1 = \frac{\partial x_2}{\partial \xi_1} \frac{\partial x_3}{\partial \xi_2} - \frac{\partial x_3}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} \quad (2.63)$$

$$g_2 = \frac{\partial x_3}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_3}{\partial \xi_2} \quad (2.64)$$

$$g_3 = \frac{\partial x_1}{\partial \xi_1} \frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_1} \frac{\partial x_1}{\partial \xi_2} \quad (2.65)$$

$$J(\xi_1, \xi_2) = \sqrt{g_1^2 + g_2^2 + g_3^2} \quad (2.66)$$

As in 2D, special shape functions are chosen in fracture mechanics analysis to simulate the $O(\sqrt{r})$ displacement variation, and the $O(\frac{1}{\sqrt{r}})$ traction variation near the crack front. Quadrilateral elements as developed by Luchi and Poggialini[82] are used in this case. If the nodes 1-5-2 are at the crack front, these shape functions were given by

$$N_d^1(\xi_1, \xi_2) = \frac{1}{2}(1 - \xi_1) \left[-\xi_1 - (1 - \frac{\sqrt{2}}{2}\xi_1)\sqrt{1 + \xi_2} + \frac{\sqrt{2}}{2}(1 + \xi_2) \right],$$

$$N_d^2(\xi_1, \xi_2) = \frac{1}{2}(1 + \xi_1) \left[\xi_1 - (1 + \frac{\sqrt{2}}{2}\xi_1)\sqrt{1 + \xi_2} + \frac{\sqrt{2}}{2}(1 + \xi_2) \right],$$

$$N_d^3(\xi_1, \xi_2) = \frac{1}{4}(1 + \xi_1) \left[\sqrt{2}(\xi_1 - 2 - \sqrt{2})\sqrt{1 + \xi_2} + (2 + \sqrt{2})(1 + \xi_2) \right],$$

$$N_d^4(\xi_1, \xi_2) = \frac{1}{4}(1 - \xi_1) \left[-\sqrt{2}(\xi_1 + 2 + \sqrt{2})\sqrt{1 + \xi_2} + (2 + \sqrt{2})(1 + \xi_2) \right],$$

$$\begin{aligned}
N_d^5(\xi_1, \xi_2) &= (1 - \xi_1^2) \left[1 - \frac{\sqrt{2}}{2} \sqrt{1 + \xi_2} \right], \\
N_d^6(\xi_1, \xi_2) &= \frac{1}{2}(1 + \xi_1) \left[(2 + \sqrt{2})\sqrt{1 + \xi_2} - (1 + \sqrt{2})(1 + \xi_2) \right], \\
N_d^7(\xi_1, \xi_2) &= \frac{\sqrt{2}}{2}(1 - \xi_1^2)\sqrt{1 + \xi_2}, \\
N_d^8(\xi_1, \xi_2) &= \frac{1}{2}(1 - \xi_1) \left[(2 + \sqrt{2})\sqrt{1 + \xi_2} - (1 + \sqrt{2})(1 + \xi_2) \right] \quad (2.67)
\end{aligned}$$

$$\begin{aligned}
N_t^1(\xi_1, \xi_2) &= N_d^1(\xi_1, \xi_2)/\sqrt{1 + \xi_2}, \\
N_t^2(\xi_1, \xi_2) &= N_d^2(\xi_1, \xi_2)/\sqrt{1 + \xi_2}, \\
N_t^3(\xi_1, \xi_2) &= \sqrt{2}N_d^3(\xi_1, \xi_2)/\sqrt{1 + \xi_2}, \\
N_t^4(\xi_1, \xi_2) &= \sqrt{2}N_d^4(\xi_1, \xi_2)/\sqrt{1 + \xi_2}, \\
N_t^5(\xi_1, \xi_2) &= N_d^5(\xi_1, \xi_2)/\sqrt{1 + \xi_2}, \\
N_t^6(\xi_1, \xi_2) &= N_d^6(\xi_1, \xi_2)/\sqrt{1 + \xi_2}, \\
N_t^7(\xi_1, \xi_2) &= \sqrt{2}N_d^7(\xi_1, \xi_2)/\sqrt{1 + \xi_2}, \\
N_t^8(\xi_1, \xi_2) &= N_d^8(\xi_1, \xi_2)/\sqrt{1 + \xi_2} \quad (2.68)
\end{aligned}$$

The corresponding shape functions for the case when the crack front is at the other sides of the element may be obtained following the same procedures as in 2D analysis method. They are given in full in Appendix A.

2.4 Evaluation of equation coefficients

Eq. (2.44) and (2.61) represents a set of linear algebraic equations when all the integrals have been carried out. Except for those elements where the singular shape functions are imposed, the integrals are non-singular if the points P^a and Q do not belong to the same element. They may therefore be calculated using Gaussian-quadrature. For the singular integrals resulting from the singular shape functions, the singularity can be removed through a co-ordinate transformation. For example, in 2D, if the first node of the element is the crack-tip, and the shape functions in Eq. (2.49) are used,

the following co-ordinate transformation is introduced

$$\xi = \frac{1}{2}(1 + \eta)^2 - 1. \quad (2.69)$$

Since the Jacobian of the transformation in Eq. (2.69) becomes zero as the 1st node of the element is approached, the integral becomes non-singular. Similarly, in 3D if the nodes (1-5-2) lie on the crack front, and the shape functions in Eq. (2.68) are used, the coordinate transformation below may be introduced:

$$\begin{aligned} \xi_1 &= \eta_1, \\ \xi_2 &= \frac{1}{2}(1 + \eta_2)^2 - 1. \end{aligned} \quad (2.70)$$

Again the Jacobian of the transformation in Eq. (2.70) becomes zero as the crack front is approached, the singularity is thereby removed. Similar co-ordinate transformations can be used when the other end or sides of the element lies on the crack front.

For numerical integration of the non-singular integrals, the standard 6-point Gaussian quadrature formula with limits of $-1 \leq \xi \leq +1$ is used in 2D, while the corresponding 6×6 -point Gaussian quadrature formula is used in 3D over quadrilateral elements. For triangle elements in 3D, the intervals of integral of ξ_1 and ξ_2 are

$$0 \leq \xi_1 \leq 1, \quad 0 \leq \xi_2 \leq (1 - \xi_1) \quad (2.71)$$

therefore the following co-ordinate transformation would first be performed.

$$\begin{aligned} \xi_1 &= \frac{1}{2}(1 + \eta_1), \\ \xi_2 &= \frac{1}{4}(1 - \eta_1)(1 + \eta_2). \end{aligned} \quad (2.72)$$

The standard Gaussian quadrature formula can then be applied under the new local co-ordinate system (η_1, η_2) .

When the point P^a is a node of the element over which the integral is to be evaluated, i.e., $P^a = P^{d(b,c)}$ in Eq. (2.44) and Eq. (2.61), singularities of the kernel functions have to be eliminated first before Gaussian quadrature formula can be employed. This is because $U_{ij}(P^a, Q)$ and $T_{ij}(P^a, Q)$ are both singular, as can be seen from Eqs. (2.21)-(2.24). The evaluation of each these singular integrals is discussed below.

2.4.1 Evaluation of singular integrals involving $U_{ij}(P^a, Q)$ in 2D

When $P^a = P^{d(b,c)}$ in Eq. (2.44) and Eq. (2.61), two basic cases need to be separately considered. They are, first, the case when the element Γ_b is not a crack-tip or crack front element, and, second, the case when it is. The reason for this is because of the singular shape functions employed in the latter case for treating crack problems. In the first case, it can be seen from Eq. (2.21) that, for 2D, a singularity of $O(\ln r)$ exists at the point P^a . If the point P^a is the 1st node of the element, the interval of integral is divided into two sub-intervals, and the following co-ordinate transformations are performed:

$$\text{between node 1 and node 2: } \xi = \frac{1}{16}(1 + \eta)^4 - 1 \quad (2.73)$$

$$\text{between node 2 and node 3: } \xi = \frac{1}{2}(1 + \eta) \quad (2.74)$$

In Eq. (2.73), since the Jacobian of the transformation becomes zero as P^a is approached, the integral becomes non-singular. If P^a is the 2nd node of the element, the interval is also divided into two sub-interval and, the corresponding co-ordinate transformations are:

$$\text{between node 1 and node 2: } \xi = -\frac{1}{16}(1 - \eta)^4 \quad (2.75)$$

$$\text{between node 2 and node 3: } \xi = \frac{1}{16}(1 + \eta)^4 \quad (2.76)$$

When P^a is the 3rd node of the element, the interval is divided into two sub-intervals as well, and the co-ordinate transformations are

$$\text{between node 1 and node 2: } \xi = -\frac{1}{2}(1 - \eta) \quad (2.77)$$

$$\text{between node 2 and node 3: } \xi = 1 - \frac{1}{16}(1 - \eta)^4 \quad (2.78)$$

In the second case, it may be recalled that Γ_b is a crack-tip element. Consider the situation when, say, the crack-tip is the first node of the element. P^a can be the 1st, the 2nd or the 3rd node of Γ_b , irrespective of which the element is divided into two

sub-elements. When P^a is the 1st node, the transformations as follows are carried out:

$$\text{between node 1 and node 2: } \xi = \frac{1}{16}(1 + \eta)^4 - 1 \quad (2.79)$$

$$\text{between node 2 and node 3: } \xi = \frac{1}{2}(1 + \eta). \quad (2.80)$$

Similarly, when P^a is the 2nd point:

$$\text{between node 1 and node 2: } \xi = -\sin^2[\frac{1}{4}\pi(1 - \eta)] \quad (2.81)$$

$$\text{between node 2 and node 3: } \xi = \frac{1}{16}(1 + \eta)^4 \quad (2.82)$$

and when P^a is the 3rd node:

$$\text{between node 1 and node 2: } \xi = \frac{1}{16}(1 + \eta)^4 - 1 \quad (2.83)$$

$$\text{between node 2 and node 3: } \xi = 1 - \frac{1}{16}(1 - \eta)^4 \quad (2.84)$$

With these transformations, it can be easily verified that the singularity in the integral involving $U_{ij}(P^a, Q)$ will be removed in the η -co-ordinate system, thereby enabling it to be integrated by standard Gaussian quadrature formula. When the crack-tip is the third node of the element, a similar set of co-ordinate transformations need also be carried out. These details may also be found in Jia, Rizzo and Shippy[50].

2.4.2 Evaluation of singular integrals involving $U_{ij}(P^a, Q)$ in 3D

As in 2D, there are also two cases to consider to eliminate the singularities of the integrals. In the first case, the integrals over the elements which are not adjacent to the crack front, are considered. If the point P^a is the corner node of the quadrilateral element, the element is divided into two triangles, as shown in Figure 2.3a. A new local co-ordinate system η_i is introduced to replace ξ_i . When P^a is the 1st node, for example, over the first triangle consisting of nodes (1-2-3), let:

$$\xi_1 = \frac{1}{2}(1 + \eta_1)(1 + \eta_2) - 1.$$

$$\begin{aligned}\xi_2 &= \eta_2, \\ d\xi_1 d\xi_2 &= \frac{1}{2}(1 + \eta_2)d\eta_1 d\eta_2\end{aligned}\quad (2.85)$$

and over the second triangle consisting of nodes (1-3-4), let

$$\begin{aligned}\xi_1 &= \eta_2, \\ \xi_2 &= \frac{1}{2}(1 - \eta_1)(1 + \eta_2) - 1, \\ d\xi_1 d\xi_2 &= \frac{1}{2}(1 + \eta_2)d\eta_1 d\eta_2.\end{aligned}\quad (2.86)$$

In Eqs. (2.85) and (2.86), it can be seen that the Jacobian of the transformations become $O(r)$ near P^a , the integrals thus become non-singular. When P^a is a mid-side node of the element, the quadrilateral is divided into three triangles, as shown in Figure 2.3b. In the case when P^a is the 5th node of the element, for example, over the first triangle consisting of the nodes (1-5-4), let

$$\begin{aligned}\xi_1 &= -\frac{1}{2}(1 + \eta_2), \\ \xi_2 &= \frac{1}{2}(1 + \eta_1)(1 + \eta_2) - 1, \\ d\xi_1 d\xi_2 &= \frac{1}{4}(1 + \eta_2)d\eta_1 d\eta_2\end{aligned}\quad (2.87)$$

and over the second triangle consisting of the nodes (3-4-5), let

$$\begin{aligned}\xi_1 &= \frac{1}{2}\eta_1(1 + \eta_2), \\ \xi_2 &= \eta_2, \\ d\xi_1 d\xi_2 &= \frac{1}{2}(1 + \eta_2)d\eta_1 d\eta_2;\end{aligned}\quad (2.88)$$

and over the third triangle consisting of the nodes (2-3-5), let

$$\begin{aligned}\xi_1 &= \frac{1}{2}(1 + \eta_2), \\ \xi_2 &= \frac{1}{2}(1 - \eta_1)(1 + \eta_2) - 1, \\ d\xi_1 d\xi_2 &= \frac{1}{4}(1 + \eta_2)d\eta_1 d\eta_2\end{aligned}\quad (2.89)$$

Again, the transformation leads to a removal of the singularity of the integral in the η -co-ordinate system. Similar co-ordinate transformations as the above need to be carried out as P^a becomes the other nodes of the element.

For a triangle element, a polar coordinate system (r, θ) is set up to replace ξ_i . If the point P is the 1st node of the element, as shown in Figure 2.3c, the coordinate transformation is given as

$$\begin{aligned}\xi_1 &= r \cos \theta, \\ \xi_2 &= r \sin \theta, \\ d\xi_1 d\xi_2 &= r dr d\theta\end{aligned}\tag{2.90}$$

where

$$0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq r \leq \frac{\sin(\pi/4)}{\sin(\pi/4 + \theta)}\tag{2.91}$$

To use the standard Gaussian quadrature formula, a further coordinate transformation is carried out as follows. Let

$$\begin{aligned}\xi_1 &= \frac{(1 + \eta_2) \cos[\frac{\pi}{4}(1 + \eta_1)]}{2\sqrt{2} \cos(\frac{\pi}{4}\eta_1)}, \\ \xi_2 &= \frac{(1 + \eta_2) \sin[\frac{\pi}{4}(1 + \eta_1)]}{2\sqrt{2} \cos(\frac{\pi}{4}\eta_1)}\end{aligned}\tag{2.92}$$

thus

$$d\xi_1 d\xi_2 = \frac{\pi(1 + \eta_2)}{32 \cos^2(\frac{\pi}{4}\eta_1)} d\eta_1 d\eta_2\tag{2.93}$$

when P^a is the 2nd node of the element, the coordinate transformation is given as

$$\begin{aligned}\xi_1 &= \frac{1}{2}(1 - \eta_2), \\ \xi_2 &= \frac{1}{2}(1 + \eta_2) \tan[\frac{\pi}{8}(1 + \eta_1)]\end{aligned}\tag{2.94}$$

and again the Jacobian of transformation becomes $O(r)$ near the point P^a , since

$$d\xi_1 d\xi_2 = \frac{\pi(1 + \eta_2)}{32 \cos^2[\frac{\pi}{8}(1 + \eta_1)]} d\eta_1 d\eta_2\tag{2.95}$$

When P^a is the 3rd node of the element, the coordinate transformation is given as

$$\begin{aligned}\xi_1 &= \frac{1}{2}(1 + \eta_2) \tan\left[\frac{\pi}{8}(1 + \eta_1)\right], \\ \xi_2 &= \frac{1}{2}(1 - \eta_2)\end{aligned}\quad (2.96)$$

the Jacobian of transformation is the same as in Eq. (2.95). When the point P^a is the 4th node of the element, the interval of integral is divided to two triangles as shown in Figure 2.3d. Over the first triangle consisting of the nodes (1-4-3), let

$$\begin{aligned}\xi_1 &= \frac{1}{4}(1 - \eta_2), \\ \xi_2 &= \frac{1}{4}(1 + \eta_2) \tan\left[\frac{1}{2}(1 + \eta_1) \arctan(2)\right]\end{aligned}\quad (2.97)$$

so

$$d\xi_1 d\xi_2 = \frac{(1 + \eta_2) \arctan(2)}{32 \cos^2\left[\frac{1}{2}(1 + \eta_1) \arctan(2)\right]} d\eta_1 d\eta_2. \quad (2.98)$$

Over the second triangle consisting of the nodes (2-3-4),

$$\begin{aligned}\xi_1 &= \frac{1}{2} + \frac{\sqrt{2}(1 + \eta_2) \cos\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2)\right]}{8 \cos\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2) - \frac{\pi}{4}\right]}, \\ \xi_2 &= \frac{\sqrt{2}(1 + \eta_2) \sin\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2)\right]}{8 \cos\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2) - \frac{\pi}{4}\right]}, \\ d\xi_1 d\xi_2 &= \frac{(1 + \eta_2)(\pi - \arctan 2)}{64 \cos^2\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2) - \frac{\pi}{4}\right]} d\eta_1 d\eta_2.\end{aligned}\quad (2.99)$$

When P^a is the 5th node, the interval of integral is divided into two triangles. Over the first triangle consisting of the nodes (5-3-1), a similar transformation is carried out by letting

$$\begin{aligned}\xi_1 &= \frac{1}{4}(1 - \eta_2), \\ \xi_2 &= \frac{1}{2} + \frac{1}{4}(1 + \eta_2) \tan\left(\frac{\pi}{4}\eta_1\right)\end{aligned}\quad (2.100)$$

so

$$d\xi_1 d\xi_2 = \frac{\pi(1 + \eta_2)}{64 \cos^2\left(\frac{\pi}{4}\eta_1\right)} d\eta_1 d\eta_2 \quad (2.101)$$

while over the second triangle consisting of the nodes (5-1-2), it is done by letting

$$\begin{aligned}\xi_1 &= \frac{1}{2} + \frac{1}{4}(1 + \eta_2) \tan\left(\frac{\pi}{4}\eta_1\right), \\ \xi_2 &= \frac{1}{4}(1 - \eta_2)\end{aligned}\quad (2.102)$$

so

$$d\xi_1 d\xi_2 = \frac{\pi(1 + \eta_2)}{64 \cos^2\left(\frac{\pi}{4}\eta_1\right)} d\eta_1 d\eta_2. \quad (2.103)$$

If P^a is the 6th node of the element, over the first triangle consisting of the nodes (6-2-3), let

$$\begin{aligned}\xi_1 &= \frac{1}{4}(1 + \eta_2) \tan\left[\frac{1}{2}(1 + \eta_1) \arctan(2)\right], \\ \xi_2 &= \frac{1}{4}(1 - \eta_2)\end{aligned}\quad (2.104)$$

the Jacobian of transformation is

$$\frac{d\xi_1 d\xi_2}{d\eta_1 d\eta_2} = \frac{(1 + \eta_2) \arctan(2)}{32 \cos^2\left[\frac{1}{2}(1 + \eta_1) \arctan(2)\right]} \quad (2.105)$$

and over the second triangle consisting of the nodes (6-1-2), let

$$\begin{aligned}\xi_1 &= \frac{\sqrt{2}(1 + \eta_2) \sin\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2)\right]}{8 \cos\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2) - \frac{\pi}{4}\right]}, \\ \xi_2 &= \frac{1}{2} + \frac{\sqrt{2}(1 + \eta_2) \cos\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2)\right]}{8 \cos\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2) - \frac{\pi}{4}\right]}\end{aligned}\quad (2.106)$$

so

$$d\xi_1 d\xi_2 = \frac{(1 + \eta_2)(\pi - \arctan 2)}{64 \cos^2\left[\frac{1}{2}(1 + \eta_1)(\pi - \arctan 2) - \frac{\pi}{4}\right]} d\eta_1 d\eta_2. \quad (2.107)$$

Now consider the second case, when the quadrilateral element is adjacent to the crack front. In addition to the above coordinate transformations to treat the singularity of the kernel functions, further transformations are needed to eliminate the singularity introduced by the singular shape functions $N_i(\xi_1, \xi_2)$ in Eq. (2.68). Referring to Fig. 2.3 and assuming that the nodes (1-5-2) lie on the crack front, these

transformations are carried out as follows. If the point P^a is the 1st node of the element, over the first triangle consisting of the nodes (1-3-4), let

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= \frac{1}{2}(1 + \zeta_2) - 1\end{aligned}\quad (2.108)$$

so

$$d\eta_1 d\eta_2 = (1 + \zeta_2) d\zeta_1 d\zeta_2 \quad (2.109)$$

and over the second triangle consisting of the nodes (1-2-3), let

$$\begin{aligned}\eta_1 &= 1 - \frac{1}{2}(1 - \zeta_1)^2, \\ \eta_2 &= \frac{1}{2}(1 + \zeta_2)^2 - 1\end{aligned}\quad (2.110)$$

so

$$d\eta_1 d\eta_2 = (1 - \zeta_1)(1 + \zeta_2) d\zeta_1 d\zeta_2. \quad (2.111)$$

If P^a is the 2nd node of the element, over the first triangle consisting of the nodes (2-4-1), let

$$\begin{aligned}\eta_1 &= \frac{1}{2}(1 + \zeta_1)^2 - 1, \\ \eta_2 &= \frac{1}{2}(1 + \zeta_2)^2 - 1\end{aligned}\quad (2.112)$$

so

$$d\eta_1 d\eta_2 = (1 + \zeta_1)(1 + \zeta_2) d\zeta_1 d\zeta_2 \quad (2.113)$$

and over the second triangle consisting of the nodes (2-3-4),

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= \frac{1}{2}(1 + \zeta_2)^2 - 1\end{aligned}\quad (2.114)$$

so

$$d\eta_1 d\eta_2 = (1 + \zeta_2) d\zeta_1 d\zeta_2. \quad (2.115)$$

When P^a is the 3rd node of the element, over the first triangle consisting of the nodes (3-4-1), let

$$\begin{aligned}\eta_1 &= \frac{1}{2}(1 + \zeta_1)^2 - 1, \\ \eta_2 &= \zeta_2\end{aligned}\tag{2.116}$$

so

$$d\eta_1 d\eta_2 = (1 + \zeta_1) d\zeta_1 d\zeta_2\tag{2.117}$$

and over the second triangle consisting of the nodes (3-1-2), let

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= 1 - \frac{1}{2}(1 - \zeta_2)^2\end{aligned}\tag{2.118}$$

so

$$d\eta_1 d\eta_2 = (1 - \zeta_2) d\zeta_1 d\zeta_2.\tag{2.119}$$

When P^a is the 4th node of the element, over the first triangle consisting of the nodes (4-1-2), let

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= 1 - \frac{1}{2}(1 - \zeta_2)^2\end{aligned}\tag{2.120}$$

so

$$d\eta_1 d\eta_2 = (1 - \zeta_2) d\zeta_1 d\zeta_2\tag{2.121}$$

and over the second triangle consisting of the nodes (4-2-3), let

$$\begin{aligned}\eta_1 &= 1 - \frac{1}{2}(1 - \zeta_1)^2, \\ \eta_2 &= \zeta_2\end{aligned}\tag{2.122}$$

so

$$d\eta_1 d\eta_2 = (1 - \zeta_1) d\zeta_1 d\zeta_2.\tag{2.123}$$

If P^a is the 5th node of the element, over the first triangle consisting of the nodes (5-4-1), let

$$\begin{aligned}\eta_1 &= \frac{1}{2}(1 + \zeta_1)^2 - 1, \\ \eta_2 &= \frac{1}{2}(1 + \zeta_2)^2 - 1\end{aligned}\quad (2.124)$$

thus

$$d\eta_1 d\eta_2 = (1 + \zeta_1)(1 + \zeta_2)d\zeta_1 d\zeta_2 \quad (2.125)$$

and over the second triangle consisting of the nodes (5-3-4), let

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= \frac{1}{2}(1 + \zeta_2)^2 - 1\end{aligned}\quad (2.126)$$

thus

$$d\eta_1 d\eta_2 = (1 + \zeta_2)d\zeta_1 d\zeta_2 \quad (2.127)$$

and over the third triangle consisting of the nodes (5-2-3), let

$$\begin{aligned}\eta_1 &= 1 - \frac{1}{2}(1 - \zeta_1)^2, \\ \eta_2 &= \frac{1}{2}(1 + \zeta_2)^2 - 1\end{aligned}\quad (2.128)$$

thus

$$d\eta_1 d\eta_2 = (1 - \zeta_1)(1 + \zeta_2)d\zeta_1 d\zeta_2. \quad (2.129)$$

If P^a is the 6th node of the element, over the first triangle consisting of the nodes (6-1-2), let

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= 1 - \frac{1}{2}(1 - \zeta_2)^2\end{aligned}\quad (2.130)$$

thus

$$d\eta_1 d\eta_2 = (1 - \zeta_2)d\zeta_1 d\zeta_2 \quad (2.131)$$

and over the second triangle consisting of the nodes (6-4-1), let

$$\begin{aligned}\eta_1 &= \frac{1}{2}(1 + \zeta_1)^2 - 1, \\ \eta_2 &= \zeta_2\end{aligned}\tag{2.132}$$

thus

$$d\eta_1 d\eta_2 = (1 + \zeta_1) d\zeta_1 d\zeta_2\tag{2.133}$$

and over the third triangle consisting of the nodes (6-3-4),

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= \zeta_2\end{aligned}\tag{2.134}$$

thus

$$d\eta_1 d\eta_2 = d\zeta_1 d\zeta_2.\tag{2.135}$$

When P^a is the 7th node of the element, over the first triangle consisting of the nodes (7-4-1), let

$$\begin{aligned}\eta_1 &= \frac{1}{2}(1 + \zeta_1)^2 - 1, \\ \eta_2 &= \zeta_2\end{aligned}\tag{2.136}$$

thus

$$d\eta_1 d\eta_2 = (1 + \zeta_1) d\zeta_1 d\zeta_2\tag{2.137}$$

and over the second triangle consisting of the nodes (7-1-2), let

$$\begin{aligned}\eta_1 &= \zeta_1, \\ \eta_2 &= 1 - \frac{1}{2}(1 - \zeta_2)^2\end{aligned}\tag{2.138}$$

thus

$$d\eta_1 d\eta_2 = (1 - \zeta_2) d\zeta_1 d\zeta_2\tag{2.139}$$

and over the third triangle consisting of the nodes (7-2-3), let

$$\begin{aligned}\eta_1 &= 1 - \frac{1}{2}(1 - \zeta_1)^2, \\ \eta_2 &= \zeta_2\end{aligned}\tag{2.140}$$

thus

$$d\eta_1 d\eta_2 = (1 - \zeta_1) d\zeta_1 d\zeta_2. \quad (2.141)$$

Finally, when P^a is the 8th node of the element, over the first triangle consisting of the nodes (8-1-2), let

$$\begin{aligned} \eta_1 &= \zeta_1, \\ \eta_2 &= 1 - \frac{1}{2}(1 - \zeta_2)^2 \end{aligned} \quad (2.142)$$

thus

$$d\eta_1 d\eta_2 = (1 - \zeta_2) d\zeta_1 d\zeta_2 \quad (2.143)$$

and over the second triangle consisting of the nodes (8-2-3), let

$$\begin{aligned} \eta_1 &= 1 - \frac{1}{2}(1 - \zeta_1)^2, \\ \eta_2 &= \zeta_2 \end{aligned} \quad (2.144)$$

thus

$$d\eta_1 d\eta_2 = (1 - \zeta_1) d\zeta_1 d\zeta_2 \quad (2.145)$$

and over the third triangle consisting of the nodes (8-3-4), let

$$\begin{aligned} \eta_1 &= \zeta_1, \\ \eta_2 &= \zeta_2 \end{aligned} \quad (2.146)$$

thus

$$d\eta_1 d\eta_2 = d\zeta_1 d\zeta_2. \quad (2.147)$$

When the crack front lies on the other directions of the element, similar coordinate transformations as the above may be obtained. In all cases it can be easily verified that the integrals of the kernel-shape function products will not be singular after the above coordinate transformations.

2.4.3 Treatment of singular integrals involving $T_{ij}(P^a, Q)$ in 2D and 3D

When $P^a = P^{d(b,c)}$ in Eq. (2.44) and Eq. (2.61), the integrals involving $T_{ij}(P^a, Q)$ are strongly singular, as these functions are $O(\frac{1}{r})$ in 2D and $O(\frac{1}{r^2})$ in 3D. No quadrature formula exists to numerically evaluate these integrals which exist only in the Cauchy principal value sense. In matrix form, Eq. (2.44) and Eq. (2.61) may be written as

$$[H] \{u\} = [G] \{t\} \quad (2.148)$$

where $[H]$ consists of $N_n \times N_n$ sub-matrices $[h]_{2 \times 2}$ in 2D and $[h]_{3 \times 3}$ in 3D. Note too that the singular integrals involving $T_{ij}(P^a, Q)$ together with the free term $C_{ij}(P^a)$ contribute to the coefficients of the leading diagonal sub-matrices of $[H]$. Following Lachat and Watson[74], these coefficients may be obtained indirectly by considering rigid-body motion in each of the i -directions. The stress state of the elastic domain is such that $\{t\} = 0$ throughout. Thus writing Eq. (2.44) and Eq. (2.61) for this state will result in

$$[h]_{ll} = - \sum_{m=1, m \neq l}^{N_n} [h]_{lm} \quad (2.149)$$

where $[h]_{ll}$ is the diagonal sub-matrix corresponding to the l -th node, and $[h]_{lm}$ is the sub-matrix at the position of the l -th "row" and the m -th "column" of matrix $[H]$.

2.5 Multi-region approach for crack problems

Consider the domain Ω being divided into two sub-domains, say, Ω^A and Ω^B . Following the procedure described above for the BIE discretization, two sets of equations can be obtained, as written in matrix form below:

$$[H^A \ H_s^A] \begin{Bmatrix} u^A \\ u_s^A \end{Bmatrix} = [G^A \ G_s^A] \begin{Bmatrix} t^A \\ t_s^A \end{Bmatrix} \quad (2.150)$$

$$[H^B \ H_s^B] \begin{Bmatrix} u^B \\ u_s^B \end{Bmatrix} = [G^B \ G_s^B] \begin{Bmatrix} t^B \\ t_s^B \end{Bmatrix} \quad (2.151)$$

where the superscripts denote the sub-domains, and the subscript s denotes the interface boundary between these sub-domains. Since the displacements along the interface are the same, and the tractions along these are equal but in the opposite directions,

$$u_s = u_s^A = u_s^B \quad (2.152)$$

$$t_s = t_s^A = -t_s^B \quad (2.153)$$

Substituting Eqs. (2.152) and (2.153) into Eqs. (2.150) and (2.151) results in the following coupled set of equations:

$$\begin{bmatrix} \mathbf{H}^A & \mathbf{0} & \mathbf{H}_s^A \\ \mathbf{0} & \mathbf{H}^B & \mathbf{H}_s^B \end{bmatrix} \begin{Bmatrix} \mathbf{u}^A \\ \mathbf{u}^B \\ \mathbf{u}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{G}^A & \mathbf{0} & \mathbf{G}_s^A \\ \mathbf{0} & \mathbf{G}^B & -\mathbf{G}_s^B \end{bmatrix} \begin{Bmatrix} \mathbf{t}^A \\ \mathbf{t}^B \\ \mathbf{t}_s \end{Bmatrix}. \quad (2.154)$$

Putting the unknowns in the above equation set on the left-hand-side, and the products of the knowns and the corresponding coefficients on the right-hand-side, Eq. (2.154) can be finally written as

$$[\mathbf{C}]\{\mathbf{x}\} = \{\mathbf{b}\} \quad (2.155)$$

where $\{\mathbf{x}\}$ contains the unknown nodal displacements and tractions. Since $[\mathbf{C}]$ in Eq. (2.155) is not fully populated, a block Gaussian elimination approach may be used to obtain the solution of the linear algebraic equations. This approach is summarized below.

First, put the unknowns in Eq. (2.150) on the left-hand-side, and the products of the knowns and the corresponding coefficients on the right-hand-side, so that Eq. (2.150) may become

$$\begin{bmatrix} \mathbf{C}^A & \mathbf{H}_s^A & -\mathbf{G}_s^A \end{bmatrix} \begin{Bmatrix} \bar{\mathbf{x}}^A \\ \mathbf{u}_s^A \\ \mathbf{t}_s^A \end{Bmatrix} = \{\mathbf{b}^A\} \quad (2.156)$$

and similarly, Eq. (2.151) may become

$$\begin{bmatrix} \bar{C}^B & \bar{H}_i^B & -\bar{G}_i^B \end{bmatrix} \begin{Bmatrix} \bar{x}^B \\ u_i^B \\ t_i^B \end{Bmatrix} = \begin{Bmatrix} \bar{b}^B \end{Bmatrix} \quad (2.157)$$

Obviously, the number of unknowns in Eq. (2.156) and Eq. (2.157) exceeds the number of the equations.

Second, perform the partial Gaussian elimination method for the sets of equations for the individual sub-domains, say, Ω^A , so Eq. (2.156) becomes

$$\begin{bmatrix} \bar{C}^A & \bar{H}_i^A & \bar{G}_i^A \\ 0 & \hat{H}_i^A & \hat{G}_i^A \end{bmatrix} \begin{Bmatrix} \bar{x}^A \\ u_i^A \\ t_i^A \end{Bmatrix} = \begin{Bmatrix} \bar{b}^A \\ \hat{b}_i^A \end{Bmatrix} \quad (2.158)$$

where the number of the rows of the matrix $\hat{H}_i^A$ or $\hat{G}_i^A$ are equal to the unknowns along the interface. Similarly, the following set of equations may thus be obtained for the sub-domain Ω^B from Eq. (2.157).

$$\begin{bmatrix} \bar{C}^B & \bar{H}_i^B & \bar{G}_i^B \\ 0 & \hat{H}_i^B & \hat{G}_i^B \end{bmatrix} \begin{Bmatrix} \bar{x}^B \\ u_i^B \\ t_i^B \end{Bmatrix} = \begin{Bmatrix} \bar{b}^B \\ \hat{b}_i^B \end{Bmatrix} \quad (2.159)$$

where the number of the rows of the matrix $\hat{H}_i^B$ or $\hat{G}_i^B$ are equal to the unknowns along the interface. The two following equation systems may then be extracted from Eq. (2.158) and Eq. (2.159), respectively.

$$\begin{bmatrix} \hat{H}_i^A & \hat{G}_i^A \end{bmatrix} \begin{Bmatrix} u_i^A \\ t_i^A \end{Bmatrix} = \{\hat{b}_i^A\} \quad (2.160)$$

$$\begin{bmatrix} \hat{H}_i^B & \hat{G}_i^B \end{bmatrix} \begin{Bmatrix} u_i^B \\ t_i^B \end{Bmatrix} = \{\hat{b}_i^B\} \quad (2.161)$$

Third, couple Eq. (2.160) and Eq. (2.161) to form the following one system of equations, by using Eq. (2.152) and Eq. (2.153).

$$\begin{bmatrix} \hat{H}_i^A & \hat{G}_i^A \\ \hat{H}_i^B & -\hat{G}_i^B \end{bmatrix} \begin{Bmatrix} u_i \\ t_i \end{Bmatrix} = \begin{Bmatrix} \hat{b}_i^A \\ \hat{b}_i^B \end{Bmatrix} \quad (2.162)$$

in the above system of linear equations. the number of the unknowns is equal to the number of the equations.

Finally, apply the standard Gaussian elimination method to solve Eq. (2.162) to obtain the unknowns along the interface of the sub-domains.

2.6 Evaluation of stress intensity factors

For a crack in a body under general conditions of loading, three independent modes of deformation of the crack surfaces may be identified, Irwin[48]. They may, with reference to Figure 2.4, be referred to as follows, for a crack lying in the $x_1 - x_3$ plane:

Opening mode—mode I: In this mode, the two crack surfaces are pulled apart in the x_2 direction, and the deformation is symmetric about the $x_1 - x_3$ and $x_1 - x_2$ planes. Under plane strain conditions, the components of stress and displacement in the crack tip region are given by

$$\begin{aligned}\sigma_{11} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right], \\ \sigma_{22} &= \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right], \\ \sigma_{12} &= \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}, \\ u_1 &= \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[1 - 2\nu + \sin^2 \frac{\theta}{2} \right], \\ u_2 &= \frac{K_I}{G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[1 - 2\nu + \sin^2 \frac{\theta}{2} \right]\end{aligned}\quad (2.163)$$

Sliding mode—mode II: In this mode, the two crack surfaces slide over each other in the x_1 direction, and the deformation is symmetric about the $x_1 - x_2$ plane, but skew symmetric about the $x_1 - x_3$ plane. Under plane strain conditions, the components of stress and displacement in the crack tip region are given by

$$\sigma_{11} = -\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left[2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right].$$

$$\begin{aligned}
\sigma_{22} &= \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2}, \\
\sigma_{12} &= \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left[1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right], \\
u_1 &= \frac{K_{II}}{G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left[1 - 2\nu + \sin^2 \frac{\theta}{2} \right], \\
u_2 &= \frac{K_{II}}{G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[1 - 2\nu - \sin^2 \frac{\theta}{2} \right]
\end{aligned} \tag{2.164}$$

Tearing mode—mode III: In the tearing mode, the two crack surfaces slide over each other in the x_3 direction, and the deformation is skew symmetric about both the $x_1 - x_2$ and $x_1 - x_3$ planes. The components of stress and displacement in the crack tip region are given by

$$\begin{aligned}
\sigma_{13} &= -\frac{K_{III}}{\sqrt{2\pi r}} \sin \frac{\theta}{2}, \\
\sigma_{23} &= \frac{K_{III}}{\sqrt{2\pi r}} \cos \frac{\theta}{2}, \\
u_2 &= \frac{K_{III}}{G} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left[2\nu - 1 + \sin^2 \frac{\theta}{2} \right], \\
u_3 &= \frac{K_{III}}{G} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2}
\end{aligned} \tag{2.165}$$

In Eqs. (2.163)–(2.165), r is the radial distance from the crack tip and θ is the angle from the plane of crack as shown in Figure 2.4. The parameters K_I , K_{II} and K_{III} are known as the *stress intensity factors* corresponding to the three modes of deformation and they characterise the magnitude of the crack tip stress field. Of significance to note is that the stress field possesses a $1/\sqrt{r}$ singularity at the crack tip, while the displacement field is $O(\sqrt{r})$.

2.6.1 Stress intensity factor formulas in 2D

The determination of stress intensity factors for cracked bodies using the BEM has been a subject of numerous studies in the past fifteen years or so, see, e.g. Cruse[25], Tan and Fenner[126], Blandford et al[16], Martinez and Dominguez[86], Luchi and

Poggialini[82], and Jia et al[50, 51]. Extrapolation techniques, in conjunction with the well-known quarter-point crack-tip element, were largely used in the earlier studies for the determination of the stress intensity factors. This followed the developments in the progress of the finite element method for crack problems. For BEM application, Blandford et al.[16] derived an expression that enables the stress intensity factor to be obtained directly from the computed nodal displacements on the quarter-point crack-tip element. The expression, commonly known in the BEM literature as the "two-point displacement formula", is obtained by equating the classical displacement field solution with the $\sqrt{r}$ -variation of the displacements formed by shifting the mid-side node of the quadratic isoparametric element to the quarter-point adjacent to the crack-tip, see, Barsoum[12]. Smith and Mason[119] also derived another expression for obtaining stress intensity factors, following essentially the same procedure as Blandford et al.[16]. But, instead of taking the displacements for all the nodes of the element, they considered just the displacements of the node closest to the crack-tip, hence name the "one-point displacement formula" for their expression.

