Tail Asymptotics and Ergodicity for the $GI/G/1$ -type Markov Chains

by

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Abstract

In this thesis, we discuss tail asymptotics of the stationary distribution, including rough decay called decay in a weak sense and exact decay, polynomial ergodicity and factorization of generating functions of the transition probability for the $GI/G/1$ -type Markov chain with finitely many background states. In chapter 3, we derive the decay rate in a weak sense for the stationary distribution which is the largest one of $\{\frac{1}{\phi_{A+}}, \frac{1}{\phi_{B+}}, \frac{1}{\alpha}\}$. The results show that the smallest singularities of $A^+(z)$, $B^+(z)$ and $[I - A(z)]^{-1}$ play an important role in determining decay rate for the stationary distribution, and both boundary components of the transition probability matrix and non-boundary components of the transition probability matrix affect the tail behavior. In chapter 4, we further analyze sufficient conditions on an exact geometric decay, and provide sufficient conditions for other exact decay when the stationary distributions is light-tailed but not exact asymptotics. Especially, we give more specific description about the effect of the boundary components of the transition probability matrix on exact decay. In chapter 5, we provide two necessary and sufficient conditions for polynomial ergodicity, and also obtain a lower bound for the first hitting time, which is useful to discuss ergodicity, especially for necessary conditions. In chapter 6, we provide factorization results for the generating functions of the transition probability matrix. In chapter 7, we provide some concluding remarks and future work based on the work in this thesis.

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Notation and glossary

I. General notation

$$\mathbb{Z}_+ = \{1, 2, \dots\}.$$

$$\mathbb{Z}_0 = \mathbb{Z}_+ \cup \{0\}.$$

$$\mathbb{Z}_- = \{-1, -2, \dots\}.$$

$$\mathbb{Z} = \mathbb{Z}_0 \cup \mathbb{Z}_-.$$

$$\mathbb{R}_+ = (0, \infty).$$

$$\mathbb{R}_0 = [0, \infty).$$

$$\mathbb{R}_- = (-\infty, 0).$$

$$\mathbb{R} = (-\infty, \infty).$$

$\mathbb{C}$ denotes the set of all complex numbers.

$\mathbb{P}(A)$ denotes the probability of an event A .

$\mathbb{E}[X]$ denotes the expectation of a random variable X .

II. Other notation

g.c.d. means the greatest common divisor.

$\lfloor x \rfloor$ denotes the largest integer equal or less than x .

adj denotes the adjoint of a matrix.

$\det$ denotes the determinant of a matrix.

$\mathbf{e}$ is unit row vector.

$\mathbf{e}^t$ is unit column vector.

$\mathbf{a}$ a row vector.

$\mathbf{a}^t/D^t$ superindex t means transpose of a vector $\mathbf{a}$ or a matrix D .

L_i denotes the set of all states in level i .

$$L_{\leq i} \triangleq \bigcup_{k=0}^i L_k.$$

$\mathbf{1}_E(x)$ denotes the indicator function.

I denotes the identity matrix.

$\pi = (\pi_0, \pi_1, \dots)$ the stationary distribution of the $GI/G/1$ -type Markov chain.

π_k denotes the stationary probability vector, which is derived by partitioning according to levels .

π_{kj} the j th element of π_k .

A_k non-boundary transition block(see page 13).

B_k boundary transition block(see page 13).

R_k see page 14.

$R_{0,k}$ see page 14.

G_k see page 14.

$G_{0,k}$ see page 14.

$$R_k^{*n} = \sum_{j=1}^{k-1} R_j^{*n-1} R_{k-j} \quad \text{for } n \geq 2.$$

$$A \triangleq \sum_{n=-\infty}^{\infty} A_n.$$

$$A(z) \triangleq \sum_{n=-\infty}^{\infty} z^n A_n.$$

$$A^+(z) \triangleq \sum_{n=1}^{\infty} z^n A_n.$$

$$B^+(z) \triangleq \sum_{n=1}^{\infty} z^n B_n.$$

$$R(z) \triangleq \sum_{n=1}^{\infty} z^n R_n.$$

$$R_0(z) \triangleq \sum_{n=1}^{\infty} z^n R_{0,n}.$$

$$G(z) \triangleq \sum_{n=1}^{\infty} z^n G_n.$$

$$\pi(z) \triangleq \sum_{n=1}^{\infty} z^n \pi_n.$$

ϕ_{A^+} denotes the convergent radius of $A^+(z)$.

ϕ_{B^+} denotes the convergent radius of $B^+(z)$.

$\gamma(z)$ denotes the maximal eigenvalue of $A(z)$.

α is a number such that $A(\alpha)$ is 1-positive recurrent.

III. Glossary

Property 1: there exists a $z > 1$ such that $\gamma(z) = 1$ and $A(z) < \infty$, that is, for the given z , $A(z)$ is 1-positive recurrent; and

Property 2: there exists no such $z > 1$ that $A(z)$ is 1-positive recurrent.

$A(\alpha)$ is 1-positive recurrent: there exists an $\alpha > 1$ such that $\gamma(\alpha) = 1$ and $A(\alpha) < \infty$.

Chapter 1

Introduction

In this chapter, we state motivations and introduce problems that we are interested in and the development of these problems in history. We also outline the background knowledge on queuing processes, the main methods used by us and the main contributions in this thesis.

1.1 Motivations

The $GI/G/1$ -type Markov chain can be found in many applications such as telecommunications, computer networks, Internet, manufacturing systems, economics and finance, etc. For example, we consider digital video sources in video traffic, which are compressed using interframe variable rate coding. The coded bit stream from each source is stored in a separate prebuffer, which assembles the data into blocks and packets the blocks. Prebuffering eliminates complicated properties in the nature of the source model. The packets from all the prebuffers join a common buffer in the multiplexer, where the pack-

ets are queued for transmission over a high-speed communication line. Decay and tail properties and ergodicity are two types of important problems in applications. Decay and tail properties can be used to deal with a series of practical problems such as stationary performance metrics, approximations, performance bounds, etc., especially there is no explicit expression for the stationary distribution, etc. Ergodicity is one of fundamental issues and plays a key role in simulation analysis.

1.2 Problem statements

We are interested in tail asymptotics and ergodicity for two-dimensional stochastic models. The type of model that we study in this thesis is called the $GI/G/1$ -type Markov chain with finitely many background states denoted by $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$, where X_n and Y_n are called the level and background(phase) variable, respectively. Throughout this thesis, the $GI/G/1$ -type Markov chain is assumed to be aperiodic, irreducible and positive recurrent. Under this assumption, its unique stationary distribution π is positive elementwise, which is partitioned into vectors $\pi = (\pi_0, \pi_1, \dots)$ according to levels.

Tail asymptotics of the $GI/G/1$ -type Markov chain that was discussed consist of light-tailed asymptotics and heavy-tailed asymptotics. In this thesis, we study the light-tailed case. With regard to the light-tailed asymptotics, two kinds of studies have been reported in the literature: rough decay, called decay in a weak sense; and exact decay, including exact geometric decay and light-tailed but not exact geometric asymptotics. Both of these are discussed in this thesis. Another important topic dealt with is ergodicity. Ordinary ergodicity, geometric ergodicity, polynomial ergodicity and uniform ergodicity, etc, are important classes of ergodicity, among which we are specifically interested in polynomial ergodicity in this thesis. Besides tail asymptotics and ergodicity, we also deal with some

factorization results regarding generating functions of the transition probability matrix.

In order to state our problems more precisely, we introduce the following definitions. Firstly, consider a sequence $\{M_n\}$ of nonnegative matrices of size $m \times m$ satisfying $\sum_{n=1}^{\infty} M_n < \infty$. $\{M_n\}$ is called light-tailed if for all $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$, the sequences $\{M_n(i, j)\}$ of nonnegative scalars are all light-tailed, that is,

$$\sum_{n=1}^{\infty} M_n(i, j) \exp\{\varepsilon n\} < \infty, \quad \text{for some } \varepsilon > 0,$$

where $M_n(i, j)$ is the (i, j) th entry of M_n and ε is independent of i and j . We call $\{M_n\}$ heavy-tailed if it is not light-tailed. More specifically, the sequence $\{M_n\}$ of nonnegative matrices of size $m \times m$ is called heavy-tailed if there exists at least one pair (i, j) , where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, m$, such that the sequence $\{M_n(i, j)\}$ of nonnegative scalars is not light-tailed, that is,

$$\sum_{n=1}^{\infty} M_n(i, j) \exp\{\varepsilon n\} = \infty, \quad \text{for all } \varepsilon > 0.$$

Secondly, the decay in a weak sense is equivalent to the decay in the logarithmic sense, which is the key issue in large deviation theory. For each $k \in \{1, 2, \dots, m\}$, the upper decay rate in a weak sense $e^{-\bar{\Lambda}_k}$ of $\{\pi_{nk}\}$ along the level direction and the lower decay rate in a weak sense $e^{-\underline{\Lambda}_k}$ of $\{\pi_{nk}\}$ along the level direction are defined by

$$e^{-\bar{\Lambda}_k} = \limsup_{n \rightarrow \infty} (\pi_{nk})^{\frac{1}{n}}$$

and

$$e^{-\underline{\Lambda}_k} = \liminf_{n \rightarrow \infty} (\pi_{nk})^{\frac{1}{n}},$$

respectively, where π_{nk} is the k th element of the vector π_n . If $\bar{\Lambda}_k = \underline{\Lambda}_k \triangleq \Lambda_k$, $e^{-\Lambda_k}$ is referred to as the decay rate in a weak sense of $\{\pi_{nk}\}$ along the level direction. In particular, if $e^{-\Lambda_k}$ is independent of k , we denote it by $e^{-\Lambda}$, which is the decay rate in a

weak sense of $\{\pi_n\}$ along the level direction. It is noted that the above definitions are equivalent to

$$-\overline{\Lambda}_k = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \pi_{nk} = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \bar{\pi}_{nk}$$

and

$$-\underline{\Lambda}_k = \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \pi_{nk} = \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \bar{\pi}_{nk},$$

where $\bar{\pi}_{nk} = \sum_{l=n}^{\infty} \pi_{lk}$. For the $GI/G/1$ -type Markov chain with finitely many background states, characterizing decay rate $e^{-\Lambda}$ in a weak sense of $\{\pi_n\}$ along the level direction is challenging and incomplete.

Thirdly, we introduce the definition for exact decay. For each $k \in \{1, 2, \dots, m\}$, $\{\pi_{nk} : n \geq 0\}$ has an exact decay along the level direction if there exists a positive function $f_k(n)$ and a positive constant c_k satisfying

$$\lim_{n \rightarrow \infty} \frac{\pi_{nk}}{f_k(n)} = c_k.$$

If $f_k(n)$ is independent of k , for all $k \in \{1, 2, \dots, m\}$, it will be denoted by $f(n)$, and called an exact decay function along the level direction for $\{\pi_n : n \geq 0\}$. Especially, if $f(n) = \zeta^n$ with $\zeta < 1$, then $\{\pi_n : n \geq 0\}$ is said to have an exact geometric decay along the level direction. If for $l \in \{1, 2, 3, \dots\}$, $f(n) = n^l$, then $\{\pi_n : n \geq 0\}$ is said to have an exact polynomial decay of degree l along the level direction. Exact geometric decay has been discussed by many researchers, but the study is still incomplete, especially for a case where the behavior of the boundary transition matrix affects the asymptotic behavior. Therefore, it is necessary to further characterize how the boundary transition matrix affects the asymptotic behavior of the tail. It should be noted that the stationary distribution $\{\pi_n\}$ does not always have an exact geometric decay in the light-tailed case. We say that it has light-tailed but not exact geometric asymptotics. Sufficient conditions for light-tailed but not exact geometric asymptotics were provided in the literature for

the QBD model with infinite block, which includes the QBD model with finite blocks as a special case. However, conditions under which $\{\pi_n : n \geq 0\}$ is light-tailed but not exact geometric asymptotics are not available for the $GI/G/1$ -type model, even with finite blocks. Furthermore, we concern about which types of asymptotics rates the stationary distribution has when it has light-tailed but not exact geometric asymptotics.

Fourthly, ergodicity characterizes the speed at which the n th step transition probability converges to the stationary distribution as n goes to infinite. It has been shown that the ordinary ergodicity of the $GI/G/1$ -type Markov chain is equivalent to $\sum_{-\infty}^{\infty} nB_n e^t < \infty$ and $\mu \sum_{-\infty}^{\infty} nA_n e^t < 0$, where μ is the stationary distribution of $\sum_{-\infty}^{\infty} A_n$ and A_n and B_n are non-boundary and boundary transition probability blocks, respectively. We aim to analyze polynomial ergodicity in this thesis. To our purpose, we define polynomial ergodicity as follows: let $l = 2, 3, \dots$, and let $\tau_0 = \inf\{n \geq 1 : (X_n, Y_n) \in L_0\}$ be the first hitting time to level 0, if for each $j \in \{1, \dots, m\}$,

$$\mathbb{E}_{(0,j)} \tau_0^l < \infty,$$

then the $GI/G/1$ -type Markov chain is called polynomial ergodicity of degree l . A necessary and sufficient condition of polynomial ergodicity for the $M/G/1$ -type was obtained in the literature. It is our goal to characterize necessary and sufficient conditions for polynomial ergodicity for the $GI/G/1$ -type model.

Finally, the factorization based on Wiener-Hopf-type equations for the transition probability matrix is important. In this thesis, we derive factorization results for generating functions of the transition probability matrix.

Both tail asymptotics and ergodicity are interesting due to their theoretical challenges and broad range of applications.

1.3 Background knowledge on queuing processes

Many queuing situations encountered in every day life such as the theory of scheduling and transportation, the design of automatic equipment and production line, etc. Here, we provide the basic queuing model, that is, the single server queuing to a description. Let t_n denote the time of arrival after $t = 0$ of the n th customer at the service station, $n = 0, 1, \dots$. The times

$$t_n - t_{n-1}, n = 1, 2, \dots,$$

with

$$t_0 = 0,$$

will be called the interarrival times. These interarrival times are assumed to be *i.i.d.* non-negative random variables. Denote by τ_n the service time of the n th customer arriving after $t = 0$, $n = 1, 2, \dots$. τ_n are assumed to be *i.i.d.* random variables. It is also assumed that the arrival process, i.e. the family of interarrival times, and service process, i.e. the family of service time, are independent. Let w_n , $n = 1, 2, \dots$ be the actual waiting time of the n th customer, i.e. the time between his arrival and the start of his service, so that $w_n = 0$ if no customers are present just before t_n . Denote the number of customers present in the system at time t by X_t (queuing length). Denote the virtual time at time t by v_t . If $X_t = 0$ then $v_t = 0$, and if $X_t = 1$ then v_t is equal to the residual service time at time t , whereas for $X_t = n > 1$ the virtual waiting time v_t is the sum of the residual service time at time t and the service times of the $n - 1$ waiting customers. From the point of view of the server the virtual waiting time v_t represents the amount of work he has still to do at time t to complete the service of all customers present at that time.

Next, if the arrival process and service process vary over time, then the queuing becomes a two-dimensional system. Let $\{Y_t : t \geq 0\}$ be a process on $\{1, 2, \dots, m\}$ or a

countable set, which controls varieties of the arrival process and service process. Then $\{(X_t, Y_t) : t \geq 0\}$ is a two-dimensional queuing system. In our model, the $GI/G/1$ -type Markov chain is an embedded Markov chain of $\{(X_t, Y_t) : t \geq 0\}$ at $\{v_{t_n}\}$.

1.4 Literature review

In the literature, two-dimensional Markov models have been extensively investigated. The $GI/M/1$ -type Markov chain with finitely many background states is a class of the two-dimensional model whose stationary distribution has a matrix-geometric structure. One of the classic references of the $GI/M/1$ -type Markov chain is Neuts [82] in which the basic structures and properties were described, and an algorithmic approach to study the models was provided. A more recent reference is Latouche and Ramaswami [57], focusing on studies of QBD process. Probabilistic arguments were used to present basic mathematical ideas and algorithms of the matrix analytic theory in this book.

Another important two-dimensional Markov model is the $M/G/1$ -type Markov chain with finitely many background states, for which the stationary distribution has a more complicated structure in contrast to that for the $GI/M/1$ -type Markov chain with finitely many background states. From the methodological viewpoint, Neuts [83] presented the matrix structure and stability conditions for the $M/G/1$ -type Markov chain. Falkenberg [21] discussed tail asymptotics of the stationary distribution of the $M/G/1$ -type Markov chain and derived sufficient conditions for an exact geometric decay when there exists an $\alpha > 1$ such that α is the Perron-Frobenius eigenvalue of $\sum_{n=0}^{\infty} \alpha^{n+1} A_n$, i.e. $A(\alpha) = \sum_{n=-1}^{\infty} \alpha^n A_n$ is 1-positive recurrent. A necessary condition was provided as well in this paper. Møller [78] studied light-tailed asymptotics of the stationary distribution of the $M/G/1$ -type Markov chain and provided conditions for an exact exponential tail

decay of the stationary distribution, which improved the result in Abate [1]. Asmussen and Møller [7] analyzed heavy-tailed asymptotics of the stationary distribution of the $M/G/1$ -type Markov chain and obtained a result for a sub-exponential decay. Takine [94] provided sufficient conditions for a geometric decay and sub-exponential decay for the stationary distribution. For the geometric decay, the work in this paper extended results to a more general case in which the $M/G/1$ -type Markov chain is assumed to be d ($d \geq 1$)-arithmetic instead of 1-arithmetic. Liu and Hou [67] studied polynomial and geometric ergodicity for the $M/G/1$ -type Markov chains. A necessary and sufficient condition for polynomial ergodicity was obtained in the paper

For the $GI/G/1$ -type Markov chain with finitely many background states, Asmussen [8] provided some results, including conditions for positive recurrence under the framework of a Markov additive process. Grassmann and Heyman [37] presented a factorization result of the off-boundary transition blocks of the $GI/G/1$ -type Markov chain, based on which the authors provided a solution for the stationary distribution for this class of Markov chains. Later, Zhao [106] obtained a factorization result for the boundary transition blocks of the $GI/G/1$ -type Markov chain. Zhao, Li and Braun [104] analyzed factorization results of some other probabilistic measures involving first passage times or expected numbers of visits to some levels while other levels are taboo, called A - and B -measures in the paper, and used these results to derive a necessary and sufficient condition for a Markov chain to be positive recurrent, recurrent or transient in terms of R - and G -measures. Zhao, Li and Braun [107] further addressed that these probabilistic measures are invariant under censoring and obtained explicit expressions of the fundamental matrix for the $GI/G/1$ -type Markov chain. Li and Zhao in [63] obtained sufficient conditions on which the stationary distribution $\{\pi_k\}$ was light-tailed or exact geometric with rate $\frac{1}{\eta}$ satisfying $\eta < \min\{\phi_{A+}, \phi_{B+}\}$ for the $GI/G/1$ -type Markov chain. In [64], Li and Zhao discussed the heavy-tailed asymptotics of the stationary distribution for the $GI/G/1$ -type Markov

chain. A necessary and sufficient condition for the heavy-tailed asymptotics and sufficient conditions for the sub-exponential asymptotics of the stationary distribution were derived in this paper.

There exist results corresponding to the two-dimensional Markov chain with countably many background states. In Takahashi, Fujimoto and Makimoto [93], a sufficient condition based on a positivity assumption was obtained for an exact geometric decay of the stationary distribution of a QBD process. Later, Haque, Liu and Zhao in [42] presented improved conditions equivalent to those in Takahashi [93]. Based on these conditions, Haque, Liu and Zhao developed a practical method for finding α and the invariant vector corresponding to α . Recently, He, Li and Zhao in [43] analyzed the exact geometric decay based on a positive invariant vector of R -measure without a positivity assumption and light-tailed but not exact geometric asymptotics for a QBD process. Foley and Macdonald [26] analyzed the decay rate in a weak sense for a modified two-node Jackson network when one node gets large. Foley and Macdonald [27] obtained exact decay of the stationary distribution for a modified two-node Jackson network. In Kroese, Scheinhardt and Taylor [56], the authors showed that the decay rate varies with changes of the boundary generator blocks for a two-node tandem Jackson model as a special example of QBD process. Motyer and Taylor [79] provided simple conditions to an geometric decay of the stationary distribution for a doubly QBD process, and also showed that changes on the boundary transition matrix possibly affected the tail behavior, which generalized the results in [56]. Miyazawa in [77] also studied the decay rate in a weak sense as the solution of a nonlinear optimization problem for a doubly QBD process and obtained exact geometric decay for positivity and non-positivity cases. In [76], Miyazawa and Zhao obtained conditions under which the stationary distribution decayed geometrically for a more general $GI/G/1$ -type Markov chain, which was a generalization of the corresponding result for the $M/G/1$ -type Markov chain in Miyazawa [75].

1.5 Methods

Different methods have been developed to analyze tail asymptotics and ergodicity. In this thesis, we mainly use the **probabilistic method**, the **matrix analytic method** and the **complex analytic method**. These methods may be used together. For the **probabilistic method**, we mainly use it to derive a lower bound for the lower decay rate in a weak sense for the stationary distribution, analyze exact geometric decay rate (specifically, twisted process, dual process and Markov renewal process) and obtain necessary and sufficient conditions for polynomial ergodicity. For the **matrix analytic method**, we mainly use it to obtain a classification for the $GI/G/1$ -type Markov chain under the light-tailed assumption, get the decay rate in a weak sense when $\alpha > 1$, and $\alpha < \phi_{B+}$, where α is a number such that $A(\alpha)$ is 1-positive recurrent and deal with sufficient conditions for exact decay. For the **complex analytic method**, we mainly use it to analyze exact decay rates and discuss the property of $\det(I - A(z))$. Besides the above three methods, we also use other techniques such as large deviation technique, properties of generating function, Tauberian theory and real analysis.

1.6 Main contributions in this thesis

In this thesis, we discuss the decay rate in a weak sense and exact decay rates of the stationary distribution, polynomial ergodicity and factorization of generating functions of the transition probability for the $GI/G/1$ -type Markov chain with finitely many background states. We have the following contributions:

- i For **decay in a weak sense**: We derive the decay rate in a weak sense for the stationary distribution in terms of the convergent parameter of $\pi(z)$, that is, the

largest one of $\{\frac{1}{\phi_{A^+}}, \frac{1}{\phi_{B^+}}, \frac{1}{\alpha}\}$ when A is irreducible. The results show that the largest singularities of $A^+(z)$, $B^+(z)$ and $[I - A(z)]^{-1}$ play an important role in determining decay rate for the stationary distribution, and both boundary components of the transition probability matrix and non-boundary components of the transition probability matrix affect the tail behavior;

- ii For **exact decay**: We obtain sufficient conditions for an exact geometric decay and light-tailed but not exact geometric asymptotics for the stationary distribution. We further provide sufficient conditions for other exact decay when the stationary distribution is light-tailed but not exact asymptotics. Especially, we give a more specific description about the effect of the boundary components of the transition probability matrix on exact decay;
- iii For **polynomial ergodicity**: We provide two necessary and sufficient conditions for polynomial ergodicity, and also obtain a lower bound for the first hitting time, which is useful for analyzing ergodicity, especially for finding necessary conditions;
- iv For **factorization**: We provide factorization results for the generating functions of the transition probability matrix.

Furthermore, we expect to discuss the following problems:

1. Analyze rough and exact asymptotics phenomenon in more general models with countably many background states or level dependent models;
2. Discuss rough and exact asymptotics phenomenon in some network models;
3. Discuss ergodicity with more general rate, for example, sub-geometric ergodicity;
4. Discuss ergodicity for more general models.

For more details, please read the concluding remarks and future work in chapter 7.

1.7 Organization of this thesis

The rest of this thesis is organized as follows. In Chapter 2, we outline some preliminary results, which includes factorization results for the $GI/G/1$ -type Markov chain, classification for the $GI/G/1$ -type Markov chain under the light-tailed assumption and renewal theory. In chapter 3, we analyze the decay rate in a weak sense for the stationary distribution. In chapter 4, we discuss exact geometric decay and light-tailed but not exact geometric asymptotics. In chapter 5, we discuss polynomial ergodicity. In chapter 6, we derive factorization for the generating functions of the transition probability matrix. In chapter 7, we provide some concluding remarks and future work.

