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Two-Dimensional BEM Modelling of Fibre-Bridging in Cracked Fibre-Metal Laminates

by

KAREN JENNIFER CAIN, B.Eng.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Master of Engineering

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

Department of
Mechanical and Aerospace Engineering
Carleton University
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June, 1996

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the degree of Master of Engineering

Thesis Supervisor

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Carleton University

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Abstract

This thesis presents a numerical study of the crack-face bridging mechanism for crack-tip shielding in fibre-metal laminates. Fibre-metal laminates are a class of structural materials which consist of fibre-reinforced preregs sandwiched between metal sheets. The fracture mechanics prediction of crack growth in these laminates involves several parameters. They include the geometry and size of the crack, the extent of fibre-bridging, and both the delamination between the prepreg and metal layers and debonding of the fibre-matrix interface. Studies of the effects of these parameters on the stresses in the bridging fibres and hence, the crack-tip stress intensity factors are carried out using boundary and finite element analyses. The present work serves to compliment numerous reported experimental research which aim to obtain a better quantitative understanding of the crack-face fibre-bridging mechanism in fibre-metal laminates.

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Nomenclature

a	- crack length
a_{ij}	- elastic compliance matrix
A	- cross-sectional area of aramid-epoxy layer
A_T	- total cross-sectional area of laminate
Al	- aluminum
ARALL	- Aramid Reinforced Aluminum Laminate
b	- fibre-bridging zone length
b_i	- body force vector
BEM	- boundary element method
BIE	- boundary integral equation
C_b	- bridging compliance
C_c	- compliance without crack bridging
d	- delamination zone length
E	- Young's modulus of an isotropic material
E_{11}	- Young's modulus in the 1 st material principal axis direction
E_{22}	- Young's modulus in the 2 nd material principal axis direction
E_f	- elastic modulus of fibre
E_m	- elastic modulus of matrix
FEM	- finite element method
$\mathcal{G}$	- strain energy release rate
G	- shear modulus of an isotropic material
G_{ij}	- shear modulus of an anisotropic material

GLARE - Glass Reinforced Aluminum Laminate

H	- plate height
J	- J-contour integral
K_B	- bridging stress intensity factor
K_{eq}	- equivalent stress intensity factor
K_I	- mode-I stress intensity factor
K_{Ic}	- material fracture toughness
K_{II}	- mode-II stress intensity factor
l	- crack-tip element length
m	- power index for fibre stress which increases towards crack-tip
n	- power index for fibre stress which decreases towards crack-tip
N^c	- shape functions
p	- closure pressure
p,P	- source point
q,Q	- field point
r	- distance between p and q (or P and Q)
R	- fibre radius
R	- BEM domain
S	- BEM boundary
t	- aramid-epoxy layer thickness
T	- total laminate thickness
t_i	- tractions
$\bar{t}_i$	- nodal 'tractions'
T_{ij}	- fundamental solution for tractions
u_i	- displacements

U_{ij}	- fundamental solution for displacements
v	- crack opening displacements
v_m	- crack opening displacement at crack-mouth
v^*	- maximum crack opening displacement at the last fibre
$\bar{v}^*$	- maximum crack opening displacement without fibre-bridging
V_f	- fibre volume fraction
V_m	- matrix volume fraction
W	- plate width
x	- distance from crack-tip
x_i	- Cartesian coordinate
δ_{ij}	- Kronecker delta
ϵ_{ij}	- strain tensor
ϵ^*	- strain in the last intact fibre
$\eta_{ij,k}$	- mutual influence coefficients
λ	- Lamé constant
μ_i	- characteristic roots of an anisotropic material
μ_{ij}	- shear modulus
ν	- Poisson's ratio of an isotropic material
ν_{ij}	- Poisson's ratio of an anisotropic material
σ	- applied stress
σ_B	- fibre-bridging stress distribution
σ_B^*	- fibre-bridging stress in the last intact fibre
σ_{ij}	- stress tensor
τ	- shear stress

Chapter 1

Introduction

With increased use of fibre-reinforced composite materials for structural applications, attention has been focused on the toughening mechanism of crack-face fibre-bridging and its effects on the crack-tip shielding process. Fibres bridging the crack faces reduce the extent of crack opening under, for example, mode-I loading. This results in slower crack growth and may even lead to crack arrest and a longer service life for critical components. This research is a numerical study of several of the parameters effecting the crack-tip stress intensity factors in crack-face bridging of fibre-metal laminates.

Fibre-metal laminate composites are made up of layers of fibre-reinforced polymer sandwiched between thin metal sheets. These are commercially available as ARALL¹ (see figure 1.1), an aramid fibre-reinforced epoxy and aluminum composite, and GLARE¹, a glass fibre-reinforced epoxy and aluminum composite. They are currently being used for fatigue-critical aircraft components, such as, lower wing skins, fuselage, horizontal and vertical stabilizers, exterior doors, floors and other secondary structures. While the strength of fibre-metal laminates is comparable to monolithic aluminum, their use can result in weight savings of up to 30%[3]. This alone makes fibre-metal composites ideal for aerospace applications. Additional

¹ARALL and GLARE are registered trademarks of the Structural Laminates Company

characteristics of fibre-metal laminates include exceptional fatigue properties, excellent machinability and formability, good heat resistance and good impact resistance and damping.

Fibre-metal laminates have been quite widely studied in literature since their introduction to industry in the early 1980s. However, most of the published works are experimental fatigue crack propagation studies on ARALL. Such studies have been performed by, for example, Rushau[1], Ritchie et al.[2], Marissen[3][4][5] and Wilson and Wilson[6], investigating the crack-face fibre-bridging length and delamination characteristics. Similar studies have also been carried out using GLARE by Roebroeks[7] and Young et al.[8]. Yeh[9] has numerically modelled the edge of an ARALL laminate with a delamination zone using singular finite elements to represent the crack-tip at the boundary of the delamination. In his studies on ARALL, Marissen[3] has developed an analytical model for determining fibre-bridging effects, which quantify the relationship between the bridging stresses and the delamination zone size as the crack grows in fatigue conditions. However, in this work, the assumption was made that the closure stress profile on the crack faces is uniform. The present study will expand on the two latter numerical studies, to include a range of bridging stress profiles as well as investigate the effects of geometry, fatigue crack size, bridging zone length, and delamination zone shape for statically loaded crack problems.

In recent years, much of the work in the area of modelling fibre-bridging has been carried out for ceramic and cementitious composites. A brief review of this research is presented in Chapter 2. These models include the shear-lag model, the compliance spring model and the stress intensity factor approach. A suitable method for treating fibre-metal composites is determined and modified for use in this study. Evidently the reduction of the three-dimensional cracked fibre-metal laminate to a two-dimensional one must involve some simplifying assumptions in the analysis. One of these being the homogenization of the laminate properties through its thickness. Also, because the crack-bridging stresses are unknown a

priori, the problem is mathematically non-linear and the numerical solution of the stress problem must be obtained via an iterative process. These are also discussed in Chapter 2.

The crack-bridging stresses and crack-tip stress intensity factors are determined numerically in two dimensions through the boundary integral equation (BIE) method, also commonly known as the boundary element method (BEM). This numerical technique is now widely recognized as an efficient computational technique for the treatment of crack problems. In Chapter 3, a brief review of the formulation of the boundary integral equation (BIE) for both elastic isotropy and anisotropy is presented. The sub-regioning technique, which allows the representation of multiple material regions of a body, is described. Also discussed is the application of the boundary element method to fracture mechanics problems. The stress and displacement crack-tip fields for anisotropic solids and the direct determination of crack-tip stress intensity factors are reviewed as well. Finally, several isotropic and orthotropic test problems with cracks are analyzed and the BEM solutions are compared with those available in the literature.

The results of the two-dimensional boundary element analysis are presented in Chapter 4. For the purpose of analysis, the properties of ARALL are used. The first problem analyzed is that of a crack which has propagated across the aluminum layers and the matrix material, leaving all of the fibres intact across the crack faces. Here, several different delamination zone lengths were varied to determine its effect on the crack opening displacements. Next, the problem of a crack which has propagated only partially across the specimen is treated. For this problem, the geometry, crack length, bridging zone lengths and corresponding bridging stress profile and the delamination size are varied in a parametric study of the bridging effects. The aim of this work is to quantify the bridging effects in fibre-metal laminates with cracks; it helps to determine the relative importance of the various parameters during fibre-bridging in reducing the crack driving force. To determine if the two dimensional analysis would be adequate for practical purposes, the actual physical problem is analyzed using the finite element method in three dimensions for the purpose of

alent stress intensity factor for the crack and K_{Ic} is the material fracture toughness. K_I can be readily obtained for many different crack geometries from the literature. K_B , on the other hand, is dependent on the stress profile on the crack-faces and it may be determined, for example, by assuming the crack-face stress distribution.

2.5.1 Power-Law Expressions for Bridging Stresses

In their work on strain-softening materials (eg. ceramics, concrete and other cementitious materials), Foote et al[35] suggested an empirical power-law equation relating the bridging stresses and the crack opening displacements as follows:

$$\sigma_B(u) = \sigma_B^* \left[1 - \frac{u(x)}{u^*} \right]^{m_i} \quad (2.6)$$

where σ_B and $u(x)$ are the bridging stress and half of the crack opening displacement, respectively, at a distance x from the crack-tip. Also, σ_B^* is the maximum value of the bridging stress occurring at the crack-tip, u^* is the crack opening displacement at the crack-mouth fibre farthest from the crack-tip and m_i is a power-law index. A similar expression was proposed by Suzuki and Sakai[22] for a distribution of bridging stresses which decrease towards the crack-tip:

$$\sigma_B(u) = \sigma_B^* \left[\frac{u(x)}{u^*} \right]^{n_d} \quad (2.7)$$

Note that a uniform bridging distribution (the Dugdale model) on the crack-face is described by $m_i = n_d = 0$.

From equation 2.6 it is clear that the fibre-bridging stress is modelled as decreasing to zero at the outermost intact fibre away from the crack-tip. For fibre-metal laminates, in practice, the crack-faces are fully bridged, or near-fully bridged. Also, it is perhaps useful to define σ_B^* in the same manner as in equation 2.7. Accordingly, for these laminates, it is more appropriate to use the expression below:

$$\sigma_B(u) = \sigma_B^* \left[1 + \left(1 - \frac{u(x)}{u^*} \right)^{m_i} \right] \quad (2.8)$$

Assuming that the crack opening displacement, $u(x)$, also follows a power-law relationship when related to the distance from the crack-tip[22], one obtains,

$$u(x) = u^* \left(\frac{x}{b} \right)^q \quad (2.9)$$

Chapter 2

Fibre-Bridging of Cracked Composites

In this chapter the various mechanisms for increasing the fracture toughness of a material are briefly described. A more detailed discussion on the process of crack-face fibre-bridging seen in fibre-metal laminates is presented.

Several different methods of determining crack-face bridging effects in fibre-reinforced ceramics have been proposed in literature. The three most commonly employed are the shear-lag model, the compliance spring model and the stress intensity approach. These models all assume that crack-face bridging is the main crack-tip shielding mechanism. They are briefly reviewed here and the model which is appropriate and perhaps most convenient for treating fibre-metal laminates is discussed.

2.1 Toughening/Crack Shielding Mechanisms

Several mechanisms can increase the intrinsic fracture toughness of a material or its ability to absorb energy before failure. These mechanisms take place in the crack wake or frontal process zone and effectively shield the crack-tip from the crack driving force. The following factors may increase the fracture toughness of materials and details of their toughness-enhancing mechanisms may be found in, for example [11][12] (see figure 2.1):

- Crack deflection and meandering
- Zone shielding
 1. Transformation toughening
 2. Microcrack toughening
 3. Crack field void formation
- Contact shielding
 1. Wedging
 - (a) Corrosion debris-induced crack closure
 - (b) Crack surface roughness-induced closure
 2. Bridging
 - (a) Ligament or fibre toughening
 3. Sliding
 - (a) Sliding crack surface interface
 4. Plasticity-induced crack closure

Note that the process of fibre-bridging is only one of many possible crack-tip shielding processes. Also, in practice it is not common to find only a single mechanism at work in a material, although, it is possible to ignore some of these effects when their contribution to shielding is small.

2.2 Fibre-Bridging Process in Fibre-Metal Laminates

Crack-face bridging is the presence of intact fibres lying across the wake of, for example, a mode-I matrix crack as shown in figure 2.2. Under fatigue crack loading, the weaker matrix material has a tendency to fail first, but the stronger fibres remain intact. Due to the discontinuity caused by the propagating crack, the matrix can

no longer transfer load except through shear at the fibre-matrix interface. This additional load on the fibres causes them to pullout of the surrounding matrix against friction. Debonding of the interface over length d , (see figures 2.2 and 2.3), may also begin to occur if the shear strength of the interface is exceeded. This pullout and debonding allow the fibres to elongate by, say, $2v^*$, bridging the opening crack-faces and providing a load transfer mechanism which essentially acts to close the crack thus shielding the crack-tip. The length of the crack-bridging zone, b , is dependent on the strength and toughness of the fibres. The matrix crack may propagate in a steady-state manner such that the stress in the fibres quickly becomes constant away from the crack-tip, as typically seen in fibre-reinforced ceramic composites.

In general, the direction of crack propagation is dependent on the type of material. The crack may propagate in a longitudinal mode in which the crack advances in the direction of the fibres. This least critical mode-II crack occurs in composites with weak interfaces. Alternatively, mode-I cracks in fibrous composites propagate in a direction perpendicular to the fibres which may result in fibre-bridging. Two possible situations can occur, in that, the crack may be fully bridged as the crack propagates through the entire matrix or, the crack may be partially bridged with fewer fibres bridging the crack. The latter may result in failure of the specimen when the bridging fibres and remaining matrix material become insufficient to sustain the applied load.

Failure in a fibre-metal laminate under fatigue loading conditions often begins with crack initiation in the metal layer which is followed by cracking in the weaker matrix material of the fibre-reinforced laminae. For fibre-metal laminate composites, the matrix cracks tend to propagate in an unsteady state manner and are usually fully bridged (see figure 2.2). The fibres also tend to debond from the matrix which disables any load transfer from the matrix to the fibres over that length[2]. Another debond also occurs, namely, the controlled delamination between the aluminum and fibre-reinforced polymer layers[3]. There has been some controversy in the literature as to which type of debond contributes the most to the decrease in crack bridging effects. However, the debonding is a necessary component, in that it redistributes the stresses allowing the fibres to remain intact. In this study, a delamination between

the fibre-reinforced polymer and metal layers has been assumed. This effectively takes a macromechanics view of the problem rather than the micromechanics view of considering the debonding of a single fibre.

2.3 Shear Lag Model

The basis of the shear-lag model for crack-face bridging of a fibre-reinforced composite consists of the stress analysis of a single axially-loaded fibre embedded in a cylindrical matrix as shown in figure 2.4. The treatment of just a single fibre simplifies the analysis considerably. However, knowledge of the load for each individual fibre in the bridging zone is required for a proper investigation of fibre-bridged cracking.

The work of Aveston and Kelly[13][14][15] employing the shear-lag model, represent perhaps the earliest studies on the effects of fibre-bridging using fracture mechanics. By considering the energy balance of the change in energy during incremental extension of the matrix crack and the changes in potential energy, strain energy, new fracture surface energy and the work done against friction, the critical loads for steady-state matrix cracking have been determined. The solutions are for composites with high strength unbonded fibres with frictional sliding between the fibre and matrix under an applied load. It should be remarked here that steady-state matrix cracking refers to the situation where the stress required to propagate a crack is not significantly dependent on crack size. In fibre reinforced ceramics, for example, this usually corresponds to large cracks which have crack-face openings asymptotically reaching the equilibrium separation of the two planes of the crack bridged by the fibres (see figure 2.5). More general solutions based on energy balance considerations using the shear-lag model have been obtained by, for example, Budiansky et al[16]. These solutions provide an insight to the critical conditions for the onset of large scale matrix cracking in ceramics. In addition, the analysis also enables estimates of the debond between the fibre and the matrix to be made.

