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**THERMO-ELASTO-PLASTIC FINITE ELEMENT
FORMULATION FOR DEFORMATION
AND RESIDUAL STRESSES DUE TO WELDS**

by
Balvant Patel, M. S.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirement for the degree of

Doctor of Philosophy

Department of
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April 1985

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The undersigned recommends to the Faculty of Graduate Studies and Research, acceptance of the thesis: *Thermo-Elasto-Plastic Finite Element Formulation for Deformation and Residual Stresses due to Welds*, submitted by Balvantrai G. Patel, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

The thesis develops and demonstrates a rigorous methodology for thermo-elasto-plastic analysis of residual stresses and distortions in welded structures. A usable theory for modelling the response is built upon the laws of continuum mechanics. Constitutive equations are developed. They account for the temperature-dependence of the material properties in a thermo-elasto-plastic medium. A nonlinear transient finite element solution scheme is proposed for solving the equations. A computer code is developed and verified using examples for which exact solution was known. The capability and usefulness of the system is demonstrated by computing the distortion due to an edge weld in an AISI 1020 and an AISI 304 steel bar. The computed vertical displacement of 4.7mm and 14.5mm is in agreement with the experimentally measured deflection of 4.9mm and 15.3mm in an AISI 1020 and an AISI 304 steel bar respectively. Finally the conclusions of the research are given and are followed by suggestions for future research.

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Issac Newton's words, "We must stand on the shoulders of giants" apply to this thesis in that it extends the work of earlier investigators. It is an integral part of a large project to develop a software system for the analysis of welds and welded structures. The heat source model originated by Chakravarti and enhanced by House, Adam and Moore was used. The 2-D transient nonlinear heat transfer code developed by Chakravarti was used. The graphics was largely written by M. Dixon. The frontal solver is due to B. M. Irons. Almost all of the stress analysis and incremental nonlinear code was developed by the candidate, B. Patel. Experiment data was provided by Professor M. Bibby and his students Hogan and Wan.

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Table of Contents

Abstract	iii
Acknowledgemnt	iv
List of Figures	ix
List of Tables	xiii
List of Symbols	xv
1 Problem Identification	1
1.1 Introduction	1
1.2 Review of past work in weld analysis	2
1.3 Scope of thesis	8
2 Development of Fundamental Equations	9
2.1 Continuum	9
2.2 Body and forces	9
2.3 Kinematics of continuum	10
2.4 Strain and deformation	12
2.5 Stress	14
2.6 Equation of motion and equilibrium	16
2.7 Constitutive equations	17
2.8 Compatibility Equations	19
2.9 Boundary conditions	20
3 Nonlinear Equations	21
3.1 Nonlinearity	21
3.2 Solution of nonlinear equations	23
3.3 Faster iteration using line search or step-length parameter	26
3.4 Line searches	26
3.5 Constitutive equations for thermo-plasticity	34
3.6 Temperature-dependent yield function	35
3.7 Particular yield function F , von Mises	38
3.8 Plastic potential function, Prandtl-Reuss equations	38
3.9 Uniaxial stress-strain curve	41

3.10	Work hardening parameter k	45
3.10.1	Kinematic work hardening rule, Prager's rule	46
3.11	Determining D'_{ijmn} and $d\lambda$	47
4	Application of Finite Element Method	50
4.1	Finite element method	50
4.2	Elements	53
4.3	Strain-displacement transformation matrix, B	56
4.4	Linearized finite element equations of motion	57
4.5	Element load matrices	59
4.5.1	Body forces, f^b	59
4.5.2	Initial stress and temperature loads	59
4.5.3	Surface traction forces	60
4.6	Numerical integration	60
4.7	Convergence criterion	61
4.7.1	Displacement criterion	62
4.7.2	Force criterion	62
4.7.3	Energy criterion	62
4.8	Stress increment calculations	63
4.9	Least square stress projection schemes	65
5	The Heat Transfer Analysis	68
5.1	Heat transfer equations	68
5.2	Finite element formulation for transient heat flow	69
5.3	Time discretization	71
5.4	Nonlinear material properties	74
5.5	Convection and radiation	75
6	Code Verification Cases and Examples	76
6.1	Code development	76
6.2	Forms of Newton-Raphson iterative schemes and line search	78
6.2.1	Results	78
6.3	Examining work hardening rules for the case of a uniaxial bar	78
6.3.1	Load specifications and finite element modelling	82
6.3.2	Results	88
6.4	Solid thin circular disk with radial temperature distribution	88