Chapter 2

Preliminaries

In this chapter, we introduce a series of definitions, concepts and ideas which will be used in the following chapters.

We firstly provide the mathematical model, which will be studied in this thesis. Let $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ be the Markov chain whose transition matrix P is expressed in block matrix form:

$$P = \left(p_{(i,u),(j,v)} \right)_{(i,u),(j,v)} = \begin{pmatrix} P_{0,0} & P_{0,1} & P_{0,2} & P_{0,3} & \cdots \\ P_{1,0} & P_{1,1} & P_{1,2} & P_{1,3} & \cdots \\ P_{2,0} & P_{2,1} & P_{2,2} & P_{2,3} & \cdots \\ P_{3,0} & P_{3,1} & P_{3,2} & P_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.1)$$

where $P_{i,j}$ is a matrix of size $m_i \times m_j$, where m_i and m_j may be infinite. We accordingly partition the state space of the chain P into

$$S = \bigcup_{i=0}^{\infty} L_i$$

with

$$L_i = \{(i, 1), (i, 2), \dots, (i, m_i)\}.$$

In state (i, u) , i is called the level variable and u the background variable. We also use the notation

$$L_{\leq i} = \bigcup_{k=0}^i L_k,$$

and

$$L_{\geq i} = \bigcup_{k=i}^{\infty} L_k.$$

Furthermore, we assume that the chain $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ has a special repeating structure with the transition matrix P in (2.1) given by

$$P = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ B_{-1} & A_0 & A_1 & A_2 & \cdots \\ B_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ B_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.2)$$

where A_i and B_i for $i = \dots, -2, -1, 0, 1, 2, \dots$, are matrices of size $m \times m$ with $m < \infty$. We usually call this type of chain the $GI/G/1$ -type Markov chain with finitely many background states. The $GI/M/1$ -type Markov chain, the $M/G/1$ -type Markov chain and the QBD are three examples whose transition matrices are given by

$$P = \begin{pmatrix} B_0 & B_1 & 0 & 0 & \cdots \\ B_{-1} & A_0 & A_1 & 0 & \cdots \\ B_{-2} & A_{-1} & A_0 & A_1 & \cdots \\ B_{-3} & A_{-2} & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.3)$$

$$P = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ B_{-1} & A_0 & A_1 & A_2 & \cdots \\ 0 & A_{-1} & A_0 & A_1 & \cdots \\ 0 & 0 & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (2.4)$$

and

$$P = \begin{pmatrix} B_0 & B_1 & 0 & 0 & \cdots \\ B_{-1} & A_0 & A_1 & 0 & \cdots \\ 0 & A_{-1} & A_0 & A_1 & \cdots \\ 0 & 0 & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (2.5)$$

respectively.

2.1 Factorization

In this section, we outline some factorization results with respect to the $GI/G/1$ -type Markov chain with finitely many background states.

For the above chain, we define R -measures and G -measures as follows. Let matrices $R_{i,j}$ for $i < j$ and $G_{i,j}$ for $i > j$ be R -measures and G -measures respectively. $R_{i,j}$ is a matrix of size $m \times m$ whose (u, v) th entry is the expected number of visits to state (j, v) before hitting any state in $L_{\leq(j-1)}$, given that the process starts in state (i, u) . $G_{i,j}$ is a matrix of size $m \times m$ whose (u, v) th entry is the probability of hitting state (j, v) when the process enters $L_{\leq(i-1)}$ for the first time, given that the process starts in state (i, u) . We call matrices $R_{i,j}$ and $G_{i,j}$, respectively, the matrices of expected number of visits to higher levels before returning to lower levels and matrices of the first passage probabilities

to lower levels. From [104], we can write $R_{n-i} = R_{i,n}$ and $G_{n-i} = G_{n,i}$ for $i > 0$ due to the property of the repeating row. To describe the properties of R - and G -measures, we define a matrix sequence $\{\Phi_i, -\infty < i < \infty\}$. For $n \geq 1$, let

$$P = \begin{matrix} & L_{\leq n} & L_{\geq (n+1)} \\ L_{\leq n} & Q_0 & U \\ L_{\geq (n+1)} & V & Q_1 \end{matrix}.$$

Set $P^{[n]} = (P_{i,j}^{[n]})_{i,j=0,\dots,n} = Q_0 + U\widehat{Q}_1V$, where $\widehat{Q}_1 = (\widehat{Q}_1(i,j))_{i,j=1,2,\dots} = \sum_{k=0}^{\infty} Q_1^k$. It is shown in [37] that the matrices $P_{n-i,n}^{[n]}$, for $0 \leq i \leq n-1$, and $P_{n,n-j}^{[n]}$, for $0 \leq j \leq n-1$, are all independent of $n \geq 1$. Hence, for $n \geq 1, 0 \leq i \leq n-1$, and $0 \leq j \leq n-1$, define

$$\Phi_i = P_{n-i,n}^{[n]}, \quad \Phi_{-j} = P_{n,n-j}^{[n]}.$$

we also know from [37] that

$$R_i = \Phi_i(I - \Phi_0)^{-1}, \quad i \geq 1,$$

and

$$G_j = (I - \Phi_0)^{-1}\Phi_{-j}, \quad i \geq 1,$$

where I is the identity matrix. From [37] and [106], R - and G -measures also satisfy the following properties:

Lemma 2.1. (see [37]) [used in Theorem 5.6 in this thesis]

$$R_i(I - \Phi_0) = A_i + \sum_{k=1}^{\infty} R_{i+k}(I - \Phi_0)G_k, \quad i \geq 1,$$

and

$$(I - \Phi_0)G_j = A_{-j} + \sum_{k=1}^{\infty} R_k(I - \Phi_0)G_{j+k}, \quad j \geq 1.$$

Lemma 2.2. (see [106]) [used in Theorem 4.13, 4.14 and 5.6 and Lemma 4.2 in this thesis]

$$R_{0,i}(I - \Phi_0) = B_i + \sum_{k=1}^{\infty} R_{0,i+k}(I - \Phi_0)G_k, \quad i \geq 1,$$

and

$$(I - \Phi_0)G_{j,0} = B_{-j} + \sum_{k=1}^{\infty} R_k(I - \Phi_0)G_{j+k,0}, \quad j \geq 1.$$

Lemma 2.3. (see [106]) [used in Theorem 3.1 (ii) in this thesis]

$$R_{0,j} = (B_j, B_{j+1}, B_{j+2}, \dots) \widehat{Q}_1(\cdot, 1),$$

$$R_i = (A_i, A_{i+1}, A_{i+2}, \dots) \widehat{Q}_1(\cdot, 1),$$

and

$$G_{i,0} = \widehat{Q}_1(1, \cdot)(B_{-i}, B_{-(i+1)}, B_{-(i+2)}, \dots)^t,$$

$$G_i = \widehat{Q}_1(1, \cdot)(A_{-i}, A_{-(i+1)}, A_{-(i+2)}, \dots)^t.$$

Lemma 2.4. (See [64]) [used in Theorem 4.13 and 4.14 in this thesis]

Let $M_i = \Phi_{-i}(I - \Phi_0)^{-1}$, $i \geq 1$, and

$$\Lambda = \begin{pmatrix} I & & & & \\ -M_1 & I & & & \\ -M_2 & -M_1 & I & & \\ -M_3 & -M_2 & -M_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$\Lambda^{-1} = \begin{pmatrix} I & & & & \\ H_1 & I & & & \\ H_2 & H_1 & I & & \\ H_3 & H_2 & H_1 & I & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$H_j = \sum_{i=1}^{\infty} \sum_{\substack{n_1+n_2+\dots+n_i=j \\ n_l \geq 1, 1 \leq l \leq i}} M_{n_1} M_{n_2} \cdots M_{n_i}, \quad j \geq 1.$$

Define the generating functions of the matrix sequences $\{R_k\}$, $\{G_k\}$ and $\{A_k\}$ by

$$R(z) = \sum_{k=1}^{\infty} z^k R_k,$$

$$G(z) = \sum_{k=1}^{\infty} z^{-k} G_k$$

and

$$A(z) = \sum_{k=-\infty}^{\infty} z^k A_k.$$

Then, the following factorization result is given by Zhao [106].

Theorem 2.1. *(See [63]) [used in Theorem 3.1 (i) and 4.4 and Lemma 4.5 in this thesis]*

For an appropriate z for which $A(z)$ is finite, the following factorization holds

$$I - A(z) = [I - R(z)](I - \Phi_0)[I - G(z)].$$

Define the generating function for the stationary distribution $\{\pi_k\}$ and the matrix sequence $\{R_{0,k}\}$ as

$$\pi(z) = \sum_{k=0}^{\infty} z^k \pi_k$$

and

$$R_0(z) = \sum_{k=1}^{\infty} z^k R_{0,k}.$$

If the $GI/G/1$ -type Markov chain is positive recurrent, then the stationary distribution $\{\pi_k\}$ can be expressed in terms of the R -measures (see [37]):

$$\pi_n = \pi_0 R_{0,n} + \sum_{k=1}^{n-1} \pi_k R_{n-k}, \quad n \geq 1. \quad (2.6)$$

Then, we have that

$$\pi(z)[I - R(z)] = \pi_0 R_0(z). \quad (2.7)$$

This relation is useful for analyzing tail properties of the stationary distribution.

2.2 Classification for the light-tailed $GI/G/1$ -type Markov chain

In this section, we give a classification for the light-tailed $GI/G/1$ -type Markov chain with finitely many background states in order to divide our study regarding light-tailed asymptotics into different cases. we assume that P is aperiodic, irreducible and positive recurrent. Let ϕ_{A+} and ϕ_{B+} be the convergent radii of

$$A^+(z) \triangleq \sum_{n=1}^{\infty} z^n A_n$$

and

$$B^+(z) \triangleq \sum_{n=1}^{\infty} z^n B_n,$$

respectively. It is clear that $\{A_n\}$ and $\{B_n\}$ are light-tailed if and only if $\phi_{A+} > 1$ and $\phi_{B+} > 1$, respectively. Let us define

$$A = \sum_{n=-\infty}^{\infty} A_n.$$

Assume that A is irreducible and aperiodic. For $0 \leq z \leq \phi_{A+}$, define $\gamma(z)$ as follows:

- (a) if $A(z) < \infty$, then $\gamma(z)$ is the maximal eigenvalue of $A(z)$;
- (b) if $A(\phi_{A+}) = \infty$, then $\gamma(\phi_{A+}) = \lim_{z \rightarrow \infty} \gamma(z)$, where $A(\phi_{A+}) = \lim_{z \rightarrow \phi_{A+}} A(z)$. $\gamma(z)$ has either Property 1 or Property 2 as stated below.

Before we state the properties, we extend the definition of s -positive recurrence given in the literature to any non-negative matrix. For a matrix $T \geq 0$ (of size $m \times m$, or infinite), if there exist an $s > 0$, a row vector $\mathbf{x}$ satisfying $\mathbf{x} \geq \mathbf{0}$ but $\mathbf{x} \neq \mathbf{0}$ and a column vector $\mathbf{y}^t$ satisfying $\mathbf{y}^t \geq \mathbf{0}$ but $\mathbf{y}^t \neq \mathbf{0}$ such that

$$\mathbf{x}T = s\mathbf{x},$$

$$T\mathbf{y}^t = s\mathbf{y}^t,$$

and

$$\mathbf{x}\mathbf{y}^t$$

then T is called s -positive recurrent. For a finite matrix T , there always exist an $s > 0$ such that T is s -positive. If T is irreducible, then we have that $\mathbf{x} > \mathbf{0}$ and $\mathbf{y}^t > \mathbf{0}$. Since $A(z)$ for $z \geq 1$, is irreducible, then, by the Perron-Frobenius Theorem, there exist a unique positive row vector $\boldsymbol{\mu}(z)$ and a unique positive column vector $\boldsymbol{\nu}^t(z)$ such that

$$\boldsymbol{\mu}(z)A(z) = \gamma(z)\boldsymbol{\mu}(z)$$

and

$$A(z)\boldsymbol{\nu}^t(z) = \gamma(z)\boldsymbol{\nu}^t(z).$$

In the following, we state that $\gamma(z)$ has one of the two properties.

Property 1: There exists a $z > 1$ such that $\gamma(z) = 1$ and $A(z) < \infty$, that is, for the given z , $A(z)$ is 1-positive recurrent; and

Property 2: There exists no such $z > 1$ that $A(z)$ is 1-positive recurrent.

The $GI/G/1$ -type Markov chain in the light-tailed case satisfies either Property 1 or Property 2. To characterize Property 1 and Property 2, we provide necessary and

sufficient conditions to Property 1 and Property 2. We first prove an equivalent condition to Property 1. Especially, when Property 1 holds, z is unique and denoted by α .

Proposition 2.1. *If either of the following conditions:*

(i) *there exists a $1 < z_0 < \phi_{A^+}$ such that $\gamma(z_0) > 1$;*

(ii) *$\phi_{A^+} < \infty$, $A(\phi_{A^+}) < \infty$ and $\gamma(\phi_{A^+}) = 1$*

is satisfied, then there exists a unique $1 < \alpha < \phi_{A^+}$ or $\alpha = \phi_{A^+}$ such that $A(\alpha)$ is 1-positive recurrent. Conversely, if for $\alpha > 1$, $A(\alpha)$ is 1-positivity, then condition (i) or condition (ii) holds.

Proof: For sufficiency, firstly, we assume that condition (i) holds. By the Perron-Frobenius Theorem and the Implicit Function Theorem, we know that $\gamma(z)$ is a continuous function. From Corollary 3.3 in [84], we know that $\gamma(z)$ is also strictly convex. Hence, if $Ae \leq e$ and $Ae \neq e$, then there exists an $\alpha > 1$ such that $\gamma(\alpha) = 1$ since $\gamma(1) < 1$; otherwise $Ae = e$, then there exists a $z_1 > 1$ such that $\gamma(z_1) < 1$ since

$$\gamma'(1) = \mathbf{h}(1) \sum_{k=1}^{\infty} k(A_k - A_{-k})e < 0, \quad (2.8)$$

where $\mathbf{h}(1)$ is the stationary probability vector of A . Therefore, there exists an $\alpha > 1$ such that $\gamma(\alpha) = 1$, which means that $A(\alpha)$ is 1-positive. Secondly, we assume that condition (ii) is true. It is easy to know that $\alpha = \phi_{A^+}$ in this case.

For necessity, assume that there exists an $\alpha > 1$ such that $\gamma(\alpha) = 1$. If $\alpha = \phi_{A^+}$, then condition (ii) holds; otherwise, if $\alpha < \phi_{A^+}$, then for all z_0 with $\alpha < z_0 < \phi_{A^+}$, we have $\gamma(z_0) > 1$ since $\gamma(z)$ is a continuous and strictly convex function. $\square$

Corollary 2.2. *$A(\alpha)$ is 1-positive for some $\alpha > 1$ if and only if there exists a finite $z_0 > 1$ such that $\gamma(z_0) = 1$. In this case $z_0 = \alpha$.*

Corollary 2.3. *If either of the following conditions:*

- (i) $A(\phi_{A+}) = \infty$ and $\gamma(\phi_{A+}) \leq 1$;
- (ii) $A(\phi_{A+}) < \infty$ and $\gamma(\phi_{A+}) < 1$

is satisfied, then we cannot find any $\alpha > 1$ such that $A(\alpha)$ is 1-positive recurrent, or $\gamma(\alpha) = 1$. Conversely, if for all $\alpha > 1$, $\gamma(\alpha) \neq 1$, then condition (i) or condition (ii) holds.

Remark 2.1. *It follows from (2.8) that for z close enough to 1, $\gamma(z) < 1$.*

Remark 2.2. *According to properties of $A(z)$ and $\gamma(z)$, we can divide our discussion into the following five cases:*

- (i) $A(\phi_{A+}) = \infty$ and $\gamma(\phi_{A+}) > 1$ (in this case, there exists a $1 < z_0 < \phi_{A+}$ such that $\gamma(z_0) > 1$);
- (ii) $A(\phi_{A+}) = \infty$ and $\gamma(\phi_{A+}) \leq 1$;
- (iii) $A(\phi_{A+}) < \infty$ and $\gamma(\phi_{A+}) < 1$ (in this case, for any $1 < z \leq \phi_{A+}$, $\gamma(z) < 1$);
- (iv) $A(\phi_{A+}) < \infty$, $\phi_{A+} < \infty$ and $\gamma(\phi_{A+}) = 1$;
- (v) $A(\phi_{A+}) < \infty$ and $\gamma(\phi_{A+}) > 1$.

Moreover, it is easy to see that (i) and (v), (iv), (ii) and (iii) are covered by Proposition 2.1 (i), Proposition 2.1 (ii), Corollary 2.3 (i) and Corollary 2.3 (ii), respectively.

Remark 2.3. *Corollary 2.3 gives an equivalent condition to Property 2. In this case, we cannot find $\alpha > 1$ such that $\gamma(\alpha) = 1$.*

Remark 2.4. (i) $\{A_n, n \geq 0\}$ and therefore $\{\pi_n : n \geq 0\}$ can be studied according to if it has Property 1 or Property 2, or if $A(z)$ is 1-positive or not;

(ii) When $A(z)$ is 1-positive, then obviously the unique $z = \alpha \leq \phi_{A+}$. In fact, $\alpha < \phi_{A+}$ unless $\gamma(\phi_{A+}) = 1$. When $\alpha = \phi_{A+}$, we have that $A(\phi_{A+}) < \infty$. Moreover, α is the smallest number z in $(1, \phi_{A+}]$ such that $\det(I - A(z)) = 0$; and

(iii) When $A(z)$ is not 1-positive, then there exists no $z > 1$ such that $\det(I - A(z)) = 0$.

Let ϕ_R and ϕ_{R_0} be the convergent radii of $R(z)$ and $R_0(z)$ respectively. For $0 \leq z \leq \phi_R$, define $r(z)$ as follows: (a) if $R(z) < \infty$, then $r(z)$ is the maximal eigenvalue of $R(z)$; (b) if $R(\phi_R) = \infty$, then $r(\phi_R) = \lim_{z \rightarrow \phi_R} r(z)$, where $R(\phi_R) = \lim_{z \rightarrow \phi_R} R(z)$.

Lemma 2.5. From Theorem 1, Lemma 3, 4 and 5 of [63], we have the following properties:

(i) $\phi_{A+} = \phi_R$ and $\phi_{B+} = \phi_{R_0}$;

(ii) $A(\alpha)$ is 1-positive recurrent if and only if $R(\alpha)$ is 1-positive recurrent, where $R(\alpha)$ may not be irreducible.

2.3 Renewal theory

In this part, we simply outline some basic results regarding renewal theory, specially for Markov renewal theory. For more details, please see the references: Feller[24]; Çinlar [15]; Asmussen [8]. Renewal theory (Markov renewal theory) is used to proofs of Theorem 4.1, 4.3, 4.6, 4.7, 4.9, 4.10 and 4.12 and Lemma 4.1 for an exact geometric decay and light-tailed but not exact geometric asymptotics in this thesis.

Consider a fixed phenomenon, and let $W_1, W_2, \dots$ be the times between its successive occurrences. Then, the epoch of the n th occurrence is given by the sum

$$Z_n = W_1 + \dots + W_n.$$

By convention $Z_0 = 0$ and 0 counts as occurrence number zero.

Definition 2.1. *The sequence $\{Z_n : n = 0, 1, \dots\}$ is called a renewal process provided that $W_1, W_2, \dots$ be independently and identically distributed nonnegative random variables with a common distribution F .*

Let $U(t)$ denote the expected number of renewals in $[0, t]$, then

$$\begin{aligned} U(t) &= \mathbb{E} \left[\sum_{k=0}^{\infty} \mathbf{1}_{[0,t]}(Z_k) \right] = \sum_{k=0}^{\infty} \mathbb{E} [\mathbf{1}_{[0,t]}(Z_k)] \\ &= \sum_{k=0}^{\infty} \mathbb{P}(Z_k \leq t) = \sum_{k=0}^{\infty} F^k(t). \end{aligned} \tag{2.9}$$

The function

$$U = \sum_{k=0}^{\infty} F^k$$

is called the renewal function corresponding to the distribution F .

Definition 2.2. *A renewal process $\{Z_n : n = 0, 1, \dots\}$ is said to be recurrent if $W_n < \infty$ almost surely for every n ; otherwise, $\{Z_n : n = 0, 1, \dots\}$ is called transient.*

Definition 2.3. *F is said to be arithmetic with span δ if all of its jumps occur at points of the form $n\delta$ for some $\delta > 0$.*

Renewal theory is the study of the so-called renewal equation

$$f = g + F * f, \tag{2.10}$$

where f and g are functions which are bounded over finite intervals and $*$ denotes the convolution. We are interested in solving for f in equation (2.10) and examine its behavior at infinity.

Theorem 2.4. *The renewal equation (2.10) has one and only one solution; it is*

$$f = U * g.$$

Let us define

$$\mathbb{E}[Z_1] = \int_0^\infty xF(dx) = \int_0^\infty [1 - F(t)]dt,$$

that is, $\mathbb{E}[Z_1]$ is the expected value of the times between successive renewals. Then, we have

Theorem 2.5. *If g is directly Riemann integrable (see Çinlar [15]), $F(\infty) = 1$, and F is not arithmetic, then*

$$\lim_{t \rightarrow \infty} U * g(t) = \frac{1}{\mathbb{E}[Z_1]} \int_0^\infty g(y)dy.$$

If $F(\infty) = 1$, F is arithmetic with span δ , and $\sum_{k=0}^\infty g(x + k\delta)$ converges, then

$$\lim_{n \rightarrow \infty} U * g(x + n\delta) = \frac{\delta}{\mathbb{E}[Z_1]} \sum_{k=0}^\infty g(x + k\delta).$$

Theorem 2.6. *If $F(\infty) < 1$, then*

$$\lim_{t \rightarrow \infty} U * g(t) = U(\infty)g(\infty)$$

provided that $g(\infty) = \lim_{t \rightarrow \infty} g(t)$ exists, where $U(\infty) = \lim_{t \rightarrow \infty} U(t)$.

Correspondingly, we can define a Markov renewal process as follows. Let Y_n be a random variable with a countable state space E and Z_n be a random variable taking values in $\mathbb{R}_0 = [0, +\infty)$ such that $0 = Z_0 \leq Z_2 \leq \dots$.

Definition 2.4. *The stochastic process $\{Z_n, Y_n : n = 0, 1, \dots\}$ is called a Markov renewal process with state space E if*

$$\mathbb{P}(Z_{n+1} - Z_n \leq t, Y_{n+1} = j | Z_0, \dots, Z_n; Y_1, \dots, Y_n) = \mathbb{P}(Z_{n+1} - Z_n \leq t, Y_{n+1} = j | Y_n)$$

for all $n = 0, 1, \dots, j \in E$, and $t \in \mathbb{R}_0$.

Here, we assume that $\{Z_n, Y_n : n = 0, 1, \dots\}$ is time-homogeneous: that is, for any $i, j \in E$, $t \in \mathbb{R}_0$,

$$\mathbb{P}(Z_{n+1} - Z_n \leq t, Y_{n+1} = j | Y_n = i) = Q(t, i, j)$$

independent of n . we call $\{Q(t, i, j) : t \in \mathbb{R}_0, i, j \in E\}$ a semi-Markov kernel over E .

