SNR-Adaptive Constellation Design for Convolutional Codes

by

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Abstract

The development of 5G communication technology has introduced new families of applications and use cases during just its standardization and some of them have brought peculiar system requirements, where existing channel coding in its current form might suffer from tackling them. As an example of this, it can be shown that deploying one of the powerful channel coding techniques might suffer in tackling stringent delay requirements while sustaining higher reliability in some mission-critical applications due to iterative decoding structure. Besides, an error-floor region might prevent turbo-coded systems from deploying in certain use cases. From this perspective, a channel coding technique, while currently less popular than the others, might have considerable potential over 5G and beyond after optimizing its modules.

Motivated by this fact, this thesis aims to present an SNR-adaptive convolutionally coded transmission model where the simplicity of a convolutional encoder has combined with current advanced optimization ability. Basically, the proposed transmission model combines one-shot decoding superiority of convolutional coder with the utilization of choosing an optimized group of symbol points, which are obtained specifically for a given convolutional encoder, channel characteristic and transmission model. The enabler of an SNR-adaptive optimization framework is the derivation of upper bound error performance expression by exploiting the product-state matrix technique, which is used in the calculation of generating function for a given convolutional encoder and it brings the superiority to work with any type of constellations. The importance of including fully arbitrary, irregular type, constellations in constellation search lies on that the uniform constellations can lead to suboptimal performance in many coded scenarios including bit-interleaved coded modulated and turbo-trellis coded modulations as already shown.

As an initial step towards the proposed SNR-adaptive transmission model, a generalized error performance calculation was first presented. Once optimized symbol

point locations via particle swarm constellation optimizer are found for each operating point, the performance comparison of the proposed SNR-adaptive convolutionally coded transmission model with SNR-independent counterparts is given in terms of BER and spectral efficiency. The superiority of the proposed transmission model, in terms of algorithmic complexity and decoding latency, is also presented.

I dedicate this thesis to my wife, Gozde, who has been with me all these years and has made them the best years of my life..

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Nomenclature

List of acronyms

5G	fifth generation
AMC	adaptive modulation coding
ARQ	automatic repeat request
AWGN	additive white Gaussian noise
BER	bit error rate
BICM	bit interleaved coded modulation
CDF	cumulative distribution function
CoMP	coordinated multi-point
DL	downlink
DVB	digital video broadcasting
DVB-T2	digital video broadcasting-second generation terrestrial
E2E	end-to-end
eMBB	enhanced mobile broadband
FEC	forward error correction
HARQ	Hybrid automatic repeat request
HSPA	high speed packet access
I/O	input-output
IoT	Internet of things

LDPC	low-density parity-check
LTE	long-term evolution
MAP	maximum a posteriori probability
MCS	modulation and coding scheme
MGF	moment generating function
MIMO	multiple input multiple output
mMTC	massive machine type communication
NOMA	non-orthogonal multiple access
OES	odd-even separator
PDF	probability distribution function
PSO	particle swarm optimization
QAM	quadrature amplitude modulation
QR	quasi-regular
QPSK	quadrature phase shift keying
SE	spectral efficiency
SCMA	sparse code multiple access
SISO	signal input single output
SNR	signal-to-noise ratio
TMRC	transmit maximum ratio combining
TCM	trellis coded modulation
TTCM	turbo trellis coded modulation
UPA	uniform planar antenna
URLLC	ultra-reliable low latency communications

List of symbols

$\mathbf{1}$	unity matrix
${}_1F_1(\cdot, \cdot; \cdot)$	Kummer confluent hypergeometric function
$G_{p,q}^{m,n} \left[z \middle \begin{matrix} (\alpha_1, \dots, \alpha_p) \\ (\beta_1, \dots, \beta_q) \end{matrix} \right]$	Meijer's G function
ρ	correlation coefficient
Ω	average fading power(or path loss)
λ	eigenvalues
$\Gamma(\cdot)$	gamma function
γ	instantaneous received SNR value
$\bar{\gamma}$	average received SNR value
$\mathcal{I}_l$	l th order modified Bessel function
$\chi(m, \bar{\gamma})$	constellation specific for m and γ
Π_s	symbol based interleaver
Π_s^{-1}	symbol based inverse interleaver
$\mathcal{B}$	bad state
$\mathcal{D}$	dummy variable in generating function calculation
b	input bits
$b(\cdot)$	mapping function
c	coded bits
D	delay element
$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$	transition probability $(u, v) \rightarrow (\bar{u}, \bar{v})$
d	Euclidean distance
$f_x(x)$	probability distribution function of variable x

$\mathcal{G}$	good state
$g(\cdot)$	bit-to-symbol mapping
h	channel coefficient
$\mathcal{I}$	dummy variable in generating function calculation
$\mathbf{I}$	identity matrix
$\text{Im}\{\cdot\}$	imaginary part
k	number of input bits
m	Nakagami- m fading parameter
N	number of states for given encoder
N_0	noise variance
$N^{(k)}$	the number of antenna for k th transmission stage
N_b	number of bits per frame
N_s	number of symbols per frame
n	number of output bits
$n_{i,j}$	noise term
P_b	bit error rate
R	code rate
r	received signal
$\text{Re}\{\cdot\}$	real part
$\mathbf{S}$	product-state matrix
s	output symbol
$T(\mathcal{D}, \mathcal{I})$	generating function
t	time index
u	initial state for a transition in the transmitter
v	final state for a transition in the transmitter

- x output symbols of TCM encoder
- $\bar{u}$ initial state for a transition in the receiver
- $\bar{v}$ final state for a transition in the receiver
- w Hamming weight

Chapter 1

Motivation and Contributions

This thesis is mainly concerned primarily with the development of an SNR-adaptive optimized convolutionally coded transmission model; which has certain superiorities over currently used coded scenarios in the context of low-complexity communications which have become a necessary characteristic on certain use cases in 5G design proposals. Instead of taking a direct step through presenting the proposed design directly, the suitability, strength and attractiveness of the proposed model are stressed out along with literature reviews of the related topics to the proposed design.

1.1 A brief history of constellation design

Constellation design is one of the most fundamental problems in digital communications. As a matter of fact, the history of constellation design goes back even before Shannon's landmark work [2]. Having said that, the constellation design has become a more structured problem only after the signal space analysis became the standard design tool with the seminal textbook of Wozencraft and Jacobs [3]. A good overview of constellation design studies until the 1980s can be found in [4], [5] and the references therein.

Interestingly, there has been a rejuvenated interest in this most classic digital communications problem in recent years since the more advanced optimization tools have become increasingly available to physical layer researchers; in particular, another seminal textbook [6] played an important role here. In the absence of optimization as a tool, most of the earlier constellation design work was confined to cases in which the possible signal point locations were on regular lattices [7] or which had other restrictive constraints. With the utilization of the optimization techniques, the constellation

design problem has increasingly been studied over the wireless systems [8–10].

Gradient-based algorithms to search optimum two-dimensional constellation over the Gaussian channels under the high SNR assumption are presented in [7] and it yields around a half dB gain compared to conventional QAM constellation for uncoded scenarios. Inspiring the behaviour of charged electrons in free space, multidimensional constellation design for uncoded scenarios is given in [11] and the same problem is taken into account in [8] that is based on maximizing Euclidean distance between signal points in the form of non-convex quadratically constrained quadratic problem. Another optimization framework with the existence of phase noise is given in [9].

Besides working with 2-D constellations, use of multidimensional constellations is proposed due to the need for better SNR efficiency, the fractional number of bits per two dimensions and already used multidimensional coded modulation [12–14]. Meanwhile, larger dimensions mean worse spectral efficiency but it can be compensated by increasing constellation size M [8]. Therefore, it is aimed to place as many signal points as possible into given N -dimensional space so lattice-based code design is proposed in [4, 15] since the lattice is the asymptotically densest structure for any given dimension size, N .

Besides mentioned studies, constellation design problem has been also considered as a joint design problem of channel coding techniques and constellations. The combination multidimensional constellation design and trellis coded modulation (TCM) brings important advantages like a smaller constituent constellation, simplified Viterbi decoder and more tolerance to phase ambiguities [16, 17]. Systematic approach about how to select multidimensional constellations, how to apply set partitioning process and how to bit to symbol mapping rule is represented in [16, 17]. After outperforming TCM coded system with more flexible encoder structure over fading channels, constellation design framework can be seen in bit-interleaved coded modulation (BICM) coded scenarios [18–20]. Also, optimization framework to maximize channel capacity for other channel coding techniques like low-density parity check code (LDPC) is represented in [21].

Sometimes instead of having a complete picture of constellation design in one study, performance evaluation framework which is valid for any type of constellations can trigger constellation design studies following. For instance, [22] offered a numerically efficient method for evaluating bit error rate expression over the uncoded scenarios. Considering its advantage of working with any type of constellations, this

method was later exploited by [23] where optimized rotated constellations varying with receiver statistic were found by adopting the method in [22] to two-way relaying model given in [23]. Another good example can be found in [24] where space-shift keying was proved to be better than conventional MIMO systems; then, it inspired constellation design framework proposed in [25].

1.2 The popularity of non-uniform constellations

The advancement in computing ability makes the researchers construct a design problem without any predefined assumption on symbol locations since most current optimization frameworks have the ability to work with very large search space without any hardware and software limitations. Within this direction, the examples of non-uniformly located signal points have emerged in the deployment of the recent wireless standards. For instance, ATSC 3.0 might be the first major broadcasting standard where non-uniform constellations are utilized [26]. Also, it was recently proven in [27] that the usage of uniformly spaced constellations can cause suboptimal coded systems in existing wireless communication standards, e.g., HSPA, IEEE.802.11.a/g/n, DVB-T2, etc. [28] shows that the difference from Shannon channel capacity tends to increase with larger SNR values in the BICM systems where uniform constellations are used. As a quick proof of tendency using various non-uniform constellations in the literature, Fig. 1.1 shows some snapshots of proposed constellations in the use of LDPC and BICM encoders.

Hierarchical constellations, multi-resolution constellations, have been received great interest since it enables transmitting different information blocks embedded into the same non-uniformly distributed constellation [29] and it has been used Qualcomm's MediaFLO and also standardized in DVB-T2 [19]. Then, constellation design studies where the optimization framework is carried out for the purpose of finding optimal symbol point locations for a given channel model can be found in [30, 31]. Considering the ability to store any dimension of symbol point locations in bigger look-up tables, there might be a reconsideration of many design problem by combining the derivation of generalized performance expressions with modern optimization techniques, and it will introduce more appearance of the non-uniform constellations in 5G and beyond networks.

Figure 1.1: Some irregular constellations examples proposed for coded scenarios [28, 32].

1.3 SNR-adaptive constellations

In addition to the current trend of employing optimized non-uniform constellations in many coded systems and wireless standards, SNR-adaptiveness is another important concept in shaping the proposed convolutionally coded model. The necessity of working with a group of constellations rather than single one stressed out after realizing the failure of strong simplex conjecture, which was commonly adopted in many years before its disproof [33]. According to the simplex conjecture, the regular simplex constellation was considered as the unique constellation which maximizes the minimum distance between the symbol points for all SNR values [34]. However, it was proven in [33] that maximizing the minimum distance does not necessarily result in having the optimum constellation for all SNR ranges over uncoded scenarios. It brings the idea of the necessity of using different constellations based on the system parameters and no fixed constellation is optimal for all SNR values.

However, it would be better to clarify how the term of SNR-adaptiveness in constellation choice differs from conventional adaptive modulation coding techniques which were widely used in wireless networks. The concept of “adaptiveness” is already widely adopted in edge wireless communication systems in order to adjust the modulation order and coding rate against the variation seen in link quality from time to time [35]. Designing rules for determining modulation order and coding rate based on different channel conditions can be found in [36, 37]. In this thesis, aside from conventional adaptive modulation and coding schemes, the proposed one allows working with different constellations even for that selected coding rate and the modulation order stay the same. There are some examples proposed over fibre optical channels

Figure 1.2: SNR-adaptive constellation examples proposed in [39] for fiber optical channels.

where the evolution of the constellations along with different channel statistics [38, 39]. The snapshots of different constellations varying with different power values presented in [39] are shown in Fig. 1.2. Compared to optical networks, the use of different constellations in wireless systems is slightly limited. SNR-adaptive constellation design over BICM coded systems is given in [40]; however, an exhaustive search algorithm is exploited to find optimized constellations in valid range search space along with predefined step size. To the best of our knowledge, the novelty of this thesis lies on the approach we adopt which applies a comprehensive optimization framework over convolutionally coded scenarios by taking into account encoder type, coding rate, and channel conditions without any predefined constraint on the symbol point locations.

1.4 Low-complexity communications

The selection of channel coding technique for a given scenario is determined by a group

of various design criteria including error performance, latency, flexibility and implementation complexity. Considering current forecast in the diversity of wireless networks, different radio access technologies can be implemented over future wireless systems. Previously, reducing the error rate was the primary goal of the channel coding techniques for several decades. The invention of the capacity-approaching/achieving codes such as the polar, LDPC, and turbo codes in the last two decades been among the key developments in this field resulting from this motivation at the expense of complex encoder/decoder structures. However, it might not possible to deploy these powerful coding techniques in any given scenario by considering budget constraint, latency requirement and the cost of implementation complexity. This trend can easily be seen in machine centric communication where the superiority of capacity-approaching/achieving codes might disappear in these cases. For example, in these scenarios, the mentioned powerful coding techniques suffer from meeting certain decoding latency constraints [41, 42] and the advantage of convolutional encoders over capacity-approaching codes was presented in the presence of strict delay constraints in [43].

Following the same manner, it can be said that it is not cost-effective to employ a polar coded for the energy-hungry applications where encoder stays active for a limited number of packet transmission so performance improvement can be reached out by designing the constellation with respect to the encoder and channel conditions. Under the umbrella of the Internet of Things (IoT) conjecture, wireless communication scenarios can span large geographical areas and each area can be exposed to its own path-loss model [44]. Rather than relocating a group of meters which causes some performance penalty, the compensation or even improvement can be yielded by using the designated constellations for such a system.

From this point of view, choosing a channel coding which gives the best error performance might not be possible in some applications regarding design parameters especially when implementation complexity is one of the existing concern. Therefore, optimizing the available system parameters for low-complex channel coding scenarios has become more important than before to make them more attractive for the deployment. From this point of view, convolutional encoders with a good design can be a potential candidate in many use cases in 5G and beyond networks considering its low implementation complexity with considerable gain from the use of proposed SNR-adaptive constellation design. Motivated by this perspective, this thesis focuses

on obtaining higher transmission reliability via simpler channel coding structure, with convolutional encoders, on purpose of satisfying lower system complexity by deploying optimized a group of constellations to obtain optimal performance at each operating point. In addition to low-complexity, lower decoding latency can be obtained from non-iterative decoding structure of convolutional decoding as compared with existing iterative decoding options.

1.5 Motivation

The motivation behind this thesis can be summarized as follows:

- **Convolutional encoders might be more popular in the near-future than today.** We aimed to strengthen the idea of the potential of using optimized SNR-adaptive constellations over convolutionally coded scenarios by emphasizing the trending low latency requirements for the next generation wireless standard. There might be a considerable potential of deploying convolutional encoders with a good design where the advantage of convolutional encoding resulting from non-iterative/one-shot decoding properties can be combined with SNR-adaptive constellation design.
- **Constellation design, or constellation shaping, mostly concentrated on uncoded and powerful error correcting codes.** Constellation design is one of the most fundamental problems in digital communications; as a matter of fact, the history of constellation design goes back even before Shannon's landmark work. In the absence of optimization as a tool, most of the earlier constellation design work has been confined to cases in which the possible signal point locations are on regular lattices or which have other restrictive constraints. With the utilization of the optimization techniques, the constellation design problem has increasingly been studied through by utilizing modern optimization techniques. However, most constellation design frameworks mentioned above concentrated on uncoded cases and cases where powerful error correcting codes are deployed. From this point view, this thesis might be the first one to implement optimization framework over convolutionally coded systems with the motivation of deployment of them in the use cases where certain implementation complexity and latency constraint exist.

- **Constellation design without any predefined constraint on the symbol locations requires utilizing the product-state matrix technique.** We aim to implement constellation design over convolutionally coded scenarios without any predefined constraint on symbol point locations. For convolutionally coded scenarios, in general, the all-zero sequences are assumed to be transmitted in the calculation of the generating function. However, it should be noted that this method can only be applicable to QR cases, while many systems can be found to be irregular, especially when encoders are paired with irregular and non-uniformly spaced constellations. Since QR is not associated with either better or worse system performance; the product-state matrix technique is chosen in error bound calculation to consider all possible candidate constellations without excluding any of them.
- **More comprehensive adaptive modulation and coding schemes are offered differently than existing ones.** To achieve peak data rates and spectral efficiencies, adaptive modulation and coding (AMC) schemes have been quite a popular technique in wireless networks for decades. The basic idea of AMC schemes is adjusting transmission power levels, coding rate and modulation order based on channel information which includes average SNR value in most cases. In this thesis, the proposed SNR-adaptive convolutionally coded transmission model has introduced generalization different from conventional AMC schemes, where the proposed one allows working with different constellations even for that selected coding rate and the modulation order stay the same.

1.6 Contributions

In the first part of this thesis, we derived error performance expression for two different convolutionally coded systems which can work with any type of constellation. Finding an error performance metric which stays valid for any possible encoder and constellation pair is a pivotal step to construct SNR-adaptive design framework in the later part of the thesis. Specifically, the error performance analysis for convolutionally coded scenarios is mainly based on the derivation of the generating function which is calculated from the state transition diagram of the encoder and it includes all possible error events. In general, all-zero sequences are assumed to be transmitted over

the calculation of generating function. However, this method can only be applicable to QR cases [45], while many systems can be found to be irregular, especially when encoders are paired with higher modulation order and irregular constellations [46, 47]. Therefore, having a performance expression, which includes irregular cases in addition to QR ones, is an essential step to construct constellation design problem where there might be better constellations which make the encoder irregular and yield better performance as compared to QR cases.

Starting from analytical framework which enables working with any type of constellation for convolutionally coded systems, the contributions of this thesis can be summarized as follows:

1.6.1 Performance analysis for irregular convolutionally coded single transmit antenna systems

To begin with, the derivation of a very tight BER upper bound for a convolutionally coded, two-transmission system (modelling CoMP, HARQ or relaying) operating in a Nakagami- m fading environment is derived for any encoder-constellation pair as long as a generating function can be written. Unlike previous approaches, the proposed method calculates the generating function based on the product-state matrix technique which does not require the constellation and encoder to satisfy the quasi-regularity.

In each of the two transmissions, the same information is sent, but through different constellations whose points can be located anywhere on the 2D space; the only two restrictions are that the average symbol energy is kept fixed as well as the constellation size (i.e., the same fixed number of bits per symbol) in both transmissions. The underlying motivation is that the explicit dependence of distances in the upper bound BER expression can be exploited towards determining good constellations. The contributions belonged by this part can be listed as follows:

- We derived the error bound expressions both for independent and correlated transmission scenarios. The existence of fading correlation over two transmission stages is an important consideration in the analysis, as it can arise for example retransmission within the coherence time of the channel (HARQ).
- We employed different and arbitrarily chosen 2D constellations during two orthogonal transmission stages while we kept the same channel encoder.

- We considered both independent and correlated (between transmission stages) Nakagami- m channels, which may not necessarily be identically distributed (i.e., different path-losses, fading parameters).
- We provided extensive simulation results to investigate the tightness of the developed BER expressions in a wide range of scenarios; randomly generated non-uniform constellations.
- The derived error bound expression can be exploited through optimized irregular constellations search.

1.6.2 Performance analysis for irregular convolutionally coded multiple transmit antenna systems

Then, performance analysis for an irregular convolutionally coded system is applied to a transmit maximum ratio combining (TMRC) system. Specifically, the mentioned model consists of multiple orthogonal transmission stages and each transmission stage can employ multiple transmit antennas. This differs from two transmission stages along with a single antenna case described in 1.6.1. There is a big difference in terms of complexity between the transmitter and the receiver side and this asymmetric complexity distribution between the transmitter and the receiver parts is inherent to the Internet of Things (IoT) ecosystem, where robustness of transmitter units against to failure and poor performance is required with increasing network intelligence. Resulting from multiple transmit antenna usage, a proper combining scheme is required. From this aspect, the transmit maximum ratio combining (TMRC) technique [48, 49] was selected to use at the transmitter side during each transmission stage to maximize the received SNR. Using new results for the statistics of sum of gamma RVs [50], we developed compact expressions for the calculation of generating function in such systems. Furthermore, we considered a very flexible system model where each stage can employ a different number of transmit antennas; where each stage can use different irregular constellations, and have different spatial correlation models, along with arbitrarily chosen Nakagami- m fading parameters; and where each stage can be modelled by co-located or distributed transmit antenna structures. The contributions belonged by this part can be listed as follows:

- We developed error performance bounds for coded systems with completely

arbitrary signal constellations with an arbitrary number of the transmissions along with arbitrary transmitter antennas for each phase.

- We considered both correlated (between the transmit antennas in each phase) and independent fading scenarios.
- Each stage can be modelled by co-located or distributed transmit antenna structures.

1.6.3 Performance analysis for irregular turbo-trellis coded systems

In addition to the presented analytical framework used for convolutionally coded systems with the existence of irregular constellations, a proper extension of the given analysis into more powerful coding technique has a considerable potential for being used in current high capacity channel coding techniques. Turbo-trellis coded modulation (TTCM) encoders emerged as a powerful coding technique which combines the powerful design of the binary-turbo coding [51] with larger constellation sizes [52].

A typical error performance of turbo coded systems tends to fall into two different characteristics which are named as waterfall region and error-floor region [51]. Considering the performance criteria inside the waterfall region is based on exit chart analysis [53], the error performance analysis which considers the irregular constellations is given only for the error-floor region. As proposed in convolutional coder case, the same analysis was extended to TTCM scenarios in which the convolutional encoders are used as the constituent codes.

1.6.4 SNR-adaptive optimized convolutionally coded model

As mentioned before, deriving error performance expression which enables working with any type of constellation is an essential part of this thesis but not the main focus. The derived expressions are seen as enablers to design coded systems which employ a set of optimized irregular constellations, which are obtained by using optimization formulation resulting from derived expressions.

Since the proposed system brings the idea of the use different irregular optimized constellations as SNR varies, the optimizer needs to perform a search for a specific range of SNR values and fading parameters. In order to search for optimized symbol

points locations, the optimizer uses a meta-heuristic evolutionary technique which is based on swarm intelligence. Specifically, particle swarm optimization is used to seek out optimized symbol point locations since it requires less computational load compared to other optimization methods and it has fewer tuning parameter as compared to the other evolutionary algorithms.

After the optimized irregular constellations are found for different average SNR values, they are stored in the look-up tables. From numerical results, it is observed that the more gains can be obtained with higher modulation order and spectral efficiency gains can be found which can vary with the modulation levels and channel characteristics for different convolutional encoders.

1.6.5 Complexity and decoding latency considerations

Even considerable performance gain is obtained from SNR-adaptive constellation design for convolutionally coded cases, it would be incomplete framework if no discussion was addressed with respect to implementation complexity of convolutionally coded system. Since the motivation of constructing mentioned framework into convolutionally coded models results from being an alternative solution in low-complexity communications where higher reliability might not satisfy network expectations alone. From this perspective, the algorithmic complexity of proposed SNR-adaptive convolutionally coded transmission model with the other powerful error correcting codes are represented to underline the advantage of convolutional coders for the use cases where low-complexity is desired as much as higher reliability. Another advantage of SNR-adaptive design lies on reaching lower decoding latencies in the process of obtaining predefined error performance metric threshold as compared with the same scenario without any constellation design. The gains in this direction are shown and the decoding latency comparisons with TTCM and LDPC coded scenarios are given as well.

1.7 Published, accepted, and submitted manuscripts

The following is a list of the publications produced during the time in enrolled PhD program.

1.7.1 Journal papers – published and accepted

- **M. C. Ilter**, P. A. Dmochowski, and H. Yanikomeroglu, “Revisiting error analysis in convolutionally coded systems: The irregular constellation case”, *IEEE Trans. Commun.*, vol. PP, no. 99, DOI: 10.1109/TCOMM.2017.2761382.
- **M. C. Ilter**, H. Yanikomeroglu and P. A. Dmochowski, “BER upper bound expressions in coded two-transmission schemes with arbitrarily spaced signal constellations,” *IEEE Commun. Lett.*, vol. 20, no. 2, pp. 248-251, Feb. 2016.

1.7.2 Journal papers – under review

- **M. C. Ilter** and H. Yanikomeroglu, “Optimized and SNR-adaptive constellations for convolutional codes in low-latency communications”, under review in *IEEE Commun. Lett.* (submission: 17 December 2017).
- **M. C. Ilter** and H. Yanikomeroglu, “Optimized and SNR-adaptive constellation design for convolutional codes in low-latency communications”, under review in *IEEE Trans. on Veh. Technol.* (submission: 30 November 2017).

1.7.3 Conference papers – published

- **M. C. Ilter**, P. A. Dmochowski, and H. Yanikomeroglu, “Arbitrary constellations with coded maximum rate transmission over downlink Nakagami-m fading channels”, in *Proc. IEEE Veh. Tech. Conf. (VTC-Fall)*, Sep. 2016, Montreal, QC, Canada.
- **M. C. Ilter** and H. Yanikomeroglu, “An upper bound on BER in a coded two-transmission scheme with same-size arbitrary 2D constellations”, in *Proc. IEEE Int. Symp. on Pers., Indoor and Mobile Radio Commun. (PIMRC)*, Sep. 2014, Washington, DC, USA, pp. 687-691.

1.7.4 Journal papers on other research topics

- T. Aldalgamouni, **M. C. Ilter**, and H. Yanikomeroglu, “Joint power allocation and constellation design for cognitive radio systems, accepted to *IEEE Trans. on Veh. Technol.* (acceptance: 29 December 2017).

- H. U. Sokun, **M. C. Ilter**, S. Ikki, and H. Yanikomeroglu, “A spectrally efficient signal space diversity-based two-way relaying system”, in *IEEE Trans. on Veh. Technol.*, vol. 66, no. 7, pp. 6215–6230, Jul. 2017.
- **M. C. Ilter**, H. U. Sokun, and H. Yanikomeroglu, “The interplay between fading severity, transmit power, and rotation angle in signal space diversity-based two-way relaying networks, under review in *IEEE Trans. on Veh. Technol.* (submission: 23 April 2017).

1.7.5 Conference papers on other research topics

- T. Aldalgamouni, **M. C. Ilter**, O. Badarneh, and H. Yanikomeroglu, “Performance analysis of Fisher-Snedecor F composite fading channels”, under review in *IEEE Middle East and North Africa Communications Conference*, 18-20 April 2018, Jounieh, Lebanon.
- H. U. Sokun, **M. C. Ilter**, S. Ikki, and H. Yanikomeroglu, “A signal space diversity based time division broadcast protocol in two-way relay systems”, in *Proc. IEEE Glob. Commun. Conf. (GLOBECOM)*, Dec. 2015, San Diego, CA, USA, pp. 1-6.

1.8 Organization

The rest of the thesis is organized as follows:

- In Chapter 2, the key definitions and the basic concepts related with the rest of the thesis are introduced in order to provide a quick introduction and guideline for the terminology and key components.
- The analytical frameworks in the derivation of performance bound expression are presented in Chapter 3. Specifically, the error performance calculation for two-orthogonal transmission stages one-transmit antenna system is first presented; then, the similar framework is carried out for more generalized system model. In addition to working on a more generalized system model, the extension of proposed analytical framework to the turbo-trellis coded cases are presented in this chapter. All derived expressions are validated via Monte Carlo simulations.

- In Chapter 4, SNR-adaptive convolutionally coded transmission model is presented along with error performance comparison curves.
- The comparisons of SNR-adaptive convolutionally coded transmission system with SNR-independent convolutionally coded system, TTCM and LDPC coded systems, are given in terms of implementation complexity and decoding latency in Chapter 5.
- In Chapter 6, final discussion about overall thesis is addressed and open research items are also given for future reference.
- In Appendix, an implementation of SNR-adaptive framework into an uncoded signal space diversity (SSD)-based two-way relaying system is presented.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, the three mainstreams of 5G standardization are first introduced. Then, starting from convolutional encoders, brief descriptions of commonly used channel coding techniques are introduced and they are followed by the section about how to classify a given convolutional encoder, the conditions of being quasi-regular. Then, the effects of the constellation on quasi-regularity is discussed over some illustrative examples. The chapter concludes with describing the product-state matrix technique which is a generalized version of mostly used conventional error performance analysis.

2.2 Current forecast in new cellular technology

Starting with 5G developments, the wireless air interface has seen revolutionary steps and completely novel enterprise use cases have been brought out in network architecture. Although performance parameters of the next generation mobile network have not been set yet in these days, a new generation is certainly going to introduce tens and thousand times higher data rates as compared to current LTE networks. Specifically, ten thousand times higher network capacity coupled with less than a millisecond latency has been targeted in mobile broadband services [54].

Based on different network requirements, 5G network access technology was divided into three broad categories [55], which are

- enhanced mobile broadband (eMBB) communication,
- massive machine type communication (mMTC),

- ultra-reliable low latency communications (URLLC).

The fundamental requirements for each category are summarized as in the following:

2.2.1 Enhanced mobile broadband communication

Enhanced mobile broad refers the most critical user demanding, human-centric, communication in the context of 5G networks. The eMBB scenarios focus on more user experienced data rate what we have today, which aims 100 Mbps at cell edge and 10 Gbps peak data rate [56]. In addition to higher data rates, a wider range of code rates and modulation orders are expected in the use cases. Based on the current assumption, the code length in this category can vary from 100 bits to 8000 bits (optionally, 12000-64000 bits) and the code rate is expected to range from 1/5 to 8/9 [57]. LDPC and polar codes seem to contend with each other for taking a role in the deployment of these scenarios [58].

2.2.2 Massive machine type communication

Autonomous devices are expected to carry out a considerable amount of total wireless traffic in near-future wireless systems. In this category, it is expected to have simple and energy efficient encoder/decoder design with a large number of devices [54]. As different from eMBB and URLLC cases, it is basically responsible for a short amount of data transmission without stringent delay requirements so small modulation orders along with short code length can be seen in the use cases belonged by this category [59].

2.2.3 Ultra-reliable low latency communication

Tactile Internet, real-time pairing, robotic motion control and driver-less car are a few examples of potential URRLC use cases, which are introduced by early-stage 5G standardization. Each case requires packet round-trip latency as low as 5 ms along with packet error rate around 10^{-6} [60]. Within this latency constraint, it has become infeasible to implement a transmission protocol which depends on retransmission for the packet (e.g. ARQ, HARQ, etc.). URLLC emphasizes stringent requirements in terms of reliability (e.g., 99.999) and latency (e.g., 4 ms) along with packet size less than 1000 bits [60]. From this point of view, higher reliability from complex

encoder/decoder modules (iterative decoding) might not be sufficient to tackle these mentioned requirements simultaneously.

2.3 Channel codes

Considering above mentioned different use cases introduced in the developments towards next-generation wireless technology, code lengths, modulation order and a choice of channel coding schemes can differ based on the scenarios. Initially, currently deployments in many wireless scenarios are based on delay-free assumption along with very large packet size [61] and considering the revolutionary effects of mission-critical communications and autonomous devices in wireless architecture, we might see different deployment scenarios in certain cases. Then, it suggests rethinking whole design framework, otherwise currently used optimized system structure might render suboptimal for many use cases [62]. Convolutional codes, turbo codes, low-density parity check codes and polar codes are being considered as potential candidates for channel coding scenarios in the new generation [57]. A short description of each channel codes and its current deployment in the wireless systems is given below:

2.3.1 Convolutional codes

The convolutional encoder has been known since the 1950s and it has been mostly used as component codes of the other error correcting codes such that trellis coded modulation, turbo codes, and turbo-trellis coded modulation while it can be deployed standalone. In convolutional coding terminology, $R = \frac{k}{n}$ corresponds to the rate of convolutional encoder where k is the number of information bits, and n is the number of the encoded bits. As an example, a rate 1/3 convolutional encoder is shown in Fig. 2.1. It has two memory elements so four possible encoder states ($N = 4$) exist and the input bits and encoded bits for each transition are presented via the state diagram as shown in Fig. 2.2. Due to the existence of the memory elements in encoder structure, the encoded bits at t are no longer only dependent on information input bit(s) at t . In other words, past information bits, $t - 1, t - 2, \dots$ have effects on the encoded bit at t . It becomes so difficult to show input-output bits relation over the state diagram when a larger number of memory elements exist; as an alternative method to refer convolutional encoder, the generator matrix definition in octal form is widely used. It describes the connections among the outputs of memory elements

and information bits by using polynomial representation. For instance, the generator matrix form for the encoder in Fig. 2.1 can be expressed as

$$G(D) = [1 + D^2, D, 1], \quad (2.1)$$

and its octal form can be written as $[5, 2, 1]_8$. Here, D refers to a delayed element and D^d corresponds to the input bits at the time $t - d$ which is used for determining the encoded output bit in a specific branch at time t . Since all codewords given in state diagram are all error sequences at the same time due to the linearity of the convolutional encoder, it can be described as error state diagram as well.

Figure 2.1: Rate 1/3 four-state convolutional encoder.

Figure 2.2: State transition diagram of rate 1/3 four-state encoder.

Specifically, tail-biting convolutional encoder (TBCC) is currently deployed in the downlink channels over LTE standards, namely, the Physical Broadcast Channel (PBCH) and the Physical Downlink Control Channel (PDCCH) [41]. In terms of backward compatibility with 2G and 3G networks, convolutional coders were utilized over traffic control channels along with cyclic redundancy check (CRC) in 4G networks [63].

Aside from their backward compatibility and popularity as component codes in other coding techniques, 5G network design might have a considerable motivation to use convolutional encoder, especially over URLLC cases. One-shot decoding structure and lower encoding/decoding complexity appear strength of convolutional coding over other powerful contenders where the low latency requirements exist. It is already shown that convolutional coding can outperform turbo coding and even LDPC coding when latency requirement is taken into consideration in evaluating system performance [41].

2.3.2 Trellis coded modulation codes

For any given convolutional encoder, encoding and modulation, assigning the encoded bits into constellation points, were considered as separate identities. The idea of designing modulation process considering given encoder type first emerged with the invention of trellis coded modulation [64]. Although trellis coded modulation still uses convolutional encoders, the set partitioning process differs it from conventional convolutional encoding. In the set partitioning process, the assigning each transition output bits into constellation points in state diagram is carried out by considering to maximize minimum free Euclidean distance among all possible error events. To do so, higher modulation order can be used with TCM encoder by compensating performance loss from higher modulation usage with coding gain.

2.3.3 Turbo codes

Turbo codes constitute of convolutional encoders combining with bit-wise and symbol-wise interleavers. They have been commonly used in LTE networks, and also took part in 3G mobile communication in Universal Mobile Telecommunications System (UMTS) and Digital Video Broadcasting (DVB). However, principal strengths of turbo coding might not apply on 5G use cases. Its high complexity makes it difficult to

tackle high data rate requirement in eMBB use cases [56]. Furthermore, having error-floor phenomenon and decoding latency resulting from iterative decoding process might fail to use turbo coded transmission protocols in URRLC and mMTC use cases. For instance, in 3GPP standard, the error-floor region for turbo coded scenarios can be seen in the BER region between 10^{-3} - 10^{-4} and to decode turbo coded information package, a max-log-MAP algorithm with eight iterations is generally used in the decoder [57].

2.3.4 Turbo-trellis coded modulation codes

TTCM encoders emerged as an attractive coding technique which combines the powerful design of binary turbo coding [51] with larger constellation sizes [52]. The block diagram of a TTCM system is shown in Fig. 2.3. Specifically, original information bits are fed into the first TCM encoder where the encoded bits are mapped into output symbols, $\mathbf{x}^{(1)}$ selected from a M -ary constellation while the interleaved version of original information bits are assigned into the output symbols, $\mathbf{x}^{(2)}$ where chosen constellation herein can be different from previous one at the second TCM encoder [65]. Also, Π_s is an odd-even separation (OES) based symbol interleaver which ensures that odd (even) indexed symbols' bits are assigned to another odd (even) indexed symbols' positions [66]. After having M -ary symbols from both the first TCM and the second TCM encoders, $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, the output symbols of TTCM encoded symbols, $\mathbf{x}$, can be selected as follows

$$\mathbf{x} = \left[x_1^{(1)} x_1^{(2)} x_2^{(1)} x_2^{(2)} x_3^{(1)} x_3^{(2)} \dots \right] \quad (2.2)$$

Figure 2.3: Block diagram of TTCM encoder.

where $\mathbf{x}^{(1)} = [x_1^{(1)} x_2^{(1)} x_3^{(1)} x_4^{(1)} \dots]$, $\mathbf{x}^{(2)} = [x_1^{(2)} x_2^{(2)} x_3^{(2)} x_4^{(2)} \dots]$, and $x_t^{(i)}$ denotes an output symbol chosen from i th TCM encoder at the time instant t ($i \in \{1, 2\}, t \geq 0$).

2.3.5 Low-density parity check codes

Although LDPC code was invented in 1962, it was not popular until the 1990s. It was proved that it outperformed turbo code and LDPC coding has been deployed in DVB-S2, 802.11n (Wi-Fi allowing MIMO) and 802.16e (Mobile WiMAX), etc [57]. As a powerful error control coding technique, it seems that LDPC codes are going to appear especially in enhanced mobile broadband cases where the code length can vary between 100-8000 along with coding rate 1/5-8/9. However, the superiority of LDPC codes might lose its superiority over URLLC cases as the same way seen in turbo coding cases [56]. In URLLC cases, the frame length is expected to be less than 400 and within this short length, standard convolutional code outperforms LDPC codes.

2.3.6 Bit-interleaved coded modulation codes

BICM encoder is proposed in [67], it outperforms TCM codes in faded scenarios while enabling flexible and simple encoder design. In BICM coded scenarios, constellation and encoder are considered separately in contrast to TCM where any change in encoder requires to redesign whole encoding structure [68]. Maximizing code diversity results in much better performance in faded scenarios; however, its simplicity and flexibility keeps it deploying in the scenarios where the fading is not considered [19]. Currently, BICM is used in HSPA, 802.11a/g/n and DVB standards [19].

2.3.7 Polar codes

Polar codes, invented by [69], is another candidate considered for the deployment over URLLC cases. With using the channel polarization, polar code is first known channel coding technique able to achieve the channel capacity over symmetric binary input discrete memory-less channels [69]. In addition to its superior performance, error-floor is not seen in the polar codes. Although decoding process of original polar code has the same complexity with successive cancellation decoding, list decoding needs to be adopted to achieve mentioned rates [70].

2.4 Viterbi decoder

Viterbi algorithm is the most used method to decode convolutional codes and it mainly aims to find the information bits by seeking the most likely path through the trellis of the encoder throughout with the help of the received signal. The use of Viterbi algorithm over the convolutional encoder results in maximum likelihood (ML) decoder and for a given bit-to-symbol mapping function $g(\cdot)$ and corresponding binary labelling for a specified transition $b(\cdot)$, a branch metric from State 'i' to State 'j' ($i, j \in \{1, \dots, N\}$) in a Viterbi decoder can be calculated as

$$m_t(y_t, h_t, \text{State 'i'}, \text{State 'j'}) = \|y_t - h_t g(b(\text{State 'k'} \rightarrow \text{State 'j'}))\|^2 \quad (2.3)$$

where h_t denotes a channel coefficient for a specific time t . In each state for each symbol interval, the decoder is set to choose to the path with the highest probability among all incoming transitions and it is called as surviving paths while the others do truncated paths. In Fig. 2.4, all surviving paths (solid lines) and truncated paths (dashed lines) are shown as an illustration based on the ML rule, which is

$$\arg \min_k m_t(y_t, h_t, \text{State 'k'}, \text{State 'j'}) \quad (2.4)$$

where State 'k' denotes the states which has a transition into the State 'j'.

