**Ionospheric gravity wave interactions and their
representation in terms of stochastic partial
differential equations**

by

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Victor Nijimbere

To my son Rumuri-King Nijimbere and to you all who try to make this world better
regardless of your race, your origin, your ethnicity or religion

Abstract

Phenomena in nature that involve diffusion and convection of matter and propagation of waves, e.g., the propagation of waves in geophysical flows, can exhibit randomness properties and thus need to be modeled by stochastic partial differential equations (SPDEs). For example in the ionosphere, the region in the upper atmosphere where there are high concentrations of ions and electrons, wave interactions are influenced by electromagnetic forces that fluctuate randomly in time, and are thus modeled by SPDEs.

In this thesis we model interactions between atmospheric waves and the ionosphere induced by upward propagating atmospheric gravity waves (AGWs) starting with the equations of conservation of mass, momentum and energy, and Maxwell's equations. Two important problems are examined: the problem in which the ionosphere is treated as a deterministic medium and the wave interactions are governed by nonlinear partial differential equations (PDEs), and the problem in which the ionosphere is a random medium and the governing equations are nonlinear stochastic partial differential equation (SPDEs) driven by the Brownian motion.

In the stochastic case we make use of numerical methods based on Wiener Chaos expansions (WCE) which are effective methods for solving SPDEs driven by Brownian motion. The accuracy of this method is accessed by comparing the results with the exact analytical or semi-analytical solutions for some problems involving stochastic evolution equations comprising the stochastic heat and stochastic advection-diffusion equations, and the stochastic Burgers' equation. In the deterministic case, we derive analytical solutions for some special simplified configurations and then carry out numerical simulations for time-dependent nonlinear configurations.

The results of the simulations of our analytical and numerical models are compared with the conclusions from previous studies which are mainly observations. Our results explain several observed phenomena arising from the interactions of the atmospheric gravity waves with the ionosphere.

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Contents

Abstract	i
Acknowledgements	ii
List of figures	vii
List of symbols	xx
List of acronyms	xxx
1 Introduction	1
1.1 Atmosphere-ionosphere interactions	1
1.2 Review of previous work	5
1.2.1 Observational studies	5
1.2.2 Analytical studies	10
1.2.3 Numerical studies	12
1.3 New ideas and original contributions	14
1.4 Overview of the thesis	16
2 Governing equations for atmosphere-ionosphere interactions	19
2.1 The effects of the ionosphere on upward propagating waves from the atmosphere	19
2.2 The effects of atmospheric waves and solar winds on the ionosphere .	20
2.3 Governing equations for gravity wave-ionosphere interactions	22

2.4	Streamfunction-vorticity formulation for AGWs and ionospheric disturbances	24
3	A simple model for the effects of the ionosphere on atmospheric gravity waves in the presence of a strong constant magnetic field	31
3.1	Steady solution of the problem where the ion drag force is due to a strong constant magnetic field in the F region	31
3.1.1	The case of the neutral atmosphere ($\nu_{ni} = 0$)	36
3.1.2	The case of the ionosphere ($\nu_{ni} \neq 0$)	37
3.2	Time-dependent solution in the long-wave limit	41
3.3	The effect of neutral gravity waves on the ionospheric F region	44
3.4	Numerical solution of the time-dependent problem for a strong constant magnetic field	46
3.4.1	Numerical implementation	46
3.4.2	Results of the simulations	51
3.5	Dimensional analysis	61
4	Wave interactions in a random ionosphere	63
4.1	Electromagnetic wave equations in a nonhomogeneous medium	63
4.2	Electromagnetic wave propagation in a random medium with weakly-random time variation	65
4.3	Approximate solution in an isotropic medium with weakly-random fluctuations in time	68
4.4	Stochastic vorticity and energy equations for the ionospheric disturbances in an isotropic random medium with weakly-random fluctuation in time	70
4.4.1	The stochastic electromagnetic field in the vertical plane . . .	71
4.4.2	Stochastic equations for the ionospheric disturbances	73
5	The Itô formulation and the Wiener Chaos Expansion (WCE) method for solving SPDEs	76

5.1	The Itô formulation and application to stochastic differential equations (SDEs)	77
5.1.1	Itô formulation	77
5.1.2	The Itô formula and application for solving SDEs	78
5.2	Wiener Chaos Expansion (WCE) method for solving SPDEs	81
5.2.1	Some properties of Hermite polynomials	83
5.2.2	The Wiener chaos expansion	87
5.2.3	Martingale application	90
5.3	Application of the WCE method to the stochastic Burgers' equation .	91
5.4	Numerical tests of the WCE method	94
5.4.1	Numerical solution of the stochastic heat equation	95
5.4.2	Numerical solution of the stochastic advection-diffusion equation	97
5.4.3	Analytical solutions of the stochastic Burgers' equation	99
5.4.4	Numerical solution of the stochastic Burgers' equation	104
6	Numerical simulations of atmospheric gravity wave-ionosphere interactions in a random ionosphere	108
6.1	Governing equations	108
6.2	Application of the WCE method	111
6.3	Results of the numerical simulations	116
6.3.1	Configuration 1: EHD-gravity wave interactions and the generation of TIDs ($\Upsilon_1 = \Upsilon_2 = 0, \Upsilon_3 \neq 0$)	117
6.3.2	Configuration 2: MHD-gravity wave interactions and the generation of TIDs ($\Upsilon_1 \neq 0, \Upsilon_2 \neq 0, \Upsilon_3 = 0$)	122
6.3.3	Configuration 3: EMHD-gravity wave interactions and generation of TIDs ($\Upsilon_1 \neq 0, \Upsilon_2 \neq 0, \Upsilon_3 \neq 0$)	130
7	General discussion and conclusions	140
7.1	Discussion	140
7.2	Future work	147
7.2.1	Rossby wave (RW) interactions in the ionosphere	147

7.2.2	Other future work	151
A	Atmospheric gravity wave propagation	153
A.1	Basic equations of fluid dynamics	153
A.1.1	Conservation of mass, momentum and energy	153
A.1.2	The Boussinesq approximation	155
A.2	Atmospheric gravity wave propagation governing equations	158
A.3	The energy equation under the Boussinesq approximation	159
A.4	Direction of propagation of gravity waves	161
B	Basics of electromagnetic wave theory	163
B.1	Time-independent electromagnetic fields	164
B.1.1	Electric field and Coulomb's law	164
B.1.2	Magnetic field and Ampere's law	166
B.2	Time-dependent electromagnetic fields and Maxwell's equations	168
B.2.1	Faraday's law of induction and Maxwell's equations	168
B.2.2	Electric and magnetic polarizations	169
B.3	Electromagnetic wave equations in a homogeneous medium	170
B.4	Energy, Poynting vector and power	171
C	Mathematical background	173
C.1	Basic probability theory concepts	173
C.1.1	Random variables, probability measure and distribution	173
C.1.2	Random vectors, expectation, covariance and independence	180
C.2	Gaussian process	184
C.2.1	The isonormal process	186
C.2.2	Regularity of a random process	188
C.3	Martingale measures	189
C.3.1	Martingales	189
C.3.2	General Martingale measures	192

Bibliography**200**

List of Figures

1.1	Atmospheric temperature and ionospheric electron density profiles. Source: www.astrosurf.com/luxorion/qs1-hf-tutorial-nm7m3.htm .	2
3.1	Linear simulation of long upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region ($\delta = 0$): contour plots of the streamfunction $\psi(x, z, t)$ obtained at $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$ and the neutral-ion collision frequency is $\nu_{ni} = 10$.	52
3.2	Linear simulation of long upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region ($\delta = 0$): plot of the the real part of the amplitude $\mathcal{R}e\{\phi(z, t)\}$ as a function of time at fixed height $z = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$ and the neutral-ion collision frequency is $\nu_{ni} = 10$.	52

- 3.3 Linear simulation of long upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region ($\delta = 0$): plot of the numerical solution $\psi(x, z, t)$ against the steady solution as a function of the height z at fixed $x = \pi/2$ and time, (a) $t = 4$, (b) $t = 10$. At $t = 10$ the numerical solution matches perfectly with the analytical steady solution derived in section 3.1 showing that the solution reaches a steady state as $t \rightarrow \infty$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$ was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$ and the neutral-ion collision frequency is $\nu_{ni} = 10$ 53
- 3.4 Linear simulation of upward propagating gravity waves (AGWs) in the neutral atmosphere ($\nu_{ni} = 0$); (a) contour plot of the streamfunction at the time $t = 15$, (b) plot of the streamfunction as function of altitude z at fixed $x = 1.43$ and time $t = 15$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$ 53
- 3.5 Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region: contour plots of the streamfunction $\psi(x, z, t)$ obtained at a $t = 30$, with the magnetic dip angle (a) $\theta = 0$, (b) $\theta = \pi/4$ and (c) $\theta = \pi/2$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$. 56
- 3.6 Analytical solution: plot of the damping rate m_I of upward propagating AGWs in the F as a function of the magnetic dip angle θ . A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$. 57

- 3.7 Analytical solution: plot of the wavenumber m_R of upward propagating AGWs in the F as a function of the magnetic dip angle θ . A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$. The straight line corresponds to the vertical wavenumber in the neutral atmosphere $m = \left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right)^{1/2}$ 57
- 3.8 Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region (effects of the ionosphere on AGWs): plot of the streamfunction $\psi(x, z, t)$ as a function of time t at fixed $x = 2.25$ and height $z = 11$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$ 58
- 3.9 Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region (effects of the ionosphere on AGWs): plot of the streamfunction $\psi(x, z, t)$ as a function of height z at fixed $x = 2.25$ and time $t = 30$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$. The solid, the dashed and dotted lines correspond to the upward propagating gravity waves in a strong magnetic field in the ionospheric F region at the magnetic dip angles $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$ respectively. 59

- 3.10 Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region: effects of atmospheric gravity wave (AGW) propagation on the ionosphere (traveling ionospheric disturbances, TIDs) in a strong magnetic field in the ionospheric F region. Plot of the perturbation number density $\mathcal{N}_i(x, z, t)$ for the ion species as a function of the height z at fixed $x = 2.7$ and time 25. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$. The solid, the dashed and dotted lines correspond to the upward propagating gravity waves in a strong magnetic field in the ionospheric F region at the magnetic dip angles $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$ respectively. 60
- 3.11 Nonlinear simulation against linear simulation ($\varepsilon = 0.05$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region (effects of the ionosphere on AGWs): Fourier spectrum $\hat{\psi}(k, z, t)$ as a function of the wavenumber k at fixed height $z = 28$ and time $t = 25$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$. The magnetic dip angle $\theta = 0$ 61
- 5.1 Plot of $u(x, t)$ as a function of position x . (a) Solution of the deterministic heat equation. (b) Solution of the stochastic heat equation. 96
- 5.2 (a) WCE solution of the stochastic heat equation against exact solution. The solid line corresponds to the exact solution while the dashed line corresponds to the WCE solution. (b) Absolute error. 97
- 5.3 Plot of $u(x, t)$ as a function of position x . (a) Numerical solution of the deterministic advection-diffusion equation. (b) Numerical solution of the stochastic advection-diffusion equation. 99

5.4	(a) WCE solution of the stochastic advection-diffusion equation against the exact solution. The solid line corresponds to the exact solution while the dashed line corresponds to the WCE solution. (b) Absolute error.	99
5.5	Burgers' equation. Plot of $u(x, t)$ as a function of position x . (a) Numerical solution of the deterministic Burgers' equation. (b) Numerical solution of the stochastic Burger's equation.	106
5.6	Burgers' equation. Plot of $u(x, t)$ as a function of time t at the point $x = 0.04$ near the boundary $x = 0$. (a) The semi-analytical solution against the WCE solution. (b) The approximate analytical solution against the WCE solution (5.67).	106
5.7	Burgers' equation. Plot of the absolute error as function of time t at the point $x = 0.04$ near the boundary $x = 0$. The semi-analytical solution error is the difference between the WCE solution and the semi-analytical solution. The approximate analytical solution error is the difference between the WCE solution and the approximate analytical error.	107
5.8	Burgers' equation. Plot of $u(x, t)$ as a function of time t at the point $x = 0.04$ near the boundary $x = 0$. (a) The semi-analytical solution against the approximate solution. (b) The absolute difference between the two approximations.	107
6.1	EHD waves: contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$	118
6.2	EHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.86$ and fixed height $z = 11$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$	118

- 6.3 EHD waves: Fourier spectrum $|\hat{\psi}_\alpha(k, z, t)|$ as a function of the wavenumber k at fixed height $z = 15$ and fixed time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 119
- 6.4 Intermittent EHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.3$ and height $z = 11$, in (a) the random effects are neglected while in (b) the random effects are taken into consideration. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 120
- 6.5 EHD-gravity wave interactions in a random ionosphere: plots of the streamfunction as a function of time t at fixed $x = 2.3$ and height $z = 11$, in (a) effects of GWs on EHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$) and (b) effects of EHD waves on GWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 120
- 6.6 Effect of gravity waves on EHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the EHD waves streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 121
- 6.7 Effects of EHD waves on GWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): Fourier spectrum of the GW streamfunction $|\hat{\psi}(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 121

- 6.8 Effect of EHD waves on gravity waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the gravity waves streamfunction $\psi(x, z, t)$ at time $t = 12$; (a) gravity waves and (b) gravity waves affected by EHD waves. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 122
- 6.9 EHD-gravity wave interactions in the ionosphere: plots of the streamfunction $\psi(x, z, t)$ as function of time t at fixed $x = 2.86$ and $z = 11$; (a) effects gravity waves on EHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$), (b) effects of EHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$) and (c) gravity and EHD waves mutual interactions ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$. . . 124
- 6.10 MHD waves: Fourier spectrum $|\hat{\psi}_\alpha(k, z, t)|$ as a function of the wavenumber k at fixed height $z = 15$, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 125
- 6.11 MHD waves: contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at the time, (a) $t = 1$, (b) $t = 3$, (c) $t = 6$ and (d) $t = 9$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 126
- 6.12 MHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.86$ and height, (a) $z = 11$ and (b) $z = 12.5$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 127
- 6.13 Effects of gravity waves on MHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): Fourier spectrum of the MHD wave streamfunction $|\hat{\psi}_\alpha(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 127

- 6.14 Effects of gravity waves on MHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 128
- 6.15 Effects of MHD waves on AGWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): Fourier spectrum of the AGW streamfunction $|\hat{\psi}(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 129
- 6.16 Effects of MHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): contour plots of the streamfunction $\psi(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 129
- 6.17 Gravity-MHD wave interactions: plots of the streamfunction as function of time t at fixed $x = 2.86$ and $z = 11$; (a) effects gravity waves on MHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$), (b) effects of MHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$) and (c) gravity and MHD waves mutual interactions ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 133
- 6.18 EMHD waves: Fourier spectrum $|\hat{\psi}_\alpha(k, z, t)|$ as a function of the wavenumber k at fixed height $z = 15$ and time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 134

- 6.19 EMHD waves: contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 134
- 6.20 EMHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.86$ and height $z = 11$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 135
- 6.21 Intermittent EMHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.3$ and height $z = 11$, in (a) the random effects are neglected while in (b) the random effects are taken into consideration. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 135
- 6.22 Effects of gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at the time $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 136
- 6.23 Effects of gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): Fourier spectrum of the EMHD wave streamfunction $|\hat{\psi}_\alpha(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 136
- 6.24 Effects of EMHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): contour plots of the streamfunction $\psi(x, z, t)$ at a the time $t = 12$; (a) gravity waves and (b) gravity waves affected by EMHD waves. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 137

- 6.25 Effects of EMHD waves on AGWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): Fourier spectrum of the AGW streamfunction $|\hat{\psi}(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 137
- 6.26 EMHD-gravity waves interaction in a random ionosphere: plots of the AGW streamfunction $\psi(x, z, t)$ as a function of time t at fixed $x = 5.7$ and height $z = 11$; (a) Effects of gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$) and (b) Effects of EMHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$). The solid line in (b) represents the AGWs in presence of the random effects while the dotted-line represents the AGWs in the absence of the random effects. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 138
- 6.27 Gravity-EMHD wave interactions: plots of the streamfunction $\psi(x, z, t)$ as function of time t at fixed $x = 2.86$ and $z = 11$; (a) effects gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$), (b) effects of EMHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{in} = 0.5$) and (c) gravity and EMHD waves mutual interactions ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$ 139

List of symbols

A. Symbols used in Basic Probability concepts

- Ω a space of random outcome
- ω a random outcome
- $\emptyset$ the empty set
- X a random variable
- $\mathcal{F}$ a σ -field
- $\mathcal{F}_t$ a filtered σ -field
- $\mathcal{C}$ a collection of subsets of Ω
- $\sigma(\mathcal{C})$ a sigma-field generated by a collection of subsets of $\mathcal{C}$
- P the probability measure or projection in L^2
- μ law of distribution or measure
- F_X a distribution function of a random variable X
- f_X a density function of a random variable X
- L^1 or $L^1(\Omega, \mathcal{F}, P)$ the family of integrable random variables
- L^2 or $L^2(\Omega, \mathcal{F}, P)$ the family of squared integrable random variables
- $\mathcal{P}$ a predictable σ -field
- $E(X)$ the expectation of a random variable X
- θ the expectation or mean of a Gaussian random variable
- Θ the expectation or mean of a Gaussian random vector

- $var(X)$ the variance of the random variable X
- $cov(X_1, X_2)$ the covariance of two random variables X_1 and X_2
- C a covariance matrix
- C_p^α possible combinations of p objects from a set of n objects
- $\varphi_X(t)$ the characteristic function of the random variable X
- $X(t, \omega)$ a stochastic process
- $\mathbf{G}$ a Gaussian process
- M a martingale or martingale measure
- W the white noise or martingale measure
- $(f \cdot M)(t, \omega)$ a transformed martingale of M by the process f
- $(f \cdot W)(t, \omega)$ the Ito stochastic integral of the process f
- $(f \circ W)(t, \omega)$ the Stratonovich stochastic integral of the process f
- $\bar{Q}$ the covariance functional of the martingale M
- $\mathbb{I}_{\{a,b\}}$ the indicator function
- *i.i.d.* independently identically distributed
- *p.d.f.* probability density function
- *a.s* almost surely or with probability one
- WCE Wiener Chaos Expansion
- $N(0, 1)$ standard normal distribution
- $U(0, 1)$ uniform zero-one distribution
- $\mathcal{G}$ a generating function

- P_n Hermite polynomials
- H_n normalized Hermite polynomials

B. Symbols used in Basics of Electromagnetic Wave theory

- q electric charge
- e^- elementary charge
- I electric current intensity
- $\mathbf{E}$ electric field vector
- $\mathbf{H}$ magnetic field vector
- $\mathbf{D}$ electric flux density
- $\mathbf{B}$ magnetic flux density
- φ electric potential
- V electric potential difference
- ϱ_{el} electrical charge density
- $\mathbf{J}$ electric current density
- $\mathcal{E}_{ind}$ the induced electromotive force
- Φ magnetic flux
- $\mathcal{C}$ curve under a surface $\mathbf{S}$
- ϵ permittivity
- ϵ_o permittivity of free space
- $\langle \epsilon \rangle$ the time-average permittivity over the time t

- $\Delta\epsilon$ the weak perturbation of permittivity
- ϵ_r relative permittivity
- μ permeability
- μ_o permeability of free space
- $\langle\mu\rangle$ the time-average permeability over the time t
- $\Delta\mu$ the weak perturbation of permeability
- μ_r relative permeability
- χ electric susceptibility
- χ_m magnetic susceptibility
- $\mathbf{P}$ electric polarization vector
- $\mathbf{M}$ magnetic polarization vector
- c celerity (or speed) of light
- c_o celerity (or speed) of light in free space (air)
- E a component of an electric field vector $\mathbf{E}$
- H a component of a magnetic field vector $\mathbf{H}$
- $\mathbf{n}$ the unit vector of an electric field
- $\mathbf{n}_T$ the unit vector tangential to the line of a magnetic field

C. General and fluid flow symbols

- x east to east or horizontal space variable
- y south to north space variable or latitude
- z vertical space variable or altitude

- t time variable
- $\hat{\mathbf{x}}$ unit vector in the x -direction
- $\hat{\mathbf{y}}$ unit vector in the y -direction
- $\hat{\mathbf{z}}$ unit vector in the z -direction
- Ψ total streamfunction
- ψ perturbation streamfunction
- $\bar{\psi}$ mean streamfunction
- ξ total vorticity
- $\bar{\zeta}$ mean vorticity
- ζ perturbation vorticity
- ϱ total stratified density
- $\bar{\rho}$ mean density or hydrostatic density
- ϱ_o constant reference density
- ρ perturbation density
- Ψ_α the plasma (α -species) total streamfunction
- ψ_α the plasma perturbation streamfunction
- $\bar{\psi}$ the plasma mean streamfunction
- ξ_α the plasma total vorticity
- $\bar{\zeta}_\alpha$ the plasma mean vorticity
- ζ_α the plasma perturbation vorticity
- ϱ_α the plasma total stratified density

- $\varrho_{\alpha,el}$ the plasma charge density
- $\varrho_{i,el}$ charge density of the ions
- ρ_{α} the plasma perturbation density
- $\bar{\rho}_{\alpha}$ the plasma mean density or hydrostatic density
- p atmospheric pressure
- $\bar{p}$ background fluid flow pressure
- T temperature
- V volume
- C_V specific heat capacity at constant volume
- C_p specific heat capacity at constant pressure
- $\mathcal{R}$ perfect gas constant
- φ amplitude of the streamfunction at the source
- ϕ amplitude of the streamfunction near the outflow boundary
- $\varnothing$ viscous dissipation
- e internal energy
- $\mathbf{q}$ heat flux vector
- $\mathcal{K}$ thermal conductivity
- $\mathcal{K}_{\alpha}$ the plasma thermal conductivity
- κ the neutral fluid heat diffusivity
- κ_{α} the plasma heat diffusivity
- $\mathbf{c}$ phase velocity

- $\mathbf{u}$ velocity of the fluid
- $\mathbf{u}_\alpha$ velocity of α species
- $\mathbf{u}_i$ ion velocity
- $\mathbf{u}_g$ group velocity
- $\mathbf{f}$ exterior force or superposition of exterior forces
- $\mathbf{F}$ force
- c_x x -component of the phase velocity
- c_z z -component of the phase velocity
- u x -component of the velocity
- $\bar{u}$ mean flow velocity
- w z -component of the velocity
- c_{gx} x -component of the group velocity
- c_{gz} z -component of the group velocity
- u_α x -component of the velocity of α -species
- $\bar{u}_\alpha$ mean velocity of α -species
- v_α y -component of the velocity of α -species
- w_α z -component of the velocity of α -species
- u_i x -component of the velocity of the ion (plasma)
- $\bar{u}_i$ mean velocity of the ions
- w_i z -component of the velocity of the ions
- k horizontal wavenumber

- m vertical wavenumber
- β_α gradient of the planetary vorticity in the y -direction
- β_d gradient of the planetary-magnetic dipole vorticity in the y -direction
- f Coriolis parameter
- $\bar{\Omega}$ angular velocity of the earth rotation
- θ the magnetic dip angle
- H scale height of the neutral atmosphere
- N Brunt Väisälä frequency of the neutral atmosphere
- ω atmospheric wave frequency
- U a typical velocity scale
- L_x a typical length scale in the x -direction
- L_z a typical length scale in the z -direction
- δ the square of the aspect ratio
- ω_{Bi} angular gyrofrequency of atomic oxygen ions
- Ω atmospheric wave intrinsic frequency
- ν kinematic viscosity
- ν_{in} ion-neutral collision frequency
- ν_{ni} neutral-ion collision frequency
- $\nu_{\alpha n}$ α species-neutral collision frequency
- $\nu_{n\alpha}$ neutral- α species collision frequency
- ρ_i ion mass density (plasma)

- $\mathcal{N}_\alpha$ concentration of α species or α species number density
- $\mathcal{N}_\alpha$ α species number density perturbation
- $\mathcal{N}_i$ ion concentration or ion number density
- $\mathcal{N}_i$ ion number density perturbation
- $\mathcal{W}_i$ vertical ion velocity perturbation
- $\mathcal{U}_i$ horizontal ion velocity perturbation
- O oxygen atom
- H hydrogen atom
- O^+ oxygen ion
- H^+ hydrogen ion
- k Fourier variable
- $\hat{f}(k) = \mathcal{F}[f(x)]$ Fourier transform of the function $f(x)$
- $f(x) = \mathcal{F}^{-1}[\hat{f}]$ Inverse Fourier transform of the function $\hat{f}(k)$
- s Laplace variable
- $F(s) = \mathcal{L}[f(t)]$ Laplace transform of the function $f(t)$
- $f(t) = \mathcal{F}^{-1}[F(s)]$ Inverse Laplace transform of the function $F(s)$
- B_{north} the northward ($\hat{\mathbf{y}}$ -direction) component of the magnetic flux density $\mathbf{B}$
- B_{east} the eastward ($\hat{\mathbf{x}}$ -direction) component of the magnetic flux density $\mathbf{B}$
- B_{vertical} the vertical ($\hat{\mathbf{z}}$ -direction) component of the magnetic flux density $\mathbf{B}$
- E_{north} the northward ($\hat{\mathbf{y}}$ -direction) component of the electric flux density $\mathbf{E}$
- E_{east} the eastward ($\hat{\mathbf{x}}$ -direction) component of the electric flux density $\mathbf{E}$

- E_{vertical} the vertical ($\hat{\mathbf{z}}$ -direction) component of the electric flux density $\mathbf{E}$
- $\mathbf{S}$ surface vector in space
- S surface in space
- A area of the surface S

List of acronyms

- AGWs atmospheric gravity waves
- GWs gravity waves
- RWs Rossby waves
- E_s sporadic E layer
- SEEK sporadic E layer experiment over Kyushu
- radar radio detector and ranging
- FAR frequency-agile radar
- FM-CW sounder (or FCS) Frequency modulation continuous waves sounder
- MU middle and upper atmosphere
- QP quasi-periodic
- POLAN polynomial analysis method
- UT universal time
- HF high frequency
- CASES Cooperative Atmosphere-Surface Exchange study
- COST Cooperative in the field of Science and Technology (Europe)
- GPS Global Positioning System
- MLT mesosphere and lower thermosphere
- EISCAT (radar) European Incoherent Scatter (radar)
- EHD electrohydrodynamic disturbances

- MHD magnetohydrodynamic disturbances
- EMHD electromagnetohydrodynamic disturbances
- TIDs traveling ionospheric disturbances
- LSTIDs large-scale traveling ionospheric disturbances
- MSTIDs medium-scale traveling ionospheric disturbances
- PDEs partial differential equations
- SDEs stochastic differential equations
- SPDEs stochastic partial differential equations
- TEC total electron content
- WCE Wiener chaos expansion method

Chapter 1

Introduction

1.1 Atmosphere-ionosphere interactions

The ionosphere is a layer of electrons and electrically charged atoms and molecules (ions) in the upper atmosphere extending from an altitude of about 50 km to more than 1000 km. It results from the effects of the ultraviolet and X-ray radiation from the sun which contains enough energy to remove electrons from the gases in the ionosphere. The density of electrons can significantly affect radio wave propagation (Hunsucker and Hargreaves, 2003), and consequently the ability to transmit radio waves over long distances and receive signals, telecommunication (Laštovička, 2006). Radio waves interact with the ionosphere via reflection and refraction from one layer to another, wave diffraction over obstacles and wave scattering (Hunsucker and Hargreaves, 2003).

The ionosphere is made up of layers that have different characteristics, electron charge density due to the type of the radiations received from the sun, locations from the sun, altitude, time of the day, day of year, sunspots, geomagnetic activities and the density of electrons. The (terrestrial) ionosphere is subdivided into three main regions called the *D*, *E* and *F* regions as shown in Figure 1.1.

The lowest of these regions extending from an altitude of about 40 km to 90 km is called the *D* region and is responsible for the absorption of radio waves. The electron density in this region is about $2.5 \times 10^9 \text{m}^{-3}$ during the day and becomes negligible

in the night.

Above the *D* region is the *E* region which extends from an altitude of about 90 km to 160 km. In this region the solar activity is more intense and the electron density is higher than in the *D* region, and depends on the solar zenith angle. The electron density can reach a value of $2 \times 10^{11} \text{m}^{-3}$ the day and decreases at about 10^{10}m^{-3} to night.

Above the *E* region is the *F* region which extends to an altitude of more than 800 km. It is strongly magnetized with a higher concentration of electrons and ions compared to the *D* and *E* regions. Most high frequency radio waves of order of gigahertz are reflected in this region of the ionosphere. The electron density in the *F* region varies from $2 \times 10^{12} \text{m}^{-3}$ the day and decreases to $2 \times 10^{11} \text{m}^{-3}$ at night.

Figure 1.1: Atmospheric temperature and ionospheric electron density profiles. Source: www.astrosurf.com/luxorion/qs1-hf-tutorial-nm7m3.htm.

The neutral atmosphere is also subdivided into several regions depending on how the temperature changes with height. The temperature in the different regions of the atmosphere is shown in Figure 1.1. Meteorological processes (Laštovička, 2006) are various processes that originate in the lower and middle atmosphere (i.e. in the troposphere and stratosphere) and affect the ionosphere; they include electrical and elec-

tromagnetic phenomena and upward propagating waves in the neutral atmosphere. The electrical and electromagnetic phenomena comprise red sprites, blue jets, and other lightning upward-induced phenomena which are responsible for changes in the global electric circuits, lightning-induced whistlers capable to reach the atmosphere. A detailed description of electrical and electromagnetic processes can be found in Singh et al. (2007). Most of the meteorological influences on the ionosphere are generated by upward propagating atmospheric waves which include gravity, tidal and planetary waves.

As a result of solar winds, e.g. magnetic storms, and other phenomena that transport random scatters and perturb the geomagnetic field, these interactions occur in a random fashion and can generate ionospheric disturbances (waves) and electromagnetic waves (Parks, 1991). Downward propagating ionospheric disturbances can reach the lower atmosphere and interact with atmospheric waves and consequently affect the general circulation of the atmosphere (Sun et al., 2003) and, hence, weather and climate (King 1975; Lästovička, 1997; Smirnov, 1984). On the other hand, waves generated in the lower atmosphere propagate upward and sometimes reach the ionosphere where they generate traveling ionospheric disturbances (Yeh and Liu, 1972; Kelley, 2006; Prikryl et al., 2009), and perturb radio wave propagation over long distances (Hunsucker and Hargreaves, 2003).

It is thus important for us to understand the properties of the ionosphere in order to ensure effective telecommunications and to adequately protect satellites (spatial stations) and astronauts who travel in the ionosphere (Prikryl et al., 2009). We can improve our understanding of the ionosphere and model and predict its evolution by using mathematical models to carry out numerical simulations analogous for those used for weather prediction and climate modeling in the lower atmosphere. The governing equations for fluid dynamics based on the law of conservation of mass momentum and energy apply in the ionosphere. However, interactions between molecules, ions and electrons must also be taken into account.

The goal of this thesis is to investigate the interactions between upward propagating atmospheric gravity waves (AGWs), the ionosphere and traveling ionospheric

disturbances (TIDs) by applying analytical and numerical methods to the governing equations. These interactions are represented by partial differential equations (PDEs) and stochastic partial differential equations (SPDEs). A mathematical model used to simulate the interactions of the AGWs waves and ionospheric disturbances is derived for both the deterministic and random media, and assessed by comparing the results of numerical simulations with analytical solutions for some simplified configurations. The results of the analytical and numerical investigations are compared with conclusions from the previous studies, and used to explain some wave phenomenon that occur in the ionosphere.

Weather prediction, climate modeling and ionospheric modeling are all affected by the presence of waves or oscillations in the atmospheric wind velocity. The effects of the earth's rotation result in the creation of planetary Rossby waves and tides, while the effects of the gravitational force and the density stratification are responsible for generating internal gravity waves. In this thesis we focus on the internal gravity waves but we note that similar types of investigations can be carried out for Rossby waves and other large-scale waves as well. Due to the density stratification of the atmosphere there is a variation in the buoyancy forces and the equilibrium is restored by generating internal gravity waves. These waves propagate upward (Nappo, 2002; Sutherland, 2010), and transport momentum and energy. Gravity waves that propagate upward and reach the ionosphere are relatively small-scale waves with horizontal wavelength less than 100 km, while those with horizontal wavelength greater than 100 km (Preusse, 2008) are reflected or absorbed by the shear in the mean flow.

Gravity waves produce large scale effects on the general circulation of the atmosphere and ocean (Nappo, 2002; Sutherland, 2010). Stratospheric sudden warming and clear air turbulence are among the observed phenomena that result from gravity wave interactions. Thus, it is important for climate modeling and weather prediction to understand gravity waves properties and their interactions in the atmosphere.

In geophysical fluid dynamics, specifically in the neutral atmosphere, the governing equations for internal gravity wave propagation are nonlinear partial differential equations (PDEs). Due to the presence of random scatters in the ionosphere, the

ionosphere becomes a random medium and the wave interactions in the ionosphere occur in a random or stochastic fashion. Thus the interactions of gravity waves and ionospheric disturbances and can be modeled by nonlinear stochastic partial differential equations (SPDEs).

In the next section a review of some of the previous studies on the interactions of the upward propagating atmospheric gravity wave and ionospheric disturbances is given. The results of my analytical and numerical investigations will be assessed in comparison with the conclusions of these studies.

1.2 Review of previous work

1.2.1 Observational studies

While analytical and numerical studies of neutral atmosphere-ionosphere interactions are found to be limited, important progress in understanding neutral atmosphere-ionosphere interactions has been made in the past 2 decades due to observations, e.g., Kelley and Miller (1997), Miller et al. (1997), Kagan et al. (2000), Šauli and Boška (2001), Sun et al. (2003), Laštovička and Bourdillon (2004), Laštovička (2006), Takahashi et al. (2006), Otsuka et al. (2013) and Song et al. (2013). These studies investigated the development and evolution of irregular perturbations in the different regions of the ionosphere.

Kelley and Miller (1997) and Miller et al. (1997) reported that the direction of propagation of the nighttime medium-scale traveling ionospheric disturbances (MTIDs) observed using GPS TEC over Arecibo in Puerto Rico and Rikubetsu and Shigaraki in Japan cannot be explained by the classical theory of upward propagating AGWs alone and suggested that electrodynamical forces could play an important role in generating these MTIDs. Kelley and Miller (1997) thus suggested calling these waves electrohydrodynamic (EHD) waves.

Kagan et al. (2000) used observations to discuss the role of the neutral motion in the formation of mid-latitude *E*-region field-aligned irregularities (sporadic E layer or

simply E_s layer). In their analysis of observational data they used the wind velocities obtained during the SEEK (Sporadic E (layer) Experiment over Kyushu) campaign as input to measure the irregularity elongations (the ratio of the wavenumber of a field perpendicular to that of a field-aligned irregularity). Since they did not have neutral wind data for the time of high-altitude echo observations they instead used the theory by Kagan and Kelley (1999) predicting that in order to induce 6.1 m irregularities the wind velocity should exceed a reasonable value of 93 m sec^{-1} , while 3.2 m irregularities need approximately 189 m sec^{-1} , and suggest that this may be the reason why the frequency-agile radar (FAR) operating at 24.5 MHz observed high-altitude backscatter but the MU radar did not. Using a multi-beam MU (Middle and upper atmosphere) radar functioning as a FM-CW (Frequency modulation-continuous waves) sounder (FCS) they observed on an FCS plot E_s layers moving southward with a mean horizontal velocity of about 125 m sec^{-1} assuring a good correspondence with the meridional neutral wind estimated to move at -129 m sec^{-1} by Kagan and Kelley (1999). According to their data analysis, some important features of the analyzed data were the 23 min periodicity exceeding the Brunt-Väisälä frequency about 5 min in the E region in both QP (Quasi-periodic) echo occurrence and the 2 MHz E_s observed for several hours with amplitude of $\pm 5 \text{ km}$. The downward phase progression of stripes of QP echoes during 20:40-22:00 allowed them to propose that the motion observed in the considered E region was generated by a single gravity wave and that if correct then while crossing the MU radar beam the sinusoidal shape E_s moving southward should give downward QP stripes with a range rate $\dot{R}$ corresponding to a line-of-sight projection of E_s velocity given by $\dot{R} = 125 \times \cos 39^\circ \cong -97 \text{ m sec}^{-1}$, a value very close to the observed range rate of about -100 m sec^{-1} , where 39° is the MU radar elevation angle.

Šauli and Boška (2001) studied the effects of acoustic gravity waves in the ionosphere connected with the passages of weather fronts based on the results of ionospheric vertical sounding. From the analysis of the results of 5-min rapid-sequence soundings on 27 October-3 November 1997 campaign, and a regular 15-min vertical ionospheric soundings collected in 1990 at mid-latitude ionospheric observatory of

Průhonice ($49.9^{\circ}N$, $14.5^{\circ}E$), they observed that the vertical component of the group velocity of the detected waves confirms the presence of acoustic gravity waves propagating upward in the ionospheric F region originated from below 180 km altitude. They converted the results of ionograms from the rapid run ionospheric soundings into a sequence of a true height electron density profiles by polynomial analysis method (POLAN) and transformed the electron density profiles to time variations of the electron concentrations at fixed height with 5 km steps from 150 to 235 km with the goal consisting of finding possible quasi-periodic oscillations of acoustic gravity waves (AGWs) using the method of correlogram allowing to compute the correlation coefficients of the time sequences of electron concentration in given altitudes with the periodic function. According to the analysis of collected data in the time interval 7-10 UT on 31 October 1997, and during the afternoon of 3 November 1997 random wave-like structures were also observed. Šauli and Boška (2001) computed the acoustic gravity waves properties (amplitude spectra and corresponding level of significance) from the vertical sounding data and their probability of occurrence as well.

Sun et al. (2003) focused their study on the turbulence intermittency generated by the solitary waves and atmospheric waves propagating horizontally and downward, and intermittent turbulence associated episodes with pressure change and wind direction shifts adjacent to the ground, using the unprecedented observational facilities deployed during the 1999 Cooperative Atmosphere-Surface Exchange study (CASES-99). They found that during the passage of both the solitary and internal gravity waves, local thermal and shear instabilities were generated as cold air and was pushed above warm air and wind gusts reached the ground and that the directional difference between the propagation of the internal gravity waves and the ambient flow led to lateral rolls. They concluded that nonlocal disturbances are responsible for local thermal and shear instabilities leading to intermittent turbulence in nocturnal boundary layers, and that the origin of these nonlocal disturbances needs to be understood to improve mesoscale numerical model performance.

Laštovička and Bourdillon (2004) described the main results achieved in COST 271 in the areas related to large-scale fluctuations of planetary and gravity waves,

development of a new type of high frequency (HF) channel simulator, geomagnetic storm effects on the ionospheric F region, the sporadic E-layer and Spread- F phenomena. They also described the HF radio wave propagation over northerly paths, the way the bit rate in ionospheric radio link can be increased. They concluded that in spite of the considerable progress achieved, various open questions must be addressed in the future, and that for instance, the planetary and gravity wave effects on the ionosphere and radio wave propagation require better quantification into possible predictions.

The investigations of Laštovička (2006) focused mainly on the study of upward propagating atmospheric waves from below the ionosphere that propagate into the ionosphere mostly directly, with the exception of planetary waves that are able to propagate to the F region height only indirectly via various potential ways like modulation of the upward propagating tides. He also mentioned that the waves may be altered during upward propagation via nonlinear interactions in the MLT region. The goal of his study was mainly to determine the origin of upward propagating waves in the neutral atmosphere both from the point of view of vertical coupling in the atmosphere-ionosphere system, and for applications in radio wave propagation and telecommunications, as they are responsible for a significant part of the radio wave propagation predictions.

Laštovička (2006) observed that gravity waves are very important in the momentum and energy budget in the MLT region and are either of meteorological origin coming to the ionosphere from below or are of auroral origin coming quasi-horizontally from the auroral zone, or are excited in situ by solar terminator or solar eclipse. In the lower ionosphere gravity wave activities result from gravity waves propagating in the neutral atmosphere from below. The profile of the electron density measured by rocket in the 1960s showed well-developed gravity wave type vertical structures, but due to their small number and irregular geographic distribution it was not possible to determine the gravity statistical characteristics and long-term changes. Laštovička (2006)'s studies of gravity waves in the E region focused mainly on the E_s layers since they play some role in the formation of these layers via enhanced neutral winds

associated with upward propagating gravity waves. Observations showed that there are very short gravity waves in echoes from blanketing E_s layers at Waltair in India. E_s layers were used as a tracer of atmospheric gravity waves and the intensity and position of these layers showed evidence of the effects of gravity waves activity. It was observed in northern Scandinavia using EISCAT enhancements of electron density in the E region attributed to periodically modulated of precipitating electrons controlled by oscillations in the magnetospheric tail. This suggests that gravity waves may be of corpuscular origin.

In the F region (Laštovička, 2006), atmosphere scientists group gravity waves into three categories according to their size, large-scale (horizontally propagating mainly of auroral origin), medium-scale (mainly coming from below) and small-scale with large frequency. Meteorological processes are a quasi-permanent source of gravity waves coming from below. Deep tropical convection, mesoscale convective complexes, hurricanes, tornados, cyclones, upper troposphere jet, or flow over topography (mountain waves) are meteorological sources of gravity waves.

Aveiro et al. (2009) inferred the vertical electric field from the Type II irregularity velocities obtained by a radar installed at the São Luiz Space Observatory. They used harmonic analysis in their inference of the vertical electric field, and their inference of the vertical field was based on the geomagnetic field and atmospheric gravity waves (AGWs). They calculated the ratio between AGW related electric field and the total vertical electric field which revealed that there was a production of an additional electric field due to gravity wave neutral wind (AGWs).

Otsuka et al. (2013) observed two dimensional structures of medium-scale traveling ionospheric disturbances (MTIDs) over Europe using the TEC (total electron content) data obtained from more than 800 GPS receiver networks. From the statistical analysis of the TEC maps data obtained 2008, they found that the observed MTIDs can be categorized into two groups, the daytime MTIDs and the nighttime MTIDs. They mentioned that most of the daytime MTIDs occur in the winter and propagate southward, and speculated that the daytime MTIDs could be caused by atmospheric gravity waves in thermosphere; whereas the nighttime MTIDs which

propagate southeastward could be caused by the polarization electric fields.

Song et al. (2013), analyzed the TEC data from the 246 GPS receivers in and around China during the medium storm on 28 May 2011, and detected two events which they called larger-scale traveling ionospheric disturbance (LSTID) events. They associated one of the events to the variations of the magnetic field while they associated the other event to the AGWs generated by the Joule heating of the equatorial electrojet.

All these observational studies indicate that there are still many open questions and some uncertainty concerning GW interactions with the ionosphere. This suggests that there is a need for rigorous and detailed analytical and numerical studies.

1.2.2 Analytical studies

Analytical studies of neutral wave-ionosphere interactions were mostly limited to the effects of the ionosphere on upward propagating waves (mostly planetary and gravity waves), e.g., Geisler (1966), Yeh and Liu (1972), Khantadze et al. (1976), Kelley (2006), Shalimov et al. (2009) and Aveiro et al. (2009).

Geisler (1966) studied atmospheric winds in the middle latitude F -region that arise from the pressure gradients associated with the diurnal bulge of the atmospheric density. His investigation dealt with calculating pressure gradients at F -region heights from a specific model atmosphere and estimating from the equation of motion the variation of magnitude and direction of wind with local time, season and epoch of solar cycle. He restricted his analysis to mid-latitudes because of the nature of the equations of motion, neglected the electrodynamic effects associated with the ions, did not take into account the effects of neutral winds on the ionosphere and made other several simplifications. He assumed that the diurnal bulge may be viewed as an atmospheric tide producing a wind system oscillating in time like $\exp(i\bar{\Omega}t)$ so that the time-derivative term of the hydrodynamic Navier-Stokes equation is multiplied by the frequency of the oscillations $\bar{\Omega}$ and therefore has a magnitude which is at most a factor of 2 less than that of the Coriolis term. He neglected the transport of momentum by fluid motion by assuming that the fluid motion in the middle latitude

F -region is less effective than molecular viscosity. He tackled the problem using two approaches: first by neglecting the viscosity leading to a geostrophic wind situation in which the ion drag dominates the Coriolis force due to a high concentration of ions, and secondly by taking into account the viscous effects and considering the special case in which the pressure gradient is directed positive along the meridian and is a linear function of height z , thus allowing him to express the velocity in terms of Bessel functions. According to Geisler (1966), the conclusion reached regarding the effects of viscosity is dependent upon the validity of the assumed lower boundary condition that the atmosphere is not in motion in the region near 140 km. He noted that there is evidence of both theoretical and observational nature that tides in the lower atmosphere penetrate the ionosphere to this level, but still the atmospheric models of the diurnal bulge are not applicable to this problem and these models could be used in the region above 180 km.

Yeh and Liu (1972) studied the effects of oxygen ions on upward propagating gravity waves in the ionospheric F by assuming the wind vertical velocity is that of a plane wave with an amplitude varying with height. In their analysis they used the fact that the atomic oxygen ion gyro-frequency is very large compared to the neutral-ion collision frequency and the gravity wave frequency. Under this condition the ions spiral about the line of the magnetic field. They solved the hydrodynamic Navier-Stokes equation taking into account the effects of the ion drag at a dip angle of zero corresponding to gravity wave propagation in the magnetic meridian plane at the magnetic equator, and obtained an explicit expression for the vertical velocity of the waves. But their results seem to be incorrect because they did not use a correct expression for the ion drag; they used the expression for the frictional force of ions on neutral wind instead of using the ion drag force (Kelley, 2006). Yeh and Liu (1972) mentioned the existence of traveling ionospheric disturbances (TIDs) and obtained a general form of the solution to the linearized continuity equation for ions in the presence of plane gravity waves. They failed in their analysis of internal wave-ionosphere interactions in the E region by trying to compute the vertical component of the electric field vector which must be known using Maxwell's equations from the

expression for the geomagnetic field vector.

Khantadze et al. (1976) carried out and generalized the results of the theoretical investigations of the horizontal wind flow in the ionosphere in the presence of vertical velocities, adopted boundary conditions by analogy with the planetary boundary layer and took into account the effects due to the electromagnetic fields but did not consider the effects of neutral wind on the ionosphere. By introducing a complex wind velocity $\Phi = u + iv$ where u and v are the x -component and the y -component of the wind velocity, they were able to combine the momentum equations into a single equation and reduce the resulting equation into a Volterra integral equation of second type and solved it using the method of successive approximations in series form to construct a Green's function. From their results they concluded that the wind velocity vector changes with height in turbulent electro-conductive atmosphere according to a logarithmic spiral rotating clockwise. According to their solution of the non-stationary problem, they found the wind field adapts to the pressure field during a specific time interval and that the transition of non-stationary wind to the stationary wind takes about one day in the E layer while it is of order of one hour in the F layer.

Shalimov et al. (2009) studied the problem of plasma instability processes for internal gravity waves propagating through the ionospheric E region. They showed that the growth rate of instabilities depends on the perturbation wavelength, the gravity wave parameters and their direction of propagation and found that the conditions for the instability are favorable when the vorticity of the neutral winds becomes antiparallel to the geomagnetic field. They observed that this does not require the presence of a sporadic E_s layer. They developed a stability condition of the E layer plasma. They evaluated the perturbed velocity of ions then used it to solve the continuity equation for the concentration of ions.

