Towards Harmonic Analysis on Locally Compact Quantum Groups

From Groups to Quantum Groups – and Back

by

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M.Sc., Sharif University, 2006

A thesis submitted to the Faculty of Graduate and Postdoctoral Affairs
in partial fulfillment of requirements of the degree of

Doctor of Philosophy

in

Pure Mathematics

Carleton University
Ottawa, Ontario

Your file Votre référence
ISBN: 978-0-494-81560-1
Our file Notre référence
ISBN: 978-0-494-81560-1

Abstract

The main mathematical objects in this thesis are locally compact quantum groups. Our approach to quantum groups is through an abstract harmonic analysis viewpoint. The starting point for the first part is the representation theory for these objects, developed in [23]. We show that "quantum point-masses" can be identified with the intrinsic group of the dual $\hat{\mathbb{G}}$. We thus assign, to each locally compact quantum group $\mathbb{G}$, a locally compact group $\tilde{\mathbb{G}}$, which is an invariant for the latter. This assignment preserves compactness as well as discreteness (hence also finiteness), and, for large classes of quantum groups, amenability. We calculate this invariant for some of the most well-known examples of non-classical quantum groups. In particular, for Woronowicz's compact matrix pseudogroups, we always obtain a compact Lie group – which in the case of $SU_q(2)$ is $\mathbb{T}$.

Conversely, we show that several structural properties of $\mathbb{G}$ are encoded by $\tilde{\mathbb{G}}$: the latter, despite being a simpler object, can carry very important information about $\mathbb{G}$.

We also derive a Heisenberg-type commutation relation for locally compact quantum groups, in terms of their associated groups. Using this relation, we define a new invariant for locally compact quantum groups which is a subgroup of $\mathbb{T}$.

The second part of the thesis is devoted to the study of various convolution-type algebras associated with a locally compact quantum group. The intersection point of the two parts of thesis, is the representation theory of [23]. Using the tools

provided in the latter, we generalize many results known in the classical situation to the quantum setting. As we shall see, the duality endows the space of trace class operators over a quantum group with two products which are operator versions of convolution and pointwise multiplication, respectively; this was first defined and studied, in the classical setting, in [30]. We investigate the relation between these two products, and derive a formula linking them.

Furthermore, we define some canonical module structures on these convolution algebras, and prove that certain topological properties of a quantum group, correspond to cohomological properties of these modules.

Acknowledgments

This thesis would not have been possible without the support of many people. I owe my deepest gratitude to my supervisor, Professor Matthias Neufang, who has been abundantly helpful and offering invaluable assistance, support and guidance in a number of ways. He has been much more than just the supervisor to me, a very good friend rather.

I would like to take this opportunity to thank Professors Amini, Badiozzaman and Rezaei for all they taught me during my undergraduate and M.Sc. studies.

I also learned a lot, during the short time that I had the chance to, from Professor Effros, to whom I would like to show my greatest gratitude.

I want to thank my friends, the ones in Iran who never forgot me during these years, and those who made my life here much more enjoyable.

I convey special acknowledgement to my family, the main source of my courage and motivation. My Father, Hamid Kalantar, is the one who helped me to build my character, and has always been, and will always be the one on whom I can unconditionally rely; thanks to him for his pure support, and thanks to my mom for her pure love.

And of course, my beloved wife, Pegah, through whom all becomes meaningful for me; thanking her is the least I can offer in return, for everything she has brought to my life.

To

Mehrnoosh, Habib, Mahtab, Hirad

and

Pegah

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Chapter 0

Introduction

Locally compact quantum groups, as introduced and studied by Kustermans–Vaes in [25], provide a category which comprises both classical group algebras and group-like objects arising in mathematical physics such as Woronowicz's famous quantum group $SU_\mu(2)$.

The mathematical objects with which we are mainly concerned, are locally compact quantum groups. Therefore, after presenting some preliminaries (Chapter 1), we shall devote a whole chapter (Chapter 2) to an introduction to the theory of locally compact quantum groups, in the sense of Kustermans and Vaes [25].

In these two chapters, which provide the background for the thesis work, we shall point the reader to the relevant literature without always giving the precise references of the results stated.

Chapter 3 – *From Quantum Groups to Groups* – is motivated by the recent work of Junge, Neufang and Ruan [23]; this paper investigates the quantum group analogue of the class of completely bounded multiplier algebras which play an important role in Fourier analysis over groups, by means of a representation theorem. The latter result enables the authors to express quantum group duality precisely in terms of a commutation relation. When $\mathbb{G} = L^\infty(G)$ for a locally compact group G, then the algebra of completely bounded multipliers defined in [23] is the measure algebra of the group G.

In [53] Wendel proved that for a locally compact group G, every positive isometric linear (left or right) $L^1(G)$-module map on $L^1(G)$, i.e., every positive isometric (right or left) multiplier, has to be the convolution by a point-mass. Moreover, the set of point-masses regarded as maps on $L^1(G)$ with the strong operator (i.e., the point-norm) topology is homeomorphic to the group G. In other words, he showed how a locally compact group can be recovered from its measure algebra.

Using the representation theory developed in [23], one can define a quantum version of positive isometric multipliers on the L^1-algebra of a group, but taking the operator space structure into account as well.

Following this path, in Chapter 3 we start with assigning to each locally compact quantum group $\mathbb{G}$, a locally compact group $\tilde{\mathbb{G}}$ that is an invariant for the latter; this assignment preserves compactness as well as discreteness (hence also finiteness), and, for large classes of quantum groups, amenability. Combining our construction with the above-mentioned commutation result, we can further assign, to each locally compact quantum group, a certain subgroup of the circle group that forms a numerical invariant generalizing Heisenberg's bi-characters.

We first prove some basic properties of this group before arriving at one of our main results, Theorem 3.2.14, establishing identifications between several different locally compact groups which can be assigned to a locally compact quantum group, including the intrinsic group of the dual quantum group. In the following sections, we calculate this associated group for some well-known examples of locally compact quantum groups. For Woronowicz's class of compact matrix pseudogroups, we always obtain a compact Lie group – which in the case of $SU_\mu(2)$ is precisely the circle group.

In the last section of this chapter, we present various applications of studying this group. In particular, we show that for a large class of locally compact quantum groups, the associated locally compact group cannot be "small", and in fact, the smallness of the latter forces the former to be of a very specific type. We also see that this group carries some natural properties inherited from the locally compact quantum group, which shows that this assignment is natural.

The second part of our original contributions in this thesis is included in Chapter 4: *Convolution Algebras over Locally Compact Quantum Groups.* This is motivated by the earlier works of Neufang [30], Pirkovskii [34], and Neufang, Ruan and Spronk [31]. In [30] Neufang defined a new product on $\mathcal{T}(L^2(G))$, the set of trace class operators on $L^2(G)$, for a locally compact group G. While the usual composition of operators can be regarded as the non-commutative counterpart of the pointwise product in $L^\infty(G)$, this new product can be viewed as a non-commutative version of the convolution product in $L^1(G)$. Also, in [30] Neufang studied various properties of the Banach algebra $\mathcal{T}(L^2(G))$ endowed with this product.

In [34] Pirkovskii investigated this Banach algebra from a (co)homological point of view. He showed that some of the most important properties of a locally compact group G, such as amenability, compactness or discreteness, are equivalent to some (co)homological properties of the Banach algebra $\mathcal{T}(L^2(G))$.

In [31] Neufang, Ruan and Spronk established the dual version of part of Neufang's work, using a similar procedure as in [30], but with the Fourier algebra $A(G)$ instead of $L^1(G)$. The corresponding dual product on $\mathcal{T}(L^2(\mathbb{G}))$ was investigated by Neufang and Runde in [32]. There is an elegant way of expressing these products by using the so-called fundamental unitaries of locally compact quantum groups. Junge, Neufang and Ruan used this idea to unify parts of the earlier works [30] and [31] in the setting of locally compact quantum groups [23].

In Chapter 4 we study these Banach algebra structures in the context of general locally compact quantum groups $\mathbb{G}$. After deriving some of the basic properties of these algebras, we investigate the relation between the (co)homological properties of $\mathcal{T}(L^2(\mathbb{G}))$ endowed with these new products, and properties of $\mathbb{G}$. In particular, we generalize some of the results proved in [34] to the case of locally compact quantum groups.

All results presented in Chapters 3 and 4, unless otherwise stated, are original work obtained in this thesis.

Chapter 1

Preliminaries

The name *locally compact quantum groups* indicates that these objects are carrying three specific mathematical structures: topological structure, group structure and quantum (non-commutative) structure; by the latter one in general means a specific algebra of operators on some Hilbert space.

So in order to get into the theory of locally compact quantum groups, we first need to have a brief introduction to the theories of topological groups and operator algebras.

1.1 Topological Groups

Among the most basic groups are the integers $\mathbb{Z}$, the unit circle $\mathbb{T}$, and the real numbers $\mathbb{R}$, which all also have natural topological structures. With this topology, the group operations are continuous. In other words these basic operations are compatible with the topology.

These groups also possess a natural measure theoretic structure given by Lebesgue measure, which is from the analytical point of view as important as their topological structures.

Definition 1.1.1. *A* topological group *is a group equipped with a topology such that the group operations and the topology are compatible. That is, the maps*

$$G \times G \to G, \ (g,h) \mapsto gh \qquad \text{and} \qquad G \to G, \ g \mapsto g^{-1}$$

are continuous. If the topology on G is a locally compact Hausdorff topology (so that there is a neighborhood base for the identity element consisting of compact sets), then G is called a locally compact group.

It is well known that Lebesgue measure is (up to a positive scalar multiple) the only positive Radon measure on $\mathbb{Z}$, $\mathbb{T}$, respectively $\mathbb{R}$, which is invariant under translation.

Definition 1.1.2. *Let G be a locally compact group. A positive Radon measure m on G is called a left/right* Haar measure *if it is left/right invariant, that is $m(gE) = m(E)$, respectively, $m(Eg) = m(E)$ for each g in G and for every Borel set E in G.*

Theorem 1.1.1. *Every locally compact group has a left/right Haar measure that is unique up to a positive scalar multiple.*

$\square$

Let G be a locally compact group. $L^\infty(G)$ will denote the algebra of all essentially bounded complex valued Borel measurable functions on G. The group algebra $L^1(G)$ is the set of all integrable functions on G. $M(G)$, the measure algebra, will denote the space of all complex (finite) Radon measures on G, and $C_0(G)$ will denote the space of (complex valued) continuous functions on G vanishing at infinity.
The Riesz representation theorem says that $M(G)$ can be identified with the dual space of $C_0(G)$. The duality is given explicitly as follows:

$$\langle \mu, f \rangle := \int_G f(x) d\mu(x) \qquad (\mu \in M(G), f \in C_0(G)).$$

One also can show that $L^\infty(G)$ can be identified with the dual space of $L^1(G)$. The duality in this case is also given by the integral

$$\langle h, f\rangle := \int_G f(x)h(x)dx \qquad (h \in L^\infty(G), f \in L^1(G)).$$

Here and in the following we will denote by $\int ... dx$, integration with respect to the left Haar measure.

Let $L^2(G)$ denote the Hilbert space of all square integrable functions on G. The left regular representation $\lambda : G \to \mathcal{B}(L^2(G))$ is defined as:

$$\lambda_g(\xi)(r) = \xi(g^{-1}r) \quad g, r \in G \ , \ \xi \in \mathcal{B}(L^2(G)).$$

Similarly the right regular representation is defined as:

$$\rho_g(\xi)(r) = \xi(rg) \quad g, r \in G \ , \ \xi \in \mathcal{B}(L^2(G)).$$

A bounded linear functional, $m : L^\infty(G) \to \mathbb{C}$, is called a *left invariant mean* if

$$\|m\| = \langle 1, m\rangle = 1,$$

and we have:

$$\langle L_g f, m\rangle = \langle f, m\rangle \qquad (g \in G, f \in L^\infty(G)),$$

where for a function $f : G \to \mathbb{C}$, we define its left translate $L_g f$ by $g \in G$ through

$$(L_g f)(h) := f(gh) \quad (h \in G).$$

A locally compact group G is called *amenable* if there is a left invariant mean on

$L^\infty(G)$.

All finite, abelian, and compact groups are amenable; however, the free group on two generators is not.

As we mentioned earlier, locally compact groups form a framework generalizing the common structure of $\mathbb{Z}$, $\mathbb{T}$, and $\mathbb{R}$.

Fourier analysis is, without doubt, one of the most important and useful theories in classical analysis. The power of this theory in applications in many different areas of science, would definitely suggest that a generalization to the case of locally compact groups is of great importance. Indeed this is true, and in the case of locally compact abelian groups the generalization is quite direct.

Let G be a locally compact group. Define

$$\hat{G} := \{\, \gamma : G \to \mathbb{T} \mid \gamma \text{ is a continuous homomorphism} \,\}$$

Theorem 1.1.2. *(Pontryagin) Let G be an abelian locally compact group. Then $\hat{G}$ with the compact-open topology becomes an abelian locally compact group, called the dual of G, and we have a canonical isomorphism*

$$\hat{\hat{G}} \cong G.$$

$\square$

Using the dual of a group we can now define the Fourier transform $\mathcal{F}(f) : \hat{G} \to \mathbb{C}$

of an integrable function f on an abelian group G in the following way:

$$\mathcal{F}(f)(\gamma) = \int_G f(r)\bar{\gamma}(r)dr.$$

1.2 Banach Algebras

The theory of Abstract Harmonic Analysis mostly deals with algebras associated to a locally compact group; some of the most important ones we have introduced in the previous section. But these algebras have also very a nice topological structure.

Definition 1.2.1. *An algebra $\mathcal{A}$, which at the same time is also a Banach space, is called a* Banach algebra *if*

$$||ab|| \leq ||a||\ ||b||$$

for all a, b in $\mathcal{A}$. $\mathcal{A}$ is called unital if there exist an element $e \in \mathcal{A}$ such that:

$$ae = ea = a$$

for all $a \in \mathcal{A}$.

For any Banach space X, we denote by X_1 its closed unit ball. In the framework of the algebra-topology correspondence, about which we shall talk more later, unital algebras correspond to compact spaces. But since we mostly work in the locally compact setting, we need a definition for approximately unital algebras.

Definition 1.2.2. *Let $\mathcal{A}$ be a Banach algebra. A bounded net $(m_\alpha)_\alpha$ in $\mathcal{A}$ is called a* bounded approximate identity *if it satisfies the following property:*

$$\lim_\alpha\ m_\alpha a = \lim_\alpha\ a m_\alpha = a \qquad (a \in \mathcal{A}).$$

Example 1.2.1. *If G is a locally compact group, then $C_0(G)$ and $L^\infty(G)$ become commutative Banach algebras with pointwise multiplication and equipped with the sup norm and the essential sup norm, respectively. The measure algebra and the group algebra become Banach algebras through the convolution product defined by:*

$$\langle \mu * \nu, f\rangle := \int_G \int_G f(xy) d\mu(x) d\nu(y) \qquad (\mu, \upsilon \in M(G), f \in C_0(G)),$$

$$f * g(r) := \int_G f(s) g(s^{-1}r) ds \qquad (f, g \in L^1(G), r \in G).$$

Note that the group algebra $L^1(G)$ can be viewed as a closed ideal of the measure algebra $M(G)$ by the map $T : L^1(G) \to M(G)$ where

$$\langle T(f). g\rangle = \int_G g(x) f(x) dx \qquad (f \in L^1(G), g \in C_0(G)).$$

□

Many of the algebras with which we will work in this thesis, also carry some canonical module structures, and in some cases we are especially interested in these module structures. But again these are more than just algebraic. In fact, they need to be compatible with the topological structure of these algebras.

Definition 1.2.3. *Let $\mathcal{A}$ be a Banach algebra. A Banach space X which is also a left $\mathcal{A}$-module, is called a left* Banach $\mathcal{A}$-module *if there is $C \geq 0$ such that*

$$\|a.x\| \leq C\|a\| \, \|x\| \qquad (a \in \mathcal{A}, \; x \in X).$$

We similarly define *right Banach $\mathcal{A}$-modules*. *Banach $\mathcal{A}$-bimodules* are defined as left and right $\mathcal{A}$-modules, such that both actions are compatible. Clearly, every Banach algebra is a Banach bimodule over itself (with $C = 1$).

Generally, the first step of the usual procedure of passing from the classical setting to the quantum one is to translate the properties of an object in the language of its function algebras.

In our case, since the product of a locally compact group G is a function from $G \times G$ into G, its counterpart, for example in the L^∞-setting, is a map from $L^\infty(G)$ into $L^\infty(G \times G)$.

But in order to stay within the proper framework, we need to express $L^\infty(G \times G)$ in terms of function algebras of G itself. This is being done by using the topological versions of the tensor product.

Suppose that X and Y are Banach spaces. Then for $u \in X \otimes Y$ we define:

$$\|u\| := \inf\left\{ \sum_{k=1}^{m} \|x_k\| \|y_k\| : \ u = \sum_{k=1}^{m} x_k \otimes y_k \right\}. \tag{1.1}$$

Then the formula 1.1 defines a norm on $X \otimes Y$, and the completion of $X \otimes Y$ with respect to this norm is called *the projective tensor product of the Banach spaces X and Y*, denoted by $X \otimes_\gamma Y$.

An involution on a Banach algebra $\mathcal{A}$ is a conjugate linear isometry $* : \mathcal{A} \to \mathcal{A}$ such that for all $a, b \in \mathcal{A}$:

- $(a^*)^* = a$
- $(ab)^* = b^* a^*$.

Definition 1.2.4. *A Banach algebra $\mathcal{A}$ with involution that satisfies $\|a^*a\| = \|a\|^2, \forall a \in \mathcal{A}$, is called a C^*-algebra.*

Theorem 1.2.1 (GNS-construction). *Let $\mathcal{A}$ be a C^*-algebra. Then there exists a Hilbert space H and an isometric *-isomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$.* $\square$

Let H be a Hilbert space. Define

$$\mathcal{T}(H) := \{ x \in \mathcal{B}(H) \,:\, tr(|x|) < \infty \};$$

then $\mathcal{T}(H)$, endowed with the norm $||x|| = tr(|x|)$, becomes a Banach space, whose elements are called trace class operators. Moreover we have:

$$\mathcal{B}(H) \cong \mathcal{T}(H)^*.$$

We define the following topologies on $\mathcal{B}(H)$:

1. $x_\alpha \to 0$ (weakly) if and only if $\langle x_\alpha \xi, \eta \rangle \to 0$ for all $\xi, \eta \in H$;

2. $x_\alpha \to 0$ (strongly) if and only if $||x_\alpha \xi|| \to 0$, for all $\xi \in H$;

3. $x_\alpha \to 0$ (weak* or σ-weakly) if and only if $x_\alpha \to 0$ in $\sigma(\mathcal{B}(H), \mathcal{T}(H))$;

4. $x_\alpha \to 0$ (σ-strongly) if and only if $\sum_i ||x_\alpha \xi_i|| \to 0$, for all $(\xi_i) \subseteq H$, with $\sum_i ||\xi_i||^2 < \infty$;

5. $x_\alpha \to 0$ (σ-strongly*) if and only if $\sum_i (||x_\alpha \xi_i|| + ||x_\alpha^* \xi_i||) \to 0$ for all $(\xi_i) \subseteq H$, with $\sum_i ||\xi_i||^2 < \infty$.

1.3 Operator Spaces

The theory of operator spaces which was mainly developed, independently by Effros–Ruan and Blecher–Paulsen, can be regarded as non-commutative functional analysis.

Activity in this area is described, e.g., in the recent monographs [12], [5], [33], [37]. The beautiful work of Ruan in [39] was in fact a break through in classical abstract harmonic analysis. It showed in a sense that the algebras associated to a locally compact group should really be considered with their operator space structures in order to capture some of the most important properties of the group. This section is devoted to a brief introduction to the theory of operator spaces. For more on this theory we refer the reader to [12].

If X is a linear space, then for each $m, n \in \mathbb{N}$, $M_{m,n}(X)$ will denote the space of all $m \times n$ matrices with entries in X. If $m = n$, then $M_{m,n}(X)$ will be denoted by $M_n(X)$, in particular, $M_n = M_n(\mathbb{C})$ will denote the space of all scalar $n \times n$ matrices.

Let X be a linear space with a norm $\|\cdot\|_n$ on $M_n(X)$, for each $n \in \mathbb{N}$, such that

$$\left\| \begin{array}{c|c} x & 0 \\ \hline 0 & y \end{array} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} \qquad (n, m \in \mathbb{N},\, x \in M_n(X),\, y \in M_m(X)) \quad \text{(R 1)}$$

and

$$\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| \qquad (n \in \mathbb{N},\, x \in M_n(X),\, \alpha, \beta \in M_n \simeq \mathcal{B}(l_n^2)). \qquad \text{(R 2)}$$

Then $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is called a matricial norm for X. Moreover, if each $\|\cdot\|_n$ is complete, then X is called an *abstract operator space.*

A linear operator $T : X \to Y$ between two abstract operator spaces X and Y induces a linear operator

$$T^{(n)} : M_n(X) \to M_n(Y), \quad (x_{i,j}) \mapsto (T(x_{i,j}))$$

for each $n \in \mathbb{N}$.

Definition 1.3.1. *Let X and Y be two abstract operator spaces, and let $T \in \mathcal{B}(X,Y)$. Then:*

- *T is completely bounded if*

$$\|T\|_{\mathsf{cb}} := \sup\left\{ \|T^{(n)}\|_{\mathcal{B}(M_n(X),M_n(Y))} : \ n \in \mathbb{N}\right\} < \infty;$$

- *T is a complete contraction if $\|T\|_{\mathsf{cb}} \le 1$;*

- *T is a complete isometry if $T^{(n)}$ is an isometry for each $n \in \mathbb{N}$.*

The set of completely bounded operators from X to Y is denoted by $\mathcal{CB}(X,Y)$.

Let X and Y be two operator spaces, and let $X \otimes Y$ denote their algebraic tensor product For each $n \in \mathbb{N}$, given an element u in $M_n(X \otimes Y)$, we define

$$\|u\| := \inf\{\|\alpha\|\|x\|\|y\|\|\beta\| : \ u = \alpha(x \otimes y)\beta\} \tag{1.2}$$

where the infimum is taken over all possible decompositions where $\alpha \in M_{n,pq}$, $\beta \in M_{pq,n}$, $x \in M_p(X)$ and $y \in M_q(Y)$, with $p, q \in \mathbb{N}$ arbitrary.
Then the formula 1.2 defines a matricial norm on $X \otimes Y$, and the completion of $X \otimes Y$, with respect to this norm is called *the projective tensor product of the operator spaces X and Y*, and is denoted by $X \widehat{\otimes} Y$. For more information on the tensor products of operator spaces, we refer the reader to [12].

1.4 von Neumann Algebras

By Theorem 1.1.1, every locally compact group has a distinguished measure, called the (left) Haar measure. This measure is compatible with both the algebraic and the topological structure of the group. In fact, having such a measure is the main advantage of locally compact groups over non locally compact ones. Also, in applications this measure proves to be the most crucial object associated to a locally compact group.

Hence, with any kind of non-commutative locally compact group theory, there should be attached a non-commutative measure theory. Of course, the best candidate for such a theory would be von Neumann algebra theory.

Definition 1.4.1. *A unital C^*-subalgebra of $\mathcal{B}(H)$ which is weakly closed is called a von Neumann algebra.*

For a subset $X \subseteq \mathcal{B}(H)$, we denote by X' the commutant of X.

A very famous result of von Neumann states the following:

Theorem 1.4.1. *Let M be a unital C^*-subalgebra of $\mathcal{B}(H)$. Then M is a von Neumann algebra if and only if $M = M''$.*

□

When X is a σ-finite measure space with measure μ, the space $L^\infty(X, \mu)$ of all essentially bounded measurable functions on X acting by multiplication on the Hilbert space of square μ-integrable functions becomes a commutative von Neumann algebra. Conversely any commutative von Neumann algebra is of the form $L^\infty(X, \mu)$, for a suitable measure space (X, μ).

As one can guess from the definitions, in contrast to C^*-algebras, the main topologies of a von Neumann algebra are weaker than the operator norm topology. Therefore the most interesting objects in this theory are those compatible with weak topologies.

Definition 1.4.2. *Let ϕ be a bounded linear functional on a von Neumann algebra M. Then ϕ is said to be normal if, whenever (x_i) is a bounded increasing net in M^+ with $x = \sup x_i$, we have $\phi(x) = \lim \phi(x_i)$.*

Theorem 1.4.2. *Let M be a von Neumann algebra on H and ϕ a bounded linear functional on M. Then the following are equivalent:*

1. *ϕ is w^* continuous on M;*
2. *ϕ is σ-strongly continuous on M;*
3. *ϕ is weakly continuous on the unit ball of M;*
4. *ϕ is strongly continuous on the unit ball of M;*
5. *ϕ is normal.*

$\square$

The predual of M, i.e., the space of all linear functionals satisfying one of the above properties, will be denoted by M_*.

Example 1.4.1. *Let H be a Hilbert space. Then the algebra $\mathcal{B}(H)$ of all bounded operators on H is a von Neumann algebra whose predual is the space $\mathcal{T}(H)$ of trace class operators.*

For $\xi \in H$, we denote by ω_ξ the bounded linear functional $\langle \omega_\xi, x\rangle = \langle x\xi, \xi\rangle$ for all $x \in \mathcal{B}(H)$. Then $\omega_\xi \in \mathcal{T}(H)^+$ and $\|\omega_\xi\|_1 = \|\xi\|_2^2$.

Example 1.4.2. *Let G be a locally compact group. We denote by $L^\infty(G)$ the space of all (complex valued) function on G that are measurable and essentially bounded with respect to (left) Haar measure. Then this space endowed with pointwise product and essentially bounded norm is a commutative von Neumann algebra, with predual $L^1(G)$.*

Using the canonical embedding via multiplication operators, $f \mapsto M_f$, $L^\infty(G)$ can be realized as a commutative sub von Neumann algebra of $\mathcal{B}(L^2(G))$.

Example 1.4.3. *[14] Let G be a locally compact group. We denote by $VN(G)$ the von Neumann algebra generated by $\{\lambda_g : g \in G\}$ in $\mathcal{B}(L^2(G))$, i.e., $VN(G) = \{\lambda_g : g \in G\}'' \subseteq \mathcal{B}(L^2(G))$. This is called the group von Neumann algebra. The predual of this von Neumann algebra can be identified with the Banach space*

$$A(G) := \{f : G \to \mathbb{C} | \ \exists \ \xi, \zeta \in L^2(G) \text{ such that } \ f(x) = \langle \lambda(x)\xi, \zeta \rangle, \ \forall x \in G\},$$

endowed with the norm

$$||f|| = \inf\{||\xi|| ||\zeta|| : f(x) = \langle \lambda(x)\xi, \zeta \rangle, \forall x \in G\}.$$

One can show that $A(G)$ is a Banach algebra under the pointwise product, called the Fourier algebra.
If G is abelian with dual group $\hat{G}$, then the adjoint of the Fourier transform gives an isomorphism of von Neumann algebras $VN(\hat{G}) \cong L^\infty(G)$.

Definition 1.4.3. *If M and N are von Neumann algebras, then a linear map $\Phi : M \to N$ is said to be normal if it is weak-*–weak-* continuous.*

Fortunately there is also a normal version of the Hahn-Banach theorem.

Theorem 1.4.3. *Let $N \subseteq M$ be von Neumann algebras. Then every normal state on N extends to a normal state on M.* □

If M_i is a von Neumann algebra on H_i $(i = 1, 2)$ then $M_1 \otimes M_2$, the algebraic tensor product of M_1 and M_2, acts naturally on $H_1 \otimes H_2$, and the weak closure is a von Neumann algebra called the von Neumann algebraic-tensor product of M_1 and

M_2, and is denoted by $M_1 \overline{\otimes} M_2$ (it can be shown that this is independent of the choice of H_i).