Remark 2.5. *If E consists of a single point, then $\{Z_n : n = 0, 1, \dots\}$ is a renewal process.*

Remark 2.6. *Let $j \in E$ be fixed, and define $Z_0^j, Z_1^j, \dots$ as the successive Z_n for which $Y_n = j$. Then $\{Z_n^j : n = 0, 1, \dots\}$ is a renewal process.*

Define the conditional probability

$$\mathbb{P}(Z_n \leq t, Y_n = j | Y_0 = i) = \mathbb{P}_i(Z_n \leq t, Y_n = j), \quad t \in \mathbb{R}_0, i, j \in E,$$

and let

$$Q^n(t, i, j) = \mathbb{P}_i(Z_n \leq t, Y_n = j), \quad t \in \mathbb{R}_0, i, j \in E,$$

for all $n = 0, 1, \dots$. Then

$$Q^0(t, i, j) = \mathbf{1}(i, j)$$

for all $t \geq 0$; and for all $n \geq 0$ we have the recursive relation:

$$Q^{n+1}(t, i, j) = \sum_{k \in E} \int_0^t Q(ds, i, k) Q^n(t - s, k, j).$$

We denote the expected number of renewals in a finite interval $[0, t]$ by $U(t, i, j)$, then

$$\begin{aligned}
 U(t, i, j) &= \mathbb{E}_i \left[\sum_{n=0}^{\infty} \mathbf{1}_j(Y_n) \mathbf{1}_{[0, t]}(Z_n) \right] \\
 &= \sum_{n=0}^{\infty} \mathbb{E}_i [\mathbf{1}_j(Y_n) \mathbf{1}_{[0, t]}(Z_n)] \\
 &= \sum_{n=0}^{\infty} \mathbb{P}_i(Z_n \leq t, Y_n = j) \\
 &= \sum_{n=0}^{\infty} Q^n(t, i, j),
 \end{aligned} \tag{2.11}$$

for any $i, j \in E$ and $t < \infty$. We call the functions $t \rightarrow U(t, i, j)$ Markov renewal functions.

Let

$$F(t, i, j) = \mathbb{P}_i(Z_0^j \leq t), \quad i \neq j,$$

that is, the distribution of the first passage time from state i to state j , and let

$$F(t, j, j) = \mathbb{P}_j(Z_1^j \leq t), \quad i \neq j,$$

that is, the distribution of the time between successive occurrences of j (note that $\mathbb{P}_j(Z_0^j = 0) = 1$). Then

$$U(t, j, j) = \sum_{n=0}^{\infty} F^n(t, j, j),$$

and

$$U(t, i, j) = \int_0^t F(ds, i, j) U(t-s, j, j), \quad i \neq j.$$

Definition 2.5. State $j \in E$ is said to be recurrent, transient, aperiodic, or periodic with period δ respectively if the corresponding renewal process $\{Z_n^j - Z_0^j : n = 0, 1, \dots\}$ is recurrent, transient, aperiodic, or periodic with period δ .

Remark 2.7. State j is recurrent if and only if j is recurrent for the Markov chain $\{Y_n\}$; similarly, j is transient if and only if j is transient for the Markov chain $\{Y_n\}$.

Next, we discuss the Markov renewal equation. Let $\mathbb{B}$ denote the class of functions $f : \mathbb{R}_0 \times E \rightarrow \mathbb{R}$ satisfying that for every $i \in E$, the function $t \rightarrow f(t, i)$ is bounded over finite intervals, and for every fixed $t \in \mathbb{R}_0$, the function $i \rightarrow f(t, i)$ is bounded. In particular, we note that for any fixed $j \in E$, the functions $(t, j) \rightarrow Q(t, i, j)$ and $(t, j) \rightarrow U(t, i, j)$ both belong to $\mathbb{B}$. A function $f \in \mathbb{B}$ is said to satisfy a Markov renewal equation if for all $t \in \mathbb{R}_0$ and $i \in E$,

$$f(t, i) = g(t, i) + \sum_{j \in E} \int_0^t Q(ds, i, j) f(t - s, j),$$

for some function $g \in \mathbb{B}$. For convenience, denote $\sum_{j \in E} \int_0^t Q(ds, i, j) f(t - s, j)$ by $Q * f(t, i)$, we then can rewrite the Markov renewal equation as

$$f = g + Q * f, \quad f, g \in \mathbb{B}. \quad (2.12)$$

To avoid certain unpleasant difficulties, we only consider the situation when f and g are nonnegative functions. We denote this class as $\mathbb{B}_+$. Similar to a renewal equation, $U * g$ is still a solution in the Markov renewal case.

Theorem 2.7. *The Markov renewal equation (2.12) has a solution $U * g$. Every solution f is of the form*

$$f = U * g + h \quad (2.13)$$

where h satisfies

$$h = U * h, \quad h \in \mathbb{B}_+.$$

Corollary 2.8. *If E is finite, then*

$$f = U * g$$

is the only solution of (2.12).

Remark 2.8. *There are some criteria for deciding when $U * g$ is the unique solution. For details, see Çinlar [15].*

Remark 2.9. *If state j is transient, the renewal function $U(t, j, j)$ approaches a finite limit $U(i, j)$ as $t \rightarrow \infty$.*

Let σ be an invariant measure for $\{Y_n\}$ and

$$\mathbb{E}_i[Z_1] = \int_0^\infty (1 - \sum_k Q(t, i, k)) dt,$$

that is, the mean sojourn time in state i .

Theorem 2.9. *Assume that $\{Z_n, Y_n\}$ is irreducible recurrent, and let σ be an invariant measure for $\{Y_n\}$ and $\mathbb{E}_i(Z_1)$ be the mean sojourn time in state i . If g is directly integrable (see Çinlar [15]) with respect to σ , then*

$$\lim_{t \rightarrow \infty} \sum_{j \in E} \int_0^t U(ds, i, j) g(t - s, j) = \frac{1}{\sigma \mathbb{E}[Z_1]} \int_0^\infty \sigma g(s) ds$$

in the aperiodic case; and

$$\lim_{n \rightarrow \infty} \sum_{j \in E} \int_0^{x+n\delta} U(ds, i, j) g(t - s, j) = \frac{\delta}{\sigma \mathbb{E}[Z_1]} \sum_{k=0}^\infty \sigma g(x - \delta_{i,j} + k\delta)$$

in the case where the states are periodic with period δ . Here, $\delta_{i,j}$ is the first jump point of the distribution $F(\cdot, i, j)$ of the first passage time from i to j .

Theorem 2.10. *If state j transient and $i \neq j$, then*

$$\lim_{t \rightarrow \infty} U(t, i, j) = F(i, j) U(j, j),$$

where $F(i, j) = \lim_{t \rightarrow \infty} F(t, i, j)$ and $U(i, j) = \lim_{t \rightarrow \infty} U(t, i, j)$.

Chapter 3

The decay rate in a weak sense

In this chapter, we consider the decay rate in a weak sense of the stationary distribution $\{\pi_n\}$ for the $GI/G/1$ -type Markov chain with finitely many background states whose transition matrix P is given by (2.2). We assume that P is aperiodic, irreducible and positive recurrent. We also assume that $\{A_n\}$ and $\{B_n\}$ are light-tailed in this chapter. We have already known that this condition is equivalent to that $\{\pi_n\}$ is light-tailed. For each $k \in \{1, 2, \dots, m\}$, the upper decay rate in a weak sense $e^{-\bar{\Lambda}_k}$ of $\{\pi_n\}$ along the level direction and the lower decay rate in a weak sense $e^{-\underline{\Lambda}_k}$ of $\{\pi_n\}$ along the level direction can be defined by

$$e^{-\bar{\Lambda}_k} = \limsup_{n \rightarrow \infty} (\pi_{nk})^{\frac{1}{n}}$$

and

$$e^{-\underline{\Lambda}_k} = \liminf_{n \rightarrow \infty} (\pi_{nk})^{\frac{1}{n}},$$

respectively, where π_{nk} is the k th element of the vector π_n . If $\bar{\Lambda}_k = \underline{\Lambda}_k \triangleq \Lambda_k$, then $e^{-\Lambda_k}$ is referred to as the decay rate in a weak sense of $\{\pi_{nk}\}$ along the level direction. In particular, if $e^{-\Lambda_k}$ is independent of k , we denote it by $e^{-\Lambda}$, which is the decay rate in a

weak sense of $\{\pi_n\}$ along the level direction. It is noted that the above definitions are equivalent to

$$-\overline{\Lambda}_k = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \pi_{nk} = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \bar{\pi}_{nk}$$

and

$$-\underline{\Lambda}_k = \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \pi_{nk} = \liminf_{n \rightarrow \infty} \frac{1}{n} \ln \bar{\pi}_{nk},$$

where $\bar{\pi}_{nk} = \sum_{l=n}^{\infty} \pi_{lk}$. In the following, we discuss the decay rate in a weak sense of the $GI/G/1$ -type Markov chain under the assumption: A is irreducible. The results are divided into two possible cases corresponding to Property 1 and Property 2, respectively.

Firstly, we assume that Property 1 is satisfied, or there exists an $\alpha > 1$ such that $\gamma(\alpha) = 1$, i.e. $A(\alpha)$ is 1-positive recurrent. Then the following conclusion can be obtained.

Theorem 3.1. *Assume that Property 1 holds with $z = \alpha$.*

(i) *If $\alpha \leq \phi_{B^+}$, then the decay rate in a weak sense of $\{\pi_n\}$ along the level direction is given by*

$$e^{-\Lambda} = \frac{1}{\alpha};$$

(ii) *If $\alpha > \phi_{B^+}$, Q_1 is irreducible, and all elements of $\pi(z)$ have an identical convergent radius, then the decay rate in a weak sense of $\{\pi_n\}$ along the level direction is given by*

$$e^{-\Lambda} = \frac{1}{\phi_{B^+}}.$$

Remark 3.1. *For the $M/G/1$ -type Markov chain under the positivity assumption and the $GI/M/1$ -type Markov chain, all elements of $\pi(z)$ have an identical convergent radius.*

Proof: (i) For an arbitrary $1 < z < \alpha$, we have that

$$\begin{aligned}\pi(z) &= \pi_0 R_0(z) [I - R(z)]^{-1} \\ &= \frac{1}{\det(I - R(z))} \pi_0 R_0(z) \operatorname{adj}(I - R(z)),\end{aligned}\tag{3.1}$$

where $\operatorname{adj}(I - R(z))$ denotes the adjoint of $I - R(z)$. Let $\mathbf{s}(z)$ and $\mathbf{t}^t(z)$ denote the left and right eigenvectors of $R(z)$ corresponding to the eigenvalue $r(z)$. Then, from Lemma 5 of [63], we know that $\mathbf{s}(\alpha) > \mathbf{0}$ and $\mathbf{t}^t(\alpha) \geq \mathbf{0}$ but $\mathbf{t}^t(\alpha) \neq \mathbf{0}$. We also denote all other eigenvalues of $R(z)$ by $r_i(z)$, $2 \leq i \leq u$, where u is the number of the eigenvalues of $R(z)$. By Lemma 6 in [63], we know that

$$\operatorname{adj}(I - R(\alpha)) = \prod_{i=2}^u [1 - r_i(\alpha)] \mathbf{t}^t(\alpha) \mathbf{s}(\alpha).$$

Let $z \uparrow \alpha$ in (3.1), we then have

$$\begin{aligned}\pi(\alpha) &= \lim_{z \rightarrow \alpha} \frac{1}{\det(I - R(z))} \pi_0 R_0(z) \operatorname{adj}(I - R(z)) \\ &= \lim_{z \rightarrow \alpha} \frac{1}{[1 - r(z)] \prod_{i=2}^u [1 - r_i(z)]} \pi_0 R_0(z) \prod_{i=2}^u [1 - r_i(z)] \mathbf{t}^t(z) \mathbf{s}(z) \\ &= \lim_{z \rightarrow \alpha} \frac{1}{1 - r(z)} \pi_0 R_0(z) \mathbf{t}^t(z) \mathbf{s}(z).\end{aligned}\tag{3.2}$$

Since $\pi_0 R_0(\alpha) > \mathbf{0}$, we have that $\pi_0 R_0(\alpha) \mathbf{t}^t(\alpha) \mathbf{s}(\alpha) > \mathbf{0}$. But $r(\alpha) = 1$, so

$$\pi(\alpha) = (\infty, \infty, \dots, \infty)$$

element-wise. Therefore, fixing $\mathbf{e} < \mathbf{a} = (a_1, a_2, \dots, a_m)$, we have that, for any n ,

$$\sum_{k=n}^{\infty} \alpha^k \pi_k > \mathbf{a}.$$

Hence, $\sum_{k=n}^{\infty} \alpha^k \pi_{kp} > a_p$, for $p = 1, 2, \dots, m$.

So,

$$\begin{aligned}
-\underline{\Lambda}_p - \ln \frac{1}{\alpha} &= \liminf \ln \sqrt[n]{\pi_{np}} - \ln \frac{1}{\alpha} \\
&= \liminf \ln \sqrt[n]{\alpha^n \pi_{np}} \\
&= \liminf \ln \sqrt[n]{\sum_{k=n}^{\infty} \alpha^k \pi_{kp}} \\
&\geq \liminf \ln \sqrt[n]{a_p} \\
&\geq 0.
\end{aligned} \tag{3.3}$$

Thus, $e^{-\underline{\Lambda}_p} \geq \frac{1}{\alpha}$. On the other hand, it is easy to know from the definition of the upper decay rate that $e^{-\bar{\Lambda}_p} = \frac{1}{\alpha}$ since α is the minimal number in the range $(1, \infty)$ such that $\sum_{k=0}^{\infty} \alpha^k \pi_{kp} = \infty$, hence $e^{-\Lambda_p} = \frac{1}{\alpha}$. Since p is arbitrary, we can obtain $e^{-\Lambda} = \frac{1}{\alpha}$.

(ii) The proof is divided into four steps.

Step 1 : Express the convergent radius of $\pi(z)$.

Claim 1a: If $\alpha > \phi_{B+}$, then ϕ_{B+} is the convergent radius of $\pi(z)$.

It is easy to know that ϕ_{B+} is the smallest singularity of $R_0(z)$, and therefore the smallest singularity of $\pi(z)$ since $\alpha > \phi_{B+}$. Then by Pringsheim theorem(see [71]), ϕ_{B+} is the convergent radius of $\pi(z)$.

Step 2 : Find an irreducible class of $R(z)$ and the process can reach any background state of the process from each state of this class. Discuss the relationship between $R(z)$ and $R_0(z)$.

Claim 2a: $R(z)$ has an irreducible class, and from each state of this class, the process can reach any background state of the process.

Let $L_i(A^+)$ be the set of states in L_i corresponding to positive elements of the vector A^+e . By Lemma 2.3,

$$R = \left(\sum_{i=1}^{\infty} A_i, \sum_{i=1}^{\infty} A_{i+1}, \sum_{i=1}^{\infty} A_{i+2}, \dots \right) \hat{Q}_1(\cdot, 1),$$

where $A^+ = A^+(1)$ and $R = R(1)$. Therefore, the j th row of R is zero vector if $(i, j) \notin L_i(A^+)$. Let R_{A^+} be the sub-matrix of R corresponding to the index set $L_i(A^+) \times L_i(A^+)$. Hence, if Q_1 is irreducible, R_{A^+} is a positive matrix and therefore irreducible since $\hat{Q}_1(\cdot, 1)$ is positive. So, we can find an irreducible class of R , and from each state of this class, the process can reach any background state of the process. Similarly, we can construct an irreducible sub-matrix $R_{A^+(z)}$ of $R(z)$ for an appropriate z if Q_1 is irreducible. Let $L_i(A^+(z))$ be the set of states in L_i corresponding to the positive elements of the vector

$$A^+(z)e$$

and

$$\bar{L}_i(A^+(z)) = L_i - L_i(A^+(z)).$$

By suitably changing the order of rows and columns, $R(z)$ can be rewritten as

$$R(z) = \begin{pmatrix} R_{A^+(z)} & \bar{R}_{A^+(z)} \\ 0 & 0 \end{pmatrix},$$

where $\bar{R}_{A^+(z)}$ is the submatrix of $R(z)$ corresponding to the index set $L_i(A^+(z)) \times \bar{L}_i(A^+(z))$.

Then, for $k \geq 1$ we have

$$R^k(z) = \begin{pmatrix} R_{A^+(z)}^k & [R_{A^+(z)}]^{k-1} \bar{R}_{A^+(z)} \\ 0 & 0 \end{pmatrix}.$$

Next, we discuss the relationship between $R(z)$ and $R_0(z)$. By suitably changing the order of rows and columns corresponding to the adjustment of $R(z)$, $R_0(z)$ can be

rewritten as

$$R_0(z) = \begin{pmatrix} R_0^{A^+(z)}(z) & \overline{R}_0^{A^+(z)}(z) \end{pmatrix},$$

where $R_0^{A^+(z)}(z)$ and $\overline{R}_0^{A^+(z)}(z)$ are the sub-matrices of $R_0(z)$, which is corresponding to the partition of $R(z)$.

Claim 2b: There is at least one element in $R_0^{A^+(z)}(z)$ whose convergent radius is the convergent radius of $R_0(z)$.

If this is not true, ϕ_{B^+} is not the convergent radius of $\pi(z)$. This contradicts *Claim 1a*.

Step 3 : Find a lower bound for the lower decay rate in a weak sense.

From (2.7), we know that for an appropriate z ,

$$\begin{aligned} \pi(z) &= \pi_0 R_0(z) [I - R(z)]^{-1} \\ &= \pi_0 R_0(z) \sum_{n=0}^{\infty} R^n(z) \\ &= \pi_0 R_0(z) \sum_{k=1}^{\infty} z^k \sum_{n=0}^{\infty} R_k^{*n}, \end{aligned} \tag{3.4}$$

where

$$R_k^{*n} = \sum_{j=1}^{k-1} R_j^{*(n-1)} R_{k-j}, \quad \text{for } n \geq 2,$$

$$R_k^{*1} = R_k,$$

and

$$R_k^{*0} = I.$$

It is easy to know from (3.4)

$$\sum_{k=l}^{\infty} z^k \pi_k \geq \pi_0 z^l R_{0,l} \sum_{k=1}^{\infty} z^k \sum_{n=0}^{\infty} R_k^{*n}.$$

Taking $z = 1$ on both sides of the above inequality, we have

$$\sum_{k=l}^{\infty} \pi_k \geq \pi_0 R_{0,l} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} R_k^{*n}.$$

Without loss of generality, we assume that $[R_{0,l}]_{ij}$ is an element such that the convergent radius of $\sum_{l=1}^{\infty} z^l [R_{0,l}]_{ij}$ is the convergent radius of $R_0(z)$. By *Claim 2b*, the location of $[R_{0,l}]_{ij}$ in $R_{0,l}$ corresponds to the partition $R_0^{A^+(z)}(z)$. Furthermore from *Claim 2a*, there some $n_0 \geq 1$ such that for each background state p , $[R^{n_0}]_{jp} > 0$, and thus $[\sum_{n=0}^{\infty} R^n]_{jp} > 0$.

Then we have

$$(\sum_{k=l}^{\infty} \pi_{kp})^{\frac{1}{n}} \geq (\pi_{0i} [R_{0,l}]_{ij} [\sum_{k=1}^{\infty} \sum_{n=0}^{\infty} R_k^{*n}]_{jp})^{\frac{1}{n}}.$$

Hence we have

$$e^{-\Lambda_p} \geq \frac{1}{\phi_{B^+}}.$$

Step 4 : Determine the decay rate in the weak sense.

From *Claim 1a*, we know that ϕ_{B^+} is the convergent radius of $\pi(z)$. Hence by the definition of the upper decay rate in the weak sense,

$$e^{-\bar{\Lambda}_p} = \frac{1}{\phi_{B^+}}.$$

So, we have for each p ,

$$e^{-\Lambda_p} = \frac{1}{\phi_{B^+}}.$$

Since p is arbitrary, we obtain

$$e^{-\Lambda} = \frac{1}{\phi_{B^+}}.$$

□

Secondly, we discuss the weak decay rate under Property 2.

Theorem 3.2. Assume that Property 2 holds, i.e., for all $z > 1$, $\gamma(z) \neq 1$.

- (i) If $\phi_{A+} \leq \phi_{B+}$, then $\phi_{\pi} = \phi_{A+}$. Moreover, if all elements of $\pi(\phi_{A+})$ are infinite, then the decay rate in the weak sense of $\{\pi_n\}$ along the level direction is given by

$$e^{-\Lambda} = \frac{1}{\phi_{A+}};$$

- (ii) If $\phi_{A+} > \phi_{B+}$, Q_1 is irreducible, and all elements of $\pi(z)$ have an identical convergent radius, then the decay rate in the weak sense of $\{\pi_n\}$ along the level direction is given by

$$e^{-\Lambda} = \frac{1}{\phi_{B+}}.$$

Proof: (i) We only need to prove ϕ_{A+} is the convergent radius of $\pi(z)$, and the other part of the proof is same to the one in Theorem 3.1(i).

It is easy to know that ϕ_{A+} is the smallest singularity of $R(z)$, and therefore the smallest singularity of $\pi(z)$ since $\phi_{A+} \leq \phi_{B+}$. Then by Pringsheim theorem, ϕ_{A+} is the convergent radius of $\pi(z)$.

- (ii) The proof can be obtained by the same arguments to the ones in Theorem 3.1(ii).

□

Chapter 4

Exact asymptotics

In this chapter, we analyze exact asymptotics for the $GI/G/1$ -type Markov chain with finitely many background states whose transition matrix P is given by (2.2). We assume that P is aperiodic, irreducible and positive recurrent. We also assume that $\{A_n\}$ and $\{B_n\}$ are light-tailed for this chapter's discussion. We have already known that this condition is equivalent to $\{\pi_n\}$ is light-tailed. We firstly discuss the exact geometric decay of the stationary distribution $\{\pi_n\}$ and obtain sufficient conditions for the exact geometric decay. However, in many cases $\{\pi_n\}$ does not has the exact geometric decay rate. Thus we also analyze conditions under which $\{\pi_n\}$ has light-tailed but not exact geometric decay, and furthermore provide sufficient conditions for other exact asymptotics when $\{\pi_n\}$ has light-tailed but not exact geometric decay.

4.1 Exact geometric decay

In this section, we discuss the exact geometric decay of the stationary distribution $\{\pi_n\}$. We divide our discussion into several cases based on whether there exists an $\alpha > 1$ such that $\gamma(\alpha) = 1$ and $A(\alpha) < \infty$, that is, for the given α , $A(\alpha)$ is 1-positive recurrent. Before giving the main result, we introduce a useful Markov additive approach. We consider the Markov additive process whose transition matrix is $\{A_k : k \in \mathbb{Z}\}$ with $\mathbb{Z}$ being the set of all integers, which was studied in [76]. Let $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ be a sequence of random vectors with the transition function:

$$\begin{aligned} & \mathbb{P}(X_{n+1} - X_n = k, Y_{n+1} = j | X_p, Y_p, p = 1, 2, \dots, n) \\ &= A_k(Y_n, j) \quad \text{for } n \geq 0, j = 1, 2, \dots, m, k \in \mathbb{Z}. \end{aligned}$$

Then, $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ is called the Markov additive process with additive component X_n and background state Y_n .

For the remaining discussion, we introduce the definition of d -arithmetic here. We call the Markov additive process $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ d -arithmetic, if $d \geq 0$ is the maximal number for which there exists a measurable function $\omega : \{1, 2, \dots, m\} \rightarrow \{0, 1, \dots, d-1\}$ (if $d = 0$, this set becomes $\{0\}$), called shift function, such that

$$\mathbb{P}(X_1 - X_0 \in \omega(i) - \omega(j) + d\mathbb{Z} | Y_0 = i, Y_1 = j) = 1, \quad \text{for all } i, j = 1, 2, \dots, m.$$

Without additional assumptions, we only discuss the case in which the Markov additive process $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ is 1-arithmetic throughout this chapter. Since X_n is integer valued, we can equivalently state for this condition that there is no integer $d > 1$ and function $w : \{1, 2, \dots, p\} \rightarrow \{0, 1, \dots, d-1\}$ such that

$$A_n(i, j) > 0 \Rightarrow n - w(i) + w(j) \in d\mathbb{Z} \quad \text{for all } i, j = 1, 2, \dots, p.$$

The proof is easy. Let $O_{A_n(i,j)}$ represent the event: $\{X_1 - X_0 = n, Y_1 = j | Y_0 = i\}$ and O_n represent the event: $\{X_1 - X_0 - w(i) + w(j) \in d\mathbb{Z}\} \cap \{X_1 - X_0 = n\}$, for all $i, j = 1, 2, \dots, p$. We then have that $\mathbb{P}(\bigcup_{n=0}^{\infty} O_n) \geq \mathbb{P}(\bigcup_{n=0}^{\infty} O_{A_n(i,j)})$, which leads to $\mathbb{P}(X_1 \in \omega(i) - \omega(j) + d\mathbb{Z} | Y_0 = i, Y_1 = j) = \mathbb{P}(\bigcup_{n=0}^{\infty} O_n) = 1$, for all $i, j = 1, 2, \dots, p$. Therefore, by the definition of 1-arithmetic, we can get the above equivalent condition.