1.2.3 Numerical studies

Numerical studies focused mainly on the effects of neutral waves on the ionosphere (e.g., Huang et al., 1998; Yokoyama et al., 2005). Huang et al. (1998) considered

a configuration similar to that of Yeh and Liu (1972) in which nonlinearities and time-dependence in the equations are neglected in the governing equations and used computer simulation to study ionospheric perturbations (TIDs) produced by gravity waves in the mid-latitude F region. They took into account the fact that the amplitude and vertical wavelength of gravity waves vary with height. In their study Huang et al. (1998) considered three cases; the gravity wave velocity being a simple harmonic function of x and z , $\mathbf{u}(x, z) = \mathbf{u}_0 e^{i(kx + mz)}$ with $\mathbf{u}_0$ being a vector of constant components, the gravity wave velocity amplitude decaying with height z , meaning that the vertical wavenumber is of the form $m = m_R + im_I$ where m_R and m_I are real and imaginary parts of m , and a dispersive case where the vertical wavenumber m is a function of height z , $m = m_0 e^{\alpha_m z}$ with α_m being a negative constant and m_0 the vertical wavenumber at the lower boundary of the domain considered. Their justification to study the case in which the vertical wavenumber m decays with height comes from the observations by Hearn and Yeh (1977) which show that the vertical wavenumber decreases by about 1 order of magnitude over the altitude range of 200-600 km. For simplification purposes Huang et al. (1998) imposed a zero boundary condition at the lower boundary of the F region and solved the steady electro-hydrodynamic Navier-Stokes equation using numerical simulations. They concluded that the presence of plasma diffusion makes the vertical profile of the TIDs remarkably different from that of the gravity waves that generate them but does not cause the ionospheric structures to become field aligned with electric field. They also concluded that the upturning of the phase surface of TIDs produced by gravity waves is caused solely by the increase with the height of the vertical wavelength of the wave independently of whether the amplitude of the gravity waves also increases or decreases with the height. Their numerical simulations of TIDs produced by gravity waves with increasing wave amplitude are consistent with the Middle and upper atmosphere (MU) radar observations. They found that theory provides a reasonable estimate of ionospheric perturbations for gravity waves that are uniform in space, but significant differences between the theory and simulations arise when the wavelength and amplitude of gravity waves vary with height. This suggests that nonlinear effects become more important for

large amplitude waves.

Yokoyama et al. (2005) performed numerical simulations of the mid-latitude ionospheric E region based on SEEK and SEEK-2 observations. They used a three-dimensional model with zero acceleration for which they impose periodic boundary conditions in both horizontal directions, and considered that the geomagnetic field is in the meridional-vertical plane with a dip angle of 45° and solved numerically the momentum equations for the metallic and molecular species of Fe^+ and NO^+ respectively. After comparing their numerical simulation results to the observational results of the SEEK and SEEK-2, they figured out that the polarization electric field calculated at the rocket launch time shows similar amplitude and structure to the measurements around the E_s layer altitude, and that the structure of the plasma density and the electric field above the E_s layer observed in the SEEK-2 showed a wave-like pattern up to an altitude of 150 km. Considering a mapping of polarization electric field generates within the E_s layer, they concluded that gravity waves are a possible source of the wave-like structure of the measured electric fields and sub-peaks of the electron density above the main E_s layers.

To study gravity waves-ionosphere interactions I solve the equations derived in chapter 2 along with exact analytical expressions for the electromagnetic waves obtained in chapter 4. I first consider configurations without random fluctuations where the equations are deterministic PDEs. I solve them using pseudo-spectral discretization with the predictor corrector method described in chapter 3. I then consider configurations with random fluctuations where the equations are SPDEs and use a method called Wiener chaos expansion (WCE) developed by Martin and Cameron (1947) and carried out some numerical tests of the method in chapter 5.

1.3 New ideas and original contributions

In this thesis I seek to investigate and explore some of the issues raised by previous observational, analytical and numerical studies:

1. How does the ion drag in the ionosphere affect the structure of gravity waves

that propagate up from the neutral atmosphere into the ionospheric F region? In particular does the ion drag affect the wavelength and the amplitude of the waves?

2. How do the electric and magnetic fields generate ionospheric disturbances?
3. Most of the wave in the ionosphere fluctuate randomly in time. What are the main factors responsible for this?
4. How does the structure of ionospheric disturbances differ from AGWs in the neutral atmosphere?
5. How do the ionospheric disturbances affect the AGWs and vice-versa?

My original contributions in the study of the interactions between the AGWs and the ionosphere are:

- In chapter 2, I write the momentum and the energy equations that model the interactions of the AGWs and the ionosphere in terms of the streamfunction and the vorticity (section 2.4).
- In chapter 3, I examine a simple model which can be used to model the effects of the ion drag on AGWs in the presence of a strong constant magnetic field. Using this model:
 1. I derive exact expressions for AGWs propagating in the ionosphere (sections 3.1 and 3.2).
 2. I write the plasma continuity equation in terms of the stream function (section 3.3).
 3. I carry out linear and nonlinear simulations to examine the effects of the ion drag on upward propagating AGWs (section 3.4).
- In chapter 4:
 1. I derive approximate solutions for the stochastic electromagnetic wave equation in an weakly time-varying isotropic random medium (section 4.2).

2. I derive the stochastic vorticity equation and the stochastic energy equation for modeling ionospheric disturbances in a random ionosphere (section 4.4).
- In chapter 5, I numerically solve some test problems involving stochastic evolution equations using the WCE method and compared numerical solutions against exact solutions.
 - In chapter 6:
 1. I present numerical simulations of the interactions of AGWs and the waves generated by the electric field, electrohydrodynamic (EHD) waves (section 6.3.1).
 2. I carry out numerical simulations of the interactions of AGWs and the waves generated by the magnetic field, magnetohydrodynamic (MHD) waves (section 6.3.2).
 3. I present numerical simulations of the interactions of AGWs and the waves generated by both the electric and magnetic fields electro-magnetohydrodynamic (EMHD) waves (section 6.3.3).

1.4 Overview of the thesis

An overview of this thesis is as follows:

In chapter 2, an overview of the interactions between the neutral atmosphere and the ionosphere is given, i.e. the effects of upward propagating gravity waves in the neutral atmosphere on the ionosphere in general and how the ionosphere responds to neutral wave-like perturbations. I present the equations of motion comprising the Navier-Stokes and continuity equations for neutral wind and the electro-magnetohydrodynamics Navier-Stokes and continuity equations for α species (ions or plasma). These equations are described in Kelley (2006). I reformulate these equations in streamfunction-vorticity form.

In chapter 3, a simple model for the effects of the ionosphere on the atmospheric gravity waves in the presence of strong constant magnetic field is investigated. This is

the configuration studied by Yeh and Liu (1972) where ions spiral about the lines of the magnetic field in the F region. I describe the gravity waves-ionosphere interactions using the perturbation methods and weakly nonlinear theory and derived an analytical exact expression for the long upward propagating atmospheric gravity waves. I describe a numerical predictor-corrector method based on the Adam-Bashforth and Adam-Moulton schemes and show the results of both linear and nonlinear simulations. The linear numerical results are compared with the analytical solutions for the long waves with corresponding numerical simulation results.

In chapter 4, I describe wave interactions in an isotropic nonhomogeneous random medium in which the electromagnetic properties, such as the permittivity and permeability, are random functions of position and time. The focus is on the case of a weakly random medium for which considerable simplifications can be made to the wave equations. I then solve analytically the wave equations which are stochastic partial differential equations (SPDEs) with random coefficients. I describe how randomness affects the electric and magnetic polarizations, and derive and solve the wave equations for both electric and magnetic fields in a weakly random medium assuming that the permittivity and permeability are random functions of time. I derive the governing equation for the interactions of AGWs with the ionospheric disturbances which are stochastic partial differential equations (SPDEs) driven by the Brownian motion.

In chapter 5, I describe the application of the Wiener chaos expansion (WCE) numerical method to some test problems involving stochastic evolution equations and compare the numerical results with analytical solutions.

In chapter 6, the numerical simulations of the interactions of the AGWs and ionospheric disturbances in a random ionosphere are described based on the WCE method, and the results are shown and discussed. These results include those obtained for the simulations of the effects of AGWs on the ionospheric disturbances, those obtained for the effects of ionospheric waves on AGWs and those obtained for the mutual interactions between AGWs and ionospheric disturbances. And the ionospheric disturbances are classified in three categories. The waves generated by the electric field and the

magnetic field are called electrohydrodynamic (EHD) waves and magnetohydrodynamic (MHD) waves respectively, while the waves generated by both the electric and magnetic fields are called electro-magnetohydrodynamic (EMHD) waves.

Finally chapter 7 discusses my results in comparison to previous work and gives a summary of future related work.

In Appendix A, the basic equations used in fluid dynamics are presented (Kundu and Cohen, 2004). These equations are the continuity equation which describes conservation of mass, the Navier-Stokes equation which describes conservation of momentum and the energy equation which describes conservation of energy of a fluid. We make use of the Boussinesq approximation (Spiegel and Veronis, 1960) to simplify and adapt the basic equations of fluid dynamics to geophysical fluid flows, and discuss about gravity wave propagation in the neutral atmosphere.

In Appendix B, I describe the basic concepts of electromagnetism; electrostatic and magneto-static concepts and time-dependent electromagnetic fields. The concepts include electric field and Coulomb's law, magnetic field and Ampere's law, Faraday's law of induction, Maxwell's equations and electric and magnetic polarizations. Electromagnetic wave equations in an isotropic homogeneous medium are discussed, as well as the Poynting theorem. This discussion is based on the books of Minoru and Fujimoto (2007), Cheng 1992 and Yeh and Liu (1972).

In Appendix C, I discuss basic concepts of stochastic analysis needed in the study of SPDEs: basic probability concepts, random variables and vectors and measures; Gaussian process, isonormal process and martingale measures. The definitions of random variables, random vectors, probability measure, sigma fields, Brownian process, covariance matrix, martingales including worthness and martingales measure are given. The proofs of some important theorems and inequalities, Markov's inequality, Tchebychev's inequality, Bulkholder's inequality, Borel-Cantalli lemma and Lebesgue Riemann theorem and other related theorems and propositions, are also given. This discussion is based on the content of several books and papers on probability and stochastic process (Zastawniak and Brzeźniak, 2003; Mikosch, 2004; Billingsley, 1986; Dalang et al. 2000).

Chapter 2

Governing equations for atmosphere-ionosphere interactions

In this chapter I examine the effects of upward propagating waves from the neutral atmosphere on the ionosphere and how the ionosphere responds to perturbations due to those waves. I present the equations that describe these interactions. The derivation of these equations can be found in Kelley (2006). They comprise the Navier-Stokes and continuity and energy equations for the neutral fluid, and the electro-magnetohydrodynamic Navier-Stokes and continuity and energy equations for the α species (ions and electrons).

2.1 The effects of the ionosphere on upward propagating waves from the atmosphere

The governing equations for upward propagating waves in the neutral atmosphere are based on the basic equations of fluid dynamics which are described in Appendix A. In order to take into account the effects of the ionosphere on the upward propagating atmospheric waves, the Navier-Stokes equation (A.4) for the neutral fluid flow has to comprise an additional term which defines the force due to neutral- α -species collisions

in the ionosphere. This equation is given by Kelley (2006) as

$$\varrho \frac{D\mathbf{u}}{Dt} = -\nabla p + \varrho \mathbf{g} + \mu \nabla^2 \mathbf{u} - 2\varrho \bar{\Omega} \times \mathbf{u} - \nu_{n\alpha} \varrho (\mathbf{u} - \mathbf{u}_\alpha), \quad (2.1)$$

where $\mathbf{u}$ is the neutral fluid velocity, ϱ is the neutral fluid density, p is the neutral fluid pressure, $\mathbf{g}$ the acceleration due to gravity, μ the neutral fluid viscosity coefficient, $\nu_{n\alpha}$ is the neutral- α species collision frequency, $\mathbf{u}_\alpha$ is the velocity of the α species and $\bar{\Omega}$ is the Earth's angular velocity. The last term on the left hand side of equation (2.1), $-\nu_{n\alpha} \varrho (\mathbf{u} - \mathbf{u}_\alpha)$, represents a frictional force of the ionospheric plasma on the neutral fluid, and is called the ion drag force.

2.2 The effects of atmospheric waves and solar winds on the ionosphere

The ionosphere is the part of the terrestrial atmosphere where different types of chemical processes permanently take place due to photo-ionization and chemical reactions (ionization due redox reaction) to maintain the high concentration of ions and electrons (see Seinfeld and Pandis (2006)). An example is the case where the oxygen ions, O^+ are produced through photo-ionization of the oxygen atom O and then interacts with the hydrogen atom H (loss of O^+) according to the equations

where h is Plank's constant and f is the frequency of the absorbed photon (particle of light). These chemical processes can be transported by collisions with other moving particles (neutral wind or random scatters) which then transfer their momentum to the α species (ions or electrons) or can be transported by electric potential differences. This can be illustrated mathematically by a continuity equation for the number of α

species per unit volume (number density)

$$\frac{\partial \mathcal{N}_\alpha}{\partial t} + \nabla \cdot (\mathcal{N}_\alpha \mathbf{u}_\alpha) = Q_\alpha - L_\alpha, \quad (2.4)$$

where $\mathcal{N}_\alpha$ is the number of α species per unit volume, $\mathbf{u}_\alpha$ is the α -species velocity, Q_α the rate of production per unit volume and L_α is the rate of loss per unit volume through chemical reactions. In terms of the density ϱ_α (the mass of α species per unit volume), equation (2.4) can be written as

$$\frac{\partial \varrho_\alpha}{\partial t} + \nabla \cdot (\varrho_\alpha \mathbf{u}_\alpha) = (Q_\alpha - L_\alpha) M_\alpha, \quad (2.5)$$

where M_α is the mass of the α species.

Conservation of energy and momentum can result in permanent disturbances of the α species (electrons and ions) of the ionosphere known as ionospheric disturbances. It is when AGWs in the neutral atmosphere propagate into the ionosphere, the collisions between the neutral fluid and the ionospheric plasma generate ionospheric oscillations known as traveling ionospheric disturbances (TIDs). According to Kelley (2006), the motion of the α species is thus modeled by the electro-hydrodynamic Navier-Stokes equation given by

$$\begin{aligned} \varrho_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = & -\nabla p_\alpha + \varrho_\alpha \mathbf{g} + \mu_\alpha \nabla^2 \mathbf{u}_\alpha - 2\varrho_\alpha \bar{\Omega} \times \mathbf{u}_\alpha \\ & - \nu_{\alpha n} \varrho_\alpha (\mathbf{u}_\alpha - \mathbf{u}) + \varrho_{\alpha,el} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}), \end{aligned} \quad (2.6)$$

where $\varrho_{\alpha,el}$ is the electric charge density (electric charge per unit of volume) of the α species and is given by $\varrho_{\alpha,el} = N_\alpha q^\pm$, $q^\pm$ being the electric charge of α species, and the superscript indicating the sign of the electric charge of the α species. ϱ_α is the mass per unit of volume of the α species, $\mathbf{E}$ and $\mathbf{B}$ are electric field vector and the magnetic flux density respectively and $\nu_{\alpha n}$ is the α species-neutral collision frequency. As before, $\mathbf{u}$ is the neutral fluid velocity which may include oscillations in the form of AGWs and $\mathbf{u}_\alpha$ is the α -species velocity which develops oscillations (TIDs) in response

to the forcing from the AGWs.

The equation of conservation of energy for the plasma can be written in terms of the plasma density ϱ_α . In contrast with the equation of conservation of energy for the neutral fluid, equation (A.33), the energy equation under the Boussinesq approximation for the α -species has to comprise an additional term that takes into account the electromagnetic energy

$$\frac{D\varrho_\alpha}{Dt} = \kappa_\alpha \nabla^2 \varrho_\alpha + \frac{\kappa_\alpha}{\mathcal{K}_\alpha} \nabla \cdot (\mathbf{E} \times \mathbf{H}), \quad (2.7)$$

where κ_α is the thermal diffusivity, $\mathcal{K}_\alpha$ is the thermal conductivity and $\mathbf{E} \times \mathbf{H}$ is the Poynting vector, which is described in Appendix B.

2.3 Governing equations for gravity wave-ionosphere interactions

In section 2.2, the equations of motion for the neutral winds and the α species are presented without looking specifically at the type of waves involved in the interactions or causing interactions. In this chapter I address this problem but limit the discussion to the interactions involving gravity waves. I discuss in details the equations describing the wave motions in the ionosphere, atmospheric gravity waves and ionospheric disturbances, and how they relate to each other. Equations describing interactions of TIDs and gravity waves in the vertical plane in terms the vorticity and the streamfunction are derived as in Appendix A for gravity waves propagating in the neutral atmosphere. My original work starts with this section where the governing equations for neutral atmospheric wave-ionosphere interactions are reformulated in streamfunction-vorticity form.

Atmospheric gravity waves (AGWs) propagate in a direction that is approximately perpendicular to the earth's tangent plane and are relatively small-scale waves. So the effect of the Coriolis force on the GW interactions is small and negligible, and can be neglected in the momentum equations (2.1) and (2.6).

The gravity wave-ionosphere interactions governing equations are

$$\varrho \frac{D\mathbf{u}}{Dt} = -\nabla p + \varrho \mathbf{g} + \mu \nabla^2 \mathbf{u} - \nu_{n\alpha} \varrho (\mathbf{u} - \mathbf{u}_\alpha), \quad (2.8)$$

$$\frac{1}{\varrho} \frac{D\varrho}{Dt} + \nabla \cdot \mathbf{u} = 0, \quad (2.9)$$

$$\frac{D\varrho}{Dt} = \kappa \nabla^2 \varrho, \quad (2.10)$$

$$\varrho_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = -\nabla p_\alpha + \varrho_\alpha \mathbf{g} + \mu \nabla^2 \mathbf{u}_\alpha - \nu_{\alpha n} \varrho_\alpha (\mathbf{u}_\alpha - \mathbf{u}) + \varrho_{\alpha,el} (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}), \quad (2.11)$$

$$\frac{1}{\varrho_\alpha} \frac{D\varrho_\alpha}{Dt} + \nabla \cdot \mathbf{u}_\alpha = \frac{1}{\varrho_\alpha} (Q_\alpha - L_\alpha) M_\alpha, \quad (2.12)$$

$$\frac{D\varrho_\alpha}{Dt} = \kappa_\alpha \nabla^2 \varrho_\alpha + \frac{\kappa_\alpha}{\mathcal{K}} \nabla \cdot (\mathbf{E} \times \mathbf{H}); \quad (2.13)$$

with the Maxwell's equations describing the electromagnetic field

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.14)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_\alpha + \frac{\partial \mathbf{D}}{\partial t}, \quad (2.15)$$

$$\nabla \cdot \mathbf{D} = \varrho_{\alpha,el}, \quad (2.16)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2.17)$$

and the continuity equation for the electric charge of the α species

$$\frac{\partial \varrho_{\alpha,el}}{\partial t} = -\nabla \cdot \mathbf{J}_\alpha. \quad (2.18)$$

The vector fields $\mathbf{E}$ and $\mathbf{D}$, and $\mathbf{H}$ and $\mathbf{B}$ are related, respectively, by

$$\mathbf{D} = \epsilon \mathbf{E} \text{ and } \mathbf{B} = \mu \mathbf{H}, \quad (2.19)$$

where ϵ and μ are, respectively, the permittivity and permeability of the medium (see Appendix B).

We can consider the configuration where $\frac{D\varrho_\alpha}{Dt} \ll \varrho_\alpha$ and $(Q_\alpha - L_\alpha)M_\alpha \ll \varrho_\alpha$, this corresponds for example to a configuration where the production rate and the rate of loss of the α species are almost equal ($Q_\alpha \approx L_\alpha$). Then if ϱ_α is enough large, the continuity equation for the α species can be reduced to

$$\nabla \cdot \mathbf{u}_\alpha = 0. \quad (2.20)$$

This form of the continuity equation (2.20) allows us to define a streamfunction for the ionospheric waves as for the gravity waves propagating in the neutral atmosphere. We can then write the momentum equation (2.12) in streamfunction-vorticity form using the same procedure as in Appendix A.

The equations presented in this section will be used in this thesis to investigate the evolution of gravity waves and ionospheric disturbances. The domain considered throughout the thesis is a rectangular domain defined in terms of cartesian coordinates x and z on a vertical plane extending in the west-to-east (x) direction and in the vertical (z) direction. The horizontal south-to-north (y) direction is ignored.

2.4 Streamfunction-vorticity formulation for AGWs and ionospheric disturbances

We consider a two-dimensional configuration where the AGWs and ionospheric disturbances propagate in a rectangular domain perpendicular to a tangent plane to the Earth's surface. We represent the horizontal (west to east) coordinate by x and the altitude by z . In two-dimensions the continuity equation under the Boussinesq approximation, equation (A.3), allows to define a streamfunction for the neutral fluid flow by $-\Psi_z = u$ and $\Psi_x = w$ where u and w are respectively the x and z components of the fluid flow velocity $\mathbf{u}$. Note that the subscripts x and z represent partial differentiation with respect to x and z .

The vorticity equation for the AGWs in the ionosphere

We use an analogous procedure as in Appendix A.2 to write the momentum equation (2.8) in streamfunction-vorticity form. We take the x and z momentum equation (2.8), differentiate the x -momentum equation with respect to z and differentiate the z -momentum equation with respect to x and subtract one from the other. This eliminates the pressure gradient terms and gives

$$\begin{aligned} \nabla^2 \Psi_t - \Psi_z \nabla^2 \Psi_x + \Psi_x \nabla^2 \Psi_z + \frac{g}{\bar{\rho}} \frac{\partial \varrho}{\partial x} - \nu \nabla^4 \Psi \\ + (\nu_{\alpha n})_x (\Psi - \Psi_\alpha)_x - (\nu_{\alpha n})_z (\Psi - \Psi_\alpha)_z + \nu_{n\alpha} \nabla^2 (\Psi - \Psi_\alpha) = 0, \end{aligned} \quad (2.21)$$

where the subscripts x and z and t represent partial differentiation with respect to x , z and t respectively, and the last three terms in this equation represent the interactions between the AGWs and the ionosphere.

In a configuration where the neutral flow temperature and that of the α species vary slowly with height, the neutral-ionosphere and ionosphere-neutral collision frequencies $\nu_{n\alpha}$ and $\nu_{\alpha n}$ (the expressions for $\nu_{\alpha n}$ and $\nu_{n\alpha}$ can be found in Kelley (2006)) can be approximated by constants. Under this condition, the effects of the ionosphere on the AGWs is simply the single term $\nu_{n\alpha} \nabla^2 (\Psi - \Psi_\alpha)$ and equation (2.21) is approximated as

$$\nabla^2 \Psi_t - \Psi_z \nabla^2 \Psi_x + \Psi_x \nabla^2 \Psi_z + \frac{g}{\bar{\rho}} \frac{\partial \varrho}{\partial x} - \nu \nabla^4 \Psi + \nu_{n\alpha} \nabla^2 (\Psi - \Psi_\alpha) = 0. \quad (2.22)$$

All the parameters and variables in the equation (2.22) can be made non dimensional on the basis of the typical length scales L_x and L_z in the horizontal and vertical directions, U a typical velocity, the typical kinematic viscosity $\mathcal{V}$, the typical neutral-plasma collision frequency $\mathcal{V}_{n\alpha}$ and ϱ_{α_0} the reference density.

Denoting the dimensional quantities by asterisks, the corresponding nondimen-

sional quantities are:

$$\begin{aligned} x &= \frac{x^*}{L_x}, \quad z = \frac{z^*}{L_z}, \quad t = \frac{t^* U}{L_x}, \quad \Psi = \frac{\Psi^*}{U L_z}, \\ g &= g^* \frac{L_z}{U^2}, \quad \nu^* = \frac{\nu}{\mathcal{V}}, \quad \rho^* = \rho \frac{V}{M}, \quad \bar{\rho} = \bar{\rho}^* \frac{V}{M} \text{ and } \alpha = \frac{\alpha^*}{\mathcal{A}}. \end{aligned} \quad (2.23)$$

For the Boussinesq approximation to be valid we consider that L_z is relatively small, i.e., $L_z \ll L_x$. The vertical to horizontal aspect ratio of the domain under consideration is a nondimensional ratio defined by $\frac{L_z}{L_x}$ and the nondimensional Laplacian operator is given by $\nabla^2 = \delta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ where $\delta = \frac{L_z^2}{L_x^2}$ is the square of the vertical to horizontal aspect ratio.

The vorticity equation (2.22) can then be written in terms of the nondimensional Laplacian operator as

$$\begin{aligned} \nabla^2 \Psi_t - \Psi_z \nabla^2 \Psi_x + \Psi_x \nabla^2 \Psi_z + \frac{g}{\bar{\rho}} \frac{\partial \varrho}{\partial x} \\ - \nu \mathcal{V} \frac{L_x}{U L_z^2} \nabla^4 \Psi + \nu_{n\alpha} \mathcal{V}_{n\alpha} \frac{L_x}{U} \nabla^2 (\Psi - \Psi_\alpha) = 0. \end{aligned} \quad (2.24)$$

Setting $\mathcal{V}_{n\alpha} = \frac{U}{L_x}$ and $\mathcal{V} = \frac{U L_z^2}{L_x}$ gives the nondimensional equation

$$\nabla^2 \Psi_t - \Psi_z \nabla^2 \Psi_x + \Psi_x \nabla^2 \Psi_z + \frac{g}{\bar{\rho}} \frac{\partial \varrho}{\partial x} - \nu \nabla^4 \Psi + \nu_{n\alpha} \nabla^2 (\Psi - \Psi_\alpha) = 0, \quad (2.25)$$

where again the nondimensional Laplacian operator is given by $\nabla^2 = \delta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ and $\delta = \frac{L_z^2}{L_x^2}$ is the square of the vertical to horizontal aspect ratio.

Equation (2.25) can be written in terms of the neutral flow vorticity ξ and the ionospheric vorticities ξ_α as

$$\xi_t - \Psi_z \xi_x + \Psi_x \xi_z + \frac{g}{\bar{\rho}} \frac{\partial \varrho}{\partial x} - \nu \nabla^2 \xi + \nu_{n\alpha} (\xi - \xi_\alpha) = 0. \quad (2.26)$$

The vorticity equation for the ionospheric disturbances

The Lorentz force, which is written in equation (2.11) in the form of a cross product, can be written in terms of the components of the α -species velocity $\mathbf{u}_\alpha$ and the

components of the magnetic flux density $\mathbf{B}$ as

$$\begin{aligned}\mathbf{u}_\alpha \times \mathbf{B} &= (\hat{\mathbf{x}}u_\alpha + \hat{\mathbf{y}}v_\alpha + \hat{\mathbf{z}}w_\alpha) \times (\hat{\mathbf{x}}B_{\text{east}} + \hat{\mathbf{y}}B_{\text{north}} + \hat{\mathbf{z}}B_{\text{vertical}}) \\ &= \hat{\mathbf{x}}(v_\alpha B_{\text{vertical}} - w_\alpha B_{\text{north}}) - \hat{\mathbf{y}}(u_\alpha B_{\text{vertical}} - w_\alpha B_{\text{east}}) + \hat{\mathbf{z}}(u_\alpha B_{\text{north}} - v_\alpha B_{\text{east}}),\end{aligned}\tag{2.27}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are unit vectors in the eastward, northward and vertical directions, B_{east} , B_{north} and B_{vertical} are the eastward, northward and vertical components of the the magnetic flux density $\mathbf{B}$ respectively.

In Cartesian coordinates, equation (2.11) becomes

$$\begin{aligned}\varrho_\alpha \frac{Du_\alpha}{Dt} &= -\frac{\partial p_\alpha}{\partial x} + \mu_\alpha \nabla^2 u_\alpha - \nu_{\alpha n} \varrho_\alpha (u_\alpha - u) \\ &\quad + \varrho_{\alpha,el}(v_\alpha B_{\text{vertical}} - w_\alpha B_{\text{north}}) + \varrho_{\alpha,el} E_{\text{east}},\end{aligned}\tag{2.28}$$

$$\begin{aligned}\varrho_\alpha \frac{Dv_\alpha}{Dt} &= -\frac{\partial p_\alpha}{\partial y} + \mu_\alpha \nabla^2 v_\alpha - \nu_{\alpha n} \varrho_\alpha (v_\alpha - v) \\ &\quad - \varrho_{\alpha,el}(u_\alpha B_{\text{vertical}} - w_\alpha B_{\text{east}}) + \varrho_{\alpha,el} E_{\text{north}}\end{aligned}\tag{2.29}$$

and

$$\begin{aligned}\varrho_\alpha \frac{Dw_\alpha}{Dt} &= -\frac{\partial p_\alpha}{\partial z} - \varrho_\alpha g + \mu_\alpha \nabla^2 w_\alpha - \nu_{\alpha n} \varrho_\alpha (w_\alpha - w) \\ &\quad + \varrho_{\alpha,el}(u_\alpha B_{\text{north}} - v_\alpha B_{\text{east}}) + \varrho_{\alpha,el} E_{\text{vertical}}.\end{aligned}\tag{2.30}$$

In the vertical plane defined in terms of x and z coordinates, we consider the ionosphere to be a plasma with velocity $\mathbf{u}_\alpha = \hat{\mathbf{x}}u_\alpha + \hat{\mathbf{z}}w_\alpha$, i.e. the north-south component of the plasma velocity v_α is assumed to be zero. Under the Boussinesq approximation

we then obtain

$$\begin{aligned} \frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} + w_\alpha \frac{\partial u_\alpha}{\partial z} = & -\frac{1}{\varrho_{\alpha_o}} \frac{\partial p_\alpha}{\partial x} + \nu_\alpha \nabla^2 u_\alpha \\ & - \nu_{\alpha n}(u_\alpha - u) - \frac{\varrho_{\alpha,el} B_{\text{north}}}{\varrho_{\alpha_o}} w_\alpha + \frac{\varrho_{\alpha,el}}{\varrho_{\alpha_o}} E_{\text{east}}, \end{aligned} \quad (2.31)$$

$$\frac{\partial p_\alpha}{\partial y} = -\varrho_{\alpha,el}(u_\alpha B_{\text{vertical}} - w_\alpha B_{\text{east}}) + \varrho_{\alpha,el} E_{\text{north}} \quad (2.32)$$

and

$$\begin{aligned} \frac{\partial w_\alpha}{\partial t} + u_\alpha \frac{\partial w_\alpha}{\partial x} + w_\alpha \frac{\partial w_\alpha}{\partial z} = & -\frac{1}{\varrho_{\alpha_o}} \frac{\partial p_\alpha}{\partial x} - \frac{\varrho_\alpha}{\bar{\rho}_\alpha} g + \nu_\alpha \nabla^2 w_\alpha \\ & - \nu_{\alpha n}(w_\alpha - w) + \frac{\varrho_{\alpha,el} B_{\text{north}}}{\varrho_{\alpha_o}} u_\alpha + \frac{\varrho_{\alpha,el}}{\varrho_{\alpha_o}} E_{\text{vertical}}, \end{aligned} \quad (2.33)$$

where $\nu_\alpha = \frac{\mu_\alpha}{\varrho_{\alpha_o}}$ is the plasma kinematic viscosity and ϱ_{α_o} is a constant reference density. Note that the y -component of the momentum equation is now uncoupled from (2.31) and (2.33).

With the Boussinesq approximation, the continuity equation is $\nabla \cdot \mathbf{u}_\alpha = u_{\alpha x} + w_{\alpha z} = 0$. In the two-dimensional configuration we can define a streamfunction for the ionospheric plasma flow by $-\Psi_{\alpha z} = u_\alpha$ and $\Psi_{\alpha x} = w_\alpha$, then use a similar procedure as for the AGWs and derive the ionospheric vorticity equation

$$\begin{aligned} & \nabla^2 \Psi_{\alpha t} - \Psi_{\alpha z} \nabla^2 \Psi_{\alpha x} + \Psi_{\alpha x} \nabla^2 \Psi_{\alpha z} + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \varrho_\alpha}{\partial x} - \nu_\alpha \nabla^4 \Psi_\alpha \\ & + (\nu_{\alpha n})_x (\Psi_\alpha - \Psi)_x - (\nu_{\alpha n})_z (\Psi_\alpha - \Psi)_z + \nu_{\alpha n} \nabla^2 (\Psi_\alpha - \Psi) - \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} B_{\text{north}})_x \Psi_{\alpha z} \\ & + \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} B_{\text{north}})_z \Psi_{\alpha x} + \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} E_{\text{east}})_z - \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} E_{\text{vertical}})_x = 0. \end{aligned} \quad (2.34)$$

In a configuration where the temperature of the α species varies slowly with height, the α species-neutral collision frequency (Kelley 2006) is approximately constant, so

equation (2.34) is approximated as

$$\begin{aligned}
& \nabla^2 \Psi_{\alpha_t} - \Psi_{\alpha_z} \nabla^2 \Psi_{\alpha_x} + \Psi_{\alpha_x} \nabla^2 \Psi_{\alpha_z} + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \varrho_\alpha}{\partial x} - \nu_\alpha \nabla^4 \Psi_\alpha \\
& + \nu_{\alpha n} \nabla^2 (\Psi_\alpha - \Psi) - \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} B_{\text{north}})_x \Psi_{\alpha_z} + \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} B_{\text{north}})_z \Psi_{\alpha_x} \\
& + \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} E_{\text{east}})_z - \varrho_{\alpha_o}^{-1} (\varrho_{\alpha,el} E_{\text{vertical}})_x = 0.
\end{aligned} \tag{2.35}$$

As done in section 2.4 for AGWs, we can nondimensionalize (2.35) on the basis of the typical length scales L_x and L_z in the horizontal and vertical directions, U a typical velocity, $\mathcal{V}$ is a typical kinematic viscosity, $\mathcal{V}_{\alpha n}$ is a typical plasma-neutral collision frequency, ϱ_{α_o} is a reference density, ϱ_{α,el_o} is a typical electric charge density, $\mathcal{B}$ is a typical magnetic field and $\mathcal{E}_{\text{east}}$ and $\mathcal{E}_{\text{vertical}}$ are typical electric fields in the horizontal and vertical directions respectively. This gives

$$\begin{aligned}
& \nabla^2 \Psi_{\alpha_t} - \Psi_{\alpha_z} \nabla^2 \Psi_{\alpha_x} + \Psi_{\alpha_x} \nabla^2 \Psi_{\alpha_z} + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \varrho_\alpha}{\partial x} - \nu_\alpha \mathcal{V}_\alpha \frac{L_x}{U_\alpha L_z^2} \nabla^4 \Psi_\alpha \\
& + \nu_{\alpha n} \mathcal{V}_n \frac{L_x}{U} \nabla^2 (\Psi_\alpha - \Psi) - \frac{\varrho_{\alpha,el_o} \mathcal{B} L_z}{U} (\varrho_{\alpha,el} B_{\text{north}})_x \Psi_{\alpha_z} + \frac{\varrho_{\alpha,el_o} \mathcal{B} L_z}{U} (\varrho_{\alpha,el} B_{\text{north}})_z \Psi_{\alpha_x} \\
& + \frac{\varrho_{\alpha,el_o} \mathcal{E}_{\text{east}} L_x}{U^2} (\varrho_{\alpha,el} E_{\text{east}})_z - \frac{\varrho_{\alpha,el_o} \mathcal{E}_{\text{east}} L_z}{U^2} (\varrho_{\alpha,el} E_{\text{vertical}})_x = 0.
\end{aligned} \tag{2.36}$$

where again the nondimensional Laplacian operator is given by $\nabla^2 = \delta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ and the aspect ratio by $\delta = \frac{L_z^2}{L_x^2}$.

We then set $\mathcal{V}_\alpha = \frac{U L_z^2}{L_x}$, $\mathcal{V}_{\alpha n} = \frac{U}{L_x}$, $\mathcal{B} = \frac{U}{\varrho_{\alpha,el_o} L_z}$, $\mathcal{E}_{\text{east}} = \frac{U^2}{\varrho_{\alpha,el_o} L_x}$ and $\mathcal{E}_{\text{vertical}} = \frac{U^2}{\varrho_{\alpha,el_o} L_z}$. This gives the nondimensional ionospheric vorticity equation

$$\begin{aligned}
& \nabla^2 \Psi_{\alpha_t} - \Psi_{\alpha_z} \nabla^2 \Psi_{\alpha_x} + \Psi_{\alpha_x} \nabla^2 \Psi_{\alpha_z} + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \varrho_\alpha}{\partial x} - \nu_\alpha \nabla^4 \Psi_\alpha \\
& + \nu_{\alpha n} \nabla^2 (\Psi_\alpha - \Psi) - (\varrho_{\alpha,el} B_{\text{north}})_x \Psi_{\alpha_z} + (\varrho_{\alpha,el} B_{\text{north}})_z \Psi_{\alpha_x} \\
& + (\varrho_{\alpha,el} E_{\text{east}})_z - (\varrho_{\alpha,el} E_{\text{vertical}})_x = 0,
\end{aligned} \tag{2.37}$$

In terms of the vorticities ξ and ξ_α and the streamfunctions Ψ and Ψ_α , equation

(2.37) can be written as

$$\begin{aligned} \xi_{\alpha_t} - \Psi_{\alpha_z} \xi_{\alpha_x} + \Psi_{\alpha_x} \xi_{\alpha_z} + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \varrho_\alpha}{\partial x} - \nu_\alpha \nabla^2 \xi_\alpha + \nu_{\alpha n} (\xi_\alpha - \xi) - (\varrho_{\alpha,el} B_{\text{north}})_x \Psi_{\alpha_z} \\ + (\varrho_{\alpha,el} B_{\text{north}})_z \Psi_{\alpha_x} + (\varrho_{\alpha,el} E_{\text{east}})_z - (\varrho_{\alpha,el} E_{\text{vertical}})_x = 0. \end{aligned} \quad (2.38)$$

Chapter 3

A simple model for the effects of the ionosphere on atmospheric gravity waves in the presence of a strong constant magnetic field

3.1 Steady solution of the problem where the ion drag force is due to a strong constant magnetic field in the F region

In this chapter we examine a simplified configuration that describes gravity wave ionosphere interactions in the F region height where the magnetic field is enough strong ($\sim 30,000$ nT at the equator and $\sim 60,000$ nT at the poles) and constant that the ions, particularly the oxygen ions O^+ , spiral about the lines of the magnetic field. This is the configuration that was examined by Yeh and Liu (1972) (in their chapter 8).

In this configuration it is possible to decouple the vorticity equations (2.25) for the waves and (2.37) for the ionosphere allowing us to solve either of them independently.

We can further simplify the equations by neglecting the nonlinearities and viscous effects, and hence derive an analytical solution for the vorticity equation (2.25).

As discussed in Neil (1978), the ionospheric F region is produced by photoionization of atomic oxygen which results in the creation of the same amount of oxygen ions O^+ and electrons e^- . Through redox reactions, the loss of oxygen ions O^+ occurs via slow chemical reactions involving nitrogen gas N_2 and oxygen gas O_2 .

In the F region altitudes, the geomagnetic field is enough strong ($\sim 30,000$ nT at the equator and $\sim 60,000$ nT at the poles) and approximately constant (Yeh and Liu, 1972; Prikryl et al., 2009), so

$$\omega_{Bi} \gg \nu_{in} \gg \omega, \quad (3.1)$$

where ω_{Bi} is the angular gyrofrequency of atomic oxygen ions and is approximately 300 rad sec^{-1} , ν_{in} is the ion-neutral collision frequency and is approximately 1 rad sec^{-1} , and ω is the angular frequency of the gravity waves.

Under this condition, the ions spiral about the lines of the magnetic field, and so the plasma (ion) velocity $\mathbf{u}_i$ can be expressed in terms of the neutral wind velocity $\mathbf{u}$ as (Yeh and Liu, 1972)

$$\mathbf{u}_i = (\mathbf{u} \cdot \hat{\mathbf{B}})\hat{\mathbf{B}}, \quad (3.2)$$

where $\hat{\mathbf{B}} = \mathbf{B}/|\mathbf{B}|$ is a unit vector in the direction of the constant magnetic flux density $\mathbf{B}$. The magnetic flux density can be expressed as $\mathbf{B} = |\mathbf{B}| \cos \theta \hat{\mathbf{x}} - |\mathbf{B}| \sin \theta \hat{\mathbf{z}}$, where θ is the magnetic dip angle measured downward from the positive horizontal x -axis in the xz -plane, and $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ denote the unit vectors in the horizontal x and vertical z directions respectively.

The ion velocity $\mathbf{u}_i = \hat{\mathbf{x}}u_i + \hat{\mathbf{z}}w_i$ can be written in terms of the magnetic dip angle θ as

$$\begin{aligned} \mathbf{u}_i &= (u \cos \theta - w \sin \theta)(\hat{\mathbf{x}} \cos \theta - \hat{\mathbf{z}} \sin \theta) \\ &= \hat{\mathbf{x}}(u \cos^2 \theta - w \sin \theta \cos \theta) - \hat{\mathbf{z}}(u \sin \theta \cos \theta - w \sin^2 \theta), \end{aligned} \quad (3.3)$$

and the ion vorticity $\xi_i = \nabla \Psi_i$ is thus given by

$$\xi_i = \nabla^2 \Psi_i = \frac{\partial w_i}{\partial x} - \frac{\partial u_i}{\partial z} = \delta \Psi_{xx} \sin^2 \theta + \Psi_{zz} \cos^2 \theta + \sqrt{\delta} \Psi_{xz} \sin 2\theta, \quad (3.4)$$

where again $\delta = \frac{L_z^2}{L_x^2}$ is the square of the vertical to horizontal aspect ratio, L_x and L_z being dimensional horizontal and vertical length scales of the domain under consideration (see chapter 2).

Substituting (3.4) into the equation (2.25) gives

$$\nabla^2 \Psi_t - \Psi_z \nabla^2 \Psi_x + \Psi_x \nabla^2 \Psi_z + \frac{g}{\bar{\rho}} \frac{\partial \varrho}{\partial x} - \nu \nabla^4 \Psi + \nu_{ni} (\delta \Psi_{xx} \cos^2 \theta + \Psi_{zz} \sin^2 \theta - \sqrt{\delta} \Psi_{xz} \sin 2\theta) = 0. \quad (3.5)$$

This equation must be solved along with the energy equation (2.10) which is written in terms of the streamfunction and density as

$$\varrho_t - \Psi_z \varrho_x + \Psi_x \varrho_z - \kappa \nabla^2 \varrho = 0. \quad (3.6)$$

To define the gravity wave perturbation, we consider each fluid variable to be the sum of an initial horizontal mean part and a time dependent perturbation. We write

$$\Psi(x, z, t) = \bar{\psi}(z) + \varepsilon \psi(x, z, t), \quad (3.7)$$

$$\varrho(x, z, t) = \bar{\rho}(z) + \varepsilon \rho(x, z, t), \quad (3.8)$$

with

$$\varepsilon = \frac{L_z U}{\varphi} \ll 1,$$

where φ is the dimensional amplitude of the wave at the source, L_z a typical dimensional vertical length scale and U a typical dimensional velocity scale. Substituting (3.7) and (3.8) into equations (3.5) and (3.6) gives

$$\begin{aligned} \nabla^2 \psi_t + \bar{u}' \nabla^2 \psi_x - \bar{u}'' \psi_x + \varepsilon (\psi_x \nabla^2 \psi_z - \psi_z \nabla^2 \psi_x) + g \frac{\rho_x}{\bar{\rho}} - \nu \nabla^4 \psi - \varepsilon^{-1} \nu \bar{u}''' \\ + \nu_{ni} (\delta \psi_{xx} \cos^2 \theta + \psi_{zz} \sin^2 \theta - \sqrt{\delta} \psi_{xz} \sin 2\theta - \bar{u}' \sin^2 \theta) = 0 \end{aligned} \quad (3.9)$$

and

$$\rho_t + \bar{u}\rho_x + \bar{\rho}'\psi_x + \varepsilon(\psi_x\rho_z - \psi_z\rho_x) - \alpha\nabla^2\rho - \varepsilon^{-1}\kappa\bar{\rho}'' = 0. \quad (3.10)$$

Note that the terms multiplied by $\varepsilon^{-1}\nu$ and $\varepsilon^{-1}\kappa$ in equations (3.9) and (3.10) are present if the mean flow speed and density do not satisfy (3.5) and (3.6) with the viscous and heat conducting terms included. The Yeh and Liu (1972) model is based on the assumption that the viscous and heat conducting terms are small enough to be neglected.

In terms of our nondimensional parameters ε , ν and κ , we require that $\kappa \ll \varepsilon \ll 1$ and $\nu \ll \varepsilon \ll 1$. In this case ε , ν , κ , $\varepsilon^{-1}\nu$ and $\varepsilon^{-1}\kappa$ are all considered to be small parameters in (3.9) and (3.10) and solutions could be obtained as perturbation series expressed in powers of these small parameters.

We examine the leading-order terms in these expansions. In order to do so we omit all the terms involving ε , ν and κ , and obtain solutions satisfying the linear inviscid vorticity and the first order transport equations

$$\begin{aligned} \nabla^2\psi_t + \bar{u}\nabla^2\psi_x - \bar{u}''\psi_x + \frac{g}{\bar{\rho}}\rho_x \\ + \nu_{ni}(\delta\psi_{xx}\cos^2\theta + \psi_{zz}\sin^2\theta - \sqrt{\delta}\psi_{xz}\sin 2\theta - \bar{u}'\sin^2\theta) = 0 \end{aligned} \quad (3.11)$$

and

$$\rho_t + \bar{u}\rho_x + \bar{\rho}'\psi_x = 0. \quad (3.12)$$

Applying the differential operator $\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)$ to equation (3.11) gives

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 \nabla^2 \psi \\ & + \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) [\nu_{ni}(\delta\psi_{xx} \cos^2 \theta + \psi_{zz} \sin^2 \theta - \sqrt{\delta}\psi_{xz} \sin 2\theta - \bar{u}' \sin^2 \theta) - \bar{u}'' \psi_x] \\ & + \frac{g}{\bar{\rho}} \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) \rho_x = 0. \end{aligned} \quad (3.13)$$

Applying the operator $\frac{g}{\bar{\rho}} \frac{\partial}{\partial x}$ to the energy equation (3.12) gives

$$\frac{g}{\bar{\rho}} \left[\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x} \right] \rho_x = N^2 \psi_{xx}, \quad (3.14)$$

where N is the Brunt-Väisälää frequency and is given by

$$N^2 = -\frac{g}{\bar{\rho}} \frac{d\bar{\rho}}{dz}. \quad (3.15)$$

We assume an exponential profile for the background density

$$\bar{\rho}(z) = \rho_o e^{-\frac{z}{H}}, \quad (3.16)$$

where H is the scale height of the atmosphere. This gives

$$N^2 = \frac{g}{H}. \quad (3.17)$$

Combining equations (3.13) and (3.14) with (3.15) gives

$$\begin{aligned} & \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 \nabla^2 \psi \\ & + \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) [\nu_{ni}(\delta\psi_{xx} \cos^2 \theta + \psi_{zz} \sin^2 \theta - \sqrt{\delta}\psi_{xz} \sin 2\theta - \bar{u}' \sin^2 \theta) - \bar{u}'' \psi_x] \\ & + N^2 \psi_{xx} = 0. \end{aligned} \quad (3.18)$$

The steady-state solution of equation (3.18) can be derived assuming the waves have the form

$$\psi(x, z, t) = \phi(z)e^{i\kappa x} + \text{c.c.}, \quad (3.19)$$

where κ is the horizontal wavenumber, $\phi(z)$ is the complex amplitude of the waves and *c.c.* denotes the complex conjugate of $\phi(z)e^{i\kappa x}$ so that $\psi(x, z, t)$ is a real-valued function.

Substituting (3.19) into the equation (3.18) yields

$$\begin{aligned} \left[1 - \frac{i\nu_{ni} \sin^2 \theta}{\kappa \bar{u}}\right] \phi_{zz} - \frac{\sqrt{\delta} \nu_{ni} \sin 2\theta}{\bar{u}} \phi_z \\ + \left(\frac{N^2}{\bar{u}^2} - \delta \kappa^2 + \frac{\kappa \bar{u}'' + i\delta \nu_{ni} \kappa^2 \cos^2 \theta}{\kappa \bar{u}}\right) \phi = 0. \end{aligned} \quad (3.20)$$

If the mean flow velocity $\bar{u}$ is constant, the solution of equation (3.20) takes the form $\phi = e^{imz}$, where m is the vertical wavenumber and satisfies the quadratic equation

$$\left[1 - \frac{i\nu_{ni} \sin^2 \theta}{\kappa \bar{u}}\right] m^2 + i \frac{\sqrt{\delta} \nu_{ni} \sin 2\theta}{\bar{u}} m - \left(\frac{N^2}{\bar{u}^2} - \delta \kappa^2 + \frac{i\delta \nu_{ni} \kappa \cos^2 \theta}{\bar{u}}\right) = 0. \quad (3.21)$$

3.1.1 The case of the neutral atmosphere ($\nu_{ni} = 0$)

We first note that $\nu_{ni} = 0$ gives the case of AGWs in the neutral atmosphere. In that case equation (3.20) becomes the well-known Taylor Goldstein equation that describes the GWs in the neutral atmosphere (Nappo, 2002; Sutherland, 2010). With a constant mean flow velocity, $\bar{u} = \text{constant}$, the Taylor Goldstein equation is

$$\phi_{zz} + \left(\frac{N^2}{\bar{u}^2} - \delta \kappa^2\right) \phi = 0, \quad (3.22)$$

and its solutions are

$$\phi(z) = e^{imz} \quad (3.23)$$

where

$$m = m^\pm = \pm \sqrt{\frac{N^2}{\bar{u}^2} - \delta \kappa^2} \quad (3.24)$$

Therefore, oscillatory solutions exist only if $N > \sqrt{\delta}|\kappa\bar{u}|$. The Brunt-Väisälää frequency N is thus the high frequency cutoff for internal gravity waves in the neutral atmosphere.