An alternative approach was suggested by Martinez and Dominguez[86] in which the crack-tip nodal traction coefficients are used instead. They obtained a simple expression, known as in the BEM literature as the "traction formula", for calculating stress intensity factors directly from the computed nodal "traction" at the crack-tip. The formula can be obtained by equating the coefficients of the $\frac{1}{\sqrt{r}}$ term in the traction variation over the quarter-point crack-tip element to that of the classical stress field solution ahead of the crack. These authors also demonstrated that the accuracy of the solutions from the traction formula is significantly less sensitive to mesh design than that obtained using the displacement formulas.

Jia et al[50] used traction-singular crack-tip elements as well in their BEM study with quadratic isoparametric elements, but with mid-side nodes instead of those with quarter-points. These elements have been discussed in Section (2.3) earlier. By equating the variations of the displacements and the tractions along the crack tip elements represented by the above shape functions to the classical field solution,

Eqs. (2.163)–(2.165), formulas for the direct evaluation of the stress intensity factors may be obtained in terms of the computed nodal quantities on these elements. They have also been referred to as the ‘displacement formula’ and the ‘traction formula’ respectively. With reference to Figure 2.5a, they are given below:

General displacement formula:

$$\begin{aligned} K_I &= \frac{G(2 + \sqrt{2})}{\kappa + 1} \sqrt{\frac{\pi}{l}} \left[2(u_1^B - u_1^D) - (u_1^C - u_1^E) \right], \\ K_{II} &= \frac{G(2 + \sqrt{2})}{\kappa + 1} \sqrt{\frac{\pi}{l}} \left[2(u_2^B - u_2^D) - (u_2^C - u_2^E) \right] \end{aligned} \quad (2.166)$$

One-point displacement formula:

$$\begin{aligned} K_I &= \frac{2G}{\kappa + 1} \sqrt{\frac{\pi}{l}} (u_1^B - u_1^D), \\ K_{II} &= \frac{2G}{\kappa + 1} \sqrt{\frac{\pi}{l}} (u_2^B - u_2^D) \end{aligned} \quad (2.167)$$

Traction formula:

$$\begin{aligned} K_I &= t_1^A \sqrt{\pi l}, \\ K_{II} &= t_2^A \sqrt{\pi l} \end{aligned} \quad (2.168)$$

In the above equations, the superscripts denote the nodal points shown in Figure 2.5a at which the quantities are used in the formulas, G is the shear modulus, l is the length of the crack tip element, and

$$\kappa = \begin{cases} (3 - 4\nu) & \text{for plane strain} \\ (3 - \nu)/(1 + \nu) & \text{for plane stress} \end{cases} \quad (2.169)$$

These special crack-tip elements with mid-side nodes, as will be evident later, are more appropriate for contact fracture mechanics analysis, and as such, are used throughout in this study.

2.6.2 Stress intensity factor formulas in 3D

By following the same procedures as in 2D analysis, namely, by equating the variations of the displacements and the tractions along the crack-front elements represented by the shape functions presented in Section (2.3), to the classical near-front field solutions, similar expressions to Eq. (2.166)–(2.168) above for the direct evaluation of the stress intensity factors may be obtained for 3D analysis, see Luchi and Poggiani[82]. As expected, the formulas obtained will be identical to the ones for 2D, except in the case when the crack-front node is a mid-side node of the quadrilateral element. In that case, the displacement formulas will have a somewhat different form. These formulas are presented, with reference to Figure 2.5b, as follows. When the crack-front node is a corner node of the crack-tip element, say node A in Figure 2.5b, they are:

General displacement formula:

$$\begin{aligned} K_I^A &= \frac{G(2 + \sqrt{2})}{\kappa + 1} \sqrt{\frac{\pi}{l}} [2(u_1^B - u_1^D) - (u_1^C - u_1^E)], \\ K_{II}^A &= \frac{G(2 + \sqrt{2})}{\kappa + 1} \sqrt{\frac{\pi}{l}} [2(u_2^B - u_2^D) - (u_2^C - u_2^E)], \\ K_{III}^A &= \frac{G(2 + \sqrt{2})}{4} \sqrt{\frac{\pi}{l}} [2(u_3^B - u_3^D) - (u_3^C - u_3^E)] \end{aligned} \quad (2.170)$$

One-point displacement formula:

$$\begin{aligned} K_I^A &= \frac{2G}{\kappa + 1} \sqrt{\frac{\pi}{l}} (u_1^D - u_1^I) \\ K_{II}^A &= \frac{2G}{\kappa + 1} \sqrt{\frac{\pi}{l}} (u_2^D - u_2^I) \\ K_{III}^A &= \frac{G}{2} \sqrt{\frac{\pi}{l}} (u_3^D - u_3^I) \end{aligned} \quad (2.171)$$

Traction formula:

$$\begin{aligned} K_I^A &= t_1^A \sqrt{\pi l}, \\ K_{II}^A &= t_2^A \sqrt{\pi l}, \\ K_{III}^A &= t_3^A \sqrt{\pi l} \end{aligned} \quad (2.172)$$

In the case of the crack-front node being a mid-side node, say node *B* in Figure 2.5b, the formulas for stress intensity factor computations are as follows:

General displacement formula:

$$\begin{aligned}
 K_I^B &= \frac{G}{\kappa+1} \sqrt{\frac{\pi}{l}} [(2+\sqrt{2})(u_1^D + u_1^H - u_1^I - u_1^M) - (1+\sqrt{2}) \\
 &\quad (u_1^E + u_1^G - u_1^J - u_1^L) + \sqrt{2}(u_1^F - u_1^K)], \\
 K_{II}^B &= \frac{G}{\kappa+1} \sqrt{\frac{\pi}{l}} [(2+\sqrt{2})(u_2^D + u_2^H - u_2^I - u_2^M) - (1+\sqrt{2}) \\
 &\quad (u_2^E + u_2^G - u_2^J - u_2^L) + \sqrt{2}(u_2^F - u_2^K)], \\
 K_{III}^B &= \frac{G}{4} \sqrt{\frac{\pi}{l}} [(2+\sqrt{2})(u_3^D + u_3^H - u_3^I - u_3^M) - (1+\sqrt{2}) \\
 &\quad (u_3^E + u_3^G - u_3^J - u_3^L) + \sqrt{2}(u_3^F - u_3^K)]
 \end{aligned} \tag{2.173}$$

One-point displacement formula:

$$\begin{aligned}
 K_I^B &= \frac{2G}{\kappa+1} \sqrt{\frac{\pi}{2l}} (u_1^F - u_1^K), \\
 K_{II}^B &= \frac{2G}{\kappa+1} \sqrt{\frac{\pi}{2l}} (u_2^F - u_2^K), \\
 K_{III}^B &= \frac{G}{2} \sqrt{\frac{\pi}{2l}} (u_3^F - u_3^K)
 \end{aligned} \tag{2.174}$$

Traction formula:

$$\begin{aligned}
 K_I^B &= t_1^B \sqrt{\pi l} \\
 K_{II}^B &= t_2^B \sqrt{\pi l} \\
 K_{III}^B &= t_3^B \sqrt{\pi l}
 \end{aligned} \tag{2.175}$$

2.7 Numerical examples

A number of crack problems have been analysed to demonstrate the veracity of the computer programs developed and to verify the validity of the formulas derived in Section (2.6.1). To this end, the sensitivity and accuracy of the solutions to changes in the sizes of the crack-tip elements have also been investigated. This aspect of the study is presented below.

2.7.1 Numerical examples in 2D

(i) The first problem considered here is a long flat plate of width $4a$ and containing a central crack of length $2a$, being subjected to a uniform uniaxial tensile stress σ at the ends, as shown in Figure 2.6. For the purpose of analysis here, the length of the plate is taken to be equal to $12a$. This problem has been analysed by several authors and the established results are compiled by, e.g. Isida in Tada, Paris and Irwin[122]. For the geometry treated, the generally accepted accurate result of K_I , to within 1 percent error, is $1.1867\sigma\sqrt{\pi a}$. This value is taken to be the reference "exact" value for comparison purpose.

Figure 2.7 shows a typical BEM mesh used to model the problem; the ratio of the crack-tip element size to half-crack length, l/a is 0.2 in this case. It should be noted that advantage can be taken of the planes of symmetry to reduce the numerical problem size. But the aim here is to demonstrate the multi-region modelling approach for crack problems. With the mesh shown, the percentage deviations of the computed values of K_I from the "exact" value above, when using the two different displacement formulas are less than 4%, while it is less than 0.5% when K_I is obtained using the traction formula. Figure 2.8 shows the variations of the percentage "error" with the relative crack-tip element size, l/a , analysed for this problem. It can be seen that the traction formula yielded K_I results which are less sensitive to the relative crack-tip element size than those obtained using either of the two displacement formulas. Indeed, even when $l/a = 0.5$, the error in K_I using the traction formula is only -0.07% .

(ii) The second problem treated is shown in Figure 2.9. It is that a long rectangular sheet of width $4a$ containing two equal edge cracks each of length a , and being subjected to a remote uniform tensile stress σ . This problem has been solved by Bowie[17] using the method of boundary collocation, and for the geometry here, his result for K_I , which is accepted to be less than 1 percent in error, is $1.151\sigma\sqrt{\pi a}$. A typical boundary element mesh used is shown in Figure 2.10. The percentage deviations between Bowie's result and those obtained using the various formulas for

different l/a ratios are shown in Figure 2.11. Again, it can be seen that while the displacement formulas gave K_I results which are less than 5% in deviations from Bowie's value of l/a up to 0.2; the traction formula produced results which are less than 3% in deviation even when the l/a ratio is up to 0.5. These features of the formulas for stress intensity factor computations, seen in problem (i) and here, are consistent with the findings of Jia et al.[50].

(iii) Figure 2.12 shows the third problem considered here. It is a rectangular plate of width $2a$ and length $5a$, containing an edge crack of length a , subjected to a bending moment per unit thickness M at the ends, as shown. The problem was analysed by Wilson in Rooke and Cartwright[109] using the method of boundary collocation, and his results for the geometry are $K_I = 0.792K_0$ and $K_{II} = 0.322K_0$, where $K_0 = 3M\sqrt{\pi a}/(2a^2)$. A typical boundary element mesh, with $l/a = 0.4$, is as shown in Figure 2.13. The percentage deviations between Wilson's results and those from the BEM analysis for different relative sizes of the crack-tip elements, l/a , are shown in Figure 2.14. For this problem, the displacement formulas gave results of K_I and K_{II} which are less than 4% in deviation with those obtained by Wilson only in the case when $l/a = 0.1$. Beyond this l/a ratio, the discrepancies progressively increase. On the other hand, the traction formula gave K_I results which are all within about 2 percentage in discrepancy from Wilson's results for $l/a = 0.1$ to 0.5. For K_{II} , the percentage deviations are also fairly small, being generally less than 5 percent for this range of l/a ratio.

(iv) The fourth problem considered is a circular arc crack in an infinite body subjected to biaxial tensile load σ , as shown in Figure 2.15. In the numerical analysis, the body was taken to be a square with width 20 times the diameter, $2a$, of the circular arc. With reference to Figure 2.15, the half arc angles θ of the crack configuration considered are 30° , 60° , 90° and 120° . Figure 2.16 shows a typical boundary element mesh used to model one-half of physical problem, taking advantage of symmetry. Note that a second subregion, a semi-circle, is introduced along the interface of which the circle lies. For a given crack size and mesh design, BEM analysis was carried out

for three crack-tip element sizes. They are $l/(a\theta) = (a\theta_c)/(a\theta) = 0.025, 0.05$ and 0.1 , where θ_c is the angular span of the crack-tip elements. Table 2.1 and Table 2.2 list the numerically obtained values of K_I/K_0 , where $K_0 = \sigma\sqrt{\pi a}$, and the corresponding exact results given by Perlman and Sih[102]. As can be seen, the magnitudes of error in the BEM solutions, using either the displacement formula or the traction formula, were generally well within 5% for all cases treated.

2.7.2 Numerical examples in 3D

(i) The first problem considered here is the well-known problem of a penny-shaped crack of radius a , embedded in an infinite domain which is subjected to remote uniform uniaxial tensile stress σ perpendicular to the crack surfaces. The exact solution for the stress intensity factor for this problem is, see, e.g. Sih[116], $K_I = 2\sigma\sqrt{a/\pi}$. In the BEM analysis, the crack is assumed to lie at the center of a large elastic cubic block with side dimensions equal to $10a$. Advantage is taken of the symmetry of the problem here and only one-quarter of the physical problem is modelled. Figure 2.17 shows the boundary element mesh employed, with $l/a = 0.2$. The differences between the results from the BEM analysis and the analytical solution for angular positions around the periphery of the modelled circle are listed in Table 2.3. It can be seen that the traction formula produced results which are all within 2 percent error from the exact analytical solution, while the errors with the displacement formula are still under 5 percent.

(ii) The second problem considered here, shown in Figure 2.18, is a semi-circular surface crack of radius a , in a plate of thickness t subjected to uniform tensile stress S_t at the ends of the plate with $h/a = b/a = 5$. The problem has been studied by Raju and Newman[106] using the finite element method with possibly the finest finite element mesh (the 6900 degrees of freedom) ever reported for the problem. Thus Raju and Newman's results are used here for the purpose of comparison with those obtained from the boundary element analysis. The ratios of the crack depth-to-plate thickness ratio, are taken to be 0.2, 0.4 and 0.6. Because of the symmetry of the problem, only

half of the physical problem is modelled. A typical boundary element mesh used is shown in Figure 2.19, with $a/t = 0.4$. The ratio of the crack-tip element size to the radius of the modelled crack, l/a , is taken to be 0.15 when $a/t = 0.2, 0.4$, and be 0.1 when $a/t = 0.6$. The stress-intensity factors as a function of the crack depth-to-plate thickness, a/t , and the circumferential angle of the modelled half-circle, ϕ , which is 0 on the free surface and $\pi/2$ at the deepest point of the crack, are contained in Table 2.4. Also, included in Table 2.4 are the discrepancies between the results from the BEM analysis and Raju and Newman's solution. It can be seen that the traction formula produced results which are within 3 percent in deviation from Raju and Newman's results when $\phi > 0$, the discrepancies being larger only at the free surface. The one-point displacement formula produced results which are all within 2.5 percent in deviation, while the general displacement formula produced results which are all within 3.5 percent in deviation, relatively the largest discrepancies being again near the free surface region. The large discrepancies between the two results near the free surface region can be explained as follows. As suggested by Hartranft and Sih[44], the stress-intensity factors in a very thin "boundary layer" near the free surface drop off rapidly and are equal to zero at the free surface. Raju and Newman[106] showed that when $\phi \leq \pi/16$, the stress intensities were affected by the refinements of element near the free surface, while the stress intensities distributions in the interior ($\phi > \pi/16$) were unaffected by the refinements. The stress-intensity factors obtained at the free surface were interpreted as average stress intensity factors near the free surface.

(iii) The third problem, treated here, shown in Figure 2.20, is a thick plate of length $4a$, width $4a$ and thickness a containing a central through-thickness crack of length $2a$, subjected to a uniform uniaxial tensile stress σ at its ends. The Poisson's ratio of the plate is taken to be 0.3. The problem has been analysed by Yagawa and Hirayama[147] by using the three-dimensional finite element method, to whose solutions, the results obtained here are compared. Because of symmetry, only quarter of the problem is modelled. The boundary element mesh employed, with $l/a = 0.2$, is shown in Figure 2.21. The comparison between Yagawa and Hirayama's results

and those obtained from the boundary element analysis with the different formulas of calculating the stress intensity factors, is shown in Figure 2.22. It can be seen from the figure that there is generally good agreement, within 4% discrepancy, between the boundary element and finite element results.

(iv) The last problem treated is that of two symmetrically placed quarter-circular corner cracks of radius a , emanating from a through-the-thickness circular hole of radius R within an elastic plate of thickness t , width $2b$ and length $2h$, with $h/b = 1.8$, $R/t = 0.5$ and $b/(R + a) \geq 5$, as shown in Figure 2.23. The ratios of the crack radius-to-plate thickness ratio, a/t , are taken to be 0.2, 0.5 and 0.8. Two types of loading for the plate are considered here. The first is the uniform tensile stress, S_t , applied on the ends of the plate, as shown in Figure 2.24a. The other is the wedge loading, with pressure σ_n acted on the whole surface of the hole, as shown in Figure 2.24b. The pressure distribution is uniform along the thickness of the plate, but possesses the following variation along the periphery of the hole.

$$\sigma_n = \frac{3}{2} S_p \sin^2 \theta \quad (2.176)$$

where $S_p = P/(2Rt)$, with P equal to the total external load.

The problem has also been studied by Raju and Newman[107] using the finite element method with 9300 degrees of freedom. Thus Raju and Newman's results are used here for the purpose of comparison with the boundary element analysis. Because of the symmetry of the problem, only quarter of the physical problem is modelled. A typical boundary element mesh used is shown in Figure 2.25, with $a/t = 0.2$. The ratio of the crack-tip element size to the radius of the modelled crack, l/a , is taken to be 0.125 when $a/t = 0.2$, 0.1 when $a/t = 0.5$, and 0.0625 when $a/t = 0.8$. The stress-intensity factors as a function of the crack radius-to-plate thickness, a/t , and the circumferential angle of the modelled half-circle, ϕ , which is defined as $\pi/2$ at the intersection of the crack-front and the hole, and zero at the intersection of the crack-front and the front surface of the plate, are contained in Table 2.5 and Table 2.6 for the cases corresponding to the remote tension and wedge loading in the hole respectively.

Also, included in Table 2.5 and Table 2.6 are the discrepancies between the results from the BEM analysis and Raju and Newman's solutions.

In the case of remote tension, it can be seen from Table 2.5 that compared with Raju and Newman's results, when $a/t = 0.2$ and $\pi/16 < \phi < 7\pi/16$, the traction formula produced results which are 3 to 5 percent higher, the one-point displacement formula produced results which are 4 to 6 percent higher, and the general displacement formula produced results which are 3 to 7 percent higher. When $a/t = 0.5$ and $\pi/16 < \phi < 7\pi/16$, all formulas produced results which are within 2 percent in deviation. When $a/t = 0.8$ and $\pi/16 < \phi < 7\pi/16$, the traction formula produced results which are within 5 percent in deviation, the one-point displacement formula produced results which are 3 percent in deviation, and the general displacement formula produced results which are within 2 percent in deviation. For remote tension, it can also be seen that near the free surface regions, especially when $\phi \geq 7\pi/16$, there are large discrepancies between Raju and Newman's results and those obtained from the present BEM analysis. These relatively large discrepancies can again be attributed by the "boundary layer" effect and the refinement of finite element mesh near those regions.

In the case of wedge loading, it can be seen from Table 2.6 that compared with Raju and Newman's results, when $a/t = 0.2$, and $\pi/16 < \phi < 7\pi/16$, the traction formula produced results which are 10 to 12 percent higher, the one-point displacement formula produced results which are 10 to 14 percent higher, and the general displacement formula produced results which are 10 to 13 percent higher. When $a/t = 0.5$ and $\pi/16 < \phi < 7\pi/16$, the traction formula produced results which are within 2 percent in deviation, the one-point displacement formula produced results which are within 5 percent in deviation, and the one-point displacement formula produced results which are within 4 percent in deviation. When $a/t = 0.8$, and $\pi/16 < \phi < 7\pi/16$, the traction formula produced results which are within 3 percent in deviation, the one-point displacement formula produced results which are 4 percent in deviation, and the general displacement formula produced results which are within 6 percent in deviation.

It should perhaps be noted here that several BEM mesh refinements were carried out for the $a/t = 0.2$ case here, but the results obtained were not significantly different.

2.8 Concluding remarks

The boundary element analysis for elastic fracture mechanics analysis in 2D and 3D has been reviewed in this chapter. Special crack-tip elements have been employed to model crack problems, and several "formulas" have been presented which enable stress intensity factors to be evaluated directly from the computed values of displacements or tractions on these elements. It is evident from the numerical examples presented above and elsewhere (see, e.g. Jia *et al.*[50, 51]) that the "traction formula" consistently gives very good results which are significantly less sensitive to the mesh design than the "displacement formulas". However the availability of both sets of formulas provide a useful cross-check of the numerical solutions obtained in a BEM analysis.

Table 2.1 Normalized stress intensity factors, $K_I^* = K_I/(\sigma\sqrt{\pi a})$, for a circular arc crack in an infinite elastic body,

(K_I^*)_D-general displacement formula result, (K_I^*)_T-traction formula result.

θ	$(K_I^*)_{\text{Exact}}$	θ_c/θ	$(K_I^*)_D$	Error Δ %	$(K_I^*)_T$	Error Δ %
30°	0.6401	0.025	0.6426	0.4	0.6387	-0.2
		0.05	0.6429	0.4	0.6409	0.1
		0.1	0.6435	0.5	0.6406	0.1
60°	0.6447	0.025	0.6470	0.4	0.6427	-0.3
		0.05	0.6473	0.4	0.6438	-0.1
		0.1	0.6478	0.5	0.6426	-0.3
90°	0.4714	0.025	0.4713	0.0	0.4657	-1.2
		0.05	0.4713	0.0	0.4678	-0.8
		0.1	0.4736	0.5	0.4671	-0.9
120°	0.2659	0.025	0.2650	-0.3	0.2625	-1.3
		0.05	0.2653	-0.2	0.2638	-0.8
		0.1	0.2690	1.2	0.2642	-0.6

Table 2.2 Normalized stress intensity factors, $K_{II}^* = K_{II}/(\sigma\sqrt{\pi a})$, for a circular arc crack in an infinite elastic body,

$(K_{II}^*)_D$ -general displacement formula result, $(K_{II}^*)_T$ -traction formula result.						
θ	$(K_{II}^*)_{Exact}$	θ_c/θ	$(K_{II}^*)_D$	Error $\Delta \%$	$(K_{II}^*)_T$	Error $\Delta \%$
30°	0.1715	0.025	0.1712	-0.2	0.1702	-0.8
		0.05	0.1701	-0.8	0.1696	-1.1
		0.1	0.1676	-2.3	0.1669	-2.7
60°	0.3722	0.025	0.3734	0.3	0.3709	-0.3
		0.05	0.3709	-0.3	0.3689	-0.9
		0.1	0.3638	-2.3	0.3609	-3.0
90°	0.4714	0.025	0.4720	0.1	0.4663	-1.1
		0.05	0.4663	-1.1	0.4628	-1.8
		0.1	0.4524	-4.0	0.4462	-5.3
120°	0.4605	0.025	0.4579	-0.6	0.4535	-1.5
		0.05	0.4484	-2.6	0.4458	-3.2
		0.1	0.4356	-5.4	0.4324	-6.1

Table 2.3 Normalized stress intensity factors, $K_I^* = K_I/(\sigma\sqrt{a})$, for a penny-shaped crack of radius a , embedded in an infinite body under a uniform tension σ at infinity: $(K_I^*)_T$ -traction formula result; $(K_I^*)_{D1}$ -one-point displacement formula result; $(K_I^*)_{DD}$ -general displacement formula result.

θ	$(K_I^*)_{Exact}$	$(K_I^*)_T$	Error $\Delta \%$	$(K_I^*)_{D1}$	Error $\Delta \%$	$(K_I^*)_{DD}$	Error $\Delta \%$
0°	1.1284	1.1166	-1.0	1.0986	-2.6	1.1655	-3.3
15°		1.1205	-0.7	1.0708	-4.7	1.1656	-3.3
30°		1.1155	-1.1	1.0985	-2.6	1.1655	-3.3
45°		1.1213	-0.6	1.0706	-4.7	1.1653	-3.3
60°		1.1155	-1.1	1.0986	-2.6	1.1655	-3.3
75°		1.1205	-0.7	1.0708	-4.7	1.1656	-3.3
90°		1.1166	-1.0	1.0993	-2.6	1.1655	-3.3

Table 2.4 Normalized stress intensity factor, $K_I^* = K_I / (S_t \sqrt{4a/\pi})$, for a semi-circular surface crack radius a in a elastic plate subjected to remote tensile stress S_t :

$(K_I^*)_R$ —result obtained by Raju and Newman; $(K_I^*)_T$ —traction formula result;

$(K_I^*)_{D1}$ —one-point displacement formula result;

$(K_I^*)_{DD}$ —general displacement formula result.

a/t	$2\phi/\pi$	$(K_I^*)_R$	$(K_I^*)_T$	Disc. Δ %	$(K_I^*)_{D1}$	Disc. Δ %	$(K_I^*)_{DD}$	Disc. Δ %
0.2	0	1.174	1.085	-7.6	1.165	-0.8	1.203	+2.5
	0.125	1.145	1.178	+2.9	1.147	+0.2	1.185	+3.5
	0.25	1.105	1.091	-1.3	1.099	-0.5	1.111	+0.5
	0.375	1.082	1.083	+0.1	1.063	-1.8	1.092	+0.9
	0.5	1.067	1.062	-0.5	1.058	-0.8	1.086	+1.8
	0.625	1.058	1.057	-0.1	1.046	-1.1	1.077	+1.8
	0.75	1.053	1.051	-0.2	1.042	-1.0	1.073	+1.9
	0.875	1.050	1.049	+0.1	1.026	-2.3	1.071	+2.0
	1.0	1.049	1.046	-0.3	1.038	-1.0	1.071	+2.1
0.4	0	1.229	1.159	-5.7	1.237	+0.7	1.269	+3.3
	0.125	1.206	1.235	+2.4	1.207	+0.1	1.240	+2.8
	0.25	1.157	1.140	-1.5	1.147	-0.9	1.158	+0.1
	0.375	1.126	1.120	-0.5	1.104	-1.9	1.129	+0.3
	0.5	1.104	1.094	-0.9	1.092	-1.1	1.117	+1.2
	0.625	1.088	1.089	+0.1	1.066	-2.0	1.103	+1.4
	0.75	1.075	1.077	+0.2	1.068	-0.7	1.094	+1.8
	0.875	1.066	1.071	+0.4	1.052	-1.3	1.090	+2.2
	1.0	1.062	1.065	+0.3	1.061	-0.1	1.088	+2.5
0.6	0	1.355	1.267	-6.5	1.369	+1.0	1.388	+2.4
	0.125	1.321	1.335	+1.1	1.314	-0.5	1.329	+0.6
	0.25	1.256	1.231	-2.0	1.238	-1.4	1.239	-1.4
	0.375	1.214	1.197	-1.4	1.189	-2.1	1.197	-1.4
	0.5	1.181	1.158	-1.9	1.162	-1.6	1.171	-0.8
	0.625	1.153	1.140	-1.1	1.135	-1.6	1.148	-0.4
	0.75	1.129	1.120	-0.8	1.123	-0.5	1.131	+0.2
	0.875	1.113	1.115	+0.2	1.110	-0.3	1.121	+0.7
	1.0	1.107	1.106	-0.1	1.110	+0.3	1.118	+1.0

Table 2.5 Normalized stress intensity factor, $K_I^* = K_I / (S_t \sqrt{4a/\pi})$, for quarter-circular corner cracks of radius a subjected to remote tensile stress S_t at the ends of the plate:

$(K_I^*)_R$ —result obtained by Raju and Newman; $(K_I^*)_T$ —traction formula result;

$(K_I^*)_{D1}$ —one-point displacement formula result;

$(K_I^*)_{DD}$ —general displacement formula result.

a/t	$2\phi/\pi$	$(K_I^*)_R$	$(K_I^*)_T$	Disc. Δ %	$(K_I^*)_{D1}$	Disc. Δ %	$(K_I^*)_{DD}$	Disc. Δ %
0.2	0	1.832	1.752	-4.4	1.890	+3.2	1.879	+2.6
	0.125	1.870	1.959	+4.8	1.970	+5.3	1.943	+3.9
	0.25	1.886	1.938	+2.8	1.972	+4.6	1.946	+3.2
	0.375	1.936	2.012	+3.9	2.027	+4.7	2.010	+3.8
	0.5	2.015	2.085	+3.5	2.103	+4.4	2.106	+4.5
	0.625	2.119	2.225	+5.0	2.212	+4.4	2.230	+5.2
	0.75	2.247	2.364	+5.2	2.375	+5.7	2.399	+6.8
	0.875	2.403	2.629	+9.4	2.569	+6.9	2.808	+16.8
	1.0	1.985	2.632	+32.6	2.821	+42.1	3.108	+56.6
0.5	0	1.699	1.606	-5.5	1.766	+3.9	1.760	+3.6
	0.125	1.699	1.754	+3.2	1.764	+3.8	1.736	+2.2
	0.25	1.675	1.675	-0.0	1.703	+1.7	1.671	-0.2
	0.375	1.687	1.687	+0.0	1.704	+1.0	1.670	-1.0
	0.5	1.733	1.707	-1.5	1.726	-0.4	1.709	-1.4
	0.625	1.817	1.800	-0.9	1.807	-0.6	1.790	-1.5
	0.75	1.951	1.919	-1.6	1.940	-0.6	1.932	-1.0
	0.875	2.137	2.257	+2.7	2.171	-1.2	2.398	+9.1
	1.0	2.072	2.632	+27.0	2.511	+21.2	2.809	+35.6
0.8	0	1.808	1.777	-1.7	1.903	+5.3	1.881	+4.0
	0.125	1.763	1.867	+5.9	1.818	+3.1	1.776	+0.7
	0.25	1.673	1.724	+3.0	1.699	+1.6	1.665	-0.5
	0.375	1.608	1.694	+5.3	1.653	+2.8	1.607	-0.1
	0.5	1.608	1.650	+2.6	1.622	+0.9	1.590	-1.1
	0.625	1.640	1.694	+3.3	1.664	+1.5	1.614	-1.6
	0.75	1.750	1.764	+0.8	1.750	+0.0	1.710	-2.3
	0.875	2.015	2.150	+6.7	2.008	-0.3	2.136	+6.0
	1.0	2.164	2.480	+14.6	2.431	+12.3	2.631	+21.6

Table 2.6 Normalized stress intensity factor, $K_I^* = K_I/(S_p\sqrt{4a/\pi})$, for quarter-circular corner cracks of radius a subjected to wedge loading σ_n in the hole:
 $(K_I^*)_R$ —result obtained by Raju and Newman; $(K_I^*)_T$ —traction formula result;
 $(K_I^*)_{D1}$ —one-point displacement formula result;
 $(K_I^*)_{DD}$ —general displacement formula result].

a/t	$2\phi/\pi$	$(K_I^*)_R$	$(K_I^*)_T$	Disc. Δ %	$(K_I^*)_{D1}$	Disc. Δ %	$(K_I^*)_{DD}$	Disc. Δ %
0.2	0	0.638	0.660	+3.4	0.729	+14.3	0.692	+8.5
	0.125	0.669	0.760	+13.6	0.786	+17.5	0.737	+10.2
	0.25	0.698	0.772	+10.6	0.794	+9.6	0.764	+9.5
	0.375	0.743	0.825	+11.0	0.847	+14.0	0.816	+9.8
	0.5	0.803	0.885	+10.2	0.899	+12.0	0.886	+10.3
	0.625	0.879	0.981	+11.6	0.983	+11.8	0.978	+11.3
	0.75	0.966	1.081	+11.9	1.088	+12.6	1.095	+13.4
	0.875	1.071	1.247	+16.4	1.214	+13.4	1.340	+25.1
	1.0	0.917	1.287	+40.3	1.369	+49.3	1.525	+66.3
0.5	0	0.462	0.454	-1.7	0.512	+10.8	0.484	+4.8
	0.125	0.480	0.510	+6.3	0.535	+11.5	0.490	+2.1
	0.25	0.498	0.503	+1.0	0.522	+4.8	0.490	-1.6
	0.375	0.530	0.531	+0.2	0.523	-1.3	0.514	-3.0
	0.5	0.581	0.570	-1.9	0.587	+1.0	0.560	-3.6
	0.625	0.657	0.650	-1.1	0.668	+1.7	0.638	-2.9
	0.75	0.762	0.754	-1.0	0.769	+0.9	0.753	-1.2
	0.875	0.922	0.976	+5.9	0.936	+1.5	1.048	+13.7
	1.0	0.978	1.133	+15.8	1.162	+18.8	1.320	+35.0
0.8	0	0.419	0.421	+0.5	0.457	+9.1	0.436	+4.1
	0.125	0.420	0.447	+6.4	0.447	+6.4	0.414	-1.4
	0.25	0.413	0.420	+1.7	0.420	+1.7	0.396	-4.1
	0.375	0.417	0.430	+3.1	0.433	+3.8	0.399	-4.3
	0.5	0.446	0.446	-0.0	0.445	-0.2	0.421	-5.6
	0.625	0.501	0.503	+0.4	0.507	+1.2	0.473	-6.0
	0.75	0.594	0.591	-0.5	0.593	-0.2	0.569	-4.2
	0.875	0.778	0.843	+8.4	0.782	+0.5	0.849	+9.1
	1.0	0.927	1.105	+19.2	1.069	+15.3	1.178	+27.1

Figure 2.1 A homogeneous elastic domain Ω with boundary Γ . P and Q are points on the boundary; P_0 and Q_0 are interior points.

a) three-node curve element in 2D

b) eight-node quadrilateral element in 3D

c) six-node triangle element in 3D

Figure 2.2 Quadratic isoparametric boundary elements

a) P^a at the 1st node b) P^a at the 5th node c) P^a at the 1st node d) P^a at the 2nd node e) P^a at the 3rd node f) P^a at the 4th node g) P^a at the 5th node h) P^a at the 6th nodeFigure 2.3 Sub-division of boundary elements in 3D analysis when $P^a = P^{a(b,c)}$

Figure 2.4 (a) Local co-ordinate system at the crack front, and
(b) modes of crack deformation.

a)

b)

Figure 2.5 (a) Crack-tip boundary elements in 2D, (b) crack-front boundary elements in 3D.

Figure 2.6 A long flat plate with a central crack subjected to remote uniform tension

Figure 2.7 A BEM mesh for the crack problem shown in Figure 2.6; $l/a=0.2$.

Figure 2.8 Variation of the percentage error of K_I with relative crack-tip element size l/a , for central crack problem shown in Figure 2.6.

Figure 2.9 A long rectangular sheet with double edge cracks

Figure 2.10 Boundary elements used for the crack problem shown in Figure 2.9; $l/a=0.2$.

Figure 2.11 Variation of the percentage error of K_I with relative crack-tip element size l/a , for edge-crack problem shown in Figure 2.9.

2

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Figure 2.12 A rectangular plate with an inclined crack subjected to bending moment at its ends.

Figure 2.13 A BEM mesh for the crack problem shown in Figure 2.12; $Va=0.4$.

Figure 2.14 Variation of the percentage error with l/a ratio for the crack problem shown in Figure 2.12.

Figure 2.15 A circular arc crack in an infinite elastic medium which is subjected to biaxial tension.

Figure 2.16 A typical BEM mesh for the problem of a circular arc crack in an infinite body as shown in Figure 2.15.

Figure 2.17 Boundary element mesh for the problem of a penny-shaped crack in an infinite body; 42 elements, 132 nodes.

Figure 2.18 Surface crack in a plate subjected to tension.

Figure 2.19a A BEM mesh for the surface crack problem as shown in Figure 2.18, with $a/t=0.4$.

Figure 2.19b Boundary elements around the interface zone for the BEM mesh as shown in Figure 2.19a.

Figure 2.20 A thick plate with a central crack subjected to remote tension.

Figure 2.21 BEM mesh for a thick plate with a central crack subjected to remote tension: 116 elements, 338 nodes.

Figure 2.22 Stress intensity factors along the thickness direction, for the crack problem shown in Figure 2.20.

Figure 2.23 Corner cracks at the edge of a hole in a finite plate.

a) Tension

b) Wedge loading

Figure 2.24 Corner crack configuration subjected to two types of loading

Figure 2.25 Boundary element mesh for the corner crack problem as shown in Figure 2.23, with $a/t=0.2$.

Chapter 3

Boundary Element Analysis of Elastic Contact Problems

3.1 Introduction

The primary interest in solving a contact problem is the determination of stress distribution at the region of contact between the elastic bodies. For most contact problems in engineering, the contact stresses are non-linear functions of load, and, furthermore, in the presence of friction, the contact stresses become load-path dependent. When the boundary element method is used to solve contact problems, the unknown contact stresses and the displacements within the contact zones are integrated into the system of boundary integral equations. Since the contact area is unknown *a priori*, the system of equations is solved iteratively until the correct solution of the contact problem is obtained. Attention should also be paid to the computational efforts involved, since the contact problem usually involves two or more bodies, and a relatively fine mesh is needed for each body as compared with that in linear structural analysis. The system of equations can thus become very large particularly for 3D contact fracture mechanics problems. It is very important therefore to develop an algorithm which converges fast and requires minimum computational effort during the iterative process. In this chapter, some of these contact algorithms for solving elastic contact problems are briefly reviewed and the one used in the present work is presented.

3.2 Fundamentals of Contact Mechanics

There are two basic formulations of the contact problem based on which numerical solutions can be developed. They are commonly referred to as the classical formulation and the variational formulation. In the former formulation, the contact problem is regarded as an ordinary problem of solid mechanics but in which the loads or displacements at the contact surface are unknown *a priori*. Some assumptions are made on the extent of contact between the two (or more) bodies and an iteration of the solution leads to the proper contact areas and the deformations. In the latter formulation, the contact problem is expressed in a variational form as an optimization problem with some inequality constraints.

3.2.1 Contact constraints

Consider two bodies in contact and let $d(\mathbf{x})$ be the distance between two opposite points at the surfaces of the two bodies, see Figure 3.1. Near the contact area it is possible to specify the following:

$$\begin{aligned} d(\mathbf{x}) &= 0 \quad \text{in the contact region,} \\ d(\mathbf{x}) &> 0 \quad \text{outside, but near, the contact region.} \end{aligned} \tag{3.1}$$

This is a displacement condition.

Now consider the surface force $\mathbf{p}(\mathbf{x})$ in and near the region of contact. For points outside the contact region,

$$\mathbf{p}(\mathbf{x}) = 0. \tag{3.2}$$

Because the contact area cannot carry a tensile normal component $p_N(\mathbf{x})$ of the force $\mathbf{p}(\mathbf{x})$,

$$p_N(\mathbf{x}) \leq 0 \tag{3.3}$$

there. If there is no friction, Eq. (3.1)–(3.3) together with the condition that the shearing component $\mathbf{p}_T(\mathbf{x})$ vanishes, constitute the conditions which determine this so-called *Normal Problem*. Note that the contact area is unknown *a priori*. In the

classical approach, an assumption is made regarding the contact area, the problem with Eq. (3.1) and Eq.(3.2) is solved, and the parameters of the contact area are determined in such a way that Eq. (3.3) is satisfied.

If frictional effects exist in the problem, it may in some instances be assumed that the contact surfaces do not move relative to each other. Let the slip of body I with respect to body II be denoted by $\mathbf{v}(\mathbf{x})$. Then

$$\begin{aligned}\mathbf{v}(\mathbf{x}) &\equiv \text{slip of body I with respect to body II} = \mathbf{0} \text{ for complete adhesion,} \\ \mathbf{p}_T(\mathbf{x}) &\equiv \text{tangential component of force } \mathbf{p} \text{ by body II exerted on body I.}\end{aligned}\tag{3.4}$$

This is also a displacement condition which has the additional complication in which the tangential load $\mathbf{p}_T$ can influence the distance $d(\mathbf{x})$.