Using the shear-lag model in the analysis of fibre-pullout from a matrix, Marshall et al[17] have also determined the relationship between the closure pressure, p , in

the bridged zone and the crack opening displacement, u , as follows:

$$p = 2\sqrt{\frac{u\tau V_f^2 E_f (1 + \eta)}{R}} \quad (2.1)$$

where

$$\eta = \frac{E_f V_f}{E_m V_m} \quad (2.2)$$

and V_f and V_m are the volume fraction of the fibres and matrix, respectively and E_f and E_m are the Young's moduli of the fibres and matrix, respectively. Also, R is the fibre radius and τ is the frictional stress at the interface between the fibre and the matrix (see figure 2.6). While equation 2.1 may be used to determine the reduction in the stress intensity factor at the crack-tip in principle (in an approach which will be discussed later), obtaining realistic values of τ may present an interesting challenge for it to be of much direct practical use in crack growth predictions of fibre-metal laminates.

Perhaps the most significant drawback of the shear-lag model is that the interaction between neighbouring fibres is not accounted for. It is evident that the fibres surrounding a broken fibre will have to bear the load that was carried by the latter, although the stresses in the fibres far-removed from it will remain unchanged. A detailed knowledge of the loading conditions on the individual fibres is required to provide an accurate crack-bridging analysis. The intra- and inter-laminar properties of hybrid-fibre composites and fibre-metal composites would make the modelling of fibre-bridging of cracks in these materials based on a single fibre in a cylindrical matrix configuration somewhat suspect, to say the least. Furthermore, steady-state cracking in the sense explained above is not a feature exhibited in fibre-metal laminate composites. It is not clear therefore, if the energy balance approach of a shear-lag model will be able to provide a convenient means to treat the fibre-bridging effects in these composites, particularly when the fracture mechanisms are more complex. Nevertheless, the model has found much support among researchers in the analysis of cracked ceramic composites due to its simplicity. It has even been extended to fatigue loading situations by, for example, Marshall and Oliver[18], Cox and Marshall[19] and Nguyen and Yang[20].

2.4 Compliance Spring Model

The spring model determines crack-bridging effects by relating changes in compliance during crack extension. The compliance technique can provide important information as to the bridging zone size and the stress distribution there. A typical model, as shown in figure 2.7, consists of springs at different locations along the crack-face to represent the compliance of the elastic body with and without bridging. A spring located along the load line, for example, is used to represent the compliance of a specimen without crack-bridging, $C_e(a)$, and a spring at half of the bridging length is used to represent the compliance due to crack-face bridging, C_B . The crack-faces are assumed to be ideally stiff rods with a flexible joint at the crack-tip.

Through load equilibrium considerations, an equation for the compliance can be obtained as follows[21]:

$$\frac{1}{C(a)} = \frac{1}{C_e(a)} + \frac{1}{C_B(b)} \left(\frac{b}{2a} \right)^2 = \frac{1}{C_e(a)} + \frac{1}{C'_B} \quad (2.3)$$

where a and b are the crack length and bridging zone length, respectively. This can be rewritten as:

$$C(a) = \frac{C_e(a)}{\left(1 + \frac{C_e(a)}{C'_B} \right)} \quad (2.4)$$

where C'_B is the effective bridging compliance. For simple geometries, the compliance of the body without crack-face bridging, $C_e(a)$, can be readily obtained from published handbooks or results in the literature. The values of b and C'_B are usually then obtained through measuring the changes in compliance by step-wise re-notching the cracked body.

The re-notching of the fibre-bridged specimen can be carried out either experimentally or by computer simulation[21][22]. The approach has been used in the study of ceramic composites with either steady-state or non-steady state fibre-bridging zones. There are several variations of the spring model/compliance technique to determine the effects of bridging on crack growth, for example, Knehans and Steinbrech[23], Hu and Wittman[24], Hu, Lutz and Swain[25], Cox and Marshall[19]

and Yang, Mura and Shah[26], but they have generally been applied to ceramic and cementitious composites only.

The compliance spring model can in principle be used to investigate the fibre-bridging effects on the crack growth in fibre-metal laminates. However, the approach may appear to be unnecessarily cumbersome for treating these materials whose fracture characteristics, as noted previously, are significantly different from those seen in ceramic composites.

2.5 Stress Intensity Approach for Fibre-Metal Laminates

The stress intensity approach considers the macro effects of the bridging fibres resisting crack opening. The primary objective here is to determine the reduction in the stress intensity factor at the crack-tip due to fibre-bridging. Therefore, the stresses in the discrete bridging fibres which restrain the crack from opening are usually assumed to be a continuous closing pressure acting over the crack-faces as shown in figure 2.8. This stress, applied to the crack surfaces, is generally not uniform and unknown initially, since it will depend on the crack opening profile. Thus, the determination of this stress is a mathematically non-linear problem even with the addition of some simplifying assumptions. These assumptions are commonly concerned with the form of the relationship between the crack closure stress distribution and the crack opening displacements. The stress intensity method of determining bridging effects has been used because of its simplicity by several authors, including; Marshall and Cox[27][28][29][30], Mai and Lawn[31], Cotterell and Mai[32], Cox[33] and Cox and Lo[34].

Quantitatively, the stress intensity approach for crack extension of a composite may be written as follows:

$$K_{eq} = K_I - K_B \leq K_{Ic} \quad (2.5)$$

where K_I is the stress intensity of the material without fibre-bridging, K_B is the reduction due to the closure effects of crack-face bridging, K_{eq} is the effective or equiv-

alent stress intensity factor for the crack and K_{Ic} is the material fracture toughness. K_I can be readily obtained for many different crack geometries from the literature. K_B , on the other hand, is dependent on the stress profile on the crack-faces and it may be determined, for example, by assuming the crack-face stress distribution.

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where σ_B and $u(x)$ are the bridging stress and half of the crack opening displacement, respectively, at a distance x from the crack-tip. Also, σ_B^* is the maximum value of the bridging stress occurring at the crack-tip, u^* is the crack opening displacement at the crack-mouth fibre farthest from the crack-tip and m_i is a power-law index. A similar expression was proposed by Suzuki and Sakai[22] for a distribution of bridging stresses which decrease towards the crack-tip:

$$\sigma_B(u) = \sigma_B^* \left[\frac{u(x)}{u^*} \right]^{n_d} \quad (2.7)$$

Note that a uniform bridging distribution (the Dugdale model) on the crack-face is described by $m_i = n_d = 0$.

From equation 2.6 it is clear that the fibre-bridging stress is modelled as decreasing to zero at the outermost intact fibre away from the crack-tip. For fibre-metal laminates, in practice, the crack-faces are fully bridged, or near-fully bridged. Also, it is perhaps useful to define σ_B^* in the same manner as in equation 2.7. Accordingly, for these laminates, it is more appropriate to use the expression below:

$$\sigma_B(u) = \sigma_B^* \left[1 + \left(1 - \frac{u(x)}{u^*} \right)^{m_i} \right] \quad (2.8)$$

Assuming that the crack opening displacement, $u(x)$, also follows a power-law relationship when related to the distance from the crack-tip[22], one obtains,

$$u(x) = u^* \left(\frac{x}{b} \right)^q \quad (2.9)$$

where q is another power index. By combining equations 2.7 and 2.8 with 2.9 and further combining the power indices (into m and n), the respective expressions for σ_B become (see figure 2.9):

$$\sigma_B = \sigma_B^* \left(\frac{x}{b} \right)^n \quad (2.10)$$

for stresses decreasing towards the crack-tip, and:

$$\sigma_B = \sigma_B^* \left[1 + \left(1 - \frac{x}{b} \right)^m \right] \quad (2.11)$$

for stresses increasing towards the crack-tip.

2.5.2 Determination of σ_B^*

Equations 2.10 and 2.11 above give the assumed bridging stresses along the crack once the stress, σ_B^* , in the farthest intact fibre is known. The magnitude of this stress may be determined through consideration of the strain in the fibres. To do this, it is also necessary to determine the extent of the debond between the fibre and the matrix, discussed earlier.

Furthermore, the closure stress is applied only to the fibre-reinforced laminae of the fibre-metal composites, since the metal layers do not contribute to closure of the crack-faces. It is evident that this problem is truly a three-dimensional one. The reduction of this problem to a two-dimensional analysis in the plane of the laminate requires the homogenization of the properties through the laminate's thickness. As well, there must be a reduction in the bridging stress applied such that the net force due to crack-bridging is identical in both two and three-dimensional models. This is accomplished by the addition of a cross-sectional area ratio, A/A_T , to the basic stress-strain relationship for the outermost fibre, as follows:

$$\sigma_B^* = E_{11} \left(\frac{A}{A_T} \right) \epsilon^* \quad (2.12)$$

where E_{11} is the Young's modulus in the fibre-direction of the reinforced composite, and A and A_T are the cross-sectional area of the fibre-reinforced composite and the total cross-sectional area of the laminate, respectively. In the case of a 2-1 layup of a fibre-metal composite, A/A_T is the thickness of the fibre-reinforced polymer

laminae divided by the total thickness of the laminate. In equation 2.12, ϵ^* is the strain in the last intact fibre:

$$\epsilon^* = \frac{v^*}{d} \quad (2.13)$$

where v^* is the half crack opening displacement at the farthest fibre and d is the delamination length between the layers at the same position. Combining equations 2.12 and 2.13 the stress, σ_B^* is:

$$\sigma_B^* = E_{11} \left(\frac{A}{A_T} \right) \left(\frac{v^*}{d} \right) \quad (2.14)$$

In general, the procedure outlined here to obtain σ_B^* , and hence σ_B , is an iterative one. The maximum half crack opening displacement with no crack bridging, $\bar{v}^*$ is first obtained using a numerical approach such as the finite element or the boundary element method, and the stress in the fibre farthest from the crack-tip is estimated using equation 2.14. A stress distribution is calculated using either equation 2.10 or 2.11 and a new v_i^* may be found. This process is repeated until the values of v_i^* and σ_B^* converge to acceptable tolerances.

For the case of a uniform crack-bridging stress distribution, that is $n = m = 0$ in equations 2.10 and 2.11, the procedure need not be iterative. Using Bueckner's principle[36] (see figure 2.10), the stress distribution can be found directly for a fully bridged crack, providing the half crack opening displacement without crack bridging is known. For no crack-face bridging under applied external load σ , half of the crack opening displacement at the crack mouth is, say:

$$v_m^* = \bar{v}^* \quad (2.15)$$

With a uniformly distributed crack-bridging stress, the half crack opening displacement becomes:

$$v_m^* = \left(\frac{\sigma - \sigma_B}{\sigma} \right) \bar{v}^* \quad (2.16)$$

Using equation 2.14 one obtains:

$$\sigma_B^* = \left(\frac{\sigma - \sigma_B}{\sigma} \right) E_{11} \left(\frac{A}{A_T} \right) \left(\frac{\bar{v}^*}{d} \right) \quad (2.17)$$

from which the following can be obtained when the applied stress, $\sigma = 1$:

$$\sigma_B = \frac{E_{11} \left(\frac{A}{A_T} \right) \left(\frac{\bar{v}^*}{d} \right)}{1 + \left(\frac{A}{A_T} \right) \left(\frac{\bar{v}^*}{d} \right)} \quad (2.18)$$

Chapter 3

The Boundary Integral Equation Method

3.1 Introduction

The boundary integral equation (BIE) method, also known as the boundary element method (BEM), has emerged in recent years as an alternative to the finite element method (FEM) for the solution of many problems in engineering analysis. The main advantage of the BIE method is that it requires only the discretization of the boundary of the solution domain. Boundary element analysis permits the use of fewer elements than FEM due to its efficiency in treating stress concentrations or regions where large stress gradients exist. For these reasons it has become widely used in the study of fracture and fatigue of materials.

This chapter reviews the formulation of the boundary integral equation for both isotropic and anisotropic solids. Quarter-point crack-tip elements and the determination of the crack-tip stress intensity factor in fracture mechanics are discussed. Finally, several examples are presented to demonstrate the accuracy of the boundary element analysis.

3.2 The Boundary Integral Equation for Isotropic Elasticity

The boundary integral equation for elasticity is obtained by transforming the governing partial differential equations, valid over the entire domain of the solid body, into an integral equation for its boundary only. Consider a two-dimensional, linear-elastic solid, R , with boundary S in a Cartesian coordinate system. Equilibrium of forces gives, in indicial notation[37]:

$$\sigma_{ij,j} + b_i = 0 \quad (3.1)$$

where σ_{ij} are components of the stress tensor, b_i are body forces and the comma index implies differentiation. Also, the relationship between the strains and displacements may be written as:

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (3.2)$$

where ϵ_{ij} are the strains and u_i are the displacements. The stress-strain relationship for a homogeneous, isotropic body is:

$$\sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \quad (3.3)$$

where δ_{ij} is the Kronecker delta and μ and λ are the Lamé constants. They are defined as follows:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (3.4)$$

$$\mu = G = \frac{E}{2(1 + \nu)} \quad (3.5)$$

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad (3.6)$$

where G is the shear modulus, E is Young's modulus and ν is Poisson's ratio.

Substitution of equation 3.2 into 3.3 and then into 3.1 results in Navier's equations for a two-dimensional isotropic solid as follows:

$$\mu \left[u_{i,jj} - \frac{1}{1 - 2\nu} u_{j,ji} \right] + f_i = 0 \quad (3.7)$$

where

$$\nu' = \begin{cases} \nu & \text{for plane strain} \\ \frac{\nu}{1+\nu} & \text{for plane stress} \end{cases} \quad (3.8)$$

To transform equation 3.7 into an integral equation, two items are required, namely, the fundamental solution to equation 3.7 and a reciprocal work theorem. The fundamental solution to Navier's equation corresponds to that for the Kelvin problem in elasticity. For displacements, it is:

$$U_{ji}(p, q) = \frac{1}{8\pi\mu(1-\nu')} \left[(3-4\nu')\delta_{ij} \log_e \frac{1}{r} + r_{,i}r_{,j} \right] \quad (3.9)$$

where $U_{ji}(p, q)$ represents the displacements in the x_i direction at point q due to a unit load in the x_j direction at point p and r is the distance between p and q (see figure 3.1). The corresponding solution for tractions is:

$$T_{ji}(p, q) = \frac{-(1-2\nu')}{4\pi(1-\nu')} \frac{1}{r} \left\{ \frac{dr}{dn} \left[\delta_{ij} + \frac{2}{1-2\nu'} \right] r_{,i}r_{,j} \right\} \quad (3.10)$$

where n is a unit outward normal at q . $T_{ji}(p, q)$ represents the tractions in the x_i direction at point q due to a unit point load in the x_j direction at p .