Using the group velocity argument of Booker and Bretherton (1967) discussed in Appendix A.4, we note that if $\kappa > 0$ and $\bar{u} > 0$ then the solution corresponding to an upward propagating wave is the one with the vertical wavenumber m^+ given by the positive square root in (3.24).

3.1.2 The case of the ionosphere ($\nu_{ni} \neq 0$)

In the ionosphere where $\nu_{ni} \neq 0$, the vertical wavenumber m is obtained by solving equation (3.21) which has complex roots

$$m^\pm = -i \frac{\sqrt{\delta} \nu_{ni} \sin \theta \cos \theta}{\bar{u} \left(1 - \frac{i \nu_{ni} \sin^2 \theta}{\kappa \bar{u}}\right)} \pm \frac{\sqrt{\left(\frac{N^2}{\bar{u}^2} - \delta \kappa^2\right) + i \frac{\nu_{ni}}{\kappa \bar{u}} \left(\frac{N^2}{\bar{u}^2} \sin^2 \theta - \delta \kappa^2\right)}}{1 - \frac{i \nu_{ni} \sin^2 \theta}{\kappa \bar{u}}}. \quad (3.25)$$

To have an upward propagating wave, we must take the positive square root in order that the solution is consistent with the neutral gravity waves given by (3.24) as $\nu_{ni} \rightarrow 0$.

We let

$$A = \frac{N^2}{\bar{u}^2} - \delta \kappa^2 \text{ and } B = \frac{\nu_{ni}}{\kappa \bar{u}} \left(\frac{N^2}{\bar{u}^2} \sin^2 \theta - \delta \kappa^2 \right), \quad (3.26)$$

and

$$a + ib = \sqrt{A + iB}. \quad (3.27)$$

Then

$$a = \frac{1}{\sqrt{2}} \sqrt{A + \sqrt{A^2 + B^2}} \text{ and } b = \pm \frac{1}{\sqrt{2}} \sqrt{-A + \sqrt{A^2 + B^2}} \quad (3.28)$$

Since $2ab = B$, with $a > 0$ the sign of b is the same as the sign of B . So we take the positive sign in (3.28) if $\sin^2 \theta = \frac{\delta \kappa^2 \bar{u}^2}{N^2}$ ($B > 0$) and take the negative sign if $\sin^2 \theta < \frac{\delta \kappa^2 \bar{u}^2}{N^2}$ ($B < 0$).

The vertical wavenumber is thus written in terms of a and b as

$$m = \frac{-i \frac{\sqrt{\delta} \nu_{ni}}{\bar{u}} \sin \theta \cos \theta + (a + ib)}{1 - \frac{i \nu_{ni} \sin^2 \theta}{\kappa \bar{u}}}. \quad (3.29)$$

So the real and imaginary parts of m are

$$m_R(\theta) = \frac{a + \left(\frac{\sqrt{\delta} \nu_{ni}}{\bar{u}} \sin \theta \cos \theta - b \right) \frac{\nu_{ni}}{\kappa \bar{u}} \sin^2 \theta}{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2} \sin^4 \theta} \quad (3.30)$$

and

$$m_I(\theta) = \frac{\frac{a \nu_{ni}}{\kappa \bar{u}} \sin^2 \theta - \frac{\sqrt{\delta} \nu_{ni}}{\bar{u}} \sin \theta \cos \theta + b}{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2} \sin^4 \theta}. \quad (3.31)$$

where

$$a = \frac{1}{\sqrt{2}} \sqrt{A + \sqrt{A^2 + B^2}} > 0 \quad (3.32)$$

and

$$b = \text{sgn}(B) \frac{1}{\sqrt{2}} \sqrt{-A + \sqrt{A^2 + B^2}}. \quad (3.33)$$

The solution takes the form

$$\phi(x, z) = e^{ikx} e^{im_R z} e^{-m_I z} \quad (3.34)$$

where m_R is given by (3.30) and m_I by (3.31). We shall see in section 3.4 that for the typical values of the nondimensional parameters N , $\bar{u}$, δ and κ used in the numerical simulations, $m_I \geq 0$ for all $0 \leq \theta \leq \pi$. This means the wave amplitude decays exponentially with increasing z . We next consider some special cases for which m_I is clearly positive.

The magnetic dip angle $\theta = 0$ or $\theta = \pi$

For the special case where the magnetic dip angle $\theta = 0$ or $\theta = \pi$, $\sin \theta = 0$. The vertical wavenumber for upward propagating waves is given by

$$m = \sqrt{\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right) + i\frac{\nu_{ni}}{\bar{u}}\delta\kappa}. \quad (3.35)$$

In this case, $B = \frac{\nu_{ni}}{\bar{u}}\delta > 0$ so we take the positive sign in (3.33) and get $b > 0$. The real part m_R and the imaginary part m_I are given by

$$m_R = \frac{1}{\sqrt{2}} \sqrt{\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right) + \sqrt{\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right)^2 + \frac{\nu_{ni}^2}{\bar{u}^2}\delta^2\kappa^2}} \quad (3.36)$$

and

$$m_I = \frac{1}{\sqrt{2}} \sqrt{-\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right) + \sqrt{\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right)^2 + \frac{\nu_{ni}^2}{\bar{u}^2}\delta^2\kappa^2}}. \quad (3.37)$$

We note that $m_I > 0$, so the waves decay in amplitude with height. Also $m_R > \frac{N^2}{\bar{u}^2} - \delta\kappa^2$, so the ion drag has the effect of damping the waves and reducing their wavelength.

The magnetic dip angle $\theta = \pi/2$

For the special case where the magnetic dip angle $\theta = \pi/2$, $\cos \theta = 0$ and $\sin \theta = 1$. The vertical wavenumber for upward propagating waves is given by

$$m = \frac{1}{1 - i\frac{\nu_{ni}}{\kappa\bar{u}}} \sqrt{\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right) + i\frac{\nu_{ni}}{\kappa\bar{u}}\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right)}. \quad (3.38)$$

In this case $B = \frac{\nu_{ni}}{\kappa\bar{u}}\left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2\right) > 0$, so again we take the positive sign in (3.33) and get $b > 0$. The real and imaginary parts m_R and m_I are respectively given by

$$m_R = \frac{1}{\sqrt{2}} \frac{\sqrt{\frac{N^2}{\bar{u}^2} - \delta\kappa^2}}{1 + \frac{\nu_{ni}^2}{\kappa^2\bar{u}^2}} \left[\sqrt{1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2\bar{u}^2}}} - \frac{\nu_{ni}}{\kappa\bar{u}} \sqrt{-1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2\bar{u}^2}}} \right] \quad (3.39)$$

and

$$m_I = \frac{1}{\sqrt{2}} \frac{\sqrt{\frac{N^2}{\bar{u}^2} - \delta \kappa^2}}{1 + \frac{\nu_{ni}^2}{k^2 \bar{u}^2}} \left[\frac{\nu_{ni}}{\kappa \bar{u}} \sqrt{1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2}}} + \sqrt{-1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2}}} \right]. \quad (3.40)$$

Again $m_I > 0$, but m_R is smaller than the vertical wavenumber in the neutral atmosphere. So the ion drag has the effect of damping the waves and increasing their wavelength compared with that of waves in the neutral atmosphere.

The long-wave limit corresponding to $\delta = 0$

For the case $\delta = 0$ (the long-wave limit), the vertical wavenumber for upward propagating waves is

$$m = \frac{N}{\bar{u}} \frac{\sqrt{1 + i \frac{\nu_{ni}}{\kappa \bar{u}} \sin^2 \theta}}{1 - \frac{i \nu_{in} \sin^2 \theta}{\kappa \bar{u}}}. \quad (3.41)$$

In this case $B = \frac{\nu_{ni}}{\kappa \bar{u}} \sin^2 \theta \geq 0$, so according to (3.33) $b \geq 0$. The real and imaginary parts of m are given respectively by

$$m_R = \frac{1}{\sqrt{2}} \frac{1}{1 + \frac{\nu_{ni}^2}{k^2 \bar{u}^2} \sin^4 \theta} \frac{N}{\bar{u}} \left\{ \sqrt{1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2} \sin^4 \theta}} - \frac{\nu_{ni} \sin^2 \theta}{\kappa \bar{u}} \sqrt{-1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2} \sin^4 \theta}} \right\} \quad (3.42)$$

and

$$m_I = \frac{1}{\sqrt{2}} \frac{1}{1 + \frac{\nu_{ni}^2}{k^2 \bar{u}^2} \sin^4 \theta} \frac{N}{\bar{u}} \left\{ \sqrt{-1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2} \sin^4 \theta}} + \frac{\nu_{ni} \sin^2 \theta}{\kappa \bar{u}} \sqrt{1 + \sqrt{1 + \frac{\nu_{ni}^2}{\kappa^2 \bar{u}^2} \sin^4 \theta}} \right\}. \quad (3.43)$$

We observe that in the long-wave limit the vertical wavenumber of upward propagating GWs is $\frac{N}{\bar{u}}$ and is larger than m_R for all $\theta > 0$. We also note that the damping rate of the wave amplitude is zero for $\theta = 0$ and positive for $\theta > 0$. So the ion drag

has the effect of damping long gravity waves if the magnetic dip angle $\theta > 0$. Long waves are not damped if $\theta = 0$, which corresponds to gravity wave propagation in the magnetic meridian plan at the magnetic equator.

3.2 Time-dependent solution in the long-wave limit

We first note that when in general the steady solution of the time-dependent equation (3.18) is of the form $e^{ikx+imz} + \text{c.c.}$. But we cannot find an exact time-dependent analytical solution for the general form of equation (3.18). However, in the special case of the long-wave limit where $\delta = 0$ we can find one. We consider a time-dependent problem given by (3.18) on the domain $0 \leq x \leq 2\pi$ and $z_1 \leq z \leq \infty$ and generate waves at the lower boundary z_1 by applying the boundary condition $\psi(x, z = z_1, t) = e^{ikx} + \text{c.c.}$, which propagate upward. We again let $\bar{u} = \text{constant}$ so that, in the long-wave limit, the streamfunction $\psi(x, z, t)$ satisfies

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 \psi_{zz} + \nu_{ni} \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) \psi_{zz} \sin^2 \theta + N^2 \psi_{xx} = 0 \quad (3.44)$$

with boundary condition

$$\psi(z = z_1, t) = e^{ikx} + \text{c.c.} \quad (3.45)$$

To derive the time-dependent solution to the problem (3.44)-(3.45) we impose the initial conditions that the vorticity and its derivative are zero at $t = 0$, i.e., $\psi_{zz}(x, z, 0) = \psi_{zzt}(x, z, 0) = 0$, and take the Laplace transform of (3.44)-(3.45) with respect to t .

The form of the boundary condition indicates that the solution take the form

$$\psi(x, z, t) = \phi(z, t)e^{ikx} + \text{c.c.} \quad (3.46)$$

In the neutral atmosphere where $\nu_{ni} = 0$ equation (3.44) becomes

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 \psi_{zz} + N^2 \psi_{xx} = 0. \quad (3.47)$$

Nadon and Campbell (2007) solved equation (3.47) and obtained the solution

$$\psi(x, z, t) = e^{i\kappa x} e^{\frac{iN(z-z_1)}{\bar{u}}} - e^{i\kappa(x-\bar{u}t)} \sum_{n=1}^{\infty} \left(\frac{i\sqrt{N(z-z_1)}}{\bar{u}\sqrt{\kappa t}} \right)^n \mathcal{J}_n(2\sqrt{N\kappa(z-z_1)t}) + \text{c.c.} \quad (3.48)$$

where $\mathcal{J}_n$ is the Bessel function of the first type of order n . The first term is the steady solution of the Taylor Goldstein equation (3.22) with $\delta = 0$. The time-dependent terms go to zero as $t \rightarrow \infty$ for finite z , so the solution approaches the steady state as $t \rightarrow \infty$.

Here we consider the case where $\nu_{ni} \neq 0$ and seek a solution of the form (3.46). Substituting (3.46) into the equation (3.44) gives

$$\left(\frac{\partial}{\partial t} + i\kappa\bar{u} \right)^2 \phi_{zz} + \nu_{ni} \sin^2 \theta \left(\frac{\partial}{\partial t} + i\kappa\bar{u} \right) \phi_{zz} - N^2 \kappa^2 \phi = 0 \quad (3.49)$$

with boundary condition

$$\phi(z_1, t) = 1. \quad (3.50)$$

We take a Laplace transform in time taking into account the initial condition, this gives

$$\tilde{\phi}_{zz} - \frac{N^2 \kappa^2}{(s + i\kappa\bar{u})(s + i\kappa\bar{u} + \nu_{ni} \sin^2 \theta)} \tilde{\phi} = 0 \quad (3.51)$$

with the boundary condition

$$\tilde{\phi}(z_1, s) = \frac{1}{s}. \quad (3.52)$$

The solution corresponding to a wave with upward group velocity is

$$\tilde{\phi}(z, s) = \frac{e^{-\frac{N\kappa(z-z_1)}{\sqrt{(s+i\kappa\bar{u})(s+i\kappa\bar{u}+\nu_{ni}\sin^2\theta)}}}}{s}. \quad (3.53)$$

This can be expanded as an infinite series to give

$$\tilde{\phi}(z, s) = \frac{1}{s} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left[\frac{N\kappa(z-z_1)}{\sqrt{(s+i\kappa\bar{u})(s+i\kappa\bar{u}+\nu_{ni}\sin^2\theta)}} \right]^j. \quad (3.54)$$

Its inverse Laplace transform is

$$\phi(z, t) = \sum_{j=0}^{\infty} \frac{(-1)^j [N\kappa(z - z_1)]^j}{j!} \mathcal{L}^{-1} \left\{ \frac{1}{s} \frac{1}{(s + i\kappa\bar{u})^{j/2} (s + i\kappa\bar{u} + \nu_{ni} \sin^2 \theta)^{j/2}} \right\}. \quad (3.55)$$

According to formula (29.3.51) in Abramowitz and Stegun (1964),

$$\begin{aligned} & \mathcal{L}^{-1} \left\{ \frac{1}{(s + i\kappa\bar{u})^{j/2} (s + i\kappa\bar{u} + \nu_{ni} \sin^2 \theta)^{j/2}} \right\} \\ &= \frac{\sqrt{\pi}}{\Gamma\left(\frac{j}{2}\right)} \left(\frac{t}{\nu_{ni} \sin^2 \theta} \right)^{\frac{j-1}{2}} e^{-i\kappa\bar{u}t} e^{-\frac{\nu_{ni} \sin^2 \theta}{2} t} \mathcal{I}_{\frac{j-1}{2}} \left(\frac{\nu_{ni} \sin^2 \theta}{2} t \right), \end{aligned} \quad (3.56)$$

where $\mathcal{I}_{\frac{j-1}{2}}$ is the modified Bessel function of order $\frac{j-1}{2}$.

We then make use the fact that $\mathcal{L}^{-1} \left\{ \frac{F(s)}{s} \right\} = \int_0^t f(t) dt$ for $F(s) = \mathcal{L} \{f(t)\}$, and obtain

$$\begin{aligned} & \phi(z, t) = 1 \\ & + \sqrt{\pi} \sum_{j=1}^{\infty} \frac{(-1)^j [N\kappa(z - z_1)]^j}{j! \Gamma\left(\frac{j}{2}\right)} \int_0^t \left(\frac{\tau}{\nu_{ni} \sin^2 \theta} \right)^{\frac{j-1}{2}} e^{-i\kappa\bar{u}\tau} e^{-\frac{\nu_{ni} \sin^2 \theta}{2} \tau} \mathcal{I}_{\frac{j-1}{2}} \left(\frac{\nu_{ni} \sin^2 \theta}{2} \tau \right) d\tau \\ & = 1 + \\ & \sqrt{2\pi} \sum_{j=1}^{\infty} \frac{(-1)^j \left[\frac{\sqrt{2}N\kappa}{\nu_{ni} \sin^2 \theta} (z - z_1) \right]^j}{j! \Gamma\left(\frac{j}{2}\right)} \int_0^{\frac{\nu_{ni} \sin^2 \theta}{2} t} \chi^{\frac{j-1}{2}} e^{-\left(\frac{i\kappa\bar{u}}{\nu_{ni} \sin^2 \theta} + 1\right)\chi} \mathcal{I}_{\frac{j-1}{2}}(\chi) d\chi. \end{aligned} \quad (3.57)$$

According to formula (11.3.7) in Abramowitz and Stegun (1964),

$$\int_0^{\gamma} e^{-p\tau} \tau^{\mu} \mathcal{I}_{\mu}(\tau) d\tau = \frac{e^{-p\gamma} \gamma^{\mu+1}}{2\mu + 1} \left[\mathcal{I}_{\mu}(\gamma) - \frac{1}{p} \mathcal{I}_{\mu+1}(\gamma) \right] \quad (3.58)$$

Setting $\gamma = \frac{\nu_{ni} \sin^2 \theta}{2} t$, $\mu = \frac{j-1}{2}$ and $p = \frac{ik\bar{u}}{\nu_{ni} \sin^2 \theta} + 1$, and making use of (3.58) gives

$$\begin{aligned}
\phi(z, t) &= 1 + e^{-\left(\frac{\nu_{ni} \sin^2 \theta}{2} + i\kappa\bar{u}\right)t} \sin \theta \sqrt{\pi \nu_{ni} t} \\
&\times \sum_{j=1}^{\infty} \frac{(-1)^j \left[\frac{N\kappa}{\sqrt{\nu_{ni} \sin \theta}} (z - z_1) \sqrt{t} \right]^j}{j\Gamma(j+1)\Gamma\left(\frac{j}{2}\right)} \left[\mathcal{I}_{\frac{j-1}{2}} \left(\frac{\nu_{ni} \sin^2 \theta}{2} t \right) - \Upsilon(\theta) \mathcal{I}_{\frac{j+1}{2}} \left(\frac{\nu_{ni} \sin^2 \theta}{2} t \right) \right] \\
&= 1 + \frac{\sin \theta}{2} \sqrt{\pi \nu_{ni} t} e^{-\left(\frac{\nu_{ni} \sin^2 \theta}{2} + i\kappa\bar{u}\right)t} \\
&\times \sum_{j=1}^{\infty} \frac{(-1)^j \left[\frac{N\kappa}{\sqrt{\nu_{ni} \sin \theta}} (z - z_1) \sqrt{t} \right]^j}{\Gamma(j+1)\Gamma\left(\frac{j}{2} + 1\right)} \left[\mathcal{I}_{\frac{j-1}{2}} \left(\frac{\nu_{ni} \sin^2 \theta}{2} t \right) - \Upsilon(\theta) \mathcal{I}_{\frac{j+1}{2}} \left(\frac{\nu_{ni} \sin^2 \theta}{2} t \right) \right],
\end{aligned} \tag{3.59}$$

where $\Upsilon(\theta) = \frac{\nu_{ni} \sin^2 \theta}{\nu_{ni} \sin^2 \theta + 2i\kappa\bar{u}}$.

From the properties of the Laplace transform, we note that as $t \rightarrow \infty$ this time-dependent solution approaches the steady-state solution that is obtained by setting $s = 0$ in (3.54), namely the steady solution $\psi(x, z) = e^{ikx+imz}$ with m given by (3.41). While this is not evident from the expression (3.59) it may be possible to show it if we express (3.59) in terms of generalized hypergeometric functions and then make use of the known asymptotic behavior of these functions (Nijimbere and Campbell, 2013).

3.3 The effect of neutral gravity waves on the ionospheric F region

We examine the effects of the waves on the ionosphere by solving the continuity equation (2.4). It is well known (e.g., Yeh and Liu, 1972) that in the F region the production rate per unit of volume Q_i of oxygen ions is almost equal to its rate of loss per unit of volume L_i , $Q_i \approx L_i$, and so the continuity equation for the concentration of the oxygen ions $\mathcal{N}_i$ can be approximated as

$$\frac{\partial \mathcal{N}_i}{\partial t} + \nabla \cdot (\mathcal{N}_i \mathbf{u}_i) = 0. \tag{3.60}$$

The waves generate perturbations in the number density $\mathcal{N}_i$. The number density can thus be written as the sum of an unperturbed $\bar{\mathcal{N}}_i$ term and a perturbed term $\mathcal{N}_i$ as

$$\mathcal{N}_i(x, z, t) = \bar{\mathcal{N}}_i(z) + \varepsilon \mathcal{N}_i(x, z, t) \quad (3.61)$$

with $\varepsilon \ll 1$.

Assuming that the background ionic velocity is in the x -direction like the background neutral velocity, the components of the ion velocity ($\mathbf{u}$) can be written as

$$u_i(x, z, t) = \bar{u}_i(z) + \varepsilon \mathcal{U}_i(x, z, t) \text{ and } w_i(x, z, t) = \varepsilon \mathcal{W}_i(x, z, t) \quad (3.62)$$

Substituting equations (3.61) and (3.62) into (3.60) and neglecting the nonlinear terms, i.e. the terms with ε , gives an equation for the perturbation in the number density

$$\frac{\partial \mathcal{N}_i}{\partial t} + \bar{u}_i \frac{\partial \mathcal{N}_i}{\partial x} = -\bar{\mathcal{N}}_i \frac{\partial \mathcal{U}_i}{\partial x} - \bar{\mathcal{N}}_i \frac{\partial \mathcal{W}_i}{\partial z} - \frac{d\bar{\mathcal{N}}_i}{dz} \mathcal{W}_i. \quad (3.63)$$

From (3.3) and (3.62), the components of the ion velocity perturbation are:

$$\mathcal{U}_i = -\psi_z \cos^2 \theta - \frac{1}{2} \psi_x \sin 2\theta \text{ and } \mathcal{W}_i = \psi_x \sin^2 \theta + \frac{1}{2} \psi_z \sin 2\theta. \quad (3.64)$$

Substituting (3.64) into the equation (3.63) gives

$$\begin{aligned} \mathcal{N}_{i_t} + \bar{u}_i \mathcal{N}_{i_x} = & -\frac{\bar{\mathcal{N}}_i}{2} [(\psi_{zz} - \psi_{xx}) \sin 2\theta - \psi_{xz} \cos 2\theta] \\ & - \bar{\mathcal{N}}_i' (\psi_z \sin \theta \cos \theta + \psi_x \sin^2 \theta). \end{aligned} \quad (3.65)$$

This equation is a first order transport equation, it shows how the ions are transported by the collision between the neutral wind and the plasma or the effects of atmospheric gravity waves on the plasma number density. Once the atmospheric gravity wave streamfunction ψ is known, then equation (3.65) can be used to compute the number density perturbation $\mathcal{N}_i$.

3.4 Numerical solution of the time-dependent problem for a strong constant magnetic field

In this section we describe the numerical solutions of the equations that describe the effects of the ion drag on upward propagating gravity waves in the ionospheric F region where there is a strong constant magnetic field on the upward propagating atmospheric gravity waves.

The exact solutions we derived in section 3.1 for the steady version of this problem showed that for certain magnetic dip angles the ion drag has the effects of damping the AGWs in the vertical direction and also reduces their wavelength. We also expect that in the time-dependent version of this problem, the solution will approach a steady state as $t \rightarrow \infty$. We now carry out time-dependent numerical simulations to confirm these predictions.

3.4.1 Numerical implementation

In this section we describe the numerical solution of the nonlinear time-dependent equations (3.9) and (3.10). We write (3.9) in terms of the perturbation vorticity $\zeta = \nabla^2 \psi$ and represent (3.9)-(3.10) in the form

$$\begin{aligned} \zeta_t + \bar{u}' \zeta_x - \bar{u}'' \psi_x + \varepsilon(\psi_x \zeta_z - \psi_z \zeta_x) + g \frac{\rho_x}{\bar{\rho}} - \nu \nabla^2 \zeta \\ + \nu_{ni}(\delta \psi_{xx} \cos^2 \theta + \psi_{zz} \sin^2 \theta - \sqrt{\delta} \psi_{xz} \sin 2\theta - \bar{u}' \sin^2 \theta) = 0, \end{aligned} \quad (3.66)$$

where

$$\zeta = \nabla^2 \psi \quad (3.67)$$

and

$$\rho_t + \bar{u} \rho_x + \bar{\rho}' \psi_x + \varepsilon(\psi_x \rho_z - \psi_z \rho_x) - \alpha \nabla^2 \rho = 0. \quad (3.68)$$

The numerical simulation is carried out on a rectangular domain in the vertical plane which is defined by $x_1 \leq x \leq x_2$ and $z_1 \leq z \leq z_2$, with the initial conditions

$$\psi(x, z, 0) = 0 \text{ and } \zeta(x, z, 0) = 0, \quad (3.69)$$

$$\rho(x, z, 0) = 0. \quad (3.70)$$

We impose at the inflow boundary the conditions

$$\psi(x, z_1, t) = e^{i\kappa x} + c.c., \quad \zeta(x, z_1, t) = 0, \quad (3.71)$$

and use time-dependent radiation condition at the outflow boundary $z = z_2$, see my MSc thesis (Victor, 2010). The linear model is obtained by simply setting $\varepsilon = 0$ in the momentum equation (3.66) and energy equation (3.68)

Equations (3.66) and (3.68) are solved using a predictor-corrector method and pseudo-spectral method since a periodic boundary condition is applied at the inflow boundary. The second-order explicit Adam-Bashforth scheme is used as a predictor method while the second-order explicit Adam-Moulton scheme is used as a corrector method. The predictor-corrector method is very stable and allows us to use relatively large values of the time and space increments with very small errors. The nonlinear terms are computed using pseudo-spectral method; this means that the nonlinear term is computed by inverting the Fourier transform at each time level. Once ζ is obtained at each time level, equation (3.67) is solved using an elliptic solver to obtain ψ which is then used in the computation of ζ at the next time level.

The numerical implementation is performed by applying the Fourier transform in x to equations (3.66) and (3.71) and discretizing the resulting equations using finite differences in z and in t . The independent variables are approximated by $x_k = k\Delta x$, $z_j = j\Delta z$ and $t_n = n\Delta t$ where Δx , Δz and Δt are the x , z and t increments respectively; and $k = 1, 2, \dots, K$, $j = 1, 2, \dots, J$ and $n = 1, 2, \dots, N$.

We approximate the dependent variables in Fourier space as

$$\hat{\psi}(k, z, t) \approx \hat{\psi}(k, j\Delta z, n\Delta t) = \hat{\psi}^{n,k,j}, \quad (3.72)$$

$$\hat{\zeta}(k, z, t) \approx \hat{\zeta}(k, j\Delta z, n\Delta t) = \hat{\zeta}^{n,k,j} \quad (3.73)$$

and

$$\hat{\rho}(k, z, t) \approx \hat{\rho}(k, j\Delta z, n\Delta t) = \hat{\rho}^{n,k,j}. \quad (3.74)$$

And we make use of the forward time finite difference approximation and the second order central difference approximation.

In the numerical simulations $\hat{\zeta}$ and $\hat{\rho}$ are computed at each time level using the predictor-corrector method as follows:

1. Euler method (for the first time step),

$$\hat{\zeta}^{n+1,k,j} = \hat{\zeta}^{n,k,j} + \Delta t f^{n,k,j} \quad (3.75)$$

and

$$\hat{\rho}^{n+1,k,j} = \hat{\rho}_i^{n,k,j} + \Delta t g^{n,k,j}. \quad (3.76)$$

2. Explicit second order Adam-Bashforth (predictor method),

$$\hat{\zeta}^{n+2,k,j} = \hat{\zeta}^{n+1,k,j} + \Delta t \left(\frac{3}{2} f^{n+1,k,j} - \frac{1}{2} f^{n,k,j} \right) \quad (3.77)$$

and

$$\hat{\rho}^{n+2,k,j} = \hat{\rho}^{n+1,k,j} + \Delta t \left(\frac{3}{2} g^{n+1,k,j} - \frac{1}{2} g^{n,k,j} \right). \quad (3.78)$$

3. Implicit second order Adam-Moulton (corrector method)

$$\hat{\zeta}^{n+2,k,j} = \hat{\zeta}^{n+1,k,j} + \Delta t \left(\frac{5}{12} f^{n+2,k,j} + \frac{2}{3} f^{n+1,k,j} - \frac{1}{12} f^{n,k,j} \right) \quad (3.79)$$

and

$$\hat{\rho}^{n+2,k,j} = \hat{\rho}^{n+1,k,j} + \Delta t \left(\frac{5}{12} g^{n+2,k,j} + \frac{2}{3} g^{n+1,k,j} - \frac{1}{12} g^{n,k,j} \right), \quad (3.80)$$

where $f^{n,k,j}$ and $g^{n,k,j}$ are respectively given by

$$\begin{aligned} f^{n,k,j} = & -ik \left(\bar{u}'^j \hat{\zeta}^{n,k,j} - \bar{u}''^j \hat{\psi}^{n,k,j} + g \frac{\hat{\rho}^{n,k,j}}{\bar{\rho}^j} \right) \\ & - \varepsilon \mathcal{F} \left\{ \mathcal{F}^{-1} \{ ik \hat{\psi} \} \mathcal{F}^{-1} \{ \hat{\zeta}_z \} - \mathcal{F}^{-1} \{ \hat{\psi}_z \} \mathcal{F}^{-1} \{ ik \hat{\zeta} \} \right\}^{n,k,j} \\ & + \nu_i \frac{\hat{\zeta}^{n,k,j+1} - \hat{\zeta}^{n,k,j} (2 + \delta k^2 \Delta z^2) + \hat{\zeta}^{n,k,j-1}}{\Delta z^2} \\ & - \nu_{ni} (i\sqrt{\delta k} \frac{\hat{\psi}^{n,k,j+1} - \hat{\psi}^{n,k,j-1}}{2\Delta z} \sin 2\theta - \bar{u}'^j \sin^2 \theta) \\ & - \nu_{ni} \frac{\hat{\psi}^{n,k,j+1} - 2\hat{\psi}^{n,k,j} (2 + \delta k^2 \Delta z^2) \cot^2 \theta + \hat{\psi}^{n,k,j-1}}{\Delta z^2} \sin^2 \theta \end{aligned} \quad (3.81)$$

and

$$\begin{aligned} g^{n,k,j} = & -ik \left(\bar{u}^j \hat{\rho}^{n,k,j} + \bar{\rho}'^j \hat{\psi}^{n,k,j} \right) \\ & - \varepsilon \mathcal{F} \left\{ \mathcal{F}^{-1} \{ ik \hat{\rho} \} \mathcal{F}^{-1} \{ \hat{\psi}_z \} - \mathcal{F}^{-1} \{ \hat{\rho}_z \} \mathcal{F}^{-1} \{ ik \hat{\psi} \} \right\}^{n,k,j} \\ & + \alpha \frac{\hat{\rho}^{n,k,j+1} - \hat{\rho}^{n,k,j} (2 + \delta k^2 \Delta z^2) + \hat{\rho}^{n,k,j-1}}{\Delta z^2}. \end{aligned} \quad (3.82)$$

Simulations are carried out using the following spectrum of wavenumbers in x , $k = \dots, -3\kappa, -2\kappa, -\kappa, 0, +\kappa, +2\kappa, +3\kappa, \dots$, where κ is the wavenumber at the forcing or inflow boundary (the lower boundary). The time step is set to $\Delta t = 0.01$ and the increment in z is set to $\Delta z = 0.1$, and these values of Δt and Δz are found being sufficient to maintain stability.

At each time level, $\hat{\psi}$ is computed using

$$\hat{\psi}_{zz} - \delta k^2 \hat{\psi}_i = \hat{\zeta}_i. \quad (3.83)$$

This is discretized using the second-order centered finite difference approximation to

obtain

$$\hat{\psi}^{n,k,j+1} - (2 + \delta\Delta z^2 k^2)\hat{\psi}^{n,k,j} + \hat{\psi}^{n,k,j-1} = \Delta z^2 \hat{\zeta}^{n,k,j}. \quad (3.84)$$

Equation (3.84) can be written in matrix form as

$$A\mathbf{U} = \mathbf{b}, \quad (3.85)$$

where the vector $\mathbf{U}$, the matrix A and the vector $\mathbf{b}$ are given respectively by

$$\mathbf{U} = \begin{pmatrix} \hat{\psi}^{n,k,1} \\ \hat{\psi}^{n,k,2} \\ \hat{\psi}^{n,k,3} \\ \dots \\ \hat{\psi}^{n,k,J-1} \\ \hat{\psi}^{n,k,J} \end{pmatrix}, \quad (3.86)$$

$$A = \begin{pmatrix} 1 & 0 & & & & \\ 1 & -2 - \delta(\Delta y)^2 \kappa^2 & & 1 & & \\ & 1 & -2 - \delta(\Delta y)^2 \kappa^2 & 1 & & \\ & & \dots & \dots & \dots & \\ & & & & \dots & \dots & 1 \\ & & & & A_{J-1,J} & A_{J,J} \end{pmatrix} \quad (3.87)$$

and

$$\mathbf{b} = \Delta z^2 \begin{pmatrix} \hat{\zeta}^{n,k,1} \\ \hat{\zeta}^{n,k,2} \\ \hat{\zeta}^{n,k,3} \\ \dots \\ \hat{\zeta}^{n,k,J-1} \\ b_J \end{pmatrix}. \quad (3.88)$$

Since A is tridiagonal and diagonally dominant, we can readily solve for $\hat{\psi}$ and then invert the transform to get ψ . The values of the elements in the J^{th} row of the matrix A , $A_{J-1,J}$ and $A_{J,J}$, and the element b_J in the J^{th} row of the vector $\mathbf{b}$ are given by the boundary condition specified and to implement the zero boundary condition we set $A_{J-1,J} = 0$, $A_{J,J} = 1$ and $b_J = 0$ so that $\hat{\psi}^{n,k,J} = 0$.

3.4.2 Results of the simulations

In this section we show the results obtained using the predictor-corrector numerical method described in section 3.4.1. Before we proceed to more complicated but more realistic configurations, especially those configurations in which nonlinear effects are taken into consideration, we assess the accuracy of the results obtained for the simple configurations for which analytical solutions were obtained in section 3.1. We first consider the configuration where the aspect ratio is $\delta = 0$ known as the long-wave limit approximation. The steady solution in this configuration is given by $\psi(x, z) = e^{ikx} e^{im_R z} e^{-m_I z}$, with m_R and m_I given respectively by (3.42) and (3.43), and the time-dependent solution is (3.59).

We use the following constants in the numerical simulations the Brunt-Väisälä frequency $N = \sqrt{2}$, the magnetic dip angle $\theta = \pi/2$, the background flow velocity $\bar{u} = 2.5$ and $z_1 = 10$. Simulations are carried out on a rectangular domain given by $0 \leq x \leq \pi$ and $10 \leq z \leq 45$, over the nondimensional time interval from $t = 0$ to $t = 10$. The waves are forced from below at $z = z_1 = 10$ by a periodic forcing $\psi(x, z = z_1, t) = e^{i\kappa x}$, $\kappa = 2$ and propagate upward toward the ionosphere.

As seen in Figure 3.1, the wave amplitude decreases with the height above $z \approx 28$ showing that the gravity waves are damped as they propagate upward in the ionosphere. Figure 3.2 shows that the waves reach a steady state as time increases while Figure 3.3 shows that the steady solution obtained in the numerical simulation matches perfectly the steady analytical solution derived in section 3.1. As seen in Figure 3.3 the wavelength of long atmospheric gravity increases as the waves propagate upward in the ionospheric F region in the configuration examined here where the magnetic dip angle $\theta = \pi/2$.

Figure 3.1: Linear simulation of long upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region ($\delta = 0$): contour plots of the streamfunction $\psi(x, z, t)$ obtained at $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$ and the neutral-ion collision frequency is $\nu_{ni} = 10$.

Figure 3.2: Linear simulation of long upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region ($\delta = 0$): plot of the the real part of the amplitude $\mathcal{Re}\{\phi(z, t)\}$ as a function of time at fixed height $z = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$ and the neutral-ion collision frequency is $\nu_{ni} = 10$.

For the general case without the long-wave limit, we set $\delta = 0.2$ and carry out simulations for the configurations where the magnetic dip angles are $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$.

Figure 3.3: Linear simulation of long upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region ($\delta = 0$): plot of the numerical solution $\psi(x, z, t)$ against the steady solution as a function of the height z at fixed $x = \pi/2$ and time, (a) $t = 4$, (b) $t = 10$. At $t = 10$ the numerical solution matches perfectly with the analytical steady solution derived in section 3.1 showing that the solution reaches a steady state as $t \rightarrow \infty$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$ was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$ and the neutral-ion collision frequency is $\nu_{ni} = 10$.

Figure 3.4: Linear simulation of upward propagating gravity waves (AGWs) in the neutral atmosphere ($\nu_{ni} = 0$); (a) contour plot of the streamfunction at the time $t = 15$, (b) plot of the streamfunction as function of altitude z at fixed $x = 1.43$ and time $t = 15$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 3.4 shows the results of the simulations of upward propagating gravity waves in the neutral atmosphere ($\nu_{ni} = 0$). The waves propagate upward and reach

the outflow boundary where a time-dependent radiation condition was applied. For details about the implementation of time-dependent radiation conditions I refer the reader to my MSc thesis (Victor, 2010). As seen in Figure 3.4 the amplitude and the wavelength of the waves are constant with altitude z .

In contrast with the neutral atmosphere ($\nu_{ni} = 0$), in the ionosphere where the neutral-ion collision $\nu_{ni} \neq 0$ there is not much wave activity at high altitude, above $z \approx 25$. In the configuration where the magnetic dip angle is $\theta = 0$ and that where the magnetic dip angle is $\theta = \pi/2$, there is not much wave activity above $z = 25$ as seen in Figures 3.5 (a) and 3.5 (c), showing that the waves are damped if the magnetic dip angle $\theta = 0$ and $\theta = \pi/2$. It is also seen that the waves decay more fast if the dip angle is $\theta = \pi/2$ than they do if $\theta = 0$. For the magnetic dip angle $\theta = \pi/4$, the wave amplitude decays less rapidly than it is for $\theta = 0$ and $\theta = \pi/2$ as shown in Figure 3.9. As shown in Figure 3.5 (b), in the configuration where $\theta = \pi/4$, the waves seem to reach the outflow boundary and it is therefore important to implement a radiation condition in order to prevent wave reflection (Victor, 2010).

The discussion in section 3.1 tells us that when the waves are steady they can be represented by the steady-state solution $\psi(x, z) = e^{ikx} e^{im_R z} e^{-m_I z}$ where $m_I(\theta)$ the decay rate is given by (3.31). This is in agreement with the predictions of the numerical solutions which show that the waves are damped. Figure 3.6 shows the plot of the decay rate of the amplitude of the waves $m_I(\theta)$ as a function of the magnetic dip angle θ , computed using (3.31). Figure 3.6 illustrates that the decay rate attains its maximum value if $\theta = \pi/2$, and 3.6 shows in particular that the decay rate of the wave amplitude vanishes for the magnetic dip angle $\theta = 0.2165\pi$ showing that the waves cannot be damped by the ion drag at this magnetic dip angle. This is the value of θ where $\frac{N^2}{\bar{u}^2} \sin^2 \theta - \delta \kappa^2 = 0$.

Figure 3.7 shows the plot of the vertical wavenumber m_R as a function of the magnetic dip angle θ , computed using (3.30). The vertical wavenumber attains its maximum value for the magnetic dip angle $\theta = 0.2165\pi$. Figure 3.7 shows that the vertical wavenumber is smaller for $\theta = \pi/2$, and so the wavelength is longer for $\theta = \pi/2$ as predicted by the numerical simulations.

Figure 3.8 shows that the waves approach the steady state solution derived in section 3.1 as the time t gets large. As shown in the figure 3.9 the vertical wavelength is longer if $\theta = \pi/2$ compared with $\theta = 0$ or $\theta = \pi/4$. This is in agreement with the conclusions from the analysis of the analytical solutions in section 3.1 as seen in Figure 3.7.

Figure 3.10 shows the results of the effects of the upward propagating atmospheric gravity waves on the ionosphere. These results were obtained by numerically solving the equation (3.65). The effects are more important if the magnetic dip angle $\theta = 0$ while they are less important for $\theta = \pi/2$.

To take into account the nonlinear effects we set $\varepsilon = 0.05$ in the equations (3.66) and (3.67) and carried out nonlinear simulations. The results shown here are for configurations for which the magnetic dip angles are $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$, as in the linear simulations ($\varepsilon = 0$). We again set the background flow velocity to be constant $\bar{u} = 1$, and examine waves forced from below at $z = z_1 = 10$ by a periodic forcing $\psi(x, z = z_1, t) = e^{i\kappa x}$, $\kappa = 2$ and propagate upward toward the ionosphere in a rectangular domain given $0 \leq x \leq \pi$ and $10 \leq z \leq 35$. We observe that for the case of the configuration with the magnetic dip angle $\theta = 0$, there is a transfer of the momentum flux from the specified wavenumbers of the gravity waves $k = \pm 2$ (at the forcing) to the mean flow corresponding to $k = 0$. This shown in Figure 3.11.

Figure 3.5: Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region: contour plots of the streamfunction $\psi(x, z, t)$ obtained at a $t = 30$, with the magnetic dip angle (a) $\theta = 0$, (b) $\theta = \pi/4$ and (c) $\theta = \pi/2$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 3.6: Analytical solution: plot of the damping rate m_I of upward propagating AGWs in the F as a function of the magnetic dip angle θ . A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 3.7: Analytical solution: plot of the wavenumber m_R of upward propagating AGWs in the F as a function of the magnetic dip angle θ . A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain, the background flow velocity is constant, $\bar{u} = 1$ and $\delta = 0.2$. The straight line corresponds to the vertical wavenumber in the neutral atmosphere $m = \left(\frac{N^2}{\bar{u}^2} - \delta\kappa^2 \right)^{1/2}$.

Figure 3.8: Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region (effects of the ionosphere on AGWs): plot of the streamfunction $\psi(x, z, t)$ as a function of time t at fixed $x = 2.25$ and height $z = 11$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$. The magnetic dip angle is $\theta = \pi/2$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$.

Figure 3.9: Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region (effects of the ionosphere on AGWs): plot of the streamfunction $\psi(x, z, t)$ as a function of height z at fixed $x = 2.25$ and time $t = 30$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$. The solid, the dashed and dotted lines correspond to the upward propagating gravity waves in a strong magnetic field in the ionospheric F region at the magnetic dip angles $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$ respectively.

Figure 3.10: Linear simulation ($\varepsilon = 0$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region: effects of atmospheric gravity wave (AGW) propagation on the ionosphere (traveling ionospheric disturbances, TIDs) in a strong magnetic field in the ionospheric F region. Plot of the perturbation number density $\mathcal{N}_i(x, z, t)$ for the ion species as a function of the height z at fixed $x = 2.7$ and time 25. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$. The solid, the dashed and dotted lines correspond to the upward propagating gravity waves in a strong magnetic field in the ionospheric F region at the magnetic dip angles $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$ respectively.

Figure 3.11: Nonlinear simulation against linear simulation ($\varepsilon = 0.05$) of upward propagating atmospheric gravity waves (AGWs) in a strong magnetic field in the ionospheric F region (effects of the ionosphere on AGWs): Fourier spectrum $\hat{\psi}(k, z, t)$ as a function of the wavenumber k at fixed height $z = 28$ and time $t = 25$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and the background flow velocity is constant, $\bar{u} = 1$, the neutral-ion collision frequency is $\nu_{ni} = 1$ and $\delta = 0.2$. The magnetic dip angle $\theta = 0$.

In both linear and nonlinear cases the results of the simulations show that the waves are damped due to the strong magnetic field in the ionospheric F region height, except at the magnetic dip angle $\theta = 0.2165\pi$. The observation of wave damping obtained in our analytical and numerical solutions is consistent with observations (Miesen, 1991)

3.5 Dimensional analysis

The solutions shown in Figures 3.1-3.11 are based on nondimensional quantities, which were defined in terms of the corresponding dimensional quantities according to (3.5).

In the neutral atmosphere the dimensional density scale height is approximately $H^* = 6.8$ km. In the numerical simulations of neutral AGWs we used a nondimensional density scale height $H = 4.9$. So the vertical length scale is $L_z = H^*/H = 1.38$ km.

The square of the aspect ratio in the numerical simulations was set to $\delta = 0.2$,

so the horizontal length scale is $L_x = L_z/\sqrt{\delta} = 3.13$ km. Therefore the dimensional horizontal wavelength is $\lambda_x^* = \lambda_x L_x = 10$ km, and so the dimensional horizontal wavenumber is then $\kappa^* = 2\pi/\lambda_x^* = 0.63 \times 10^{-3} \text{m}^{-1}$. We also have that the height of the rectangular domain is about 27.6 km while the width is about 10 km.

In the numerical simulations the nondimensional acceleration due to gravity was set to $g = 9.8$. Since the dimensional acceleration due to gravity is approximately $g^* = 9.8 \text{ m sec}^{-2}$, then the acceleration scale is simply 1 m sec^{-2} . This implies that the velocity scale is $U = \sqrt{L_z} = 37 \text{ m sec}^{-1}$, and so the dimensional mean flow velocity is $\bar{u}^* = 37 \text{ m sec}^{-1}$. The time scale is then $L_x/U = 85 \text{ sec}$.

According to Figure 3.8, the gravity wave amplitude increases initially and then about $t = 10$, it reaches an almost steady state with small amplitude oscillations about a value of about 0.1. The dimensional time frame of the waves to reach this state is about 15 minutes. We also note that the dimensional neutral-ion collision frequency is $\nu_{ni}^* = 12 \times 10^{-3} \text{ sec}^{-1}$.

According to (3.17) the dimensional Brunt-Väisälä frequency is approximately $N^* = \sqrt{g^*/H^*} = 0.03 \text{ sec}^{-1}$. The dimensional vertical wavenumber is then

$$m^* = \sqrt{\frac{N^{*2}}{u^{*2}} - \kappa^{*2}} = 0.81 \cdot 10^{-3} \text{m}^{-1}. \quad (3.89)$$

Hence the dimensional vertical wavelegth is $\lambda_z^* = 2\pi/m^* = 7.8$ km.

The properties of gravity waves described here are in good agreement with the properties of gravity waves generated in the neutral atmosphere (below the ionosphere) or in the lower ionosphere, which propagate vertically upward in the ionosphere and in particular in the ionospheric F region (Preusse et al., 2008).

Chapter 4

Wave interactions in a random ionosphere

4.1 Electromagnetic wave equations in a nonhomogeneous medium

In this section we derive the wave equations for electromagnetic waves propagating in an isotropic nonhomogeneous medium. An isotropic medium is a medium with similar properties in all directions. In a nonhomogeneous medium the dielectric permittivity ϵ and the permeability μ can be functions of time or space or both. A random medium is a typical example of a nonhomogeneous medium where ϵ and μ are random functions of time or space or both. The phenomenon of wave propagation in a nonhomogeneous medium arises in several context in nature, for example, acoustic-gravity waves in the ocean and atmosphere, seismic waves and radio waves in the ionosphere, and continuous pulsation electromagnetic waves in the ionosphere (Parks, 2005).

In this thesis we consider an isotropic source-free medium with random fluctuations in time. We examine a configuration that is representative of the ionosphere which has the property of an electric dynamo. An electric dynamo generates a direct current (DC); therefore the charge density is independent of time. Hence, $\frac{\partial \rho_{el}}{\partial t} = 0$.