Figure 2.4: Viterbi decoding algorithm with surviving and truncated paths.

2.5 Classification of codes

Uniform constellations have been very popular for decades because of their symmetrical distributions on the space, which provides easier physical implementation and the use of uniform constellations also results in computational advantages in the calculation of upper bound on BER calculation for any given convolutional encoder. The upper bound on BER expression is obtained from generating function (also called transfer function or enumerating function) which provides the sum of all pairwise error event probabilities [71]. For any given pair of N -state convolutional encoder and a uniform constellation, 2^N -state diagram can be enough for the calculation of generating function while it might require up to 2^{N^2} -state diagram with the existence of an irregular constellation [72]. At this point, certain classifications of convolutional encoder are introduced in order to understand how to select the method of calculating generating function based on any given convolutional encoder and constellation.

2.5.1 Linear codes

For any linear convolutional encoder, the sum of any codewords results in one of an existing codeword. For example, the 8-state 8-PSK TCM coder proposed in [73] is linear codes since any encoded bits can be expressed as the sum of eight possible codewords which can be seen in Fig. 2.5.

2.5.2 Quasi-regular codes

Now, the conditions for that 2^N -state diagram can handle error performance calculation for any given convolutional encoder. These conditions are referred as quasi-regularity (QR) [45] and they are a generalized version of symmetry conditions for

Figure 2.5: 8-state convolutional encoder

computing free Euclidean distance for TCM codes [73]. Being QR makes the analysis fully independent from transmitted sequences so the assumption of all zeros codeword sent can be used in the analysis.

In order to categorize any given convolutional encoder as QR, the convolutional encoder, along with a choice of constellation, needs to satisfy two conditions, which are [45]:

- The encoder needs to be linear,
- For any possible error vector, $\mathbf{e}$, its the error weight profile, $P_{(.,.),\mathbf{e}}(\mathcal{D})$, must be independent of starting and ending states of any possible transition corresponding for that vector. In other words, two possible erroneous events resulting from the same $\mathbf{e}$, which are occurring from state- u to state- v and from state- $\bar{u}$ to state- $\bar{v}$ in a state diagram, are required to satisfy:

$$P_{(u,v),\mathbf{e}}(\mathcal{D}) = P_{(\bar{u},\bar{v}),\mathbf{e}}(\mathcal{D}), \quad (2.5)$$

where, $P_{(u,v),\mathbf{e}}(\mathcal{D})$ and $P_{(\bar{u},\bar{v}),\mathbf{e}}(\mathcal{D})$ denote the distance polynomials, and they are explicitly given by

$$P_{(u,v),\mathbf{e}}(\mathcal{D}) = \sum_{\mathbf{c}|u} p(\mathbf{c}|u) \mathcal{D}^{d^2[g(\mathbf{c}),g(\mathbf{c}\oplus\mathbf{e})]}, \quad (2.6a)$$

$$P_{(\bar{u},\bar{v}),\mathbf{e}}(\mathcal{D}) = \sum_{\bar{\mathbf{c}}|\bar{u}} p(\bar{\mathbf{c}}|\bar{u}) \mathcal{D}^{d^2[g(\bar{\mathbf{c}}),g(\bar{\mathbf{c}}\oplus\mathbf{e})]}, \quad (2.6b)$$

where $p(\mathbf{c}|u)$ and $p(\bar{\mathbf{c}}|\bar{u})$ are the probabilities of correct codewords, c and $\bar{c}$, respectively. Also, $d^2[g(\mathbf{c}),g(\mathbf{c}\oplus\mathbf{e})]$ and $d^2[g(\bar{\mathbf{c}}),g(\bar{\mathbf{c}}\oplus\mathbf{e})]$ denote the Euclidean distances between the symbol pair of $g(\mathbf{c})$ and $g(\mathbf{c}\oplus\mathbf{e})$, and $g(\bar{\mathbf{c}})$ and $g(\bar{\mathbf{c}}\oplus\mathbf{e})$, under the bit-to-symbol mapping rule, $g(\cdot)$.

Satisfying above given conditions make the error analysis of convolutional coded system independent of transmitted sequence since the error weight profile stays the same regardless of transmitted codeword, either $\mathbf{c}$ or $\bar{\mathbf{c}}$. Note that (2.5) must be validated each pair of states and admissible error vector $\mathbf{e}$ in a given state diagram.

2.5.3 Regular codes

The definitions of regular codes are introduced in the context of trellis coded modulation (TCM) [45] where the subsets of constellations resulting from set partitioning

process are considered in this definition.

A given convolutional encoder can be classified as regular if and only if [45]

- The encoder is linear,
- A minimum Euclidean distance between two subsets only depends on binary labels of the subsets.

2.5.4 Geometrically uniform codes

The strongest classification for a given encoder in the context of regularity is defined in [74] as geometrically uniform codes (GUC); however the algebraic process in this categorization is beyond the scope of this thesis.

2.5.5 Irregular codes

Any coded system lacks of satisfying (2.5) is described as irregular codes [45]. Without considering non-uniform constellations, puncturing and erasure can also cause irregular coders [75] and irregular coders might show better performance than QR one. Note that the constellation and encoder can be irregular with the existence of with geometrically uniform conventional M -QAM and M -PSK constellations, for instance, the rate-1/3 convolutional encoder specified LTE standard used in the downlink control information [76] is not irregular when it uses a 64-QAM constellation.

2.6 The effect of constellation on quasi-regularity

As mentioned above, the choice of constellation determines the class of encoder and depending on the class of encoder, the analytical framework constructs on either 2^N or 2^{N^2} state diagram. In order to understand the effects of choice of constellation, some numerical examples are given and the way how to determine the class of convolutional coding along with given constellations is presented in this section.

Example 1. The rate-2/3 8-state convolutional encoder given in Fig. 2.5 with 8-PSK is first considered. Its state diagram and corresponding codewords for each transition are shown in Fig. 2.6. Now, we would like to check that this coder is QR or not based on (2.5). For this purpose, the error weight profile matrix of each possible error vector, $\{000, 001, 010, 011, 100, 101, 110, 111\}$, is calculated and $\mathcal{D}$ is

assumed as 2 (arbitrarily chosen value) over the calculations in order to facilitate the understanding without losing generality and it is presented in Table 2.1 for each error codeword. For instance, in the case of $P_{(\cdot,\cdot),001}$, it can be seen that the error weight profile does not vary with starting and ending state if transition exists, as shown in the following:

$$P_{(1,2),001} = P_{(2,1),001} = P_{(3,4),001} = P_{(4,3),001} = P_{(5,6),001} = P_{(6,5),001} = P_{(7,8),001} = P_{(8,7),001}. \quad (2.7)$$

The same phenomenon is seen in each possible codeword, then; it can be said that the rate-2/3 8-state convolutional encoder along with 8-PSK is a QR code based on (2.5).

Figure 2.6: State transition diagram of convolutional encoder given in Fig. 2.5

Example 2. Now we would like to give another example where the 2nd QR condition, (2.5), is violated due to the choice of constellation for the same encoder. Let us use the 4-state rate-1/2 convolutional encoder, $[2, 1]_8$ which is shown in Fig. 2.7. In the case of 4-QAM is employed with this encoder, all possible error weight profile can be calculated as follows:

Figure 2.7: The rate-1/2 convolutional encoder, $[2, 1]_8$.

$$P_{(\cdot,\cdot),\mathbf{00}} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, P_{(\cdot,\cdot),\mathbf{01}} = \begin{bmatrix} 0 & 1.3324 \\ 1.3324 & 0 \end{bmatrix}, \quad (2.8a)$$

$$P_{(\cdot,\cdot),\mathbf{10}} = \begin{bmatrix} 1.3324 & 0 \\ 0 & 1.3324 \end{bmatrix}, P_{(\cdot,\cdot),\mathbf{11}} = \begin{bmatrix} 0 & 1.9986 \\ 1.9986 & 0 \end{bmatrix}. \quad (2.8b)$$

From (2.8), it can be seen that $[2, 1]_8$ along with 4-QAM constellation yields QR coder since the error weight profiles are invariant based on any starting and ending states. Now let consider an arbitrarily located constellation use along with the same encoder. For this purpose, Constellation-I, shown in Fig. 2.8, is generated and its symbol points are listed as $\{-2.17+0.14i, 0.66-0.84i, -0.06+0.29i, -0.62-0.13i\}$. The newly calculated error weight profiles for Constellation-I are calculated as,

$$P_{(\cdot,\cdot),\mathbf{00}} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, P_{(\cdot,\cdot),\mathbf{01}} = \begin{bmatrix} 0 & 1.6641 \\ 1.6641 & 0 \end{bmatrix}, \quad (2.9a)$$

$$P_{(\cdot,\cdot),\mathbf{10}} = \begin{bmatrix} 1.5988 & 0 \\ 0 & 1.1173 \end{bmatrix}, P_{(\cdot,\cdot),\mathbf{11}} = \begin{bmatrix} 0 & 1.1153 \\ 1.1153 & 0 \end{bmatrix}. \quad (2.9b)$$

where the error weight profile has turned into being dependent of starting and ending states for the error vector, $\mathbf{10}$, since $P_{(1,1),\mathbf{10}} \neq P_{(2,2),\mathbf{10}}$.

2.7 Product-state matrix technique

Once the independence of the error weight profiles exists, error weight profile becomes independent from an actually transmitted sequence, a choice of either $\mathbf{c}$ or $\bar{\mathbf{c}}$ seen in

Figure 2.8: Symbol point locations of 4-QAM and Constellation-I.

(2.6). This situation leads to a considerable reduction in error analysis by assuming all zero codeword is sent which will be described later. However, the transmitted sequence has direct effect on the value of error weight profile so the comprehensive analysis is required to take into consideration of all possible transmitted codewords.

For this purpose, [77] introduced a product-state matrix technique which is generalized version of Viterbi's conventional analysis [71]. The main difference of product-state matrix technique can be seen in Fig. 2.9 and state-0-to-state-0 transitions occur during the transmission in the method of [71]. In this figure, the product states are defined as (u, v) and $(\bar{u}, \bar{v})$, which are corresponding to the starting and ending states at the encoder and decoder, respectively. Considering all possible encoded transitions leads to N^2 ordered pairs of product-states for an N -state convolutional encoder.

The product-states are classified into two categories based on the equivalence of encoded and decoded states [47, 78]. The classification of a given product-state is determined by a simple rule where it is called as “good state ($\mathcal{G}$)” when the encoder and decoder's either starting or ending states are assumed to be the same, otherwise it is called as “bad state ($\mathcal{B}$)”, which is shown in Table 2.2. Using this classification and suitably ordering the product-states, product-state matrix, $\mathbf{S}$, can be written in

Figure 2.9: The comparison of (a) conventional error analysis and (b) product-state matrix technique.

the $N^2 \times N^2$ matrix form of [77]

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_{\mathcal{G}\mathcal{G}} & \mathbf{S}_{\mathcal{G}\mathcal{B}} \\ \mathbf{S}_{\mathcal{B}\mathcal{G}} & \mathbf{S}_{\mathcal{B}\mathcal{B}} \end{bmatrix}. \quad (2.10)$$

A particular entry of $\mathbf{S}$, $\mathbf{S}_{(u,v),(\bar{u},\bar{v})}$ can be expressed by

$$\mathbf{S}_{(u,v),(\bar{u},\bar{v})} = \Pr(u \rightarrow \bar{u}|u) \sum_n p_n \mathcal{I}^{W(u \rightarrow \bar{u}) \oplus W(v \rightarrow \bar{v})} \mathcal{D}_{(u,v),(\bar{u},\bar{v})}, \quad (2.11)$$

where the summation in (2.11) is over possible n parallel transitions depending on a given encoder, p_n denotes the probability of n th parallel transition between $(u \rightarrow \bar{u})$ if it exists, otherwise $p_n = 1$. $\Pr(u \rightarrow \bar{u}|u)$ is the conditional probability of a transition from state u to state $\bar{u}$ given state u and $W(i \rightarrow j)$ denotes the Hamming weight of information sequence for the transition from i to j where $i \in \{u, v\}$ and $j \in \{\bar{u}, \bar{v}\}$ [78].

In the error analysis of any give convolutional encoder, the calculation of the generating function, $T(\mathcal{D}, \mathcal{I})$, is the pivotal step. After obtaining each entry of $\mathbf{S}$,

Table 2.2: Classification of the product states.

$\mathbf{S}$	$u = v$	$u \neq v$
$\bar{u} = \bar{v}$	$S_{\mathcal{G}\mathcal{G}}$	$S_{\mathcal{B}\mathcal{G}}$
$\bar{u} \neq \bar{v}$	$S_{\mathcal{G}\mathcal{B}}$	$S_{\mathcal{B}\mathcal{B}}$

the generating function, $T(\mathcal{D}, \mathcal{I})$, can be computed by [77]

$$T(\mathcal{D}, \mathcal{I}) = \mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{G}} \mathbf{1} + (\mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{B}})^T [\mathbf{I} - \mathbf{S}_{\mathcal{B}\mathcal{B}}]^{-1} \mathbf{S}_{\mathcal{B}\mathcal{G}} \mathbf{1}, \quad (2.12)$$

where $\mathbf{1}$ and $\mathbf{I}$ denote the unity and identity matrices, respectively. Using (2.12), one can compute the upper bound BER [79]

$$P_b \leq \frac{1}{k} \frac{\partial T(\mathcal{D}, \mathcal{I})}{\partial \mathcal{I}} \bigg|_{\mathcal{I}=1}. \quad (2.13)$$

The derivative of (2.13) can be easily shown to be [77]

$$\begin{aligned} \frac{\partial T(\mathcal{D}, \mathcal{I})}{\partial \mathcal{I}} = & \frac{1}{2N} \left((\mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{G}})' + (\mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{B}})'^T [\mathbf{I} - \mathbf{S}_{\mathcal{B}\mathcal{B}}]^{-1} \mathbf{S}_{\mathcal{B}\mathcal{G}} \mathbf{1} \right. \\ & + (\mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{B}})^T [\mathbf{I} - \mathbf{S}_{\mathcal{B}\mathcal{B}}]^{-1} (\mathbf{S}_{\mathcal{B}\mathcal{G}} \mathbf{1})' \\ & \left. + (\mathbf{1}^T \mathbf{S}_{\mathcal{G}\mathcal{B}})^T [\mathbf{I} - \mathbf{S}_{\mathcal{B}\mathcal{B}}]^{-1} \mathbf{S}_{\mathcal{B}\mathcal{B}}' [\mathbf{I} - \mathbf{S}_{\mathcal{B}\mathcal{B}}]^{-1} \mathbf{S}_{\mathcal{B}\mathcal{G}} \mathbf{1} \right), \end{aligned} \quad (2.14)$$

where $(\cdot)'$ denotes element-wise derivative with respect to $\mathcal{I}$.

The importance of product-state matrix technique lies on being an analysis fully independent from a transmitted sequence where the assumption of the all-zero code-word over QR scenarios has reduced the complexity of analysis considerably. While facilitating analysis, QR is not associated with improved performance [80] and many systems can be found to be irregular, especially when encoders are paired with non-uniformly distributed constellations [72]. For this reason, utilizing the product-state matrix technique enables to work with any pair of convolutional encoder and constellation in the search process of optimized constellation without any predefined structure.

2.8 Performance analysis for QR scenarios

In QR cases, the complexity of analysis has reduced considerably due to assuming all

Figure 2.10: Error state diagram of the $[2, 1]_8$ convolutional encoder.

zero codeword is sent. Following this manner leads to $u = 0$ and $v = 0$ in (2.11) and all possible decoder transitions, $\bar{u}$ and $\bar{v}$, should be only considered. Then, the size of product-state matrix, $\mathbf{S}$ for QR scenario has reduced from $N^2 \times N^2$ to $N \times N$.

Let consider a pair of convolutional encoder and constellation given in Example 2 in Section 2.6. Since the chosen constellation consists of four possible output symbols, e.g., $s_n \in \{s_1, s_2, s_3, s_4\}$, there are four possible pairs of the original and the erroneous symbol $(s, \hat{s})$ for the error sequence of $\{01\}$ (i.e., $n = 2$). These are $\{00, 01\}$, $\{01, 00\}$, $\{10, 11\}$, and $\{11, 10\}$, assuming the natural bit-to-symbol mapping is used. The generating function can be found as

$$T(\mathcal{D}, \mathcal{I}) = \frac{\mathcal{D}_2 \mathcal{D}_1 I}{1 - \mathcal{D}_3 I}, \quad (2.15)$$

where $\mathcal{D}_n$ corresponds to the error weight profile for the error sequence e_n , $n \in \{1, 2, 3\}$. For $\mathcal{D}_2$, averaging the squared distance between these possible pairs gives

$$d_2 = \frac{\frac{1}{4}(|s_1 - s_2|^2 + |s_2 - s_1|^2 + |s_3 - s_4|^2 + |s_4 - s_3|^2)}{4N_0}. \quad (2.16)$$

Then, $\mathcal{D}_2(\cdot)$ is can be calculated while $\mathcal{D}_2$ and $\mathcal{D}_3$ can be found by following the similar steps.

2.9 Conclusion

This chapter aims to give supporting material about the concepts and technique which will be exploited in later parts of this thesis. To do so, the terminology, channel coding techniques, the classification of any given convolutionally coded system and the required steps towards to analytical framework are introduced.

Chapter 3

The Proposed Performance Bounds

3.1 Introduction

In this chapter, error performance calculation for two-orthogonal transmission stages and one-transmit antenna system is presented; then, the same framework is extended into more generalized system architecture where multiple transmission stages exist along with multiple transmit antennas. The extension of proposed analytical framework to the turbo-trellis coded cases are also presented. All proposed bound expressions are validated via Monte Carlo simulations. In addition, the analysis is extended to turbo-trellis coded modulation (TTCM) scenarios in which the convolutional encoders are used as the constituent codes. We demonstrate, via simulation, that commonly used performance analysis techniques in the literature fail to provide a valid BER bound in coded cases where quasi-regularity is not satisfied. In contrast, the technique proposed herein does not require the chosen pair of constellation and encoder to be QR. Simulation results demonstrate the accuracy of the derived analytical results for a wide range of system scenarios.

3.2 Single transmission stage with single transmit antenna

We consider a system architecture consisting of single transmission stage. During the transmission stage, the information bit sequence is first coded by a rate R convolutional encoder, and the resulting bits are assigned a signal point from a given M -ary constellation based on a bit-to-symbol mapping rule.

Specifically, in the transmitter, the information bits for the l th frame $\mathbf{b}_l = [b_{l,1} \cdots b_{l,N_b}]$ are encoded (one frame has N_b information bits) and the encoded bits, $\mathbf{c}_l = [c_{l,1} \cdots c_{l,N_c}]$, are fed to the bit-to-symbol mapper in which the optimized M -ary irregular constellation for a specific $\bar{\gamma}$, $\chi(\bar{\gamma})$, is ready to be used for transmitting length N_s symbols output frame, $\mathbf{s}_l = [s_{l,1}, \cdots, s_{l,N_s}]$, over the wireless channel. Note that throughout this thesis, the natural mapping is employed as a bit-to-symbol mapping function, $g(\cdot)$. For instance, in a 64-ary signalling scheme, s_{60} corresponds to the following 6 bits: 111100. Then, the received signal for the i^{th} symbol in the l^{th} frame can be written as

$$r_{l,i} = s_{l,i} + n_{l,i}, \quad (3.1)$$

where $n_{l,i}$ is the additive white Gaussian noise (AWGN) sample with zero-mean and $N_0/2$ variance per dimension, $s_{l,i} \in \chi(\bar{\gamma})$ where the average received SNR can be explicitly defined as $\bar{\gamma} = E_s/N_0$. Here, E_s denotes the average symbol energy of $\chi(\bar{\gamma})$. In the receiver side, soft-decision Viterbi decoding is carried out by assuming that constellation $\chi(\bar{\gamma})$ is used for a specific $\bar{\gamma}$, is known.

3.2.1 AWGN channel

The probability of decoding an erroneous symbol at the receiver in place of the transmitted symbol can be expressed in terms of the probability of an erroneous decoder transition ($v \rightarrow \bar{v}$) for an actual transition state at encoder ($u \rightarrow \bar{u}$) and it is denoted by $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ seen in (2.11). For a given and corresponding binary label of a specified transition $b(\cdot)$, $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ is a function of the distances between the output symbols $s = g(b(u \rightarrow \bar{u}))$ and the erroneously decoded symbols $\hat{s} = g(b(v \rightarrow \bar{v}))$. $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be expressed as [47]

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \Pr(|r - s|^2 - |r - \hat{s}|^2 \geq 0). \quad (3.2)$$

After some mathematical manipulations, Chernoff bound expression for (3.2) can be written as [81]

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = e^{-d}, \quad (3.3)$$

where $d = \frac{|s - \hat{s}|^2}{4N_0}$.

3.2.2 Rayleigh channel

In the case of fading channels where the fading coefficients of the channel during the l th frame transmission is denoted by $\mathbf{h}_l = [h_{l,1}, \dots, h_{l,N_s}]$, the received signal for the i^{th} symbol in the l^{th} frame can be rewritten as

$$r_{l,i} = h_{l,i}s_{l,i} + n_{l,i}, \quad (3.4)$$

where $n_{l,i}$ is the additive white Gaussian noise (AWGN) sample with zero-mean and $N_0/2$ variance per dimension, $s_{l,i} \in \chi(\bar{\gamma})$ where the average received SNR can be explicitly defined as $\bar{\gamma} = \Omega E_s / N_0$. Here, E_s denotes the average symbol energy of $\chi(\bar{\gamma})$ and Ω can interpreted as the path-loss term. Note that we assume that the channel coefficient stays constant during one symbol transmission and that each symbol is exposed to a different fading coefficient. In the receiver side, soft-decision Viterbi decoding is carried out by using the perfect the channel state information (CSI) and assuming that constellation $\chi(\bar{\gamma})$ used for a specific $\bar{\gamma}$, is known.

For a given channel coefficient, h , along with the output symbol, $s = g(b(u \rightarrow \bar{u}))$, and the erroneously decoded symbol, $\hat{s} = g(b(v \rightarrow \bar{v}))$, the conditional $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be expressed as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})|h} = \Pr [|r - hs|^2 - |r - h\hat{s}|^2 \geq 0 | h]. \quad (3.5)$$

After some mathematical manipulations, Chernoff bound expression for (3.5) can be written as [81]

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})|\gamma} = e^{-d\gamma}, \quad (3.6)$$

where $d = |s - \hat{s}|^2 / 4N_0$ and $\gamma = |h|^2$. Averaging (3.6) over the channel statistics yields $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ which we use to obtain each entry of (2.10) which in turn will be used to compute the upper bound BER using (2.13) and (2.14).

We now derive an unconditional $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ expression for Rayleigh fading scenarios. From (3.6), $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be written as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \int_0^\infty d\gamma e^{-d\gamma} f_\gamma(\gamma), \quad (3.7)$$

where $f_\gamma(\gamma)$ denotes the PDF of γ . For the Rayleigh fading case, $f_\gamma(\gamma)$ can be written

as

$$f_{\gamma}(\gamma) = \frac{1}{\Omega} e^{-\frac{\gamma}{\Omega}}, \quad (3.8)$$

then; utilizing [82, eq. 3.35.2] yields

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \frac{1}{1 + \Omega d}. \quad (3.9)$$

3.2.3 Nakagami- m channel

In this section, channel characteristic is modeled by Nakagami- m distribution. Modeling the channel with Nakagami- m distribution brings the advantage of working with more practical urban radio multi-path channel scenarios [83] and derived expressions for Nakagami- m fading case are able to cover Rayleigh fading case when $m = 1$ and AWGN channel when $m \rightarrow \infty$. In the case of Nakagami- m channels, $f_{\gamma}(\gamma)$ denotes the probability density function of the squared envelope of Nakagami-faded channel coefficient, known to follow the gamma distribution [84]

$$f_{\gamma}(\gamma) = \frac{\gamma^{m-1} e^{-\gamma \frac{m}{\Omega}}}{\Omega^m m^{-m} \Gamma(m)}, \quad (3.10)$$

where Ω is the average fading power, $m \geq 0.5$, and $\Gamma(\cdot)$ is the gamma function [82]. It was shown in [85] that the uniform phase distribution only holds for the Rayleigh case ($m_k = 1$), otherwise it is given by [85, 86]

$$f_{\theta}(\theta) = \frac{\Gamma(m) |\sin 2\theta|^{m-1}}{2^{m+1/2} \Gamma^2(0.5m)}, \quad -\pi \leq \theta \leq \pi. \quad (3.11)$$

Using (3.10) and [82, eq. 3.35.2] one can obtain

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \left(\frac{\Omega d}{m} + 1 \right)^{-m}, \quad (3.12)$$

where $h = |h|e^{-j\theta}$ after some manipulations.

3.3 Two transmission stage with single transmit antenna

In this section, we consider a system architecture consisting of two orthogonal transmission stages as shown in Fig. 3.1 where each stage employs the same convolutional

encoder, but not necessarily the same constellation used for mapping encoded bits to transmitted symbols. Two-orthogonal transmission stages can be considered as the realizations of the relaying, HARQ, or CoMP which are shown in Fig. 3.2.

During each transmission stage, the same information bit sequence is first coded by a rate R convolutional encoder, and the resulting bits are assigned a signal point from an arbitrary M -ary constellation based on a bit-to-symbol mapping rule.

The mapper output symbols $\{s^{(k)}\}$, $k = 1, 2$, where k refers to the transmission stage, are transmitted. The corresponding received signals are given by

$$r_k = h_k s^{(k)} + n_k, \quad (3.13)$$

where h_k denotes frequency non-selective Nakagami- m fading coefficient with shaping parameter m_k and average fading power Ω_k ; n_k is the additive white Gaussian noise (AWGN) sample with zero-mean and $N_0/2$ variance per dimension $k \in \{1, 2\}$. Note that we allow the channel parameters to vary between the transmission stages and that each symbol is exposed to a different fading coefficient. Independent fading between the stages is considered first followed by the analysis for the correlated case, where the correlation coefficient between h_1 and h_2 , is defined as

$$\rho = \frac{\text{cov}(|h_1|^2, |h_2|^2)}{\sqrt{\text{var}(|h_1|^2) \text{var}(|h_2|^2)}}, \rho \in [0, 1]. \quad (3.14)$$

Herein, $\rho = 0$ corresponds to the independent fading case and $\rho = 1$ shows the fully correlated scenario.

For a given channel coefficient, h_1, h_2 , the conditional $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be expressed as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})|h_1,h_2} = \Pr \left[\sum_{k=1}^2 |y_k - h_k s^{(k)}|^2 - |y_k - h_k \hat{s}^{(k)}|^2 \geq 0 \mid h_1, h_2 \right]. \quad (3.15)$$

After some mathematical manipulations, Chernoff bound expression for (3.15) can be written as [81]

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})|\{\gamma_1,\gamma_2\}} = e^{\sum_{k=1}^2 -d_k \gamma_k}, \quad (3.16)$$

where $d_k = |s^{(k)} - \hat{s}^{(k)}|^2 / 4N_0$ and $\gamma_k = |h_k|^2$. Averaging (3.16) over the channel statistics yields $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ which we use to obtain each entry of (2.10) which in turn will be

Figure 3.1: Transmitter block diagram of two-orthogonal transmissions.

used to compute the upper bound BER using (2.13) and (2.14).

We now derive $D_{(u,v),(\bar{u},\bar{v})}$ expression for Nakagami- m fading scenarios with and without correlation between transmission stages. From (3.16), the unconditional Bhattacharyya parameter for the case of independent transmission stages can be written as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \int_0^\infty \int_0^\infty d\gamma_1 d\gamma_2 e^{-d_1\gamma_1 - d_2\gamma_2} f_{\gamma_1, \gamma_2}(\gamma_1, \gamma_2), \quad (3.17)$$

where $f_\gamma(\gamma_1, \gamma_2)$ denotes the joint PDF of γ_1 and γ_2 .

3.3.1 AWGN channels

For AWGN channel case, the resulting unconditional $D_{(u,v),(\bar{u},\bar{v})}$ expression can be found as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = e^{-d_1 - d_2}, \quad (3.18)$$

where $d_k = \frac{|s^{(k)} - \hat{s}^{(k)}|^2}{4N_0}$. $D_{(u,v),(\bar{u},\bar{v})}$ in (3.18) is first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

Figure 3.2: Possible system realizations for the two-orthogonal transmission scheme.

3.3.2 Rayleigh channels

Independent Case

Here, the joint PDF is simply $f_{\gamma}(\gamma_1, \gamma_2) = f_{\gamma}(\gamma_1) f_{\gamma}(\gamma_2)$, $f_{\gamma}(\gamma)$ can be written as

$$f_{\gamma}(\gamma_1, \gamma_2) = \frac{1}{\Omega_1 \Omega_2} e^{-\frac{\gamma_1}{\Omega_1}} e^{-\frac{\gamma_2}{\Omega_2}}, \quad (3.19)$$

then; utilizing [82, eq. 3.35.2] into (3.17) yields

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \left(\frac{1}{1 + \Omega_1 d_1} \right) \left(\frac{1}{1 + \Omega_2 d_2} \right). \quad (3.20)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ in (3.20) is first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

Correlated Case

Fading correlation in the two transmission stages is an important consideration, as it can arise, for example, in an HARQ retransmission within the coherence time of the channel [87]. For Rayleigh faded case, $f_{\gamma}(\gamma_1, \gamma_2)$ is given by [88]

$$f_{\gamma}(\gamma_1, \gamma_2) = \frac{e^{-\frac{1}{1-\rho}(\frac{\gamma_1}{\Omega_1} + \frac{\gamma_2}{\Omega_2})}}{\Omega_1 \Omega_2 (1-\rho)} \mathcal{I}_0 \left(\frac{2\sqrt{\gamma_1 \gamma_2 \rho}}{(1-\rho)\sqrt{\Omega_1 \Omega_2}} \right), \quad (3.21)$$

where $\mathcal{I}_0(\cdot)$ denotes the zeroth order modified Bessel function [82]. For $\Omega_1 = \Omega_2 = 1$, substituting (3.25) into (3.36) and using [82, (6.643.2), (7.621.2)] lead to

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \frac{1}{1 + d_2 + d_1(1 + d_2 - d_2\rho)} \quad (3.22)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ in (3.20) and (3.22), are first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

3.3.3 Nakagami- m channels

Independent Case

Here, the joint PDF is simply $f_\gamma(\gamma_1, \gamma_2) = f_\gamma(\gamma_1) f_\gamma(\gamma_2)$, where $f_\gamma(\gamma_k)$ denotes the probability density function of the squared envelope of Nakagami-faded channel coefficient, known to follow the Gamma distribution [84]

$$f_{\gamma_k}(\gamma) = \frac{\gamma^{m_k-1} e^{-\gamma/\Omega_k}}{\Omega_k^{m_k} m_k^{-m_k} \Gamma(m_k)}, \quad (3.23)$$

where Ω_k is the average fading power, $m_k \geq 0.5$, and $\Gamma(\cdot)$ is the gamma function [82]. Combining (3.36)-(3.23) and using [82, eq. 3.35.2] one can, after some manipulation, obtain

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \left(\frac{\Omega_1 d_1}{m_1} + 1 \right)^{-m_1} \left(\frac{\Omega_2 d_2}{m_2} + 2 \right)^{-m_2}. \quad (3.24)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ in (3.24) is first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

Correlated Case

We start by considering a special case, followed by a more general result. For $m_1 = m_2 = m$, $f_\gamma(\gamma_1, \gamma_2)$ is given by [88]

$$f_\gamma(\gamma_1, \gamma_2) = \frac{(\gamma_1 \gamma_2)^{0.5m-0.5} m^{m+1} e^{-\frac{m}{1-\rho}(\frac{\gamma_1}{\Omega_1} + \frac{\gamma_2}{\Omega_2})}}{\Gamma(m) \Omega_1 \Omega_2 (1-\rho) (\sqrt{\rho \Omega_1 \Omega_2})^{m-1}} \mathcal{I}_{m-1} \left(\frac{2m\sqrt{\gamma_1 \gamma_2 \rho}}{(1-\rho)\sqrt{\Omega_1 \Omega_2}} \right), \quad (3.25)$$

where $\mathcal{I}_{m-1}(\cdot)$ denotes the modified Bessel function of order $m-1$ [82]. For $\Omega_1 = \Omega_2 = 1$, substituting (3.25) into (3.36) and using [82, (6.643.2), (7.621.2)]

leads to

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \frac{m^{2m} \left(\left(d_1 + \frac{m}{1-\rho} \right) \left(d_2 + \frac{m}{1-\rho} \right) - \frac{m^2 \rho}{(1-\rho)^2} \right)^{-m}}{(1-\rho)^m}. \quad (3.26)$$

For the more general case of $m_1 \neq m_2$, $\Omega_1 \neq \Omega_2$, the joint PDF $f_\gamma(\gamma_1, \gamma_2)$ is given by [88]

$$f_\gamma(\gamma_1, \gamma_2) = (1-\rho)^{m_2} \sum_{k=0}^{\infty} \left\{ \frac{(m_1)_k \rho^k}{k!} \left(\frac{m_1}{\Omega_1(1-\rho)} \right)^{m_1+k} \left(\frac{m_2}{\Omega_2(1-\rho)} \right)^{m_2+k} \frac{\gamma_1^{m_1+k-1}}{\Gamma(m_1+k)} \right. \\ \left. \times \frac{\gamma_2^{m_2+k-1}}{\Gamma(m_2+k)} e^{\frac{-1}{(1-\rho)} \left(\frac{m_1 \gamma_1}{\Omega_1} + \frac{m_2 \gamma_2}{\Omega_2} \right)} {}_1F_1 \left(m_2 - m_1, m_2 + k; \frac{\rho m_2 \gamma_2}{\Omega_2(1-\rho)} \right) \right\}, \quad (3.27)$$

where ${}_1F_1(\cdot, \cdot; \cdot)$ is the Kummer confluent hypergeometric function [82], and $(m_1)_k$ denotes the Pochhammer symbol. Substituting (3.27) into (3.36) and using [82, (3.351.2)] to integrate over γ_1 , result in

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \sum_{k=0}^{\infty} C \left\{ \int_0^{\infty} d\gamma_2 e^{-\left(\frac{m_2}{\Omega_2(1-\rho)} + \frac{d_2}{\sin^2 \Phi} \right) \gamma_2} \gamma_2^{m_2+k-1} {}_1F_1 \left(m_2 - m_1, m_2 + k; \frac{\rho m_2 \gamma_2}{\Omega_2(1-\rho)} \right) \right. \\ \left. \times \left(\frac{m_1}{\Omega_1(1-\rho)} + \frac{d_1}{\sin^2 \Phi} \right)^{-m_1-k} \Gamma(m_1 + k) \right\}, \quad (3.28)$$

where C is a constant. After some rearrangement and the utilization of [82, (7.522.9)], (3.28) can be rewritten as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \sum_{k=0}^{\infty} C \left\{ \left(\frac{m_2}{\Omega_2(1-\rho)} + \frac{d_2}{\sin^2 \Phi} \right)^{-m_2-k} {}_2F_1(m_2 - m_1, m_2 + k; m_2 + k; M) \right. \\ \left. \times \Gamma(m_2 + k) \left(\frac{m_1}{\Omega_1(1-\rho)} + \frac{d_1}{\sin^2 \Phi} \right)^{-m_1-k} \Gamma(m_1 + k) \right\}. \quad (3.29)$$

In (3.29), ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ denotes the Gauss hypergeometric function [82] and M is defined as

$$M = \frac{\rho m_2}{\Omega_2(1-\rho) \left(\frac{m_2}{\Omega_2(1-\rho)} + \frac{d_{l,2}}{\sin^2 \Phi} \right)}. \quad (3.30)$$

Finally, utilizing the identity ${}_2F_1(a, b; b; c) = (1-c)^{-a}$ [82, (9.121.1)], we obtain

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ as

$$\begin{aligned} \mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \sum_{k=0}^{\infty} (m_2 \rho)^{k+m_2} \left(\frac{m_1}{m_1 + \Omega_1 d_1 - \Omega_1 d_1 \rho} \right)^{m_1+k} & (- (m_2 + \Omega_2 d_2) (\rho - 1))^{m_1-m_2} \\ & \times (m_1)_k \frac{1}{k!} (m_2 - \Omega_2 d_2 (\rho - 1))^{-m_1-k} \left(\frac{1}{\rho} - 1 \right)^{m_2}. \end{aligned} \quad (3.31)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ expressions in (3.12), (3.26) and (3.31) are first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

3.4 Multiple transmission stage with multiple transmit antenna

Now, we extend the given analysis to more flexible TMRC system model which is consisting of K orthogonal transmission stages. The some potential scenarios covered by this model can be listed as follows:

- Coded SISO link system model.
- Coded diversity system models including multiple transmitter antenna scenarios.
- Multiple retransmission HARQ system with single or multiple transmit antennas.
- Coordinated multipoint (CoMP) transmission and reception with multiple transmit and receive antennas from multiple antenna deployed at different locations.
- The downlink (DL) of non-cooperative multi-cellular multiple orthogonal transmission systems, assuming with multiple antennas per base station (BS) and multiple receiver at the user terminal (UT) can be assumed [89]. Our system model includes an arbitrary path loss and antenna correlation for each orthogonal stage.

In the described TMRC system model, the receiver side only performs processing of the received data for decoding, while co-phasing of transmitting symbol and feeding

the same information into each distributed antenna in each stage take place in the transmitter. This asymmetric complexity distribution between the transmitter and the receiver parts is inherent to Internet of Things (IoT) ecosystem where robustness of transmitter units against to failure and poor performance is required with increasing network intelligence.

Each transmission stage, k , employs $N^{(k)}$ transmit antennas, and one receive antenna as shown in Fig. 3.3. The channels between the transmit antennas and the receive antenna are an $N^{(k)} \times 1$ vector, $\mathbf{h}_k$. In a distributed antenna transmitter and faded scenarios, we assume that antenna elements co-located in a given cluster experience fading governed by a common parameter m_k for Nakagami- m cases and fading power Ω_k . We assume independent fading between the K orthogonal transmission stages, with possibly different shaping parameters $m_{k,i}$ for Nakagami- m cases and link powers $\Omega_{k,i}$.

Within each transmission stage, we consider spatial correlation governed by a matrix $\mathbf{R}_k$. The amount of correlation between co-located antennas i and j , i.e., the element $[\mathbf{R}_k]_{i,j} = \rho_{ij}$ follows an exponential decay, where some r_k , $\rho_{ij} = r_k^{|i-j|}$ [90, 91]. It is assumed that no correlation is present, ($\rho_{ij} = 0$), for antennas belonging to different clusters. While we have assumed the simple exponential decay correlation model, other models, such as the one-ring scattering model [92, 93] can also be used to compute the correlation coefficients ρ_{ij} . In the k -th transmission stage, the information symbols are convolutionally encoded and the resulting bits are assigned an output symbol, $s^{(k)}$, based on a specific constellation $\chi^{(k)}$. Each transmitter employs a precoder $\mathbf{w}_k$, matched to the normalized channel vector $\mathbf{h}_k$,

$$\mathbf{w}_k = \frac{\mathbf{h}_k^H}{\|\mathbf{h}_k\|}, \quad (3.32)$$

where $\|\cdot\|$ denotes the Euclidean norm. The received signal during k th transmission stage is given by

$$\begin{aligned} r_k &= \mathbf{h}_k^H \mathbf{w}_k s^{(k)} + n_k \\ &= \alpha_k s^{(k)} + n_k, \end{aligned} \quad (3.33)$$

where $\alpha_k = \frac{\mathbf{h}_k^H \mathbf{h}_k}{\|\mathbf{h}_k\|} = \|\mathbf{h}_k\|$ and n_k is the additive white Gaussian noise (AWGN) sample with zero-mean and $N_0/2$ variance per dimension.