If M_1, M_2, N_1, N_2 are von Neumann algebras and $\Phi : M_1 \to M_2$ and $\Psi : N_1 \to N_2$ are normal completely bounded maps, then the map $\Phi \otimes \Psi : M_1 \otimes N_1 \to M_2 \otimes N_2$ given on elementary tensors by

$$(\Phi \otimes \Psi)(x \otimes y) = \Phi(x) \otimes \Psi(y),$$

extends to a normal completely bounded map, from $M_1 \overline{\otimes} N_1$ to $M_2 \overline{\otimes} N_2$, also denoted by $\Phi \otimes \Psi$.

As an important special case, we get the so called Tomiyama's slice maps which play a key role in the theory of locally compact quantum groups. We denote by ι the identity map. For von Neumann algebras M and N, and a normal functional ϕ on M, the the right slice map $(\phi \otimes \iota)$ given by

$$(\phi \otimes \iota)(x \otimes y) = \phi(x) y$$

extends to a normal map from $M \overline{\otimes} N$ to N. Similarly we can define left slice maps.

As we have already mentioned, the theory of von Neumann algebras can be regarded as non-commutative measure theory. The non-commutative version of measures are called *weights*.

Example 1.4.4. *When X is a σ-finite measure space with measure μ, we can assign to every non-negative function f in the von Neumann algebra $L^\infty(X, \mu)$ the value $\varphi(f) \in [0, \infty]$ as follows:*

$$\varphi(f) = \int_X f d\mu.$$

Such a φ will be a typical example of a (normal, semi-finite faithful) weight on the *commutative von Neumann algebra $L^\infty(X, \mu)$.*

Definition 1.4.4. *A weight on a von Neumann algebra M is a function $\varphi : M^+ \to [0, \infty]$ such that:*

1. *$\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in M^+$;*

2. *$\varphi(rx) = r\varphi(x)$ for all $r \in \mathbb{R}^+$ and $x \in M^+$.*

Let φ be a weight on a von Neumann algebra M. We will use the following standard notations:

- ${\mathcal{M}_\varphi}^+ = \{a \in M^+ : \varphi(a) < \infty\}$;

- $\mathcal{N}_\varphi = \{a \in M : \varphi(a^*a) < \infty\}$;

- $\mathcal{M}_\varphi = \text{span}\, {\mathcal{M}_\varphi}^+ = {\mathcal{N}_\varphi}^* \mathcal{N}_\varphi$, where ${\mathcal{N}_\varphi}^* \mathcal{N}_\varphi = \text{span}\, \{\, x^*y : x, y \in \mathcal{N}_\varphi \,\}$.

Then have $\mathcal{N}_\varphi$ is a left ideal in M, $\mathcal{M}_\varphi$ is a *-subalgebra of M, and ${\mathcal{M}_\varphi}^+ = \mathcal{M}_\varphi \cap M^+$.

In the classical picture, $\mathcal{M}_\varphi^+$ consists of all essentially bounded φ-integrable non-negative functions, $\mathcal{N}_\varphi$ consists of all essentially bounded square integrable functions, and $\mathcal{M}_\varphi$ of all essentially bounded integrable functions.

Let φ be a weight on the von Neumann algebra M. We say that φ is *semi-finite* if one of the following equivalent conditions is satisfied:

- $\mathcal{M}_\varphi^+$ is weakly dense in M^+;

- $\mathcal{M}_\varphi$ is weakly dense in M;

- $\mathcal{N}_\varphi$ is weakly dense in M.

The weight φ is called *faithful* if the following holds:

$$x \in M^+, \varphi(x) = 0 \Rightarrow x = 0.$$

φ is called *normal* if one of the following equivalent conditions is satisfied:

- for every $\lambda \in \mathbb{R}^+$, the set $\{ a \in M^+ : \varphi(a) \leq \lambda \}$ is σ-weakly closed;

- whenever $(x_\imath)_{\imath \in I}$ is an increasing net in M^+, converging strongly to $x \in M^+$, we have:

$$\varphi(x) = \lim_{\imath \in I} \varphi(x_\imath);$$

- there exists a family $(\omega_\imath)_{\imath \in I}$ in M_*^+ such that

$$\varphi(x) = \sum_{\imath \in I} \omega_\imath(x) \quad \forall x \in M^+;$$

- if $(x_\imath)$ is a net in M^+, converging σ-weakly to $x \in M^+$, then

$$\varphi(x) \leq \liminf_{\imath} \varphi(x_\imath).$$

By an n.s.f. weight we mean a normal semi-finite faithful weight.

Let M be a von Neumann algebra and φ an n.s.f. weight on M. Define:

$$\langle x, y \rangle := \varphi(y^* x) \qquad (x, y \in \mathcal{N}_\varphi).$$

Since φ is faithful, this obviously defines an inner product on $\mathcal{N}_\varphi$.
Let the Hilbert space H_φ be the completion of $\mathcal{N}_\varphi$ with respect to this inner product.
We denote by

$$\Lambda_\varphi : \mathcal{N}_\varphi \to H_\varphi$$

the canonical embedding. For $x \in M$ define $\pi_\varphi(x) : \Lambda_\varphi(\mathcal{N}_\varphi) \to \Lambda_\varphi(\mathcal{N}_\varphi)$ by

$$(\pi_\varphi(x))(\Lambda_\varphi(a)) = \Lambda_\varphi(xa).$$

Then $\pi_\varphi(x)$ extends to a bounded operator on H_φ which we also denote by $\pi_\varphi(x)$. The triple $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ is called the GNS-construction for φ.

Consider a normal semi-finite weight φ on a von Neumann algebra M. Then we define the set

$$G_\varphi := \{\omega \in (M_*)^+ \ : \ \exists \epsilon > 0 \text{ s.t. } (1+\epsilon)\omega(x) \leq \varphi(x), \forall x \in M\}.$$

Then one can show that G_φ is an upward directed subset of $(M_*)^+$, and we have:

$$\varphi(x) = \lim_{\omega \in G_\varphi} \omega(x) \quad \forall x \in M^+. \tag{1.3}$$

Let φ be a normal semi-finite weight on the von Neumann algebra M, and let $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ be the GNS-construction for φ. Then the following hold:

- the map $\Lambda_\varphi : \mathcal{N}_\varphi \to H_\varphi$ is σ-weak–weak closed;
- the representation π_φ is normal and unital.

The rest of this section will be devoted to a brief overview of the modular theory of von Neumann algebras. The importance of this theory for locally compact quantum groups cannot be overestimated.
Here we recall some standard definitions and statements regarding the theory of weights on von Neumann algebras which we will need later. For details we refer the reader to [45].

Let φ be an n.s.f. weight on a von Neumann algebra M. Then there exists a unique strongly continuous one-parameter group of automorphisms $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ of M satisfying:

1. $\varphi \circ \sigma_t^\varphi = \varphi$ for all $t \in \mathbb{R}$;
2. for every $x \in \mathcal{D}(\sigma_{\frac{i}{2}}^\varphi)$, we have $\varphi(x^*x) = \varphi(\sigma_{\frac{i}{2}}^\varphi(x)\sigma_{\frac{i}{2}}^\varphi(x)^*)$.

We call $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ the modular automorphism group of φ.

Lemma 1.4.4. *Let φ be an n.s.f. weight on a von Neumann algebra M and $\alpha \in Aut(M)$ such that $\varphi \circ \alpha = \varphi$. Then:*

1. *$x \in \mathcal{N}_\varphi \Leftrightarrow \alpha(x) \in \mathcal{N}_\varphi$;*
2. *the map:*
$$\Lambda_\varphi(x) \to \Lambda_\varphi(\alpha(x)) \ , \ x \in \mathcal{N}_\varphi$$
extends to a unitary map $u_\alpha : H_\varphi \to H_\varphi$;
3. *we have:*
$$\alpha(x) = u_\alpha x u_\alpha^*$$
for $x \in M$.

□

Let $(H_\varphi, \pi_\varphi, \Lambda_\varphi)$ be the GNS-construction for φ. Then there exists a unique closed, anti-linear operator S on H_φ satisfying:

1. $S(\Lambda_\varphi(x)) = \Lambda_\varphi(x^*)$ for all $x \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$;
2. $\Lambda_\varphi(\mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*)$ is a core for S.

Let $S = J_\varphi \Delta_\varphi^{\frac{1}{2}}$ be the polar decomposition of S. Then J is an anti-unitary on H_φ, and Δ_φ is a strictly positive operator on H_φ. We call J the *modular conjugation* of φ, and Δ_φ the *modular operator* of φ.

Lemma 1.4.5. *For an n.s.f. weight on a von Neumann algebra M we have the following relations:*

1. *for all* $x \in \mathcal{N}_\varphi$ *and* $t \in \mathbb{R}$, *we have:*

$$\Lambda_\varphi(\sigma_t^\varphi(x)) = \Delta_\varphi^{\imath t}\Lambda_\varphi(x);$$

2. *for all* $x \in \mathcal{N}_\varphi \cap \mathcal{D}(\sigma_{\frac{\imath}{2}}^\varphi)$, *we have:*

$$J\Lambda_\varphi(x) = \Lambda_\varphi(\sigma_{\frac{\imath}{2}}^\varphi(x)^*);$$

3. *if* $x \in \mathcal{N}_\varphi$ *and* $y \in \mathcal{D}(\sigma_{\frac{\imath}{2}}^\varphi)$, *then* $xy \in \mathcal{N}_\varphi$, *and*

$$\Lambda(xy) = J_\varphi\pi_\varphi(\sigma_{\frac{\imath}{2}}^\varphi(y))^* J\Lambda_\varphi(x);$$

4. *if* $a \in \mathcal{D}(\sigma_{-\imath}^\varphi)$ *and* $x \in \mathcal{M}_\varphi$, *then* ax *and* $x\sigma_{-\imath}^\varphi(a)$ *belong to* $\mathcal{M}_\varphi$ *and*

$$\varphi(ax) = \varphi(x\sigma_{-\imath}^\varphi(a));$$

5. *if* $x \in \mathcal{N}_\varphi \cap \mathcal{N}_\varphi^*$ *and* $a \in \mathcal{N}_\varphi^* \cap \mathcal{D}(\sigma_{-\imath}^\varphi)$ *such that* $\sigma_{-\imath}^\varphi(a) \in \mathcal{N}_\varphi$, *then*

$$\varphi(ax) = \varphi(x\sigma_{-\imath}^\varphi(a)).$$

□

Remark 1.4.6. *From Lemmas 1.4.4 and 1.4.5 we have* $\sigma_t^\varphi = Ad(\Delta_\varphi^{\imath t})$. *Also, it is easy to see that* φ *is tracial if and only if* $\sigma_t^\varphi = \iota$, *if and only if* $\Delta_\varphi = \iota$.

Chapter 2

Locally Compact Quantum Groups

As mentioned in the introduction, locally compact quantum groups are the main objects in this thesis. Therefore we devote this chapter to introduce the fundamentals of the theory of locally compact quantum groups in the sense of Kustermans and Vaes [25].

Drienfeld was the first who used the term quantum groups for the objects known as Hopf algebras at the time. There are mainly three different approaches leading to these structures. One arises in Number Theory and Algebra, known as algebraic quantum group theory. The second comes from mathematical physics where the most interesting examples appear. And the third aspect of the theory is of interest to abstract harmonic analysts and operator algebraists. Since our approach and interest is lying in the last two, we shall not consider the first case.

Right after the success in generalizing Fourier analysis to the case of locally compact abelian groups which beside its vast variety of applications, led to a nice duality theorem within this category, the efforts to find a similar theorem for non-abelian locally compact groups began. There were some partial successes in finding a dual object, mainly in the compact case; developing the representation theory of compact groups helped arriving at some duality theorems such as the Peter–Weyl and Tannaka–Krein theorems. But the main problem in all these theories was that the dual object was itself not a locally compact group. Despite all those efforts, a perfect generalization of the Pontryagin duality theorem turned out to be hopeless. This pushed the people in search of another category containing all locally compact groups and just big enough to provide a perfect duality theory.

This chapter is devoted to a quick introduction to the theory of locally compact quantum groups developed by Kustermans and Vaes [25]. For details and proofs we refer the reader to [51]; see also [25] and [26].

2.1 Definition

Our approach to locally compact quantum groups will be via von Neumann algebras.

Definition 2.1.1. *A Hopf–von Neumann algebra is a pair (M, Γ) where M is a von Neumann algebra and Γ is a co-multiplication, i.e., a unital, injective, normal *-homomorphism $\Gamma : M \to M \overline{\otimes} M$ which is co-associative: that is, the diagram*

$$\begin{tikzcd} M \arrow[r, "\Gamma"] \arrow[d, "\Gamma"'] & M\overline{\otimes}M \arrow[d, "\Gamma\otimes\iota"] \\ M\overline{\otimes}M \arrow[r, "\iota\otimes\Gamma"'] & M\overline{\otimes}M\overline{\otimes}M \end{tikzcd}$$

commutes, in short, $\Gamma(\iota \otimes \Gamma) = (\iota \otimes \Gamma)\Gamma$.

Definition 2.1.2. *A locally compact quantum group is a quadruple $\mathbb{G} := (M, \Gamma, \varphi, \psi)$, where:*

1. *(M, Γ) is a Hopf-von Neumann algebra;*

2. φ *is an n.s.f. weight on* M*, called a* ***left Haar weight****, satisfying:*

$$\varphi((\omega \otimes \iota)\Gamma(x)) = \omega(1)\varphi(x) \quad \forall x \in \mathcal{M}_\varphi, \omega \in M_*;$$

3. ψ *is a n.s.f. weight on* M*, called a* ***right Haar weight****, satisfying:*

$$\psi((\iota \otimes \omega)\Gamma(x)) = \omega(1)\psi(x) \quad \forall x \in \mathcal{M}_\psi, \omega \in M_*.$$

If $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ is a locally compact quantum group, then Γ_*, the pre-adjoint of Γ, induces a product on M_*, turning it into a completely contractive Banach algebra (i.e., $\Gamma_* : M_* \widehat{\otimes} M_* \to M_*$ is a complete contraction).

Example 2.1.1. *Let G be a locally compact group. Define:*

$$\Gamma_G : L^\infty(G) \to L^\infty(G)\overline{\otimes}L^\infty(G) \cong L^\infty(G \times G)$$

$$\Gamma_G(f)(s,t) = f(st) \quad \forall f \in L^\infty(G),\ s,t \in G.$$

Then Γ_G is a co-multiplication on $L^\infty(G)$, and $(L^\infty(G), \Gamma_G, \varphi_c, \psi_c)$ is a commutative locally compact quantum group, where φ_c and ψ_c are left and right Haar integrals, respectively. Moreover, one can show that any commutative or co-commutative locally compact quantum group is necessarily a Kac algebra, and we have the following.

Theorem 2.1.1. *[44, Theorem 2] Let $\mathbb{G} := (M, \Gamma, \varphi, \psi)$ be a locally compact quantum group with M commutative. Then there exists a locally compact group G such that $M = L^\infty(G)$ and*

$$\mathbb{G} = (L^\infty(G), \Gamma_G, \varphi_c, \psi_c).$$

The category of locally compact quantum groups provides a unified framework for both $L^\infty(G)$ and the group von Neumann algebra $VN(G)$.

Example 2.1.2. *Let G be a locally compact group. Define:*

$$\hat{\Gamma}_G : VN(G) \to VN(G)\overline{\otimes}VN(G) \cong VN(G \times G)$$

$$\hat{\Gamma}_G(\lambda_g) = \lambda_g \otimes \lambda_g.$$

Then $\hat{\Gamma}_G$ is a (symmetric) co-multiplication on $VN(G)$. One can show (see [45, Section VII.3]) that there exists an n.s.f. weight φ_s on $VN(G)$, both left and right invariant with respect to $\hat{\Gamma}_G$, such that $G = (VN(G), \hat{\Gamma}_G, \varphi_s)$ is a co-commutative locally compact quantum group, i.e., $(\hat{\Gamma}_G)_*$ is commutative.

In the sequel, for a von Neumann algebra M, we denote by χ the flip map

$$\begin{aligned} \chi : M\overline{\otimes}M &\to M\overline{\otimes}M \\ a\otimes b &\to b\otimes a. \end{aligned}$$

If (M, Γ) is a Hopf-von Neumann algebra, then co-commutativity is equivalent to $\chi \circ \Gamma = \Gamma$.

As a consequence of Theorem 2.1.1, one also has:

Theorem 2.1.2. *If $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ is a co-commutative locally compact quantum group, then there exists a locally compact group G such that $\mathbb{G} = (VN(G), \hat{\Gamma}_G, \varphi_s, \varphi_s)$. In fact, G is the group (endowed with the weak*-topology) of all unitaries u in M such that $\Gamma(u) = u \otimes u$.* □

In complete analogy to the classical case we have:

Theorem 2.1.3. *The left and right Haar weights on a locally compact quantum group are unique up to a multiple of a positive scalar.*

□

With the above examples in mind it is natural to transfer concepts from abstract harmonic analysis to the setting of locally compact quantum groups.

Definition 2.1.3. *Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a locally compact quantum group.*

- $\mathbb{G}$ *is called* discrete *if the Banach algebra M_* is unital;*
- $\mathbb{G}$ *is called* compact *if φ is finite.*

Remark 2.1.4. *If $\mathbb{G}$ is a compact quantum group, then φ is also a right Haar weight; in this case, we always choose φ to be a state.*

Definition 2.1.4. *Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a locally compact quantum group.*

- $\mathbb{G}$ *is called* co-amenable *if M_* has a bounded approximate identity;*
- $\mathbb{G}$ *is called* amenable *if there exist a state F on M such that*

$$\langle (f \otimes \iota)\Gamma(x), F\rangle = \langle (\iota \otimes f)\Gamma(x), F\rangle = \langle f, 1\rangle\langle x, F\rangle$$

  for all $f \in M_*$ and $x \in M$.

2.2 The Fundamental Unitary

Let G be a locally compact group. Define

$$W_G : L^2(G \times G) \to L^2(G \times G)$$

$$W_G\xi(r, s) = \xi(r, r^{-1}s)$$

for all $\xi \in L^2(G \times G)$. Then W_G is a unitary on $L^2(G \times G) \cong L^2(G) \otimes_2 L^2(G)$ (where $\otimes_2$ denotes the Hilbert space tensor product), and we have:

1. $\Gamma_G(M_f) = W_G^*(1 \otimes M_f)W_G \quad \forall f \in L^\infty(G)$;
2. $W_G \in L^\infty(G)\overline{\otimes}VN(G)$;
3. $L^\infty(G) = \overline{\{(\iota \otimes \omega)W_G : \omega \in \mathcal{T}(L^2(G))\}}^{w^*}$;
4. $C_0(G) = \overline{\{(\iota \otimes \omega)W_G : \omega \in \mathcal{T}(L^2(G))\}}^{\|\cdot\|}$;
5. $\lambda(f) = (f \otimes \iota)W_G \quad \forall f \in L^1(G)$;
6. $VN(G) = \overline{\{(\omega \otimes \iota)W_G : \omega \in \mathcal{T}(L^2(G))\}}^{w^*}$;
7. $C_r^*(G) = \overline{\{(\omega \otimes \iota)W_G : \omega \in \mathcal{T}(L^2(G))\}}^{\|\cdot\|}$.

We see from the above that almost all information about the group G is carried by the unitary operator W_G. There also exists a similar unitary operator $\hat{W}_G$ for the quantum group $(VN(G), \hat{\Gamma}_G, \varphi_s)$, which has the same properties as W_G:

1. $\hat{\Gamma}_G(x) = \hat{W}_G^*(1 \otimes x)\hat{W}_G \quad x \in VN(G)$;
2. $\hat{W}_G \in VN(G)\overline{\otimes}L^\infty(G)$;
3. $VN(G) = \overline{\{(\iota \otimes \omega)\hat{W}_G : \omega \in \mathcal{T}(L^2(G))\}}^{w^*}$;

4. $C_r^*(G) = \overline{\{(\iota \otimes \omega)\hat{W}_G) \, : \, \omega \in \mathcal{T}(L^2(G))\}}^{\|\cdot\|}$;

5. $M_f = (f \otimes \iota)\hat{W}_G \quad \forall f \in A(G)$;

6. $L^\infty(G) = \overline{\{(\omega \otimes \iota)\hat{W}_G \, : \, \omega \in \mathcal{T}(L^2(G))\}}^{w^*}$;

7. $C_0(G) = \overline{\{(\omega \otimes \iota)\hat{W}_G \, : \, \omega \in \mathcal{T}(L^2(G))\}}^{\|\cdot\|}$.

We also have the relation:

$$\hat{W}_G = \chi(W_G^*).$$

For every locally compact quantum group $\mathbb{G}$ there exists such an object W, called the left fundamental unitary. It is a unitary on $H_\varphi \otimes_2 H_\varphi$, defined by

$$W^*(\Lambda_\varphi(a) \otimes \Lambda_\varphi(b)) = \Lambda_{\varphi\otimes\varphi}(\Gamma(b)(a \otimes 1))$$

for all $a, b \in \mathcal{N}_\varphi$. This operator satisfies the so called *pentagonal* relation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}.$$

Here we used the leg notation:

$$W_{12} = W \otimes 1 \quad , \quad W_{23} = 1 \otimes W \quad , \quad W_{13} = (\chi \otimes 1)W_{23}.$$

The co-multiplication Γ on M can be written as:

$$\Gamma(x) = W^*(1 \otimes x)W.$$

Just as in the classical case, this unitary turns out to carry almost all the information about $\mathbb{G}$. This fact motivated people to try to find an alternative definition for locally compact quantum groups, requiring the existence of such unitaries [56], [43].

In analogy to the commutative case, we define:

Definition 2.2.1. *The left regular representation* $\lambda : M_* \to \mathcal{B}(H_\varphi)$ *is defined by*

$$\lambda(f) = (f \otimes \iota)(W).$$

It is easy to verify that λ is an injective completely contractive homomorphism from M_* into $\mathcal{B}(H_\varphi)$.

Similarly, the right Haar weight ψ induces an inner product

$$\langle \Lambda_\psi(x), \Lambda_\psi(y) \rangle_\psi = \psi(y^* x)$$

on $\mathcal{N}_\psi = \{x \in M : \psi(x^* x) < \infty\}$. We denote by H_ψ the Hilbert space completion of $\mathcal{N}_\psi$. There exists a right fundamental unitary operator V on $H_\psi \otimes_2 H_\psi$ associated with ψ, which is defined by

$$V(\Lambda_\psi(x) \otimes \Lambda_\psi(y)) = (\Lambda_\psi \otimes \Lambda_\psi)(\Gamma(x)(1 \otimes y))$$

for all $x, y \in \mathcal{N}_\psi$. This operator V also satisfies the pentagonal relation, i.e.,

$$V_{12} V_{13} V_{23} = V_{23} V_{12}.$$

Now, the co-multiplication Γ on M can be written as

$$\Gamma(x) = V(x \otimes 1)V^*.$$

2.3 The Antipode

Existence of the inverse of every element obviously plays a very important role in group theory, being the property which distinguishes groups from semigroups. This suggests the importance of a similar object as the "inverse" in the quantum setting. In fact, for many years, this was the main obstacle in achieving a perfect axiomatization for locally compact quantum groups. Even though the theory of Kac algebras had completely solved the duality problem for locally compact groups, it was still not a completely satisfactory theory for topological quantum groups, as it did not contain some of natural examples arising in physics. The main problem stemmed from the unboundedness of the quantum counterpart of the inverse in a group. While this operation is bounded in classical function algebras, as well as Kac algebras, it becomes unbounded for example, when deforming a compact Lie group. In fact, this is the phenomenon that distinguishes the category of locally compact quantum group from Kac algebras. For a locally compact quantum group $\mathbb{G}$, the antipode, which is the object corresponding to the inverse in a group, is defined in the following way.

Proposition 2.3.1. *For every $x, y \in \mathcal{N}_\varphi^* \mathcal{N}_\psi$ we have $(\psi \otimes \iota)(\Gamma(x^*)(y \otimes 1)) \in \mathcal{N}_\varphi$, and if we define*

$$G\Lambda_\varphi((\psi \otimes \iota)(\Gamma(x^*)(y \otimes 1))) = \Lambda_\varphi((\psi \otimes \iota)(\Gamma(y^*)(x \otimes 1)))$$

then G is a closed densely defined anti-linear operator on H_φ. Moreover if $G = IN^{\frac{1}{2}}$ is the polar decomposition of G, we have:

$$I = I^*, \ I^2 = 1, \ INI = N^{-1}.$$

□

Definition and Proposition 2.3.2. *Define the operators*

$$\begin{aligned} R : M \ni x &\to Ix^*I \in M; \\ \tau_t : M \ni x &\to N^{-it}xN^{it} \in M \quad (t \in \mathbb{R}). \end{aligned}$$

Then R is an anti-automorphism on M such that $R^2 = 1$, and $\{\tau_t\}$ is a strongly continuous one-parameter group of automorphisms on M.
R is called the unitary antipode *of $\mathbb{G}$, and $\{\tau_t\}$ is called the* scaling group *of $\mathbb{G}$.*
We also have

$$\tau_t R = R\tau_t$$

for all $t \in \mathbb{R}$.

The antipode is defined via the operators in the last theorem.

Definition 2.3.1. *$S := R\tau_{-\frac{i}{2}}$ is called the antipode of $\mathbb{G}$.*

Theorem 2.3.3. *The antipode S has the following properties:*

1. *S is densely defined and has dense range;*
2. *S is injective and $S^{-1} = R\tau_{\frac{i}{2}} = \tau_{\frac{i}{2}}R$;*
3. *S is anti-multiplicative, that is, for all $x, y \in \mathcal{D}(S)$, we have $xy \in \mathcal{D}(S)$ and*

$$S(xy) = S(y)S(x);$$

4. *for all $x \in \mathcal{D}(S)$, we have $S(x)^* \in \mathcal{D}(S)$ and*

$$S(S(x)^*) = x^*;$$

5. $S^2 = \tau_{-i}$;

6. $RS = SR$;

7. $\tau_t S = S\tau_t \quad \forall t \in \mathbb{R}$.

2.4 The Modular Element

Let μ and ν be a left and a right Haar measure on a locally compact group G. Then there exists a continuous homomorphism $\Delta : G \to \mathbb{R}_+^\times$, such that

$$\int_G f(rs)d\mu(r) = \Delta(s^{-1}) \int_G f(r)d\mu(r)$$

for all $s \in G$ and $f \in L^1(G)$. The function Δ which is the Radon-Nikodym derivative of μ with respect to ν, is called the *modular function* of G.
In every locally compact quantum group there exists a version of the modular function, called the modular element, which links left and right Haar weights.

Definition and Proposition 2.4.1. *There exist a strictly positive operator δ, called* modular element, *affiliated with M, such that*

$$\psi(x) = \varphi(\delta^{1/2} x \delta^{1/2})$$

for all $x \in \mathcal{M}_\psi$. The operator δ satisfies the following properties:

1. $\sigma_t^\varphi(\delta) = \sigma_t^\psi(\delta) = \delta \quad \forall t \in \mathbb{R}$;

2. $\Gamma(\delta^{it}) = \delta^{it} \otimes \delta^{it} \quad \forall t \in \mathbb{R}$;

3. $\tau_t(\delta) = \delta \quad \forall t \in \mathbb{R}$;

4. $R(\delta) = \delta^{-1}$;

5. $\sigma_t^\psi(x) = \delta^{it}\sigma_t^\varphi(x)\delta^{-it} \quad t \in \mathbb{R},\ x \in M$.

$\square$

Theorem 2.4.2. *The following hold:*

1. *The automorphism groups σ^φ, σ^ψ and τ commute pairwise.*

2. *We have:*

$$\begin{aligned} \Gamma\tau_t &= (\tau_t \otimes \tau_t)\Gamma & \qquad \Gamma\sigma_t^\varphi &= (\tau_t \otimes \sigma_t^\varphi)\Gamma \\ \Gamma\tau_t &= (\sigma_t^\varphi \otimes \sigma_{-t}^\psi)\Gamma & \qquad \Gamma\sigma_t^\psi &= (\sigma_t^\psi \otimes \tau_{-t})\Gamma \end{aligned}$$

for all $t \in \mathbb{R}$.