We derive the wave equations from the Maxwell's equations (B.32)-(B.35) which

are given in Appendix B. We first multiply equation (B.32) by $\frac{1}{\mu}$ and then take the curl of the resulting equation to obtain

$$\begin{aligned}\nabla \times \left(\frac{\nabla \times \mathbf{E}}{\mu} \right) &= -\nabla \times \left(\frac{1}{\mu} \frac{\partial \mathbf{B}}{\partial t} \right) \\ &= -\nabla \times \left[\left(\frac{\partial}{\partial t} \ln |\mu| \right) \mathbf{H} \right] - \frac{\partial \mathbf{J}}{\partial t} - \frac{\partial^2 (\epsilon \mathbf{E})}{\partial t^2}.\end{aligned}\quad (4.1)$$

From vector calculus we know that if φ is a scalar and $\mathbf{V}$ is a vector, then

$$\nabla \times (\varphi \mathbf{V}) = (\nabla \varphi) \times \mathbf{V} + \varphi \nabla \times \mathbf{V} \quad (4.2)$$

and

$$\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}. \quad (4.3)$$

Making use of (4.2) with $\varphi = \frac{1}{\mu}$ and $\mathbf{V} = \nabla \times \mathbf{E}$ and applying (4.3) with $\mathbf{V} = \mathbf{E}$, we find that the left-hand side of equation (4.1) is

$$\begin{aligned}\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) &= \left(\nabla \frac{1}{\mu} \right) \times \nabla \times \mathbf{E} + \frac{1}{\mu} \nabla \times \nabla \times \mathbf{E} \\ &= -\frac{1}{\mu} \nabla \ln |\mu| \times \nabla \times \mathbf{E} + \frac{1}{\mu} \nabla \times \nabla \times \mathbf{E} \\ &= \frac{1}{\mu} \left[-\nabla \ln |\mu| \times (\nabla \times \mathbf{E}) + \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \right].\end{aligned}\quad (4.4)$$

Making use of (4.2) with $\varphi = \frac{\partial}{\partial t} \ln |\mu|$ and $\mathbf{V} = \mathbf{H}$ we find that the first term on the right-hand side of (4.1) is

$$\nabla \times \left[\left(\frac{\partial}{\partial t} \ln |\mu| \right) \mathbf{H} \right] = \left(\frac{\partial}{\partial t} \nabla \ln |\mu| \right) \times \mathbf{H} + \frac{\partial}{\partial t} \ln |\mu| \nabla \times \mathbf{H}. \quad (4.5)$$

Substituting (4.4) and (4.5) into (4.1) yields

$$\begin{aligned}-\nabla \ln |\mu| \times (\nabla \times \mathbf{E}) + \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\left(\frac{\partial}{\partial t} \nabla \ln |\mu| \right) \times (\mu \mathbf{H}) \\ &\quad - \mu \left(\frac{\partial}{\partial t} \nabla \ln |\mu| \right) \nabla \times \mathbf{H} - \mu \frac{\partial^2 (\epsilon \mathbf{E})}{\partial t^2} - \frac{\partial (\mu \mathbf{J})}{\partial t}.\end{aligned}\quad (4.6)$$

Making use of (B.33) and rearranging terms gives

$$\begin{aligned} \nabla^2 \mathbf{E} - \mu \frac{\partial^2(\epsilon \mathbf{E})}{\partial t^2} - \nabla(\nabla \cdot \mathbf{E}) + \nabla \ln |\mu| \times (\nabla \times \mathbf{E}) - \frac{\partial \mu}{\partial t} \frac{\partial(\epsilon \mathbf{E})}{\partial t} \\ = \left(\frac{\partial}{\partial t} \nabla \ln |\mu| \right) \times (\mu \mathbf{H}) + \frac{\partial(\mu \mathbf{J})}{\partial t}. \end{aligned} \quad (4.7)$$

Following an analogous procedure, multiplying equation by (B.33) by $\frac{1}{\epsilon}$ and taking the curl of the resulting equation, and then making use of the relations (4.2) and (4.3) gives

$$\begin{aligned} \nabla^2 \mathbf{H} - \epsilon \frac{\partial^2(\mu \mathbf{H})}{\partial t^2} - \nabla(\nabla \cdot \mathbf{H}) + \nabla \ln |\epsilon| \times (\nabla \times \mathbf{H}) - \frac{\partial \epsilon}{\partial t} \frac{\partial(\mu \mathbf{H})}{\partial t} \\ = - \left(\frac{\partial}{\partial t} \nabla \ln \epsilon \right) \times (\epsilon \mathbf{E}) + \nabla \ln |\epsilon| \times \mathbf{J} - \nabla \times \mathbf{J}. \end{aligned} \quad (4.8)$$

Equations (4.7) and (4.8) are the governing equations for electromagnetic wave propagation. We consider a source free-region where $\rho = 0$ and $\mathbf{J} = 0$ and assume that ϵ and μ only depend on time. Under these conditions, equations (4.7) and (4.8) become

$$\nabla^2 \mathbf{E} - \mu \frac{\partial^2(\epsilon \mathbf{E})}{\partial t^2} - \frac{d\mu}{dt} \frac{\partial(\epsilon \mathbf{E})}{\partial t} = 0 \quad (4.9)$$

and

$$\nabla^2 \mathbf{H} - \epsilon \frac{\partial^2(\mu \mathbf{H})}{\partial t^2} - \frac{d\epsilon}{dt} \frac{\partial(\mu \mathbf{H})}{\partial t} = 0. \quad (4.10)$$

4.2 Electromagnetic wave propagation in a random medium with weakly-random time variation

A random medium is a medium whose properties are random functions of time and position. Thus the properties of random media are characterized by statistical quantities. The randomness may be due to the fluctuations of thermodynamic quantities of the medium, or due to the irregular scatterers in the medium as discussed in Yeh

and Liu (1972). When electromagnetic waves propagate in a random medium, there are complex interferences and interactions between the fields which change the polarization of the fields (spatial configuration of a vector field). The polarization of the electromagnetic field is characterized by the electric and magnetic polarization vectors (Cheng, 1992).

There are two ways to study properties of wave propagation in a random medium. One way is to consider the medium as a continuum and the other is to consider waves scattered by randomly distributed discrete scatters. In the continuum medium the properties of the medium are characterized by its dielectric permittivity (ϵ) and permeability (μ). This allows to consider a weak random variation of the dielectric permittivity from its mean value and a weak random variation of the permeability from its mean value and leads to a significant simplification of the wave equations (4.9) and (4.10). This is the approach I will take here.

Three configurations may be taken into consideration: the first case is that for which the permittivity and permeability of a medium are random may be considered to be random functions of both position and time (which leads to extremely complicated mathematical problems), of space only (Yeh and Liu, 1972) or of time only, the case we consider here. The common property in all these three cases is that the electromagnetic fields are governed by SPDEs with random coefficients.

In this thesis we consider a weakly-random medium in which the electric permittivity and magnetic permeability are random functions of time t . The permittivity $\epsilon(t)$ and the permeability $\mu(t)$ in a random medium can be respectively represented as

$$\epsilon = \epsilon_o \epsilon_r = \epsilon_o \langle \epsilon_r \rangle + \Delta \epsilon(t) \text{ and } \mu = \mu_o \mu_r = \mu_o \langle \mu_r \rangle + \Delta \mu(t), \quad (4.11)$$

where $\epsilon_o \langle \epsilon_r \rangle$ and $\mu_o \langle \mu_r \rangle$ are the average permittivity and permeability over t , $\langle \epsilon_r \rangle$ and $\langle \mu_r \rangle$ are the mean relative permittivity average and mean relative permeability respectively, and $\Delta \epsilon(t)$ and $\Delta \mu(t)$ are the fluctuating parts of $\epsilon(t)$ and $\mu(t)$ respectively. We assume that $\Delta \epsilon(t)$ and $\Delta \mu(t)$ are stochastic variables with mean zero and that $\Delta \epsilon(0) = 0$ and $\Delta \mu(0) = 0$. Therefore $\epsilon(0) = \epsilon_o \langle \epsilon_r \rangle$ and $\mu(0) = \mu_o \langle \mu_r \rangle$. We

consider the case where $|\Delta\epsilon(t)| \ll |\epsilon_o\langle\epsilon_r\rangle|$ and $|\Delta\mu(t)| \ll |\mu_o\langle\mu_r\rangle|$ which means that the medium is weakly-random. If we choose the variance of $\Delta\epsilon(t)$ and $\Delta\mu(t)$ is t for example, then $\Delta\epsilon(t)$ and $\Delta\mu(t)$ are simply white noise.

The electric susceptibility of the medium is thus given by

$$\chi(t) = \frac{1}{\epsilon_o}(\epsilon(t) - \epsilon_o) = (\langle\epsilon_r\rangle - 1) + \frac{\Delta\epsilon(t)}{\epsilon_o}. \quad (4.12)$$

This implies the electric polarization vector is given by

$$\mathbf{P} = \epsilon_o\chi(t)\mathbf{E} = [(\langle\epsilon_r\rangle - 1) + \Delta\epsilon(t)]\mathbf{E} = \langle\mathbf{P}\rangle + \Delta\mathbf{P}, \quad (4.13)$$

where

$$\langle\mathbf{P}\rangle = (\langle\epsilon_r\rangle - 1)\mathbf{E} \quad (4.14)$$

is the average of $\mathbf{P}$ and the fluctuating part is

$$\Delta\mathbf{P} = \Delta\epsilon(t)\mathbf{E}. \quad (4.15)$$

On the other hand the magnetic susceptibility is given

$$\chi_m(t) = \frac{1}{\mu_o}[\mu(t) - \mu_o] = (\langle\mu_r\rangle - 1) + \frac{\Delta\mu(t)}{\mu_o}. \quad (4.16)$$

The magnetic polarization vector is thus given by

$$\mathbf{M} = \mu_o\chi_m(t)\mathbf{H} = [(\langle\mu_r\rangle - 1) + \Delta\mu(t)]\mathbf{H} = \langle\mathbf{M}\rangle + \Delta\mathbf{M}, \quad (4.17)$$

where

$$\langle\mathbf{M}\rangle = (\langle\mu_r\rangle - 1)\mathbf{H} \quad (4.18)$$

is the average of $\mathbf{M}$ and the fluctuating part is

$$\Delta\mathbf{M} = \Delta\mu(t)\mathbf{H}. \quad (4.19)$$

4.3 Approximate solution in an isotropic medium with weakly-random fluctuations in time

Considering that the medium is isotropic, each individual component E of the electric field vector and H of the magnetic field vector respectively satisfies

$$\nabla^2 E - \mu \frac{\partial^2(\epsilon E)}{\partial t^2} - \frac{d\mu}{dt} \frac{\partial(\epsilon E)}{\partial t} = 0 \quad (4.20)$$

and

$$\nabla^2 H - \epsilon \frac{\partial^2(\mu H)}{\partial t^2} - \frac{d\epsilon}{dt} \frac{\partial(\mu H)}{\partial t} = 0. \quad (4.21)$$

Equations (4.20) and (4.21) are of the same form, so once the expression for the electric field E is derived, it is straightforward to obtain that of the magnetic field H by interchanging ϵ and μ . We thus focus on the solution of (4.20). We derive exact solutions using separation of variables, i.e. writing E as $E(\mathbf{r}, t) = R(\mathbf{r})T(t)$ where $\mathbf{r}$ is the position vector and t the time. The wave equation (4.20) then gives

$$\frac{\nabla^2 R}{R} = \frac{(\mu(\epsilon T)')'}{T} = \lambda, \quad \text{a constant}, \quad (4.22)$$

where the prime stands for differentiation with respect to time t . From equation (4.22), $R(\mathbf{r})$ satisfies

$$\nabla^2 R - \lambda R = 0 \quad (4.23)$$

while $T(t)$ satisfies

$$(\mu(\epsilon T)')' - \lambda T = 0. \quad (4.24)$$

The general solution $E(\mathbf{r}, t)$ may be constructed as an infinite series of products of the eigenfunctions $R(\mathbf{r})$ and $T(t)$ satisfying (4.23) and (4.24).

The eigenvalues $\lambda \neq 0$ give eigenfunctions representing waves that oscillate in both space and time and vary randomly in time. Approximate solutions for (4.22) can be derived using the WKB method (Bender and Orszag, 1978). These oscillatory solutions resemble regular pulsation electromagnetic waves in the ionosphere and

their interactions with the gravity waves and TIDs can be studied. This is one of my proposals for future work (see section 7.2).

In this thesis we only focus on the solutions corresponding to $\lambda = 0$. The eigenvalue $\lambda = 0$ gives oscillations in space which are random in time. In the two-dimensional ionospheric configuration in a domain given by $0 < x < 2\pi$ and $z_1 < z < \infty$, if we consider periodic boundary conditions with sinusoidal oscillations in the x -direction and exponential variation in the z -direction, then the solution which is finite as $z \rightarrow \infty$ is

$$E_{\lambda=0}(x, z, t) = \frac{e^{-\eta z}}{\epsilon(t)} \left[A_1 + A_2 \int \frac{1}{\mu(t)} dt \right] (B_1 \cos \eta x + B_2 \sin \eta x), \quad (4.25)$$

where η , A_1 , A_2 , B_1 and B_2 are constants.

Observe that in a weakly-random

$$\frac{1}{\epsilon(t)} = \frac{1}{\epsilon_o \langle \epsilon_r \rangle + \Delta \epsilon(t)} = \frac{1}{\epsilon_o \langle \epsilon_r \rangle} \frac{1}{1 + \frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle}} \sim \frac{1}{\epsilon_o \langle \epsilon_r \rangle} \left[1 - \frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} + O \left(\frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} \right)^2 \right] \quad (4.26)$$

and

$$\frac{1}{\mu(t)} \frac{1}{\mu_o \langle \mu_r \rangle + \Delta \mu(t)} = \frac{1}{\mu_o \langle \mu_r \rangle} \frac{1}{1 + \frac{\Delta \mu(t)}{\mu_o \langle \mu_r \rangle}} \sim \frac{1}{\mu_o \langle \mu_r \rangle} \left[1 - \frac{\Delta \mu(t)}{\mu_o \langle \mu_r \rangle} + O \left(\frac{\Delta \mu(t)}{\mu_o \langle \mu_r \rangle} \right)^2 \right], \quad (4.27)$$

and so $\int \frac{1}{\epsilon(\tau)} d\tau \sim O(t)$ and $\int \frac{1}{\mu(\tau)} d\tau \sim O(t)$.

In order for $E_{\lambda=0}$ to be finite for large time $t \gg 1$, A_2 must be zero. Hence

$$E_{\lambda=0}(x, z, t) = \frac{e^{-\eta z}}{\epsilon(t)} (B_1 \cos \eta x + B_2 \sin \eta x) \quad (4.28)$$

$$\sim \frac{e^{-\eta z}}{\epsilon_o \langle \epsilon_r \rangle} (B_1 \cos \eta x + B_2 \sin \eta x) \left[1 - \frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} + O \left(\frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} \right)^2 \right]. \quad (4.29)$$

Since equations (4.9) and (4.10) are symmetric, we can obtain the expression for the magnetic field H corresponding to $\lambda = 0$ by simply interchanging ϵ and μ in the expression for the electric field (4.25). This gives

$$H_{\lambda=0}(x, z, t) = \frac{e^{-\eta z}}{\mu(t)} \left[C_1 + C_2 \int \frac{1}{\epsilon(t)} dt \right] (D_1 \cos \eta x + D_2 \sin \eta x), \quad (4.30)$$

where η , C_1 , C_2 , D_1 and D_2 are constants.

In order for $H_{\lambda=0}$ to be finite for large time $t \gg 1$, C_2 must be zero. Hence

$$\begin{aligned} H_{\lambda=0}(x, z, t) &= \frac{e^{-\eta z}}{\mu(t)} (D_1 \cos \eta x + D_2 \sin \eta x) \\ &\sim \frac{e^{-\eta z}}{\mu_o \langle \mu_r \rangle} (D_1 \cos \eta x + D_2 \sin \eta x) \left[1 - \frac{\Delta \mu(t)}{\mu_o \langle \mu_r \rangle} + O \left(\left(\frac{\Delta \mu(t)}{\mu_o \langle \mu_r \rangle} \right)^2 \right) \right]. \end{aligned} \quad (4.31)$$

(4.32)

We can then compute the electric flux density D from (4.28) and the magnetic flux density B from (4.31). This gives

$$D(x, z, t) = \epsilon(t)E(x, z, t) = e^{-\eta(z-z_1)} (B_1 \cos \eta x + B_2 \sin \eta x) \quad (4.33)$$

and

$$B(x, z, t) = \mu(t)H(x, z, t) = e^{-\eta(z-z_1)} (D_1 \cos \eta x + D_2 \sin \eta x). \quad (4.34)$$

We note that in this configuration E and H fluctuate randomly in time but D and B are time-independent.

4.4 Stochastic vorticity and energy equations for the ionospheric disturbances in an isotropic random medium with weakly-random fluctuation in time

In this section we derive the stochastic vorticity equation which will be used to model ionospheric disturbances in the vertical plane in a random ionosphere. This is the

vorticity equation (2.37) derived in chapter 2.3 in which the magnetic field B_{north} , and electric fields E_{east} and E_{vertical} are random functions of time and are represented by the solutions (4.29) and (4.32) of the wave equations (4.20) and (4.21) derived in section 4.3 corresponding to the eigenvalue $\lambda = 0$.

4.4.1 The stochastic electromagnetic field in the vertical plane

Considering the ionosphere to be isotropic means that each component of the electromagnetic field vectors $\mathbf{E}$ and $\mathbf{H}$ satisfies the same wave equation (4.20) and (4.21) and so each component can be represented in the form (4.29) or (4.32).

We impose the initial conditions

$$E_{\text{east}}(x, z, 0) = \frac{e^{-\eta_e z}}{\epsilon_o \langle \epsilon_r \rangle} \sin \eta_e x; \quad x_1 \leq x \leq x_2, \quad z_1 \leq z < +\infty \quad (4.35)$$

$$E_{\text{vertical}}(x, z, 0) = \frac{e^{-\eta_v z}}{\epsilon_o \langle \epsilon_r \rangle} \cos \eta_v x; \quad x_1 \leq x \leq x_2, \quad z_1 \leq z < +\infty \quad (4.36)$$

and

$$H_{\text{north}}(x, z, 0) = \frac{e^{-\eta_n z}}{\mu_o \langle \mu_r \rangle} \cos \eta_n x; \quad x_1 \leq x \leq x_2, \quad z_1 \leq z < +\infty \quad (4.37)$$

and the boundary conditions

$$E_{\text{east}}(x, z = z_1, t) = \frac{1}{\epsilon(t)} \sin \eta_e x, \quad t \geq 0 \quad (4.38)$$

$$E_{\text{vertical}}(x, z = z_1, t) = \frac{1}{\epsilon(t)} \cos \eta_v x, \quad t \geq 0. \quad (4.39)$$

and

$$H_{\text{north}}(x, z = z_1, t) = \frac{1}{\mu(t)} \cos \eta_n x, \quad t \geq 0. \quad (4.40)$$

for E_{east} , E_{vertical} and H_{north} respectively.

According to section 4.3, these conditions give the solutions

$$E_{\text{east}}(x, z, t) = \frac{e^{-\eta_e(z-z_1)}}{\epsilon(t)} \sin \eta_e x \quad (4.41)$$

$$\sim \frac{e^{-\eta_e(z-z_1)}}{\epsilon_o \langle \epsilon_r \rangle} \sin \eta_e x \left[1 - \frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} + O \left(\frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} \right)^2 \right], \quad (4.42)$$

$$E_{\text{vertical}}(x, z, t) = \frac{e^{-\eta_v(z-z_1)}}{\epsilon(t)} \cos \eta_v x \quad (4.43)$$

$$\sim \frac{e^{-\eta_v(z-z_1)}}{\epsilon_o \langle \epsilon_r \rangle} \cos \eta_v x \left[1 - \frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} + O \left(\frac{\Delta \epsilon(t)}{\epsilon_o \langle \epsilon_r \rangle} \right)^2 \right] \quad (4.44)$$

and

$$H_{\text{north}}(x, z, t) = \frac{e^{-\eta_n(z-z_1)}}{\mu(t)} \cos \eta_n x \quad (4.45)$$

$$\sim \frac{e^{-\eta_n(z-z_1)}}{\mu_o \langle \mu_r \rangle} \cos \eta_n x \left[1 - \frac{\Delta \mu(t)}{\mu_o \langle \mu_r \rangle} + O \left(\frac{\Delta \mu(t)}{\mu_o \langle \mu_r \rangle} \right)^2 \right]. \quad (4.46)$$

The components of the electric flux and magnetic flux densities are

$$D_{\text{east}}(x, z, t) = e^{-\eta_e(z-z_1)} \sin \eta_e x, \quad (4.47)$$

$$D_{\text{vertical}}(x, z, t) = e^{-\eta_v(z-z_1)} \cos \eta_v x \quad (4.48)$$

and

$$B_{\text{north}}(x, z, t) = e^{-\eta_n(z-z_1)} \cos \eta_n x. \quad (4.49)$$

Therefore, according to (B.34) the charge density $\varrho_{\alpha,el}$ is

$$\begin{aligned} \varrho_{\alpha,el} &= \nabla \cdot \mathbf{D} = \frac{\partial D_{\text{east}}}{\partial x} + \frac{\partial D_{\text{vertical}}}{\partial z} \\ &= \frac{\partial}{\partial x} e^{-\eta_e(z-z_1)} \sin \eta_e x + \frac{\partial}{\partial z} e^{-\eta_v(z-z_1)} \cos \eta_v x \\ &= \eta_e e^{-\eta_e(z-z_1)} \cos \eta_e x - \eta_v e^{-\eta_v(z-z_1)} \cos \eta_v x, \end{aligned} \quad (4.50)$$

and thus $\frac{\partial \varrho_{\alpha,el}}{\partial t} = 0$.

From (B.22) this means that $\nabla \cdot \mathbf{J} = 0$ and so the current density $\mathbf{J}$ is constant. Therefore the random ionosphere in this configuration is an electric dynamo.

4.4.2 Stochastic equations for the ionospheric disturbances

We use the vorticity equation (2.37) and the energy equation (2.13) to model ionospheric disturbances in a random ionosphere. Since the medium is weakly-random, we omit the terms of order $O\left(\frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle}\right)^2$ in the $\mathbf{E}$ components, and write the vorticity equation (2.37) as

$$\begin{aligned} & \nabla^2 \Psi_{\alpha_t} - \Psi_{\alpha_z} \nabla^2 \Psi_{\alpha_x} + \Psi_{\alpha_x} \nabla^2 \Psi_{\alpha_z} + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \varrho_\alpha}{\partial x} - \nu_\alpha \nabla^4 \Psi_\alpha \\ & + \nu_{\alpha n} \nabla^2 (\Psi_\alpha - \Psi) - \Upsilon_1(x, z) \Psi_{\alpha_z} + \Upsilon_2(x, z) \Psi_{\alpha_x} + \Upsilon_3(x, z) \left[1 - \frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle} \right] = 0, \end{aligned} \quad (4.51)$$

where $\Upsilon_1(x, z)$, $\Upsilon_2(x, z)$ and $\Upsilon_3(x, z)$ are given respectively by

$$\begin{aligned} \Upsilon_1(x, z) &= (\varrho_{\alpha,el} B_{\text{north}})_x \\ &= \eta_e \frac{\partial}{\partial x} \left[e^{-(\eta_e + \eta_n)(z - z_i)} \cos \eta_e x \cos \eta_n x \right] \\ &\quad - \eta_n \frac{\partial}{\partial x} \left[e^{-(\eta_v + \eta_n)(z - z_i)} \cos \eta_v x \cos \eta_n x \right] \\ &= -\eta_e e^{(\eta_e + \eta_n)(z - z_i)} (\eta_e \sin \eta_e x \cos \eta_n x + \eta_n \cos \eta_e x \sin \eta_n x) \\ &\quad + \eta_v e^{(\eta_v + \eta_n)(z - z_i)} (\eta_v \sin \eta_v x \cos \eta_n x + \eta_n \cos \eta_v x \sin \eta_n x), \end{aligned} \quad (4.52)$$

$$\begin{aligned} \Upsilon_2(x, z) &= (\varrho_{\alpha,el} B_{\text{vertical}})_z \\ &= \eta_e \frac{\partial}{\partial z} \left[e^{-(\eta_e + \eta_n)(z - z_i)} \cos \eta_e x \cos \eta_n x \right] \\ &\quad - \eta_v \frac{\partial}{\partial z} \left[e^{-(\eta_v + \eta_n)(z - z_i)} \cos \eta_v x \cos \eta_n x \right] \\ &= -\eta_e (\eta_e + \eta_n) e^{-(\eta_e + \eta_n)(z - z_i)} \cos \eta_e x \cos \eta_n x \\ &\quad + \eta_v (\eta_v + \eta_n) e^{-(\eta_v + \eta_n)(z - z_i)} \cos \eta_v x \cos \eta_n x \end{aligned} \quad (4.53)$$

and

$$\begin{aligned}
\Upsilon_3(x, z) &= \frac{\eta_e}{\langle \epsilon \rangle} \frac{\partial}{\partial z} [e^{-2\eta_e(z-z_i)} \cos \eta_e x \sin \eta_n x] + \frac{\eta_v}{\langle \epsilon \rangle} \frac{\partial}{\partial z} [e^{-(\eta_e+\eta_v)(z-z_i)} \cos \eta_v x \sin \eta_e x] \\
&\quad - \frac{\eta_e}{\langle \epsilon \rangle} \frac{\partial}{\partial x} [e^{-(\eta_e+\eta_v)(z-z_i)} \cos \eta_e x \cos \eta_v x] - \frac{\eta_v}{\langle \epsilon \rangle} \frac{\partial}{\partial x} [e^{-2\eta_v(z-z_i)} \cos^2 \eta_v x] \\
&= -\frac{2\eta_w^2}{\langle \epsilon \rangle} e^{-2\eta_e(z-z_i)} \cos \eta_e x \sin \eta_v x - \frac{\eta_v(\eta_e + \eta_v)}{\langle \epsilon \rangle} e^{-(\eta_e+\eta_v)(z-z_i)} \cos \eta_v x \sin \eta_e x \\
&\quad + \frac{\eta_e}{\langle \epsilon \rangle} e^{-(\eta_e+\eta_n)(z-z_i)} (\eta_e \sin \eta_e x \cos \eta_v x + \eta_v \cos \eta_e x \sin \eta_v x) \\
&\quad + \frac{2\eta_v^2}{\langle \epsilon \rangle} e^{2\eta_v(z-z_i)} \cos \eta_v x \sin \eta_v x.
\end{aligned} \tag{4.54}$$

Since $\Delta\epsilon(t)$ is a random variable with mean zero and $\Delta\epsilon(0) = 0$, $\frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle}$ can be written as $\sigma\dot{W}(t)$ where $\dot{W}(t)$ is a Wiener process and σ is a small constant. Hence equation (4.51) becomes

$$\begin{aligned}
&\nabla^2 \Psi_{\alpha t} - \Psi_{\alpha z} \nabla^2 \Psi_{\alpha x} + \Psi_{\alpha x} \nabla^2 \Psi_{\alpha z} + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \varrho_\alpha}{\partial x} - \nu_\alpha \nabla^4 \Psi_\alpha + \nu_{\alpha n} \nabla^2 (\Psi_\alpha - \Psi) \\
&\quad - \Upsilon_1(x, z) \Psi_{\alpha z} + \Upsilon_2(x, z) \Psi_{\alpha x} = -\Upsilon_3(x, z) + \sigma \Upsilon_3(x, z) \dot{W}(t).
\end{aligned} \tag{4.55}$$

A detailed description of Wiener integrals and the necessary background notation is given in Appendix C.

On the other hand, the divergence of the Poynting vector $\nabla \cdot (\mathbf{E} \times \mathbf{H})$ (the electromagnetic energy per unit of volume) is given by

$$\begin{aligned}
\nabla \cdot (\mathbf{E} \times \mathbf{H}) &= - \left\{ e^{-(\eta_v+\eta_n)(z-z_1)} [\eta_v \sin \eta_v x \cos \eta_n x + \eta_n \cos \eta_v x \sin \eta_n x] + \right. \\
&\quad \left. (\eta_e + \eta_n) e^{-(\eta_e+\eta_n)(z-z_1)} \cos \eta_e x \cos \eta_n x \right\} \left[1 - \frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle} + \left(\frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle} \right)^2 \right] \\
&\quad \left[1 - \frac{\Delta\mu(t)}{\mu_o\langle\mu_r\rangle} + \left(\frac{\Delta\mu(t)}{\mu_o\langle\mu_r\rangle} \right)^2 \right].
\end{aligned} \tag{4.56}$$

Substituting 4.56 into the energy equation (2.13) and neglecting the terms of order

$O\left(\frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle}\right)^2$, $O\left(\frac{\Delta\mu(t)}{\mu_o\langle\mu_r\rangle}\right)^2$ and $O\left(\frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle}\frac{\Delta\mu(t)}{\mu_o\langle\mu_r\rangle}\right)$, the energy equation is thus written as

$$\varrho_{\alpha_t} - \Psi_{\alpha_z}\varrho_{\alpha_x} + \Psi_{\alpha_x}\varrho_{\alpha_z} - \kappa_\alpha \nabla^2 \varrho_\alpha = \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) \left[1 - \frac{\Delta\epsilon(t)}{\epsilon_o\langle\epsilon_r\rangle} - \frac{\Delta\mu(t)}{\mu_o\langle\mu_r\rangle} \right], \quad (4.57)$$

where

$$\begin{aligned} \Upsilon_4(x, z) = & -e^{-(\eta_v+\eta_n)(z-z_i)} (\eta_v \sin \eta_v x \cos \eta_n x + \eta_n \cos \eta_v x \sin \eta_n x) \\ & - (\eta_e + \eta_n) e^{-(\eta_e+\eta_n)(z-z_i)} \cos \eta_e x \cos \eta_n x. \end{aligned} \quad (4.58)$$

We also write $\frac{\Delta\mu(t)}{\mu_o\langle\mu_r\rangle}$ as $\tilde{\sigma}\dot{\tilde{W}}(t)$ where $\dot{\tilde{W}}(t)$ is a Wiener process and $\tilde{\sigma}$ is a small constant. This gives the stochastic energy equation

$$\varrho_{\alpha_t} - \Psi_{\alpha_z}\varrho_{\alpha_x} + \Psi_{\alpha_x}\varrho_{\alpha_z} - \kappa_\alpha \nabla^2 \varrho_\alpha = \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) - \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) [\sigma \dot{W}(t) + \tilde{\sigma} \dot{\tilde{W}}(t)]. \quad (4.59)$$

In the next chapter, we discuss a numerical method that can be used to numerically solve stochastic PDEs such as (4.55) and (4.59). In chapter 6 we solve these SPDEs along with the equations for AGWs.

Chapter 5

The Itô formulation and the Wiener Chaos Expansion (WCE) method for solving SPDEs

In this chapter, the Itô stochastic integral is defined in terms of the Riemann-Stieltjes sum. Our description follows that given in Mikosch (2004). We present and apply the Itô lemma to derive analytical solutions to some stochastic evolution equations: the stochastic heat equation, the stochastic advection-diffusion equation and the stochastic Burgers' equation. We then present a numerical method based on Wiener chaos expansions (Martin and Cameron, 1947) and apply it to obtain numerical solutions of the three test problems. The accuracy of the numerical method is assessed by comparing the numerical solutions of each of the test problems with the analytical solutions. This method will be used in chapter 6 to solve the SPDEs (4.55) and (4.59) in our ionospheric problems.

5.1 The Itô formulation and application to stochastic differential equations (SDEs)

5.1.1 Itô formulation

In this section we present the Itô formula for SDEs (Mikosch, 2004). In (classical) calculus the integral of a real-valued function $h(t)$ $h : [0, T] \rightarrow \mathbb{R}$ with respect to a real function $g(t)$ is defined in terms of the Riemann sum by

$$\int_0^T h(t) dg(t) = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} h(t_j^*) [g(t_{j+1}) - g(t_j)], \quad (5.1)$$

where t_j^* is any point in the interval $[t_j, t_{j+1}]$ for each j . In stochastic calculus the stochastic integral $\int_0^T hW(dt)$ of a stochastic process $h(\omega, t) \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$ is defined in an analogous form. We write

$$\int_0^T h(\omega, t) W(dt) = \lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} h(\omega, t_j^*) [W(t_{j+1}) - W(t_j)]. \quad (5.2)$$

With the choice of $t_j^* = t_j$, we have

$$\lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} h(\omega, t_j) [W(t_{j+1}) - W(t_j)] = \int_0^T h(\omega, t) \cdot W(dt). \quad (5.3)$$

This integral is called the Itô stochastic integral and is sometimes denoted as $(h \cdot W)(t \in [0, T])(\omega)$. It was first developed by Itô in the 1940s.

On the other hand if $t_j^* = t_{j+1/2} = \frac{t_j + t_{j+1}}{2}$, equation (5.2) becomes

$$\lim_{n \rightarrow \infty} \sum_{j=0}^{n-1} h(\omega, t_{j+1/2}) [W(t_{j+1}) - W(t_j)] = \int_0^T h(\omega, t) \circ W(dt), \quad (5.4)$$

where we use the notation “ $\circ$ ” to distinguish the integral (5.4) from the integral 5.3. This integral is called the Stratonovich stochastic integral and is sometimes denoted as

$(h \circ W)(t \in [0, T])(\omega)$. Each of the integrals (5.3) and (5.4) is a particular form of the Riemann-Stieltjes integral. Both (5.3) and (5.4) converge in the sense of mean square (convergence in L^2), but we can verify that (5.3) is a martingale with respect to the filtration $\mathcal{F}_t$ but (5.4) is not. Moreover the integral (5.4) is not an isonormal process so it does not satisfy the isometric property (C.11). Although the integrals (5.3) and (5.4) are meant to represent the same result, their properties are very different.

The isometric property of the Itô integral (see Appendix C) allows us to expand white noise as a Fourier series. The Wiener chaos expansion method (Hermite-Fourier expansion) developed by Martin and Cameron (1947) that allows us to obtain a martingale solution of a stochastic partial differential equation (SPDE) is possible because of the nice properties of the Itô integral. This method discussed in section 5.2.

5.1.2 The Itô formula and application for solving SDEs

Here we follow (Mikosch, 2004) and derive the simplest versions of the Itô formula. Suppose that f is a function of the Brownian motion path $W(t)$, $f = f(W(t))$ and assume that f is twice differentiable. Let $\xi = W(t)$. Then f can be approximated by the random Taylor series

$$f((W(t) + dW(t)) - f(W(t)) = f_\xi(W(t))dW(t) + \frac{1}{2}f_{\xi\xi}(W(t))(dW(t))^2 + \dots \quad (5.5)$$

Since $E(dW(t))^2 = dt$ (see Appendix C), then $(dW(t))^2$ can be replaced by dt .

Taking the $dt \rightarrow 0$ and integrating both sides of equation (5.5) gives the following lemma:

Lemma 1. *Let f be continuously differentiable. Then*

$$\int_{t_0}^t df(W(s)) := f(W(t)) - f(W(t_0)) = \int_{t_0}^t f_\xi(W(s))dW(s) + \frac{1}{2} \int_{t_0}^t f_{\xi\xi}(W(s))ds, \quad t > t_0. \quad (5.6)$$

This is the simple form of the Itô lemma.

Now consider $f = f(t, W(t))$ and suppose that the first and second partial derivatives of f exist and are continuous. Let $\xi = W(t)$. Then

$$\begin{aligned} f(t+dt, W(t+dt)) - f(t, W(t)) &= f_t(t, W(t))dt + f_\xi(t, W(t))dW(t) \\ &+ \frac{1}{2}f_{tt}(t, W(t))dt^2 + \frac{1}{2}f_{t\xi}(t, W(t))dt dW(t) + \frac{1}{2}f_{\xi\xi}(t, W(t))(dW(t))^2 + \dots \end{aligned} \quad (5.7)$$

Non taking the limit $dt \rightarrow 0$ and integrating both sides of the equation (5.7) gives the following lemma:

Lemma 2. *Let $f(t, \xi)$ be a function whose first and second partial derivatives are continuous. Then*

$$\begin{aligned} f(t, W(t)) - f(t_0, W(t_0)) &= \int_{t_0}^t f_t(s, W(s))ds \\ &+ \frac{1}{2} \int_{t_0}^t f_{\xi\xi}(s, W(s))ds + \int_{t_0}^t f_\xi(s, W(s))dW(s), \quad t > t_0. \end{aligned} \quad (5.8)$$

This is the first extension of the Itô lemma.

Example 1. *Consider $f(\xi) = \xi^2$ so that $f_\xi(\xi) = 2\xi$ and $f_{\xi\xi}(\xi) = 2$. According to Lemma 1 we get*

$$W^2(t) - W^2(t_0) = 2 \int_{t_0}^t W(s)dW(s) + \int_{t_0}^t ds = 2 \int_{t_0}^t W(s)dW(s) + (t - t_0). \quad (5.9)$$

Hence

$$\int_{t_0}^t W(s)dW(s) = \frac{1}{2} [W^2(t) - W^2(t_0)] - (t - t_0). \quad (5.10)$$

Example 2. *Consider $f(t, \xi) = e^{\xi - \frac{1}{2}t}$ so that $f_t(t, \xi) = -\frac{1}{2}f(t, \xi)$, $f_\xi(\xi) = f(\xi)$ and*

$f_{\xi\xi}(\xi) = f(\xi)$. According to Lemma 2 we get

$$e^{-\frac{1}{2}t}e^{W(t)} - e^{-\frac{1}{2}t_0}e^{W(t_0)} = \int_{t_0}^t e^{-\frac{1}{2}s}e^{W(s)}dW(s). \quad (5.11)$$

Hence,

$$e^{-\frac{1}{2}t}e^{W(t)} = \int_{t_0}^t e^{-\frac{1}{2}s}e^{W(s)}dW(s) + e^{-\frac{1}{2}t_0}e^{W(t_0)}. \quad (5.12)$$

The expression (5.12) is called the Itô exponential.

Example 3. This example is the continuation of Example 2. Let $A(t) = e^{-\frac{1}{2}t}e^{W(t)}$. Differentiating (5.11) with respect to t then yields

$$\dot{A}(t)dt = A(t)\dot{W}(t)dt, \quad (5.13)$$

Hence,

$$dA(t) = A(t)dW(t). \quad (5.14)$$

The stochastic process

$$A(t) = A_0e^{-\frac{1}{2}t}e^{W(t)} \quad (5.15)$$

is the solution of the SDE (5.14) with initial condition $A(0) = A_0$.

Lemma 3. Consider the Itô stochastic process given by

$$A(t) - A(t_0) = \int_{t_0}^t g(s)ds + \int_{t_0}^t h(s)dW(s). \quad (5.16)$$

If $f(t, \xi)$ is a function whose first and second order partial derivatives are continuous, then

$$\begin{aligned} f(t, A(t)) - f(t_0, A(t_0)) \\ = \int_{t_0}^t [f_t(s, A(s)) + \frac{1}{2}h^2(s)f_{\xi\xi}(s, A(s))]ds + \int_{t_0}^t f_{\xi}(s, A(s))dA(s). \end{aligned} \quad (5.17)$$

Example 4. Consider the Langevin stochastic differential equation

$$dA(t) = cA(t) + \sigma dW(t), \quad c, \sigma = \text{constants} \quad (5.18)$$

with initial condition

$$A(0) = A_0. \quad (5.19)$$

To solve (5.18) and (5.19) for $A(t)$, let $\mathcal{A}(t) = e^{-ct}A(t)$ and observe that $A(0) = A_0 = \mathcal{A}_0$. Applying Lemma 3 with $f(t, \xi) = e^{-ct}\xi$, $f_t(t, \xi) = -ce^{-ct}\xi$, $f_\xi(t, \xi) = e^{-ct}$, $f_{\xi\xi}(t, \xi) = 0$, $g(s) = cA(s)$ and $h(s) = \sigma$ gives

$$\mathcal{A}(t) - \mathcal{A}_0 = \sigma \int_0^t e^{-cs} dW(s). \quad (5.20)$$

Substituting $\mathcal{A}(t) = e^{-ct}A(t)$ in (5.20) gives

$$A(t) = e^{ct}A_0 + \sigma e^{ct} \int_0^t e^{-cs} dW(s). \quad (5.21)$$

5.2 Wiener Chaos Expansion (WCE) method for solving SPDEs

In this section we describe a numerical method based on the concepts of the Wiener Chaos Expansion (WCE) or Fourier-Hermite expansion. It is an extension of the polynomial chaos method developed by Norbert Wiener (Mikulevicius and Rozovskii, 2004, Hou et al., 2006). The polynomial chaos method uses Hermite polynomials as an orthonormal random basis for expanding homogeneous chaos depending on white noise. Based on Wiener's polynomial chaos method, Martin and Cameron (1947) developed a more explicit and intuitive formulation for the Wiener-Hermite expan-

sion. They proposed to discretize the white noise process through its Fourier-Hermite expansion to elegantly represent the solution of the SPDEs. Their work simplified the numerical study of SPDEs. It is commonly called the Wiener Chaos Expansion (WCE) method and is now a useful tool in stochastic analysis that involves the study of white noise. The solution of the SPDE is then a random process depending on the realizations of the Brownian motion forcing. We seek a solution $u = u(x, t; \mathbf{W}_0^t)$ for a parabolic SPDE of the form

$$u_t = \mathcal{L}(u) + \sigma(x, t, u) \cdot \dot{\mathbf{W}}, \quad (5.22)$$

where $\mathcal{L}$ is a differential operator acting on u in space and $\mathbf{W}_t$ a Brownian motion vector, σ a vector function and $\mathbf{W}_0^t$ is the Brownian motion paths up to time t .

To introduce the WCE method we define Hermite polynomials, define the Wiener chaos expansion of a function of the form $u(x, t; W_0^t)$ and then give some properties of Hermite polynomials that are needed to define the method.

Definition 1. *Hermite polynomials P_n are eigenfunctions of the Hermite equation*

$$\mathcal{L}(P) = P'' - xP = -EP. \quad (5.23)$$

The eigenvalues of the Hermite equation are $E_n = n + \frac{1}{2}$ where n is a nonnegative integer (Eisberg and Resnick, 1985).

The generating function of Hermite polynomials is

$$\mathcal{G}(x, z) = \exp\left(-\frac{z^2}{2} + xz\right). \quad (5.24)$$

And so Hermite polynomials can be obtained using

$$P_n(x) = \left[\frac{\partial^n \mathcal{G}(x, z)}{\partial z^n} \right]_{z=0}. \quad (5.25)$$

Therefore

$$\mathcal{G}(x, z) = \exp\left(-\frac{z^2}{2} + xz\right) = \sum_{n=0}^{\infty} P_n(x) \frac{z^n}{n!}. \quad (5.26)$$

Definition 2. *The Fourier-Hermite expansion (or Wiener chaos expansion) of $u(x, t; W_0^t)$ is the infinite series given by*

$$u(x, t; W_0^t) = \sum_{\alpha} u_{\alpha}(x, t) T_{\alpha}, \quad (5.27)$$

where $u_{\alpha}(x, t)$ are deterministic functions and T_{α} are multi-variable Hermite polynomials of Gaussian random variables.

5.2.1 Some properties of Hermite polynomials

In this sections we describe some properties of Hermite polynomials which are helpful to understand the WCE method.

Property 1. *Hermite polynomials are orthogonal with respect to the Gaussian measure.*

Proof. Let us write the Hermite equation (5.23) in its self-Adjoint form to obtain

$$\left(\exp\left(-\frac{x^2}{2}\right) P'\right)' + E \exp\left(-\frac{x^2}{2}\right) P = 0. \quad (5.28)$$

According to equation (5.28) Hermite polynomials satisfy the orthogonality property with respect to the measure $\mu(dx) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} dx$,

$$\int_{-\infty}^{+\infty} P_n(x) P_m(x) \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} dx = n! \delta_{n,m}. \quad (5.29)$$

Definition 3. *The normalized Hermite polynomials are the functions $H_n = \frac{P_n}{\sqrt{n!}}$, where P_n are Hermite polynomials and n is nonnegative integer.*

The measure $\mu(dx) = f(x)dx = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}} dx$ is the Gaussian measure on the real axis $\mathbb{R} = (-\infty, +\infty)$ with $f(x) = \frac{\exp\left(-\frac{x^2}{2}\right)}{\sqrt{2\pi}}$ being the density function for the distribu-

tion $\mu\{(-\infty, x)\}$ and dx the Lebesgue measure. $H(x)$ are square-integrable functions with respect to the Gaussian measure $\mu(dx)$ as

$$\mathcal{L}^2(\mathbb{R}, \mu) = \left\{ H(x) : \int_{-\infty}^{+\infty} H^2(x) \mu(dx) < \infty \right\}. \quad (5.30)$$

The inner product is defined as

$$\langle H, G \rangle_\mu = \int_{-\infty}^{+\infty} H(x)G(x) \mu(dx) = \int_{-\infty}^{+\infty} H(x)G(x) f(x) dx, \quad (5.31)$$

where H and G are Hermite polynomials. Since x is a standard Gaussian random variable with $N(0, 1)$ distribution, it implies that

$$\langle H, G \rangle_\mu = E[H(x)G(x)] \quad (5.32)$$

is the covariance of H and G . According to the orthonormal property of Hermite polynomials we have

$$\langle H_n(x), H_m(x) \rangle_\mu = E[H_n(x)H_m(x)] = \delta_{n,m}. \quad (5.33)$$

Also,

$$E[H_n(x)] = \int_{-\infty}^{+\infty} H(x) \mu(dx) = \langle H_n(x), 1 \rangle_\mu = 0 \text{ if } n \neq 0. \quad (5.34)$$

Therefore the Hermite polynomials H_n with order $n > 0$ are standard Gaussian variables. □

Property 2. *Hermite polynomials are eigenfunctions of the Fourier transform.*

Proof. Consider the generating function of Hermite polynomials

$$\mathcal{G}(x, z) = \exp\left(-\frac{z^2}{2} + xz\right). \quad (5.35)$$

Multiplying $\mathcal{G}(x, z)$ with $\exp\left(-\frac{x^2}{2}\right)$ and taking the Fourier transform with respect

to x yields

$$\begin{aligned}\mathcal{F} \left\{ \mathcal{G}(x, z) \exp \left(-\frac{x^2}{2} \right) \right\} &= \mathcal{F} \left\{ \exp \left(-\frac{x^2}{2} + xz - \frac{z^2}{2} \right) \right\} \\ &= \mathcal{F} \left\{ \sum_{n=0}^{\infty} \exp \left(-\frac{x^2}{2} \right) P_n(x) \frac{z^n}{n!} \right\},\end{aligned}\quad (5.36)$$

according to (5.35). On the other hand we have

$$\begin{aligned}\mathcal{F} \left\{ \exp \left(-\frac{x^2}{2} + xz - \frac{z^2}{2} \right) \right\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp(ikx) \exp \left(-\frac{x^2}{2} + xz - \frac{z^2}{2} \right) dx \\ &= \exp \left(-\frac{k^2}{2} - ikz - \frac{z^2}{2} \right) \\ &= \sum_{n=0}^{\infty} \exp \left(-\frac{k^2}{2} \right) P_n(k) \frac{(-iz)^n}{n!}.\end{aligned}\quad (5.37)$$

Thus,

$$\begin{aligned}\mathcal{F} \left\{ \sum_{n=0}^{\infty} \exp \left(-\frac{x^2}{2} \right) P_n(x) \frac{z^n}{n!} \right\} &= \sum_{n=0}^{\infty} \mathcal{F} \left\{ \exp \left(-\frac{x^2}{2} \right) P_n(x) \frac{z^n}{n!} \right\} \\ &= \sum_{n=0}^{\infty} \exp \left(-\frac{k^2}{2} \right) P_n(k) \frac{(-iz)^n}{n!}.\end{aligned}\quad (5.38)$$

Therefore

$$\mathcal{F} \left\{ \exp \left(-\frac{x^2}{2} \right) P_n(x) \frac{z^n}{n!} \right\} = \exp \left(-\frac{k^2}{2} \right) P_n(k) \frac{(-iz)^n}{n!}.\quad (5.39)$$

Hence

$$\mathcal{F} \left\{ \exp \left(-\frac{x^2}{2} \right) P_n(x) \right\} = (-i)^n \exp \left(-\frac{k^2}{2} \right) P_n(k).\quad (5.40)$$

□

Property 3. For nonnegative integers α and β we have

$$H_\alpha(x)H_\beta(x) = \sum_{p \leq \alpha \wedge \beta} B(\alpha, \beta, p) H_{\alpha+\beta-2p}(x), \quad x \in \mathbb{R},\quad (5.41)$$

where $\alpha \wedge \beta = \min(\alpha, \beta)$ and $B(\alpha, \beta, p) = \left(C_p^\alpha C_p^\beta C_{\alpha-p}^{\alpha+\beta-2p} \right)^{1/2}$.