The probability of decoding an erroneous codeword vector $\hat{\mathbf{s}} = [\hat{s}^{(1)} \hat{s}^{(2)} \dots \hat{s}^{(K)}]$ at the receiver in place of the transmitted codeword $\mathbf{s} = [s^{(1)} s^{(2)} \dots s^{(K)}]$ where

Figure 3.3: TMRC with multiple orthogonal transmission stage downlink system model.

$\exists k, s^{(k)} \neq \hat{s}^{(k)}$ can be expressed in terms of the probability of an erroneous decoder transition ($v \rightarrow \bar{v}$) for an actual transition state at encoder ($u \rightarrow \bar{u}$) and it is denoted by $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$. For a given channel coefficient set, $\mathbf{h}_1, \dots, \mathbf{h}_K$, the conditional $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be expressed as [47]

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})|\mathbf{h}_1 \dots \mathbf{h}_K} = \Pr \left(\sum_{k=1}^K |r_k - \alpha_k s^{(k)}|^2 - |r_k - \alpha_k \hat{s}^{(k)}|^2 \geq 0 | \mathbf{h}_1 \dots \mathbf{h}_K \right). \quad (3.34)$$

After some mathematical manipulations, Chernoff bound expression for (3.34) can be written as [81]

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})|\mathbf{h}_1, \dots, \mathbf{h}_K} = e^{-\sum_{k=1}^K d_k X_k}, \quad (3.35)$$

where $d_k = \frac{|s^{(k)} - \hat{s}^{(k)}|^2}{4N_0}$ and $X_k = \alpha_k^2$ is a sum of squared envelope Nakagami- m faded

coefficients or, equivalently, the sum of gamma variables. The unconditional $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be calculated using

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \int_0^{\infty} e^{-d_k X_k} f_{X_k}(X_k) dX_k, \quad (3.36)$$

where $f_{X_k}(X_k)$ denotes the PDF of X_k .

3.4.1 Rayleigh channels

For Rayleigh faded scenario, $f_{X_k}(X_k)$ can be obtained as [50]

$$f_{X_k}(X_k) = \sum_{i=1}^{N^{(k)}} \left(\prod_{p \neq i} \frac{\frac{1}{\Omega_{k,p}}}{\frac{1}{\Omega_{k,p}} - \frac{1}{\Omega_{k,i}}} \right) \frac{1}{\Omega_{k,i}} e^{-\frac{X_k}{\Omega_{k,i}}}, \quad (3.37)$$

then; utilizing [82, eq. 3.35.2] along with substituting (3.37) into (3.36) gives

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \sum_{i=1}^{N^{(k)}} \left(\prod_{p \neq i} \frac{\frac{1}{\Omega_{k,p}}}{\frac{1}{\Omega_{k,p}} - \frac{1}{\Omega_{k,i}}} \right) \frac{1}{1 + d_k \Omega_{k,i}}. \quad (3.38)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ in (3.38) is first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

3.4.2 Nakagami- m channels

We begin with the closed-form expression for the PDF of X_k for non-integer Nakagami fading parameters which can be written in terms of the Fox's $\bar{H}$ function [50], as

$$f_{X_k}(X_k) = \prod_{i=1}^{N^{(k)}} \left(\frac{m_{k,i}}{\Omega_{k,i}} \right)^{m_{k,i}} \bar{H}_{N^{(k)}, N^{(k)}}^{0, N^{(k)}} \left[e^{X_k} \left| \begin{array}{c} \Xi_{N^{(k)}}^{(1)} \\ \Xi_{N^{(k)}}^{(2)} \end{array} \right. \right], \quad (3.39)$$

where the explicit definition of Fox's $\bar{H}$ function is given in Table 3.1. The coefficient sets $\Xi_{N^{(k)}}^{(1)}$ and $\Xi_{N^{(k)}}^{(2)}$ are defined as [50]

$$\begin{aligned} \Xi_{N^{(k)}}^{(1)} &= \overbrace{\left(1 - \frac{m_{k,1}}{\Omega_{k,1}}, 1, m_{k,1} \right), \dots, \left(1 - \frac{m_{k,N^{(k)}}}{\Omega_{k,N^{(k)}}}, 1, m_{k,N^{(k)}} \right)}^{N^{(k)}\text{-bracketed terms}}, \\ \Xi_{N^{(k)}}^{(2)} &= \overbrace{\left(-\frac{m_{k,1}}{\Omega_{k,1}}, 1, m_{k,1} \right), \dots, \left(-\frac{m_{k,N^{(k)}}}{\Omega_{k,N^{(k)}}}, 1, m_{k,N^{(k)}} \right)}^{N^{(k)}\text{-bracketed terms}}. \end{aligned} \quad (3.40)$$

Substituting (3.39) into (3.36) gives

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \prod_{i=1}^{N^{(k)}} \left(\frac{m_{k,i}}{\Omega_{k,i}} \right)^{m_{k,i}} Z_k, \quad (3.41)$$

where Z_k is defined by

$$Z_k = \int_0^\infty e^{-d_k X_k} \bar{H}_{N^{(k)}, N^{(k)}}^{0, N^{(k)}} \left[e^{X_k} \left| \begin{array}{c} \Xi_{N^{(k)}}^{(1)} \\ \Xi_{N^{(k)}}^{(2)} \end{array} \right. \right] dX_k. \quad (3.42)$$

Using the explicit definition of Fox's $\bar{H}$ function given in Table 3.1, (3.42) can be rewritten in the form of Mellin-Barnes contour integral [50],

$$Z_k = \int_0^\infty e^{-d_k X_k} \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^{N^{(k)}} \Gamma(1 - \alpha_j + A_j)^{a_j}}{\prod_{j=1}^{N^{(k)}} \Gamma(1 - \beta_j + B_j)^{b_j}} e^{sX_k} ds dX_k. \quad (3.43)$$

Here, (α_j, A_j, a_j) and (β_j, B_j, b_j) correspond to the elements of j -th coefficient in (3.40). Utilizing the gamma function definition [82, 8.331.1], we obtain the following expression for the first integral in (3.43)

$$\int_0^\infty e^{-d_k X_k} e^{sX_k} dX_k = \frac{1}{d_k - s} = -\frac{\Gamma(s - d_k)}{\Gamma(s - d_k + 1)}. \quad (3.44)$$

Interchanging the order of integrals in (3.43) and using (3.44) and [50, (A.2)], Z_k can be expressed as

$$Z_k = -\oint_C \frac{\prod_{j=1}^{N^{(k)}} \Gamma(1 - \alpha_j + A_j)^{a_j}}{\prod_{j=1}^{N^{(k)}} \Gamma(1 - \beta_j + B_j)^{b_j}} \frac{\Gamma(s - d_k)}{\Gamma(s - d_k + 1)} ds = -\bar{H}_{N^{(k)}+1, N^{(k)}+1}^{0, N^{(k)}+1} \left[1 \left| \begin{array}{c} \Xi_{N^{(k)}}^{(1)}, (1 + d_k, 1, 1) \\ \Xi_{N^{(k)}}^{(2)}, (d_k, 1, 1) \end{array} \right. \right], \quad (3.45)$$

giving the result for $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ of

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \prod_{i=1}^{N^{(k)}} - \left(\frac{m_{k,i}}{\Omega_{k,i}} \right)^{m_{k,i}} \bar{H}_{N^{(k)}+1, N^{(k)}+1}^{0, N^{(k)}+1} \left[1 \left| \begin{array}{c} \Xi_{N^{(k)}}^{(1)}, (1 + d_k, 1, 1) \\ \Xi_{N^{(k)}}^{(2)}, (d_k, 1, 1) \end{array} \right. \right]. \quad (3.46)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ in (3.46) is first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

Special case: $D_{(u,v),(\bar{u},\bar{v})}$ for integer m parameter

While (3.39) gives the distribution characteristics of X_k for any value of $m_{k,i}$, it includes the Fox's $\bar{H}$ function which currently has limited availability in the standard mathematical packages. Considering integer valued m parameters for Nakagami- m fading channels simplifies the analysis in a significant way, where the results can be expressed as fractional products.

In the case of integer valued $m_{k,i}$, (3.39) simplifies to a closed-form PDF expression given by [94]

$$f_{X_k}(X_k) = \prod_{i=1}^{N^{(k)}} \left(\frac{m_{k,i}}{\Omega_{k,i}} \right)^{m_{k,i}} G_{\kappa,\kappa}^{\kappa,0} \left(e^{-X_k} \left| \begin{matrix} \psi_{\kappa}^{(1)} \\ \psi_{\kappa}^{(2)} \end{matrix} \right. \right), \quad (3.47)$$

where $G_{p,q}^{m,n} \left(x \left| \begin{matrix} \psi_{\kappa}^{(1)} \\ \psi_{\kappa}^{(2)} \end{matrix} \right. \right)$ denotes the Meijer's G function [82], $\kappa = \sum_{i=1}^{N^{(k)}} m_{k,i}$ is an integer, and the coefficient sets $\psi_{\kappa}^{(1)}$ and $\psi_{\kappa}^{(2)}$ are defined as [94]

$$\begin{aligned} \psi_{\kappa}^{(1)} &= \overbrace{\left(1 + \frac{m_{k,1}}{\Omega_{k,1}} \right), \dots, \left(1 + \frac{m_{k,N^{(k)}}}{\Omega_{k,N^{(k)}}} \right)}^{\kappa\text{-bracketed terms}}, \dots, \\ \psi_{\kappa}^{(2)} &= \overbrace{\left(\frac{m_{k,1}}{\Omega_{k,1}} \right), \dots, \left(\frac{m_{k,N^{(k)}}}{\Omega_{k,N^{(k)}}} \right)}^{\kappa\text{-bracketed terms}}, \dots. \end{aligned} \quad (3.48)$$

Averaging (3.35) over the PDFs of X_k in (3.47), $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be expressed by

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \prod_{i=1}^{N^{(k)}} \left(\frac{m_{k,i}}{\Omega_{k,i}} \right)^{m_{k,i}} Z_k, \quad (3.49)$$

where Z_k is given by

$$Z_k = \int_0^{\infty} e^{-d_k X_k} G_{\kappa,\kappa}^{\kappa,0} \left(e^{-X_k} \left| \begin{matrix} \psi_{\kappa}^{(1)} \\ \psi_{\kappa}^{(2)} \end{matrix} \right. \right) dX_k. \quad (3.50)$$

Beginning with (3.50), we first perform the change of variable $y = e^{-X_k}$ giving

$$Z_k = \int_0^1 y^{d_k-1} G_{\kappa,\kappa}^{\kappa,0} \left(y \left| \begin{matrix} \psi_\kappa^{(1)} \\ \psi_\kappa^{(2)} \end{matrix} \right. \right) dy. \quad (3.51)$$

Using [82, (9.31.5)] and the definition of the Meijer's G function [95], changing the order of integrals allows us to write

$$\begin{aligned} Z_k &= \int_0^1 y^{d_k-1} G_{\kappa,\kappa}^{\kappa,0} \left(y \left| \begin{matrix} \psi_\kappa^{(1)} \\ \psi_\kappa^{(2)} \end{matrix} \right. \right) dy \\ &= \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^{\kappa} \Gamma(\psi_{\kappa,j}^{(2)} - s)}{\prod_{j=1}^{\kappa} \Gamma(\psi_{\kappa,j}^{(1)} - s)} \frac{1}{s + d_k} ds. \end{aligned} \quad (3.52)$$

Using [82, 8.331.1] in (3.52) gives

$$Z_k = \frac{1}{2\pi i} \oint_C \frac{1}{\prod_{i=1}^{\kappa} \left(\frac{m_{k,i}}{\Omega_{k,i}} - s \right)} \frac{1}{s + d_k} ds, \quad (3.53)$$

and implementing the Cauchy's integral formula [96], which is

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz, \quad (3.54)$$

into (3.53) where

$$f(z) = \prod_{i=1}^{\kappa} \frac{1}{\frac{m_{k,i}}{\Omega_{k,i}} - z}, \quad a = -d_k, \quad (3.55)$$

results in

$$Z_k = \prod_{i=1}^{\kappa} \left(d_k + \frac{m_{k,i}}{\Omega_{k,i}} \right)^{-m_{k,i}}, \quad (3.56)$$

which when combined with (3.49) results in (3.57). Using the result, the final expression for $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ can be expressed as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \prod_{i=1}^{N^{(k)}} \left(1 + d_k \frac{\Omega_{k,i}}{m_{k,i}} \right)^{-m_{k,i}}. \quad (3.57)$$

Simplifying (3.57) for $N^{(1)} = 1, K = 1$ reduces to a SISO case derived in [47]¹.

Correlated case for general m with correlated antennas

We now consider correlation between the co-located transmit antenna elements. Considering transmit antennas deployed in a cluster, we take $\Omega_k = \Omega_{k,i}$ and $m_k = m_{k,i}, \forall i$. Here, the moment generating function (MGF) of X_k is given by [50]

$$\mathcal{M}_{X_k}(s) = \prod_{i=1}^{N^{(k)}} (1 - s\lambda_{k,i})^{m_k}, \quad (3.58)$$

where $\{\lambda_{k,i}\}_i^{N^{(k)}}$ are the eigenvalues of the matrix governing the correlation between the channel powers for the antenna elements. We denote this matrix by $\mathbf{A}_k$ where $\mathbf{A}_k = \mathbf{D}_k \times \mathbf{C}_k$ where $\mathbf{C}_k$ is a $N^{(k)} \times N^{(k)}$ symmetric positive definite matrix and $\mathbf{D}_k$ is a $N^{(k)} \times N^{(k)}$ diagonal matrix with entries Ω_k/m_k , which are given by [84]

$$\mathbf{C}_k = \begin{bmatrix} 1 & \sqrt{\nu_{12}} & \cdots & \sqrt{\nu_{1N^{(k)}}} \\ \sqrt{\nu_{21}} & 1 & & \sqrt{\nu_{2N^{(k)}}} \\ \vdots & & \ddots & \vdots \\ \sqrt{\nu_{N^{(k)}1}} & \cdots & \cdots & 1 \end{bmatrix}, \quad (3.59)$$

and

$$\mathbf{D}_k = \begin{bmatrix} \Omega_k/m_k & 0 & \cdots & 0 \\ 0 & \Omega_k/m_k & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \Omega_k/m_k \end{bmatrix}. \quad (3.60)$$

It is important to note that the correlation coefficients ν_{ij} seen in (3.59) are not the same with ρ_{ij} where the former one corresponds to power correlation, $|h_{k,i}|^2$ and $|h_{k,j}|^2$, while the latter governs the correlation between the envelopes, $|h_{k,i}|$ and $|h_{k,j}|$.

¹Note that [47, (9)] contains a typographical error, where the second Ω in each factor should be replaced by “1”.

Their relation is given by [97]

$$\rho_{ij} = \varphi(i, j) \left\{ {}_2F_1 \left(-\frac{1}{2}, -\frac{1}{2}; m_k; \nu_{ij} \right) - 1 \right\}, \quad (3.61)$$

where $\rho_{ij} = r_k^{|i-j|}$, and

$$\varphi(i, j) = \frac{\Gamma^2 \left(i + \frac{j}{2} \right)}{\Gamma(i) \Gamma(i+j) - \Gamma^2 \left(i + \frac{j}{2} \right)}, \quad (3.62)$$

$\Gamma(\cdot)$ is the gamma function and ${}_2F_1(\cdot, \cdot; \cdot; \cdot)$ denotes the Gauss hypergeometric function [82]. The PDF of X_k can be found via the inverse Laplace transform of (3.58) [50] and is given for non-integer m_k values by [98]

$$f_{X_k}(X_k) = \prod_{i=1}^{N^{(k)}} \left(\frac{1}{\lambda_{k,i}} \right)^{m_k} \bar{H}_{N^{(k)}, N^{(k)}}^{0, N^{(k)}} \left[e^{X_k} \left| \begin{array}{c} \Xi_{N^{(k)}}^{(1)} \\ \Xi_{N^{(k)}}^{(2)} \end{array} \right. \right], \quad (3.63)$$

where the coefficient sets $\Xi_{N^{(k)}}^{(1)}$ and $\Xi_{N^{(k)}}^{(2)}$ are defined as [98]

$$\begin{aligned} \Xi_{N^{(k)}}^{(1)} &= \overbrace{\left(1 - \frac{1}{\lambda_{k,1}}, 1, m_k \right), \dots, \left(1 - \frac{1}{\lambda_{k, N^{(k)}}}, 1, m_k \right)}^{N^{(k)}\text{-bracketed terms}}, \\ \Xi_{N^{(k)}}^{(2)} &= \overbrace{\left(-\frac{1}{\lambda_{k,1}}, 1, m_k \right), \dots, \left(-\frac{1}{\lambda_{k, N^{(k)}}}, 1, m_k \right)}^{N^{(k)}\text{-bracketed terms}}. \end{aligned} \quad (3.64)$$

Utilizing (3.44) and [50, (A.2)], $\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ for non-integer m_k values can be expressed as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \prod_{i=1}^{N^{(k)}} \left(\frac{1}{\lambda_{k,i}} \right)^{m_k} Z_k, \quad (3.65)$$

where Z_k is given as

$$Z_k = -\bar{H}_{N^{(k)}+1, N^{(k)}+1}^{0, N^{(k)}+1} \left[1 \left| \begin{array}{c} \Xi_{N^{(k)}}^{(1)}, (1 + d_k, 1, 1) \\ \Xi_{N^{(k)}}^{(2)}, (d_k, 1, 1) \end{array} \right. \right]. \quad (3.66)$$

Correlated case for integer m with correlated antenna

For integer m_k , (3.47) can be modified to include correlation using the same approach as in general m cases. The resulting PDF is given by

$$f_{X_k}(X_k) = \prod_{i=1}^{N^{(k)}} \left(\frac{1}{\lambda_{k,i}} \right)^{m_k} G_{\kappa,\kappa}^{\kappa,0} \left(e^{-X_k} \left| \begin{matrix} \psi_{\kappa}^{(1)} \\ \psi_{\kappa}^{(2)} \end{matrix} \right. \right), \quad (3.67)$$

where

$$\psi_{\kappa}^{(1)} = \overbrace{\left(1 + \frac{1}{\lambda_{k,1}} \right), \dots, \left(1 + \frac{1}{\lambda_{k,N^{(k)}}} \right)}^{N^{(k)} m_k \text{--bracketed terms}} \quad \psi_{\kappa}^{(2)} = \overbrace{\left(\frac{1}{\lambda_{k,1}} \right), \dots, \left(\frac{1}{\lambda_{k,N^{(k)}}} \right)}^{N^{(k)} m_k \text{--bracketed terms}}. \quad (3.68)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ for integer m_k values can be expressed as

$$\mathcal{D}_{(u,v),(\bar{u},\bar{v})} = \prod_{k=1}^K \prod_{i=1}^{N^{(k)}} \left(\frac{1}{\lambda_{k,i}} \right)^{m_k} Z_k, \quad (3.69)$$

where

$$Z_k = \left(d_k + \frac{1}{\lambda_{k,i}} \right)^{-m_k}. \quad (3.70)$$

$\mathcal{D}_{(u,v),(\bar{u},\bar{v})}$ expressions in (3.46), (3.57), (3.65), and (3.69) are first substituted into (2.11) and then used in (2.14) and (2.13) to obtain the BER upper bound.

Other correlation models

The exponential model was selected in this thesis due to its simplicity. It was also shown that exponential correlation model can be directly applicable to massive-MIMO scenarios where a uniform planar antenna (UPA) array is considered [99]. On the other hand, different correlation models have also been developed for other physical conditions. For example, the one ring spatial correlation model has an advantage of fitting the power angular spectrum and is widely used in different standards, including IEEE 802.11 wireless standards [100]. For example, in the case of the one ring model with a Laplacian angular distribution, the correlation, ρ_{ij} , can be calculated from

$$\rho_{ij} = \frac{1}{K} \int_{\phi_0^l - \pi}^{\phi_0^l + \pi} e^{-\frac{\sqrt{2}}{\theta} |\phi_l - \phi_0^l - i2\pi d_s(i,j) \sin(\phi_l)|} d\phi_l, \quad (3.71)$$

where K is the normalization constant, $d_s(i, j)$ is normalized distance between the i th and j th antenna, ϕ_l is actual direction of arrival (DoA), and ϕ_0^l denotes the central azimuth angle. However, since this modification is trivial and the thesis focus is on the BER bound computation, which already includes a complex system model, the exponential correlation model is selected as only correlation model. In other words, the presented framework is based on the eigenvalues of $\mathbf{A}_k$. Any numeric value of ρ_{ij} generated by an exponential correlation model can be obtained from other correlation models as well; then, the same analytical framework can be easily adopted for any given correlation model.

3.5 Extension to turbo-trellis coded cases

TTCM encoders were emerged as a powerful coding technique which combines the powerful design of the binary-turbo coding [51] with larger constellation sizes [52]. Basically, information bits are fed into two convolutional encoders, where a bit-interleaver and a symbol-based interleaver are employed before and after the second encoder, respectively. Output symbols from the first encoder, $\mathbf{s}^{(1)}$, is selected from a M -ary constellation while the interleaved version of original information bits are assigned into the output symbols, $\mathbf{s}^{(2)}$ [52, 66]. After having M -ary symbols from both the first TCM and the second TCM encoders, $\mathbf{s}^{(1)}$ and $\mathbf{s}^{(2)}$, the output symbols of TTCM encoded symbols, $\mathbf{s}$, can be selected as follows

$$\mathbf{s} = \left[s_1^{(1)} s_2^{(2)} s_3^{(1)} s_4^{(2)} s_5^{(1)} s_6^{(2)} \dots \right], \quad (3.72)$$

where $\mathbf{s}^{(1)} = \left[s_1^{(1)} s_2^{(1)} s_3^{(1)} s_4^{(1)} \dots \right]$, $\mathbf{s}^{(2)} = \left[s_1^{(2)} s_2^{(2)} s_3^{(2)} s_4^{(2)} \dots \right]$, and $s_t^{(i)}$ denotes an output symbol chosen from i th TCM encoder at the time instant t ($i \in \{1, 2\}, t \geq 0$).

A typical error performance of turbo coded systems tends to fall into two different characteristics which are named as “waterfall region” and “error-floor region” [51]. Considering the performance criteria inside the waterfall region is based on EXIT chart analysis [53], the error performance analysis which considers the irregular constellation for TTCM cases is given only for error-floor region. In connection with the process of obtaining a generating function in convolutionally coded cases, the concept of a hyper-trellis, a pair of product state of each encoder, was introduced in [101], which combines the analysis of two encoders. Following the same manner with the

analysis for QR cases in [66], each entry of product state matrix for the case of TTCM encoder can be written as

$$\mathbf{S}_{(u_1, u_2, v_1, v_2), (\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2)} = \Pr(u_1 \rightarrow \bar{u}_1 | u_1) \mathcal{I}^{\mathcal{W}(u_1 \rightarrow \bar{u}_1 \oplus v_1 \rightarrow \bar{v}_1)} \mathcal{D}_{(u_1, u_2, v_1, v_2), (\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2)}, \quad (3.73)$$

where

$$\mathcal{D}_{(u_1, u_2, v_1, v_2), (\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2)} = \mathcal{D}_{(u_1, v_1), (\bar{u}_1, \bar{v}_1)} \mathcal{D}_{(u_2, v_2), (\bar{u}_2, \bar{v}_2)} P_{\Pi}^N. \quad (3.74)$$

Herein, $\mathbf{S}_{(u_1, u_2, v_1, v_2), (\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2)}$ is hyper-trellis version of two product states, $\mathbf{S}_{(u_1, v_1), (\bar{u}_1, \bar{v}_1)}$ and $\mathbf{S}_{(u_2, v_2), (\bar{u}_2, \bar{v}_2)}$ already given in (2.11). Here, P_{Π}^N accounts for interleaver effect used in the encoder and is related to the number of possible error events having w_o Hamming weight in odd-indices and w_e for even indices with existence of N information bits where l is the number of information bit per one symbol. In order to take further step in the analysis, it is assumed that the interleavers used in the encoder are uniform as in [101]. Then, the explicit definition of P_{Π}^N can be given by [102]

$$P_{\Pi}^N = \frac{1}{\binom{N/2}{w_{o,1}} \binom{N/2 - w_{o,1}}{w_{o,2}} \dots \binom{N/2 - w_{o,l-1}}{w_{o,l}}} \times \frac{1}{\binom{N/2}{w_{e,1}} \binom{N/2 - w_{e,1}}{w_{e,2}} \dots \binom{N/2 - w_{e,l-1}}{w_{e,l}}}. \quad (3.75)$$

Once $D_{(u_1, v_1), (\bar{u}_1, \bar{v}_1)}$ and $D_{(u_2, v_2), (\bar{u}_2, \bar{v}_2)}$ are obtained from convolutional encoder analysis, entry of the product-state of hyper-trellis for TTCM encoders is calculated by (3.73). To reduce the complexity of calculating the generating function, it is assumed that j -times single error event occurring from a good state and ending a good state [103]. As a result, the probability of j -times error probability can be expressed as

$$[E_i(I)]^j = \left[(\mathbf{1}^T \mathbf{S}_{\mathcal{GB}})^T [\mathbf{I} - \mathbf{S}_{\mathcal{BB}}]^{-1} \mathbf{S}_{\mathcal{BG}} \mathbf{1} \right]^j. \quad (3.76)$$

The approximated bit error probability of the TTCM encoder in the error floor region can be obtained as

$$P_b \approx \sum_{j_1}^N \sum_{j_2}^N \frac{N^{(j_1+j_2)} [E_i(I)]^{j_1} [E_i(I)]^{j_2}}{j_1! j_2!}, \quad (3.77)$$

where the approximation term in (3.77) is a result of the Stirling approximation which is used in the calculation for each convolutional encoder, as in the similar one given in [103].

3.6 Simulation results

In this section, the proposed performance bounds are plotted along with Monte Carlo simulations with different system model and channel scenarios.

3.6.1 Two transmission stage with single transmit antenna

In this section, simulation results are presented to validate the upper bound BER expressions derived in Section 3.3. 4-ary and 64-ary signalling are considered and the selected convolutional encoders, along with the values of other key parameters are listed in Table 3.2. The infinite sum seen in (3.31) for the case of correlated fading with $m_1 \neq m_2$ was truncated after 15 terms.

We first consider Scenarios I and II with 4-ary signalling and a rate $R = 1/2$ convolutional encoder $[2, 1]_8$. The arbitrary 4-ary constellations given in Table 3.2 were obtained from a standard uniform distribution generator under an average symbol energy constraint. By means of that, the locations of each constellation point for a given modulation order, M , can be totally arbitrary on the 2D space. Based on the QR criteria described in [45], Constellation-I in conjunction with the above encoder results in a irregular system. In contrast, to enable a comparison to the previous bounds in the literature which are obtained based on the generating function [104, 105], Constellation-II was chosen to satisfy QR criteria. Fig. 3.4 shows the BER obtained by simulation and the developed upper bound BER expression using (2.13) with (3.12) as well as the BER bound obtained by the conventional generating function calculation [104, 105].

As it can be seen from Fig. 3.4, the proposed BER bound expression is very tight for both QR and irregular scenarios. For the QR case (Scenario II), the conventional and proposed methods are almost identical, but as expected, the conventional bound fails to predict the performance of the irregular scenario.

Scenarios III and IV consider 64-ary signalling and a rate $R = 1/3$ convolutional encoder, $[133, 171, 165]_8$, deployed in downlink control information (DCI) specified

in the 3GPP LTE standard [76]. To generate arbitrary 64-ary constellations, the rotation technique in [79] was used, with $\theta = 0$ and $\pi/5$, and $\theta = 0$ and $\pi/8$, for transmission stages 1 and 2, respectively. Note that we were unable to generate a 64-ary constellation which satisfies QR when paired with above coder. Scenario III features a 0.46 dB SNR difference between the received signal levels in the transmission stages. As seen in Fig. 3.4, the upper bounds obtained by (3.26) and (3.31) show good agreement with the simulated BER results in moderate and high SNR regions. It is seen that for the scenarios considered, the BER upper bounds are fairly tight in the high SNR regions with $\text{BER} \leq 10^{-2}$.

Simulation results herein consider only a few specific scenarios due to space limitations. To give more solid validity check for the upper bound expression and measure its tightness over different realizations, the proposed BER bound is evaluated accuracy for a large number of randomly generated 4-ary constellations by using the given Algorithm 1. Specifically, for a BER target, $P_{b,th}$, we calculate the error $\Delta\gamma$ between the actual, simulated SNR required for $P_{b,th}$, and the SNR obtained using the bound derived in 3.3. By plotting the CDF of $\Delta\gamma$, the characteristics of both the independent and correlated expressions can be investigated for a broad range of constellation pairs. As seen in Fig. 3.5, the proposed upper bound BER expressions show better agreement for independent cases with both equal and unequal Nakagami- m shaping parameters. The SNR discrepancy obtained is limited to around 0.3 dB. Thus, the proposed method performs well over most of the realizations.

3.6.2 Multiple transmission stage with multiple transmit antenna

In order to test the proposed BER bound expressions in 3.4 over multiple transmit multi-orthogonal transmission stage cases, four different irregular 4-ary constellation scenarios with a rate $R = 1/2$ convolutional encoder $[2, 1]_8$ for non-integer m fading are first considered. The 2D irregular constellations are generated by a random number generator and are listed in Table 3.3, along with the other simulation parameters. Independent fading is assumed in the first two scenarios, while Scenario-III and Scenario-IV include spatial correlation. Scenario II represents distributed antenna

Algorithm 1: Testing accuracy of the upper bound BER expression over wide range of realizations.

Input: A number of the realizations N ,
a specific P_b threshold value as $P_{b,th}$,
channel parameters $\Omega_1, \Omega_2, \rho, m_1, m_2$,
step size τ (dB)

Output: the CDF of variable X measuring SNR (dB) discrepancy between simulation and analytical values

```

1 for  $n = 1 : N$  do
2   Generate a Gaussian distributed constellation pair
3   Set  $E_b/N_{0,sim} = 0$  dB,  $E_b/N_{0,appr} = 0$  dB
4   Find & save  $P_{b,sim}$  at 0 dB
5   Calculate & save  $P_{b,appr}$  at 0 dB
6   while  $P_{b,sim} > P_{b,th}$  do
7     Generate  $h_{l,1}$  and  $h_{l,2}$ 
8     Increase  $E_b/N_{0,sim} \rightarrow E_b/N_{0,sim} + \tau$  & save
9     Increase  $E_b/N_{0,appr} \rightarrow E_b/N_{0,appr} + \tau$  & save
10    Find  $P_{b,sim}$  & save
11    Calculate  $P_{b,appr}$  & save
12  Get  $(E_b/N_0)_{sim,th}, (E_b/N_0)_{appr,th}$  yielding  $P_{b,th}$ 
13  from  $(E_b/N_{0,sim}, P_{b,sim})$  &  $(E_b/N_{0,appr}, P_{b,appr})$  values
14  Then,  $\Delta\gamma[n] = \left| (E_b/N_0)_{sim,th} - (E_b/N_0)_{appr,th} \right|$ 
15 return Plot the CDF of  $\Delta\gamma$ ,  $F(\Delta\gamma)$ 

```

transmission, where the fading parameters $m_{k,i}$ and $\Omega_{k,i}$ can vary between antennas. For instance, it features 1.1 dB SNR difference between 2nd transmit and 1st transmit antennas in the 1st transmission stage.

For Scenarios III and IV, the eigenvalues of the matrix of $\mathbf{A}_k$ are required for the calculation of (3.65). Using $\rho_{ij} = r_k^{|i-j|}$ and (3.61), $\mathbf{C}_k$ is first calculated based on the values of r_k given in Table 3.3, and then $\mathbf{C}_k$ is multiplied with (3.60) which is defined as $\mathbf{A}_k$. In Scenario III, $\mathbf{A}_1$ can be calculated as

$$\mathbf{A}_1 = \begin{bmatrix} 0.3125 & 0.2806 & 0.2517 \\ 0.2806 & 0.3125 & 0.2806 \\ 0.2517 & 0.2806 & 0.3125 \end{bmatrix}, \quad (3.78)$$

where its eigenvalues are given by $\{\lambda_{1,i}\}_i^3 = \{0.0220, 0.0608, 0.8547\}$. In Scenario-IV, the matrices of $\mathbf{A}_1$ and $\mathbf{A}_2$ given by

$$\mathbf{A}_1 = \begin{bmatrix} 0.3333 & 0.2811 \\ 0.2811 & 0.3333 \end{bmatrix}, \quad (3.79a)$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.4667 & 0.0 \\ 0.0 & 0.6667 \end{bmatrix}, \quad (3.79b)$$

with eigenvalues $\{\lambda_{1,i}\}_i^2 = \{0.0523, 0.6144\}$ and $\{\lambda_{2,i}\}_i^2 = \{0.4667, 0.6667\}$, respectively. Note that in some scenarios, the correlation characteristic requires the use of (3.46) and (3.65) in the BER bound calculation. For instance, although 1st orthogonal stage shows correlation between its transmit antennas, there is only one transmit antenna deployed in the second transmission stage.

For the above scenarios, Fig. 3.6 shows the BER values generating by Monte Carlo simulations and the derived upper bound BER expressions resulting from (3.46) for independent and (3.65) for correlated cases. Fig. 3.6 demonstrates that the developed upper bound BER expressions yield good agreement with simulations for both independent and correlated fading cases.

In the case of 64-ary signalling, a rate $R = 1/3$ convolutional encoder [133, 171, 165]₈, specified in the 3GPP LTE standard for use in downlink control

information (DCI) [76], is used. Integer valued Nakagami- m values are only considered for these scenarios for simplicity. To generate irregular 64-ary constellations, the technique given in [79], originally proposed for generating non-uniform M -ary constellations. Specifically, conventional 64-QAM is first separated into two subsets, similar to the first step of the set partitioning process in conventional TCM design, then one of these subsets is rotated by a specific angle θ_k for each orthogonal stage. Instead of generating 64-ary irregular constellations for all cases, 64-ary nonuniform constellations given in [106] are also used in Scenario VI and Scenario VII. These, along with the values of the number of orthogonal stages K , Nakagami- m fading parameters $m_{k,i}$ and the number of transmitting antenna $N^{(k)}$, correlation parameter r_k , and average fading powers $\Omega_{k,i}$ for each stage are listed in Table 3.4. The correlation matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ for Scenario-VII are given by

$$\mathbf{A}_1 = \begin{bmatrix} 0.2500 & 0.2243 & 0.2011 \\ 0.2243 & 0.2500 & 0.2243 \\ 0.2011 & 0.2243 & 0.2500 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 0.3333 & 0.2806 \\ 0.2806 & 0.3333 \end{bmatrix}, \quad (3.80)$$

with eigenvalues $\{\lambda_{1,i}\}_i^3 = \{0.0178, 0.0489, 0.6834\}$ and $\{\lambda_{2,i}\}_i^2 = \{0.0527, 0.6140\}$, respectively. In Scenario-VIII, $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$ can be calculated as

$$\mathbf{A}_1 = \begin{bmatrix} 0.3000 & 0.2855 \\ 0.2855 & 0.3000 \end{bmatrix}, \quad (3.81a)$$

$$\mathbf{A}_2 = \begin{bmatrix} 0.8000 & 0.0000 \\ 0.0000 & 0.8000 \end{bmatrix}, \quad (3.81b)$$

$$\mathbf{A}_3 = \begin{bmatrix} 1.0000 \end{bmatrix}, \quad (3.81c)$$

with eigenvalues $\{\lambda_{1,i}\}_i^2 = \{0.0145, 0.5855\}$, $\{\lambda_{2,i}\}_i^2 = \{0.8000, 0.8000\}$, and $\{\lambda_{3,i}\}_i^1 = \{1.0000\}$, respectively. Fig. 3.7 demonstrates that the developed upper bound BER expressions yield good agreement with simulations in the moderate and high SNR regions, where P_b drops below 10^{-2} .

Having presented two bound comparisons and eight different numerical scenarios

working with 4-ary and 64-ary signalling cases along with different types of the encoders illustrates that tightness of the upper bound curves can vary with the numerical scenarios. On the other hand, the proposed bound are valid for all cases regardless of whether the combination convolutional encoder and constellation are QR or not. From this point of view, the proposed error analysis gives a comprehensive tool for evaluating convolutional coded system without any restriction on the encoder and constellation used.

3.6.3 Turbo-trellis coded cases

Besides the convolutional coded cases, the bound for TTCM encoders given in (3.77) in Section 3.5 is validated over the AWGN channel with a parameter set of $(K = 1, N^{(1)} = 1)$. Here, the TTCM encoder is considered [52] along with 8-ary signalling. At each constituent convolutional decoder, a symbol-by-symbol maximum a posteriori probability (MAP) decoder operates in the log domain, the information block length is chosen as 1024 and the number of turbo decoder iterations, $Q = 8$. As seen in Fig. 3.8, the proposed analysis for the error floor region, (3.77), shows good agreement with simulated results for all cases. In these simulations, three different 8-ary signalling cases are considered, which are $\{1 - 1i, 1, 1 + 1i, 1i, -1 + 1i, -1, -1 - 1i, -1i\}$, $\{-1.6 + 0.8i, -0.6 + 0.8i, 0.6 + 0.8i, 1.6 + 0.8i, -1.6 - 0.8i, -0.6 - 0.8i, 0.6 - 0.8i, 1.6 - 0.8i\}$, and $\{0.28 - 1.1i, -0.01 - 0.16i, 0.13 + 0.82i, 1.19 + 0.48i, -0.80 - 0.59i, -1.20 + 0.69i, 0.98 - 0.54i, -0.14 - 0.40i\}$, respectively.

3.6.4 Discussion over QR and irregular cases

For the purpose of highlighting the necessity of the proposed product-state matrix technique for irregular cases, multiple transmit antenna cases, TMRC with a rate $R = 1/2$ convolutional encoder $[2, 1]_8$, are used for one QR and one irregular cases. The QR case is considered using the regular QAM constellation, $\{1 + 1i, -1 + 1i, 1 - 1i, -1 - 1i\}$ while the irregular scenario uses a 4-ary signal constellation, $\chi^{(1)}$, given in Table 3.3. It is important to note that each $\chi^{(i)}$ was chosen one which shows irregularity along with $[2, 1]_8$ convolutional encoder, based on (2.5), among the sample set generated by a standard uniform distribution generator under an average symbol

energy constraint ($i \in \{1, 2, 3\}$). For this scenario we consider a single transmission stage with two transmit antennas ($K = 1, N^{(1)} = 2$) with channel parameters $m_1 = m_2 = 2.5$ and unit fading power. Unlike the single transmit antenna cases, the generating function calculation utilizes (3.46) in both conventional [81, (13.50)] and product-state matrix methods, since (3.46) is required for the transition probability for the TMRC system considered. It is important to note that upper bound BER curves for conventional method are plotted based on (2.13) without any tightening constants such that given in [104] so the comparison herein aims to demonstrate that the conventional method no longer produces a valid bound in irregular cases, rather than which method gives the tighter bound.

In Fig. 3.9, the proposed BER bound expression is very tight for both QR and irregular cases while the method using conventional generating function is valid only for the QR case. This demonstrates that in the irregular cases, the conventional BER bound does not represent a reliable error performance metric. This implies that the product-state matrix technique offers the important flexibility to be used independently of the constellation and encoder employed.