3. *We have:*

$$\begin{aligned} \varphi\sigma_t^\psi &= \varphi & \qquad \varphi\tau_t &= \varphi \\ \psi\sigma_t^\varphi &= \psi & \qquad \psi\tau_t &= \psi \end{aligned}$$

for all $t \in \mathbb{R}$.

$\square$

Up to now, we have referred to the notion of Kac algebras several times, without giving any definition for these structures. Actually we are not going to do so and just refer the reader to [13] for that. But one can show that:

Theorem 2.4.3. *A locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ is a Kac algebra if and only if the antipode S is bounded and the modular element δ is affiliated with the center of M.*

Hence in this thesis, we define a Kac algebra to be a locally compact quantum group with the above properties.

2.5 The Dual Quantum Group

Now we are at the step to define the dual of a locally compact quantum group, and stating a biduality theorem. The following theorem gives us the motivation for that.

Theorem 2.5.1. *For an abelian locally compact group G, the following hold:*

1. *G is compact if and only if $\hat{G}$ is discrete;*
2. *$L^2(G) \cong L^2(\hat{G})$;*
3. *$L^\infty(G) \cong VN(\hat{G})$;*
4. *$L^1(G) \cong A(\hat{G})$.*

□

From Theorem 2.5.1 we see that if we want to extend the duality of locally compact abelian groups to some larger category including function algebras of all locally compact groups, in the von Neumann algebraic setting, we must assign $L^\infty(G)$ and $VN(G)$ as dual to each other. We saw that one can construct each of these von Neumann algebras from the other one, via the unitary W_G.
Similarly, in general, it is the fundamental unitary which plays the role of a bridge in passing from a quantum group to its dual.

Definition and Proposition 2.5.2. *Let $\mathbb{G}$ be a locally compact quantum group with the left fundamental unitary W. Define $\hat{M} := \overline{\{\lambda(f) : f \in M_*\}}^{w^*}$. Furthermore, we define $\hat{\Gamma} : \hat{M} \to \hat{M}\bar{\otimes}\hat{M}$ given by*

$$\hat{\Gamma}(\hat{x}) = \hat{W}^*(1 \otimes \hat{x})\hat{W}$$

where the fundamental unitary operator $\hat{W}$ is given by

$$\hat{W} = \chi(W^*).$$

There exist a left Haar weight $\hat{\varphi}$ and a right Haar weight $\hat{\psi}$ such that $\hat{\mathbb{G}} := (\hat{M}, \hat{\Gamma}, \hat{\varphi}, \hat{\psi})$ is a locally compact quantum group.

We also have $W \in M\bar{\otimes}\hat{M}$ and $H_\varphi \cong H_{\hat{\varphi}}$. The completely contractive (left) regular representation is defined by

$$\hat{\lambda} : \hat{M}_* \ni \hat{f} \to (\hat{f} \otimes \iota)(\hat{W}) \in M.$$

Theorem 2.5.3. *We have the perfect (Pontryagin-type) duality theorem:*

$$\hat{\hat{\mathbb{G}}} \cong \mathbb{G}.$$

□

As in the case of locally compact abelian groups, discreteness and compactness are also dual properties in the category of locally compact quantum groups.

Theorem 2.5.4. *A locally compact quantum group $\mathbb{G}$ is discrete if and only if its dual is compact.* □

Similarly, the right regular representation $\rho : M_* \to \mathcal{B}(H_\varphi)$ can be defined by

$$\rho(f) = (\iota \otimes f)(V) \in \hat{M}' \subseteq \mathcal{B}(H_fee).$$

The corresponding dual quantum group can be expressed as $\hat{\mathbb{G}}' = (\hat{M}', \hat{\Gamma}', \hat{\varphi}', \hat{\psi}')$, where the co-multiplication $\hat{\Gamma}'$ is given by

$$\hat{\Gamma}'(\hat{x}') = \hat{V}'(\hat{x}' \otimes 1)\hat{V}'^{*}.$$

In this case (by considering the duality of $\mathbb{G}$ and $\hat{\mathbb{G}}'$), we obtain: $\hat{V}' \in M\overline{\otimes}\hat{M}'$, and also $\hat{V} = \chi(V^{*})$.

In analogy to the commutative case, we will use the following notation for a locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$:

- $L^{\infty}(\mathbb{G}) := M$;
- $L^{1}(\mathbb{G}) := M_{*}$
- $L^{2}(\mathbb{G}) := H_{\varphi} \cong H_{\psi} \cong H_{\hat{\varphi}} \cong H_{\hat{\psi}}$;
- $C_0(\mathbb{G}) := \overline{\{(\omega \otimes \iota)V \, : \, \omega \in \mathcal{T}(L^2(\mathbb{G}))\}}^{\|\cdot\|}$;
- $M(\mathbb{G}) := C_0(\mathbb{G})^{*}$

One can show that $C_0(\mathbb{G})$ is a C^{*}-algebra, and the co-multiplication gives rise to a Banach algebra structure on $M(\mathbb{G})$. This multiplication extends the convolution product of $L^{1}(\mathbb{G})$ via the canonical embedding $L^{1}(\mathbb{G}) \hookrightarrow M(\mathbb{G})$, and $L^{1}(\mathbb{G})$ becomes a two-sided ideal of $M(\mathbb{G})$.
Also, the left regular representation λ extends from $L^{1}(\mathbb{G})$ to $M(\mathbb{G})$.

As in the classical case we have the following.

Proposition 2.5.5. *A locally compact quantum group $\mathbb{G}$ is compact if and only if $C_0(\mathbb{G})$ is unital.* $\square$

We also have several equivalent formulation of co-amenability.

Theorem 2.5.6. *[4, Theorem 3.1] Let $\mathbb{G}$ be a locally compact quantum group, and W its left fundamental unitary. Then the following are equivalent:*

1. *$\mathbb{G}$ is co-amenable;*

2. *there exists a positive linear functional ε on $C_0(\mathbb{G})$ such that $\lambda(\varepsilon) = 1$;*

3. *there exists a net (ξ_i) of unit vectors in $L^2(\mathbb{G})$ such that*

$$\lim_i \|W(\xi_i \otimes \eta) - \xi_i \otimes \eta\|_2 = 0$$

for every $\eta \in L^2(\mathbb{G})$. □

2.6 Relations Between a Quantum Group and its Dual

We now present a series of results proved in [25], concerning relations between a locally compact quantum group and its dual, which we will need later in our work.

Theorem 2.6.1. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have:*

$$L^\infty(\mathbb{G}) \cap L^\infty(\hat{\mathbb{G}}) = L^\infty(\mathbb{G}) \cap L^\infty(\hat{\mathbb{G}})' = L^\infty(\mathbb{G})' \cap L^\infty(\hat{\mathbb{G}}) = L^\infty(\mathbb{G})' \cap L^\infty(\hat{\mathbb{G}})' = \mathbb{C}1.$$

Theorem 2.6.2. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have:*

$$\mathcal{B}(L^2(\mathbb{G})) = \overline{span}^{w^*}\{x\hat{x} : x \in L^\infty(\mathbb{G}), \hat{x} \in L^\infty(\hat{\mathbb{G}})\}.$$

For the following, recall that N and δ have been introduced in Proposition 2.3.1, and Definition and Proposition 2.4.1, respectively; for the definition of τ_t and R, see

Definition and Proposition 2.3.2. Also, note that the scaling constant for a locally compact quantum group is equal to 1, by [9, Corollary 2.7].

Proposition 2.6.3. *We have:* $N = \hat{N}$ *and* $\Delta_{\hat{\varphi}}^{it} = N^{it} J_\varphi \delta^{it} J_\varphi$.

$\square$

Proposition 2.6.4. *The following hold:*

$$\begin{aligned} \tau_t(x) &= \Delta_{\hat{\varphi}}^{it} x \Delta_{\hat{\varphi}}^{-it} & \qquad R(x) &= J_{\hat{\varphi}} x J_{\hat{\varphi}} \\ \hat{\tau}_t(\hat{x}) &= \Delta_{\varphi}^{it} \hat{x} \Delta_{\varphi}^{-it} & \qquad \hat{R}(\hat{x}) &= J_\varphi \hat{x} J_\varphi \end{aligned}$$

for all $x \in L^\infty(\mathbb{G})$, $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ *and* $t \in \mathbb{R}$.

$\square$

Proposition 2.6.5. *The operators* $\Delta_{\varphi}^{it_1}$, $\Delta_{\psi}^{it_2}$, $\Delta_{\hat{\varphi}}^{it_3}$, $\Delta_{\hat{\psi}}^{it_4}$, N^{it_5} *and* δ^{it_6} *commute pairwise, for any* $t_1, ..., t_6 \in \mathbb{R}$.

$\square$

Finally, we should note that there is also an equivalent C^*-algebraic approach to this theory. In the classical picture, we can pass from the category of locally compact groups to the category of C^*-algebras by assigning $C_0(G)$ to every locally compact group G. By a theorem of Weil (see [52]), in the standard Borel setting, the Haar measure on G and its topology are uniquely determined by each other. In other words, considering the topological aspect of the theory of locally compact groups, i.e., working with the C^*-algebra $C_0(G)$, is equivalent to studying the measure theory of G, equipped with its Haar measure, or equivalently, studying the von Neumann algebra $L^\infty(G)$.

The same phenomenon occurs in the quantum case. One can start with a C^*-algebra with a co-multiplication and left and right Haar weights, where all objects have the appropriate properties in the category of C^*-algebras. This is the point of view taken in [54], [55] and [25].

Chapter 3

From Quantum Groups to Groups

To find a Pontryagin-type duality theorem which holds for all locally compact groups rather than just abelian ones, one has to pass to a larger category than locally compact groups, namely locally compact quantum groups.

In order to embed locally compact groups in this larger category, we have to work with the algebras associated with a group. So, instead of working with a locally compact group G, we study $L^\infty(G)$, and we consider $VN(G)$ as its Pontryagin dual.

There are many other occasions in which one prefers or even has to pass from a group to an associated function algebra. But there are several equivalent ways to recover the initial group, from these algebras: G is topologically isomorphic to

- the spectrum of $C_0(G)$, as well as $A(G)$;
- the set of all group-like elements in $VN(G)$ with its canonical co-multiplication;
- ...

One can also recover the group from its measure algebra $M(G)$. Indeed we can consider the elements of $M(G)$ as maps on $L^1(G)$, which commute with convolution;

Wendel proved in [53] that any such map which is also isometric, has to be a (multiple of a) point-mass.

In [23] Junge, Neufang and Ruan defined an analogue of measure algebra for locally compact quantum groups, and studied its structure and representation theory. This work motivated us to look for objects similar to point-masses in the classical case, so-to-speak "quantum point-masses".

3.1 Completely Bounded Multipliers of $L^1(\mathbb{G})$

As mentioned earlier, the series of works [30], [31] and [23] have been the chief motivation for this thesis. In [23] the authors succeeded in unifying all the earlier results, by developing a representation theory for completely bounded multipliers of the L^1-algebra of a locally compact quantum group. In the classical case, where $\mathbb{G} = L^\infty(G)$ for a locally compact group G, this algebra is just the measure algebra $M(G)$. So we can regard the algebra $M_{cb}(L^1(\mathbb{G}))$ as a quantization of the measure algebra. Following this idea, we shall study in this chapter the quantum analogue of point-masses.

We start with a brief review of some of the results in [23].

Definition and Proposition 3.1.1. *[23, Section 4] Let $\mathbb{G}$ be a locally compact quantum group. An operator $\hat{b}' \in L^\infty(\hat{\mathbb{G}}')$ is called a completely bounded right multiplier of $L^1(\mathbb{G})$ (associated with the right fundamental operator V) if we have*

$$\rho(f)\hat{b}' \in \rho(L^1(\mathbb{G}))$$

for all $f \in L^1(\mathbb{G})$,and the induced map

$$m^r_{\hat{b}'} : L^1(\mathbb{G}) \ni f \longmapsto \rho^{-1}(\rho(f)\hat{b}') \in L^1(\mathbb{G})$$

is completely bounded. We denote by $M^r_{cb}(L^1(\mathbb{G}))$ the space of all completely bounded right multipliers of $L^1(\mathbb{G})$. Then $M^r_{cb}(L^1(\mathbb{G}))$ is (algebraically) a unital subalgebra of $L^\infty(\hat{\mathbb{G}}')$ and we can identify $L^1(\mathbb{G})$ with the subalgebra $\rho(L^1(\mathbb{G}))$ in $M^r_{cb}(L^1(\mathbb{G}))$.

In the sequel we denote by $b.f$ the canonical action of $b \in L^\infty(\mathbb{G})$ on $f \in L^1(\mathbb{G})$.

Proposition 3.1.2. *[23, Proposition 4.1] Let $\hat{b}'$ be an operator in $L^\infty(\hat{\mathbb{G}}')$. Then the following are equivalent:*

1. *$\hat{b}' \in M^r_{cb}(L^1(\mathbb{G}))$ satisfies*
$$\|m^r_{\hat{b}'}\|_{cb} \leq 1;$$

2. *we have*
$$\|[\hat{b}'.\hat{f}'_{ij}]\|_n \leq \|[\hat{f}'_{ij}]\|_n$$
for all $[\hat{f}'_{ij}] \in M_n(L^1(\hat{\mathbb{G}}'))$ and $n \in \mathbb{N}$;

3. *there exists a complete contraction $M^r_{\hat{b}'}$ on $C_0(\mathbb{G})$ such that*
$$M^r_{\hat{b}'}(\hat{\rho}'(\hat{f}')) = \hat{\rho}'(\hat{b}'.\hat{f}')$$
for all $\hat{f}' \in L^1(\hat{\mathbb{G}}')$. In this case, we actually have $M^r_{\hat{b}'} = (m^r_{\hat{b}'})^*|_{C_0(\mathbb{G})}$;

4. *there exists a normal complete contraction $\Phi^r_{\hat{b}'}$ on $L^\infty(\mathbb{G})$ such that*
$$(\iota \otimes \Phi^r_{\hat{b}'})(V) = V(\hat{b}' \otimes 1).$$

□

Definition 3.1.1. *[23, Section 4] A completely bounded map T on $L^1(\mathbb{G})$ is said to be a completely bounded* right centralizer *on $L^1(\mathbb{G})$ if it satisfies*

$$T(f * g) = f * T(g)$$

for all $f, g \in L^1(\mathbb{G})$. We denote by $C^r_{cb}(L^1(\mathbb{G}))$ the space of all completely bounded right centralizers of $L^1(\mathbb{G})$.

A normal completely bounded map Φ on $L^\infty(\mathbb{G})$ is called right covariant *if it satisfies:*

$$(\iota \otimes \Phi) \circ \Gamma = \Gamma \circ \Phi.$$

We denote by $\mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G}))$ the algebra of all normal completely bounded right covariant maps on $L^\infty(\mathbb{G})$.

It is easy to see that a normal completely bounded map Φ on $L^\infty(\mathbb{G})$ is in $\mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G}))$ if and only if it is a right $L^1(\mathbb{G})$-module map on $L^\infty(\mathbb{G})$. Therefore, a map T is in $C^r_{cb}(L^1(\mathbb{G}))$ if and only if T^* is in $\mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G}))$.

The main point of the work [23] was to show that the algebra $M^r_{cb}(L^1(\mathbb{G}))$ can be identified with the algebra of normal completely bounded maps Φ on $\mathcal{B}(L^2(\mathbb{G}))$ satisfying the following two properties:

- $\Phi(L^\infty(\mathbb{G})) \subseteq L^\infty(\mathbb{G})$;
- $\Phi(\hat{a}x\hat{b}) = \hat{a}\Phi(x)\hat{b} \quad \forall \hat{a}, \hat{b} \in L^\infty(\hat{\mathbb{G}}), x \in \mathcal{B}(L^2(\mathbb{G}))$.

We denote the algebra of all such maps Φ by $\mathcal{CB}^{\sigma, L^\infty(\mathbb{G})}_{L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G})))$.

More precisely, we have the following.

Theorem 3.1.3. *[23, Theorem 4.5] Let $\mathbb{G}$ be a locally compact quantum group. Then there exists a completely isometric algebra isomorphism $\Theta_r : M^r_{cb}(L^1(\mathbb{G})) \to \mathcal{CB}^{\sigma, L^\infty(\mathbb{G})}_{L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G})))$, and we can completely isometrically and algebraically identify the following completely contractive Banach algebras:*

$$M^r_{cb}(L^1(\mathbb{G})) \cong \mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G})) \cong C^r_{cb}(L^1(\mathbb{G})) \cong \mathcal{CB}^{\sigma, L^\infty(\mathbb{G})}_{L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G}))).$$

We denote the identification $C^r_{cb}(L^1(\mathbb{G})) \cong \mathcal{CB}^{\sigma, L^\infty(\mathbb{G})}_{L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G})))$ by $T \mapsto \Phi(T)$.

As we noted earlier, a theorem by Wendel motivated part of this work. The statement of his result is as follows.

Theorem 3.1.4. *[53, Theorem 3] Let G be a locally compact group, and T a bounded linear map on $L^1(G)$ Then $T = \lambda r_g$, for some $\lambda \in \mathbb{T}$ and $g \in G$, where r_g is right translation by g, if and only if T is an isometric left centralizer.*

The next result, which can be seen as a quantum version of Wendel's theorem above, is the starting point for our work. In fact, this theorem will suggest how to define the objects that should be called quantum point-masses.

In the sequel we shall denote by $Ad(u)$ the map $x \mapsto uxu^*$, for a unitary u.

Theorem 3.1.5. *[23, Theorem 4.7] Let $\mathbb{G}$ be a locally compact quantum group, and let T be a complete contraction in $C^r_{cb}(L^1(\mathbb{G}))$. Then the following are equivalent:*

1. *T is a completely isometric linear isomorphism on $L^1(\mathbb{G})$;*
2. *T has a completely contractive inverse in $C^r_{cb}(L^1(\mathbb{G}))$;*
3. *there exist a unitary operator $\hat{u}' \in L^\infty(\hat{\mathbb{G}}')$ and a complex number $\lambda \in \mathbb{T}$ such that $\Phi_T(x) = \lambda Ad(\hat{u}')(x)$. If, in addition, T is completely positive, then so is T^{-1}. In this case, we have $\Phi_T = Ad(\hat{u}')$.* □

A beautiful result in [23] states that one can capture the Pontryagin-type duality of quantum groups as a commutation relation

In the following, for $S \subseteq \mathcal{CB}(\mathcal{B}(L^2(\mathbb{G})))$, we denote by S^c the commutant taken in $\mathcal{CB}(\mathcal{B}(L^2(\mathbb{G})))$.

Theorem 3.1.6. *[23, Theorem 5.1 and Corollary 5.3] Let $\mathbb{G}$ be a locally compact quantum group with dual quantum group $\hat{\mathbb{G}}$. Then we have:*

1. $\hat{\Theta}_r(M_{cb}^r(L^1(\hat{\mathbb{G}}))) = \Theta_r(M_{cb}^r(L^1(\mathbb{G})))^c \cap \mathcal{CB}^{\sigma}_{L^\infty(\mathbb{G})}(\mathcal{B}(L^2(\mathbb{G})))$;

2. *the following bi-commutant theorem holds:*

$$\Theta_r(M_{cb}^r(L^1(\mathbb{G})))^{cc} = \Theta_r(M_{cb}^r(L^1(\mathbb{G}))).$$

Theorem 3.1.7. *Let $\mathbb{G}$ be a co-amenable locally compact quantum group. Then we have $M_{cb}^r(L^1(\mathbb{G})) \cong M(\mathbb{G})$, via the map:*

$$M_{cb}^r(L^1(\mathbb{G})) \ni \hat{b}' \mapsto \phi_{\hat{b}'} \in M(\mathbb{G})$$

where $\langle \hat{\lambda}(\hat{\omega}'), \phi_{\hat{b}'} \rangle = \langle \hat{\omega}', \hat{b}' \rangle$ for all $\hat{\omega}' \in L^1(\hat{\mathbb{G}}')$.

3.2 Assigning a Group to a Quantum Group

Bearing Wendel's result, Theorem 3.1.4, in mind we are led to the following definition.

Definition 3.2.1. *Let $\mathbb{G}$ be a locally compact quantum group. Define $\tilde{\mathbb{G}}$ to be the set of all completely positive maps $m \in C_{cb}^r(L^1(\mathbb{G}))$ which satisfy one of the equivalent conditions of Theorem 3.1.5. We endow $\tilde{\mathbb{G}}$ with the strong operator topology.*

Example 3.2.1. *As a consequence of Theorem 3.1.4, we see that if* $\mathbb{G} = L^\infty(G)$ *for a locally compact group* G*, then* $\tilde{\mathbb{G}}$ *is topologically isomorphic to* G*.*

Example 3.2.2. *[38, Theorem 2.] If* $\mathbb{G} = VN(G)$ *for an amenable locally compact group* G*, then* $\tilde{\mathbb{G}}$ *is topologically isomorphic to* $\hat{G}$*, the set of all continuous characters on* G *with the compact-open topology.*

As we shall see later, amenability is not necessary.

We thus obtain an assignment $\mathbb{G} \longrightarrow \tilde{\mathbb{G}}$, from the category of locally compact quantum groups to the category of groups, which is inverse to the usual embedding of the latter category into the former. The main purpose of this chapter is to investigate how much information about $\mathbb{G}$ one can get from studying $\tilde{\mathbb{G}}$, and also study the preservation of several natural properties under this assignment

Note that in the classical case, for $m \in \tilde{\mathbb{G}}$, $m^* : L^\infty(G) \to L^\infty(G)$ is just the right translation. The next proposition (which was proved for the Kac algebra case in [7, Lemma 2.2.]) shows that for a locally compact quantum group $\mathbb{G}$ and $m \in \tilde{\mathbb{G}}$, m^* can be regarded as quantum right translation.

Proposition 3.2.1. *Let* $m \in \tilde{\mathbb{G}}$ *and* ψ *be the right Haar weight on* $\mathbb{G}$*. Then*

$$\psi \circ m^* = \psi.$$

Proof. For $x \in L^\infty(\mathbb{G})^+$ we have $m^*(x) \in L^\infty(\mathbb{G})^+$ as well, and

$$\begin{aligned}
(\psi \circ m^*(x))1 &= \psi(m^*(x))1 = (\psi \otimes \iota)\Gamma(m^*(x)) \\
&= (\psi \otimes m^*)\Gamma(x) = \psi(x)m^*(1) \\
&= \psi(x)1.
\end{aligned}$$

□

Definition and Proposition 3.2.2. *[19, Proposition 3.7.] Let M be a von Neumann algebra. On the set of all bounded, weak* continuous operators on M, we define the p-topology to be the one induced by the semi-norms*

$$T \to \langle Tx, \omega \rangle \qquad (x \in M,\, \omega \in M_*),$$

and the u-topology by the semi-norms

$$T \to \|T_*\omega\| \qquad \omega \in M_*.$$

Let φ be an n.s.f. weight on M. Then the p-topology and the u-topology coincide on $Aut_\varphi(M)$, the set of all *-automorphism on M under which φ is invariant. $\square$

Corollary 3.2.3. *The p-topology and the u-topology coincide on $(\tilde{\mathbb{G}})^* := \{m^* : m \in \tilde{\mathbb{G}}\}$, and $\tilde{\mathbb{G}}$ equipped with any of these becomes a topological group.*

Proof. That both topologies coincide on $(\mathbb{G})^*$, is clear from Proposition 3.2.1 and Proposition 3.2.2.

To see that $\tilde{\mathbb{G}}$ is a topological group, let $m_\alpha^* \xrightarrow{u} \iota$ and $n_\alpha^* \xrightarrow{u} \iota$, where (m_α) and (n_α) are nets in $\tilde{\mathbb{G}}$. Then, for all $f \in L^1(\mathbb{G})$, we have:

$$\begin{aligned} \|(m_\alpha n_\alpha - \iota)f\| &\leq \|(m_\alpha n_\alpha - m_\alpha)f\| + \|(m_\alpha - \iota)f\| \\ &\leq \|m_\alpha\|\|(n_\alpha - \iota)f\| + \|(m_\alpha n_\alpha - \iota)f\| \to 0. \end{aligned}$$

We also have:

$$\|m_\alpha^{-1}f - f\| = \|m_\alpha^{-1}f - m_\alpha^{-1}(m_\alpha f)\| = \|m_\alpha^{-1}(f - m_\alpha f)\| \leq \|f - m_\alpha(f)\| \to 0.$$

Definition 3.2.2. *Let (M, Γ) be a Hopf-von Neumann algebra. We define*

$$Gr(M, \Gamma) := \{x \in M : \Gamma(x) = x \bar{\otimes} x \text{ and } x \text{ is invertible}\}.$$

$Gr(M, \Gamma)$ obviously forms a group, and we endow this group with the induced weak* topology.

Remark 3.2.4. *As is easily seen (cf. [13, Proposition 1.2.3]) we have $Gr(M, \Gamma) = \{x \in M : \Gamma(x) = x \bar{\otimes} x$ and x is unitary$\}$.*

For a locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ we denote $Gr(M, \Gamma)$ simply by $Gr(\mathbb{G})$.

Proposition 3.2.5. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have:*

$$Gr(\mathbb{G}) = Gr(\mathcal{B}(L^2(\mathbb{G})), \Gamma).$$

Proof. Obviously $Gr(\mathbb{G}) \subseteq Gr(\mathcal{B}(L^2(\mathbb{G})), \Gamma)$. If $x \in Gr(\mathcal{B}(L^2(\mathbb{G})), \Gamma)$, then $\Gamma(x) = x \otimes x$; but since $\Gamma(\mathcal{B}(L^2(\mathbb{G}))) \subseteq \mathcal{B}(L^2(\mathbb{G})) \bar{\otimes} L^\infty(\mathbb{G})$, we have $x \in L^\infty(\mathbb{G})$. □

Lemma 3.2.6. *Let (m_α) be a net in $\tilde{\mathbb{G}}$ and (u_α) a net of unitaries in $\mathcal{B}(L^2(\mathbb{G}))$. Then we have:*

1. *$m_\alpha \to 1$ if and only if $\|m_\alpha(\omega_{\Lambda_\psi(x)}) - \omega_{\Lambda_\psi(x)}\| \to 0$ for all $x \in \mathcal{N}_\psi$;*
2. *$u_\alpha \xrightarrow{w^*} 1$ if and only if $\|u_\alpha \Lambda_\psi(x) - \Lambda_\psi(x)\| \to 0$ for all $x \in \mathcal{N}_\psi$.*

Proof. 1. Since $L^\infty(\mathbb{G})$ is in standard form on $L^2(\mathbb{G})$, every $f \in L^1(\mathbb{G})^+$ is of the form ω_ξ, for some $\xi \in L^2(\mathbb{G})$. Let $\epsilon > 0$, and assume

$$\|m_\alpha(\omega_{\Lambda_\psi(x)}) - \omega_{\Lambda_\psi(x)}\| \to 0 \quad \forall x \in \mathcal{N}_\psi.$$

Since $\Lambda_\psi(\mathcal{N}_\psi)$ is dense in $L^2(\mathbb{G})$, we can find $x \in \mathcal{N}_\psi$ such that $\|\omega_\xi - \omega_{\Lambda_\psi(x)}\| < \epsilon$. Then we have:

$$\begin{aligned}
\|m_\alpha(\omega_\xi) - \omega_\xi\| &\leq \|m_\alpha(\omega_\xi) - m_\alpha(\omega_{\Lambda_\psi(x)})\| + \|m_\alpha(\omega_{\Lambda_\psi(x)}) - \omega_{\Lambda_\psi(x)}\| + \|\omega_{\Lambda_\psi(x)} - \omega_\xi\| \\
&\leq \|m_\alpha\|\|\omega_\xi - \omega_{\Lambda_\psi(x)}\| + \|m_\alpha(\omega_{\Lambda_\psi(x)}) - \omega_{\Lambda_\psi(x)}\| + \|\omega_{\Lambda_\psi(x)} - \omega_\xi\| \\
&\leq 2\epsilon + \|m_\alpha(\omega_{\Lambda_\psi(x)}) - \omega_{\Lambda_\psi(x)}\|.
\end{aligned}$$

Since ϵ was arbitrary, the conclusion follows.