Proof. From equation (5.35) we can show that

$$\mathcal{G}(x, z)\mathcal{G}(x, w) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{P_{\alpha}(x)P_{\beta}(x)}{\alpha!\beta!} z^{\alpha}w^{\beta}. \quad (5.42)$$

On the other hand

$$\begin{aligned} \mathcal{G}(x, z)\mathcal{G}(x, w) &= \exp \left[-\frac{z^2 + w^2}{2} + x(z + w) \right] \\ &= \exp(zw) \exp \left(\frac{x^2}{2} \right) \exp \left[-\frac{(z + w - x)^2}{2} \right] \\ &= \sum_{p=0}^{\infty} \frac{(zw)^p}{p!} \sum_{l=0}^{\infty} \frac{P_l(x)}{l!} (z + w)^l \\ &= \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} \frac{P_l(x)}{p!l!} \sum_{j=0}^l C_j^l z^{j+p} w^{l+p-j}. \end{aligned} \quad (5.43)$$

Setting $l = j + m$ implies $m \geq 0$ if $j \leq l$, then

$$\mathcal{G}(x, z)\mathcal{G}(x, w) = \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{P_{j+m}(x)}{p!j!m!} z^{j+p} w^{m+p}. \quad (5.44)$$

Now let $j+p = \alpha$, $m+p = \beta$, then $j = \alpha - p$ and $m = \beta - p$ imply $p \leq \min(\alpha, \beta) = \alpha \wedge \beta$ and $j + m = \alpha + \beta - 2p$. Therefore

$$\mathcal{G}(x, z)\mathcal{G}(x, w) = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \sum_{p \leq \alpha \wedge \beta}^{\infty} \frac{P_{\alpha+\beta-2p}(x)}{p!(\alpha-p)!(\beta-p)!} z^{\alpha}w^{\beta}. \quad (5.45)$$

Comparing this equation with equation (5.43) implies

$$P_{\alpha}(x)P_{\beta}(x) = \sum_{p \leq \alpha \wedge \beta}^{\infty} \frac{\alpha!\beta!}{p!(\alpha-p)!(\beta-p)!} P_{\alpha+\beta-2p}(x). \quad (5.46)$$

Hence substituting $P_n(x) = \sqrt{n!}H_n(x)$ in this equation we obtain exactly equation (5.41). $\square$

5.2.2 The Wiener chaos expansion

We now consider the SPDE,

$$u_t(x, t) = \mathcal{L}[u(x, t)] + \sigma(x, t, u(x, t))\dot{W}(t), \quad (5.47)$$

where σ , W are scalar functions and $\dot{W}$ is the derivative of W with respect to time t . Since the forcing depends on all possible Brownian motion paths up to time t , this makes the solution $u = u(x, t)$ of equation (5.47) a functional of $\{W(s), 0 < s < t\}$. We can generalize the Fourier-Hermite expansion to the stochastic solution $u(x, t)$. To do so we define a set of orthonormal bases $\{m_i(s)\}_{i=1}^n$, $t > 0$ in the Hilbert space $L^2\{[0, t]\}$ as defined in Appendix C.2. If $\{\dot{W}(s)\}$ is expanded in its Fourier series then the Fourier coefficients are

$$\xi_i = \int_0^t m_i(s)\dot{W}(s)ds = \int_0^t m_i(s)dW(s), \quad i = 1, 2, \dots \quad (5.48)$$

We also have using the isometric property of martingale (stochastic) integrals that

$$E \left[\dot{W}(m_i)\dot{W}(m_j) \right] = E[\xi_i\xi_j] = \langle \xi_i, \xi_j \rangle = \int_0^t m_i(s)m_j(s)ds = \delta_{i,j}. \quad (5.49)$$

According to Appendix C.2, Brownian motion can be written as a linear combination of the random variables ξ_i , known as the Lévy-Ciesielski series representation, given in the following theorem :

Theorem 1. *Let $\{m_i(s)\}_{i=1}^n$, $0 \leq s \leq t$ be an orthonormal basis in the Hilbert space $L^2\{[0, t]\}$, then*

$$W(s) = \sum_{i=1}^{\infty} \xi_i \int_0^s m_i(\tau)d\tau, \quad 0 \leq s \leq t \quad (5.50)$$

and the series (5.50) converges uniformly for $\forall s \leq t$.

Proof. The interval $[0, s]$ can be characterized by the indicator function $\mathbb{I}_{[0,s]}(\tau)$, $\tau \in$

$[0, t]$. Since $\mathbb{I}_{[0,s]}(\tau)$ is square-integrable it has the following expansion

$$\mathbb{I}_{[0,s]}(\tau) = \sum_{i=1}^{\infty} \langle \mathbb{I}_{[0,s]}(\tau), m_i(\tau) \rangle m_i(\tau) = \sum_{i=1}^{\infty} m_i(\tau) \int_0^s m_i(\tau) d\tau. \quad (5.51)$$

It follows that

$$\begin{aligned} W(s) &= \int_0^s dW_\tau = \int_0^s \mathbb{I}_{[0,s]}(\tau) dW_\tau = \int_0^s \left[\sum_{i=1}^{\infty} m_i(\tau) \int_0^s m_i(\tau) d\tau \right] dW_\tau \\ &= \sum_{i=1}^{\infty} m_i(\tau) dW_\tau \int_0^s m_i(\tau) d\tau = \sum_{i=1}^{\infty} \xi_i \int_0^s m_i(\tau) d\tau. \end{aligned} \quad (5.52)$$

□

In particular, if the orthonormal basis is given by the trigonometric function $m_1(t) = \frac{1}{\sqrt{T}}, m_i(t) = \frac{2}{\sqrt{T}} \cos\left(\frac{i-1}{T}\pi t\right), i = 2, 3, \dots, 0 \leq t \leq T$, then

$$W(t) = \frac{t}{\sqrt{T}} + \frac{\sqrt{2T}}{\pi} \sum_{i=2}^{\infty} \frac{\xi_i}{i-1} \sin\left(\frac{(i-1)\pi t}{T}\right). \quad (5.53)$$

The expression (5.53) is called the Paley-Wiener representation of $W(t)$.

Since the stochastic solution is a function of $\{W(s) : 0 \leq s \leq t\}$ it can be written as a linear combination of the Gaussian random variable $\xi_i, i = 1, 2, \dots$ and thus can be interpreted as

$$u(x, t) = u(x, t; \xi_1, \dots, \xi_n, \dots). \quad (5.54)$$

Since the product of Hermite polynomials can be expressed as a linear combination of Hermite polynomials according to Property 3, we can then write $u(x, t)$ as

$$u(x, t) = \sum_{\alpha} u_{\alpha}(x, t) T_{\alpha}(\xi), \quad (5.55)$$

where $\xi = (\xi_1, \dots, \xi_n, \cdot)$ and

$$T_\alpha(\xi) = \prod_{i=1}^{\infty} H_{\alpha_i}(\xi_i) \quad (5.56)$$

are called Wick polynomials with α_i defined within the set

$$\mathcal{G} = \left\{ \alpha = (\alpha_i, i \geq 1) \mid \alpha_i \in \{1, 2, 3, \dots\}, |\alpha| = \sum_1^{\infty} \alpha_i < \infty \right\}. \quad (5.57)$$

The expansion (5.55) is the Fourier-Hermite expansion or Wiener chaos expansion.

The following theorem is very important for the analysis of nonlinear SPDEs.

Theorem 2. *Suppose u, v have Wiener chaos expansion*

$$u(x, t) = \sum_{\alpha} u_{\alpha}(x, t) T_{\alpha}(\xi), \quad v(x, t) = \sum_{\beta} v_{\beta}(x, t) T_{\beta}(\xi).$$

If $E(|uv|^2) < \infty$, then the product uv has the Wiener chaos expansion

$$uv = \sum_{\theta \in \mathcal{G}} \left(\sum_{p \in \mathcal{G}} \sum_{0 < \beta < \theta} C(\theta, \beta, p) u_{\alpha - \beta - p} v_{\beta + p} \right) T_{\theta}(\xi), \quad (5.58)$$

where

$$C(\theta, \beta, p) = (C_{\beta}^{\theta} C_p^{\theta - \beta + p} C_p^{\beta + p}).$$

The expansion for the product uv given in (5.58) is analogous to the expression for the convolution in the context of the Fourier transform.

Proof.

$$\begin{aligned} uv &= \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} u_{\alpha} v_{\beta} T_{\alpha} T_{\beta} \\ &= \sum_{\alpha \in \mathcal{G}} \sum_{\beta \in \mathcal{G}} u_{\alpha} v_{\beta} \sum_{p \leq \alpha \wedge \beta} C_p^{\alpha} C_p^{\beta} p! \frac{\sqrt{(\alpha + \beta - 2p)!}}{\sqrt{\alpha! \beta!}} T_{\alpha + \beta - 2p}. \end{aligned} \quad (5.59)$$

Let $j = \alpha - p$, $l = \beta - p$, then $p \leq \alpha \wedge \beta$. This implies $j, l \geq 0$. Using the fact $\alpha = j + p$ and $\beta = l + p$

$$uv = \sum_{j \in \mathcal{G}} \sum_{l \in \mathcal{G}} \sum_{p \in \mathcal{G}} u_{j+p} v_{l+p} C_p^{j+p} C_p^{l+p} p! \frac{\sqrt{(j+l)!}}{\sqrt{(j+p)!(l+p)!}} T_{j+l}. \quad (5.60)$$

Now setting $\alpha = j$, $\beta = l$ and $\alpha = \theta - \beta$ gives

$$\begin{aligned} uv &= \sum_{\alpha \in \mathcal{G}} \sum_{\alpha + \beta \leq \theta} \sum_{p \in \mathcal{G}} u_{\alpha+p} v_{\beta+p} C_p^{\alpha+p} C_p^{\beta+p} p! \frac{\sqrt{\theta!}}{\sqrt{(\alpha+p)!(\beta+p)!}} T_{\theta} \\ &= \sum_{\alpha \in \mathcal{G}} \sum_{0 < \beta < \theta} \sum_{p \in \mathcal{G}} u_{\theta-\beta+p} v_{\beta+p} C_p^{\theta-\beta+p} C_p^{\beta+p} p! \frac{\sqrt{\theta!}}{\sqrt{(\theta-\beta+p)!(\beta+p)!}} T_{\theta} \\ &= \sum_{\theta \in \mathcal{G}} \left(\sum_{p \in \mathcal{G}} \sum_{0 < \beta < \theta} C(\theta, \beta, p) u_{\alpha-\beta-p} v_{\beta+p} \right) T_{\theta}(\xi). \end{aligned} \quad (5.61)$$

□

5.2.3 Martingale application

For $0 \leq s \leq t \leq T$ the σ -field $\mathcal{F}_t$ generated by the Brownian motion $\{W(t), 0 \leq t \leq T\}$ is a filtration in the sense that if $s \leq t$ we have $\mathcal{F}_t \supset \mathcal{F}_s$. The set of all square-integrable functions defined as

$$\mathcal{L}^2(\Omega, \mathcal{F}_t, P) = \{f(W(s) : 0 \leq s \leq t) | E(f^2) < \infty\} \quad (5.62)$$

is a Hilbert space with respect to the Brownian motion measure P defined in Appendix C.1 (Definition 5). Since $\mathcal{F}_s \subset \mathcal{F}_t$ then

$$\mathcal{L}^2(\Omega, \mathcal{F}_s, P) \subset \mathcal{L}^2(\Omega, \mathcal{F}_t, P). \quad (5.63)$$

Therefore,

$$u(x, t) \in \mathcal{L}^2(\Omega, \mathcal{F}_t, P). \quad (5.64)$$

Thus, $u(x, t)$ is martingale with respect to the filtration $\mathcal{F}_t$.

The following theorem is quite important and allows us to prove that the stochastic solution $u(x, t)$ of equation (5.47) is a martingale with respect to the σ -field $\mathcal{F}_t$. The detailed proof can be found in the paper by Cameron and Martin (1947).

Theorem 3. [*Cameron-Martin*] *Assume that for any $x \in \mathbb{R}$ and $s \leq t \in \mathbb{R}$, the solution of $u(x, s)$ of equation (5.47) is a functional of the Brownian motion $\{W(t), 0 \leq t \leq T\}$ with $E|u(x, s)|^2 < \infty$, then $u(x, s)$ has the following Fourier-Hermite expansion:*

$$u(x, s) = \sum_{\alpha \in \mathcal{G}} u_{\alpha}(x, s) T_{\alpha}(\xi), \quad u_{\alpha}(x, s) = E[u(x, s) T_{\alpha}(\xi)], \quad (5.65)$$

where $T_{\alpha}(\xi)$ are wick polynomials defined by equation (5.56). Furthermore, the mean and variance of $u(x, s)$ are respectively given by $E[u(x, s)] = u_0(x, s)$ and $E[u^2(x, s)] = \sum_{\alpha \in \mathcal{G}, \alpha \neq 0} |u_{\alpha}(x, s)|^2$.

Now taking the expectation of $u(x, t)$ with respect to the σ -field $\mathcal{F}_s$ gives

$$E[u(x, t) | \mathcal{F}_s] = \sum_{\alpha \in \mathcal{G}} E\{E[u(x, t) T_{\alpha}(\xi)] | \mathcal{F}_s\} E[T_{\alpha} | \mathcal{F}_s] = \sum_{\alpha \in \mathcal{G}} u_{\alpha}(x, s) T_{\alpha}(\xi) = u(x, s). \quad (5.66)$$

Thus $u(x, t)$ is a martingale with respect to the σ -field $\mathcal{F}_t$.

5.3 Application of the WCE method to the stochastic Burgers' equation

In this section, the WCE method is applied to numerically solve some test problems involving stochastic evolution equations: the stochastic heat equation, the stochastic advection-diffusion equation and the stochastic Burgers equation. We describe the implementation of the method to Burgers' equation and in the next section we present the results of the test.

The stochastic Burgers' equation driven by the Brownian motion is often used in modeling fluid flow, and is a simplified equation that models some of the features of

the Navier-Stokes equation, namely advection, dispersion and diffusion (Kraichnan, 1999). To solve the stochastic Burgers' equation numerically we first apply to it the WCE method and this leads to a system of deterministic PDEs for the coefficients $u_\alpha(x, t)$ that can be solved using classical numerical techniques.

We consider the stochastic Burgers' equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \sigma(x) \dot{W}, \quad t > 0, x \in (0, L) \quad (5.67)$$

with boundary conditions

$$u(0, t) = u(L, t) = g(t), \quad t \geq 0 \quad (5.68)$$

and initial condition

$$u(x, 0) = u_0(x), \quad x \in [0, L] \quad (5.69)$$

with $u_0(0) = u_0(L)$. Now let $u(x, t) = \sum_\alpha u_\alpha(x, t) T_\alpha$ and $u_\alpha(x, t) = E[u(x, t) T_\alpha]$ where T_α are Wick polynomials.

Burgers' equation can be written in integral form as

$$u(x, t) = u_0(x) + \int_0^t \left[\nu \frac{\partial^2 u}{\partial x^2}(x, \tau) - u(x, \tau) \frac{\partial u}{\partial x}(x, \tau) \right] d\tau + \sigma(x) W(t). \quad (5.70)$$

Multiplying both sides by T_α and taking the expectation gives

$$u_\alpha(x, t) = u_0(x) \mathbb{I}_{\alpha=0} + \int_0^t \left[\nu \frac{\partial^2 u_\alpha}{\partial x^2}(x, \tau) - \frac{1}{2} \frac{\partial E[u^2 T_\alpha]}{\partial x}(x, \tau) \right] d\tau + \sigma(x) E[W(t) T_\alpha]. \quad (5.71)$$

From Theorem 2 in section 5.2.2,

$$E[u^2 T_\alpha] = \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \alpha} C(\alpha, \beta, p) u_{\alpha-\beta+p} u_{\beta+p}. \quad (5.72)$$

Theorem 1 [Lévy-Ciesielski series representation of $W(t)$] implies

$$E[W(t)T_\alpha] = \sum_{i=1}^{\infty} \int_0^t m_i(\tau) d\tau E[\xi_i T_\alpha]. \quad (5.73)$$

And so

$$E[\xi_i T_\alpha] = \text{cov} \left[H_1(\xi), \prod_{i=1}^{\infty} H_{\alpha_i}(\xi_i) \right] = \mathbb{I}_{\alpha_j = \delta_{i,j}}.$$

Therefore

$$E[W(t)T_\alpha] = \sum_{i=1}^{\infty} \mathbb{I}_{\alpha_j = \delta_{i,j}} \int_0^t m_i(\tau) d\tau. \quad (5.74)$$

Substituting (5.74) into the equation (5.71) gives

$$\begin{aligned} u_\alpha(x, t) &= u_0(x) \mathbb{I}_{\alpha=0} + \int_0^t \nu \frac{\partial^2 u_\alpha}{\partial x^2}(x, \tau) d\tau + \sigma(x) \sum_{i=1}^{\infty} \mathbb{I}_{\alpha_j = \delta_{i,j}} \int_0^t m_i(\tau) d\tau \\ &\quad - \frac{1}{2} \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \alpha} C(\alpha, \beta, p) \int_0^t \frac{\partial}{\partial x} [u_{\alpha-\beta+p}(x, \tau) u_{\beta+p}(x, \tau)] d\tau. \end{aligned} \quad (5.75)$$

Differentiating this with respect to t gives

$$\begin{aligned} \frac{\partial u_\alpha}{\partial t}(x, t) &= \nu \frac{\partial^2 u_\alpha}{\partial x^2}(x, t) + \sigma(x) \sum_{i=1}^{\infty} \mathbb{I}_{\alpha_j = \delta_{i,j}} m_i(t) \\ &\quad - \frac{1}{2} \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \alpha} C(\alpha, \beta, p) \frac{\partial}{\partial x} [u_{\alpha-\beta+p}(x, t) u_{\beta+p}(x, t)]. \end{aligned} \quad (5.76)$$

We then take the Fourier transform of both sides of equation (5.76) with $\hat{u}_\alpha(k, t) = \mathcal{F}\{u_\alpha\}$ and obtain

$$\begin{aligned} \frac{d}{dt} \hat{u}_\alpha(k, t) &= -\nu k^2 \hat{u}_\alpha(k, t) + \hat{\sigma}(x) \sum_{i=1}^{\infty} \mathbb{I}_{\alpha_j = \delta_{i,j}} m_i(t) \\ &\quad - \frac{ik}{2} \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \alpha} C(\alpha, \beta, p) \mathcal{F}\{u_{\alpha-\beta+p}(x, t) u_{\beta+p}(x, t)\}. \end{aligned} \quad (5.77)$$

For each α we solve equation (5.77); it is deterministic and can be solved using

classical numerical methods. Since it is nonlinear with periodic boundary conditions the spectral method is a suitable procedure, and the nonlinear terms can be calculated using a pseudo-spectral approximation.

We set

$$f(t, \hat{u}_\alpha(k, t)) = -\nu k^2 \hat{u}_\alpha(k, t) + \hat{\sigma}(k) \sum_{i=1}^{\infty} \mathbb{I}_{\alpha_j=\delta_{i,j}} m_i(t) - \frac{ik}{2} \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \alpha} C(\alpha, \beta, p) \mathcal{F}\{u_{\alpha-\beta+p}(x, t) u_{\beta+p}(x, t)\} \quad (5.78)$$

The dependent variable is approximated by

$$\hat{u}_\alpha(k, t) \approx \hat{u}_\alpha(k, n\Delta t) = \hat{u}_\alpha^{k,n} \quad (5.79)$$

so that

$$f(t, \hat{u}_\alpha(k, t)) \approx f(n\Delta t, \hat{u}_\alpha(k, n\Delta t)) = f(n\Delta t, \hat{u}_\alpha^{k,n}) = f^{k,n} = -\nu k^2 \hat{u}_\alpha^{k,n} + \hat{\sigma}(k) \sum_{i=1}^{\infty} \mathbb{I}_{\alpha_j=\delta_{i,j}} m_i^n - \frac{ik}{2} \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \alpha} C(\alpha, \beta, p) \mathcal{F}\{\mathcal{F}^{-1}\{\hat{u}_{\alpha-\beta+p}\} \mathcal{F}^{-1}\{\hat{u}_{\beta+p}\}\}^{k,n}. \quad (5.80)$$

We numerically compute $\hat{u}_\alpha$ using the predictor-corrector method described in section 3.4, which is a combination of the second order explicit Adam-Bashforth used for the predictor step and the second order implicit Adam-Moulton scheme used for corrector step. The use of an implicit scheme for corrector step allows us to avoid stiffness if there is any.

5.4 Numerical tests of the WCE method

In this section we present the results obtained for the numerical simulations of three test problems involving the stochastic heat equation, the stochastic advection-diffusion equation and the stochastic Burgers' equation. To assess the accuracy of

the numerical simulations, the results for the heat equation and advection diffusion equation are compared with exact solutions. For the stochastic Burgers' equation the results are compared with a semi-analytical solution and an approximate analytical solution.

5.4.1 Numerical solution of the stochastic heat equation

In this section we obtain an exact analytical solution of the stochastic heat equation driven by the Brownian motion and compare it with the numerical solution obtained using the WCE numerical method described in section 5.3.

We consider the diffusion problem given by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + e^{ikx} \dot{W} \quad t > 0, x \in (0, 2\pi) \quad (5.81)$$

with initial and boundary conditions

$$u(x, 0) = u_0(x) = A_0 e^{ikx}, \quad u(0, t) = u(2\pi, t), \quad (5.82)$$

where $k \in \mathbb{N}$ and $\nu \in \mathbb{R}$.

The initial and boundary conditions suggest that solution of this problem must be of the form

$$u(x, t) = A(t) e^{ikx}. \quad (5.83)$$

Substituting (5.83) into equations (5.81)-(5.82) tells that $A(t)$ must satisfy the Langevin stochastic differential equation

$$dA(t) = -\nu k^2 A(t) dt + dW(t) \quad t \geq 0 \quad (5.84)$$

with initial condition

$$A(0) = A_0. \quad (5.85)$$

According to Example 4, the exact solution of the problem (5.84)-(5.84) is given by

$$A(t) = A_0 e^{-\nu k^2 t} + e^{-\nu k^2 t} \int_0^t e^{\nu k^2 \tau} dW(\tau). \quad (5.86)$$

Hence, the analytical solution of the problem (5.81)-(5.82) is

$$u(x, t) = A(t) e^{ikx} = e^{ikx} e^{-\nu k^2 t} \left[A_0 + \int_0^t e^{\nu k^2 \tau} dW(\tau) \right]. \quad (5.87)$$

In the numerical simulations shown here we set $k = 2$ so that the amplitude of the random forcing is given by $\sigma(x) = e^{i2x}$ and the initial condition is $u_0(x) = e^{i2x}$. We set the viscous coefficient $\nu = 0.02$, the time step $\Delta t = 0.005$ and $L = 2\pi$. The results of the simulations are shown in the figures 5.1 and 5.2. Figure 5.1 (a) shows the results obtained for the deterministic heat equation ($\sigma(x) = 0$) while those obtained for the stochastic heat equation (5.82) are shown in Figures 5.1 (b) and 5.2. A comparison between the numerical and analytical solutions is shown in Figure 5.2. The results obtained for the stochastic heat equation show that there is good agreement between the numerical and analytical solutions.

Figure 5.1: Plot of $u(x, t)$ as a function of position x . (a) Solution of the deterministic heat equation. (b) Solution of the stochastic heat equation.

Figure 5.2: (a) WCE solution of the stochastic heat equation against exact solution. The solid line corresponds to the exact solution while the dashed line corresponds to the WCE solution. (b) Absolute error.

5.4.2 Numerical solution of the stochastic advection-diffusion equation

We compare the analytical solution with the numerical solution obtained using the numerical method described in section 5.3 which based on the WCE method.

In this section we consider the advection-diffusion problem given by

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + e^{ikx} \dot{W} \quad t > 0, x \in (0, 2\pi) \quad (5.88)$$

with initial and boundary conditions

$$u(x, 0) = u_0(x) = e^{ikx}, \quad u(0, t) = u(2\pi, t), \quad (5.89)$$

where $k \in \mathbb{N}$ and $c, \nu \in \mathbb{R}$.

The initial and boundary conditions suggest that solution to this problem must be of the form

$$u(x, t) = A(t)e^{ikx}. \quad (5.90)$$

Substituting this into (5.88) and (5.89) tells us that $A(t)$ must satisfy the Langevin

stochastic differential equation

$$dA(t) = -(\nu k^2 + ikc)A(t)dt + dW(t) \quad t \geq 0 \quad (5.91)$$

with initial condition

$$A(0) = A_0. \quad (5.92)$$

The exact solution is

$$A(t) = e^{-(\nu k^2 + ikc)t} \left[A_0 + \int_0^t e^{(\nu k^2 + ikc)\tau} dW(\tau) \right]. \quad (5.93)$$

Therefore, the solution of (5.88)-(5.89) is

$$u(x, t) = e^{ik(x-ct)} e^{-\nu k^2 t} \left[A_0 + \int_0^t e^{(\nu k^2 + ikc)\tau} dW(\tau) \right]. \quad (5.94)$$

In the numerical simulations shown here we use the same parameters as in section 5.4.1, e.i. $k = 2$, $\nu = 0.02$, $\Delta t = 0.005$ and $L = 2\pi$. The results are shown in Figures 5.3 and 5.3. The results obtained for the stochastic advection-diffusion equation show that there is good agreement between the numerical and analytical solutions.

Figure 5.3: Plot of $u(x,t)$ as a function of position x . (a) Numerical solution of the deterministic advection-diffusion equation. (b) Numerical solution of the stochastic advection-diffusion equation.

Figure 5.4: (a) WCE solution of the stochastic advection-diffusion equation against the exact solution. The solid line corresponds to the exact solution while the dashed line corresponds to the WCE solution. (b) Absolute error.

5.4.3 Analytical solutions of the stochastic Burgers' equation

In this section we derive an approximate analytical solution of the stochastic Burgers' equation valid for small t and a semi-analytical solution. We use these in section 5.4.4 to assess the accuracy of the WCE numerical method. We consider the stochastic

Burgers' equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \sigma \dot{W}, \quad t > 0, \quad x \in (0, L), \quad (5.95)$$

with initial and boundary conditions

$$u(x, 0) = f(x) = A = \text{a constant}, \quad x \in [0, L], \quad (5.96)$$

$$u(0, t) = u(L, t) = 0, \quad t > 0. \quad (5.97)$$

An approximate analytical solution to this problem near $t = 0$, and a semi-analytical solution can be obtained by making use of the following theorem:

Theorem 4. *Consider the stochastic Burgers' equation*

$$u_t + uu_x = \nu u_{xx} + \sigma \dot{W}, \quad x \in (0, L), \quad t > 0, \quad (5.98)$$

where σ is a constant, with initial condition

$$u(x, 0) = u_0(x), \quad x \in [0, L], \quad (5.99)$$

and boundary conditions

$$u(0, t) = u(L, t) = g(t), \quad t > 0. \quad (5.100)$$

Let

$$\xi = X(x, t) = x - \sigma \int_0^t W(s) ds. \quad (5.101)$$

The solution of (5.98)-(5.100) is

$$u(x, t) = v \left(x - \sigma \int_0^t W(s) ds, t \right) + \sigma W(t), \quad (5.102)$$

where $v(x, t)$ is the solution of the deterministic Burgers' equation

$$v_t + vv_\xi = \nu v_{\xi\xi}, \quad t > 0 \quad (5.103)$$

with

$$\xi \in \left(-\sigma \int_0^t W(s) ds, L - \sigma \int_0^t W(s) ds \right) = (X(0, t), X(L, t)),$$

and initial condition

$$v(\xi, 0) = u_0(\xi), \quad \xi \in \left(-\sigma \int_0^t W(s) ds, L - \sigma \int_0^t W(s) ds \right) = (X(0, t), X(L, t)) \quad (5.104)$$

and stochastic boundary conditions

$$v(X(0, t), t) = v(X(L, t), t) = g(t) - \sigma W(t), \quad t > 0. \quad (5.105)$$

Proof. Let

$$\xi = X(x, t) = x - \sigma \int_0^t W(s) ds, \quad (5.106)$$

and consider $u(x, t) = v(\xi, t) + \sigma W(t)$. Then

$$u_t = v_t + v_\xi X_t + \sigma \dot{W} = v_t - \sigma v_\xi W + \sigma \dot{W} \quad (5.107)$$

and

$$u_x = v_\xi X_x = v_\xi \quad \text{and} \quad u_{xx} = v_{\xi\xi}. \quad (5.108)$$

Substituting these derivatives into (5.98) gives

$$v_t - \sigma v_\xi W + \sigma \dot{W} + vv_\xi + \sigma v_\xi W = \nu v_{\xi\xi} + \sigma \dot{W} \quad (5.109)$$

which is equation (5.103). Also $u(x, 0) = v(X(x, 0), 0) = v(\xi, 0) = u_0(\xi)$ since at

$t = 0$, $\xi = X(x, 0) = x$. And the boundary conditions (5.100) become

$$g(t) = u(0, t) = v(X(0, t), t) + \sigma W(t) \text{ and } g(t) = u(L, t) = v(X(L, t), t) + \sigma W(t). \quad (5.110)$$

Hence

$$v(X(0, t), t) = v(X(L, t), t) = g(t) - \sigma W(t), \quad (5.111)$$

which is (5.105) □

Note that Theorem 4 is based on the Theorem 3.1 in Hou et al. (2006). However their theorem is incorrect because instead of the correct stochastic boundary condition (5.105) they give incorrect deterministic boundary conditions, and their solution is thus only approximate not exact.

Semi-analytical solution

Using our corrected version of the theorem, a semi-analytical solution to problem (5.95)-(5.97) can be obtained by solving the deterministic Burgers' equation (5.103) numerically subject to the initial condition (5.104) and boundary condition (5.105). Any suitable classical numerical technique can be used such as the predictor-corrector method described in section 3.4. Once we obtain $v(\xi, t)$ numerically we can then use (5.100) to obtain $u(x, t)$ as $u(x, t) = v(\xi(x, t), t) + \sigma W(t)$.

Approximate solution

Alternatively we can obtain an approximation for u by noting that for small t , the end-points of the domain $\xi = X(0, t) = -\sigma \int_0^t W(s)ds$ and $\xi = X(L, t) = L - \sigma \int_0^t W(s)ds$ which vary as stochastic functions of time of t can be approximated by the fixed points $\xi = 0$ and $\xi = L$, and the stochastic boundary condition (5.105) can be approximated by the deterministic boundary condition

$$v(0, t) = v(L, t) = 0, t \geq 0. \quad (5.112)$$

We can thus solve (5.103) with this fixed domain with the deterministic boundary condition to set an approximate solution to the stochastic problem (5.95)-(5.97).

To solve (5.103) on the fixed domain (5.112) we make a Hopf-Cole transformation (Hopf, 1950; Cole, 1951). We let

$$v(\xi, t) = -2\nu \frac{\frac{\partial \eta(\xi, t)}{\partial \xi}}{\eta(\xi, t)} \quad (5.113)$$

so that

$$\frac{\partial v}{\partial t} = -2\nu \frac{\frac{\partial^2 \eta}{\partial \xi \partial t}}{\eta} + 2\nu \frac{\frac{\partial \eta}{\partial \xi} \frac{\partial \eta}{\partial t}}{\eta^2}, \quad (5.114)$$

$$\frac{\partial v}{\partial \xi} = -2\nu \frac{\frac{\partial^2 \eta}{\partial \xi^2}}{\eta} + 2\nu \frac{\left(\frac{\partial \eta}{\partial \xi}\right)^2}{\eta^2} \quad (5.115)$$

and

$$\frac{\partial^2 v}{\partial \xi^2} = -2\nu \frac{\frac{\partial^3 \eta}{\partial \xi^3}}{\eta} + 6\nu \frac{\frac{\partial \eta}{\partial \xi} \frac{\partial^2 \eta}{\partial \xi^2}}{\eta^2} - 2\nu \frac{\left(\frac{\partial \eta}{\partial \xi}\right)^3}{\eta^3}. \quad (5.116)$$

Substituting (5.113)-(5.116) into (5.103) and rearranging terms yields

$$0 = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial \xi} - \nu \frac{\partial^2 v}{\partial \xi^2} = -\frac{\partial \eta}{\partial \xi} \left(\frac{\partial \eta}{\partial t} - \nu \frac{\partial^2 \eta}{\partial \xi^2} \right) + \frac{\partial}{\partial \xi} \left(\frac{\partial \eta}{\partial t} - \nu \frac{\partial^2 \eta}{\partial \xi^2} \right) = 0. \quad (5.117)$$

This implies that if η is the solution of the problem

$$\frac{\partial \eta}{\partial t} = \nu \frac{\partial^2 \eta}{\partial \xi^2}, \quad t > 0, \quad x \in [0, L] \quad (5.118)$$

with initial condition

$$\eta(\xi, 0) = \eta_0(\xi) = C e^{-\frac{A}{2\nu} \xi}, \quad t \geq 0 \quad (5.119)$$

and boundary condition

$$\frac{\partial \eta}{\partial \xi}(0, t) = \frac{\partial \eta}{\partial \xi}(L, t) = 0, \quad \xi \in [0, L] \quad (5.120)$$

where C is an arbitrary constant, then $v(\xi, t)$ is an approximate solution to the

problem (5.103)-(5.105) for small t .

Since C is an arbitrary constant, we set $C = 1$ and solve the problem (5.118) using separation of variables and obtain

$$\eta(\xi, t) = \frac{\nu}{AL} e^{-\frac{AL}{2\nu}} (e^{\frac{AL}{2\nu}} - 1) + 2\nu AL e^{-\frac{AL}{2\nu}} \sum_{n=1}^{\infty} \frac{e^{\frac{AL}{2\nu}} - (-1)^n}{A^2 L^2 + 4\nu^2 n^2 \pi^2} e^{-\left(\frac{n\pi}{L}\right)^2 \nu t} \cos\left(\frac{n\pi\xi}{L}\right). \quad (5.121)$$

Using (5.113) we evaluate v to obtain

$$\begin{aligned} v(\xi, t) &= -2\nu \frac{\frac{\partial \eta(\xi, t)}{\partial \xi}}{\eta(\xi, t)} \\ &= \frac{2\pi\nu^2 A e^{-\frac{AL}{2\nu}} \sum_{n=1}^{\infty} n \frac{e^{\frac{AL}{2\nu}} - (-1)^n}{A^2 L^2 + 4\nu^2 n^2 \pi^2} e^{-\left(\frac{n\pi}{L}\right)^2 \nu t} \sin\left(\frac{n\pi\xi}{L}\right)}{\frac{\nu}{AL} e^{-\frac{AL}{2\nu}} (e^{\frac{AL}{2\nu}} - 1) + 2\nu AL e^{-\frac{AL}{2\nu}} \sum_{n=1}^{\infty} \frac{e^{\frac{AL}{2\nu}} - (-1)^n}{A^2 L^2 + 4\nu^2 n^2 \pi^2} e^{-\left(\frac{n\pi}{L}\right)^2 \nu t} \cos\left(\frac{n\pi\xi}{L}\right)}. \end{aligned} \quad (5.122)$$

We then use Theorem 4 to evaluate u . Hence

$$\begin{aligned} u(x, t) &= v\left(x - \sigma \int_0^t W(s) ds, t\right) + \sigma W(t) \\ &= \frac{2\pi\nu^2 A e^{-\frac{AL}{2\nu}} \sum_{n=1}^{\infty} n \frac{e^{\frac{AL}{2\nu}} - (-1)^n}{A^2 L^2 + 4\nu^2 n^2 \pi^2} e^{-\left(\frac{n\pi}{L}\right)^2 \nu t} \sin\left[\frac{n\pi}{L} \left(x - \sigma \int_0^t W(s) ds\right)\right]}{\frac{\nu}{AL} e^{-\frac{AL}{2\nu}} (e^{\frac{AL}{2\nu}} - 1) + 2\nu AL e^{-\frac{AL}{2\nu}} \sum_{n=1}^{\infty} \frac{e^{\frac{AL}{2\nu}} - (-1)^n}{A^2 L^2 + 4\nu^2 n^2 \pi^2} e^{-\left(\frac{n\pi}{L}\right)^2 \nu t} \cos\left[\frac{n\pi}{L} \left(x - \sigma \int_0^t W(s) ds\right)\right]} \\ &\quad + \sigma W(t). \end{aligned} \quad (5.123)$$

This is an approximate solution of the problem (5.95)-(5.97).

5.4.4 Numerical solution of the stochastic Burgers' equation

We now apply the WCE method to solve the stochastic Burgers' problem (5.95)-(5.96). Applying the WCE method gives a system of deterministic equations for the

coefficient of the WCE. Each coefficient satisfying the nonlinear PDE

$$\begin{aligned} \frac{\partial u_\alpha}{\partial t}(x, t) = & \nu \frac{\partial^2 u_\alpha}{\partial x^2}(x, t) + \sigma(x) \sum_{i=1}^{\infty} \mathbb{I}_{\alpha_j = \delta_{i,j}} m_i(t) \\ & - \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \alpha} C(\alpha, \beta, p) u_{\alpha - \beta + p}(x, t) \frac{\partial}{\partial x} u_{\beta + p}(x, t), \end{aligned} \quad (5.124)$$

where the indices α are defined within the set

$$\mathcal{G} = \left\{ \alpha = (\alpha_i, i \geq 1) \mid \alpha_i \in \{1, 2, 3, \dots\}, |\alpha| = \sum_1^\infty \alpha_i < \infty \right\}, \quad (5.125)$$

and with initial and boundary conditions

$$u_{\alpha=0}(x) = A \text{ and } u_{\alpha \neq 0}(x) = 0, \quad x \in [0, L], \quad (5.126)$$

$$u_\alpha(0, t) = u_\alpha(L, t) = 0, \quad t \geq 0. \quad (5.127)$$

The WCE coefficient corresponding to $\alpha = 0$ satisfies the deterministic PDE

$$\frac{\partial u_0}{\partial t}(x, t) + u_0 \frac{\partial u_0}{\partial x} + \sum_{\alpha \neq 0} u_\alpha \frac{\partial u_\alpha}{\partial x} = \nu \frac{\partial^2 u_0}{\partial x^2} \quad (5.128)$$

with the deterministic initial and boundary conditions from the original problem i.e.,

$$u_{\alpha=0}(0, x) = A \text{ and } u_{\alpha=0}(0, t) = u_{\alpha=0}(L, t) = 0. \quad (5.129)$$

The numerical method described in section 5.3 is then used to compute the WCE coefficients $u_\alpha(x, t)$. The following constants and parameters were used: the diffusion constant $\nu = 0.005$, the time step $\Delta t = 0.005$ and the spatial step $\Delta x = 0.008$. The results are shown in Figures 5.5-5.8.

The results obtained for the stochastic Burgers' equation show that there is a good agreement between the WCE numerical, semi-analytical and approximate analytical solutions for $t \leq 0.2$. The difference observed between the semi-analytical

Figure 5.5: Burgers' equation. Plot of $u(x, t)$ as a function of position x . (a) Numerical solution of the deterministic Burgers' equation. (b) Numerical solution of the stochastic Burger's equation.

Figure 5.6: Burgers' equation. Plot of $u(x, t)$ as a function of time t at the point $x = 0.04$ near the boundary $x = 0$. (a) The semi-analytical solution against the WCE solution. (b) The approximate analytical solution against the WCE solution (5.67).

and approximate analytical solutions is caused by the stochastic boundary condition of the semi-analytical problem. This difference may be minimized by using a higher order scheme and a very small time step.

Figure 5.7: Burgers' equation. Plot of the absolute error as function of time t at the point $x = 0.04$ near the boundary $x = 0$. The semi-analytical solution error is the difference between the WCE solution and the semi-analytical solution. The approximate analytical solution error is the difference between the WCE solution and the approximate analytical error.

Figure 5.8: Burgers' equation. Plot of $u(x, t)$ as a function of time t at the point $x = 0.04$ near the boundary $x = 0$. (a) The semi-analytical solution against the approximate solution. (b) The absolute difference between the two approximations.

Chapter 6

Numerical simulations of atmospheric gravity wave-ionosphere interactions in a random ionosphere

6.1 Governing equations

In this chapter we describe numerical simulations of gravity wave-ionosphere interactions in a random ionosphere by solving the system of SPDEs (2.10), (2.26), (4.55) and (4.59) using the WCE method (see section 5.3). To represent the gravity wave perturbations we consider each fluid variable to be the sum of an initial horizontal mean part and a time dependent perturbation as in section 3.4.1. The gravity wave perturbation and the Lorentz force in (2.26) and (2.38) generate a perturbations in the ionosphere (TIDs). To represent the ionospheric perturbations we consider each ionospheric variable to be the sum of a horizontal mean part and a time-dependent perturbation as done in section (3.1). We write

$$\Psi(x, z, t) = \bar{\psi}(z) + \varepsilon\psi(x, z, t) \text{ and } \Psi_{\alpha}(x, z, t) = \bar{\psi}_{\alpha}(z) + \varepsilon\psi_{\alpha}(x, z, t) \quad (6.1)$$

and

$$\varrho(x, z, t) = \bar{\rho}(z) + \varepsilon \rho(x, z, t) \quad \text{and} \quad \varrho_\alpha(x, z, t) = \bar{\rho}_\alpha(z) + \varepsilon \rho_\alpha(x, z, t) \quad (6.2)$$

with

$$\varepsilon = \frac{L_z U}{\varphi} \ll 1,$$

where φ is the dimensional amplitude of the wave at the source, L_z a typical dimensional vertical length scale and U a typical dimensional velocity scale.

Substituting (6.1) and (6.2) into equations (2.10), (2.13), (2.26) and (2.38) gives

$$\zeta_t + \bar{u}' \zeta_x - \bar{u}'' \psi_x + \varepsilon(\psi_x \zeta_z - \psi_z \zeta_x) + \frac{g}{\bar{\rho}} \frac{\partial \rho}{\partial x} - \nu \nabla^2 \zeta + \nu_{n\alpha}(\zeta - \zeta_\alpha) = 0 \quad (6.3)$$

$$\zeta = \nabla^2 \psi \quad (6.4)$$

$$\rho_t + \bar{u} \rho_x + \bar{\rho}' \psi_x + \varepsilon(\psi_x \rho_z - \psi_z \rho_x) - \kappa \nabla^2 \rho_\alpha = 0 \quad (6.5)$$

for the gravity waves and

$$\begin{aligned} \zeta_{\alpha t} + \bar{u}'_\alpha \zeta_{\alpha x} - \bar{u}''_\alpha \psi_{\alpha x} + \varepsilon(\psi_{\alpha x} \zeta_{\alpha z} - \psi_{\alpha z} \zeta_{\alpha x}) + \frac{g}{\bar{\rho}_\alpha} \frac{\partial \rho_\alpha}{\partial x} - \nu_\alpha \nabla^2 \zeta_\alpha + \nu_{\alpha n}(\zeta_\alpha - \zeta) \\ - \Upsilon_1(x, z)(-\varepsilon^{-1} \bar{u}_\alpha + \psi_{\alpha z}) + \Upsilon_2(x, z) \psi_{\alpha x} = -\Upsilon_3(x, z) + \sigma \Upsilon_3(x, z) \dot{W}(t) \end{aligned} \quad (6.6)$$

$$\zeta_\alpha = \nabla^2 \psi_\alpha \quad (6.7)$$

$$\begin{aligned} \rho_{\alpha t} + \bar{u}_\alpha \rho_{\alpha x} + \bar{\rho}'_\alpha \psi_{\alpha x} + \varepsilon(\psi_{\alpha x} \rho_{\alpha z} - \psi_{\alpha z} \rho_{\alpha x}) - \kappa_\alpha \nabla^2 \rho_\alpha = \\ \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) - \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) [\sigma \dot{W}(t) + \tilde{\sigma} \dot{\tilde{W}}(t)] \end{aligned} \quad (6.8)$$

for the ionosphere.

The functions $\Upsilon_1(x, z)$, $\Upsilon_2(x, z)$, $\Upsilon_3(x, z)$ and $\Upsilon_4(x, z)$ represent the effects due to the magnetic and electric fields and are given in equations (4.52), (4.53) and (4.54) and (4.56).

The numerical simulations are carried out on a rectangular domain in the vertical

plane defined by $0 \leq x \leq 2\pi$ and $z_1 \leq z \leq z_2$ with the initial conditions

$$\psi(x, z, 0) = \zeta(x, z, 0) = \rho(x, z, 0) = 0, \quad (6.9)$$

$$\psi_\alpha(x, z, 0) = \zeta_\alpha(x, z, 0) = \rho_\alpha(x, z, 0) = 0 \quad (6.10)$$

and the boundary conditions,

$$\psi(x, z_1, t) = e^{i\kappa x} + c.c., \quad \zeta(x, z_1, t) = \rho(x, z_1, t) = 0, \quad (6.11)$$

$$\psi_\alpha(x, z_1, t) = \zeta_\alpha(x, z_1, t) = \rho_\alpha(x, z_1, t) = 0, \quad (6.12)$$

$$\psi(x, z_2, t) = \zeta(x, z_2, t) = \rho(x, z_2, t) = 0, \quad (6.13)$$

and

$$\psi_\alpha(x, z_2, t) = \zeta_\alpha(x, z_2, t) = \rho_\alpha(x, z_2, t) = 0. \quad (6.14)$$

We consider 3 configurations:

1. We solve equations (6.6)-(6.8) with $\Upsilon_1 = \Upsilon_2 = 0$ and $\Upsilon_3 \neq 0$ to simulate waves generated by the electric field (electrohydrodynamic (EHD) waves).
2. We solve equations (6.6)-(6.8) with $\Upsilon_1 \neq 0$, $\Upsilon_2 \neq 0$ and $\Upsilon_3 = 0$ to simulate waves generated by the magnetic field (magnetohydrodynamic (MHD) waves).
3. We solve equations (6.6)-(6.8) with $\Upsilon_1 \neq 0$, $\Upsilon_2 \neq 0$ and $\Upsilon_3 \neq 0$ to simulate waves generated by both the electric and magnetic fields (electromagnetohydrodynamic (EMHD) waves).

In each of the 3 configurations we examine:

(a) **E/MHD waves only:** We first set $\nu_{\alpha n} = 0$ in the equation (6.6). This means there are no gravity wave effects and there are only EHD, MHD or EMHD waves present.

(b) **E/MHD wave effects on GWs:** Having computed ψ_α and ρ_α for the E/MHD

waves we also solve equations (6.3), (6.4) and (6.5) with $\nu_{n\alpha} \neq 0$ adding the computed ψ_α as a nonhomogeneous forcing term. This gives the effects of the waves on the GWs.

(c) **GW effects on TIDs:** We solve (6.3), (6.4) and (6.5) with $\nu_{n\alpha} = 0$ to simulate the neutral GWs. Having computed ψ and ρ for GWs we solve (6.6), (6.7) and (6.8) with $\nu_{\alpha n} \neq 0$ adding the computed ψ as a nonhomogeneous term and adding electromagnetic effects to either 1, 2 or 3. The resulting disturbances are driven by GWs and by E/MHD waves. These are traveling ionospheric disturbances (TIDs).

(d) We solved equations (6.3)-(6.8) as a coupled system of equations with $\nu_{\alpha n} \neq 0$ and $\nu_{n\alpha} \neq 0$ and including the electromagnetic effects according to either 1, 2 or 3. Equations (6.3)-(6.5) give ψ and ρ for the GWs. Equations (6.5)-(6.8) give ψ_α and ρ_α for the TIDs. We simulate the coupled neutral atmosphere ionosphere system and examine the interactions between the GWs and TIDs.

6.2 Application of the WCE method

In this section we describe the implementation of the WCE method to numerically solve equations (6.6) and (6.8) using a similar procedure to that used in section 5.4. To keep the description simple we set $\nu_{\alpha n} = 0$ and $\nu_{n\alpha} = 0$ in this section. To do so, we let

$$\psi_\alpha(x, z, t) = \sum_{\gamma} \psi_{\alpha\gamma}(x, z, t)T_\gamma \text{ and } \psi_{\alpha\gamma}(x, z, t) = E[\psi_\alpha(x, z, t)T_\gamma], \quad (6.15)$$

$$\zeta_\alpha(x, z, t) = \sum_{\gamma} \zeta_{\alpha\gamma}(x, z, t)T_\gamma \text{ and } \zeta_{\alpha\gamma}(x, z, t) = E[\zeta_\alpha(x, z, t)T_\gamma] \quad (6.16)$$

and

$$\rho_\alpha(x, z, t) = \sum_{\gamma} \rho_{\alpha\gamma}(x, z, t)T_\gamma \text{ and } \rho_{\alpha\gamma}(x, z, t) = E[\rho_\alpha(x, z, t)T_\gamma], \quad (6.17)$$

where $T_\gamma = T_{\gamma_1} \tilde{T}_{\gamma_2}$ are Wick polynomials and are obtained using Theorem 3 (see section 5.2); and the Wick polynomials T_{γ_1} are function of $W(t)$ while T_{γ_2} are function of $\tilde{W}(t)$.