The Betti-Rayleigh reciprocal work theorem relates the equilibrium states of an elastic body under different loads. The theorem states that the work done by the primed tractions, t'_i , acting through the unprimed displacement, u_i , is equal to the work done by the unprimed tractions, t_i , acting through the primed displacements u'_i . In the absence of body forces, it may be mathematically written as:

$$\int_S t_i u'_i dS = \int_S t'_i u_i dS \quad (3.11)$$

Without loss of generality, the primed state may be taken as corresponding to the fundamental solution to Navier's equation and the unprimed state as unknown. By moving the interior point p to a point, P , on the surface, and following the usual limiting process (see, for example, Tan[38]), the boundary integral equation can be obtained as follows:

$$C_{ji}(P)u_i(P) + \int_S u_i(Q)T_{ji}(P, Q)dS(Q) = \int_S t_i(Q)U_{ji}(P, Q)dS(Q) \quad (3.12)$$

C_{ji} is dependent on the geometry of the surface at point P and is:

$$C_{ji}(P) = \lim_{\epsilon \rightarrow 0} \int_{S_\epsilon} T_{ji}(P, Q) dS(Q) \quad (3.13)$$

where S_ϵ is the boundary of an excluded region around point P .

3.3 Numerical Solution of the Boundary Integral Equation

To numerically solve the boundary integral equation, equation 3.12, the boundary, S , of the two-dimensional solution domain, R , is divided into M line elements. In the quadratic isoparametric element formulation (see, for example, Tan and Gao[39][40]), each element has three equally spaced nodes and the assemblage of elements and nodes, also known as a mesh, is shown schematically in figure 3.1. Over each element, the geometry, displacements and tractions are expressed in terms of the quadratic shape functions, $N^c(\zeta)$, of the intrinsic coordinate, ζ , as follows:

$$x_j(\zeta) = N^c(\zeta)x_j^c \quad (3.14)$$

$$u_j(\zeta) = N^c(\zeta)u_j^c \quad (3.15)$$

$$t_j(\zeta) = N^c(\zeta)t_j^c \quad (3.16)$$

where $c = 1, 3$ and:

$$\begin{aligned} N^{(1)}(\zeta) &= -\frac{1}{2}\zeta(1-\zeta) \\ N^{(2)}(\zeta) &= 1-\zeta^2 \\ N^{(3)}(\zeta) &= \frac{1}{2}\zeta(1+\zeta) \quad -1 \leq \zeta \leq +1 \end{aligned} \quad (3.17)$$

Substitution of equations 3.14 to 3.16 into equation 3.12 results in the discretized form of the boundary integral equation:

$$\begin{aligned} &C_{ji} (P^a) u_i (P^a) \\ &+ \sum_{b=1}^M \sum_{c=1}^3 u_i (P^{d(b,c)}) \int_S T_{ji} (P^a, Q) N^c(\zeta) |J(\zeta)| dS \\ &= \sum_{b=1}^M \sum_{c=1}^3 t_i (P^{d(b,c)}) \int_S U_{ji} (P^a, Q) N^c(\zeta) |J(\zeta)| dS \quad P^a = 1, N \end{aligned} \quad (3.18)$$

where P^a is the a -th node of the mesh, $P^{d(b,c)}$ is the c -th node of the b -th element and $|J(\zeta)| = \frac{ds}{d\zeta}$ is the Jacobian of transformation. Equation 3.18 represents a set of $2N$ algebraic equations which may be solved by standard matrix methods for the unknown displacements and tractions on the boundary, S .

3.3.1 Sub-Regioning Technique

For problems involving piecewise-different materials, the body may be divided into several sub-regions. The BIE may be written for each sub-region in turn with the associated material properties. At the interfaces between the sub-regions, continuity of displacements and equilibrium require that:

$$u_i^{(m)}(x) = u_i^{(n)}(x) \quad (3.19)$$

$$t_i^{(m)}(x) = -t_i^{(n)}(x) \quad (3.20)$$

where the superscripts denote the m -th and n -th sub-regions, respectively (see figure 3.2).

3.4 The Boundary Integral Equation for Anisotropic Elasticity

The formulation of the boundary integral equation for two-dimensional anisotropic elasticity is identical to that for isotropic materials discussed in the previous section. The basic difference lies primarily in the stress-strain relations, with appropriate changes to the fundamental solutions.

The stress-strain relations for a homogeneous anisotropic elastic material under plane stress conditions are:

$$\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \quad (3.21)$$

where a_{ij} are elastic compliances which are related to the engineering constants by [41]:

$$a_{11} = \frac{1}{E_{11}} \quad (3.22)$$

$$a_{22} = \frac{1}{E_{22}} \quad (3.23)$$

$$a_{12} = -\frac{\nu_{12}}{E_{11}} = -\frac{\nu_{21}}{E_{22}} \quad (3.24)$$

$$a_{16} = \frac{\eta_{12,1}}{E_{11}} = \frac{\eta_{1,12}}{G_{12}} \quad (3.25)$$

$$a_{26} = \frac{\eta_{12,2}}{E_{22}} = \frac{\eta_{2,12}}{G_{12}} \quad (3.26)$$

$$a_{66} = \frac{1}{G_{12}} \quad (3.27)$$

where E_{11} and E_{22} are the Young's moduli in the first and second material principal axis directions, G_{12} is the shear modulus, ν_{ij} is the Poisson's ratio, and $\eta_{ij,1}$ and $\eta_{ij,2}$ are the coefficients of mutual influence of the first and second kind, respectively. Both mutual influence coefficients are zero in the special case of an orthotropic material.

The characteristic equation for an anisotropic material in stable equilibrium has been shown to be [42]:

$$a_{11}\mu^4 - a_{16}\mu^3 + (2a_{12} + a_{66})\mu^2 - a_{26}\mu + a_{22} = 0 \quad (3.28)$$

The four roots are distinct and never real so long as the material is non-isotropic. The corresponding fundamental solution to Navier's equation are [43]:

$$U_{ji} = 2\text{Re} [r_{i1}A_{j1}\log_e(z_1) + r_{i2}A_{j2}\log_e(z_2)] \quad i, j = 1, 2 \quad (3.29)$$

$$\begin{aligned} T_{ji} = & 2n_1\text{Re} \left[\mu_1^2 \left(\frac{A_{j1}}{z_1} \right) + \mu_2^2 \left(\frac{A_{j2}}{z_2} \right) \right] \\ & - 2n_2\text{Re} \left[\mu_1 \left(\frac{A_{j1}}{z_1} \right) + \mu_2 \left(\frac{A_{j2}}{z_2} \right) \right] \quad i, j = 1, 2 \end{aligned} \quad (3.30)$$

where

$$z_j = x_1 + \mu_j x_2 \quad (3.31)$$

$$r_{1j} = a_{11}\mu_j^2 + a_{12} - a_{16}\mu_j \quad (3.32)$$

and:

$$r_{2j} = a_{12}\mu_j + \frac{a_{22}}{\mu_j} - a_{26} \quad (3.33)$$

In equation 3.29 and 3.30, A_{ji} are complex constants which can be obtained from the following equations:

$$\begin{bmatrix} \text{Im}\{B_1\} & \text{Re}\{B_1\} & \text{Im}\{B_2\} & \text{Re}\{B_2\} \end{bmatrix} \begin{bmatrix} \text{Re}\{A_{j1}\} \\ \text{Im}\{A_{j1}\} \\ \text{Re}\{A_{j2}\} \\ \text{Re}\{A_{j2}\} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4}\pi\delta_{j2} \\ \frac{1}{4}\pi\delta_{j1} \\ 0 \\ 0 \end{bmatrix} \quad j = 1, 2 \quad (3.34)$$

where

$$\left\{ B_i \right\} = \left[i \quad \mu_i \quad r_{1i} \quad r_{2i} \right]^T \quad i = 1, 2 \quad (3.35)$$

The discretized form of the BIE has exactly the same form as equation 3.18 and the numerical solution process also follows the same steps as in isotropic elasticity.

3.5 Application of BIE to Linear-Elastic Fracture Mechanics

The characteristic parameter for the stress fields in the vicinity of the crack-tip is the mode-I stress intensity factor, K_I . Once K_I reaches a critical value which equals the fracture toughness of the material, K_{Ic} , fast fracture occurs and the specimen fails in a catastrophic manner. The goal in linear-elastic fracture mechanics (LEFM) analysis is to determine the stress intensity factor at the crack-tip. The relationship between the stress fields ahead of the crack and the crack-tip stress intensity factor for an anisotropic body has the same form as in isotropic elasticity and is as follows:

$$\sigma_{yy} = K_I / \sqrt{2\pi r} \quad (3.36)$$

$$\sigma_{xy} = K_{II} / \sqrt{2\pi r} \quad (3.37)$$

where $x - y$ and $r - \theta$ are local Cartesian and polar coordinate systems, respectively, with the origin at the crack-tip (see figure 3.3). Equation 3.37 is not a necessary consideration in the present work, since only mode-I loading will be treated.

The displacements on the crack-faces ($\theta = 180^\circ$) in the vicinity of the crack are [39]:

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \sqrt{\frac{2r}{\pi}} \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} \quad (3.38)$$

where

$$\begin{aligned} D_{11} &= \operatorname{Im} \left[\frac{s_2 p_1 - s_1 p_2}{s_1 - s_2} \right] \\ D_{12} &= \operatorname{Im} \left[\frac{p_1 - p_2}{s_1 - s_2} \right] \\ D_{21} &= \operatorname{Im} \left[\frac{s_2 q_1 - s_1 q_2}{s_1 - s_2} \right] \\ D_{22} &= \operatorname{Im} \left[\frac{q_1 - q_2}{s_1 - s_2} \right] \end{aligned} \quad (3.39)$$

The material property parameters, s_j , are related to the roots of the characteristic equation for an anisotropic material (equation 3.28), that is[42]:

$$s_j = \frac{\mu_j \cos \phi + \sin \phi}{\cos \phi - \mu_j \sin \phi} \quad j = 1, 2 \quad (3.40)$$

where ϕ is defined in figure 3.3. In equation 3.39, p_j and q_j are defined as follows:

$$p_j = a'_{11} s_j^2 + a'_{12} s_j \quad j = 1, 2 \quad (3.41)$$

$$q_j = \frac{a'_{12} s_j^2 + a'_{22} - a'_{26} s_j}{s_j} \quad j = 1, 2 \quad (3.42)$$

The local compliances above are obtained by the following transformation from the global x_1 - x_2 coordinate system (see figure 3.3):

$$\begin{aligned} a'_{11} &= a_{11} \cos^4 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi \\ &\quad + a_{22} \sin^4 \phi + (a_{16} \cos^2 \phi + a_{26} \sin^2 \phi) \sin 2\phi \\ a'_{22} &= a_{11} \sin^4 \phi + (2a_{12} + a_{66}) \sin^2 \phi \cos^2 \phi \\ &\quad + a_{22} \cos^4 \phi - (a_{16} \cos^2 \phi + a_{26} \sin^2 \phi) \sin 2\phi \\ a'_{12} &= a_{12} + (a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 \phi \cos^2 \phi \\ &\quad + \frac{1}{2} (a_{26} - a_{16}) \sin 2\phi \cos 2\phi \\ a'_{66} &= a_{66} + 2(a_{11} + a_{22} - 2a_{12} - a_{66}) \sin^2 2\phi \end{aligned}$$

$$\begin{aligned}
& + 2(a_{26} - a_{16}) \sin 2\phi \cos 2\phi \\
a'_{16} &= \left[a_{22} \sin^2 \phi - a_{11} \cos^2 \phi + \frac{1}{2}(2a_{12} + a_{66}) \cos 2\phi \right] \sin 2\phi \\
& + (a_{16} \cos^2 \phi) (\cos^2 \phi + 3 \sin^2 \phi) + (a_{16} \sin^2 \phi) (3 \cos^2 \phi - \sin^2 \phi) \\
a'_{26} &= \left[a_{22} \sin^2 \phi - a_{11} \cos^2 \phi - \frac{1}{2}(2a_{12} + a_{66}) \cos 2\phi \right] \sin 2\phi \\
& + (a_{16} \sin^2 \phi) (\cos^2 \phi - \sin^2 \phi) + (a_{26} \cos^2 \phi) (\cos^2 \phi - 3 \sin^2 \phi) \quad (3.43)
\end{aligned}$$

3.5.1 Quarter-Point Crack-Tip Element and Stress Intensity Factor Determination

The practice of shifting the mid-point nodes of quadratic isoparametric elements adjacent to the crack-tip to quarter points, allows the proper $\sqrt{r}$ variation of the displacement field near the crack-tip. In the BIE method, the traction field in the element ahead of the crack-tip may also have the correct $\frac{1}{\sqrt{r}}$ variation if the traction shape functions are multiplied by $\sqrt{\frac{l}{r}}$, where l is the length of the element. Details of these can be found, for example, in Tan and Gao[39], Cruse and Wilson[46], and Martinez and Dominguez[47]. The computed 'traction' values, $\bar{t}_j$, are related to the physical tractions, t_j , as follows:

$$\bar{t}_j^E = t_j^E \quad (3.44)$$

$$\bar{t}_j^D = \frac{1}{2} t_j^D \quad (3.45)$$

$$\bar{t}_j^A = \lim_{r \rightarrow 0} t_j^A \sqrt{\frac{r}{l}} \quad (3.46)$$

where r is the distance from the crack-tip, l is the length of the element ahead of the crack-tip and the superscripts denote the nodal points shown in figure 3.4.

Combining equations 3.46 and 3.36 for the stress field ahead of the crack-tip, a relationship between the calculated 'tractions' and the crack-tip stress intensity factor can be obtained:

$$\bar{t}_2^A = \lim_{r \rightarrow 0} t_2^A \sqrt{\frac{r}{l}} = \lim_{r \rightarrow 0} \sigma_{22} \sqrt{\frac{r}{l}} = \frac{K_I}{\sqrt{2\pi l}} \quad (3.47)$$

or:

$$K_I = \bar{t}_2^A \sqrt{2\pi l} \quad (3.48)$$

This formula allows the crack-tip stress intensity factor to be calculated directly in a simple and effective manner from the BEM-calculated nodal 'tractions'. It has been well established in literature that this "traction formula" (equation 3.48) is simpler and more insensitive to the boundary element mesh than similar formulas based on the computed crack-face displacements (see Martinez and Dominguez[47] and Tan and Gao[39]).

3.5.2 Other Crack-tip Field Characterizing Parameters in LEFM

In linear-elastic fracture mechanics (LEFM), there are other approaches to quantify crack growth, such as, the energy balance and contour integral approaches, respectively.

According to Griffith's theory, the energy release rate, $\mathcal{G}$, and the fracture resistance must balance at the onset of crack growth. However, $\mathcal{G}$, can be related to the stress intensity factor, K_I at the crack-tip by the following relation for orthotropic materials:

$$\mathcal{G} = K_I^2 \sqrt{\frac{a_{11}a_{22}}{2}} \left[\sqrt{\frac{a_{22}}{a_{11}}} + \left(\frac{a_{66} + 2e_{12}}{2a_{11}} \right) \right]^{\frac{1}{2}} \quad (3.49)$$

where the coefficients a_{ij} are defined in equations 3.22 to 3.24 and 3.27.

The development of the J-contour integral approach was originally intended for use in elastic-plastic fracture mechanics, but can be appropriately used in LEFM as well. The J-integral is based on the principle of conservation of energy around a closed loop, thus making the contours path independent. This is seen, for example, from [45]:

$$\oint w dy - t \frac{\partial u}{\partial x} dS = 0 \quad (3.50)$$

where w is the strain energy density, u is the displacement and t is the tension perpendicular to a surface element dS . When a crack is introduced into the material, equation 3.50 becomes:

$$\int_S w dy - t \frac{\partial u}{\partial x} dS = J \quad (3.51)$$

In LEFM, J is essentially equivalent to $\mathcal{G}$, the strain energy release rate. Many finite element software packages, such as ABAQUS[10], have the ability to evaluate J for crack problems. Once J has been determined it can be related to the stress intensity factor, K_I , according to equation 3.49.