2. We have:

$$\|u_\alpha\xi - \xi\| \leq \|u_\alpha\xi - u_\alpha\Lambda_\psi(x)\| + \|u_\alpha\Lambda_\psi(x) - \Lambda_\psi(x)\| + \|\Lambda_\psi(x) - \xi\|$$

for all $\xi \in L^2(\mathbb{G})$ and $x \in \mathcal{N}_\psi$. This implies $u_\alpha \xrightarrow{sot} 1$ if and only if $\|u_\alpha\Lambda_\psi(x) - \Lambda_\psi(x)\| \to 0$, and since the strong operator and the weak* topologies coincide on unitaries, the conclusion follows. □

The following theorem generalizes a result by de Cannière [7, Definition 2.5.] from the case of Kac algebras to all locally compact quantum groups.

Theorem 3.2.7. *Let $\mathbb{G}$ be a locally compact quantum group. Then there is a group homeomorphism:*

$$\Phi : \tilde{\mathbb{G}} \to Gr(\hat{\mathbb{G}}').$$

Proof. Let $m \in \tilde{\mathbb{G}}$. Since $\psi \circ m^* = \psi$ (by Proposition 3.2.1), we can, in view of Lemma 1.4.4, extend the map

$$\Lambda_\psi(x) \mapsto \Lambda_\psi(m^*(x)) \quad (x \in \mathcal{N}_\psi)$$

to a unitary $\hat{u}'_m$ on $L^2(\mathbb{G})$, such that:

$$m^* = Ad(\hat{u}'_m).$$

We shall prove that $\hat{u}'_m \in Gr(\hat{\mathbb{G}}')$. Let V be the right fundamental unitary. For all $x, y \in \mathcal{N}_\psi$, we have:

$$\begin{aligned}
(1 \otimes \hat{u}'_m)V(\Lambda_\psi(x) \otimes \Lambda_\psi(y)) &= (1 \otimes \hat{u}'_m)\Lambda_{\psi\otimes\psi}(\Gamma(x))(1 \otimes y)) \\
&= \Lambda_{\psi\otimes\psi}((\iota \otimes m^*)(\Gamma(x)(1 \otimes y))) \\
&= \Lambda_{\psi\otimes\psi}(\Gamma(m^*(x))(1 \otimes m^*(y))) \\
&= V(\Lambda_\psi(m^*(x)) \otimes \Lambda_\psi(m^*(y)) \\
&= V(\hat{u}'_m \otimes \hat{u}'_m)(\Lambda_\psi(x) \otimes \Lambda_\psi(y)).
\end{aligned}$$

Hence, we obtain $V^*(1 \otimes \hat{u}'_m)V = \hat{u}'_m \otimes \hat{u}'_m$ which implies:

$$\begin{aligned}
\hat{V}'(\hat{u}'_m \otimes 1)\hat{V}'^* &= \chi(V^*)(\hat{u}'_m \otimes 1)\chi(V) \\
&= \chi(V^*(1 \otimes \hat{u}'_m)V) \\
&= \chi(\hat{u}'_m \otimes \hat{u}'_m) \\
&= \hat{u}'_m \otimes \hat{u}'_m.
\end{aligned}$$

Thus $\hat{u}'_m \in Gr(\mathcal{B}(L^2(\mathbb{G})), \hat{\Gamma}')$, and so $\hat{u}'_m \in Gr(\hat{\mathbb{G}}')$ by Proposition 3.2.5.

Now define

$$\begin{aligned}
\Phi : \tilde{\mathbb{G}} &\to Gr(\hat{\mathbb{G}}') \\
m &\mapsto \hat{u}'_m.
\end{aligned}$$

It is easily seen that Φ is a well-defined group homomorphism.

If $\Phi(m) = 1$, we have $m^* = Ad1 = \iota$, which implies $m = \iota$. Hence Φ is injective.

To see that Φ is also surjective, let $\hat{u}' \in Gr(\hat{\mathbb{G}}')$. Then the above calculations show that for all $\omega \in \mathcal{T}(L^2(\mathbb{G}))$, we have:

$$\begin{aligned} Ad(\hat{u}')((\omega\otimes\iota)V) &= (\omega\otimes\iota)((1\otimes\hat{u}')V(1\otimes\hat{u}'^*)) \\ &= (\omega\otimes\iota)(V(\hat{u}'\otimes\hat{u}')(1\otimes\hat{u}'^*)) \\ &= (\omega\otimes\iota)(V(\hat{u}'\otimes 1)) \\ &= ((\hat{u}'.\omega)\otimes\iota)V \in L^\infty(\mathbb{G}). \end{aligned}$$

Since $\{(\omega\otimes\iota)V : \omega \in \mathcal{T}(L^2(\mathbb{G}))\}$ is weak* dense in $L^\infty(\mathbb{G})$, we see that

$$Ad(\hat{u}')(L^\infty(\mathbb{G})) \subseteq L^\infty(\mathbb{G}),$$

whence $Ad(\hat{u}') \in \mathcal{CB}^{\sigma, L^\infty(\mathbb{G})}_{L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G})))$. Hence, by Theorem 3.1.5, there exists $m \in \tilde{\mathbb{G}}$ such that $m^* = Ad(\hat{u}')$, which implies $\Phi(m) = \hat{u}'$.

We now show that Φ is a homeomorphism with respect to the corresponding topologies. For all $x \in \mathcal{N}_\psi$, $y \in L^\infty(\mathbb{G})$ and $m \in \tilde{\mathbb{G}}$, with $m^* = Ad(\hat{u}'_m)$, we have:

$$\begin{aligned} \langle m(\omega_{\Lambda_\psi(x)}), y\rangle &= \langle \omega_{\Lambda_\psi(x)}, m^*(y)\rangle \langle \omega_{\Lambda_\psi(x)}, \hat{u}'_m y \hat{u}'^*_m\rangle \\ &= \langle \hat{u}'_m y \hat{u}'^*_m \Lambda_\psi(x), \Lambda_\psi(x)\rangle \\ &= \langle y \hat{u}'^*_m \Lambda_\psi(x), \hat{u}'^*_m \Lambda_\psi(x)\rangle \\ &= \langle \omega_{(\hat{u}'_m)^* \Lambda_\psi(x)}, y\rangle, \end{aligned}$$

which implies:

$$m(\omega_{\Lambda_\psi(x)}) = \omega_{(\hat{u}'_m)^* \Lambda_\psi(x)}.$$

Now, let (m_α) be a net in $\mathbb{G}$ such that $m_\alpha \to 1$, with $m_\alpha^* = Ad(\hat{u}'_\alpha)$. By Lemma 3.2.6, we have:

$$\begin{aligned} m_\alpha \to 1 \quad &\Leftrightarrow \quad \|m_\alpha(\omega_{\Lambda_\psi(x)}) - \omega_{\Lambda_\psi(x)}\| \to 0 \quad \forall x \in \mathcal{N}_\psi \\ &\Leftrightarrow \quad \|\omega_{(\hat{u}'_\alpha)^*\Lambda_\psi(x)} - \omega_{\Lambda_\psi(x)}\| \to 0 \quad \forall x \in \mathcal{N}_\psi \\ &\Leftrightarrow \quad \|(\hat{u}'_\alpha)^*\Lambda_\psi(x) - \Lambda_\psi(x)\| \to 0 \quad \forall x \in \mathcal{N}_\psi \\ &\Leftrightarrow \quad (\hat{u}'_\alpha)^* \xrightarrow{sot} 1 \Leftrightarrow \hat{u}'_\alpha \xrightarrow{sot} 1 \Leftrightarrow \hat{u}'_\alpha \xrightarrow{w^*} 1. \end{aligned}$$

□

Our next result is a generalization of the Heisenberg commutation relation, which has been known to hold for Kac algebras [13, Corollary 4.6.6].

Corollary 3.2.8. *Let $\mathbb{G}$ be a locally compact quantum group. Let $u \in Gr(\mathbb{G})$ and $\hat{u}' \in Gr(\hat{\mathbb{G}}')$. Then there exists $\lambda \in \mathbb{T}$ such that*

$$u\hat{u}' = \lambda \hat{u}' u.$$

Proof. We obtain from Theorems 3.1.5 and 3.2.7 that $\Theta_r(u) = Ad(u)$ and $\hat{\Theta}'_r(\hat{u}') = Ad(\hat{u}')$. Also we know from Theorem 3.1.6 that $\Theta_r(u)$ and $\hat{\Theta}'_r(\hat{u}')$ commute. Therefore we have:

$$Ad(u)Ad(\hat{u}') = Ad(\hat{u}')Ad(u) \Longrightarrow Ad(u)Ad(\hat{u}')Ad(u^*)Ad(\hat{u}'^*) = \iota \Longrightarrow u\hat{u}'u^*\hat{u}'^* \in \mathbb{C}1,$$

which yields the conclusion. □

Our Theorem 3.2.11 below, is as well a generalization of a result known in the Kac algebra case [13]. But the proof in Kac algebras setting does not work for the case of locally compact quantum groups. Here we present a different argument for the general case. We should thank Professor Vainerman for his valuable help with this proof.

We shall use the notion of extended positive part of a von Neumann algebra, introduced by Haagerup [17].

Definition and Proposition 3.2.9. *The extended positive part $\overline{M}^+$ of a von Neumann algebra M is the set of all maps $\{m : {M_*}^+ \to [0, \infty]\}$ that satisfy:*

1. *$m(\lambda f) = \lambda m(f)$ for all $f \in {M_*}^+$ and $\lambda \in \mathbb{R}^+$;*
2. *$m(f + g) = m(f) + m(g)$ for all $f, g \in {M_*}^+$;*
3. *m is lower semi-continuous.*

The set $\overline{M}^+$ is closed under addition, multiplication by a positive number, increasing limits, and M^+ is naturally embedded in $\bar{M}^+$.

Using the above notion, we have the following stronger version of left (right) invariance.

Proposition 3.2.10. *[26, Proposition 3.1] Let $\mathbb{G}$ be a locally compact quantum group. then we have*

$$(\iota \otimes \varphi)\Gamma(x) = \varphi(x)1 \quad (x \in L^\infty(\mathbb{G})),$$

where the equation is to be read in $\overline{L^\infty(\mathbb{G})}^+$. □

For a Banach algebra $\mathcal{A}$ we denote by $sp(\mathcal{A})$ its spectrum, i.e, the set of all non-zero bounded multiplicative linear functionals on $\mathcal{A}$.

Theorem 3.2.11. *For any locally compact quantum group $\mathbb{G}$ we have $Gr(\mathbb{G}) = sp(L^1(\mathbb{G}))$.*

Proof. Let $x \in Gr(\mathbb{G})$. Then $\Gamma(x) = x \otimes x$ and $x \neq 0$, and for $\omega, \omega' \in L^1(\mathbb{G})$ we have

$$\langle \omega * \omega', x \rangle = \langle \omega \bar{\otimes} \omega', \Gamma(x) \rangle = \langle \omega \bar{\otimes} \omega', x \bar{\otimes} x \rangle = \langle \omega, x \rangle \langle \omega', x \rangle,$$

which implies $x \in sp(L^1(\mathbb{G}))$. Hence, $Gr(\mathbb{G}) \subseteq sp(L^1(\mathbb{G}))$.

To show the inverse inclusion, let $x \in sp(L^1(\mathbb{G}))$, i.e., $\Gamma(x) = x \otimes x$ and $x \neq 0$. We need to show that x is then invertible. We first prove that x^*x is invertible.

Suppose x^*x is not invertible; then there exists a state f on $L^\infty(\mathbb{G})$ such that $f(x^*x) = 0$. Let φ be the left Haar weight, and (ω_α) a net of normal positive linear functionals on $L^\infty(\mathbb{G})$ such that $\sup_\alpha \omega_\alpha(y) = \varphi(y)$ for all $y \in L^\infty(\mathbb{G})$; see equation (1.3). Also, by Proposition 3.2.10 we have:

$$(\iota \otimes \varphi)\Gamma(y) = \varphi(y)1$$

for all $y \in L^\infty(\mathbb{G})$, where both sides correspond to some element in $\overline{L^\infty(\mathbb{G})^+}$, the extended positive part of $L^\infty(\mathbb{G})$. So we have:

$$\begin{aligned} \sup_\alpha \omega_\alpha(x^*x)x^*x &= \sup_\alpha (\iota \otimes \omega_\alpha)(x^*x \otimes x^*x) \\ &= \sup_\alpha (\iota \otimes \omega_\alpha)\Gamma(x^*x) \\ &= (\iota \otimes \varphi)\Gamma(x^*x) \\ &= \varphi(x^*x)1 \\ &= \sup_\alpha \omega_\alpha(x^*x)1. \end{aligned}$$

Now, let $\overline{f}$ be the unique extension of f to $\overline{L^\infty(\mathbb{G})^+}$. Then we obtain:

$$\begin{aligned} 0 &= \sup_\alpha \omega_\alpha(x^*x)\overline{f}(x^*x) &&= \sup_\alpha \overline{f}(\omega_\alpha(x^*x)x^*x) \\ &= \overline{f}(\sup_\alpha \omega_\alpha(x^*x)x^*x) &&= \overline{f}(\sup_\alpha \omega_\alpha(x^*x)1) \\ &= \sup_\alpha \omega_\alpha(x^*x)\overline{f}(1) &&= \varphi(x^*x). \end{aligned}$$

Since φ is faithful, $x = 0$. Hence, for any $x \in sp(L^1(G))$, x^*x is invertible. Similarly, we can show that xx^* is invertible as well, and so x is invertible. □

Theorem 3.2.12. *For any locally compact quantum group $\mathbb{G}$ we have $Gr(\mathcal{B}(L^2(\mathbb{G})),\Gamma) = sp(\mathcal{T}_*(\mathbb{G}))$.*

Proof. Recall that $Gr(\mathbb{G}) = Gr(\mathcal{B}(L^2(\mathbb{G})),\Gamma)$ by Proposition 3.2.5.

Obviously $Gr(\mathcal{B}(L^2(\mathbb{G})),\Gamma) \subseteq sp(\mathcal{T}_*(\mathbb{G}))$. Conversely, let $x \in sp(\mathcal{T}_*)$, i.e., $\Gamma(x) = x \otimes x$ and $x \neq 0$. Then we have $x \in L^\infty(\mathbb{G})$, and so, by Theorem 3.2.11, $x \in Gr(\mathbb{G})$. □

Theorem 3.2.13. *If $\mathbb{G}$ is co-amenable then we have a group homeomorphism:*

$$Gr(\hat{\mathbb{G}}') \cong sp(C_0(\mathbb{G})).$$

Proof. If $\mathbb{G}$ is co-amenable, we can identify $M^r_{cb}(L^1(\mathbb{G}))$ with $M(\mathbb{G}) = C_0(\mathbb{G})^*$ by Theorem 3.1.7, and, then using the same notation, we have:

$$\begin{aligned} \hat{b}' \in Gr(\hat{\mathbb{G}}') &\Leftrightarrow \hat{\Gamma}'(\hat{b}') = \hat{b}' \otimes \hat{b}' \Leftrightarrow \langle \hat{\omega}_1' * \hat{\omega}_2', \hat{b}'\rangle = \langle \hat{\omega}_1', \hat{b}'\rangle\langle \hat{\omega}_2', \hat{b}'\rangle \ \forall \hat{\omega}_1', \hat{\omega}_2' \in L^1(\hat{\mathbb{G}}') \\ &\Leftrightarrow \langle \hat{\lambda}'(\hat{\omega}_1' * \hat{\omega}_2'), \phi_{\hat{b}'}\rangle = \langle \hat{\lambda}'(\hat{\omega}_1'), \phi_{\hat{b}'}\rangle\langle \hat{\lambda}'(\hat{\omega}_2'), \phi_{\hat{b}'}\rangle \ \forall \hat{\omega}_1', \hat{\omega}_2' \in L^1(\hat{\mathbb{G}}') \\ &\Leftrightarrow \phi_{\hat{b}'} \in sp(C_0(\mathbb{G})). \end{aligned}$$

Since $\hat{\lambda}'(L^1(\hat{\mathbb{G}}'))$ is norm dense in $C_0(\mathbb{G})$, the map

$$Gr(\hat{\mathbb{G}}) \ni \hat{b}' \mapsto \Phi_{\hat{b}'} \in sp(C_0(\mathbb{G}))$$

is a weak*–weak* homeomorphism. □

Our next result combines all the above identifications.

Theorem 3.2.14. *The following can be identified as locally compact groups:*

1. $\tilde{\mathbb{G}}$ *with the strong operator topology;*
2. $Gr(\hat{\mathbb{G}}')$ *with the weak* topology;*
3. $sp(L^1(\hat{\mathbb{G}}'))$ *with the weak* topology;*
4. $sp(\mathcal{T}_{\bullet}(\mathbb{G}))$ *with the weak* topology;*
5. $Gr(\mathcal{B}(L^2(\mathbb{G})), \hat{\Gamma}')$ *with the weak* topology;*

   moreover, when $\mathbb{G}$ *is co-amenable, these can also be identified with:*

6. $sp(C_0(\mathbb{G}))$ *with the weak* topology.*

Proof. Since the spectrum of a Banach algebra is locally compact with weak* topology, all the above groups are locally compact groups. □

Remark 3.2.15. *Applying Theorem 3.2.14 to the case where* $\mathbb{G} = VN(G)$ *for a locally compact group* G*, we obtain a generalization of a Renault's result (cf. [38, Theorem 2]), in which* G *is assumed amenable.*

Theorem 3.2.16. *The assignment $\mathbb{G} \to \tilde{\mathbb{G}}$ preserves compactness, discreteness and hence finiteness.*

Proof. 1. Let $\mathbb{G}$ be compact. In view of Theorem 3.2.14, we may equivalently show that $Gr(\hat{\mathbb{G}}')$ is compact. Let $\hat{e}' \in L^1(\hat{\mathbb{G}}')$ be the unit ($\hat{\mathbb{G}}'$ is discrete by Theorem 2.5.4). Then, for any $\hat{x}' \in Gr(\hat{\mathbb{G}}')$, we have

$$\begin{aligned}
\langle \hat{f}', \hat{x}' \rangle &= \langle \hat{f}' * \hat{e}', \hat{x}' \rangle \\
&= \langle \hat{f}' \hat{\otimes} \hat{e}', \hat{\Gamma}'(\hat{x}') \rangle \\
&= \langle \hat{f}' \otimes \hat{e}', \hat{x}' \otimes \hat{x}' \rangle \\
&= \langle \hat{f}', \hat{x}' \rangle \langle \hat{e}', \hat{x}' \rangle
\end{aligned}$$

for all $\hat{f}' \in L^1(\hat{\mathbb{G}}')$. So $\langle \hat{e}', \hat{x}' \rangle = 1$ for all $\hat{x}' \in \hat{\mathbb{G}}'$ and since $Gr(\hat{\mathbb{G}}') = sp(L^1(\hat{\mathbb{G}}'))$ by Theorem 3.2.11, this implies that the constant function $\hat{e}'|_{Gr(\hat{\mathbb{G}}')} \equiv 1$ lies in $C_0(Gr(\hat{\mathbb{G}}'))$. Therefore $Gr(\hat{\mathbb{G}}')$ is compact.

2. Now let $\mathbb{G}$ be discrete. Then $\hat{\mathbb{G}}'$ is compact, by Theorem 2.5.4, and again by Theorem 3.2.14 we need to show that $Gr(\hat{\mathbb{G}}')$ is discrete. Let $\hat{x}' \in Gr(\hat{\mathbb{G}}')$ and $\hat{\varphi}' \in L^1(\hat{\mathbb{G}}')$ be the Haar state. Then we have

$$\langle \hat{f}', 1 \rangle \langle \hat{\varphi}', \hat{x}' \rangle = \langle \hat{\varphi}' * \hat{f}', \hat{x}' \rangle = \langle \hat{\varphi}' \otimes \hat{f}', \hat{x}' \otimes \hat{x}' \rangle = \langle \hat{\varphi}', \hat{x}' \rangle \langle \hat{f}', \hat{x}' \rangle$$

for all $\hat{f}'$. So if $\hat{x}' \neq 1$, we must have $\langle \hat{\varphi}', \hat{x}' \rangle = 0$, and since $\langle \hat{\varphi}', 1 \rangle = 1$, we see $\hat{\varphi}'$, as a function on $Gr(\hat{\mathbb{G}}')$ is the characteristic function of $\{1\}$. But since $\hat{\varphi}'$ is continuous on $Gr(\hat{\mathbb{G}}')$, the latter must be discrete.

$\square$

In the following, we shall investigate the relation between the operation $\mathbb{G} \to \tilde{\mathbb{G}}$ and $\mathbb{G} \to \hat{\mathbb{G}}$. We start with some preparation.

Lemma 3.2.17. *Let G and H be two locally compact groups in duality, i.e., there exists a continuous bi-homomorphism*

$$\langle \cdot, \cdot \rangle : G \times H \to \mathbb{T}.$$

Define the sets

$$G_1 := \{g \in G : \langle g, H \rangle = 1\},$$
$$H_1 := \{h \in H : \langle G, h \rangle = 1\}.$$

Then G_1 and H_1 are closed normal subgroups of G and H, containing the commutator subgroups, and we have:

$$\frac{G}{G_1} \cong \widehat{\left(\frac{H}{H_1}\right)}.$$

Proof. For all $g \in G$, $g_1 \in G_1$ and $h \in H$ we have:

$$\langle g^{-1} g_1 g, h \rangle = \langle g^{-1}, h \rangle \langle g_1, h \rangle \langle g, h \rangle = \langle g^{-1}, h \rangle \langle g, h \rangle = \langle e, h \rangle = 1.$$

Therefore, G_1 is normal in G. For all $g_1, g_2 \in G$ and $h \in H$ we have:

$$\langle g_1 g_2 g_1^{-1} g_2^{-1}, h \rangle = \langle g_1, h \rangle \langle g_2, h \rangle \langle g_1^{-1}, h \rangle \langle g_2^{-1}, h \rangle = 1.$$

Thus, $[G, G] \subseteq G_1$; similarly, we see that H_1 is normal in H, and $[H, H] \subseteq H_1$. Now it just remains to show the last assertion. Define

$$\phi : G \to \widehat{\left(\frac{H}{H_1}\right)}, \phi(g)(\overline{h}) = \langle g, h \rangle.$$

The definition of H_1 implies that $\phi(g)$ is well-defined for each $g \in G$. Obviously, ϕ is a group homomorphism, and we have $Ker(\phi) = G_1$. Hence we have an injective group homomorphism

$$\overline{\phi} : \frac{G}{G_1} \hookrightarrow \widehat{\left(\frac{H}{H_1}\right)}.$$

Similarly, by exchanging the roles of G and H we obtain:

$$\frac{H}{H_1} \hookrightarrow \widehat{\left(\frac{G}{G_1}\right)}$$

whence

$$\widehat{\widehat{\left(\frac{G}{G_1}\right)}} \twoheadrightarrow \widehat{\left(\frac{H}{H_1}\right)}.$$

If we compose the last surjection with the identification of $\frac{G}{G_1}$ with its second dual, we get $\overline{\phi}$. Hence $\overline{\phi}$ is onto, and

$$\frac{G}{G_1} \cong \widehat{\left(\frac{H}{H_1}\right)}.$$

□

Applying Lemma 3.2.17 to our situation, in which we have a duality between $\tilde{\mathbb{G}}$ and $\tilde{\hat{\mathbb{G}}}$ (by Corollary 3.2.8), we obtain the following.

Theorem 3.2.18. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have a group homeomorphism:*

$$\frac{Gr(\hat{\mathbb{G}})}{Gr(\hat{\mathbb{G}}) \cap Gr(\mathbb{G})'} \cong \left(\frac{\widehat{Gr(\mathbb{G})}}{Gr(\mathbb{G}) \cap Gr(\hat{\mathbb{G}})'}\right).$$

3.3 Some Invariants for a Quantum Group

In the last section, to a locally compact quantum group $\mathbb{G}$, we have assigned the locally compact group $Gr(\mathbb{G})$, homeomorphic to $Gr(\hat{\mathbb{G}}')$, which is easily seen to be an invariant for $\mathbb{G}$. Using $Gr(\mathbb{G})$, we shall now assign two further invariants to $\mathbb{G}$.

Let $\mathbb{G}$ be a locally compact quantum group, $v \in Gr(\mathbb{G})$ and $\hat{v} \in Gr(\hat{\mathbb{G}})$. By Corollary 3.2.8, there exists $\lambda_{v,\hat{v}} \in \mathbb{T}$ such that

$$v\hat{v} = \lambda_{v,\hat{v}}\hat{v}v.$$

Also, for any $v \in Gr(\mathbb{G})$, the map $\gamma_v : \hat{v} \to \lambda_{v,\hat{v}}$ defines a character on $Gr(\mathbb{G})$. So we obtain a bi-homomorphism $\gamma : Gr(\mathbb{G}) \times Gr(\hat{\mathbb{G}}) \to \mathbb{T}$.

Definition and Proposition 3.3.1. *Let $\mathbb{G}$ be a locally compact quantum group. We denote by $G_{\mathbb{T}}$, the image of γ, which is a subgroup of $\mathbb{T}$.*

Proof. Let

$$K := \frac{Gr(\mathbb{G})}{Gr(\mathbb{G}) \cap Gr(\hat{\mathbb{G}})'} \qquad \text{and} \qquad \hat{K} := \frac{Gr(\hat{\mathbb{G}})}{Gr(\hat{\mathbb{G}}) \cap Gr(\mathbb{G})'} .$$

Then γ induces a bi-character $\gamma_1 : K \times \hat{K} \to \mathbb{T}$, with $Im(\gamma_1) = Im(\gamma)$ (see the proof of Lemma 3.2.17). Since K and $\hat{K}$ are abelian, by the universal property of the tensor product, there exists a homomorphism $\gamma_2 : K \otimes_{\mathbb{Z}} \hat{K} \to \mathbb{T}$, with $Im(\gamma_2) = Im(\gamma_1)$; $\otimes_{\mathbb{Z}}$ denotes the algebraic tensor product of the abelian groups K and $\hat{K}$ over the ring of integers. But $Im(\gamma_2)$ is a subgroup of $\mathbb{T}$ since γ_2 is a group homomorphism. $\square$

Since there is a good classification of subgroups of $\mathbb{T}$ (cf. [22, Theorem 25.13]), this invariant may be helpful towards some sort of classification of locally compact quantum groups.

We now proceed to define the third invariant. Let $v \in Gr(\mathbb{G})$, and define $\varphi_v = \varphi(v \,.\, v^*)$, where φ denotes the left Haar weight of $\mathbb{G}$. It is then clear that φ_v is an n.s.f. weight on $L^\infty(\mathbb{G})$. For all $\omega \in L^1(\mathbb{G})$ we have:

$$\begin{aligned}
x \in \mathcal{M}_{\varphi_v} &\Rightarrow vxv^* \in \mathcal{M}_\varphi \\
&\Rightarrow (\omega \otimes \iota)\Gamma(vxv^*) \in \mathcal{M}_\varphi \\
&\Rightarrow v((v^*\omega v \otimes \iota)\Gamma(x))v^* \in \mathcal{M}_\varphi \\
&\Rightarrow (v^*\omega v \otimes \iota)\Gamma(x) \in \mathcal{M}_{\varphi_v},
\end{aligned}$$

and

$$\begin{aligned}
\varphi_v((\omega \otimes \iota)\Gamma(x)) &= \varphi((v\omega v^* \otimes \iota)\Gamma(vxv^*)) \\
&= v\omega v^*(1)\varphi(vxv^*) \\
&= \omega(1)\varphi_v(x).
\end{aligned}$$

Hence, φ_v is a left Haar weight on $\mathbb{G}$, and so there exists $\lambda_v > 0$ such that $\varphi_v = \lambda_v\varphi$ (see also the proof of [3, Proposition 4.2.]).

Then we clearly have the following.