In the case between complete adhesion and complete absence of friction, a frictional law must be specified. For most instances Coulomb's law is appropriate. It reads:

$$|\mathbf{p}_T(\mathbf{x})| \leq f|p_N(\mathbf{x})| \quad (\text{Limitation of tangential load}),\tag{3.5}$$

where f is the coefficient of friction. Furthermore,

$$\mathbf{v}(\mathbf{x}) \neq \mathbf{0} \Rightarrow \mathbf{p}_T(\mathbf{x}) = -f|p_N(\mathbf{x})| \frac{\mathbf{v}(\mathbf{x})}{|\mathbf{v}(\mathbf{x})|}.\tag{3.6}$$

Eq. (3.6) is a rather complicated, nonlinear condition. Even in simple cases it is satisfied only approximately, and that is sufficient when the angle between $(-\mathbf{p}_T)$ and $\mathbf{v}$ is small.

Under the influence of friction law and the deformability of the bodies, the contact area may be divided into a region of adhesion where $\mathbf{v}(\mathbf{x}) = \mathbf{0}$, and $|\mathbf{p}_T| \leq f|p_N|$, and a region of slip where $\mathbf{v}(\mathbf{x}) \neq \mathbf{0}$, and when Eq. (3.6) holds. The regions of adhesion and slip are unknown *a priori*, and often are of a more irregular and complicated form than the total contact area itself. These complications that arise in frictional problems are the reasons why classical solutions of these problems are relatively rare[56].

3.2.2 Load incremental theory

When solving non-linear problems with linear methods the specific problem usually has to be considered in increments. The total quantities are then calculated as the sum of increments, i.e. for surface load and distance

$$\begin{aligned} \mathbf{p}^n(\mathbf{x}) &= \Delta \mathbf{p}^n(\mathbf{x}) + \mathbf{p}^{n-1}(\mathbf{x}) = \sum_{k=1}^n \Delta \mathbf{p}^k(\mathbf{x}), \\ d^n(\mathbf{x}) &= \Delta d^n(\mathbf{x}) + d^{n-1}(\mathbf{x}) = \sum_{k=1}^n \Delta d^k(\mathbf{x}) \end{aligned} \quad (3.7)$$

where the superscripts represent the time steps, $\Delta \mathbf{p}(\mathbf{x})$ and $\Delta d(\mathbf{x})$ are the increments of $\mathbf{p}(\mathbf{x})$ and $d(\mathbf{x})$, respectively. This is true for the frictional contact problems where the total quantities are path-dependent. A demonstration of this was given by, e.g. Tsuta and Yamaji[137]. Consider a problem as shown in Figure 3.2, in which three bodies **A**, **B** and **C** are subjected to two independent external loads p and q . The loading path is partitioned into three steps. In the first step, the load p is applied and kept at the constant value p_0 . With p_0 , the load q is then applied up to the value q_0 . In the last step, the load q is removed. Figure 3.2(b) shows the hysteresis of the tangential contact force t along the interface between **A** and **B**, where t is assumed to be zero in the first step. In the second step, t increases up to the critical frictional force $-fp_0$ where slip take place beginning at point S_1 , t is kept constant at the critical value until the lateral load q reaches the value q_0 . In the third step, when q decreases from q_0 to zero, t decreases, changes its sign and increases up to fp_0 , the critical value in the opposite direction. Thus, at the point S_2 , slip takes place in the opposite direction. Consequently, the final state S_3 is remarkably different from the initial state S_0 in spite of the same loading condition.

In the absence of friction, however, the stresses and strains do not have to be calculated by increments. Instead, they can be calculated directly from the total quantities. That this is possible can be proved using the uniqueness theorem, see, e.g. Fung[34].

3.3 Review of BEM in Contact Mechanics

Early contributions of the direct BEM to contact problems include those by Andersson *et al*[9], and Tsuta and Yamaji[138]. The former authors presented a formulation for 2-D frictionless elastic contact problems and demonstrated its veracity with some examples. The latter authors presented the multi-region boundary element method suitable for 2-D and 3-D frictionless and reversible friction elastic contact problems. The validity of their algorithm was also demonstrated by some simple examples in 2-D; the algorithm is based on the classical formulation and they used “constant” boundary elements. Since then, many publications on boundary element analysis of contact problems have appeared, either on various aspects of improving the algorithm, or dealing with new applications. These works are now reviewed below.

3.3.1 Contact algorithms derived from classical formulations

Andersson[7] extended the classical contact algorithm to solve 2-D elastic contact problems with non-classical friction law, but again used the constant boundary element formulation. He later employed linear and quadratic boundary elements to analyse 2-D elastic contact problems with friction[8]. Andersson and Allan-Persson[10] later described their approach in a more systematic way, and pointed out the three main advantages of this algorithm when compared to other numerical methods, namely: (i) the boundary element method only treats the boundary, which is of primary interest in the contact solution procedure, (ii) it is possible to couple normal and tangential tractions, and (iii) the contact pressure, determined directly from the traction distribution is a primary unknown quantity and hence in boundary element analysis, is obtained with the same accuracy as the displacement unknowns. The approach proposed by them can be applied to a variety of 2-D elastic contact problems. However, the iteration process of their algorithm is relatively slow, since in each iteration, the system of equations which contains all the unknowns of the elastic

bodies that are in contact, must be solved.

Paris and Garrido[98] proposed the concept of using discontinuous linear elements in 2-D contact mechanics analysis, and the technique of condensing the non-active degrees of freedom during the iteration process. By using their condensation technique during the iteration process, the number of equations needed to be solved iteratively may be reduced to be equal to the number of unknowns within the contact zone. But a large amount of matrix multiplications is still required using this technique. This basic condensation technique was also employed by several authors, e.g. see Margenov *et al.*[85] and Jin *et al.*[54].

Kuich[68] was perhaps the first to propose a flexibility approach for solving contact problems using discontinuous boundary elements. This technique was further investigated using the BEM by Takahashi and Brebbia[123, 124]. It essentially consists of forming the flexibility matrices which establish the relations between the tractions and displacements at the nodes inside the contact zone for the individual elastic bodies coming into contact. Once this is done, the contact analysis can progress always with the same matrices. This approach has the added advantage that the system of equations required in the iteration process is usually small, involving the nodes possibly in contact only. However, this technique also has an inherent drawback that it is almost impossible to calculate the flexibility matrix and deformation matrix for the elastic body which is subjected to no displacement constraint but with the contact compatibility condition specified. Besides, it appears to be limited to the use of the discontinuous elements. Further publications of basically similar BEM formulations for 2-D elastic contact problems but with more examples, include those by Zheng[155], Paris and Garrido[99, 100], Jiang *et al.*[52, 53], Sun and Huston[121], Dandekar and Conant[27, 28].

Three-dimensional elastic contact problem without friction has been treated by Liu *et al.*[75] using the linear and quadratic elements. A special Gaussian elimination approach was developed in the work whereby only the system of equations which contains the unknowns in the contact area is needed to be solved during the iterative

process. The technique was modified for analysing contact problem with Coulomb friction by Liu and Tan[76] using quadratic isoparametric elements. The description of a similar technique was also very recently reported by Dandekar and Conant[29]. The BEM formulations for 3-D frictionless elastic contact problems using triangular constant elements, along with two examples of conforming contact problems were reported by Garrido *et al*[38]. Gao and Wen[37] have recently also applied the boundary element method to solve 3-D elastic contact problems with isotropic Coulomb friction, using quadratic elements. Two examples were included in the analysis but the analysis was only verified for the frictionless case. Forces *et al*[33] have also reported the analysis of 3-D frictional conforming contact using triangular constant elements, with the isotropic Coulomb friction law. The system of equations was solved by an iterative method, namely, the Newton-Raphson technique with relaxation, and the applicability of the proposed procedure is shown by solving the problem of a punch on an elastic foundation.

Tanaka[129] was perhaps the first to apply the BEM to axisymmetric elastic contact problems without friction. The kernel functions in the integral equations were the line integrals of the kernel functions in 3-D BEM analysis. The numerical examples presented include a Hertz contact problem and the stress analysis of the threaded connections. In spite of the approximate kernel functions, good agreement between his BEM results and the Hertz solution, and the finite element solutions was obtained. Abdul-Mishein *et al*[1] have also presented a BEM formulation for treating axisymmetric elastic contact problems with friction using isoparametric quadratic elements. But the veracity of their approach was only demonstrated with several examples without friction.

Several practical applications and new results obtained using the BEM based in classical formulations have also been reported over the recent years. For example, Yin[150] presented a BEM contact mechanics analysis of roller bearings; Adey and Niku[3] investigated the deformation of a turbine blade dovetail in a disc; and Ulmet[139] studied the 2-D contact problem of bolted joints of inhomogeneous ma-

terials. Based on the present work, Liu and Tan[77] have also recently carried out a parametric study which analysed the stress concentrations for pin-loaded lugs. Finally, it should be noted here that none of the references mentioned above has dealt with frictional contact problems with cracks present. In this area of application, only a very limited amount of work using BEM has hitherto been reported in the literatures. They will be addressed later in the next chapter.

3.3.2 Contact algorithms based on optimization techniques

Several BEM contact algorithms based on optimization principles have been proposed in recent years. They are briefly reviewed below. Lu[81] used the mathematical programming technique and constant boundary elements to treat 2-D elastoplastic contact problems without friction. In his analysis, the initial stress algorithm, the Von-Mises yielding criterion and the Prandtl-Reuss flow rules were employed. The objective functions were the external loads, contact stresses, and the gaps between the elements within the possible contact area in each body. Among the constraints was the force balance equation, and the final solution procedure for the optimal problem was the simplex-type algorithm first proposed by Conry and Seireg[22]. This approach is suitable for the analysis of contact problems involving the bodies with simple shapes within the contact area. However, if the contact area is complex or if the contact stresses vary rapidly, it is very difficult to apply the constraints of force-balance.

Yamazaki and Mori[148] has applied the penalty function method with the BEM to analyse 2-D elastic contact problems with Coulomb friction. In their formulation, the non-zero contact stresses were equal to the products of the corresponding displacement expressions and the penalty parameter. The parameter was a large number, thus the displacement expressions was forced to be equal to zero. These displacement expressions were written such that they satisfy the displacement constraint conditions when they are equal to zero within the contact area. Two contact problems were analysed using this approach. Adey and Niku[3] described in detail how the penalty function method may be implemented into a general boundary element code to solve

2-D contact problems. The advantages of the penalty function method are, first, the simplicity of the optimization algorithm, and second, the system of equations required during the iteration process is relatively small. However, the choice of the penalty function remains *ad hoc*, and a systematic approach to determine this function for the general case continues to be a subject of current research.

Kwak and Lee[70] developed a complementary problem formulation based on mathematical programming for the solution of 2-D contact problems with Coulomb friction using the BEM. The constraint inequalities were transformed to equalities by introducing slack variables so that a system of linear equations was formed. Among the equations were the equilibrium equations for the external force and the contact stresses. A modified simplex method was used to obtain the unknowns which were subjected to the complementary conditions. Several 2-D elastic contact problems were analysed with this technique and the dependency of the contact stress state on loading path was clearly illustrated by the examples.

Based on the finite element work of Klarbring *et al*[64, 65, 66], Gakwaya and Lambert *et al*[35, 36] have presented a mathematical programming formulation with the BEM to solve 3-D contact problems with Coulomb friction. No numerical examples were provided in the reference, however. A BEM analysis of an elastic punch on elastic foundation using this approach was carried out in 1991 by Kong *et al*[69].

Šimunović and Saigal[117] derived the formulations for the problem of frictionless contact between an elastic body and a rigid surface using the quadratic programming (QP) technique with BEM. Three objective functions were defined in terms of the unknown displacements and tractions on the contact area, and expressed as quadratic objective functions that are to be minimized. But only one objective function was convex. The constraints demand that the pressure is non-negative, and the elastic body does not penetrate the rigid surface within the possible contact area. The solution was obtained using the standard quadratic programming techniques of optimization. A number of example problems in 2D with straight and curved contact boundaries were solved.

Starting from a boundary variational inequality expressed in terms of the contact pressures which were assumed to balance the external forces, Tralli *et al.* [135] formulated the elastic frictionless contact problems as linear complementary problems. However, the coefficient matrices of this parametric complementary problem could not be calculated using the boundary element method, but was instead approximated. This approach thus cannot be considered to be very reliable. Similar approaches were also developed by Alliney *et al.*[5], Panagiotopoulos[96, 97], Bui[18, 19] for contact related problems.

The above variational-optimization algorithms provide a good alternative to the classical formulation with the BEM for solving the contact problems. Indeed, the uniqueness and convergence of the algorithms are provable, see, e.g. Wang[144]. However, there are also two basic drawbacks of the variational-optimization technique. The first is that the contact stresses and displacements within the contact area were never obtained simultaneously, see, e.g., Yamazaki[148], Kwak and Lee[70], Šimunović and Saigal[117]. For problems in which both the unknowns within and outside the contact area are needed, this means that a further linear analysis has to be carried out with either the contact stresses or the displacements within the contact area as the known boundary conditions, to obtain the displacements and tractions outside the contact area. As a result, this technique is not computational efficient for this type of problems. The second is that often some additional boundary conditions, see, e.g. Kwak and Lee[70], or assumptions, see, e.g. Gakwaya *et al.*[35, 36] have to be made to generate a proper optimization problem. Consequently, the stresses and the displacements within the contact area may not satisfy the boundary integral equations. This, may result in inaccurate results in some cases.

The primary objective in the present work is to efficiently solve frictional contact problems in which a crack or cracks are present. In these problems, both the load distribution in the contact area as well as the stress state outside it are of interest. The classical formulation appears to be an appropriate choice for treating this class of problems and is adopted for the present study. For convenience in the analysis

that follows, the BIE, Eq. (2.32), for zero body forces may be re-written as

$$\int_{\Gamma} \hat{T}_{ij}(P, Q) u_j(Q) d\Gamma(Q) = \int_{\Gamma} U_{ij}(P, Q) t_j(Q) d\Gamma(Q) \quad (3.8)$$

where

$$\hat{T}_{ij}(P, Q) = \delta_{ij}(P, Q) + T_{ij}(P, Q) \quad (3.9)$$

The quantity $\delta_{ij}(P, Q)$ is a Dirac delta function defined by

$$\int_{\Gamma} \delta_{ij}(P, Q) u_j(Q) d\Gamma = \begin{cases} C_{ij}(P) u_j(P) & \text{if } P \in \Gamma \\ 0 & \text{if } P \notin (\Gamma + \Omega) \end{cases} \quad (3.10)$$

where it is to be understood that Eq. (3.8) represents Eq. (3.9) only in the case when $P \in \Gamma$.

3.4 Incremental BEM analysis for contact problems in 2D

As has been mentioned previously, the first application of the BEM in the analysis of 2-D contact problems was due to Andersson, Persson and Fredriksson[9]. However, their formulation was relatively inefficient and is computationally very time-consuming for treating practical problems. Several relatively more efficient algorithms have been developed by, e.g. Zheng[155], Alcantud, Doblaré, Gracia and Dominguez[4], Dandekar and Conant[29]. Improvements over these algorithms are implemented here; they reduce the central memory and CPU time requirements in the contact mechanics analysis.

3.4.1 Contact Conditions

Consider the two elastic bodies Ω^A and Ω^B , in Figure 3.3, bounded by surface Γ^A and Γ^B , respectively, and are in contact with each other along the contact zone Γ_c . The

boundaries of the bodies are divided into three paths corresponding to the boundary conditions and the contact zone.

$$\Gamma^A = \Gamma_t^A + \Gamma_u^A + \Gamma_c^A \quad (3.11)$$

$$\Gamma^B = \Gamma_t^B + \Gamma_u^B + \Gamma_c^B \quad (3.12)$$

where the superscripts represent the two bodies; Γ_t is that part of the boundary over which tractions are prescribed; Γ_u is that part of the boundary where displacements are prescribed; and Γ_c^A and Γ_c^B are the contact boundaries in the undeformed state. According to small displacement theory, they have approximately the same length as the contact boundary Γ_c .

A new local co-ordinate system (η_1, η_2) , located on Γ_c is introduced. From the initial state or a previous load step, the gap g_i between the two contact boundaries are calculated, and the traction and displacement components are re-defined, with original symbols for these quantities still used, under this local co-ordinate system.

When friction is considered, the relation between normal and tangential tractions is defined by the coefficient of friction f . The contact zone is divided into three paths Γ_{cp} , Γ_{cs} and Γ_{ca} corresponding to the following friction state: open, slip and adhesion, for which the traction relations are

$$\begin{aligned} t_2 = t_1 = 0 & \quad \text{on } \Gamma_{cp} & (\text{open}) \\ t_2 = \pm f t_1 & \quad \text{on } \Gamma_{cs} & (\text{slip}) \\ |t_2| < f |t_1| & \quad \text{on } \Gamma_{ca} & (\text{adhesion}) \end{aligned} \quad (3.13)$$

where t_1 and t_2 are the normal and tangential tractions, respectively. For each load increment the contact conditions for the contact zones are written as

$$\begin{aligned} \Gamma_{cp}: \quad \Delta u_1^A - \Delta u_1^B < g_1^n \quad \Delta u_2^A - \Delta u_2^B < g_2^n \\ \Delta t_1^A = \Delta t_1^B = 0 \quad \Delta t_2^A = \Delta t_2^B = 0 \end{aligned} \quad (3.14)$$

$$\begin{aligned} \Gamma_{cs}: \quad \Delta u_1^A = \Delta u_1^B + g_1^n = \Delta u_1 \quad \Delta t_2^A (\Delta u_2^A - \Delta u_2^B) \leq 0 \\ \Delta t_1^A = -\Delta t_1^B = \Delta t_1 \quad \Delta t_2^A = -\Delta t_2^B = \pm f \Delta t_1 \\ t_1^A < 0 \end{aligned} \quad (3.15)$$

$$\begin{aligned}
\Gamma_{ca} : \quad \Delta u_1^A &= \Delta u_1^B + g_1^n = \Delta u_1 & \Delta u_2^A &= \Delta u_2^B + g_2^n = \Delta u_2 \\
\Delta t_1^A &= -\Delta t_1^B = \Delta t_1 & \Delta t_2^A &= -\Delta t_2^B = \Delta t_2 \\
t_1^A &< 0 & |t_2^A| &< -f t_1^A
\end{aligned} \tag{3.16}$$

where u_1 and u_2 are the normal and tangential displacements, respectively. The sign for t_2^A and t_2^B are chosen according to the fact that friction acts against motion. For example, if the initial state is the adhesive state and the next state is the slip state, then the friction force is in the same direction of the previous adhesive force.

3.4.2 Boundary integral equation formulation

Consider again the two elastic bodies of Figure 3.3. To solve the contact problem, assume a possible maximum contact area which is divided into one path Γ_{ca} , or two paths, Γ_{ca} and Γ_{cp} . Since friction is involved, a load incremental theory is used. The total quantities are then calculated as the sum of increments, i.e. for displacements and tractions

$$t_i^n = \Delta t_i^n + t_i^{n-1} = \sum_{k=1}^n \Delta t_i^k \tag{3.17}$$

$$u_i^n = \Delta u_i^n + u_i^{n-1} = \sum_{k=1}^n \Delta u_i^k \tag{3.18}$$

Writing the boundary integral equations in an incremental form for both bodies and using Eq. (3.14)–(3.16) at the n -load step results in the following two equations.

$$\begin{aligned}
& \int_{\Gamma^A - \Gamma_c^A} \hat{T}_{ji}^A \Delta u_i^A d\Gamma + \int_{\Gamma_{ca}^A} \hat{T}_{jk}^A n_{ik} \Delta u_i^A d\Gamma + \int_{\Gamma_{ca}^A} (\hat{T}_{jk}^A n_{1k} \Delta u_1 + \hat{T}_{jk}^A n_{2k} \Delta u_2^A) d\Gamma \\
& + \int_{\Gamma_{cp}^A} \hat{T}_{jk}^A n_{ik} \Delta u_i^A d\Gamma = \int_{\Gamma^A - \Gamma_c^A} U_{ji}^A \Delta t_i^A d\Gamma \\
& + \int_{\Gamma_{ca}^A} U_{jk}^A n_{ik} \Delta t_i^A d\Gamma + \int_{\Gamma_{ca}^A} (U_{jk}^A n_{1k} \pm f U_{j2}^A n_{2k}) \Delta t_1^A d\Gamma
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
& \int_{\Gamma^B - \Gamma_c^B} \hat{T}_{ji}^B \Delta u_i^B d\Gamma + \int_{\Gamma_{cp}^B} \hat{T}_{jk}^B n_{ik} \Delta u_i^B d\Gamma + \int_{\Gamma_a^B} \hat{T}_{jk}^B n_{ik} (\Delta u_i - g_i^n) d\Gamma \\
& + \int_{\Gamma_a^B} [\hat{T}_{jk}^B n_{1k} (\Delta u_1 - g_1^n) + \hat{T}_{jk}^B n_{2k} \Delta u_2^B] d\Gamma \\
& = \int_{\Gamma^B - \Gamma_c^B} U_{ji}^B \Delta t_i^B d\Gamma - \int_{\Gamma_a^B} U_{jk}^B n_{ik} \Delta t_i d\Gamma - \int_{\Gamma_a^B} (U_{jk}^B n_{1k} \pm f U_{jk}^B n_{2k}) \Delta t_1 d\Gamma \quad (3.20)
\end{aligned}$$

where n_{ij} in the above equations is the unit vector of η_i in the global coordinate system x_j . If there are m bodies ($m > 2$) involving contact, m set of equations as the above may be obtained. They may however be very complicated in form, since one body may be in contact with two or three other bodies simultaneously

3.4.3 Boundary element discretization techniques

The boundary element discretization of Eqs. (3.19) and (3.20) follows established techniques of the isoparametric formulation. Here the two- and three-node isoparametric elements are used. This will result in the following matrix form of the linear equations in terms of the increments.

$$\begin{bmatrix} \hat{T}^A & 0 & \hat{T}_a^A & \hat{T}_{1,}^A & \hat{T}_{2,}^A & 0 & \hat{T}_{cp}^A & 0 \\ 0 & \hat{T}^B & \hat{T}_a^B & \hat{T}_{1,}^B & 0 & \hat{T}_{2,}^B & 0 & \hat{T}_{cp}^B \end{bmatrix} \begin{Bmatrix} \Delta u^A \\ \Delta u^B \\ \Delta u_a \\ \Delta u_{1,} \\ \Delta u_{2,}^A \\ \Delta u_{2,}^B \\ \Delta u_p^A \\ \Delta u_p^B \end{Bmatrix}$$

$$- \begin{bmatrix} U^A & 0 & U_a^A & (U_{1a}^A \pm f U_{2a}^A) \\ 0 & U^B & -U_a^B & -(U_{1a}^B \pm f U_{2a}^B) \end{bmatrix} \begin{Bmatrix} \Delta t^A \\ \Delta t^B \\ \Delta t_a \\ \Delta t_{1a} \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ \hat{T}_a^B & \hat{T}_{1a}^B \end{bmatrix} \begin{Bmatrix} g_a^n \\ g_{1a}^n \end{Bmatrix} \quad (3.21)$$

where the superscripts denote the intrinsic bodies; Δu and Δt denote the vectors of displacements and tractions, respectively, outside the contact zone; Δu_a and Δt_a denote the vectors of displacements and tractions, respectively, within Γ_{ca} ; and Δu_p denotes the vectors of displacements within Γ_{cp} . Also Δt_{1a} is the vector of tractions in η_1 direction within Γ_{ca} ; Δu_{1a} denotes the vector of displacements in the η_1 direction within Γ_{ca} ; g_a^n denotes the vector of gap between Γ_{ca}^A and Γ_{ca}^B at the n -load step; and g_{1a}^n denotes the vector of gap at the η_1 direction between Γ_{ca}^A and Γ_{ca}^B at the n -load step. Furthermore, the symbols U and $\hat{T}$ in the above equations represent submatrices containing the coefficients which are obtained from integrating the fundamental solutions U_{ij} and $\hat{T}_{ij}$, respectively, and the shape function products over the boundary elements.

3.4.4 Evaluation of the minimum load increment

The amount of the external load increment Δq_i^n at the n -load step can be written as

$$\Delta q_i^n = \beta \Delta q_i \quad (3.22)$$

where Δq_i is a certain load increment, and β is the load incremental ratio to be determined. The solution to Eq. (3.21) is scaled by the value of β to give a zero normal traction inside the contact zone or at new element pairs coming into contact with each other outside the assumed contact zone. This scaling procedure is done for every element pair inside or outside (near) the contact zone. The value of β should be chosen as the smallest of the values to preclude overlapping outside the contact zone, or the largest of the values (then $\beta < 0$) calculated for zero normal tractions inside the contact zone (to preclude tensile contact pressure).

Suppose that at the end of the previous n step, $\bar{t}_1^n$ is the average element traction within Γ_c ; $\bar{g}_1^n$ is the average gap between the element pair within Γ_c ; $\Delta\bar{t}_1^n$ is the average element traction increment due to Δq_i ; $\Delta\bar{u}_1^A$ and $\Delta\bar{u}_1^B$ are the average element displacement increments for bodies A and B respectively, due to Δq_i . The elements related to contact may be in one of the three possible states listed below:

From open state to contact state: In such a circumstance, the following equation must be satisfied:

$$\bar{g}_1^n - \beta(\Delta\bar{u}_1^A - \Delta\bar{u}_1^B) = 0. \quad (3.23)$$

Therefore β is given by

$$\beta = \frac{\bar{g}_1^n}{\Delta\bar{u}_1^A - \Delta\bar{u}_1^B}. \quad (3.24)$$

From contact state to open state: In this case, the normal traction of the element is zero, i.e.

$$\bar{t}_1^n - \beta(\Delta\bar{t}_1^n) = 0. \quad (3.25)$$

Thus β is obtained as

$$\beta = \frac{\bar{t}_1^n}{\Delta\bar{t}_1^n} \quad (3.26)$$

From adhesion state to slip state: When the tangential traction reaches its limit, sliding between the two contact surfaces will begin. This means that

$$|\bar{t}_2^n + \beta\Delta\bar{t}_2^n| + f(\bar{t}_1^n + \beta\Delta\bar{t}_1^n) = 0. \quad (3.27)$$

Consequently,

$$\beta = -\frac{\bar{t}_2^n - f\bar{t}_1^n}{\Delta\bar{t}_2^n - f\Delta\bar{t}_1^n} \quad \text{if } \Delta\bar{t}_1^n \cdot \bar{t}_2^n \leq \Delta\bar{t}_2^n \cdot \bar{t}_1^n; \quad (3.28)$$

$$\beta = -\frac{\bar{t}_2^n + f\bar{t}_1^n}{\Delta\bar{t}_2^n + f\Delta\bar{t}_1^n} \quad \text{if } \Delta\bar{t}_1^n \cdot \bar{t}_2^n > \Delta\bar{t}_2^n \cdot \bar{t}_1^n. \quad (3.29)$$

3.4.5 Solving the system equations

After rearranging the coefficients of the matrices shown in Eq. (3.21) corresponding to the unknown quantities and multiplying those to the corresponding known variables, a linear equation system is established. The solution for the unknown boundary variables may be obtained by Gaussian elimination. However, a direct Gaussian elimination procedure is very time-consuming since in each iteration process, the whole set of the linear system of equations must be solved. Therefore, a special Gaussian elimination approach, which is the extension of the one developed by the author, see Liu *et al*[75], is used here to solve Eq. (3.21). Using this approach, in the iteration process, only a small system of equations involving only the unknowns over the contact zones needs to be solved. This procedure is described below.

Step (1) Form an uncoupled linear system of equations for each contacting body, i.e.

$$\begin{bmatrix} \mathbf{H}^A & \mathbf{H}_u^A & \mathbf{H}_t^A \\ \mathbf{G}^A & \mathbf{G}_u^A & \mathbf{G}_t^A \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}^A \\ \Delta \mathbf{u}_c^A \\ \Delta \mathbf{t}_c^A \end{Bmatrix} = \begin{Bmatrix} \Delta \mathbf{b}^A \\ \Delta \mathbf{p}^A \end{Bmatrix} \quad (3.30)$$

$$\begin{bmatrix} \mathbf{H}^B & \mathbf{H}_u^B & \mathbf{H}_t^B \\ \mathbf{G}^B & \mathbf{G}_u^B & \mathbf{G}_t^B \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}^B \\ \Delta \mathbf{u}_c^B \\ \Delta \mathbf{t}_c^B \end{Bmatrix} = \begin{Bmatrix} \Delta \mathbf{b}^B \\ \Delta \mathbf{p}^B \end{Bmatrix} \quad (3.31)$$

where the superscripts denote the intrinsic bodies; $\Delta \mathbf{x}^A$ denotes the vector of the unknowns outside Γ_c^A and $\mathbf{H}^A$, $\mathbf{G}^A$ the corresponding submatrices; $\Delta \mathbf{u}_c^A$ denotes the vector of the displacements over Γ_c^A and $\mathbf{H}_u^A$, $\mathbf{G}_u^A$ the corresponding submatrices; $\Delta \mathbf{t}_c^A$ denotes the vector of the tractions over Γ_c^A and $\mathbf{H}_t^A$, $\mathbf{G}_t^A$ the corresponding submatrices, $\Delta \mathbf{b}^A$ and $\Delta \mathbf{p}^A$ are two known vectors obtained from Eq. (3.21). Two points of detail need be made here. The first is that for the common node formed by two different elements, there are four traction components for such a node. This is due to the fact that the two elements may have different contact conditions, and the two sets of tractions are therefore not equal to each other. As such, the number of columns of the submatrix $\mathbf{H}_t^A$ or

$\mathbf{G}_t^A$ is larger than twice the number of the nodes on Γ_c^A . But the number of columns of the submatrix of $\mathbf{H}_u^A$ or $\mathbf{G}_u^A$ is always equal to twice the number of the nodes on Γ_c^A in 2D. Second, the number of rows for the submatrix $\mathbf{H}^A$ is always equal to the number of unknowns outside Γ_c^A .

Step (2) Carry out a partial Gaussian elimination for the above two uncoupled system so that

$$\begin{bmatrix} \dot{\mathbf{H}}^A & \dot{\mathbf{H}}_u^A & \dot{\mathbf{H}}_t^A \\ 0 & \dot{\mathbf{G}}_u^A & \dot{\mathbf{G}}_t^A \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}^A \\ \Delta \mathbf{u}_c^A \\ \Delta \mathbf{t}_c^A \end{Bmatrix} = \begin{Bmatrix} \Delta \dot{\mathbf{b}}^A \\ \Delta \dot{\mathbf{p}}^A \end{Bmatrix} \quad (3.32)$$

$$\begin{bmatrix} \dot{\mathbf{H}}^B & \dot{\mathbf{H}}_u^B & \dot{\mathbf{H}}_t^B \\ 0 & \dot{\mathbf{G}}_u^B & \dot{\mathbf{G}}_t^B \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}^B \\ \Delta \mathbf{u}_c^B \\ \Delta \mathbf{t}_c^B \end{Bmatrix} = \begin{Bmatrix} \Delta \dot{\mathbf{b}}^B \\ \Delta \dot{\mathbf{p}}^B \end{Bmatrix} \quad (3.33)$$

Step (3) Use the contact conditions Eq. (3.14)–(3.16) and submatrices $\dot{\mathbf{G}}_u^A, \dot{\mathbf{G}}_t^A, \dots, \Delta \dot{\mathbf{p}}^B$ to form a system of equations so that

$$[\mathbf{C}] \{\Delta \mathbf{x}_c\} = \{\Delta \mathbf{b}_c\} \quad (3.34)$$

where $\Delta \mathbf{x}_c$ is the vector of unknowns over Γ_c . If $\Delta \mathbf{x}_c$ is valid, upon substituting it into Eq. (3.32) and (3.33) and satisfying the contact conditions Eq. (3.14)–(3.16), $\Delta \mathbf{x}^A$ and $\Delta \mathbf{x}^B$ are obtained without involving much computational effort since $\dot{\mathbf{H}}^A$ in Eq. (3.32) and $\dot{\mathbf{H}}^B$ in Eq. (3.33) are upper triangular matrices.

It may also be worthwhile mentioning that if more than one coefficient of friction is involved in the contact problem, the solution process which considers the different coefficient of frictions are directly involved in step (3) only.

3.4.6 Convergence criteria

An assumed contact zone should come to the correct one if the iterative process converges. The various convergence criteria are tabulated below in Table 3.1.

Table 3.1 Criterion of convergence in 2-D contact mechanics analysis

Condition		Criteria
Before	After	
Open	Open	$\bar{g}_1^n > 0$
	Contact	$\bar{g}_1^n \leq 0$
Adhesion	Adhesion	$\bar{t}_1^n \leq 0, \quad \bar{t}_2^n \leq -f\bar{t}_1^n$
	Open	$\bar{t}_1^n > 0$
	Slip	$\bar{t}_1^n \leq 0, \quad \bar{t}_2^n \geq -f\bar{t}_1^n$
Slip	Adhesion	$\bar{t}_1^n \leq 0, \quad \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) > 0$
	Open	$\bar{t}_1^n > 0$
	Slip	$\bar{t}_1^n \leq 0, \quad \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) \leq 0$

3.5 Numerical Examples in 2D

Although the main focus of this thesis is the development, implementation and application of the above formulations to treat crack problems, a number of non-crack contact problems have also been analysed. Four of them are presented here primarily for the purpose of illustration. The first is a classical Hertz contact problem while the second problem is a non-Hertz contact problem, both involving smooth contact conditions. They are chosen so that the BEM results may be compared with those obtained using other methods of analysis. The third and fourth problems are contact problems of pin-lug joints, without and with a bushing, respectively. They are part of a more comprehensive study on these joints carried out in conjunction with the work reported in this thesis. The significant results of the third problem, in particular, have been published in Liu and Tan[77]. These solutions have much practical significance in, e.g., the aircraft industry.

Example 1

An elastic roller on an elastic foundation was considered, as shown in Figure 3.4. Both the roller and the foundation are of the same material. The BEM mesh used is shown in Figure 3.5. The contact pressure, as a function of the contact zone, compared with the Hertz solution[133], is shown in Figure 3.6, where E is the Young's modulus, and ν is the Poisson's ratio which was taken to be equal to 0.3 in the BEM analysis. From Figure 3.6 it can be seen that the agreement between the BEM and exact solution is excellent indeed.

Example 2

The second problem considered was that of an elastic pin in an infinite plate with a hole of nearly the same radius as the pin, as shown in Figure 3.7. Both the pin and the infinite plate are of the same material. The problem was treated as an exterior problem in the BEM analysis in which the pin was divided into 40 elements and the infinite plate into 20 elements as shown in Figure 3.8. The external load was applied over the 20 small elements at the center of the pin which was modelled as a small central hole with radius 6% of that of the pin. The Poisson ratio was taken as 0.3 in the BEM analysis. The results for the contact stresses for the different clearances treated are shown in Figure 3.9 together with those obtained by Persson[103]. It can be seen that the two solutions are in good agreement, their discrepancies being generally less than 5%.

Example 3

Figure 3.10 shows a round-ended straight symmetric lug of Young's modulus of E_1 , loaded axially via a circular pin of Young's modulus E_2 . Such joints are commonly used in, for example, aircraft structures to connect major structural components. In these joints, both the contact stresses and the stress concentration factors for the lugs are of practical interest. Although the analysis of the stresses in pin-loaded lugs/plates has been a subject of much analytical and experimental investigations over the past

few decades, surprisingly, relatively little about the contact stress distributions have been made available in the literature. A set of FEM results of the pin-loaded lug problem for both the contact stress distributions and the tangential stresses along the periphery of the lug-hole has also been given by Kathiresan *et al.*[58] but their analysis assumed that there is no friction between the neat-fitted pin and the lug ($R_1 = R_2$).

A typical BEM mesh employed in this study is shown in Figure 3.11. To simulate the radial point force F in the BEM model, a small central cavity was introduced in the pin, over part of the surface of which a radial pressure was applied. The boundary element analysis was first carried out to compare with the finite element results obtained by Kathiresan *et al.*[58] in which $E_2/E_1 = 3$, $H/W = 2$, $(R_1 - R_2) = 0$ (see Figure 3.10) and the coefficient of friction $f = 0$. The Poisson's ratio was taken to be equal to 0.3 for both the pin and lug throughout. Figure 3.12 and Figure 3.13 show the normalized contact pressure distributions, p/σ_{br} and the normalised tangential stresses, $\sigma_{\varphi\varphi}/\sigma_{br}$ at the edge of the pin-hole in the lug, where $\sigma_{br} = F/(2R_1b)$, b being the thickness of the lug. From these figures, it can be seen that the agreement between the boundary element analysis and the finite element results are excellent indeed, the discrepancies being generally less than 5%.

The effects of pin-fit, the coefficient of friction and the rigidity ratio, E_2/E_1 , were then investigated in this study for the lug geometry with $H/W = 2$ and $R_1/W = 0.5$. The cases treated were:

$$(R_1 - R_2)/R_1 = -0.0006, 0 \text{ and } 0.0006$$

$$f = 0, 0.2, 0.4, 0.6, 0.8 \text{ and } 1$$

$$E_2/E_1 = 3 \text{ and } 1.$$

Clearly, in the above, $R_1 > R_2$ corresponds to clearance fit and $R_1 < R_2$ corresponds to interference fit. It should be noted, however, that, for a given lug, friction condition and the extent of contact, the resultant stresses in the lug, when normalised by σ_{br} , will not be affected by the actual magnitude of the initial value of $(R_1 - R_2)/R_1$.

These normalised stresses are instead influenced by the extent of contact between the two components, as measured by the angle ϕ shown in Figure 3.10. Also, in the case of neat-fit condition ($R_1 = R_2$), the magnitude of ϕ is not affected by that of the applied load F , being equal to about 90° throughout.

Figure 3.14 shows the variations of the computed normalised contact pressure, namely, p/σ_{br} , with the angular position φ (see Figure 3.10) for the cases treated under initial neat-fit condition. It is evident from these results that friction has a very significant effect on the distribution of the contact pressure between the pin and the lug. The value of p/σ_{br} at a given position φ at the edge of the pin-hole can be seen to decrease with increasing friction. Of interest to note too is that for the geometry treated, the distribution of the contact pressure around the pin-hole is not generally a simple cosine function in terms of φ as has sometimes been assumed in the literature for treating lug joints using conventional linear analysis. It is also worth noting that except for the frictionless case, the contact pressure distribution is significantly affected by the E_2/E_1 ratio, as to be expected. The corresponding distributions of the normalised contact shear stress, $\sigma_{r\varphi}/\sigma_{br}$, between the pin and the hole surface are as shown in Figure 3.15. As expected, the shear stress is zero at the plane of symmetry, remaining relatively small in magnitude in the region of the contact zone where the surfaces are in the "stick" mode, then increasing to a peak value near the ends of the contact zone before decreasing rapidly to zero again.