3.7 Conclusion

In this chapter, the derivation of proposed BER upper bound expressions for single transmit antenna two-orthogonal transmission stage and multiple transmit antenna multi-orthogonal transmission stage are presented. Then, the validity of these expressions is validated by Monte Carlo simulations. It has been seen that the proposed BER bound expressions based on the generating function calculation from the product-state matrix are compatible with any convolutional encoder and non-uniform signal constellation where symbol locations can be completely arbitrary. This is in contrast to previous methods, which can only be used when the encoder and constellation satisfy the quasi-regularity condition, likely to be violated in future systems where optimized irregular constellations are used. In addition, the presented analysis is extended to the turbo-trellis coded systems to enable the irregular constellation optimization in the presence of more advanced error correcting coding techniques.

Table 3.1: Definition of the Fox's $\bar{H}$ and Meijer's G functions.

The definition of the Fox's $\bar{H}$ function results from the following contour integral which is given as [50]	
$\bar{H}_{p,q}^{m,n} \left[z \middle \begin{matrix} (\alpha_j, A_j, a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j, b_j)_{m+1,q} \end{matrix} \right]$	$= \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^n \{\Gamma(1-\alpha_j+A_j s)\}^{a_j} \prod_{j=1}^m \Gamma(\beta_j-B_j s)}{\prod_{j=n+1}^p \Gamma(\alpha_j-A_j s) \prod_{j=m+1}^q \{\Gamma(1-\beta_j+B_j s)\}^{b_j}} z^s ds,$
where $\Gamma(\cdot)$ denotes the gamma function [82]. Herein, z can take real or complex values except $z = 0$ and the following inequalities are required to satisfy:	
$1 \leq m \leq q, 0 \leq n \leq p, A_j > 0$ for $j = 1, \dots, p, B_j > 0$ for $j = 1, \dots, q, \alpha_j, \beta_j \in \mathbb{C}, a_j, b_j \notin \mathbb{Z}.$	
As the special case of the Fox's $\bar{H}$ function where $A_j = B_j = 1$ and $a_j = b_j = 1$ for all j , the Meijer's G function is defined as [82]	
$G_{p,q}^{m,n} \left[z \middle \begin{matrix} (\alpha_1, \dots, \alpha_p) \\ (\beta_1, \dots, \beta_q) \end{matrix} \right]$	$= \frac{1}{2\pi i} \oint_C \frac{\prod_{j=1}^n \Gamma(1-\alpha_j+s) \prod_{j=1}^m \Gamma(\beta_j-s)}{\prod_{j=n+1}^p \Gamma(\alpha_j-s) \prod_{j=m+1}^q \Gamma(1-\beta_j+s)} z^s ds.$
where $\Gamma(\cdot)$ denotes the gamma function [82]. Herein, z can take real or complex values except $z = 0$ and the following inequalities are required to satisfy:	
$1 \leq m \leq q, 0 \leq n \leq p, \alpha_j, \beta_j \in \mathbb{C}, a_j, b_j \notin \mathbb{Z}.$	

Table 3.2: Monte Carlo simulation parameters for two-orthogonal transmission stage scenarios.

	Scenario	Ω_1	Ω_2	m_1	m_2	ρ
4-ary w/ [2, 1] ₈	I	1	1	1.5	1.5	0
	II	1	1	1.5	1	0
64-ary w/ [133, 171, 165] ₈	III	1	0.9	1	1.5	0.7
	IV	1	1	3.5	3.5	0.5
Constellation-I	$\{-2.17 + 0.14i, 0.66 - 0.84i, -0.06 + 0.29i, -0.62 - 0.13i\}$					
Constellation-II	$\{1.06 + 2.64i, -1.72 - 1.03i, -0.34 - 0.59i, 1.34 + 0.72i\}$

Figure 3.4: Bit error probability of the two-transmission scheme for arbitrary 4-ary and 64-ary signalling.

Table 3.3: Simulation parameters for multiple transmit antenna multi-orthogonal transmission stages for 4-ary signalling.

Scenarios	K	$N^{(k)}$	$m_{k,i}$	r_k	$\Omega_{k,i}$
I	2	$N^{(1)} = 2$	$m_{1,1} = m_{1,2} = 2.5$	$r_k = 0, \forall k$	$\Omega_{1,1} = \Omega_{1,2} = 1.0$
		$N^{(2)} = 3$	$m_{2,1} = m_{2,2} = m_{2,3} = 1.5$		$\Omega_{2,1} = \Omega_{2,2} = \Omega_{2,3} = 1.0$
II	3	$N^{(1)} = 2$	$m_{1,1} = 2.5, m_{1,2} = 1.2$	$r_k = 0, \forall k$	$\Omega_{1,1} = 0.9, \Omega_{1,2} = 0.7$
		$N^{(2)} = 1$	$m_{2,1} = 2.4$		$\Omega_{2,1} = 1.0$
		$N^{(3)} = 3$	$m_{3,1} = 1.8, m_{3,2} = 2.8, m_{3,3} = 2.5$		$\Omega_{3,1} = 0.8, \Omega_{3,2} = 0.9, \Omega_{3,3} = 1.0$
III	1	$N^{(1)} = 2$	$m_{1,1} = m_{1,2} = m_{1,3} = 3.2$	$r_1 = 0.8$	$\Omega_{1,1} = \Omega_{1,2} = \Omega_{1,3} = 1.0$
IV	2	$N^{(1)} = 2$	$m_{1,1} = m_{1,2} = 2.4$	$r_1 = 0.7$	$\Omega_{1,1} = \Omega_{1,2} = 0.8$
		$N^{(2)} = 2$	$m_{2,1} = 1.5, m_{2,2} = 1.8$	$r_2 = 0.0$	$\Omega_{2,1} = 0.7, \Omega_{2,2} = 1.0$
$\chi^{(1)} = \{-1.3295 - 0.5003i, -0.1550 + 1.8850i, -0.3707 + 0.8251i, -1.2487 + 0.1667i\}$					
$\chi^{(2)} = \{2.2094 - 0.8762i, -0.8014 + 0.7153i, -0.2460 - 0.0759i, -0.7526 + 0.7512i\}$					
$\chi^{(3)} = \{-1.9607 - 0.3067i, -0.7829 - 1.0952i, -0.5734 + 1.1526i, -1.0863 + 0.7867i\}$

Figure 3.5: CDF of $\Delta\gamma$ over 500 random constellation realizations for 4-ary signalling.

Table 3.4: Simulation parameters for multi-orthogonal transmission stages 64-ary signalling.

Scenarios	K	$N^{(k)}$	$m_{k,i}$	r_k	θ_k	$\Omega_{k,i}$
V	3	$N^{(1)} = 2$	$m_{1,1} = 2.0, m_{1,2} = 1.0$	$r_k = 0 \forall k$	$\theta_1 = 0$	$\Omega_{1,1} = 1.0, \Omega_{1,2} = 0.6$
		$N^{(2)} = 1$	$m_{2,1} = 2.0$		$\theta_2 = \pi/3$	$\Omega_{2,1} = 1.0$
		$N^{(3)} = 2$	$m_{3,1} = 3.0, m_{3,2} = 1.0$		$\theta_3 = \pi/6$	$\Omega_{3,1} = 0.8, \Omega_{3,2} = 1.0$
VI	2	$N^{(1)} = 2$	$m_{1,1} = 1.0, m_{1,2} = 2.0$	$r_k = 0, \forall k$	N/A	$\Omega_{1,1} = 1.0, \Omega_{1,2} = 0.8$
		$N^{(2)} = 2$	$m_{2,1} = m_{2,2} = 1.0$		Constellations in [106]	$\Omega_{2,1} = \Omega_{2,2} = 1.0$
VII	2	$N^{(1)} = 3$	$m_{1,1} = m_{1,2} = m_{1,3} = 4.0$	$r_1 = 0.8$	N/A	$\Omega_{k,i} = 1, \forall k, i$
		$N^{(2)} = 2$	$m_{2,1} = m_{2,2} = 3.0$	$r_2 = 0.7$	Constellations in [106]	
VIII	3	$N^{(1)} = 2$	$m_{1,1} = m_{1,2} = 2.0$	$r_1 = 0.9$	$\theta_1 = 0$	$\Omega_{1,1} = 0.6, \Omega_{1,2} = 0.6$
		$N^{(2)} = 2$	$m_{2,1} = 1.0, m_{2,2} = 2.0$	$r_2 = 0.0$	$\theta_2 = \pi/3$	$\Omega_{2,1} = 0.8, \Omega_{2,2} = 1.0$
		$N^{(3)} = 1$	$m_{3,1} = 1.0$	$r_3 = 0.0$	$\theta_3 = \pi/6$	$\Omega_{3,1} = 1.0$

Figure 3.6: Bit error probability for 4-ary signalling over different scenarios.

Figure 3.7: Bit error probability for 64-ary signalling over different scenarios.

Figure 3.8: Bit error probability for 8-ary signalling over different scenarios.

Figure 3.9: Bound comparison for 4-ary signalling with TMRC system.

Chapter 4

SNR-Adaptive Convolutionally Coded Transmission Model

4.1 Introduction

Due to the advent of powerful forward error correction (FEC) codes, reaching the Shannon limit has no longer been the main challenge for the physical-layer researchers. At the same time, some of emerging new family of applications in wireless networks bring new requirements not only for reliability, but also for massive connectivity, latency, and energy efficiency and only concentrating on particular parameter in network design might not be efficient way for tackling these requirements. For instance, the LDPC and turbo coded design might suffer from the latency perspective due to their iterative decoders and the non-iterative/one-shot decoding algorithms might become strong alternative where low-complexity communication is considered. As a support of this projection, it has been already observed that the well-known convolutional codes may result in a superior performance in comparison to the more advanced coding techniques when the latency requirements do not allow iterative decoding. In the meanwhile, the increased computing and data storage capacities encourage working with highly adaptive transmission techniques with constellations optimized for different channel conditions and SNR values. Motivated by this, we propose an SNR-adaptive convolutionally coded transmission model in which constellations are specifically designed for the given channel conditions and coding rates. Numerical examples demonstrate that promising performance improvements in terms of bit-error-rate and spectral efficiency can be obtained when constellations optimized specifically for the given convolutional encoders and the channel condition are used.

4.2 SNR-adaptive constellation optimizer

Since the proposed system bring the idea of the use different irregular optimized constellations as $\bar{\gamma}$ varies, the optimizer needs to perform search for SNR specific constellation procedure before the transmission starts. Therefore, the following parameters are essential for constructing a look-up table in order to choose the optimized output symbol points at the encoder:

- *Encoder characteristics*: Convolutional encoder and puncturing pattern (if exists).
- *Channel characteristics*: Fading parameter m .
- *Signalling characteristics*: Modulation order M .
- *Energy constraints*: Average symbol energy constraint, E_s .

Figure 4.1: SNR-adaptive constellation optimizer block diagram.

The block diagram of constellation optimizer is given in Fig. 4.1 to give general structure of the optimizer. The detailed description of how the optimizer works is presented in the next subsection.

4.3 Particle swarm optimization

In order to search for optimized symbol points locations, the optimizer uses an meta-heuristic evolutionary technique which is based on swarm intelligence. PSO technique is chosen here since it requires less computational load and it has fewer tuning parameter as compared to other evolutionary algorithms; it makes easy to implement the optimization framework in wireless communication fields including radio planning in LTE systems [107].

Algorithm 2: PSO algorithm for constellation search

Input: Encoder characteristics $T(D, \mathcal{I})$, fading parameter m , average fading power Ω , modulation order M , swarm size P , and optimizer parameters $(r_1, r_2, w, N_{\text{iter}})$.
Output: $\chi(m, \bar{\gamma}) = \{s_0, s_1, \dots, s_{M-1}\}$.

- 1 Initialize P particles' positions, x_i^0 , and the velocity of each particle, v_i^0 .
- 2 Evaluate fitness value of each particle from (4).
- 3 Set g_i^0 as the particle best value.
- 4 Calculate $g_p^0 = \min\{g_i^0\}$ as the swarm best value.
- 5 **for** $n = 1 : N_{\text{iter}}$ **do**
- 6 **for** $i = 1 : P$ **do**
- 7 $x_i^n = x_i^{n-1} + v_i^n$
- 8 $v_i^n = v_i^{n-1} + r_1 c_1 (g_i^{n-1} - x_i^{n-1}) + r_2 c_2 (g_p^{n-1} - x_i^{n-1})$
- 9 **if** $x_i^n \leq g_i^n$ **then**
- 10 $g_i^n = x_i^n$
- 11 Find g_p^n
- 12 **return** $x_i^{N_{\text{iter}}} \rightarrow \chi(m, \bar{\gamma}) = \{s_0, s_1, \dots, s_{M-1}\}$

Before the mathematical proof presented in [33], the regular simplex constellation was considered as the unique constellation which maximizes the minimum distance between symbol points for a given constellation [34]. Then, it was proven that one constellation might not be optimal for all SNR values. From this point of view, the optimizer in this work needs to perform its search for optimized constellations, $\chi(m, \bar{\gamma})$ as $\bar{\gamma}$ varies in addition to encoder type, coding rate and fading characteristics, m . The PSO algorithm used in finding SNR-adaptive irregular constellations is summarized in Algorithm 2 and MATLAB PSO optimizer is utilized in the optimization framework [1]. The detailed list of optimizer parameters for tuning the optimization framework can be seen in Table 4.1 and the values of the parameters assigned differently from default values are given in Table 4.2

Table 4.1: MATLAB PSO optimizer configuration parameters [1].

OPTIONS-STRUCTURE (OPTIONS).	
PopInitRange:	Range of random initial population
PopulationSize:	Number of particles in swarm
Generations:	Maximum number of generations
TimeLimit:	Maximum time before pso terminates
FitnessLimit:	Fitness value at which pso terminates
StallGenLimit:	Terminate if fitness value changes less than TolFun over StallGenLimit
StallTimeLimit:	Terminate if fitness value changes less than TolFun over StallTimeLimit
TolFun:	Tolerance on fitness value
TolCon:	Acceptable constraint violation
HybridFcn:	Function called after pso terminates
Display:	Display output in command window
OutputFcns:	User specified output function called after each generation
PlotFcns:	User specified plot function called after each generation
Vectorized	Specify whether fitness function is vectorized
InitialPopulation:	Initial position of particles
InitialVelocities:	Initial velocities of particles
InitialGeneration:	Initial generation number
PopInitBest:	Initial personal best of particles
CognitiveAttraction:	Attraction towards personal best
SocialAttraction:	Attraction towards global best
VelocityLimit:	Limit absolute velocity of particles
BoundaryMethod:	Set method of enforcing constraints

Table 4.2: SNR-adaptive constellation optimizer user-defined parameters.

CognitiveAttraction: 0.5000
ConstrBoundary: 'absorb'
AccelerationFcn: @psoiterate
DemoMode: 'off'
Display: 'final'
FitnessLimit: -Inf
Generations: 1000
HybridFcn: @fmincon
InitialPopulation: Gray-mapped M-QAM
PlotInterval: 1
PopInitRange: [2x2M double]
PopulationSize: 100
PopulationType: 'doubleVector'
SocialAttraction: 1.2500
StallGenLimit: 50
StallTimeLimit: Inf
TimeLimit: Inf
TolCon: 1.0000e-12
TolFun: 1.0000e-12

The optimization process aims for minimizing (2.13), while the following constraint on the symbol point locations is only taken into account, which guarantees the average transmitted symbol energy cannot exceed the average energy constraint, E_s ; it is given by,

$$\frac{1}{M} \sum_{i=0}^{M-1} |s_i|^2 < E_s, \quad s_i \in \mathbb{C}, \forall i. \quad (4.1)$$

Since the optimized constellations $\chi(m, \bar{\gamma})$ vary depending on $(\bar{\gamma}, m)$ and the encoder properties, each constellation need to be stored along with a label of corresponding $\bar{\gamma}$ and m values for future use in the transmission. By this way, an appropriate $\chi(m, \bar{\gamma})$ can be used in the transmitter based on the channel information from the receiver when transmission occurs in a given convolutionally coded scenario. The overall optimization process is summarized in Fig. 4.2.

Figure 4.2: The working principle of SNR-based constellation optimizer.

4.4 SNR-adaptive convolutionally coded transmission model

Once a set of optimized irregular constellations, $\chi(m, \bar{\gamma})$, is available for the transmission by storing them inside the look-up tables, SNR-adaptive convolutionally coded transmission model can be described in a detailed way. Basically, the SNR-adaptive convolutionally coded transmission model is given in Fig. 4.3. The information bits belonged by l th frame $\mathbf{b}_l = [b_{l,1} \cdots b_{l,N_b}]$ are encoded (one frame has N_b information bits) and the encoded bits, $\mathbf{c}_l = [c_{l,1} \cdots c_{l,N_c}]$, are fed to the bit-to-symbol mapper in which transmitting symbols with a length of N_s , $\mathbf{s}_l = [s_{l,1}, \cdots, s_{l,N_s}]$, are assigned from $\chi(m, \bar{\gamma})$. The fading coefficients of the channel for l th frame, $\mathbf{h}_l = [h_{l,1}, \cdots, h_{l,N_s}]$, is modeled by frequency non-selective Nakagami- m fast fading model with a shaping parameter m and an average fading power Ω . Then, the received signal for the i^{th}

Figure 4.3: The block diagram of SNR-adaptive convolutionally coded system.

symbol in the l^{th} frame can be written as

$$r_{l,i} = h_{l,i}s_{l,i} + n_{l,i}, \quad (4.2)$$

where $n_{l,i}$ is the additive white Gaussian noise (AWGN) sample with zero-mean and $N_0/2$ variance per dimension, $s_{l,i} \in \chi(m, \bar{\gamma})$ where the average received SNR can be explicitly defined as $\bar{\gamma} = \Omega E_s / N_0$. Here, E_s denotes the average symbol energy of $\chi(m, \bar{\gamma})$ and Ω can be interpreted as the path-loss term. Note that we assume that the channel coefficient stays constant during one symbol transmission and that each symbol is exposed to a different fading coefficient. This model fits for ITU pedestrian B channel model, 5G applications for transportation systems and 5G channel model up to 100 GHz [55, 108]. In the receiver side, soft-decision Viterbi decoding is carried out by assuming the perfect channel state information (CSI) and $\chi(m, \bar{\gamma})$ is used for a specific $\bar{\gamma}$, is known.

4.5 Simulation results

We have discussed SNR-adaptive convolutionally coded transmission model in the previous sections but practically it is required to test in terms of commonly used

performance metrics; such as, BER and spectral efficiency. We consider a SISO system in which $N_b = 1200$ and a rate-1/2 convolutional encoder $[5, 7]_8$ is used along with three different modulation and coding schemes (MCSs), which are MCS-1 (rate-1/2 and 16-ary signaling), MCS-2 (rate-3/4 and 16-ary signaling), and MCS-3 (rate-3/4 and 64-ary signaling), respectively. To increase the coding rate from the original rate 1/2 to 3/4, the puncturing patterns $[1\ 1\ 0; 0\ 1\ 1]$ and $[1\ 1\ 0\ 1; 0\ 1\ 1\ 1]$ are used over the second and third MCSs, respectively [109]. The detailed lists of optimized irregular constellation of each MCS scheme over different m and $\bar{\gamma}$ values are given in Table 4.4–4.9. The performance gain obtained from SNR-adaptive constellation optimization is measured by comparing the conventional M -QAM constellations and optimized constellations given in [10], which were originally proposed for the uncoded SISO system.

Figure 4.4: Simulated BER comparison: SNR-adaptive M -ary, SNR-independent conventional M -QAM, and SNR-independent [10] constellations.

In Fig. 4.4, simulated P_b curves are presented for three different MCSs for $m = 1$,

$m = 2$, and $m = 4$. After the optimization process is carried out within $\bar{\gamma} \in [8, 25]$ dB along with different m values and the coding rate, the optimized irregular constellations are stored with these values in the look-up tables. Fig. 4.4 also underscores the importance of encoder-based design since optimized constellation given in [10] for uncoded scenario gave poor performance, more than 2 dB loss, as compared to SNR-adaptive optimized irregular constellations. It can be seen in Table 4.3 that the symbol locations are changing with not only $\bar{\gamma}$ values but also the puncturing pattern and m values. In addition to the P_b curves, the spectral efficiencies (SEs) are also plotted in Fig. 4.5 for different m values, $m \in \{2, 4, \infty\}$. The SE values are calculated by considering coding rate, R , modulation order, M , and frame length, N_b , for each case, and SE can be expressed as [110, 111]

$$SE = \log_2(M) (1 - (1 - P_b)^{N_b}) R \quad (4.3)$$

For all cases, the use of irregular constellation yields better throughput performance as compared to the conventional QAM cases and constellations given in [10].

Table 4.3: Optimized irregular constellations $\chi(m, \bar{\gamma} [\text{dB}])$ for 16-ary signaling cases for $\Omega = 1$.

$s_i \in \chi(m, \bar{\gamma} [\text{dB}])$	MCS-1 $\chi(2, 12 \text{ dB})$	MCS-1 $\chi(2, 18 \text{ dB})$	MCS-2 $\chi(2, 12 \text{ dB})$	MCS-2 $\chi(4, 18 \text{ dB})$	MCS-1 & MCS-2 χ -conventional
$s_0 - '0000'$	$0.1358 + 0.6934i$	$0.1344 + 0.7968i$	$0.4180 + 0.2218i$	$0.4011 - 0.2021i$	$0.3162 + 0.3162i$
$s_1 - '0001'$	$0.2277 + 1.1093i$	$0.1378 + 1.0616i$	$0.3164 + 0.9437i$	$0.4224 + 0.8205i$	$0.3162 + 0.9487i$
$s_2 - '0010'$	$0.7812 + 0.2156i$	$0.8691 + 0.1748i$	$1.0867 + 0.2580i$	$1.2082 - 0.1436i$	$0.9487 + 0.3162i$
$s_3 - '0011'$	$1.1138 + 0.2908i$	$1.0965 + 0.1874i$	$0.7577 + 0.9934i$	$0.8400 + 1.0048i$	$0.9487 + 0.9487i$
$s_4 - '0100'$	$0.1536 - 0.6739i$	$0.0952 - 0.7598i$	$0.4231 - 0.1860i$	$0.5018 + 0.1137i$	$0.3162 - 0.3162i$
$s_5 - '0101'$	$0.2018 - 1.1258i$	$0.1499 - 1.0601i$	$0.3569 - 0.9393i$	$0.5327 - 0.8708i$	$0.3162 - 0.9487i$
$s_6 - '0110'$	$0.7925 - 0.2395i$	$0.9095 - 0.1928i$	$1.1119 - 0.2179i$	$1.0799 + 0.0848i$	$0.9487 - 0.3162i$
$s_7 - '0111'$	$1.1134 - 0.2804i$	$1.0940 - 0.1970i$	$0.8077 - 0.9658i$	$0.8946 - 0.8777i$	$0.9487 - 0.9487$
$s_8 - '1000'$	$-0.1485 + 0.7277i$	$-0.1192 + 0.7983i$	$-0.4002 + 0.1637i$	$-0.3726 - 0.1073i$	$-0.3162 + 0.3162i$
$s_9 - '1001'$	$-0.2129 + 1.1209i$	$-0.1313 + 1.0539i$	$-0.3536 + 0.8912i$	$-0.4074 + 0.8057i$	$-0.3162 + 0.9487i$
$s_{10} - '1010'$	$-0.7882 + 0.2351i$	$-0.8882 + 0.1854i$	$-1.0934 + 0.1829i$	$-1.1578 - 0.1654i$	$-0.9487 + 0.3162i$
$s_{11} - '1011'$	$-1.1099 + 0.3083i$	$-1.0766 + 0.2004i$	$-0.8106 + 0.9501i$	$-0.8566 + 0.9038i$	$-0.9487 + 0.9487i$
$s_{12} - '1100'$	$-0.1830 - 0.7536i$	$-0.1663 - 0.8178i$	$-0.3945 - 0.2104i$	$-0.4892 + 0.0895i$	$-0.3162 - 0.3162i$
$s_{13} - '1101'$	$-0.1653 - 1.1270i$	$-0.1139 - 1.0711i$	$-0.3249 - 0.9387i$	$-0.4336 - 0.8546i$	$-0.3162 - 0.9487i$
$s_{14} - '1110'$	$-0.8090 - 0.2594i$	$-0.8722 - 0.2025i$	$-1.0691 - 0.2598i$	$-1.0589 + 0.1394i$	$-0.9487 - 0.3162i$
$s_{15} - '1111'$	$-1.1028 - 0.2412i$	$-1.1185 - 0.1574i$	$-0.7736 - 0.9970i$	$-0.8248 - 0.9107i$	$-0.9487 - 0.9487i$

Figure 4.5: Spectral efficiency comparison with different rates: SNR-adaptive optimized constellations vs. SNR-independent conventional M -QAM.

4.6 SNR dependency

In this section, we would like to give some insight about the manner of irregular optimized constellations follow over the different SNR and m values. In order to focus on the changes seen in the symbol point locations, MCS-3 given in the previous section is considered. Within this direction, some snapshots of optimized irregular constellations only for 64-ary signalling cases are shown in Fig. 4.9. From this figure, it is not straightforward to get a conclusion on how the constellation shape is changing with respect to the value of m .

To facilitate the understanding of the analysis, two extreme points are given where the first one is defined as where P_b has become less than 10^{-2} and a chosen high average SNR value ($\bar{\gamma} = 50\text{dB}$) is the second one. The snapshots of the constellations are presented over Fig. 4.6 and Fig. 4.7 for the $m = 1, 2, 3, 4, 30$, respectively.

It can be seen from Fig. 4.6 that all optimized irregular constellations at the first operating points has almost identical shape which seems to be Gaussian distributed

symbol locations. However; the characteristics of the optimized irregular constellations show considerable difference at the final operating point $\bar{\gamma}$ as seen in Fig. 4.7. For instance, the constellation shape for $m = 1$ has reached out circular shape and the hole are between the origin and symbol location points is declining with each increase on m values and at the end, circular shape is disappeared. By considering the almost identical constellation shape at the first operating point, it can be said that the change of the symbol point locations are inversely proportional to an increase in m value.

After all uncertainties for the trend of the symbol point locations with the change of $\bar{\gamma}$ and m values, it would be good to have a indicator term which gives an idea how the optimized irregular constellations are sensitive to the above mentioned design parameters.

From this point of view, we start to define the indicator ΔXY for observing the average spatial differences within $X-Y$ axis with the different values of $\Delta\bar{\gamma} = |\bar{\gamma}_2 - \bar{\gamma}_1|$ between any two operation points ($\bar{\gamma}_1$ and $\bar{\gamma}_2$) where $\bar{\gamma} = \frac{E_s}{N_0}$. The average changes in symbol locations overall in M -ary signalling cases, ΔXY , can be defined as

$$\Delta XY = \sum_{i=1}^M \frac{|\chi(m, \bar{\gamma}_2)_i - \chi(m, \bar{\gamma}_1)_i|}{M}, \quad (4.4)$$

where $\chi(m, \bar{\gamma}_j)_i$ denotes the i th symbol point in the constellation designed for the average SNR value of $\bar{\gamma}_j$, $j \in \{1, 2\}$. The value of $\bar{\gamma}_1$ is set as fixed at the operating point which gives the first P_b value less than 10^{-2} ; then, $\Delta\bar{\gamma}$ is calculated based on it. For 64-ary signalling cases, ΔXY values along with varying m ($m = \{1, 2, 3, 30\}$) are plotted in Fig. 4.8. It can be seen from that the higher symbol locations variations are seen in lower m values. In the limit case where $m = 30$, the SNR dependency is almost non-exist.

4.7 Physical impairments

Testing the proposed irregular optimized constellations with the existence of I/Q imbalance, which mainly occurs because of the existence of the finite tolerances of capacitors and resistors in the TX/RX units, can develop the context of the manuscript in a considerable way. In order to model this imbalance, we have utilized the RX I/Q imbalance model given in [112]. Within the existence of I/Q imbalance at the receiver, the received signal for the i^{th} symbol in the l^{th} frame is rewritten as

$$\hat{r}_{l,i} = K_1 r_{l,i} + K_2 r_{l,i}^*, \quad (4.5)$$

where $r_{l,i}$ is the received signal under perfect I/Q balance. The coefficients K_1 and K_2 are defined as

$$K_1 = (1 + g_R e^{-j\phi_R})/2, \text{ and } K_2 = (1 - g_R e^{j\phi_R})/2. \quad (4.6)$$

Herein, g_R and ϕ_R are the RX amplitude and phase mismatch, respectively [112]. The measure of RX IQ imbalance can be expressed in terms of *image rejection ratios* (IRR_R), which is [112]

$$IRR_R = |K_2|^2 / |K_1|^2. \quad (4.7)$$

Note that $K_1 = 1$ and $K_2 = 0$ correspond to the perfect I/Q matching. In the simulated scenarios, IRR_R is chosen as -20 dB which corresponds to the values of $20\log(g_R) = 1.58$ dB and $\phi = 5^\circ$. In order to test irregular optimized constellation with the existence of the RX I/Q imbalance, third modulation and coding scheme, MCS-3, which employs 64-ary signaling cases is considered and the performance curves are presented in Fig. 4.10.

4.8 SNR mismatch

Since the proposed model requires to have exact knowledge of which optimized irregular constellations are used at both the transmitter and the receiver side during the transmission. However it would be interesting to investigate what happens if there is a mismatch in SNR values, in other words, the receiver side could sense a different

SNR value from actual SNR value used in the transmitter. For this purpose, the simulated results over MCS-3 are generated over different m values and the cases of that assuming only one constellation used in the transmitter at the receiver, 1-dB SNR mismatch in the optimized irregular constellations, for instance 14 dB optimized constellation is assumed in fact 15-dB optimized constellation is used and perfect SNR matching are investigated. From Fig. 4.11, it can be seen that 1 dB SNR mismatch does not cause any performance loss, while supposing 9 dB and 11 dB optimized constellation for all other $\bar{\gamma}$ values causes the performance loss for $m = 2$ and $m = 4$, respectively.

4.9 Conclusion

In this chapter, SNR-adaptive optimized irregular constellations are proposed for convolutionally encoded systems over fading channels. In this adaptive construct, the symbol locations vary with the received SNR, channel characteristics, as well as the encoder properties. To find the optimized irregular constellations for each SNR value, an upper bound on the bit error rate calculated from the product-state matrix method is used. By doing so, a large search space has become available to the optimizer by including irregular constellation and encoder pairs. From the simulations, it is observed that the more gains can be obtained with higher modulation order and spectral efficiency gains can be found in the order of 0.5 – 1.5dB depending on the modulation level and channel characteristics.

Figure 4.6: Optimized irregular constellations at the first operating point ($P_b \leq 10^{-2}$) over different m values ($m = 1, 2, 3, 4, 30$).

Figure 4.7: Optimized irregular constellations at high SNR over different m values ($m = 1, 2, 3, 4, 30$).

Figure 4.8: Symbol point locations variations over different m values.

Figure 4.9: Optimized irregular constellations over different m values (from left to right $m = 2, 4, 30$).

Figure 4.10: Testing irregular optimized constellations with existence of I/Q imbalance.

Figure 4.11: Simulated BER results in the case of SNR value mismatch at the transmitter and the receiver for MCS-3.

Table 4.4: Optimized irregular constellations proposed for MCS-1 along with $m = 1$ and $\Omega = 1$.

Rate 1/2 [5, 7] _s convolutional encoder w/o puncturing	
$\overline{\gamma}$	$\chi(1, \overline{\gamma}) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}$
7 dB	0.3689 + 0.3512i, 0.2350 + 0.9910i, 1.0137 + 0.2187i, 0.9265 + 0.8718i, 0.3634 - 0.3422i, 0.2219 - 0.9901i, 1.0046 - 0.2400i, 0.8977 - 0.8956i, -0.3378 + 0.3325i, -0.2480 + 1.0335i, -1.0259 + 0.2619i, -0.9004 + 0.8593i, -0.3471 - 0.3110i, -0.2592 - 1.0269i, -1.0175 - 0.2564i, -0.8957 - 0.8577i
8 dB	0.3653 + 0.3497i, 0.2344 + 0.9816i, 1.0129 + 0.2177i, 0.9269 + 0.8868i, 0.3598 - 0.3381i, 0.2056 - 0.9827i, 0.9983 - 0.2404i, 0.8945 - 0.9080i, -0.3284 + 0.3324i, -0.2367 + 1.0190i, -1.0158 + 0.2633i, -0.9022 + 0.8630i, -0.3481 - 0.3049i, -0.2518 - 1.0196i, -1.0192 - 0.2616i, -0.8954 - 0.8581i
9 dB	0.3456 + 0.3465i, 0.2295 + 0.9757i, 0.9899 + 0.2133i, 0.9216 + 0.8929i, 0.3561 - 0.3408i, 0.1949 - 0.9580i, 0.9904 - 0.2396i, 0.8938 - 0.9108i, -0.3162 + 0.3302i, -0.2244 + 1.0078i, -1.0092 + 0.2607i, -0.9133 + 0.8641i, -0.3352 - 0.3042i, -0.2310 - 1.0161i, -1.0005 - 0.2551i, -0.8921 - 0.8667i
10 dB	0.3395 + 0.3449i, 0.2103 + 0.9647i, 0.9890 + 0.2114i, 0.9267 + 0.8906i, 0.3473 - 0.3309i, 0.1915 - 0.9584i, 0.9865 - 0.2429i, 0.8944 - 0.9180i, -0.3210 + 0.3289i, -0.2224 + 1.0053i, -0.9949 + 0.2591i, -0.9117 + 0.8925i, -0.3254 - 0.3075i, -0.2117 - 1.0140i, -1.0001 - 0.2571i, -0.8979 - 0.8688i
11 dB	0.3346 + 0.3395i, 0.1923 + 0.9549i, 0.9915 + 0.2269i, 0.9276 + 0.9031i, 0.3364 - 0.3310i, 0.1856 - 0.9388i, 0.9805 - 0.2319i, 0.8989 - 0.9290i, -0.3211 + 0.3262i, -0.1995 + 1.0028i, -0.9919 + 0.2558i, -0.9184 + 0.8872i, -0.3231 - 0.3086i, -0.1954 - 1.0090i, -0.9939 - 0.2668i, -0.9043 - 0.8814i
12 dB	0.3274 + 0.3393i, 0.1837 + 0.9401i, 0.9910 + 0.2214i, 0.9222 + 0.9149i, 0.3278 - 0.3352i, 0.1795 - 0.9394i, 0.9807 - 0.2300i, 0.8984 - 0.9310i, -0.3138 + 0.3292i, -0.1835 + 0.9989i, -0.9916 + 0.2594i, -0.9160 + 0.8916i, -0.3255 - 0.3102i, -0.1850 - 0.9988i, -0.9915 - 0.2649i, -0.9039 - 0.8854i
13 dB	0.3310 + 0.3387i, 0.1842 + 0.9413i, 1.0073 + 0.2209i, 0.9082 + 0.9484i, 0.3326 - 0.3349i, 0.1773 - 0.9294i, 0.9832 - 0.2298i, 0.8709 - 0.9811i, -0.3095 + 0.3299i, -0.1778 + 0.9972i, -0.9952 + 0.2578i, -0.9154 + 0.8925i, -0.3188 - 0.3107i, -0.1822 - 0.9884i, -0.9932 - 0.2650i, -0.9027 - 0.8874i
14 dB	0.3356 + 0.3398i, 0.1652 + 0.9136i, 1.0033 + 0.2081i, 0.9037 + 0.9581i, 0.3332 - 0.3314i, 0.1569 - 0.9267i, 0.9828 - 0.2247i, 0.8602 - 0.9857i, -0.2971 + 0.3326i, -0.1672 + 0.9897i, -0.9940 + 0.2582i, -0.9123 + 0.8968i, -0.3163 - 0.3036i, -0.1643 - 0.9864i, -0.9909 - 0.2454i, -0.8987 - 0.8930i
15 dB	0.3362 + 0.3334i, 0.1650 + 0.9065i, 1.0029 + 0.2098i, 0.8994 + 0.9693i, 0.3326 - 0.3314i, 0.1521 - 0.9058i, 0.9823 - 0.2199i, 0.8382 - 0.9940i, -0.3055 + 0.3279i, -0.1371 + 0.9814i, -0.9951 + 0.2504i, -0.9076 + 0.9029i, -0.3207 - 0.3026i, -0.1540 - 0.9832i, -0.9925 - 0.2457i, -0.8963 - 0.8992i
16 dB	0.3422 + 0.3304i, 0.1638 + 0.9002i, 0.9981 + 0.2108i, 0.8734 + 0.9817i, 0.3365 - 0.3212i, 0.1495 - 0.9010i, 0.9888 - 0.2175i, 0.8201 - 1.0213i, -0.3057 + 0.3177i, -0.1314 + 0.9787i, -0.9863 + 0.2558i, -0.9001 + 0.9084i, -0.3175 - 0.2969i, -0.1502 - 0.9761i, -0.9784 - 0.2420i, -0.9027 - 0.9077i
17 dB	0.3470 + 0.3219i, 0.1557 + 0.8813i, 0.9996 + 0.2090i, 0.8506 + 0.9689i, 0.3451 - 0.3172i, 0.1443 - 0.8716i, 0.9846 - 0.1936i, 0.7326 - 1.0652i, -0.3098 + 0.3102i, -0.1063 + 0.9764i, -0.9898 + 0.2597i, -0.8827 + 0.9023i, -0.3087 - 0.2911i, -0.1401 - 0.9629i, -0.9724 - 0.2124i, -0.8497 - 0.9158i
18 dB	0.3542 + 0.3162i, 0.1529 + 0.8748i, 0.9927 + 0.2051i, 0.8166 + 0.9947i, 0.3768 - 0.3061i, 0.1409 - 0.8708i, 0.9846 - 0.2015i, 0.6754 - 1.0781i, -0.3101 + 0.2814i, -0.0863 + 0.9383i, -0.9760 + 0.2595i, -0.8590 + 0.8974i, -0.3263 - 0.2588i, -0.1241 - 0.9184i, -0.9781 - 0.2117i, -0.8341 - 0.9220i
19 dB	0.3607 + 0.3236i, 0.1585 + 0.8206i, 1.0164 + 0.1954i, 0.8007 + 1.0246i, 0.4619 - 0.3033i, 0.1360 - 0.8480i, 0.9853 - 0.1966i, 0.4622 - 1.1508i, -0.3215 + 0.2836i, -0.0658 + 0.9330i, -0.9736 + 0.2578i, -0.8489 + 0.9122i, -0.3260 - 0.2528i, -0.1107 - 0.9145i, -0.9777 - 0.1646i, -0.7574 - 0.9204i
20 dB	0.4014 + 0.2584i, 0.1608 + 0.8263i, 1.0453 + 0.1706i, 0.4606 + 1.0804i, 0.5226 - 0.2536i, 0.1426 - 0.8154i, 1.0081 - 0.1940i, 0.4478 - 1.1366i, -0.3095 + 0.2503i, -0.0380 + 0.9352i, -0.9468 + 0.2610i, -0.7942 + 0.9108i, -0.3149 - 0.2555i, -0.0954 - 0.9081i, -0.9723 - 0.1712i, -0.7183 - 0.9585i
21 dB	0.4366 + 0.2404i, 0.1731 + 0.8101i, 0.9832 + 0.1649i, 0.3411 + 1.0930i, 0.5506 - 0.2405i, 0.1330 - 0.8220i, 0.9687 - 0.1151i, 0.2672 - 1.1214i, -0.3472 + 0.2516i, -0.0541 + 0.8566i, -0.9590 + 0.2493i, -0.7102 + 0.9361i, -0.3220 - 0.2431i, -0.0639 - 0.8887i, -0.9449 - 0.1891i, -0.4520 - 0.9821i
22 dB	0.5446 + 0.2401i, 0.1359 + 0.7882i, 0.9714 + 0.2009i, 0.0945 + 1.1110i, 0.5559 - 0.2276i, 0.1709 - 0.8026i, 0.9873 - 0.1052i, 0.2644 - 1.1044i, -0.3568 + 0.1348i, -0.0478 + 0.8662i, -0.9167 + 0.2183i, -0.6426 + 0.9278i, -0.3069 - 0.2239i, -0.0489 - 0.8652i, -0.9551 - 0.1741i, -0.4502 - 0.9842i
23 dB	0.5213 + 0.2192i, 0.0544 + 0.6998i, 0.9333 + 0.1462i, 0.0369 + 1.0710i, 0.6218 - 0.0844i, 0.1015 - 0.7981i, 0.9579 - 0.1455i, 0.0842 - 1.0137i, -0.5089 + 0.1011i, -0.0171 + 0.7985i, -0.8846 + 0.1389i, -0.3651 + 0.9008i, -0.3563 - 0.1775i, -0.0772 - 0.6566i, -0.8954 - 0.1842i, -0.2069 - 1.0156i
24 dB	0.5670 + 0.1661i, 0.0798 + 0.6979i, 0.9297 + 0.1389i, 0.0456 + 1.0006i, 0.6594 - 0.1017i, 0.1073 - 0.7667i, 0.9807 - 0.1114i, 0.1092 - 0.9983i, -0.4951 + 0.0414i, -0.0065 + 0.8007i, -0.8800 + 0.1312i, -0.2942 + 0.9362i, -0.6721 - 0.1382i, -0.0638 - 0.6622i, -0.8995 - 0.1966i, -0.1675 - 0.9381i
25 dB	0.6065 + 0.1617i, 0.0818 + 0.6742i, 0.9009 + 0.0981i, -0.0107 + 0.9634i, 0.6703 - 0.0763i, 0.0997 - 0.7689i, 0.9777 - 0.1155i, 0.0901 - 0.9653i, -0.5410 + 0.0051i, -0.0132 + 0.7902i, -0.8415 + 0.1476i, -0.2504 + 0.9237i, -0.6736 - 0.1075i, -0.0546 - 0.6607i, -0.9048 - 0.1256i, -0.1373 - 0.9442i

Table 4.5: Optimized irregular constellations proposed for MCS-1 along with $m = 2$ and $\Omega = 1$.