Definition and Proposition 3.3.2. *Let $\mathbb{G}$ be a locally compact quantum group. Then the map $\beta : v \mapsto \lambda_v$ from $Gr(\mathbb{G})$ into $\mathbb{R}_+^\times$ is a continuous homomorphism. We denote by $G_\mathbb{R}$ the image of β which is a subgroup of $\mathbb{R}_+^\times$.* $\square$

3.4 Examples

3.4.1 Woronowicz's Compact Matrix Pseudogroups

Let A be a C^*-algebra with unit, $U_N = [u_{ij}]$ an $N \times N$ ($N \in \mathbb{N}$) matrix with entries belonging to A, and $\mathcal{A}$ be the *-subalgebra of A generated by the entries of U_N. Then $\mathbf{G} = (A, U_N)$ is called a *compact matrix pseudogroup* [54, Definition 1.1.] if the following hold:

1. $\mathcal{A}$ is dense in A;

2. there exists a C^*-homomorphism Γ from A to $A \otimes_{min} A$, the minimal tensor product of A with itself, such that:

$$\Gamma(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj} \qquad (i, j = 1, 2, ..., N);$$

3. there exists a linear anti-multiplicative map $\kappa : \mathcal{A} \to \mathcal{A}$ such that

$$\kappa((\kappa(a^*)^*)) = a$$

for all $a \in \mathcal{A}$, and

$$\begin{aligned} \textstyle\sum_k \kappa(u_{ik}) u_{kj} &= \delta_{ij} 1, \\ \textstyle\sum_k u_{ik} \kappa(u_{kj}) &= \delta_{ij} 1. \end{aligned}$$

for all $i.j = 1, 2, ..., N$.

Theorem 3.4.1. *[55, Remark 2.] There exists a compact quantum group $\mathbb{G} = (L^\infty(\mathbb{G}), \tilde{\Gamma}, \tilde{\varphi})$ such that $C_0(\mathbb{G}) = A$ and $\tilde{\Gamma}|_{C_0(\mathbb{G})} = \Gamma$.*

Theorem 3.4.2. *Let* $\mathbf{G} = (A, U_N)$ *be a compact matrix pseudogroup. Then* $\tilde{\mathbb{G}}$ *is homeomorphic to a compact subgroup of* $GL_N(\mathbb{C})$*, hence a compact Lie group.*

Proof. Define the map

$$\Phi : sp(A) \to M_N(\mathbb{C}) \quad , \quad f \mapsto [f(u_{ij})]_{ij}.$$

Then Φ is injective since $\{u_{ij}\}$ generates A. For all $f, g \in sp(A) \subseteq M(\mathbb{G})$, we have:

$$\begin{aligned}
\Phi(f * g) &= [f * g(u_{ij})]_{ij} \\
&= [(f \otimes g)\Gamma(u_{ij})]_{ij} \\
&= [(f \otimes g) \sum_{k=1}^{N} (u_{ik} \otimes u_{kj})]_{ij} \\
&= [\sum_{k=1}^{N} f(u_{ik}) g(u_{kj})]_{ij} \\
&= [f(u_{ij})]_{ij} [g(u_{ij})]_{ij} \\
&= \Phi(f)\Phi(g).
\end{aligned}$$

So, Φ is an injective group homomorphism. Obviously $Im(\Phi) \subseteq GL_N(\mathbb{C})$.
Since each of the maps $f \mapsto f(u_{ij})$ is continuous, Φ is also continuous. By Theorem 3.2.16, $\tilde{\mathbb{G}}$ is compact, and therefore Φ is a homeomorphism onto its image. □

3.4.2 $SU_\mu(N)$

Definition 3.4.1. *The quantum group* $SU_\mu(N)$*,* $\mu \in (0, 1]$ *and* $N \geq 2$*, introduced and studied by Woronowicz [58], is the compact matrix pseudogroup* (A, U_N) *with the following properties:*

1. $\sum_{k=1}^{N} u_{ik} u_{jk}^{*} = \delta_{ji} 1;$

2. $\sum_{k=1}^{N} u_{ki}{}^{*} u_{kj} = \delta_{ij} 1$;

3. $\sum_{\sigma \in S_N} (-\mu)^{I(\sigma)} u_{\sigma(1),\tau(1)} \dots u_{\sigma(N),\tau(N)} = (-\mu)^{I(\tau)} \quad \forall \tau \in S_N$;

4. $\sum_{\sigma \in S_N} (-\mu)^{I(\sigma)} u_{\sigma(1),k_1} \dots u_{\sigma(N),k_N} = 0$ *if* $k_i = k_j$ *for some* i, j,

where $I(\sigma) = |\{(i,j) : i < j \,,\, \sigma(j) < \sigma(i)\}|$ *is the number of inversions.*

If $\mu = 1$ we have $C_0(SU_\mu(N)) = C(SU(N))$. If $\mu \neq 1$, then $C_0(SU_\mu(N))$ is a non-commutative C*-algebra which is a deformation of $C(SU(N))$.

In the above definition, properties (1) and (2) should be understood as the U part, and (3) and (4) as the S part of $SU_\mu(N)$.

The proof of the next theorem mimics the proof of [29, Proposition 17] for the special case $n = 2$. In this case, the argument is simpler, but the statement is stronger.

Theorem 3.4.3. *Let* $\mathbf{G} = SU_\mu(2)$.

1. *If* $\mu = 1$, *then* $\tilde{\mathbb{G}} = SU(2)$;

2. *if* $\mu \in (0,1)$, *then* $\tilde{\mathbb{G}} = \mathbb{T}$.

Proof. The first statement follows from Example 3.2.1. To show the second statement, let

$$\Phi : sp(C_0(SU_\mu(2))) \to GL_2(\mathbb{C})$$

be as in the proof of Theorem 3.4.2, and $f \in sp(C_0(SU_\mu(2)))$. It is easy to verify that $\mathbb{T} \subseteq Im(\Phi)$, under the identification of $\mathbb{T}$ with the matrices of the form

$$\Phi(f) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

where $\lambda \in \mathbb{T}$. We now show that any element in $Im(\Phi)$ is of this form. It is clear from properties (1) and (2) in definition 3.4.1 that $\Phi(f)$ is unitary. Let

$$\Phi(f) = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}.$$

Then property (3) implies the equations:

$$\begin{cases} \lambda_{11}\lambda_{22} - \mu\lambda_{12}\lambda_{21} = 1 \\ -\mu\lambda_{11}\lambda_{22} + \lambda_{12}\lambda_{21} = -\mu \end{cases}$$

which have the unique solutions:

$$\lambda_{12}\lambda_{21} = 0 \quad \text{and} \quad \lambda_{11}\lambda_{22} = 1.$$

But since any 2×2 unitary matrix is of the form:

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \alpha \end{pmatrix}$$

where $\alpha\bar{\alpha} + \beta\bar{\beta} = 1$, we must have:

$$\lambda_{11} = \bar{\lambda}_{22} \quad \text{and} \quad \lambda_{12} = \lambda_{21} = 0.$$

So we obtain:

$$\Phi(f) = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \bar{\lambda}_{11} \end{pmatrix}.$$

□

The following result requiers some basic Lie theory.

Theorem 3.4.4. *[29, Proposition 17.] There is a neighborhood Ω of 1 such that for all $1 \neq \mu \in \Omega$, we have $\tilde{\mathbb{G}} = \mathbb{T}^{N-1}$, where $\mathbb{G} \cong SU_\mu(N)$.*

3.4.3 $E_\mu(2)$ and its Dual

In [57] Woronowicz defined the non-compact quantum group $E_\mu(2)$ for a parameter $\mu \in (0, 1)$, and studied some of its properties and those of its dual. This quantum group is regarded as one of the simplest examples of a non-compact, non-Kac, locally compact quantum group.

We first present the definition of this quantum group.

Let $(e_{k,l})_{k,l\in\mathbb{Z}}$ be the canonical basis for $l^2(\mathbb{Z}\times\mathbb{Z})$. Define operators v and n on $l^2(\mathbb{Z}\times\mathbb{Z})$ as follows:

$$\begin{cases} ve_{k,l} &= e_{k-1,l} \\ ne_{k,l} &= \mu^k e_{k,l+1}\,. \end{cases}$$

Then v is a unitary and n is a normal operator with $sp(n)\subseteq \overline{C}^{\mu}$, where

$$\overline{\mathbb{C}}^{\mu} := \{z\in\mathbb{C} : z=0 \text{ or } |z|\in\mu^{\mathbb{Z}}\}.$$

Definition and Theorem 3.4.5. *[57, Section 1.] $C_0(E_\mu(2))$ is defined to be the non-unital C^*-algebra generated by the operators of the form $\Sigma v^k f_k(n)$, where k runs over a finite set of integers, and $f_k\in C_0(\overline{\mathbb{C}}^{\mu})$.*

The co-multiplication Γ is defined on $E_\mu(2)$ in the following way:

$$\begin{cases} \Gamma(v) &= v\otimes v \\ \Gamma(n) &= v\otimes n + n\otimes v\,. \end{cases}$$

Then there exists a locally compact quantum group $\mathbb{G}=(L^\infty(\mathbb{G}),\tilde{\Gamma},\tilde{\varphi},\tilde{\psi})$, such that $C_0(\mathbb{G})=C_0(E_\mu(2))$, and $\tilde{\Gamma}|_{C_0(\mathbb{G})}=\Gamma$.

Note that neither v nor n belong to $C_0(E_\mu(2))$, but from the definition is clear how to define Γ on elements in $C_0(E_\mu(2))$.

The following result gives a universal property for $E_\mu(2)$, which proves useful in studying its structure.

Theorem 3.4.6. *[57, Theorem 1.1.]*

1. *Let π be a *-representation of $C_0(E_\mu(2))$ on a Hilbert space H, and put $\tilde{v} = \pi(v)$, $\tilde{n} = \pi(n)$. Then $\tilde{v}$ is unitary, $\tilde{n}$ is normal with $sp(\tilde{n}) \subseteq \overline{C}^\mu$, and we have:*

$$\tilde{v}\tilde{n}\tilde{v} = \mu\tilde{n}.$$

2. *Any pair $(\tilde{v}, \tilde{n})$ of operators acting on a Hilbert space H, which satisfy the above properties, is of the form $\tilde{v} = \pi(v)$, $\tilde{n} = \pi(n)$, where π is a *-representation of $C_0(E_\mu(2))$ on H.*

□

Using theorem 3 4.6, we can easily calculate $\tilde{\mathbb{G}}_{E_\mu(2)}$.

Theorem 3.4.7. *We have a group homeomorphism·*

$$\tilde{\mathbb{G}}_{E_\mu(2)} \cong \mathbb{T}.$$

Proof. By Theorem 3.4.6, there is a character f_1 on $C_0(E_\mu(2))$ such that $f_1(v) = 1$ and $f_1(n) = 0$. Then for every $f \in C_0(E_\mu(2))^*$ we have:

$$\begin{aligned} f_1 * f(v) &= (f_1 \otimes f)\Gamma(v) = (f_1 \otimes f)(v \otimes v) \\ &= f_1(v)f(v) = f(v), \\ f_1 * f(n) &= (f_1 \otimes f)\Gamma(n) = (f_1 \otimes f)(v \otimes n + n \otimes v) \\ &= f_1(v)f(n) + f_1(n)f(v) = f(n). \end{aligned}$$

Similarly, $f * f_1 = f$ and so $C_0(E_\mu(2))^*$ is unital. Hence, by Theorem 2.5.6, we see that $E_\mu(2)$ is co-amenable, whence by Theorem 3.2.13 we obtain:

$$\tilde{\mathbb{G}} \cong sp(C_0(E_\mu(2))).$$

In view of Theorem 3.4.6, for any $z \in \mathbb{T}$, we can define a character f_z on $C_0(E_\mu(2))$ such that $f_z(v) = z$ and $f_z(n) = 0$.

Conversely, if $f \in sp(C_0(E_\mu(2)))$, then by Theorem 3.4.6 again, $f(v)$ is a unitary and $f(n)$ is a normal operator on $\mathbb{C}$, and we have: $f(n) = f(v)f(n)\overline{f(v)} = \mu f(n)$. Since $\mu \neq 1$, we must have $f(n) = 0$. Put $z := f(v) \in \mathbb{T}$. So $f = f_z$. Moreover, for $z, z' \in \mathbb{T}$ we have:

$$\begin{aligned}
f_z * f_{z'}(v) &= (f_z \otimes f_{z'})\Gamma(v) = (f_z \otimes f_{z'})(v \otimes v) \\
&= f_z(v)f_{z'}(v) = zz' = f_{zz'}(v), \\
f_z * f_{z'}(n) &= (f_z \otimes f_{z'})\Gamma(n) = (f_z \otimes f_{z'})(v \otimes n + n \otimes v) \\
&= f_z(v)f_{z'}(n) + f_z(n)f_{z'}(v) = 0 = f_{zz'}(n).
\end{aligned}$$

Hence $f_z * f_{z'} = f_{zz'}$. □

In [57] Woronowicz has also described $\widehat{E_\mu(2)}$, the dual quantum group of $E_\mu(2)$. Similarly to $E_\mu(2)$, this quantum group is also determined by two operators $\hat{n}$ and $\hat{v}$, with co-multiplication determined by $\hat{\Gamma}(\hat{n}) = \hat{n} \otimes 1 + 1 \otimes \hat{n}$, $\hat{\Gamma}(\hat{v}) = \hat{v} \otimes \mu^{\frac{\hat{n}}{2}} + \mu^{\frac{-\hat{n}}{2}} \otimes \hat{v}$. The dual quantum group $\widehat{E_\mu(2)}$ has a universal property as well, which makes it easy to find our group. In the following, we rewrite this property for the one dimensional unitary representations of $C_0(E_\mu(2))$.

Theorem 3.4.8. *[57, theorem 3.1., special case] $\hat{f} \in sp(C_0(\widehat{E_\mu(2)}))$ if and only if $\hat{f}(\hat{v}) = 0$ and $\hat{f}(\hat{n}) \in \mathbb{Z}$.* □

Theorem 3.4.9. *We have the group homeomorphism:* $\tilde{\mathbb{G}}_{\widehat{E_\mu(2)}} \cong \mathbb{Z}$.

Proof. For any $s \in \mathbb{Z}$ we define $\hat{f}_s \in Mor(C_0(\widehat{E_\mu(2)}), \mathbb{C})$ by $\hat{f}_s(\hat{v}) = 0$, $\hat{f}_s(\hat{n}) = s$. Then it is clear from Theorem 3.4.8 that the map $s \mapsto \hat{f}_s \in sp(C_0(\widehat{E_\mu(2)}))$ is a bi-continuous bijection.

For $s, s' \in \mathbb{Z}$ we have:

$$\begin{aligned}
\hat{f}_s * \hat{f}_{s'}(\hat{v}) &= (\hat{f}_s \otimes \hat{f}_{s'})\Gamma(\hat{v}) = (\hat{f}_s \otimes \hat{f}_{s'})(\hat{v} \otimes \mu^{\frac{\hat{n}}{2}} + \mu^{-\frac{\hat{n}}{2}} \otimes \hat{v}) \\
&= \hat{f}_s(\hat{v})\hat{f}_{s'}(\mu^{\frac{\hat{n}}{2}}) + \hat{f}_s(\mu^{-\frac{\hat{n}}{2}})\hat{f}_{s'}(\hat{v}) = 0 \\
&= \hat{f}_{s+s'}(\hat{v}), \\
\hat{f}_s * \hat{f}_{s'}(\hat{n}) &= (\hat{f}_s \otimes \hat{f}_{s'})\Gamma(\hat{n}) = (\hat{f}_s \otimes \hat{f}_{s'})(\hat{n} \otimes 1 + 1 \otimes \hat{n}) \\
&= \hat{f}_s(\hat{n})\hat{f}_{s'}(1) + \hat{f}_s(1)\hat{f}_{s'}(\hat{n}) = s + s' \\
&= \hat{f}_{s+s'}(\hat{n}).
\end{aligned}$$

So $\hat{f}_s * \hat{f}_{s'} = \hat{f}_{s+s'}$. □

3.4.4 Behaviour of $\tilde{\mathbb{G}}$ under Natural Constructions

There are several ways of constructing new locally compact quantum groups from existing ones, such as tensor products, crossed products and bi-crossed products (see, e.g., [42], [8], [59], [3]).

The relation between the intrinsic group of the resulting structure and the original one, has already been studied for most of the above-mentioned types of constructions. Here we discuss some examples of these constructions and the results regarding the behavior of the intrinsic groups.

1. Tensor product

Let $\mathbb{G}_i = (M_i, \Gamma_i, \varphi_i, \psi_i)$, $i = 1, 2$ be Kac algebras. Then $M_1 \overline{\otimes} M_2$ can be endowed with a Kac algebra structure with co-multiplication

$$\Gamma = (\iota \otimes \chi \otimes \iota)(\Gamma_1 \otimes \Gamma_2)$$

where $\chi : M_1 \overline{\otimes} M_2 \to M_2 \overline{\otimes} M_1$ is the flip map. We also have:

$$(\widehat{M_1 \overline{\otimes} M_2}) = \hat{M}_1 \overline{\otimes} \hat{M}_2.$$

Proposition 3.4.10. *[7, Proposition 3.2.] Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be Kac algebras. Then we have:*

$$Gr(L^\infty(\mathbb{G}_1) \otimes L^\infty(\mathbb{G}_2)) \cong Gr(\mathbb{G}_1) \times Gr(\mathbb{G}_2),$$

i.e., the intrinsic group of the tensor product of two Kac algebras is homeomorphic to the product of their intrinsic groups. □

2. Crossed product of a Kac algebra by a locally compact group

Let $\mathbb{G} = (L^\infty(\mathbb{G}), \Gamma, \varphi, \psi)$ be a Kac algebra and G a locally compact group. A σ-weakly continuous homomorphism $\alpha : G \to Aut(L^\infty(\mathbb{G}))$ is called an *action* of G on $\mathbb{G}$ if we have

$$\Gamma \circ \alpha_g = (\alpha_g \otimes \alpha_g) \circ \Gamma$$

and

$$R \circ \alpha_g = \alpha_g \circ R$$

for all $g \in G$. We define a normal injective map $\pi : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(G)$ by

$$(\pi(x)\xi)(g) = \alpha_{g^{-1}}(x)\xi(g) \quad (x \in L^\infty(\mathbb{G}),\, \xi \in L^2(\mathbb{G}) \otimes L^2(G),\, g \in G).$$

We denote by $L^\infty(\mathbb{G}) \ltimes_\alpha G$, the von Neumann algebra generated by $\pi(L^\infty(\mathbb{G}))$ and $1_{\mathcal{B}(L^2(\mathbb{G}))} \otimes VN(G)$ in $\mathcal{B}(L^2(\mathbb{G}) \otimes_2 L^2(G))$, and call it the *crossed product* of $L^\infty(\mathbb{G})$ by G via α. The algebra $L^\infty(\mathbb{G}) \ltimes_\alpha G$ can be endowed with a Kac algebra structure (see [8]), denoted by $\mathbb{G} \ltimes_\alpha G$, with co-multiplication:

$$\begin{aligned}
\hat{\Gamma}(\pi(x)) &= (\pi \otimes \pi)(\Gamma(x)) \qquad (x \in L^\infty(\mathbb{G})), \\
\hat{\Gamma}(1 \otimes y) &= (1_{\mathcal{B}(L^2(\mathbb{G}))} \otimes \chi \otimes 1_G)(1 \otimes 1 \otimes \hat{\Gamma}_G(y)) \quad (y \in VN(G))
\end{aligned}$$

where $\hat{\Gamma}_G$ denote the co-multiplication of $VN(G)$.

Proposition 3.4.11. *[7, Proposition 3.4.] Let $\mathbb{G}$ be a Kac algebra, G a locally compact group, and let α be an action of G on $\mathbb{G}$. Then we have a group homeomorphism:*

$$Gr(\mathbb{G} \ltimes_\alpha G) \cong Gr(\mathbb{G}) \ltimes_\alpha G,$$

i.e., the intrinsic group of the crossed product Kac algebra is homeomorphic to the semi-direct product of the original intrinsic group and the acting group G, where in the latter, the action is the restriction of the original action. □

3. Matched pairs of locally compact groups

Let G and H be locally compact groups. Consider a left action α of G on H, and a right action β of H on G. Suppose that the following maps are continuous:

$$G \times H \to H, \quad (g, s) \mapsto \alpha_g(s)$$
$$G \times H \to G, \quad (g, s) \mapsto \beta_s(g).$$

Suppose further that

$$\beta_{st}(g) = \beta_s(\beta_t(g)), \quad \alpha_g(st) = \alpha_{\beta_t(g)}(s)\alpha_g(t) \quad \text{for all } (s,t,g) \in H \times H \times G,$$
$$\alpha_{gh}(s) = \alpha_g(\alpha_h(s)), \quad \beta_s(gh) = \beta_{\alpha_h(s)}(g)\beta_s(h) \quad \text{for all } (g,h,s) \in G \times G \times H.$$

Then we call (G, H) a *matched pair* of groups. Now we define

$$\begin{aligned} \alpha : L^\infty(H) &\to L^\infty(G \times H), \\ (\alpha f)(g,s) &= f(\alpha_g(s)). \end{aligned}$$

Then we have:

$$(\Gamma_G \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$$

where Γ_G denotes the co-multiplication of $L^\infty(G)$. We define the crossed product von Neumann algebra M as the von Neumann algebra generated by $\alpha(L^\infty(H))$ and $VN(G) \otimes 1_{\mathcal{B}(L^2(H))}$ in $\mathcal{B}(L^2(G) \otimes_2 L^2(H))$ We further define the unitary W on $L^2(G \times H \times G \times H)$ by

$$(W\xi)(g,s,h,t) = \xi(\beta_{\alpha_g(s)^{-1}t}(h)g, s, h, \alpha_g(s)^{-1}t).$$

Using W, we define a co-multiplication on M as follows:

$$\Gamma(x) = W^*(1_{\mathcal{B}(L^2(G \times H))} \otimes x)W \quad (x \in M).$$

One can show that this yields a locally compact quantum group – in fact, a Kac algebra – which we shall denote by $\mathbb{G}_{H \ltimes_\alpha G}$ (see [28]).

Theorem 3.4.12. *[59, Theorem 2.5] Let (G, H, α, β) be a matched pair of groups. Then we have a group homeomorphism:*

$$Gr(\mathbb{G}_{H \ltimes_\alpha G}) \cong \hat{H} \ltimes_\alpha G^\beta$$

where G^β is the set of elements in G which are fixed under the action β.

□

3.5 Applications

In this section, we investigate the relation between the structure of $\mathbb{G}$ and that of $\tilde{\mathbb{G}}$.

3.5.1 Intrinsic Group and Unimodularity of the Quantum Group

Let $\mathbb{G}$ be a locally compact quantum group. The main goal of this section (Theorem 3.5.11) is to show that if both $\tilde{\mathbb{G}}$ and $\tilde{\tilde{\mathbb{G}}}$ are small, then $\mathbb{G}$ is of a very specific type, namely a unimodular Kac algebra.

Recall that $\delta^{it} \in Gr(\mathbb{G})$ for all $t \in \mathbb{R}$, where δ is the modular element of $\mathbb{G}$ (cf. Theorem 2.4.1). Hence, we obviously have the following.

Proposition 3.5.1. *If $Gr(\mathbb{G}) = \{1\}$, then $\mathbb{G}$ is unimodular.* □

But since the map $t \to \delta^{it}$ is strongly continuous, we can even say more:

Proposition 3.5.2. *If $Gr(\mathbb{G})$ is discrete, then $\mathbb{G}$ is unimodular.*

Proof. The map

$$\mathbb{R} \ni t \mapsto \delta^{it} \in Gr(\mathbb{G})$$

being continuous, its range must be connected. But since $Gr(\mathbb{G})$ is discrete, the range must be a single point. Therefore, we obtain $\delta^{it} = 1$ for all $t \in \mathbb{R}$, which implies $\delta = 1$. $\square$

Combining Proposition 3.5.2 with Theorem 3.4.9, we obtain the following.

Corollary 3.5.3. *The quantum group $E_\mu(2)$ is unimodular.* $\square$

It turns out that we can still do one more step in strengthening Propositions 3.5.2. We denote the center of a group G by $\mathbf{Z}(G)$.

Lemma 3.5.4. *[3, Proposition 4.2.] We have: $\delta^{it} \in \mathbf{Z}(Gr(\mathbb{G}))$ for all $t \in \mathbb{R}$.* $\square$

Hence, we arrive at the following.

Theorem 3.5.5. *Let $\mathbb{G}$ be a locally compact quantum group. If $\mathbf{Z}(Gr(\mathbb{G}))$ is discrete, then $\mathbb{G}$ is unimodular.* $\square$

Lemma 3.5.6. *Let Φ, Ψ be weak* continuous linear maps on $L^\infty(\mathbb{G})$. If*

$$(\iota \otimes \Phi) \circ \Gamma = (\iota \otimes \Psi) \circ \Gamma \quad \text{or} \quad (\Phi \otimes \iota) \circ \Gamma = (\Psi \otimes \iota) \circ \Gamma$$

then $\Phi = \Psi$.

Proof. Assume that $(\iota\otimes\Phi)\circ\Gamma = (\iota\otimes\Psi)\circ\Gamma$. Then, for all $x \in L^\infty(\mathbb{G})$ and $\omega \in L^1(\mathbb{G})$, we have:

$$\Phi((\omega \otimes \iota)\Gamma(x)) = \Psi((\omega \otimes \iota)\Gamma(x)).$$

Since the set $\{ (\omega \otimes \iota)\Gamma(x) \, : \, \omega \in L^1(\mathbb{G}), x \in L^\infty(\mathbb{G}) \}$ is weak* dense in $L^\infty(\mathbb{G})$, the conclusion follows. The argument assuming the second relation is analogous. $\square$

Lemma 3.5.7. *If $\delta = 1$ and $\sigma_t^\varphi = \tau_t$ for all $t \in \mathbb{R}$, then $\tau_t = \sigma_t^\varphi = \iota$ for all $t \in \mathbb{R}$, and $\mathbb{G}$ is a Kac algebra.*

Proof. Since $\delta = 1$, we have $\sigma_t^\psi = \sigma_t^\varphi$ for all $t \in \mathbb{R}$. Moreover, since $\sigma_t^\varphi = \tau_t$, we have, by Theorem 2.4.2:

$$\begin{aligned} \Gamma\tau_t &= (\tau_t \otimes \tau_t)\Gamma = (\sigma_t^\varphi \otimes \sigma_t^\psi)\Gamma, \\ \Gamma\tau_t &= (\sigma_t^\varphi \otimes \sigma_{-t}^\psi)\Gamma. \end{aligned}$$

So we obtain:

$$(\sigma_t^\varphi \otimes \iota)(\iota \otimes \sigma_{-t}^\psi)\Gamma = (\sigma_t^\varphi \otimes \iota)(\iota \otimes \sigma_t^\psi)\Gamma,$$

which implies that $(\iota \otimes \sigma_{-t}^\psi)\Gamma = (\iota \otimes \sigma_t^\psi)\Gamma$. Now, Lemma 3.5.6 yields $\sigma_{-t}^\psi = \sigma_t^\psi$, i.e., $\sigma_{2t}^\psi = \iota$, for all $t \in \mathbb{R}$. Hence, $\tau_t = \sigma_t^\varphi = \sigma_t^\psi = \iota$ for all $t \in \mathbb{R}$, and therefore $\mathbb{G}$ is a Kac algebra, by Remark 2.4.3. □

Proposition 3.5.8. *If $\mathbb{G}$ and $\hat{\mathbb{G}}$ are both unimodular, then $\mathbb{G}$ is a Kac algebra.*

Proof. Since $\hat{\delta} = 1$, Proposition 2.6.3 implies that $N^{it} = \Delta_\varphi^{it}$, hence $\sigma_t^\varphi = \tau_t$, for all $t \in \mathbb{R}$. Since, in addition, $\delta = 1$, Lemma 3.5.7 yields the claim. □

In particular, combining Theorem 3.5.5 and Proposition 3.5.8 we see that for a unimodular locally compact quantum group $\mathbb{G}$, the smallness of the group $\tilde{\mathbb{G}}$ forces the quantum group $\mathbb{G}$ to be of Kac type:

Theorem 3.5.9. *Let $\mathbb{G}$ be a unimodular locally compact quantum group. If $Z(\tilde{\mathbb{G}})$ is discrete, then $\mathbb{G}$ is a Kac algebra.* □

Since every compact quantum group is unimodular, we obtain the following.