In the case of interference fit of the pin, partial loss of contact between the surfaces of the pin and the pin-hole occurs when a sufficiently high magnitude of F is applied. The size of the contact zone decreases as F increases. For the treatment here of the problem, the pin force F was increased until the extent of contact $\phi = \pm 135^\circ$. Figures 3.16 and 3.17 show the computed variation of the normalised contact pressure, p/σ_{br} , and the normalised shear stress $\sigma_{r\varphi}/\sigma_{br}$, with the angular position φ . Again, friction can be seen to affect these distributions significantly. It is worth noting too that the magnitude of the external load F to cause $\phi = 135^\circ$ increases with friction, although the normalised contact pressure at a given angular position decreases with

increasing value of f . As for the rigidity ratio E_2/E_1 , it has generally the same effect on the contact stress distributions as in the neat-fit case above.

In the clearance-fit case treated, the extent of contact zone considered here is $\phi = \pm 45^\circ$. The corresponding results for the normalised contact pressure and normalised shear stress distributions are shown in Figure 3.18 and 3.19, respectively. In contrast to the interference-fit case, the magnitude of F to cause $\phi = \pm 45^\circ$ of the extent of contact increases as friction decreases, as to be expected. Because of the relatively small extent of the contact zone, friction does not appear to affect the contact pressure distributions as dramatically as in the case seen earlier, although its effects on the contact shear stress are very evident. In both these clearance-fit and the interference-fit case, the contact zone and the contact stress distributions will approach those for the neat-fit cases when the pin-load F is increased further.

Example 4

The fourth problem considered here is as shown in Figure 3.20. It is the same pin-loaded lug as in the previous example except that an interference-fitted bushing has been inserted into the hole of the lug. Such joints have been shown to give better fatigue life performance than those without the bushing[111]. However, there are very few studies on the stress distributions in these components that have been reported in the literature.

As before, let E_1 be the Young's modulus of the lug, and E_2 be the Young's modulus of the pin. For the purpose of illustration here, let the pin and the bushing be made of the same material, as is often the case. Poisson's ratio for all the material was taken to be 0.3. Referring to Figure 3.20, the geometric case considered corresponds to $H/W = 2$, $R_3/R_1 = 1.25$, and $R_1/W = 0.5$. Also, the initial radial interference between the bushing and the hole in the lug was taken to be $0.0006R_3$ and the pin was treated to be in clearance-fit with the internal radius of the bushing, the clearance $(R_1 - R_2)$ being $0.0006R_1$. As before, two values of E_2/E_1 were chosen, namely, 3 and 1.

Figure 3.21 shows a typical BEM mesh used in the analysis, advantage being taken of the plane of symmetry of the problem. To simulate the pin-loading, a small cavity was again introduced at the centre of the pin where a radial pressure was applied. Results for the stress distribution at the periphery of the hole in the lug at these different stages of loading are presented here for the frictionless case. They are, first, the prestresses induced by the interference-fit of the bushing on the lug alone; second, the stresses produced as a result of pin-loading up to a contact zone of $\phi = \pm 45^\circ$ between the pin and the bushing; and third, the stresses produced when a sufficiently high pin-load was applied such that partial separation between the bushing and the lug occurred. In this last case, the maximum extent of contact between the pin and the bushing has been reached and was at about $\phi = \pm 90^\circ$. Also, the total angular span of the loss of contact between the bushing and the lug considered was about 90° over the whole circumference.

Figure 3.22 shows the normalised interfacial pressure, $p/(\delta E_1)$, between the bushing and the lug hole due to the interference-fit alone of the bushing. The quantity δ here is the ratio of the magnitude of the radial interference to the radius R_3 . Of interest to note is that the interfacial pressure is fairly uniform along the periphery of the lug hole despite the non-axisymmetric nature of the problem. For $E_2/E_1 = 1$, the value of $p/(\delta E_1)$ was about 0.16 while for $E_2/E_1 = 3$, it was about 0.28. Comparing these to the corresponding interfacial pressure for a shrink fitted compound ring with same radius ratios W/R_3 and R_3/R_1 , they are 0.203 and 0.295, respectively, for the two different Young's modulus ratio. Figure 3.23 shows the computed distribution of the normalized tangential stress, $\sigma_{\phi\phi}/(\delta E_1)$, along the periphery of the lug hole due to the interference-fit. The trends for both the cases when $E_2/E_1 = 1$ and $E_2/E_1 = 3$ are similar; these stresses are however not uniform around the hole periphery, as to be expected.

With the pin-load applied and the zone of contact between the pin and the bushing spanning over $\phi = \pm 45^\circ$, the normalised interfacial pressure, p/σ_{br} , between the bushing and that between the pin and the bushing are as shown in Figure 3.24.

σ_{br} here is again the "bearing" stress on the lug defined previously, namely, $\sigma_{br} = F/(2R_1 B)$, where B is the thickness of the lug. Figure 3.25 shows the corresponding distributions of the normalised tangential stress, $\sigma_{\varphi\varphi}/\sigma_{br}$, along the periphery of the lug hole. It can be seen from these figures and Figure 3.22 and 3.23, the pre-stresses due to the interference-fit are still predominant in the distributions of the final stresses when the extent of contact between the pin and the bushing is relatively small.

Figure 3.26 shows the distributions of the normalised contact pressure between the pin and the bushing and that between the bushing and the lug for the third load case considered here. Of significance to note is that separation of the bushing and the lug did not occur at $\varphi = 180^\circ$. Indeed, for the loads applied to cause 90° total angular span of separation between the two components, the lug remained in full contact with the bushing at $\varphi = \pm 180^\circ$ for both $E_2/E_1 = 1$ and $E_2/E_1 = 3$. When $E_2/E_1 = 1$, the separation occurred at about $\varphi = \pm 115^\circ$ to $\varphi = \pm 160^\circ$; when $E_2/E_1 = 3$, it occurred at about $\varphi = \pm 125^\circ$ to $\varphi = \pm 170^\circ$. Figure 3.27 shows the corresponding variation of the normalised tangential stress, $\sigma_{\varphi\varphi}/\sigma_{br}$, around the periphery of the lug hole. The qualitative features of the normalised tangential stress distributions, for $E_2/E_1 = 1$ and $E_2/E_1 = 3$ are both very similar, and the computed peak values occurred at $\varphi = \pm 97.5^\circ$, their magnitudes being 2.75 for the former Young's modulus ratio and 2.6 for the latter.

3.6 Incremental BEM Formulations for Contact Problems in 3D

Very few publications are available for 3-D contact problems using BEM. Moreover, the formulations in these published studies have limitations in terms of their applicability to more general 3-D contact problems. The procedures outlined in the previous Section are extended here to solve such problems. The isotropic Coulomb's friction law is again used and the special Gaussian elimination algorithm, as outlined in Section 3.4, is adopted for use in the 3-D analysis.

3.6.1 Contact Conditions

Consider the two elastic bodies Ω^A and Ω^B , in Figure 3.30, bounded by the surface Γ^A and Γ^B respectively in three dimensions, and being in contact with each other along the contact zone Γ_c . The surfaces of the bodies are again divided into three regions corresponding to the boundary conditions and the contact zone.

$$\begin{aligned}\Gamma^A &= \Gamma_t^A + \Gamma_u^A + \Gamma_c^A, \\ \Gamma^B &= \Gamma_t^B + \Gamma_u^B + \Gamma_c^B\end{aligned}\quad (3.35)$$

where the superscripts denotes the respective bodies; Γ_t is that part of the surface on which tractions are prescribed; and Γ_u is that part of the surface where displacements are prescribed. Also, Γ_c^A and Γ_c^B are the contact surfaces and according to small displacement theory, they have approximately the same magnitude as the contact surface Γ_c .

A local co-ordinate system (η_1, η_2, η_3) is introduced on Γ_c . From the initial state or a previous load step, the gap g_i between any candidate points in two contact surfaces is calculated, and the traction and displacement components are re-defined with original symbols still used, under this co-ordinate system.

When friction is considered, let f be the coefficient of friction and let f_i be the "pseudo-coefficient of friction" in the η_i direction. The need to introduce the latter will be apparent later in Section 3.6.3. For the analysis, the contact zone is further divided into four subregions corresponding to the following contact conditions: open, slip, adhesion and mixed (assuming that slip occurs in η_2 direction), on which the tractions are

$$\begin{aligned}t_1 &= 0, \quad t_2 = t_3 = 0 \quad \text{on } \Gamma_{cp} \quad (\text{open}) \\ t_2 &= \pm f_2 t_1, \quad t_3 = \pm f_3 t_1 \quad \text{on } \Gamma_{cs} \quad (\text{slip}) \\ |t_2| &< f|t_1|, \quad |t_3| < f|t_1| \quad \text{on } \Gamma_{ca} \quad (\text{adhesion}) \\ t_2 &= \pm f t_1, \quad |t_3| < f|t_1| \quad \text{on } \Gamma_{cm} \quad (\text{mixed})\end{aligned}\quad (3.36)$$

where t_1 is the normal traction, and t_2 and t_3 are the two tangential tractions. Also, the additional subscripts p , s , a and m on Γ_c denote the open, slip, adhesion and

mixed contact conditions on Γ_c , respectively. For each load increment the contact conditions for the above contact zones are written as

$$\begin{aligned}\Gamma_{cp} : \quad & \Delta t_1^A = \Delta t_2^A = \Delta t_3^A = 0, \quad \Delta t_1^B = \Delta t_2^B = \Delta t_3^B = 0 \\ & \Delta u_1^A - \Delta u_1^B < g_1^n, \quad \Delta u_2^A - \Delta u_2^B < g_2^n \\ & \Delta u_3^A - \Delta u_3^B < g_3^n\end{aligned}\quad (3.37)$$

$$\begin{aligned}\Gamma_{ca} : \quad & \Delta u_1^A = \Delta u_1^B + g_1^n = \Delta u_1, \quad \Delta t_2^A (\Delta u_2^A - \Delta u_2^B) \leq 0 \\ & \Delta t_3^A (\Delta u_3^A - \Delta u_3^B) \leq 0, \quad \Delta t_1^A = -\Delta t_1^B = \Delta t_1 \\ & \Delta t_2^A = -\Delta t_2^B = \pm f_2 \Delta t_1, \quad \Delta t_3^A = -\Delta t_3^B = \pm f_3 \Delta t_1 \\ & t_1^A < 0\end{aligned}\quad (3.38)$$

$$\begin{aligned}\Gamma_{ca} : \quad & \Delta u_1^A = \Delta u_1^B + g_1^n = \Delta u_1, \quad \Delta u_2^A = \Delta u_2^B + g_2^n = \Delta u_2 \\ & \Delta u_3^A = \Delta u_3^B + g_3^n = \Delta u_3, \quad \Delta t_1^A = -\Delta t_1^B = \Delta t_1 \\ & \Delta t_2^A = -\Delta t_2^B = \Delta t_2, \quad \Delta t_3^A = -\Delta t_3^B = \Delta t_3 \\ & t_1^A < 0, \quad |t_2^A|, |t_3^A| < -f t_1^A\end{aligned}\quad (3.39)$$

$$\begin{aligned}\Gamma_{cm} : \quad & \Delta u_1^A = \Delta u_1^B + g_1^n = \Delta u_1, \quad \Delta u_3^A = \Delta u_3^B + g_3^n = \Delta u_3 \\ & \Delta t_2^A (\Delta u_2^A - \Delta u_2^B) \leq 0, \quad \Delta t_1^A = -\Delta t_1^B = \Delta t_1 \\ & \Delta t_3^A = -\Delta t_3^B = \Delta t_3, \quad \Delta t_2^A = -\Delta t_2^B = \pm f \Delta t_1^A \\ & t_1^A < 0, \quad |t_3^A| < -f t_1^A\end{aligned}\quad (3.40)$$

where u_1 is the normal displacement, and u_2 and u_3 are the tangential displacements. As in 2-D analysis, the sign for the traction in the contact region should be chosen according to the fact that friction acts against the motion.

3.6.2 Boundary element analysis

Consider again the two elastic bodies of Figure 3.28. As in 2-D analysis described in Section 3.4.2, an incremental-iterative scheme must be followed to solve the frictional contact problem. The boundary integral equations in incremental form for both of

the bodies may then be written using Eq. (3.37)–(3.40) as follows

$$\begin{aligned}
& \int_{\Gamma^A - \Gamma_c^A} \hat{T}_{ij}^A \Delta u_j^A d\Gamma + \int_{\Gamma_{ca}^A} \hat{T}_{ik}^A n_{kj} \Delta u_j d\Gamma + \int_{\Gamma_{ca}^A} (\hat{T}_{ik}^A n_{1k} \Delta u_1 + \hat{T}_{ik}^A n_{2k} \Delta u_2^A + \hat{T}_{ik}^A n_{3k} \Delta u_3^A) d\Gamma \\
& + \int_{\Gamma_{cm}^A} (\hat{T}_{ik}^A n_{1k} \Delta u_1 + \hat{T}_{ik}^A n_{3k} \Delta u_3 + \hat{T}_{ik}^A n_{2k} \Delta u_2^A) d\Gamma + \int_{\Gamma_{cp}^A} \hat{T}_{ik}^A n_{jk} \Delta u_k^A d\Gamma \\
& = \int_{\Gamma^A - \Gamma_c^A} U_{ij}^A \Delta t_j^A d\Gamma + \int_{\Gamma_{ca}^A} U_{ik}^A n_{jk} \Delta t_j d\Gamma + \int_{\Gamma_{ca}^A} (U_{ik}^A n_{1k} \pm f_2 U_{ik}^A n_{2k} \pm f_3 U_{ik}^A n_{3k}) \Delta t_1 d\Gamma \\
& + \int_{\Gamma_{cm}^A} [U_{ik}^A n_{3k} \Delta t_3 + (U_{ik}^A n_{1k} \pm f U_{ik}^A n_{2k}) \Delta t_1] d\Gamma \quad (3.41)
\end{aligned}$$

$$\begin{aligned}
& \int_{\Gamma^B - \Gamma_c^B} \hat{T}_{ij}^B \Delta u_j^B d\Gamma + \int_{\Gamma_{ca}^B} [\hat{T}_{ik}^B n_{1k} (\Delta u_1 - g_1^n) + \hat{T}_{ik}^B n_{2k} \Delta u_2^B + \hat{T}_{ik}^B n_{3k} \Delta u_3^B] d\Gamma \\
& + \int_{\Gamma_{ca}^B} \hat{T}_{ik}^B n_{jk} (\Delta u_j - g_j^n) d\Gamma + \int_{\Gamma_{cp}^B} \hat{T}_{ik}^B n_{jk} \Delta u_j^B d\Gamma \\
& + \int_{\Gamma_{cm}^B} [\hat{T}_{ik}^B n_{1k} (\Delta u_1 - g_1^n) + \hat{T}_{ik}^B n_{3k} (\Delta u_3 - g_3^n) + \hat{T}_{ik}^B n_{2k} \Delta u_2^B] d\Gamma \\
& = \int_{\Gamma^B - \Gamma_c^B} U_{ij}^B \Delta t_j^B d\Gamma - \int_{\Gamma_{ca}^B} U_{ik}^B n_{jk} \Delta t_j d\Gamma - \int_{\Gamma_{ca}^B} (U_{ik}^B n_{1k} \pm f_2 U_{ik}^B n_{2k} \pm f_3 U_{ik}^B n_{3k}) \Delta t_1 d\Gamma \\
& - \int_{\Gamma_{cm}^B} [U_{ik}^B n_{3k} \Delta t_3 + (U_{ik}^B n_{1k} \pm f U_{ik}^B n_{2k}) \Delta t_1] d\Gamma \quad (3.42)
\end{aligned}$$

If there are m bodies involving contact, m set of equations as the above may then be obtained.

3.6.3 Boundary element discretization techniques

In the boundary element discretization considered here, four- and eight-node isoparametric quadrilateral, and three- and six-node isoparametric triangular elements are used. Using matrix notation, the following set of linear equations for solving the

increments can be obtained from Eq. (3.41) and Eq. (3.42):

$$\begin{aligned}
 & \begin{bmatrix} \hat{T}^A & 0 & \hat{T}_a^A & \hat{T}_{1s}^A & \hat{T}_{2s}^A & \hat{T}_{3s}^A & 0 & 0 & \hat{T}_{1m}^A & \hat{T}_{3m}^A & \hat{T}_{2m}^A & 0 & \hat{T}_p^A & 0 \\ 0 & \hat{T}^B & \hat{T}_a^B & \hat{T}_{1s}^B & 0 & 0 & \hat{T}_{2s}^B & \hat{T}_{3s}^B & \hat{T}_{1m}^B & \hat{T}_{3m}^B & 0 & \hat{T}_{2m}^B & 0 & \hat{T}_p^B \end{bmatrix} \begin{Bmatrix} \Delta u^A \\ \Delta u^B \\ \Delta u_a \\ \Delta u_{1s} \\ \Delta u_{2s}^A \\ \Delta u_{3s}^A \\ \Delta u_{2s}^B \\ \Delta u_{3s}^B \\ \Delta u_{1m} \\ \Delta u_{3m} \\ \Delta u_{2m}^A \\ \Delta u_{2m}^B \\ \Delta u_p^A \\ \Delta u_p^B \end{Bmatrix} \\
 & - \begin{bmatrix} U^A & 0 & U_a^A & (U_{1s}^A \pm f_2 U_{2s}^A \pm f_3 U_{3s}^A) & (U_{1m}^A \pm f U_{2m}^A) & U_{3m}^A \\ 0 & U^B & -U_a^B & -(U_{1s}^B \pm f_2 U_{2s}^B \pm f_3 U_{3s}^B) & -(U_{1m}^B \pm f U_{2m}^B) & -U_{3m}^B \end{bmatrix} \begin{Bmatrix} \Delta t^A \\ \Delta t^B \\ \Delta t_a \\ \Delta t_{1s} \\ \Delta t_{1m} \\ \Delta t_{3m} \end{Bmatrix} \\
 & = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \hat{T}_a^B & \hat{T}_{1s}^B & \hat{T}_{1m}^B & \hat{T}_{2m}^B \end{bmatrix} \begin{Bmatrix} g_a^n \\ g_{1s}^n \\ g_{1m}^n \\ g_{2m}^n \end{Bmatrix} \quad (3.43)
 \end{aligned}$$

where the superscripts denote the two bodies; Δu^A and Δt^A denote the vectors of displacements and tractions, respectively, for body **A** outside the contact zone; Δu_a and Δt_a denote the vectors of displacements and tractions, respectively, within Γ_{ca} ;

$\Delta \mathbf{u}_p$ denotes the vectors of displacements within Γ_{cp} ; $\Delta \mathbf{t}_{1s}$ is the vector of tractions in η_1 direction within Γ_{cs} ; $\Delta \mathbf{u}_{is}$ denote the vector of displacements in the η_i direction within Γ_{cs} ; $\Delta \mathbf{u}_{im}$ and $\Delta \mathbf{t}_{im}$ denote the vectors of displacements and tractions, respectively, in the η_i direction within Γ_{cm} ; $\mathbf{g}_a^n$ denotes the vector of gap between Γ_{ca}^A and Γ_{ca}^B at the n -load step $\mathbf{g}_{is}^n$ denotes the vector of gap at the η_i direction between Γ_{cs}^A and Γ_{cs}^B at the n -load step. Finally, the symbols $\mathbf{U}$ and $\hat{\mathbf{T}}$ denote the submatrices containing the coefficients obtained from integrating the fundamental solutions U_{ij} and $\hat{T}_{ij}$, respectively, with the shape function products over the boundary elements.

3.6.4 Decomposition of the tangential tractions

At the P^n -th node-pair on Γ_{cs} , the tangential contact traction is equal to the product of the normal traction and the coefficient of friction, according to the Coulomb law of friction. In the analysis of 3D contact problems, additional complications arise because the tangential traction in the contact zone can be decomposed into the two component tractions $t_2(P^n)$ and $t_3(P^n)$ in the η_2 and the η_3 directions, respectively. Thus

$$\begin{aligned} t_2(P^n) &= -f_2(P^n)\Delta t_1(P^n), \\ t_3(P^n) &= -f_3(P^n)\Delta t_1(P^n). \end{aligned} \quad (3.44)$$

The coefficients of friction $f_2(P^n)$ and $f_3(P^n)$ must satisfy the following condition:

$$f|\Delta t_1(P^n)| = \sqrt{[f_2(P^n)\Delta t_1(P^n)]^2 + [f_3(P^n)\Delta t_1(P^n)]^2} \quad (3.45)$$

It can be seen from Eq. (3.45) that f_2 and f_3 may not be equal to f . But, the direction of the tangential traction should always be opposite to the direction of the relative movement for the P^n -th node pair in contact. Let value of f_2 be proportional to that of the relative slip between the P^n -th node-pair in the η_2 direction, namely [37],

$$f_2(P^n) = \frac{f|\Delta u_2^A(P^n) - \Delta u_2^B(P^n)|}{\sqrt{[\Delta u_2^A(P^n) - \Delta u_2^B(P^n)]^2 + [\Delta u_3^A(P^n) - \Delta u_3^B(P^n)]^2}} \quad (3.46)$$

$$f_3(P^n) = \frac{f_i |\Delta u_3^A(P^n) - \Delta u_3^B(P^n)|}{\sqrt{[\Delta u_2^A(P^n) - \Delta u_2^B(P^n)]^2 + [\Delta u_3^A(P^n) - \Delta u_3^B(P^n)]^2}} \quad (3.47)$$

where $\Delta u_i^A(P^n)$ and $\Delta u_i^B(P^n)$ are the increments of displacement for the P^n -th node-pair in contact. Substituting Eq. (3.46) and (3.47) into Eq. (3.45), it can be seen that the pseudo-coefficients of friction defined according to Eq. (3.46) and (3.47), satisfy Eq. (3.45). On Γ_{cm} and Γ_{ca} , for the sake of simplicity, the coefficients of friction in the η_1 and η_2 directions may be assumed to be equal to f .

3.6.5 Determination of the load increment

As in the 2-D case, the amount of the external load increment Δq_i^n at the n -load step can be written as

$$\Delta q_i^n = \beta \Delta q_i \quad (3.48)$$

where Δq_i is a certain load increment, and β is the load incremental ratio to be determined. To obtain β , a scaling procedure is done for every element pair inside and outside (near) the contact zone. The value of β should be chosen as the smallest of the values to preclude overlapping of the elements outside the contact zone, or the largest of the values (then $\beta < 0$) to preclude normal tensile tractions inside the contact zone.

Assume that at the end of the previous n step, $\bar{t}_i^n$ is the average element traction within Γ_c ; $\bar{g}_i^n$ is the average gap between the element pair within Γ_c ; $\Delta \bar{t}_i^n$ is the average element traction increment due to Δq_i ; $\Delta \bar{u}_i^A$ and $\Delta \bar{u}_i^B$ are the average element displacement increments for bodies A and B respectively, due to Δq_i . These elements on Γ_c , or just outside it, may undergo any one of the three possible changes of contact state below:

From open condition to contact condition:

In this case, the equation as follows must be satisfied:

$$\bar{g}_1^n - \beta(\Delta \bar{u}_1^A - \Delta \bar{u}_1^B) = 0.$$

Therefore β is given by

$$\beta = \frac{\bar{g}_1^n}{\Delta \bar{u}_1^A - \Delta \bar{u}_1^B}. \quad (3.49)$$

From contact condition to open condition:

In this case, the normal traction of the element becomes zero, thus

$$\bar{t}_1^n - \beta (\Delta \bar{t}_1^n) = 0$$

Therefore

$$\beta = \frac{\bar{t}_1^n}{\Delta \bar{t}_1^n} \quad (3.50)$$

From adhesion condition to slip condition:

Sliding between the two contact surfaces will begin to slide when the tangential traction reaches the limit governed by the coefficient of friction. This means that in the η_2 -direction, sliding occurs when

$$|\bar{t}_2^n + \beta \Delta \bar{t}_2^n| + f (\bar{t}_1^n + \beta \Delta \bar{t}_1^n) = 0.$$

Consequently,

$$\begin{aligned} \beta &= -\frac{\bar{t}_2^n - f \bar{t}_1^n}{\Delta \bar{t}_2^n - f \Delta \bar{t}_1^n} & \text{if } \Delta \bar{t}_1^n \cdot \bar{t}_2^n \leq \Delta \bar{t}_2^n \cdot \bar{t}_1^n \\ \beta &= -\frac{\bar{t}_2^n + f \bar{t}_1^n}{\Delta \bar{t}_2^n + f \Delta \bar{t}_1^n} & \text{if } \Delta \bar{t}_1^n \cdot \bar{t}_2^n > \Delta \bar{t}_2^n \cdot \bar{t}_1^n \end{aligned} \quad (3.51)$$

For the sliding condition in the η_3 -direction, the suffix 3 is used in place of the suffix 2 in Eq. (3.51).

3.6.6 Convergence criteria

If the iterative process converges, the assumed contact zone will come to the corrected one. The convergence criteria are listed in Table 3.2 for the different changes in contact state.

Table 3.2 Convergence criteria for 3-D contact problems

Condition		Criteria		
Before	After			
Open	Open	$\bar{g}_1^n > 0$		
	Contact	$\bar{g}_1^n \leq 0$		
Adhesion	Adhesion	$\bar{t}_1^n \leq 0,$	$ \bar{t}_2^n \leq -f\bar{t}_1^n,$	$ \bar{t}_3^n \leq -f\bar{t}_1^n$
	Open	$\bar{t}_1^n > 0$		
	Slip	$\bar{t}_1^n \leq 0,$	$ \bar{t}_2^n \geq -f\bar{t}_1^n,$	$ \bar{t}_3^n \geq -f\bar{t}_1^n$
	Mixed	$\bar{t}_1^n \leq 0,$	$ \bar{t}_2^n \geq -f\bar{t}_1^n,$	$ \bar{t}_3^n < -f\bar{t}_1^n$
Slip	Adhesion	$\bar{t}_1^n \leq 0, \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) > 0, \Delta\bar{t}_3^n (\Delta\bar{u}_2^A - \Delta\bar{u}_3^B) > 0$		
	Open	$\bar{t}_1^n > 0$		
	Slip	$\bar{t}_1^n \leq 0, \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) \leq 0, \Delta\bar{t}_3^n (\Delta\bar{u}_3^A - \Delta\bar{u}_3^B) \leq 0$		
	Mixed	$\bar{t}_1^n \leq 0, \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) \leq 0, \Delta\bar{t}_3^n (\Delta\bar{u}_3^A - \Delta\bar{u}_3^B) > 0$		
Mixed	Adhesion	$\bar{t}_1^n \leq 0, \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) > 0, \bar{t}_3^n \leq -f\bar{t}_1^n$		
	Open	$\bar{t}_1^n > 0$		
	Slip	$\bar{t}_1^n \leq 0, \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) \leq 0, \bar{t}_3^n > -f\bar{t}_1^n$		
	Mixed	$\bar{t}_1^n \leq 0, \Delta\bar{t}_2^n (\Delta\bar{u}_2^A - \Delta\bar{u}_2^B) \leq 0, \bar{t}_3^n \leq -f\bar{t}_1^n$		

3.7 Numerical Examples in 3D

Three examples in 3D are chosen here to demonstrate the validity of the formulation described above.

Example 1

The first problem considered was the determination of the contact stresses between two identical elastic spheres in frictionless contact when subjected to an axial load, as shown in Figure 3.29. This is a classical Hertz problem with an exact solution[133] and the shape of the zone of contact is a circle. Figure 3.30 shows the boundary element mesh employed; it has 408 elements and 388 nodes. The computed distribution of

the normalised contact pressure between the two spheres along a radial ray in the contact zone is shown in Figure 3.31, together with the exact Hertz solution. As can be seen, the accuracy of the BEM results is excellent indeed, with errors all within 2%.

Example 2

The next problem treated was that of frictionless contact between two identical circular flat plates each with a circular hole and subjected to a transverse axisymmetric pressure load, as shown in Figure 3.32. The diameter of the hole in the plate was $2a$, its thickness was $3a$ and the outer diameter was $24a$. An axisymmetric pressure p was applied around the hole, from $r = a$ to $r = 1.5a$, on each plate, as shown, giving rise to contact stresses at the mating surfaces of the plates. This problem has also been studied experimentally as well as analytically, using approximate series solution techniques, by Minakuchi *et al*[89]. The BEM mesh employed here is shown in Figure 3.33. Figure 3.34 shows the computed normalised contact stress, $(-\sigma_{zz}/p)$, with the relative radial position r/a along the interface between the two plates. Also shown are the experimental and predicted results of Minakuchi *et al*[89]. It can be seen that the BEM results are generally in much better agreement with the experimental results than those predicted by these authors. It should also be mentioned here that, according to their analysis, no separation of the plates occur at the interface even though the contact pressure becomes zero beyond about $r = 5a$. Separation of the plates was in actual fact observed in the experiment, at their outer peripheries. This was verified in the BEM analysis here.

Example 3

The third problem treated was the pin-loaded lug again, as shown in Figure 3.35. To investigate the effect of the pin bending on the stress solutions, two lengths of the pin were considered. Let E_1 be the Young's modulus of the lug, and E_2 be the Young's modulus of the pin. Poisson's ratio was taken to be 0.3 for both the pin and lug

throughout. A typical BEM mesh is shown in Figure 3.36. The external load F was applied as shear forces uniformly distributed on the two ends of the pin. The effects of the friction and the rigidity ratio E_2/E_1 were also investigated. The geometrical parameters of the lug considered were $H/W = 4$, and $R_1/W = 0.5$. Three values of the coefficient of friction were analysed with, namely, 0, 0.5 and 1. Also two values of E_2/E_1 were chosen; they were 3 and 1. Neat fit between the pin and the lug was assumed in the analysis.

First, the length of the pin was taken to be equal to the thickness of the pin ($d = 0$ in Figure 3.35). In this case, the extent of contact zone between the pin and the lug was not affected by the magnitude of the applied load, and was approximately 180° in angular span. Figure 3.37 to 3.42 show qualitatively the relative contact pressure distributions in three dimensions; the results for the normalised contact pressure across the thickness of the lug for three different positions $z/t = 0, 0.25$ and 0.5 , are shown in Figure 3.43 to 3.45. It can be seen that the contact stresses at the edge of the hole are higher than the ones at the central periphery of the lug. This could be attributed to the effect of pin-bending. As a result, a large portion of the external load was applied near the edges of the pin-hole. This became even more evident when the coefficient of friction was increased. As expected, the normal contact stresses decreased and the contact zone became larger when the coefficient of friction was increased. The effect of the rigidity ratio E_2/E_1 on the contact stress distributions is clearly shown by Figure 3.43 to 3.45. When $E_2/E_1 = 3$, the normal contact stresses near the edges of the pin-hole are significantly smaller compared with the case of $E_2/E_1 = 1$. The normalised tangential stress distribution at the periphery of the pin-hole for three different positions z/t across the thickness of the lug are shown in Figure 3.46 to 3.48. As expected, the normalised tangential stresses increased in magnitude with the coefficient of the friction f . Of interest to note is that the tangential stresses were fairly uniform across the thickness of the lug, and are close to those obtained from 2D analysis[77].

Next, the length of the pin was taken to be larger than the thickness of the pin,

more specifically, $d = 0.5t$. The contact zone was again not affected by the magnitude of the applied load, and was approximately 180° in angular span. Figure 3.49 to 3.54 show the relative contact pressure distributions in three dimensions; the results for the normalised contact pressure across the thickness of the lug for three different positions $z/t = 0, 0.25$ and 0.5 , are shown in Figure 3.55 to 3.57. It can be seen that the contact stresses at the edges of the hole were again higher than the ones at the mid-thickness plane of the lug. This higher stresses were more pronounced than those seen in the case of $d = 0$. The effect of the rigidity ratio E_2/E_1 on the contact stress distributions here is shown by Figure 3.55 to 3.57. When $E_2/E_1 = 3$, the normal contact stresses near the edges of the pin-hole were significantly smaller compared with the case of $E_2/E_1 = 1$. The normalised tangential stress distribution at the periphery of the pin-hole for three different positions z/t across the thickness of the lug are shown in Figure 3.58 to 3.60. As in the $d = 0$ case, the normalised tangential stresses increased with the magnitude of the coefficient of the friction f . Also, they were fairly uniform across the thickness of the lug, but were about 5% higher than the corresponding values in the $d = 0$ case.

3.8 Concluding Remarks

The formulation of the boundary element method for contact mechanics analysis in both 2D and 3D has been presented in this chapter. The incremental-iterative techniques employed have been described and for the problems analysed, there were no difficulties in the convergence of the solutions with well graded meshes. The results for a number of 2D and 3D non-crack problems in contact mechanics have been presented to demonstrate the veracity of the formulations, and very good agreement with available solutions was obtained for the test problems treated.

Figure 3.1 Two bodies in contact

(a)

(b)

Figure 3.2 Irreversibility of tangential force in the presence of friction

Figure 3.3 Two elastic bodies in contact along the boundary Γ_c in 2-D.

Figure 3.4 A roller lying on a foundation.

Figure 3.5 BEM mesh for the problem shown in Figure 3.4; 48 elements, 96 nodes.

Figure 3.6 Variation of the normalised contact pressure with contact length for the problem of an elastic roller on an elastic foundation.

Figure 3.7 An infinite plate with a hole loaded by a circular pin.

Figure 3.8 Boundary element mesh for the problem shown in Figure 3.7.

Figure 3.9 Normalised contact pressure distribution between an elastic pin and a circular hole in an infinite elastic plate.

Figure 3.10 A lug loaded by a pin along the plane of symmetry of the lug.

Figure 3.11 BEM mesh for the pin-lug joint shown in Figure 3.10.

Figure 3.12 Pin-bearing pressure distributions along the contact surface of a lug subjected to axial pin loading under frictionless, neat-fit conditions.

Figure 3.13 Variation of the normalised tangential stress along the periphery of the hole of a lug, for axial pin loading under the conditions of neat-fit and no friction.

Figure 3.14 Variation of the normalised contact pressure, p/σ_{br} , with the angular position φ at the periphery of the pin-hole: $H/W = 2$, $R_1/W = 0.5$ and $\phi \approx \pm 90^\circ$.
 $\sigma_{br} = F/[2(W - R_1)b]$.

Figure 3.15 Variation of the normalised shear stress, $\sigma_{r\varphi}/\sigma_{br}$, with the angular position φ at the periphery of the pin-hole; $H/W = 2$, $R_1/W = 0.5$, initial neat-fit. $\sigma_{br} = F/[2(W - R_1)b]$.

Figure 3.16 Variation of the normalised contact pressure, p/σ_{br} , with the angular position φ at the periphery of the pin-hole; $H/W = 2$, $R_1/W = 0.5$ and $\phi = \pm 135^\circ$.
 $\sigma_{br} = F/[2(W - R_1)b]$.

Figure 3.17 Variation of the normalised shear stress, $\sigma_{r\phi}/\sigma_{br}$, with the angular position ϕ at the periphery of the pin-hole when the contact zone $\phi = \pm 135^\circ$; $H/W = 2$, $R_1/W = 0.5$, $\sigma_{br} = F/[2(W - R_1)b]$.

Figure 3.18 Variation of the normalised contact pressure, p/σ_{br} , with the angular position φ at the periphery of the pin-hole; $H/W = 2$, $R_1/W = 0.5$ and $\phi = \pm 15^\circ$.

Figure 3.19 Normalised shear stress, $\sigma_{\tau\varphi}/\sigma_{br}$, at the periphery of the pin-hole when the contact zone is $\phi = \pm 45^\circ$; $H/W = 2$, $R_1/W = 0.5$, $\sigma_{br} = F/[2(W - R_1)b]$.

Figure 3.20 A lug having an interference-fit loaded by a pin along the plane of symmetry of the lug.

Figure 3.21 BEM mesh for the pin-loaded lug problem shown in Figure 3.20.

Figure 3.22 Normalised contact pressure distribution between a lug and an interference-fit bushing: $R_1/W = 0.5$, $R_3/R_1 = 1.25$, $f = 0$.

Figure 3.23 Normalised tangential stress along the periphery of the lug hole due to interference-fit of the bushing; $R_1/W = 0.5$, $R_3/R_1 = 1.25$, $f = 0$.

Figure 3.24 Distributions of the normalised contact pressure for the pin-loaded lug problem with a bushing when the contact zone between the pin and the bushing spans $\phi = \pm 45^\circ$; $H/W = 2$, $R_1/W = 0.5$, $R_3/R_1 = 1.25$, $f = 0$, $\sigma_{br} = F/[2(W - R_1)b]$.

Figure 3.25 Distribution of the normalised tangential stress along the periphery of the lug hole when the contact zone between the pin and the bushing spans $\phi = \pm 45^\circ$; $R_1/W = 0.5$, $R_3/R_1 = 1.25$, $f = 0$.

Figure 3.26 Distributions of the normalised contact pressure for the pin-loaded lug problem with a bushing when partial separation between the bushing and the lug has occurred: $H/W = 2$, $R_1/W' = 0.5$, $R_3/R_1 = 1.25$, $f = 0$, $\sigma_{br} = F/[2(W' - R_1)b]$.

Figure 3.27 Tangential stress distributions along the periphery of the lug hole for a pin-loaded lug problem with bushing when partial separation between the bushing and the lug has occurred; $R_1/W = 0.5$, $R_3/R_1 = 1.25$, $f = 0$.

Figure 3.28 Two elastic bodies in contact in 3-D.

Figure 3.29 Two elastic spheres in contact when loaded by an axial force.

Figure 30 BEM mesh for two spheres in contact when subjected to an axial load.
(Note: The elements used are curved quadratic isoparametric elements. Straight-sided elements shown here are due to the limitations of the graphics software employed.)

Figure 3.31 Contact pressure distribution along the radius of the contact zone

Figure 3.32 Diametral section of two circular plates with a circular hole subjected to transverse axisymmetric pressure loads around the holes.

Figure 3.33 BEM mesh for two circular finite plates in contact when subjected to transverse axisymmetric loads around the hole.

Figure 3.34 Contact stress at the interface of two circular plates with a hole.

Figure 3.35 A pin-loaded attachment lug assembly.

Figure 3.36 A typical BEM mesh for the problem shown in Figure 3.35: 224 elements, 654 nodes.

Figure 3.37 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0$, $f = 0$, and $E_2/E_1 = 3$. $\sigma_{br} = F/[2(W - R_1)\ell]$.

Figure 3.38 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0$, $f = 0$, and $E_2/E_1 = 1$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.39 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0$, $f = 0.5$, and $E_2/E_1 \doteq 3$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.40 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0$, $f = 0.5$, and $E_2/E_1 = 1$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.41 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0$, $f = 1$, and $E_2/E_1 = 3$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.42 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0$, $f = 1$, and $E_2/E_1 = 1$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.43 Variation of the normalised contact pressure, $p/\{F/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0$, $f = 0$.

Figure 3.44 Variation of the normalised contact pressure, $p/\{F/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0$, $f = 0.5$.

Figure 3.45 Variation of the normalised contact pressure, $p/[F/(2(W - R_1)t)]$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0$, $f = 1$.

Figure 3.46 Variation of the normalised tangential stress, $\sigma_{\varphi\varphi}/\{\sigma_{br}/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0$, $f = 0$.