Let

$$\tau_k = \inf\{n > 0 : X_n - X_0 \leq k\}, \quad k \in \mathbb{Z},$$

where $\tau_k = \infty$ if there is no such n that $X_n - X_0 \leq k$. It is easy to know that τ_k is the hitting time of X_n at the set $(-\infty, k]$ when $X_0 = 0$. Then, we have that

$$\begin{aligned} R_k(i, j) &= \mathbb{E} \left[\mathbf{1}(\tau_{k-1} < \infty) \sum_{n=1}^{\tau_{k-1}-1} \mathbf{1}(X_n - X_0 = k, Y_n = j | Y_0 = i) \right], \quad k \geq 1, \\ G_k(i, j) &= \mathbb{P}(X_{\tau_{-1}} - X_0 = k, Y_{\tau_{-1}} = j | Y_0 = i), \quad k \leq -1, \end{aligned}$$

where $\mathbf{1}(\cdot)$ is the indicator function. We now assume that Property 1 holds, and let α , given in Proposition 2.1, be the number such that $A(\alpha)$ is 1-positive recurrent. Define

$$A_n^{(\alpha)} = \alpha^n \Delta_{\nu(\alpha)}^{-1} A_n \Delta_{\nu(\alpha)},$$

where $\nu^t(\alpha)$ is the right eigenvector of $A(\alpha)$ and $\Delta_{\nu(\alpha)}$ is the diagonal matrix whose i th diagonal entry is the corresponding entry of the vector $\nu(\alpha)$. Then, we have

$$A^{(\alpha)} \triangleq \sum_{n=-\infty}^{\infty} A_n^{(\alpha)}$$

is stochastic and positive recurrent. We call $\{(X_n^{(\alpha)}, Y_n^{(\alpha)}) : n = 0, 1, 2, \dots\}$, whose transition matrix is $\{A_k^{(\alpha)} : k \in \mathbb{Z}\}$, the twisted Markov additive process. From Lemma 4.1 of [76], we know that $R^{(\alpha)}$ - and $G^{(\alpha)}$ -measures have the following properties:

$$R_k^{(\alpha)} = \alpha^k \Delta_{\nu(\alpha)}^{-1} R_k \Delta_{\nu(\alpha)}, \quad k \geq 1$$

and

$$G_k^{(\alpha)} = \alpha^k \Delta_{\nu(\alpha)}^{-1} G_k \Delta_{\nu(\alpha)}, \quad k \leq -1.$$

Let $\{(\tilde{X}_n^{(\alpha)}, \tilde{Y}_n^{(\alpha)}) : n = 0, 1, 2, \dots\}$ be the dual process of the Markov additive process $\{(X_n^{(\alpha)}, Y_n^{(\alpha)}) : n = 0, 1, 2, \dots\}$ whose transition matrix is

$$\tilde{A}_n^{(\alpha)} = \alpha^n \Delta_{\mu(\alpha)}^{-1} A_n^t \Delta_{\mu(\alpha)},$$

where $\mu(\alpha)$ is the left eigenvector of $A(\alpha)$ and $\Delta_{\mu(\alpha)}$ is the diagonal matrix whose i th diagonal entry is the corresponding entry of the vector $\mu(\alpha)$. It is easy to know that $\{(\tilde{X}_n^{(\alpha)}, \tilde{Y}_n^{(\alpha)}) : n = 0, 1, 2, \dots\}$ is stochastically equivalent to $\{(-X_{-n}^{(\alpha)}, Y_{-n}^{(\alpha)}) : n = 0, 1, 2, \dots\}$ under the stationary measure of $A^{(\alpha)}$. Let

$$\tilde{\tau}_k^{(\alpha)} = \inf\{n > 0 : \tilde{X}_n^{(\alpha)} - \tilde{X}_0^{(\alpha)} \geq k\}, \quad k \geq 0$$

be the first hitting time at $[k, \infty)$. Let

$$\tilde{G}_k^{(\alpha)}(i, j) = \mathbb{P}(\tilde{X}_{\tilde{\tau}_1^{(\alpha)}}^{(\alpha)} - \tilde{X}_0^{(\alpha)} = k, \tilde{Y}_{\tilde{\tau}_1^{(\alpha)}}^{(\alpha)} = j | \tilde{Y}_0^{(\alpha)} = i), \quad k \geq 1,$$

then $\{\tilde{G}_k^{(\alpha)}(i, j) : j = 1, 2, \dots, m, k \geq 1\}$ is the ladder height distribution for each $i = 1, 2, \dots, m$. Similarly, we can define the occupation measure $\tilde{R}_k^{(\alpha)}$. Then, by Lemmas 3.2 and 4.1 of [76],

$$\tilde{G}_k^{(\alpha)} = \alpha^k \Delta_{\mu(\alpha)}^{-1} R_k^t \Delta_{\mu(\alpha)}, \quad k \geq 1$$

and

$$\tilde{R}_k^{(\alpha)} = \alpha^k \Delta_{\mu(\alpha)}^{-1} G_k^t \Delta_{\mu(\alpha)}, \quad k \leq -1.$$

Let $\tilde{\pi}_0^{(\alpha)} = \pi_0^t$, $\tilde{\pi}_k^{(\alpha)} = \alpha^k \Delta_{\mu(\alpha)}^{-1} \pi_k^t$ for $k \geq 1$, and $\tilde{G}_{0,k}^{(\alpha)} = \alpha^k \Delta_{\mu(\alpha)}^{-1} R_{0,k}^t$. Then, we have that

$$\tilde{\pi}_n^{(\alpha)} = \tilde{G}_{0,n}^{(\alpha)} \tilde{\pi}_0^{(\alpha)} + \sum_{k=1}^{n-1} \tilde{G}_{n-k}^{(\alpha)} \tilde{\pi}_k^{(\alpha)}, \quad n \geq 1. \quad (4.1)$$

From Lemma 4.2 of [76], we know that $\{\tilde{G}_k^{(\alpha)} : k \geq 1\}$ is a stochastic, positive recurrent and 1-arithmetic kernel. Hence, the equation (4.1) is a proper Markov renewal equation.

Our purpose is to discuss conditions of exact geometric decay of π_n . To this end, let us review Proposition 2.1. By Proposition 2.1, we have α such that $A(\alpha)$ is 1-positive recurrent. we first assume that $\alpha > \phi_{B+}$. Let $\tilde{\pi}_0^{(\phi_{B+})} = \pi_0^t$, $\tilde{\pi}_k^{(\phi_{B+})} = \phi_{B+}^k \Delta_{\mu(\alpha)}^{-1} \pi_k^t$ for $k \geq 1$ and $\tilde{G}_{0,k}^{(\phi_{B+})} = \phi_{B+}^k \Delta_{\mu(\alpha)}^{-1} R_{0,k}^t$. Then, we have that

$$\tilde{\pi}_n^{(\phi_{B+})} = \tilde{G}_{0,n}^{(\phi_{B+})} \tilde{\pi}_0^{(\phi_{B+})} + \sum_{k=1}^{n-1} \tilde{G}_{n-k}^{(\phi_{B+})} \tilde{\pi}_k^{(\phi_{B+})}, \quad n \geq 1. \quad (4.2)$$

Then, equation (4.2) is a defective Markov renewal equation since $\alpha > \phi_{B+}$ and $\{\tilde{G}_k^{(\alpha)}\}$ is stochastic. In this case, the exact geometric decay rate of π_n depends on the limit $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 R_{0,n}$. We have the following conclusions.

Theorem 4.1. *Assume that Property 1 holds with $z = \alpha$. And if $\alpha > \phi_{B+}$ and $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = d > 0$, then π_n has an exact geometric decay.*

The proof will be divided into several lemmas.

Lemma 4.1. *Assume that Property 1 holds with $z = \alpha$. If $\alpha > \phi_{B+}$ and $\lim_{n \rightarrow \infty} \phi_{B+}^n B_n < \infty$, then we have*

$$\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_n = \pi_0 R_{0,\infty}^{(\phi_{B+})} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\phi_{B+})}]^{*p},$$

where $*$ denotes convolution, $R_{0,\infty}^{(\phi_{B+})} = \lim_{n \rightarrow \infty} \phi_{B+}^n R_{0,n}$, $R_n^{(\phi_{B+})} = \phi_{B+}^n R_n$ and all limits are taken element-wise.

Before proving Lemma 4.1, we need the following conclusions.

Lemma 4.2. *If, for a number $\theta > 1$, $\lim_{n \rightarrow \infty} \theta^n B_n = C \geq 0$ but $\neq 0$, then we have $\lim_{n \rightarrow \infty} \theta^n R_{0,n} = C' \geq 0$ but $\neq 0$; The converse is also true.*

Proof: For sufficiency, from Lemma 2.2, we have

$$R_{0,i}(I - \Phi_0) = B_i + \sum_{k=1}^{\infty} R_{0,i+k}(I - \Phi_0)G_k, \quad i \geq 1. \quad (4.3)$$

It is easy to know from equation (4.3) that

$$R_{0,i}(I - \Phi_0) \geq B_i.$$

Therefore, for $\theta > 1$,

$$\lim_{i \rightarrow \infty} \theta^i R_{0,i} \geq \lim_{i \rightarrow \infty} \theta^i B_i (I - \Phi_0)^{-1}.$$

On the other hand, from Lemma 2.4, we have

$$R_{0,i} = B_i(I - \Phi_0)^{-1} + \sum_{k=1}^{\infty} B_{k+i}(I - \Phi_0)^{-1}H_k.$$

Since $\sum_{k=1}^{\infty} (I - \Phi_0)^{-1}H_k$ is finite, it follows by the dominated convergence theorem that for $\theta > 1$,

$$\lim_{i \rightarrow \infty} \theta^i R_{0,i} \leq \lim_{i \rightarrow \infty} \theta^i B_i (I - \Phi_0)^{-1} + \sum_{k=1}^{\infty} \lim_{i \rightarrow \infty} \theta^{k+i} B_{k+i} (I - \Phi_0)^{-1}H_k.$$

Hence, if $\lim_{i \rightarrow \infty} \theta^i B_i$ is finite, so is $\lim_{i \rightarrow \infty} \theta^i R_{0,i}$. So, if, for $\theta > 1$, $\lim_{n \rightarrow \infty} \theta^n B_n = C \geq 0$ but $\neq 0$, then

$$\lim_{n \rightarrow \infty} \theta^n R_{0,n} = C' \geq 0 \text{ but } \neq 0,$$

since $[I - \Phi_0]^{-1}$ exists.

For necessity, assume that for a number θ , $\lim_{n \rightarrow \infty} \theta^n R_{0,n} = C' \geq 0$ but $\neq 0$. Then, we have that there exists a matrix $N > 0$ such that for all n ,

$$\theta^n R_{0,n} < N.$$

Premultiplying the equation (4.3) by θ^i , we get that

$$\theta^i R_{0,i}(I - \Phi_0) = \theta^i B_i + \sum_{k=1}^{\infty} \theta^{i+k} R_{0,i+k}(I - \Phi_0)\theta^{-k}G_k, \quad i \geq 1. \quad (4.4)$$

Since for all $i \geq 1$,

$$\sum_{k=1}^{\infty} \theta^{i+k} R_{0,i+k} (I - \Phi_0) \theta^k G_k \leq N \sum_{k=1}^{\infty} (I - \Phi_0) \theta^{-k} G_k < \infty,$$

we take the limit over n on both sides of the equation (4.4) to obtain

$$C'(I - \Phi_0) = \lim_{n \rightarrow \infty} \theta^i B_i + \sum_{k=1}^{\infty} C'(I - \Phi_0) \theta^{-k} G_k, \quad i \geq 1.$$

Therefore,

$$\lim_{n \rightarrow \infty} \theta^i B_i = C'(I - \Phi_0) [I - \sum_{k=1}^{\infty} \theta^{-k} G_k].$$

Letting $C = C'(I - \Phi_0) [I - \sum_{k=1}^{\infty} \theta^{-k} G_k]$, we obtain

$$\lim_{n \rightarrow \infty} \theta^n B_n = C.$$

Moreover, since $\{B_n\}$ is always nonnegative, so is C . □

Corollary 4.2. *For a number θ , $\lim_{n \rightarrow \infty} \theta^n \pi_0 B_n = c > 0$ if and only if $\lim_{n \rightarrow \infty} \theta^n \pi_0 R_{0,n} = c' > 0$.*

By the same procedure, we can obtain the following conclusions.

Lemma 4.3. *For a number θ , $\lim_{n \rightarrow \infty} \theta^n B_n = 0$ if and only if $\lim_{n \rightarrow \infty} \theta^n R_{0,n} = 0$.*

Lemma 4.4. *For a number θ , $\lim_{n \rightarrow \infty} \theta^n B_n = \infty$ if and only if $\lim_{n \rightarrow \infty} \theta^n R_{0,n} = \infty$.*

Remark 4.1. *Lemma 4.1, 4.3 and 4.4 provide a relationship between the R -measure $\{R_{0,n} : n \geq 1\}$ and the boundary transition block $\{B_n : n \geq 1\}$. In practice, the condition on $\{B_n\}$ can be verified.*

Now, we give a proof of Lemma 4.1.

Proof: For convenience, we replace $\tilde{\pi}_n^{(\phi_{B^+})}$, $\tilde{G}_{0,n}^{(\phi_{B^+})}$ and $\tilde{G}_n^{(\phi_{B^+})}$ with $\pi(n)$, $g_0(n)$ and $g(n)$ in equation (4.2), respectively. Then, we can rewrite equation (4.2) as:

$$\pi(n) = g_0(n)\pi(0) + \sum_{k=1}^{n-1} g(k)\pi(n-k) \quad (4.5)$$

$$= g_0(n)\pi(0) + g * \pi(n), \quad (4.6)$$

where $*$ denotes convolution. By successive iteration in (4.5), we get the solution to (4.5):

$$\pi(n) = g_0(n)\pi(0) + g * g_0(n)\pi(0) + g^{*2} * g_0(n)\pi(0) + \cdots \quad (4.7)$$

$$= \left(\sum_{p=0}^{\infty} g^{*p} \right) * g_0(n)\pi(0), \quad (4.8)$$

where g^{*2} represents $g * g$. Since the inverse of $I - \tilde{G}(\phi_{B^+})$ exists, where $\tilde{G}(\phi_{B^+}) = \sum_{k=1}^{\infty} \tilde{G}_k^{(\phi_{B^+})}$, the Markov chain corresponding to $\tilde{G}(\phi_{B^+})$ is transient. From page 125 and 328 of [15], we know that $\sum_{k=1}^{\infty} \sum_{p=0}^{\infty} g^{*p}(k)$ is finite. Hence, provided that $\lim_{n \rightarrow \infty} \phi_{B^+}^n R_{0,n} < \infty$, it follows by the dominated convergence theorem that

$$\lim_{n \rightarrow \infty} \pi(n) = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} g^{*p}(k) g_0(n)\pi(0).$$

Substitute $\pi(n)$, $g_0(n)$ and $g(n)$ with $\tilde{\pi}_n^{(\phi_{B^+})}$, $\tilde{G}_{0,n}^{(\phi_{B^+})}$ and $\tilde{G}_n^{(\phi_{B^+})}$, respectively, then it follows that

$$\lim_{n \rightarrow \infty} \tilde{\pi}_n^{(\phi_{B^+})} = \lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [\tilde{G}_k^{(\phi_{B^+})}]^{*p} \tilde{G}_{0,n}^{(\phi_{B^+})} \tilde{\pi}_0^{(\phi_{B^+})}. \quad (4.9)$$

Multiplying both sides of the equation (4.9) by $\Delta_{\mu(\alpha)}$ and transposing it, we get

$$\lim_{n \rightarrow \infty} \phi_{B^+}^n \pi_n = \pi_0 R_{0,\infty}^{(\phi_{B^+})} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\phi_{B^+})}]^{*p},$$

where $R_{0,\infty}^{(\phi_{B^+})} = \lim_{n \rightarrow \infty} \phi_{B^+}^n R_{0,n}$ and $R_n^{(\phi_{B^+})} = \phi_{B^+}^n R_n$. Further, from Lemmas 4.3 and 4.4, we get that $\lim_{n \rightarrow \infty} \phi_{B^+}^n R_{0,n} < \infty$ is equivalent to $\lim_{n \rightarrow \infty} \phi_{B^+}^n B_n < \infty$. Therefore,

provided that $\lim_{n \rightarrow \infty} \phi_{B^+}^n B_n < \infty$, we still have that

$$\lim_{n \rightarrow \infty} \phi_{B^+}^n \pi_n = \pi_0 R_{0,\infty}^{(\phi_{B^+})} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\phi_{B^+})}]^{*p}.$$

□

By the above lemmas, the proof of Theorem 4.1 can be obtained as follows.

Proof: It is easy to know from Lemma 4.1 that if $\lim_{n \rightarrow \infty} \phi_{B^+}^n \pi_0 B_n = \mathbf{d} > 0$, we have

$$\lim_{n \rightarrow \infty} \phi_{B^+}^n \pi_n = \mathbf{d}' \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\phi_{B^+})}]^{*p} > \mathbf{0},$$

where $\lim_{n \rightarrow \infty} \phi_{B^+}^n \pi_0 R_{0,n} = \mathbf{d} > 0$. Hence, π_n has exact geometric decay with rate $\frac{1}{\phi_{B^+}}$.

□

Next, we set $\alpha \leq \phi_{B^+}$. We know that $\frac{1}{\alpha}$ is the decay rate of π_n in the weak sense in this case. Combining the Implicit Function Theorem and Theorem 4.1 of [76], we can easily get that $\frac{1}{\alpha}$ is also the exact geometric decay rate. Now we state the conclusion as follows.

Theorem 4.3. *Assume that Property 1 holds with $z = \alpha$. Moreover, if $\alpha < \phi_{A^+}$, $\alpha \leq \phi_{B^+}$ and $B^+(\alpha) < \infty$, then we have*

$$\lim_{n \rightarrow \infty} \alpha^n \pi_n = \mathbf{c},$$

where $\mathbf{c}$ is a positive vector and the limit is taken element-wise.

Proof: By the assumption, we know that $A(\alpha)$ is 1-positive recurrent and $B^+(\alpha) < \infty$. Hence, from Theorem 4.1 of [76], we know that there exists $\mathbf{c}' : \mathbf{0} < \mathbf{c}' < \infty$ such that

$$\lim_{n \rightarrow \infty} \alpha^n \pi_n = \frac{\mathbf{c}'}{\alpha \mu(\alpha) A'(\alpha) \nu(\alpha)},$$

where $A'(\alpha)$ is the first order derivative of $A(z)$ at α and the limit is taken element-wise. It is easy to know that by the Implicit Function Theorem, $\gamma'(\alpha) = \mu(\alpha)A'(\alpha)$ exists. Therefore, let $\mathbf{c} = \frac{\mathbf{c}'}{\alpha\mu(\alpha)A'(\alpha)\nu(\alpha)}$, we get that

$$\lim_{n \rightarrow \infty} \alpha^n \pi_n = \mathbf{c} > \mathbf{0}.$$

□

Remark 4.2. From Theorem 4.1 of [76], we can also know the explicit expression of $\mathbf{c}$.

Thirdly, we obtain another sufficient condition on the exact geometric decay by complex analysis. We state it as follows.

Theorem 4.4. Assume that Property 1 holds with $z = \alpha$. If $\alpha = \phi_{A+}$, and $\alpha < \phi_{B+}$, then π_n has an exact geometric decay.

Before we prove the above theorem, we give a lemma. In order to make the result more general, we temporarily relax the 1-arithmetic assumption. Instead, we assume that the Markov additive process $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ is d -arithmetic for this lemma. Let $\mathbb{C}$ be the set of all complex numbers.

Lemma 4.5. $\det(I - R(z)) = 0$ has d zeros on $|z| = \alpha$ ($z \in \mathbb{C}$), and they are simple zeros.

Proof: In order to prove this lemma, we show that $\det(I - R(z))$ has only finitely many zeros on $|z| = \alpha$, and they are simple zeros. Since for $|z| > 1$,

$$\det(I - R(z)) = 0 \text{ if and only if } \det(I - A(z)) = 0,$$

we show equivalently that $\det(I - A(z))$ has only finitely many zeros on $|z| = \alpha$, and they are simple zeros. For this purpose, we define several concepts. A path from background

state i ($i = j_1$) to background state j ($j = j_{s+1}$) is a set of triples (j_r, j_{r+1}, n_r) , $r = 1, \dots, s$, where $j_r \in \{1, \dots, m\}$ is a background state, $n_r \in \{\dots, -1, 0, 1, \dots\}$ is a level, and $A_{n_r}(j_r, j_{r+1}) > 0$. The displacement of the path is defined as

$$g = n_1 + \dots + n_s.$$

A cycle is a path for which $i = j$. Let $\mathcal{D}$ be the set of all cycle displacements g . Then

$$d = g.c.d.\mathcal{D}$$

is the greatest common divisor of the cycle displacements. We claim

Claim $\det(I - A(z))$ is a function of z^d , that is, $z^d = w^d$ implies $\det(I - A(z)) = \det(I - A(w))$.

We prove the claim as follows. Expand the function $\det(I - A(z))$ in a power series in z by using the definition of the determinant and expanding the entries $[A(z)]_{ij} = \sum_{k=-\infty}^{\infty} z^k A_k(i, j)$ in a power series. The result is a sum of monomials each of which has the form

$$\pm \left[\prod_{i \in \Theta} A_{n(i)}(i, i) \right] \left[\prod_{\sigma(i) \neq i} A_{n(i)}(i, \sigma(i)) \right] z^g,$$

where σ is a permutation of indices $\{1, \dots, m\}$, Θ is a subset of $\{i : \sigma(i) = i\}$, n is a function from $\Theta \cup \{i : \sigma(i) \neq i\}$ to the non-negative integers, and $g = n_1 + \dots + n_s$. Such a term is non-zero only if all the coefficients $A_n(i, j)$ that appear in the product are non-zero. Since every permutation can be written as a product of disjoint cycles, the integer $g \in \mathcal{D}$. Hence, g is a multiple of d , so that $\det(I - A(z))$ is clearly a function of z^d .

Next, we claim

Claim $\det(I - A(\alpha e^{i\theta})) = 0$ if and only if $\alpha e^{i\theta}$ is a d th root of $|z| = \alpha$, that is, $e^{i\theta}$ is a d th root of $|z| = 1$.