Rate 1/2 [5, 7] _s convolutional encoder w/o puncturing	
$\bar{\gamma}$	$\chi(2, \bar{\gamma}) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}$
5 dB	0.2784 + 0.6900i, 0.4133 + 1.0976i, 0.7527 + 0.2095i, 1.1148 + 0.3846i, 0.2564 - 0.6996i, 0.3821 - 1.0980i, 0.7638 - 0.2325i, 1.1071 - 0.4211i -0.2656 + 0.7598i, -0.3713 + 1.1158i, -0.7607 + 0.2487i, -1.0890 + 0.3975i, -0.2989 - 0.7546i, -0.3843 - 1.1048i, -0.7940 - 0.2392i, -1.1049 - 0.3537i
6 dB	0.2495 + 0.6918i, 0.3823 + 1.1105i, 0.7435 + 0.2191i, 1.1183 + 0.3661i, 0.2218 - 0.6889i, 0.3398 - 1.1179i, 0.7664 - 0.2474i, 1.1095 - 0.4047i -0.2250 + 0.7445i, -0.3476 + 1.1267i, -0.7542 + 0.2415i, -1.0911 + 0.3909i, -0.2721 - 0.7437i, -0.3443 - 1.1162i, -0.7861 - 0.2413i, -1.1105 - 0.3312i
7 dB	0.2389 + 0.6828i, 0.3785 + 1.1202i, 0.7375 + 0.2156i, 1.1188 + 0.3559i, 0.2105 - 0.6917i, 0.3107 - 1.1170i, 0.7672 - 0.2450i, 1.1169 - 0.3903i -0.2161 + 0.7325i, -0.3142 + 1.1230i, -0.7518 + 0.2523i, -1.1009 + 0.3859i, -0.2673 - 0.7438i, -0.3257 - 1.1176i, -0.7935 - 0.2442i, -1.1097 - 0.3184i
8 dB	0.2192 + 0.6894i, 0.3476 + 1.1211i, 0.7385 + 0.2085i, 1.1184 + 0.3505i, 0.1981 - 0.6707i, 0.2917 - 1.1244i, 0.7583 - 0.2439i, 1.1141 - 0.3832i -0.2030 + 0.7285i, -0.2904 + 1.1329i, -0.7561 + 0.2513i, -1.1067 + 0.3651i, -0.2325 - 0.7661i, -0.2970 - 1.1153i, -0.7966 - 0.2410i, -1.1036 - 0.3028i
9 dB	0.1745 + 0.6866i, 0.2929 + 1.1299i, 0.7513 + 0.2148i, 1.1173 + 0.3490i, 0.1911 - 0.6778i, 0.2874 - 1.1314i, 0.7613 - 0.2494i, 1.1103 - 0.3389i -0.1863 + 0.7301i, -0.2466 + 1.1492i, -0.7551 + 0.2503i, -1.1030 + 0.3323i, -0.2129 - 0.7674i, -0.2854 - 1.1159i, -0.7947 - 0.2487i, -1.1020 - 0.3128i
10 dB	0.1566 + 0.6836i, 0.2804 + 1.1110i, 0.7544 + 0.2158i, 1.1088 + 0.3157i, 0.1637 - 0.6736i, 0.2723 - 1.1480i, 0.7830 - 0.2458i, 1.1121 - 0.3076i -0.1632 + 0.7232i, -0.2434 + 1.1431i, -0.7780 + 0.2499i, -1.1063 + 0.3251i, -0.2064 - 0.7610i, -0.2258 - 1.1234i, -0.8062 - 0.2518i, -1.1019 - 0.2560i
11 dB	0.1610 + 0.6873i, 0.2405 + 1.1166i, 0.7659 + 0.2148i, 1.1105 + 0.3166i, 0.1609 - 0.6686i, 0.2286 - 1.1299i, 0.7884 - 0.2492i, 1.1159 - 0.3049i -0.1545 + 0.7157i, -0.2375 + 1.1371i, -0.7769 + 0.2429i, -1.1096 + 0.3195i, -0.1951 - 0.7588i, -0.1943 - 1.1262i, -0.8044 - 0.2496i, -1.0995 - 0.2634i
12 dB	0.1358 + 0.6934i, 0.2277 + 1.1093i, 0.7812 + 0.2156i, 1.1138 + 0.2908i, 0.1536 - 0.6739i, 0.2018 - 1.1258i, 0.7925 - 0.2395i, 1.1134 - 0.2804i -0.1485 + 0.7277i, -0.2129 + 1.1209i, -0.7882 + 0.2351i, -1.1099 + 0.3083i, -0.1830 - 0.7536i, -0.1653 - 1.1270i, -0.8090 - 0.2594i, -1.1028 - 0.2412i
13 dB	0.1346 + 0.6889i, 0.1961 + 1.0872i, 0.8152 + 0.2116i, 1.1011 + 0.2964i, 0.1265 - 0.6773i, 0.1845 - 1.1174i, 0.8172 - 0.2330i, 1.1066 - 0.2840i -0.1502 + 0.7284i, -0.1548 + 1.1233i, -0.8233 + 0.2438i, -1.1095 + 0.2623i, -0.1786 - 0.7441i, -0.1540 - 1.1287i, -0.8135 - 0.2466i, -1.0981 - 0.2109i
14 dB	0.1420 + 0.7252i, 0.1711 + 1.0723i, 0.8531 + 0.2037i, 1.0996 + 0.2603i, 0.1256 - 0.7081i, 0.1828 - 1.0999i, 0.8232 - 0.2166i, 1.1019 - 0.2527i -0.1392 + 0.7376i, -0.1612 + 1.1096i, -0.8365 + 0.2213i, -1.0912 + 0.2418i, -0.1897 - 0.7575i, -0.1448 - 1.1042i, -0.8269 - 0.2371i, -1.1097 - 0.1957i
15 dB	0.1532 + 0.7547i, 0.1684 + 1.0651i, 0.8730 + 0.1931i, 1.1103 + 0.2095i, 0.1043 - 0.7128i, 0.1702 - 1.0921i, 0.8283 - 0.2089i, 1.0975 - 0.2602i -0.1234 + 0.7681i, -0.1425 + 1.1000i, -0.8831 + 0.2177i, -1.0780 + 0.2426i, -0.1773 - 0.7724i, -0.1456 - 1.1012i, -0.8331 - 0.2153i, -1.1225 - 0.1880i
16 dB	0.1429 + 0.7721i, 0.1588 + 1.0546i, 0.8721 + 0.1827i, 1.0981 + 0.2056i, 0.0964 - 0.7279i, 0.1595 - 1.0711i, 0.8781 - 0.2111i, 1.0916 - 0.2482i -0.1210 + 0.7620i, -0.1473 + 1.0882i, -0.8933 + 0.2191i, -1.0834 + 0.2364i, -0.1730 - 0.7766i, -0.1207 - 1.0877i, -0.8394 - 0.2083i, -1.1195 - 0.1896i
17 dB	0.1395 + 0.7910i, 0.1557 + 1.0572i, 0.8685 + 0.1759i, 1.1048 + 0.2040i, 0.0919 - 0.7282i, 0.1515 - 1.0692i, 0.9109 - 0.2000i, 1.0923 - 0.2313i -0.1197 + 0.7689i, -0.1395 + 1.0860i, -0.8949 + 0.2031i, -1.0818 + 0.2039i, -0.1663 - 0.7974i, -0.1125 - 1.0764i, -0.8737 - 0.2069i, -1.1267 - 0.1808i
18 dB	0.1344 + 0.7968i, 0.1378 + 1.0616i, 0.8691 + 0.1748i, 1.0965 + 0.1874i, 0.0952 - 0.7598i, 0.1499 - 1.0601i, 0.9095 - 0.1928i, 1.0940 - 0.1970i -0.1192 + 0.7983i, -0.1313 + 1.0539i, -0.8882 + 0.1854i, -1.0766 + 0.2004i, -0.1663 - 0.8178i, -0.1139 - 1.0711i, -0.8722 - 0.2025i, -1.1185 - 0.1574i
19 dB	0.1325 + 0.8002i, 0.1260 + 1.0268i, 0.9061 + 0.1648i, 1.0886 + 0.1767i, 0.0828 - 0.7689i, 0.1302 - 1.0299i, 0.9298 - 0.1799i, 1.0945 - 0.1964i -0.1137 + 0.8132i, -0.1243 + 1.0465i, -0.9161 + 0.1884i, -1.0735 + 0.1921i, -0.1538 - 0.8427i, -0.1077 - 1.0537i, -0.9029 - 0.1955i, -1.0987 - 0.1419i
20 dB	0.1277 + 0.8244i, 0.1218 + 1.0450i, 0.8992 + 0.1575i, 1.0856 + 0.1726i, 0.0959 - 0.7981i, 0.1340 - 1.0280i, 0.9298 - 0.1597i, 1.0749 - 0.1885i -0.1079 + 0.8243i, -0.1187 + 1.0460i, -0.9248 + 0.1605i, -1.0713 + 0.1682i, -0.1477 - 0.8532i, -0.1011 - 1.0481i, -0.9042 - 0.1888i, -1.0930 - 0.1340i
21 dB	0.1251 + 0.8268i, 0.1160 + 1.0320i, 0.9006 + 0.1493i, 1.0851 + 0.1588i, 0.0909 - 0.8061i, 0.1249 - 1.0205i, 0.9295 - 0.1487i, 1.0728 - 0.1463i -0.1103 + 0.8224i, -0.1184 + 1.0398i, -0.9196 + 0.1436i, -1.0664 + 0.1513i, -0.1397 - 0.8573i, -0.0837 - 1.0445i, -0.9167 - 0.1643i, -1.0899 - 0.1365i
22 dB	0.1257 + 0.8552i, 0.0951 + 1.0297i, 0.9125 + 0.1509i, 1.0820 + 0.1399i, 0.1013 - 0.8208i, 0.1216 - 1.0144i, 0.9357 - 0.1462i, 1.0734 - 0.1497i -0.0991 + 0.8286i, -0.1166 + 1.0335i, -0.9345 + 0.1370i, -1.0630 + 0.1501i, -0.1377 - 0.8556i, -0.0903 - 1.0429i, -0.9194 - 0.1603i, -1.0868 - 0.1349i
23 dB	0.1109 + 0.8576i, 0.0897 + 1.0233i, 0.9418 + 0.1446i, 1.0783 + 0.1343i, 0.1007 - 0.8386i, 0.1185 - 1.0112i, 0.9368 - 0.1407i, 1.0696 - 0.1478i -0.0977 + 0.8298i, -0.1231 + 1.0350i, -0.9357 + 0.1325i, -1.0630 + 0.1333i, -0.1253 - 0.8606i, -0.0958 - 1.0253i, -0.9221 - 0.1414i, -1.0837 - 0.1249i
24 dB	0.1059 + 0.8820i, 0.0883 + 1.0149i, 0.9409 + 0.1396i, 1.0753 + 0.1281i, 0.0992 - 0.8511i, 0.1173 - 1.0119i, 0.9377 - 0.1281i, 1.0552 - 0.1279i -0.0950 + 0.8291i, -0.1073 + 1.0300i, -0.9390 + 0.1269i, -1.0636 + 0.1260i, -0.1200 - 0.8671i, -0.0859 - 1.0228i, -0.9244 - 0.1484i, -1.0847 - 0.1192i
25 dB	0.1052 + 0.8962i, 0.0908 + 1.0069i, 0.9470 + 0.1216i, 1.0730 + 0.1256i, 0.1044 - 0.8806i, 0.1060 - 0.9965i, 0.9460 - 0.1253i, 1.0568 - 0.1115i -0.0974 + 0.8481i, -0.0926 + 1.0212i, -0.9409 + 0.1157i, -1.0700 + 0.1156i, -0.1169 - 0.8763i, -0.0880 - 1.0234i, -0.9410 - 0.1140i, -1.0825 - 0.1234i

Table 4.6: Optimized irregular constellations proposed for MCS-2 along with $m = 1$ and $\Omega = 1$.

Rate 1/2 [5, 7] _s convolutional encoder w/ puncturing	
$\bar{\gamma}$	$\chi(1, \bar{\gamma}) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}$
12 dB	0.3584 + 0.2519i, 0.2887 + 0.9583i, 1.0221 + 0.2673i, 0.7770 + 0.9461i, 0.3528 - 0.2659i, 0.3174 - 0.9849i, 1.0491 - 0.2342i, 0.8052 - 0.9121i -0.3530 + 0.2611i, -0.3170 + 0.9747i, -1.0398 + 0.2410i, -0.8015 + 0.9201i, -0.3561 - 0.2589i, -0.2895 - 0.9644i, -1.0320 - 0.2620i, -0.7818 - 0.9383i
13 dB	0.3676 + 0.2428i, 0.3006 + 0.9574i, 1.0244 + 0.2540i, 0.7729 + 0.9476i, 0.3716 - 0.2482i, 0.3331 - 0.9722i, 1.0508 - 0.2239i, 0.8021 - 0.9191i -0.3734 + 0.2446i, -0.3311 + 0.9637i, -1.0402 + 0.2273i, -0.8011 + 0.9248i, -0.3678 - 0.2447i, -0.2988 - 0.9556i, -1.0391 - 0.2544i, -0.7716 - 0.9442i
14 dB	0.3838 + 0.2278i, 0.3140 + 0.9401i, 1.0416 + 0.2444i, 0.7707 + 0.9492i, 0.3921 - 0.2268i, 0.3412 - 0.9460i, 1.0548 - 0.2132i, 0.7961 - 0.9260i -0.3859 + 0.2265i, -0.3451 + 0.9538i, -1.0491 + 0.2130i, -0.8010 + 0.9236i, -0.3876 - 0.2291i, -0.3165 - 0.9494i, -1.0393 - 0.2406i, -0.7699 - 0.9474i
15 dB	0.3983 + 0.2074i, 0.3205 + 0.9317i, 1.0238 + 0.2334i, 0.7658 + 0.9504i, 0.4057 - 0.2239i, 0.3601 - 0.9369i, 1.0693 - 0.2110i, 0.7962 - 0.9302i -0.3881 + 0.2199i, -0.3535 + 0.9508i, -1.0586 + 0.2064i, -0.7949 + 0.9127i, -0.4017 - 0.2081i, -0.3315 - 0.9108i, -1.0454 - 0.2380i, -0.7662 - 0.9537i
16 dB	0.4041 + 0.1955i, 0.3278 + 0.9102i, 1.0282 + 0.2013i, 0.7728 + 0.9458i, 0.4059 - 0.2047i, 0.3549 - 0.9248i, 1.0894 - 0.2013i, 0.7901 - 0.9154i -0.3923 + 0.2015i, -0.3689 + 0.9447i, -1.0640 + 0.2050i, -0.7985 + 0.9190i, -0.4118 - 0.1872i, -0.3465 - 0.9087i, -1.0234 - 0.2301i, -0.7680 - 0.9508i
17 dB	0.4022 + 0.1442i, 0.3866 + 0.8967i, 1.0296 + 0.1975i, 0.7725 + 0.9253i, 0.4185 - 0.1954i, 0.3862 - 0.9226i, 1.0991 - 0.1874i, 0.7797 - 0.8994i -0.4516 + 0.1946i, -0.3670 + 0.9218i, -1.0618 + 0.1947i, -0.7997 + 0.9137i, -0.4350 - 0.1612i, -0.3583 - 0.8740i, -1.0336 - 0.1999i, -0.7675 - 0.9484i
18 dB	0.4072 + 0.1136i, 0.3899 + 0.8269i, 1.0288 + 0.1772i, 0.7693 + 0.9011i, 0.4259 - 0.1819i, 0.3951 - 0.8697i, 1.1050 - 0.1558i, 0.7792 - 0.9076i -0.4631 + 0.1882i, -0.3766 + 0.9055i, -1.0745 + 0.1958i, -0.7748 + 0.9196i, -0.4444 - 0.1428i, -0.3580 - 0.8577i, -1.0416 - 0.1869i, -0.7674 - 0.9256i
19 dB	0.4142 + 0.1052i, 0.3879 + 0.8300i, 1.0321 + 0.1657i, 0.7732 + 0.8987i, 0.4475 - 0.1377i, 0.4059 - 0.8719i, 1.1110 - 0.1040i, 0.7879 - 0.8691i -0.4567 + 0.1221i, -0.4025 + 0.8969i, -1.0780 + 0.1526i, -0.7778 + 0.9136i, -0.4628 - 0.1417i, -0.3565 - 0.8657i, -1.0598 - 0.1801i, -0.7657 - 0.9146i
20 dB	0.4199 + 0.0837i, 0.4173 + 0.7986i, 1.0283 + 0.1805i, 0.7744 + 0.9049i, 0.4684 - 0.1385i, 0.4313 - 0.8488i, 1.0894 - 0.0778i, 0.7976 - 0.8454i -0.4568 + 0.1099i, -0.4462 + 0.8505i, -1.0888 + 0.1221i, -0.7813 + 0.9002i, -0.4702 - 0.1216i, -0.3975 - 0.8482i, -1.0432 - 0.1660i, -0.7426 - 0.9041i
21 dB	0.4756 + 0.0866i, 0.4247 + 0.7901i, 1.0276 + 0.1676i, 0.7642 + 0.8970i, 0.4805 - 0.1296i, 0.4741 - 0.8322i, 1.0986 - 0.0679i, 0.7993 - 0.8480i -0.4503 + 0.0919i, -0.5008 + 0.7689i, -1.0762 + 0.1213i, -0.7803 + 0.8983i, -0.4930 - 0.1142i, -0.4506 - 0.8283i, -1.0366 - 0.1046i, -0.7567 - 0.8969i
22 dB	0.4784 + 0.0594i, 0.4609 + 0.7623i, 1.0200 + 0.1555i, 0.7523 + 0.8821i, 0.4733 - 0.1077i, 0.5205 - 0.8186i, 1.1027 - 0.0635i, 0.8186 - 0.7989i -0.4557 + 0.0762i, -0.5167 + 0.7348i, -1.0783 + 0.1211i, -0.7877 + 0.8837i, -0.5019 - 0.1005i, -0.4761 - 0.7932i, -1.0408 - 0.0921i, -0.7693 - 0.9006i
23 dB	0.4742 - 0.0512i, 0.5323 + 0.7629i, 1.0261 + 0.0910i, 0.7617 + 0.8755i, 0.3891 + 0.2006i, 0.5062 - 0.7735i, 1.0582 - 0.0833i, 0.8147 - 0.8307i -0.4556 + 0.0272i, -0.5181 + 0.7296i, -1.0731 + 0.0812i, -0.8094 + 0.8615i, -0.5128 - 0.1114i, -0.5375 - 0.7414i, -1.0387 - 0.0853i, -0.6171 - 0.9525i
24 dB	0.4506 - 0.0923i, 0.4857 + 0.7709i, 0.9363 - 0.1866i, 0.7362 + 0.8481i, 0.4660 + 0.2110i, 0.5274 - 0.6908i, 1.0190 + 0.1515i, 0.7144 - 0.8946i -0.3453 - 0.2257i, -0.5546 + 0.6996i, -1.0640 + 0.0499i, -0.7721 + 0.8435i, -0.4056 + 0.1514i, -0.6445 - 0.6788i, -0.9755 - 0.0592i, -0.5740 - 0.8981i
25 dB	0.4305 - 0.1181i, 0.4931 + 0.7749i, 0.8072 - 0.2305i, 0.7295 + 0.8042i, 0.4605 + 0.2404i, 0.5636 - 0.6969i, 0.9407 + 0.1061i, 0.5672 - 0.8747i -0.3585 - 0.2717i, -0.5378 + 0.7556i, -0.8528 - 0.1000i, -0.7787 + 0.7117i, -0.3877 + 0.2849i, -0.6203 - 0.6676i, -0.8893 + 0.1816i, -0.5672 - 0.8997i

Table 4.7: Optimized irregular constellations proposed for MCS-2 along with $m = 2$ and $\Omega = 1$.

Rate 1/2 [5, 7] ₈ convolutional encoder w/ puncturing	
$\overline{\gamma}$	$\chi(2, \overline{\gamma}) = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, s_{11}, s_{12}, s_{13}, s_{14}, s_{15}, s_{16}\}$
9 dB	0.3653 + 0.2495i, 0.2937 + 0.9634i, 1.0646 + 0.2719i, 0.7722 + 0.9852i, 0.3685 - 0.2445i, 0.3293 - 0.9661i, 1.0779 - 0.2352i, 0.8073 - 0.9546i -0.3665 + 0.2465i, -0.3320 + 0.9704i, -1.0781 + 0.2323i, -0.8103 + 0.9502i, -0.3622 - 0.2482i, -0.2951 - 0.9641i, -1.0619 - 0.2721i, -0.7721 - 0.9841i
10 dB	0.3820 + 0.2304i, 0.3093 + 0.9471i, 1.0735 + 0.2605i, 0.7714 + 0.9870i, 0.3834 - 0.2274i, 0.3407 - 0.9504i, 1.0880 - 0.2256i, 0.8037 - 0.9562i -0.3862 + 0.2278i, -0.3401 + 0.9497i, -1.0888 + 0.2277i, -0.8014 + 0.9601i, -0.3840 - 0.2313i, -0.3074 - 0.9468i, -1.0744 - 0.2609i, -0.7703 - 0.9889i
11 dB	0.3957 + 0.2242i, 0.3187 + 0.9477i, 1.0766 + 0.2580i, 0.7780 + 0.9905i, 0.4095 - 0.2159i, 0.3525 - 0.9512i, 1.0874 - 0.2119i, 0.8044 - 0.9591i -0.3859 + 0.2116i, -0.3426 + 0.9453i, -1.0869 + 0.2065i, -0.7976 + 0.9543i, -0.3828 - 0.2153i, -0.3208 - 0.9356i, -1.0725 - 0.2620i, -0.7694 - 0.9931i
12 dB	0.4180 + 0.2218i, 0.3164 + 0.9437i, 1.0867 + 0.2580i, 0.7577 + 0.9934i, 0.4231 - 0.1860i, 0.3569 - 0.9393i, 1.1119 - 0.2179i, 0.8077 - 0.9658i -0.4002 + 0.1637i, -0.3536 + 0.8912i, -1.0934 + 0.1829i, -0.8106 + 0.9501i, -0.3945 - 0.2104i, -0.3249 - 0.9387i, -1.0691 - 0.2598i, -0.7736 - 0.9970i
13 dB	0.4467 + 0.2075i, 0.3255 + 0.9459i, 1.0834 + 0.2426i, 0.7685 + 0.9902i, 0.4434 - 0.1841i, 0.3704 - 0.9027i, 1.1374 - 0.1852i, 0.8221 - 0.9629i -0.4144 + 0.1637i, -0.3693 + 0.8799i, -1.0986 + 0.1807i, -0.8152 + 0.9480i, -0.4124 - 0.2019i, -0.3316 - 0.9325i, -1.0782 - 0.2019i, -0.7695 - 0.9704i
14 dB	0.4876 + 0.1890i, 0.3298 + 0.9089i, 1.1071 + 0.2404i, 0.7792 + 1.0030i, 0.4453 - 0.1320i, 0.3816 - 0.8662i, 1.1317 - 0.1858i, 0.8199 - 0.9557i -0.4222 + 0.1543i, -0.4151 + 0.8758i, -1.1077 + 0.1687i, -0.8068 + 0.9328i, -0.4302 - 0.1696i, -0.3752 - 0.8944i, -1.0893 - 0.1881i, -0.7897 - 0.9820i
15 dB	0.5059 + 0.1588i, 0.3419 + 0.8903i, 1.1191 + 0.2347i, 0.7768 + 1.0043i, 0.4982 - 0.1333i, 0.4251 - 0.8388i, 1.1471 - 0.1676i, 0.8336 - 0.9212i -0.3878 + 0.0716i, -0.4700 + 0.8278i, -1.1443 + 0.1677i, -0.8240 + 0.9260i, -0.4716 - 0.1722i, -0.3959 - 0.8878i, -1.0880 - 0.1802i, -0.8021 - 0.9273i
16 dB	0.5187 + 0.1445i, 0.3408 + 0.8410i, 1.1037 + 0.2341i, 0.7591 + 0.9900i, 0.4940 - 0.1020i, 0.4226 - 0.8667i, 1.1514 - 0.1600i, 0.8519 - 0.9205i -0.3791 - 0.0135i, -0.4683 + 0.8163i, -1.1573 + 0.1820i, -0.8158 + 0.9308i, -0.5312 - 0.1509i, -0.3919 - 0.9005i, -1.1092 - 0.0688i, -0.8234 - 0.9122i
17 dB	0.5847 + 0.1415i, 0.4500 + 0.8412i, 1.1270 + 0.2252i, 0.7073 + 1.0020i, 0.4924 - 0.0445i, 0.4603 - 0.8158i, 1.1805 - 0.1516i, 0.8150 - 0.9074i -0.3751 - 0.1987i, -0.4898 + 0.8275i, -1.1510 + 0.1839i, -0.8068 + 0.9193i, -0.4706 + 0.0337i, -0.3926 - 0.9232i, -1.0902 - 0.0508i, -0.8303 - 0.8985i
18 dB	0.5891 + 0.0407i, 0.4745 + 0.8222i, 1.1593 + 0.1260i, 0.7032 + 0.9746i, 0.4704 - 0.0310i, 0.5498 - 0.8097i, 1.2047 - 0.0754i, 0.8524 - 0.8876i -0.3937 - 0.2040i, -0.4869 + 0.8019i, -1.1579 + 0.1605i, -0.7790 + 0.9361i, -0.4613 + 0.0631i, -0.4502 - 0.9252i, -1.0668 - 0.0023i, -0.8115 - 0.8922i
19 dB	0.5852 - 0.0403i, 0.4725 + 0.8094i, 1.1123 + 0.0453i, 0.7194 + 0.9790i, 0.4684 + 0.0044i, 0.5628 - 0.8378i, 1.1980 - 0.0660i, 0.8655 - 0.8868i -0.4036 - 0.2162i, -0.4906 + 0.8074i, -1.1251 - 0.2397i, -0.7842 + 0.8857i, -0.4536 + 0.0981i, -0.4563 - 0.9374i, -1.0539 + 0.0402i, -0.8251 - 0.9640i
20 dB	0.5591 - 0.0973i, 0.5369 + 0.8205i, 1.0336 - 0.2237i, 0.7113 + 1.0177i, 0.4654 + 0.2524i, 0.5592 - 0.7542i, 1.2114 - 0.0089i, 0.8809 - 0.8329i -0.4789 - 0.2679i, -0.5040 + 0.8136i, -1.0989 - 0.2443i, -0.7967 + 0.8954i, -0.5282 + 0.1822i, -0.4871 - 0.9165i, -1.0877 + 0.1029i, -0.7835 - 0.9161i
21 dB	0.5797 - 0.1953i, 0.5684 + 0.8012i, 1.0497 - 0.2071i, 0.7004 + 0.9906i, 0.4639 + 0.2652i, 0.5324 - 0.7506i, 1.2016 + 0.2663i, 0.8713 - 0.8349i -0.4185 - 0.3711i, -0.5442 + 0.8006i, -0.9905 - 0.3103i, -0.8194 + 0.8687i, -0.5856 + 0.2121i, -0.5461 - 0.9173i, -1.0673 + 0.1660i, -0.7735 - 0.9460i
22 dB	0.6067 - 0.2405i, 0.5674 + 0.7969i, 1.0780 - 0.2110i, 0.6543 + 1.0489i, 0.4764 + 0.3428i, 0.5404 - 0.7534i, 0.9997 + 0.4150i, 0.8650 - 0.8165i -0.4505 - 0.4014i, -0.5405 + 0.8315i, -0.9733 - 0.4025i, -0.8494 + 0.8623i, -0.5834 + 0.2658i, -0.5448 - 0.9007i, -1.0026 + 0.2698i, -0.7961 - 0.9688i
23 dB	0.6274 - 0.2453i, 0.5643 + 0.8693i, 1.0286 - 0.2653i, 0.6410 + 1.0303i, 0.4891 + 0.3756i, 0.6408 - 0.7997i, 0.8874 + 0.4637i, 0.8386 - 0.8376i -0.5182 - 0.4362i, -0.5581 + 0.8389i, -0.8819 - 0.4102i, -0.8694 + 0.8284i, -0.5979 + 0.3430i, -0.5027 - 0.9236i, -0.9828 + 0.2676i, -0.8263 - 0.9330i
24 dB	0.6932 - 0.2500i, 0.5558 + 0.8748i, 0.9286 - 0.2905i, 0.6285 + 1.0188i, 0.6290 + 0.3819i, 0.6536 - 0.8139i, 0.8790 + 0.4678i, 0.8414 - 0.8676i -0.5219 - 0.4495i, -0.6345 + 0.8520i, -0.8689 - 0.4105i, -0.8724 + 0.8241i, -0.5911 + 0.3671i, -0.5546 - 0.9055i, -0.8995 + 0.3079i, -0.7558 - 0.9307i
25 dB	0.6904 - 0.3666i, 0.5432 + 0.8648i, 0.9361 - 0.3288i, 0.6442 + 1.0015i, 0.6455 + 0.3959i, 0.6756 - 0.8211i, 0.8175 + 0.4683i, 0.8480 - 0.8752i -0.5393 - 0.4754i, -0.6383 + 0.8276i, -0.8661 - 0.4417i, -0.8507 + 0.8193i, -0.6065 + 0.3656i, -0.5579 - 0.8998i, -0.8961 + 0.3089i, -0.7409 - 0.9209i

Table 4.8: Optimized irregular constellations proposed for MCS-3 along with $m = 2$ and $\Omega = 1$.