3

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Figure 3.47 Variation of the normalised tangential stress, $\sigma_{\varphi\varphi}/\{\sigma_{br}/[2(W-R_1)]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t ; $d=0$, $f=0.5$.

Figure 3.48 Variation of the normalised tangential stress, $\sigma_{\varphi\varphi}/\{F/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0$, $f = 1$.

Figure 3.49 Contact pressure distribution, around the periphery of the pin hole in three dimensions: $d = 0.5t$, $f = 0$, and $E_2/E_1 = 3$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.50 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0.5t$, $f = 0$, and $E_2/E_1 = 1$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.51 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0.5t$, $f = 0.5$, and $E_2/E_1 = 3$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.52 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0.5t$, $f = 0.5$, and $E_2/E_1 = 1$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.53 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0.5t$, $f = 1$, and $E_2/E_1 = 3$. $\sigma_{br} = F/[2(W - R_1)t]$.

Figure 3.54 Contact pressure distribution around the periphery of the pin hole in three dimensions: $d = 0.5t$, $f = 1$, and $E_2/E_1 = 1$. $\sigma_{br} = F/[2(W - R_1)d]$.

Figure 3.55 Variation of the normalised contact pressure, $p/\{F/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0.5t$, $f = 0$.

Figure 3.56 Variation of the normalised contact pressure, $p/\{F/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0.5t$, $f = 0.5$.

Figure 3.57 Variation of the normalised contact pressure, $p/\{F/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0.5t$, $f = 1$.

Figure 3.58 Variation of the normalised tangential stress, $\sigma_{\varphi\varphi}/\{F/[2(W - R_1)\ell]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0.5t$, $f = 0$.

Figure 3.59 Variation of the normalised tangential stress, $\sigma_{\varphi\varphi}/\{\sigma_{br}/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0.5t$, $f = 0.5$.

Figure 3.60 Variation of the normalised tangential stress, $\sigma_{\varphi\varphi}/\{F/[2(W - R_1)t]\}$, with the angular position φ at the periphery of the pin-hole for different values of z/t : $d = 0.5t$, $f = 1$.

Chapter 4

Boundary Element Analysis of 2-D Contact Fracture Mechanics Problems

4.1 Introduction

Problems in contact fracture mechanics typically involve one or both of the following two cases: (i) the two faces of a crack are in contact under applied loads, and (ii) a cracked component is loaded by another component through contact. These problems have been quite extensively studied using the finite element, and experimental methods, see, e.g., Smith and Smith[118], Newman Jr. and Armen Jr.[91], Fredriksson[32], Oda and Nishikawa[93], Thiagarajan and Alwar[132]. In recent years, there has been an increasing interest in using the BEM to analyze contact fracture mechanics problems. Karami and Fenner[57] were perhaps the first to use the BEM to analyse crack closure problems. They investigated rectangular strips with edge, and inclined central cracks using the multiregion crack modelling technique in which only the proper $\sqrt{r}$ variation for the crack-tip displacement field was represented in the quarter-point crack-tip element. Similar works were reported by Selvadurai and Au[113], Cen, Qin and Du[21], but with the singular crack-tip element having the $\sqrt{r}$ variation for the displacements and the $1/\sqrt{r}$ variation for the tractions near the crack tips, irrespective of whether the tractions are truly singular or not. For example, for a partially closed crack, as shown in Figure 4.1, K_I is equal to zero on the

right-hand-side crack-tip. The stress $[\sigma_{22}]_{\theta=0}$ at $x_1 = a_2$ is not singular, although the stress state at $x_1 = -a_1$ has the $1/\sqrt{r}$ singularity. The problem of interface between the cracked layer and the elastic medium, which exhibits nonlinear frictional effects and separation, has also been investigated by Selvadurai and Au[114]. As the problem is symmetrical and loading is due to the cracked layer, no contact between the surfaces of crack is involved. This is a special case in engineering practice, so the application of the approach in their work is somewhat limited.

The general analysis of contact fracture problems by the BEM should consider the multibody contact at a given nodal point in the contact zone and the appropriate stress fields at the crack tips. This may involve setting up extra equations at such a node in order to present an efficient and accurate treatment of 2-D contact fracture mechanics problems. The implementation of this has been successfully carried out in this study and several numerical examples are given in this chapter to verify the new approach. Some of these results have been published in a recent paper[76].

4.2 Supplementary Equations

In Chapter 3, it was mentioned that the contact stresses and the displacements in the contact zone are usually computed in the local coordinate system in the numerical analysis. Each node pair in the contact zone then has four unknowns, and four boundary integral equations are available. However, when a cracked body is in contact with another elastic body, as shown in Figure 4.2, the points Q^I , Q^{II} and Q^{III} provide three node pairs for which only six boundary integral equations are available when the multiregion crack modelling approach is used. Six extra equations are therefore required for the solution of the problem. These supplementary equations may be obtained from two considerations. The first is deformation compatibility. In the case shown in Figure 4.2, for example, if all the three node pairs are in slip contact states, the following six extra equations can be written:

$$u_1^I n_{11}^I + u_2^I n_{12}^I = u_1^{II} n_{11}^{II} + u_2^{II} n_{12}^{II}$$

$$\begin{aligned}
u_1^I n_{21}^I + u_2^I n_{22}^I &= u_1^{II} n_{21}^{II} + u_2^{II} n_{22}^{II} \\
u_1^I n_{11}^I + \hat{u}_2^I n_{12}^I &= u_1^{III} n_{11}^{III} + u_2^{III} n_{12}^{III} \\
u_1^I n_{21}^I + \hat{u}_2^I n_{22}^I &= u_1^{III} n_{21}^{III} + u_2^{III} n_{22}^{III} \\
u_1^{II} n_{11}^{II} + \hat{u}_2^{II} n_{12}^{II} &= u_1^{III} n_{11}^{III} + \hat{u}_2^{III} n_{12}^{III} \\
u_1^{II} n_{21}^{II} + \hat{u}_2^{II} n_{22}^{II} &= u_1^{III} n_{21}^{III} + \hat{u}_2^{III} n_{22}^{III}
\end{aligned} \tag{4.1}$$

where the superscripts denote the intrinsic node pairs; and u_2^I and $\hat{u}_2^I$ are the two tangential displacements for the first node pair. If the above equations are not fully independent, the second consideration, namely, equilibrium involving the tractions may be used. For example, if the three node pairs are in the adhesive contact states, the following two equations may be written.

$$\begin{aligned}
t_1^{II} &= t_1^{III}, \\
t_2^{II} &= t_2^{III}
\end{aligned} \tag{4.2}$$

Note that the above equations are only approximately true. The exact expressions associated with the tractions can be developed from the stress-strain relations, but for most problems this is not necessary.

4.3 Solution of the System Equations

The solver of the system of linear equations described in Chapter 3, can be directly applied to solve the problems presented here if the interface resulting from the multi-region modelling, is treated as a part of the contact zone with all the nodes on the interface satisfying the adhesive condition. However, the solution process is slowed to some extent by the unknowns related to the nodes on the interface. This is because the number of the nodes on the interface can be very large compared with the number of the nodes in the the contact zone. Therefore a new solver for the system of equations which involves only the unknowns in the contact zone was developed. For the sake of clarity, only the subregions *A* and *B* are considered here, as shown in Figure 4.3.

although the algorithm was developed for problems involving multi-regions and multi-bodies in contact. This algorithm is described below:

Step (1) Form an uncoupled linear system of equations for each contact body, i.e.

$$\begin{bmatrix} H^A & H_s^A & H_c^A \\ D^A & D_s^A & D_c^A \\ G^A & G_s^A & G_c^A \end{bmatrix} \begin{Bmatrix} \Delta x^A \\ \Delta x_s^A \\ \Delta x_c^A \end{Bmatrix} = \begin{Bmatrix} \Delta b^A \\ \Delta b_s^A \\ \Delta b_c^A \end{Bmatrix} \quad (4.3)$$

$$\begin{bmatrix} H^B & H_s^B & H_c^B \\ D^B & D_s^B & D_c^B \\ G^B & G_s^B & G_c^B \end{bmatrix} \begin{Bmatrix} \Delta x^B \\ \Delta x_s^B \\ \Delta x_c^B \end{Bmatrix} = \begin{Bmatrix} \Delta b^B \\ \Delta b_s^B \\ \Delta b_c^B \end{Bmatrix} \quad (4.4)$$

Let Γ_s represent the interface boundary of the respective subregion and Γ_c be that part of the boundary of the subregion which is in contact with the adjoining subregion or the other body. In Eqs. (4.3) and (4.4) the superscripts denote the intrinsic bodies; Δx^A denotes the vector of the unknowns outside Γ_s^A and Γ_c^A , and H^A , D^A and G^A the corresponding submatrices; Δx_s^A denotes the vector of the unknowns over Γ_s^A , and H_s^A , D_s^A and G_s^A the corresponding submatrices; Δx_c^A denotes the vector of the unknowns over Γ_c^A and H_c^A , D_c^A and G_c^A the corresponding submatrices. Three points of detail should perhaps be made here. First, for the common node between two different elements in contact, there are four traction components. This is because the two elements may have different contact conditions, and thus the two sets of tractions are not equal to one other. Accordingly, the number of columns of the submatrices H_c^A , D_c^A , and G_c^A is greater than four times of the number of the nodes on Γ_c^A . But the number of columns in the submatrices H_s^A , D_s^A and G_s^A is equal to four times of the number of the nodes on Γ_c^A . Second, the number of rows of the submatrix H^A equals the number of unknowns outside Γ_s^A and Γ_c^A , and the number of rows of the submatrix D^A equals the number of unknowns on Γ_s^A . Third, the nodes which are shared by the elements on Γ_s^A and the elements on Γ_c^A are treated as the nodes on Γ_c^A in Eq. (4.4).

Step (2) Carry out a partial Gaussian elimination for the above two uncoupled

systems so that

$$\begin{bmatrix} \hat{H}^A & \hat{H}_s^A & \hat{H}_c^A \\ \mathbf{o} & \hat{D}_s^A & \hat{D}_c^A \\ \mathbf{o} & \hat{G}_s^A & \hat{G}_c^A \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}^A \\ \Delta \mathbf{x}_s^A \\ \Delta \mathbf{x}_c^A \end{Bmatrix} = \begin{Bmatrix} \Delta \hat{\mathbf{b}}^A \\ \Delta \hat{\mathbf{b}}_s^A \\ \Delta \hat{\mathbf{b}}_c^A \end{Bmatrix} \quad (4.5)$$

$$\begin{bmatrix} \hat{H}^B & \hat{H}_s^B & \hat{H}_c^B \\ \mathbf{o} & \hat{D}_s^B & \hat{D}_c^B \\ \mathbf{o} & \hat{G}_s^B & \hat{G}_c^B \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}^B \\ \Delta \mathbf{x}_s^B \\ \Delta \mathbf{x}_c^B \end{Bmatrix} = \begin{Bmatrix} \Delta \hat{\mathbf{b}}^B \\ \Delta \hat{\mathbf{b}}_s^B \\ \Delta \hat{\mathbf{b}}_c^B \end{Bmatrix} \quad (4.6)$$

Step (3) Form a coupled linear system of equations using Eq. (4.5) and Eq. (4.6), i.e.

$$\begin{bmatrix} \hat{D}_s^A & \hat{D}_c^A \\ \hat{G}_s^A & \hat{G}_c^A \\ \bar{D}_s^B & \bar{D}_c^B \\ \bar{G}_s^B & \bar{G}_c^B \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}_s \\ \Delta \mathbf{x}_c \end{Bmatrix} = \begin{Bmatrix} \Delta \hat{\mathbf{b}}_s^A \\ \Delta \hat{\mathbf{b}}_c^A \\ \Delta \hat{\mathbf{b}}_s^B \\ \Delta \hat{\mathbf{b}}_c^B \end{Bmatrix} \quad (4.7)$$

where $\Delta \mathbf{x}_s = \Delta \mathbf{x}_s^A$, and $\Delta \mathbf{x}_c$ is the vector of unknowns in the contact zone. The corresponding matrices are obtained from the matrices in Eq. (4.5) and Eq. (4.6) using Eq. (2.152) and Eq. (2.153), or Eqs. (3.14)–(3.16). For example, the difference between the matrices $\bar{G}_c^B$ and $\hat{G}_c^B$, lies in that the elements corresponding to the tractions having opposite signs. It should also be mentioned here that the number of columns in matrices $\bar{D}_c^A$ and $\bar{G}_c^A$ is four times of number of the node-pairs in the contact zone.

Step (4) Carry out a further partial Gaussian elimination with partial pivoting and store the permutation matrix $\mathbf{P}$ [145] and the unit lower triangular matrix $\mathbf{L}$.

$$\begin{bmatrix} \mathbf{U}_s & \mathbf{U}_c \\ \mathbf{o} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \Delta \mathbf{x}_s \\ \Delta \mathbf{x}_c \end{Bmatrix} = \begin{Bmatrix} \Delta \mathbf{b}_s \\ \Delta \mathbf{b}_c \end{Bmatrix} \quad (4.8)$$

The number of rows in the matrix $\mathbf{C}$ is equal to the dimensions of the vector $\Delta \mathbf{x}_c$. The matrices $\mathbf{L}$ and $\mathbf{P}$ are thus obtained by the above Gaussian elimination process:

$$\begin{bmatrix} U_s & U_c \\ 0 & C \end{bmatrix} = L^{-1} P \begin{bmatrix} \bar{D}_s^A & \bar{D}_c^A \\ \bar{G}_s^A & \bar{G}_c^A \\ \bar{D}_s^B & \bar{D}_c^B \\ \bar{G}_s^B & \bar{G}_c^B \end{bmatrix} = L^{-1} A \quad (4.9)$$

$$\begin{Bmatrix} \Delta b_s \\ \Delta b_c \end{Bmatrix} = L^{-1} P \begin{Bmatrix} \Delta \hat{b}_s^B \\ \Delta \hat{b}_c^B \end{Bmatrix} = L^{-1} b \quad (4.10)$$

Note that L and P are $(n_s + n_c) \times (n_s + n_c)$ square matrices where n_s and n_c are the dimension of Δx_s , and Δx_c , respectively. Also, in Eq. (4.9) and Eq. (4.10), it is not intended to multiply any matrices by P . A and b are in fact obtained directly after interchanging the rows of coefficient matrix and the right-hand-side vector, in Eq. (4.7), respectively, so that the diagonal terms are just the pivots from the first step to the n_s step elimination procedure. The elements of L thus consist of the multipliers used for the factorization of A by the Gaussian elimination, i.e.

$$L = \begin{bmatrix} 1 & & & & & & & \\ l_{21} & 1 & & & & & & \\ l_{31} & l_{32} & 1 & & & & 0 & \\ \vdots & \vdots & \vdots & \ddots & & & & \\ \vdots & \vdots & \vdots & \vdots & 1 & & & \\ \vdots & \vdots & \vdots & \vdots & 0 & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 1 & \\ l_{(n_s+n_c)1} & l_{(n_s+n_c)2} & \cdots & l_{(n_s+n_c)n_s} & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$l_{ij} = 1, \quad i = j,$$

$$l_{ij} = 0, \quad j > i,$$

$$l_{ij} = 0, \quad j > n_s, \quad i > j \quad (4.11)$$

where l_{ij} is the multiplier for the i -th row at the j -th step elimination procedure to create zero at the i -th row and the j -th column of A .

It is not necessarily to perform the Gaussian elimination as in Eq. (4.8) twice to obtain A and L . Instead, in the implementation of the algorithm, a one-dimensional array was introduced to track the row interchanges, and the multipliers were stored at the positions occupied by zero elements and they were also subjected to the row interchanges as with the elements in the same row, during the course of the elimination process. In the end, arranging the rows of the coefficient matrix and right-hand-side vector in Eq (4.7) in the order as indicated by the array, will produce A and b . The elements l_{ij} , $j = 1, \dots, n_s$, $i = (j + 1), \dots, (n_s + n_c)$, were located at the same position in the coefficient matrix in Eq. (4.8) as in L in Eq. (4.11). The inverse matrix of L can be easily obtained as shown below:

$$L^{-1} = M = \begin{bmatrix} 1 & & & & & & \\ m_{21} & 1 & & & & & \\ m_{31} & m_{32} & 1 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \vdots & \vdots & \vdots & \vdots & & 1 & \\ \vdots & \vdots & \vdots & \vdots & & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & 1 \\ m_{(n_s+n_c)1} & m_{(n_s+n_c)2} & \cdots & m_{(n_s+n_c)n_s} & & & \end{bmatrix},$$

$$\begin{aligned} m_{ij} &= -\sum_k^{i-1} l_{ik} m_{kj}, \quad \begin{pmatrix} j = 1, \dots, n_s \\ i = j + 1, \dots, (n_s + n_c) \end{pmatrix} \\ m_{ij} &= 1, \quad i = j, \\ m_{ij} &= 0, \quad j > i, \\ m_{ij} &= 0, \quad j > n_s, \quad i > j. \end{aligned} \tag{4.12}$$

Let

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n_c} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n_c} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n_c} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n_c1} & a_{n_c2} & a_{n_c3} & \cdots & a_{n_cn_c} \end{bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{(n_s+n_c)} \end{Bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n_c} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n_c} \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3n_c} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n_c1} & c_{n_c2} & c_{n_c3} & \cdots & c_{n_cn_c} \end{bmatrix}, \quad \Delta \mathbf{b}_c = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{n_c} \end{Bmatrix}$$

then

$$c_{ij} = a_{(n_s+i)(n_s+j)} + \sum_{k=1}^{n_s} m_{(i+n_s)k} a_{k(j+n_s)}, \quad i, j = 1, \dots, n_c \quad (4.13)$$

$$q_i = b_{(n_s+i)} + \sum_{k=1}^{n_s} m_{(i+n_s)k} b_k, \quad i = 1, \dots, n_c \quad (4.14)$$

Step (5) Use Eq. (4.8)–(4.14) to form the following system of equations:

$$\mathbf{C} \Delta \mathbf{x}_c = \Delta \mathbf{b}_c. \quad (4.15)$$

In every iteration, the new matrices $\mathbf{A}$ and $\mathbf{b}$ are formed, and the new $\mathbf{C}$ and $\Delta \mathbf{b}_c$ matrices are calculated using Eq. (4.13) and Eq. (4.14). Substituting $\Delta \mathbf{x}_c$ into Eq. (4.8), $\Delta \mathbf{x}_s$ can be easily obtained, since $\mathbf{U}_s$ is an upper triangular matrix. Let

$$\mathbf{U}_c = \begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1n_s} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2n_s} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3n_s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n_c1} & v_{n_c2} & v_{n_c3} & \cdots & v_{n_cn_s} \end{bmatrix}, \quad \Delta \mathbf{b}_s = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_{n_c} \end{Bmatrix}$$

where

$$v_{ij} = a_{i(n_s+j)} + \sum_{k=1}^{n_s} m_{ik} a_{k(n_s+j)}, \quad \begin{pmatrix} i = 1, \dots, n_s \\ j = 1, \dots, n_s \end{pmatrix} \quad (4.16)$$

$$t_i = b_i + \sum_{k=1}^{i-1} m_{ik} b_k, \quad i = 1, \dots, n_s \quad (4.17)$$

from which,

$$U_s \Delta x_s = \Delta b_s - U_c \Delta x_c. \quad (4.18)$$

4.4 Numerical Examples

The boundary element formulation described in the previous chapters and above has been applied to five angled crack problems in this study. The first two problems considered were those of a rectangular plate with a slanted central crack, subjected to remote uniform compression and to pure bending, respectively. The physical geometry for the third and fourth problems was also a rectangular plate, but with an inclined edge crack, the loads applied being the same as in the first two problems. Finally, the problem of a simply supported beam with an inclined edge crack loaded by a rigid stamp was considered. All these problems involved crack closure, and the effects of friction between the surfaces in contact were also investigated for a wide range of geometric parameters. To this end, the coefficient of friction f was varied from the perfectly smooth case when $f = 0$ to the case when $f = 1$. Plane strain conditions were assumed and Poisson's ratio was taken to be 0.3 throughout.

4.4.1 Compression of the strip with a central angled crack (Problem I)

Figure 4.4a shows a rectangular plate, length $2H$ and width $2W$, with a central angled crack of length $2a$, subjected to a uniform compressive stress σ_0 . In this study, H/W was taken to be 3, but the following relative crack sizes, a/W , and angles of inclination

of the crack, θ , were treated:

$$\begin{aligned}\theta &= 22.5^\circ, 45^\circ, 67.5^\circ \\ a/W &= 0.2, 0.4, 0.6, 0.8\end{aligned}\tag{4.19}$$

It is evident that in this problem, the crack surfaces will remain closed under compressive load: K_I is thus zero. For each cracked geometry, only the mode II stress intensity factor K_{II} was thus obtained. A typical BEM mesh used for this problem is shown in Figure 4.5. As can be seen, it is made up of two sub-regions, with the crack plane coinciding with their interface boundaries. Figures 4.6–4.8 show the variations of normalized stress intensity factors, $K_{II}/\sigma_0\sqrt{\pi a}$, with the coefficient of friction for the different relative crack sizes when $\theta = 22.5^\circ, 45^\circ$ and 67.5° , respectively. Of interest to note is that the stress intensity factor varies linearly with f in all cases. Also, when the angle of inclination of the crack $\theta = 22.5^\circ$, the BEM analysis showed that there would be no tendency for the crack surfaces to slide relatively to one another when f is greater than about 0.4; thus $K_{II} = 0$ then. This is true for all crack sizes considered. The same situation would occur for the case of $\theta = 45^\circ$, but only when f is, or exceeds unity. It is also worth comparing the BEM results with those calculated from an analytical expression derived by Melville[87] for an inclined central crack, or an edge crack, in a plane infinite medium subjected to a uniaxial compressive stress σ_0 . This expression is:

$$K_{II} = \lambda \sigma_0 \sqrt{\pi a} \cos \theta (\sin \theta - f \cos \theta)\tag{4.20}$$

where $\lambda = 1.0$ for a central crack of length $2a$, and $\lambda = 1.1215$ for an edge crack of length a . The comparison for the $a/W = 0.2$ case is shown in Figure 4.9. It was found that for this relative crack length, the deviations between the BEM results and those using this equation were generally within 1.5% for all three values of θ considered. This suggests that the effects of finite boundaries on K_{II} are minimal for relative crack lengths up to this value. When the cracks are longer, however, this is evidently not so. For example, when $\theta = 45^\circ$ and $f = 0$, the discrepancy between the analytical and BEM solutions increases from about 7% when $a/W = 0.4$ to about 29% when $a/W = 0.8$.

4.4.2 A rectangular plate with an inclined central crack subjected to in-plane bending (Problem II)

This problem, shown in Figure 4.4b, is similar to the first problem, except that the plate is now subject to an in-plane bending moment M instead. The geometry and the range of geometric parameters considered were the same as in the previous problem. Referring to Figure 4.4b, the crack surfaces near the crack tip A will remain closed under the bending moment M shown, while those at crack tip B will open. Table 4.1 lists the normalized mode II stress intensity factors, $(K_{II})_A/[(6M\sqrt{\pi a})/W^2]$, at the point A for the different parameters treated. As can be seen, the normalized stress intensity factors are generally quite small in magnitude, but they are not insignificant, especially for longer cracks. Also, it is evident that increasing the coefficient of friction between the crack surfaces which are in contact has the effect of decreasing, in a linear manner, the stress intensity factor there. At crack tip B , it was found from the BEM analysis that increasing the coefficient of friction did not, to all intents and purposes, have any influence on K_I . Figure 4.10 shows the variations of the normalized stress intensity factor, $(K_I)_B/[(6M\sqrt{\pi a})/W^2]$, there with relative crack length, a/W , for the various angles of inclination of the crack treated. Note that an additional set of results for the case of $\theta = 0$ has been included in the figure. The analysis for this case was carried out in order to compare the BEM results with those recently obtained by Woo *et al.*[146] using the boundary collocation method. Agreement between the two sets of solutions obtained by the different solution methods was found to be excellent, with discrepancies of less than 2% for the crack sizes considered. The BEM computed values of $(K_{II})_B/[(6M\sqrt{\pi a})/W^2]$ at crack tip B are presented in Table 4.2. From this table, it can be seen that friction between the closed part of the crack surfaces did appear to have some effect on the magnitude of this stress intensity factor, although it is not as significant as for K_{II} at crack tip A . This particular problem has also been solved by Karami and Fenner[57] and by Cen *et al.*[21], but for the case of $\theta = 45^\circ$ and $a/W = 1/\sqrt{2}$. Although the quadratic isoparametric boundary element formulation was also employed by these authors, the treatment of the near-tip fields for the crack

problem and the determination of the stress intensity factors, as has been mentioned earlier, differ somewhat between these two studies, and are slightly different from the present work as well. This geometric case was therefore also investigated for comparison purposes and the results are shown together in Table 4.3. As can be seen, the present results are in good agreement with those by Cen *et al.*[21], the deviations all being within 5%, less for K_I than for K_{II} . However, the discrepancies are significantly greater when comparing the same results with those of Karami and Fenner[57]

4.4.3 A rectangular plate with an inclined edge crack subject to uniaxial compression (Problem III)

Figure 4.11a shows the problem considered, namely that of a finite size plate with an edge-angled crack, subjected to the compression loading. The different geometries and friction coefficients treated are:

$$\begin{aligned}\theta &= 22.5^\circ, 45^\circ, 67.5^\circ \\ f &= 0.0, 0.1, 0.2, \dots, 1.0 \\ a/W &= 0.2, 0.4, 0.6, 0.8 \\ H/W &= 2.\end{aligned}\tag{4.21}$$

A typical BEM mesh employed for the analysis of this problem is shown in Figure 4.12.

Again, only mode II crack deformation exists for this problem. The BEM results of $K_{II}/\sigma_0\sqrt{\pi a}$ are shown plotted against the coefficient of friction, f , in Figures 4.13–4.15, for different crack sizes and crack inclination angles. The variation of K_{II} with f was again found to be linear in all the cases considered. Interestingly, when $\theta = 22.5^\circ$ and 45° , there is no tendency for the crack faces to slide relatively to one another at the crack tip if f is greater than about 0.4 and 1.0, respectively. These are the same corresponding values of f for $K_{II} = 0$ as in Section 4.4.1. A comparison between the results obtained using Eq. 4.20 and the BEM results in the case of $a/W = 0.2$, is shown in Figure 4.16. It can be seen from the figure that the present solution is

in excellent agreement with the analytical one. However, for larger values of a/W , the discrepancies become significant. For example, when $\theta = 45^\circ$ and $f = 0$, the deviation was about 9% for $a/W = 0.4$; this increases to about 36% for $a/W = 0.8$.

4.4.4 A rectangular plate with an inclined edge crack subject to in-plane bending (Problem IV)

The same physical problem as in Section 4.4.3 was treated here, except that the applied load was an in-plane bending moment M at the ends, as shown in Figure 4.11b. Indeed, the same BEM mesh discretization was also used for a given cracked geometry. However, for this problem, an additional set of cases corresponding to $\theta = 0$ was included in the analysis. Under the bending load, the crack faces will remain closed as long as the crack has not propagated beyond the longitudinal centre line of the plate. Thus, in Table 4.4, which shows the computed normalized mode I stress intensity factors, $K_I/[(6M\sqrt{\pi a})/W^2]$, for the crack, only the non-zero values are presented for the range of geometries considered. The corresponding results for $K_{II}/[(6M\sqrt{\pi a})/W^2]$ are listed in Table 4.5. It can be seen that the value of K_{II} is significantly influenced by the friction between the crack faces in contact, decreasing again in a linear manner with f .

4.4.5 Simply supported beam with an inclined edge crack loaded by a rigid stamp (Problem V)

Figure 4.17 shows a simply supported beam, length $2L$, height W , with an inclined edge crack of length a , subject to a total force P which is transmitted via a rigid stamp of width H . The different geometries considered were as follows:

$$\begin{aligned}
 L/W &= 3; \\
 \theta &= 0^\circ \text{ and } 45^\circ; \\
 H/W &= 0.5 \text{ and } 1.0; \\
 a/W &= 0.2, 0.4, 0.6 \text{ and } 0.8
 \end{aligned}
 \tag{4.22}$$

For this analysis, the Young's modulus of the stamp was taken to be 1000 times that of the beam. Also, the coefficient of friction between the stamp and the beam was taken to be the same as that between the crack faces. A typical BEM mesh used is shown in Figure 4.18. It has three sub-regions, one for the stamp and the other two for the cracked beam.

Evidently, in this problem, the crack faces near the crack tip will remain closed when $a \cos \theta < W/2$, in which case only mode II crack deformation exists when θ is not zero. Table 4.6 lists the normalized values of the mode I stress intensity factors, $K_I/[3PL\sqrt{\pi a})/W^2]$, obtained for the cases when the crack faces are open near the crack tip. The influence of friction on this mode of crack deformation did not appear to be very significant. The computed results of $K_I/[3PL\sqrt{\pi a})/W^2]$ when $\theta = 45^\circ$ are presented in Table 4.7 for the different geometries analysed. It can be seen that given a cracked configuration, K_{II} decreases fairly rapidly with f , and the relation is not linear. Furthermore, the extent of the contact between the stamp and the beam also has the effect of increasing the magnitude of this stress intensity factor; the influence is greater the smaller the size of the crack, as expected.

4.5 Concluding Remarks

The multi-region boundary element method for contact mechanics analysis of crack problems has been applied to five angled crack problems involving crack closure in this chapter. Special crack tip elements have been employed and the algorithm has been designed to accurately represent the state of stress near the tip of a crack whose faces may be in contact with one another. Supplementary equations have been obtained for nodes common to three (sub)regions in contact. An improved solver for the system of linear equations arising from contact fracture mechanics problems has been presented. Stress intensity factor results have been presented for a range of geometric parameters and different coefficients of friction for the five problems treated. The study has shown that the magnitude of the stress intensity factors can be significantly influenced by

the frictional conditions between the cracks faces when they are in contact.

Table 4.1 Normalized stress intensity factors, $(K_{II})_A/[(6M\sqrt{\pi a})/W^2]$, for crack tip *A* of an inclined crack in a rectangular plate subjected to in-plane bending moment *M* at its ends (Problem II).

θ	a/W	Coeff. of friction, f					
		0.0	0.2	0.4	0.6	0.8	1.0
22.5°	0.2	0.0330	0.0173	0.0016	0.0000	0.0000	0.0000
	0.4	0.0673	0.0358	0.0042	0.0000	0.0000	0.0000
	0.6	0.1058	0.0574	0.0085	0.0000	0.0000	0.0000
	0.8	0.1576	0.0864	0.0135	0.0000	0.0000	0.0000
45.0°	0.2	0.0354	0.0284	0.0214	0.0144	0.0074	0.0000
	0.4	0.0722	0.0581	0.0442	0.0303	0.0164	0.0024
	0.6	0.1101	0.0894	0.0686	0.0476	0.0264	0.0050
	0.8	0.1536	0.1255	0.0968	0.0677	0.0380	0.0079
67.5°	0.2	0.0134	0.0123	0.0112	0.0101	0.0089	0.0078
	0.4	0.0272	0.0249	0.0227	0.0205	0.0183	0.0161
	0.6	0.0411	0.0378	0.0346	0.0314	0.0280	0.0247
	0.8	0.0562	0.0518	0.0497	0.0431	0.0387	0.0343

Table 4.2 Normalized stress intensity factors, $(K_{II})_B / [(6M\sqrt{\pi a})/W^2]$, for crack tip B of an inclined crack in a rectangular plate subjected to in-plane bending moment M at its ends (Problem II).

θ	a/W	Coeff. of friction, f					
		0.0	0.2	0.4	0.6	0.8	1.0
22.5°	0.2	0.0327	0.0341	0.0356	0.0358	0.0358	0.0358
	0.4	0.0647	0.0681	0.0715	0.0722	0.0722	0.0722
	0.6	0.0970	0.1033	0.1098	0.1118	0.1118	0.1118
	0.8	0.1364	0.1479	0.1596	0.1644	0.1652	0.1652
45.0°	0.2	0.0355	0.0360	0.0366	0.0373	0.0379	0.0386
	0.4	0.0714	0.0726	0.0740	0.0754	0.0768	0.0782
	0.6	0.1078	0.1100	0.1122	0.1145	0.1168	0.1191
	0.8	0.1493	0.1524	0.1556	0.1588	0.1621	0.1655
67.5°	0.2	0.0135	0.0136	0.0137	0.0138	0.0139	0.0140
	0.4	0.0272	0.0273	0.0275	0.0278	0.0280	0.0282
	0.6	0.0412	0.0416	0.0419	0.0423	0.0426	0.0429
	0.8	0.0566	0.0570	0.0575	0.0579	0.0584	0.0588

Table 4.3 Comparison of BEM computed normalized stress intensity factors, $(K_I^* = K_I/(3M\sqrt{2\pi a}/W^2))$, for an inclined central angled crack under action of in-plane bending moment M ; $\theta = 45^\circ$, $a/W = 1/\sqrt{2}$; subscripts A and B denote the crack-tip (Problem II).

f	0.0	0.1	0.2	0.3	1.0	Reference
$(K_{II}^*)_A$	0.250	0.224	0.198	0.175	0.000	(Karami <i>et al.</i>)
	0.196	0.178	0.159	0.140	N/A	(Cen <i>et al.</i>)
	0.187	0.170	0.153	0.135	0.000	Present
$(K_I^*)_B$	0.244	0.244	0.243	0.243	0.242	(Karami <i>et al.</i>)
	0.229	0.228	0.229	0.229	N/A	(Cen <i>et al.</i>)
	0.223	0.223	0.223	0.223	0.220	Present
$(K_{II}^*)_B$	0.240	0.243	0.245	0.248	0.264	(Karami <i>et al.</i>)
	0.192	0.194	0.196	0.198	N/A	(Cen <i>et al.</i>)
	0.183	0.185	0.187	0.188	0.203	Present

Table 4.4 Normalized stress intensity factors, $K_I/[6M\sqrt{\pi a}/W^2]$, for an inclined edge-crack in a rectangular plate subjected to in-plane bending moment M at its ends (Problem IV).

θ	a/W	Coeff. of friction, f					
		0.0	0.2	0.4	0.6	0.8	1.0
0.0°	0.6	0.0467	0.0468	0.0468	0.0468	0.0468	0.0468
	0.8	0.2220	0.2220	0.2220	0.2220	0.2220	0.2220
22.5°	0.6	0.0188	0.0186	0.0186	0.0186	0.0186	0.0186
	0.8	0.1565	0.1467	0.1371	0.1363	0.1363	0.1363
45.0°	0.8	0.0220	0.0203	0.0186	0.0168	0.0150	0.0132

Table 4.5 Normalized stress intensity factors, $K_{II}/(6M\sqrt{\pi a}/W^2)$, for an inclined edge-crack in a rectangular plate subjected to in-plane bending moment M at its ends (Problem IV).

θ	a/W	Coeff. of friction, f					
		0.0	0.2	0.4	0.6	0.8	1.0
22.5°	0.2	0.3075	0.1591	0.0102	0.0000	0.0000	0.0000
	0.4	0.2353	0.1195	0.0070	0.0000	0.0000	0.0000
	0.6	0.1686	0.0825	0.0000	0.0000	0.0000	0.0000
	0.8	0.1084	0.0293	0.0000	0.0000	0.0000	0.0000
45.0°	0.2	0.4590	0.3666	0.2741	0.1819	0.0904	0.0000
	0.4	0.4042	0.3180	0.2340	0.1526	0.0742	0.0000
	0.6	0.3505	0.2721	0.1977	0.1272	0.0611	0.0000
	0.8	0.2801	0.2153	0.1537	0.0952	0.0401	0.0000
67.5°	0.2	0.3693	0.3336	0.2985	0.2641	0.2304	0.1977
	0.4	0.3580	0.3220	0.2868	0.2525	0.2192	0.1872
	0.6	0.3508	0.3144	0.2790	0.2447	0.2117	0.1801
	0.8	0.3509	0.3131	0.2766	0.2417	0.2084	0.1767

Table 4.6 Normalized stress intensity factors, $K_I/(3Pl\sqrt{\pi a}/W^2)$, for an edge crack in a simply-supported beam loaded by a rigid stamp (Problem V).

θ	a/W	H/W	Coeff. of friction, f					
			0.0	0.2	0.4	0.6	0.8	1.0
0.0°	0.6	0.5	0.0413	0.0415	0.0418	0.0420	0.0422	0.0425
		1.0	0.0329	0.0343	0.0356	0.0370	0.0383	0.0395
	0.8	0.5	0.1852	0.1839	0.1831	0.1822	0.1801	0.1791
		1.0	0.1629	0.1621	0.1612	0.1604	0.1597	0.1590
45.0°	0.6	0.5	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
		1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.8	0.5	0.0387	0.0352	0.0325	0.0299	0.0282	0.0260
		1.0	0.0342	0.0279	0.0226	0.0183	0.0154	0.0139

Table 4.7 Normalized stress intensity factors, $K_{II}/(3PL\sqrt{\pi a}/W^2)$, for an edge-crack in a simply-supported beam loaded by a rigid stamp; $\theta = 45^\circ$ (Problem V).

a/W	H/W	Coeff. of friction, f					
		0.0	0.2	0.4	0.6	0.8	1.0
0.2	0.5	0.2456	0.1576	0.0675	0.0000	0.0000	0.0000
	1.0	0.3069	0.2382	0.1798	0.1314	0.0792	0.0290
0.4	0.5	0.1874	0.1162	0.0475	0.0000	0.0000	0.0000
	1.0	0.2393	0.1764	0.1150	0.0535	0.0000	0.0000
0.6	0.5	0.1524	0.0942	0.0487	0.0000	0.0000	0.0000
	1.0	0.1798	0.1272	0.0751	0.0244	0.0000	0.0000
0.8	0.5	0.1114	0.0577	0.0247	0.0000	0.0000	0.0000
	1.0	0.1236	0.0850	0.0464	0.0000	0.0000	0.0000

Figure 4.3 Definition of Γ_s and Γ_c for a contact fracture mechanics problem.

Figure 4.4 A rectangular plate with an inclined central crack subject to: (a) uniform compressive stress σ_0 ; (b) in-plane bending moment M .

Figure 4.5 BEM mesh for Problems (I) and (II).

Figure 4.6 Variation of the normalised stress intensity factor, $K_{II}/\sigma_0\sqrt{\pi a}$, with the coefficient of friction f for $\theta = 22.5^\circ$, Problem (I).

Figure 4.7 Variation of the normalised stress intensity factor, $K_{III}/\sigma_0\sqrt{\pi a}$, with the coefficient of friction f for $\theta = 45^\circ$. Problem (I).

Figure 4.8 Variation of the normalised stress intensity factor, $K_{II}/\sigma_0\sqrt{\pi a}$, with the coefficient of friction f for $\theta = 67.5^\circ$, Problem (1)

Figure 4.9 Comparison between Melville's solution and BEM for the stress intensity factors of an angled central closed crack in an infinite domain.

Figure 4.10 Variation of the normalised stress intensity factor, $(K_I)_B / (6M\sqrt{\pi a}/W^2)$, with relative crack length, a/W , at crack tip B. Problem (II).

(a)

(b)

Figure 4.11 A rectangular plate with an inclined edge crack subject to: (a) uniform compressive stress σ_0 ; (b) in-plane bending moment M .