3.6 Example Problems

To demonstrate the effectiveness of the BIE method for treating fracture problems, several test problems involving isotropic and orthotropic materials have been analyzed. Results for these problems are also available in the literature, thus the BIE-obtained solutions can be checked with them.

3.6.1 Test Problems in Isotropy

Three sample crack problems have been analyzed. They are long finite-width plates with (i) an edge crack (see figure 3.5), (ii) a double edge crack (see figure 3.6) and (iii) a central crack (see figure 3.7) subjected to remote tension, σ , respectively. The size of the crack was 'a' while the height-to-width ratio of the edge cracked plates, $H/2W$, were taken to be 1.0 and the height-to-width ratio for the double edge crack and the central crack plates, H/W , were taken to be 3.0. Relative crack sizes of $a/2W = 0.5$ for case (i) and $a/W = 0.5$ for cases (ii) and (iii) were arbitrarily chosen for the study.

The boundary element meshes used are shown in figure 3.8. Note that in all these cases, advantage has been taken of the planes of symmetry in modelling the problems. The stress intensity factors obtained from the BIE analysis using the traction formula, equation 3.48, are compared with those available in Tada et al [48] which have been obtained either by conformal mapping or the eigenfunction asymptotic expansion approach. These literature solutions are claimed to be accurate to within 0.5% error for cases (i) and (iii) and to within 2% error for case (ii).

Table 3.1 lists the normalized stress intensity factors, $K_I/\sigma\sqrt{\pi a}$, for the three cases treated. It can be seen that even with relatively coarse meshes, the BIE method produced results which are in excellent agreement with those in the literature.

3.6.2 Test Problems in Orthotropy

The problems of the plate with the double edge cracks and the central cracks treated in the previous section have also been analyzed with orthotropic material properties instead of isotropy. These problems with $H/W = 3.0$ have been investigated by Snyder and Cruse[49] using the BIE method with a Green's function approach to determine the stress intensity factors. Linear boundary elements were employed in their study. For the sake of comparison, the same set of orthotropic material properties as those treated in [49], was chosen for the present BIE analysis, that is:

$$E_{11} = 144.8 \text{ GPa} \quad E_{22} = 11.7 \text{ GPa} \quad G_{12} = 9.65 \text{ GPa} \quad \nu_{12} = 0.21 \quad (3.52)$$

Several crack sizes were analyzed, namely, $a/W = 0.4, 0.6, 0.8$. A typical BEM mesh employed in this study is shown in figure 3.8, where again advantage has been taken of the planes of symmetry. Table 3.2 lists the resulting normalized stress intensity factors obtained from the present BIE analyses and those obtained by Snyder and Cruse[49]. Agreement between the two sets of results for all the crack sizes treated is again excellent indeed.

Chapter 4

Analysis of Cracked Fibre-Metal Laminates

The increased use of ceramic composites and composite/metal hybrids for structural applications in aerospace and other industries has led to greater attention being focused on the crack-tip shielding effects of crack-face fibre-bridging. To fully investigate the bridging effects of a fibre-metal laminate, such as ARALL or GLARE, a three-dimensional study is really necessary. This is intractable analytically and is relatively expensive to carry out numerically, particularly when repeated analyses are required. Simplification of the problems into two-dimensional ones is thus highly desirable and is the main subject of this chapter.

Simplifying the study into two dimensions requires the treatment of the problem in both the x_1 - x_2 plane as well as in the x_1 - x_3 plane (see figures 4.1, 4.2 and 4.3). In the latter, the crack is fully bridged and extends across both the metal layers and the matrix entirely. Each of the metal layers is treated as isotropic whereas the fibre-reinforced laminae are orthotropic. This problem is analyzed in order to provide an estimate of the maximum crack opening displacement, $2v$, for a given height of delamination zone between the layers.

The problem to be treated in the x_1 - x_2 plane is more complex due to the larger number of parameters involved. These include the fatigue crack length, a , the bridging zone length, b , the delamination zone size, d , and n , and m , which are the power-law indices in equations 2.10 and 2.11, respectively. In this study, several

different crack geometries, subjected to remote tension, were treated, namely, a rectangular orthotropic plate with edge, double edge and central cracks, respectively (see figures 4.1, 4.2 and 4.3). All of the two-dimensional analyses were carried out using the BEM.

It is also necessary to perform a three-dimensional study to determine the applicability of a two-dimensional solution to the three-dimensional problem of crack-face bridging in fibre-metal laminates. To this end, a three-dimensional finite element analysis for the case of a fully bridged crack was performed (see figure 2.3) and the fibre-bridging stresses and stress intensity factor solutions are compared with those obtained from the two-dimensional BEM analyses.

4.1 Material Properties

Although the procedures are applicable to general fibre-metal laminates, for the purpose of analysis here, the material properties used in this study correspond to the in-plane (x_1 - x_2) properties of a 2-1 layup of ARALL. They are given below, where the 1, 2 and 3 denote the fibre direction, the transverse direction and through the thickness direction of the aramid-epoxy lamina, respectively.

For the aramid-epoxy layers, the mechanical properties are[9]:

$$\begin{aligned} E_{11} &= 62.7 \text{ GPa} & E_{22} &= 4.56 \text{ GPa} & \nu_{12} &= 0.355 \\ \mu_{12} &= 1.63 \text{ GPa} & \mu_{21} &= 1.53 \text{ GPa} & \nu_{21} &= 0.493 \end{aligned} \quad (4.1)$$

and for the aluminum layers:

$$E = 71.7 \text{ GPa} \quad \mu_{12} = 27.6 \text{ GPa} \quad \nu_{12} = 0.3 \quad (4.2)$$

For the analysis in the x_1 - x_3 plane, the following material properties were assumed[50]:

$$E_{33} = E_{22} \quad (4.3)$$

$$\mu_{13} = \mu_{12} \quad \mu_{23} = \frac{E_{22}}{2(1 + \nu_{32})} \quad (4.4)$$

$$\nu_{21} = \nu_{31} \quad \nu_{23} = \nu_{32} \quad -\frac{\nu_{21}}{E_{22}} = -\frac{\nu_{12}}{E_{11}} \quad (4.5)$$

These properties are also required in the three-dimensional finite element analysis.

For the two-dimensional analysis in the x_1 - x_2 plane the properties were homogenized through the thickness of the laminate. This was accomplished using classical laminate plate theory. The resulting properties are listed below:

$$E_{11} = 69.5 \text{ GPa} \quad E_{22} = 55.9 \text{ GPa} \quad \mu_{12} = 20.7 \text{ GPa} \quad \nu_{12} = 0.3 \quad (4.6)$$

4.2 BEM Analysis of the Laminate in the x_1 - x_3 Plane

As a first step in the two-dimensional treatment of a cracked fibre-metal laminate, a fully bridged crack which has propagated through both the metal layer and the epoxy-matrix across the entire specimen was analyzed. The model is set up such that it is in the x_1 - x_3 plane (see figures 4.1 to 4.3). The "out-of-plane" thickness is relatively large, thus, plane strain conditions were assumed to prevail under the remote uniform tensile stress, σ . As has been mentioned previously, the purpose of this analysis was to obtain an estimate of the 'maximum' crack opening displacement of the crack faces. However, this value was subsequently found to be less than the true maximum because of bending effects which cause the crack-opening displacement in the x_1 - x_2 plane to be greater.

With reference to the right-hand side figures of figures 4.1, 4.2 and 4.3, only one parameter, the delamination zone, d , between the metal and fibre-reinforced layers, was varied. The height to fibre-reinforced laminae thickness ratio, H/t , and the total laminate to fibre-reinforced laminae thickness ratio, T/t were taken to be 20 and 4, respectively.

A typical boundary element mesh employed is shown in figure 4.4. Note that only a quarter of the problem was modelled, taking advantage of the planes of symmetry. The BEM sub-regioning technique was used in order to model the metal and fibre-reinforced polymer layers separately. Also, quarter-point crack-tip elements were used at the delamination boundary.

The variation of the normalized crack opening displacement, $v / \left(\frac{\sigma t}{E_{11}} \right)$, with relative delamination zone size, d/H , is shown in figure 4.5. Of interest to note is that the normalized crack opening displacement increases linearly with increasing delamination zone size from a value of zero for $d/H = 0$ to 4.06 for $d/H = 0.5$. Also shown for comparison, are the values for the three-dimensional finite element analysis of fully bridged cracks which have propagated only half way through the specimen. The larger crack opening displacements seen from this analysis can be attributed to bending of the specimen about the crack-front as mentioned previously. This suggests that the two-dimensional analysis of the problem in this plane will significantly underestimate the largest magnitude of the bridging stresses in the fibres.

4.3 BEM Analysis of the Laminate in the x_1 - x_2 Plane

4.3.1 Problem Statement and Scope

The in-plane, x_1 - x_2 plane, boundary element analysis of fibre-bridged cracked fibre-metal laminates was performed on three different crack problems: (i) an edge crack, (ii) a double edge crack and (iii) a central crack in a rectangular ARALL sheet under remote tension, σ as shown in figures 4.1, 4.2 and 4.3. Since the laminate is very thin through the thickness, plane stress conditions were assumed.

The overall size of the laminates treated was $H/2W = 1$ for problem (i) and $H/W = 1$ for problems (ii) and (iii). The relative crack lengths treated for the former problem were $a/2W = 0.1$ to 0.5 and for the latter problems, $a/W = 0.1$ to 0.5. For each crack geometry, the fibre-bridging stress distribution was determined and the corresponding stress intensity factor obtained. The extent of the bridging length, b , (see figures 4.1 to 4.3) was varied from the fully bridged case to the unbridged case. Also, the bridging stress profiles were changed using the power-law indices n and m in equations 2.10 and 2.11 with n and m taking values of 0, 0.5, and 1. Two delamination zone shapes were considered as well, namely, a quarter-circular zone

and a quarter-elliptical zone with ellipse aspect ratio 2:1. The latter is a more realistic shape in practice and the former represents a more severe situation than would likely be encountered. It is nevertheless of interest to investigate these delamination shapes to see their effects on the fibre-bridging stresses and the corresponding stress intensity factor at the crack-tip.

As has been mentioned in Chapter 2, the nature of this problem is non-linear since the continuous bridging stresses are not known initially. An iterative scheme has to be followed, the details of which, with a numerical example, are given in Appendix A. The orthotropic material properties are the same as those of ARALL with a 2-1 layup and are given in equation 4.6. Figure 4.6 shows a typical boundary element mesh used for the analysis. Again, advantage was taken of symmetry planes and quarter-point crack-tip elements were employed.

A three-dimensional FEM study was undertaken for the case of fully bridged crack-faces using the commercial software package ABAQUS[10] to determine if, in fact, this problem can be adequately modelled as a two-dimensional problem. The crack length and the delamination zone size were varied as in the BIE analyses. Unlike the two-dimensional problem however, the bridging stresses can be determined directly without following an iterative process. An example of a three-dimensional FEM mesh used, taking into account the symmetric geometry and load conditions, is shown in figure 4.7. The FEM elements used are 20-node quadratic isoparametric brick elements with the mid-side points shifted to quarter-points at the crack-front in the metal layer. Quarter-point crack-front elements were not used at the delamination boundary however, being labour intensive. However, this is not expected to greatly affect the bridging stresses or the resulting crack-tip stress intensity factor for the crack in the aluminum layer. The stress intensity factors were calculated from the J-contour integral computed by ABAQUS.

4.3.2 Crack-face Bridging Stresses

FEM Results for the Fully Bridged Case

The variation of the normalized stress distribution with relative distance from the crack-tip, x/a , along the centre-line of the uncracked fibre-reinforced epoxy layer in ARALL for the edge crack problem with an elliptical delamination zone is shown in figure 4.8. The corresponding distributions for the double edge and central crack problems are shown in figures 4.9 and 4.10, respectively. The results for all the crack sizes treated in the region $x > 0$ are also shown in figures 4.11, 4.12 and 4.13, as are the results for the respective crack problems with a circular delamination zone.

In general, the bridging stresses increase as x , the distance from the crack-tip, increases. The bridging stresses also increase in the region directly adjacent to the crack-tip in the aluminum layer. For example, for the fully bridged edge crack problem with a quarter-elliptical delamination zone and $a/2W = 0.5$, the normalized bridging stress at the crack mouth is 3.45. This decreases to 2.82 at a distance of $x/a = 0.208$ and then reaches a value of 4.51 at $x/a = 0$. Again these values are along the centre of the aramid-epoxy layer. The reason for this increase in the bridging stresses near the crack-tip is likely due in part to the induced singularity caused by the use of quarter-point crack-tip elements in the aluminum layer and the single layer of elements in the mesh of the aramid-epoxy layer. Both the edge and double edge crack problems exhibit an increase in normalized bridging stress, σ_B^*/σ , similar to that mentioned above as the distance from the crack-tip, x/a , increases. However, for the central crack problem, the normalized bridging stresses remain relatively constant away from the crack-tip. Thus, the assumption of uniform crack-face bridging stresses by Marissen[3], is only true in the central crack configuration that he has studied.

The normalized closure stresses increase in magnitude from, for example, a value of 2.44 for $a/2W = 0.1$ to 3.75 for $a/2W = 0.5$ adjacent to the crack mouth of the edge crack problem with quarter-elliptical delamination as the relative crack size increases. As the corresponding delamination zone sizes, d/H , increase, these values decrease to 1.82 and 3.22 respectively. The trends are similar for the double edge

and central crack problems. Thus, a relatively small decrease in the fibre-bridging stress can cause quite an extensive amount of delamination between the aluminum and fibre-reinforced epoxy layers. Ignoring the near-tip region, the profiles of the problems treated lie between a linearly increasing function, which would correspond to $n = 1$ in equation 2.10, and a uniform bridging stress function, corresponding to $n = 0$. Thus, it remains to determine whether a two-dimensional solution, with the appropriate power-law index assumed, could be used to predict the effects of bridging on the stresses and stress intensity factors in fibre-metal laminates.

Two-Dimensional BIE Results

The variations of normalized bridging stress at the last intact fibre, σ_B^*/σ , with relative crack length, $a/2W$, for the edge crack problem with quarter-circular and quarter-elliptical delaminations are shown in figures 4.14 and 4.15, respectively, for different values of bridging length, b , and fibre-bridging stress indices, n and m . The corresponding results for the variations of σ_B^*/σ with normalized crack length, a/W , for the double edge crack and central crack problems are shown in figures 4.16 and 4.17, and 4.18 and 4.19, respectively. Also shown in these figures are the three-dimensional finite element results. Many of the same results can be seen in Tables 4.1 to 4.12.

The effect of changing the geometry from that of the edge crack problem to the double edge crack and then to the central crack problem, is that the stress in the fibres decreases. For example, for the fully bridged edge crack problem with a quarter-elliptical delamination, a relative crack size of $a/2W = 0.5$, and the power-law index $n = 0.5$, the normalized bridging stress in the last intact fibre, σ_B^*/σ , was found to be 4.08. In comparison, for the same crack size in the double edge and central crack problems the normalized bridging stresses, σ_B^*/σ , were significantly lower at 3.04 and 2.71, as to be expected. Also, as the relative crack size increases, the magnitude of the bridging stresses increases because a larger crack has the ability to open more which in turn may lead to higher strains in the fibres. The increase is evidently non-linear for both the edge and central crack problems and approximately linear for the double edge crack problem.