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
11 dB	<p>0.8689 + 0.5675i, 0.5161 + 0.4218i, 0.0689 + 0.4581i, 0.1847 + 0.1899i, 0.0564 + 1.1963i, 0.3924 + 1.0563i, -0.0195 + 0.9000i, 0.6385 + 1.2393i, 0.4850 + 0.5374i, 0.7351 + 0.2198i, 1.3110 + 0.2763i, 1.2191 - 0.0742i, 0.2479 + 0.7533i, 0.6687 + 0.9847i, 1.1668 + 0.5728i, 0.9399 + 0.8908i, 0.9069 - 0.6748i, 0.7527 - 0.3812i, 0.2284 - 0.1852i, 0.3092 + 0.0781i, 0.3746 - 0.7928i, 0.3017 - 1.1135i, 0.2168 - 0.4660i, 0.0537 - 0.9945i, 0.5779 - 0.3107i, 0.7685 - 0.0804i, 1.3220 - 0.3118i, 1.0107 + 0.1027i, 0.4374 - 0.5051i, 0.6037 - 1.0336i, 1.1462 - 0.5983i, 0.9194 - 0.9837i, -0.9302 + 0.6925i, -0.7002 + 0.5113i, -0.2214 + 0.1013i, -0.2687 - 0.1759i, -0.1808 + 0.6907i, -0.3592 + 0.9325i, -0.1882 + 0.3381i, -0.2684 + 1.2512i, -0.5528 + 0.2611i, -0.7632 + 0.1125i, -1.3362 + 0.3098i, -1.0559 - 0.0808i, -0.3978 + 0.4613i, -0.6186 + 1.1025i, -1.1195 + 0.5963i, -0.8309 + 0.9459i, -0.9234 - 0.5395i, -0.6526 - 0.4922i, -0.1109 - 0.4530i, -0.1185 - 0.1949i, -0.1988 - 0.9849i, -0.4314 - 1.1556i, -0.0986 - 0.7472i, -0.0566 - 1.4358i, -0.5143 - 0.4360i, -0.7355 - 0.1330i, -1.3227 - 0.2426i, -1.1400 + 0.1329i, -0.4652 - 0.7846i, -0.7004 - 1.0730i, -1.1717 - 0.5602i, -0.9774 - 0.8900i</p>
12 dB	<p>0.8666 + 0.5691i, 0.5169 + 0.4250i, 0.0758 + 0.4651i, 0.1790 + 0.2135i, 0.0604 + 1.1873i, 0.4042 + 1.0492i, -0.0240 + 0.8977i, 0.6443 + 1.2362i, 0.5255 + 0.5298i, 0.7369 + 0.2222i, 1.2919 + 0.2660i, 1.2150 - 0.0705i, 0.2554 + 0.7923i, 0.6654 + 0.9915i, 1.1616 + 0.5773i, 0.9336 + 0.8832i, 0.9144 - 0.6736i, 0.7551 - 0.3930i, 0.2363 - 0.1942i, 0.3103 + 0.0541i, 0.3831 - 0.7904i, 0.3019 - 1.1043i, 0.2120 - 0.4757i, 0.0414 - 0.9888i, 0.5833 - 0.3072i, 0.7654 - 0.0834i, 1.3197 - 0.3046i, 1.0081 + 0.0955i, 0.4426 - 0.5223i, 0.5984 - 1.0250i, 1.1413 - 0.5958i, 0.9143 - 0.9757i, -0.9317 + 0.6802i, -0.7052 + 0.5181i, -0.2229 + 0.1054i, -0.2459 - 0.1346i, -0.1967 + 0.6991i, -0.3703 + 0.9313i, -0.1748 + 0.3650i, -0.2680 + 1.2358i, -0.5482 + 0.2581i, -0.7635 + 0.1118i, -1.3253 + 0.3092i, -1.0562 - 0.0774i, -0.4068 + 0.4817i, -0.6218 + 1.0970i, -1.1168 + 0.5966i, -0.8306 + 0.9281i, -0.9292 - 0.5379i, -0.6543 - 0.4973i, -0.1285 - 0.4531i, -0.1076 - 0.2184i, -0.2009 - 0.9837i, -0.4312 - 1.1500i, -0.1111 - 0.7455i, -0.0506 - 1.4355i, -0.5267 - 0.4351i, -0.7628 - 0.1516i, -1.3132 - 0.2376i, -1.1401 + 0.1306i, -0.4665 - 0.8204i, -0.6960 - 1.0767i, -1.1643 - 0.5574i, -0.9685 - 0.8862i</p>
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Table 4.8 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
13 dB	<p>0.8695 + 0.5702i, 0.5242 + 0.4259i, 0.0783 + 0.4653i, 0.1754 + 0.2151i, 0.0616 + 1.1867i, 0.4039 + 1.0077i, -0.0218 + 0.9060i, 0.6434 + 1.2288i, 0.5364 + 0.4894i, 0.7378 + 0.2233i, 1.2918 + 0.2660i, 1.2120 - 0.0687i, 0.2553 + 0.7929i, 0.6662 + 0.9929i, 1.1623 + 0.5784i, 0.9274 + 0.8821i, 0.9133 - 0.6833i, 0.7558 - 0.3903i, 0.2362 - 0.1944i, 0.3090 + 0.0551i, 0.3824 - 0.7891i, 0.3025 - 1.0867i, 0.2109 - 0.4757i, 0.0405 - 0.9715i, 0.5843 - 0.3064i, 0.7657 - 0.0792i, 1.3139 - 0.3040i, 1.0060 + 0.0962i, 0.4464 - 0.5238i, 0.5985 - 1.0288i, 1.1420 - 0.5979i, 0.9147 - 0.9723i, -0.9310 + 0.6809i, -0.7067 + 0.5186i, -0.2231 + 0.1064i, -0.2465 - 0.1192i, -0.1972 + 0.7026i, -0.3697 + 0.9288i, -0.1741 + 0.3667i, -0.2666 + 1.2354i, -0.5476 + 0.2583i, -0.7635 + 0.1120i, -1.3242 + 0.3096i, -1.0550 - 0.0772i, -0.4069 + 0.4801i, -0.6215 + 1.0765i, -1.1152 + 0.5998i, -0.8316 + 0.9276i, -0.9283 - 0.5368i, -0.6616 - 0.4979i, -0.1258 - 0.4471i, -0.1070 - 0.2179i, -0.2042 - 0.9832i, -0.4377 - 1.1246i, -0.1097 - 0.7563i, -0.0489 - 1.4311i, -0.5318 - 0.4303i, -0.7619 - 0.1512i, -1.3091 - 0.2424i, -1.1401 + 0.1377i, -0.4765 - 0.8242i, -0.6955 - 1.0785i, -1.1630 - 0.5570i, -0.9646 - 0.8751i</p>
14 dB	<p>0.8679 + 0.5792i, 0.5247 + 0.4256i, 0.0852 + 0.4688i, 0.1750 + 0.2203i, 0.0590 + 1.1861i, 0.4004 + 1.0064i, -0.0203 + 0.9000i, 0.6423 + 1.2288i, 0.5634 + 0.4872i, 0.7359 + 0.2283i, 1.2906 + 0.2630i, 1.2224 - 0.0575i, 0.2618 + 0.8051i, 0.6640 + 0.9910i, 1.1612 + 0.5760i, 0.9250 + 0.8800i, 0.9103 - 0.6826i, 0.7570 - 0.3936i, 0.2336 - 0.1960i, 0.3065 + 0.0532i, 0.3813 - 0.7893i, 0.3020 - 1.0803i, 0.2080 - 0.4994i, 0.0342 - 0.9741i, 0.5765 - 0.3013i, 0.7527 - 0.0959i, 1.3164 - 0.3080i, 0.9991 + 0.0973i, 0.4449 - 0.5214i, 0.5983 - 0.9916i, 1.1304 - 0.5955i, 0.8988 - 0.9732, -0.9319 + 0.6781i, -0.7070 + 0.5202i, -0.2314 + 0.1230i, -0.2518 - 0.1219i, -0.2017 + 0.7042i, -0.3706 + 0.9260i, -0.1388 + 0.3659i, -0.2694 + 1.2356i, -0.5485 + 0.2562i, -0.7677 + 0.1153i, -1.3067 + 0.3105i, -1.0241 - 0.0699i, -0.4080 + 0.4897i, -0.6246 + 1.0752i, -1.1183 + 0.5950i, -0.8321 + 0.9505i, -0.9284 - 0.5355i, -0.6645 - 0.4975i, -0.1267 - 0.4481i, -0.1076 - 0.2304i, -0.2060 - 0.9836i, -0.4402 - 1.1250i, -0.1100 - 0.7610i, -0.0492 - 1.4289i, -0.5332 - 0.4318i, -0.7715 - 0.1521i, -1.3025 - 0.2478i, -1.1416 + 0.1336i, -0.4759 - 0.8714i, -0.6969 - 1.0789i, -1.1506 - 0.5561i, -0.9716 - 0.8759i</p>
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Table 4.8 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
15 dB	<p>0.8649 + 0.5796i, 0.5316 + 0.4225i, 0.0904 + 0.4727i, 0.1631 + 0.2289i, 0.0649 + 1.1846i, 0.4263 + 0.9871i, -0.0219 + 0.9036i, 0.6436 + 1.1833i, 0.5812 + 0.4891i, 0.7383 + 0.2280i, 1.2897 + 0.2644i, 1.2240 - 0.0561i, 0.2568 + 0.8102i, 0.6022 + 1.0040i, 1.1445 + 0.5774i, 0.9177 + 0.8889i, 0.9142 - 0.6864i, 0.7543 - 0.4139i, 0.2281 - 0.1944i, 0.3095 - 0.0020i, 0.3604 - 0.7809i, 0.2830 - 1.0811i, 0.1799 - 0.4846i, 0.0296 - 0.9616i, 0.5820 - 0.3039i, 0.7614 - 0.0925i, 1.3072 - 0.3022i, 1.0050 + 0.1082i, 0.4470 - 0.5313i, 0.5592 - 0.9770i, 1.1388 - 0.5839i, 0.8792 - 0.9511i, -0.9465 + 0.6785i, -0.7007 + 0.5268i, -0.2341 + 0.1243i, -0.2488 - 0.1191i, -0.2071 + 0.7083i, -0.3524 + 0.9259i, -0.1225 + 0.3605i, -0.2604 + 1.2377i, -0.5448 + 0.2597i, -0.7697 + 0.1148i, -1.3025 + 0.3263i, -1.0220 - 0.0219i, -0.4027 + 0.4922i, -0.6200 + 1.0755i, -1.1257 + 0.5972i, -0.8331 + 0.9247i, -0.8652 - 0.5447i, -0.6651 - 0.4979i, -0.1329 - 0.4494i, -0.1052 - 0.2534i, -0.1936 - 0.9814i, -0.4411 - 1.1016i, -0.1060 - 0.7636i, -0.0494 - 1.4295i, -0.5353 - 0.4317i, -0.7661 - 0.1566i, -1.2748 - 0.2431i, -1.1368 + 0.0914i, -0.4740 - 0.8721i, -0.7061 - 1.0713i, -1.1490 - 0.5553i, -0.9630 - 0.8812i</p>
16 dB	<p>0.8637 + 0.5762i, 0.5316 + 0.4207i, 0.0887 + 0.4730i, 0.1574 + 0.2420i, 0.0632 + 1.1813i, 0.4333 + 0.9793i, -0.0198 + 0.9180i, 0.6395 + 1.1344i, 0.5818 + 0.4878i, 0.7408 + 0.2314i, 1.2590 + 0.2635i, 1.2212 - 0.0434i, 0.2569 + 0.8056i, 0.6043 + 1.0023i, 1.1342 + 0.5780i, 0.9134 + 0.8918i, 0.9071 - 0.6784i, 0.7543 - 0.4126i, 0.2277 - 0.2021i, 0.3100 - 0.0228i, 0.3607 - 0.7966i, 0.2847 - 1.0833i, 0.1812 - 0.4936i, 0.0183 - 0.9664i, 0.5822 - 0.3043i, 0.7612 - 0.0927i, 1.3086 - 0.3003i, 1.0056 + 0.1016i, 0.4474 - 0.5332i, 0.5580 - 0.9775i, 1.1257 - 0.5858i, 0.8786 - 0.9472i, -0.9219 + 0.6766i, -0.7048 + 0.5254i, -0.2341 + 0.1252i, -0.2463 - 0.0851i, -0.2433 + 0.7046i, -0.3516 + 0.9258i, -0.1223 + 0.4009i, -0.2633 + 1.2346i, -0.5438 + 0.2605i, -0.7661 + 0.1144i, -1.2581 + 0.3225i, -1.0207 - 0.0225i, -0.4023 + 0.4908i, -0.6199 + 1.0738i, -1.0648 + 0.5960i, -0.8252 + 0.9213i, -0.8670 - 0.5369i, -0.6670 - 0.5002i, -0.1330 - 0.4462i, -0.1069 - 0.2748i, -0.1939 - 0.9870i, -0.4432 - 1.1002i, -0.1037 - 0.7655i, -0.0521 - 1.4190i, -0.5310 - 0.4306i, -0.7686 - 0.1560i, -1.2763 - 0.2426i, -1.1440 + 0.0906i, -0.4812 - 0.8703i, -0.7098 - 1.0233i, -1.1501 - 0.5690i, -0.9643 - 0.8804i</p>
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Table 4.8 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
17 dB	<p>0.8738 + 0.5800i, 0.5170 + 0.4044i, 0.0946 + 0.4760i, 0.1668 + 0.2436i, 0.0811 + 1.1863i, 0.4349 + 0.9729i, -0.0122 + 0.9215i, 0.6427 + 1.1369i, 0.5948 + 0.4850i, 0.7404 + 0.2269i, 1.2083 + 0.2770i, 1.1681 - 0.0385i, 0.2544 + 0.8072i, 0.5890 + 1.0304i, 1.1267 + 0.5772i, 0.8659 + 0.8966i, 0.9238 - 0.6750i, 0.7554 - 0.4092i, 0.2392 - 0.1870i, 0.3040 - 0.0248i, 0.3711 - 0.7932i, 0.2902 - 1.0778i, 0.1868 - 0.4955i, 0.0285 - 0.9655i, 0.5844 - 0.3154i, 0.7644 - 0.0892i, 1.3090 - 0.2964i, 0.9983 + 0.0907i, 0.4590 - 0.5392i, 0.5626 - 0.9743i, 1.1372 - 0.5859i, 0.8805 - 0.9496i, -0.9171 + 0.6660i, -0.6981 + 0.5284i, -0.2452 + 0.1318i, -0.2190 - 0.0480i, -0.2496 + 0.7298i, -0.3820 + 0.9277i, -0.1152 + 0.4045i, -0.2482 + 1.1085i, -0.5423 + 0.2427i, -0.7596 + 0.1318i, -1.2540 + 0.3266i, -1.0229 - 0.0203i, -0.4490 + 0.4902i, -0.6030 + 1.0773i, -1.0573 + 0.5989i, -0.8208 + 0.9228i, -0.8606 - 0.5360i, -0.6624 - 0.4950i, -0.1299 - 0.4899i, -0.1047 - 0.2730i, -0.2153 - 0.9981i, -0.4460 - 1.1020i, -0.0956 - 0.7628i, -0.0426 - 1.4006i, -0.5707 - 0.4295i, -0.7609 - 0.1458i, -1.2752 - 0.2514i, -1.1371 + 0.0982i, -0.4788 - 0.8968i, -0.7056 - 1.0271i, -1.1453 - 0.5661i, -0.9268 - 0.8392i</p>
18 dB	<p>0.8754 + 0.5960i, 0.5195 + 0.4047i, 0.1076 + 0.4780i, 0.1649 + 0.2470i, 0.0830 + 1.1894i, 0.4259 + 0.9771i, -0.0081 + 0.8498i, 0.5868 + 1.1419i, 0.5999 + 0.4511i, 0.7380 + 0.3056i, 1.2094 + 0.2789i, 1.1669 - 0.0361i, 0.2550 + 0.8318i, 0.5931 + 1.0313i, 1.1263 + 0.5834i, 0.8759 + 0.8632i, 0.9259 - 0.6795i, 0.7554 - 0.4293i, 0.2380 - 0.1823i, 0.3084 - 0.0519i, 0.3746 - 0.7930i, 0.2992 - 1.0614i, 0.1829 - 0.4779i, 0.0280 - 0.9642i, 0.5864 - 0.3052i, 0.7675 - 0.0825i, 1.3079 - 0.3007i, 1.0046 + 0.0955i, 0.4593 - 0.5714i, 0.5679 - 0.9671i, 1.1415 - 0.5887i, 0.8821 - 0.9588i, -0.9113 + 0.6850i, -0.6823 + 0.5333i, -0.2429 + 0.1336i, -0.2216 - 0.0412i, -0.2558 + 0.7314i, -0.3641 + 0.9149i, -0.0585 + 0.4044i, -0.2392 + 1.1026i, -0.5369 + 0.2521i, -0.7825 + 0.1380i, -1.2643 + 0.3086i, -1.0238 + 0.0075i, -0.4488 + 0.4938i, -0.6030 + 1.0716i, -1.0551 + 0.5802i, -0.8179 + 0.8665i, -0.8603 - 0.5356i, -0.6795 - 0.4924i, -0.2156 - 0.4868i, -0.1030 - 0.3529i, -0.2233 - 0.9848i, -0.4538 - 1.0920i, -0.0982 - 0.7609i, -0.0249 - 1.3768i, -0.5588 - 0.4183i, -0.7748 - 0.1429i, -1.2513 - 0.2535i, -1.1411 + 0.0981i, -0.4749 - 0.8904i, -0.7026 - 1.0075i, -1.1421 - 0.5644i, -0.9371 - 0.7959i</p>
Continued on next page	

Table 4.8 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
19 dB	<p>0.8758 + 0.5922i, 0.5170 + 0.3879i, 0.1052 + 0.4681i, 0.1633 + 0.2394i, 0.0875 + 1.1819i, 0.4253 + 0.9685i, -0.0347 + 0.8669i, 0.5985 + 1.0392i, 0.6036 + 0.2578i, 0.7406 + 0.2979i, 1.2017 + 0.2798i, 1.1576 + 0.0552i, 0.2664 + 0.8107i, 0.5674 + 0.9369i, 1.1221 + 0.5833i, 0.7681 + 0.8703i, 0.9153 - 0.6927i, 0.7549 - 0.4626i, 0.2290 - 0.1885i, 0.3064 - 0.0857i, 0.3779 - 0.8018i, 0.3017 - 1.0313i, 0.1617 - 0.4850i, 0.0311 - 0.9737i, 0.5893 - 0.2793i, 0.7537 - 0.0755i, 1.2632 - 0.2965i, 1.0059 + 0.1058i, 0.4463 - 0.5727i, 0.5451 - 0.8557i, 1.1309 - 0.5938i, 0.8735 - 0.9480i, -0.9059 + 0.6809i, -0.6837 + 0.5337i, -0.2416 + 0.1306i, -0.2224 + 0.0324i, -0.2607 + 0.7325i, -0.3535 + 0.9279i, -0.0524 + 0.4056i, -0.2309 + 1.1057i, -0.5400 + 0.2499i, -0.7753 + 0.1399i, -1.1877 + 0.3147i, -0.9395 + 0.0058i, -0.4492 + 0.5070i, -0.6039 + 1.0745i, -1.0646 + 0.5617i, -0.8183 + 0.8669i, -0.8596 - 0.5307i, -0.6363 - 0.4975i, -0.2163 - 0.4960i, -0.1021 - 0.3528i, -0.1945 - 0.9633i, -0.3952 - 1.0936i, -0.1004 - 0.7613i, -0.0264 - 1.3526i, -0.5585 - 0.2667i, -0.7756 - 0.1309i, -1.2086 - 0.2566i, -1.1422 + 0.0942i, -0.4765 - 0.8949i, -0.7059 - 1.0159i, -1.1431 - 0.5664i, -0.9805 - 0.7843i</p>
20 dB	<p>0.8776 + 0.5888i, 0.5154 + 0.3904i, 0.1028 + 0.4698i, 0.1634 + 0.4073i, 0.1067 + 1.0761i, 0.4163 + 0.9644i, -0.0364 + 0.8704i, 0.5945 + 1.0318i, 0.6211 + 0.2527i, 0.7483 + 0.3098i, 1.1979 + 0.2821i, 1.1573 + 0.0603i, 0.2559 + 0.8284i, 0.5589 + 0.9386i, 1.1212 + 0.5874i, 0.7023 + 0.8710i, 0.9067 - 0.6969i, 0.7561 - 0.4663i, 0.2397 - 0.1990i, 0.3062 - 0.0995i, 0.3740 - 0.7672i, 0.2949 - 1.0430i, 0.1560 - 0.4873i, 0.0330 - 0.9993i, 0.5991 - 0.2692i, 0.7860 - 0.0720i, 1.2560 - 0.2970i, 1.0045 + 0.1114i, 0.4512 - 0.5744i, 0.5382 - 0.8527i, 1.1305 - 0.5917i, 0.8686 - 0.9380i, -0.9211 + 0.6792i, -0.6634 + 0.5384i, -0.2394 + 0.1406i, -0.2233 + 0.0410i, -0.2657 + 0.7281i, -0.3603 + 0.9079i, -0.0518 + 0.4108i, -0.2002 + 1.0953i, -0.5421 + 0.2481i, -0.7744 + 0.1470i, -1.1781 + 0.3087i, -0.9479 + 0.0009i, -0.4586 + 0.4986i, -0.6061 + 0.9185i, -1.0644 + 0.5666i, -0.8284 + 0.8601i, -0.8588 - 0.5431i, -0.6324 - 0.4987i, -0.2192 - 0.4954i, -0.1117 - 0.3647i, -0.1812 - 0.9609i, -0.3947 - 1.0562i, -0.1049 - 0.7628i, -0.0072 - 1.3135i, -0.5663 - 0.2584i, -0.7776 - 0.1627i, -1.2127 - 0.2554i, -1.1243 + 0.0761i, -0.4692 - 0.8973i, -0.7050 - 0.9842i, -1.1337 - 0.5148i, -0.9801 - 0.7850i</p>
Continued on next page	

Table 4.8 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
21 dB	<p>0.8562 + 0.4843i, 0.5175 + 0.4213i, 0.1142 + 0.4751i, 0.1545 + 0.4097i, 0.1099 + 1.0862i, 0.4212 + 0.9569i, -0.0250 + 0.8957i, 0.5997 + 1.0314i, 0.6470 + 0.2614i, 0.7626 + 0.3064i, 1.1933 + 0.2147i, 1.1117 + 0.0675i, 0.2663 + 0.8244i, 0.5605 + 0.8668i, 1.0495 + 0.5896i, 0.6867 + 0.8716i, 0.9106 - 0.6909i, 0.7462 - 0.4659i, 0.2481 - 0.2072i, 0.3063 - 0.1821i, 0.3630 - 0.7608i, 0.2808 - 0.8997i, 0.1093 - 0.4938i, 0.0363 - 0.9296i, 0.5922 - 0.2598i, 0.7893 - 0.0881i, 1.1480 - 0.2017i, 1.0009 + 0.1122i, 0.4541 - 0.5595i, 0.5192 - 0.8164i, 1.1070 - 0.5927i, 0.8445 - 0.9397i, -0.9180 + 0.6836i, -0.6770 + 0.5130i, -0.2392 + 0.1226i, -0.2116 + 0.0572i, -0.2656 + 0.7292i, -0.3500 + 0.9025i, -0.0467 + 0.4441i, -0.1894 + 1.0832i, -0.5543 + 0.2484i, -0.7818 + 0.1489i, -1.1665 + 0.1560i, -0.9488 - 0.0000i, -0.4656 + 0.4934i, -0.5992 + 0.9126i, -0.9925 + 0.6054i, -0.8267 + 0.8578i, -0.8586 - 0.4828i, -0.6221 - 0.4833i, -0.2187 - 0.5002i, -0.0911 - 0.3667i, -0.1843 - 0.9394i, -0.3742 - 1.0523i, -0.0982 - 0.7591i, -0.0013 - 1.2361i, -0.5682 - 0.2624i, -0.7770 - 0.1761i, -1.1830 - 0.2395i, -1.1150 - 0.0657i, -0.4612 - 0.9094i, -0.6985 - 0.9798i, -1.0233 - 0.5373i, -0.9737 - 0.7552i</p>
22 dB	<p>0.8513 + 0.4915i, 0.5200 + 0.4251i, 0.1226 + 0.4785i, 0.1564 + 0.4120i, 0.1123 + 1.0892i, 0.4051 + 0.9567i, -0.0200 + 0.8866i, 0.6043 + 1.0545i, 0.6620 + 0.2547i, 0.7685 + 0.3171i, 1.1843 + 0.2349i, 1.1146 + 0.0676i, 0.2699 + 0.8278i, 0.5656 + 0.8719i, 1.0576 + 0.5916i, 0.7048 + 0.8777i, 0.9087 - 0.6968i, 0.7495 - 0.4880i, 0.2565 - 0.2147i, 0.2962 - 0.1827i, 0.3561 - 0.7693i, 0.2970 - 0.9066i, 0.0929 - 0.4941i, 0.0464 - 0.8779i, 0.6260 - 0.2525i, 0.7911 - 0.0907i, 1.1076 - 0.2070i, 0.9965 + 0.1129i, 0.4584 - 0.5557i, 0.5121 - 0.8065i, 1.0426 - 0.5478i, 0.8400 - 0.9187i, -0.9149 + 0.6472i, -0.6743 + 0.5088i, -0.2365 + 0.1271i, -0.2166 + 0.0642i, -0.2604 + 0.7623i, -0.3518 + 0.8899i, -0.0474 + 0.4490i, -0.1877 + 1.0871i, -0.5541 + 0.2480i, -0.7737 + 0.1496i, -1.1640 + 0.0846i, -0.9439 + 0.1506i, -0.4595 + 0.4975i, -0.5941 + 0.9155i, -1.0073 + 0.6063i, -0.8304 + 0.8643i, -0.8622 - 0.4788i, -0.6164 - 0.5725i, -0.2191 - 0.4971i, -0.1374 - 0.3683i, -0.2766 - 0.9506i, -0.3756 - 1.0449i, -0.0931 - 0.7544i, 0.0033 - 1.2266i, -0.5907 - 0.2599i, -0.7776 - 0.1821i, -1.1526 - 0.2351i, -1.0430 - 0.0649i, -0.4667 - 0.9183i, -0.6761 - 0.9887i, -1.0186 - 0.6985i, -0.9378 - 0.7526i</p>
Continued on next page	

Table 4.8 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
23 dB	<p>0.8496 + 0.4704i, 0.5164 + 0.4230i, 0.1888 + 0.4733i, 0.1521 + 0.4056i, 0.1148 + 1.0850i, 0.3997 + 0.9320i, -0.0155 + 0.8409i, 0.5958 + 1.0469i, 0.6509 + 0.2150i, 0.7574 + 0.3133i, 1.1793 + 0.2403i, 1.0676 + 0.0710i, 0.2737 + 0.8113i, 0.5683 + 0.8790i, 1.0452 + 0.5766i, 0.7259 + 0.8453i, 0.9031 - 0.6844i, 0.7278 - 0.4917i, 0.2605 - 0.2216i, 0.2983 - 0.1905i, 0.2959 - 0.7846i, 0.2884 - 0.9078i, 0.0850 - 0.4999i, 0.0307 - 0.8763i, 0.6456 - 0.2283i, 0.6825 - 0.0998i, 1.1132 - 0.1994i, 0.9974 + 0.1148i, 0.4607 - 0.5717i, 0.5291 - 0.8059i, 1.0417 - 0.5628i, 0.8386 - 0.7967i, -0.9108 + 0.6456i, -0.6759 + 0.5049i, -0.2558 + 0.1426i, -0.2514 + 0.0972i, -0.2733 + 0.7842i, -0.3461 + 0.8853i, -0.0189 + 0.4498i, -0.1879 + 1.0846i, -0.5997 + 0.2151i, -0.7835 + 0.1505i, -1.1492 + 0.0636i, -0.9433 + 0.1452i, -0.4635 + 0.5366i, -0.5899 + 0.9108i, -1.0150 + 0.5973i, -0.8524 + 0.8486i, -0.8739 - 0.4924i, -0.6116 - 0.5473i, -0.2282 - 0.5029i, -0.1359 - 0.3726i, -0.2772 - 0.8960i, -0.3621 - 1.0394i, -0.0953 - 0.7200i, 0.0151 - 1.2030i, -0.5970 - 0.2002i, -0.7848 - 0.1778i, -1.1552 - 0.2357i, -1.0453 - 0.0799i, -0.4641 - 0.9356i, -0.6814 - 1.0227i, -0.8664 - 0.7028i, -0.7885 - 0.7558i</p>
24 dB	<p>0.8552 + 0.5077i, 0.5180 + 0.5177i, 0.1868 + 0.4618i, 0.1705 + 0.3890i, 0.0875 + 1.0576i, 0.4028 + 0.8726i, -0.0332 + 0.8255i, 0.6003 + 0.9724i, 0.6693 + 0.1753i, 0.7570 + 0.3453i, 1.1834 + 0.2201i, 1.0537 + 0.0532i, 0.2845 + 0.7996i, 0.5777 + 0.8684i, 1.0424 + 0.5508i, 0.7268 + 0.8395i, 0.9058 - 0.6921i, 0.7361 - 0.5215i, 0.2663 - 0.2317i, 0.3036 - 0.2017i, 0.2805 - 0.7898i, 0.2944 - 0.9088i, 0.0874 - 0.5085i, 0.0504 - 0.9349i, 0.6509 - 0.2398i, 0.6838 - 0.1147i, 1.0516 - 0.2154i, 1.0009 + 0.0932i, 0.4628 - 0.5905i, 0.5279 - 0.8118i, 0.9925 - 0.5740i, 0.8412 - 0.7999i, -0.9048 + 0.6351i, -0.6896 + 0.5007i, -0.2603 + 0.1502i, -0.2487 + 0.2301i, -0.2754 + 0.7779i, -0.3397 + 0.8723i, 0.0251 + 0.4623i, -0.1903 + 1.0717i, -0.5759 + 0.1950i, -0.8110 + 0.1469i, -1.1248 + 0.0533i, -0.9323 + 0.1424i, -0.4753 + 0.5266i, -0.5793 + 0.8897i, -1.0159 + 0.5782i, -0.8443 + 0.8163i, -0.8713 - 0.5091i, -0.6122 - 0.5659i, -0.3346 - 0.5109i, -0.1316 - 0.3869i, -0.2905 - 0.9049i, -0.3518 - 0.9476i, -0.0911 - 0.7248i, 0.0185 - 0.7119i, -0.5919 - 0.2170i, -0.7872 - 0.2049i, -1.1549 - 0.2493i, -1.0427 - 0.0907i, -0.4655 - 0.9411i, -0.6404 - 1.0466i, -0.8648 - 0.7360i, -0.7643 - 0.7158i</p>
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Table 4.8 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
25 dB	<p> $0.8615 + 0.5041i, 0.5130 + 0.5147i, 0.2151 + 0.4606i, 0.2540 + 0.4020i, 0.0895 + 1.0534i, 0.4048 + 0.8442i, -0.0309 + 0.8256i, 0.6010 + 0.9507i,$ $0.6690 + 0.1759i, 0.7546 + 0.3407i, 1.1657 + 0.2058i, 1.0475 + 0.0579i, 0.2878 + 0.7944i, 0.5715 + 0.8656i, 1.0386 + 0.5505i, 0.7276 + 0.8344i,$ $0.9043 - 0.6933i, 0.7201 - 0.5232i, 0.2662 - 0.2399i, 0.3354 - 0.2014i, 0.2849 - 0.7935i, 0.2942 - 0.9063i, 0.0882 - 0.4991i, 0.0484 - 0.9408i,$ $0.6457 - 0.2378i, 0.6778 - 0.1169i, 1.0539 - 0.2112i, 0.9896 + 0.0839i, 0.4545 - 0.5867i, 0.5198 - 0.7918i, 0.9627 - 0.5767i, 0.8415 - 0.8000i,$ $-0.8963 + 0.6360i, -0.6933 + 0.5088i, -0.3301 + 0.1697i, -0.2513 + 0.2209i, -0.2611 + 0.7700i, -0.3443 + 0.8604i, 0.0268 + 0.4582i, -0.1805 + 1.0562i,$ $-0.5783 + 0.1861i, -0.8090 + 0.1474i, -1.1290 + 0.0567i, -0.9312 + 0.1459i, -0.4770 + 0.6011i, -0.5434 + 0.8953i, -1.0146 + 0.5805i, -0.8484 + 0.8179i,$ $-0.8735 - 0.5193i, -0.6264 - 0.6025i, -0.3380 - 0.5084i, -0.1303 - 0.3890i, -0.2881 - 0.9122i, -0.3389 - 0.9191i, -0.0949 - 0.7287i, 0.0030 - 0.7273i,$ $-0.5944 - 0.2200i, -0.8142 - 0.2035i, -1.1572 - 0.2568i, -1.0371 - 0.0927i, -0.4429 - 0.9369i, -0.6315 - 0.9856i, -0.8546 - 0.7366i, -0.7776 - 0.7183i$ </p>

Table 4.9: Optimized irregular constellations proposed for MCS-2 along with $m = 4$ and $\Omega = 1$.