Figure 4.12 BEM mesh for Problems (III) and (IV).

Figure 4.13 Variation of the normalised stress intensity factor, $K_{II}/\sigma_0\sqrt{\pi a}$, with the coefficient of friction f for $\theta = 22.5^\circ$. Problem (III).

Figure 4.14 Variation of the normalised stress intensity factor, $K_{II}/\sigma_o\sqrt{\pi a}$, with the coefficient of friction f for $\theta = 45^\circ$. Problem (III).

Figure 4.15 Variation of the normalised stress intensity factor, $K_{II}/\sigma_o\sqrt{\pi a}$, with the coefficient of friction f for $\theta = 67.5^\circ$. Problem (III).

Figure 4.16 Comparison between Melville's solution and BEM for the stress intensity factors of an angled edge closed crack in an infinite domain.

Figure 4.17 A simply supported cracked beam loaded via a rigid stamp, Problem V.

Figure 4.18 BEM mesh for the problem (Problem V) when $\theta = 45^\circ$, $a/W = 0.6$.

Chapter 5

Boundary Element Analysis of 3-D Crack Closure Problems

5.1 Introduction

Contact mechanics analysis is often required in the treatment of 3-D crack problems. For example, the crack faces of a through-the-thickness crack in a thick rectangular plate, come into contact on the compression side when the plate is subjected to bending. Without considering crack closure, the calculated transverse deflections of the plate are significantly higher than the actual values[45]. Another example is the corner crack problem at the hole in a straight attachment lug commonly seen in aircraft structures. The lug is usually loaded by a pin inserted into the hole and stress intensity factor solutions for the corner cracks can vary significantly depending upon the bearing pressure distribution assumed in linear static analysis[58].

Three-dimensional contact fracture mechanics problems have been a subject of study by quite a number of researchers using the finite element method (FEM) in the last two decades. In this regard, Jones and Swedlow[55] were perhaps the first to take into account the effects of crack closure along the compressive edges of the faces, in their finite element analysis of a through-the-thickness central crack in a plate subjected to remote bending loads based on Kirchhoff theory. The same problem was also treated using FEM by Heming[45] who improved upon their solution using

sixth order Mindlin theory of plate bending. Similar crack closure problems involving finite and infinite plates subjected to biaxial and/or uniaxial bending loads have been analysed by, e.g., Kwon[71], Alwar and Nambissan[6] and Yagawa and Hirayama[147], as well using FEM. Hitherto, however, the author is not aware of any published work, using the BEM, on the analysis of general 3-D frictional contact problems with cracks present. The treatment of such contact problems using BEM has hitherto been limited to two-dimensions. These have been reviewed in the previous chapter.

In this chapter, the multi-domain boundary element method developed for the analysis of three-dimensional contact crack problems as discussed in Chapter 3 is extended to treat fracture problems involving crack closure. Special crack-tip elements that simulate the singular stress fields in the vicinity of the crack tip, as discussed in Chapter 2, are used. Thus, the stress intensity factors can be obtained in a quick and direct manner. The results of four problems involving crack closure are presented; they are considered primarily for comparison with those available in the literature. The first three problems are concerned with through-the-thickness crack in thick plates subjected to different modes of bending. The fourth problem is that of a circumferential edge crack in a cylindrical bar subjected to pure bending.

As in the two-dimensional case, for a partially closed crack, K_I is zero at the closed end. Thus non-singular shape functions for the tractions in the directions normal to the crack faces were used for the elements adjacent to the closed crack front. The stress state at the open crack front has, however, a $1/\sqrt{r}$ singularity, and the shape functions in Eq. (2.68) were used. The solver for the system of equations developed in Chapter 4 were employed as well to solve the three-dimensional contact fracture mechanics problems.

5.2 Supplementary equations

Similar to the treatment in two-dimensions, the contact stresses and the displacements in the contact zone are usually computed in the local coordinate system in

the numerical analysis. However, each node pair in the contact zone now has six unknowns, and six boundary integral equations are generally available. But when a cracked body comes into contact with another elastic body, as shown in Figure 5.1, the points Q_I , Q_{II} and Q_{III} provide three node pairs for which only nine independent boundary integral equations are available when the multiregion crack modelling approach is used. Nine extra equations, must therefore be provided. The present work is limited to the case that the intersection formed by multi-domains only involves one crack, as shown in Figure 5.1. The supplementary equations may again be obtained from two considerations. The first is deformation compatibility. For example, if all the three node pairs are in slip contact states, the following nine extra equation can be written:

$$\begin{aligned}
 u_1^I n_{11}^I + u_2^I n_{12}^I + u_3^I n_{13}^I &= u_1^{II} n_{11}^{II} + u_2^{II} n_{12}^{II} + u_3^{II} n_{13}^{II}, \\
 u_1^I n_{21}^I + u_2^I n_{22}^I + u_3^I n_{23}^I &= u_1^{II} n_{21}^{II} + u_2^{II} n_{22}^{II} + u_3^{II} n_{23}^{II}, \\
 u_1^I n_{31}^I + u_2^I n_{32}^I + u_3^I n_{33}^I &= u_1^{II} n_{31}^{II} + u_2^{II} n_{32}^{II} + u_3^{II} n_{33}^{II}, \\
 u_1^I n_{11}^I + \hat{u}_2^I n_{12}^I + \hat{u}_3^I n_{13}^I &= u_1^{III} n_{11}^{III} + u_2^{III} n_{12}^{III} + u_3^{III} n_{13}^{III}, \\
 u_1^I n_{21}^I + \hat{u}_2^I n_{22}^I + \hat{u}_3^I n_{23}^I &= u_1^{III} n_{21}^{III} + u_2^{III} n_{22}^{III} + u_3^{III} n_{23}^{III}, \\
 u_1^I n_{31}^I + \hat{u}_2^I n_{32}^I + \hat{u}_3^I n_{33}^I &= u_1^{III} n_{31}^{III} + u_2^{III} n_{32}^{III} + u_3^{III} n_{33}^{III}, \\
 u_1^{II} n_{11}^{II} + \hat{u}_2^{II} n_{12}^{II} + \hat{u}_3^{II} n_{13}^{II} &= u_1^{III} n_{11}^{III} + \hat{u}_2^{III} n_{12}^{III} + \hat{u}_3^{III} n_{13}^{III}, \\
 u_1^{II} n_{21}^{II} + \hat{u}_2^{II} n_{22}^{II} + \hat{u}_3^{II} n_{23}^{II} &= u_1^{III} n_{21}^{III} + \hat{u}_2^{III} n_{22}^{III} + \hat{u}_3^{III} n_{23}^{III}, \\
 u_1^{II} n_{31}^{II} + \hat{u}_2^{II} n_{32}^{II} + \hat{u}_3^{II} n_{33}^{II} &= u_1^{III} n_{31}^{III} + \hat{u}_2^{III} n_{32}^{III} + \hat{u}_3^{III} n_{33}^{III}
 \end{aligned} \tag{5.1}$$

where the superscripts denote the intrinsic node pairs; and u_2^I and $\hat{u}_2^I$ are the two tangential displacements for the first node pair. If the above equations are not fully independent, the second consideration, namely, equilibrium of the tractions may be used. For example, if the three node pairs are in the adhesive contact states, the following two equations can be written:

$$\begin{aligned}
 t_1^{II} &= t_1^{III}, \\
 t_2^{II} &= t_2^{III},
 \end{aligned}$$

$$t_3^{\text{II}} = t_3^{\text{III}} \quad (5.2)$$

Note that, as before, the above equations are only approximately true. The exact expressions associated with the tractions can be developed from the stress-strain relations. They are relatively more complex in form and for most problems in practice, Eq. (5.2) is adequate.

5.3 Numerical Examples

The four fracture problems involving crack closure presented below have, as was mentioned earlier, been treated by various authors. The effects of friction between the crack faces was ignored in all these previous studies. For the purpose of comparison, therefore, the BEM analysis of these problems were carried out assuming frictionless conditions as well. A 3-D frictional contact fracture mechanics problem treated in the present work will be presented in the next chapter instead. In all the cases considered here, Poisson's ratio was taken to be 0.3. Also the stress intensity factors presented have all been obtained using the traction formula as discussed in Section 2.6.2 in Chapter 2. The discrepancies of the corresponding results obtained using the displacement formulas were found to be relatively small in general for all the problems analysed, hence they will not be shown.

5.3.1 Example 1: Uniform bending of a rectangular plate with a central through-the-thickness crack

Figure 5.2 shows a rectangular plate, length $2L$, width $2W$, and thickness h , containing a central through-the-thickness crack of length $2a$ and subjected to a uniform bending moment M_0 at the boundaries. The ratio of the crack length to the thickness, $2a/h$, was taken to be 4; and L/a and W/a were taken to be 4 and 4.625, respectively. Taking advantage of the symmetry of the problem, only half of the physical domain was discretized, and the boundary element mesh is shown in Figure 5.3; it has a total

of 200 elements and 590 nodes for each subdomain.

This problem with identical geometrical parameters was first analysed by Heming[45] using the finite element method and the Mindlin sixth order plate theory. In his analysis, only the contact between the crack faces at the compressive edge was considered, i.e. the contact zone was assumed to be a line. Among the results presented by Heming[45] were the transverse deflections of crack flank, and the crack opening displacements with and without the crack closure. The same problem, except with $L/a = 4.625$ and $W/a = 4$, was later analysed by Alwar and Nambissan[6], also using the three-dimensional finite element method. The crack closure was considered to be a zone in their analysis. In addition to the deflections, the distributions of the stress intensity factor with and without the crack closure were also included in their analysis. The results obtained by Heming[45] and Alwar and Nambissan[6] are used for comparison with those obtained by the present boundary element analysis.

Figure 5.4 shows the transverse deflection w along the crack flank obtained by the different methods, the abscissa (r/a) being the normalised distance from the crack front. The deflection has been normalized by the factor $a^2 M_o / 2D(1 + \nu)$, which is the transverse deflection at $x = a$ and $y = 0$ for an uncracked plate subjected to uniform pure bending[134]. As can be seen, when crack closure was ignored, there was an excellent agreement between the boundary element and the finite element results. When crack closure was considered, the computed set of deflections by Heming[45] and Alwar and Nambissan[6] showed some significant deviations from one another. The boundary element results were generally in between these two finite element solutions. It is evident from these results that crack closure does significantly reduce the transverse deflection.

Figure 5.5 shows the normalized crack-opening displacements $2D(1 + \nu)v / (a^2 M_o)$, along the crack flank on the tensile edge computed by the different methods. For positions near the crack front, there is an again excellent agreement between the results from boundary element analysis and those from the finite element analysis carried out by Heming[45] and Alwar and Nambissan[6], where crack closure was not

considered. When the crack closure was considered, the results obtained from the boundary element analysis and those from the finite element analysis carried out by Alwar and Nambissan[6] were also in generally very good agreement. However, there was a significant discrepancy between the boundary element result and the finite element result of Heming[45].

Figure 5.6 shows the BEM computed variations of the normalized stress intensity factor, $K_I/[6M_o\sqrt{\pi a}/h^2]$, across the plate thickness, with and without the crack closure considered. Also shown are the FEM results obtained by Alwar and Nambissan[6]. As can be seen, there was generally good agreement between the two sets of results except at the intersection of the crack and the free surface, i.e. at $z/h = 0.5$. The discrepancies there are not really of much significance, not least because of the uncertainty in interpreting the value of the stress intensity factor at such an intersection, as the stress singularity is not $O(\frac{1}{\sqrt{r}})$. Finally, the thickness of the zone of crack closure, i.e. in the z -direction, was found in the BEM analysis to be $0.27h$ for most parts along the length of the crack, decreasing gradually as the center of the crack was reached, to about $0.21h$. These values are in excellent agreement with the results of Alwar and Nambissan[6].

5.3.2 Example 2: A square plate with a central through-the-thickness crack subjected to uniaxial bending

This problem, shown in Figure 5.7, is similar to the previous one, except that the plate is square and now subject to the bending moment M_o at just two ends instead. The geometry and the range of geometric parameters considered were $W/h = 4$, and $a/W = 0.1, 0.3$ and 0.5 . By virtue of symmetry, only one half of the domain was discretized, and a typical boundary element mesh was shown in Figure 5.8.

Figure 5.9 to 5.11 show the transverse deflections w along the crack flanks on the tensile side of the plate (i.e. $z = h/2$) for the three relative crack sizes treated. These deflections have been normalized by the quantity $\nu M_o a^2/2D(1 - \nu^2)$, which is the transverse deflection at $x = a$ according to simple plate bending theory of an

uncracked plate subjected to the same bending load. The effects of crack closure can be seen in these figures to be very significant, reducing the normalised transverse deflections by about 15 percent at the crack front when the relative crack size $a/W = 0.1$, to about 74% when $a/W = 0.5$. Of interest to note too is that, when crack closure is considered for the two larger cracks treated, the transverse deflection along the crack flanks or this tensile edge of the plate were found to be significantly lower in magnitude than the corresponding values of an uncracked plate.

The variations of the normalised crack-opening-displacements, $v/[\nu M_o a^2/2D(1 - \nu^2)]$, along the crack flanks at $z = h/2$ (tensile edge) are as shown in Figure 5.12 to Figure 5.14. As to be expected, crack closure on the compressive region of the plate has the effect of reducing the amount which the crack flanks would open under the bending load at this edge. Also the height of the region of contact was found from the BEM analysis to be about $0.36h$ for $a/W = 0.1$, about $0.20h$ for $a/W = 0.3$ and about $0.16h$ for $a/W = 0.5$.

Figure 5.15 to 5.17 show the distributions of the normalised stress intensity factor, $K_I h^2/6M_o \sqrt{\pi a}$, along the crack front, through the thickness of the plate, for the three crack sizes analysed. Comparing the case when crack closure was considered and that when it was not, it was found that the stress intensity factors at the crack front towards $z = h/2$ were lower in the former case than in the latter. At most points along the crack front, however, it was interesting to note that the reverse of this was true, although the magnitudes of the stress intensity factors were lower because of the lower bending stress in the inner fibres. Yagawa and Hirayama[147] has also solved this problem with consideration for crack closure using FEM, but only for one crack size, namely, $a/W = 0.5$. Their results are shown plotted with the BEM results in Figure 5.17. Agreement between the BEM and finite element solutions are generally quite good (within about 10 percent discrepancy) except at positions near the crack front-free surface intersection.

5.3.3 Example 3: A square plate with a central through-the-thickness crack subject to transverse loading

Figure 5.18 shows the problem considered, namely, that of a finite size square plate of side-length L , with a central through-the-thickness crack of length $2a$, clamped on all sides and subjected to uniform transverse pressure q . The geometries treated were $a/L = 0.25$, and $h/L = 0.05, 0.1$ and 0.2 . This problem has been solved by Kwon[71] previously for the same geometric cases. In Ref.[71], the finite element method with a plate bending element degenerated from a three-dimensional solid element to model the crack closure was employed. The plate element has only translational displacement degrees of freedom and no rotational ones. Figure 5.19 shows a typical BEM mesh used in the present study. Only one half of the physical domain was modelled, as, again, advantage can be taken of the symmetry of the problem.

For the purpose of comparison with the results of Kwon[71], the BEM computed values for the transverse deflections and crack opening displacements were all normalised with respect to the corresponding displacements at the center of the plate obtained without considering crack closure. These latter displacements will be denoted by the subscripts "nc". Figure 5.20 to 5.22 show the variations of the normalised transverse deflections, w/w_{nc} , along the flanks of the crack for the different plate thicknesses. Also shown in the figures are the corresponding finite element solutions of Kwon[71] with the surface closure model. The agreement between the BEM and finite element results is excellent indeed, the discrepancies being less than 3 percent for these transverse displacements. The results show the significant effects of crack closure and the need to consider it when it does indeed occur.

Figure 5.23 to 5.25 show the effects of crack closure on the crack opening displacements for the three plate thicknesses respectively as obtained from the BEM analysis. Also shown in these figures are the corresponding normalised crack-opening displacements obtained by Kwon[71]. Unlike the results for the transverse deflections, there were significant discrepancies between the BEM and finite element solutions here of up to about 10 percent. This can be attributed to the deficiencies in the plate bend-

ing element used in the finite element analysis and is further evidenced by the crack closure model depth computed in Kwon's analysis[71]. The ratio of the crack surface closure depth to the plate thickness computed from the finite element study was 0.40 for $h/L = 0.05$, 0.33 when $h/L = 0.1$ and 0.26 when $h/L = 0.2$. The decrease in the relative crack closure depth with increase of plate thickness is in fact opposite to the experimental findings of Smith and Smith[118], albeit that the clamped plates used in the experimental study were circular instead of square in shape. The BEM results for the relative crack closure depths were 0.20, 0.25 and 0.30 respectively; the trend was evidently more consistent with the experimental results of Ref.[118].

The BEM computed stress intensity factors along the crack front for the three plate thickness treated are shown in Figure 5.26 to 5.28, respectively, for the case when crack closure was considered and that when it was not. It is very clear from the results that closure of the crack surfaces on the compressive side of the plate in bending must be taken into consideration when computing the stress intensity factors for such cracks. Unfortunately, for the set of results presented, no finite element (or other) stress intensity factor solutions are available for comparison. Kwon, for example, did not present these results in Ref.[71].

5.3.4 Example 4: A cylindrical bar with a circumferential edge crack subjected to pure bending

The last problem considered in this Chapter is as shown in Figure 5.29. It is that of a long cylindrical bar of radius b with a circumferential edge crack of length a in the radial direction in a plane normal to the axis, subjected to a bending moment M . Three different relative crack lengths were considered, namely, $a/b = 0.1$, 0.3 and 0.5. This problem has also been analysed by Benthem and Koiter[14] using the method of asymptotic expansion. The effects of crack closure was ignored in their analysis and a formula was also obtained for the maximum stress intensity factors in terms of the bending moment and relative crack size.

For the BEM analysis, the length of the cylinder was taken to be eight times

its radius. A typical mesh used is shown in Figure 5.30. Figure 5.31 shows the computed variations of the normalized stress intensity factor $K_I/\{4M\sqrt{a/\pi bc}/c^2\}$, with the angular position ψ at the crack periphery as defined in Figure 5.29, when crack closure was ignored in the analysis. The computed maximum values of the normalised stress intensity factors for $a/b = 0.1$, 0.3 and 0.5 , were 0.930 , 0.669 and 0.526 , respectively, all occurring at $\psi = 0^\circ$. The corresponding values obtained from Benthem and Koiter's formula[14] were 0.928 , 0.673 and 0.531 , respectively. It can be easily verified that the deviations between these two sets of results are less than 1 percent.

When crack closure was considered, Figure 5.32 shows the variations of the normalised stress intensity factor, $K_I/\{4M\sqrt{a/\pi bc}/c^2\}$, with the angular position ψ at the crack periphery. Comparing these results to those shown in Figure 5.31, it is evident that the effect of crack closure in reducing the stress intensity factors for this circumferential crack is very significant indeed, particularly for the deeper cracks. When $a/b = 0.1$, the reduction in the stress intensity factor at $\psi = 0^\circ$ was found to be only about 3 percent. But when $a/b = 0.3$ and $a/b = 0.5$, the decrease in the stress intensity factor at $\psi = 0^\circ$ was about 20 percent, and 47 percent, respectively. Of interest to note too is that as a/b became larger, the angular span, as measured by ψ , in which the crack faces would open under the bending load, would increase. In contrast, if crack closure is ignored, crack opening would always be at $\psi = 0^\circ$ to $\psi = \pm 90^\circ$.

5.4 Concluding Remarks

The multi-region boundary element method for 3-D contact mechanics analysis of crack problems has been developed in this study. As before in 2D, special crack tip elements have been employed to accurately represent the state of stress near the tip of a crack whose faces may be in contact with one another. Supplementary equations that need to be provided for treating some crack problems have also been derived.

Four crack closure problems have been analysed and the stress intensity factor results have been compared with those obtained previously by various authors using other numerical or analytical methods. It should be noted that the BEM formulation developed can be used for analysing frictional contact problems, although in the four examples treated, friction was ignored. This is because of the symmetry of these problems and the mode of crack deformation being that of only Mode I. The effects of friction on K_I can therefore be expected to be negligible. This was in fact verified in the study. The study has also shown the significant influence of crack closure on the stress intensity factor when it occurs and the need to use contact mechanics analysis for treating such problems.

Figure 5.1 Sectional view of the intersection of an elastic body in contact with another cracked body at the mouth of the crack.

Figure 5.2 A cracked plate subjected to bending moment M_0 along all its edges.

Figure 5.3 Boundary element mesh for Example 1.

Figure 5.4 Variation of the normalised transverse deflection.

$w / \left\{ \frac{a^2 M_0}{2D(1 + \nu)} \right\}$, along the crack flanks at $z = h/2$, r being

measured from the crack front: Example 1.

Figure 5.5 Variation of the normalised crack-opening-displacement,

$$v / \left\{ \frac{a^2 M_o}{2D(1 + \nu)} \right\}, \text{ along the crack flanks at } z = h/2, r \text{ being}$$

measured from the crack front: Example 1.

Figure 5.6 Variation of the normalised stress intensity factor, $K_I h^2 / \{6 M_o \sqrt{\pi a}\}$, across the plate thickness: $2a/h = 4$, $L/h = 4$, $W/b = 4.625$; Example 1.

Figure 5.7 A thick plate with a central through-the-thickness crack under the action of bending moment M_0 .

Figure 5.8 Boundary element mesh for Example 2.

Figure 5.9 Variation of the normalised transverse deflection, $\frac{2D(1 - \nu^2)}{\nu M_0 a^2} w$,

along the crack flanks at $z = h/2$: Example 2, $a/W = 0.1$.

Figure 5.10 Variation of the normalised transverse deflection, $\frac{2D(1-\nu^2)}{\nu M_o a^2} w$, along the crack flanks at $z = h/2$: Example 2. $a/W = 0.3$.

Figure 5.11 Variation of the normalised transverse deflection, $\frac{2D(1 - \nu^2)}{\nu M_o a^2} w$, along the crack flanks at $z = h/2$: Example 2. $a/W = 0.5$.

Figure 5.12 Variation of the normalised crack-opening-displacement, $v / \left\{ \frac{\nu M_o a^2}{2D(1-\nu^2)} \right\}$, along the crack flanks at $z = h/2$: Example 2, $a/W = 0.1$.

Figure 5.13 Variation of the normalised crack-opening-displacement, $v / \left\{ \frac{\nu M_o a^2}{2D(1 - \nu^2)} \right\}$, along the crack flanks at $z = h/2$: Example 2. $a/W = 0.3$.

Figure 5.14 Variation of the normalised crack-opening-displacement, $v / \left\{ \frac{\nu M_o a^2}{2D(1-\nu^2)} \right\}$, along the crack flanks at $z = h/2$: Example 2. $a/W = 0.5$.

Figure 5.15 Variation of the normalised stress intensity factor, $K_I / \left\{ \frac{6M_o}{h^2} \sqrt{\pi a} \right\}$, along the crack front, through the thickness of the plate : Example 2. $a/W = 0.1$.

Figure 5.16 Variation of the normalised stress intensity factor, $K_I / \left\{ \frac{6 M_o}{h^2} \sqrt{\pi a} \right\}$, along the crack front, through the thickness of the plate : Example 2, $a/W = 0.3$.

Figure 5.17 Variation of the normalised stress intensity factor, $K_I / \left\{ \frac{6M_o}{h^2} \sqrt{\pi a} \right\}$, along the crack front, through the thickness of the plate: Example 2, $a/W = 0.5$.

Figure 5.18 A clamped plate with a central through-the-thickness crack subjected to uniform pressure q .

Figure 5.19 Boundary element mesh for the problem shown in Figure 5.18 with $h/L_z=0.2$.

Figure 5.20 Variation of the normalised transverse displacement, w/w_{nc} , along

the crack flanks at $z = -h/2$: Example 3, $h/L = 0.05$.

Figure 5.21 Variation of the normalised transverse displacement, w/w_{nc} along the crack flanks at $z = -h/2$: Example 3, $h/L = 0.1$.

Figure 5.22 Variation of the normalised transverse displacement, w/w_{nc} , along the crack flanks at $z = -h/2$: Example 3. $h/L = 0.2$.

Figure 5.23 Variation of the normalised crack-opening-displacement, v/v_{nc} , along

the crack flanks at $z = -h/2$: Example 3, $h/L = 0.05$.

Figure 5.24 Variation of the normalised crack-opening-displacement, v/v_{nc} , along the crack flanks at $z = -h/2$: Example 3. $h/L = 0.1$.

Figure 5.25 Variation of the normalised crack-opening-displacement, v/v_{nc} , along the crack flanks at $z = -h/2$: Example 3, $h/L = 0.2$.

Figure 5.26 Variation of the normalised stress intensity factor, $K_I/q\sqrt{\pi a}$, along the crack front through the plate thickness: Example 3. $h/L = 0.05$.

Figure 5.27 Variation of the normalised stress intensity factor, $K_I / q \sqrt{\pi a}$, along the crack front through the plate thickness: Example 3, $h/L = 0.1$.

Figure 5.28 Variation of the normalised stress intensity factor, $K_I / q\sqrt{\pi a}$, along the crack front through the plate thickness: Example 3, $h/L = 0.2$.

Figure 5.29 A cylindrical bar with a circumferential edge crack subjected to a bending moment M .

Figure 5.30 Boundary element mesh for Example 4 with $a/b=0.5$; total 384 elements, 1092 nodes.

Figure 5.31 Variation of the normalised stress intensity factor, $K_I / [4M\sqrt{a}/(\pi bc)/c^2]$, with angular position ψ around the crack periphery (crack closure ignored).

Figure 5.32 Variation of the normalised stress intensity factor, $K_I / [4M\sqrt{a/(\pi bc)}/c^2]$, with angular position ψ around the crack periphery (crack closure considered).

Chapter 6

Boundary Element Contact Mechanics Analysis of Pin-Loaded Lugs with Cracks

6.1 Introduction

Pin-loaded attachment lugs are, with respect to failure by fracture, some of the most critical components in aircraft structures. Stress intensity factor solutions for cracks in these components are often needed to establish their structural integrity in damage tolerance design. Although the lug joint appears to be a rather simple joint, there are a number of factors which result in surprisingly complex response and fatigue behaviour[111]. These factors include: i) friction between the pin and the lug; ii) clearance between pin and hole; iii) prestressing or repairing by interference fits. To examine these factors, various experimental and analytical techniques have been employed. For example, Schijve and Hoeymaker[112], following the procedure developed by James and Anderson[49], obtained the stress intensity factors from experimental fatigue crack growth rate data for through-the-thickness cracks in lugs under constant amplitude loading. Nicoletto[92], on the other hand, used the frozen stress photoelasticity method to determine approximate stress intensity factor solutions for both through-the-thickness and corner cracks in these components. The range of geometries considered in his study was very limited, but the results were assessed against those obtained from other studies and the factors affecting the stress intensity factors

were discussed. Moon[90] investigated the effects of friction on the crack propagation rate of the lug with a corner crack, loaded by the pin with or without the flat. Analytically, a relatively simple compounding method, which involves the superposition of known solutions, was applied to cracks in lugs by Kirkby and Rooke[61] and by Liu and Kan[80]. The Green's function method[59] and the weight function method[47, 143] have also been employed to treat the cracked lug problem, although in reference[143], the numerical boundary element method was used to compute the weight functions.

The finite element method has hitherto been the computational technique almost exclusively being used for investigating the problem of cracks in lugs. Examples include the works of Pian *et al*[104], Saouma and Zatz[110], Van der Velden and Louwaard[141] and Kathiresan *et al*[58]. In reference[104], the pin-loading was assumed to be distributed either in a uniform manner or in a cosine form around the 180° of the hole circumference. A significant finding of the study was that for a given crack length, the difference in the stress intensity factor obtained from the two different pin-load distribution can be as much as 30 percent. A similar linear analysis was carried out assuming uniform pin-load distribution by Saouma and Zatz[110]. However, Nicoletto[92] in his experimental study mentioned earlier, has shown that the contact pressure distribution due to the pin-loading is, in general, neither uniform nor sinusoidal in form. Furthermore, this distribution changes significantly in the presence of a crack at the hole in the lug.

It is evident, therefore, that an accurate stress analysis of the pin-loaded lug problem must involve contact mechanics. This is because the distribution of the pin-load at the hole periphery changes in a non-linear fashion as the load increases. Using 2-D FEM contact mechanics, Van der Velden and Louwaard[141] obtained the contact pressure distributions and the values of stress intensity factors for a rounded-ended lug containing a through-the-thickness crack of various lengths, subjected to pin-loading. Although the investigation was limited somewhat in scope, it showed that the pressure distribution and the stress intensity factors depend on pin to hole

clearance, crack length and external loads. The results also helped to explain some seemingly contradictory data in the literature.

The work carried out by Kathiresan *et al*[58], perhaps, represent the most comprehensive treatment of the problem as one of contact mechanics using the finite element method in the open literature. In their study, constant strain triangular and quadrilateral elements were used to model the pin and the lug while spring elements were employed for the pin and hole interface. The contact area under progressive loading was determined iteratively and stress intensity factor results were obtained for a wide range of crack sizes and several lug geometries. The authors also showed the relatively large discrepancies between the previous results of reference[101] and those which they have obtained using contact mechanics analysis. However, the solutions presented are only valid for the neat-fit case and when the conditions at the pin lug assembly are frictionless. More recently, Man *et al*[83, 84], using the 2-D incremental BEM, treated a related problem, namely, that of a pin-loaded plate with symmetric edge cracks at the clearance pin hole. The stress intensity factor results were obtained using the *J*-integral approach and it was shown that the presence of friction has a significant effect on the absolute values of these fracture parameters.

Although numerous studies have been made to the analysis of cracks in the pin loaded lugs, there is still a lack of contact mechanics analysis in the cases where an interference-fit bushing is present or where there are corner cracks emanating from the edge of the pin-hole. Hsu and Kathiresan[46], for example, estimated the stress intensity factors for cracks at attachment lugs with an interference-fit bushing by superposition of two solutions. The first solution is to the contact problem involving the cracked lug and the shrunken bushing. The second solution is to the contact problem involving the pin and the cracked lug having a bonded bushing. Friction and the proper contact pressure distribution between the pin and the bushing were not taken into account in their analysis. For a corner crack at the edge of pin hole in a round-ended lug, the results of stress intensity factors given by Kathiresan *et al*[58] were based on the uniform pressure distribution over the 180° angular range of

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the hole periphery. For two symmetric corner cracks at the edge of the pin hole in a plate, Raju and Newman[107] proposed that the stress intensity factor solutions be the linear superposition of two solutions which are to the problem of cracked plate subjected to far-end pull, and to the wedge loading on the hole, respectively.

In this chapter, the multi-domain boundary element method which has been developed in the previous chapters, is applied to pin-loaded attachment lugs with either a single through-the-thickness crack or double symmetric corner cracks at the pin-hole. For the single through-the-thickness cracks, the work consists of two parts. The first part deals with a crack in the lug without a bushing, the second part deals with a crack in the lug which has an interference-fit bushing. Stress intensity factor results are obtained for several cracked geometries and directions of pin-loading. The effects of friction, the types of fit, thickness of the bushing and the difference in material properties between the pin and the lug are investigated for the single through-the-thickness cracks. For the symmetric corner cracks, a neat-fit or full load assumption was made for the pin-lug assembly. Stress intensity factor results from the 3D BEM contact mechanics analysis are obtained for a few crack sizes. The effects of friction, and the difference in material properties between the pin and the lug are also investigated.

6.2 2-D Analysis of edge cracks in an attachment lug

Figure 6.1 shows the pin-loaded straight, symmetric attachment lug with a single crack of length a , considered in this study. The Young's modulus of the lug is E_1 and that of the circular pin is E_2 . The pin-hole has a radius R_1 while that of the pin is R_2 . With reference to this figure, the value of H/W was taken to be equal to 2, but three different values of R_1/W were treated. They were $R_1/W = 0.5, 0.375$ and 0.25 . For each of these values of R_1/W , the problem was analysed for relative crack lengths, $a/(W - R_1)$, ranging from 0.1 to 0.5, and in each of these geometric cases,

the coefficient of friction considered was from $f = 0.1$ to $f = 1$. Also, each of the analyses was carried out for two different values of E_2/E_1 , namely, $E_2/E_1 = 1$ and 3; but Poisson's ratio was taken to be 0.3 for both the lug and the pin throughout.

The study was further separated into two parts according to the direction of pin-loading. In the first part, Part (i), the pin force F was applied along the symmetric plane of the lug, i.e. $\theta = 0^\circ$ in Figure 6.1. All three types of pin-fit conditions were considered, namely, clearance-fit, where $R_1 > R_2$; neat fit, where $R_1 = R_2$; and interference fit, where $R_1 < R_2$. For a given lug geometry and friction condition, the absolute magnitudes of the contact stresses in the lug will vary with the magnitude of the ratio $(R_1 - R_2)/R_1$, and the extent of contact between the pin and the hole periphery. However, as has been verified in a previous study[77], if these stresses are normalised by, say, the nominal stress, $\sigma_o = F/(2Wb)$, where b is the thickness of the lug, at the remote gross section, the normalised stresses are only influenced by the extent of the contact zone, as measured by angle ϕ shown in Figure 6.1, and they are independent of the ratio $(R_1 - R_2)/R_1$. As to be expected then, the same was found to be also true for the stress intensity factors when they are normalised by the quantity $\sigma_o\sqrt{\pi a}$. These normalised stress intensity factors were obtained for $\phi = 90^\circ$, 180° and 270° in the study. It is perhaps useful to note here that the case of $\phi = 180^\circ$ can be reached from all initial conditions of pin-fit. However, the extent of contact with $\phi < 180^\circ$ can only be reached if $R_1 > R_2$ (i.e. clearance fit) while $\phi > 180^\circ$ can only be attained under progressive loading if $R_1 < R_2$ (i.e. interference fit). For completeness, stress intensity factors were also obtained for the cracked geometries due to prestresses alone which are set up due to the interference fit of the pin. In this case the results are shown herein to be for $\phi = 360^\circ$ and they would be normalised by the quantity $\sigma^*\sqrt{\pi a}$ where $\sigma^* = \{(R_2 - R_1)/R_1\}E_1\sqrt{\pi a}$. It should also be mentioned here that for the load case with $\theta = 0^\circ$, the entire analysis has been repeated for $H/W = 4$ and $H/W = 6$ as well, and no significant differences were detected when the results were compared to the corresponding sets obtained for $H/W = 2$. Thus the stress intensity factor solutions presented here should be valid for similar lug

configurations with $H/W \geq 2$.

In Part (ii) of this study, the force F was taken to act along the line of action which formed an angle θ to the plane of symmetry of the lug as shown in Figure 6.1. Two different values of θ were treated, namely, 22.5° and 45° . The same geometric cases as in Part (i) were considered, but the BEM load-incremental analysis in each case was carried out only for the extent of contact reaching $\phi = 180^\circ$, and when coefficient of friction $f = 0$.

Figure 6.2a and Figure 6.2b show two typical BEM meshes used in the study. The pin was represented by one subregion while the cracked lug was modelled with two subregions such that their interface boundaries passed through the plane containing the edge crack at the pin-hole. Relatively small elements were used for the pin and the hole boundaries, and in the vicinity of the crack-tip. Adjacent to the crack-tip, special crack-tip elements as discussed in Chapter 2 were employed. A small central square cavity with side lengths of $0.05R_2$ was introduced in order to simulate the radial point force F in the numerical model. This small cavity was modelled with 20 very small elements and a uniform pressure was applied to just one of these elements such that the line of action of the resultant force passed through the centre of the pin. In the case of clearance fit, the initial coordinates of the node-pair A-B as shown in Figure 6.3, were determined using the following equations:

$$\begin{aligned} x_1(A) &= R_1 \cos \varphi, \\ x_2(A) &= h_1 \sin \varphi \end{aligned} \quad (6.1)$$

$$\begin{aligned} x_1(B) &= (R_1 - R_2) + R_2 \cos \varphi', \\ x_2(B) &= R_2 \sin \varphi' \end{aligned} \quad (6.2)$$

where the angles φ and φ' are as defined in Figure 6.3, and where the clearance between the pin and the hole is exaggerated for the purpose of illustration. For the neat-fit case, point A and point B have the same initial coordinates, while in the case of interference-fit, the initial distance between the points A and B are $(R_2 - R_1)$.

6.2.1 Part (i): Cracked lug subjected to axial load

The case of the pin-loaded lug with $H/W = 2$, $R_1/W = 0.5$ and $E_2/E_1 = 3$ has also been treated by Van der Velden and Louwaard[141] and by Kathiresan *et al*[58] using FEM with the assumption of perfectly smooth contact condition, and by Geier[39] experimentally. Thus it may be useful here to first compare the results obtained from the BEM analysis with those obtained by these authors. For the conditions mentioned, the extent of pin contact at the hole periphery as measured by ϕ was 180° . Table 6.1a lists the normalised mode I stress intensity factors, $K_I/\sigma_0\sqrt{\pi a}$, for the range of relative crack sizes considered here. It can be seen that the agreement between the BEM and FEM solutions was very good indeed; their discrepancies, indicated in parentheses in the table, were less than 2 percent in all but one case where it was slightly higher. The relatively higher discrepancies between the experimental results and the numerical analysis may be due to the effects of friction neglected in the computations[141]. Figure 6.4 shows the contact pressure distribution in the presence of the edge crack, when $H/W = 2$, $R_1/W = 0.5$ and $a/(W - R_1) = 0.5$. Agreement between the BEM and FEM computed distributions was excellent indeed as well. Table 6.1b shows the corresponding results, obtained using FEM and BEM, when the problem was treated as a linear non-contact mechanics problem in which the pin-load was assumed to be uniformly distributed over the $\phi = 180^\circ$ angular span at the pin hole. Although the discrepancies between these results and those obtained from contact mechanics analysis were less than 5 percent for $a/(W - R_1) \leq 0.2$ for the frictionless case shown here, they were considerably larger for larger crack sizes and when friction was present. This can be attributed to the fact that the real pressure distribution in the presence of a crack, as shown in Figure 6.4, is very different from a uniform distribution. As can be verified from the results which will be presented later, discrepancies of as much as 35 percent were found for this $R_1/W = 0.5$ case when $a/(W - R_1) = 0.5$, at high friction. This further emphasizes the need to carry out proper contact mechanics stress analysis if accurate stress intensity factors for the cracked pin-loaded lug problem are required.