From the analysis here, the normalized bridging stress, σ_B^*/σ , increases with a decreased relative bridging zone size, because a decrease in the number of fibres bridging the crack increases the stresses in the remaining intact fibres. Although fibre-metal laminates tend to be fully or nearly-fully bridged, comparisons have been made between the fully bridged normalized stresses and the 75% and 50% bridging cases treated, in order to determine the possible range of errors that may result from, for example, an incorrect estimation of the bridging zone length. The results for the edge crack problem with quarter-circular delamination for different values of n and m and different values of relative crack length are shown in Table 4.1 and for the elliptical delamination case, in Table 4.2. The corresponding results for the double edge and central crack problems with both quarter-circular and quarter-elliptical delaminations are listed in Tables 4.3 and 4.4 and Tables 4.5 and 4.6, respectively. For the edge crack problem with quarter-elliptical delamination and $n = 0.5$, for example, the assumption of a 75% or a 50% bridged crack, will result in errors in normalized bridging stress, σ_B^*/σ , of 38.2% and 83.3%, respectively, when compared to the fully bridged situation.

The normalized crack-face bridging stresses also increase with increasing power-law index, n . However, the magnitude of the normalized bridging stresses, σ_B^*/σ , is relatively invariant with m greater than zero. Again, for a fully bridged edge crack problem with a quarter-elliptical delamination zone and $a/2W = 0.5$, $n = 1.0$, the normalized bridging stress in the last intact fibre is 4.72, whereas for $n = m = 0$, $\sigma_B^*/\sigma = 3.33$ and for $m = 1.0$, $\sigma_B^*/\sigma = 3.75$.

As the relative delamination zone size increases, the stress in the fibres decreases due to the smaller area for shear stress transfer between the layers. In fact, a small decrease in the bridging stress results in a much larger delamination zone size as mentioned previously. This is evident when comparing the fully bridged edge crack problem with a quarter-elliptical delamination, $a/2W = 0.5$ and $n = 0.5$, with that of the same problem but with a quarter-elliptical delamination zone. The latter results in a normalized bridging stresses in the last intact fibre of 4.08 while in the former, the normalized stress is 3.37. This trend is in agreement with the experimental studies by Marissen[5].

Comparing the two-dimensional BIE and three-dimensional FEM normalized fibre-bridging stresses leads to an estimate in the power-law index which determines the bridging stress profile. From the three-dimensional bridging stress distribution, the profile lies between a linearly increasing function and a uniform bridging stress as mentioned in the previous subsection. From the two-dimensional analysis, it is evident that the profile is that of a "strain-hardening" rather than a "strain-softening" material, in that, in general, the bridging stress decreases toward the crack-tip. The power index n , for agreement with the three-dimensional results, is approximately 0.5 for quarter-circular delamination zones and slightly lower, between 0 and 0.5, for quarter-elliptical delamination zones. Therefore, considering only the fibre stresses, the two-dimensional BIE solution can provide fairly accurate estimates for the three-dimensional problem. However, determination of the correct value of n is not complete without ensuring the same agreement within the stress intensity factor solutions.

4.3.3 Crack-tip Stress Intensity Factors

Figures 4.20 and 4.21, show the variations of normalized crack-tip stress intensity factor, $K_I/\sigma\sqrt{\pi a}$, with relative crack size for the edge crack problem with quarter-circular and quarter-elliptical delamination zones, respectively. The results from the BIE analyses of the double edge and central crack problems are shown in figures 4.22 and 4.23 and figures 4.24 and 4.25, respectively. In each of these figures, the bridging lengths, b , and power-law indices, n and m , have been varied in a similar manner to the fibre-bridging stress results shown in the previous section. The stress intensity factors, K_I , were calculated using equation 3.48.

The geometry with the highest normalized crack-tip stress intensity factors is the edge crack, followed by the central crack and finally the double edge crack. For example, the edge crack problem with a quarter-elliptical delamination zone, a relative crack size of 0.5 and $n = 0.5$ has a crack-tip stress intensity factor of 0.868. The corresponding values for the double edge and central crack problems are 0.771 and 0.841 respectively. Values of normalized stress intensity factor generally

increase with relative crack length. In all cases of crack-face bridging, the increase in the magnitude of $K_I/\sigma\sqrt{\pi a}$ with relative crack length was found to be most rapid when $n = 1$. The normalized stress intensity factors approach a constant value as the power-indices n and m approach zero and for the case of cracks with a quarter-elliptical delamination zone the normalized stress intensity factors may even decrease slightly with increasing relative crack size for $m > 0$.

The results also show the significant advantages of crack-face bridging in reducing the stress intensity factor at the crack-tip and hence, crack growth. Less extensive bridging with fewer fibres intact across the crack wake causes higher stress intensity factors because the crack faces are less restrained. For example, the normalized crack-tip stress intensity factor, $K_I/\sigma\sqrt{\pi a}$, is 0.868 for the fully bridged edge crack problem with a quarter-elliptical delamination zone, $a/2W = 0.5$ and $n = 0.5$. For a similar problem with only 50% of the crack bridged by fibres, the corresponding value is 1.35. This value increases even more dramatically for the case of an unbridged crack where $K_I/\sigma\sqrt{\pi a}$ reaches a value of 2.82. The largest difference in the magnitudes obtained when comparing a partially bridged crack to a fully bridged one was 1630% for an unbridged edge crack problem with a quarter-elliptical delamination zone, $m = 1.0$ and $a/W = 0.5$. Comparisons of the numerical values of $K_I/\sigma\sqrt{\pi a}$ for partially bridged crack problems with those for the case of a fully bridged crack are shown in Tables 4.7 and 4.8 for the edge crack problem with quarter-circular and quarter-elliptical delamination zones, respectively. Similarly, Tables 4.9 and 4.10 and Tables 4.11 and 4.12 show the comparisons for the double edge and central crack problems, respectively.

The variations of normalized stress intensity factor, $K_I/\sigma\sqrt{\pi a}$, with power-law indices n and m are very similar to those found for the normalized fibre-bridging stresses in the previous section, in that, it increases with n but only changes by a relatively small amount with m greater than zero. For example, for a fully bridged edge crack with a quarter-elliptical delamination zone and $a/2W = 0.5$, $K_I/\sigma\sqrt{\pi a} = 0.479$ when $n = 0$. This increases to a value of 1.07 for $n=1.0$. Whereas $K_I/\sigma\sqrt{\pi a} = 0.182$ when $m = 0.5$ and is 0.160 when $m = 1.0$ for the same crack geometry.

An increase in the relative delamination zone size has the effect of increasing the normalized stress intensity factor thus making bridging less effective. An example of this can be seen by comparing the normalized stress intensity factor for the case of a fully bridged edge crack with a quarter-circular delamination zone with that of one with a quarter-elliptical delamination zone for the same fibre-bridging stress profile. This means that, although delamination between the layers and debonding between the fibres and matrix, may be essential to the ability of the fibres to displace and bridge the crack faces in fibre-metal laminates, the increase in the size of the delamination zone will lessen the beneficial effects of the fibre-bridging, all else being equal. It is also noteworthy to mention again that extensive delamination may be the result of only relatively minor increases in normalized bridging stress.

By comparing the two-dimensional BIE results with the three-dimensional FEM results for the normalized stress intensity factors, one can obtain again numerical estimates for the power-law index, m or n , which can be used in the two-dimensional analysis of these problems. A helpful situation for predictive studies of the cracked laminates would be for these estimates to be the same in value as those determined through similar comparisons with bridging stresses in the previous section. For the fully bridged edge crack problem with a quarter-circular delamination zone and $a/2W = 0.5$, the value of n is about 0.5. For the quarter-elliptical delamination case, n is found to be slightly lower. This is consistent with the values of n obtained from considering the crack-face fibre-bridging stresses before. Thus, it appears to be possible to treat the three-dimensional fibre-bridged crack problem of fibre-metal laminates using two-dimensional models for obtaining the stress intensity factors, provided some care is given to the choice of the assumed bridging stress profile. Of interest to note too is that for a given cracked geometry of the specimen, the discrepancies in the stress intensity factors between the case of $n = 0.5$ and $n = 1.0$ is less than 15% in the fully bridged case, although the corresponding discrepancies between the $n=0$ case and $n = 0.5$ case are somewhat larger.

4.4 Concluding Remarks

The first problem considered, that of a through crack in the aluminum layers of ARALL bridged only by the fibre-reinforced lamina, gave the result that the crack opening displacement increases linearly with increased delamination zone size. However, the 'maximum' values were found to be significantly smaller than those obtained from the three-dimensional FEM analysis with the crack in the aluminum layer being a fraction of the width of the specimen. This can be attributed to the substantial amount of bending in the x_1 - x_2 plane which cannot be accounted for in the two-dimensional analysis in the x_1 - x_3 plane. This analysis is unlikely to provide much useful information for crack growth studies of the fibre-metal laminate.

A number of parameters have been investigated in the two-dimensional study of the laminate in the x_1 - x_2 plane. They are the relative crack length, a/W or $a/2W$, the bridging zone size, b , the bridging stress profile as defined by the power index n or m and the delamination zone size, d . However, it is evident from the results obtained that some of these variables are of greater importance than others in the prediction of crack-face fibre-bridging stresses and crack-tip stress intensity factors. The most important parameter in this regard is the size of the bridging zone, b , even though in the case of ARALL, for example, the cracks are usually fully or near-fully bridged in practice, as has been previously mentioned. Thus, this may not be a critical issue to deal with in the case of ARALL, although, it may not be true for other fibre-metal laminates.

Chapter 5

Summary and Conclusions

In this study, many of the mechanisms responsible for toughening a material, and specifically toughening due to fibre-bridging, have been discussed. The process by which fibre-metal laminates exhibit crack-face fibre-bridging has also been described as well as the parameters which would influence the crack-tip shielding effects of the bridging. Some models that have been applied in literature, such as, the shear-lag model, the compliance spring model and the stress intensity approach have been reviewed in order to assess their applicability to the present work on fibre-metal laminate composites. The stress intensity approach was chosen due to its simplicity and the procedure for the determination of bridging stresses and stress intensity factors have then been outlined. As the mathematical problem is non-linear, since the bridging stresses are initially unknown, the process of determining the fibre-bridging stresses in two dimensions is an iterative one.

For the purpose of analysis, the properties of ARALL have been used in this study. The stress and displacement fields for bridged cracks in fibre-metal composites were obtained through the use of two well-known numerical methods, namely, the finite element method (FEM) and the boundary integral equation method (BIE), also known as the boundary element method (BEM). The FEM has been and is currently widely used in literature and several computer software packages are available, for example, ABAQUS[10]. The BIE method, although not as popular as the FEM, has been well established in the literature to be a particularly efficient numerical tool for stress analysis problems in which large stress gradients exist. The BIE for-

mulation for isotropic and anisotropic elasticity has been briefly reviewed and its application to fracture mechanics discussed in this work. A traction formula was used which enabled the crack-tip stress intensity factors to be obtained directly from the computed tractions. Several crack problems in isotropic and orthotropic plates subjected to remote tension have been analyzed using BEM to test its efficiency and accuracy.

The primary objectives of the present work were: (a) to carry out a study on the effects of changing the various parameters in the crack-face fibre-bridging of fibre-metal laminates on the crack-tip stress intensity factor, and (b) to establish if two-dimensional analysis of this evidently three-dimensional problem is adequate in providing accurate solutions for the crack-bridging stresses and the stress intensity factor. A series of two-dimensional BIE analyses have been performed both in the x_1 - x_3 plane and in the x_1 - x_2 plane (see figures 4.1, 4.2 and 4.3). In the former, a fully bridged crack with varying delamination zone sizes between the aluminum and fibre-reinforced epoxy layers was studied. In the latter, a single edge, double edge and central crack problems were modelled. In addition three-dimensional FEM analyses were performed to provide results for comparison with the two-dimensional BEM analysis.

The results of the maximum crack opening displacement for a fully bridged through crack in ARALL in the x_1 - x_3 plane for a specific delamination height have been discussed. The normalized crack opening displacement increases linearly with increasing relative delamination zone size. However, this problem with its severe cracking did not result in the determination of the 'maximum' value for the crack opening displacement; rather, the values obtained from the x_1 - x_2 plane analysis were found to be higher due to large extent of bending of the crack-faces in this plane. Two-dimensional analysis in this x_1 - x_3 plane is thus unlikely to provide fruitful information for crack growth predictions in the x_1 - x_2 plane.

The trends in calculated bridging stresses and crack-tip stress intensity factor results have been discussed for differing crack size, bridging zone size, bridging stress profile and delamination zone size for several crack problems in the x_1 - x_2 plane. The bridging stresses and crack-tip stress intensity factors generally increase with

increasing crack size and the power-law index, n , where bridging stress decreases towards the crack-tip. These values have been found to vary by relatively small amounts with the power-law index, m , where the bridging stress increases towards the crack-tip. As confirmed by the three-dimensional FEM analysis, fibre-bridged cracking of ARALL exhibits fibre-bridging stresses that decrease toward the crack-tip. Thus, the power-index, m , does not accurately describe the stress profile on the crack-faces.

The bridging stress profile, examined here through a power-law relationship involving the index n or m , has the effect of increasing the crack-tip stress intensity factors as n increases but is almost constant with m greater than zero. The agreement of the two-dimensional BIE and three-dimensional FEM analyses suggests that a power-law index with $n = 0.5$ approximately would be suitable to model fibre-bridging effects in two dimensions for the edge crack and the double edge crack problems, while a value of between 0 and 0.5 would be more suitable for the central crack problem. The appropriate values of n are not constant, however, but vary with crack length and extent of fibre-bridging and delamination.

In addition to the profile of the bridging stresses, the length of the bridging zone also inherently affects the bridging mechanism. The larger the bridging zone, the larger the stress in the intact fibres, resulting in more extensive shielding of the crack-tip and lower stress-intensity factors. Therefore, cracks in ARALL which are usually fully or near-fully bridged have lower crack-tip stress intensity factors and have cracks that are less likely to be critical at a specified stress level than in, say, the aluminum alloy alone. This accounts for the excellent fatigue life properties of ARALL. The study has also confirmed the experimental finding that the delamination zone size has the opposite effect to that of the crack-bridging length, decreasing the effectiveness of the fibre-bridging as its size increases.

Comparison of the BEM results of the stress intensity factors with those obtained by the three-dimensional FEM analysis also indicate that the power-law index n of approximately 0.5 to be an appropriate value for the two-dimensional analysis of the problem. These results also validate the use of a two-dimensional model even though the problem is clearly a three-dimensional one.

The relative importance of the factors which affect the fibre-bridging mechanism of crack-tip shielding in fibre-metal laminates has been found. An incorrect estimation of the amount of fibre-bridging, for example, can result in very high errors indeed in the bridging stresses and crack-tip stress intensity factors. Relatively speaking, smaller errors would from the choice of one of the power-law relationships, that is, the choice of either m or n . Knowledge of the magnitudes of these possible errors is useful in crack growth prediction studies when many of the parameters are often difficult to measure.

5.1 Recommendations for Future Work

This present research can be further developed into a wider study of the bridging mechanism in fibre-metal composites. Hitherto, very few studies which quantify the effects of residual stresses such as those caused by the curing process and pre- or post-cure stretching to obtain favourable stress states in fibre-metal laminates, have been reported. Also, a more detailed study of the interdependency of residual stresses, crack length, bridging zone and delamination size is necessary. To this end, it may be necessary to consider a more detailed treatment of the inter-related effects of the debonding at the fibre-matrix interface and the delamination between the fibre-reinforced epoxy and aluminum layers as the crack in the metal layer grows. These studies, including the one presented here, would strongly compliment much of the experimental work in this area which have been reported or are in progress.