$\bar{\gamma}$	$\chi(4, \bar{\gamma})$
9 dB	<p>0.8450 + 0.5588i, 0.5746 + 0.3912i, -0.0629 + 0.2694i, 0.2324 + 0.0694i, 0.1812 + 0.8675i, 0.1633 + 1.1595i, 0.0746 + 0.6052i, 0.5246 + 1.2625i, 0.7245 + 0.1604i, 0.8154 + 0.1003i, 1.2813 + 0.3939i, 1.2007 - 0.1352i, 0.2383 + 0.6572i, 0.6163 + 1.0221i, 1.1062 + 0.6882i, 0.8249 + 0.8893i, 0.8993 - 0.5887i, 0.6210 - 0.5083i, -0.0314 - 0.0474i, 0.2664 + 0.0774i, 0.0412 - 0.6857i, 0.2772 - 1.0023i, 0.1801 - 0.3610i, 0.3904 - 1.2363i, 0.6192 - 0.3120i, 0.7439 - 0.1717i, 1.3096 - 0.3541i, 1.2604 + 0.1202i, 0.2813 - 0.6047i, 0.6157 - 1.0413i, 1.1368 - 0.6799i, 0.8001 - 0.9329i, -0.9618 + 0.5376i, -0.6673 + 0.4823i, 0.3068 + 0.4951i, -0.2447 + 0.1766i, -0.2274 + 0.7232i, -0.4342 + 0.8442i, 0.0142 + 1.3063i, -0.3159 + 1.2249i, -0.6944 + 0.2247i, -0.7256 + 0.0966i, -1.3271 + 0.2848i, -1.2191 - 0.1750i, -0.4075 + 0.4539i, -0.6191 + 1.0465i, -1.1722 + 0.6413i, -0.8677 + 0.9416i, -0.8886 - 0.5498i, -0.4795 - 0.4177i, 0.0320 - 0.4279i, -0.2107 - 0.1949i, -0.0562 - 1.1315i, -0.3579 - 0.8570i, -0.0096 - 1.3457i, -0.6269 - 1.1962i, -0.6737 - 0.2619i, -0.6948 - 0.1374i, -1.2532 - 0.3786i, -1.2266 + 0.1043i, -0.2913 - 0.5821i, -0.5171 - 0.9438i, -1.1240 - 0.7000i, -0.8109 - 0.9154i</p>
10 dB	<p>0.8548 + 0.5178i, 0.5516 + 0.3792i, -0.0218 + 0.2975i, 0.2799 + 0.3356i, 0.0494 + 1.0124i, 0.2429 + 1.2092i, 0.0276 + 0.7192i, 0.6395 + 1.1181i, 0.5885 - 0.0455i, 0.7999 + 0.1018i, 1.2766 + 0.3463i, 1.1655 - 0.1182i, 0.1341 + 0.6995i, 0.6709 + 1.0149i, 1.1264 + 0.6197i, 0.7237 + 0.7787i, 0.9031 - 0.6006i, 0.6130 - 0.5877i, 0.0309 + 0.0123i, 0.3290 + 0.0857i, 0.0381 - 0.6759i, 0.3565 - 0.9138i, 0.1589 - 0.3377i, 0.2851 - 1.2155i, 0.5659 - 0.3797i, 0.6733 - 0.1796i, 1.2339 - 0.3203i, 1.1388 + 0.1781i, 0.2823 - 0.5653i, 0.6449 - 1.1132i, 1.1486 - 0.5972i, 0.8073 - 0.9287i, -1.0003 + 0.4715i, -0.7301 + 0.4738i, 0.3318 + 0.5675i, -0.2207 + 0.1253i, -0.2548 + 0.6218i, -0.4380 + 0.8167i, 0.0208 + 1.3479i, -0.3078 + 1.1855i, -0.7219 + 0.1897i, -0.5815 + 0.0837i, -1.2282 + 0.1666i, -1.1723 - 0.2701i, -0.4477 + 0.5008i, -0.5968 + 0.9994i, -1.1294 + 0.6183i, -0.8601 + 0.9154i, -0.8178 - 0.4888i, -0.5032 - 0.4445i, -0.0422 - 0.4302i, -0.1069 - 0.1708i, -0.1004 - 1.1510i, -0.3757 - 0.8841i, 0.0543 - 1.2210i, -0.6969 - 1.1478i, -0.6023 - 0.1583i, -0.6505 - 0.1670i, -1.2074 - 0.2925i, -1.2970 + 0.0692i, -0.2628 - 0.6824i, -0.5199 - 0.9574i, -1.0634 - 0.6620i, -0.7900 - 0.8721i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
11 dB	<p>0.8530 + 0.5155i, 0.5515 + 0.3772i, -0.0219 + 0.2965i, 0.2824 + 0.3362i, 0.0509 + 1.0117i, 0.2502 + 1.2065i, 0.0244 + 0.7171i, 0.6395 + 1.1150i, 0.5928 - 0.0471i, 0.7998 + 0.1241i, 1.2745 + 0.3349i, 1.1664 - 0.1192i, 0.1342 + 0.7216i, 0.6695 + 1.0142i, 1.1270 + 0.6168i, 0.7218 + 0.7732i, 0.9047 - 0.6009i, 0.6128 - 0.5899i, 0.0286 + 0.0102i, 0.3294 + 0.0815i, 0.0399 - 0.6804i, 0.3568 - 0.9212i, 0.1590 - 0.3391i, 0.2828 - 1.2174i, 0.5657 - 0.3862i, 0.6739 - 0.1820i, 1.2297 - 0.3151i, 1.1385 + 0.1761i, 0.2823 - 0.5650i, 0.6454 - 1.1150i, 1.1456 - 0.5980i, 0.8075 - 0.9322i, -0.9979 + 0.4629i, -0.7327 + 0.4754i, 0.3243 + 0.5661i, -0.2202 + 0.1278i, -0.2587 + 0.6196i, -0.4413 + 0.8140i, 0.0170 + 1.3467i, -0.3096 + 1.1830i, -0.7209 + 0.1879i, -0.5811 + 0.0837i, -1.2281 + 0.1653i, -1.1754 - 0.2749i, -0.4491 + 0.5007i, -0.5802 + 0.9934i, -1.1170 + 0.6167i, -0.8588 + 0.9166i, -0.8165 - 0.4898i, -0.5048 - 0.4390i, -0.0424 - 0.4280i, -0.1061 - 0.1742i, -0.1046 - 1.1533i, -0.3752 - 0.8864i, 0.0534 - 1.1621i, -0.6990 - 1.1495i, -0.6042 - 0.1607i, -0.6485 - 0.1664i, -1.2080 - 0.2914i, -1.3002 + 0.0679i, -0.2577 - 0.6789i, -0.5232 - 0.9605i, -1.0608 - 0.6604i, -0.7910 - 0.8716i</p>
12 dB	<p>0.8536 + 0.5119i, 0.5529 + 0.3763i, -0.0219 + 0.2952i, 0.2833 + 0.3599i, 0.0532 + 1.0132i, 0.2520 + 1.2061i, 0.0252 + 0.7180i, 0.6355 + 1.1105i, 0.5936 - 0.0481i, 0.8028 + 0.1238i, 1.2431 + 0.3345i, 1.1651 - 0.1137i, 0.1295 + 0.7244i, 0.6614 + 1.0123i, 1.1351 + 0.6176i, 0.7232 + 0.7736i, 0.9042 - 0.5952i, 0.6138 - 0.5957i, 0.0301 + 0.0109i, 0.3283 + 0.0778i, 0.0393 - 0.6840i, 0.3548 - 0.9257i, 0.1523 - 0.3381i, 0.2896 - 1.2092i, 0.5685 - 0.3851i, 0.6730 - 0.1904i, 1.2240 - 0.3151i, 1.1384 + 0.1730i, 0.2802 - 0.5662i, 0.6477 - 1.1165i, 1.1391 - 0.5984i, 0.8068 - 0.9322i, -0.9977 + 0.4502i, -0.7322 + 0.4774i, 0.3235 + 0.5646i, -0.1960 + 0.1272i, -0.2617 + 0.6191i, -0.4434 + 0.8086i, 0.0145 + 1.3461i, -0.3085 + 1.1820i, -0.7106 + 0.1839i, -0.5824 + 0.0838i, -1.2322 + 0.1549i, -1.1530 - 0.2683i, -0.4621 + 0.4967i, -0.5821 + 0.9929i, -1.1158 + 0.6162i, -0.8448 + 0.9144i, -0.8181 - 0.4896i, -0.5052 - 0.4571i, -0.0483 - 0.4302i, -0.0999 - 0.1847i, -0.1006 - 1.1528i, -0.3874 - 0.8843i, 0.0535 - 1.1320i, -0.6985 - 1.1504i, -0.6103 - 0.1624i, -0.6463 - 0.1672i, -1.1987 - 0.2900i, -1.3000 + 0.0654i, -0.2555 - 0.6802i, -0.5240 - 0.9663i, -1.0619 - 0.6560i, -0.7920 - 0.8372i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
13 dB	<p>0.8494 + 0.5058i, 0.5573 + 0.3833i, -0.0218 + 0.3072i, 0.2890 + 0.3628i, 0.0578 + 0.9587i, 0.2497 + 1.1927i, 0.0297 + 0.7178i, 0.6204 + 1.0998i, 0.5882 - 0.0493i, 0.8061 + 0.1282i, 1.2401 + 0.3276i, 1.1220 - 0.1155i, 0.1343 + 0.7817i, 0.6681 + 1.0151i, 1.1676 + 0.5810i, 0.7322 + 0.7678i, 0.8617 - 0.5897i, 0.6305 - 0.5821i, 0.0324 + 0.0014i, 0.3331 + 0.0689i, 0.0390 - 0.6842i, 0.3540 - 0.9239i, 0.1372 - 0.3436i, 0.2894 - 1.2129i, 0.5585 - 0.3871i, 0.6557 - 0.1872i, 1.2204 - 0.3058i, 1.0957 + 0.1649i, 0.2943 - 0.5667i, 0.6481 - 1.1157i, 1.1371 - 0.5751i, 0.8017 - 0.9273i, -0.9968 + 0.4522i, -0.7387 + 0.4801i, 0.3164 + 0.5652i, -0.1806 + 0.1342i, -0.2639 + 0.6159i, -0.4243 + 0.8096i, 0.0126 + 1.3497i, -0.3127 + 1.1226i, -0.6978 + 0.1865i, -0.5771 + 0.0895i, -1.2280 + 0.1557i, -1.1553 - 0.2665i, -0.4616 + 0.5496i, -0.5670 + 0.9878i, -1.1146 + 0.6154i, -0.8377 + 0.9123i, -0.8203 - 0.4814i, -0.5036 - 0.4526i, -0.0502 - 0.4309i, -0.1031 - 0.1919i, -0.1024 - 1.1525i, -0.3907 - 0.8839i, 0.0501 - 1.1310i, -0.6993 - 1.1522i, -0.6092 - 0.1587i, -0.6491 - 0.1644i, -1.1963 - 0.2864i, -1.2896 + 0.0665i, -0.2542 - 0.6806i, -0.5246 - 0.9701i, -1.0204 - 0.6585i, -0.7886 - 0.8299i</p>
14 dB	<p>0.8474 + 0.5064i, 0.5546 + 0.4039i, -0.0106 + 0.3235i, 0.3031 + 0.3746i, 0.0574 + 0.9585i, 0.2537 + 1.1849i, 0.0309 + 0.7220i, 0.6299 + 1.0802i, 0.5848 - 0.0518i, 0.8137 + 0.1851i, 1.2399 + 0.3162i, 1.1172 - 0.1225i, 0.1251 + 0.7843i, 0.6693 + 1.0107i, 1.1579 + 0.5770i, 0.7334 + 0.7734i, 0.8585 - 0.5409i, 0.6431 - 0.5978i, 0.0297 + 0.0014i, 0.3298 + 0.0361i, 0.0375 - 0.6840i, 0.3500 - 0.9232i, 0.1343 - 0.3911i, 0.2871 - 1.2150i, 0.5582 - 0.3853i, 0.6581 - 0.1882i, 1.2039 - 0.2717i, 1.0910 + 0.1633i, 0.2985 - 0.5650i, 0.6446 - 1.1147i, 1.1168 - 0.5777i, 0.8008 - 0.9354i, -1.0073 + 0.4491i, -0.7486 + 0.4786i, 0.3083 + 0.5751i, -0.1822 + 0.1555i, -0.2672 + 0.6143i, -0.4248 + 0.8045i, 0.0300 + 1.3437i, -0.3182 + 1.1217i, -0.7049 + 0.1993i, -0.5725 + 0.0976i, -1.2166 + 0.1522i, -1.1575 - 0.2507i, -0.4556 + 0.5452i, -0.5560 + 0.9850i, -1.1233 + 0.6104i, -0.8340 + 0.9205i, -0.8252 - 0.4802i, -0.5147 - 0.4693i, -0.0507 - 0.4618i, -0.0999 - 0.1919i, -0.0999 - 1.1522i, -0.3930 - 0.8955i, 0.0454 - 1.1032i, -0.6742 - 1.1461i, -0.6089 - 0.1649i, -0.6500 - 0.1668i, -1.1895 - 0.3150i, -1.2804 + 0.0724i, -0.2574 - 0.7291i, -0.5286 - 0.9667i, -1.0207 - 0.6473i, -0.7715 - 0.8220i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
15 dB	<p>0.8224 + 0.4903i, 0.5436 + 0.3869i, -0.0088 + 0.3252i, 0.3030 + 0.3712i, 0.0573 + 0.9490i, 0.2447 + 1.1954i, 0.0337 + 0.7117i, 0.6282 + 1.0790i, 0.5749 - 0.0523i, 0.8167 + 0.1846i, 1.2415 + 0.3633i, 1.1055 - 0.1194i, 0.1255 + 0.7900i, 0.6705 + 1.0198i, 1.1984 + 0.5699i, 0.7483 + 0.7645i, 0.8607 - 0.5435i, 0.6432 - 0.5999i, 0.0312 + 0.0013i, 0.3324 - 0.0084i, 0.0535 - 0.6876i, 0.3571 - 0.9021i, 0.1118 - 0.4200i, 0.2867 - 1.2230i, 0.5527 - 0.3902i, 0.6578 - 0.1954i, 1.1940 - 0.2711i, 1.0903 + 0.1599i, 0.2989 - 0.5714i, 0.6452 - 1.1198i, 1.1224 - 0.5825i, 0.8155 - 0.8927i, -0.9830 + 0.4422i, -0.7514 + 0.5074i, 0.2987 + 0.5761i, -0.1830 + 0.1533i, -0.2651 + 0.6114i, -0.4211 + 0.7750i, 0.0195 + 1.3436i, -0.3182 + 1.1279i, -0.7036 + 0.1967i, -0.5674 + 0.0994i, -1.2213 + 0.1574i, -1.1559 - 0.2544i, -0.4863 + 0.5582i, -0.5724 + 0.9779i, -1.1101 + 0.6066i, -0.8249 + 0.9206i, -0.8245 - 0.4781i, -0.5463 - 0.4849i, -0.0430 - 0.4812i, -0.1005 - 0.1969i, -0.1003 - 1.1752i, -0.3941 - 0.8678i, 0.0376 - 1.0934i, -0.6836 - 1.1079i, -0.6111 - 0.0304i, -0.6491 - 0.1533i, -1.1876 - 0.3132i, -1.2254 + 0.0715i, -0.2482 - 0.8099i, -0.5300 - 0.9785i, -1.0157 - 0.6510i, -0.7914 - 0.8320i</p>
16 dB	<p>0.8085 + 0.4929i, 0.5507 + 0.3889i, -0.0014 + 0.3283i, 0.3038 + 0.3806i, 0.0571 + 0.9575i, 0.2544 + 1.2028i, 0.0362 + 0.7113i, 0.6310 + 1.0734i, 0.5745 - 0.0496i, 0.8201 + 0.1839i, 1.2487 + 0.3676i, 1.0962 - 0.1163i, 0.1278 + 0.8263i, 0.6569 + 1.0015i, 1.2031 + 0.5715i, 0.7497 + 0.7671i, 0.8641 - 0.5208i, 0.6318 - 0.5971i, 0.0396 - 0.0364i, 0.3338 - 0.0060i, 0.0701 - 0.6851i, 0.3595 - 0.8944i, 0.1136 - 0.4410i, 0.2941 - 1.1969i, 0.5602 - 0.3846i, 0.6618 - 0.1910i, 1.2081 - 0.2642i, 1.0892 + 0.1704i, 0.2996 - 0.5725i, 0.6304 - 1.1138i, 1.1274 - 0.5789i, 0.8090 - 0.8821i, -0.9824 + 0.4389i, -0.7833 + 0.5126i, 0.2703 + 0.5824i, -0.1778 + 0.1574i, -0.2585 + 0.6134i, -0.4193 + 0.7775i, 0.0212 + 1.3474i, -0.3202 + 1.1255i, -0.7172 + 0.1783i, -0.5649 + 0.0935i, -1.2227 + 0.1606i, -1.1522 - 0.2573i, -0.4833 + 0.5878i, -0.5620 + 0.9802i, -1.1089 + 0.5364i, -0.8182 + 0.9203i, -0.8159 - 0.4771i, -0.5495 - 0.5236i, -0.0477 - 0.4666i, -0.1054 - 0.2436i, -0.1014 - 1.1632i, -0.3943 - 0.8667i, 0.0402 - 1.0865i, -0.6839 - 1.1072i, -0.6189 - 0.0172i, -0.6477 - 0.1552i, -1.2067 - 0.3027i, -1.2222 + 0.0713i, -0.2464 - 0.8149i, -0.5245 - 0.9712i, -1.0165 - 0.6467i, -0.7896 - 0.8771i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
17 dB	<p>0.8096 + 0.4909i, 0.5511 + 0.4022i, -0.0119 + 0.3215i, 0.3120 + 0.3790i, 0.0578 + 0.9517i, 0.2551 + 1.0822i, 0.0358 + 0.7099i, 0.6357 + 1.0337i, 0.5560 - 0.0553i, 0.8218 + 0.1863i, 1.2037 + 0.3605i, 1.0862 - 0.1139i, 0.1326 + 0.8993i, 0.6299 + 0.9973i, 1.2032 + 0.5563i, 0.7490 + 0.7851i, 0.8632 - 0.5267i, 0.6418 - 0.5679i, 0.0479 - 0.0187i, 0.3355 - 0.0101i, 0.0701 - 0.6769i, 0.3669 - 0.8985i, 0.1149 - 0.4435i, 0.2368 - 1.1867i, 0.5664 - 0.3813i, 0.6628 - 0.1932i, 1.1535 - 0.2660i, 1.0861 + 0.1755i, 0.3008 - 0.5649i, 0.6386 - 1.0987i, 1.1249 - 0.5802i, 0.8116 - 0.8817i, -0.9765 + 0.4370i, -0.8358 + 0.5111i, 0.2643 + 0.5785i, -0.1733 + 0.1559i, -0.2858 + 0.6335i, -0.4006 + 0.7766i, 0.0233 + 1.3488i, -0.3498 + 1.1212i, -0.7154 + 0.1930i, -0.5383 + 0.0866i, -1.1321 + 0.1582i, -1.0534 - 0.2878i, -0.4827 + 0.5738i, -0.5641 + 0.9749i, -1.1083 + 0.5357i, -0.8158 + 0.9074i, -0.8182 - 0.4504i, -0.5494 - 0.5252i, -0.0516 - 0.4706i, -0.1090 - 0.2506i, -0.0991 - 1.1600i, -0.4077 - 0.8738i, 0.0495 - 1.0479i, -0.6875 - 1.0969i, -0.6204 - 0.0201i, -0.6444 - 0.1552i, -1.2078 - 0.3080i, -1.2219 + 0.0363i, -0.2418 - 0.8202i, -0.5167 - 0.9756i, -1.0113 - 0.6451i, -0.7681 - 0.8084i</p>
18 dB	<p>0.8041 + 0.4917i, 0.5388 + 0.4241i, -0.0053 + 0.3236i, 0.3119 + 0.3978i, 0.0593 + 0.9617i, 0.3193 + 1.0549i, 0.0334 + 0.7164i, 0.6273 + 1.0202i, 0.5551 - 0.0502i, 0.8026 + 0.1953i, 1.1590 + 0.3613i, 1.0891 - 0.0711i, 0.1345 + 0.9121i, 0.6337 + 0.9084i, 1.2104 + 0.5625i, 0.7452 + 0.7918i, 0.8611 - 0.5356i, 0.6406 - 0.5653i, 0.0538 - 0.0140i, 0.3428 - 0.0156i, 0.0696 - 0.6758i, 0.3728 - 0.9227i, 0.0656 - 0.4393i, 0.2364 - 1.1491i, 0.5669 - 0.3612i, 0.6571 - 0.1871i, 1.1496 - 0.2502i, 1.0911 + 0.1794i, 0.2696 - 0.5859i, 0.6410 - 1.0695i, 1.1129 - 0.5768i, 0.7700 - 0.9045i, -0.9703 + 0.4487i, -0.8388 + 0.5480i, 0.2193 + 0.5878i, -0.1845 + 0.1627i, -0.2946 + 0.6508i, -0.4074 + 0.7616i, 0.0251 + 1.3597i, -0.3598 + 1.1185i, -0.7080 + 0.1276i, -0.5169 + 0.0883i, -1.0854 + 0.1567i, -1.0247 - 0.2466i, -0.4874 + 0.6400i, -0.5200 + 0.9685i, -1.1162 + 0.5419i, -0.8270 + 0.9118i, -0.8173 - 0.4501i, -0.5497 - 0.5479i, -0.0640 - 0.5329i, -0.1025 - 0.2485i, -0.1056 - 1.1893i, -0.4459 - 0.8756i, 0.0295 - 1.0477i, -0.6895 - 1.0775i, -0.6141 - 0.0123i, -0.6419 - 0.1451i, -1.2023 - 0.2996i, -1.1891 - 0.0440i, -0.2479 - 0.8167i, -0.5169 - 0.9899i, -0.9494 - 0.6640i, -0.7163 - 0.8122i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
19 dB	<p>0.8076 + 0.4941i, 0.5357 + 0.4323i, -0.0071 + 0.3167i, 0.3093 + 0.3973i, 0.0565 + 0.9566i, 0.3121 + 1.0548i, 0.0259 + 0.7247i, 0.5804 + 1.0125i, 0.5624 - 0.0549i, 0.7996 + 0.2052i, 1.1557 + 0.3583i, 1.0774 - 0.0744i, 0.1214 + 0.9095i, 0.6333 + 0.9040i, 1.2232 + 0.5613i, 0.7244 + 0.8196i, 0.8561 - 0.5230i, 0.6248 - 0.5735i, 0.0565 - 0.0155i, 0.3398 - 0.0086i, 0.0674 - 0.6813i, 0.3613 - 0.8897i, 0.0625 - 0.4491i, 0.2496 - 1.1491i, 0.5586 - 0.3410i, 0.6603 - 0.1887i, 1.1370 - 0.2526i, 1.0647 + 0.1787i, 0.2714 - 0.5897i, 0.6362 - 1.0656i, 1.1128 - 0.5808i, 0.7723 - 0.8841i, -0.9741 + 0.4501i, -0.8475 + 0.5481i, 0.2198 + 0.5795i, -0.1881 + 0.1937i, -0.3131 + 0.6325i, -0.3595 + 0.7514i, 0.0705 + 1.3670i, -0.3597 + 1.1361i, -0.7058 + 0.1253i, -0.5209 + 0.0411i, -1.0881 + 0.1577i, -1.0197 - 0.2482i, -0.4683 + 0.6416i, -0.5224 + 0.9005i, -1.1153 + 0.5404i, -0.8359 + 0.9218i, -0.8214 - 0.4478i, -0.5487 - 0.5877i, -0.0807 - 0.5330i, -0.1090 - 0.3049i, -0.0764 - 1.1918i, -0.4495 - 0.8573i, 0.0258 - 1.0476i, -0.6913 - 1.0814i, -0.6195 + 0.0304i, -0.6475 - 0.1492i, -1.0820 - 0.3007i, -1.1766 - 0.0399i, -0.2487 - 0.8357i, -0.5536 - 1.0057i, -0.9247 - 0.6653i, -0.7173 - 0.7254i</p>
20 dB	<p>0.8028 + 0.4968i, 0.5229 + 0.4306i, 0.0039 + 0.3233i, 0.3132 + 0.3986i, 0.0638 + 0.9650i, 0.3210 + 0.9954i, 0.0279 + 0.7255i, 0.5843 + 0.9903i, 0.5647 - 0.0577i, 0.8018 + 0.2083i, 1.1537 + 0.3569i, 1.0436 - 0.0708i, 0.1362 + 0.9191i, 0.6405 + 0.9008i, 1.2125 + 0.5597i, 0.7204 + 0.8232i, 0.8418 - 0.5229i, 0.5955 - 0.5763i, 0.0713 - 0.0605i, 0.3470 - 0.0080i, 0.0718 - 0.6875i, 0.3613 - 0.7852i, 0.0122 - 0.4460i, 0.2537 - 1.1525i, 0.5671 - 0.3307i, 0.6604 - 0.1999i, 1.0437 - 0.2573i, 1.0512 + 0.1843i, 0.2615 - 0.5819i, 0.6262 - 1.0226i, 1.1131 - 0.5971i, 0.7782 - 0.8654i, -0.9042 + 0.4380i, -0.8503 + 0.5518i, 0.1929 + 0.5781i, -0.1864 + 0.1949i, -0.3125 + 0.6283i, -0.3665 + 0.7509i, 0.0390 + 1.3561i, -0.3521 + 1.1398i, -0.7073 + 0.1233i, -0.5196 + 0.0961i, -1.0828 + 0.1623i, -1.0049 - 0.2968i, -0.4589 + 0.6565i, -0.5153 + 0.8390i, -1.1171 + 0.5290i, -0.7935 + 0.8636i, -0.8185 - 0.4474i, -0.5523 - 0.5769i, -0.0930 - 0.5369i, -0.1278 - 0.3055i, -0.0733 - 1.1920i, -0.4517 - 0.8463i, 0.0253 - 1.0203i, -0.6610 - 1.0736i, -0.6315 + 0.0327i, -0.6324 - 0.1442i, -1.0795 - 0.3152i, -1.1676 - 0.0263i, -0.2405 - 0.8357i, -0.5447 - 0.9963i, -0.8642 - 0.6692i, -0.7169 - 0.7134i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
21 dB	<p>0.7859 + 0.4830i, 0.5254 + 0.4246i, 0.0090 + 0.3201i, 0.3178 + 0.3982i, 0.0649 + 0.9648i, 0.3207 + 0.9639i, 0.0291 + 0.7287i, 0.5939 + 0.8991i, 0.5693 - 0.0593i, 0.8183 + 0.2559i, 1.1416 + 0.3735i, 1.0565 - 0.0694i, 0.1283 + 0.9468i, 0.6631 + 0.9052i, 1.2172 + 0.5544i, 0.7191 + 0.8165i, 0.8426 - 0.5057i, 0.5998 - 0.5784i, 0.0734 - 0.0921i, 0.3588 - 0.0104i, 0.0921 - 0.6867i, 0.3660 - 0.7771i, 0.0136 - 0.4484i, 0.2595 - 1.1249i, 0.5841 - 0.3248i, 0.6597 - 0.2008i, 1.0445 - 0.2582i, 0.9786 + 0.1852i, 0.2616 - 0.5834i, 0.6256 - 1.0201i, 1.0999 - 0.5953i, 0.7714 - 0.8654i, -0.9104 + 0.4363i, -0.8484 + 0.5616i, 0.1851 + 0.5761i, -0.1887 + 0.1934i, -0.2762 + 0.6382i, -0.3625 + 0.7519i, 0.0142 + 1.3270i, -0.3461 + 1.1400i, -0.7039 + 0.1179i, -0.5167 + 0.0864i, -1.0765 + 0.1602i, -0.9771 - 0.2949i, -0.4605 + 0.6576i, -0.5221 + 0.8330i, -1.1246 + 0.5274i, -0.7966 + 0.8793i, -0.8174 - 0.3632i, -0.5452 - 0.5940i, -0.1528 - 0.5486i, -0.1681 - 0.3274i, -0.0868 - 1.1926i, -0.4484 - 0.8406i, 0.0466 - 1.0208i, -0.6523 - 1.0736i, -0.6311 + 0.0328i, -0.6235 - 0.1771i, -1.0765 - 0.2669i, -1.1693 - 0.0261i, -0.2381 - 0.8352i, -0.5405 - 0.9907i, -0.8604 - 0.6617i, -0.7167 - 0.7254i</p>
22 dB	<p>0.7920 + 0.4785i, 0.5182 + 0.4216i, 0.0923 + 0.3197i, 0.3419 + 0.3994i, 0.0475 + 0.9549i, 0.3064 + 0.9713i, 0.0241 + 0.7302i, 0.5893 + 0.9030i, 0.5643 - 0.0584i, 0.8033 + 0.2554i, 1.1128 + 0.3890i, 1.0382 - 0.0838i, 0.1223 + 0.9456i, 0.6452 + 0.8978i, 1.2051 + 0.5437i, 0.7175 + 0.8110i, 0.8701 - 0.5086i, 0.5894 - 0.5796i, 0.0995 - 0.0900i, 0.3695 - 0.0250i, 0.0862 - 0.6948i, 0.3664 - 0.7849i, 0.0030 - 0.4554i, 0.2559 - 1.1033i, 0.5957 - 0.3207i, 0.6379 - 0.2262i, 0.9942 - 0.2668i, 1.0019 + 0.1251i, 0.2569 - 0.5829i, 0.5886 - 0.8964i, 0.9779 - 0.5866i, 0.7618 - 0.8614i, -0.9123 + 0.4358i, -0.8544 + 0.5501i, 0.1778 + 0.5675i, -0.2060 + 0.2087i, -0.2788 + 0.6326i, -0.3633 + 0.7683i, 0.0094 + 1.3302i, -0.3341 + 1.1189i, -0.6848 + 0.1296i, -0.5336 + 0.0848i, -1.0855 + 0.1577i, -0.8375 - 0.2884i, -0.4759 + 0.6658i, -0.5259 + 0.7876i, -1.1291 + 0.5156i, -0.7795 + 0.8803i, -0.8248 - 0.3651i, -0.5608 - 0.5825i, -0.1672 - 0.5480i, -0.1895 - 0.3302i, -0.0881 - 1.1892i, -0.4518 - 0.8448i, 0.0433 - 1.0267i, -0.6162 - 1.0808i, -0.6372 + 0.0277i, -0.6211 - 0.1849i, -1.0832 - 0.2521i, -1.1708 - 0.0258i, -0.2403 - 0.8400i, -0.5173 - 0.9239i, -0.8689 - 0.6755i, -0.5680 - 0.7244i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
23 dB	<p>0.7701 + 0.4776i, 0.5167 + 0.4160i, 0.1010 + 0.3120i, 0.3376 + 0.3958i, 0.1089 + 0.9613i, 0.3078 + 0.9647i, 0.0348 + 0.7247i, 0.5905 + 0.8970i, 0.5765 - 0.0663i, 0.7667 + 0.2562i, 1.0130 + 0.4288i, 1.0412 - 0.0744i, 0.1163 + 0.9592i, 0.6480 + 0.8513i, 1.1968 + 0.5427i, 0.7202 + 0.8007i, 0.8663 - 0.5060i, 0.5953 - 0.5727i, 0.1033 - 0.1022i, 0.3718 - 0.0286i, 0.0966 - 0.6915i, 0.3673 - 0.7835i, 0.0050 - 0.4785i, 0.2619 - 1.1052i, 0.6089 - 0.3204i, 0.6400 - 0.2248i, 1.0006 - 0.2613i, 0.9087 + 0.1199i, 0.2503 - 0.6042i, 0.5788 - 0.8849i, 0.9822 - 0.5871i, 0.7736 - 0.8577i, -0.8954 + 0.4284i, -0.8483 + 0.5542i, 0.1205 + 0.5680i, -0.2515 + 0.2069i, -0.2631 + 0.6403i, -0.3600 + 0.7840i, 0.0104 + 1.3243i, -0.3310 + 1.1122i, -0.6835 + 0.1297i, -0.5400 + 0.0835i, -1.0698 + 0.1567i, -0.8102 - 0.2883i, -0.4690 + 0.6929i, -0.5096 + 0.8465i, -1.1211 + 0.5266i, -0.7784 + 0.8573i, -0.8129 - 0.3522i, -0.5607 - 0.5830i, -0.2231 - 0.5490i, -0.1990 - 0.3434i, -0.0773 - 1.1876i, -0.4283 - 0.8468i, 0.0436 - 0.9509i, -0.5724 - 1.0936i, -0.6306 + 0.0289i, -0.6275 - 0.1701i, -1.0805 - 0.2645i, -1.1160 - 0.1564i, -0.2342 - 0.8320i, -0.5228 - 0.9171i, -0.8721 - 0.6483i, -0.5429 - 0.7157i</p>
24 dB	<p>0.7747 + 0.4717i, 0.5174 + 0.4220i, 0.1034 + 0.3076i, 0.3402 + 0.3999i, 0.1116 + 0.9588i, 0.3040 + 0.9616i, 0.0351 + 0.7185i, 0.5910 + 0.8392i, 0.5648 - 0.0324i, 0.7649 + 0.2478i, 1.0110 + 0.4176i, 1.0453 - 0.0756i, 0.1164 + 0.9568i, 0.6366 + 0.8611i, 1.1887 + 0.5673i, 0.7209 + 0.7890i, 0.8665 - 0.5046i, 0.6107 - 0.5620i, 0.1156 - 0.1263i, 0.4119 - 0.0468i, 0.1253 - 0.6947i, 0.3570 - 0.6655i, 0.0011 - 0.4959i, 0.2592 - 1.0921i, 0.6179 - 0.3216i, 0.6251 - 0.2260i, 1.0003 - 0.2643i, 0.9077 + 0.1162i, 0.2548 - 0.6162i, 0.5603 - 0.8961i, 0.9814 - 0.6062i, 0.7740 - 0.8339i, -0.8927 + 0.4414i, -0.8599 + 0.5548i, 0.0998 + 0.5675i, -0.2648 + 0.2055i, -0.2668 + 0.6494i, -0.3532 + 0.7852i, -0.0111 + 1.2835i, -0.3311 + 1.1141i, -0.6745 + 0.1209i, -0.5363 + 0.0891i, -1.0659 + 0.1018i, -0.8066 - 0.2806i, -0.4700 + 0.6916i, -0.5106 + 0.8488i, -1.1123 + 0.5211i, -0.7780 + 0.8495i, -0.8140 - 0.3543i, -0.6208 - 0.5872i, -0.2319 - 0.5430i, -0.2344 - 0.3425i, -0.0723 - 1.1636i, -0.4110 - 0.8517i, 0.0416 - 0.8867i, -0.5718 - 1.0738i, -0.6278 + 0.0252i, -0.6378 - 0.1646i, -1.0736 - 0.2681i, -1.0745 - 0.1620i, -0.2325 - 0.8426i, -0.5221 - 0.9184i, -0.8508 - 0.6506i, -0.5270 - 0.7348i</p>
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Table 4.9 – continued from previous page

$\bar{\gamma}$	$\chi(2, \bar{\gamma})$
25 dB	<p>0.7648 + 0.4517i, 0.5153 + 0.4161i, 0.1414 + 0.2932i, 0.3446 + 0.4020i, 0.1051 + 0.9522i, 0.3050 + 0.9576i, 0.0566 + 0.7197i, 0.5845 + 0.8262i, 0.5660 - 0.0304i, 0.7504 + 0.2625i, 0.9491 + 0.4158i, 1.0313 - 0.0129i, 0.1170 + 0.9596i, 0.6577 + 0.8545i, 1.1865 + 0.5491i, 0.7238 + 0.7727i, 0.8729 - 0.5147i, 0.6138 - 0.5612i, 0.1182 - 0.1353i, 0.4185 - 0.0581i, 0.1257 - 0.7017i, 0.3584 - 0.6674i, 0.0032 - 0.5073i, 0.2585 - 1.0713i, 0.6281 - 0.3215i, 0.6316 - 0.2288i, 0.9996 - 0.2559i, 0.9047 + 0.1187i, 0.2515 - 0.6572i, 0.5670 - 0.8387i, 0.9867 - 0.6191i, 0.7280 - 0.8380i, -0.8937 + 0.4488i, -0.8630 + 0.5596i, 0.0554 + 0.5794i, -0.2631 + 0.2188i, -0.2774 + 0.6460i, -0.3379 + 0.7869i, -0.0167 + 1.2937i, -0.3275 + 1.1098i, -0.6798 + 0.1260i, -0.5367 + 0.0914i, -1.0700 + 0.1559i, -0.8070 - 0.2838i, -0.4529 + 0.7029i, -0.5036 + 0.7995i, -1.1144 + 0.5039i, -0.7351 + 0.8479i, -0.8069 - 0.3474i, -0.6229 - 0.5929i, -0.2397 - 0.5583i, -0.2430 - 0.3494i, -0.0736 - 1.1755i, -0.4083 - 0.8520i, 0.0291 - 0.9215i, -0.5680 - 0.9988i, -0.6291 + 0.0105i, -0.6221 - 0.1657i, -1.0739 - 0.2682i, -1.0691 - 0.1628i, -0.2221 - 0.8673i, -0.5083 - 0.8788i, -0.8539 - 0.6575i, -0.5302 - 0.7333i</p>

Chapter 5

Decoding Latency and Complexity

5.1 Introduction

In the previous section, the advantage of SNR-adaptive convolutionally coded transmission model is presented over SNR-independent conventional constellation cases in terms of error performance and spectral efficiency without any consideration from implementation complexity and other interpretation of the enhancement seen in error performance. In the cases of neglecting decoding delay or algorithmic complexity in the evaluation of convolutionally coded system, it is quite certain that convolutional encoders eventually show inferior error performance compared to powerful channel coding techniques. Interestingly, it has been observed that the well-known convolutional encoders may result in more superior performance than advanced coding techniques when the latency requirements do not allow iterative decoding structure of mentioned powerful techniques [62]. Then, it might be said that non-iterative/one-shot decoding algorithms, as seen in convolutional decoding, might have great potential to use for these cases with the bonus of lower system complexity [56]. Following this manner, this section aims to evaluate the proposed SNR-adaptive convolutionally coded scenario in terms of the decoding latency and implementation complexity by comparing it with TCM and LDPC coded cases.

5.2 Decoding latency

Latency can be defined as the time interval from the moment the information sent from the transmitter to the completion of decoding process [60]. If the time period of encoding process and delay in transmission are neglected, the term of latency is

deducted into decoding latency which results from the delay occurred in decoding process [113]. The iterative decoding algorithm has been using over TTCM, BICM and LDPC coded scenarios where very small capacity gap exists due to the powerful decoding process. However; this iterative decoding structure might turn into a disadvantage because of causing to exceed predefined latency requirement. To illustrate this relationship, Fig. 5.1 shows that after certain latency threshold is exceeded, the link is considered as an outage even it still keeps higher reliability.

Figure 5.1: The relation between reliability and latency [60].

As an initial step, evaluating different channel coding techniques in the same context, a common latency measure is required for considering scenarios which are convolutional coding, TTCM coding, and LDPC coding scenarios. Considering soft-decision Viterbi decoding algorithm is used during the decoding process, the window-length of Viterbi decoder (τ), backsearch limit or path memory [43], is selected for comparing decoding latencies.

For convolutional coded scenarios, the decoding delay, $t_{\text{delay,conv}}$, can be expressed as a function of only τ , which is

$$t_{\text{delay,conv}} = f(\tau). \quad (5.1)$$

Considering the iterative decoding property shown in Fig. 5.2, the iteration number of occurring between two encoders, Q , has become another factor to determine the

Figure 5.2: Decoding structure of turbo decoder.

decoding latency. Then, the decoding delay for TCM cases can be written as

$$t_{\text{delay,TCM}} = f(Q, \tau). \quad (5.2)$$

Following the same manner, the decoding delay for LDPC cases can be formulated as

$$t_{\text{delay,LDPC}} = f(Q, \tau). \quad (5.3)$$

After defining the decoding latencies for all considered scenarios, required SNR values to achieve a given BER/SER threshold is considered as target performance metric with respect to different τ values.

5.2.1 Latency comparison with SNR-independent convolutionally coded cases

To begin with, the advantage of SNR-adaptive design as compared to convolutionally coded model without any constellation framework is presented. For this purpose, we aim to find required SNR value to reach the BER of $P_{b,th} = 10^{-4}$. The choice of convolutional encoder and puncturing pattern is the same as the one used in the MCS-3 scheme in Chapter 4. The required SNR values for SNR-independent convolutionally coded model is first determined; then the SNR-adaptive constellation optimizer introduced in Section 4.2 yields optimized irregular constellation based on these SNR values. The constellation used in SNR-adaptive convolutionally coded model can be seen in Fig. 5.3 over different $\bar{\gamma}$ and m values. Then, the decoding latencies of SNR-adaptive convolutionally coded model are obtained via the simulations where optimized irregular constellations are employed.

Figure 5.3: Optimized irregular constellations for SNR-adaptive convolutionally coded scenarios.

In Fig. 5.4, it can be seen that SNR-adaptive design also offers lower decoding latencies, the order of hundred bits, at the same required SNR values. It means that the error performance gains obtained from SNR-adaptive design can be translated into decoding latency superiority in the context of low-latency communications. From the other aspect, 1-2 dB SNR gain can be obtained by using SNR-adaptive optimized irregular constellation for the same decoding delay value.

5.2.2 Latency comparison with TTCM cases

Now, we would like to compare SNR adaptive convolutionally coded system with TTCM coded scenario. For this purpose, we choose the convolutional encoder from the component code of the first proposed TTCM encoder given in [52] and the TTCM coded scenario with 8-PSK using this convolutional encoder is compared with SNR-adaptive scenario along with optimized irregular 8-ary signalling cases over AWGN channel.

Following the similar steps as in convolutionally coded cases, the required SNR values are first obtained via simulations over TTCM scenario; then optimized irregular constellations are found for convolutionally coded cases based on these values. The some snapshots of optimized irregular constellation can be seen in Fig. The required SNR values to reach $P_{b,th} = 10^{-3}$ over different decoding latencies are plotted in Fig. 5.5. As it can be seen from the figure, higher decoding iteration number leads better performance at expense of higher decoding latency in TTCM scenarios. From this point of view, SNR-adaptive convolutionally coded model can be potential alternative to TTCM scenarios over low-complexity communications coupled with some

Figure 5.4: Decoding latency comparison of the conv. coded SNR-adaptive optimized model (64-ary signalling) with conv. coded SNR-independent 64-QAM.

decoding latency requirements.

5.2.3 Latency comparison with LDPC cases

Now, SNR-adaptive convolutionally coded system is compared with rate-1/2 LDPC code along with 16-QAM deployed in WiMAX standard [114]. For the LDPC coded scenarios, different iteration number, Q is considered, $Q = \{1, 2, 3, 4\}$, over AWGN channels in order to reach $P_{b,th} = 10^{-4}$.

As it can be seen from the Fig. 5.6, convolutionally coded SNR-adaptive scenario yields better decoding latency performance for the iterations, $Q = \{1, 2\}$. For higher iteration numbers, the superiority of LDPC coded scenarios can be observed at expense of higher decoding latencies.

Figure 5.5: Decoding latency comparison of the conv. coded SNR-adaptive optimized model (8-ary signalling) with TTCM 8-PSK [52].

Figure 5.6: Decoding latency comparison of the conv. coded SNR-adaptive optimized model (16-ary signalling) with LDPC 16-QAM.

5.3 Algorithmic complexity

In order to characterize the decoding complexity of different channel coding techniques, algorithmic complexity is commonly used. Algorithmic complexity is mainly based on finding total equivalent number of each operation used in the encoding/decoding process. In the context of this thesis, algorithmic complexity only in the decoder part is considered. In the decoding process, various operators are used and they are listed in Table 5.1 along with the number of equivalent addition for each operation [57]. Note that such complexity calculations aim to only give guidance about implementation complexity by observing from software implementations of the considered scenarios without any hardware implementation [56]. Since there is no difference between the proposed convolutionally coded SNR-adaptive system and convolutionally coded SNR-independent system in terms of algorithmic complexity at the decoder, the comparisons with TTCM and LDPC coded scenarios are considered

in the following parts.

Table 5.1: The numbers of equivalent addition per each operation.

Operation	Corresponding number of equivalent addition
Addition, subtraction	1
$\pm$ Multiplication, division	1
Comparison	2
Maximum, minimum	1
Parallel list	1
Look-up table	6

5.3.1 Decoding complexity comparison with TTCM coded cases

In this section, we would like to compare the algorithmic complexity of SNR adaptive convolutionally coded system with TTCM coded scenario considered in Section 5.2.2. In Table 5.2, the algorithmic complexity of SNR-adaptive convolutionally coded transmission and TTCM coded cases are obtained based on Table 5.1 along with different values τ . The corresponding values are given with the ratio of the algorithmic complexity of TTCM coder where $Q = 4$. Table 5.2 emphasizes the simplicity of convolutionally coded schemes compared to TTCM cases and the snapshots of optimized 8-ary irregular constellation are shown in Fig. 5.7.

Figure 5.7: Optimized 8-ary irregular constellations over AWGN channels.

5.3.2 Decoding complexity comparison with LDPC coded cases

Now, SNR-adaptive convolutionally coded system corresponding *MCS-1* is compared with rate-1/2 LDPC code along with 16-QAM deployed in WiMAX standard [114] over AWGN channels. For the LDPC coded scenario, Q is considered as 4. The similar characteristic seen in TTCM comparison also appears in the LDPC coded case, shown in Table 5.3. Even LDPC coded scheme has less decoding complexity as compared to TTCM cases, the decoding complexity of convolutionally coded cases is still quite low for considered LDPC case with $Q = 4$. The snapshots of optimized 16-ary irregular constellations are given in Fig.5.8.

Figure 5.8: Optimized 16-ary irregular constellations for MCS-1 over AWGN channels.

5.4 Conclusion

In this section, the proposed convolutionally SNR-adaptive system model has been investigated in terms of decoding latency and implementation complexity. From the decoding latency aspect, it can be observed that SNR-adaptive design also results in lower decoding latency values when reaching target BER values. In order to evaluate implementation complexity, algorithmic complexity is used to determine the system complexity. The system complexity results support the idea of using convolutional coded techniques especially for low-complexity use cases.

Table 5.2: Algorithmic decoding complexity comparison of SNR-adaptive convolutionally coded with TTCM systems.

Coding Scheme	τ [bits]	Total number of summation	Percentage (w.r.t. TTCM (Q=4))
SNR-adaptive conv. coded 8-ary signalling	128	1.22×10^5	1.47%
	384	3.69×10^5	0.73%
	640	6.16×10^5	0.49%
	896	8.64×10^5	0.37%
	1152	1.11×10^6	0.47%
	1408	1.35×10^6	0.25%
	1664	1.60×10^6	0.21%
Turbo-trellis coded 8-PSK $Q = 1$	128	8.31×10^6	25.05%
	384	5.01×10^7	25.02%
	640	4.26×10^7	25.01%
	896	5.68×10^7	25.01%
	1152	7.10×10^7	25.01%
	1408	8.52×10^7	25.01%
	1664	9.94×10^7	25.01%
Turbo-trellis coded 8-PSK $Q = 4$	128	1.45×10^7	100%
	384	2.89×10^7	100%
	640	4.34×10^7	100%
	896	5.78×10^7	100%
	1152	7.23×10^7	100%
	1408	8.68×10^7	100%
	1664	1.01×10^8	100%
Turbo-trellis coded 8-PSK $Q = 8$	128	1.50×10^7	199.92%
	384	3.02×10^7	199.96%
	640	4.49×10^7	199.97%
	896	5.99×10^7	199.98%
	1152	7.49×10^7	199.98%
	1408	8.99×10^7	199.98%
	1664	1.94×10^8	199.99%

Table 5.3: Algorithmic decoding complexity comparison of SNR-adaptive convolutionally coded with LDPC coded systems.

Coding Scheme	τ [bits]	Total number of summation	Percentage (w.r.t. TTCM (Q=4))
SNR-adaptive conv. coded 16-ary signalling	128	3.07×10^4	3.36%
	384	9.29×10^4	3.4%
	640	1.55×10^5	3.4%
	896	2.17×10^5	3.4%
	1152	3.41×10^5	4.16%
	1408	4.03×10^5	4.02%
	1664	4.66×10^5	3.93%
LDPC coded 16-QAM $Q = 4$	128	9.12×10^5	100%
	384	2.73×10^6	100%
	640	4.56×10^6	100%
	896	6.38×10^6	100%
	1152	8.21×10^6	100%
	1408	1.03×10^7	100%
	1664	1.18×10^7	100%

Chapter 6

Conclusions and Future Work

6.1 Summary and contributions

As main difference between 5G and previous standards, new design parameters have been introduced to the system design starting from an early phase of its standardization. Minimizing error rates has been the principal goal in wireless system design for decades. The invention of the capacity-approaching/achieving codes such as; turbo code, low-density parity-check code (LDPC), and polar codes have been among the key milestones to reach this goal. However; most powerful channel coding codes might suffer from higher decoding latency while their reliability does not produce any concern over low-latency communications. From this point of view, there might be considerable potential for convolutional coded cases considering its one-shot decoding property and its simplicity. Motivated by this fact, we propose a convolutionally coded SNR-adaptive transmission model which aims to combine the simplicity of convolutional coding with performance gain obtained from SNR-adaptive irregular optimized constellations. The main contributions of this thesis can be summarized as follows:

- Conventional error performance analysis for convolutionally coded systems is only limited to use in quasi-regular cases. The quasi-regularity implies that the error performance analysis is independent from which transmitted sequence is sent; on the other hand, more comprehensive error performance framework is required especially with the existence of irregular constellation. As a pivotal step, upper bound on bit error rate (BER) for a two-transmission system (modelling CoMP, HARQ or relaying) is presented which can work with any given pair of convolutional encoder and constellation. Unlike previous approaches,

the proposed method calculates the generating function of the convolutional encoder based on the product-state matrix technique which does not require the constellation and encoder to satisfy the quasi-regularity.

- Next, the error performance analysis is extended to a well-rounded system model which includes multiple transmission stages and multiple transmit antennas in each transmission stage. On behalf of being resilient on the system design, it is allowed to use a different number of the transmit antennas and fading characteristics, and constellation in each stage. Due to the use of multiple transmit antenna at the transmitter side, transmit maximum ratio combining schemes are employed to combine the signals coming from the different transmit antenna in order to maximize received SNR at the end of each transmission stage. This asymmetric complexity distribution between the transmitter and the receiver parts is inherent to the Internet of Things (IoT) ecosystem where robustness of transmitter units against to failure and poor performance is required with increasing network intelligence. The correlation is taken into consideration as well but it is assumed that it exists only between the transmit antennas and that there is no correlation between the stages.
- Although the main motivation in the thesis results from low-complexity communications where convolutional encoders have a potential deployment along with SNR-adaptive optimized irregular constellations, the analytical framework has been also extended to the turbo-trellis coded scenarios where the use of the irregular constellations can be also exploited through powerful coding techniques.
- An SNR-adaptive convolutionally coded transmission model which uses different optimized constellations for any given SNR and fading parameter values is proposed. To the best of our knowledge, the novelty of this study lies on the approach we adopt which applies a comprehensive optimization framework over convolutionally coded scenarios by taking into account encoder type, coding rate, and channel conditions without any predefined constraint on symbol locations. From the simulations, it is observed that the more gains can be obtained with higher modulation order and spectral efficiency gains can be found in the order of 0.5 – 2.5 dB depending on the modulation level and channel characteristics.

- To achieve peak data rates and spectral efficiencies, adaptive modulation and coding (AMC) schemes have been quite popular techniques in wireless networks for decades. The basic idea of AMC schemes is adjusting transmission power levels, coding rate and modulation order based on channel information which includes average SNR value in most cases. In this study, the proposed SNR-adaptive convolutionally coded transmission model has added another dimension different from conventional AMC schemes, which is putting average SNR value as constellation changer parameter with a predefined variation on its value even other parameters stay the same. In other words, in the case of that conventional AMC technique is used, only a given constellation selection, coding rate, and modulation order are used for a chosen range of average SNR values while different choices of constellations might be appeared along with the same coding rate and modulation order in the same interval over the proposed design.
- The proposed SNR-adaptive convolutionally coded framework has been compared with the other powerful error channel coding techniques in terms of decoding latency and implementation complexity. The superiority of the proposed transmission model resulting from the simplicity of convolutional encoding and non-iterative decoding structure can be observed in the calculation of algorithmic complexity.

6.2 Future research directions

The work presented in this thesis brings some interesting questions for future research.

- Recently, there is an existing trend to deploy bit-interleaved coded modulation which has simpler and flexible encoder structure as compared to trellis coded modulation and turbo codes. Fundamentally, bit-interleaved coded modulation uses convolutional encoder along with bit-level interleaver and it yields better performance over faded scenarios as compared to TCM coded counterparts. Currently, some geometrical shaping studies are available and they aim to construct hierarchically modulated symbols over BICM systems. As stated at the beginning of this thesis, it was recently shown that the usage of uniformly spaced constellations can cause suboptimal coded systems in existing

wireless communication standards, e.g., HSPA, IEEE.802.11.a/g/n, DVB-T2, etc. It was also shown that the difference between the capacity of the conventional uniform QAM constellations and the Shannon capacity increases with larger SNR values in the systems where BICM is used. From this point of view, we consider the BICM coded systems as another good candidate to extend the proposed SNR-adaptive constellation framework with the existence of optimized irregular constellations.