The stress intensity factor solutions obtained from the DEM contact mechanics analysis will now be presented for the different values of ϕ treated, starting with $\phi = 90^\circ$ which can only be attained from clearance pin-fit conditions. With this extent of pin contact with the lug, it was found from the analysis that although the mode II stress intensity factors K_{II} were small, being typically one order of magnitude smaller than the values of K_I , they were not always insignificant. Figure 6.5(a) and 6.5(b) show the BEM computed variations of the normalised stress intensity factors, $K_I/\sigma_0\sqrt{\pi a}$ and $K_{II}/\sigma_0\sqrt{\pi a}$, respectively, with the relative crack length $a/(W - R_1)$ for the three R_1/W ratios considered. The results shown are for $E_2/E_1 = 3$ and $f = 0$. It was found from the analysis that for a given crack size, friction has a very negligible effect on the magnitude of the pin force F to cause this angular span of pin contact with the lug, and on the mode I stress intensity factor. The influence on K_{II} was more apparent, however, although it was still small. The computed values of K_{II} were generally no more than 5 percent greater when f was increased from zero to unity for the cases of $R_1/W = 0.5$ and 0.375 , but they were 10 percent greater when $R_1/W = 0.25$. On the effect of the relative rigidity of the pin and the lug, it was found that the normalised stress intensity factors were, to all intents and purposes, virtually unchanged when the ratio of the Young's moduli E_2/E_1 was decreased from 3 to 1. The stress intensity factor results corresponding to mode I and mode II for the two ratios are tabulated in Table 6.2a and Table 6.2b, respectively. It should be noted that with the lower value of E_2/E_1 , a smaller magnitude of the pin force F would be required to obtain the same extent of contact between the pin and the lug.

Figure 6.6, 6.7 and 6.8 show, for the angular span of contact $\phi = 180^\circ$, the computed variations of $K_I/\sigma_0\sqrt{\pi a}$ with the coefficient of friction f , when $R_1/W = 0.5$, 0.375 and 0.25 , respectively, for the three relative crack sizes $a/(W - R_1) = 0.1$, 0.3 and 0.5 . These plotted results are for $E_2/E_1 = 3$; they and the corresponding results for $E_2/E_1 = 1$ are also listed in Table 6.3a. The mode II stress intensity factors were found to be very small compared to those for mode I, and are presented in Table 6.3b. Given the cracked lug geometry, there was a noticeable change in the value of the

normalised stress intensity factor, $K_I/\sigma_0\sqrt{\pi a}$, with increasing friction. This change was very gradual, and in some cases it was very small indeed. For example, when $E_2/E_1 = 3$, the difference in the value of $K_I/\sigma_0\sqrt{\pi a}$, when f was increased from zero to unity, was as low as -2.0 percent in the case of $R_1/W = 0.5$ and $a/(W - R_1) = 0.1$, to -22.4 percent in the case of $R_1/W = 0.375$ and $a/(W - R_1) = 0.5$. Changing the relative rigidity of the pin and the lug from $E_2/E_1 = 3$ to $E_2/E_1 = 1$ did not appear to alter the respective values of the normalised stress intensity factors very significantly, although they were, as expected, higher in all cases for the latter E_2/E_1 ratio. The increase in the values of $K_I/\sigma_0\sqrt{\pi a}$ with the decrease in the value of E_2/E_1 ratio from 3 to 1 was more pronounced when $R_1/W = 0.5$ at high coefficients of friction, where the differences ranged from about 10 to 16 percent for the crack sizes treated.

The computed variations of $K_I/\sigma_0\sqrt{\pi a}$ with the coefficient of friction f for all the geometric cases treated, when $\phi = 270^\circ$ are shown in Figures 6.9-6.11. The results are for $E_2/E_1 = 3$. The results for $E_2/E_1 = 3$ and $E_2/E_1 = 1$ are also presented in Table 6.4a for the same three relative crack sizes, and it can be seen that the trends are similar for the lower E_2/E_1 ratio and the higher E_2/E_1 ratio. It is perhaps worth mentioning here again that the value of ϕ here would correspond to the angular span over which the pin remains in contact with the hole periphery when the load is progressively increased from an initial interference fit condition. In this case, the pin load F to cause the $\phi = 270^\circ$ span of contact was found to increase with increasing magnitudes of the coefficient of friction. This was to be expected. Of significance to note from the results here is that the presence of friction has evidently a very significant effect on the normalised value of the mode I stress intensity factor for every of the cracked geometries treated, with it decreasing as f increased in value. It is also interesting to note that the computed magnitudes of K_{II} remained very small relative to K_I under perfectly smooth conditions. But as the value of f was increased, K_{II} was found also to increase. Although their magnitudes were still quite small compared to K_I , they were not all insignificant. Thus, their normalised values are listed in Table 6.4b for the extreme values of f considered, namely, $f = 0$ and

$$f = 1.0.$$

Figure 6.12 to 6.14 show, for interference pin-fits, the computed variations of $K_I/\sigma^*\sqrt{\pi a}$ with the coefficient of friction due to shrink-fit stresses alone, for the different geometric cases treated. As the presence of friction would reduce the crack opening displacements, the mode I stress intensity factor was expected to decrease with increasing amounts of friction. However, as can be seen from the figures, when f reached a certain value, typically 0.4-0.6 depending on the geometry, the pin and hole boundaries attained an adhesive state of contact. Increasing friction beyond this value would have no effect on the contact stresses and K_I . Results obtained for the Young's modulus ratio $E_2/E_1 = 1$ showed also similar trends; their numerical values are presented in Table 6.5. In all the cases treated, the shrink-fit stresses did not produce any significant amounts of mode II crack deformation.

6.2.2 Part (ii): Cracked lug subjected to an inclined load

Stress intensity factors were obtained for the cases $H/W = 2, 4$ and 6 when the pin-load was applied at $\theta = 22.5^\circ$ and $\theta = 45^\circ$, to the axis of symmetry of the lug. The computed normalized values of $K_I/\sigma_o\sqrt{\pi a}$, are listed in Table 6.6a for the range of geometries, and the Young's modulus ratios considered when $f = 0$. As seen earlier in the $\theta = 0^\circ$ case above, the effect of changing the E_2/E_1 ratio from 3 to 1 on the normalised stress intensity factor was again quite small in general, given the cracked lug geometry. The changes in these normalised quantities with H/W were, quite significant, however. An inference of this result is that a tapered lug would be a preferable design if the pin-load acts in an inclined direction, as the bending effects may then be reduced. Table 6.6b lists the normalized mode II stress intensity factors, $K_{II}/\sigma_o\sqrt{\pi a}$ for the case considered. As can be seen, although in many instances K_{II} was negligible when compared to the magnitude of K_I , this was not always so. Finally, it is perhaps worth mentioning that for both inclined directions of pin-loading considered here, the area of contact between the pin and the lug remained, for all practical purposes, symmetrical about the plane of the pin force F in spite of the

presence of the edge crack.

6.3 2-D Analysis of edge cracks in an attachment lug with a bushing

Figure 6.15 shows the pin-loaded straight, symmetric lug having an interference-fit bushing and containing a single edge crack of length a , emanating from the interface between the lug and the bushing. The material of the pin and that of the bushing were taken to be the same in the analysis. As in previously, the Young's modulus of the lug is E_1 and that of the circular pin and the bushing is E_2 . Also, the external and internal radii of the bushing are R_3 and R_1 , respectively, and δ_1 is the difference between the radius R_1 and that of the pin, R_2 .

In the BEM analysis, the ratios of R_3/R_1 considered were 1.125 and 1.25. Also, with reference to Figure 6.15, the value of H/W treated were 2, 4 and 6; those of R_1/W considered were 0.5, 0.375 and 0.25. In each of these geometric cases, the problem was analysed for three relative crack lengths, namely, $a/(W - R_3) = 0.1$, 0.3 and 0.5. Again, the coefficient of friction between the pin and the bushing was valued from $f = 0.1$ to $f = 1$. Furthermore, each of the analyses was carried out for $E_2/E_1 = 1$ and $E_2/E_1 = 3$, as before; but Poisson's ratio was taken to be 0.3 throughout.

The study was separated into two parts according to the direction of pin-loading. In the first part, Part (i), the pin force F was applied along the symmetric plane of the lug, i.e., $\theta = 0^\circ$ in Figure 6.15. In the second part, Part (ii), the pin-force F was applied at $\theta = 22.5^\circ$ and at $\theta = 45^\circ$ to the symmetric plane. Also, the analysis was carried out, in this latter part, for frictionless conditions only, and under the limiting elastic contact load. This will be explained later. Two typical BEM meshes are shown in Figure 6.16a and Figure 6.16b for the axial and inclined directions of loading, respectively. Four subregions were used to model the problem and, again, a small cavity was introduced in the pin to apply the load F in the

numerical analysis.

6.3.1 Part (i): Cracked lug with a bushing subjected to an axial load

The problem was analysed for the initial conditions of interference-fit between the bushing and the lug and a clearance-fit between the pin and the bushing as well. The radial interference between the outer radius R_3 of the bushing and the radius of the hole in the lug, R'_3 , is δ_2 . When the load F is applied, the contact zone between the pin and the bushing would increase. However, there is also a tendency for bushing to separate from the surface of the lug-hole if the load is sufficiently large. This is evident in Figure 6.17 which shows, for example, the typical interfacial pressure distribution between the bushing and the hole in the presence of the crack and no friction in such a load situation. The angular position around the lug-hole is denoted in φ and is measured from the plane of symmetry of the lug and the origin at the centre of the hole (see Figure 6.15). It can be seen in Figure 6.17 that the interfacial pressure at $|\varphi| > 90^\circ$ is relatively small indeed.

The crack problem may therefore be analysed for four basic load levels as follows which were considered in the study:

1. No external load was applied to the lug, but the crack faces were opened by the prestresses due to the interference-fit.
2. A small pin-load was applied to the lug so that the contact zone between the pin and the bushing, ϕ_1 , spanned 90° . In this case, the bushing remained in full interfacial contact with the lug.
3. A relative large pin-load was applied to the lug so that the limiting angular span of the contact zone between the pin and the bushing was reached, i.e. at about $\phi_1 = 180^\circ$, and that there was partial separation between the bushing and the lug. If their angular extent of contact is denoted by ϕ_2 , then in this case, $\phi_2 < 360^\circ$.

4. The pin-load was further increased beyond that in Case 3 above so that the loss of contact between the bushing and the lug also reached its limit at about $\phi_2 = 180^\circ$.

The stress intensity factor solutions which are presented here are all for the lugs with $H/W = 2$. Also was seen earlier as well in Section 6.2, repeat analyses of the problems with $H/W = 4$ and $H/W = 6$ showed no significant differences in the corresponding stress intensity factors when the pin-load was applied along the plane of symmetry. Also, as to be expected, the mode II stress intensity factors were negligible when compared to K_I .

Table 6.7 lists the normalised stress intensity factors $K_I/\{E_1\delta_2/R_3\}\sqrt{\pi a}$ for the edge crack due to the interference-fit between the bushing and the lug alone, i.e. Case 1 above. The effect of friction between the bushing and the lug-hole can be seen to significantly reduce the value of K_I for a given cracked geometry. This is to be expected as friction would resist the mouth of the crack from opening. As a matter of information, the value of the ratio δ_2/R_3 used in the study was 6×10^{-4} , although the normalised results listed in Table 6.7 are valid for any other positive values of this ratio.

With the value of δ_2/R_3 above for the interference-fit, and a pin-load F then applied to the lug so that the contact zone between the pin and the bushing $2\phi_1 = 90^\circ$, the BEM computed normalised stress intensity factor, $K_I/\{F/(2Wb)\}\sqrt{\pi a}$, are as shown in Table 6.8a. Note that the magnitudes of these normalised stress intensity factors are relatively large. This is because the value of F needed to achieve the $\phi_1 = 90^\circ$ contact between the pin and the bushing is relatively small. Table 6.8b lists the values of the same stress intensity factors but normalised with respect to $\{E_1\delta_2/R_3\}\sqrt{\pi a}$. Comparing these normalised quantities with the corresponding ones in Table 6.7, it can be seen that K_I has only been increased by a small amount when F was applied in this case, for a given cracked geometry and friction condition.

In case 3, the pin force F in the analysis was increased to a sufficiently higher level so that there was partial separation between the bushing and the lug; the angular

span of contact between them being $\phi_2 = 270^\circ$. Table 6.9a and 6.9b show the stress intensity factors normalised with respect to $(F/2Wb)\sqrt{\pi a}$ and by $\{E_1\delta_2/R_3\}\sqrt{\pi a}$, respectively, for the various cracked geometries and friction conditions treated. Except for one cracked configuration, the normalised stress intensity factors were reduced by the presence of friction, but not as dramatically as in Case 2. The fact that the contact zone between the bushing and the lug-hole still covering the mouth of the crack here continued to inhibit the mouth of the crack from opening with friction present. Comparing the results shown in Table 6.9b and those in Table 6.7, it is evident that the absolute values of K_I have been increased by a very significant amount, and the contribution to K_I due to the pin-load F becomes predominant over the pre-stresses due to the shrink-fit.

In the last load case considered, Case 4, the region of contact between pin and the bushing, and the separation between the bushing and the lug-hole have both reached the limiting angular extents, i.e. $\phi_1 \simeq 180^\circ$ and $\phi_2 \simeq 180^\circ$. Table 6.10 lists the computed stress intensity factors normalized by $(F/2Wb)\sqrt{\pi a}$ for the pin-loaded lugs analysed. In contrast to the previous three cases, 1 to 3, the stress intensity factors were found to increase with friction in general. Comparing the results of Tables 6.9a and 6.10, it can be seen that the normalised values of K_I increased as ϕ_2 decreased from 270° to about 180° here, even though the contribution to these normalised quantities by the interference-fit of the bushing could be neglected.

6.3.2 Part (ii): Cracked lug with a bushing subjected to an inclined load

In the part of the study on the cracked lug with a bushing subjected to inclined loads, frictionless contact conditions between the lug and the pin, and the bushing and the lug were assumed in the analysis. The primary objective, as before in Section 6.2.2 was to examine the effect of bending of the lug on the stress intensity factors. Also, only the most severe pin-loading condition that caused limiting contact conditions was considered. This corresponds to Case 4 in section 6.3.1 above in which the angular

span of contact between the pin and the bushing, ϕ_1 , was about 180° ; and so was that between the bushing and the lug-hole, i.e. $\phi_2 \simeq 180^\circ$.

Table 6.11 shows the normalised stress intensity factors, $K_I/\{F/2Wb\}\sqrt{\pi a}$, for the different crack sizes and lug geometries when the pin-load was applied at $\theta = 22.5^\circ$. The corresponding values for pin-loading at $\theta = 45^\circ$ are listed in Table 6.12. It was found from the BEM analysis that the mode II stress intensity factors were small when compared to K_I , thus they are not presented here. As expected, the stress intensity factor for a given crack size in the lug decreased as angle θ increased in value, when $H/W = 2$. Of interest to note however is that it increased quite significantly as the relative length of the lug, H/W , was increased. This was similar in trend to that seen in Section 6.2.2 for the lug without the bushing, and could be attributed to the effect of lug bending causing a re-distribution of the transmitted pin force.

6.4 3-D analysis of corner cracks in a pin-loaded rectangular plate

The 3-D problem considered here is that of a rectangular plate subjected to pin loading and containing two symmetric quarter-circular corner cracks of radius a , emanating from the edge of the circular pin-hole of radius R , as shown in Figure 6.18. The plate has thickness t , width $2b$ and length $2h$. For the analysis, h/b was taken to be 1.8, $R/t = 0.5$ and 0.8. The neat-fit clearance condition between the pin and the hole was assumed and two values of the rigidity ratio E_2/E_1 were treated, namely, 3 and 1, where E_2 and E_1 are the Young's moduli of the plate and pin, respectively. Stress intensity factors were obtained for smooth contact conditions between the pin and the surface of the pin-hole, that is, when the coefficient of friction $f = 0$, as well as when $f = 0.5$.

Figure 6.19 shows a typical BEM mesh employed for the problem. By virtue of symmetry, only a half of the physical problem was modelled; the BEM mesh has three subregions, one for the pin and two for the cracked plate. The discretisation of

one of the subdomains of the plate was the same as that shown in Figure 2.25 from which the mesh near the crack front can be clearly seen. For the sake of simplicity in the analysis, the length of the pin was taken to be equal to the thickness of the plate and the external load F was assumed to be uniformly distributed across the two cross-sections of the pin modelled. It was seen in Chapter 3 that the length of the pin has little influence on the stress distributions in the plate. It was therefore assumed that this would be true also for the stress intensity factors in the crack problem.

Similar three dimensional problems have been analysed using FEM by Kathiresan *et al*[58] and Raju and Newman[107]. Instead of treating the pin-loaded plate as a contact problem, the former authors assumed a uniform pressure distribution of the pin-load over the upper half of the surface of the hole. Raju and Newman[107] also treated the as a linear one, but suggested instead, the following superposition formula to obtain the stress intensity factors for the corner cracks:

$$K_I = \frac{1}{2}S_p\sqrt{\frac{4a}{\pi}}\left[\frac{R}{b}(K_I^*)_t + (K_I^*)_w\right] \quad (6.3)$$

In Eq. (6.3), $(K_I^*)_t$ is the normalized stress intensity factor, $K_I/(S_t\sqrt{4a/\pi})$, when the plate is subjected to remote tensile stress S_t . Also, $(K_I^*)_w$ is the normalized stress intensity factor, $K_I/(S_p\sqrt{4a/\pi})$, when the plate is subjected to a normal pressure distribution of $\sigma_n = 1.5S_p \cos^2 \varphi$ at the pin hole, with φ measured from the direction of the applied load F and $S_p = F/(2Rt)$.

Figure 6.20 shows the computed normalised contact pressure distribution, p/S_p , between the pin and the surface of the hole in the cracked plate when $a/t = 0.5$, $E_2/E_1 = 3$ and $f = 0$. The qualitative trends for the other crack sizes were basically similar to those shown. It is evident that the transmitted load to the cracked plate is not uniform over the upper half of the hole surface in the angular direction φ . The distribution of the contact pressure is also not uniform across the thickness of the plate.

The BEM computed values of the stress intensity factors at the various positions around the corner crack, when $f = 0$ and with $E_2/E_1 = 3$ are listed in Table 6.13.

They have been normalised with respect to $S_p\sqrt{4a/\pi}$, and the angle φ is as defined in Figure 6.18. Also shown in the Table are the corresponding results obtained using the linear analysis approaches of Kathiresan *et al*[58] and Raju and Newman[107]. It can be seen that for this frictionless case, there was fairly good agreement between the results of the BEM contact mechanics analysis and those computed from using the superposition method of Raju and Newman[107]. The discrepancies were generally within about 8 percent along the crack front, except at points near the intersection of the crack front and the surface of the hole. There were, however, very significant discrepancies when the BEM results were compared with those obtained using a linear analysis treatment with the uniform pressure assumption at the pin hole.

When friction between the pin and the surface of the pin-hole was considered, it was found from the BEM analysis that almost all the elements in contact became adhesive at the value of $f = 0.5$ treated. Table 6.14 shows the corresponding results of the normalised stress intensity factor around the crack front. For a given crack size, the stress intensity factors were higher than the corresponding values at the crack front for the frictionless case, as to be expected. This was particularly so when the crack was relatively small and the influence of the contact load distribution was more significant. Also, because of the adhesive contact state between the pin and the surface of the hole at $\psi = \pi/2$, the order of stress singularity there was higher than $r^{-1/2}$. The computed stress intensity factors at this point were in fact very high indeed, but caution should again be exercised in interpreting the results there.

Figure 6.21 shows the variation of the stress intensity factors K_I normalised by $S_p\sqrt{4a/\pi}$, around the crack front where $E_2/E_1 = 1$ in the frictionless case. The corresponding variations for $f = 0.5$ are shown in Figure 6.22. Comparing these results to those presented in Table 6.13 and 6.14, it can be seen that the rigidity ratio E_2/E_1 has very little effect on the normalised stress intensity factors.

6.5 Conclusions

To obtain accurate stress intensity factors in pin-loaded lugs with cracks, contact mechanics stress analysis should be employed. This is because, for a given lug joint geometry, these stress intensity factors are influenced by the friction between the pin and the lug and are a function of the angular span of contact ϕ between the two components. The angular span of contact ϕ that be reached for a given friction condition, in turn, depends on the initial condition of the pin-fit and the applied force. The results obtained here have further shown the discrepancies that may arise in the solutions when using linear-type analysis for the problem.

Table 6.1a Comparison of present BEM normalised stress intensity factor solutions, $K_I/\sigma_o\sqrt{\pi a}$, with other contact mechanics analysis results for single edge cracks in an attachment lug: $E_2/E_1 = 3$, $H/W = 2$, $R_1/W = 0.5$, $f = 0$, $\theta = 0^\circ$ and $\phi = 180^\circ$, $\sigma_o = F/(2Wb)$

$a/(W - R_1)$	$K_I/\sigma_o\sqrt{\pi a}$					
	(BEM) _{present}	(FEM) _{Ref.[141]}		(FEM) _{Ref.[58]}		(EXP) _{Ref.[39]}
0.1	4.205	4.156	(-1.2%)	4.474	(+6.0%)	4.375 (+4.0%)
0.2	3.557	3.594	(+1.0%)	3.578	(+0.6%)	3.638 (+2.3%)
0.3	3.217	3.250	(+1.0%)	3.277	(+1.9%)	3.300 (+2.6%)
0.4	3.015	3.040	(+0.8%)	3.012	(-0.1%)	3.156 (+4.7%)
0.5	2.913	2.937	(+0.8%)	2.943	(+1.0%)	3.125 (+7.3%)

Table 6.1b Comparison of present BEM solutions, $K_I/\sigma_o\sqrt{\pi a}$, with other non-contact mechanics analysis results: $E_2/E_1 = 3$, $H/W = 2$, $R_1/W = 0.5$, $f = 0$, $\theta = 0^\circ$, and $\phi = 180^\circ$, $\sigma_o = F/(2Wb)$

$a/(W - R_1)$	$K_I/\sigma_o\sqrt{\pi a}$			
	(BEM) _{present}	(FEM) _L		(BEM) _L
0.1	4.205	4.175	(-0.7%)	4.072 (-3.2%)
0.2	3.557	3.788	(+6.5%)	3.702 (+4.1%)
0.3	3.217	3.550	(+10.4%)	3.478 (+8.1%)
0.4	3.015	3.425	(+13.6%)	3.370 (+11.8%)
0.5	2.913	3.375	(+15.9%)	3.322 (+14.0%)

(FEM)_L-FEM results from linear non-contact mechanics analysis in Ref.[141].

(BEM)_L-BEM results from linear non-contact mechanics analysis in Ref.[143].

Table 6.2a Normalized stress intensity factors, $K_1/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug. $H/W = 2$, $\phi = 90^\circ$, $\theta = 0^\circ$

E_2/E_1	R_1/W	$a/(W - R_1)$	Coefficient of Friction, f					
			0	0.2	0.4	0.6	0.8	1.0
1	0.5	0.1	6.262	6.277	6.294	6.302	6.308	6.314
		0.3	4.865	4.875	4.879	4.884	4.889	4.894
		0.5	4.195	4.196	4.196	4.200	4.199	4.203
	0.375	0.1	4.796	4.818	4.851	4.860	4.866	4.872
		0.3	3.327	3.343	3.353	3.355	3.359	3.362
		0.5	2.732	2.738	2.740	2.741	2.744	2.746
	0.25	0.1	3.918	3.962	3.981	3.992	3.999	4.005
		0.3	2.335	2.351	2.358	2.360	2.362	2.364
		0.5	1.841	1.846	1.848	1.848	1.849	1.850
3	0.5	0.1	6.238	6.262	6.267	6.274	6.275	6.288
		0.3	4.857	4.856	4.855	4.860	4.865	4.871
		0.5	4.183	4.087	4.162	4.166	4.170	4.174
	0.375	0.1	4.792	4.831	4.839	4.851	4.858	4.865
		0.3	3.320	3.328	3.337	3.340	3.344	3.348
		0.5	2.725	2.721	2.723	2.725	2.728	2.730
	0.25	0.1	3.916	3.959	3.981	3.997	4.005	4.013
		0.3	2.332	2.343	2.351	2.353	2.356	2.358
		0.5	1.839	1.840	1.841	1.843	1.844	1.845

Table 6.2b Normalized stress intensity factors, $K_{II}/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug. $H/W = 2$, $\phi = 90^\circ$, $\theta = 0^\circ$

E_2/E_1	R_1/W	$a/(W - R_1)$	Coefficient of Friction, f					
			0	0.2	0.4	0.6	0.8	1.0
1	0.5	0.1	0.324	0.327	0.332	0.334	0.335	0.336
		0.3	0.669	0.676	0.681	0.684	0.688	0.691
		0.5	0.887	0.892	0.897	0.901	0.899	0.903
	0.375	0.1	0.297	0.306	0.317	0.321	0.323	0.325
		0.3	0.523	0.541	0.552	0.557	0.560	0.563
		0.5	0.615	0.631	0.639	0.644	0.648	0.652
	0.25	0.1	0.386	0.415	0.427	0.435	0.439	0.444
		0.3	0.481	0.511	0.524	0.530	0.533	0.537
		0.5	0.474	0.496	0.506	0.511	0.514	0.517
3	0.5	0.1	0.317	0.323	0.324	0.325	0.325	0.328
		0.3	0.666	0.667	0.666	0.669	0.673	0.676
		0.5	0.881	0.863	0.861	0.744	0.871	0.876
	0.375	0.1	0.296	0.310	0.313	0.318	0.321	0.323
		0.3	0.520	0.531	0.541	0.547	0.551	0.555
		0.5	0.609	0.614	0.622	0.628	0.633	0.637
	0.25	0.1	0.386	0.414	0.428	0.440	0.445	0.450
		0.3	0.480	0.506	0.520	0.528	0.532	0.537
		0.5	0.473	0.491	0.499	0.509	0.512	0.516

Table 6.3a Normalized stress intensity factors, $K_I/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug. $H/W = 2$, $\phi = 180^\circ$, $\theta = 0^\circ$

E_2/E_1	R_1/W	$a/(W - R_1)$	Coefficient of Friction, f					
			0	0.2	0.4	0.6	0.8	1.0
1	0.5	0.1	4.342	4.353	4.432	4.528	4.646	4.772
		0.3	3.358	3.344	3.329	3.307	3.228	3.155
		0.5	3.028	2.972	2.927	2.845	2.772	2.709
	0.375	0.1	3.790	3.864	3.951	4.046	4.131	4.147
		0.3	2.632	2.648	2.643	2.609	2.565	2.526
		0.5	2.265	2.248	2.210	2.166	2.127	2.093
	0.25	0.1	3.371	3.485	3.569	3.632	3.666	3.636
		0.3	2.052	2.069	2.061	2.038	2.016	1.998
		0.5	1.681	1.674	1.659	1.640	1.624	1.611
3	0.5	0.1	4.205	4.156	4.140	4.149	4.136	4.182
		0.3	3.217	3.162	3.106	3.043	2.958	2.835
		0.5	2.913	2.763	2.751	2.655	2.555	2.464
	0.375	0.1	3.700	3.723	3.768	3.813	3.872	3.899
		0.3	2.554	2.543	2.523	2.485	2.422	2.359
		0.5	2.205	2.175	2.131	2.078	2.026	1.981
	0.25	0.1	3.309	3.407	3.478	3.533	3.576	3.516
		0.3	2.018	2.027	2.023	2.002	1.971	1.942
		0.5	1.658	1.647	1.629	1.594	1.563	1.535

Table 6.3b Normalized stress intensity factors, $K_{II}/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug. $H/W = 2$, $\phi = 180^\circ$, $\theta = 0^\circ$

E_2/E_1	R_1/W	$a/(W - R_1)$	Coefficient of Friction, f					
			0	0.2	0.4	0.6	0.8	1.0
1	0.5	0.1	0.601	0.091	0.009	0.090	0.149	0.318
		0.3	0.125	0.049	0.005	0.040	0.048	0.052
		0.5	0.055	0.005	0.020	0.020	0.020	0.020
	0.375	0.1	0.436	0.064	0.019	0.080	0.122	0.205
		0.3	0.055	0.010	0.045	0.054	0.057	0.059
		0.5	0.017	0.062	0.070	0.068	0.069	0.069
	0.25	0.1	0.095	0.004	0.078	0.129	0.169	0.176
		0.3	0.025	0.085	0.107	0.109	0.112	0.113
		0.5	0.083	0.121	0.128	0.130	0.132	0.132
3	0.5	0.1	0.666	0.135	0.001	0.125	0.238	0.310
		0.3	0.161	0.111	0.023	0.034	0.092	0.110
		0.5	0.099	0.055	0.016	0.019	0.036	0.044
	0.375	0.1	0.507	0.100	0.012	0.110	0.185	0.348
		0.3	0.085	0.023	0.030	0.070	0.082	0.090
		0.5	0.012	0.031	0.061	0.074	0.075	0.074
	0.25	0.1	0.126	0.028	0.064	0.139	0.192	0.252
		0.3	0.005	0.063	0.106	0.126	0.133	0.137
		0.5	0.068	0.106	0.128	0.135	0.138	0.139

Table 6.4a Normalized stress intensity factors, $K_1/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug. $H/W = 2$, $\phi = 270^\circ$, $\theta = 0^\circ$

E_2/E_1	R_1/W	$a/(W - R_1)$	Coefficient of Friction, f					
			0	0.2	0.4	0.6	0.8	1.0
1	0.5	0.1	3.421	3.040	2.515	2.061	1.754	1.700
		0.3	2.956	2.566	2.276	2.042	1.844	1.670
		0.5	2.864	2.499	2.226	2.076	1.902	1.767
	0.375	0.1	3.173	2.750	2.334	1.984	1.762	1.663
		0.3	2.434	2.142	1.945	1.782	1.648	1.539
		0.5	2.237	1.984	1.822	1.692	1.586	1.492
	0.25	0.1	3.000	2.687	2.530	2.121	1.927	1.754
		0.3	2.031	1.830	1.703	1.600	1.512	1.434
		0.5	1.740	1.586	1.488	1.407	1.341	1.285
3	0.5	0.1	3.216	2.964	2.470	2.063	1.716	1.465
		0.3	2.829	2.446	2.171	1.942	1.750	1.604
		0.5	2.766	2.406	2.164	1.968	1.810	1.678
	0.375	0.1	2.998	2.681	2.301	1.928	1.678	1.497
		0.3	2.344	2.047	1.855	1.698	1.575	1.458
		0.5	2.180	1.922	1.754	1.619	1.511	1.423
	0.25	0.1	2.916	2.621	2.326	2.066	1.878	1.711
		0.3	1.995	1.776	1.651	1.539	1.448	1.377
		0.5	1.720	1.563	1.453	1.364	1.293	1.236

Table 6.4b Normalized stress intensity factors, $K_{II}/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug. $H/W = 2$, $\phi = 270^\circ$, $\theta = 0^\circ$

E_2/E_1	R_1/W	$a/(W - R_1)$	Coefficient of Friction, f					
			0	0.2	0.4	0.6	0.8	1.0
1	0.5	0.1	0.010	0.131	0.160	0.181	0.179	0.197
		0.3	0.038	0.069	0.100	0.121	0.138	0.147
		0.5	0.073	0.067	0.084	0.109	0.111	0.121
	0.375	0.1	0.021	0.118	0.159	0.188	0.219	0.217
		0.3	0.049	0.060	0.085	0.106	0.121	0.133
		0.5	0.079	0.065	0.075	0.085	0.093	0.102
	0.25	0.1	0.047	0.132	0.178	0.206	0.234	0.252
		0.3	0.082	0.083	0.101	0.111	0.122	0.134
		0.5	0.100	0.086	0.093	0.098	0.101	0.105
3	0.5	0.1	0.100	0.176	0.234	0.258	0.268	0.283
		0.3	0.037	0.102	0.142	0.168	0.188	0.204
		0.5	0.067	0.084	0.108	0.123	0.133	0.145
	0.375	0.1	0.019	0.184	0.236	0.264	0.294	0.176
		0.3	0.042	0.084	0.115	0.136	0.162	0.181
		0.5	0.066	0.070	0.086	0.099	0.108	0.116
	0.25	0.1	0.047	0.192	0.249	0.291	0.337	0.366
		0.3	0.071	0.097	0.120	0.135	0.148	0.159
		0.5	0.085	0.088	0.097	0.103	0.108	0.112

Table 6.5 Normalized stress intensity factors, $K_I/\{E_1(R_2 - R_1)/R_1\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug. $H/W = 2$, $\phi = 360^\circ$, $\theta = 0^\circ$

E_2/E_1	R_1/W	$a/(W - R_1)$	Coefficient of Friction, f					
			0	0.2	0.4	0.6	0.8	1.0
1	0.5	0.1	0.597	0.497	0.409	0.410	0.410	0.410
		0.3	0.501	0.424	0.382	0.368	0.366	0.364
		0.5	0.465	0.398	0.366	0.351	0.340	0.341
	0.375	0.1	0.493	0.401	0.372	0.351	0.365	0.347
		0.3	0.357	0.301	0.281	0.270	0.262	0.255
		0.5	0.303	0.261	0.242	0.234	0.229	0.230
	0.25	0.1	0.382	0.312	0.292	0.284	0.251	0.275
		0.3	0.224	0.192	0.180	0.176	0.173	0.170
		0.5	0.170	0.149	0.140	0.135	0.134	0.133
3	0.5	0.1	0.694	0.569	0.518	0.478	0.441	0.437
		0.3	0.573	0.486	0.438	0.422	0.412	0.409
		0.5	0.525	0.451	0.413	0.390	0.382	0.376
	0.375	0.1	0.582	0.471	0.427	0.407	0.395	0.384
		0.3	0.416	0.351	0.321	0.309	0.300	0.304
		0.5	0.349	0.301	0.277	0.262	0.255	0.259
	0.25	0.1	0.457	0.369	0.335	0.317	0.323	0.320
		0.3	0.264	0.226	0.209	0.201	0.201	0.200
		0.5	0.198	0.174	0.161	0.156	0.154	0.153

Table 6.6a Normalized stress intensity factors, $K_I/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug subjected to an inclined load, $\phi = 180^\circ$, $f = 0$

θ	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			H/W			H/W		
			2	4	6	2	4	6
22.5°	0.5	0.1	3.819	4.375	4.623	3.667	4.226	4.443
		0.3	3.180	3.484	3.619	3.054	3.356	3.429
		0.5	2.907	3.163	3.272	2.802	3.054	3.159
	0.375	0.1	3.554	3.905	4.042	3.463	3.798	3.948
		0.3	2.590	2.766	2.820	2.522	2.689	2.735
		0.5	2.239	2.143	2.428	2.187	2.083	2.366
	0.25	0.1	3.353	3.560	3.625	3.301	3.493	3.544
		0.3	2.079	2.173	2.200	2.054	2.140	2.164
		0.5	1.698	1.781	1.804	1.682	1.759	1.779
45°	0.5	0.1	3.389	4.828	5.525	3.240	4.720	5.364
		0.3	2.916	3.986	4.417	2.846	3.886	4.274
		0.5	2.727	3.633	4.007	2.675	3.551	3.887
	0.375	0.1	3.240	4.484	4.932	3.207	4.431	4.831
		0.3	2.492	3.238	3.502	2.491	3.190	3.424
		0.5	2.171	2.508	3.022	2.172	2.471	2.964
	0.25	0.1	3.238	4.180	4.489	3.287	4.162	4.442
		0.3	2.087	2.585	2.751	2.119	2.579	2.727
		0.5	1.696	2.116	2.259	1.722	2.115	2.242

Table 6.6b Normalized stress intensity factors, $K_{II}/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug subjected to an inclined load, $\phi = 180^\circ$, $f = 0$

θ	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			H/W			H/W		
			2	4	6	2	4	6
22.5°	0.5	0.1	0.152	0.209	0.705	0.203	0.268	0.768
		0.3	0.247	0.173	0.145	0.266	0.201	0.175
		0.5	0.219	0.106	0.061	0.227	0.130	0.094
	0.375	0.1	0.234	0.192	0.509	0.259	0.192	0.579
		0.3	0.231	0.115	0.082	0.227	0.129	0.100
		0.5	0.170	0.089	0.003	0.160	0.102	0.019
	0.25	0.1	0.274	0.165	0.351	0.280	0.179	0.153
		0.3	0.177	0.041	0.005	0.157	0.041	0.012
		0.5	0.103	0.028	0.062	0.082	0.031	0.057
45°	0.5	0.1	0.153	0.218	0.270	0.142	0.281	0.312
		0.3	0.312	0.282	0.225	0.297	0.314	0.266
		0.5	0.345	0.225	0.135	0.322	0.236	0.163
	0.375	0.1	0.237	0.288	0.255	0.237	0.324	0.279
		0.3	0.383	0.241	0.156	0.363	0.236	0.171
		0.5	0.364	0.186	0.052	0.329	0.182	0.071
	0.25	0.1	0.386	0.297	0.240	0.387	0.272	0.242
		0.3	0.403	0.165	0.072	0.349	0.140	0.063
		0.5	0.321	0.073	0.002	0.271	0.046	0.031

Table 6.7 Normalized stress intensity factors, $K_I/\{E_1\delta_2/R_3\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug due to interference fit of bushing and the lug only:
 $H/W = 2$, $\phi_2 = 360^\circ$ and $\theta = 0^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			f			f		
			0.0	0.5	1.0	0.0	0.5	1.0
1.125	0.5	0.1	0.162	0.107	0.107	0.330	0.227	0.227
		0.3	0.169	0.120	0.108	0.345	0.240	0.216
		0.5	0.182	0.123	0.112	0.366	0.243	0.221
	0.375	0.1	0.128	0.086	0.085	0.276	0.180	0.180
		0.3	0.110	0.080	0.079	0.238	0.163	0.152
		0.5	0.104	0.075	0.070	0.221	0.152	0.141
	0.25	0.1	0.098	0.068	0.067	0.218	0.146	0.143
		0.3	0.065	0.050	0.046	0.144	0.103	0.098
		0.5	0.054	0.041	0.039	0.117	0.085	0.081
1.25	0.5	0.1	0.290	0.230	0.200	0.493	0.364	0.350
		0.3	0.313	0.230	0.208	0.515	0.387	0.342
		0.5	0.347	0.239	0.219	0.554	0.393	0.359
	0.375	0.1	0.227	0.154	0.154	0.406	0.278	0.278
		0.3	0.208	0.147	0.137	0.368	0.256	0.239
		0.5	0.203	0.139	0.132	0.352	0.240	0.226
	0.25	0.1	0.174	0.119	0.119	0.327	0.220	0.217
		0.3	0.124	0.091	0.085	0.230	0.161	0.155
		0.5	0.105	0.077	0.073	0.190	0.135	0.129

Table 6.8a Normalized stress intensity factors, $K_I/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug with a bushing: $H/W = 2$, $\phi_1 = 90^\circ$, $\phi_2 = 360^\circ$, and $\theta = 0^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			f			f		
			0.0	0.5	1.0	0.0	0.5	1.0
1.125	0.5	0.1	32.28	11.87	11.26	20.86	8.33	7.05
		0.3	43.87	13.96	11.20	23.24	8.54	7.10
		0.5	41.06	15.82	11.58	23.36	8.60	6.90
	0.375	0.1	40.23	26.04	15.84	17.52	8.01	6.50
		0.3	80.63	22.26	19.43	20.01	7.41	6.24
		0.5	74.57	21.78	16.92	18.20	7.00	5.66
	0.25	0.1	300.3	47.50	45.27	19.03	8.01	7.02
		0.3	212.9	33.44	32.46	16.75	6.33	5.87
		0.5	46.75	43.87	41.73	12.61	5.24	4.72
1.25	0.5	0.1	31.75	15.38	12.22	20.66	7.95	6.33
		0.3	40.79	15.21	12.09	22.77	8.42	6.07
		0.5	44.76	14.41	13.14	34.84	8.52	7.09
	0.375	0.1	43.33	16.74	15.92	18.61	7.00	5.81
		0.3	38.05	17.16	15.72	19.49	6.95	5.97
		0.5	44.18	18.43	17.10	18.52	6.98	6.71
	0.25	0.1	96.21	32.69	33.69	17.11	8.43	6.88
		0.3	66.36	39.19	33.78	19.69	6.72	5.88
		0.5	55.34	64.53	46.09	16.91	5.79	4.93