Corollary 3.5.10. *Let $\mathbb{G}$ be a compact quantum group. If $Z(\tilde{\mathbb{G}})$ is discrete, then $\mathbb{G}$ is a Kac algebra.* □

The class of non-Kac compact quantum groups is one of the most important and well-known classes of non-classical quantum groups. Some important examples of

such objects are deformations of compact Lie groups, such as Woronowicz's famous $SU_\mu(N)$, which we have discussed in Section 3.4.2. This shows the significance of our Corollary 3.5.10:
there is some richness of classical information in these classes of quantum structures!

Now we are at the point of stating the main theorem of this section, which shows in particular that for a non-Kac locally compact quantum group, both $\tilde{\mathbb{G}}$ and $\tilde{\hat{\mathbb{G}}}$ cannot be small at the same time.

Theorem 3.5.11. *Let $\mathbb{G}$ be a locally compact quantum group. If $\mathbf{Z}(\tilde{\mathbb{G}})$ and $\mathbf{Z}(\tilde{\hat{\mathbb{G}}})$ are both discrete, then $\mathbb{G}$ is a unimodular Kac algebra.*

Proof. Since $\mathbf{Z}(\tilde{\mathbb{G}})$ is discrete, $\hat{\mathbb{G}}$ is unimodular by Theorem 3.5.5, and hence our assertion follows from Theorem 3.5.9, applied to $\hat{\mathbb{G}}$. □

3.5.2 Intrinsic Group and Traciality of the Haar Weight

If φ is a left Haar weight of a locally compact quantum group $\mathbb{G}$, then $\psi = \varphi \circ R$ is easily seen to be a right Haar weight for $\mathbb{G}$. Therefore, the left Haar weight is tracial if and only if the right Haar weight is. There is a strong connection between the group $\tilde{\mathbb{G}}$, assigned to $\mathbb{G}$, and traciality of the Haar weights, especially in the Kac algebra case. Since the scaling group τ_t in this case is trivial, Theorem 2.4.2 implies:

$$\Gamma\sigma_t^\varphi = (\iota \otimes \sigma_t^\varphi)\Gamma \quad \forall t \in \mathbb{R}, \tag{3.1}$$

and so $\sigma_t^\psi \in \mathcal{CB}^\sigma_{cov}(L^\infty(\mathbb{G})) \cong Gr(\hat{\mathbb{G}})$. Thus, we obtain, by Remark 1.4.6, the following.

Proposition 3.5.12. *[7, Corollary 2.4] Let $\mathbb{G}$ be a Kac algebra, and Δ_ψ be the modular operator associated to ψ. Then $\Delta_\psi^{it} \in Gr(\hat{\mathbb{G}}')$ for all $t \in \mathbb{R}$.* $\square$

Combining Proposition 3.5.12 with Remark 1.4.6 we see that if $\mathbb{G}$ is a Kac algebra such that $\tilde{\mathbb{G}}$ is trivial, then φ is tracial. Similarly to Proposition 3.5.2, the fact that the map $t \mapsto \sigma_t^\varphi$ is strongly continuous, allows us to further generalize this result.

Theorem 3.5.13. *Let $\mathbb{G}$ be a Kac algebra. If $\tilde{\mathbb{G}}$ is discrete, then φ is tracial.* $\square$

Remark 3.5.14. *Since equation (3.1) holds also for any locally compact quantum group with trivial scaling group, Theorem 3.5.13 also holds for any such locally compact quantum group.*

Proposition 2.6.3 implies that if a locally compact quantum group $\mathbb{G}$ is unimodular, i.e., $\delta = 1$, with trivial scaling group, i.e., $N = 1$, then we have $\Delta_{\hat{\varphi}} = 1$, i.e., $\hat{\varphi}$ is tracial, and vice versa (cf. 1.4.6). In particular, we can state the following.

Proposition 3.5.15. *[13, Proposition 6.1.2] Let $\mathbb{G}$ be a Kac algebra. Then $\mathbb{G}$ is unimodular if and only if $\hat{\varphi}$ is tracial.*

Remark 3.5.16. *Applying Proposition 3.5.2, we can also derive Theorem 3.5.13 Proposition 3.5.15.*

We have seen in Theorem 3.5.9 that for a unimodular locally compact quantum group $\mathbb{G}$, the smallness of $\tilde{\mathbb{G}}$ forces the quantum group to be a Kac algebra. The situation is similar for traciality:

Proposition 3.5.17. *Let $\mathbb{G}$ be a locally compact quantum group with tracial Haar weight. If $Gr(\mathbb{G})$ is discrete, then $\mathbb{G}$ is a Kac algebra.*

Proof. Since $Gr(\mathbb{G})$ is discrete, we have $\delta = 1$. Also, traciality of φ implies that

$$\Gamma = \Gamma\sigma_t^{\varphi} = (\tau_t \otimes \sigma_t^{\varphi})\Gamma = (\tau_t \otimes \iota)\Gamma,$$

for all $t \in \mathbb{R}$, which implies that $\tau = \iota$, by Lemma 3.5.6. Hence, $\mathbb{G}$ is a Kac algebra, by Theorem 2.4.3. $\square$

Moreover, combining Proposition 3.5.15 with Theorem 3.5.11, we obtain a stronger version of the latter.

Theorem 3.5.18. *Let $\mathbb{G}$ be a locally compact quantum group. If $\mathbf{Z}(\tilde{\mathbb{G}})$ and $\mathbf{Z}(\tilde{\hat{\mathbb{G}}})$ are both discrete, then $\mathbb{G}$ is a unimodular Kac algebra with tracial Haar weight.* $\square$

3.5.3 Intrinsic Group and Amenability of the Quantum Group

In this last section we investigate the question of whether amenability passes from $\mathbb{G}$ to $\tilde{\mathbb{G}}$.

Definition 3.5.1. *Let $\mathbb{G}$ be a locally compact quantum group. We define $N_{\mathbb{G}}$ to be the sub von Neumann algebra of $L^\infty(\mathbb{G})$ generated by $Gr(\mathbb{G})$.*

For the following, recall that we denote by S the antipode of $\mathbb{G}$ (see Definition 2.3.1).

Lemma 3.5.19. *[25, Proposition 5.33.] Let $a, b \in L^\infty(\mathbb{G})$ be such that $a \otimes 1 = \Gamma(x)(1 \otimes y)$ and $b \otimes 1 = (1 \otimes x)\Gamma(y)$ for some $x, y \in L^\infty(\mathbb{G})$. Then $a \in \mathcal{D}(S)$ and $S(a) = b$.*

Theorem 3.5.20. *Let $\mathbb{G}$ be a locally compact quantum group such that $\mathcal{M}_\varphi \cap N_{\mathbb{G}}$ is weakly dense in $N_{\mathbb{G}}$. Then we have the identification:*

$$N_{\mathbb{G}} \cong VN(Gr(\mathbb{G})).$$

Proof. It is obvious that the restriction of Γ to $N_{\mathbb{G}}$ defines a co-multiplication on $N_{\mathbb{G}}$, and by our assumption, on $N_{\mathbb{G}}$, the restriction of φ to $N_{\mathbb{G}}$ is an n.s.f. left invariant weight on $N_{\mathbb{G}}$. Now, let $v \in Gr(\mathbb{G})$, then, by Lemma 3.5.19 (with $a = x = v, b = y = v^*$), we have $v \in \mathcal{D}(S)$, and $S(v) = v^*$. This implies $\tau_{-\imath}(v) = S^2(v) = v$, and it follows that $\tau_t(v) = v$ for all $v \in Gr(\mathbb{G})$ and $t \in \mathbb{R}$. Hence, we obtain $R(v) = v^*$ for any $v \in Gr(\mathbb{G})$, and so $R(N_{\mathbb{G}}) \subseteq N_{\mathbb{G}}$, which implies that $\varphi \circ R$ defines a right Haar weight on $N_{\mathbb{G}}$. Therefore, $N_{\mathbb{G}}$ can be given a locally compact quantum group structure. Obviously, it is co-commutative, and so $N_{\mathbb{G}} \cong VN(Gr(\mathbb{G}))$, by Theorem 2.1.2. $\square$

Corollary 3.5.21. *If $\mathbb{G}$ is a compact quantum group, then*

$$N_{\mathbb{G}} \cong VN(Gr(\mathbb{G})).$$

$\square$

Let $\imath : N_{\mathbb{G}} \hookrightarrow L^\infty(\mathbb{G})$ be the canonical injection. Obviously, i is weak*-continuous, so we have the pre-adjoint map $i_* : L^1(\mathbb{G}) \twoheadrightarrow (N_{\mathbb{G}})_*$.

Lemma 3.5.22. *The map $i_* : L^1(\mathbb{G}) \twoheadrightarrow (N_{\mathbb{G}})_*$ is a completely bounded algebra homomorphism.*

Proof. Let $\omega_1, \omega_2 \in L^1(\mathbb{G})$ and $y \in N_{\mathbb{G}}$. Then we have:

$$\begin{aligned} \langle i_*(\omega_1 * \omega_2), y\rangle &= \langle \omega_1 * \omega_2, i(y)\rangle = \langle \omega_1 * \omega_2, y\rangle \\ &= \langle \omega_1 \otimes \omega_2, \Gamma(y)\rangle = \langle \omega_1 \otimes \omega_2, (i \otimes i)\Gamma(y)\rangle \\ &= \langle i_*(\omega_1) \otimes i_{*'}(\omega_2), \Gamma(y)\rangle = \langle i_*(\omega_1) * i_*(\omega_2), y\rangle. \end{aligned}$$

So $i_*(\omega_1 * \omega_2) = i_*(\omega_1) * i_*(\omega_2)$. $\square$

If a locally compact quantum group $\mathbb{G}$ satisfies the condition of Theorem 3.5.20, then $VN(Gr(\mathbb{G})) \cong Gr(\mathbb{G})'' \cap L^\infty(\mathbb{G})$, and by the above we have a surjective continuous algebra homomorphism $i_* : L^1(\mathbb{G}) \to A(Gr(\mathbb{G}))$. Therefore, in this case, many of the algebraic properties of $L^1(\mathbb{G})$ will be satisfied by the Fourier algebra of the intrinsic group as well.

There are many different equivalent characterizations of amenability for a locally compact group. The question of whether the quantum counterpart of these conditions are equivalent as well, remains unsolved in many important instances.

In the following, we present a few of those equivalent characterizations in the group case which can be found for instance in [41].

Theorem 3.5.23. *For a locally compact group G, the following are equivalent:*

1. *G is amenable;*

2. *$L^1(G)$ is an amenable Banach algebra;*

3. *$A(G)$ has a bounded approximate identity (BAI);*

4. *$A(G)$ is operator amenable.* □

The equivalence (1) $\Leftrightarrow$ (2) is due to Johnson (1972), (1) $\Leftrightarrow$ (3) is Leptin's theorem (1968), and (1) $\Leftrightarrow$ (4) is due to Ruan (1995).

Proposition 3.5.24. *Let $\mathbb{G}$ be a discrete quantum group. If $L^1(\hat{\mathbb{G}})$ has a BAI, then $\tilde{\mathbb{G}}$ is amenable (and discrete).*

Proof. Since $\hat{\mathbb{G}}$ is compact, by Corollary 3.5.21, we have $N_{\mathbb{G}} \cong VN(Gr(\mathbb{G}))$. If $(\hat{\omega}_\alpha)$ is a BAI for $L^1(\hat{\mathbb{G}})$, then as easily seen $(i_*(\hat{\omega}_\alpha))$ is a BAI for $A(Gr(\mathbb{G}))$, whence $Gr(\mathbb{G})$ is amenable, by Theorem 3.5.23. □

Proposition 3.5.25. *Let $\mathbb{G}$ be a discrete quantum group. If $L^1(\mathbb{G})$ is operator amenable, then $\tilde{\mathbb{G}}$ is amenable.*

Proof. Since $\mathbb{G}$ is discrete, we have, by Lemma 3.5.22, a completely bounded surjective algebra homomorphism from $L^1(\hat{\mathbb{G}})$ onto $A(\tilde{\mathbb{G}})$. Since $L^1(\hat{\mathbb{G}})$ is operator amenable, then so is $A(\tilde{\mathbb{G}})$. Hence, by Theorem 3.5.23, $\tilde{\mathbb{G}}$ is amenable. $\square$

Proposition 3.5.26. *Let $\mathbb{G}$ be a discrete quantum group. If $L^1(\hat{\mathbb{G}})$ is an amenable Banach algebra, then $\tilde{\mathbb{G}}$ is almost abelian.*

Proof. The argument is analogous to the one given in Proposition 3.5.25, but instead of Theorem 3.5.23, we use [16, Theorem 2.3], stating that $A(G)$ is amenable if and only if G is almost abelian. $\square$

Theorem 3.5.27. *[47, Theorem 3.8.] A discrete quantum group $\mathbb{G}$ is amenable if and only if $\hat{\mathbb{G}}$ is co-amenable.* $\square$

Combining Theorem 3.5.27 and Proposition 3.5.24 we obtain the following.

Theorem 3.5.28. *Let $\mathbb{G}$ be a discrete quantum group. If $\mathbb{G}$ is amenable, then so is $\tilde{\mathbb{G}}$.* $\square$

Chapter 4

Convolution Algebras over Locally Compact Quantum Groups

In [30] Neufang defined a new product on $\mathcal{T}(L^2(G))$, the space of trace class operators on $L^2(G)$ for a locally compact group G. While the usual composition of operators can be regarded as a non-commutative version of the pointwise product, this new product can be viewed as non-commutative convolution. This is justified by the fact that the canonical quotient map from $\mathcal{T}(L^2(\mathbb{G}))$ with this product onto $L^1(\mathbb{G})$ with convolution becomes a homomorphism. Neufang in [30] studied various properties of the Banach algebra $\mathcal{T}(L^2(G))$ endowed with this product.

In this chapter we investigate these Banach algebra structures in the context of general locally compact quantum groups. After deriving some of the basic properties, we study the relation between cohomological properties of $\mathcal{T}(L^2(\mathbb{G}))$ and properties of the locally compact quantum group $\mathbb{G}$. In particular, we generalize some of the results proved by Pirkovskii in [34] to the case of locally compact quantum groups.

4.1 Convolution and Pointwise Product for Locally Compact Quantum Groups

In this section we define a quantum analogous of the convolution and pointwise products for a locally compact quantum group.

Definition 4.1.1. *Let $\mathbb{G}$ be a locally compact quantum group, and $V \in L^\infty(\hat{\mathbb{G}}')\overline{\otimes}L^\infty(\mathbb{G})$ its right fundamental unitary. We can lift the co-products Γ and $\hat{\Gamma}$ to $\mathcal{B}(L^2(\mathbb{G}))$, still using the same notation, as follows:*

$$\begin{aligned}
\Gamma \ &: \ \mathcal{B}(L^2(\mathbb{G})) \longrightarrow \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}L^\infty(\mathbb{G}) \ : \ x \to V(x\otimes 1)V^*, \\
\hat{\Gamma} \ &: \ \mathcal{B}(L^2(\mathbb{G})) \longrightarrow \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}L^\infty(\hat{\mathbb{G}}') \ : \ x \to \hat{V}'(x\otimes 1)\hat{V}'^*.
\end{aligned}$$

Then $\Gamma_*, \hat{\Gamma}_* : \mathcal{T}(L^2(\mathbb{G}))\widehat{\otimes}\mathcal{T}(L^2(\mathbb{G})) \longrightarrow \mathcal{T}(L^2(\mathbb{G}))$ define two different products on $\mathcal{T}(L^2(\mathbb{G}))$. We denote them by $*, \bullet$ respectively. We also denote by $\mathcal{T}_*(\mathbb{G})$ and $\mathcal{T}_\bullet(\mathbb{G})$ the Banach algebras $(\mathcal{T}(L^2(\mathbb{G})), *)$ and $(\mathcal{T}(L^2(\mathbb{G})), \bullet)$, respectively.

If $\mathbb{G} = L^\infty(G)$ for a locally compact group G, then $\mathcal{T}_*(\mathbb{G})$ is the convolution algebra introduced by Neufang in [30]. Also, the canonical quotient map from $\mathcal{T}_\bullet(\mathbb{G})$ onto $A(G)$ with pointwise product becomes a Banach algebra homomorphism.

Lemma 4.1.1. *The canonical quotient map $\pi : \mathcal{T}_*(\mathbb{G}) \twoheadrightarrow L^1(\mathbb{G})$ and the trace map $tr : \mathcal{T}_*(\mathbb{G}) \longrightarrow \mathbb{C}$ are Banach algebra homomorphisms.*

Proof. For all $\rho, \eta \in \mathcal{T}_*(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$ we have:

$$\langle \pi(\rho * \eta), x\rangle = \langle \rho * \eta, x\rangle = \langle \rho\otimes\eta, \Gamma(x)\rangle.$$

Since $\Gamma(x) \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$, we obtain:

$$\langle \rho \otimes \eta, \Gamma(x) \rangle = \langle \pi(\rho) \otimes \pi(\eta), \Gamma(x) \rangle = \langle \pi(\rho) * \pi(\eta), x \rangle.$$

Hence, $\pi(\rho * \eta) = \pi(\rho) * \pi(\eta)$.

Also, we have:

$$\begin{aligned} tr(\rho * \eta) &= \langle \rho * \eta, 1 \rangle &&= \langle \rho \otimes \eta, \Gamma(1) \rangle \\ &= \langle \rho \otimes \eta, 1 \otimes 1 \rangle &&= \langle \rho, 1 \rangle \langle \eta, 1 \rangle \\ &= tr(\rho) tr(\eta). \end{aligned}$$

□

The above proposition allows us to define (right) $\mathcal{T}_*(\mathbb{G})$-module structures on $L^1(\mathbb{G})$ and $\mathbb{C}$ as follows:

$$f.\rho = f * \pi(\rho) \quad \text{and} \quad \lambda.\rho = \lambda tr(\rho) \quad (\rho \in \mathcal{T}_*(\mathbb{G}), f \in L^1(\mathbb{G}), \lambda \in \mathbb{C}).$$

In the following, we will show that some of the topological properties of $\mathbb{G}$ can be deduced from these module structures.

First we prove some properties of the lifted co-products and their induced multiplications.

Lemma 4.1.2. *Let $x \in \mathcal{B}(L^2(\mathbb{G}))$. If $\Gamma(x) = y \otimes 1$ for some $y \in \mathcal{B}(L^2(\mathbb{G}))$ then we have $x = y \in L^\infty(\hat{\mathbb{G}})$.*

Proof. For every $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$ we have:

$$((\iota \otimes \omega)V)x = (\iota \otimes \omega)(V(x \otimes 1)) = (\iota \otimes \omega)((y \otimes 1)V) = y((\iota \otimes \omega)V).$$

Since $L^\infty(\hat{\mathbb{G}}') = \overline{\{(\iota \otimes \omega)V : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}}^{w^*}$, we get $\hat{a}'x = y\hat{a}'$ for all $\hat{a}' \in L^\infty(\hat{\mathbb{G}}')$. In particular for $\hat{a}' = 1$, it follows that $x = y$, and since $\hat{a}'x = x\hat{a}'$ for all $\hat{a}' \in L^\infty(\hat{\mathbb{G}}')$, we have $x \in L^\infty(\hat{\mathbb{G}})$.

□

Lemma 4.1.3. *Let $x \in \mathcal{B}(L^2(\mathbb{G}))$. If $\Gamma(x) = 1 \otimes y$, for some $y \in \mathcal{B}(L^2(\mathbb{G}))$, then $x \in L^\infty(\hat{\mathbb{G}}')$ and $y \in L^\infty(\mathbb{G})$.*

Proof. Recall that $V \in L^\infty(\hat{\mathbb{G}}')\overline{\otimes}L^\infty(\mathbb{G})$. So $(1\otimes y) = V(x\otimes 1)V^* \in \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}L^\infty(\mathbb{G})$, which implies $y \in L^\infty(\mathbb{G})$. Similarly, $x \otimes 1 = V^*(1 \otimes y)V \in L^\infty(\hat{\mathbb{G}}')\overline{\otimes}L^\infty(\mathbb{G})$ implies that $x \in L^\infty(\hat{\mathbb{G}}')$.

□

Corollary 4.1.4. *Let $x \in \mathcal{B}(L^2(\mathbb{G}))$. If $\Gamma(x) = 1 \otimes x$, then $x \in \mathbb{C}1$.*

Proof. If $\Gamma(x) = 1 \otimes x$, the previous lemma implies that $x \in L^\infty(\mathbb{G}) \cap L^\infty(\hat{\mathbb{G}}')$, which equals $\mathbb{C}1$, by Theorem 2.6.1.

□

4.2 Basic Relations Between $\mathbb{G}$ and its Convolution Algebras

In this section we investigate topological and amenability-type relations between a quantum group and its convolution algebras.

The following proposition is known and has been stated in many different places. But we include the proof in order to let the reader compare this with the result we prove immediately thereafter.

Proposition 4.2.1. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following hold:*

1. *$L^1(\mathbb{G})$ has a left (right) identity if and only if $\mathbb{G}$ is discrete.*
2. *$L^1(\mathbb{G})$ has a bounded left (right) approximate identity if and only if $\mathbb{G}$ is co-amenable.*

Proof. 1. Assume e be a left identity for $L^1(\mathbb{G})$ and let $e^r = e \circ R$, where R is the unitary antipode of $\mathbb{G}$. Also, for each $f \in L^1(\mathbb{G})$, let $f' = f \circ R^{-1}$. Then, for $x \in L^\infty(\mathbb{G})$, we have:

$$\begin{aligned} \langle f * e^r, x\rangle &= \langle f \otimes e^r, \Gamma(x)\rangle \\ &= \langle f' \otimes e, (R \otimes R)\Gamma(x)\rangle \\ &= \langle e \otimes f', \Gamma(R(x))\rangle \\ &= \langle e * f', R(x)\rangle \\ &= \langle f', R(x)\rangle \\ &= \langle f, x\rangle. \end{aligned}$$

Therefore, e^r is a right identity and so $e = e^r$ is an identity for $L^1(\mathbb{G})$.

2. Let e_α be a bounded left approximate identity for $L^1(\mathbb{G})$, and let $f' = f \circ R^{-1}$ and $e^r_\alpha = e_\alpha \circ R$. Then, for all $f \in L^1(\mathbb{G})$ we have

$$\begin{aligned} \lim_\alpha \|f * e^r_\alpha - f\| &= \lim_\alpha \|(e_\alpha * f') \circ R - f' \circ R\| \\ &= \lim_\alpha \|e_\alpha * f' - f'\| \longrightarrow 0. \end{aligned}$$

Therefore, e^r_α is a bounded right approximate identity, and so $\mathbb{G}$ has a bounded approximate identity. $\square$

Proposition 4.2.2. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following hold:*

1. *$\mathcal{T}_*(\mathbb{G})$ does not have a left identity, unless $\mathbb{G}$ is trivial, and it has a right identity if and only if $\mathbb{G}$ is discrete;*

2. *$\mathcal{T}_*(\mathbb{G})$ does not have a left approximate identity, unless $\mathbb{G}$ is trivial, and it has a bounded right approximate identity if and only if $\mathbb{G}$ is co-amenable.*

Proof. 1. Let $\omega_0 \in \mathcal{B}(L^2(\mathbb{G}))_*$ be a non-zero normal functional, whose restriction to $L^\infty(\mathbb{G})$ is zero. Since $\Gamma(\mathcal{B}(L^2(\mathbb{G}))) \subseteq \mathcal{B}(L^2(\mathbb{G}))\bar{\otimes}L^\infty(\mathbb{G})$, we have

$$\langle \rho * \omega_0, x\rangle = \langle \omega_0, (\rho\bar{\otimes}\iota)\Gamma(x)\rangle = 0,$$

for every $\rho \in \mathcal{B}(L^2(\mathbb{G}))_*$ and $x \in \mathcal{B}(L^2(\mathbb{G}))$, which obviously implies that there does not exist a left identity, unless $\mathbb{G}$ is trivial.

Now, let $\mathbb{G}$ be discrete, $e \in L^1(\mathbb{G})$ be the unit element, and $\tilde{e} \in \mathcal{T}(L^2(\mathbb{G}))$ be a norm preserving weak*-extension of e. Then, for all $\rho \in \mathcal{T}_*(\mathbb{G})$, $x \in L^\infty(\mathbb{G})$ and $\hat{x} \in L^\infty(\hat{\mathbb{G}})$, we have

$$\begin{aligned}
\langle \rho * \tilde{e}, x\hat{x}\rangle &= \langle \rho \otimes \tilde{e}, \Gamma(x)(\hat{x} \otimes 1)\rangle = \langle \hat{x}\rho \otimes \tilde{e}, \Gamma(x)\rangle \\
&= \langle \pi(\hat{x}\rho) \otimes \pi(\tilde{e}), \Gamma(x))\rangle = \langle \pi(\hat{x}\rho) * e, x\rangle \\
&= \langle \pi(\hat{x}\rho), x\rangle = \langle \hat{x}\rho, x\rangle = \langle \rho, x\hat{x}\rangle,
\end{aligned}$$

where $\pi : \mathcal{T}(L^2(\mathbb{G})) \twoheadrightarrow L^1(\mathbb{G})$ is the canonical quotient map. Since the span of $\{x\hat{x} : x \in L^\infty(\mathbb{G}), \hat{x} \in L^\infty(\hat{\mathbb{G}})\}$ is weak* dense in $\mathcal{B}(L^2(\mathbb{G}))$ (Theorem 2.6.2), the assertion follows.

Conversely, assume that $\mathcal{T}_*(\mathbb{G})$ has a right identity $\tilde{e}$. Then, since the canonical quotient map $\pi : \mathcal{T}_*(\mathbb{G}) \to L^1(\mathbb{G})$ is a surjective homomorphism, $\pi(\tilde{e})$ is clearly a right identity for $L^1(G)$, whence $\mathbb{G}$ is discrete by Theorem 4.2.1.

2. Similarly to the first part, one can show that $\mathcal{T}_*(\mathbb{G})$ cannot possess a left approximate identity, unless it is trivial (equal to $\mathbb{C}$).

Let $\mathbb{G}$ be co-amenable. Then, by Theorem 2.5.6, there exists a net $(\xi_\imath)$ of unit vectors in $L^2(\mathbb{G})$ such that $\|V^*(\eta\otimes\xi_\imath) - \eta\otimes\xi_\imath\| \to 0$ for all unit vector $\eta \in L^2(\mathbb{G})$. Now, for all $x \in \mathcal{B}(H)$ and $\eta \in H$ with $\|x\| = \|\eta\| = 1$ we have

$$\begin{aligned}
|\langle\omega_\eta * \omega_{\xi_\imath} - \omega_\eta, x\rangle| &= |\langle V(x\otimes 1)V^*(\eta\otimes\xi_\imath), \eta\otimes\xi_\imath\rangle - \langle(x\otimes 1)(\eta\otimes\xi_\imath), \eta\otimes\xi_\imath\rangle| \\
&= |\langle(x\otimes 1)(V^*(\eta\otimes\xi_\imath) - \eta\otimes\xi_\imath), V^*(\eta\otimes\xi_\imath)\rangle + \\
&\qquad \langle(x\otimes 1)(\eta\otimes\xi_\imath), V^*(\eta\otimes\xi_\imath) - \eta\otimes\xi_\imath\rangle| \\
&\leq 2\|V^*(\eta\otimes\xi_\imath) - \eta\otimes\xi_\imath\| \to 0.
\end{aligned}$$

Since the span of $\{\omega_\eta : \eta \in L^2(\mathbb{G})\}$ is norm dense in $\mathcal{T}(L^2(\mathbb{G}))$ it follows that $\omega_{\xi_\imath}$ is a right bounded approximate identity for $\mathcal{T}_*(\mathbb{G})$.