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Crack Type	$\left(\frac{K_I}{\sigma\sqrt{\pi a}}\right)_{BIE}$	$\left(\frac{K_I}{\sigma\sqrt{\pi a}}\right)_{Tada}$	%Difference
Edge	2.8313	2.8269	0.156
Double Edge	1.1878	1.1635	2.09
Central	1.2056	1.1826	1.94

Table 3.1: Comparison of BIE normalized stress intensity factors for an isotropic plate with values from Tada et al. for $a/2W = 0.5$ for the edge crack problem and $a/W = 0.5$ for the double edge and central crack problems[48].

Crack Type	$\frac{a}{W}$	$\left(\frac{K_I}{\sigma\sqrt{\pi a}}\right)_{BIE}$	$\left(\frac{K_I}{\sigma\sqrt{\pi a}}\right)_{Snyder}$	%Difference
Double Edge	0.4	1.1533	1.1329	1.80
	0.6	1.2715	-	-
	0.8	1.6272	-	-
Central	0.4	1.1370	1.1120	2.25
	0.6	1.3207	1.2909	2.31
	0.8	1.7717	1.7552	0.940

Table 3.2: Comparison of BIE normalized stress intensity factors for an orthotropic plate with values from Snyder and Cruse[49].

Index	$\frac{a}{2W}$	Normalized Fibre-Bridging Stress, σ_B^*/σ		
		Fully Bridged	75% Bridged	50% Bridged
n=0.0	0.1	1.736	1.948 (12.2%)	2.052 (18.2%)
	0.2	1.904	2.168 (13.9%)	2.336 (22.7%)
	0.3	2.156	2.532 (17.4%)	2.800 (29.9%)
	0.4	2.480	3.032 (22.3%)	3.492 (40.8%)
	0.5	2.840	3.684 (29.7%)	4.560 (60.6%)
n=0.5	0.1	1.892	2.076 (9.73%)	2.144 (13.3%)
	0.2	2.092	2.336 (11.7%)	2.456 (17.4%)
	0.3	2.428	2.772 (14.2%)	2.972 (22.4%)
	0.4	2.860	3.392 (18.6%)	3.760 (31.5%)
	0.5	3.372	4.240 (25.7%)	4.960 (47.1%)
n=1.0	0.1	2.008	2.164 (7.77%)	2.204 (9.76%)
	0.2	2.252	2.456 (9.06%)	2.536 (12.6%)
	0.3	2.636	2.944 (11.7%)	3.088 (17.1%)
	0.4	3.164	3.656 (15.5%)	3.944 (24.7%)
	0.5	3.812	4.640 (21.7%)	5.320 (39.6%)
m=0.5	0.1	1.428	1.688 (18.2%)	1.860 (30.3%)
	0.2	1.536	1.848 (20.3%)	2.088 (35.9%)
	0.3	1.680	2.092 (24.5%)	2.456 (46.2%)
	0.4	1.684	2.416 (43.5%)	2.984 (77.2%)
	0.5	2.056	2.812 (36.8%)	3.724 (81.1%)
m=1.0	0.1	1.524	1.768 (16.0%)	1.920 (26.0%)
	0.2	1.648	1.944 (18.0%)	2.164 (31.3%)
	0.3	1.824	2.220 (21.7%)	2.560 (40.4%)
	0.4	2.036	2.592 (27.3%)	3.136 (54.0%)
	0.5	2.272	3.048 (34.2%)	3.948 (73.8%)

Table 4.1: Normalized bridging stress in the last intact fibre, σ_B^*/σ , for different relative crack sizes, $a/2W$, and power-law indices: Edge crack problem with quarter-circular delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged normalized bridging stresses.

		Normalized Fibre-Bridging Stress, σ_B^*/σ		
Index	$\frac{a}{2W}$	Fully Bridged	75% Bridged	50% Bridged
n=m=0.0	0.1	2.420	3.016 (24.6%)	3.424 (41.5%)
	0.2	2.580	3.268 (26.7%)	3.816 (47.9%)
	0.3	2.800	3.652 (30.4%)	4.440 (58.6%)
	0.4	3.060	4.160 (35.9%)	5.320 (73.9%)
	0.5	3.328	4.720 (41.8%)	6.520 (95.9%)
n=0.5	0.1	2.736	3.340 (22.1%)	3.684 (34.6%)
	0.2	2.956	3.664 (24.0%)	4.160 (40.7%)
	0.3	3.276	4.160 (27.0%)	4.880 (49.0%)
	0.4	3.600	4.840 (32.2%)	5.960 (62.8%)
	0.5	4.080	5.640 (38.2%)	7.480 (83.3%)
n=1.0	0.1	2.984	3.596 (20.5%)	3.868 (29.6%)
	0.2	3.260	3.964 (21.6%)	4.400 (35.0%)
	0.3	3.668	4.560 (24.3%)	5.200 (41.8%)
	0.4	4.160	5.400 (29.8%)	6.440 (54.8%)
	0.5	4.720	6.400 (35.6%)	8.200 (73.7%)
m=0.5	0.1	1.872	2.444 (30.6%)	2.920 (56.0%)
	0.2	1.944	2.584 (32.9%)	3.200 (64.6%)
	0.3	2.056	2.800 (36.2%)	3.636 (76.8%)
	0.4	2.176	3.072 (41.2%)	4.240 (94.9%)
	0.5	2.292	3.376 (47.3%)	5.000 (118.%)
m=1	0.1	2.032	2.612 (27.9%)	3.072 (49.1%)
	0.2	2.212	2.780 (27.8%)	3.384 (51.0%)
	0.3	2.264	3.036 (31.4%)	3.868 (70.8%)
	0.4	2.416	3.360 (39.1%)	4.520 (87.1%)
	0.5	2.568	3.724 (45.0%)	5.400 (110.%)

Table 4.2: Normalized bridging stress in the last intact fibre, σ_B^*/σ , for different relative crack sizes, $a/2W$, and power-law indices: Edge crack problem with quarter-elliptical delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged normalized bridging stresses.

Index	$\frac{a}{2W}$	Normalized Fibre-Bridging Stress, σ_B^*/σ		
		Fully Bridged	75% Bridged	50% Bridged
n=m=0.0	0.1	1.688	1.884 (11.6%)	1.980 (17.3%)
	0.2	1.740	1.952 (12.2%)	2.064 (18.6%)
	0.3	1.812	2.048 (13.0%)	2.180 (20.3%)
	0.4	1.892	2.156 (14.0%)	2.312 (22.2%)
	0.5	1.968	2.268 (15.2%)	2.440 (24.0%)
n=0.5	0.1	1.836	2.004 (9.15%)	2.064 (12.4%)
	0.2	1.896	2.080 (9.70%)	2.156 (13.7%)
	0.3	1.984	2.192 (10.5%)	2.284 (15.1%)
	0.4	2.084	2.320 (11.3%)	2.424 (16.3%)
	0.5	2.180	2.448 (12.2%)	2.568 (17.8%)
n=1.0	0.1	1.944	2.084 (7.20%)	2.120 (9.05%)
	0.2	2.012	2.172 (7.95%)	2.216 (10.1%)
	0.3	2.112	2.296 (8.71%)	2.352 (11.4%)
	0.4	2.228	2.436 (9.34%)	2.504 (12.4%)
	0.5	2.340	2.576 (10.1%)	2.656 (13.5%)
m=0.5	0.1	1.400	1.644 (17.4%)	1.800 (28.6%)
	0.2	1.432	1.692 (18.2%)	1.868 (30.4%)
	0.3	1.476	1.760 (19.2%)	1.964 (33.1%)
	0.4	1.528	1.840 (20.4%)	2.068 (35.3%)
	0.5	1.576	1.916 (21.6%)	2.176 (38.1%)
m=1.0	0.1	1.492	1.720 (15.3%)	1.856 (24.4%)
	0.2	1.528	1.772 (16.0%)	1.928 (26.2%)
	0.3	1.584	1.848 (16.7%)	2.032 (28.3%)
	0.4	1.640	1.936 (18.0%)	2.144 (30.7%)
	0.5	1.696	2.024 (19.3%)	2.256 (33.0%)

Table 4.3: Normalized bridging stress in the last intact fibre, σ_B^*/σ , for different relative crack sizes, a/W , and power-law indices: Double edge crack problem with quarter-circular delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged normalized bridging stresses.

Index	$\frac{a}{2W}$	Normalized Fibre-Bridging Stress, σ_B^*/σ		
		Fully Bridged	75% Bridged	50% Bridged
n=m=0	0.1	2.376	2.948 (24.1%)	3.320 (39.7%)
	0.2	2.424	3.020 (24.6%)	3.436 (41.7%)
	0.3	2.492	3.128 (25.5%)	3.600 (44.5%)
	0.4	2.568	3.252 (26.6%)	3.788 (47.5%)
	0.5	2.640	3.376 (27.9%)	3.968 (50.3%)
n=0.5	0.1	2.672	3.248 (21.6%)	3.564 (33.4%)
	0.2	2.740	3.340 (21.9%)	3.700 (35.0%)
	0.3	2.832	3.480 (22.9%)	3.892 (37.4%)
	0.4	2.936	3.640 (24.0%)	4.120 (40.3%)
	0.5	3.036	3.796 (25.0%)	4.320 (42.3%)
n=1.0	0.1	2.908	3.468 (19.3%)	3.732 (28.3%)
	0.2	2.988	3.580 (19.8%)	3.884 (30.0%)
	0.3	3.104	3.740 (20.5%)	4.080 (31.4%)
	0.4	3.236	3.932 (21.5%)	4.320 (33.5%)
	0.5	3.356	4.116 (22.6%)	4.560 (35.9%)
m=0.5	0.1	1.840	2.404 (30.7%)	2.840 (54.3%)
	0.2	1.868	2.440 (30.6%)	2.924 (56.5%)
	0.3	1.904	2.504 (31.5%)	3.048 (60.1%)
	0.4	1.940	2.580 (33.0%)	3.180 (63.9%)
	0.5	1.980	2.656 (34.1%)	3.316 (67.5%)
m=1.0	0.1	2.004	2.564 (27.9%)	2.988 (49.1%)
	0.2	2.040	2.608 (27.8%)	3.080 (51.0%)
	0.3	2.080	2.684 (29.0%)	3.212 (54.4%)
	0.4	2.128	2.776 (30.5%)	3.360 (57.9%)
	0.5	2.172	2.864 (31.9%)	3.508 (61.5%)

Table 4.4: Normalized bridging stress in the last intact fibre, σ_B^*/σ , for different relative crack sizes, a/W , and power-law indices: Double edge crack problem with quarter-elliptical delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged normalized bridging stresses.

Index	$\frac{a}{2W}$	Normalized Fibre-Bridging Stress, σ_B^*/σ		
		Fully Bridged	75% Bridged	50% Bridged
n=m=0	0.1	1.352	1.552 (14.8%)	1.684 (24.6%)
	0.2	1.396	1.612 (15.5%)	1.756 (25.8%)
	0.3	1.468	1.704 (16.1%)	1.868 (27.2%)
	0.4	1.564	1.828 (16.9%)	2.028 (29.7%)
	0.5	1.684	1.992 (18.3%)	2.308 (37.1%)
n=0.5	0.1	1.448	1.632 (12.7%)	1.748 (20.7%)
	0.2	1.500	1.700 (13.3%)	1.828 (21.9%)
	0.3	1.588	1.804 (13.6%)	1.948 (22.7%)
	0.4	1.704	1.952 (14.6%)	2.120 (24.4%)
	0.5	1.852	2.140 (15.6%)	2.344 (26.6%)
n=1.0	0.1	1.516	1.684 (11.1%)	1.792 (18.2%)
	0.2	1.576	1.760 (11.7%)	1.872 (18.8%)
	0.3	1.668	1.876 (12.5%)	2.000 (19.9%)
	0.4	1.804	2.036 (12.9%)	2.184 (21.1%)
	0.5	1.972	2.244 (13.8%)	2.420 (22.7%)
m=0.5	0.1	1.156	1.380 (19.4%)	1.548 (33.9%)
	0.2	1.188	1.428 (20.2%)	1.608 (35.4%)
	0.3	1.236	1.496 (21.0%)	1.700 (37.5%)
	0.4	1.300	1.588 (22.2%)	1.828 (40.6%)
	0.5	1.376	1.704 (23.8%)	1.992 (44.8%)
m=1.0	0.1	1.220	1.432 (17.4%)	1.592 (30.5%)
	0.2	1.252	1.484 (18.5%)	1.656 (32.3%)
	0.3	1.308	1.560 (19.3%)	1.752 (33.9%)
	0.4	1.380	1.660 (20.2%)	1.892 (37.1%)
	0.5	1.468	1.788 (21.8%)	2.064 (40.6%)

Table 4.5: Normalized bridging stress in the last intact fibre, σ_B^*/σ , for different relative crack sizes, a/W , and power-law indices: Central crack problem with semi-circular delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged normalized bridging stresses.

Index	$\frac{a}{2W}$	Normalized Fibre-Bridging Stress, σ_B^*/σ		
		Fully Bridged	75% Bridged	50% Bridged
n=m=0	0.1	2.020	2.504 (24.0%)	2.872 (42.2%)
	0.2	2.068	2.580 (24.8%)	2.976 (43.9%)
	0.3	2.148	2.696 (25.5%)	3.140 (46.2%)
	0.4	2.248	2.844 (26.5%)	3.364 (49.6%)
	0.5	2.372	3.032 (27.8%)	3.608 (52.1%)
n=0.5	0.1	2.244	2.724 (21.4%)	3.060 (36.4%)
	0.2	2.308	2.820 (22.2%)	3.180 (37.8%)
	0.3	2.408	2.960 (22.9%)	3.372 (40.0%)
	0.4	2.548	3.152 (23.7%)	3.632 (42.5%)
	0.5	2.712	3.392 (25.1%)	3.964 (46.2%)
n=1.0	0.1	2.412	2.880 (19.4%)	3.192 (32.3%)
	0.2	2.488	2.992 (20.3%)	3.324 (33.6%)
	0.3	2.612	3.156 (20.8%)	3.532 (35.2%)
	0.4	2.780	3.380 (21.6%)	3.820 (37.4%)
	0.5	2.984	3.660 (22.7%)	4.200 (40.8%)
m=0.5	0.1	1.612	2.084 (29.3%)	2.488 (54.3%)
	0.2	1.640	2.136 (30.2%)	2.568 (56.6%)
	0.3	1.684	2.208 (31.1%)	2.688 (59.6%)
	0.4	1.740	2.300 (32.2%)	2.852 (63.9%)
	0.5	1.804	2.412 (33.7%)	3.052 (69.2%)
m=1.0	0.1	1.736	2.212 (27.4%)	2.608 (50.2%)
	0.2	1.772	2.272 (28.2%)	2.696 (52.1%)
	0.3	1.820	2.352 (29.2%)	2.824 (55.2%)
	0.4	1.888	2.456 (30.1%)	3.004 (59.1%)
	0.5	1.964	2.588 (31.8%)	3.228 (64.4%)

Table 4.6: Normalized bridging stress in the last intact fibre, σ_B^*/σ , for different relative crack sizes, a/W , and power-law indices: Central crack problem with semi-elliptical delamination. Values in parenthesis indicate percentage the differences between partially bridged and fully bridged normalized bridging stresses.