We prove the claim as follows. Since $\det(I - A(\alpha)) = 0$ and $\det(I - A(z))$ is a function of z^d , it is clear that any d th root of $|z| = \alpha$ is a singularity of $I - A(z)$. To prove the converse, it is enough to show that if $|\xi| = 1$, $\det(I - A(\alpha\xi)) = 0$, and $g \in \mathcal{D}$, then $\xi^g = 1$. When $I - A(\alpha\xi)$ is singular, the right null space has dimension at least 1, and we let $\mathbf{c}^t$ be any non-trivial right null vector of $I - A(\alpha\xi)$. Let i be the index of $\mathbf{c}^t$ with the largest modulus. Then, the equation $(I - A(\alpha\xi))\mathbf{c}^t = \mathbf{0}^t$ implies

$$c_i = \sum_{j=1}^m \sum_{n=-\infty}^{\infty} \alpha^n A_n(i, j) \xi^n c_j.$$

Because $\sum_{j=1}^m \sum_{n=-\infty}^{\infty} \alpha^n A_n(i, j) = 1$ (without loss of generality), $|\xi| = 1$, $|c_j| \leq |c_i| \neq 0$, and $A_n(i, j) \geq 0$, it follows that $\frac{\xi^n c_j}{c_i} = 1$ when $A_n(i, j) > 0$. since $A(\alpha)$ is irreducible, by repeating the argument we obtain $|c_j| = |c_i|$ for $j = 1, \dots, m$, which shows that the entries of $\mathbf{c}^t$ are of constant modulus. If the background states $j_1, \dots, j_{s+1}$ determine a path with displacement g , then it follows that

$$\xi^g = \xi^{\sum_{r=1}^s n_r} = \frac{c_{j_1}}{c_{j_{s+1}}} \prod_{r=1}^s \xi^{n_r} \frac{c_{j_{r+1}}}{c_{j_r}} = \frac{c_{j_1}}{c_{j_{s+1}}}.$$

If the path is a cycle, then $\xi^g = 1$. Hence, we have $c_j = c_0 \xi^{g_j}$, where g_j is the total level displacement along any path from background state j to background state 1, and so the right null space has dimension 1. $\square$

Corollary 4.5. *In particular, if we assume that the Markov additive process $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ is 1-arithmetic, then $\det(I - R(z)) = 0$ has 1 zero at α on $|z| = \alpha$ ($z \in \mathbb{C}$) when $\alpha \leq \phi_{A+}$, and it is a simple zero.*

To prove Theorem 4.4, we need the following Lemma.

Lemma 4.6. (see [19]) Suppose that $y(z)$ is a meromorphic functions that is analytic at $z = 0$ and has polar singularities at the points $q_1, \dots, q_r$ in the circle $|z| < R$ ($z \in \mathbb{C}$), where $R = (\limsup y_n^{1/n})^{-1}$, that is, $y(z)$ has a representation of the form

$$y(z) = \sum_{j=1}^r \sum_{k=1}^{\lambda_j} \frac{B_{jk}}{(1 - z/q_j)^k} + T(z),$$

where $T(z)$ is analytic in the region $|z| < R$. then let $[z^n]y(z)$ is coefficient extractor which is coefficient of z^n in $y(z)$, for every $\varepsilon > 0$

$$[z^n]y(z) = \sum_{j=1}^r \sum_{k=1}^{\lambda_j} B_{jk} \binom{n+k-1}{k-1} q_j^{-n} + O(R^{-n}(1+\varepsilon)^n).$$

In the following, we provide a proof to Theorem 4.4.

Proof: Let $s(z)$ and $t^t(z)$ denote the left and right eigenvectors of $R(z)$ corresponding to the eigenvalue $r(z)$, where $z \in \mathbb{C}$. Then, from Lemma 5 of [63], we know that $s(\alpha) > 0$ and $t^t(\alpha) \geq 0$ but $t^t(\alpha) \neq 0$. We also denote all other eigenvalues of $R(z)$ by $r_i, 2 \leq i \leq u$, where u is the number of the eigenvalue of $R(z)$. By Lemma 6 in [63], we know that

$$\text{adj}(I - R(z)) = \prod_{i=2}^u [1 - r_i(\alpha)] t^t(\alpha) s(\alpha) + (\alpha - z)H(z),$$

where $\text{adj}(I - R(z))$ denotes the adjoint of $I - R(z)$ and $H(z)$ is an analytic function for $z, 1 < |z| < \alpha$. In this case, we thus have that for $1 < |z| < \alpha$,

$$\begin{aligned} \pi(z) &= \pi_0 R_0(z) [I - R(z)]^{-1} \\ &= \frac{1}{\det(I - R(z))} \pi_0 R_0(z) \text{adj}(I - R(z)) \\ &= \frac{1}{[1 - r(z)] \prod_{i=2}^u [1 - r_i(z)]} \pi_0 R_0(z) \left[\prod_{i=2}^u [1 - r_i(\alpha)] t^t(\alpha) s(\alpha) + (\alpha - z)H(z) \right] \\ &= \frac{1}{(\alpha - z)h(z) \prod_{i=2}^u [1 - r_i(z)]} \pi_0 R_0(z) \left[\prod_{i=2}^u [1 - r_i(\alpha)] t^t(\alpha) s(\alpha) + (\alpha - z)H(z) \right], \end{aligned} \tag{4.10}$$

where $h(z)$ is analytic at $z = \alpha$.

Let

$$L(z) = \frac{\prod_{i=2}^u [1 - r_i(\alpha)] \mathbf{t}^{\mathbf{t}}(\alpha) \mathbf{s}(\alpha)}{h(z) \prod_{i=2}^u [1 - r_i(z)]}.$$

It is easy to know that $L(z)$ is analytic at $z = \alpha$. Then, $L(z)$ can be expanded at $z = \alpha$

$$L(z) = \frac{\mathbf{t}^{\mathbf{t}}(\alpha) \mathbf{s}(\alpha)}{h(\alpha)} + \sum_{k=1}^{\infty} \frac{L^{(k)}(\alpha)}{k!} (\alpha - z)^k,$$

where $L^{(k)}(\alpha) = \frac{d^k}{dz^k} L(z)|_{z=\alpha}$. Therefore, we have

$$\pi(z) = \pi_0 \frac{R_0(z) \mathbf{t}^{\mathbf{t}}(\alpha) \mathbf{s}(\alpha)}{\alpha - z} \frac{1}{h(\alpha)} + \pi_0 R_0(z) \sum_{k=1}^{\infty} \frac{L^{(k)}(\alpha)}{k!} (\alpha - z)^{k-1} + \frac{\pi_0 R_0(z) H(z)}{h(z) \prod_{i=2}^u [1 - r_i(z)]}.$$

Since $\pi_0 R_0(z) \frac{\mathbf{t}^{\mathbf{t}}(\alpha) \mathbf{s}(\alpha)}{h(\alpha)} > 0$, we know by Corollary 4.5 that $\pi(z)$ only has a singular point α on $|z| = \alpha$, and it is a pole of order 1. Hence, by Lemma 4.6, we have

$$\pi_n \sim \alpha^{-n} \mathbf{c}, \quad n \rightarrow \infty.$$

□

In the rest of this section, we assume that Property 2 (see page 19) holds. Under this assumption, we discuss exact geometric decay of stationary distribution $\{\pi_n\}$. To this end, we construct a defective Markov renewal equation and then discuss conditions of exact geometric decay by the renewal approach. Firstly, we assume that $A(\phi_{A+}) < \infty$ and for all $1 < z \leq \phi_{A+}$, $\gamma(z) < 1$ and $\phi_{B+} \leq \phi_{A+}$. Since there do not exist a positive row eigenvector and a column eigenvector for 1 under this assumption, we will construct a dual process by a superregular vector. Let $\boldsymbol{\mu}(\phi_{B+})$ and $\boldsymbol{\nu}^{\mathbf{t}}(\phi_{B+})$ be a row superregular vector and a column superregular vector, respectively. It is easy to know that

$$[I - A(\phi_{B+})]^{-1} < \infty.$$

Thus, all states of the chain associated with the transition matrix $A(\phi_{B+})$ are transient. Hence, we can take $\mu(\phi_{B+})$ and $\nu^t(\phi_{B+})$ to be positive vectors. Let $\{(\tilde{X}_n^{(\phi_{B+})}, \tilde{Y}_n^{(\phi_{B+})}) : n = 0, 1, 2, \dots\}$ be the dual process of the Markov additive process $\{(X_n^{(\phi_{B+})}, Y_n^{(\phi_{B+})}) : n = 0, 1, 2, \dots\}$, whose transition matrix is

$$\tilde{A}_n^{(\phi_{B+})} = \phi_{B+}^n \Delta_{\mu(\phi_{B+})}^{-1} A_n^t \Delta_{\mu(\phi_{B+})}.$$

Let $\tilde{\pi}_0^{(\phi_{B+})} = \pi_0^t$, $\tilde{\pi}_k^{(\phi_{B+})} = \phi_{B+}^k \Delta_{\mu(\phi_{B+})}^{-1} \pi_k^t$ for $k \geq 1$, and $\tilde{G}_{0,k}^{(\phi_{B+})} = \phi_{B+}^k \Delta_{\mu(\phi_{B+})}^{-1} R_{0,k}^t$. Then, we have that

$$\tilde{\pi}_n^{(\phi_{B+})} = \tilde{G}_{0,n}^{(\phi_{B+})} \tilde{\pi}_0^{(\phi_{B+})} + \sum_{k=1}^{n-1} \tilde{G}_{n-k}^{(\phi_{B+})} \tilde{\pi}_k^{(\phi_{B+})}, \quad n \geq 1. \quad (4.11)$$

From the dual version of Theorem 2.1, it is easy to know that $\{\tilde{G}_k^{(\phi_{B+})} : k \geq 1\}$ is sub-stochastic. Then, the equation (4.11) is a defective Markov renewal equation. In this situation, the exact geometric decay of π_n depends on the limit $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n$. The following result can be yielded.

Theorem 4.6. *Assume that $A(\phi_{A+}) < \infty$ and for all $1 < z \leq \phi_{A+}$, $\gamma(z) < 1$, $\phi_{B+} \leq \phi_{A+}$ and $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = d > 0$. Then we have*

$$\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_n = d' \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\phi_{B+})}]^{*p} > 0,$$

where $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 R_{0,n} = d'$, i.e., π_n has an exact geometric decay.

Proof: Similar to Lemma 4.1. □

But, if $\phi_{B+} > \phi_{A+}$, we have the following conclusion.

Theorem 4.7. *Assume that $A(\phi_{A+}) < \infty$ and for all $1 < z \leq \phi_{A+}$, $\gamma(z) < 1$, $\phi_{B+} > \phi_{A+}$. Then we have*

$$\lim_{n \rightarrow \infty} \phi_{A+}^n \pi_n = \pi_0 \times 0 \times \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\phi_{A+})}]^{*p} = 0,$$

$\frac{1}{\phi_{A+}}$ is not exact geometric decay rate of π_n .

Proof: Since $\lim_{n \rightarrow \infty} \phi_{A+}^n B_n = 0$ and $\sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\phi_{A+})}]^{*p} < \infty$, the result can easily be obtained by the dominated convergent theorem. $\square$

However, when $A(\phi_{A+}) = \infty$ and $\gamma(\phi_{A+}) \leq 1$, it is hard to use this method to analyze the exact decay.

4.2 Conditions regarding light-tailed but not exact geometric asymptotics

We characterized exact geometric decay of the stationary vector π_n in Section 4. However, π_n does not always have exact geometric asymptotics under the light-tailed assumption. In this section, we discuss light-tailed but not exact geometric asymptotics and provide an exact light-tailed asymptotic rate. Firstly, we extend Theorem 2.6 of [43] to the $GI/G/1$ -type model. The proof is similar to the one in [43]. We omit it here.

Theorem 4.8. *For the $GI/G/1$ -type Markov chain with finitely many background states, if $\pi_{n,j}$, for $j = 1, 2, \dots, m$, satisfies either of the following two conditions:*

- (i) $\lim_{n \rightarrow \infty} \theta^n \pi_{n,j} = 0$ for all $\theta \leq \zeta$ and $\limsup_{n \rightarrow \infty} \theta^n \pi_{n,j} = \infty$ for all $\theta > \zeta$;
- (ii) $\lim_{n \rightarrow \infty} \theta^n \pi_{n,j} = 0$ for all $\theta < \zeta$ and $\limsup_{n \rightarrow \infty} \theta^n \pi_{n,j} = \infty$ for all $\theta \geq \zeta$,

then, $\pi_{n,j}$ has light-tailed but not exact geometric asymptotics with rate ζ .

In the following, we will provide sufficient conditions for light-tailed but not exact geometric asymptotics. We assume that Property 1 (see page 19) holds, then there exists

an $\alpha > 1$ such that $A(\alpha)$ is 1-positive recurrent. Firstly, we set $\alpha = \phi_{B^+}$. Under some conditions, we can know that $\alpha^n \pi_n$ trends to infinity, which yields the following conclusion.

Theorem 4.9. *Assume that $\alpha = \phi_{B^+}$, $B^+(\alpha) = \infty$, and for any $\theta \geq \alpha$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 B_n = c > 0$; or $= \infty$. Then, we have that π_n has light-tailed but not exact geometric asymptotic with rate α .*

Proof: From the proof of Lemma 4.1, we know that

$$\alpha^n \pi_n = \pi_0 R_{0,n}^{(\alpha)} * \sum_{p=0}^{\infty} [R^{(\alpha)}]^{*p},$$

where $R_{0,n}^{(\alpha)} = \alpha^n R_{0,n}$ and $R_n^{(\alpha)} = \alpha^n R_n$. Then, by Fatou's lemma, we have that

$$\begin{aligned} \liminf \alpha^n \pi_n &= \liminf \pi_0 R_{0,n}^{(\alpha)} * \sum_{p=0}^{\infty} [R^{(\alpha)}]^{*p} \\ &\geq \liminf \alpha^n \pi_0 R_{0,n} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\alpha)}]^{*p}. \end{aligned} \quad (4.12)$$

But, $\sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\alpha)}]^{*p} = \infty$. Hence, if, for any $\theta \geq \alpha$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 R_{0,n} = c' > 0$; or $= \infty$, we have that $\lim_{n \rightarrow \infty} \alpha^n \pi_n = \infty$. From Lemmas 4.2 and 4.4, we know that for a number $\theta \geq \alpha$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 R_{0,n} = c' > 0$; or $= \infty$ is equivalent to that for a number $\theta \geq \alpha$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 B_n = c > 0$; or $= \infty$, respectively. Since for any $\theta < \alpha$, $\lim_{n \rightarrow \infty} \theta^n \pi_n = 0$, by Theorem 4.8, we have that if for any $\theta \geq \alpha$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 B_n = c > 0$; or $= \infty$, π_n has light-tailed but not exact geometric asymptotic with rate α . $\square$

But if $\lim_{n \rightarrow \infty} \alpha^n B_n = 0$ in Theorem 4.9, we give the following result without a proof.

Theorem 4.10. *Assume that $\alpha = \phi_{B^+}$, $B^+(\alpha) = \infty$, $\lim_{n \rightarrow \infty} \alpha^n \pi_n = 0$, and for any $\theta > \alpha$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 B_n = c > 0$; or $= \infty$. Then, we have that π_n has light-tailed but not exact geometric asymptotic with rate α .*

So far, we have discussed sufficient conditions of light-tailed but not exact geometric asymptotics if $\alpha \leq \phi_{B+}$. Next, we will deal with the case: $\alpha > \phi_{B+}$

Theorem 4.11. *If $\alpha > \phi_{B+}$, and $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = \infty$, then we have that π_n has light-tailed but not exact geometric asymptotic with rate $\frac{1}{\phi_{B+}}$.*

Proof: Since $\pi_n = \pi_0 R_{0,n} + \sum_{k=1}^{n-1} \pi_k R_{n-k}$, $n \geq 1$, we have that

$$\pi_n \geq \pi_0 R_{0,n}.$$

Multiplying the above inequality by ϕ_{B+}^n , it follows that

$$\phi_{B+}^n \pi_n \geq \pi_0 \phi_{B+}^n R_{0,n}.$$

Hence, if $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 R_{0,n} = \infty$, we have that

$$\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_n = \infty.$$

Since $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 R_{0,n} = \infty$ is equivalent to $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = \infty$, we have that if $\lim_{n \rightarrow \infty} \phi_{B+}^n B_n = \infty$,

$$\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_n = \infty.$$

But, for all $\theta < \phi_{B+}$, we have that

$$\lim_{n \rightarrow \infty} \theta^n \pi_n = 0.$$

So, Theorem 4.8 yields that if $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = \infty$, then π_n has light-tailed but not exact geometric asymptotic with rate $\frac{1}{\phi_{B+}}$. $\square$

Theorem 4.12. *If $\alpha > \phi_{B+}$, $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = 0$, and for any $\theta > \phi_{B+}$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 B_n = \infty$, then we have that π_n has light-tailed but not exact geometric asymptotic with rate $\frac{1}{\phi_{B+}}$.*

Proof: If $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = 0$, we know from Theorem 4.1 that

$$\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_n = \mathbf{0}.$$

For any $\theta > \phi_{B+}$, by Fatou's lemma, we have

$$\liminf \theta^n \pi_n \geq \liminf \theta^n \pi_0 R_{0,n} \sum_{k=1}^{\infty} \sum_{p=0}^{\infty} [R_k^{(\theta)}]^{*p}.$$

Hence, if for any $\theta > \phi_{B+}$, $\lim_{n \rightarrow \infty} \theta^n \pi_0 R_{0,n} = \infty$, we have that by Theorem 4.8 and Lemma 4.4 that π_n has light-tailed but not exact geometric asymptotic with rate $\frac{1}{\phi_{B+}}$. $\square$

Remark 4.3. *If Property 2 holds, we can, by analogy with exact geometric decay under Property 2, analyze light-tailed but not exact geometric asymptotics.*

4.3 Other exact asymptotic rates of π_n when it has not exact geometric decay rate

The stationary distribution does not always have exact geometric decay rate under the light-tailed assumption. Then, which kinds of asymptotic rates does it have when it has light-tailed but not exact geometric asymptotics? Tauberian theory is a useful tool to analyze asymptotic rates. In this section, we use Tauberian theory to discuss the above questions. Let us cite two theorems as follows.

Lemma 4.7. *Let $f_n \geq 0$ and suppose that*

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

converges for $0 \leq z < 1$. If L varies slowly at infinity and $0 \leq \rho < \infty$, then the properties

$$f(z) \sim \frac{1}{(1-z)^\rho} L\left(\frac{1}{1-z}\right), \quad z \rightarrow 1-, \quad (4.13)$$

and

$$f_0 + f_1 + \cdots + f_{n-1} \sim \frac{1}{\Gamma(\rho+1)} n^\rho L(n), \quad n \rightarrow \infty,$$

are equivalent.

Furthermore, if the sequence $\{f_n\}$ is (ultimately) monotonic and $0 < \rho < \infty$, then (4.13) is equivalent to

$$f_n \sim \frac{1}{\Gamma(\rho)} n^{\rho-1} L(n), \quad n \rightarrow \infty.$$

Lemma 4.8. *Let α be an arbitrary complex number in $\mathbb{C} \setminus \mathbb{Z}_-$. The coefficient of z^n in the function*

$$f(z) = (1-z)^{-\alpha} \left(\frac{1}{z} \log \frac{1}{1-z} \right)^\beta, \quad z \in \mathbb{C},$$

admits for large n a full asymptotic expansion in descending powers of n ,

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} (\log n)^\beta \left(1 + \frac{C_1}{\log n} + \frac{C_2}{\log^2 n} + \frac{C_3}{\log^3 n} + \cdots \right),$$

where $C_k = \binom{\beta}{k} \Gamma(\alpha) \frac{d^k}{ds^k} \frac{1}{\Gamma(s)}|_{s=\alpha}$, $\mathbb{Z}_- = \{-1, -2, \dots\}$ and $[z^n]f(z)$ is coefficient extractor which is coefficient of z^n in $f(z)$.

To discuss our questions, we firstly assume Proposition 2.1 holds with α , that is, $A(\alpha)$ is 1-positive recurrent. We have already known that π_n has exact geometric decay when $\alpha < \phi_{A+}$, $\alpha \leq \phi_{B+}$ and $B^+(\alpha) < \infty$; $\alpha = \phi_{A+}$ and $\alpha < \phi_{B+}$; or $\alpha > \phi_{B+}$ and $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n = \mathbf{d} > 0$ under this assumption. Then we have the following conclusions.

Theorem 4.13. *Assume that Property 1 holds with $z = \alpha$. If $\alpha = \phi_{B+}$, $B^+(\alpha) = \infty$,*

$$B^+(z) \sim \frac{1}{(1-z/\alpha)^\beta} L\left(\frac{1}{1-z/\alpha}\right) D, \quad z \rightarrow \alpha$$

and the sequence $\{B_n\}$ is (ultimately) monotonic and $0 < \beta < \infty$, where D is nonnegative matrix and there is at least one positive element in each column of D , and L a slowly varying function, then we have

$$\pi_n \sim \frac{1}{\Gamma(\beta + 1)} \alpha^{-n} n^\beta L(n) \mathbf{c}, \quad n \rightarrow \infty,$$

where $\mathbf{c}$ is a positive vector.

Proof: Let $\mathbf{s}(z)$ and $\mathbf{t}^t(z)$ denote the left and right eigenvectors of $R(z)$ corresponding to the eigenvalue $r(z)$. Then, from Lemma 5 of [63], we know that $\mathbf{s}(\alpha) > \mathbf{0}$ and $\mathbf{t}^t(\alpha) \geq \mathbf{0}$ but $\mathbf{t}^t(\alpha) \neq \mathbf{0}$. We also denote all other eigenvalues of $R(z)$ by $r_i, 2 \leq i \leq u$, where u is the number of the eigenvalue of $R(z)$. By Lemma 6 in [63], we know that

$$\text{adj}(I - R(z)) = \prod_{i=2}^u [1 - r_i(\alpha)] \mathbf{t}^t(\alpha) \mathbf{s}(\alpha) + (\alpha - z) H(z),$$

where $\text{adj}(I - R(z))$ denotes the adjoint of $I - R(z)$ and $H(z)$ is an analytic function for $z, 1 < |z| < \alpha$. In this case, we thus have that for $1 < |z| < \alpha$,

$$\begin{aligned} \pi(z) &= \pi_0 R_0(z) [I - R(z)]^{-1} \\ &= \frac{1}{\det(I - R(z))} \pi_0 R_0(z) \text{adj}(I - R(z)) \\ &= \frac{1}{[1 - r(z)] \prod_{i=2}^u [1 - r_i(z)]} \pi_0 R_0(z) \left[\prod_{i=2}^u [1 - r_i(\alpha)] \mathbf{t}^t(\alpha) \mathbf{s}(\alpha) + (\alpha - z) H(z) \right] \\ &= \frac{1}{(\alpha - z) h(z) \prod_{i=2}^u [1 - r_i(z)]} \pi_0 R_0(z) \left[\prod_{i=2}^u [1 - r_i(\alpha)] \mathbf{t}^t(\alpha) \mathbf{s}(\alpha) + (\alpha - z) H(z) \right], \end{aligned} \tag{4.14}$$

where $h(z)$ is analytic at $z = \alpha$.