Conversely, assume that $\mathcal{T}_*(\mathbb{G})$ has a bounded right approximate identity $\tilde{e}_\alpha$. Then, since $\pi : \mathcal{T}_*(\mathbb{G}) \to L^1(\mathbb{G})$ is a surjective homomorphism, $\pi(\tilde{e}_\alpha)$ is clearly a bounded right approximate identity for $L^1(\mathbb{G})$. Hence, $\mathbb{G}$ is co-amenable by Theorem 4.2.1.

□

The following is a slight generalization of [4, Proposition 3.1].

Proposition 4.2.3. *Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{G}$ is compact if and only if there exists a non-zero left (right) invariant positive functional on $C_0(\mathbb{G})$.*

Proof. For the non-trivial implication let $f \in M(\mathbb{G})$ be a non-zero left invariant positive functional on $C_0(\mathbb{G})$, and let $\omega \in L^1(\mathbb{G})$ be a normal state. From the left invariance of f, we have:

$$\omega * f = \langle \omega, 1\rangle f = f.$$

Since $L^1(\mathbb{G})$ is an ideal in $M(\mathbb{G})$, this implies that $f \in L^1(\mathbb{G})$. Now, by equation (1.3), there exists a net of normal functionals $(\omega_\imath)$ on $L^\infty(\mathbb{G})$, such that

$$\psi(x) = \lim_\imath \langle \omega_\imath, x\rangle \qquad \forall x \in L^\infty(\mathbb{G}).$$

For $x \in \mathcal{M}_\varphi$ we then have:

$$\begin{aligned}
\langle f, 1\rangle \psi(x) &= \psi((\iota\otimes f)\Gamma(x)) = \lim_\imath \langle \omega_\imath, (\iota\otimes f)\Gamma(x)\rangle \\
&= \lim_\imath \langle f, (\omega_\imath\otimes\iota)\Gamma(x)\rangle = \lim_\imath \langle f, x\rangle\langle \omega_\imath, 1\rangle \\
&= \langle f, x\rangle \lim_\imath \langle \omega_\imath, 1\rangle = \langle f, x\rangle \psi(1).
\end{aligned}$$

Hence $\psi(1) < \infty$, and $\mathbb{G}$ is therefore compact. □

The convolution product of $L^1(\mathbb{G})$ can be extended to an action of $L^1(\mathbb{G})$ on $L^\infty(\mathbb{G})^*$. Let $f \in L^1(\mathbb{G})$ and $F \in L^\infty(\hat{\mathbb{G}})^*$. Define:

$$\langle x, \rho * F\rangle := \langle (\rho\otimes\iota)\Gamma(x), F\rangle$$

for all $x \in L^\infty(\hat{\mathbb{G}})$. Similarly, the convolution product of $\mathcal{T}_*(\mathbb{G})$ is extended to an action of $\mathcal{T}_*(\mathbb{G})$ on $\mathcal{B}(L^2(\mathbb{G}))^*$.

Proposition 4.2.4. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following hold:*

1. *$\mathbb{G}$ is compact if and only if there exists a state $\tilde{\varphi} \in \mathcal{T}_*(\mathbb{G})$ such that*

$$\langle \rho * \tilde{\varphi}, x\hat{x}\rangle = \langle \rho * \tilde{\varphi}, \hat{x}x\rangle = \langle \rho, \hat{x}\rangle\langle \tilde{\varphi}, x\rangle \tag{4.1}$$

for all $x \in L^\infty(\mathbb{G})$, $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ *and* $\rho \in \mathcal{T}_*(\mathbb{G})$;

2. $\mathbb{G}$ *is amenable if and only if there exists a state* $\tilde{F} \in \mathcal{T}_*(\mathbb{G})^{**}$ *such that*

$$\langle x\hat{x}, \rho * \tilde{F}\rangle = \langle \hat{x}x, \rho * \tilde{F}\rangle = \langle \rho, \hat{x}\rangle\langle x, \tilde{F}\rangle$$

for all $x \in L^\infty(\mathbb{G})$, $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ *and* $\rho \in \mathcal{T}_*(\mathbb{G})$.

Proof. 1. Suppose that $\mathbb{G}$ is compact with normal Haar state φ, and $\tilde{\varphi} \in \mathcal{T}_*(\mathbb{G})$ is a norm preserving extension of φ. Then $\tilde{\varphi}$ is a state (since $\|\tilde{\varphi}\| = \tilde{\varphi}(1) = 1$), and we have:

$$\begin{aligned}
\langle \rho * \tilde{\varphi}, x\hat{x}\rangle &= \langle \rho \otimes \tilde{\varphi}, \Gamma(x)(\hat{x} \otimes 1)\rangle = \langle \hat{x}\rho \otimes \tilde{\varphi}, \Gamma(x)\rangle \\
&= \langle \pi(\hat{x}\rho) \otimes \pi(\tilde{\varphi}), \Gamma(x)\rangle = \langle \pi(\hat{x}\rho) * \pi(\tilde{\varphi}), x\rangle \\
&= \langle \pi(\hat{x}\rho), 1\rangle\langle \varphi, x\rangle = \langle \hat{x}.\rho, 1\rangle\langle \varphi, x\rangle \\
&= \langle \rho, \hat{x}\rangle\langle \varphi, x\rangle = \langle \rho, \hat{x}\rangle\langle \tilde{\varphi}, x\rangle.
\end{aligned}$$

In a similar way, we can show that $\langle \rho * \tilde{\varphi}, \hat{x}x\rangle = \langle \rho, \hat{x}\rangle\langle \tilde{\varphi}, x\rangle$.

Conversely, suppose such a state $\tilde{\varphi} \in \mathcal{T}_*(\mathbb{G})$ exists. Let $\varphi = \pi(\tilde{\varphi}) \in L^1(\mathbb{G})^+$ and $f \in L^1(\mathbb{G})$, and let $\tilde{f} \in \mathcal{T}_*(\mathbb{G})$ be a weak*-extension of f. Then, by putting $\hat{x} = 1$ in equation (4.1), we have

$$\langle f * \varphi, x\rangle = \langle \pi(\tilde{f}) * \pi(\tilde{\varphi}), x\rangle = \langle \tilde{f} * \tilde{\varphi}, x\rangle = \langle \tilde{f}, 1\rangle\langle \tilde{\varphi}, x\rangle = \langle f, 1\rangle\langle \varphi, x\rangle$$

for all $x \in L^\infty(\mathbb{G})$. Hence, φ is a left invariant state in $L^1(\mathbb{G})$, and so $\mathbb{G}$ is compact, by Proposition 4.2.3.

2. Suppose $\mathbb{G}$ is amenable. Let $F \in L^\infty(\mathbb{G})^*$ be an invariant mean, and $\tilde{F} \in \mathcal{B}(L^2(\mathbb{G}))^*$ an state extension of F. We can find a net (ρ_α) in $\mathcal{T}_*(\mathbb{G})$, converging

to $\tilde{F}$ in the weak*-topology. Then we have:

$$\begin{aligned}
\langle x\hat{x}, \rho * \tilde{F}\rangle &= \langle(\rho \otimes \iota)\Gamma(x\hat{x}), \tilde{F}\rangle \\
&= \lim_{\alpha} \langle \rho_{\alpha}, (\rho \otimes \iota)(\Gamma(x)(\hat{x} \otimes 1))\rangle \\
&= \lim_{\alpha} \langle \rho_{\alpha}, (\hat{x}\rho \otimes \iota)\Gamma(x)\rangle \\
&= \lim_{\alpha} \langle \pi(\rho_{\alpha}), (\pi(\hat{x}\rho) \otimes \iota)\Gamma(x)\rangle \\
&= \langle(\pi(\hat{x}\rho) \otimes \iota)\Gamma(x), F\rangle \\
&= \langle \pi(\hat{x}\rho, 1\rangle\langle x, F\rangle \\
&= \langle \hat{x}\rho, 1\rangle\langle x, F\rangle \\
&= \langle \rho, \hat{x}\rangle\langle x, F\rangle \\
&= \langle \rho, \hat{x}\rangle\langle x, \tilde{F}\rangle.
\end{aligned}$$

An analogous argument yields that $\langle \rho * \tilde{F}, \hat{x}x\rangle = \langle \rho, \hat{x}\rangle\langle x, \tilde{F}\rangle$.

Conversely, if such $\tilde{F}$ exists, then, similarly to part 1, we can show that the restriction of $\tilde{F}$ to $L^{\infty}(\mathbb{G})$ is an invariant mean on $\mathbb{G}$. □

4.3 Cohomological Properties of Convolution Algebras

In this section, we shall consider various module structures associated with convolution algebras over a locally compact quantum group. In [34] it was shown that topological properties of a locally compact group G – such as compactness and discreteness – are equivalent to cohomological properties – such as projectivity and flatness – of certain convolution algebras over the group G. Our goal here is to prove similar results in the quantum setting.

First, we briefly recall some standard definitions and notations from the cohomology theory of Banach algebras. All objects are defined in the category of Banach spaces, with bounded linear maps as morphisms.

A bounded linear map $\sigma : X \to Y$ from a Banach space X into a Banach space Y is called *admissible* if it has a bounded right inverse.

Definition 4.3.1. *Let $\mathcal{A}$ be a Banach algebra and P a right $\mathcal{A}$-module. P is called* projective *if for all $\mathcal{A}$-modules X and Y, any admissible morphism $\sigma : X \to Y$ and any morphism $\rho : P \to Y$, there exists a morphism $\phi : P \to X$ such that $\sigma \circ \phi = \rho$.*

Denote by $X.\mathcal{A} \subseteq X$ the closed linear span of the set $\{x.a : a \in \mathcal{A}, x \in X\}$. Then X is called *essential* if $X.\mathcal{A} = X$.

Theorem 4.3.1. *[21, IV.I] An essential right $\mathcal{A}$-module X is projective if and only if there exists a morphism $\psi : X \to X\widehat{\otimes}\mathcal{A}$ such that $m \circ \psi = \iota_X$, where $m : X\widehat{\otimes}\mathcal{A} \to X$ is the canonical module action morphism, and $X\widehat{\otimes}\mathcal{A}$ is regarded as a right $\mathcal{A}$-module, via the action $(x \otimes a).b = x \otimes ab$.* □

Note that if $\phi : \mathcal{B} \to \mathcal{A}$ is a homomorphism of Banach algebras, then any right $\mathcal{A}$-module X can be endowed with a $\mathcal{B}$-action, $x.b := x.\phi(b)$, making it a right $\mathcal{B}$-module. In this way, if ϕ is surjective, and X is a projective right $\mathcal{B}$-module, then X is also a projective right $\mathcal{A}$-module.

The case of left modules and bi-modules are analogous.

The following was proved in the more general setting of Hopf-von Neumann algebras by Aristov in [1, Theorem 2.3]

Proposition 4.3.2. *$\mathbb{C}$ is a projective $\mathcal{T}_*(\mathbb{G})$-module if and only if $\mathbb{G}$ is compact.*

Proof. Let $\mathbb{G}$ be compact with Haar state φ, and $\tilde{\varphi} := \omega_{\Lambda_\varphi(1)} \in \mathcal{T}_*(\mathbb{G})$. Define

$$\Phi : \mathbb{C} \to \mathbb{C} \otimes \mathcal{T}_*(\mathbb{G}) \ , \ \lambda \mapsto \lambda \otimes \tilde{\varphi}.$$

Then it is easy to see that Φ is a module map and $m \circ \Phi = 1_{\mathbb{C}}$.

Conversely, if $\mathbb{C}$ is $\mathcal{T}_*(\mathbb{G})$-projective, then $\mathbb{C}$ is $L^1(\mathbb{G})$-projective, so there exists an $L^1(\mathbb{G})$-module map $\Phi : \mathbb{C} \to \mathbb{C} \otimes L^1(\mathbb{G})$ such that $m \circ \Phi = 1_{\mathbb{C}}$. Now, let $\Phi(1) = 1 \otimes \varphi$, for some non-zero $\varphi \in L^1(\mathbb{G})$. Then, for any $f \in L^1(\mathbb{G})$, we have:

$$1 \otimes \langle f, 1\rangle\varphi = \langle f, 1\rangle 1 \otimes \varphi = \Phi(1.f) = \Phi(1).f = 1 \otimes \varphi * f.$$

Hence, $\langle f, 1\rangle\varphi = \varphi * f$ for all $f \in L^1(\mathbb{G})$ Therefore, φ is a non-zero normal invariant functional, and so by the proof of Proposition 4.2.3 $\mathbb{G}$ is compact. $\square$

Lemma 4.3.3. *For ρ, ξ and $\eta \in \mathcal{T}(L^2(\mathbb{G}))$, the following two relations hold:*

$$\begin{aligned} \rho * (\xi \bullet \eta) &= \eta(1)\rho * \xi; \\ \rho \bullet (\xi * \eta) &= \eta(1)\rho \bullet \xi. \end{aligned}$$

Proof. Let $x \in L^\infty(\mathbb{G})$ and $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. Then we have:

$$\begin{aligned} \langle \rho * (\xi \bullet \eta), x\hat{x}\rangle &= \langle \rho \otimes (\xi \bullet \eta), \Gamma(x\hat{x})\rangle \\ &= \langle \rho \otimes \xi \otimes \eta, (\iota \otimes \hat{\Gamma})(\Gamma(x)(\hat{x} \otimes 1)\rangle \\ &= \langle \rho \otimes \xi \otimes \eta, (\Gamma(x) \otimes 1)(\hat{x} \otimes 1 \otimes 1)\rangle \\ &= \langle \eta, 1\rangle\langle \rho \otimes \xi, \Gamma(x)(\hat{x} \otimes 1)\rangle \\ &= \langle \eta, 1\rangle\langle \rho * \xi, x\hat{x}\rangle. \end{aligned}$$

The second relation follows along similar lines.

$\square$

Since there are two different multiplications on $\mathcal{T}(L^2(\mathbb{G}))$ arising from $\mathbb{G}$ and $\hat{\mathbb{G}}$, it is tempting to consider the corresponding two actions at the same time by defining a bi-module structure on $\mathcal{T}(L^2(\mathbb{G}))$, using these two products. But one can deduce from the above, that multiplication from the left and right via these products, is not associative, and so we cannot turn $\mathcal{T}(L^2(\mathbb{G}))$ into a $\mathcal{T}_*(\mathbb{G}) - \mathcal{T}_\bullet(\mathbb{G})$ bimodule in this fashion. However, the next theorem will provide us with a way of doing so.

Theorem 4.3.4. *For ρ, ξ and $\eta \in \mathcal{T}(L^2(\mathbb{G}))$, the following relation holds:*

$$(\rho * \xi) \bullet \eta = (\rho \bullet \eta) * \xi.$$

Proof. Let $x \in L^\infty(\mathbb{G})$ and $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. Then we have:

$$\begin{aligned}
\langle (\rho * \xi) \bullet \eta, x\hat{x} \rangle &= \langle (\rho * \xi) \otimes \eta, \hat{\Gamma}(x\hat{x}) \rangle \\
&= \langle (\rho * \xi) \otimes \eta, (x \otimes 1)\hat{\Gamma}(\hat{x}) \rangle \\
&= \langle \rho \otimes \xi \otimes \eta, (\Gamma \otimes \iota)[(x \otimes 1)\hat{\Gamma}(\hat{x})] \rangle \\
&= \langle \rho \otimes \xi \otimes \eta, (\Gamma(x) \otimes 1)(\hat{\Gamma}(\hat{x})_{13}) \rangle \\
&= \langle \rho \otimes \eta \otimes \xi, \Gamma(x)_{13}(\hat{\Gamma}(\hat{x}) \otimes 1) \rangle \\
&= \langle \rho \otimes \eta \otimes \xi, (\hat{\Gamma} \otimes \iota)[\Gamma(x)(\hat{x} \otimes 1)] \rangle \\
&= \langle (\rho \bullet \eta) \otimes \xi, \Gamma(x\hat{x}) \rangle \\
&= \langle (\rho \bullet \eta) * \xi, x\hat{x} \rangle.
\end{aligned}$$

□

The last theorem has even more significance: it may be possible to encode quantum group duality by this relation. In fact, one might be also able to start from this relation on trace class operators on a Hilbert space with some extra conditions to define a locally compact quantum group.

Proposition 4.3.5. *$\mathcal{T}(L^2(\mathbb{G}))$ becomes a $\mathcal{T}_*(\mathbb{G})^{op} - \mathcal{T}_\bullet(\mathbb{G})$ bimodule via the actions*

$$\eta \cdot \rho = \rho * \eta \quad \text{and} \quad \rho \cdot \xi = \rho \bullet \xi,$$

where $\rho \in \mathcal{T}(L^2(\mathbb{G}))$, $\eta \in \mathcal{T}_*(\mathbb{G})$ and $\xi \in \mathcal{T}_\bullet(\mathbb{G})$.

Proof. We only need to check the associativity of the left-right action:

$$\begin{aligned}(\eta \cdot \rho) \cdot \xi &= (\eta \cdot \rho) \bullet \xi = (\rho * \eta) \bullet \xi \\ &= (\rho \bullet \xi) * \eta = \eta \cdot (\rho \bullet \xi) \\ &= \eta \cdot (\rho \cdot \xi).\end{aligned}$$

□

It is easy to show that a topological group (or more generally, a topological space) which is both compact and discrete, has to be finite. This is also true in the non-commutative setting. One can use the structure theory of discrete quantum groups to prove this fact (cf. [13, Theorem 6.6.1] for the proof in the Kac algebra case). Here we give a different proof for the general case of locally compact quantum groups. But we first need some preparation.

Proposition 4.3.6. *[27, Proposition 1.11.7] Let M be a von Neumann algebra. If M is reflexive as a Banach space, then M is finite-dimensional.*

□

Theorem 4.3.7. *[40, Theorem 3.8] Let $\mathbb{G}$ be a locally compact quantum group. Then $\mathbb{G}$ is compact if and only if $L^1(\mathbb{G})$ is an ideal in $L^1(\mathbb{G})^{**}$, with the left Arens product.*

□

Theorem 4.3.8. *If $\mathbb{G}$ is both compact and discrete, then $\mathbb{G}$ is finite (dimensional).*

Proof. If $\mathbb{G}$ is compact, then $L^1(\mathbb{G})$ is an ideal in $L^1(\mathbb{G})^{**}$ with the left Arens product, by Theorem 4.3.7. But since $\mathbb{G}$ is also discrete, $L^1(\mathbb{G})$ is unital, and its unit is obviously also an identity element for the left Arens product of $L^1(\mathbb{G})^{**}$. Being a unital ideal (via the canonical embedding), $L^1(\mathbb{G})$ must be equal to $L^1(\mathbb{G})^{**}$. So $L^1(\mathbb{G})$ is reflexive, hence $L^\infty(\mathbb{G})$ is, which implies that $L^\infty(\mathbb{G})$ is finite-dimensional, by Proposition 4.3.6. □

Theorem 4.3.9. *Let $\mathbb{G}$ be a bi-co-amenable locally compact quantum group, i.e., both $\mathbb{G}$ and $\hat{\mathbb{G}}$ are co-amenable. Then the following are equivalent:*

1. *$\mathcal{T}(L^2(\mathbb{G}))$ is $\mathcal{T}_*(\mathbb{G})^{op} - \mathcal{T}_\bullet(\mathbb{G})$ bi-projective;*
2. *$\mathbb{C}$ is $\mathcal{T}_*(\mathbb{G})^{op} - \mathcal{T}_\bullet(\mathbb{G})$ bi-projective;*
3. *$\mathbb{G}$ is finite.*

Proof. We show that $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. Since both $\mathbb{G}$ and $\hat{\mathbb{G}}$ are co-amenable, Proposition 4.2.2 implies that $\mathcal{T}_*\mathbb{G})^{op}$ and $\mathcal{T}_\bullet(\mathbb{G})$ have bounded left, respectively, right approximate identities. Since $\mathbb{C}$ is essential, (1) implies (2) by [20, 7.1.60].

(2) implies that both $\mathbb{G}$ and $\hat{\mathbb{G}}$ are compact, by Proposition 4.3.2 which implies that $\mathbb{G}$ is both compact and discrete and therefore finite by Theorem 4.3.8.

$(3) \Rightarrow (1)$: Obvious. □

Theorems 4.3.2 and 4.3.8 show that compactness and finiteness of $\mathbb{G}$ can be deduced from some cohomological properties of its convolution algebras. We now proceed to complete this picture by proving a similar statement for discreteness of $\mathbb{G}$. However, as one might expect, here we need to take the quantum (operator space) structure of the underlying Banach spaces into account as well. So we work in the category of operator spaces. We shall define our module structures in the quantized

Banach space category, so the modules are operator spaces, and morphisms are completely bounded linear maps. But here the situation is even more subtle. There are some technical difficulties which appear when one wants to link the quantum group structure to the quantum Banach space structure. This happens mainly because the operator space structure is essentially defined based on the Banach space structure of these algebras, and do not seem to see all aspects of the quantum group structure. These technical issues appear also in some of the open problems in this theory, and seem to be a major subtle point [6].

To avoid such difficulties, we define every object in the category of operator spaces with completely contractive maps as morphisms. But as Theorem 4.3.14 shows, this might be actually the right setting!

Many categorical statements which hold in the category of Banach spaces, also hold in this setting with an obvious slight categorical modification. In particular, Theorem 4.3.1, which we will need in the sequel, also holds in this setting.

Definition 4.3.2. *Let N be a von Neumann algebra and $N_1 \subseteq N$ a sub von Neumann algebra. A map $E : N \to N_1$ is called a* conditional expectation *if it is a surjective norm-one projection.*

Theorem 4.3.10. *[27, Theorem 4.1.5] If $E : N \to N_1$ is a conditional expectation, then it is completely bounded (in fact, completely positive), and we have*

$$E(axb) = aE(x)b$$

for all $a, b \in N_1$ and $x \in N$.

$\square$

Theorem 4.3.11. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following are equivalent:*

1. *there exists a normal conditional expectation $E : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\mathbb{G})$ which satisfies $\Gamma \circ E = (E \otimes E)\Gamma$;*
2. *there exists a normal conditional expectation $E : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\mathbb{G})$ which satisfies $\Gamma \circ E = (E \otimes \iota)\Gamma$;*
3. *there exists a normal conditional expectation $E : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\mathbb{G})$ which satisfies $E(L^\infty(\hat{\mathbb{G}})) \subseteq \mathbb{C}1$;*
4. *$\mathbb{G}$ is discrete.*

Proof. (1) $\Leftrightarrow$ (2) : This follows from the fact that $\Gamma(\mathcal{B}(L^2(\mathbb{G}))) \subseteq \mathcal{B}(L^2(\mathbb{G}))\overline{\otimes}L^\infty(\mathbb{G})$, and $E = \iota$ on $L^\infty(\mathbb{G})$.

(1) $\Rightarrow$ (3) : Let $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. We have:

$$\begin{aligned}\Gamma(E(\hat{x})) &= (E \otimes E)\Gamma(\hat{x}) \\ &= (E \otimes E)(\hat{x} \otimes 1) = E(\hat{x}) \otimes 1,\end{aligned}$$

which implies that $E(\hat{x}) \in \mathbb{C}1$, by Corollary 4.1.4.

(3) $\Rightarrow$ (4) : (3) with Theorem 4.3.10 imply $E \in \mathcal{CB}^{\sigma, L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})}(\mathcal{B}(L^2(\mathbb{G})))$, and hence it follows by Theorem 3.1.3 that there exists $\hat{m} \in M_{cb}(L^1(\hat{\mathbb{G}}))$ such that $E = \hat{\Theta}^r(\hat{m})$. Now, define a complex-valued map $\hat{f}$ on $L^\infty(\hat{\mathbb{G}})$ such that $E(\hat{x}) = \hat{f}(\hat{x})1$ for all $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. Since E is a unital linear normal positive map, $\hat{f}$ is a normal state on $L^\infty(\hat{\mathbb{G}})$, and for every $\hat{\omega} \in L^1(\hat{\mathbb{G}})$ and $\hat{x} \in L^\infty(\hat{\mathbb{G}})$ we have:

$$\begin{aligned}\langle \hat{m}(\hat{\omega}), \hat{x} \rangle &= \langle \hat{\omega}, \hat{\Theta}^r(\hat{m})(\hat{x}) \rangle = \langle \hat{\omega}, E(\hat{x}) \rangle \\ &= \langle \hat{\omega}, \hat{f}(\hat{x})1 \rangle = \langle \hat{\omega}, 1 \rangle \hat{f}(\hat{x}).\end{aligned}$$

Hence $\hat{m}(\hat{\omega}) = \langle \hat{\omega}, 1 \rangle \hat{f}$. Let $\hat{\omega}_0$ be a normal state on $L^\infty(\hat{\mathbb{G}})$. Then we have

$$\hat{\omega} * \hat{f} = \hat{\omega} * \hat{m}(\hat{\omega}_0) = \hat{m}(\hat{\omega} * \hat{\omega}_0) = \langle \hat{\omega} * \hat{\omega}_0, 1 \rangle \hat{f} = \langle \hat{\omega}, 1 \rangle \hat{f}.$$

Hence, $\hat{f}$ is a normal left invariant state on $L^\infty(\hat{\mathbb{G}})$, and therefore $\hat{G}$ is compact by Proposition 4.2.3, which implies 4.

(4) $\Rightarrow$ (2) : Let e be the identity of $L^1(\mathbb{G})$, and $\tilde{e} \in \mathcal{T}_*(\mathbb{G})$ a norm-preserving extension of e. Define:

$$\begin{array}{rccc} E: & \mathcal{B}(L^2(\mathbb{G})) & \to & L^\infty(\mathbb{G}) \\ & x & \to & (\tilde{e} \otimes \iota)\Gamma(x). \end{array}$$

Then E is normal, unital and completely contractive, since both $(\tilde{e} \otimes \iota)$ and Γ are, which also implies $\|E\| = 1$. For all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$ we have:

$$\langle f, E(x) \rangle = \langle f, (\tilde{e} \otimes \iota)\Gamma(x) \rangle = \langle (\tilde{e} \otimes f), x \rangle = \langle f, x \rangle,$$

which implies that $E^2 = E$, and E is surjective. Hence, E is a conditional expectation. Now, for all $x \in \mathcal{B}(L^2(\mathbb{G}))$, we have:

$$\begin{aligned} \Gamma(E(x)) &= \Gamma((\tilde{e} \otimes \iota)\Gamma(x)) \\ &= (\tilde{e} \otimes \iota \otimes \iota)((\iota \otimes \Gamma) \circ \Gamma(x)) \\ &= (\tilde{e} \otimes \iota \otimes \iota)((\Gamma \otimes \iota) \circ \Gamma(x)) \\ &= (((\tilde{e} \otimes \iota)\Gamma) \otimes \iota) \circ \Gamma(x) \\ &= (E \otimes \iota) \circ \Gamma(x). \end{aligned}$$

Hence, $\Gamma \circ E = (E \otimes \iota) \circ \Gamma$, and (2) follows.

$\square$

For the following recall that $\pi : \mathcal{T}_*(\mathbb{G}) \twoheadrightarrow L^1(\mathbb{G})$ is the canonical quotient map.

Corollary 4.3.12. *For a locally compact quantum group* $\mathbb{G}$ *the following are equivalent:*

1. *there exists an isometric algebra homomorphism* $\Phi : L^1(\mathbb{G}) \to \mathcal{T}_*(\mathbb{G})$ *such that* $\pi \circ \Phi = \iota_{L^1(\mathbb{G})}$*;*
2. $\mathbb{G}$ *is discrete.*

Proof. If $\mathbb{G}$ is discrete, then Φ may be taken to be the pre-adjoint of the map E constructed in the proof of the implication (4) $\Rightarrow$ (2) in Theorem 4.3.11.