We now write equations (6.6) and (6.8) in the integral form as

$$\begin{aligned} \zeta_\alpha(x, z, t) = & \zeta_\alpha(x, z, 0) + \int_0^t \left[-\bar{u}'_\alpha \zeta_{\alpha_x}(x, z, \tau) + \bar{u}''_\alpha \psi_{\alpha_x}(x, z, \tau) - g \frac{\rho_{\alpha_x}(x, z, \tau)}{\bar{\rho}_\alpha} \right] d\tau \\ & + \int_0^t \{ \nu_\alpha \nabla^2 \zeta_\alpha(x, z, \tau) - \varepsilon [\psi_{\alpha_x}(x, z, \tau) \zeta_{\alpha_z}(x, z, \tau) - \psi_{\alpha_z}(x, z, \tau) \zeta_{\alpha_x}(x, z, \tau)] \} d\tau \\ & + \int_0^t [-\nu_{in} \zeta_\alpha(x, z, \tau) + \Upsilon_1(x, z) \psi_{\alpha_z}(x, z, \tau) + \Upsilon_2(x, z) \psi_{\alpha_x}(x, z, \tau)] d\tau \\ & - \bar{u}_\alpha \Upsilon_1(x, z) t + \Upsilon_3(x, z) t - \sigma \Upsilon_3(x, z) W(t) \end{aligned} \quad (6.18)$$

and

$$\begin{aligned} \rho_\alpha(x, z, t) = & \rho_\alpha(x, z, 0) - \int_0^t [\bar{u}_\alpha \rho_{\alpha_x}(x, z, \tau) + \bar{\rho}'_\alpha \psi_{\alpha_x}(x, z, \tau)] d\tau \\ & - \int_0^t \{ \varepsilon [\psi_{\alpha_x}(x, z, \tau) \rho_{\alpha_z}(x, z, \tau) - \psi_{\alpha_z}(x, z, \tau) \rho_{\alpha_x}(x, z, \tau)] - \alpha \nabla^2 \rho_\alpha(x, z, \tau) \} d\tau \\ & + \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) t - \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) [\sigma W(t) + \tilde{\sigma} \tilde{W}(t)]. \end{aligned} \quad (6.19)$$

Multiplying equations (6.18) and (6.19) by the Wick polynomials T_γ and taking

the expectation gives

$$\begin{aligned}
\zeta_{\alpha_\gamma}(x, z, t) &= \zeta_\alpha(x, z, 0)\mathbb{I}_{\gamma=0} \\
&- \int_0^t \left[\bar{u}'_\alpha \zeta_{\alpha_{\gamma_x}}(x, z, \tau) - \bar{u}''_\alpha \psi_{\alpha_{\gamma_x}}(x, z, \tau) - \nu_\alpha \nabla^2 \zeta_{\alpha_\gamma}(x, z, \tau) + g \frac{\rho_{\alpha_{\gamma_x}}(x, z, \tau)}{\bar{\rho}_\alpha} \right] d\tau \\
&- \varepsilon \int_0^t \{ E[\psi_{\alpha_x}(x, z, \tau) \zeta_{\alpha_z}(x, z, \tau) T_\gamma] - E[\psi_{\alpha_z}(x, z, \tau) \zeta_{\alpha_x}(x, z, \tau) T_\gamma] \} d\tau \\
&- \int_0^t [\nu_{\alpha n} \zeta_{\alpha_\gamma}(x, z, \tau) - \Upsilon_1(x, z) \psi_{\alpha_{\gamma_z}}(x, z, \tau) + \Upsilon_2(x, z) \psi_{\alpha_{\gamma_x}}(x, z, \tau)] d\tau \\
&- [\bar{u}_\alpha \Upsilon_1(x, z) t - \Upsilon_3(x, z) t] \mathbb{I}_{\gamma=0} - \sigma \Upsilon_3(x, z) E[W(t) T_\gamma] \tag{6.20}
\end{aligned}$$

and

$$\begin{aligned}
\rho_{\alpha_\gamma}(x, z, t) &= \rho_\alpha(x, z, 0)\mathbb{I}_{\gamma=0} \\
&- \int_0^t [\bar{u}_\alpha \rho_{\alpha_{\gamma_x}}(x, z, \tau) + \bar{\rho}'_\alpha \psi_{\alpha_{\gamma_x}}(x, z, \tau) + \alpha \nabla^2 \rho_{\alpha_\gamma}(x, z, \tau)] d\tau \\
&- \varepsilon \int_0^t \{ E[\psi_{\alpha_x}(x, z, \tau) \rho_{\alpha_z}(x, z, \tau) T_\gamma] - E[\psi_{\alpha_z}(x, z, \tau) \rho_{\alpha_x}(x, z, \tau) T_\gamma] \} + d\tau \\
&+ \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) t \mathbb{I}_{\gamma=0} - \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) \{ \sigma E[W(t) T_\gamma] + \tilde{\sigma} E[\tilde{W}(t) T_\gamma] \}. \tag{6.21}
\end{aligned}$$

Using Theorem 2 in section 5.2 gives

$$E[\psi_{\alpha_x}(x, z, \tau) \zeta_{\alpha_z}(x, z, \tau) T_\gamma] = \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha_{\gamma-\beta+p_x}} \zeta_{\alpha_{\beta+p_z}}, \tag{6.22}$$

$$E[\psi_{\alpha_z}(x, z, \tau) \zeta_{\alpha_x}(x, z, \tau) T_\gamma] = \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha_{\gamma-\beta+p_z}} \zeta_{\alpha_{\beta+p_x}}, \tag{6.23}$$

$$E[\psi_{\alpha_x}(x, z, \tau) \rho_{\alpha_z}(x, z, \tau) T_\gamma] = \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha_{\gamma-\beta+p_x}} \rho_{\alpha_{\beta+p_z}} \tag{6.24}$$

and

$$E[\psi_{\alpha_z}(x, z, \tau)\rho_{\alpha_x}(x, z, \tau)T_\gamma] = \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha_{\gamma-\beta+p_z}} \rho_{\alpha_{\beta+p_x}}. \quad (6.25)$$

The Fourier expansion of the white noise $W(t)$ (Lévy-Ciesielski series representation, Theorem 1) implies that the covariance of $W(t)$ and T_γ is given by

$$E[W(t)T_\gamma] = \sum_{l=1}^{\infty} \gamma_l \mathbb{I}_{\gamma_j=\delta_{l,j}} \int_0^t m_l(\tau) d\tau, \quad (6.26)$$

and that of $\tilde{W}(t)$ and T_γ is

$$E[\tilde{W}(t)T_\gamma] = \sum_{l=1}^{\infty} \gamma_l \tilde{\mathbb{I}}_{\gamma_i=\delta_{l,i}} \int_0^t m_l(\tau) d\tau. \quad (6.27)$$

where $\mathbb{I}$ and $\tilde{\mathbb{I}}$ are indicator functions. Therefore

$$\begin{aligned} \zeta_{\alpha_\gamma}(x, z, t) &= \zeta_\alpha(x, z, 0) \mathbb{I}_{\gamma=0} \\ &- \int_0^t \left[\bar{u}'_\alpha \zeta_{\alpha_{\gamma_x}}(x, z, \tau) - \bar{u}''_\alpha \psi_{\alpha_{\gamma_x}}(x, z, \tau) - \nu_\alpha \nabla^2 \zeta_{\alpha_\gamma}(x, z, \tau) + g \frac{\rho_{\alpha_{\gamma_x}}(x, z, \tau)}{\bar{\rho}_\alpha} \right] d\tau \\ &- \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \int_0^t \psi_{\alpha_{\gamma-\beta+p_x}}(x, z, \tau) \zeta_{\alpha_{\beta+p_z}}(x, z, \tau) d\tau \\ &+ \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \int_0^t \psi_{\alpha_{\gamma-\beta+p_z}}(x, z, \tau) \zeta_{\alpha_{\beta+p_x}}(x, z, \tau) d\tau \\ &- \int_0^t [\nu_{\alpha n} \zeta_{\alpha_\gamma}(x, z, \tau) - \Upsilon_1(x, z) \psi_{i_{\gamma_z}}(x, z, \tau) + \Upsilon_2(x, z) \psi_{\alpha_{\gamma_x}}(x, z, \tau)] d\tau \\ &- [\bar{u}_\alpha \Upsilon_1(x, z) - \Upsilon_3(x, z)] t \mathbb{I}_{\gamma=0} - \sigma \Upsilon_3(x, z) \sum_{l=1}^{\infty} \gamma_l \mathbb{I}_{\gamma_j=\delta_{l,j}} \int_0^t m_l(\tau) d\tau \quad (6.28) \end{aligned}$$

and

$$\begin{aligned}
\rho_{\alpha_\gamma}(x, z, t) &= \rho_\alpha(x, z, 0)\mathbb{I}_{\gamma=0} \\
&- \int_0^t [\bar{u}_\alpha \rho_{\alpha_{\gamma_x}}(x, z, \tau) + \bar{\rho}'_\alpha \psi_{\alpha_{\gamma_x}}(x, z, \tau) + \alpha \nabla^2 \rho_{\alpha_\gamma}(x, z, \tau)] d\tau \\
&- \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \int_0^t \psi_{\alpha_{\gamma-\beta+p_x}}(x, z, \tau) \rho_{\alpha_{\beta+p_z}}(x, z, \tau) d\tau \\
&+ \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \int_0^t \psi_{\alpha_{\gamma-\beta+p_z}}(x, z, \tau) \rho_{\alpha_{\beta+p_x}}(x, z, \tau) d\tau \\
&+ \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) t \mathbb{I}_{\gamma=0} - \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) \sum_{l=1}^{\infty} \gamma_l (\sigma \mathbb{I}_{\gamma_j=\delta_{l,j}} + \tilde{\sigma} \mathbb{I}_{\gamma_i=\delta_{l,i}}) \int_0^t m_l(\tau) d\tau.
\end{aligned} \tag{6.29}$$

Differentiating equations (6.28) and (6.29) with respect to time gives

$$\begin{aligned}
\zeta_{\alpha_{\gamma_t}}(x, z, t) &= -\bar{u}'_\alpha \zeta_{\alpha_{\gamma_x}}(x, z, t) + \bar{u}''_\alpha \psi_{\alpha_{\gamma_x}}(x, z, t) - g \frac{\rho_{\alpha_{\gamma_x}}(x, z, t)}{\bar{\rho}_\alpha} + \nu_\alpha \nabla^2 \zeta_{\alpha_\gamma}(x, z, t) \\
&- \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha_{\gamma-\beta+p_x}}(x, z, t) \zeta_{\alpha_{\beta+p_z}}(x, z, t) \\
&+ \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha_{\gamma-\beta+p_z}}(x, z, t) \zeta_{\alpha_{\beta+p_x}}(x, z, t) \\
&- \nu_{in} \zeta_{\alpha_\gamma}(x, z, t) + \Upsilon_1(x, z) \psi_{\alpha_{\gamma_z}}(x, z, t) - \Upsilon_2(x, z) \psi_{\alpha_{\gamma_x}}(x, z, t) \\
&- [\bar{u}_\alpha \Upsilon_1(x, z) - \Upsilon_3(x, z)] \mathbb{I}_{\gamma=0} - \sigma \Upsilon_3(x, z) \sum_{l=1}^{\infty} \gamma_l \mathbb{I}_{\gamma_j=\delta_{l,j}} m_l(t) \tag{6.30}
\end{aligned}$$

and

$$\begin{aligned}
\rho_{\alpha\gamma_t}(x, z, t) = & -\bar{u}_\alpha \rho_{\alpha\gamma_x}(x, z, t) - \bar{\rho}'_\alpha \psi_{\alpha\gamma_x}(x, z, t) - \alpha \nabla^2 \rho_{\alpha\gamma}(x, z, t) \\
& - \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha\gamma-\beta+p_x}(x, z, t) \rho_{\alpha\beta+p_z}(x, z, t) \\
& + \varepsilon \sum_{p \in \mathcal{G}} \sum_{0 \leq \beta \leq \gamma} C(\gamma, \beta, p) \psi_{\alpha\gamma-\beta+p_z}(x, z, t) \rho_{\alpha\beta+p_x}(x, z, t) \\
& + \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) \mathbb{I}_{\gamma=0} + \frac{\kappa_\alpha}{\mathcal{K}} \Upsilon_4(x, z) \sum_{l=1}^{\infty} \gamma_l (\sigma \mathbb{I}_{\gamma_j=\delta_{l,j}} + \tilde{\sigma} \mathbb{I}_{\gamma_i=\delta_{l,i}}) m_l(t). \quad (6.31)
\end{aligned}$$

Equations (6.30) and (6.31) are deterministic and can thus be solved for ζ_{i_γ} and ρ_{i_γ} , for each γ , using the predictor-corrector method described in section 3.4.

6.3 Results of the numerical simulations

In this section we solve the initial boundary value problem (6.3)-(6.14) using the WCE numerical method described (sections 5.2 and 6.2). Simulations are carried out over the nondimensional time interval from $t = 0$ to $t = 20$ on a rectangular domain given by $0 < x < 2\pi$ and $10 < z < 30$. The following parameters are used: $N = \sqrt{2}$, $\delta = 0.2$, $\nu = 10^{-6}$ and $\text{Pr} = 0.72$.

The mean flow velocity is set to a constant $\bar{u} = 1$. In the configurations with GWs, we consider that the GWs are generated by a source at the lower boundary of the domain, $z_1 = 10$, given by $\psi(x, z_1, t) = e^{i\kappa x} + c.c.$ with $\kappa = 2$, and propagate upward into the ionosphere.

In the simulations presented here, we neglect the nonlinear wave-wave interactions (higher wavenumber), and consider only the nonlinear interactions between the electromagnetic field and the waves because when the nonlinear wave-wave interactions are taken into consideration, the WCE method is accurate for simulations carried out over a short time frame only ($t < 1$). This is shown in section 5.3 where the WCE method was applied to solve numerically the stochastic Burgers' equation.

6.3.1 Configuration 1: EHD-gravity wave interactions and the generation of TIDs ($\Upsilon_1 = \Upsilon_2=0$, $\Upsilon_3 \neq 0$)

To simulate EHD waves we solve (6.6)-(6.6) with $\Upsilon_1 = \Upsilon_2=0$. We consider the case where $\eta_e = \eta_n = 0$ and $\eta_v = \eta$ in the expression for $\Upsilon_3(x, z)$ and $\Upsilon_4(x, z)$ so that

$$\Upsilon_3(x, z) = \eta^2 e^{-2\eta(z-z_1)} \sin 2\eta x \quad (6.32)$$

and

$$\Upsilon_4(x, z) = \eta e^{-\eta(z-z_1)} \sin \eta x. \quad (6.33)$$

Therefore the electric field generates the horizontal wavenumbers $\pm 2\eta$ while the energy flux generates the horizontal wavenumbers $\pm \eta$. We set $\langle \epsilon \rangle = 1.5$ and $\eta = 1$. Therefore we expect that the horizontal wavenumber of the EHD waves to be approximately $k_x = 2\eta = 2$ so that the wavelength is approximately $2\pi/2 = \pi \approx 3.14$ as seen Figure 6.1. Figure 6.2 shows the Fourier spectrum of the EHD wave streamfunction $\hat{\psi}_\alpha$ as a function of the wavenumber k . The Fourier spectrum is zero except for the wavenumbers $k = \pm 1$ (medium-scale) and $k = \pm 2$ (small-scale) as expected. This shows that EHD waves are medium-scale disturbances.

On the other hand there are similarities in the results obtained from the simulations of atmospheric gravity waves propagating upward in the ionospheric F region in the configuration where the magnetic dip angle $\theta = \pi/2$ (Figures 3.5 (c) and 3.8), and those obtained from the simulations of the EHD waves propagating upward in the ionosphere (Figures 6.1, 6.2 and 6.3). For example at $z = 11$ both the GWs and EHD waves reach the steady state, their intrinsic frequencies are regular and approximately the same or multiple of each other, and their horizontal wavenumber are practically the same. The important difference observed between GWs and EHD waves is that the gravity waves are damped while the EHD waves are not. This straightforward comparison shows that during an experimental expedition (measurements in the ionosphere) it may be hard to establish the difference between the TIDs associated to the atmospheric gravity waves and EHD waves (Otsuka et al., 2013)

Figure 6.1: EHD waves: contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$

Figure 6.2: EHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.86$ and fixed height $z = 11$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$

Figure 6.4 shows the results of the simulations of the upward propagating EHD waves in a weakly random ionosphere. In a weakly-random time varying ionosphere the EHD wave streamfunction ψ is a random function of time. Figure 6.5 (a) shows the results of the simulations of the effects of the EHD waves on the GWs in random ionosphere while figure 6.5 (b) shows the results of the simulations of the effects of

Figure 6.3: EHD waves: Fourier spectrum $|\hat{\psi}_\alpha(k, z, t)|$ as a function of the wavenumber k at fixed height $z = 15$ and fixed time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$

the GWs on the EHD waves in random ionosphere. As seen in Figure 6.5 (a) the random effects are amplified. On the other hand it is seen in Figure 6.5 (b) that the weak random effects are attenuated by the collisions. The amplification of the random effects comes from the term $\nu_{\alpha n}\psi_\alpha$ in the equation (6.6). The intermittent wave-like structure observed on 3 November 1997 in Průhonice and analyzed by Šauli and Boška (2001) is a typical example of the effects of GWs on the intermittent EHD waves.

Figure 6.6 shows the results of the simulations of the effects of the GWs on the EHD waves. A comparison of Figure 6.6 with Figure 6.1 reveals that as the EHD waves propagate upward in the ionosphere, they resemble GWs, showing that the collision effects become more important than the effects produced by the electric field. This is also seen in the Figure 6.9 (a) which shows that the temporal characteristic features of the EHD waves affected by the GWs are comparable to those of the GWs. An example of this situation can be found in Yokohama et al. (2005).

On the other hand Figure 6.8 shows the results obtained for the simulations of the effects of the EHD waves on the GWs. As seen in Figure 6.8, the GWs resemble EHD waves at later time, showing that the effects due to the collisions of the plasma on the neutral fluid flow become more important compared to the gravity wave-like

Figure 6.4: Intermittent EHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.3$ and height $z = 11$, in (a) the random effects are neglected while in (b) the random effects are taken into consideration. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$

Figure 6.5: EHD-gravity wave interactions in a random ionosphere: plots of the streamfunction as a function of time t at fixed $x = 2.3$ and height $z = 11$, in (a) effects of GWs on EHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$) and (b) effects of EHD waves on GWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

perturbations. This is also seen in Figure 6.9 (b) which shows that the temporal characteristic features of the GWs affected by the EHD waves are comparable to those of the EHD waves at later time.

Figure 6.6: Effect of gravity waves on EHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the EHD waves streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.7: Effects of EHD waves on GWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): Fourier spectrum of the GW streamfunction $|\hat{\psi}(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

As shown in Figure 6.7 the wavenumber of the leading-order of (linear) solution is $k = \pm 2$. By $t = 10$, a nonzero contribution corresponding to the wavenumbers $k = \pm 1$ has developed indicating that there is a transfer of energy from the plasma to the neutral flow.

Figure 6.9 (c) shows the configuration where $\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$ meaning

Figure 6.8: Effect of EHD waves on gravity waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the gravity waves streamfunction $\psi(x, z, t)$ at time $t = 12$; (a) gravity waves and (b) gravity waves affected by EHD waves. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

that the equations for the GWs and the ionosphere are coupled. The results show that both the gravity and EHD waves evolve to some steady states but their amplitudes are different.

6.3.2 Configuration 2: MHD-gravity wave interactions and the generation of TIDs ($\Upsilon_1 \neq 0$, $\Upsilon_2 \neq 0$, $\Upsilon_3 = 0$)

We let $\eta_e = \eta_n = 0$ and $\eta_v = \eta$, which means

$$\Upsilon_1(x, z) = \eta^2 e^{-\eta(z-z_1)} \sin \eta x \quad (6.34)$$

and

$$\Upsilon_2(x, z) = \eta^2 e^{-\eta(z-z_1)} \cos \eta x. \quad (6.35)$$

We set $\Upsilon_3 = 0$ so there is no electric force and we have MHD waves. In contrast to EHD waves, we expect that MHD waves develop other harmonics as shown in Figure 6.10 which arise from the nonlinear interactions of the waves and the ionosphere via the Lorentz force which is directly proportional to the cross product $\mathbf{u} \times \mathbf{B}$ ($F = \varrho_{\alpha,el} \mathbf{u} \times \mathbf{B}$). This explains why we observe a huge transfer of the momentum flux to

the mean flow in Figure 6.10 (b).

As seen in Figure 6.11, for the time $t \lesssim 3$ we observe that the horizontal wavelength is approximately $2\pi/\eta \approx 2\pi/1 = 2\pi$. But the mean flow develops rapidly compared with the other harmonics while the horizontal wavenumber decreases fast and approaches zero ($k_x = 1$ at $t = 0$ and $k_x \rightarrow 0$ at a later time) at a later time. This explains why the MHD waves transform into wave packets in time rather than evolving to a steady state as seen in Figure 6.12, and indicate that MHD waves are large-scale waves.

Figure 6.9: EHD-gravity wave interactions in the ionosphere: plots of the streamfunction $\psi(x, z, t)$ as function of time t at fixed $x = 2.86$ and $z = 11$; (a) effects gravity waves on EHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$), (b) effects of EHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$) and (c) gravity and EHD waves mutual interactions ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.10: MHD waves: Fourier spectrum $|\hat{\psi}_\alpha(k, z, t)|$ as a function of the wavenumber k at fixed height $z = 15$, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.11: MHD waves: contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at the time, (a) $t = 1$, (b) $t = 3$, (c) $t = 6$ and (d) $t = 9$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.12: MHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.86$ and height, (a) $z = 11$ and (b) $z = 12.5$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.13: Effects of gravity waves on MHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): Fourier spectrum of the MHD wave streamfunction $|\hat{\psi}_\alpha(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.14: Effects of gravity waves on MHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figures 6.13 and 6.14 show the results of the simulations of the effects of the GWs on the MHD waves. As seen in Figure 6.14 the upward propagating MHD waves resemble GWs at later time, showing that the collision effects become more important than the effects of the magnetic field. Figure 6.17 (a) shows that the temporal characteristics of the MHD waves affected by the GWs are comparable to those of the GWs. This explains why the harmonics corresponding to the wavenumber $k = \pm 2$ dominate that of the mean flow ($k = 0$) in the figure 6.13.

Figures 6.15 and 6.16 show the results of the simulations of the effects of the MHD waves on the GWs. We observe from Figure 6.16 (b) that the GWs resemble MHD waves at later time, indicating that the collision effects become more important than the GW-like perturbations. As seen in Figure 6.16 the harmonic corresponding to the wavenumber $k = 0$ are large compared to that corresponding $k = \pm 2$ at later time, showing that at a later time there a significant development of the mean neutral mean which is comparable to that of the plasma mean flow.

Figure 6.15: Effects of MHD waves on AGWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): Fourier spectrum of the AGW streamfunction $|\hat{\psi}(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.16: Effects of MHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): contour plots of the streamfunction $\psi(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.17 (c) shows the configuration where $\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$. We observe that both the AGWs and MHD waves evolve to steady states with different amplitudes at later time, while the MHD waves which are not affected at all by gravity waves become wave packets at a later time. On the other hand the gravity waves reach a steady different from that of gravity waves propagating upward in the neutral atmosphere once they are affected by MHD waves.

6.3.3 Configuration 3: EMHD-gravity wave interactions and generation of TIDs ($\Upsilon_1 \neq 0$, $\Upsilon_2 \neq 0$, $\Upsilon_3 \neq 0$)

In this section we show the results obtained for the simulations of gravity-EMHD waves interactions. In contrast with the EHD waves (section 6.3.1) and MDH waves (section 6.3.2) where the waves are generated by the electric field (EHD waves) or the magnetic field (MHD waves), the EMHD waves are generated by both the electric and magnetic fields, $\Upsilon_1(x, z)$, $\Upsilon_2(x, z)$ and $\Upsilon_3(x, z)$ are all taken into account in the equation (6.6).

Letting $\eta_e = \eta_n = 0$ and $\eta_v = \eta$ implies that

$$\Upsilon_1(x, z) = \eta^2 e^{-\eta(z-z_1)} \sin \eta x, \quad (6.36)$$

$$\Upsilon_2(x, z) = \eta^2 e^{-\eta(z-z_1)} \cos \eta x, \quad (6.37)$$

$$\Upsilon_3(x, z) = \eta^2 e^{-2\eta(z-z_1)} \sin 2\eta x \quad (6.38)$$

and

$$\Upsilon_4(x, z) = \eta e^{-\eta(z-z_1)} \sin \eta x. \quad (6.39)$$

Figure 6.18 shows that there is a more rapid development of the mean flow as a result of the interactions of the ionosphere with the waves due to the magnetic field, as observed for the MHD waves (section 6.3.2). The harmonics which correspond to $k = \pm 2$ are generated by the electric field as for EHD waves (section 6.3.1), while the harmonics corresponding to the wavenumber $k = \pm 3$ and $k = \pm 4$ result from the

contributions of both the electric and magnetic fields.

In Figure 6.19 (a) we observe that at $t = 3$ the horizontal wavelength is approximately π . But by $t = 12$ it is seen in Figure 6.19 (b) that the horizontal wavelength at $z = 12$ is different from that at $z = 23$ showing that for the EMHD waves the horizontal wavelength may change with altitude z .

Figure 6.20 shows that the EMHD wave pulsation is irregular. Therefore EMHD waves can be classified in the category of irregular pulsation waves. These are the waves mentioned by Parks (1991) as irregular pulsation MHD waves.

The results for the simulations of intermittent EMHD waves are shown in the figure 6.21. The intermittency features of the EMHD waves are only due to the electric field.

Figures 6.22, 6.23 and 6.27 (a) show the results of simulations of the effects of gravity waves on the EMHD waves. As seen in Figure 6.23, the harmonics corresponding to $k = \pm 2$ are comparable to that of the plasma mean flow ($k = 0$) showing that the GWs constitute an extra source of oscillations of period π . This explains why the horizontal wavelength of the EMHD is uniform along z as shown in Figure 6.22. Moreover, as observed in Figure 6.27 (a), the temporal characteristics of EMHD waves become more regular because of the collisions between the plasma and the neutral fluid flow.

Figures 6.24, 6.25 and 6.27 (b) show the results of the simulations of the effects of EMHD waves on GWs. Figure 6.24 show that the GWs resemble EMHD waves at a later time. As seen in Figure 6.25, the harmonic corresponding to the wavenumber $k = 0$ becomes larger than those corresponding to the wavenumbers $k = \pm 2$ at later time, showing that at later time there is a huge development of the neutral mean flow. As observed in the figure 6.24 the horizontal wavelength is not uniform along z .

Figure 6.26 illustrates the effects of the intermittent EMHD waves on the GWs. As seen in Figure 6.26 (b), the perturbations resulting from the collisions of the intermittent EMHD waves are deterministic, so the random effects are attenuated during the collisions. This somewhat surprising as we would expect to see intermittent perturbations of the neutral flow after the collisions.

On the other hand by comparing, Figure 6.21 and Figure 6.26 (a), the random effects seem to be amplified. In fact, the amplification of the randomness comes from the term $\nu_{\alpha n}\psi_{\alpha}$ in the equation (6.6).

Figures 6.27 (c) shows that both the AGWs and EMHD waves evolve to some steady states with different at later time, as observed for EHD waves in section 6.3.1 and MHD waves in section 6.3.2. This explains why upward propagating gravity waves are responsible for part of the uncertainties of ionospheric radio wave propagations and predictions, and forecasts for telecommunications purposes (Lřstovička and Bourdillon, 2004; Lřstovička, 2006).

Figure 6.17: Gravity-MHD wave interactions: plots of the streamfunction as function of time t at fixed $x = 2.86$ and $z = 11$; (a) effects gravity waves on MHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$), (b) effects of MHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$) and (c) gravity and MHD waves mutual interactions ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.18: EMHD waves: Fourier spectrum $|\hat{\psi}_\alpha(k, z, t)|$ as a function of the wavenumber k at fixed height $z = 15$ and time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.19: EMHD waves: contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at a the time, (a) $t = 1$ and (b) $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.20: EMHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.86$ and height $z = 11$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.21: Intermittent EMHD waves: plots of the streamfunction $\psi_\alpha(x, z, t)$ as a function of time t at fixed $x = 2.3$ and height $z = 11$, in (a) the random effects are neglected while in (b) the random effects are taken into consideration. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.22: Effects of gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): contour plots of the streamfunction $\psi_\alpha(x, z, t)$ at the time $t = 12$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.23: Effects of gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$): Fourier spectrum of the EMHD wave streamfunction $|\hat{\psi}_\alpha(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.24: Effects of EMHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): contour plots of the streamfunction $\psi(x, z, t)$ at a the time $t = 12$; (a) gravity waves and (b) gravity waves affected by EMHD waves. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.25: Effects of EMHD waves on AGWs ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$): Fourier spectrum of the AGW streamfunction $|\hat{\psi}(k, z, t)|$ plotted as a function of the wavenumber k at fixed height $z = 15$ and time, (a) $t = 1$ (b) $t = 10$. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.26: EMHD-gravity waves interaction in a random ionosphere: plots of the AGW streamfunction $\psi(x, z, t)$ as a function of time t at fixed $x = 5.7$ and height $z = 11$; (a) Effects of gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$) and (b) Effects of EMHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0$). The solid line in (b) represents the AGWs in presence of the random effects while the dotted-line represents the AGWs in the absence of the random effects. A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Figure 6.27: Gravity-EMHD wave interactions: plots of the streamfunction $\psi(x, z, t)$ as function of time t at fixed $x = 2.86$ and $z = 11$; (a) effects gravity waves on EMHD waves ($\nu_{n\alpha} = 0$ and $\nu_{\alpha n} = 0.5$), (b) effects of EMHD waves on gravity waves ($\nu_{n\alpha} = 0.5$ and $\nu_{in} = 0.5$) and (c) gravity and EMHD waves mutual interactions ($\nu_{n\alpha} = 0.5$ and $\nu_{\alpha n} = 0.5$). A periodic forcing $e^{i\kappa x}$, $\kappa = 2$, was applied at the lower boundary of the domain and, the background flow is constant, $\bar{u} = 1$ and $\delta = 0.2$.

Chapter 7

General discussion and conclusions

7.1 Discussion

It is well known (Laštovička, 2006) that when upward propagating waves in the neutral atmosphere, which include gravity waves, planetary (Rossy) and tides, reach the ionosphere they interact with the ionosphere via conservation of mass, momentum and energy and thus produce oscillations or interact with ionospheric disturbances excited by the ionospheric electromagnetic fields. In this study we focused on mathematical investigations of the ionospheric disturbances and their interactions with the upward propagating atmospheric gravity waves (AGWs). First we derived a mathematical model, equations (6.3)-(6.14), for these interactions in terms of the vorticity and streamfunction, starting with the equations for conservation of mass, momentum and energy in a 2-D vertical plane and Maxwell's equations (Kelley, 2006). The advantage of our mathematical model is that it is independent of the pressure, meaning that we do not need to take into consideration the pressure terms in the computations like Geisler (1966) and Khantadze, Gvelesiani and Kurtskhalia (1976).

Once we obtained the expressions for the magnetic and electric fields we combined these with the momentum and the energy equations, used the continuity equation (mass conservation) to define the streamfunction and derived the stochastic vorticity equation (SPDE) for the ionospheric waves by combining the horizontal and the vertical components of the momentum equation.

We focused on two important problems. The first was the problem studied by Yeh and Liu (1972) where the upward propagating atmospheric gravity waves interact with the ionospheric F region where the ions are collisionless ($\nu_{in} \approx 0$), the magnetic field is enough strong that the ions are constrained to spiral about the lines of the magnetic field, and the atomic oxygen ion gyro-frequency is very large compared to the neutral-ion collision frequency and the gravity wave frequency (Hunsucker and Hargreaves 2003). We considered a configuration where the temperature varies very slowly so that the neutral-collision frequency ν_{in} is approximately constant. We assumed that each fluid variable is a sum of a mean flow quantity and a perturbation quantity (wave), and therefore used the weakly nonlinear theory to linearize the equations. Yeh and Liu (1972) investigated the ionospheric effects of the oxygen ions O^+ on the upward propagating gravity waves in a configuration without nonlinear and viscous effects. They derived an explicit expression for the neutral vertical wind velocity taking into account the effects of the ion drag for the dip angle $\theta = 0$ corresponding to the gravity wave propagation in the magnetic meridian plan at the magnetic equator only. But their results seem to be incorrect because they used the expression for the frictional force of ions on the neutral wind in the momentum equation instead of using the ion drag force.

In our investigations we studied the same problem as Yeh and Liu (1972) but considered a general magnetic dip angle and examined two configurations: a configuration where the nonlinear effects are neglected and a configuration where the nonlinear effects are taken into consideration.

We solved the problem for the case where the wave amplitude is steady and found that the wave amplitude varies exponentially with height. We obtained an expression for the rate of change of the amplitude with height as a function of the magnetic dip angle. From this expression we noted that the rate of change of the amplitude depends on the buoyancy frequency and the mean velocity of the neutral background flow, the neutral-ion collision frequency, the horizontal wavenumber of the gravity waves and the vertical to horizontal aspect ratio of the domain, as well as on the magnetic dip angle. In the long-wave limit of zero aspect ratio the AGW amplitude

decreases with height for all dip angles. Also for typical values of the mean flow and wave parameters, there is an exponential decrease in the general case without the long wave limit. This wave damping is consistent with the predictions of analyses of (Gupta and Nagpal, 1982). We observed that in the long-wave limit of zero aspect ratio GWs are not damped at the magnetic dip $\theta = 0$ corresponding to the gravity wave propagation in the magnetic meridian plan at the magnetic equator. We also observed that the wavelength of the atmospheric gravity waves in the ionosphere is longer than for AGWs in the neutral atmosphere for some magnetic dip angle (e.g., longer for $\theta = \pi/2$ than in the neutral atmosphere but shorter for $\theta = 0$ than in the neutral atmosphere). This is consistent with the observations of Hearn and Yeh (1977).

We then solved the time-dependent linear equations for the special case of long-wave limit using a Fourier-Laplace transform and obtained an explicit expression for the amplitude of the atmospheric gravity wave amplitude. Our results showed that upward propagating long GWs approach a steady state at late time. We also derived, using the weakly nonlinear theory, a more general equation (equation 3.65) for the number density perturbation which is different from that in Hooke (1968).

We also performed numerical simulations to extend this study to general wavelengths and to include the nonlinear effects. The accuracy of our numerical method was accessed by comparing the results obtained for the simulations of the long gravity waves (case of the long-wave limit) with the analytical results and there was very close agreement between the results of the simulations and the analytical results. We found that the vertical wavelength of the GWs varies with the height and that the variation of the wavelength differs from one magnetic dip angle to another. For example, the increase in the wavelength is larger for the magnetic angle $\theta = \pi/2$ than for $\theta = 0$ at the magnetic equator, while the wavelength is shortest for $\theta = \pi/4$. We also found that the waves are damped in the vertical direction and that the damping rates vary with the magnetic dip angle as predicted by Miesen (1991).

We also computed numerically the number density for the dip angles $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$ and found that the effects of the upward propagating atmospheric

gravity are more significant at the magnetic equator $\theta = 0$ and become negligible as $\theta \rightarrow \pi/2$. This is consistent with the measurement and analysis of the TEC data at the magnetic dip angle of $\theta = 72^\circ$ observed from the 800 GPS receivers of the European receiver networks (Otsuka et al., 2013).

With the nonlinear effects taken into consideration, we observed a development of the mean flow with height (as z increases) for the configuration where the magnetic dip angle $\theta = 0$, showing that there is a transfer of the momentum flux to the mean at high altitudes. We also found that the waves are damped and that the damping rates vary with the magnetic dip angle as in the linear simulations. The wavelength variation with height which depends on one magnetic angle to another was observed in the nonlinear simulations as well. The results for both linear and nonlinear simulations showed that AGWs in the F region reach some steady states at later time and that the magnitude of the group velocity of AGWs in the ionosphere may exceed that of AGWs in the neutral atmosphere.

The second important ionospheric problem considered in this thesis was a more realistic configuration in which the electric as well as magnetic forces were taken into account. A major problem in magnetohydrodynamic (MHD) modeling is that there are no exact expressions for the electric and magnetic fields. In magnetohydrodynamics (MHD) the substitution $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ is commonly used for the electric field $\mathbf{E}$ in Maxwell's equations and the energy equations. This essentially implies that $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$ in the momentum equation (Kelley, 2006 chapter 6). Therefore the momentum equation becomes simply the Navier-Stokes equation because it no longer contains the electric and magnetic forces and thus there is no a magnetohydrodynamic problem to solve. To get around this the electric field $\mathbf{E}$ is usually neglected while the term $\mathbf{u} \times \mathbf{B}$ is kept in the momentum equation. However this contradicts the substitution $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ in Maxwell's and energy equations. Therefore these simplifications are not consistent. The equations based on these simplifications are what are generally called MHD equations. They cannot be solved analytically but there are many numerical methods that have been specifically developed for these MHD equations.

In our investigations we use a consistent representation for the electric and the magnetic fields. To avoid the inconsistencies in the standard MHD formulation, we first derived the electromagnetic wave equations from Maxwell's equations and solved them for $\mathbf{E}$ and $\mathbf{B}$ for a configuration corresponding to an electric dynamo and then substituted these into the momentum equation for the ionospheric plasma.

We classified the ionospheric disturbances into three categories, the electrohydrodynamic (EHD) waves, the magnetohydrodynamic (MHD) waves and the electromagnetohydrodynamic waves according to the form of the electromagnetic field under consideration and quantified these using our mathematical model (6.3)-(6.14). If the ionospheric oscillations (waves) were induced by the electric field, we called these EHD waves, as suggested by Miller et. al (1997) and Kelley and Miller (1997). On the other hand if the ionospheric disturbances were generated by the magnetic field we called them MHD waves and those generated by both the electric and magnetic fields were called EMHD waves.

We also investigated the ionospheric problem where the electromagnetic field varies randomly in time and thus generates intermittent ionospheric waves (Sauvaud et al., 2001; Shiokawa et. al, 2003; Aveiro et al., 2009; Martinis et al. 2010; Qian et al., 2013; Otsuka et al., 2013). In that case, the relative permittivity and relative permeability become random variables or statistical quantities in time or space or both. The configuration where these quantities are considered to be random functions of space is discussed in Yeh and Liu (1972). In the present study we focused on the configuration where the ionosphere is a random medium with the permittivity and permeability being functions of time. Thus, the electromagnetic wave equations are stochastic partial differential equations (SPDEs) with random coefficients. We were able to solve equations these equations for the electric field and magnetic field by considering that the randomness of the medium (ionosphere) is weak so that the relative permittivity and relative permeability are given by the sum of a mean quantity and a random quantity with mean zero with the mean part being the solution of the deterministic wave equation.

We considered the situation where the mean quantity is very large compared to

the random quantity. This allowed us to obtain expressions for the magnetic and electric fields, each as a sum of a mean part and a random stochastic part. The assumption of weak-random variation is justified by observational data presented by other researchers. Using the total electron content (TEC) data observed from the 246 GPS receivers in and around China during the medium storm on 28 May 2011, Song et al. (2013) generated a plot of the magnetic field as a function of time which reveals the magnetic field is a weakly-random function.

Having obtained expressions for the electromagnetic field quantities (chapter 4), we then added these as forcing in the equations for the ionospheric plasma. The solutions of these equations represented ionospheric disturbances, either electrohydrodynamic (EHD), magnetohydrodynamic (MHD) or electromagnetohydrodynamic (EMHD) waves. We observed that according to our mathematical model (6.3)-(6.14) only EHD and EMHD waves can fluctuate randomly, and that the electric field is the main factor responsible for the random fluctuations.

We carried out numerical simulations of EHD and EMHD waves first without and then with the random effects. By comparing the results of the simulations obtained for the EHD waves and those obtained for the upward propagating gravity in the ionosphere we found these waves have similar properties. This suggests that EHD waves could be mistakenly interpreted as traveling ionospheric disturbances. For example, Otsuka et al. (2013) observed EHD waves and called them medium-scale traveling ionospheric disturbances (MSTIDs). We found that the MHD waves become wave packets and their horizontal wavelength increases at later time because of the nonlinear interactions of these waves and the ionosphere. We found that the EMHD waves fit in the category of irregular pulsation waves because of their unpredictable temporal characteristic feature. Moreover, the EMHD wave horizontal wavelength varies with height, their wavelength shortens with the height.

In addition to the study of the ionospheric waves (EHD, MHD and EMHD waves), we analyzed the interactions of these waves with the upward AGWs. We studied three configurations, the configuration where the AGWs affect the ionospheric waves ($\nu_{\alpha n} \neq 0$, $\nu_{n\alpha} = 0$), the configuration where the ionospheric waves affect the AGWs

($\nu_{\alpha n} = 0, \nu_{n\alpha} \neq 0$) and the configuration where there are reciprocal interactions ($\nu_{\alpha n} \neq 0, \nu_{n\alpha} \neq 0$). We found that in the configuration where the GWs affect the ionospheric waves, the collisional effects of the AGWs on the ionospheric wave become important relative to the electromagnetic field effects to the extent that the characteristic features of the ionospheric waves resemble those of the AGWs and vice-versa. On the other hand, if the interactions are reciprocal, both the ionospheric waves and the AGWs evolve to states which are categorically different from the original waves. As mentioned before, this can provide an explanation for the fact that upward propagating atmospheric gravity waves are responsible for a part of the uncertainty of ionospheric radio wave propagations and predictions and forecasts for telecommunication purposes (Lästovička and Bourdillon, 2004; Lästovička, 2006).

A typical example of the effects the AGWs on MHD waves can be found in Song et al. (2013). After analyzing the TEC data from the 246 GPS receivers in and around China during the medium storm on 28 May 2011, they detected two events which they called larger-scale traveling ionospheric disturbance (LSTID) events. One of the events was associated with the variation of the magnetic field while the other event was associated with the AGWs generated by the Joule heating of the equatorial electrojet. The event associated with the magnetic field represents the MHD waves while the other LSTID event arises from the effects of the AGWs on the MHD waves.

Different studies (experiments, analysis of measurements and observations) performed around the world (e.g., Hunsaker and Hargreaves, 2003; Yeh and Liu 1972; Gupta and Nagpal, 1982; Lästovička and Bourdillon, 2004; Lästovička, 2006; Zolesi and Cander, 2004; Song et al., 2013) revealed that gravity wave interactions constitute an important source for some of the uncertainties in the ionospheric radio wave propagation and predictions and in telecommunication. If we are able to predict their effects then we can improve accuracy of predictions and forecasts (Lästovička and Bourdillon, 2004; Lästovička, 2006). With similar perspectives we hope that the investigations carried out here can be helpful in giving us insight that can improve the radio wave propagation predictions and telecommunications.

7.2 Future work

A number of interesting related projects arise from the investigations that were described in this thesis. In this section we describe some possible directions for future research as well as some current work in progress that we did not include in the thesis.

7.2.1 Rossby wave (RW) interactions in the ionosphere

Rossby waves are planetary-scale waves that are generated in the atmosphere as a results of the effects of the rotation of the earth (Coriolis force). Like ionospheric gravity waves, planetary ionospheric waves can be generated by electric and magnetic fields (Sauvaud et al., 2001). In contrast to ionospheric gravity waves which propagate vertically, ionospheric Rossby waves frequently propagate in a horizontal plane tangent to the surface of the earth, and in that case the effects due to the gravity potential on Rossby wave (RWs) can be neglected in the momentum equation and only the effects due to the Coriolis force need to be taken into consideration. These types of waves are called barotropic Rossby waves and they can be modeled using equation (2.6).

The ionospheric Rossby waves are governed by

$$\varrho_\alpha \frac{D\mathbf{u}_\alpha}{Dt} = -\nabla p_\alpha + \mu \nabla^2 \mathbf{u}_\alpha - 2\varrho \Omega \times \mathbf{u}_\alpha + \varrho_{\alpha,el}(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) \quad (7.1)$$

$$\frac{1}{\varrho_\alpha} \frac{D\varrho_\alpha}{Dt} + \nabla \cdot \mathbf{u}_\alpha = \frac{1}{\varrho_\alpha} (Q_\alpha - L_\alpha) M_\alpha \quad (7.2)$$

$$\frac{\partial \varrho_{\alpha,el}}{\partial t} = \nabla \cdot \mathbf{n}_\alpha \quad (7.3)$$

and Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (7.4)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_\alpha + \frac{\partial \mathbf{D}}{\partial t} \quad (7.5)$$

$$\nabla \cdot \mathbf{D} = \varrho_{\alpha,el} \quad (7.6)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (7.7)$$

where again Ω is the angular velocity of rotation of the Earth, Q_α and L_α are the rates of production and loss per unit of volume respectively, $\mathbf{E}$ and $\mathbf{H}$ the electric and magnetic fields vectors, $\mathbf{D}$ and $\mathbf{B}$ the electric and magnetic flux densities of the electromagnetic field and $\mathbf{J}_\alpha$ is the current density of the α species.

Considering the case where $\frac{D\rho_\alpha}{Dt} \ll \rho_\alpha$ and $(Q_\alpha - L_\alpha)M_\alpha \ll \rho_\alpha$ (for example, $Q_\alpha \approx L_\alpha$), then the continuity equation for α species in the two-dimensional horizontal plane can be approximated by

$$\nabla \cdot \mathbf{u}_\alpha = 0 \quad (7.8)$$

The barotropic vorticity equation for ionospheric Rossby waves

To model ionospheric RWs, we consider a two-dimensional configuration in a rectangular domain parallel to a plane tangent to the Earth's surface. We represent the horizontal (west to east) coordinate by x and the latitude by y . The Lorentz force per unit of volume is given by (2.27). In the configuration under consideration here where the z -component of the velocity is set to zero ($w_\alpha = 0$), the Lorentz force per unit of volume is then given by

$$\begin{aligned} \mathbf{u}_\alpha \times \mathbf{B} &= (\hat{\mathbf{x}}u_\alpha + \hat{\mathbf{y}}v_\alpha) \times (\hat{\mathbf{x}}B_{\text{east}} + \hat{\mathbf{y}}B_{\text{north}} + \hat{\mathbf{z}}B_{\text{vertical}}) \\ &= \hat{\mathbf{x}}v_\alpha B_{\text{vertical}} - \hat{\mathbf{y}}u_\alpha B_{\text{vertical}} + \hat{\mathbf{z}}(u_\alpha B_{\text{north}} - v_\alpha B_{\text{east}}). \end{aligned} \quad (7.9)$$

In Cartesian coordinates each component of the momentum equation (A.4) can thus be written as

$$\begin{aligned} \frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} + v_\alpha \frac{\partial u_\alpha}{\partial y} &= -\frac{1}{\rho_\alpha} \frac{\partial p_\alpha}{\partial x} \\ &+ \nu_\alpha \nabla^2 u_\alpha + f v_\alpha + \varrho_{\alpha o}^{-1} \varrho_{\alpha, el} B_{\text{vertical}} v_\alpha + \varrho_{\alpha o}^{-1} \varrho_{\alpha, el} E_{\text{east}}, \end{aligned} \quad (7.10)$$

$$\begin{aligned} \frac{\partial v_\alpha}{\partial t} + u_\alpha \frac{\partial v_\alpha}{\partial x} + v_\alpha \frac{\partial v_\alpha}{\partial y} = & -\frac{1}{\varrho_\alpha} \frac{\partial p_\alpha}{\partial y} \\ & + \nu_\alpha \nabla^2 v_\alpha - f u_\alpha - \varrho_{\alpha_o}^{-1} \varrho_{\alpha,el} B_{\text{vertical}} u_\alpha + \varrho_{\alpha_o}^{-1} \varrho_{\alpha,el} E_{\text{north}} \end{aligned} \quad (7.11)$$

and

$$\frac{\partial p_\alpha}{\partial z} = -\frac{\varrho_\alpha}{\bar{\varrho}_\alpha} g - \frac{\varrho_{\alpha,el}}{\bar{\varrho}_\alpha} (u_\alpha B_{\text{north}} - v_\alpha B_{\text{east}}) + \frac{\varrho_{\alpha,el}}{\bar{\varrho}_\alpha} E_{\text{vertical}}, \quad (7.12)$$

where ϱ_{α_o} is a constant reference density and, $\nu_\alpha = \frac{\mu_\alpha}{\varrho_\alpha}$ is the plasma kinematic viscosity and f is the Coriolis factor of a beta-plane in the ionosphere and is given by

$$f \sim f_{\alpha_o} + \left(\frac{df_\alpha}{d\theta} \right)_{\theta=\theta_o} (\theta - \theta_o) \simeq f + \left(\frac{df_\alpha}{dy} \right)_{y=0} \frac{y}{R_\alpha} = f_{\alpha_o} + \beta_\alpha y \quad (7.13)$$

where $\beta_\alpha = \left(\frac{\partial f_\alpha}{\partial y} \right)_{y=0} = \frac{1}{R_\alpha} \left(\frac{df_\alpha}{d\theta} \right)_{\theta=\theta_o} = \frac{2\Omega}{R_\alpha} \cos \theta_o$ is the gradient of planetary vorticity in y -direction, $R_\alpha = R + z_\alpha$ with R being the earth's radius and $z_\alpha \ll R_\alpha$ is the distance from the earth to the beta-plane in the ionosphere.