Let

$$L(z) = \frac{\prod_{i=2}^u [1 - r_i(\alpha)] \mathbf{t}^t(\alpha) \mathbf{s}(\alpha)}{h(z) \prod_{i=2}^u [1 - r_i(z)]}.$$

It is easy to know that $L(z)$ is analytic at $z = \alpha$. Then, $L(z)$ can be expanded at $z = \alpha$

$$L(z) = \frac{t^t(\alpha)s(\alpha)}{h(\alpha)} + \sum_{k=1}^{\infty} \frac{L^{(k)}(\alpha)}{k!} (\alpha - z)^k,$$

where $L^{(k)}(\alpha) = \frac{d^k}{dz^k} L(z)|_{z=\alpha}$. Therefore, we have

$$\pi(z) = \pi_0 \frac{R_0(z)}{\alpha - z} \frac{t^t(\alpha)s(\alpha)}{h(\alpha)} + \pi_0 R_0(z) \sum_{k=1}^{\infty} \frac{L^{(k)}(\alpha)}{k!} (\alpha - z)^{k-1} + \frac{\pi_0 R_0(z) H(z)}{h(z) \prod_{i=2}^u [1 - r_i(z)]}.$$

Next, from Lemma 2.2 and Lemma 2.4, we have

$$R_{0,k} = B_k(I - \Phi_0)^{-1} + \sum_{i=1}^{\infty} B_{k+i}(I - \Phi_0)^{-1} H_i.$$

Hence,

$$R_0(z) = B^+(z)(I - \Phi_0)^{-1} + B^+(z)(I - \Phi_0)^{-1} H(z),$$

where $H(z) = \sum_{i=1}^{\infty} z^{-i} H_i$. Therefore,

$$R_0(z) = B^+(z)(I - \Phi_0)^{-1}(I + H(z)).$$

But from the assumption, we know

$$B^+(z) \sim \frac{1}{(1 - z/\alpha)^\beta} L\left(\frac{1}{1 - z/\alpha}\right) D, \quad z \rightarrow \alpha.$$

Hence,

$$R_0(z) \sim \frac{1}{(1 - z/\alpha)^\beta} L\left(\frac{1}{1 - z/\alpha}\right) D(I - \Phi_0)^{-1}(I + H(\alpha)), \quad z \rightarrow \alpha.$$

so

$$\pi(z) \sim \frac{1}{(1 - z/\alpha)^{\beta+1}} L\left(\frac{1}{1 - z/\alpha}\right) \frac{\pi_0 D(I - \Phi_0)^{-1}(I + H(\alpha)) t^t(\alpha) s(\alpha)}{h(\alpha)}.$$

Hence, by Lemma 4.7, we can obtain

$$\pi_n \sim \frac{1}{\Gamma(\beta + 1)} \alpha^{-n} n^\beta L(n) c, \quad n \rightarrow \infty,$$

since

$$\frac{\pi_0 D(I - \Phi_0)^{-1}(I + H(\alpha)) t^t(\alpha) s(\alpha)}{h(\alpha)} > 0.$$

□

Theorem 4.14. *Assume that Property 1 holds with $z = \alpha$. If $\alpha = \phi_{B^+}$, $B^+(\alpha) = \infty$,*

$$B^+(z) = (1 - z/\alpha)^{-\rho} \left(\frac{1}{z/\alpha} \log \frac{1}{1 - z/\alpha} \right)^\beta D, \quad z \in \mathbb{C},$$

where D is nonnegative matrix and there is at least one positive element in each column of D , admits for large n a full asymptotic expansion in descending powers of n , then we have

$$\pi_n \sim \alpha^{-n} \frac{n^{\rho-1}}{\Gamma(\rho)} (\log n)^\beta \left(1 + \frac{C_1}{\log n} + \frac{C_2}{\log^2 n} + \frac{C_3}{\log^3 n} + \dots \right) \mathbf{c},$$

where $C_k = \begin{pmatrix} \beta \\ n \end{pmatrix} \Gamma(\rho) \frac{d^k}{ds^k} \frac{1}{\Gamma(s)}|_{s=\rho}$ and $\mathbf{c}$ is positive vector.

Proof: From the proof of Theorem 4.13, we know

$$\pi(z) = \pi_0 \frac{R_0(z) \mathbf{t}^t(\alpha) \mathbf{s}(\alpha)}{\alpha - z} \frac{1}{h(\alpha)} + \pi_0 R_0(z) \sum_{k=1}^{\infty} \frac{L^{(k)}(\alpha)}{k!} (\alpha - z)^{k-1} + \frac{\pi_0 R_0(z) H(z)}{h(z) \prod_{i=2}^u [1 - r_i(z)]}.$$

Next, from Lemma 2.2 and Lemma 2.4, we have

$$R_{0,k} = B_k(I - \Phi_0)^{-1} + \sum_{i=1}^{\infty} B_{k+i}(I - \Phi_0)^{-1} H_i.$$

Hence,

$$R_0(z) = B^+(z)(I - \Phi_0)^{-1} + B^+(z)(I - \Phi_0)^{-1} H(z),$$

where $H(z) = \sum_{i=1}^{\infty} z^{-i} H_i$. Therefore,

$$R_0(z) = B^+(z)(I - \Phi_0)^{-1}(I + H(z)).$$

But from the assumption, we know

$$B^+(z) = (1 - z/\alpha)^{-\rho} \left(\frac{1}{z/\alpha} \log \frac{1}{1 - z/\alpha} \right)^\beta D, \quad z \rightarrow \alpha, \quad z \in \mathbb{C}.$$

Hence,

$$R_0(z) = (1 - z/\alpha)^{-\rho} \left(\frac{1}{z/\alpha} \log \frac{1}{1 - z/\alpha} \right)^\beta D(I - \Phi_0)^{-1}(I + H(\alpha)), \quad z \rightarrow \alpha, \quad z \in \mathbb{C}.$$

So

$$\pi(z) \sim \frac{1}{(1 - z/\alpha)^{\rho+1}} \left(\frac{1}{z/\alpha} \log \frac{1}{1 - z/\alpha} \right)^\beta \frac{\pi_0 D(I - \Phi_0)^{-1} (I + H(\alpha)) \mathbf{t}^t(\alpha) \mathbf{s}(\alpha)}{h(\alpha)}.$$

Hence, by Lemma 4.8, we can obtain

$$\pi_n \sim \alpha^{-n} \frac{n^{\rho-1}}{\Gamma(\rho)} (\log n)^\beta \left(1 + \frac{C_1}{\log n} + \frac{C_2}{\log^2 n} + \frac{C_3}{\log^3 n} + \dots \right) \mathbf{c},$$

where $C_k = \binom{\beta}{n} \Gamma(\rho) \frac{d^k}{ds^k} \frac{1}{\Gamma(s)} \big|_{s=\rho}$ since

$$\frac{\pi_0 D(I - \Phi_0)^{-1} (I + H(\alpha)) \mathbf{t}^t(\alpha) \mathbf{s}(\alpha)}{h(\alpha)} > 0.$$

□

Remark 4.4. If $\alpha > \phi_{B+}$ but $\lim_{n \rightarrow \infty} \phi_{B+}^n \pi_0 B_n \nrightarrow \mathbf{d} > 0$, we can get corresponding results by a similar argument.

Remark 4.5. If Property 2 holds, and $\phi_{A+} > \phi_{B+}$, we can get similar results.

4.4 Summary

In this section, we briefly outline our results in this chapter. We draw several trees for the outline. We denote an exact geometric decay by E and light-tailed but not exact asymptotics by L.

1. Parent tree under Property 1

1(a). Child tree 1

Note*: Some results can be reached, which are not included in this thesis.

1(b). Child tree 2

Note**: Some results can be reached, which are not included in this thesis.

2. Parent tree under Property 2

Depends on properties of B_n or $B(z)$ #

Note#: Some results can be reached, which are not included in this thesis.

2(a). Child tree 1

Note##: Some results can be reached, which are not included in this thesis.

2(b). Child tree 2

Note###: Some results can be reached, which are not included in this thesis.

Chapter 5

Ergodicity for $GI/G/1$ -type Markov chain

In this chapter, we discuss polynomial ergodicity for the $GI/G/1$ -type Markov chain with finitely many background states whose transition matrix P is given by (2.2). In Højgaard and Møller [44], a sufficient condition is given for the $GI/G/1$ -type Markov chain with finitely many background states. In this chapter, we mainly provide two necessary and sufficient conditions to polynomial ergodicity for this kind of Markov chain. We assume that P is aperiodic, irreducible and positive recurrent.

Let $\tau_i = \inf\{n \geq 1 : (X_n, Y_n) \in L_i\}$, for $i = 0, 1, \dots$, which is the first time for the chain to enter level i . We define polynomial ergodicity of degree l of the $GI/G/1$ -type Markov chain as follows: given $l \in \{2, 3, \dots\}$, if for $j = 1, \dots, m$,

$$\mathbb{E}_{(0,j)} [\tau_0^l] < \infty. \quad (5.1)$$

Remark 5.1. If $l = 1$ in equation 5.1, then the $GI/G/1$ -type Markov chain is called ordinary ergodicity.

In the following, we firstly derive lower bounds for first hitting times, and then obtain two necessary and sufficient conditions for polynomial ergodicity for the $GI/G/1$ -type Markov chain.

5.1 Lower bounds for first hitting times

In this section, we derive lower bounds for the first hitting time τ_i , for any finite i , which are useful to discuss necessary conditions for different types of ergodicity. We provide a lemma based on τ_0 . Similar results can be obtained for any $i \neq 0$ by the same argument. In fact, we can get a corresponding result for the first hitting time of a finite set instead of a level.

Lemma 5.1. *If $\sum_{k=0}^{\infty} k B_k e^t < \infty^t$ and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then for i sufficiently large, we have, for each sample on an specified event to be given in the proof*

$$\tau_0 - 1 \geq \frac{i - 1}{4\mu}, \quad (5.2)$$

where $\mu > 0$ is some fixed constant.

Proof: Since we consider the level independent $GI/G/1$ -type Markov chain, we have, for $i \geq 1$ and $j = 1, \dots, m$,

$$\mathbb{P}_{(i,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=i}^{i+(k-1)} L_u \right)^c \right) = \mathbb{P}_{(1,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right), \quad \text{for any } k > 0,$$

where L^c denotes the complement of L and $\mathbb{P}_{(i,j)}((X_1, Y_1) \in \cdot) = \mathbb{P}((X_1, Y_1) \in \cdot | (X_0, Y_0) = (i, j))$. Without loss of generality, we assume that

$$\begin{aligned} & \mathbb{P}_{(0,j_1)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right) \\ &= \max_j \left\{ \mathbb{P}_{(0,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right), \quad j = 1, \dots, m \right\}, \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}_{(1,j_2)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right) \\ &= \max_j \left\{ \mathbb{P}_{(1,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right), \quad j = 1, \dots, m \right\}. \end{aligned}$$

Let

$$h(k) = \max \left\{ \mathbb{P}_{(0,j_1)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right), \mathbb{P}_{(1,j_2)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right) \right\}.$$

Notice that

$$\sum_{k=0}^{\infty} \mathbb{P}_{(0,j_1)} \left((X_1, Y_1) \in \left(\bigcup_{u=0}^{k-1} L_u \right)^c \right) < \infty,$$

and

$$\sum_{k=0}^{\infty} \mathbb{P}_{(1,j_2)} \left((X_1, Y_1) \in \left(\bigcup_{u=1}^{1+(k-1)} L_u \right)^c \right) < \infty$$

since $\sum_{k=0}^{\infty} k B_k e^t < \infty^t$ and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$. Hence

$$\sum_{k=1}^{\infty} h(k) < \infty.$$

It is easy to know that for all $i \geq 0$, $j = 1, \dots, m$ and $k > 0$,

$$\mathbb{P}_{(i,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=i}^{i+(k-1)} L_u \right)^c \right) \leq h(k),$$

and $h(k)$ is a non-increasing function. Therefore there exists a sequence of *i.i.d.* random variables $W_n > 0$ with mean $\mu = \mathbb{E}[W_n]$ such that for all $i \geq 0$ and all $k \geq 0$,

$$\mathbb{P}_{(i,j)} \left((X_1, Y_1) \in \left(\bigcup_{u=i}^{i+(k-1)} L_u \right)^c \right) \leq \mathbb{P}(W_n \geq k).$$

By the weak law of large numbers, we get for any $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}(S_n \leq (\mu + \epsilon)n) = 1,$$

where $S_n = W_1 + \dots + W_n$. Hence there exists N so large that for $n \geq N$,

$$\mathbb{P}(S_n < 2\mu n) \geq \frac{1}{2}.$$

Then a stochastic comparison argument yields for all $i \geq 0$ and all $n \geq N$,

$$\mathbb{P}_{(i,j)} \left((X_s, Y_s) \in \left(\bigcup_{u=i}^{i+2\mu n} L_u \right) \quad \text{for } s = 0, 1, \dots, n \right) \geq \frac{1}{2}. \quad (5.3)$$

For i large enough such that $\frac{i}{4\mu} \geq N$, we get from (5.3) with $n = \lfloor \frac{i}{4\mu} \rfloor \geq N$, where $\lfloor x \rfloor$ denote the largest integer equal to or large than x that

$$\mathbb{P}_{(i,j)} \left((X_s, Y_s) \in \left(\bigcup_{u=i}^{i+\frac{i}{2}} L_u \right) \quad \text{for } s = 0, 1, \dots, \lfloor \frac{i}{4\mu} \rfloor \right) \geq \frac{1}{2}.$$

Hence for i large enough such that $\bigcup_{u=i}^{i+\frac{i}{2}} L_u \subsetneq L_0$, we have, for each sample on the above event

$$\tau_0 - 1 \geq \lfloor \frac{i}{4\mu} \rfloor \geq \frac{i-1}{4\mu}.$$

□

Corollary 5.1. *Given $l \in \{2, 3, \dots\}$, if for $j = 1, 2, \dots, m$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$, $\sum_{k=0}^{\infty} k B_k e^t < \infty^t$, and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then for i sufficiently large, we have*

$$\mathbb{E}_{(i,j)} [(\tau_0 - 1)^l] \geq \frac{1}{2} \left(\frac{i-1}{4\mu} \right)^l, \quad (5.4)$$

where $\mu > 0$ is some fixed constant and $\mathbb{E}_{(i,j)} [(\tau_0 - 1)^l] = \mathbb{E} [(\tau_0 - 1)^l | (X_0, Y_0) = (i, j)]$.

Proof: From Lemma 5.1, we know that for i large enough such that $\lfloor \frac{i}{4\mu} \rfloor \geq N$,

$$\mathbb{P}_{(i,j)} \left((X_s, Y_s) \in \left(\bigcup_{u=i}^{i+\frac{i}{2}} L_u \right) \text{ for } s = 0, 1, \dots, \lfloor \frac{i}{4\mu} \rfloor \right) \geq \frac{1}{2}. \quad (5.5)$$

Hence for i large enough such that $\bigcup_{u=i}^{i+\frac{i}{2}} L_u \subsetneq L_0$, we have on the above event

$$\tau_0 - 1 \geq \lfloor \frac{i}{4\mu} \rfloor \geq \frac{i-1}{4\mu},$$

and therefore

$$(\tau_0 - 1)^l \geq \lfloor \frac{i}{4\mu} \rfloor \geq \left(\frac{i-1}{4\mu} \right)^l.$$

For i sufficiently large, this event has probability at least $\frac{1}{2}$ by (5.5) and therefore for i sufficiently large,

$$\mathbb{E}_{(i,j)} [(\tau_0 - 1)^l] \geq \frac{1}{2} \left(\frac{i-1}{4\mu} \right)^l,$$

where $\mu = \mathbb{E}[W_n] > 0$ is some fixed constant. $\square$

Corollary 5.2. *Given any non-negative and non-decreasing function f , if for $j = 1, 2, \dots, m$, $\mathbb{E}_{(0,j)}[f(\tau_0)] < \infty$, $\sum_{k=0}^{\infty} k B_k e^t < \infty^t$, and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then for i sufficiently large, we have*

$$\mathbb{E}_{(i,j)}[f(\tau_0 - 1)] \geq \frac{1}{2} f \left(\frac{i-1}{4\mu} \right), \quad (5.6)$$

where $\mu > 0$ is some fixed constant.

Proof: From Lemma 5.1, we know that for i large enough such that $\lfloor \frac{i}{4\mu} \rfloor \geq N$,

$$\mathbb{P}_{(i,j)} \left((X_s, Y_s) \in \left(\bigcup_{u=i}^{i+\frac{i}{2}} L_u \right) \text{ for } s = 0, 1, \dots, \lfloor \frac{i}{4\mu} \rfloor \right) \geq \frac{1}{2}. \quad (5.7)$$

Hence for i large enough such that $\bigcup_{u=i}^{i+\frac{i}{2}} L_u \subsetneq L_0$, we have on the above event

$$\tau_0 - 1 \geq \lfloor \frac{i}{4\mu} \rfloor \geq \frac{i-1}{4\mu},$$

and therefore

$$f(\tau_0 - 1) \geq f\left(\lfloor \frac{i}{4\mu} \rfloor\right) \geq f\left(\frac{i-1}{4\mu}\right).$$

For i sufficiently large, this event has probability at least $\frac{1}{2}$ by (5.7) and therefore for i sufficiently large,

$$\mathbb{E}_{(i,j)}[f(\tau_0 - 1)] \geq \frac{1}{2}f\left(\frac{i-1}{4\mu}\right),$$

where $\mu = \mathbb{E}[W_n] > 0$ is some fixed constant. $\square$

Secondly, we discuss the relationship between first hitting time and one step transition probability.

Lemma 5.2. *Given $l \in \{2, 3, \dots\}$, if for $j = 1, \dots, m$, $\mathbb{E}_{(0,j)}[\tau_0^l] < \infty$, we have*

$$\mathbb{E}_{(0,j)}[\tau_0^l] = \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)}[(\tau_0 + 1)^l] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)), \quad (5.8)$$

and

$$\mathbb{E}_{(1,j)}[\tau_0^l] = \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)}[(\tau_0 + 1)^l] \mathbb{P}_{(1,j)}((X_1, Y_1) = (k, i)), \quad (5.9)$$

where $\mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) = \mathbb{P}((X_1, Y_1) = (k, i) | (X_0, Y_0) = (0, j))$ and $\mathbb{P}_{(1,j)}((X_1, Y_1) = (k, i)) = \mathbb{P}((X_1, Y_1) = (k, i) | (X_0, Y_0) = (1, j))$.

Proof: Given $l \in \{2, 3, \dots\}$, we have that for $j = 1, \dots, m$,

$$\begin{aligned}
& \mathbb{E}_{(0,j)} [\tau_0^l] \\
&= \mathbb{E}_{(0,j)} [\mathbb{E} [\tau_0^l | (X_1, Y_1)]] \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E} [\tau_0^l | (X_1, Y_1) = (k, i)] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \sum_{n=1}^{\infty} n^l \mathbb{P}(\tau_0 = n | (X_1, Y_1) = (k, i)) \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \sum_{n=1}^{\infty} n^l \mathbb{P}(\tau_0 = n-1 | (X_0, Y_0) = (k, i)) \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \sum_{n=1}^{\infty} (n+1)^l \mathbb{P}(\tau_0 = n | (X_0, Y_0) = (k, i)) \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\
&= \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)} [(\tau_0 + 1)^l] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)).
\end{aligned}$$

Similarly, we can prove the other case. $\square$

Corollary 5.3. *Given $l \in \{2, 3, \dots\}$, if for $j = 1, \dots, m$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$, we have that for $i = 1, 2, \dots, m$,*

$$\mathbb{E}_{(0,j)} [\tau_0^l] \geq \sum_{k=0}^{\infty} \mathbb{E}_{(k,i)} [(\tau_0 + 1)^l] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)), \quad (5.10)$$

and

$$\mathbb{E}_{(1,j)} [\tau_0^l] \geq \sum_{k=0}^{\infty} \mathbb{E}_{(k,i)} [(\tau_0 + 1)^l] \mathbb{P}_{(1,j)}((X_1, Y_1) = (k, i)). \quad (5.11)$$

Remark 5.2. *In fact, for any non-negative and measurable function f , we can get decompositions corresponding to Lemma 5.2 for $f(\tau_0)$.*

5.2 Necessary and sufficient conditions for polynomial ergodicity

In this section, we characterize necessary and sufficient conditions for polynomial ergodicity for the $GI/G/1$ -type Markov chain. We firstly use the lemmas from the above section to obtain a necessary condition to polynomial ergodicity.

Theorem 5.4. *Given $l \in \{2, 3, \dots\}$, if for $j = 1, \dots, m$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$, and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then*

$$\sum_k k^l A_k e^t < \infty^t,$$

and

$$\sum_k k^l B_k e^t < \infty^t.$$

Proof: By Corollary 5.1, given $l \in \{2, 3, \dots\}$, for $j = 1, \dots, m$, taking N large enough such that for all $k > N$,

$$\mathbb{E}_{(k,j)} [(\tau_0 - 1)^l] \geq \frac{1}{2} \left(\frac{k-1}{4\mu} \right)^l. \quad (5.12)$$

Then by Lemma 5.2,

$$\begin{aligned} \mathbb{E}_{(0,j)} [\tau_0^l] &= \sum_{k=0}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)} [(\tau_0 + 1)^l] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\ &= \sum_{k=0}^N \sum_{i=1}^m \mathbb{E}_{(k,i)} [(\tau_0 + 1)^l] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)) \\ &\quad + \sum_{k=N+1}^{\infty} \sum_{i=1}^m \mathbb{E}_{(k,i)} [(\tau_0 + 1)^l] \mathbb{P}_{(0,j)}((X_1, Y_1) = (k, i)). \\ &\geq \left[\sum_{k=N+1}^{\infty} k^l B_k e^t \right]_j, \end{aligned}$$

where $\left[\sum_{k=N+1}^{\infty} k^l B_k e^t \right]_j$ denotes the j th element of vector $\sum_{k=N+1}^{\infty} k^l B_k e^t$. Hence

$$\sum_{k=N+1}^{\infty} k^l B_k e^t < \infty^t,$$

and therefore

$$\sum_k k^l B_k e^t < \infty^t.$$

Similarly, we can obtain

$$\sum_k k^l A_k e^t < \infty^t.$$

□

Remark 5.3. *This theorem can be extended to a class of more general non-negative and non-decreasing rate functions.*

Furthermore, Højgaard and Møller [44] showed that the converse of Theorem 5.4 is also true. Therefore, necessary and sufficient condition for polynomial ergodicity can be obtained. For convenience, we state it as follows:

Theorem 5.5. *If $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then given $l \in \{2, 3, \dots\}$, for $j = 1, \dots, m$, $\mathbb{E}_{(0,j)} [\tau_{\mathbf{0}}^l] < \infty$, if and only if $\sum_k k^l A_k e^t < \infty^t$ and $\sum_k k^l B_k e^t < \infty^t$.*

Proof: For necessity, it follows from Theorem 5.6. And Theorem 3.3 in Højgaard and Møller [44] showed that this condition were also sufficient. Hence we obtain the theorem.

□

Next, we give another necessary and sufficient condition for polynomial ergodicity based on the relationship between ergodicity and tail behavior of the stationary distribution for the $GI/G/1$ -type Markov chain with finitely many background states. Firstly, we cite a relevant lemma from Jarner and Tweedie [49].

Lemma 5.3. *Given $l \in \{2, 3, \dots\}$, for $j = 1, \dots, m$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$, there exists a finite set (without loss of generality, we assume that this finite set is L_0) such that*

$$\sum_{i=0}^{\infty} \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} < \infty, \quad (5.13)$$

where π_{ij} is the j th element of π_i .

Using Lemma 5.2, lemma 5.3 and factorization results, we have the following result.