For the converse, note that $\Phi^* : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\mathbb{G})$ is a normal surjective norm-one projection, i.e., a normal conditional expectation. For all $x \in \mathcal{B}(L^2(\mathbb{G}))$ and $\rho, \eta \in \mathcal{T}_*(\mathbb{G})$, then we have:

$$\begin{aligned}
\langle \rho \otimes \eta, \Gamma(\Phi^*(x)) \rangle &= \langle \rho * \eta, \Phi^*(x) \rangle \\
&= \langle \Phi(\rho * \eta), x \rangle \\
&= \langle \Phi(\rho) * \Phi(\eta), x \rangle \\
&= \langle \Phi(\rho) \otimes \Phi(\eta), \Gamma(x) \rangle \\
&= \langle \rho \otimes \eta, (\Phi^* \otimes \Phi^*) \circ \Gamma(x) \rangle,
\end{aligned}$$

which implies $\Gamma \circ \Phi^* = (\Phi^* \otimes \Phi^*) \circ \Gamma$, and hence the theorem follows from Theorem 4.3.11. $\square$

As we promised earlier in this section, in the following we prove that discreteness of a locally compact quantum group $\mathbb{G}$, can also be characterized in terms of projectivity of its convolution algebras. We recall that here the morphisms are completely contractive maps.

We start with a preliminary lemma.

Lemma 4.3.13. *If $\Phi : L^1(\mathbb{G}) \to \mathcal{T}_*(\mathbb{G})$ is a right inverse to the canonical quotient map $\pi : \mathcal{T}_*(\mathbb{G}) \twoheadrightarrow L^1(\mathbb{G})$, i.e., $\pi \circ \Phi = \iota_{L^1(\mathbb{G})}$, then for all $\eta, \rho \in \mathcal{T}_*(\mathbb{G})$ we have:*

$$\eta * \Phi(\pi(\rho)) = \eta * \rho.$$

Proof. Recall that $\Gamma(x) \in \mathcal{B}(H)\overline{\otimes}L^\infty(\mathbb{G})$ for all $x \in \mathcal{B}(H)$. Therefore, we clearly obtain that $\eta * \rho = \eta * \pi(\rho)$. Hence, we have

$$\eta * \Phi(\pi(\rho)) = \eta * \pi(\Phi(\pi(\rho))) = \eta * \pi(\rho) = \eta * \rho.$$

$\square$

Theorem 4.3.14. *For a locally compact quantum group $\mathbb{G}$, the following are equivalent:*

1. *$L^1(\mathbb{G})$ is projective $\mathcal{T}_*(\mathbb{G})$-module;*
2. *$\mathbb{G}$ is discrete.*

Proof. (1) $\Rightarrow$ (2) : assume that $L^1(\mathbb{G})$ is a projective $\mathcal{T}_*(\mathbb{G})$-module. So there exists a $\mathcal{T}_*(\mathbb{G})$-module map $\Psi : L^1(\mathbb{G}) \to L^1(\mathbb{G})\widehat{\otimes}\mathcal{T}_*(\mathbb{G})$ such that $m \circ \Psi = \iota_{L^1(\mathbb{G})}$. Define $\Phi : L^1(\mathbb{G}) \to \mathcal{T}_*(\mathbb{G})$ to be $\Phi := \Gamma_* \circ \Psi$. We shall prove that Φ satisfies the conditions of Corollary 4.3.12. First note that since Ψ and Γ_* are both contractions, so is Φ. Now, for all $f \in L^1(\mathbb{G})$ and $x \in L^\infty(\mathbb{G})$, we get

$$\langle \Phi(f), x\rangle = \langle \Gamma_* \circ \Psi(f), x\rangle = \langle \Psi(f), \Gamma(x)\rangle = \langle f, x\rangle.$$

Hence, $\Phi(f)$ extends f, and since Φ is contractive, it is an isometry.

Now, for all $\omega, \omega' \in L^1(\mathbb{G})$ and $x \in \mathcal{B}(L^2(\mathbb{G}))$, we have:

$$\begin{aligned}
\langle \Phi(\omega * \omega'), x \rangle &= \langle \Gamma_* \circ \Psi(\omega * \omega'), x \rangle \\
&= \langle \Psi(\omega * \omega'), \Gamma(x) \rangle \\
&= \langle \Psi\left(\omega * (\pi \circ \Phi(\omega'))\right), \Gamma(x) \rangle \\
&= \langle \Psi(\omega.\Phi(\omega')), \Gamma(x) \rangle \\
&= \langle \Psi(\omega).\Phi(\omega'), \Gamma(x) \rangle \\
&= \langle \Psi(\omega) \otimes \Phi(\omega'), (\iota \otimes \Gamma) \circ \Gamma(x) \rangle \\
&= \langle \Psi(\omega) \otimes \Phi(\omega'), (\Gamma \otimes \iota) \circ \Gamma(x) \rangle \\
&= \langle (\Gamma_* \otimes \iota)(\Psi(\omega) \otimes \Phi(\omega')), \Gamma(x) \rangle \\
&= \langle (\Gamma_* \circ \Psi(\omega)) \otimes \Phi(\omega'), \Gamma(x) \rangle \\
&= \langle \Phi(\omega) \otimes \Phi(\omega'), \Gamma(x) \rangle \\
&= \langle \Phi(\omega) * \Phi(\omega'), x \rangle.
\end{aligned}$$

Thus, Φ satisfies the condition of Corollary 4.3.12, whence $\mathbb{G}$ is discrete.

(2) $\Rightarrow$ (1) : let $e \in L^1(\mathbb{G})$ be the identity element, and $\Phi : L^1(\mathbb{G}) \to \mathcal{T}_*(\mathbb{G})$, as in Corollary 4.3.12. Define the map $\Psi : L^1(\mathbb{G}) \to L^1(\mathbb{G}) \widehat{\otimes} \mathcal{T}_*(\mathbb{G})$ by

$$\Psi(f) = e \otimes \Phi(f) \qquad (f \in L^1(\mathbb{G})).$$

Since $\pi \circ \Phi = \iota_{L^1(\mathbb{G})}$, we have $m \circ \Psi = \iota_{L^1(\mathbb{G})}$, where $m : L^1(\mathbb{G}) \widehat{\otimes} \mathcal{T}_*(\mathbb{G}) \to L^1(\mathbb{G})$ is the canonical map associated with the module action.

Moreover, using lemma 4.3.13, we have

$$\begin{aligned}\Psi(f.\rho) = e\otimes\Phi(f.\rho) &= e\otimes\Phi(f * \pi(\rho)) \\ &= e\otimes\Phi(f) * \Phi(\pi(\rho)) \\ &= e\otimes\Phi(f) * \rho \\ &= \Psi(f).\rho\end{aligned}$$

for all $f \in L^1(\mathbb{G})$ and $\rho \in \mathcal{T}_*(\mathbb{G})$. Therefore Ψ is a morphism, and so $L^1(\mathbb{G})$ is projective. $\square$

We can also define a right $L^1(\mathbb{G})$-module structure on $\mathcal{T}(L^2(\mathbb{G}))$, as follows:

$$\rho \diamond f := (\rho \otimes f) \circ \Gamma \quad (\rho \in \mathcal{T}(L^2(\mathbb{G})), f \in L^1(\mathbb{G})).$$

Theorem 4.3.15. *For a locally compact quantum group $\mathbb{G}$, the following are equivalent:*

1. *there exists an isometric $L^1(\mathbb{G})$-module map $\Phi : L^1(\mathbb{G}) \to \mathcal{T}(L^2(\mathbb{G}))$ such that $\pi \circ \Phi = \iota_{L^1(\mathbb{G})}$, where $\pi : \mathcal{T}(L^2(\mathbb{G})) \twoheadrightarrow L^1(\mathbb{G})$ is the canonical quotient map;*

2. *$\mathbb{G}$ is discrete.*

Proof. If $\mathbb{G}$ is discrete, then the predual of the map E constructed in the proof of the implication (4) $\Rightarrow$ (2) in Theorem 4.3.11, is easily seen to satisfy the desired conditions.

Conversely, if such a map Φ exists, then it is straightforward to see that $E := \Phi^* : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\mathbb{G})$ enjoys the properties in part (2) of Theorem 4.3.11, and so $\mathbb{G}$ is discrete.

$\square$

In the following, we shall consider another important cohomology-type property for convolution algebras, namely amenability.

Our next theorem is a generalization of a result due to Hulanicki who considered the case $\mathbb{G} = L^\infty(G)$ [36, Theorem 2.4.], to the setting of general locally compact quantum groups.

Theorem 4.3.16. *Let $\mathbb{G}$ be a co-amenable locally compact quantum group. Then the following are equivalent:*

1. *The left regular representation $\lambda : L^1(\mathbb{G}) \to L^\infty(\hat{\mathbb{G}})$ is isometric on $L^1(\mathbb{G})^+$;*

2. *$\hat{\mathbb{G}}$ is co-amenable.*

Proof. $(1) \Rightarrow (2)$: We first show that for $f \in L^1(\mathbb{G})^+$ we have $\|1+\lambda(f)\| = 1 + \|f\|$. Let $(e_\alpha) \in L^1(\mathbb{G})^+$ be a bounded approximate identity, and $g \in L^1(\mathbb{G})^+$ with $\|e_\alpha\| = \|g\| = 1$ for all α. Then we have

$$\begin{aligned} \|f\| + 1 = f(1) + 1 &= f(1)g(1) + e_\alpha(1)g(1) = \langle f * g + e_\alpha * g, 1\rangle \\ &= \|f * g + e_\alpha * g\| \longrightarrow \|f * g + g\| = \|\lambda(f * g) + \lambda(g)\| \\ &= \|(\lambda(f) + 1)\lambda(g)\| \leq \|\lambda(f) + 1\|\|\lambda(g)\| = \|\lambda(f) + 1\| \\ &\leq \|\lambda(f)\| + 1 = \|f\| + 1, \end{aligned}$$

which implies our claim. Since $\mathbb{G}$ is co-amenable, there exists $\varepsilon \in M(\mathbb{G})_1^+$ such that $\lambda(\varepsilon) = 1$, by Theorem 2.5.6. Let

$$F_0 = \{E \cup \{\varepsilon\} : E \subseteq L^1(\mathbb{G})_1^+ \text{ and } E \text{ is finite}\},$$

and fix $E \in F_0$. Then we have

$$\Big\|\sum_{f\in E} f\Big\|_1 = \langle \sum_{f\in E} f, 1\rangle = |E|.$$

So $\big\|\sum_{f\in E}\lambda(f)\big\| = |E|$, and therefore there exists a sequence (ξ_n) of unit vectors in $L^2(\mathbb{G})$ such that

$$\lim_n \Big\|\sum_{f\in E}\lambda(f)\xi_n\Big\|_2 = |E|.$$

Now fix $f_0 \in E$, and let $E' = E\backslash\{\varepsilon, f_0\}$. Then we have

$$\lim_n \Big\|(\lambda(f_0)\xi_n + \xi_n) + \sum_{f\in E'}\lambda(f)\xi_n\Big\|_2 = |E|.$$

But since

$$\Big\|\sum_{f\in E'}\lambda(f)\xi_n\Big\|_2 \leq |\,E\,| - 2 \quad \forall n \in \mathbb{N},$$

we obtain $\lim_n \|\lambda(f_0)\xi_n + \xi_n\|_2 = 2$, which implies $\lim_n \|\lambda(f_0)\xi_n - \xi_n\|_2 = 0$. Since $f_0 \in E$ and $E \in F_0$ were arbitrary, there exists a net $(\xi_\imath)$ of unit vectors in $L^2(\mathbb{G})$ such that

$$\|\lambda(f)\xi_i - \xi_i\|_2 \to 0 \quad \forall f \in L^1(\mathbb{G})_1^+,$$

and since $L^\infty(\mathbb{G})$ is standard on $L^2(\mathbb{G})$ we have:

$$\|W(\eta\otimes\xi_i) - \eta\otimes\xi_i\|_2 \to 0$$

for all unit vectors $\eta \in L^2(\mathbb{G})$. Hence, $\hat{\mathbb{G}}$ is co-amenable, by Theorem 2.5.6.

(2) $\Rightarrow$ (1) : Since $\hat{\mathbb{G}}$ is co-amenable and $\chi(W)$ is an isometry, Theorem 2.5.6 ensures

the existence of a net $(\xi_\imath)$ of unit vectors in $L^2(\mathbb{G})$ such that

$$\lim_{\imath} \|W(\eta \otimes \xi_\imath) - \eta \otimes \xi_\imath\|_2 = 0 \quad \forall \eta \in L^2(\mathbb{G}).$$

Now, let $f \in L^1(\mathbb{G})^+$. Since $L^\infty(\mathbb{G})$ is in standard form in $\mathcal{B}(L^2(\mathbb{G}))$, we have $f = \omega_\zeta$, for some $\zeta \in L^2(\mathbb{G})$ with $\|f\|_1 = \|\zeta\|_2$. Assuming $\|\lambda(f)\| \leq 1$, we obtain:

$$\begin{aligned} 1 &\geq \lim_{\imath} |\langle \lambda(f)\xi_\imath, \xi_\imath\rangle| = \lim_{\imath} |\langle (f \otimes \iota)W\xi_\imath, \xi_\imath\rangle| \\ &= \lim_{\imath} |\langle W(\zeta \otimes \xi_\imath), \zeta \otimes \xi_\imath\rangle| = \lim_{\imath} \|\zeta \otimes \xi_\imath\|_2^2 \\ &= \|\zeta\|_2^2 = \|f\|_1^2. \end{aligned}$$

So $\|\lambda(f)\| \leq 1$ implies $\|f\|_1 \leq 1$, therefore the conclusion follows. $\square$

Remark 4.3.17. *Note that the assumption of co-amenability of $\mathbb{G}$ is not necessary for the implication* (2) $\Rightarrow$ (1).
Also, this condition is not necessary if $\mathbb{G}$ is a Kac algebra, as an easy modification of our argument shows. In the Kac algebra case the result was obtained by Kraus and Ruan in [24, Theorem 7.6], through a different proof.

4.4 Radon-Nikodym Property for Convolution Algebras

In this section we study the Radon-Nikodym property (in short: RNP) for the convolution algebra $L^1(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$.

Definition 4.4.1. *A Banach space X has the Radon-Nikodym property if for each finite measure space (Ω, S, μ) and each bounded linear operator $T : L^1(\Omega, S, \mu) \to X$,* *there is a bounded μ-measurable function $\phi : \Omega \to X$ such that*

$$Tf = \int_\Omega f\phi \, d\mu \qquad (f \in (L^1(\Omega, S, \mu)).$$

Proposition 4.4.1.

1. *[10, Theorem 4.2] The RNP is inherited by closed subspaces and stable under isomorphism.*

2. *[10, Page 268] If H is a Hilbert space then $\mathcal{T}(H)$ has the RNP.*

3. *[11, III.1] Let G be a locally compact group. Then $L^1(G)$ has the RNP if and only if G is discrete.*

$\square$

Theorem 4.4.2. *Let $\mathbb{G}$ be a locally compact quantum group. If there exists $f \in L^1(\mathbb{G})$ such that the map*

$$L^1(\mathbb{G}) \ni \omega \mapsto f * \omega \in L^1(\mathbb{G})$$

is injective, then $L^1(\mathbb{G})$ has the RNP.

Proof. Let $\tilde{f} \in \mathcal{T}(L^2(\mathbb{G}))$ be a weak*-extension of f. Then the map

$$\imath : L^1(\mathbb{G}) \ni \omega \mapsto \tilde{f} * \omega \in \mathcal{T}_*(\mathbb{G})$$

is injective. To see this, let $\tilde{f} * \omega = 0$. Then

$$0 = \pi(\tilde{f} * \omega) = \pi(\tilde{f}) * \omega = f * \omega.$$

Whence, $\omega = 0$ by our assumption. This implies that $L^1(\mathbb{G})$ is isomorphic to a subspace of $\mathcal{T}(L^2(\mathbb{G}))$, hence the claim follows from parts (1) and (2) of Proposition 4.4.1. $\square$

In [53] Wendel proved that, given a locally compact group G, if for some $\mu \in M(G)$ the map $L^1(G) \ni \omega \mapsto \omega * \mu \in L^1(G)$ is isometric, then μ is a point-mass. Obviously, Wendel's result does not hold if we replace "isometric" by "injective" ($M(G)$ is a unital Banach algebra, and so the set of invertible measures is open). However, as an immediate corollary of Wendel's result, we have the following.

Proposition 4.4.3. *Let G be a locally compact group. If there exists $\mu \in L^1(G)$ such that the map $L^1(G) \ni \omega \mapsto \omega * \mu \in L^1(G)$ is isometric, then G is discrete.*

This result in turn can be generalized by using Theorem 4.4.2, as follows.

Proposition 4.4.4. *Let G be a locally compact group. Then the following are equivalent*

1. *there exists $f \in L^1(G)$ such that the following map is injective:*

$$L^1(G) \ni \omega \mapsto \omega * f \in L^1(G);$$

2. *G is discrete.*

Proof. (2) $\Rightarrow$ (1) : this is trivial since $L^1(G)$ is unital in this case.
(1) $\Rightarrow$ (2) : the claim follows Theorem 4.4.2 combined with part 3 of Proposition 4.4.1. $\square$

Part (3) of Proposition 4.4.1, at first glance, suggests that one might have a dual version of this statement, saying that the Fourier algebra $A(G)$ has the RNP if and only if G is compact. But in fact, this is not the case. A counter-example is given by the Fell group (see [46, Remark 4.6]) which is non-compact, but its Fourier algebra has the RNP.

Analogously to our discussion of amenability, one may need to take the operator space structures of the L^1-algebras into account as well. Indeed, there is an operator space version of the RNP, due to Pisier (see [35]), which may be useful in this context.

We shall close the section by pointing out another way of looking at this problem. We start with a general result.

Theorem 4.4.5. *Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. Then the following are equivalent:*

1. *M is atomic;*

2. *M_* has the RNP;*

3. *there is a normal conditional expectation from $B(H)$ onto M.*

Proof. (1) and (2) are equivalent by [46, Theorem 3.5].

Now assume (1), i.e., M is an l_∞-direct sum of $\mathcal{B}(H_i)$'s for some Hilbert spaces H_i. So $M = N^{**}$ where $N = \oplus_\infty K(H_i)$ is an ideal in M. Then, we have (3) by [49, Theorem 5].

Finally, if (3) holds, then the pre-adjoint map of the conditional expectation defines an isometric embedding of M_* into $\mathcal{T}(H)$. In view of parts (1) and (2) of Proposition 4.4.1, we obtain (2). □

Corollary 4.4.6. *Let G be locally compact quantum group. Then the following are equivalent:*

1. *there exists a normal conditional expectation E from $\mathcal{B}(L^2(\mathbb{G}))$ to $L^\infty(\mathbb{G})$;*

2. *$L^1(\mathbb{G})$ has RNP.* □

In Theorem 4.3.11 we gave a characterization of discreteness of $\mathbb{G}$ in terms of existence of a normal and covariant conditional expectation. By comparing this result with Corollary 4.4.6 above, we see that the covariance accounts precisely for the difference between the RNP and discreteness, for L^1-algebras of locally compact quantum groups.

Bibliography

[1] O. Y. Aristov, *Amenability and compact type for Hopf-von Neumann algebras from the homological point of view,* Banach algebras and their applications, Contemp. Math., Amer. Math. Soc., Providence, RI. **363** (2004), no. 1, 15–37.

[2] S. Baaj and G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisès de C^*-algèbres,* Ann. Sci. Ecole Norm. Sup. **26** (1993), 425–488.

[3] S. Baaj and S. Vaes, *Double crossed products of locally compact quantum groups,* J. Inst. Math. Jussieu **4** (2005), no. 1, 135–173.

[4] E. Bedos and L. Tuset, *Amenability and co-amenability for locally compact quantum groups,* Internat. J. Math. **14** (2003), 865–884.

[5] D. P. Blecher and C. Le Merdy, *Operator Algebras and Their Modules – an Operator Space Approach,* London Mathematical Society Monographs, New Series, 30, Oxford Science Publications, The Clarendon Press, Oxford University Press, Oxford, 2004.

[6] M. Daws, *Operator biprojectivity of compact quantum groups,* Proc. Amer. Math. Soc. **138** (2010), no. 4, 1349–1359.

[7] J. de Cannière, *On the intrinsic group of a Kac algebra,* Proc. London Math. Soc. (3) **40** (1980), no. 1, 1–20.

[8] J. de Cannière, *Produit croisé d'une algèbre de Kac par un groupe localement compact,* Bull. Soc. Math. France **107** (1979), no. 4, 337–372.

[9] K. De Commer, A. Van Daele, *Algebraic quantum groups imbedded in C^*-algebraic quantum groups,* Rocky Mountain J. Math. **40** (2010), no. 4, 1149–1182.

[10] J. Diestel and B. Faires, *On vector measures,* Trans. Amer.Math. Soc. **198** (1974), 253–271.

[11] J. Diestel and J. Uhl Jr., *Vector Measures,* American Mathematical Society, Providence, RI., 1977.

[12] E. G. Effros and Z.-J. Ruan, *Operator Spaces,* London Mathematical Society Monographs, New Series, 23, The Clarendon Press, Oxford University Press, New York, 2000.

[13] M. Enock and J. M Schwartz, *Kac Algebras and Duality of Locally Compact Groups,* Springer-Verlag, Berlin, 1992.

[14] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact,* Bull. Soc. Math. France **92** (1964), 181–236.

[15] P. Fima and L. Vainerman, *Twisting and Rieffel's deformation of locally compact quantum groups: deformation of the Haar measure,* Comm. Math. Phys. **286** (2009), no. 3, 1011–1050.

[16] B. Forrest and V. Runde, *Amenability and weak amenability of the Fourier algebra,* Math. Z. **250** (2005), no. 4, 731–744.

[17] U. Haagerup, *Operator-valued weights in von Neumann algebras. I,* J. Funct. Anal. **32** (1979), no. 2, 175–206.

[18] U. Haagerup, *Operator-valued weights in von Neumann algebras. II,* J. Funct. Anal. **33** (1979), no. 3, 339–361.

[19] U. Haagerup, *The standard form of von Neumann algebras,* Math. Scand. **37** (1975), no. 2, 271–283.

[20] A. Ya. Helemskii, *Banach and Polynormed Algebras: General Theory, Representations, Homology,* Nauka, Moscow, 1989 (Russian); English transl.: Oxford University Press, 1993.

[21] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras,* Moscow University Press, 1986 Nauka, Moscow, 1989 (Russian); English transl.: Kluwer Academic Publishers, Dordrecht, 1989.

[22] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Vol. I. Structure of Topological Groups, Integration Theory, Group Representations,* Second edition, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 115, Springer-Verlag, Berlin-New York, 1979.

[23] M. Junge, M. Neufang and Z.-J. Ruan, *A representation theorem for locally compact quantum groups.* Internat. J. Math. **20** (2009), no. 3, 377–400.

[24] J. Kraus and Z.-J. Ruan, *Multipliers of Kac algebras,* Internat. J. Math. **8** (2009), no. 2, 213–248.

[25] J. Kustermans and S. Vaes, *locally compact quantum groups,* Ann. Sci. Ecole Norm. Sup. **33** (2000), 837–934.

[26] J. Kustermans and S. Vaes, *Locally compact quantum groups in the von Neumann algebraic setting,* Math. Scand. **92** (2003), no. 1, 68–92.

[27] B.-R. Li, *Introduction to Operator Algebras,* World Scientific Pub Co Inc, 1992.

[28] S. Majid, *Hopf-von Neumann algebra bicrossproducts, Kac algebra bicrossproducts, and the classical Yang-Baxter equations,* J. Funct. Anal. **95** (1991), no. 2, 291–319.

[29] A. Müller, *Classifying spaces for quantum principal bundles,* Comm. Math. Phys. **149** (1992), no. 3, 495–512.

[30] M. Neufang, *Abstrakte harmonische Analyse und Modulhomomorphismen über von Neumann-Algebren,* Dissertation zur Erlangung des Grades des Doktors der Naturwissenschaften, Saarbrücken, 2000.

[31] M. Neufang, Z.-J. Ruan and N. Spronk, *Completely isometric representations of* $M_{\mathrm{cb}}A(G)$ *and* $\mathrm{UCB}(\hat{G})^*$, Trans. Amer. Math. Soc. **360** (2008), no. 3, 1133–1161.

[32] M. Neufang and V. Runde, *Harmonic operators: the dual perspective,* Math. Z. **255** (2007), no. 3, 669–690.

[33] V. Paulsen, *Completely Bounded Maps and Operator Algebras,* Cambridge Studies in Advanced Mathematics, 78, Cambridge University Press, Cambridge, 2002.

[34] A. Y. Pirkovskii, *Biprojectivity and biflatness for convolution algebras of nuclear operators,* Canad. Math. Bull. **47** (2004), no. 3, 445–455.

[35] G. Pisier, *Non-Commutative Vector Valued L_p-Spaces and Completely p-Summing Maps,* Astérisque, no. 247, (1998).

[36] G. Pisier, *Similarity Problems and Completely Bounded Maps,* Lecture Notes in Mathematics 1618, Springer-Verlag, Berlin, 1996.

[37] G. Pisier, *Introduction to Operator Space Theory,* London Mathematical Society Lecture Note Series 294, Cambridge University Press, Cambridge, 2003.

[38] P. F. Renaud, *Centralizers of Fourier algebra of an amenable group,* Proc. Amer. Math. Soc. **32** (1972), 539–542.

[39] Z.-J. Ruan, *The operator amenability of $A(G)$,* Amer. J. Math. **117** (1995), no. 6, 1449–1474.

[40] V. Runde, *Characterizations of compact and discrete quantum groups through second duals,* J. Operator Theory, **60** (2008), no. 2, 415–428.

[41] V. Runde, *Lectures on Amenability,* Lecture Notes in Mathematics 1774, Springer-Verlag, Berlin, 2002.

[42] J.-M. Schwartz, *Sur la structure des algèbres de Kac. II,* Proc. London Math. Soc.(3) **41** (1980), no. 3, 465–480.

[43] P. M. Soltan and S. L. Woronowicz, *From multiplicative unitaries to quantum groups. II,* J. Funct. Anal. **252** (2007), no. 1, 42–67.

[44] M. Takesaki, *A characterization of group algebras as a converse of Tannaka–Stinespring–Tatsuuma duality theorem,* Amer. J. Math. **91** (1969), 529–564.

[45] M. Takesaki, *Theory of Operator Algebras. II.* Encyclopaedia of Mathematical Sciences 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag. Berlin, 2003.

[46] K. F. Taylor, *Geometry of the Fourier algebras and locally compact groups with atomic unitary representations,* Math. Ann. **262** (1983), no. 2, 183–190.

[47] R. Tomatsu, *Amenable discrete quantum groups,* J. Math. Soc. Japan **58** (2006), no. 4, 949–964.

[48] J. Tomiyama, *On the projection of norm one in W^*-algebras,* Proc. Japan Acad. **33** (1957), 608–612.

[49] J. Tomiyama, *On the projection of norm one in W^*-algebras. III,* Tôhoku Math. J. **2** (1959), 11, 125–129.

[50] S. Vaes, *Examples of locally compact quantum groups through the bicrossed product construction,* in: Proceedings of the XIIIth International Congress of Mathematical Physics, Imperial College, London, 2000. International Press of Boston, Somerville MA (2001), 341–348.

[51] S. Vaes, *Locally Compact Quantum Groups,* Ph.D. thesis, K.U. Leuven, 2000.

[52] A. Weil, *L'intégration dans les groupes topologiques et ses applications,* Act. Sc. Ind. Hermann, Paris, 1953.

[53] J. G. Wendel, *Left centralizers and isomorphisms of group algebras,* Pacific J. Math. (1952), 251–261.

[54] S. L. Woronowicz, *Compact matrix pseudogroups,* Comm. Math. Phys. **111** (1987), no. 4, 613–665.

[55] S. L. Woronowicz, *Compact quantum groups,* Symétries quantiques (Les Houches, 1995), 845–884, North-Holland, Amsterdam, 1998.

[56] S. L. Woronowicz, *From multiplicative unitaries to quantum groups,* Int. J. Math. **7** (1996), 127–149.

[57] S. L. Woronowicz, *Quantum $E(2)$ group and its Pontryagin dual,* Lett. Math. Phys. **23** (1991), no. 4, 251–263.

[58] S. L. Woronowicz, *Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(N)$ groups,* Invent. Math. **93** (1988), no. 1, 35–76.

[59] T. Yamanouchi, *The intrinsic group of Majid's bicrossproduct Kac algebra,* Pacific J. Math. **159** (1993), no. 1, 185–199.