The dipole magnetic field effects on the ionospheric Rossby waves

For a dipole magnetic field, the divergence of the electric flux density is constant

$$\nabla \cdot \mathbf{D} = \varrho_{el,\alpha} = \text{constant}. \quad (7.14)$$

Accordingly, the eastern and the northern components of the electric field vector per unit volume are respectively given by

$$E_{\text{east}} = \frac{\varrho_{el,\alpha}}{4\pi\epsilon} \frac{x - x_1}{[(x - x_1)^2 + (y - y_1)^2]^{3/2}} \quad (7.15)$$

and

$$E_{\text{north}} = \frac{\varrho_{el,\alpha}}{4\pi\epsilon} \frac{y - y_1}{[(x - x_1)^2 + (y - y_1)^2]^{3/2}}. \quad (7.16)$$

Therefore the electric field is conserved

$$\frac{\partial}{\partial y} E_{\text{east}} = \frac{\partial}{\partial x} E_{\text{north}} = -\frac{\partial^2 V}{\partial x \partial y} = -\frac{\partial^2 V}{\partial y \partial x}, \quad (7.17)$$

where V is the electric potential per unit volume and is given by

$$V = -\frac{1}{4\pi\epsilon} \frac{\varrho_{el,i}}{[(x - x_1)^2 + (y - y_1)^2]^{1/2}}. \quad (7.18)$$

Moreover due to the rotation of the earth there is an induction of a dipole magnetic field B_d (Parks, 1991) for which the vertical component is given by

$$B_d = -2B_o \frac{R^3}{(R + z_\alpha)^{5/2}} \sin \theta, \quad (7.19)$$

where B_o is the magnetic field measured at the Earth's surface.

Observe that from equations (7.10) and (7.11) we can define a Coriolis factor due the dipole magnetic field as

$$f_d = \varrho_{\alpha_o}^{-1} B_{\text{vertical}}. \quad (7.20)$$

For a beta plane, a similar analysis were done by Kaladze et al. (2004) and Kaladze et al. (2006). They showed that the Coriolis factor which is due to the dipole magnetic field can be approximated as

$$\begin{aligned} f_d &= -2\varrho_{\alpha_o}^{-1} B_o \frac{R^3}{R_\alpha^{5/2}} \sin \theta_o - 2\varrho_{\alpha_o}^{-1} B_o \frac{R^3}{R_\alpha^{5/2}} \cos \theta_o (\theta - \theta_o) \\ &= -2\varrho_{\alpha_o}^{-1} B_o \frac{R^3}{R_\alpha^{5/2}} \sin \theta_o - 2 \left[\varrho_{\alpha_o}^{-1} B_o \frac{R^3}{R_\alpha^{7/2}} \cos \theta_o \right] R_\alpha (\theta - \theta_o) \end{aligned} \quad (7.21)$$

$$= f_{d_o} + \beta_d y, \quad (7.22)$$

where $f_{d_o} = -2B_o \frac{R^3}{R_\alpha^{5/2}} \sin \theta_o$, $\beta_d = \left(\frac{\partial}{\partial y} f_d \right)_{y=0} = -2B_o \frac{R^3}{R_\alpha^{7/2}} \cos \theta_o$ and β_d is the gradient of planetary-magnetic dipole vorticity and again $R_\alpha = R + z_\alpha$.

The continuity equation (7.8) allows us to define a streamfunction for the planetary ionospheric disturbances. The streamfunction is defined in terms of the the horizontal components of the plasma velocity by $u_\alpha = -\Psi_{\alpha_y}$ and $v_\alpha = \Psi_{\alpha_x}$ and the plasma

vorticity is given by $\xi_\alpha = \Psi_{\alpha_{xx}} + \Psi_{\alpha_{yy}}$.

Since the x and z momentum equations are uncoupled with the y momentum equation, we can derive the vorticity equation for ionospheric Rossby waves in the horizontal plane. To do so, we differentiate equation (7.10) with respect to y and equation (7.11) with respect to x , subtract the first equation from the second and then combine with the continuity equation (7.8) to obtain the barotropic vorticity equation for the plasma,

$$\nabla^2 \Psi_{\alpha_t} + \Psi_{\alpha_x} \nabla^2 \Psi_{\alpha_y} - \Psi_{\alpha_y} \nabla^2 \Psi_{\alpha_x} + (\beta_\alpha \pm \beta_d) \Psi_{\alpha_x} - \nu \nabla^4 \Psi_\alpha = 0 \quad (7.23)$$

where the $\pm$ sign in the equation (7.23) indicates that the magnetic dipole field may change sign when the geomagnetic field B_o changes sign.

We can analytically solve equation (7.23) for simplified configurations where nonlinear and viscous effects are neglected. For example in (2014, manuscript in preparation), we derived an exact time-dependent solution for barotropic Rossby waves in long-wave limit (zero meridional to zonal aspect ratio). A similar procedure can be used to solve equation (7.23) (with $\nu = 0$).

We can also investigate a configuration where the electromagnetic field fluctuate randomly in time as in chapters 4 and 6.

7.2.2 Other future work

In this thesis we examined the interactions of the upward propagating atmospheric gravity waves with the ionosphere and limited our discussion on the interactions involving nonlinear effects arising from the interaction of waves and ionosphere which are generated by the Lorentz force. Several questions related to observations and measurements of gravity wave-ionosphere interactions were addressed but still other important questions remain. For example we could extend our discussion to include the nonlinear effects arising from wave-wave interactions.

In our investigation of EHD, MHD and EMHD waves, we limited our discussion to a configuration where the ionosphere is an electric dynamo, meaning that only DC

currents are generated in the medium. We derived expressions for the electric and magnetic field vectors for an electric dynamo. These are the solutions corresponding to the zero eigenvalue of the wave equations (4.9) and (4.10). On the other hand, the solutions corresponding to the nonzero eigenvalues represent regular (continuous) pulsation electromagnetic waves (Parks, 1991). Approximate solutions can be obtained using the WKB method. We have described these in more details in a manuscript that we are preparing for publication. These electromagnetic wave solutions could also be added to the ionospheric equations to investigate interactions between regular continuous pulsation waves and AGWs or RWs.

In our discussion we also included random (intermittent) ionospheric disturbances but limited our discussion to waves in a nonhomogeneous isotropic medium. It would be interesting to extend our investigation to wave propagation in a nonhomogeneous anisotropic medium and to include random variation in space.

The study of the interactions of the ionosphere with the atmosphere is not limited to gravity wave interactions and Rossby waves, but also includes tides (Lästovička and Bourdillon, 2004; Lästovička, 2006). This study can also be extended to the interactions of the tides with the ionosphere.

In geophysical flows when there is a shear in the mean flow, it sometimes happens that the phase velocity of the waves equals the mean flow velocity (Booker and Bretherton, 1967), the level at which this occurs is called the critical level or critical layer. In the critical layer the waves can be absorbed or reflected and thus exchange momentum and energy with the mean flow (Béland, 1976; Campbell, 2004; Campbell and Maslowe, 2003). It is important that we investigate ionospheric wave interactions across critical layers.

In this thesis we examined the cases where the collision frequencies $\nu_{\alpha n}$ and $\nu_{n\alpha}$ are both constants. It would also be important to examine the configurations where $\nu_{\alpha n}$ and $\nu_{n\alpha}$ vary with altitude or temperature.

Appendix A

Atmospheric gravity wave propagation

In this chapter the basic equations of fluid dynamics are described, and adapted to geophysical fluid flows. These are the momentum conservation (Navier-Stokes), mass conservation (continuity) and energy conservation equations. We derive the governing equations for the upward propagating atmospheric gravity waves, which sometimes are able reach the ionosphere and affect it dynamically, starting with the basic equations of fluid dynamics.

A.1 Basic equations of fluid dynamics

A.1.1 Conservation of mass, momentum and energy

In this section we present the basic equations of fluid dynamics which are based on the physical principles of conservation of mass, momentum and energy and then we describe the Boussinesq approximation which is a commonly used approximation for studying geophysical flows. The description of the equations given in this chapter follows chapter 2 of my Masters thesis (Victor 2010) and is based on the derivation of equations that can be found in fluid dynamics texts such as Batchelor (1967) and Kundu and Cohen (2004). We consider fluid with density ρ in some 3-dimensional

region in space which can be described in terms of rectangular coordinates x , y and z .

The principle of conservation of mass of a fluid leads to the continuity equation which is given in rectangular coordinates by

$$\frac{1}{\varrho} \frac{D\varrho}{Dt} + \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] = 0, \quad (\text{A.1})$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}, \quad (\text{A.2})$$

and u , v and w are the 3 components of the fluid velocity vector $\mathbf{u}$, e.i. $\mathbf{u} = (u, v, w)$.

In conditions where $\frac{D\varrho}{Dt}$ is enough small relative to the density ϱ , we can neglect the first term so that the continuity equation can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (\text{A.3})$$

The momentum equation for incompressible fluid flows is the Navier-Stokes equation

$$\varrho \frac{D\mathbf{u}}{Dt} = -\nabla p + \varrho \mathbf{g} + \mu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (\text{A.4})$$

where $\mathbf{f}$ is an exterior force or a superposition of two or more exterior forces that could include the Coriolis force, the centrifugal or centripetal force, and or a force due to electric field or magnetic field, depending on the problem that one needs to model. And $\mathbf{g} = (0, 0, -g)$ is the acceleration due to gravity and p the pressure. The term $\mu \nabla^2 \mathbf{u}$ represents the effects of viscosity or friction in the fluid, and μ is the dynamic viscosity coefficient.

In Cartesian coordinates, equation (A.4) is written as a system of equations

$$\varrho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u + \mathbf{f}_1, \quad (\text{A.5})$$

$$\varrho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \mu \nabla^2 v + \mathbf{f}_2 \quad (\text{A.6})$$

and

$$\varrho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} - \varrho g + \mu \nabla^2 w + \mathbf{f}_3. \quad (\text{A.7})$$

The conservation of energy is governed by

$$\varrho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} - p \nabla \cdot \mathbf{u} + \phi, \quad (\text{A.8})$$

where e represents the internal energy, ϕ is the viscous dissipation and $\mathbf{q}$ is the heat flux vector (per unit of area). At a constant volume the energy of a fluid is simply given by

$$e = C_V T, \quad (\text{A.9})$$

where $C_V = \left(\frac{\partial e}{\partial T}\right)_V$ is the specific heat capacity at constant volume V .

The thermodynamic state of a perfect gas is characterized by the equation of state

$$p = \mathcal{R} \varrho T, \quad (\text{A.10})$$

where $\mathcal{R} = C_p - C_V$ is a constant for a given perfect gas, and $C_p = \left(\frac{\partial h}{\partial T}\right)_p$ is the specific heat capacity at constant pressure p and h is the enthalpy of the fluid, see Kundu and Cohen (2004) for a detailed explanation.

A.1.2 The Boussinesq approximation

The Boussinesq approximation is a commonly used approximation for studying geophysical flows. It is described in details by Spiegel and Veronis (1960). Under the Boussinesq approximation the fluid density and temperature are treated as constant except in the terms involving the effects of gravity in the governing equations.

The assumptions made in the Boussinesq approximation are:

1. The variation of the density ϱ in time and space is small enough that we can set $\frac{1}{\varrho} \frac{D\varrho}{Dt} \approx 0$.
2. We also assume $\frac{\partial \varrho}{\partial p}$ is small, i.e. the fluid is almost incompressible or the density does not change much with the pressure p .

3. The vertical scale height is small enough that the density varies by only a small amount in the vertical direction.
4. The fluid properties such as μ the viscosity coefficient and $\mathcal{K}$ the thermal conductivity are assumed to be constant and the viscous dissipation ϕ can be neglected.

Under the Boussinesq approximation the term $\frac{1}{\rho} \frac{D\rho}{Dt}$ is considered small relative to the fluid acceleration, so we approximate the continuity equation by $\nabla \cdot \mathbf{u} = 0$. However, although we write the continuity equation as $\nabla \cdot \mathbf{u} = 0$ we can not justify neglecting $p\nabla \cdot \mathbf{u}$ term in the equation (A.8) because although the $\nabla \cdot \mathbf{u}$ is small the pressure can be large and the product $p\nabla \cdot \mathbf{u}$ can be of the same order of magnitude as the other terms in (A.8). So we keep the $p\nabla \cdot \mathbf{u}$ term in the equation (A.8) and rewrite it in terms of ρ and T .

We can write the continuity equation (A.1) as

$$\frac{1}{\rho} \frac{D\rho}{Dt} = -\nabla \cdot \mathbf{u}, \quad (\text{A.11})$$

assume that the density is both function of pressure and temperature and multiply both sides of the equation (A.11) by the pressure p to obtain

$$-p\nabla \cdot \mathbf{u} = \frac{p}{\rho} \frac{D\rho}{Dt} = \frac{p}{\rho} \left[\left(\frac{\partial \rho}{\partial T} \right) \frac{DT}{Dt} + \left(\frac{\partial \rho}{\partial p} \right) \frac{Dp}{Dt} \right]. \quad (\text{A.12})$$

For incompressible fluids, the second term in the equation (A.12) vanishes. Then

$$-p\nabla \cdot \mathbf{u} = \frac{p}{\rho} \frac{D\rho}{Dt} \approx \frac{p}{\rho} \left(\frac{\partial \rho}{\partial T} \right)_p \frac{DT}{Dt}, \quad (\text{A.13})$$

where the subscript p denotes differentiation of ρ with respect to T with the pressure being kept fixed.

Using the equation of state for a perfect gas (A.10), equation (A.13) can be written as,

$$-p\nabla \cdot \mathbf{u} \approx \frac{p}{\rho} \left(\frac{\partial}{\partial T} \left(\frac{p}{\mathcal{R}T} \right) \right)_p \frac{DT}{Dt} = \frac{p}{\rho} \frac{p}{\mathcal{R}} \left(-\frac{1}{T^2} \right) \frac{DT}{Dt} = -\mathcal{R}\rho \frac{DT}{Dt}. \quad (\text{A.14})$$

Substituting this into the equation (A.8) yields

$$\rho \frac{De}{Dt} = -\nabla \cdot \mathbf{q} - \mathcal{R} \rho \frac{DT}{Dt} + \phi. \quad (\text{A.15})$$

Using equation (A.9) and the fact that $\mathcal{R} = C_p - C_V$, equation (A.15) can be written as

$$C_p \rho \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \phi. \quad (\text{A.16})$$

According to Fourier's law of heat conduction the heat flux $\mathbf{q}$ is given by

$$\mathbf{q} = -\mathcal{K} \nabla T, \quad (\text{A.17})$$

where $\mathcal{K}$ is the thermal conductivity.

For geophysical flows under the Boussinesq approximation the ratio of the magnitude of viscous dissipation ϕ to the term $C_p \rho \frac{DT}{Dt}$ is extremely small ($\sim 10^{-7}$). This can be shown using dimensional analysis (Kundu and Cohen, 2004., p 120). We are thus justified in neglecting the viscous dissipation in the equation (A.16). We then substitute (A.17) into equation (A.16) to get

$$C_p \rho \frac{DT}{Dt} = -\nabla \cdot (-\mathcal{K} \nabla T) = \mathcal{K} \nabla^2 T. \quad (\text{A.18})$$

Rearranging the terms gives

$$\frac{DT}{Dt} = \alpha \nabla^2 T, \quad (\text{A.19})$$

where

$$\alpha = \frac{\mathcal{K}}{C_p \rho} \quad (\text{A.20})$$

is the thermal diffusivity.

A.2 Atmospheric gravity wave propagation governing equations

In this section we derive the governing equations for gravity waves propagating in the neutral atmosphere in a 2 dimensional rectangular configuration with a single horizontal coordinate x and a vertical coordinate z , starting from the momentum equations (A.5) and (A.7), the continuity equation (A.3) and the energy equation (A.19). We make use of the Boussinesq approximation described in section A.1.2.

The Momentum equation under the Boussinesq approximation

Under the Boussinesq approximation it is assumed that the total density $\varrho(x, z, t)$ is given by

$$\varrho(x, z, t) = \varrho_o + \varrho'(x, z, t), \quad (\text{A.21})$$

where ϱ_o is constant reference density and ϱ' is the variation of the density from this constant value. The amplitude of the density variations over the scale height is small, which means $\varrho'(x, z, t) \ll \varrho_o$. Substituting (A.21) into the momentum equations (A.5) and (A.7) and dividing both sides by ϱ_o yields

$$\left[1 + \frac{\varrho'(x, z, t)}{\varrho_o}\right] \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z}\right] = -\frac{1}{\varrho_o} \frac{\partial p}{\partial x} + \frac{1}{\varrho_o} \mu \nabla^2 u \quad (\text{A.22})$$

and

$$\left[1 + \frac{\varrho'(x, z, t)}{\varrho_o}\right] \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z}\right] = -\frac{1}{\varrho_o} \frac{\partial p}{\partial z} - \frac{\varrho}{\varrho_o} g + \frac{1}{\varrho_o} \mu \nabla^2 w. \quad (\text{A.23})$$

Since $\frac{\varrho'}{\varrho_o} \ll 1$, we can neglect the factor of $\frac{\varrho'}{\varrho_o}$ in the equations (A.22) and (A.23). In reality, ϱ_o changes with altitude z , so in the gravity term in (A.23), a substitution which is commonly used is to replace ϱ_o by the hydrostatic density $\bar{\rho}(z)$. This is the version of the Boussinesq approximation that we use here.

Of course these assumptions are only valid if it is assumed that the density $\bar{\rho}(z)$ does not change very rapidly with altitude, i.e. the range of altitudes must be of the

same order of magnitude as the scale height. Making these approximations leads to the Boussinesq equations

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho_o} \frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (\text{A.24})$$

and

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho_o} \frac{\partial p}{\partial z} - \frac{\rho}{\bar{\rho}} g + \nu \nabla^2 w, \quad (\text{A.25})$$

where $\nu = \frac{\mu}{\rho_o}$ is called the kinetic viscosity.

Under the Boussinesq approximation, the continuity equation in the xz -plane is simply $u_x + w_z = 0$. For a two dimensional fluid with velocity components in the x and z directions only, the only nonzero component of the vorticity is in the y direction and is given by $\xi = -\hat{\mathbf{y}} \cdot \nabla \times \mathbf{u} = w_x - u_z$. We can then define a streamfunction Ψ by $u = -\Psi_z$, $w = \Psi_x$ and combine equations (A.24) and (A.25) into a single equation for the streamfunction and the vorticity. To do so we differentiate (A.24) with respect to z and (A.25) with respect to x and subtract the first equation from the second to obtain

$$\xi_t - \Psi_z \xi_x + \Psi_x \xi_z + \frac{g}{\bar{\rho}} \frac{\partial \rho}{\partial x} - \nu \nabla^2 \xi = 0, \quad (\text{A.26})$$

where

$$\xi = \Psi_{xx} + \Psi_{zz} = \nabla^2 \Psi. \quad (\text{A.27})$$

is the vorticity in the y -direction ($\hat{\mathbf{y}}$).

A.3 The energy equation under the Boussinesq approximation

Under the Boussinesq approximation the energy equation (A.19) can be written in terms of the density. And Since the pressure is assumed to be almost constant under

the Boussinesq approximation, the equation of state (A.10) can be approximated as

$$\varrho T = \text{constant}. \quad (\text{A.28})$$

We can then write

$$\varrho T = \varrho_o T_o. \quad (\text{A.29})$$

A slight change in temperature T' induces a slight change in density ϱ' , in such a way $T = T_o + T'$ and $\varrho = \varrho_o + \varrho'$ and $\varrho' \ll \varrho$, $T' \ll T$. Making this substitution for ϱ and T into (A.29) yields

$$\varrho' T_o + \varrho_o T_o + \varrho' T' + \varrho_o T' = \varrho_o T_o. \quad (\text{A.30})$$

The third term on the left hand side is negligible compared with other terms since it is a product of perturbation quantities. Thus (A.30) becomes

$$\varrho' T_o = -\varrho_o T'. \quad (\text{A.31})$$

If we substitute for ϱ' and T' in the equation (A.31), use the fact that $\varrho - \varrho_o = \varrho'$ and $T - T_o = T'$, and rearrange the terms, we obtain

$$\varrho = \varrho_o \left[1 - \frac{1}{T_o} (T - T_o) \right]. \quad (\text{A.32})$$

So according to (A.32), there is a linear relation between the total density ϱ and the temperature T and so the energy equation (A.18) can be written in terms of the density as

$$\frac{D\varrho}{Dt} = \alpha \nabla^2 \varrho. \quad (\text{A.33})$$

This equation can be written in terms of the streamfunction and the density as

$$\varrho_t - \Psi_z \varrho_x + \Psi_x \varrho_z - \alpha \nabla^2 \varrho = 0. \quad (\text{A.34})$$

Our gravity wave model comprises equation (A.26) coupled with equation (A.34).

A.4 Direction of propagation of gravity waves

In this section we describe how to determine the direction of propagation of GWs. The procedure described here is due to Booker and Bretherton (1967). We consider the vorticity equation for AGWs in the neutral atmosphere (equation (3.18) with $\nu_{ni} = 0$)

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)^2 \nabla^2 \psi - \bar{u}''(z) \left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right) \psi_x + N^2 \psi_{xx} = 0, \quad (\text{A.35})$$

where $\bar{u}$ is the mean flow velocity, ψ is the streamfunction perturbation and N is the Brunt Väisälä frequency. The Laplacian operator is given by $\nabla^2 = \delta \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ where δ is the square of the aspect ratio as defined in section 2.4.

To derive the dispersion relation we consider the case where $\bar{u} = \text{constant}$ and write the solution of (A.35) as

$$\psi(x, z, t) = A e^{i(kx + mz - \omega t)}, \quad (\text{A.36})$$

where k and m are the horizontal and vertical wavenumbers and ω is the wave frequency. Substituting this into equation (A.35) yields

$$(\omega - \bar{u}k)^2 (\delta k^2 + m^2) - N^2 k^2 = 0. \quad (\text{A.37})$$

Solving for ω gives

$$\omega(k, m) = \bar{u}k \pm \frac{Nk}{(\delta k^2 + m^2)^{1/2}}. \quad (\text{A.38})$$

The group velocity is given by

$$\begin{aligned} \mathbf{c}_g &= \hat{\mathbf{x}} c_{gx} + \hat{\mathbf{z}} c_{gz} = \hat{\mathbf{x}} \frac{\partial \omega}{\partial k} + \hat{\mathbf{z}} \frac{\partial \omega}{\partial m} \\ &= \hat{\mathbf{x}} \left\{ \bar{u} \pm \frac{N[(\delta k^2 + m^2)^{1/2} - \delta k^2]}{(\delta k^2 + m^2)^{3/2}} \right\} + \hat{\mathbf{z}} \frac{\mp Nkm}{(\delta k^2 + m^2)^{3/2}} \end{aligned} \quad (\text{A.39})$$

while the horizontal phase speed is

$$c = \frac{\omega}{k} = \bar{u} \pm \frac{N}{(\delta k^2 + m^2)^{1/2}}. \quad (\text{A.40})$$

From (A.40), we have

$$\left(\frac{c - \bar{u}}{N}\right)^3 = \pm \frac{1}{(\delta k^2 + m^2)^{3/2}}. \quad (\text{A.41})$$

Thus we can write the vertical component of group velocity in terms of the horizontal phase speed

$$c_{gz} = \frac{(\bar{u} - c)^3 km}{N^2}. \quad (\text{A.42})$$

where

$$m = \sqrt{\frac{N^2}{(\bar{u} - \frac{\omega}{k})^2} - \delta k^2} = \sqrt{\frac{N^2}{(\bar{u} - c)^2} - \delta k^2}. \quad (\text{A.43})$$

For fixed ω and k there are two possible choices for the vertical wavenumber: m and $-m$. These choices give two possible solutions $e^{i(kx+mz-\omega t)}$ and $e^{i(kx-mz-\omega t)}$. We wish to determine which of these corresponds to an upward propagating wave and which corresponds to a downward propagating wave. The group velocity of the wave proportional to e^{-imz}

$$c_{gz} = \frac{(\bar{u} - c)^3 km}{N^2}$$

as given in (A.42). The group velocity of the wave proportional to e^{-imz} is

$$c_{gz} = \frac{\partial \omega}{\partial(-m)} = -\frac{(\bar{u} - c)^3 km}{N^2}$$

If we define m so that $\text{sgn}(m) = \text{sgn}(\bar{u} - c)$ then $k > 0$, $c_{gz}(m) > 0$ and $c_{gz}(-m) < 0$. So for $k > 0$, the wave e^{imz} is upward propagating, and the wave e^{-imz} is downward propagating. For $k < 0$, $c_{gz}(m) < 0$ and $c_{gz}(-m) > 0$. So for $k < 0$, the wave e^{imz} is downward propagating, and the wave e^{-imz} is upward propagating.

In the steady solution in chapter 3.1 where we have k and $\bar{u}$ positive and $c_x = 0$, we define m to be the positive square root of $\frac{N^2}{\bar{u}^2} - \delta k^2$. Thus, for $k = \kappa > 0$ the upward propagating wave is $e^{ikx+imz}$. For $k = -\kappa < 0$ the upward propagating wave is $e^{ikx-imz}$. These form a complex conjugate pair and their sum gives the real function ψ .

Appendix B

Basics of electromagnetic wave theory

In this chapter the basics of electromagnetism are discussed starting from electric and magnetic fields to electromagnetic waves following Cheng (1972) and Minoru and Fujimoto (2003). The theory of propagation of waves in a random medium is also described but not in details, a fascinating discussion of this topic can be found in Yeh and Liu (1972).

Electromagnetic waves are defined as a simultaneous propagation of both electric and magnetic fields in both space and time. The electric field is defined as the properties of a space around an electric charge while the magnetic field is defined as the properties of the space around a magnet or a wire with electric current. The fact that electric current generates a magnetic field was discovered by the Danish physicist Hans Christian Oersted in 1820's.

The electric charge is given by

$$q = n \times e, \tag{B.1}$$

where n is a natural number and $e = 1.6 \times 10^{-18}$ coulomb is the elementary charge. So the electric charge is quantized and is measured in Coulomb (C). The electric current intensity is the amount of charge passing through a wire per unit of time and is given

by

$$I = \lim_{\Delta t \rightarrow 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}, \quad (\text{B.2})$$

and measured in the units Ampère (A).

B.1 Time-independent electromagnetic fields

In this section we describe electric and magnetic fields which do not change with time but vary in space (Cheng, 1972; Minoru and Fujimoto, 2003).

B.1.1 Electric field and Coulomb's law

The electric field generated by an electric charge q at a point which is at position $\mathbf{r}$ from q is characterized by the electric field vector

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \frac{q}{r^2} \mathbf{n}, \quad (\text{B.3})$$

where $\mathbf{n} = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{\mathbf{r}}{r}$ and $\epsilon = \epsilon_o \epsilon_r(r, t)$ is the permittivity, $\epsilon_o = 8.654 \times 10^{12} \text{Faraday m}^{-1}$ is the permittivity of the free space and $\epsilon_r(\mathbf{r}, t)$ is the relative permittivity. In free space the permittivity is independent of location and time, and is $\epsilon(r, t) = 1$. The electric flux density is given by

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_o \epsilon_r(\mathbf{r}, t) \mathbf{E}. \quad (\text{B.4})$$

Another charge $\tilde{q}$ on position $\mathbf{r}$ from q feels some amount of force

$$\mathbf{F} = \tilde{q} \mathbf{E} = \frac{1}{4\pi\epsilon} \frac{\tilde{q}q}{r^2} \mathbf{n}. \quad (\text{B.5})$$

Equation (B.5) is known as Coulomb's law.

In free space, the electric potential is given by

$$\varphi = \frac{q}{4\pi\epsilon_o} \int_{-\infty}^r \frac{1}{\tilde{r}^2} d\tilde{r} = -\frac{q}{4\pi\epsilon_o} \frac{1}{r}. \quad (\text{B.6})$$

So the potential difference is

$$V = (\varphi_2 - \varphi_1) = \frac{q}{4\pi\epsilon_o} \int_{1 \rightarrow 2} \frac{1}{\tilde{r}^2} d\tilde{r} = -\frac{q}{4\pi\epsilon_o} \left(\frac{1}{r_2} - \frac{1}{r_1} \right). \quad (\text{B.7})$$

Therefore, the work done by the electric field $\mathbf{E}$ generated by the charge q to move the charge $\tilde{q}$ from position $\mathbf{r}_1$ to position $\mathbf{r}_2$ is given by

$$W_{1 \rightarrow 2} = \int_{1 \rightarrow 2} \mathbf{F} \cdot d\mathbf{r} = \tilde{q} \int_{1 \rightarrow 2} \mathbf{E} \cdot d\mathbf{r} = -\tilde{q}(\varphi_2 - \varphi_1) = -\tilde{q}V. \quad (\text{B.8})$$

Now let us consider an electric charge q at the center of the sphere of radius r . Then

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \frac{q}{4\pi} \oint_{\text{sphere}} \frac{r^2 d\vartheta}{r^2} = \frac{q}{4\pi} \oint_{\text{sphere}} d\vartheta = q, \quad (\text{B.9})$$

where ϑ is the spherical solid angle. Using the fact that the charge density (the charge per unit of volume V) is given by $\varrho_{el} = \frac{\partial q}{\partial V}$, equation B.9 becomes

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_V \varrho_{el} dv. \quad (\text{B.10})$$

Using Gauss' divergence theorem gives

$$\nabla \cdot \mathbf{D} = \varrho_{el}. \quad (\text{B.11})$$

This equation is known as Gauss' law.

If the relative permittivity is constant, $\epsilon = \text{constant}$, and Stokes' theorem is applied then,

$$\oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = 0, \quad (\text{B.12})$$

where $\mathcal{C}$ is the boundary of the surface S . Therefore

$$\nabla \times \mathbf{E} = 0, \quad (\text{B.13})$$

showing that the electric field is conserved.

B.1.2 Magnetic field and Ampere's law

The magnetic field generated by an electric wire at a point with position $\mathbf{r}$ in the space is characterized by the magnetic field vector

$$\mathbf{H} = \frac{1}{2\pi\mu} \frac{I}{r} \mathbf{n}_T, \quad (\text{B.14})$$

where $\mu(r, t) = \mu_o \mu_r(\mathbf{r}, t)$ is the magnetic permeability (or simply permeability), $\mu_o = 4\pi 10^{-7} \text{Henry m}^{-1}$ is the permeability in free space, $\mu_r(\mathbf{r}, t)$ is the relative permeability, I is the electric current and $\mathbf{n}_T$ is a vector tangential to the magnetic field. The force per unit of length that is felt by another wire with electric current $\tilde{I}$ at the position $\mathbf{r}$ is given by the cross product

$$\mathbf{F}/l = \tilde{I} \times \mathbf{B}, \quad (\text{B.15})$$

where $\mathbf{B}$ is the magnetic flux density. Equation (B.15) tells us that the direction of the electric current is perpendicular to that of the magnetic field since the direction of the vector $\mathbf{B}$ is tangential to the line of the magnetic field. Accordingly, the direction of the force $\mathbf{F}$ is obtained using the right hand rule.

The magnetic flux density $\mathbf{B}$ is given

$$\mathbf{B} = \mu \mathbf{H}. \quad (\text{B.16})$$

From equation (B.15) and according to the definition of the electric current, equation (B.2), we obtain the Lorentz force

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \quad (\text{B.17})$$

where the magnetic field is seen to be moving with velocity $\mathbf{v}$.

Let $\mathbf{J}$ be the volume current density (or simply current density), i.e. the current

per unit of surface over some open surface S with area A , then

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r} = \frac{I}{2\pi} \int_0^{2\pi} \frac{r d\theta}{r} = I = \int_A \mathbf{J} \cdot \mathbf{n} dS, \quad (\text{B.18})$$

where $\mathcal{C}$ is the boundary of the surface S and $\mathbf{n}$ is a vector normal to the surface S . This is known as the integral form of Ampere's law.

Now using Stokes' theorem together with equation (B.16) gives the differential form of Ampere's law

$$\nabla \times \mathbf{H} = \mathbf{J}. \quad (\text{B.19})$$

Since $\mathbf{B}$ is tangential to the concentric lines of the magnetic field we have that

$$\nabla \cdot \mathbf{B} = 0. \quad (\text{B.20})$$

On the other hand, if the surface S in B.18 is closed and $\mathcal{V}$ is the volume enclosed by the surface S then

$$\oint_S \mathbf{J} \cdot \mathbf{n} dS = I = \frac{dq}{dt} = -\frac{d}{dt} \iiint_{\mathcal{V}} \varrho_{el} dv = -\iiint_{\mathcal{V}} \frac{\partial \varrho_{el}}{\partial t} dV. \quad (\text{B.21})$$

Using the Gauss' divergence theorem yields the continuity equation

$$\frac{\partial \varrho_{el}}{\partial t} = -\nabla \cdot \mathbf{J}. \quad (\text{B.22})$$

This equation means that the electric charge is conserved anywhere in the electric wire.

So far we have all the necessary ingredients to describe time-independent electromagnetic fields which are the time-independent Maxwell's equations

$$\nabla \times \mathbf{E} = 0, \quad (\text{B.23})$$

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad (\text{B.24})$$

$$\nabla \cdot \mathbf{D} = \varrho_{el} \quad (\text{B.25})$$

and

$$\nabla \cdot \mathbf{B} = 0, \quad (\text{B.26})$$

where again $\mathbf{E}$ is the electric field vector, $\mathbf{B}$ is the magnetic field vector and $\mathbf{J}$ is the current density. In the next section, we describe the generalization of Maxwell's equations to time-dependent electromagnetic fields.

B.2 Time-dependent electromagnetic fields and Maxwell's equations

B.2.1 Faraday's law of induction and Maxwell's equations

The physicist Maxwell was the first to include time-dependence in the description of electromagnetic fields. The great physicist Faraday in 1830's discovered the way to induce electric power from the magnetic field. The process of generating electric power using magnetic fields is known as Faraday's law of induction.

The induced electromotive force on some surface S in the space is thus given by

$$\mathcal{E}_{ind} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \oint_S \mathbf{B} \cdot d\mathbf{S}, \quad (\text{B.27})$$

where Φ is the magnetic flux and $\mathbf{B}$ is the magnetic flux density. Let the surface S be bounded by the curve $\mathcal{C}$, then

$$-\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} = \oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r}. \quad (\text{B.28})$$

Using Stokes' theorem yields

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \quad (\text{B.29})$$

Now using the fact that the magnetic field is changing in time, the electric current is given by $I = I_{static} + I_{induced}$ where I_{static} is obtained using equation (B.18), while

$I_{induced}$ is obtained using equation (B.9) so that

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r} = I = I_{static} + I_{induced} = \oint_S \mathbf{J} \cdot d\mathbf{S} + \frac{d}{dt} \oint_S \mathbf{D} \cdot d\mathbf{S} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}, \quad (\text{B.30})$$

where $\frac{\partial \mathbf{D}}{\partial t}$ is called the displacement current.

Again Stokes' theorem gives

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad (\text{B.31})$$

Equations (B.29) and (B.31) together with (B.25) and (B.26) completely describe the electromagnetic field in both space and time. They are known as Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{B.32})$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad (\text{B.33})$$

$$\nabla \cdot \mathbf{D} = \rho_{el} \quad (\text{B.34})$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{B.35})$$

B.2.2 Electric and magnetic polarizations

In the presence of an electromagnetic field, the microscopic properties of the medium may change. Microscopic charges associated distributions of molecules and atoms are distorted, consequently producing electric and magnetic dipole moments. Electric and magnetic dipole moments are characterized by electric and magnetic polarization vectors. The electric flux density $\mathbf{E}$ can be expressed as

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_o \epsilon_r \mathbf{E} = \epsilon_o (1 + \chi) \mathbf{E} = \epsilon_o \mathbf{E} + \mathbf{P}, \quad (\text{B.36})$$

where $\mathbf{P}$ is the electric dipole moment and $\chi = \epsilon - 1$ is the electric susceptibility of the electric field $\mathbf{E}$. The electric dipole moment $\mathbf{P}$ is, in term of the electric field vector $\mathbf{E}$, given by

$$\mathbf{P} = \epsilon_o(\epsilon_r - 1)\mathbf{E} = \epsilon_o\chi\mathbf{E}. \quad (\text{B.37})$$

Similarly the magnetic flux density $\mathbf{B}$ can be expressed as

$$\mathbf{B} = \mu\mathbf{H} = \mu_o\mu_r\mathbf{H} = \mu_o(1 + \chi_m)\mathbf{H} = \mu_o\mathbf{E} + \mathbf{M}, \quad (\text{B.38})$$

where $\mathbf{M}$ is the magnetic dipole moment and $\chi_m = \mu_r - 1$ is the magnetic susceptibility of the magnetic field $\mathbf{H}$. The magnetic dipole moment $\mathbf{M}$ is, in term of the magnetic field vector $\mathbf{H}$, given by

$$\mathbf{M} = \mu_o(\mu_r - 1)\mathbf{H} = \mu_o\chi_m\mathbf{H}. \quad (\text{B.39})$$

Equations (B.36) and (B.38) mean that a slight change of electromagnetic field induces a slight change of the electric and magnetic susceptibilities. This is an important feature that allows us to characterize small changes in the properties of a medium under small changes in the electromagnetic field.

B.3 Electromagnetic wave equations in a homogeneous medium

In electromagnetism, a homogeneous medium is understood to mean a medium in which both the dielectric constant and the magnetic permeability ϵ and μ are constant, independent of space and time. Free space is a typical example of a homogeneous medium.

Now assuming ϵ and μ are constants and applying the curl to equation (B.32) gives

$$\nabla \times \nabla \times \mathbf{E} = \frac{\partial \nabla \times \mathbf{B}}{\partial t}. \quad (\text{B.40})$$

Now using the fact that $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ implies that

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \frac{\partial}{\partial t} \nabla \times \mathbf{B}. \quad (\text{B.41})$$

Combining this with equation (B.33) and (B.34) gives

$$\nabla \rho_{el} - \nabla^2 \mathbf{E} = \mu \frac{\partial}{\partial t} \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right). \quad (\text{B.42})$$

If we consider a source-free region where $\rho_{el} = 0$ and $\mathbf{J} = 0$, and substitute $\mathbf{D} = \epsilon \mathbf{E}$ into (B.42) gives the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad (\text{B.43})$$

where

$$c = \frac{1}{\sqrt{\epsilon_o \mu_o}} \frac{1}{\sqrt{\epsilon_r \mu_r}} = \frac{c_o}{n},$$

and $n = \sqrt{\epsilon_r \mu_r}$ is called the index of refraction and $c_o = \frac{1}{\sqrt{\epsilon_o \mu_o}}$ is the celerity of light in free space.

Applying the same procedure to equation (B.33) yields a similar equation for $\mathbf{H}$,

$$\nabla^2 \mathbf{H} - \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0. \quad (\text{B.44})$$

One can obtain $\mathbf{E}$ from (B.43) and apply (B.33) to get $\mathbf{H}$, or on the other hand use (B.44) to solve for $\mathbf{H}$ and then compute $\mathbf{E}$ from (B.32).

B.4 Energy, Poynting vector and power

Wave propagation involves transportation of momentum and energy (Cheng, 1992). I already discussed momentum in terms of the force due to the electric field and Lorentz force. I supposed entities of charges moving from place to place or charges being in electrostatic and magneto-static fields. Instead of analyzing the dynamics of these particles, let us focus on the effects on the environment. Observation shows that the

particles emit a spectrum of energy per unit of surface and unit time, i.e a flux of energy per unit of time. This flux of energy per unit of time is called the Poynting vector and is given by the Poynting theorem (Cheng, 1992). A typical example of electromagnetic waves is the light emitted by a lamp in a room. If the lamp has a high power the room becomes hot.

In order to derive the Poynting theorem we take the scalar product of $\mathbf{H}$ with equation (B.32) and the scalar product of $\mathbf{E}$ with equation (B.33) and subtract the second of the resulting equations from the first to obtain

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = - \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right) - \mathbf{E} \cdot \mathbf{J}. \quad (\text{B.45})$$

Using the fact that

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = - \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right) - \mathbf{E} \cdot \mathbf{J},$$

and applying the divergence theorem yields

$$\iiint_V \left(\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right) dV = \oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} - \iiint_V \mathbf{E} \cdot \mathbf{J} dV. \quad (\text{B.46})$$

This called the Poynting theorem. The vector $\mathbf{E} \times \mathbf{H}$ is the Poynting vector. If there is no loss of energy, the left hand side of equation (B.46) represents the rate of change of the magnetic and electric energies while the second term on the right hand is the power emitted by the source. Thus (B.46) represents the energy conservation in the absence of loss of energy.

Appendix C

Mathematical background

In this chapter some probability theory background important for the understanding of stochastic partial differential equations (SPDEs) is described. We focus on Gaussian processes, related important theorems and lemmas; we describe isonormal processes and martingale measures and use these to describe the stochastic integral and the Wiener chaos expansion (WCE). We mainly follow the books by Wilcox and Myers (1994), Brzeźniak and Zastawniak (2003), Mikosch (2004), Billingsley (1986) and the course by Dalang et al. (2000).

C.1 Basic probability theory concepts

In this section the description of some basic concepts in probability theory which are useful in the analysis of SPDEs is given. The references used are Zastawniak and Brzeźniak (2003), Mikosch (2004) and Billingsley (1986).

C.1.1 Random variables, probability measure and distribution

Random variables characterize the outcome of a random experiment. Let Ω be the set of the possible outcomes ω of some random experiment. The elements ω are called random outcomes. An event of a random experiment is a subset of Ω ; it can

be represented by the empty set $\emptyset$ if it does not occur, it could be Ω itself, or a union or an intersection of subsets of Ω . A random variable X is a function of the random outcomes $\omega \in \Omega$, i.e., $X = X(\omega)$.

Let us define the concept of a σ -field, and use that to define the concepts of Borel set, probability measure, distribution and density function.

Definition 4. Let Ω be a non-empty set. A σ -field $\mathcal{F}$ on Ω is a collection of subsets of Ω satisfying the following properties:

1. The empty set $\emptyset \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ then its complement $A^c \in \mathcal{F}$. This implies $\Omega = \emptyset^c \in \mathcal{F}$.
3. If $A_1, A_2, \dots$ are subsets in $\mathcal{F}$, then the union $A_1 \cup A_2 \cup \dots$ belongs to $\mathcal{F}$. From property 2, $(A_1 \cup A_2 \cup \dots)^c \in \mathcal{F}$, and since $(A_1 \cup A_2 \cup \dots)^c = A_1^c \cap A_2^c \cap \dots \in \mathcal{F}$ therefore $A_1 \cap A_2 \cap \dots$ also belongs to $\mathcal{F}$.

If for a given collection $\mathcal{C}$ of subsets of Ω , there exists a smallest σ -field $\sigma(\mathcal{C})$ on Ω containing $\mathcal{C}$, $\sigma(\mathcal{C})$ is called the σ -field generated by $\mathcal{C}$.

The smallest σ -field that contains all intervals in $\mathbb{R}$ is denoted by $\beta(\mathbb{R})$, and is called Borel set or Borel σ -field.

Example 5. Let $\Omega = \{a, b, c\}$. $\Sigma_1 = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$ and

$\Sigma_2 = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ are σ -fields on Ω . The σ -field generated by the subset $\{a\}$ is $\sigma\{\{a\}\} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} = \Sigma_1$. The σ -field that contains all subsets of Ω is the σ -field generated by the following collection of subsets of Ω , $\mathcal{C} = \{\{a\}, \{b\}, \{c\}\}$ that is given by

$$\Sigma_\sigma = \sigma(\mathcal{C}) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

The σ -field Σ_σ defines all possible events on the set Ω .

Definition 5. Let Ω be a non-empty set and $\mathcal{F}$ be a σ -field on Ω . A probability measure P is a function

$$P : \mathcal{F} \longrightarrow [0, 1]$$

such that:

1. $P(\Omega) = 1$ and $P(\emptyset) = 1 - P(\Omega) = 0$.

2. If $A_1, A_2 \dots$ are subsets in $\mathcal{F}$, then

$$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots - 2 \sum_{i \neq j} P(A_i \cap A_j)$$

If the sets $A_1, A_2 \dots$ are pairwise disjoint (i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$), then

$$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space. The subsets belonging to $\mathcal{F}$ are called events and are characterized by their measure P . An event A is said to occur almost surely (a.s) whenever $P(A) = 1$.

The probability measure P is also denoted as μ and μ is called the law of distribution or distribution.

Example 6. Let $\Omega = \{a, b, c\}$ as in the example 5. If an element of Ω is chosen randomly, then each element of Ω has equal chance to be chosen that is given by

$$P(\{a\}) = P(\{b\}) = P(\{c\}) = 1/3.$$

The probability of choosing c or b is

$$P(\{b, c\}) = P(\{b\}) + P(\{c\}) = 1/3 + 1/3 = 2/3.$$

The probability of choosing a or b or c is

$$P(\{a, b, c\}) = P(\Omega) = P(\{a\}) + P(\{b\}) + P(\{c\}) = 1.$$

Many more probabilities can be computed;

$$P(\{a\} \cap \{b, c\}) = P(\emptyset) = 0,$$

$$P(\{a\} \cap \{a, b, c\}) = P(\{a\}) = 1/3,$$

$$P(\{b, c\} \cap \{a, b, c\}) = P(\{b, c\}) = 2/3, \text{ and so}$$

$$P(\{a\} \cup \{b, c\} \cup \{a, b, c\}) = P(\{a, b, c\}) = P(\{a\}) + P(\{b, c\}) + P(\{a, b, c\})$$

$$- P(\{a\} \cap \{b, c\}) - P(\{a\} \cap \{a, b, c\}) - P(\{b, c\} \cap \{a, b, c\})$$

$$= 1/3 + 2/3 + 1 - 0 - 1/3 - 2/3 = 1.$$

Definition 6. Let $\Omega \subset \mathbb{R}$ and $\omega \in \Omega$. The collection of the probabilities

$$P(X \leq x) = P(\{\omega : X(\omega) \leq x\}) = \mu\{(-\infty, x)\}, \quad x \in \mathbb{R}$$

is called the *distribution of the random variable* $X = X(\omega)$ and is denoted as function of the *observation* x , $F_X : \mathbb{R} \longrightarrow [0, 1]$. The function F_X is called the *distribution function* of the random variable X . A distribution function $F_X(x)$ can be either continuous or discrete. If $X \in (a, b]$, then

$$P(\{\omega : a < X(\omega) \leq b\}) = \mu\{(a, b]\} = F_X(b) - F_X(a).$$

Moreover $F_X(x)$ is a non-decreasing function of x that satisfies $\lim_{x \rightarrow -\infty} F_X(x) = 0$ and $\lim_{x \rightarrow +\infty} F_X(x) = 1$.