Theorem 5.6. *If $\sum_{k=0}^{\infty} k A_k e^t < \infty$, then given $l \in \{2, 3, \dots\}$, for $j = 1, \dots, m$, $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$ if and only if $\sum_i i^l \pi_{ij} < \infty$.*

Proof: For necessity, by Corollary 5.1, given $l \in \{2, 3, \dots\}$, $j = 1, \dots, m$, taking N so large that for all $i > N$,

$$\mathbb{E}_{(i,j)} [(\tau_0 - 1)^l] \geq \frac{1}{2} \left(\frac{i-1}{4\mu} \right)^l. \quad (5.14)$$

Then by Lemma 5.3,

$$\begin{aligned} \infty &> \sum_{i=0}^{\infty} \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} \\ &= \sum_{i=0}^N \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} + \sum_{i=N+1}^{\infty} \mathbb{E}_{(i,j)} [\tau_0^l] \pi_{ij} \\ &\geq \sum_{i=N+1}^{\infty} i^l \pi_{ij}. \end{aligned}$$

Thus

$$\sum_i i^l \pi_{ij} < \infty.$$

For sufficiency, we use some factorization results and techniques of generating function to analyze the result. From (2.7) we have that for $0 < z < 1$,

$$\pi(z)[I - R(z)] = \pi_0 R_0(z),$$

and thus

$$\pi(z) = \pi_0 R_0(z) [I - R(z)]^{-1}, \quad (5.15)$$

since $I - R(z)$ is invertible. Taking the l th ($l = 2, 3, \dots$) derivative on both sides of the equation (5.15), we have

$$\begin{aligned} & \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) z^{k-l} \pi_k \\ = & \pi_0 \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) z^{k-l} R_{0,k} \sum_{k=1}^{\infty} z^k \sum_{n=0}^{\infty} R_k^{n*} \\ & + \pi_0 \sum_{k=1}^{\infty} z^k R_{0,k} \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) z^{k-l} \sum_{n=0}^{\infty} R_k^{n*} \\ & + c_{l-1}(z), \end{aligned} \quad (5.16)$$

where $c_{l-1}(z)$ is the summation of all terms of the form

$$\pi_0 \sum_{k=p-1}^{\infty} k(k-1) \cdots (k-p+1) z^{k-p} R_{0,k} \sum_{k=q-1}^{\infty} k(k-1) \cdots (k-q+1) z^{k-q} \sum_{n=0}^{\infty} R_k^{n*}$$

and

$$p + q \leq l - 1.$$

There are only finitely many such terms of this kind of items. Let $z \rightarrow 1-$ in (5.16), we get

$$\begin{aligned} & \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \pi_k \\ = & \pi_0 \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) R_{0,k} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} R_k^{n*} \\ & + \pi_0 \sum_{k=1}^{\infty} R_{0,k} \sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \sum_{n=0}^{\infty} R_k^{n*} \\ & + c_{l-1}(1). \end{aligned}$$

Since $\sum_i i^l \pi_i < \infty$ and

$$\lim_{k \rightarrow \infty} \frac{k(k-1) \cdots (k-l+1)}{k^l} = 1,$$

it follows that

$$\sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \pi_k < \infty.$$

Therefore

$$\sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) R_{0,k} < \infty$$

and

$$\sum_{k=l-1}^{\infty} k(k-1) \cdots (k-l+1) \sum_{n=0}^{\infty} R_k^{n*} < \infty.$$

Thus

$$\sum_{k=1}^{\infty} k^l R_{0,k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^l \sum_{n=0}^{\infty} R_k^{n*} < \infty.$$

From Lemma 2.1 and Lemma 2.2, we have for $k \geq 1$, $R_{0,k} \geq B_k$ and $R_k \geq A_k$. Hence,

$$\sum_{k=1}^{\infty} k^l B_k e^t \leq \sum_{k=1}^{\infty} k^l R_{0,k} e^t < \infty^t$$

and

$$\sum_{k=1}^{\infty} k^l A_k e^t \leq \sum_{k=1}^{\infty} k^l \sum_{n=0}^{\infty} A_k^{n*} e^t \leq \sum_{k=1}^{\infty} k^l \sum_{n=0}^{\infty} R_k^{n*} e^t < \infty^t.$$

So by Theorem 5.5, we know that $\mathbb{E}_{(0,j)} [\tau_0^l] < \infty$. □

Chapter 6

The factorization for $GI/G/1$ -type Markov chain

In this chapter, we discuss RG -factorizations of the generating functions and properties with respect to R - and G -measures for the $GI/G/1$ -type Markov chain with finitely many background states whose transition matrix P is given by (2.2). We assume that P is aperiodic and irreducible. In [37], the authors provided decompositions for R -, G - and Φ -measures based on level variables without considering time variables. In the following, we extend their results to the case considering both level variables and time variables. Results given in this chapter have their own interest, and could be also used for studying ergodicity.

For the above chain, we define the R -measure, the G -measure and Φ -measure respectively as follows. For $i < j$, let $R_{i,j}^{[n]}$ be a $m \times m$ matrix whose (u, v) entry $R_{(i,u),(j,v)}^{[n]}$ is the probability that starting in (i, u) , the chain $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ reaches (j, v)

for the first time at the n th step without visiting $L_{\leq j-1}$ in the intermediate steps; or

$$R_{(i,u),(j,v)}^{[n]} = \mathbb{P}(X_n = (j, v), X_t \notin L_{\leq j-1}, t = 1, \dots, n-1 | X_0 = (i, u)).$$

We see $R_{(i,u),(j,v)} = \sum_{n=1}^{\infty} R_{(i,u),(j,v)}^{[n]}$

For $i > j$, let $G_{i,j}^{[n]}$ be a $m \times m$ matrix whose (u, v) entry $G_{(i,u),(j,v)}^{[n]}$ is the probability that starting in (i, u) , the chain $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ reaches (j, v) at the n th step without visiting $L_{\leq i-1}$ in the intermediate steps; or

$$G_{(i,u),(j,v)}^{[n]} = \mathbb{P}(X_n = (j, v), X_t \notin L_{\leq i-1}, t = 1, \dots, n-1 | X_0 = (i, u)).$$

Similarly, $G_{(i,u),(j,v)} = \sum_{n=1}^{\infty} G_{(i,u),(j,v)}^{[n]}$.

For $i \leq j$, let $\Phi_{i,j}^{[n]}$ be a $m \times m$ matrix whose (u, v) entry $\Phi_{(i,u),(j,v)}^{[n]}$ is the probability that starting in (i, u) , the chain $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ reaches (j, v) at the n th step without visiting $L_{\leq j}$ in the intermediate steps; or

$$\Phi_{(i,u),(j,v)}^{[n]} = \mathbb{P}(X_n = (j, v), X_t \notin L_{\leq j}, t = 1, \dots, n-1 | X_0 = (i, u)).$$

For $i \geq j$, let $\Phi_{i,j}^{[n]}$ be a $m \times m$ matrix whose (u, v) entry $\Phi_{(i,u),(j,v)}^{[n]}$ is the probability that starting in (i, u) , the chain $\{(X_n, Y_n) : n = 0, 1, 2, \dots\}$ reaches (j, v) at the n th step without visiting $L_{\leq i}$ in the intermediate steps; or

$$\Phi_{(i,u),(j,v)}^{[n]} = \mathbb{P}(X_n = (j, v), X_t \notin L_{\leq i}, t = 1, \dots, n-1 | X_0 = (i, u)).$$

Similarly, $\Phi_{(i,u),(j,v)} = \sum_{n=1}^{\infty} \Phi_{(i,u),(j,v)}^{[n]}$.

Because the $GI/G/1$ -type Markov chain has row repeating structure, we can refine the expression of the above definitions. In the following, we do this in steps. From Lemma 4 in [105] and Lemma 7 in [60], we get a special case as follows:

Lemma 6.1. *For $j > i > 0$ and for fixed n , $R_{i,j}^{[n]}$ and $\Phi_{i,j}^{[n]}$ only depend on the value of $j - i$ respectively; or $R_{i,j}^{[n]} = R_{i+k,j+k}^{[n]}$ and $\Phi_{i,j}^{[n]} = \Phi_{i+k,j+k}^{[n]}$ for any $k \geq 0$. For $i > j > 0$ and for fixed n , $G_{i,j}^{[n]}$ and $\Phi_{i,j}^{[n]}$ only depend on the value of $i - j$ respectively; or $G_{i,j}^{[n]} = G_{i+k,j+k}^{[n]}$ and $\Phi_{i,j}^{[n]} = \Phi_{i+k,j+k}^{[n]}$ for any $k \geq 0$.*

This lemma says that the non-boundary components of the R -, G - and Φ -measures inherit the repeating property of P . Hence, we can define

$$R_{j-i}^{[n]} = R_{i,j}^{[n]} \quad \text{and} \quad \Phi_{j-i}^{[n]} = \Phi_{i,j}^{[n]}, \quad \text{for } j > i > 0$$

and

$$G_{i-j}^{[n]} = G_{i,j}^{[n]} \quad \text{and} \quad \Phi_{i-j}^{[n]} = \Phi_{i,j}^{[n]}, \quad \text{for } i > j > 0.$$

Without any additional assumption, we will use R_i , G_j , Φ_i and Φ_{-j} to denote the non-boundary R -, G - and Φ -measures respectively, and represent the boundary R -, G - and Φ -measures by $R_{0,i}$, $G_{j,0}$, $\Phi_{0,i}$ and $\Phi_{j,0}$ respectively in the following part. In the remaining part, we discuss RG factorization and the relationship between R - and G -measures, and furthermore give an example for a special $M/G/1$ -type Markov chain.

6.1 The RG -factorization of generating functions

In this section, we discuss the RG -factorization of the generating functions of R -, G - and Φ -measures for $GI/G/1$ -type Markov chain. We firstly give decompositions for the non-boundary components of the R -, G - and Φ -measures, which is based on the transition steps.

Lemma 6.2. *For $i \geq 0$, we have*

$$R_i^{[1]} = \Phi_i^{[1]} = A_i,$$

$$R_i^{[2]} - R_i^{[1]} \Phi_0^{[1]} = \Phi_i^{[2]} = \sum_{k=1}^{\infty} R_{i+k}^{[1]} G_k^{[1]}$$

and

$$\begin{aligned} R_i^{[l]} - \sum_{n=1}^{l-1} R_i^{[n]} \Phi_0^{[l-n]} &= \Phi_i^{[l]} \\ &= \sum_{n=1}^{l-1} \sum_{k=1}^{\infty} R_{i+k}^{[n]} G_k^{[l-n]} - \sum_{n=1}^{l-2} \sum_{r=1}^{l-n-1} \sum_{k=1}^{\infty} R_{i+k}^{[n]} \Phi_0^{[r]} G_k^{[l-n-r]}. \end{aligned}$$

Proof: We prove the Lemma by a probabilistic argument. Firstly, we show that the first equality in each equation holds. By the definitions of R - and Φ -measures, it is easy to know

$$R_i^{[1]}(r, s) = \mathbb{P}(X_1 = (i+k, s) | X_0 = (k, r)) = \Phi_i^{[1]}(r, s). \quad (6.1)$$

For $l \geq 2$, by conditioning on L_{i+k} and Markov property, we have

$$\begin{aligned} &\left[R_i^{[l]} - \sum_{n=1}^{l-1} R_i^{[n]} \Phi_0^{[l-n]} \right] (r, s) \\ &= R_i^{[l]}(r, s) - \sum_{v=1}^p \sum_{n=1}^{l-1} R_i^{[n]}(r, v) \Phi_0^{[l-n]}(v, s) \\ &= \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\ &\quad - \sum_{v=1}^p \sum_{n=1}^{l-1} \mathbb{P}(X_n = (i+k, v), X_t \notin L_{\leq i+k-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\ &\quad \mathbb{P}(X_{l-n} = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, n-1 | X_0 = (i+k, v)) \\ &= \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, n-1 | X_0 = (k, r)) \\ &\quad + \sum_{v=1}^p \sum_{n=1}^{l-1} \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k}, t = n+1, \dots, l-1, \\ &\quad \quad X_n = (i+k, v), X_t \notin L_{\leq i+k-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\ &\quad - \sum_{v=1}^p \sum_{n=1}^{l-1} \mathbb{P}(X_n = (i+k, v), X_t \notin L_{\leq i+k-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\ &\quad \mathbb{P}(X_{l-n} = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, l-n-1 | X_0 = (i+k, v)) \end{aligned} \quad (6.2)$$

$$\begin{aligned}
 &= \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &\quad + \sum_{v=1}^p \sum_{n=1}^{l-1} \mathbb{P}(X_n = (i+k, v), X_t \notin L_{\leq i+k-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &\quad \quad \mathbb{P}(X_{l-n} = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, l-n-1 | X_0 = (i+k, v)) \\
 &\quad - \sum_{v=1}^p \sum_{n=1}^{l-1} \mathbb{P}(X_n = (i+k, v), X_t \notin L_{\leq i+k-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &\quad \quad \mathbb{P}(X_{l-n} = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, l-n-1 | X_0 = (i+k, v)) \\
 &= \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &= \Phi_i^{[l]}(r, s)
 \end{aligned}$$

Secondly, we prove that the second equality in each equation holds. Since

$$A_i(r, s) = \mathbb{P}(X_1 = (i+k, s) | X_0 = (k, r)) = \Phi_i^{[1]}(r, s), \quad (6.3)$$

we know that for $l = 1$, the right equality holds. For $l = 2$, we have

$$\begin{aligned}
 &\left[\sum_{k=1}^{\infty} R_{i+k}^{[1]} G_k^{[1]} \right] (r, s) \\
 &= \sum_{v=1}^p \sum_{k=1}^{\infty} R_{i+k}^{[1]}(r, v) G_k^{[1]}(v, s) \\
 &= \sum_{v=1}^p \sum_{k=1}^{\infty} \mathbb{P}(X_1 = (i+k+h, v) | X_0 = (h, r)) \mathbb{P}(X_1 = (i+h, s) | X_0 = (i+k+h, v)) \quad (6.4) \\
 &= \sum_{v=1}^p \sum_{k=1}^{\infty} \mathbb{P}(X_2 = (i+h, s), X_1 = (i+k+h, v) | X_0 = (h, r)) \\
 &= \mathbb{P}(X_2 = (i+h, s), X_1 \notin L_{i+h, v} | X_0 = (h, r)) \\
 &= \Phi_i^{[2]}(r, s).
 \end{aligned}$$

Hence, for $l = 2$, the second equality holds. For $l \geq 3$, by conditioning on $L_{i+k+1}, L_{i+k+2}, \dots$

and Markov property, we have

$$\begin{aligned}
 & \left[\sum_{n=1}^{l-1} \sum_{q=1}^{\infty} R_{i+q}^{[n]} G_q^{[l-n]} - \sum_{n=1}^{l-2} \sum_{u=1}^{l-n-1} \sum_{q=1}^{\infty} R_{i+q}^{[n]} \Phi_0^{[u]} G_q^{[l-n-u]} \right] (r, s) \\
 &= \sum_{v=1}^p \sum_{n=1}^{l-1} \sum_{q=1}^{\infty} R_{i+q}^{[n]}(r, v) G_q^{[l-n]}(v, s) - \sum_{v=1}^p \sum_{h=1}^p \sum_{n=1}^{l-2} \sum_{u=1}^{l-n-1} \sum_{q=1}^{\infty} R_{i+q}^{[n]}(r, v) \Phi_0^{[u]}(v, h) G_q^{[l-n-u]}(h, s) \\
 &= \sum_{v=1}^p \sum_{n=1}^{l-1} \sum_{q=1}^{\infty} \mathbb{P}(X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q-1}, t=1, \dots, n-1 | X_0 = (k, r)) \\
 & \quad \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t=n+1, \dots, l-n-1 | X_n = (i+k+q, r)) \\
 & \quad - \sum_{v=1}^p \sum_{h=1}^p \sum_{n=1}^{l-2} \sum_{u=1}^{l-n-1} \sum_{q=1}^{\infty} \mathbb{P}(X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q-1}, t=1, \dots, n-1 | X_0 = (k, r)) \\
 & \quad \mathbb{P}(X_{n+u} = (i+k+q, h), X_t \notin L_{\leq i+k+q}, t=n+1, \dots, n+u-1 | X_n = (i+k+q, v)) \\
 & \quad \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t=n+u+1, \dots, l-1 | X_n = (i+k+q, h)) \\
 &= \sum_{v=1}^p \sum_{n=1}^{l-1} \sum_{q=1}^{\infty} \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t=n+1, \dots, l-n-1, \\
 & \quad X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q-1}, t=1, \dots, n-1 | X_0 = (k, r)) \\
 & \quad - \sum_{v=1}^p \sum_{h=1}^p \sum_{n=1}^{l-2} \sum_{u=1}^{l-n-1} \sum_{q=1}^{\infty} \mathbb{P}(X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q-1}, t=1, \dots, n-1 | X_0 = (k, r)) \\
 & \quad \mathbb{P}(X_{n+u} = (i+k+q, h), X_t \notin L_{\leq i+k+q}, t=n+1, \dots, n+u-1 | X_n = (i+k+q, v)) \\
 & \quad \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t=n+u+1, \dots, l-1 | X_n = (i+k+q, h))
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{v=1}^p \sum_{n=1}^{l-1} \sum_{q=1}^{\infty} \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t = n+1, \dots, l-n-1, \\
 &\quad X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &\quad + \sum_{v=1}^p \sum_{h=1}^p \sum_{n=1}^{l-2} \sum_{u=1}^{l-n-1} \sum_{q=1}^{\infty} \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t = n+u+1, \dots, l-1, \\
 &\quad X_{n+u} = (i+k+q, h), X_t \notin L_{\leq i+k+q-1}, t = n+1, \dots, n+u-1, \\
 &\quad X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &\quad - \sum_{v=1}^p \sum_{h=1}^p \sum_{n=1}^{l-2} \sum_{u=1}^{l-n-1} \sum_{q=1}^{\infty} \mathbb{P}(X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q-1}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &\quad \mathbb{P}(X_{n+u} = (i+k+q, h), X_t \notin L_{\leq i+k+q}, t = n+1, \dots, n+u-1 | X_n = (i+k+q, v)) \\
 &\quad \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t = n+u+1, \dots, l-1 | X_n = (i+k+q, h)) \\
 &= \sum_{v=1}^p \sum_{n=1}^{l-1} \sum_{q=1}^{\infty} \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k+q-1}, t = n+1, \dots, l-n-1, \\
 &\quad X_n = (i+k+q, v), X_t \notin L_{\leq i+k+q}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &= \mathbb{P}(X_l = (i+k, s), X_t \notin L_{\leq i+k}, t = 1, \dots, n-1 | X_0 = (k, r)) \\
 &= \Phi_i^{[l]}(r, s), \tag{6.5}
 \end{aligned}$$

then the second equality is true for $l \geq 3$. So, by (6.1), (6.2), (6.3), (6.4), and (6.5), we know that the lemma holds. $\square$

By a similar argument, we can obtain the following result.

Lemma 6.3. *For $j \geq 0$, we have*

$$\begin{aligned}
 G_j^{[1]} &= \Phi_{-j}^{[1]} = A_{-j}, \\
 G_j^{[2]} - \Phi_0^{[1]} G_j^{[1]} &= \Phi_{-j}^{[2]} = \sum_{k=1}^{\infty} R_k^{[1]} G_{j+k}^{[1]}
 \end{aligned}$$

and

$$\begin{aligned} G_j^{[l]} - \sum_{n=1}^{l-1} \Phi_0^{[l-n]} G_j^{[n]} &= \Phi_{-j}^{[l]} \\ &= \sum_{n=1}^{l-1} \sum_{k=1}^{\infty} R_k^{[n]} G_{j+k}^{[l-n]} - \sum_{n=1}^{l-2} \sum_{r=1}^{l-n-1} \sum_{k=1}^{\infty} R_k^{[n]} \Phi_0^{[r]} G_{j+k}^{[l-n-r]}. \end{aligned}$$

Now, we define the generating functions as follows: $R_i(s) = \sum_{n=1}^{\infty} s^n R_i^{[n]}$, $G_j(s) = \sum_{n=1}^{\infty} s^n G_j^{[n]}$, $\Phi_i(s) = \sum_{n=1}^{\infty} s^n \Phi_i^{[n]}$ and $\Phi_0(s) = \sum_{n=1}^{\infty} s^n \Phi_0^{[n]}$. From Lemma 6.2 and Lemma 6.3, we can get the RG -factorization of the generating functions.

Theorem 6.1. For $i, j > 0$,

$$R_i(s)(I - \Phi_0(s)) = sA_i + \sum_{k=1}^{\infty} R_{i+k}(s)(I - \Phi_0(s))G_k(s), \quad (6.6)$$

$$(I - \Phi_0(s))G_j(s) = sA_{-j} + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_{j+k}(s) \quad (6.7)$$

and

$$\Phi_0(s) = sA_0 + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_k(s) \quad (6.8)$$

or equivalently,

$$\Phi_i(s) = sA_i + \sum_{k=1}^{\infty} \Phi_{i+k}(s)(I - \Phi_0(s))^{-1}\Phi_{-k}(s), \quad (6.9)$$

$$\Phi_{-j}(s) = sA_{-j} + \sum_{k=1}^{\infty} \Phi_k(s)(I - \Phi_0(s))^{-1}\Phi_{-j-k}(s) \quad (6.10)$$

and

$$\Phi_0(s) = sA_0 + \sum_{k=1}^{\infty} \Phi_k(s)(I - \Phi_0(s))^{-1}\Phi_k(s) \quad (6.11)$$

Proof: By taking summations over i and j in Lemma 6.2 and Lemma 6.3 respectively and changing the order of the summations, the results can easily be yielded. We omit the details here. $\square$

Further, we define the generating functions regarding level as follows: $R(z, s) = \sum_{i=1}^{\infty} z^i R_i(s)$, $G(z, s) = \sum_{j=1}^{\infty} z^j G_j(s)$ and $A(z, s) = s \sum_{i=-\infty}^{\infty} z^i A_i$. Then, we can obtain another style of RG -factorization.

Theorem 6.2.

$$I - A(z, s) = (I - R(z, s))(I - \Phi_0(s))(I - G(z, s)). \quad (6.12)$$

Proof: By taking summations over i and j in Theorem 6.1, respectively, we have that

$$\begin{aligned} & R(z, s)(I - \Phi_0(s)) \\ &= \sum_{i=1}^{\infty} z^i R_i(s)(I - \Phi_0(s)) \\ &= s \sum_{i=1}^{\infty} z^i A_i + \sum_{i=1}^{\infty} z^i \sum_{k=1}^{\infty} R_{i+k}(s)(I - \Phi_0(s))G_k(s) \\ &= s \sum_{i=1}^{\infty} z^i A_i + \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} z^{i+k} R_{i+k}(s)(I - \Phi_0(s))z^{-k}G_k(s) \\ &= s \sum_{i=1}^{\infty} z^i A_i + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} z^{i+k} R_{i+k}(s)(I - \Phi_0(s))z^{-k}G_k(s) \\ &= s \sum_{i=1}^{\infty} z^i A_i + \sum_{k=1}^{\infty} \sum_{i=k+1}^{\infty} z^i R_i(s)(I - \Phi_0(s))z^{-k}G_k(s) \\ &= s \sum_{i=1}^{\infty} z^i A_i + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} z^i R_i(s)(I - \Phi_0(s))z^{-k}G_k(s) - \sum_{k=1}^{\infty} \sum_{i=1}^k z^i R_i(s)(I - \Phi_0(s))z^{-k}G_k(s) \\ &= s \sum_{i=1}^{\infty} z^i A_i + \sum_{k=1}^{\infty} R(z, s)(I - \Phi_0(s))z^{-k}G_k(s) - \sum_{k=1}^{\infty} \sum_{i=1}^k z^i R_i(s)(I - \Phi_0(s))z^{-k}G_k(s) \\ &= s \sum_{i=1}^{\infty} z^i A_i + R(z, s)(I - \Phi_0(s))G(z, s) - \sum_{k=1}^{\infty} \sum_{i=1}^k z^i R_i(s)(I - \Phi_0(s))z^{-k}G_k(s), \end{aligned} \quad (6.13)$$

and

$$\begin{aligned}
 & (I - \Phi_0(s))G(z, s) \\
 &= \sum_{j=1}^{\infty} z^{-j} (I - \Phi_0(s))G_j(s) \\
 &= s \sum_{j=1}^{\infty} z^{-j} A_{-j} + \sum_{j=1}^{\infty} z^{-j} \sum_{k=1}^{\infty} R_k(s) (I - \Phi_0(s))G_{j+k}(s) \\
 &= s \sum_{j=1}^{\infty} z^{-j} A_{-j} + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} z^k R_k(s) (I - \Phi_0(s)) z^{-(j+k)} G_{j+k}(s) \\
 &= s \sum_{j=1}^{\infty} z^{-j} A_{-j} + \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} z^k R_k(s) (I - \Phi_0(s)) z^{-j} G_j(s) \\
 &= s \sum_{j=1}^{\infty} z^{-j} A_{-j} + R(z, s) (I - \Phi_0(s))G(z, s) - \sum_{k=1}^{\infty} \sum_{j=1}^k z^k R_k(s) (I - \Phi_0(s)) z^{-j} G_j(s).
 \end{aligned}
 \tag{6.14}$$

Then, (6.13)+ (6.14)+ (6.8) on both sides respectively yields

$$\begin{aligned}
 & R(z, s)(I - \Phi_0(s)) + (I - \Phi_0(s))G(z, s) + \Phi_0(s) \\
 &= s \sum_{i=1}^{\infty} z^i A_i + s \sum_{j=1}^{\infty} z^{-j} A_{-j} + sA_0 + R(z, s)(I - \Phi_0(s))G(z, s) + R(z, s)(I - \Phi_0(s))G(z, s) \\
 &\quad - \sum_{k=1}^{\infty} \sum_{i=1}^k z^i R_i(s)(I - \Phi_0(s))z^{-k} G_k(s) - \sum_{k=1}^{\infty} \sum_{j=1}^k z^k R_k(s)(I - \Phi_0(s))z^{-j} G_j(s). \\
 &\quad + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_k(s) \\
 &= A(z, s) + R(z, s)(I - \Phi_0(s))G(z, s) + R(z, s)(I - \Phi_0(s))G(z, s) \\
 &\quad - \sum_{j=1}^{\infty} \sum_{k=1}^j z^k R_k(s)(I - \Phi_0(s))z^{-j} G_j(s) - \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} z^k R_k(s)(I - \Phi_0(s))z^{-j} G_j(s). \\
 &\quad + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_k(s) \\
 &= A(z, s) + R(z, s)(I - \Phi_0(s))G(z, s). \tag{6.15}
 \end{aligned}$$

Hence, simplifying (6.15), we get

$$I - A(z, s) = (I - R(z, s))(I - \Phi_0(s))(I - G(z, s)).$$

□

Next, we give the similar decompositions to the boundary components. Similar to Lemma 6.2, we can prove the following lemmas.