Table 6.8b Normalized stress intensity factors, $K_1/\{E_1\delta_2/R_3\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug with a bushing: $H/W = 2$, $\phi_1 = 90^\circ$, $\phi_2 = 360^\circ$ and $\theta = 0^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			f			f		
			0.0	0.5	1.0	0.0	0.5	1.0
1.125	0.5	0.1	0.208	0.141	0.134	0.472	0.337	0.286
		0.3	0.196	0.150	0.133	0.443	0.325	0.279
		0.5	0.196	0.151	0.138	0.445	0.338	0.279
	0.375	0.1	0.149	0.093	0.096	0.375	0.248	0.232
		0.3	0.115	0.088	0.113	0.286	0.212	0.193
		0.5	0.107	0.081	0.077	0.260	0.200	0.175
	0.25	0.1	0.100	0.068	0.067	0.272	0.191	0.167
		0.3	0.066	0.050	0.048	0.168	0.131	0.122
		0.5	0.056	0.042	0.040	0.135	0.112	0.101
1.25	0.5	0.1	0.378	0.256	0.232	0.689	0.530	0.422
		0.3	0.381	0.272	0.245	0.651	0.501	0.433
		0.5	0.373	0.275	0.250	0.622	0.507	0.422
	0.375	0.1	0.258	0.171	0.171	0.532	0.367	0.304
		0.3	0.227	0.163	0.150	0.441	0.315	0.284
		0.5	0.210	0.154	0.143	0.397	0.299	0.304
	0.25	0.1	0.183	0.125	0.128	0.407	0.291	0.246
		0.3	0.126	0.093	0.089	0.258	0.192	0.182
		0.5	0.106	0.078	0.075	0.207	0.166	0.153

Table 6.9a Normalized stress intensity factors, $K_1/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug with a bushing: $H/W = 2$, $\phi_1 \simeq 90^\circ$, $\phi_2 = 270^\circ$ and $\theta = 0^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			f			f		
			0.0	0.5	1.0	0.0	0.5	1.0
1.125	0.5	0.1	4.502	4.683	4.755	4.215	4.339	4.345
		0.3	3.796	3.637	3.544	3.580	3.369	3.297
		0.5	3.506	3.281	3.167	3.352	3.108	2.975
	0.375	0.1	3.864	3.660	3.400	3.660	3.491	3.385
		0.3	2.874	2.711	2.657	2.782	2.575	2.513
		0.5	2.524	2.371	2.351	2.428	2.222	2.196
	0.25	0.1	3.421	3.321	3.330	3.325	3.258	3.323
		0.3	2.177	2.067	2.044	2.136	2.018	1.969
		0.5	1.815	1.721	1.683	1.779	1.702	1.656
1.25	0.5	0.1	4.855	4.706	4.643	4.498	4.480	4.109
		0.3	4.287	4.006	3.785	3.985	3.645	3.424
		0.5	4.126	3.743	3.517	3.984	3.531	3.272
	0.375	0.1	3.963	3.950	3.938	3.704	3.600	3.480
		0.3	3.128	2.888	2.809	2.947	2.684	2.567
		0.5	2.808	2.563	2.462	2.677	2.416	2.300
	0.25	0.1	3.462	3.387	3.414	3.310	3.177	3.091
		0.3	2.333	2.138	2.118	2.253	2.074	1.984
		0.5	1.958	1.820	1.774	1.910	1.765	1.718

Table 6.9b Normalized stress intensity factors, $K_1/\{E_1\delta_2/R_3\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug with a bushing: $H/W = 2$, $\phi_1 \simeq 90^\circ$, $\phi_2 = 270^\circ$ and $\theta = 0^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			f			f		
			0.0	0.5	1.0	0.0	0.5	1.0
1.125	0.5	0.1	2.047	2.091	1.998	1.731	1.908	1.863
		0.3	1.600	1.345	1.272	1.364	1.281	1.225
		0.5	1.294	1.015	0.980	1.149	0.973	0.957
	0.375	0.1	1.867	1.848	1.830	1.536	1.561	1.538
		0.3	1.208	0.926	0.889	0.994	0.868	0.847
		0.5	0.918	0.610	0.610	0.804	0.637	0.625
	0.25	0.1	1.397	1.169	1.177	1.164	1.102	1.101
		0.3	0.752	0.541	0.545	0.636	0.529	0.516
		0.5	0.519	0.377	0.373	0.466	0.362	0.353
1.25	0.5	0.1	1.942	1.962	1.857	1.644	1.814	1.663
		0.3	1.582	1.502	1.352	1.376	1.302	1.223
		0.5	1.375	1.114	1.089	1.186	1.051	1.013
	0.375	0.1	1.817	1.630	1.594	1.508	1.479	1.433
		0.3	1.281	1.005	0.956	1.095	0.962	0.934
		0.5	1.023	0.763	0.715	0.892	0.759	0.723
	0.25	0.1	1.434	1.185	1.195	1.190	1.157	1.104
		0.3	0.778	0.611	0.605	0.697	0.592	0.591
		0.5	0.583	0.438	0.427	0.523	0.441	0.409

Table 6.10 Normalized stress intensity factors, $K_1/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a pin-loaded lug with a bushing: $H/W = 2$, $\phi_1 \simeq \phi_2 \simeq 180^\circ$ and $\theta = 0^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			f			f		
			0.0	0.5	1.0	0.0	0.5	1.0
1.125	0.5	0.1	4.747	5.041	5.897	4.572	5.107	4.815
		0.3	3.926	3.897	3.695	3.764	3.592	3.398
		0.5	3.576	3.442	3.224	3.422	3.233	3.001
	0.375	0.1	3.989	4.459	4.730	3.871	4.342	4.802
		0.3	2.937	2.932	2.900	2.841	2.797	2.682
		0.5	2.546	2.474	2.400	2.460	2.378	2.263
	0.25	0.1	3.488	3.840	3.910	3.419	3.806	3.999
		0.3	2.203	2.239	2.221	2.162	2.154	2.089
		0.5	1.828	1.819	1.800	1.785	1.748	1.692
1.25	0.5	0.1	5.279	5.973	6.622	5.052	5.066	5.080
		0.3	4.538	4.828	4.715	4.334	4.086	3.750
		0.5	4.237	4.073	3.770	4.036	3.747	3.486
	0.375	0.1	4.240	4.745	5.019	4.079	4.573	5.112
		0.3	3.248	3.374	3.291	3.119	3.047	2.861
		0.5	2.859	2.771	2.604	2.744	2.631	2.448
	0.25	0.1	3.602	3.967	4.060	3.516	3.934	4.154
		0.3	2.360	2.401	2.375	2.284	2.300	2.218
		0.5	1.967	1.966	1.948	1.923	1.876	1.809

Table 6.11 Normalized stress intensity factors, $K_I/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a lug with a bushing subjected to inclined pin-load: $\phi_1 \simeq \phi_2 \simeq 180^\circ$ and $\theta = 22.5^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			H/W			H/W		
			2	4	6	2	4	6
1.125	0.5	0.1	4.054	4.791	5.003	3.758	4.463	4.810
		0.3	3.560	3.946	4.076	3.355	3.794	4.001
		0.5	3.349	3.673	3.783	3.159	3.479	3.645
	0.375	0.1	3.667	4.152	4.258	3.485	4.025	4.125
		0.3	2.825	3.040	3.104	2.713	2.934	2.995
		0.5	2.488	2.665	2.718	2.403	2.583	2.633
	0.25	0.1	3.417	3.693	3.752	3.330	3.618	3.671
		0.3	2.210	2.320	2.369	2.169	2.265	2.304
		0.5	1.828	1.927	1.948	1.802	1.882	1.917
1.25	0.5	0.1	4.356	4.933	5.495	3.959	4.765	5.259
		0.3	3.969	4.405	4.566	3.685	4.300	4.580
		0.5	3.854	4.230	4.369	3.624	4.119	4.280
	0.375	0.1	3.780	4.363	4.499	3.529	4.128	4.330
		0.3	3.067	3.315	3.392	2.913	3.200	3.278
		0.5	2.962	2.771	3.026	2.635	2.867	2.932
	0.25	0.1	3.466	3.800	3.868	3.338	3.706	3.789
		0.3	2.347	2.470	2.520	2.287	2.417	2.451
		0.5	1.967	2.069	2.113	1.925	2.035	2.064

Table 6.12 Normalized stress intensity factors, $K_1/\{F/2Wb\}\sqrt{\pi a}$, for a single crack in a lug with a bushing subjected to inclined pin-load: $\phi_1 \simeq \phi_2 \simeq 180^\circ$ and $\theta = 45^\circ$

R_3/R_2	R_1/W	$a/(W - R_3)$	$E_2/E_1 = 1$			$E_2/E_1 = 3$		
			H/W			H/W		
			2	4	6	2	4	6
1.125	0.5	0.1	3.566	5.024	6.040	3.292	4.890	5.844
		0.3	3.208	4.414	4.999	3.068	4.306	4.825
		0.5	3.095	4.148	4.655	2.999	4.049	4.497
	0.375	0.1	3.310	4.634	5.242	3.220	4.567	5.118
		0.3	2.678	3.543	3.857	2.640	3.453	3.736
		0.5	2.388	3.094	3.384	2.361	3.047	3.288
	0.25	0.1	3.262	3.343	4.664	3.275	4.327	4.601
		0.3	2.197	2.747	2.934	2.219	2.724	2.895
		0.5	1.814	2.280	2.441	1.833	2.271	2.409
1.25	0.5	0.1	3.816	5.380	6.548	3.498	4.929	6.293
		0.3	3.562	4.895	5.576	3.325	4.596	5.463
		0.5	3.543	4.747	5.354	3.388	4.507	5.238
	0.375	0.1	3.404	4.713	5.518	3.236	4.687	5.346
		0.3	2.864	3.825	4.239	2.786	3.727	4.074
		0.5	2.617	3.429	3.782	2.561	3.351	3.650
	0.25	0.1	3.278	4.448	4.803	3.265	4.412	4.716
		0.3	2.309	3.140	2.520	2.319	2.892	3.076
		0.5	1.936	2.637	2.113	1.948	2.433	2.591

Table 6.13 Normalized stress intensity factor, $K_I^* = K_I / (S_p \sqrt{4a/\pi})$, for the quarter-circular corner cracks of radius a subjected to pin-loading when the coefficient of friction $f = 0$:

$(K_I^*)_T$ —Traction formula result from BEM contact mechanics analysis;

$(K_I^*)_R$ —BEM linear analysis by using Raju and Newman formula;

$(K_I^*)_U$ —BEM linear analysis using the uniform pressure distribution;

a/t	$2\psi/\pi$	$(K_I^*)_T$	$(K_I^*)_U$	Disc. Δ %	$(K_I^*)_R$	Disc. Δ %
0.2	0	0.474	0.633	+33.5	0.447	-5.7
	0.125	0.526	0.693	+31.7	0.511	-2.9
	0.25	0.510	0.676	+32.5	0.515	+1.0
	0.375	0.528	0.698	+32.2	0.547	+3.6
	0.5	0.559	0.719	+28.6	0.582	+4.1
	0.625	0.617	0.764	+23.8	0.639	+3.6
	0.75	0.706	0.806	+14.2	0.698	-1.1
	0.875	0.908	0.885	-2.5	0.799	-12.0
	1.0	1.195	0.882	-26.1	0.819	-31.5
0.5	0	0.300	0.454	+51.3	0.307	+2.3
	0.125	0.333	0.490	+47.1	0.325	+3.8
	0.25	0.322	0.470	+46.0	0.333	+3.4
	0.375	0.334	0.479	+43.4	0.350	+4.8
	0.5	0.353	0.498	+41.1	0.370	+4.8
	0.625	0.390	0.545	+39.7	0.415	+6.4
	0.75	0.447	0.602	+34.7	0.473	+5.8
	0.875	0.574	0.728	+26.8	0.601	+4.7
	1.0	0.756	0.788	+4.2	0.698	-7.7
0.8	0	0.267	0.407	+52.4	0.279	+4.5
	0.125	0.281	0.421	+49.8	0.295	+5.0
	0.25	0.264	0.393	+48.9	0.276	+4.5
	0.375	0.263	0.389	+47.9	0.280	+6.5
	0.5	0.272	0.399	+46.7	0.287	+5.5
	0.625	0.293	0.435	+48.5	0.317	+8.2
	0.75	0.339	0.495	+46.0	0.363	+7.1
	0.875	0.577	0.655	+13.5	0.504	-12.7
	1.0	0.752	0.790	+5.1	0.648	-13.8

Table 6.14 Normalized stress intensity factor, $K_I^* = K_I / (S_p \sqrt{4a/\pi})$, for the quarter-circular corner cracks of radius a subjected to pin-loading when the coefficient of friction $f = 0.5$:

$2\psi/\pi$	a/t		
	0.2	0.5	0.8
0	0.558	0.329	0.288
0.125	0.624	0.362	0.300
0.25	0.618	0.346	0.277
0.375	0.651	0.355	0.271
0.5	0.701	0.374	0.276
0.625	0.748	0.396	0.286
0.75	0.923	0.490	0.357
0.875	2.919	1.753	1.462
1.0	8.267	4.925	4.052

Figure 6.1 A pin-loaded, symmetric, attachment lug with a single crack.

Figure 6.2a A typical BEM mesh when $\theta = 0^\circ$. $R_1/W = 0.5$ and $a/(W - R_1) = 0.5$.

Figure 6.2b A typical BEM mesh when $\theta = 22.5^\circ$, $R_1/W = 0.5$ and $a/(W - R_1) = 0.5$.

Figure 6.3 Typical positions for the contact node pair: point A and point B.

Figure 6.4 Variation of the normalised contact pressure, $p / \{F / [2(W - R_1)b]\}$, along the hole in the presence of an edge crack; $H/W = 2$, $R_1/W = 0.5$, $a/(W - R_1) = 0.5$, $\phi \simeq 180^\circ$ and $f = 0$.

Figure 6.5a Variation of the normalised stress intensity factor, $K_I / \{[F/(2Wb)]\sqrt{\pi a}\}$, with relative crack length, $a/(W - R_1)$. $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.5$, $\phi = 90^\circ$, $\theta = 0^\circ$, $f = 0$.

Figure 6.5b Variation of the normalised stress intensity factor, $K_{II} / \{[F/(2Wb)]\sqrt{\pi a}\}$, with relative crack length, $a/(W - R_1)$. $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.5$, $\phi = 90^\circ$, $\theta = 0^\circ$, $f = 0$.

Figure 6.6 Variation of the normalised stress intensity factor, $K_I / \{[F/(2Wb)]\sqrt{\pi a}\}$, with the coefficient of friction, f : $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.5$, $\phi = 180^\circ$ and $\theta = 0^\circ$.

Figure 6.7 Variation of the normalised stress intensity factor, $K_I / \{[F/(2Wb)]\sqrt{\pi a}\}$, with the coefficient of friction. f : $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.375$, $\phi = 180^\circ$ and $\theta = 0^\circ$.

Figure 6.8 Variation of the normalised stress intensity factor, $K_I / \{[F/(2Wb)]\sqrt{\pi a}\}$, with the coefficient of friction, f : $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.25$, $\phi = 180^\circ$ and $\theta = 0^\circ$.

Figure 6.9 Variation of the normalised stress intensity factor, $K_I/\sigma_o\sqrt{\pi a}$, with the coefficient of friction f : $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.5$, $\theta = 0^\circ$, $\phi = 270^\circ$.
 $\sigma_o = F/(2Wb)$

Figure 6.10 Variation of the normalised stress intensity factor, $K_I / \sigma_0 \sqrt{\pi a}$, with the coefficient of friction f : $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.375$, $\theta = 0^\circ$, $\phi = 270^\circ$.
 $\sigma_0 = F/(2Wb)$

Figure 6.11 Variation of the normalised stress intensity factor, $K_I / \sigma_o \sqrt{\pi a}$, with the coefficient of friction f : $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.25$. $\theta = 0^\circ$, $\phi = 270^\circ$.
 $\sigma_o = F/(2Wb)$

Figure 6.12 Variation of the normalised stress intensity factor, $K_I / \sigma^* \sqrt{\pi a}$, with the coefficient of friction, f . $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.5$, $\phi = 360^\circ$.
 $\sigma^* = E_1(R_2 - R_1)/R_1$

Figure 6.13 Variation of the normalised stress intensity factor, $K_I / \sigma^* \sqrt{\pi a}$, with the coefficient of friction, f . $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.375$, $\phi = 360^\circ$.

$$\sigma^* = E_1(R_2 - R_1)/R_1$$

Figure 6.14 Variation of the normalised stress intensity factor, $K_I / \sigma^* \sqrt{\pi a}$, with the coefficient of friction, f . $H/W = 2$, $E_2/E_1 = 3$, $R_1/W = 0.25$, $\phi = 360^\circ$.

$$\sigma^* = E_1(R_2 - R_1)/R_1$$

Figure 6.15 A pin-loaded, symmetric, attachment lug with an interference-fit bushing and a single crack.

Figure 6.16a A typical BEM mesh for the pin-loaded lug joint having a bushing:
 $\theta = 0^\circ$, $R_1/W = 0.5$, $R_3/R_1 = 1.25$ and $a/(W - R_3) = 0.3$.

Figure 6.16b A typical BEM mesh for the pin-loaded lug joint having a bushing:
 $\theta = 22.5^\circ$, $R_1/W = 0.5$, $R_3/R_1 = 1.25$ and $a/(W - R_3) = 0.3$.

Figure 6.17 Interfacial pressure distribution, $p/\{F/(2Wb)\}$, along the hole in the presence of a crack at $\varphi = 90^\circ$ when $H/W = 2$, $R_2/W = 0.5$, $R_3/R_1 = 1.25$, $a/(W - R_3) = 0.3$, $\varphi_1 \simeq 180^\circ$, $\varphi_2 = 360^\circ$ and $f = 0$.

Figure 6.18 A plate with two symmetrical corner cracks loaded by a neat-fit pin.

Figure 6.19 A typical LEM mesh for the problem as shown in **Figure 6.18**.

Figure 6.20 Variation of the normalised contact pressure between the pin and the surface of the hole for a pin-loaded plate with symmetric corner cracks: $E_2/E_1 = 3$, $a/t = 0.5$, $f = 0$, $S_p = F/(2Rt)$.

Figure 6.21 Variation of the normalised stress intensity factor, $K_I/S_p\sqrt{4a/\pi}$, with angular position ψ around the periphery of the corner crack in a pin-loaded plate: $f = 0$; $S_p = F/2Rt$.

Figure 6.22 Variation of the normalised stress intensity factor, $K_I / S_p \sqrt{4a/\pi}$, with angular position ψ around the periphery of the corner crack in a pin-loaded plate: $f = 0.5$; $S_p = F/2Rt$.

Chapter 7

Conclusions and Recommendations for Future Work

7.1 Conclusions

The analytical and numerical formulations of the boundary element method (BEM) for fracture mechanics and contact mechanics analysis in 2D and 3D have been reviewed. Special crack-tip elements have been employed to model crack problems, and several "formulas" have been presented which enable stress intensity factors to be evaluated directly from the computed values of displacements or tractions on these elements. Results of the numerical examples presented have re-confirmed the findings of previous studies that the "traction formula" consistently gives very good results which are significantly less sensitive to the mesh design than the "displacement formulas". However, the availability of both sets of formulas provide a useful cross-check of the numerical solutions obtained in a BEM analysis.

The incremental-iterative techniques employed for the BEM contact mechanics analysis have been described and for all the cases analysed, there were no difficulties in the convergence of the solutions with well graded meshes. The results for several 2D and 3D non-crack problems in contact mechanics have also been presented to demonstrate the veracity of the formulations. Very good agreement with available solutions was obtained for the problems treated.

The 2D multi-region BEM for elastic frictional contact mechanics analysis has also been employed to determine the stress distributions in uncracked straight attachment lugs with and without a bushing, subjected to axial loading. Typical contact stress distributions and tangential stress distributions at the periphery of the pin-hole for different extents of contact between the components have been presented for coefficients of friction f ranging from 0 to 1 and for different ratios of the Young's modulus of the pin to that of the lug. The BEM analysis has confirmed that for a given lug configuration and lubrication condition between it and a given pin, the normalised contact stresses and the tangential stresses vary only with the extent of contact between the components. It has also been shown that everything else being equal, the contact stress distribution is not very significantly influenced by the difference between the Young's modulus of the pin and that of the lug, except at relatively high friction coefficients.

In this study, the three-dimensional BEM for frictional contact mechanics analysis has also been applied to treat the same lug as in the 2D case, but with two different lengths of the pin and under neat-fit conditions. It was found that although the contact stresses vary across the thickness of the lug, the tangential stresses remain almost the same across the thickness. Also, these tangential stresses were very close in magnitudes to the corresponding results obtained from the two-dimensional BEM contact mechanics analysis.

The multi-region boundary element method developed for contact mechanics analysis of crack problems has been applied to five angled crack problems involving crack closure in two dimensions. The algorithm has been designed to accurately represent the state of stress near the tip of a crack whose faces may be in contact with one another. Supplementary equations which are required for the BEM analysis have been obtained for nodes common to, for example, three (sub)regions in contact, and an improved solver for the system of linear equations arising from contact fracture mechanics problems has been presented. Stress intensity factor results have been presented for a range of geometric parameters and different coefficients of friction for the five problems treated. The study has shown that the magnitude of the stress intensity factors can be significantly influenced by the frictional conditions between

the cracks faces when they are in contact.

The multi-region boundary element method for 3-D contact mechanics analysis of crack problems has also been developed in this study. Four crack closure problems involving bending of cracked plates, and a cylindrical bar with a circumferential edge crack subjected to pure bending, have been investigated. Stress intensity factor results have been presented for a range of geometric parameters. Because of the symmetry of the problems, the effects of friction on the stress intensity factors were not significant in these examples. There was good agreement between the results obtained from the boundary element analysis and those presented in the literature. The study has also shown that the magnitude of the stress intensity factors can be significantly influenced by crack closure in such problems.

The problems of pin-loaded attachment lugs with either a single through-the-thickness crack or double symmetric corner cracks at the pin-hole have also been analysed in the study. For the single through-the-thickness crack, the analysis was carried out in 2D and was divided into two parts. The first part dealt with a crack in the lug without a bushing, the second part dealt with one which has an interference-fit bushing. Stress intensity factor results have been obtained for several cracked geometries and directions of pin-loading. The effects of friction, the types of fit, thickness of the bushing and the difference in material properties between the pin and the lug have been investigated. For the symmetric corner cracks, a neat-fit or full load assumption was made for the pin-lug assembly. Stress intensity factor results have been obtained from three-dimensional BEM contact mechanics analysis for a few crack sizes. The effects of friction, difference in material properties between the pin and the lug were also investigated. The contact stress distributions along the periphery of the lug in the presence of a through-the-thickness crack or two corner cracks have also been analysed. Where possible, the BEM results were compared with those available in the literature.

The investigation of cracked attachment lugs here has shown that for a given geometry and frictional condition between the pin and the lug, the final normalised elastic contact stresses and the normalised stress intensity factors are again a function of the angular span of contact between the two components. Also, friction has

generally a significant effect on the stress intensity factors as well. The influence of the difference in Young's modulus between the pin and the lug on the stress intensity factors has been found to be relatively small except at high coefficients of friction. Furthermore, the study has shown that significant discrepancies in the stress intensity factor solution can occur between those obtained using proper contact mechanics treatment and the corresponding ones obtained using linear analysis with simplifying assumptions on the load distribution at the pin hole.

The achievements and contributions of the present research may thus be summarized below. They represent further developments on the BEM for treating contact problems in the presence of cracks which have not been previously reported by others in the literature.

- (i) The development and implementation of the BEM formulations for contact mechanics analysis in 2D and 3D capable of treating multi-body frictional contact problems, with or without cracks present.
- (ii) The implementation of special crack-tip elements in both 2D and 3D to represent the proper near-tip fields in the BEM contact formulations for treating fracture problems.
- (iii) The derivation and treatment of the supplementary equations arising specifically from the use of the multi-region BEM technique in the contact mechanics analysis of crack problems. The implementation of these equations in the BEM computer codes is not a straight-forward matter.
- (iv) The development of an efficient solver for the system equations resulting from BEM contact fracture mechanics analysis.
- (v) The determination of new stress intensity factor solutions, using BEM contact mechanics analysis, for a number of angled-crack problems in 2D, and for some crack closure problems in 3D.
- (vi) The determination of new stress intensity factor results for the problem of a pin-loaded lug, with and without a bushing, for a range of geometries, loading and frictional contact conditions.

7.2 Recommendations for Future Work

Finally, some suggestions for future work in BEM contact fracture mechanics are perhaps in order. They are as follows:

- (i) The present work can be extended to include non-linear material behaviour (e.g. elasto-plasticity) and/or dynamic load situations (e.g. impact loading). The former effect is particularly important in, for example, high load conditions with materials of relatively high ductility involved. Under very high loads and impact conditions, it may also be necessary to consider geometric non-linearities as well.
- (ii) The present work can also be extended to investigate the effects of thin-film lubrication between cracked bodies in contact. In this case, the hydrodynamic effects of the lubrication may be important in the load transfer.
- (iii) Although the classical contact algorithm has been used throughout this study, it is worthwhile to investigate further, the potential use of optimisation techniques in BEM contact fracture mechanics analysis.

Appendix A

Shape Functions for Crack Tip Elements

For the quadrilateral eight-node element shown in Figure 2.2, if the nodes 1-5-2 are on the crack front, the displacement and traction shape functions, N_d^c and N_t^c , respectively are as follows: were as below:

$$\begin{aligned} N_d^1(\xi_1, \xi_2) &= \frac{1}{2}(1 - \xi_1) \left[-\xi_1 - (1 - \frac{\sqrt{2}}{2}\xi_1)\sqrt{1 + \xi_2} + \frac{\sqrt{2}}{2}(1 + \xi_2) \right], \\ N_d^2(\xi_1, \xi_2) &= \frac{1}{2}(1 + \xi_1) \left[\xi_1 - (1 + \frac{\sqrt{2}}{2}\xi_1)\sqrt{1 + \xi_2} + \frac{\sqrt{2}}{2}(1 + \xi_2) \right], \\ N_d^3(\xi_1, \xi_2) &= \frac{1}{4}(1 + \xi_1) \left[\sqrt{2}(\xi_1 - 2 - \sqrt{2})\sqrt{1 + \xi_2} + (2 + \sqrt{2})(1 + \xi_2) \right], \\ N_d^4(\xi_1, \xi_2) &= \frac{1}{4}(1 - \xi_1) \left[-\sqrt{2}(\xi_1 + 2 + \sqrt{2})\sqrt{1 + \xi_2} + (2 + \sqrt{2})(1 + \xi_2) \right], \\ N_d^5(\xi_1, \xi_2) &= (1 - \xi_1^2) \left[1 - \frac{\sqrt{2}}{2}\sqrt{1 + \xi_2} \right], \\ N_d^6(\xi_1, \xi_2) &= \frac{1}{2}(1 + \xi_1) \left[(2 + \sqrt{2})\sqrt{1 + \xi_2} - (1 + \sqrt{2})(1 + \xi_2) \right], \\ N_d^7(\xi_1, \xi_2) &= \frac{\sqrt{2}}{2}(1 - \xi_1^2)\sqrt{1 + \xi_2}, \\ N_d^8(\xi_1, \xi_2) &= \frac{1}{2}(1 - \xi_1) \left[(2 + \sqrt{2})\sqrt{1 + \xi_2} - (1 + \sqrt{2})(1 + \xi_2) \right] \end{aligned} \quad (A.1)$$

$$N_t^1(\xi_1, \xi_2) = N_d^1(\xi_1, \xi_2)/\sqrt{1 + \xi_2},$$

$$N_t^2(\xi_1, \xi_2) = N_d^2(\xi_1, \xi_2)/\sqrt{1 + \xi_2},$$

$$N_t^3(\xi_1, \xi_2) = \sqrt{2}N_d^3(\xi_1, \xi_2)/\sqrt{1 + \xi_2},$$

$$\begin{aligned}
N_t^4(\xi_1, \xi_2) &= \sqrt{2}N_d^4(\xi_1, \xi_2)/\sqrt{1+\xi_2}, \\
N_t^5(\xi_1, \xi_2) &= N_d^5(\xi_1, \xi_2)/\sqrt{1+\xi_2}, \\
N_t^6(\xi_1, \xi_2) &= N_d^6(\xi_1, \xi_2)/\sqrt{1+\xi_2}, \\
N_t^7(\xi_1, \xi_2) &= \sqrt{2}N_d^7(\xi_1, \xi_2)/\sqrt{1+\xi_2}, \\
N_t^8(\xi_1, \xi_2) &= N_t^8(\xi_1, \xi_2)/\sqrt{1+\xi_2}
\end{aligned} \tag{A.2}$$

If the nodes 2-6-3 are on the crack front, the shape functions N_d^i and N_t^i are:

$$\begin{aligned}
N_d^1(\xi_1, \xi_2) &= \frac{1}{4}(1-\xi_2) \left[-\sqrt{2}(\xi_2+2+\sqrt{2})\sqrt{1-\xi_1} + (2+\sqrt{2})(1-\xi_1) \right], \\
N_d^2(\xi_1, \xi_2) &= \frac{1}{2}(1-\xi_2) \left[-\xi_2 - (1-\frac{\sqrt{2}}{2}\xi_2)\sqrt{1-\xi_1} + \frac{\sqrt{2}}{2}(1-\xi_1) \right], \\
N_d^3(\xi_1, \xi_2) &= \frac{1}{2}(1+\xi_2) \left[\xi_2 - (1+\frac{\sqrt{2}}{2}\xi_2)\sqrt{1-\xi_1} + \frac{\sqrt{2}}{2}(1-\xi_1) \right], \\
N_d^4(\xi_1, \xi_2) &= \frac{1}{4}(1+\xi_2) \left[\sqrt{2}(\xi_2-2-\sqrt{2})\sqrt{1-\xi_1} + (2+\sqrt{2})(1-\xi_1) \right], \\
N_d^5(\xi_1, \xi_2) &= \frac{1}{2}(1-\xi_2) \left[(2+\sqrt{2})\sqrt{1-\xi_1} - (1+\sqrt{2})(1-\xi_1) \right], \\
N_d^6(\xi_1, \xi_2) &= (1-\xi_2^2) \left[1 - \frac{\sqrt{2}}{2}\sqrt{1-\xi_1} \right], \\
N_d^7(\xi_1, \xi_2) &= \frac{1}{2}(1+\xi_2) \left[(2+\sqrt{2})\sqrt{1-\xi_1} - (1+\sqrt{2})(1-\xi_1) \right], \\
N_d^8(\xi_1, \xi_2) &= \frac{\sqrt{2}}{2}(1-\xi_2^2)\sqrt{1-\xi_1}
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
N_t^1(\xi_1, \xi_2) &= \sqrt{2}N_d^1(\xi_1, \xi_2)/\sqrt{1-\xi_1}, \\
N_t^2(\xi_1, \xi_2) &= N_d^2(\xi_1, \xi_2)/\sqrt{1-\xi_1}, \\
N_t^3(\xi_1, \xi_2) &= N_d^3(\xi_1, \xi_2)/\sqrt{1-\xi_1}, \\
N_t^4(\xi_1, \xi_2) &= \sqrt{2}N_d^4(\xi_1, \xi_2)/\sqrt{1-\xi_1}, \\
N_t^5(\xi_1, \xi_2) &= N_d^5(\xi_1, \xi_2)/\sqrt{1-\xi_1}, \\
N_t^6(\xi_1, \xi_2) &= N_d^6(\xi_1, \xi_2)/\sqrt{1-\xi_1}, \\
N_t^7(\xi_1, \xi_2) &= N_d^7(\xi_1, \xi_2)/\sqrt{1-\xi_1}, \\
N_t^8(\xi_1, \xi_2) &= \sqrt{2}N_t^8(\xi_1, \xi_2)/\sqrt{1-\xi_1}
\end{aligned} \tag{A.4}$$

If the nodes 3-7-4 are on the crack front, the shape functions N_d^c and N_t^c are:

$$\begin{aligned}
 N_d^1(\xi_1, \xi_2) &= \frac{1}{4}(1 - \xi_1) \left[-\sqrt{2}(\xi_1 + 2 + \sqrt{2})\sqrt{1 - \xi_1} + (2 + \sqrt{2})(1 - \xi_2) \right], \\
 N_d^2(\xi_1, \xi_2) &= \frac{1}{4}(1 + \xi_1) \left[\sqrt{2}(\xi_1 - 2 - \sqrt{2})\sqrt{1 - \xi_2} + (2 + \sqrt{2})(1 - \xi_2) \right], \\
 N_d^3(\xi_1, \xi_2) &= \frac{1}{2}(1 + \xi_1) \left[\xi_1 - (1 + \frac{\sqrt{2}}{2}\xi_1)\sqrt{1 - \xi_2} + \frac{\sqrt{2}}{2}(1 - \xi_2) \right], \\
 N_d^4(\xi_1, \xi_2) &= \frac{1}{2}(1 - \xi_1) \left[-\xi_1 - (1 - \frac{\sqrt{2}}{2}\xi_1)\sqrt{1 - \xi_2} + \frac{\sqrt{2}}{2}(1 - \xi_2) \right], \\
 N_d^5(\xi_1, \xi_2) &= \frac{\sqrt{2}}{2}(1 - \xi_1^2)\sqrt{1 - \xi_2}, \\
 N_d^6(\xi_1, \xi_2) &= \frac{1}{2}(1 + \xi_1) \left[(2 + \sqrt{2})\sqrt{1 - \xi_2} - (1 + \sqrt{2})(1 - \xi_2) \right], \\
 N_d^7(\xi_1, \xi_2) &= (1 - \xi_1^2) \left[1 - \frac{\sqrt{2}}{2}\sqrt{1 - \xi_2} \right], \\
 N_d^8(\xi_1, \xi_2) &= \frac{1}{2}(1 - \xi_1) \left[(2 + \sqrt{2})\sqrt{1 - \xi_2} - (1 + \sqrt{2})(1 - \xi_2) \right] \quad (A.5)
 \end{aligned}$$

$$\begin{aligned}
 N_t^1(\xi_1, \xi_2) &= \sqrt{2}N_d^1(\xi_1, \xi_2)/\sqrt{1 - \xi_2}, \\
 N_t^2(\xi_1, \xi_2) &= \sqrt{2}N_d^2(\xi_1, \xi_2)/\sqrt{1 - \xi_2}, \\
 N_t^3(\xi_1, \xi_2) &= N_d^3(\xi_1, \xi_2)/\sqrt{1 - \xi_2}, \\
 N_t^4(\xi_1, \xi_2) &= N_d^4(\xi_1, \xi_2)/\sqrt{1 - \xi_2}, \\
 N_t^5(\xi_1, \xi_2) &= \sqrt{2}N_d^5(\xi_1, \xi_2)/\sqrt{1 - \xi_2}, \\
 N_t^6(\xi_1, \xi_2) &= N_d^6(\xi_1, \xi_2)/\sqrt{1 - \xi_2}, \\
 N_t^7(\xi_1, \xi_2) &= N_d^7(\xi_1, \xi_2)/\sqrt{1 - \xi_2}, \\
 N_t^8(\xi_1, \xi_2) &= N_d^8(\xi_1, \xi_2)/\sqrt{1 - \xi_2} \quad (A.6)
 \end{aligned}$$

If the nodes 4-8-1 are on the crack front, the shape functions N_d^c and N_t^c are:

$$\begin{aligned}
 N_d^1(\xi_1, \xi_2) &= \frac{1}{2}(1 - \xi_2) \left[-\xi_2 - (1 - \frac{\sqrt{2}}{2}\xi_2)\sqrt{1 + \xi_1} + \frac{\sqrt{2}}{2}(1 + \xi_1) \right], \\
 N_d^2(\xi_1, \xi_2) &= \frac{1}{4}(1 - \xi_2) \left[-\sqrt{2}(\xi_2 + 2 + \sqrt{2})\sqrt{1 + \xi_1} + (2 + \sqrt{2})(1 + \xi_1) \right], \\
 N_d^3(\xi_1, \xi_2) &= \frac{1}{4}(1 + \xi_2) \left[\sqrt{2}(\xi_2 - 2 - \sqrt{2})\sqrt{1 + \xi_1} + (2 + \sqrt{2})(1 + \xi_1) \right], \\
 N_d^4(\xi_1, \xi_2) &= \frac{1}{2}(1 + \xi_2) \left[\xi_2 - (1 + \frac{\sqrt{2}}{2}\xi_2)\sqrt{1 + \xi_1} + \frac{\sqrt{2}}{2}(1 + \xi_1) \right], \\
 N_d^5(\xi_1, \xi_2) &= \frac{1}{2}(1 - \xi_2) \left[(2 + \sqrt{2})\sqrt{1 + \xi_1} - (1 + \sqrt{2})(1 + \xi_1) \right], \\
 N_d^6(\xi_1, \xi_2) &= \frac{\sqrt{2}}{2}(1 - \xi_2^2)\sqrt{1 + \xi_1}, \\
 N_d^7(\xi_1, \xi_2) &= \frac{1}{2}(1 + \xi_2) \left[(2 + \sqrt{2})\sqrt{1 + \xi_1} - (1 + \sqrt{2})(1 + \xi_1) \right], \\
 N_d^8(\xi_1, \xi_2) &= (1 - \xi_2^2) \left[1 - \frac{\sqrt{2}}{2}\sqrt{1 + \xi_1} \right]
 \end{aligned} \tag{A.7}$$

$$\begin{aligned}
 N_t^1(\xi_1, \xi_2) &= N_d^1(\xi_1, \xi_2)/\sqrt{1 + \xi_1}, \\
 N_t^2(\xi_1, \xi_2) &= \sqrt{2}N_d^2(\xi_1, \xi_2)/\sqrt{1 + \xi_1}, \\
 N_t^3(\xi_1, \xi_2) &= \sqrt{2}N_d^3(\xi_1, \xi_2)/\sqrt{1 + \xi_1}, \\
 N_t^4(\xi_1, \xi_2) &= N_d^4(\xi_1, \xi_2)/\sqrt{1 + \xi_1}, \\
 N_t^5(\xi_1, \xi_2) &= N_d^5(\xi_1, \xi_2)/\sqrt{1 + \xi_1}, \\
 N_t^6(\xi_1, \xi_2) &= \sqrt{2}N_d^6(\xi_1, \xi_2)/\sqrt{1 + \xi_1}, \\
 N_t^7(\xi_1, \xi_2) &= N_d^7(\xi_1, \xi_2)/\sqrt{1 + \xi_1}, \\
 N_t^8(\xi_1, \xi_2) &= N_d^8(\xi_1, \xi_2)/\sqrt{1 + \xi_1}
 \end{aligned} \tag{A.8}$$

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