- In the context of this thesis, we employ two dimensional (2D) constellations in convolutionally coded transmission model since the optimization framework is carried out on 2D space to give the idea along with a simpler illustration. However, the optimization framework can be reformulated for multidimensional space. The advantage of multidimensional constellations comes from the flexibility between the symbol point locations in comparison to the 2D constellations.
- Although some simulated results are presented for the case of that SNR value mismatch exists at the transmitter and the receiver side, it would be interesting to investigate this issue by using existing channel estimation error models before the constellation design.
- The proposed design aims to find optimized irregular constellations over different average SNR values for a given convolutional encoder; in other words, it does not include the joint design of the convolutional encoder and the constellation. Including the encoder design along with search for optimized symbol locations simultaneously will be a more complicated research problem since it requires to find the best possible connections between the memory elements in the construction of an optimal convolutional encoder and optimized constellation simultaneously. We hope that this proposed framework can trigger a research in that direction.
- Considering the big amount of increase in Internet traffic, optical networks are utilized by many telecommunication operators in order to carry out their data-load. Within the same manner existing in wireless systems, channel coding techniques are very common to deal with existing impairments over optical channels. Actually, it was underlined that more efficient channel coding codes are required to tackle low bit error rate values at long distances due to as

uncompensated chromatic and polarization mode dispersions. We demonstrated that SNR-adaptive convolutionally coded system yields considerable gain with a smart design so the same adaptive design can be extended into optical networks by using suitable performance analysis along with optimization framework over generalized analysis.

- Differently from applying existing analytical framework to the other channel coding techniques, another dimension can be added to optimization framework by seeking out optimal bit-to-symbol mapping rule for each SNR value and fading parameter. By doing so, SNR-adaptiveness of bit-to-symbol mapping can be investigated and a different pattern of labelling rules can be obtained.
- Gaussian noise is assumed throughout this thesis as a destructive factor in the receiver. However, a non-Gaussian model for the noise may be also used to for the cases where the transmitter data unknown to the receiver. Considering the irregular type of constellation on the transmitter side, we believe that non-Gaussian noise assumption can suit well for this type of communications.
- Software defined radios are intelligent radios that can reconfigure its transmission parameters like modulation type and transmit power without the need to change the hardware. Many studies can be found in the literature where constellation design has been addressed for different systems along with different objectives and constraints. Following the same manner, the design can be formulated as an optimization problem that minimizes error performance subject to transmit power constraints and symbol point locations.
- Sparse code multiple access (SCMA) techniques have attracted great attention in the context of non-orthogonal multiple access (NOMA) schemes, especially in 5G networks. Specifically, modulation design and spreading are jointly taking into account where multidimensional constellation is usually considered. An interesting research problem can emerge SNR adaptive design which also takes into account adaptive spreading factor design over multiple access scenarios.
- With the availability of smarter network components resulting from modern channel sensing techniques and rapid adaptiveness against transmission environment, the concept of auto-encoder has been introduced in the concept of deep learning (DL). Mainly, DL in physical layer introduced in [115] which focuses

on intelligent transmitter and receiver units. Considering the SNR adaptive convolutionally coded transmission model in this thesis, we think that after studying the concept of DL in a physical layer, the presented framework can be categorized under this concept.

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Appendix A

SNR-Adaptive Design for Two-Way Relaying Systems

A.1 Introduction

In this appendix, signal space diversity (SSD)-based two-way relaying system is presented as an extension of the concept of SNR-adaptive design to an uncoded scenario. In the same manner as presented in the convolutionally coded scenarios, SNR-adaptive design also brings considerable performance gain to SSD-based two-way relaying system model, where more reliable end-to-end error performance can be obtained by using SNR-adaptive constellation rotation along with power allocation between the users and the relay.

To do so, the interaction between transmit power, rotation angle, fading severity, and bit-to-symbol mapping in signal space diversity-based three time-slots decode-and-forward two-way relaying networks is studied. To model different severities of fading, Nakagami distribution is adopted herein. In particular, a joint design of rotation angle selection and power allocation is developed, while taking into account the influence the fading severities of the channels. The objective is to promote transmission reliability, while satisfying power budget, and average error probability constraints. To this end, average error probabilities of end-sources for arbitrary constellations, which capture all possible signal constellations produced by using different rotation angles are derived; then, the joint design problem is formulated in an optimization form. Unfortunately, the resultant formulation is a non-convex programming problem. Hence, numerical optimization is resorted to find the solution. For various scenarios, with varying fading severity levels and bit-to-symbol mapping rules, the

gains offered by the joint design problem are investigated. Numerical results not only corroborate the analyses, but also demonstrate the efficiency of the proposed framework, and provides useful insights on the interplay between rotation angle, transmit power, symbol mapping, and fading severity of the channels.

A.2 Literature review

To improve the network capacity, one of the fundamental approaches is to improve spectral efficiency (SE) at link level. For instance, LTE-Advanced networks adopt cooperative communication [116], and more complicated versions of this technology might be a potential candidate for 5G networks.

To achieve a higher SE, signal space diversity (SSD) [117] has been incorporated into cooperative schemes, see, e.g., [118–120], called signal space cooperation, which takes advantage of not only spatial diversity, but also modulation diversity. In these schemes, original data symbols are first rotated by a certain angle before transmission. Thereby, the information of original data symbols is distributed over the in-phase and quadrature components of the respective data symbols. Then, the components of the rotated symbols are sent via the cooperation of the end-sources and the relay(s) so as to ensure that these components experience different fading coefficients. To further enhance SE, the idea of signal space cooperation has been applied to three time-slots two-way relaying networks with time division broadcast (TDBC) protocol [121, 122]. Particularly, in [121], the optimization of rotation angle at each SNR values has been considered. Furthermore, in [122], the optimization of rotation angle along with power allocation has been investigated. A common assumption in the aforementioned works is that all channels are modeled only using Rayleigh distribution, i.e., no difference in severity of fading channels.

A.3 Contributions

In this part, joint optimization of rotation angle and power allocation is investigated, while taking into account the influence of the fading severity of the channels. To account for the fading severities of the channels, a general distribution, Nakagami distribution, is utilized. Specifically, the main contributions of the paper are as follows:

- For the SSD-based two-way relaying scheme, first a closed-form expression for the probability density function (PDF) of the output SNR at the end-sources is derived over the Nakagami- m fading channels.
- Using the derived PDF expressions, a closed-form expression for the average error probabilities of the end-sources for arbitrary constellations is obtained.
- Next, using the average error probability, an optimization framework that explores the interaction of rotation angle selection, and transmit power allocation for various SNR values, fading severity levels, and bit-to-symbol mapping rules is proposed. The design objective is to minimize the average error probability of one of the end-sources, while meeting the power budget constraints, and the predefined threshold for the average error probability of the other end-source.
- Differently from existing works, the performance of mapping rules can vary based on the rotation angle used in transmission and the fading severities of channels—e.g., Gray mapping is not necessarily the best mapping rule for all range of rotation angles.

A.4 Signal space diversity-based TDBC protocol in two-way relaying systems

The mentioned system that consists of two end-sources ($\mathcal{A}$ and $\mathcal{B}$), and an intermediate relay ($\mathcal{R}$) is considered. It is assumed that the channels are reciprocal, and are modeled by a Nakagami- m random variable with shaping parameters of $\{m_{\mathcal{AB}}, m_{\mathcal{AR}}, m_{\mathcal{BR}}\}$, and average fading powers of $\{\Omega_{\mathcal{AB}}, \Omega_{\mathcal{AR}}, \Omega_{\mathcal{BR}}\}$ for the links of $\mathcal{A} \leftrightarrow \mathcal{R}$, $\mathcal{B} \leftrightarrow \mathcal{R}$, and $\mathcal{A} \leftrightarrow \mathcal{B}$, respectively. Moreover, the additive white Gaussian noise (AWGN) at each node is assumed to have zero-mean and equal variance (N_0).

To enhance both the performance and spectral efficiency of the system, combining the SSD technique with TDBC protocol (the so-called SSD-based TDBC protocol) is considered. In the conventional TDBC protocol [123], the transmission of two symbols needs three time slots, where the end-source $\mathcal{A}$ and the end-source $\mathcal{B}$ transmit to the relay $\mathcal{R}_\ell$ over the first and second time slots, respectively, and in the third time slot, the the relay $\mathcal{R}$ transmits a function of the received signals to the end-source $\mathcal{A}$ and the end-source $\mathcal{B}$.

However, using SSD-based TDBC protocol, the number of symbols that are transmitted over three time slots can be doubled, i.e., four symbols over three time slots. The basic idea behind SSD-based TDBC protocol is that the original symbols are rotated by a certain angle before being transmitted from both the end-source $\mathcal{A}$ and the end-source $\mathcal{B}$, and then, the end-source $\mathcal{A}$ and the end-source $\mathcal{B}$ cooperate with the relay $\mathcal{R}$ to send the real and imaginary parts of the rotated symbols. In the two-dimensional signal space, there exists rotations in which the in-phase component and the quadrature component of the transmitted signal carry enough information to uniquely represent the original signal [118].

Let χ be a constellation generated by applying a transformation Θ to an ordinary constellation shown in Fig. A.1, and the transformation Θ be given as

$$\Theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad (\text{A.1})$$

where θ is the rotation angle in two-dimensional signal space.

Figure A.1: An example of rotated constellation that is generated by applying a transformation Θ to the quadrature phase shift keying (QPSK) constellation.

Then, let's assume that $\mathbf{s}^A = (s_1^A; s_2^A)$ be a pair of signal points from the rotated

constellation, i.e., $s_1^A, s_2^A \in \chi$, which corresponds to the end-source $\mathcal{A}$'s message. Note that $s_1^A = \Re\{s_1^A\} + j\Im\{s_1^A\}$ and $s_2^A = \Re\{s_2^A\} + j\Im\{s_2^A\}$, where $\Re\{\cdot\}$ and $\Im\{\cdot\}$ represent the in-phase and the quadrature components of the corresponding signal points, respectively. After interleaving the components of s_1^A and s_2^A , the new constellation point that will be sent from the end-source $\mathcal{A}$ can be written as

$$\lambda_A = \Re\{s_1^A\} + j\Im\{s_2^A\}. \quad (\text{A.2})$$

Let's next assume that $\mathbf{s}^B = (s_1^B, s_2^B)$ be a pair of signal points from the rotated constellation, i.e., $s_1^B, s_2^B \in \chi$, which corresponds to the end-source $\mathcal{B}$'s message. Similar to the end-source $\mathcal{A}$, the constellation point that will be transmitted from the end-source $\mathcal{B}$ is formed by interleaving the components of s_1^B and s_2^B as follows:

$$\lambda_B = \Re\{s_1^B\} + j\Im\{s_2^B\}. \quad (\text{A.3})$$

It is worth mentioning that both λ_A and λ_B do not belong to the rotated constellation any more; rather, they belong to the expanded constellation, Λ , defined as

$$\Lambda = \Re\{\chi\} \times \Im\{\chi\}, \quad (\text{A.4})$$

where $\times$ denotes the Cartesian product of two sets. In this expanded constellation, all members consist of two components each of which uniquely identifies a particular member of χ . Thus, decoding a member of the expanded constellation results in decoding two different members of the original constellation.

In the first time slot, the received signals at the end-source $\mathcal{B}$ and the relay $\mathcal{R}$ can be written as

$$y^{A \rightarrow B} = h_{AB} \sqrt{P_A} \lambda_A + n_B, \quad (\text{A.5})$$

$$y^{A \rightarrow R} = h_{AR} \sqrt{P_A} \lambda_A + n_R, \quad (\text{A.6})$$

where P_A denotes the transmit power at the end-source $\mathcal{A}$.

In the second time slot, the received signals at the end-source $\mathcal{A}$ and the relay $\mathcal{R}_\ell$ can be given as

$$y^{B \rightarrow A} = h_{AB} \sqrt{P_B} \lambda_B + n_A, \quad (\text{A.7})$$

$$y^{B \rightarrow R} = h_{BR} \sqrt{P_B} \lambda_B + \tilde{n}_R, \quad (\text{A.8})$$

where P_B denotes the transmit power at the end-source $\mathcal{B}$.

The detection of the end-sources' signals at the relay, i.e., s_1^A and s_2^A from λ_A , and s_1^B and s_2^B from λ_B , is given as

$$\hat{\lambda}_A = \arg \min_{\lambda_A \in \Lambda} \left[y^{A \rightarrow R} - h_{AR} \sqrt{P_A} \lambda_A \right], \quad (\text{A.9})$$

$$\hat{\lambda}_{\mathcal{B}} = \arg \min_{\lambda_{\mathcal{B}} \in \Lambda} \left[y^{\mathcal{B} \rightarrow \mathcal{R}} - h_{\mathcal{B}\mathcal{R}} \sqrt{P_{\mathcal{B}} \lambda_{\mathcal{B}}} \right]. \quad (\text{A.10})$$

It is important to note that knowing $\lambda_{\mathcal{A}}$ and $\lambda_{\mathcal{B}}$ leads to knowing $(s_1^{\mathcal{A}}, s_2^{\mathcal{A}})$ and $(s_1^{\mathcal{B}}, s_2^{\mathcal{B}})$, respectively.

A.5 Performance analysis over Nakagami- m fading cases

The end-to-end (E2E) error probability for i th user is expressed as

$$\bar{P}_i(e) = \bar{P}_{off} \bar{P}_{direct}^{i \rightarrow j} + (1 - \bar{P}_{off}) \bar{P}_i(e|R), \quad (\text{A.11})$$

such that $i \neq j$, $j \in \{\mathcal{A}, \mathcal{B}\}$. Here, $\bar{P}_{off}$ is the probability of the case where the relay remains silent because of erroneous reception of information from a single user or both, $\bar{P}_{direct}^{i \rightarrow j}$ gives the average error probability belonged by $\mathcal{A} \leftrightarrow \mathcal{B}$ link, and $\bar{P}_i(e|R)$ considers the error probability where the relay is actively used at the end user as a cooperative manner.

In a direct link scenario, e.g., $\mathcal{A} \rightarrow \mathcal{B}$, the received instantaneous SNR at end-source $\mathcal{B}$ can be given by $\gamma_{\mathcal{B}}^{\text{direct}} = P_{\mathcal{A}} |h_{\mathcal{AB}}|^2 / N_0$ and the instantaneous error probability expression at end-source $\mathcal{B}$ can be written in the form of a function of $\gamma_{\mathcal{B}}^{\text{direct}}$:

$$P_{\text{direct}}^{\mathcal{A} \rightarrow \mathcal{B}} = \sum_{k=0}^{M^2-1} \sum_{\substack{l=0 \\ l \neq k}}^{M^2-1} P_k \Pr \left[y^{\mathcal{A} \rightarrow \mathcal{B}} \in D_{\Lambda(l)} \left(\gamma_{\mathcal{B}}^{\text{direct}} \right) | \Lambda(k) \right], \quad (\text{A.12})$$

where P_k is the probability of transmitting the k -th symbol, $\Lambda(k)$ is the k -th symbol in the expanded constellation, and $D_{\Lambda(k)}$ is the decision region of the symbol $\Lambda(k)$. Note that $P_{\text{direct}}^{\mathcal{A} \rightarrow \mathcal{B}}$ consists of the sum all possibilities that the transmitted $\Lambda(l)$ symbol drops into $D_{\Lambda(k)}$. By utilizing the geometric trajectory on 2-D space shown in [124], $\Pr \left[y^{\mathcal{A} \rightarrow \mathcal{B}} \in D_{\Lambda(l)} (\gamma_{\mathcal{B}}) | \Lambda(k) \right]$ can be formulated as

$$\Pr \left[y^{\mathcal{A} \rightarrow \mathcal{B}} \in D_{\Lambda(l)} (\gamma_{\mathcal{B}}) | \lambda_{\mathcal{A}} = \Lambda(k) \right] = \sum_{t=1}^{T_l} \pm \mathbf{Q} \left(\pm \mathcal{L}_{l,p_t} (\Lambda(k)) \sqrt{2\gamma_{\mathcal{B}}}, \pm \mathcal{L}_{l,p_{t+1}} (\Lambda(k)) \sqrt{2\gamma_{\mathcal{B}}^{\text{direct}}}, \right. \\ \left. \pm \Re \left[c_{l,p_t}, c_{l,p_t}^* \right] \right), \quad (\text{A.13})$$

where T_l denotes the lines bounding the decision region $D_{\Lambda(l)}$. In (A.13), the neighbour decision regions of the symbol $\Lambda(l)$ are expressed by $\Lambda(p_t)$ and $\Lambda(p_{t+1})$ and $\mathbf{Q}(\cdot, \cdot; \cdot)$

denotes the complementary cumulative density function (CCDF) of a bivariate Gaussian variable [125]. The detailed information about the sign $\pm$ and summation terms can be found in [126].

The $\bar{P}_{\text{direct}}^{A \rightarrow B}$ can be obtained by taking the average of instantaneous error probability of $P_{\text{direct}}^{A \rightarrow B}$ with respect to $\gamma_{\mathcal{B}}^{\text{direct}}$, such as

$$\bar{P}_{\text{direct}}^{A \rightarrow B} = \sum_{k=0}^{M^2-1} \sum_{\substack{l=0 \\ l \neq k}}^{M^2-1} \sum_{t=1}^{T_l} \pm \int_0^\infty Q(a, b; \rho) f_{\text{direct}}^{A \rightarrow B}(\gamma) d\gamma, \quad (\text{A.14})$$

where $a = \pm \bar{\mathcal{L}}_{l, p_t}(\Lambda(k)) \sqrt{2\gamma}$, $b = \pm \bar{\mathcal{L}}_{l, p_{t+1}}(\Lambda(k)) \sqrt{2\gamma}$, $\rho = \pm \Re \left[c_{l, p_t}, c_{l, p_t}^* \right]^1$. Since the channels are modeled as Nakagami- m distribution with the fading parameter m_{AB} and average fading power Ω_{AB} , the resulting $\bar{P}_{\text{direct}}^{A \rightarrow B}$ can be expressed as

$$\bar{P}_{\text{direct}}^{A \rightarrow B} = \sum_{k=0}^{M^2-1} \sum_{\substack{l=0 \\ l \neq k}}^{M^2-1} \sum_{t=1}^{T_l} \int_0^{v(a, b, \rho)} \frac{d\theta}{2\pi} \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + a/2m_{AB}} \right)^{m_{AB}} + \int_0^{v(b, a, \rho)} \frac{d\theta}{2\pi} \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + b/2m_{AB}} \right)^{m_{AB}}, \quad (\text{A.15})$$

where $\alpha_1 = \sqrt{2}\bar{\mathcal{L}}_{l, p_t}(\Lambda(k))$, $\alpha_2 = \sqrt{2}\bar{\mathcal{L}}_{l, p_{t+1}}(\Lambda(k))$, and the definition of $v(\alpha_1, \alpha_2, \rho)$ is given in [22]. Using [Eq.(5A.24), [81]], (A.15) results in

$$\bar{P}_{\text{direct}}^{A \rightarrow B} = \sum_{k=0}^{M^2-1} \sum_{\substack{l=0 \\ l \neq k}}^{M^2-1} \sum_{t=1}^{T_l} \left[\sum_{i=1}^2 \frac{\mathcal{A}(v(\alpha_i, \alpha_{3-i}, \rho), \alpha_i)}{2\pi} \right], \quad (\text{A.16})$$

under the assumptions of that $0 \leq v(\alpha_1, \alpha_2, \rho) < 2\pi$ and $0 \leq v(\alpha_2, \alpha_1, \rho) < 2\pi$. The auxiliary functions in (A.16), $\mathcal{A}(\cdot, \cdot)$ and $\varphi(x, y)$, are defined as

$$\begin{aligned} \mathcal{A}(x, y) = & x - \frac{1}{2} ((1 + \text{sign}(x - \pi)) \pi + 2\varphi(x, y)) \sqrt{\frac{y^2}{2m_{AB} + y^2}} \sum_{l=1}^{m_{AB}-1} \binom{2l}{l} \left[4 \left(1 + \frac{y^2}{2m_{AB}} \right) \right]^{-l} \\ & - 2 \sqrt{\frac{y^2}{2m_{AB} + y^2}} \sum_{l=0}^{m_{AB}-1} \sum_{p=0}^l \binom{2l}{p} \frac{(-1)^{l+p}}{\left[4 \left(1 + \frac{y^2}{2m_{AB}} \right) \right]^l} \frac{\sin(2(l-p)\varphi(x, y))}{2(l-p)}, \end{aligned} \quad (\text{A.17})$$

where

$$\begin{aligned} \varphi(x, y) = & \frac{1}{2} \arctan \left(\frac{2 \sqrt{\frac{y^2}{2m_{AB}} \left(1 + \frac{y^2}{2m_{AB}} \right)} \sin(2x)}{\left(1 + \frac{y^2}{m_{AB}} \right) \cos(2x) - 1} \right) \\ & + \frac{\pi}{2} \left[1 - \frac{1}{2} \left(1 + \text{sign} \left(\left(1 + \frac{y^2}{m_{AB}} \right) \cos(2x) - 1 \right) \right) \text{sign}(\sin(2x)) \right]. \end{aligned} \quad (\text{A.18})$$

¹Refer to [122] for detailed definitions and intermediate steps for the analysis.

Noting that $\bar{P}_{\text{direct}}^{A \rightarrow B}$ can be obtained just replacing $\gamma_{\mathcal{B}}^{\text{direct}}$ with $\gamma_{\mathcal{A}}^{\text{direct}} = P_{\mathcal{B}}\Omega_{\mathcal{AB}}/N_0$.

The relay operations depend on decoding process of received signals in the first and second time slots from $\mathcal{A} \leftrightarrow \mathcal{R}$ and $\mathcal{B} \leftrightarrow \mathcal{R}$, respectively. The probability of remaining silent in third time slot for the relay is expressed as $\bar{P}_{\text{off}} = 1 - \left(1 - \bar{P}_{\text{direct}}^{A \rightarrow R}\right) \left(1 - \bar{P}_{\text{direct}}^{B \rightarrow R}\right)$ where $\bar{P}_{\text{direct}}^{A \rightarrow R}$ and $\bar{P}_{\text{direct}}^{B \rightarrow R}$ can be obtained following the same steps already given in (A.15)-(A.16) just replacing $(m_{\mathcal{AB}}, \gamma_{\mathcal{B}}^{\text{direct}})$ with $(m_{\mathcal{AR}}, \gamma_{\mathcal{AR}}^{\text{direct}} = P_{\mathcal{A}}\Omega_{\mathcal{AR}}/N_0)$ for $\mathcal{A} \rightarrow \mathcal{R}$ link and $(m_{\mathcal{BR}}, \gamma_{\mathcal{BR}}^{\text{direct}} = P_{\mathcal{B}}\Omega_{\mathcal{BR}}/N_0)$ for $\mathcal{B} \rightarrow \mathcal{R}$ link, respectively.

In a cooperative scenario, the average SER of the cooperative link, e.g., $A \rightarrow R \rightarrow B$, can be given as

$$\bar{P}_B(e|R) = \sum_{k=0}^{M-1} \sum_{l=0}^{M-1} \sum_{t=1}^{T_l} \pm \frac{1}{2\pi} \underbrace{\int_0^\infty Q(a, b; \rho) f_{\gamma_{\text{coop}}}(\gamma) d\gamma}_{\tau}, \quad (\text{A.19})$$

$l \neq k$

where $a = \pm\sqrt{2\gamma}\bar{L}_{l,p_t}(\chi(k))$, $b = \pm\sqrt{2\gamma}\bar{L}_{l,p_{t+1}}(\chi(k))$, $\rho = \pm \Re[c_{l,p_t}, c_{l,p_t}^*]$. Since two-way relaying system is considered, the PDF of $\gamma_{\mathcal{RB}}$ is dependent on the average powers used in the first and the second time slots. Therefore, $f_{\gamma_{\mathcal{RB}}}(\gamma)$, can obtained from the derivative of the joint CDF of $(\mathcal{Z}, \gamma_{\mathcal{RB}}(z, \gamma))$, $F_{\mathcal{Z}, \gamma_{\mathcal{RB}}}(z, \gamma)$, which is defined as

$$\begin{aligned} F_{\mathcal{Z}, \gamma_{\mathcal{RB}}}(z, \gamma) &= \Pr[\mathcal{Z} \leq z, \gamma_{\mathcal{RB}} \leq \gamma] \\ &= \Pr[\gamma_{\mathcal{AR}} > \gamma_{\mathcal{BR}}] \Pr[\gamma_{\mathcal{BR}} \leq z, \gamma_{\mathcal{RB}} \leq \gamma | \gamma_{\mathcal{AR}} > \gamma_{\mathcal{BR}}] \\ &\quad + \Pr[\gamma_{\mathcal{BR}} > \gamma_{\mathcal{AR}}] \Pr[\gamma_{\mathcal{AR}} \leq z, \gamma_{\mathcal{RB}} \leq \gamma | \gamma_{\mathcal{BR}} > \gamma_{\mathcal{AR}}], \end{aligned} \quad (\text{A.20})$$

where $\mathcal{Z}$ is a bottleneck term, defined as $\mathcal{Z} = \min(\gamma_{\mathcal{A} \rightarrow \mathcal{R}}, \gamma_{\mathcal{B} \rightarrow \mathcal{R}})$. By using the order statistics, $F_{\mathcal{Z}, \gamma_{\mathcal{RB}}}(z, \gamma)$ can be rewritten as

$$\begin{aligned} F_{\mathcal{Z}, \gamma_{\mathcal{RB}}}(z, \gamma) &= \frac{\bar{\gamma}_{\mathcal{AR}}}{\bar{\gamma}_{\mathcal{BR}} + \bar{\gamma}_{\mathcal{AR}}} F_{\gamma_{\mathcal{AR}}}(z) F_{\gamma_{\mathcal{RB}}}(\gamma) + \frac{\bar{\gamma}_{\mathcal{BR}}}{\bar{\gamma}_{\mathcal{BR}} + \bar{\gamma}_{\mathcal{AR}}} F_{\gamma_{\mathcal{BR}}}(z) u(\gamma/\mathcal{K}_{\mathcal{AB}} - z) \\ &\quad + F_{\gamma_{\mathcal{BR}}}(\gamma/\mathcal{K}_{\mathcal{AB}}) u(z - \gamma/\mathcal{K}_{\mathcal{AB}}) - F_{\gamma_{\mathcal{BR}}}(z) \delta(z - \gamma/\mathcal{K}_{\mathcal{AB}}), \end{aligned} \quad (\text{A.21})$$

where $\bar{\gamma}_{\mathcal{RB}} = \mathcal{K}_{\mathcal{AB}}\bar{\gamma}_{\mathcal{BR}}$ and $\mathcal{K}_{\mathcal{AB}} = P_{\mathcal{R}}/P_{\mathcal{B}}$. Herein, $u(\cdot)$ and $\delta(\cdot)$ denote the unit step function and dirac delta function, respectively. After using the identity $F_{\gamma_{\mathcal{RB}}}(\gamma) = \lim_{z \rightarrow \infty} F_{\mathcal{Z}, \gamma_{\mathcal{RB}}}(z, \gamma)$ and taking the derivative of $F_{\gamma_{\mathcal{RB}}}(\gamma)$ with respect to γ , the PDF of $\gamma_{\mathcal{RB}}$ is obtained as

$$\begin{aligned} f_{\gamma_{\mathcal{RB}}}(\gamma) &= \frac{\Gamma(m_{\mathcal{RB}})^{-1} \bar{\gamma}_{\mathcal{AR}}}{\bar{\gamma}_{\mathcal{BR}} + \bar{\gamma}_{\mathcal{AR}}} \left(\frac{m_{\mathcal{RB}}}{\bar{\gamma}_{\mathcal{RB}}^{\text{direct}}} \right)^{m_{\mathcal{RB}}} e^{-\gamma \frac{m_{\mathcal{RB}}}{\bar{\gamma}_{\mathcal{RB}}^{\text{direct}}}} \\ &\quad + \frac{\mathcal{K}_{\mathcal{AB}} \bar{\gamma}_{\mathcal{BR}}}{\Gamma(m_{\mathcal{RB}})(\bar{\gamma}_{\mathcal{BR}} + \bar{\gamma}_{\mathcal{AR}})} \left(\frac{m_{\mathcal{RB}}}{\mathcal{K}_{\mathcal{AB}} \bar{\gamma}_{\mathcal{BR}}^{\text{direct}}} \right)^{m_{\mathcal{BR}}} e^{-\gamma \frac{m_{\mathcal{RB}}}{\mathcal{K}_{\mathcal{AB}} \bar{\gamma}_{\mathcal{BR}}^{\text{direct}}}}. \end{aligned} \quad (\text{A.22})$$

The PDF of $f_{\gamma_B^{\mathcal{A} \rightarrow \mathcal{R} \rightarrow \mathcal{B}}}(\gamma)$ is a convolution of $f_{\gamma_{\mathcal{RB}}}(\gamma)$ and $f_{\gamma_{\mathcal{AB}}}(\gamma)$, i.e., $f_{\gamma_B^{\mathcal{A} \rightarrow \mathcal{R} \rightarrow \mathcal{B}}}(\gamma) = f_{\gamma_{\mathcal{RB}}}(\gamma) * f_{\gamma_{\mathcal{AB}}}(\gamma)$ and substituting $f_{\gamma_B^{\mathcal{A} \rightarrow \mathcal{R} \rightarrow \mathcal{B}}}(\gamma)$ into (A.19), τ in (A.19) is written as

$$\begin{aligned} \tau = & \sum_{i=1}^2 \int_0^{v(\alpha_i, \alpha_p, \rho)} \frac{\bar{\gamma}_{\mathcal{AR}}}{\bar{\gamma}_{\mathcal{BR}} + \bar{\gamma}_{\mathcal{AR}}} \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + \alpha_i^2 \gamma_{\mathcal{RB}} / 2m_{\mathcal{RB}}} \right)^{m_{\mathcal{RB}}} \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + \alpha_i^2 \gamma_{\mathcal{AB}} / 2m_{\mathcal{AB}}} \right)^{m_{\mathcal{AB}}} d\theta \\ & + \int_0^{v(\alpha_i, \alpha_p, \rho)} \frac{\mathcal{K}_{\mathcal{AB}} \bar{\gamma}_{\mathcal{BR}}}{\bar{\gamma}_{\mathcal{BR}} + \bar{\gamma}_{\mathcal{AR}}} \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + \alpha_i^2 \mathcal{K}_{\mathcal{AB}} \gamma_{\mathcal{BR}} / 2m_{\mathcal{RB}}} \right)^{m_{\mathcal{RB}}} \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + \alpha_i^2 \gamma_{\mathcal{AB}} / 2m_{\mathcal{AB}}} \right)^{m_{\mathcal{AB}}} d\theta, \end{aligned} \quad (\text{A.23})$$

where $p = 3-i$, $\alpha_1 = \sqrt{2}\mathcal{L}_{l,p_t}(\chi(k))$, and $\alpha_2 = \sqrt{2}\mathcal{L}_{l,p_t+1}(\chi(k))$. Considering integer-valued $m_{\mathcal{AB}}$ and $m_{\mathcal{RB}}$ Nakagami fading parameters enables to utilize from the residue theorem, which is given by [(5.70), [81]]

$$\prod_{l=1}^L \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + c_l} \right) = \sum_{l=1}^L \sum_{k=1}^{m_l} A_{k,l} \left(\frac{\sin^2(\theta)}{\sin^2(\theta) + c_l} \right)^k, \quad (\text{A.24})$$

where $A_{k,l}$ is given by [81]

$$A_{k,l} = \frac{\left\{ \frac{d^{m_l-k}}{dx^{m_l-k}} \prod_{n=1, n \neq l}^L \left(\frac{1}{1+c_n x} \right)^{m_n} \right\} \Big|_{x=c_l^{-1}}}{c_l^{m_l-k} (m_l - k)!}. \quad (\text{A.25})$$

Thereby, a closed-form expression for terms seen in (A.23) can be rewritten as in the same form as the ones for $\bar{P}_{\text{direct}}^{\mathcal{A} \rightarrow \mathcal{B}}$, which are given (A.16)-(A.17), by putting the suitable variables into (A.24).

A.6 Transmission reliability maximization

In this section, it is observed that additional performance gains are possible by considering joint optimization of rotation angle and transmit power allocation. To this end, the constraints, and the design objective are introduced.

The growing demand for data traffic increases energy consumption in the system. To limit the total energy consumption over three time slots, the following constraint is introduced: $P_{\mathcal{A}} + P_{\mathcal{B}} + P_{\mathcal{R}} \leq P_T$, where $P_i^{\max} \leq P_T$, $i \in \{\mathcal{A}, \mathcal{B}, \mathcal{R}\}$, and $P_T \leq P_{\mathcal{A}}^{\max} + P_{\mathcal{B}}^{\max} + P_{\mathcal{R}}^{\max}$. In addition, since each node has a limited battery life in practical systems, the power consumption at the nodes is constrained as $P_i \leq P_i^{\max}$, $i \in \{\mathcal{A}, \mathcal{B}, \mathcal{R}\}$. Furthermore, the average error probabilities at the end-sources are constrained by a predefined threshold, $\bar{P}_{\text{th}}(e)$, to ensure that transmission reliability for the end-sources is higher than a certain threshold. The design objective is to minimize the

average error probability of one of the end-source, i.e., to maximize the transmission reliability for this end-source. Then, the joint rotation angle and power allocation problem for transmission reliability maximization is casted as

$$\min_{P_{\mathcal{A}}, P_{\mathcal{R}}, P_{\mathcal{B}}, \theta \in (0^\circ, 45^\circ)} \bar{P}_{\mathcal{B}}(e) \quad (\text{A.26a})$$

$$\text{subject to} \quad \bar{P}_{\mathcal{A}}(e) \leq \bar{P}_{\text{th}}(e), \quad (\text{A.26b})$$

$$0 < P_i \leq P_i^{\max}, \quad i \in \{\mathcal{A}, \mathcal{B}, \mathcal{R}\}, \quad (\text{A.26c})$$

$$P_{\mathcal{A}} + P_{\mathcal{B}} + P_{\mathcal{R}} \leq P_T. \quad (\text{A.26d})$$

Noting that the constraints (A.26c), and (A.26d) are linear, and the objective function and the constraint (A.26b) are non-convex. Hence, the optimization problem in (A.26) is non-convex. To obtain the solution for this problem, numerical optimization using Matlab command *fmincon* is implemented.

A.7 Numerical results

We investigate the performance of proposed joint design approach, and provide numerical results to illustrate the merits of this approach, while considering various fading scenarios, and mapping rules. Throughout the numerical analysis, QPSK signalling is considered, and the following scenarios: Scenario-1 ($m_{\mathcal{AB}} = 1, m_{\mathcal{AR}} = 1, m_{\mathcal{BR}} = 1$), Scenario-2 ($m_{\mathcal{AB}} = 2, m_{\mathcal{AR}} = 2, m_{\mathcal{BR}} = 2$), and Scenario-3 ($m_{\mathcal{AB}} = 1, m_{\mathcal{AR}} = 2, m_{\mathcal{BR}} = 3$) along with $3\Omega_{\mathcal{AB}} = \Omega_{\mathcal{AR}} = 1.5\Omega_{\mathcal{BR}}$.

A.7.1 Validity of performance bounds

In Fig. A.2, the E2E error probability of user $\mathcal{B}$ in the proposed TDBC scheme (P-TDBC) is plotted, and the fixed- θ and variable- θ_{opt} cases are compared. In this figure, the total power budget is assumed to be equally distributed over the end-sources and the relay. It can be seen that the analytical results are in good agreement with the simulation values in both cases and the results demonstrate that the transmission reliability of the system depends on not only the chosen value of the rotation angle, but also the considered fading scenario. Noting that the optimum values of θ_{opt} in degree for different P_T/N_0 values are presented in Table A.1.

Table A.1: Optimum rotation angles for different fading severity

P_T/N_0 (dB)	Scenario-1 $\theta_{\text{opt}} (^{\circ})$	Scenario-2 $\theta_{\text{opt}} (^{\circ})$	Scenario-3 $\theta_{\text{opt}} (^{\circ})$
0	30.42	30.67	30.44
3	29.04	29.01	28.88
6	28.32	28.15	28.05
9	27.94	27.69	27.61
12	27.74	27.45	27.39
15	27.63	27.32	27.28
18	27.57	27.25	27.23
21	27.54	27.21	27.23
24	27.53	27.19	27.23
27	27.52	27.18	27.22
30	27.51	27.18	27.21

Table A.2: Optimum values from joint optimization

P_T/N_0 (dB)	Scenario-1 $\{\theta_{\text{opt}}(^{\circ}), P_A, P_B, P_R\}$	Scenario-3 $\{\theta_{\text{opt}}(^{\circ}), P_A, P_B, P_R\}$
15	{27.58, 0.2671, 0.6419, 0.0900}	{27.12, 0.5545, 0.2992, 0.2274}
18	{27.55, 0.2684, 0.6406, 0.0900}	{26.81, 0.5025, 0.2431, 0.2246}
21	{27.53, 0.4415, 0.3767, 0.1808}	{26.77, 0.3932, 0.2445, 0.2164}
24	{27.52, 0.5538, 0.3144, 0.1308}	{26.73, 0.3485, 0.2402, 0.2304}
27	{27.52, 0.5808, 0.3139, 0.1042}	{26.59, 0.3384, 0.2204, 0.2639}
30	{27.51, 0.5753, 0.3146, 0.1048}	{26.52, 0.3259, 0.2202, 0.3572}

A.7.2 Impact of joint optimization and symbol mapping

Here the rotation angle selection problem along with the power allocation problem is investigated. The improvement achieved by the joint optimization in the transmission reliability is illustrated in Fig. A.3. The optimal values of θ_{opt} and $\{P_A, P_B, P_R\}$ for various values of P_T/N_0 are shown in Table A.2. As a benchmark model, the E2E error probability of the conventional TDBC scheme, called C-TDBC, is also included to Fig. A.3. For a fair comparison, 16-QAM modulation is considered for C-TDBC,

Figure A.2: Simulation and analytical results for the E2E error probability.

which achieves the same spectral efficiency over three time slots. From this figure, it can be seen that P-TDBC with optimal rotation angles outperforms the C-TDBC over the entire range of SNR values due to both signal space and spatial diversities.

In Fig. A.4, the interaction between mapping rule, rotation angle and fading severity is examined, while P_T/N_0 is assumed to be fixed at 18 dB. For comparison, natural and Gray mappings are used at the end-sources for Scenario-2 and Scenario-3. It can be seen that Gray mapping does not show the best performance for all rotation angles in all fading scenarios. Based on this observation, an adaptive bit-to-symbol mapping rule might be consider to further enhance the performance, especially for high-order M -QAM.

Figure A.3: Simulation and analytical results for the E2E error probability.

A.8 Conclusion

This work aims to discuss the interplay between rotation angle, transmit power, symbol mapping, and the fading severities of the channels in the SSD-based two-way relaying scheme. Towards that end, a closed-form expression for the average error probabilities of the end-sources for arbitrary constellations is first derived. In such analysis, the channel fading is assumed to be Nakagami distributed. Subsequently, using the average error probabilities, an optimization framework that jointly optimizes rotation angle selection and power allocation is developed so as to enhance transmission reliability subject to power budget and average error probability constraints. Lastly, using this optimization framework, useful insights on the interplay between rotation angle, transmit powers, fading severity and bit-to-symbol mapping are provided. Numerical results confirm the analysis, and show the possible gains that can be obtained by the joint optimization in various scenarios.

Figure A.4: Performance in terms of different mapping rules and rotation angles.