Lemma 6.4. *For $i > 0$, we have*

$$\begin{aligned}
 R_{0,i}^{[1]} &= \Phi_{0,i}^{[1]} = B_i, \\
 R_{0,i}^{[2]} - R_{0,i}^{[1]}\Phi_0^{[1]} &= \Phi_{0,i}^{[2]} = \sum_{k=1}^{\infty} R_{0,i+k}^{[1]} G_k^{[1]}
 \end{aligned}$$

and

$$\begin{aligned} R_{0,i}^{[l]} - \sum_{n=1}^{l-1} R_{0,i}^{[n]} \Phi_0^{[l-n]} &= \Phi_{0,i}^{[l]} \\ &= \sum_{n=1}^{l-1} \sum_{k=1}^{\infty} R_{0,i+k}^{[n]} G_k^{[l-n]} - \sum_{n=1}^{l-2} \sum_{r=1}^{l-n-1} \sum_{k=1}^{\infty} R_{0,i+k}^{[n]} \Phi_0^{[r]} G_k^{[l-n-r]}. \end{aligned}$$

Lemma 6.5. For $j > 0$, we have

$$\begin{aligned} G_{j,0}^{[1]} &= \Phi_{j,0}^{[1]} = B_{-j}, \\ G_{j,0}^{[2]} - \Phi_0^{[1]} G_{j,0}^{[1]} &= \Phi_{j,0}^{[2]} = \sum_{k=1}^{\infty} R_k^{[1]} G_{j+k,0}^{[1]} \end{aligned}$$

and

$$\begin{aligned} G_{j,0}^{[l]} - \sum_{n=1}^{l-1} \Phi_0^{[l-n]} G_{j,0}^{[n]} &= \Phi_{j,0}^{[l]} \\ &= \sum_{n=1}^{l-1} \sum_{k=1}^{\infty} R_k^{[n]} G_{j+k,0}^{[l-n]} - \sum_{n=1}^{l-2} \sum_{r=1}^{l-n-1} \sum_{k=1}^{\infty} R_k^{[n]} \Phi_0^{[r]} G_{j+k,0}^{[l-n-r]}. \end{aligned}$$

Lemma 6.6.

$$\begin{aligned} \Phi_{0,0}^{[1]} &= B_0, \\ \Phi_{0,0}^{[2]} &= \sum_{k=1}^{\infty} R_{0,k}^{[1]} G_{k,0}^{[1]} \end{aligned}$$

and

$$\Phi_{0,0}^{[l]} = \sum_{n=1}^{l-1} \sum_{k=1}^{\infty} R_{0,k}^{[n]} G_{k,0}^{[l-n]} - \sum_{n=1}^{l-2} \sum_{r=1}^{l-n-1} \sum_{k=1}^{\infty} R_{0,k}^{[n]} \Phi_0^{[r]} G_{k,0}^{[l-n-r]}.$$

We also define the generating function for the boundary matrix: $R_{0,i}(s) = \sum_{n=1}^{\infty} s^n R_{0,i}^{[n]}$, $G_{j,0}(s) = \sum_{n=1}^{\infty} s^n G_{j,0}^{[n]}$, and $\Phi_{0,0}(s) = \sum_{n=1}^{\infty} s^n \Phi_{0,0}^{[n]}$. From Lemma 6.4, Lemma 6.5 and Lemma 6.6, we can the RG -factorization of the generating functions.

Theorem 6.3. For $i, j > 0$,

$$R_{0,i}(s)(I - \Phi_0(s)) = sB_i + \sum_{k=1}^{\infty} R_{0,i+k}(s)(I - \Phi_0(s))G_k(s), \quad (6.16)$$

$$(I - \Phi_0(s))G_{j,0}(s) = sB_{-j} + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_{j+k,0}(s) \quad (6.17)$$

and

$$\Phi_{0,0}(s) = sB_0 + \sum_{k=1}^{\infty} R_{0,k}(s)(I - \Phi_0(s))G_{k,0}(s). \quad (6.18)$$

6.2 The relationship between R - and G -components

In this section, we first deal with the relationship between R - and G -components. Furthermore, we give the solutions to the equations in Theorem 6.1 and Theorem 6.3 in terms of G measures. From Theorem 6.1, we can reach a representation of $R_i(s)(I - \Phi_0(s))$ in terms of $G_k(s)$.

Proposition 6.1. *For $i > 0$,*

$$\begin{aligned}
 R_i(s)(I - \Phi_0(s)) &= sA_i + \sum_{k_1=1}^{\infty} sA_{i+k_1}G_{k_1}(s) + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sA_{i+k_1+k_2}G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sA_{i+k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sA_{i+k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) + \cdots
 \end{aligned} \tag{6.19}$$

Proof: From Theorem 6.1, we know, for $i > 0$,

$$R_i(s)(I - \Phi_0(s)) = sA_i + \sum_{k=1}^{\infty} R_{i+k}(s)(I - \Phi_0(s))G_k(s).$$

By successively substituting $R_{i+k}(s)(I - \Phi_0(s))$ with the above equation, we get

$$\begin{aligned}
 R_i(s)(I - \Phi_0(s)) &= sA_i + \sum_{k_1=1}^{\infty} R_{i+k_1}(s)(I - \Phi_0(s))G_{k_1}(s) \\
 &= sA_i + \sum_{k_1=1}^{\infty} [sA_{i+k_1} + \sum_{k_2=1}^{\infty} R_{i+k_1+k_2}(s)(I - \Phi_0(s))G_{k_2}(s)]G_{k_1}(s) \\
 &= sA_i + \sum_{k_1=1}^{\infty} sA_{i+k_1}G_{k_1}(s) + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} R_{i+k_1+k_2}(s)(I - \Phi_0(s))G_{k_2}(s)G_{k_1}(s) \\
 &= sA_i + \sum_{k_1=1}^{\infty} sA_{i+k_1}G_{k_1}(s) + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sA_{i+k_1+k_2}G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sA_{i+k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sA_{i+k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) + \cdots
 \end{aligned}$$

□

Similarly, from Theorem 6.3, we can get the representation of $R_{0,i}(s)(I - \Phi_0(s))$

Proposition 6.2. For $i > 0$,

$$\begin{aligned}
 R_{0,i}(s)(I - \Phi_0(s)) &= sB_i + \sum_{k_1=1}^{\infty} sB_{i+k_1}G_{k_1}(s) + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sB_{i+k_1+k_2}G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sB_{i+k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sB_{i+k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) + \cdots
 \end{aligned} \tag{6.20}$$

Proof: By Theorem 6.3,

$$R_{0,i}(s)(I - \Phi_0(s)) = sB_i + \sum_{k=1}^{\infty} R_{0,i+k}(s)(I - \Phi_0(s))G_k(s), \quad \text{for } i > 0.$$

Hence, successively replace $R_{0,i+k_1}(s)(I - \Phi_0(s))$ in above equation to yield

$$\begin{aligned}
 R_{0,i}(s)(I - \Phi_0(s)) &= sB_i + \sum_{k_1=1}^{\infty} R_{0,i+k_1}(s)(I - \Phi_0(s))G_{k_1}(s) \\
 &= sB_i + \sum_{k_1=1}^{\infty} [sB_{i+k_1} + \sum_{k_2=1}^{\infty} R_{0,i+k_1+k_2}(s)(I - \Phi_0(s))G_{k_2}(s)]G_{k_1}(s) \\
 &= sB_i + \sum_{k_1=1}^{\infty} sB_{i+k_1}G_{k_1}(s) + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} R_{0,i+k_1+k_2}(s)(I - \Phi_0(s))G_{k_2}(s)G_{k_1}(s) \\
 &= sB_i + \sum_{k_1=1}^{\infty} sB_{i+k_1}G_{k_1}(s) + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sB_{i+k_1+k_2}G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sB_{i+k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sB_{i+k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) + \cdots
 \end{aligned}$$

□

Therefore, we solve all equations in Theorem 6.1 and Theorem 6.3 by substituting the representations of $R_i(s)$ and $R_{0,j}(s)$ into Theorem 6.3.

Proposition 6.3.

$$\begin{aligned}
 \Phi_0(s) &= sA_0 + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_k(s) \\
 &= sA_0 \\
 &\quad + \sum_{k_1=1}^{\infty} sA_{k_1}G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sA_{k_1+k_2}G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sA_{k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sA_{k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1}(s) \\
 &\quad + \cdots,
 \end{aligned} \tag{6.21}$$

$$\begin{aligned}
 G_j(s) &= sA_{-j} + \Phi_0(s)G_j(s) + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_{j+k}(s) \\
 &= sA_{-j} \\
 &+ sA_0G_j(s) \\
 &+ \sum_{k_1=1}^{\infty} sA_{k_1}G_{k_1}(s)G_j(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sA_{k_1+k_2}G_{k_2}(s)G_{k_1}(s)G_j(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sA_{k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1}(s)G_j(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sA_{k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1}(s)G_j(s) \quad (6.22) \\
 &+ \dots \\
 &+ \sum_{k_1=1}^{\infty} sA_{k_1}G_{j+k_1}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sA_{k_1+k_2}G_{k_2}(s)G_{j+k_1}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sA_{k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{j+k_1}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sA_{k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{j+k_1}(s) \\
 &+ \dots,
 \end{aligned}$$

$$\begin{aligned}
 \Phi_{0,0}(s) &= sB_0 + \sum_{k=1}^{\infty} R_{0,k}(s)(I - \Phi_0(s))G_{k,0}(s) \\
 &= sB_0 \\
 &+ \sum_{k_1=1}^{\infty} sB_{k_1}G_{k_1,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sB_{k_1+k_2}G_{k_2}(s)G_{k_1,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sB_{k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sB_{k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1,0}(s) \\
 &+ \dots
 \end{aligned} \tag{6.23}$$

and

$$\begin{aligned}
 G_{j,0}(s) &= sB_{-j} + \Phi_0(s)G_{j,0}(s) + \sum_{k=1}^{\infty} R_k(s)(I - \Phi_0(s))G_{j+k,0}(s) \\
 &= sB_{-j} \\
 &+ sA_0G_{j,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} A_{k_1}G_{k_1}(s)G_{j,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sA_{k_1+k_2}G_{k_2}(s)G_{k_1}(s)G_{j,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sA_{k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{k_1}(s)G_{j,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sA_{k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{k_1}(s)G_{j,0}(s) \quad (6.24) \\
 &+ \cdots \\
 &+ \sum_{k_1=1}^{\infty} sA_{k_1}G_{j+k_1,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} sA_{k_1+k_2}G_{k_2}(s)G_{j+k_1,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} sA_{k_1+k_2+k_3}G_{k_3}(s)G_{k_2}(s)G_{j+k_1,0}(s) \\
 &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \sum_{k_3=1}^{\infty} \sum_{k_4=1}^{\infty} sA_{k_1+k_2+k_3+k_4}G_{k_4}(s)G_{k_3}(s)G_{k_2}(s)G_{j+k_1,0}(s) \\
 &+ \cdots .
 \end{aligned}$$

For convenience, we can reorganize the above representations. For example, we can

rewrite $\Phi_0(s)$ as follows.

$$\begin{aligned}
 \Phi_0(s) &= sA_0 \\
 &+ sA_1G_1(s) \\
 &+ sA_2[G_2(s) + G_1^2(s)] \\
 &+ sA_3[G_3(s) + G_1(s)G_2(s) + G_2(s)G_1(s) + G_1^3(s)] \\
 &+ sA_4[G_4(s) + G_1(s)G_3(s) + G_2^2(s) + G_3(s)G_1(s) + \cdots + G_1^4(s)] \\
 &+ \cdots .
 \end{aligned} \tag{6.25}$$

6.3 An example

We suppose that the transition matrix P is given by

$$P = \left(p_{(i,u),(j,v)} \right)_{(i,u),(j,v)} = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \cdots \\ B_{-1} & A_0 & A_1 & A_2 & \cdots \\ 0 & A_{-1} & A_0 & A_1 & \cdots \\ 0 & 0 & A_{-1} & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6.26}$$

where A_i and B_i for $i = -1, 0, 1, 2, \dots$, are matrices of size $m \times m$ with $m < \infty$. This kind of chain is called the $M/G/1$ -type Markov chain, which possesses the property of repeating rows and skip-free to left for levels. The $M/G/1$ -type Markov chain is a special case of the $GI/G/1$ -type Markov chain.

Due to the special structure of the transition probability matrix of the $M/G/1$ -type Markov chain, we can obtain a simpler version of Theorem 6.1.

$$R_i(s)(I - \Phi_0(s)) = sA_i + R_{i+1}(s)(I - \Phi_0(s))G_1(s), \quad \text{for } i \geq 0, \tag{6.27}$$

$$\Phi_0(s) = sA_0 + R_1(s)(I - \Phi_0(s))G_1(s) \tag{6.28}$$

and

$$(I - \Phi_0(s))G_1(s) = sA_{-1}. \quad (6.29)$$

Similarly, we also have a representation for the boundary components corresponding to Theorem 6.3.

$$R_{0,i}(s)(I - \Phi_0(s)) = sB_i + R_{0,i+1}(s)(I - \Phi_0(s))G_1(s), \quad \text{for } i \geq 0, \quad (6.30)$$

$$\Phi_{0,0}(s) = sB_0 + R_{0,1}(s)(I - \Phi_0(s))G_{1,0}(s) \quad (6.31)$$

and

$$(I - \Phi_0(s))G_{1,0}(s) = sB_{-1}. \quad (6.32)$$

From the equations (6.27) and (6.30), we can get the following conclusions.

Proposition 6.4.

$$\begin{aligned} R_i(s) &= \left[\sum_{k=i}^{\infty} sA_k G_{k-i}(s) \right] (I - \Phi_0(s))^{-1}, \quad \text{for } i \geq 0, \\ &= \left[\sum_{k=i}^{\infty} sA_k G_1^{k-i}(s) \right] (I - \Phi_0(s))^{-1}. \end{aligned} \quad (6.33)$$

Similarly, we can obtain

Proposition 6.5.

$$\begin{aligned} R_{0,i}(s) &= \left[\sum_{k=i}^{\infty} sB_k G_{k-i}(s) \right] (I - \Phi_0(s))^{-1}, \quad \text{for } i \geq 0, \\ &= \left[\sum_{k=i}^{\infty} sB_k G_1^{k-i}(s) \right] (I - \Phi_0(s))^{-1}. \end{aligned} \quad (6.34)$$

Therefore, we can reach the following conclusion.

Proposition 6.6.

$$\Phi_{0,0}(s) = sB_0 + \left[\sum_{k=1}^{\infty} sB_k G_1^{k-1}(s) \right] G_{1,0}(s), \quad (6.35)$$

$$G_{1,0}(s) = sB_{-1} + \left[\sum_{k=0}^{\infty} sA_k G_1^k(s) \right] G_{1,0}(s) \quad (6.36)$$

and

$$G_1(s) = sA_{-1} + \sum_{k=0}^{\infty} sA_k G_1^{k+1}(s), \quad (6.37)$$

where we say $G_1^0(s) = I$.

Proof: Firstly, Setting $i = 1$ in (6.34), we get

$$R_{0,1}(s) = \left[\sum_{k=1}^{\infty} sB_k G_1^{k-1}(s) \right] (I - \Phi_0(s))^{-1}. \quad (6.38)$$

By replacing (6.38) into (6.35), we have

$$\begin{aligned} \Phi_{0,0}(s) &= sB_0 + \left[\sum_{k=1}^{\infty} sB_k G_1^{k-1}(s) \right] (I - \Phi_0(s))^{-1} (I - \Phi_0(s)) G_{1,0}(s), \\ &= sB_0 + \left[\sum_{k=1}^{\infty} sB_k G_1^{k-1}(s) \right] G_{1,0}(s). \end{aligned} \quad (6.39)$$

Next, by the similar method we set $i = 1$ in (6.33) to obtain

$$R_1(s) = \left[\sum_{k=1}^{\infty} sA_k G_1^{k-1}(s) \right] (I - \Phi_0(s))^{-1}. \quad (6.40)$$

Replace (6.40) into (6.31), then we yield

$$\begin{aligned} \Phi_0(s) &= sA_0 + \left[\sum_{k=1}^{\infty} sA_k G_1^{k-1}(s) \right] (I - \Phi_0(s))^{-1} (I - \Phi_0(s)) G_1(s), \\ &= sA_0 + \left[\sum_{k=1}^{\infty} sA_k G_1^{k-1}(s) \right] G_1(s). \end{aligned} \quad (6.41)$$

Substitute Φ_0 in (6.32) with (6.41), then

$$G_{1,0}(s) = sB_{-1} + \left[\sum_{k=0}^{\infty} sA_k G_1^k(s) \right] G_{1,0}(s), \quad (6.42)$$

where we say $G_1^0(s) = I$.

By replacing (6.41) into (6.29), we have

$$G_1(s) = \sum_{k=-1}^{\infty} sA_k G_1^{k+1}(s), \quad (6.43)$$

where we say $G_1^0(s) = I$. □

Chapter 7

Concluding remarks and future work

In chapter 3, we aim to characterize decay in a weak sense of $\{\pi_n\}$ along level direction for the $GI/G/1$ -type Markov chain with finitely many background states under the assumption: A is irreducible. In Theorem 3.1 (i), we obtain the decay rate in a weak sense of $\{\pi_n\}$ along level direction is $\frac{1}{\alpha}$ if $A(\alpha)$ is 1-positive recurrent, and $\alpha \leq \phi_{B+}$, however, for other cases, we need impose additional assumptions to get the decay rate in a weak sense of $\{\pi_n\}$ along level direction. For example, we assume that all elements of $\pi(z)$ have identical convergent radius in Theorem 3.1 (ii). But for the $M/G/1$ -type Markov chain with finitely many background states, we know that all elements of $\pi(z)$ have identical convergent radius ϕ_{B+} if $\phi_{B+} < \alpha$ since we have the representation

$$\pi(z) = (\pi_0 B(z) - \pi_1 A_{-1})[I - A(z)/z]^{-1}.$$

Then we have a refined result for the $M/G/1$ -type Markov chain. Can we remove additional assumptions from the $GI/G/1$ -type Markov chain to refine the results? What are results if A is reducible? On the other hand, the method used in Theorem 3.1 (i) can not be used to models with countably many background states since we can not express the

inverse of $I - R(z)$ in the same way. But the method in Theorem 3.1 (ii) may be available to models with countably many background states. By the Markov renewal theory (see Theorem 2.7), $\{\pi_n\}$ is one of the solutions of the equation

$$g_n = g_0 R_{0,n} + \sum_{k=1}^{n-1} g_k R_{n-k}, \quad n \geq 1,$$

which is unique under appropriate assumptions and convolutions of R -measures (Is it possible that $\{\pi_n\}$ is the minimal solution of the above equation). Once we have it, we can define the generating function of $\{\pi_n\}$. Then we may use the method in Theorem 3.1 (ii) to analyze decay rates in a weak sense for models with countably many background states. Next, we want to emphasize that $e^{-\Lambda} : 0 < e^{-\Lambda} < 1$ in chapter 3. But $e^{-\Lambda}$ may take values 0 and 1. If $e^{-\Lambda} = 1$, it corresponds to $\{\pi_n\}$ is heavy-tailed; if $e^{-\Lambda} = 0$, it corresponds to the case that $\{\pi_n\}$ decays faster than geometric speed. Then do we have the decay rate in a weak sense is 1 in heavy-tailed case, and 0 in the case that $\{\pi_n\}$ decays faster than geometric speed? It is possible to use the method in this chapter to analyze this question. In chapter 4, we analyze exact decay of $\{\pi_n\}$ along level direction for the $GI/G/1$ -type Markov chain with finitely many background states under light-tailed assumption. What are the results in heavy-tailed case? Li and Zhao [64] characterized sub-exponential decay. We probably have results for polynomial decay by taking derivative on both sides of the equation

$$z^n \pi_n = \sum_{j=1}^{n-1} \pi_0 z^j R_{0,j} \sum_{k=0}^{\infty} z^{n-j} R_{n-j}^{*k}.$$

The conjectured results:

if $\lim \pi_0 n^l B_n = c > 0$, $\phi_{B^+} < \phi_{A^+}$, and $\{A_n\}$ is light-tailed, then $\{\pi_n\}$ has polynomial decay; if $\lim \pi_0 n^l B_n = c > 0$, $\phi_{B^+} < \phi_{A^+}$, and $\sum_{n=1}^{\infty} n^l A_n$ is light-tailed, then $\{\pi_n\}$ has polynomial decay. On the other hand, Markov renewal theory is available for models with countably many background states. Miyazawa and Zhao [76] dealt with geometric

decay by Markov renewal theory. Is it possible for us to use Markov renewal theory to analyze an exact geometric decay corresponding to Theorem 4.1. Furthermore, we expect to discuss the following problems:

1. Discuss large deviation and exact asymptotic phenomena in some network models. For example, [26] obtained the large deviation result for modified Jackson Model.
2. Discuss large deviation and exact asymptotic phenomena in level-dependent models.

In chapter 5, we obtain lower bounds of the first hitting times to levels, which leads to a necessary condition to polynomial ergodicity. In fact, this result can be extended to more general cases. If we know that for any non-negative and non-decreasing function f , $\mathbb{E}_{(0,j)}[f(\tau_0)] < \infty$, $\sum_{k=0}^{\infty} k B_k e^t < \infty^t$, and $\sum_{k=0}^{\infty} k A_k e^t < \infty^t$, then we can get $\sum_k f(k) A_k e^t < \infty^t$ and $\sum_k f(k) B_k e^t < \infty^t$. We hope to know that the inverse of this result is true, for example, f is a sub-geometric rate function? Højgaard and Møller [44] showed that this was true for both geometric and polynomial cases by coupling method and stopped random walk. Is this method available for more general cases? Next, can the method by Højgaard and Møller [44] be used to discuss ergodicity for models with countably many background states? In Højgaard and Møller [44], the authors borrowed a geometric upper bound of phase-type distribution. Since phase-type distribution is defined by a finite states chain, we do not have it for models with countably many background states. In [22], by a drift condition to ordinary ergodicity, the authors gave a geometric upper bound to the first passage probability for a special model with countably many background states. Is it possible to combine this bound and the method by Højgaard and Møller [44] to provide sufficient condition to ergodicity (with different rate)? On the other hand, we obtain necessary condition to general ergodicity based on a random walk typed structure (see [49]). Do we have that models with countably many background states are random walk type? If it is random walk type or a special case is random walk

type (Intuitively, the model in [22] is random walk type), then we can use the method in chapter 5 to discuss necessary condition on it. If it is not random walk type, can we use this method or some other methods together with this method to analyze necessary conditions?

Results of the RG -factorization obtained in chapter 6 can be used to do analysis of ergodicity. For example, we can provide a different proof to Theorem 3.1 in [67]. It is interesting to investigate if this approach remains valid for ergodicity analysis for the $GI/G/1$ -type Markov chains.

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