		Normalized Stress Intensity Factor, $K_I/\sigma(\pi a)$			
Index	$\frac{a}{2W}$	Fully Bridged	75% Bridged	50% Bridged	Unbridged
n=m=0.0	0.1	0.680	0.729 (7.21%)	0.820 (20.6%)	1.200 (76.5%)
	0.2	0.726	0.792 (9.09%)	0.914 (25.9%)	1.380 (90.1%)
	0.3	0.775	0.873 (12.6%)	1.050 (35.5%)	1.640 (112.%)
	0.4	0.810	0.953 (17.7%)	1.220 (50.6%)	2.120 (162.%)
	0.5	0.817	1.010 (23.6%)	1.430 (75.0%)	2.820 (245.%)
n=0.5	0.1	0.880	0.924 (5.00%)	0.987 (12.2%)	1.200 (36.4%)
	0.2	0.955	1.020 (6.81%)	1.110 (16.2%)	1.380 (44.5%)
	0.3	1.050	1.150 (9.52%)	1.290 (22.9%)	1.640 (56.2%)
	0.4	1.140	1.300 (14.0%)	1.540 (35.1%)	2.120 (86.0%)
	0.5	1.210	1.450 (19.8%)	1.870 (54.5%)	2.820 (133.%)
n=1.0	0.1	0.958	0.999 (4.28%)	1.050 (9.60%)	1.200 (25.3%)
	0.2	1.050	1.10 (5.71%)	1.190 (13.3%)	1.380 (31.4%)
	0.3	1.170	1.270 (8.55%)	1.390 (18.8%)	1.640 (40.2%)
	0.4	1.300	1.460 (12.3%)	1.680 (29.2%)	2.120 (63.1%)
	0.5	1.400	1.670 (19.3%)	2.060 (47.1%)	2.820 (101.%)
m=0.5	0.1	0.461	0.489 (6.07%)	0.597 (29.6%)	1.200 (160.%)
	0.2	0.477	0.518 (8.60%)	0.653 (36.9%)	1.380 (189.%)
	0.3	0.487	0.546 (12.1%)	0.725 (48.9%)	1.640 (237.%)
	0.4	0.484	0.558 (15.3%)	0.802 (65.7%)	2.120 (338.%)
	0.5	0.454	0.535 (17.8%)	0.862 (89.9%)	2.820 (521.%)
m=1.0	0.1	0.471	0.510 (8.28%)	0.621 (31.8%)	1.200 (155.%)
	0.2	0.489	0.543 (11.0%)	0.682 (39.5%)	1.380 (182.%)
	0.3	0.503	0.576 (14.5%)	0.762 (51.5%)	1.640 (226.%)
	0.4	0.500	0.592 (18.4%)	0.849 (69.8%)	2.120 (324.%)
	0.5	0.467	0.576 (23.3%)	0.926 (98.3%)	2.820 (504.%)

Table 4.7: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi a)^{1/2}$, for different relative crack sizes, $a/2W$, and power-law indices: Edge crack problem with quarter-circular delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged cases.

		Normalized Stress Intensity Factor, $K_I/\sigma(\pi a)$			
Index	$\frac{a}{2W}$	Fully Bridged	75% Bridged	50% Bridged	Unbridged
n=m=0.0	0.1	0.477	0.472 (1.05%)	0.568 (19.1%)	1.200 (152.%)
	0.2	0.494	0.497 (0.607%)	0.618 (25.1%)	1.380 (178.%)
	0.3	0.506	0.519 (2.57%)	0.680 (34.4%)	1.640 (224.%)
	0.4	0.503	0.517 (2.78%)	0.741 (47.3%)	2.120 (321.%)
	0.5	0.479	0.499 (4.18%)	0.778 (62.4%)	2.820 (489.%)
n=0.5	0.1	0.738	0.758 (2.71%)	0.836 (13.3%)	1.200 (62.6%)
	0.2	0.783	0.815 (4.09%)	0.922 (17.8%)	1.380 (76.2%)
	0.3	0.829	0.887 (7.00%)	1.050 (26.7%)	1.640 (97.8%)
	0.4	0.864	0.940 (8.80%)	1.200 (38.9%)	2.120 (145.%)
	0.5	0.868	0.994 (14.5%)	1.350 (55.0%)	2.820 (225.%)
n=1.0	0.1	0.841	0.869 (3.33%)	0.938 (11.5%)	1.200 (42.7%)
	0.2	0.902	0.945 (4.77%)	1.040 (15.3%)	1.380 (53.0%)
	0.3	0.971	1.040 (7.11%)	1.200 (23.6%)	1.640 (68.9%)
	0.4	1.030	1.140 (10.7%)	1.390 (35.0%)	2.120 (106.%)
	0.5	1.070	1.240 (15.9%)	1.620 (51.4%)	2.820 (164.%)
m=0.5	0.1	0.239	0.172 (28.0%)	0.254 (6.28%)	1.200 (402.%)
	0.2	0.236	0.175 (25.8%)	0.266 (12.7%)	1.380 (485.%)
	0.3	0.227	0.165 (27.3%)	0.269 (18.5%)	1.640 (622.%)
	0.4	0.209	0.133 (36.4%)	0.246 (17.8%)	2.120 (914.%)
	0.5	0.182	0.076 (58.5%)	0.190 (4.40%)	2.820 (1450%)
m=1.0	0.1	0.229	0.182 (20.5%)	0.274 (19.7%)	1.200 (424.%)
	0.2	0.228	0.183 (19.7%)	0.288 (26.3%)	1.380 (505.%)
	0.3	0.220	0.172 (21.8%)	0.295 (34.1%)	1.640 (645.%)
	0.4	0.197	0.138 (29.9%)	0.287 (45.7%)	2.120 (976.%)
	0.5	0.160	0.078 (51.2%)	0.228 (42.5%)	2.820 (1630%)

Table 4.8: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi a)^{1/2}$, for different relative crack sizes, $a/2W$, and power-law indices: Edge crack problem with quarter-elliptical delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged cases.

		Normalized Stress Intensity Factors, $K_I/\sigma(\pi a)$			
Index	$\frac{a}{2W}$	Fully Bridged	75% Bridged	50% Bridged	Unbridged
n=m=0.0	0.1	0.666	0.710 (6.61%)	0.793 (19.1%)	1.150 (72.7%)
	0.2	0.682	0.730 (7.04%)	0.822 (20.5%)	1.200 (76.0%)
	0.3	0.697	0.753 (8.03%)	0.856 (22.8%)	1.270 (82.2%)
	0.4	0.707	0.769 (8.77%)	0.884 (25.0%)	1.340 (89.5%)
	0.5	0.710	0.776 (9.30%)	0.904 (27.3%)	1.390 (95.8%)
n=0.5	0.1	0.857	0.896 (4.55%)	0.953 (11.2%)	1.150 (34.2%)
	0.2	0.883	0.926 (4.87%)	0.990 (12.1%)	1.200 (35.9%)
	0.3	0.910	0.961 (5.60%)	1.030 (13.2%)	1.270 (39.6%)
	0.4	0.932	0.990 (6.22%)	1.080 (15.9%)	1.340 (43.8%)
	0.5	0.947	1.010 (6.65%)	1.110 (17.2%)	1.390 (46.8%)
n=1.0	0.1	0.931	0.966 (3.76%)	1.010 (8.49%)	1.150 (23.5%)
	0.2	0.961	1.000 (4.06%)	1.050 (9.26%)	1.200 (24.9%)
	0.3	0.996	1.040 (4.42%)	1.100 (10.4%)	1.270 (27.5%)
	0.4	1.020	1.080 (5.88%)	1.150 (12.7%)	1.340 (31.4%)
	0.5	1.040	1.110 (6.73%)	1.190 (14.4%)	1.390 (33.7%)
m=0.5	0.1	0.453	0.479 (5.74%)	0.579 (27.8%)	1.150 (154.%)
	0.2	0.460	0.489 (6.30%)	0.597 (29.8%)	1.200 (161.%)
	0.3	0.466	0.498 (6.87%)	0.616 (32.2%)	1.270 (173.%)
	0.4	0.464	0.499 (7.54%)	0.628 (35.3%)	1.340 (189.%)
	0.5	0.458	0.493 (7.64%)	0.633 (38.2%)	1.390 (203.%)
m=1.0	0.1	0.463	0.499 (7.78%)	0.602 (30.0%)	1.150 (148.%)
	0.2	0.470	0.510 (8.51%)	0.576 (7.66%)	1.200 (124.%)
	0.3	0.477	0.520 (9.01%)	0.642 (22.6%)	1.270 (155.%)
	0.4	0.475	0.523 (10.1%)	0.657 (38.3%)	1.340 (182.%)
	0.5	0.424	0.516 (21.7%)	0.662 (56.1%)	1.390 (228.%)

Table 4.9: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi a)^{1/2}$, for different relative crack sizes, a/W , and power-law indices: Double edge crack problem with quarter-circular delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged cases.

Index	$\frac{a}{2W}$	Normalized Stress Intensity Factor, $K_I/\sigma(\pi a)$			
		Fully Bridged	75% Bridged	50% Bridged	Unbridged
n=m=0	0.1	0.470	0.463 (1.49%)	0.553 (17.7%)	1.150 (145.%)
	0.2	0.477	0.472 (1.05%)	0.569 (19.3%)	1.200 (152.%)
	0.3	0.482	0.480 (0.415%)	0.586 (21.6%)	1.270 (163.%)
	0.4	0.482	0.481 (0.207%)	0.595 (23.4%)	1.340 (178.%)
	0.5	0.477	0.474 (0.629%)	0.597 (25.2%)	1.390 (191.%)
n=0.5	0.1	0.712	0.740 (3.93%)	0.810 (13.8%)	1.150 (61.5%)
	0.2	0.740	0.759 (2.57%)	0.837 (13.1%)	1.200 (62.2%)
	0.3	0.756	0.780 (3.17%)	0.870 (15.1%)	1.270 (68.0%)
	0.4	0.767	0.794 (3.52%)	0.895 (16.7%)	1.340 (74.7%)
	0.5	0.771	0.802 (4.02%)	0.915 (18.7%)	1.390 (80.3%)
n=1.0	0.1	0.823	0.846 (2.79%)	0.908 (10.3%)	1.150 (39.7%)
	0.2	0.844	0.870 (3.08%)	0.940 (11.4%)	1.200 (42.2%)
	0.3	0.867	0.898 (3.58%)	0.981 (13.1%)	1.270 (46.5%)
	0.4	0.883	0.921 (4.30%)	1.020 (15.5%)	1.340 (51.8%)
	0.5	0.873	0.935 (7.10%)	1.040 (19.1%)	1.390 (55.7%)
m=0.5	0.1	0.235	0.173 (26.4%)	0.250 (6.38%)	1.150 (389.%)
	0.2	0.234	0.173 (26.1%)	0.255 (8.97%)	1.200 (413.%)
	0.3	0.233	0.172 (26.2%)	0.255 (9.44%)	1.270 (445.%)
	0.4	0.228	0.162 (28.9%)	0.249 (9.21%)	1.340 (488.%)
	0.5	0.218	0.146 (33.0%)	0.234 (7.34%)	1.390 (538.%)
m=1.0	0.1	0.228	0.179 (21.5%)	0.269 (18.0%)	1.150 (404.%)
	0.2	0.226	0.183 (19.7%)	0.274 (20.2%)	1.200 (426.%)
	0.3	0.226	0.181 (19.9%)	0.276 (22.1%)	1.270 (462.%)
	0.4	0.219	0.169 (22.8%)	0.271 (23.7%)	1.340 (512.%)
	0.5	0.210	0.153 (27.1%)	0.257 (22.4%)	1.390 (562.%)

Table 4.10: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi a)^{1/2}$, for different relative crack sizes, a/W , and power-law indices: Double edge crack problem with quarter-elliptical delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged cases.

Index	$\frac{a}{2W}$	Normalized Stress Intensity Factor, $K_I/\sigma(\pi a)$			
		Fully Bridged	75% Bridged	50% Bridged	Unbridged
n=m=0	0.1	0.685	0.697 (1.75%)	0.743 (8.47%)	1.030 (50.4%)
	0.2	0.705	0.720 (2.13%)	0.773 (9.65%)	1.080 (53.2%)
	0.3	0.735	0.755 (2.72%)	0.816 (11.0%)	1.160 (57.8%)
	0.4	0.770	0.797 (3.51%)	0.871 (13.1%)	1.260 (63.6%)
	0.5	0.806	0.841 (4.34%)	0.934 (15.9%)	1.390 (72.5%)
n=0.5	0.1	0.834	0.848 (1.68%)	0.877 (5.16%)	1.030 (23.5%)
	0.2	0.863	0.879 (1.85%)	0.913 (5.79%)	1.080 (25.1%)
	0.3	0.904	0.926 (2.43%)	0.968 (7.08%)	1.160 (28.3%)
	0.4	0.957	0.985 (3.24%)	1.040 (8.67%)	1.260 (31.7%)
	0.5	1.010	1.050 (3.96%)	1.120 (10.9%)	1.390 (37.6%)
n=1.0	0.1	0.890	0.903 (1.46%)	0.926 (4.04%)	1.030 (15.7%)
	0.2	0.921	0.937 (1.74%)	0.965 (4.78%)	1.080 (17.3%)
	0.3	0.970	0.991 (2.16%)	1.030 (6.19%)	1.160 (19.6%)
	0.4	1.030	1.060 (2.91%)	1.100 (6.80%)	1.260 (22.3%)
	0.5	1.100	1.140 (3.64%)	1.200 (9.09%)	1.390 (26.4%)
m=0.5	0.1	0.510	0.507 (0.588%)	0.563 (10.4%)	1.030 (102.%)
	0.2	0.521	0.520 (0.192%)	0.582 (11.7%)	1.080 (107.%)
	0.3	0.537	0.539 (0.372%)	0.611 (13.8%)	1.160 (116.%)
	0.4	0.555	0.560 (0.901%)	0.646 (16.4%)	1.260 (127.%)
	0.5	0.571	0.579 (1.40%)	0.681 (19.3%)	1.390 (143.%)
m=1.0	0.1	0.520	0.525 (0.962%)	0.582 (11.9%)	1.030 (98.1%)
	0.2	0.523	0.538 (0.938%)	0.602 (12.9%)	1.080 (103.%)
	0.3	0.550	0.559 (1.64%)	0.634 (15.3%)	1.160 (111.%)
	0.4	0.570	0.583 (2.28%)	0.670 (17.5%)	1.260 (121.%)
	0.5	0.587	0.605 (3.07%)	0.709 (20.8%)	1.390 (137.%)

Table 4.11: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi a)^{1/2}$, for different relative crack sizes, a/W , and power-law indices: Central crack problem with semi-circular delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged cases.