Proof. Let $x \in \mathbb{R}$. Since $F_X(x) = \mu\{(-\infty, x)\}$, $x \rightarrow -\infty \Rightarrow \mu\{(-\infty, x)\} \rightarrow \mu\{\emptyset\} = 0$. Suppose $x_1 \leq x \leq x_2$ and $x_1, x_2 \in \mathbb{R}$. Then $(-\infty, x_1) \subseteq (-\infty, x) \subseteq (-\infty, x_2)$ and so $\mu\{(-\infty, x_1)\} \leq \mu\{(-\infty, x)\} \leq \mu\{(-\infty, x_2)\}$ because μ is a measure. Therefore $F_X(x)$ is non-decreasing in x . Moreover, $x \rightarrow \infty \Rightarrow \mu\{(-\infty, x)\} \rightarrow \mu\{\Omega\} = 1$. $\square$

Definition 7. Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable and suppose that $F_X(x)$ is continuous and differentiable with respect to $x \in \mathbb{R}$. The function $f_X(x) \geq 0$ defined on $\mathbb{R}$ that is given by

$$\frac{dF_X(x)}{dx} = f_X(x)$$

is called a *probability density function (p.d.f)* or *density function* of the random variable X .

In the context of general measures we have that

$$dP = d\mu = dF_X(x) = f_X dx, \tag{C.1}$$

where dx is called the Lebesgue measure. $d\mu$ is sometimes denoted as $\mu(dx)$, meaning that the measure $d\mu$ is a function of the Lebesgue measure dx and that the measure of a measure is also a measure.

Definition 8. An important continuous distribution is the normal or Gaussian distribution $N(\theta, \sigma^2)$ with parameters $\theta \in \mathbb{R}, \sigma^2 > 0$, where θ is the expectation of the random variable X and σ^2 its variance. It has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{(x - \theta)^2}{2\sigma^2} \right\}, \quad x \in \mathbb{R}. \quad (\text{C.2})$$

In the context of probability theory, $dP = d\mu = \mu(dx) = f_X dx$ is called the Gaussian measure with respect to the Lebesgue measure dx and f_X the density function of the distribution μ .

Definition 9. Let Ω be a set. If $\mathcal{F}$ is a σ -field on Ω , then a function $\xi : \Omega \longrightarrow \mathbb{R}$ is said to be $\mathcal{F}$ -measurable if

$$\xi \in B \quad \text{and} \quad B \in \mathcal{F}$$

for every Borel set $B \in \beta(\mathbb{R})$. If $(\Omega, \mathcal{F}, P)$ is a probability space, then such a function ξ is called a random variable.

Example 7. Consider the random variable ξ given by $\xi = F_X : \mathbb{R} \longrightarrow [0, 1]$, $\xi \in [0, 1]$ and $[0, 1] \in \beta([0, 1])$. Then $([0, 1], \beta([0, 1]), P)$ is the probability space for the random variable $\xi = F_X$. Therefore $\xi = F_X$ is measurable with respect to the Lebesgue measure $d\xi$. F_X has a uniform zero-one distribution, $U(0, 1)$. If $a \in [0, 1]$,

$$P\{\xi \leq a\} = \int_0^a d\xi = a.$$

Hence the distribution function F_X is a random variable.

Definition 10. A random variable $\xi : \Omega \longrightarrow \mathbb{R}$ is said to be integrable if the Lebesgue integral of $|\xi|$ with respect to the probability measure satisfies

$$\int_{\Omega} |\xi| dP < \infty.$$

Then

$$E(\xi) = \int_{\Omega} \xi dP$$

exists and is called the expectation of ξ .

The family of integrable random variable $\xi : \Omega \longrightarrow \mathbb{R}$ is denoted by L^1 or $L^1(\Omega, \mathcal{F}, P)$.

Definition 11. A random variable $\xi : \Omega \longrightarrow \mathbb{R}$ is said to be square-integrable if

$$\int_{\Omega} |\xi|^2 dP < \infty.$$

Then the variance of ξ is defined as

$$\text{var}(\xi) = \int_{\Omega} [\xi - E(\xi)]^2 dP.$$

The family of square-integrable random variables $\xi : \Omega \longrightarrow \mathbb{R}$ is denoted by L^2 or $L^2(\Omega, \mathcal{F}, P)$.

If $l \geq 1$ and

$$\int_{\Omega} |\xi|^l dP < \infty$$

then the l^{th} moments of X can be defined by

$$E(\xi^l) = \int_{\Omega} \xi^l dP.$$

So the first moment of X is its expectation.

Example 8. This example is a continuation of example 7. Let $\xi = F_X : \mathbb{R} \rightarrow [0, 1]$ be a random variable, then the l^{th} moment of ξ is

$$E(|\xi|^l) = E(\xi^l) = \int_0^1 \xi^l d\xi = \frac{1}{l+1}.$$

Therefore $\xi = F_X$ has finite moments, and hence the random variable $\xi = F_X$ is both

integrable and square-integrable.

Definition 12. Let $X : \Omega \rightarrow \mathbb{R}$ and $t \in \mathbb{R}$, then the characteristic function of X is

$$\varphi_X(t) = E(e^{ixt}) = \int_{\Omega} e^{ixt} \mu(dx), \quad t \in \mathbb{R}.$$

The characteristic function $\varphi_X(t)$ exists $\forall t \in \mathbb{R}$ since e^{ixt} is integrable, i.e., $E(|e^{ixt}|) = E(1) = 1 < \infty$. If the characteristic function is finite as $|t| \rightarrow \infty$, then the measure P has a density function, this is known as the Riemann-Lebesgue theorem (Billingsley, 1986). But before we state and prove the Riemann-Lebesgue theorem, let us first give the definition elementary and simple functions which are used in the proof of the Riemann-Lebesgue theorem.

Definition 13. A function $f : \mathbb{R}^N \times \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$ is elementary if

$$f(x, t, \omega) = X(\omega) \mathbb{I}_{(a,b]}(t) \mathbb{I}_A(x),$$

where $X : \Omega \rightarrow \mathbb{R}$ is bounded and $\mathcal{F}_a$ -measurable where $\mathcal{F}_a = \mathcal{F}_{t \leq a}$, and $A \in \beta(\mathbb{R}^N)$. Finite linear combinations of elementary functions are called simple functions.

Theorem 5. [Riemann-Lebesgue theorem] If the distribution μ has a density function, then $|\varphi_X(t)| \rightarrow 0$ as $|t| \rightarrow \infty$.

Proof. Let f be the density function of μ so that $\mu(dx) = f dx$. $\forall \epsilon > 0$ there exists a function g simple such that $\int |f - g| dx < \epsilon$.

$$\begin{aligned} \varphi_X(t) &= \int_{-\infty}^{\infty} e^{ixt} f(x) dx = \int_{-\infty}^{\infty} e^{ixt} [f(x) - g(x)] dx + \int_{-\infty}^{\infty} e^{ixt} g(x) dx \\ \Rightarrow |\varphi_X(t)| &\leq \int_{-\infty}^{\infty} |f(x) - g(x)| dx + \left| \int_{-\infty}^{\infty} e^{ixt} g(x) dx \right| = \epsilon + \left| \int_{-\infty}^{\infty} e^{ixt} g(x) dx \right|. \end{aligned}$$

Since $g(x)$ is simple we can approximate it as a sum of step functions, i.e., $g(x) = \sum_{k=1}^M c_k \mathbb{I}_{(a_k, b_k]}(x)$. Then

$$\int e^{ixt} g(x) dx = \int \sum_{k=1}^M c_k \mathbb{I}_{(a_k, b_k]}(x) e^{ixt} dx = \sum_{k=1}^M c_k \frac{e^{ib_k t} - e^{ia_k t}}{it}.$$

Now let $\alpha = \max_{k \leq M} c_k$, then

$$\left| \int e^{ixt} g(x) dx \right| \leq \frac{2\alpha M}{t} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Therefore $|\varphi_X(t)| \leq \epsilon$ and hence must vanish as $t \rightarrow \infty$. $\square$

C.1.2 Random vectors, expectation, covariance and independence

Definition 14. $\mathbf{X} = (X_1, \dots, X_n)$ is an n -dimensional random vector if its components $X_1, \dots, X_n$ are one-dimensional real-valued random variables.

Definition 15. Let $\mathbf{X}$ be a random vector. Then the collection of the probabilities

$$\mu\{(-\infty, x_1], \dots, (-\infty, x_n]\} = F_{\mathbf{X}}(x_1, \dots, x_n) = P(X_1 \leq x_1, \dots, X_n \leq x_n)$$

$$= P(\{\omega : X_1(\omega) \leq x_1, \dots, X_n(\omega) \leq x_n\}), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$$

is the distribution of the random vector $\mathbf{X}$.

If the distribution μ has density $f_{\mathbf{X}}$ according to Theorem 5, then the distribution function $F_{\mathbf{X}}$ is given by

$$F_{\mathbf{X}}(x_1, \dots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{\mathbf{X}}(y_1, \dots, y_n) dy_1 \cdots dy_n,$$

where $f_{\mathbf{X}}(\mathbf{x}) \geq 0 \quad \forall \mathbf{x} \in \mathbb{R}^n$

Definition 16. The expectation of a random vector $\mathbf{X} = (X_1, \dots, X_n)$ is defined as

$$E(\mathbf{X}) = (E(X_1), \dots, E(X_n)).$$

Definition 17. The covariance matrix of a random vector $\mathbf{X} = (X_1, \dots, X_n)$ is defined as

$$C = [\text{cov}(X_i, X_j); i, j = 1, \dots, n],$$

where

$$\text{cov}(X_i, X_j) = E[(X_i - EX_i)(X_j - EX_j)] = E(X_i X_j) - EX_i EX_j$$

is the covariance of X_i and X_j .

If $i = j$, then

$$\text{cov}(X_i, X_j) = \text{cov}(X_j, X_j) = E[(X_j - EX_j)^2] = EX_j^2 - E^2 X_j = \text{var}(X_j).$$

Definition 18. Two random variables $X_1 : \Omega \rightarrow \mathbb{R}$ and $X_2 : \Omega \rightarrow \mathbb{R}$ are independent if

$$P(X_1 \in B_1, X_2 \in B_2) = P(X_1 \in B_1)P(X_2 \in B_2)$$

for all subsets $B_1, B_2 \in \beta(\mathbb{R})$.

This can be generalized to n independent random variables and for any functions $g_1 \cdots g_n : \Omega \rightarrow \mathbb{R}$ or $\mathbb{C}$ as

$$E[g_1(X_1) \cdots g_n(X_n)] = E[g_1(X_1)] \cdots E[g_n(X_n)].$$

It follows that the characteristic function is

$$\varphi_{\mathbf{X}_n(t_1 \cdots t_n)} = \varphi_{X_1}(t_1) \cdots \varphi_{X_n}(t_n), \quad (\text{C.3})$$

where $t_1, \dots, t_n \in \mathbb{R}$, $\mathbf{X}_n = \sum_{j=1}^n X_j$ and $g_i = e^{ix_i t_i}$, $i = 1 \dots n$. This leads to the following definition.

Definition 19. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a random vector. The distribution of $\mathbf{X}$ is Gaussian if $\mathbf{t} \cdot \mathbf{X} = \sum_{i=1}^n t_i X_i$ is a Gaussian random variable (Definition 8) for all $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n$. There exists a n -dimensional vector Θ and a $n \times n$ symmetric nonnegative definite matrix C such that

$$\varphi_{\mathbf{X}_n(t_1 \dots t_n)} = \exp \left(i \mathbf{t} \cdot \Theta - \frac{1}{2} \mathbf{t} \cdot C \mathbf{t} \right). \quad (\text{C.4})$$

The following lemmas known as the first and the second Borel-Cantalli lemmas are important in probability theory and are given in books such as Billingsley (1986). The second concerns independent sequences of events.

Theorem 6. [First Borel-Cantalli lemma] Let Ω be a non-empty set, $\mathcal{F}$ a σ -field on Ω and $\{A_n\}$ a sequence of events in $\mathcal{F}$. If $\sum_i P(A_n)$ converges, then $P(\limsup_n A_n) = 0$.

Proof. Since $\limsup_n A_n \subset \cup_{k=m}^\infty A_k$ it follows that $P(\limsup_n A_n) \leq P(\cup_{k=m}^\infty A_k) \leq \sum_{k=m}^\infty P(A_k)$. And this sum approaches zero as $m \rightarrow \infty$ if $\sum_n P(A_n)$ converges. $\square$

We are going to state the second Borel-Cantalli lemma without proving it, we refer the reader to Billingsley (1986) for the detailed proof.

Theorem 7. [Second Borel-Cantalli lemma] Let Ω be a non-empty set, $\mathcal{F}$ a σ -field on Ω and $\{A_n\}$ a sequence of events in $\mathcal{F}$. If $\{A_n\}$ is an independent sequence of events and $\sum_n P(A_n)$ diverges, then $P(\limsup_n A_n) = 1$.

Two important inequalities are often used in basic probability theory; these are Markov's inequality and Tchebychev's inequality and are the following:

Markov's inequality: Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable with mean $E(X)$. Then for some $\epsilon > 0$

$$P(X \geq \epsilon) \leq \frac{E(X)}{\epsilon}. \quad (\text{C.5})$$

Proof. Suppose X is a continuous random variable with *p.d.f.* $f_X(x)$ then for some $\epsilon > 0$ we have $E(X) = \int_{x < \epsilon} x f_X(x) dx + \int_{x \geq \epsilon} x f_X(x) dx \geq \epsilon P(X \geq \epsilon)$. Hence $P(X \geq \epsilon) \leq \frac{EX}{\epsilon}$. $\square$

Tchebychev's inequality: Let $X : \Omega \rightarrow \mathbb{R}$ be a random variable with mean $E(X)$ and variance $var(X)$. Then for some $\epsilon > 0$

$$P(|X - E(X)| \geq \epsilon) \leq \frac{var(X)}{\epsilon^2}. \quad (C.6)$$

Proof. From Markov's inequality we have $P(X - EX \geq \epsilon) \leq \frac{E(X-EX)}{\epsilon}$ and this implies $P(X - EX \geq \epsilon) \leq P((X - EX)^2 \geq \epsilon^2) \leq \frac{E(X-EX)^2}{\epsilon^2} = \frac{var X}{\epsilon^2}$. Hence

$$P(|X - EX| \geq \epsilon) \leq \frac{var X}{\epsilon^2}. \quad (C.7)$$

$\square$

More generally for a given real or complex-valued function g we have

$$P(X \geq \epsilon) \leq P[g(X) \geq g(\epsilon)] \leq \frac{E[g(X)]}{g(\epsilon)}. \quad (C.8)$$

One can consider random variables which depend on a variable t representing time. These random variables are known as stochastic processes and will be the focus of this document.

Definition 20. A stochastic process $X : \Omega \times T \rightarrow \mathbb{R}$, where $T \subset \mathbb{R}$, is a collection of random variables

$$(X(\omega, t), t \in T, \omega \in \Omega),$$

defined on a set Ω .

For a fixed instant of time t , $X = X(\omega)$ is a random variable:

$$X(\omega, t) = X(\omega), \quad \omega \in \Omega.$$

For a fixed random outcome $\omega \in \Omega$, X is a function of time:

$$X = X(t), \quad t \in T.$$

$X(t)$ is called a realization, a trajectory or a sample path of the process X .

C.2 Gaussian process

In this section we follow Dalang et al (2000) to describe some important concepts that will be needed to define and analyze SPDEs, including brownian motion, white noise, isonormal processes, regularity of random processes and martingale measures.

Definition 21. Let $T \subset \mathbb{R}$ be a set and consider $\mathbf{G} = \{G(t, \omega)\}_{t \in T}$ a collection of random variables indexed by T . We may refer to $\mathbf{G}$ as either a random field or a stochastic process. $\mathbf{G}$ is a Gaussian process if $(G(t_1, \omega), \dots, G(t_n, \omega))$ is a n -dimensional Gaussian random vector of events $t_1, \dots, t_n$ (see Definition 19). The distribution of the process $\mathbf{G}$ is given by

$$\mu_{t_1, \dots, t_n}(A_1, \dots, A_n) = P\{G(t_1) \in A_1, \dots, G(t_n) \in A_n\}.$$

The mean and the covariance between events t_1 and t_2 are respectively $\theta(t) = E[G(t)]$ and $C(t_1, t_2) = \text{cov}[G(t_1), G(t_2)]$.

According to Khoshnevisan (2000) there is a theorem due to Herglotz, Bochner, Minilos which says that a nonnegative definite function $f : T \times T \rightarrow \mathbb{R}^n$ is symmetric $f(t_1, t_2) = f(t_2, t_1)$ for all $t_1, t_2 \in T$ iff f is a covariance function. (C.4) is an example. It implies that given a function $\Theta : T \rightarrow \mathbb{R}^n$ and a nonnegative definite function $C : T \times T \rightarrow \mathbb{R}^n \times \mathbb{R}^n$, there exists a Gaussian process $\{G(t)\}_{t \in T}$ whose the mean function is Θ and the covariance function is C .

Example 9. Suppose $T = \mathbb{R}_+$ and $\theta(t) = 0$ and $C(s, t) = \min(s, t)$. C is nonnegative definite.

Proof. consider $z_1, \dots, z_k \in \mathbb{C}$ and $t_1, \dots, t_k \geq 0$

$$\sum_{j=1}^k \sum_{l=1}^k \min(t_j, t_l) z_j \bar{z}_l = \sum_{j=1}^k \sum_{l=1}^k z_j \bar{z}_l \int_0^\infty \mathbb{I}_{[0, t_j]}(x) \mathbb{I}_{[0, t_l]}(x) dx = \int_0^\infty \left| \sum_{j=1}^k \mathbb{I}_{[0, t_j]}(x) z_j \right|^2 dx.$$
 Hence this is greater than zero. $\square$

Since C is symmetric it must be a covariance function of some zero mean Gaussian process, $B = \{B(t)\}_{t \geq 0}$. That process is called the Brownian process. The Brownian process has the Markov property.

Markov property Let $t \geq s \geq 0$ be fixed. Then the process $\{B(t+s) - B(s)\}_{t \geq 0}$ is independent of $\{B(u)\}_{0 \leq u \leq s}$.

Let us prove the Markov property.

Proof. To prove this it is sufficient to show that $\forall t \geq 0$ and $0 \leq u \leq s$, the covariance of $\{B(t+s) - B(s)\}_{t \geq 0}$ and $\{B(u)\}_{0 \leq u \leq s}$ is zero, i.e., $E[(B(t+s) - B(s))B(u)] = 0$. We have

$$E[(B(t+s) - B(s))B(u)] = E[B(t+s)B(u)] - E[B(s)B(u)] = \text{cov}[B(t+s), B(u)] - \text{cov}[B(s), B(u)] = \min(t+s, u) - \min(s, u) = u - u = 0,$$

where

$$\begin{aligned} \text{cov}[B(t), B(s)] &= \text{cov}(t, s) = E\{[(B(t) - B(s)) + B(s)]B(s)\} = E\{[B(t) - B(s)]B(s)\} + \\ &E[B^2(s)] = 0 + s = s = \min(s, t). \end{aligned} \quad \square$$

Definition 22. A Brownian process $B = \{B(t)\}_{t \geq 0}$ is a Gaussian process which has the following properties:

1. the mean $E[B(t)] = 0$ and the covariance $C(s, t) = \text{cov}[B(s), B(t)] = \min(s, t)$ for some $t \geq 0$ and $s \geq 0$,
2. and given t and s such that $t \geq s \geq 0$, the process $B = \{B(u)\}_{0 \leq u \leq s}$ is independent of the process $\{B(t+s) - B(s)\}_{t \geq 0}$ (Markov property).

White noise

Definition 23. Let $t \in \mathbb{R}_+$ and $A, B \in \beta(\mathbb{R}^N)$ where $\beta(\mathbb{R}^N)$ is the collection of all Borel-measurable subsets of $\mathbb{R}^N$. Let $C(A, B) = \lambda^N(A \cap B)$ be the covariance

of some Gaussian process on $\mathbb{R}^N$, where λ^N denotes the N -dimensional Lebesgue measure. C is symmetric since $C(A, B) = \lambda^N(A \cap B) = \lambda^N(B \cap A) = C(B, A)$. C is also nonnegative definite because it is the covariance of some Gaussian process. The resulting Gaussian process $W = \{W(A; t)\}$, where $A \in \beta(\mathbb{R}^N)$ and $t \in \mathbb{R}_+$, is called white noise.

If A is fixed the white noise $W = W(t)$ is a function of t only and can be expressed in terms of the Brownian process $\{B(t)\}_{t \geq 0}$ as follows

$$W(t) = \frac{B(t+h) - B(t)}{h}, t \geq 0 \quad (\text{C.9})$$

where $h > 0$ is some fixed constant.

C.2.1 The isonormal process

Let W denote white noise on $\mathbb{R}^N$ and define $W(h; t)$ where $h : A \rightarrow \mathbb{R}$ is a $\mathbb{C}_0^\infty$ function for all $A \in \beta(\mathbb{R}^N)$. We can also denote $W(h; t)$ by $W(A; t)$. Let W_t be the partial derivative of W with respect to time t . Consider disjoint sets $A_1, \dots, A_k \in \beta(\mathbb{R}^N)$ so that $W(A_1; t), \dots, W(A_k; t)$ are independent. Then there exists $c_1, \dots, c_k \in \mathbb{R}$ such that $W_t \left(\sum_{j=1}^k c_j \mathbb{I}_{A_j}; t \right) = \sum_{j=1}^k c_j W_t(A_j; t)$. If $h \in L^2(\mathbb{R}^N)$ there exists a function h_n of the form $\sum_{j=1}^{k(n)} c_{j,n} \mathbb{I}_{A_{j,n}}$ such that $A_{1,n}, \dots, A_{j,n} \in \beta(\mathbb{R}^N)$ are disjoint and $\|h - h_n\|_{L^2(\mathbb{R})} \rightarrow 0$ as $n \rightarrow \infty$. Moreover $\left\| W_t \left(\sum_{j=1}^k c_j \mathbb{I}_{A_j}; t \right) \right\|^2 = \sum_{j=1}^k c_j^2 |A_j| = \left\| \sum_{j=1}^k c_j \mathbb{I}_{A_j} \right\|^2$. Hence $\{W_t(h_n; t)\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(P)$.

Definition 24. The limit of $\{W_t(h_n; t)\}_{n=1}^\infty$ as $n \rightarrow \infty$ that is given by $W_t(h; t)$ is called the Wiener integral of $h \in L^2(\mathbb{R})$

Sometimes it is written as

$$W_t(h; t) = \int h dW = \int h W_t(h; t) dt. \quad (\text{C.10})$$

The white noise $W_t(h)$ has the following key feature

$$\|W_t(h; t)\| = \|h\|_{L^2(\mathbb{R}^N)} \quad (\text{C.11})$$

known as the isometric property. Therefore $W_t : L^2(\mathbb{R}^N) \longrightarrow L^2(P)$ is an isometry and is called the Wiener's isometry. The process $\{W_t(h; t)\}_{h \in L^2(\mathbb{R}^N)}$ is called the isonormal process. Its mean function is zero and its covariance function is given by the $L^2(\mathbb{R}^N)$ inner product for all $h, g \in L^2(\mathbb{R}^N)$.

$$C(h, g) = \langle W_t(h; t), W_t(g; t) \rangle = \int_{\mathbb{R}^N} h(x)g(x)dx. \quad (\text{C.12})$$

Proposition 1. *If $\{h_j\}_{j=1}^\infty$ is a complete orthonormal system in $\mathbb{R}^N$ then so is $\{W_t(h_j; t)\}_{j=1}^\infty$ in $L^2(P)$. For all Gaussian random variables $Z \in L^2(P)$ measurable with respect to the white noise*

$$Z = \sum_{j=1}^{\infty} a_j W_t(h_j; t) \text{ a.s., with } a_j = \text{cov}[Z, W_t(h_j; t)].$$

Proof. Using the isometric property leads

$$\langle W_t(h_i; t), W_t(h_j; t) \rangle = \langle h_i, h_j \rangle = \delta_{i,j}. \quad (\text{C.13})$$

Now let us define $\text{Proj } Z = E(Z|A_j) = \text{cov}(Z, W_t(h_j; t))$ where $A_j \in \beta(\mathbb{R}^N)$. Then,

1. $\text{Proj } Z = E(Z|A_j) = E[E(Z|A_j)|A_j] = \text{Proj}^2 Z$ (Riesz representation theorem).
2. $\langle E(Z|A_j), Z - E(Z|A_j) \rangle = E\{(Z|A_j)[Z - E(Z|A_j)]\} = E\{E[E(Z|A_j)(Z - E(Z|A_j))]|A_j\} = E[E(Z|A_j)]E\{[Z - E(Z|A_j)]|A_j\} = E(0) = 0$.

Therefore the projection given by $a_j = \text{cov}[Z, W_t(h_j; t)]$ is orthonormal. Hence $L^2(P)$ is a Hilbert space of square-integrable random variables on $(\Omega, \mathcal{F}, P)$. $\square$

C.2.2 Regularity of a random process

Definition 25. Let X and X' be two stochastic processes indexed by some set $T \subset \mathbb{R}$. We say that X' is a modification of X if $P\{X'(t) = X(t)\} = 1$ for all $t \in T$. If $X(t)$ is continuous in t , then $X'(t)$ is called a continuous modification.

An important theorem by Wiener states that we can always find a continuous modification of a Brownian motion which is also a Brownian motion. Thus a Wiener process is defined as a Brownian motion such that the random function $t \longrightarrow B(t)$ is continuous. Hence a Wiener process is also known as the standard Brownian motion.

Kolmogorov's continuity theorem

When does a stochastic process have a continuous modification?. A theorem that gives a sufficient condition for a stochastic process to have a continuous modification is due to Kolmogorov and is the following:

Theorem 8. [Kolmogorov's continuity theorem] Suppose $\{X(t)\}_{t \in T}$ is a stochastic process indexed by a compact cube $T = [a_1, b_1] \times \dots \times [a_N, b_N] \subset \mathbb{R}^N$. Suppose that there exist constants $c > 0, p > 0$ and $\gamma > N$ such that uniformly for all $s, t \in T$

$$E[|X(t) - X(s)|^p] \leq c|t - s|^\gamma.$$

Then X has a continuous modification X' . Moreover if $0 \leq \theta < \frac{\gamma - N}{p}$ then

$$\left\| \sup_{s \neq t} \frac{|X'(t) - X'(s)|}{|s - t|^\theta} \right\|_{L^p(P)} < \infty.$$

The proof to this theorem can be found in Khoshnevisan (2000).

Definition 26. [Hölder continuity] A function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is said to be globally Hölder continuous with index α if there exists a constant A such that for all $x, y \in \mathbb{R}^N$

$$|f(x) - f(y)| \leq A|x - y|^\alpha.$$

It is said to be [locally] Hölder continuous with index α if for all compact sets $K \subset \mathbb{R}^N$ there exists for all $x, y \in \mathbb{R}^N$ a constant A_K such that

$$|f(x) - f(y)| \leq A_K |x - y|^\alpha.$$

The Wiener process is a Gaussian random process, it has a modification that is Hölder continuous of any given order $\alpha < 1/2$. This result is due to Wiener and can be found in Dozzi (2002).

C.3 Martingale measures

In this section we mainly follow Mikosch (2004), Billingsley (1986) and the course by Dalang et al. (2000).

C.3.1 Martingales

Definition 27. *Let Ω be a nonempty set. A collection $(\mathcal{F}_t, t \geq 0)$ of σ -fields on Ω is called a filtration if*

$$\mathcal{F}_s \subset \mathcal{F}_t \quad \forall 0 \leq s \leq t.$$

This makes a filtration an increasing stream of information.

If $(\mathcal{F}_n, n = 0, 1, \dots)$ is a sequence of σ -fields on Ω and $\mathcal{F}_{n+1} \supset \mathcal{F}_n$ for all n , $\mathcal{F}_n$ is also called a filtration.

And a stochastic process $X = (X(\omega, t), t \geq 0)$ is said to be adapted to the filtration $(\mathcal{F}_t, t \geq 0)$ if

$$\sigma(X) \subset \mathcal{F}_t \quad \forall t \geq 0.$$

The stochastic process X is always adapted to the natural filtration generated by X , $\mathcal{F}_t = \sigma(X(\omega, s), s \geq t)$. This simply means that X is $\mathcal{F}_t$ -measurable.

Definition 28. [*Martingale*] A stochastic process $X = (X(\omega, t), t \geq 0)$ is called a continuous-time martingale with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ if

1. X is $\mathcal{F}_t$ -measurable.
2. $E(|X|) < \infty \forall t \geq 0$ (X is integrable).
3. $E[X(\omega, t)|\mathcal{F}_s] = E[X(\omega, s)] \forall 0 \leq s < t$ a.s.

Definition 29. A stochastic process $X = (X_n, n = 0, 1, \dots)$ is called a discrete-time martingale with respect to the filtration $(\mathcal{F}_n, n = 0, 1, \dots)$ if

1. X_n is $\mathcal{F}_n$ -measurable.
2. $E(|X_n|) < \infty \forall n \geq 0 \in \mathbb{N}$ (X_n is integrable).
3. $E(X_{n+1}|\mathcal{F}_n) = E(X_n) \forall n \in \mathbb{N}$ a.s.

Example 10. The Brownian motion $B = (B(t), t \geq 0)$ is martingale in the sense that $(B(t), t \geq 0) \in \sigma(B(t), t \geq 0) \subset \mathcal{F}_t$, $E(|B|) \geq |E(B)| = 0 < \infty$ and $E[B(t)|\mathcal{F}_s] = E(B(s))$ almost surely.

Definition 30. [*Submartingale*] $Y = (Y(\omega, t \geq 0))$ is $\mathcal{F}_t$ -submartingale if Y is $\mathcal{F}_t$ -measurable; Y is integrable and $E(Y|\mathcal{F}_s) \geq E[Y(\omega, s)]$.

The white noise $W = (W(t), t \geq 0)$ is martingale since $B(B(t), t \geq 0)$ is martingale.

Let $W = W(A; t)$ be white noise on $\mathbb{R}^N$, we can verify that $W_t = W_t(A; t)$ has the following properties:

1. $W_t(\emptyset; t) = 0$ a.s.
2. For all disjoint (nonrandom) sets $A_1, A_2, \dots \in \beta(\mathbb{R}^N)$,

$$P \left\{ W_t \left(\bigcup_{i=1}^{\infty} A_i; t \right) = \sum_{i=1}^{\infty} W_t(A_i; t) \right\} = 1,$$

where the infinite sum converges in $L^2(P)$.

The following proposition is then necessary (Khoshnevisan, 2000),

Proposition 2. *White noise is an $L^2(P)$ -valued σ -finite signed measure.*

By finite we mean that the measure is not allowed to equal infinity (always $< \infty$), σ stands for the fact that P is a σ -field while signed means that the measure is allowed to be negative (< 0).

Proof. We need to show these two statement hold:

1. If $A_1 \supset A_2 \supset \dots$ are all in $\beta(\mathbb{R}^N)$ and $\bigcap A_n = \emptyset$, then for some fixed $t > 0$ $W_t(A_n; t) \rightarrow 0$ as $n \rightarrow \infty$ in $L^2(P)$.
2. For all compact sets K , $E[(W_t(K; t))^2] < \infty$.

We prove them as follows:

1. Since $E[(W_t(A_n; t))^2] = t|A_n|$ where $|A_n|$ is the Lebesgue measure of A_n , and $|A_n| \rightarrow 0$, therefore $W_t(A_n; t) \rightarrow 0$ in $L^2(P)$ because Lebesgue measure is a measure.
2. For some fixed $t > 0$, $E[(W_t(K; t))^2] = t|K| < \infty$ because K is compact and so Lebesgue measure is σ -finite.

□

Let $\mathcal{F}$ be the filtration of the process $\{W_t\}_{t \geq 0}$, by filtration we understand the following: For all $t \geq 0$, $\mathcal{F}_t$ is a σ -field generated by $\{W_s(A) : 0 \leq s \leq t, A \in \beta(\mathbb{R}^{N-1})\}$.

Lemma 4. $\{W_t(A; t)\}_{t \geq 0}, A \in \beta(\mathbb{R}^{N-1})$ is a martingale measure in the sense that

1. For all $A \in \beta(\mathbb{R}^{N-1})$, $W_t(A; t = 0) = 0$ a.s.
2. If $t \geq 0$ then W_t is σ -finite, $L^2(P)$ -valued signed measure.
3. For all $A \in \beta(\mathbb{R}^{N-1})$, $\{W_t(A; t)\}_{t \geq 0}$ is a mean-zero martingale.

Proof. 1. $E[(W_t(A; t))^2] = t|A|$ where $|A|$ denotes the $(N - 1)$ -dimensional Lebesgue measure of A . Therefore $W_t(A; t = 0) = 0$

2. Similar proof as in Proposition 2.

3. Let $A \in \beta(\mathbb{R}^{N-1})$ be fixed. Then $\forall t, s, u \in \mathbb{R}$ such that $t \geq s \geq u \geq 0$, $E[(W_t(A; t) - W_s(A; s))W_u(A; u)] = E[(\dot{W}([0, t] \times A) - \dot{W}([0, s] \times A))\dot{W}([0, u] \times A)] = \min(t, u)|A| - \min(s, u)|A| = 0$. Therefore $W_t(A; t) - W_s(A; s)$ is independent of $\mathcal{F}_s$. Moreover $E[W_t(A; t)|\mathcal{F}_t] = E[(W_t(A; t) - W_s(A; s) + W_s(A; s))|\mathcal{F}_t] = E[(W_t(A; t) - W_s(A; s))|\mathcal{F}_t] + E[W_s(A; s)|\mathcal{F}_t] = 0 + E[W_s(A; s)|\mathcal{F}_t] = W_s(A; s)$. Hence $\{W_t(A; t)\}_{t \geq 0}$ satisfies the martingale property. $\square$

C.3.2 General Martingale measures

The description of the concepts of martingale measures is taken from Khoshnevisan (2000). Let $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$ be a filtration of σ -algebra and assume that $\mathcal{F}$ is right continuous, i.e., $\mathcal{F} = \bigcap_{s > t} \mathcal{F}_s \quad \forall t \geq 0$.

Definition 31. A process $\{M(A; t) : A \in \beta(\mathbb{R}^N)\}$ is a martingale measure with respect to $\mathcal{F}$ if

1. $M(A; 0) = 0 \quad a.s.$
2. If $t > 0$ then M is σ -finite, $L^2(P)$ -valued signed measure.
3. For all $A \in \beta(\mathbb{R}^N)$, $\{M(A; t)\}_{t \geq 0}$ is a mean-zero martingale with respect to the filtration $\mathcal{F}$.

Definition 32. Let φ denote the class of all simple functions. If M is a martingale measure and $f : \mathbb{R} \times \mathbb{R}_+ \times \Omega \rightarrow \mathbb{R}$, an elementary function of the form

$$f(x, t, \omega) = X(\omega)\mathbb{I}_{[a, b]}(t)\mathbb{I}_A(x),$$

where $X : \Omega \rightarrow \mathbb{R}$, $\omega \in \Omega$ is a random outcome and $x \in \mathbb{R}$. Then the stochastic

integral process of f is defined as

$$\begin{aligned} \iint_{[a,b] \times [A \cap B]} f(x, t, \omega) dM &= \iint X(\omega) \mathbb{I}_{[a,b]}(t) \mathbb{I}_{\{A \cap B\}}(x) dM \\ &= X(\omega) \iint_{[a,b] \times [A \cap B]} dM = X(\omega) [M(A \cap B; t \wedge b) - M(A \cap B; t \wedge a)](\omega), \end{aligned} \quad (\text{C.14})$$

where $t \wedge a = \min(t, a)$. If $t \leq a$, a is called the stopping time and $M(A; t \wedge a)$ is called a stopped martingale with respect to $\mathcal{F}_a = \mathcal{F}_{t \leq a}$.

Definition 33. Let $f(x, t, \omega)$ be a stochastic process and M be a martingale with respect to the filtration $\mathcal{F}$. The stochastic process denoted as $(f \cdot M)(B; t \in [0, T])(\omega)$ that is given by

$$(f \cdot M)(B; t \in [0, T])(\omega) = \iint_{[0, T] \times [A \cap B]} f(x, t, \omega) dM \quad (\text{C.15})$$

is called the transformed martingale of M by f and the transformed martingale $(f \cdot M)(B; t \in [0, T])(\omega)$ is a martingale.

Example 11. If f is an elementary function, then $(f \cdot M)(B; t \in [0, T])(\omega)$ is a stochastic integral process. We can verify that it is a martingale with respect to the filtration $\mathcal{F}$.

Definition 34. Let Ω be a nonempty set, φ be a space of functions that map Ω into $\mathbb{R}$ and $\mathcal{P}$ the σ -field generated by all functions in φ . Then $\mathcal{P}$ is called the predictable σ -field.

In order to define the stochastic integral for random functions the concept of worthiness on martingale measure is important. But let us first define what a covariance functional of a martingale is.

Definition 35. Let M be a martingale measure. The covariance functional of M is defined as

$$\overline{Q}_t(A, B) = \langle M(A; t), M(B; t) \rangle \quad \forall t \geq 0, A, B \in \beta(\mathbb{R}^N) \quad (\text{C.16})$$

and has the following properties:

1. $\overline{Q}_t(A, B) = \overline{Q}_t(B, A)$, almost surely;
2. If $A \cap C = \emptyset$ then $\overline{Q}_t(A, B \cup C) = \overline{Q}_t(A, B) + \overline{Q}_t(A, C)$, almost surely;
3. $|\overline{Q}_t(B, A)|^2 \leq \overline{Q}_t(A, A)\overline{Q}_t(B, B)$, almost surely; and
4. $t \mapsto \overline{Q}_t(A, A)$ is almost surely non-decreasing.

- Proof.*
1. $\overline{Q}_t(A, B) = \langle M(A; t), M(B; t) \rangle = \langle M(B; t), M(A; t) \rangle = \overline{Q}_t(B, A)$, almost surely.
 2. If $A \cap C = \emptyset$, $\overline{Q}_t(A, B \cup C) = \overline{Q}_t(A, B) + \overline{Q}_t(A, C) - 2\overline{Q}_t(A, B \cap C) = \overline{Q}_t(A, B) + \overline{Q}_t(A, C) - 2\langle M(A; t), M(\emptyset; t) \rangle = \overline{Q}_t(A, B) + \overline{Q}_t(A, C) - 2\langle M(A; t), 0 \rangle = \overline{Q}_t(A, B) + \overline{Q}_t(A, C)$, almost surely.
 3. The proof follows directly from the Cauchy-Schwartz inequality.
 4. $t \mapsto \overline{Q}_t(A, A) \iff t \mapsto \langle M(A; t), M(A; t) \rangle \iff t \mapsto |M(A; t)|_t^2$ which is a sub-martingale since M is a martingale.

□

For all $t \geq s \geq 0$ and $A, B \in \beta(\mathbb{R}^N)$ let

$$Q(A, B; (s, t]) = \overline{Q}_t(A, B) - \overline{Q}_s(A, B). \quad (\text{C.17})$$

If $A_i \times B_i \times (s_i, t_i]$ are disjoint then for $(1 \leq i \leq m)$ we have that

$$Q\left(\bigcup_{i=1}^m A_i, B_i; (s_i, t_i]\right) = \sum_{i=1}^m Q(A_i, B_i; (s_i, t_i]). \quad (\text{C.18})$$

This extends the definition of the definition of Q to rectangles. In general, one can go further in searching a complete theory of stochastic integrals but it is not an easy task at all. However this works well if M is worth.

Definition 36. [*Worthiness*] A martingale measure M is worthy if there exists a random σ -field finite measure $K(A \times B \times C, \omega)$ where $A, B \in \beta(\mathbb{R}^N)$, $C \in \beta(\mathbb{R}_+)$ and $\omega \in \Omega$ such that

1. $A \times B \longmapsto K(A \times B \times C, \omega)$ is nonnegative definite and symmetric.
2. $\{K(A \times B \times (0, t])\}_{t \geq 0}$ is a predictable (i.e., $\mathcal{P}$ -measurable) for $A, B \in \beta(\mathbb{R}^N)$.
3. For all compact sets $A, B \in \beta(\mathbb{R}^N)$ and $t > 0$

$$E[K(A \times B \times (0, t])] < \infty.$$

4. For all compact sets $A, B \in \beta(\mathbb{R}^N)$ and $t > 0$

$$|Q(A \times B \times (0, t])| < K(A \times B \times (0, t]).$$

Remark: If M is worthy then Q_M can be extended to a measure on $\beta(\mathbb{R}^N) \times \beta(\mathbb{R}^N) \times \beta(\mathbb{R}_+)$ (Khoshnevisan, 2000).

Proposition 3. Suppose $f \in \varphi$ and M is a worthy martingale measure and $(f \cdot M)_t$ is the stochastic integral process of f (Definition 32). Then

$$E[((f \cdot M)(B; t \in (0, T]))^2] = E \left[\iiint_{B \times B \times (0, T]} f(x, t) f(y, t) Q(dx, dy, dt) \right]. \quad (\text{C.19})$$

Although Q is not a proper measure the triple integral is well defined.

Proof. Since X is $\mathcal{F}_a$ -measurable,

$$\begin{aligned} E[((f \cdot M)(B; t \in [a, b])(\omega))^2] &= E[X^2(M(A \cap B; t \wedge b) - M(A \cap B; t \wedge a))^2] \\ &= E[X^2 M^2(A \cap B; t \wedge b)] - 2E[M(A \cap B; t \wedge b)M(A \cap B; t \wedge a)] \\ &\quad + E[M^2(A \cap B; t \wedge a)] \\ &= 2E[X^2 M^2(A \cap B; t \wedge a) - \langle M(A \cap B; t \wedge a), M(A \cap B; t \wedge a) \rangle]. \end{aligned} \quad (\text{C.20})$$

Moreover if f is simple, i.e., $f = \sum_{i=1}^k c_i f_i$ where $f_1 \cdots f_k$ are elementary and $c_1 \cdots c_k$ are real,

$$\begin{aligned}
E[((f \cdot M)(B; t)(\omega))^2] &= E \left[\left(\left(\sum_{i=1}^k c_i^2 f_i \right) \cdot M \right)^2 (B; t) \right] \\
&= E \left[\left(\sum_{i=1}^k c_i^2 (f_i \cdot M)^2 \right) (B; t) \right] \\
&\quad + 2E \left[\left(\sum_{i=1}^k \sum_{j=1, j \neq i}^k c_i c_j (f_i \cdot M)^2 (B; t) (f_j \cdot M)^2 (B; t) \right) \right] \\
&= E \left[\left(\sum_{i=1}^k c_i^2 (f_i \cdot M)^2 \right) (B; t) \right] \\
&\quad + 2 \sum_{i=1}^k \sum_{j=1, j \neq i}^k c_i c_j \overline{Q}((f_i \cdot M), (f_j \cdot M))(B; t) = E \left[\left(\sum_{i=1}^k c_i^2 (f_i \cdot M)^2 \right) (B; t) \right].
\end{aligned} \tag{C.21}$$

If the support for each of $f_1 \cdots f_k$ is disjoint to each other. Hence the triple integral is well defined. $\square$

Proposition 4. *If M is a worthy martingale and $f \in \varphi$ then $(f \cdot M)$ is a worth martingale measure. If Q_M and K_M respectively define the covariance functional and the dominant measure of a worthy martingale M , then*

$$Q_{f \cdot M}(dxdydt) = f(x, t)f(y, t)Q_M(dxdydt). \tag{C.22}$$

$$K_{f \cdot M}(dxdydt) = |f(x, t)f(y, t)|K_M(dxdydt). \tag{C.23}$$

The detailed proof of this proposition can be found in Khoshnevisan (2000).

Now if $t \in (0, T]$ for some finite time T and if K_M is the dominant measure for the

worth martingale M , then for all predictable $f \in \varphi$ $\|f\|_M$ is defined as

$$\|f\|_M^2 = E \left[\iiint_{\mathbb{R}^N \times \mathbb{R}^N \times (0, T]} |f(x, t)f(y, t)| K_M(dx, dy, dt) \right]. \quad (\text{C.24})$$

Theorem 9. *Let M be a worthy martingale measure. Then for all $f \in \mathcal{P}_M$, $(f \cdot M)_t$ is worthy martingale measure that satisfies equation (C.23). Moreover, for all $t \in (0, T]$ and $A, B \in \beta(\mathbb{R}^N)$,*

$$\langle (f \cdot M)(A; t), (f \cdot M)(B; t) \rangle_{t \in (0, T]} = \iiint_{A \times B \times (0, T]} f(x, t)f(y, t) Q_M(dx, dy, dt), \quad (\text{C.25})$$

and

$$E[(f \cdot M)^2(B; t \in (0, T))] \leq \|f\|_M^2.$$

Proof. We use the fact that φ is dense in $\mathcal{P}_M$.

From Propositions 3 and 4,

$$E[(f \cdot M)^2(B; t)] \leq \|f\|_M^2 \quad \forall t \in (0, T], f \in \varphi, B \in \beta(\mathbb{R}^N) \quad (\text{C.26})$$

So if $\{f_m\}_{m=1}^\infty$ is a Cauchy sequence in $(\varphi, \|\cdot\|_M)$ then the sequence $\{(f_m \cdot M)(B; t)\}_{m=1}^\infty$ is Cauchy in $L^2(P)$. If $f_m \rightarrow f$ in $\|\cdot\|_M$ then $\{(f_m \cdot M)(B; t)\}_{m=1}^\infty \rightarrow \{(f \cdot M)(B; t)\}$ in $L^2(P)$ and since $A, B \in \beta(\mathbb{R}^N)$ consequently

$$\langle \{(f_m \cdot M)(A; t)\}_{m=1}^\infty, \{(f_m \cdot M)(B; t)\}_{m=1}^\infty \rangle \rightarrow \langle (f \cdot M)(A; t), (f \cdot M)(B; t) \rangle_t.$$

□

An extension from $L^2(P)$ to $L^p(P)$, where $p \geq 2$, is given by Burkholder's inequality.

Theorem 10. *[Burkholder's inequality] For all $p \geq 2$ there exists $c_p \in (0, \infty)$ such*

that for all predictable f and all $t > 0$

$$E[|(f \cdot M)(B; t \in (0, T])|^p] \leq c_p E \left[\left(\iiint_{\mathbb{R}^N \times \mathbb{R}^N \times (0, T]} |f(x, t)f(y, t)| K_M(dx, dy, dt) \right)^{p/2} \right]. \quad (\text{C.27})$$

This theorem is a very useful tool to prove the uniqueness theorem for SPDEs and its proof can be found in Khoshnevisan (2000).

The following notation is often used for stochastic integrals:

$$(f \cdot M)(A; s \in (0, t]) = \int \int_{A \times (0, t]} f dM = \int \int_{A \times (0, t]} f(x, s) M(dx, ds). \quad (\text{C.28})$$

Since M is a martingale measure, stochastic integrals are often called martingale integrals. Since the white noise is a martingale, the Wiener integral given by $W_t = \int h dW$ is also a stochastic integral or a martingale integral where f is replaced by h and dM by dW . Therefore W_t is a transformed martingale of W by h , and can be written as

$$W_t = \int h dW = (h \cdot W). \quad (\text{C.29})$$

Stochastic (or martingale) integrals have the Fubini-Tonelli property stated in the following theorem:

Theorem 11. *Suppose M is a worthy martingale measure with dominant measure K . Let $(A, \mathcal{A}, \mu)$ be a measure space and $f : \mathbb{R}^N \times \mathbb{R}_+ \times A$ measurable such that the following expectation is finite:*

$$\int \iiint \int_{\Omega \times \mathbb{R}^N \times \mathbb{R}^N \times (0, T] \times A} |f(x, t, \omega, u)f(y, t, \omega, u)\mu(du)| K(dx, dy, dt) P(d\omega). \quad (\text{C.30})$$

Then almost surely,

$$\begin{aligned}
& \int_A \left(\iint_{\mathbb{R}^N \times (0,t]} f(x, s, \bullet, u) M(dx, ds) \right) \mu(du) \\
&= \iint_{\mathbb{R}^N \times (0,t]} \left(\int_A f(x, s, \bullet, u) \mu(du) \right) M(dx, ds). \tag{C.31}
\end{aligned}$$

The proof of this theorem follows from Fubini's theorem if equation (C.30) holds.

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