Index	$\frac{a}{2W}$	Normalized Stress Intensity Factor, $K_I/\sigma(\pi a)$			
		Fully Bridged	75% Bridged	50% Bridged	Unbridged
n=m=0	0.1	0.513	0.493 (3.90%)	0.542 (5.65%)	1.030 (101.%)
	0.2	0.525	0.504 (4.00%)	0.559 (6.48%)	1.080 (106.%)
	0.3	0.539	0.521 (3.34%)	0.584 (8.35%)	1.160 (115.%)
	0.4	0.555	0.539 (2.88%)	0.614 (10.6%)	1.260 (127.%)
	0.5	0.569	0.555 (2.46%)	0.657 (15.5%)	1.390 (144.%)
n=0.5	0.1	0.726	0.725 (0.138%)	0.762 (4.96%)	1.030 (41.9%)
	0.2	0.745	0.746 (0.134%)	0.789 (5.91%)	1.080 (45.0%)
	0.3	0.774	0.778 (0.517%)	0.830 (7.24%)	1.160 (49.9%)
	0.4	0.806	0.816 (1.24%)	0.882 (9.43%)	1.260 (56.3%)
	0.5	0.841	0.856 (1.78%)	0.940 (11.8%)	1.390 (65.3%)
n=1.0	0.1	0.806	0.811 (0.620%)	0.843 (4.59%)	1.030 (27.8%)
	0.2	0.830	0.838 (0.964%)	0.875 (5.42%)	1.080 (30.1%)
	0.3	0.865	0.877 (1.39%)	0.924 (6.82%)	1.160 (34.1%)
	0.4	0.908	0.926 (1.98%)	0.987 (8.70%)	1.260 (38.8%)
	0.5	0.954	0.979 (2.62%)	1.060 (11.1%)	1.390 (45.7%)
m=0.5	0.1	0.304	0.240 (21.1%)	0.279 (8.22%)	1.030 (239.%)
	0.2	0.308	0.241 (21.8%)	0.285 (7.47%)	1.080 (251.%)
	0.3	0.313	0.244 (22.0%)	0.294 (6.07%)	1.160 (271.%)
	0.4	0.316	0.246 (22.2%)	0.301 (4.75%)	1.260 (299.%)
	0.5	0.317	0.243 (23.3%)	0.304 (4.10%)	1.390 (338.%)
m=1.0	0.1	0.303	0.249 (17.8%)	0.296 (2.31%)	1.030 (240.%)
	0.2	0.306	0.250 (18.3%)	0.302 (1.31%)	1.080 (253.%)
	0.3	0.313	0.255 (18.5%)	0.313 (0.00%)	1.160 (271.%)
	0.4	0.316	0.258 (18.4%)	0.322 (1.90%)	1.260 (299.%)
	0.5	0.317	0.254 (19.9%)	0.326 (2.84%)	1.390 (338.%)

Table 4.12: Normalized crack-tip stress intensity factors, $K_I/\sigma(\pi a)^{1/2}$, for different relative crack sizes, a/W , and power-law indices: Central crack problem with semi-elliptical delamination. Values in parenthesis indicate the percentage differences between partially bridged and fully bridged cases.

Figure 1.1: 3-2 layup of ARALL[9].

Crack deflection and meandering

Corrosion debris-induced crack closure

Crack surface roughness-induced closure

Transformation toughening

Ligament or fiber toughening

Microcrack toughening

Sliding crack surface interference

Crack field void formation

Plasticity-induced crack closure

Figure 2.1: Mechanisms which increase material toughness[11].

Figure 2.2: Mechanism of fibre-bridging in composites.

Figure 2.3: Schematic of the delaminations between the aramid-epoxy and aluminum layers and fibre-bridging[3].

Figure 2.4: Shear-lag model for fibre-bridging.[16]

Figure 2.5: Steady-state matrix cracking in composites.

Figure 2.6: Shear stress at the fibre-matrix interface.[18]

Equivalent Spring System

Figure 2.7: Compliance spring model for fibre-bridging.[21]

Figure 2.8: Continuous closure pressure or bridging stress.

Figure 2.9: Crack-face bridging stress distributions.[22]

Figure 2.10: Bueckner's principle.

Figure 3.1: BIE definitions.

Figure 3.2: Sub-regioning of the numerical solution domain into sub domains, n and m .

Figure 3.3: Definition of co-ordinate systems at the crack-tip[40].

Figure 3.4: Quarter-point crack-tip elements[40].

Figure 3.5: Edge crack problem for isotropic and orthotropic test cases. In the case of orthotropy, the fibres are taken to be aligned in the x_1 direction.

Figure 3.6: Double edge crack problem for isotropic and orthotropic test cases. In the case of orthotropy, the fibres are taken to be aligned in the x_1 direction.

Figure 3.7: Central crack problem for isotropic and orthotropic test cases. In the case of orthotropy, the fibres are taken to be aligned in the x_1 direction.

Figure 3.8: Typical BIE mesh for isotropic and orthotropic test problems.

Figure 4.1: Schematic of the edge crack problem.

Figure 4.2: Schematic of the double edge crack problem.

Figure 4.3: Schematic of the central crack problem

Figure 4.11b: Variations of normalized bridging stress, σ_B / σ with relative distance from the crack-tip, x / a , along the centreline of the aramid epoxy layer, for the fully bridged edge crack problem (quarter-elliptical delamination zone)

Figure 4.4: Typical BIE mesh for the fully bridged through-crack problem in the x_1 - x_3 plane.

Figure 4.5: Variation of normalized crack opening displacement, $v / (\frac{\sigma t}{E_{11}})$, with relative delamination zone size, d / H , for a fully bridged through-crack in the $x_1 - x_3$ plane.

Figure 4.6: Typical mesh for crack problems in the x_1, x_2 plane of the ARALL laminate, $a/W = 0.4$.

Figure 4.7: A typical three-dimensional FEM mesh for the analysis of the fully bridged cracked ARALL laminate: a. $W = 0.1$, $d/H = 0.25$. No. of elements = 2616. No. of nodes = 20571.

Figure 4.8: Variations of normalized bridging stress, σ_B / σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged edge crack problem (quarter-elliptical delamination zone).

Figure 4.9: Variations of normalized bridging stress, σ_B / σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged double edge crack problem (quarter-elliptical delamination zone).

Figure 4.10: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid epoxy layer, for the fully bridged central crack problem (semi-elliptical delamination zone)

Figure 4.11a. Variations of normalized bridging stress, σ_B / σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged edge crack problem (quarter-circular delamination zone).

Figure 4.11b: Variations of normalized bridging stress, σ_B / σ with relative distance from the crack-tip, x/a , along the centreline of the aramid epoxy layer, for the fully bridged edge crack problem (quarter-elliptical delamination zone)

Figure 4.12a: Variations of normalized bridging stress, σ_B / σ , with relative distance from the crack tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged double edge crack problem (quarter-circular delamination zone).

Figure 4.12b: Variations of normalized bridging stress, σ_B / σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged double edge crack problem (quarter-elliptical delamination zone)

Figure 4.13a: Variations of normalized bridging stress, σ_B/σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid-epoxy layer, for the fully bridged central crack problem (semi-circular delamination zone).

Figure 4.13b: Variations of normalized bridging stress, σ_B / σ , with relative distance from the crack-tip, x/a , along the centreline of the aramid epoxy layer, for the fully bridged central crack problem (semi-elliptical delamination zone).

Figure 4.14: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, $a/2W$, for the edge crack problem (quarter-circular delamination).

Figure 4.15: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, $a/2W$, for the edge crack problem (quarter-elliptical delamination).

Figure 4.16: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, a/W , for the double edge crack problem (quarter-circular delamination).

Figure 4.17: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, a/W , for the double edge crack problem (quarter-elliptical delamination).

Figure 4.18: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, a/W , for the central crack problem (semi-circular delamination).

Figure 4.19: Variations of normalized bridging stress, σ_B^*/σ , with relative crack length, a/W , for the central crack problem (semi-elliptical delamination).

Figure 4.20: Variations of normalized crack-tip stress intensity factor, $K_I / \sigma(\pi a)^{1/2}$, with relative crack length, $a/2W$, for the edge crack problem (quarter-circular delamination).

Figure 4.21: Variations of normalized crack-tip stress intensity factor, $K_I / (\sigma(\pi a)^{1/2})$, with relative crack length, $a/2W$, for the edge crack problem (quarter elliptical delamination).

Figure 4.22: Variations of normalized crack-tip stress intensity factor, $K_I / (\sigma(\pi a)^{1/2})$, with relative crack length, a/W , for the double edge crack problem (quarter-circular delamination).

Figure 4.23: Variations of normalized crack tip stress intensity factor, $K_I / (\sigma(\pi a)^{1/2})$, with relative crack length, a/W , for the double edge crack problem (quarter-elliptical delamination).

Figure 4.24: Variations of normalized crack-tip stress intensity factor, $K_I / \sigma (\pi a)^{1/2}$, with relative crack length, a/W , for the central crack problem (semi-circular delamination)

Figure 4.25: Variations of normalized crack-tip stress intensity factor, $K_I / (\sigma(\pi a)^{1/2})$, with relative crack length, a/W for the central crack problem (semi-elliptical delamination).

Appendix A

Numerical Determination of σ_B^* – Example

A numerical example which highlights the iterative process in which σ_B^* is determined is given below. The numbers used are for the fully bridged edge crack problem with a quarter- elliptical delamination between the aluminum and aramid-epoxy layers. The relative crack length is $a/2W = 0.5$ with a taken to be 10 mm. for illustration. The bridging stress profile is taken to be of the form of equation 2.10 with $n = 0.5$. For the analysis, the applied stress, σ , is 1 MPa and $d = 5$ mm. since the ellipse aspect ratio is 2:1. Also, the Young's modulus in the direction of the fibres, E_{11} , is 62.7 GPa.

Initialization:

Step 1: Arbitrarily set the initial value of σ_B^* , say $\sigma_B^* = 1.05 \text{ MPa}$.

Step 2: Calculate the crack-face bridging stresses to be applied at the node points from equation 2.10 (or 2.11 if the analysis involves this stress profile). The results are in Table A.1.

Step 3: Carry out the BEM analysis. Obtain the crack opening displacement, v^* , and compute the corresponding value of σ_B^* from equation 2.14. For this case, the BEM computed values are:

$$v^* = 0.28304 \times 10^{-3} \text{ mm} \quad (\text{A.1})$$

Element	Node	ξ	Bridging Stresses (MPa)
mouth	1	1.000	1.05
mouth	2	0.875	0.982
mouth	3	0.750	0.909
2	1	0.750	0.909
2	2	0.625	0.830
2	3	0.500	0.742
3	1	0.500	0.742
3	2	0.410	0.672
3	3	0.320	0.594
4	1	0.320	0.594
4	2	0.240	0.514
4	3	0.160	0.420
tip	1	0.160	0.420
tip	2	0.040	0.210
tip	3	0.000	0.000

Table A.1: Nodal Stresses for the Initialization

$$\begin{aligned}\sigma_B^* &= E_{11} \left(\frac{A}{A_T} \right) \left(\frac{v^*}{d} \right) \\ &= (62.7 \times 10^9) \left(\frac{1}{4} \right) \left(\frac{0.28304 \times 10^{-6}}{5 \times 10^{-3}} \right) = 0.887 \text{ MPa}\end{aligned}\quad (\text{A.2})$$

Step 4: Compare the above value of σ_B^* with that used initially for this iteration. To speed up the iterative process one may take, for example, the average of these values to use as the σ_B^* for the next iteration. For this case:

$$\sigma_B^* = \frac{1}{2} (1.05 + 0.887) = 0.969 \text{ MPa} \quad (\text{A.3})$$

The results for the iterations are presented below. It is terminated when successive BEM computed values of σ_B^* converge to within a set tolerance.

Iteration #1:

Take σ_B^* from the end of the previous iteration:

$$\sigma_B^* = 0.969 \text{ MPa} \quad (\text{A.4})$$

and repeat steps 2 and 3 above. The calculated bridging stress according to equation 2.10 are shown in Table A.2. The results for the second BEM run with this set of bridging stresses give:

$$v^* = 0.38264 \times 10^{-3} \text{ mm} \quad \sigma_B^* = 1.20 \text{ MPa} \quad (\text{A.5})$$

According to step 4 above, the new value of σ_B^* becomes:

$$\sigma_B^* = \frac{1}{2} (0.969 + 1.20) = 1.01 \text{ MPa} \quad (\text{A.6})$$

Iteration #2:

Take σ_B^* from the previous iteration:

$$\sigma_B^* = 1.01 \text{ MPa} \quad (\text{A.7})$$

Steps 2 and 3 above are repeated. See Table A.3 for the calculated bridging stresses according to equation 2.10. The results from the third BEM computer run give:

$$v^* = 0.33190 \times 10^{-3} \text{ mm} \quad \sigma_B^* = 1.04 \text{ MPa} \quad (\text{A.8})$$

Element	Node	$\frac{x}{a}$	Bridging Stresses (MPa)
mouth	1	1.000	0.969
mouth	2	0.875	0.906
mouth	3	0.750	0.839
2	1	0.750	0.839
2	2	0.625	0.766
2	3	0.500	0.685
3	1	0.500	0.685
3	2	0.410	0.620
3	3	0.320	0.548
4	.	0.320	0.548
4	2	0.240	0.475
4	3	0.160	0.388
tip	1	0.160	0.388
tip	2	0.040	0.194
tip	3	0.000	0.000

Table A.2: Nodal Stresses for Iteration #1

Element	Node	$\frac{\varepsilon}{\alpha}$	Bridging Stresses (MPa)
mouth	1	1.000	1.01
mouth	2	0.875	0.945
mouth	3	0.750	0.875
2	1	0.750	0.875
2	2	0.625	0.798
2	3	0.500	0.714
3	1	0.500	0.714
3	2	0.410	0.647
3	3	0.320	0.571
4	1	0.320	0.571
4	2	0.240	0.495
4	3	0.160	0.404
tip	1	0.160	0.404
tip	2	0.040	0.202
tip	3	0.000	0.000

Table A.3: Nodal Stresses for Iteration #2

Element	Node	$\frac{z}{z_0}$	Bridging Stresses (MPa)
mouth	1	1.000	1.03
mouth	2	0.875	0.963
mouth	3	0.750	0.892
2	1	0.750	0.892
2	2	0.625	0.814
2	3	0.500	0.728
3	1	0.500	0.728
3	2	0.410	0.660
3	3	0.320	0.583
4	1	0.320	0.583
4	2	0.240	0.505
4	3	0.160	0.412
tip	1	0.160	0.412
tip	2	0.040	0.206
tip	3	0.000	0.000

Table A.4: Nodal Stresses for Iteration #3

which gives a new value of σ_B^* , according to step 4, as follows:

$$\sigma_B^* = \frac{1}{2} (1.01 + 1.04) = 1.03 \text{ MPa} \quad (\text{A.9})$$

Iteration #3:

Take σ_B^* from the previous iteration:

$$\sigma_B^* = 1.03 \text{ MPa} \quad (\text{A.10})$$

Repeat steps 2 and 3 above with Table A.4 listing the calculated bridging stress according to equation 2.10. The BEM results from the fourth computer run give:

$$v^* = 0.30752 \times 10^{-3} \text{ mm} \quad \sigma_B^* = 0.964 \text{ MPa} \quad (\text{A.11})$$

from which a new value of σ_B^* is obtained according to step 4, as follows:

$$\sigma_B^* = \frac{1}{2} (1.03 + 1.01) = 1.02 \text{ MPa} \quad (\text{A.12})$$

Element	Node	$\frac{z}{z_0}$	Bridging Stresses (MPa)
mouth	1	1.000	1.02
mouth	2	0.875	0.954
mouth	3	0.750	0.883
2	1	0.750	0.883
2	2	0.625	0.806
2	3	0.500	0.721
3	1	0.500	0.721
3	2	0.410	0.653
3	3	0.320	0.577
4	1	0.320	0.577
4	2	0.240	0.500
4	3	0.160	0.408
tip	1	0.160	0.408
tip	2	0.040	0.204
tip	3	0.000	0.000

Table A.5: Nodal Stresses for Iteration #4**Iteration #4:**

Take σ_B^* from the previous iteration:

$$\sigma_B^* = 1.02 \text{ MPa} \quad (\text{A.13})$$

Steps 2 and 3 above are repeated. See Table A.5 lists the calculated bridging stress according to equation 2.10. The BEM results for the fifth computer run give:

$$v^* = 0.31988 \times 10^{-3} \text{ mm} \quad \sigma_B^* = 1.00 \text{ MPa} \quad (\text{A.14})$$

Therefore, the final result for σ_B^* is:

$$\sigma_B^* = 1.02 \text{ MPa} \quad (\text{A.15})$$

END

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FIN