6.4.1	<i>Example definition and solutions</i>	95
6.4.2	<i>Results</i>	95
6.5	Thick cylinder subjected to internal pressure and a temperature gradient	99
6.6	Thermo-Elasto-plastic stresses in a thin flat strip	108
6.6.1	<i>Theoretical solution for a thin flat strip</i>	109
6.6.2	<i>Example definition</i>	113
6.6.3	<i>Numerical solution of analytical formulation</i>	113
6.6.4	<i>Finite element solution.</i>	123
7	Computing the Distortion to Edge Welds in Steel Bars	126
7.1	Heat transfer analysis.	128
7.2	Geometry and boundary condition for heat transfer analysis.	130
7.3	Results of heat transfer analysis.	133
7.4	Deflection and stress analysis of the bar	135
7.5	Temperature mapping and time step subdivision	135
7.6	Elasto-thermo-plastic stress analysis of the 3-D slice.	141
7.6.1	<i>Geometry and boundary conditions for the 3-D slice.</i>	141
7.7	Results of the 3-D slice analysis.	144
7.8	Elastic 2-D deflection analysis of the bar	148
7.9	Comparison of the bar deflections	154
7.10	Residual stress field	155
8	Conclusions and Future Research	157
8.1	Conclusions	157
8.2	Future research	158
	References	159
	APPENDIX A	166
A.1	Temperature dependent material properties for 0.23% carbon steel	166
A.2	Temperature dependent material properties for 304 stainless steel	167
A.3	Temperature dependent material properties for fine grain steel	168
B.	Least Square Stress Projection in FEM	169
B.1	Introduction	169
B.2	Theory	171
B.2.1	<i>Bilinear projector</i>	171

B.2.2 Trilinear projection for brick elements	172
B.2.3 Quadratic projector	172
B.2.4 Cubic projector	173
B.2.5 Least squares	174
B.2.6 Weighted least squares	174
B.2.7 Stress space	175
B.3 Numerical Tests and Results	175
B.3.1 Tests using bilinear projector	183
B.3.2 Tests using quadratic projector	187
B.3.3 Tests using cubic projector	187
B.4 Discussion	188
B.5 Conclusions	190
APPENDIX C	192
C. Projection Matrices.	192
C.1 Weighted least square bilinear projection matrices $P_{\sqrt{w}}$	193
C.2 Weighted least square quadratic projection matrices $P_{\sqrt{w}}$	193
C.3 Weighted least square cubic projection matrices $P_{\sqrt{w}}$	195
C.4 Weighted least square trilinear projection matrices for brick elements. $P_{\sqrt{w}}$	196
APPENDIX D	199
D. Gauss Point Weights and Coordinates	200

List of Figures

- Figure 2.1: Motion of body in Total Lagrangian reference system.
- Figure 2.2: Stress vectors.
- Figure 3.1: Interpolation and extrapolation techniques in line search.
- Figure 3.2: Young's Modulus E vs. temperature for various steels
- Figure 3.3: Poisson's ratio (ν) vs. temperature for various steels
- Figure 3.4: Linear coefficient of thermal expansion (α) vs. temperature for various steels
- Figure 3.5: Uniaxial yield stress (σ_y) vs. temperature for various steels
- Figure 3.6: Hardening parameter (E_p) temperature for various steels
- Figure 3.7: Yield surface and stress vectors in stress space
- Figure 3.8: The uniaxial engineering stress-strain curve for typical metal
- Figure 3.9: Idealized uniaxial true stress-strain curve for metal plasticity
- Figure 3.10: Types of most frequently used idealized stress- strain curves in metal plasticity
- Figure 3.11: Idealized uniaxial stress- strain curve in loading and unloading
- Figure 4.1: Typical two and three dimensional isoparametric elements and their local coordinate axis.
- Figure 4.2: Types of elements used in present work.
- Figure 4.3: Incremental stress in elasto-plastic medium.
- Figure 5.1: Body subjected to heat transfer
- Figure 5.2: Linear time marching basis function
- Figure 6.1: Iterative solution using modified Newton-Raphson method
- Figure 6.2: Iterative solution using modified Newton-Raphson method and line search
- Figure 6.3: Iterative solution using Newton-Raphson method
- Figure 6.4: Total prescribed deflection vs. time. Loading conditions for examining work hardening rules in case of uniaxial bar test.
- Figure 6.5: Geometry and boundary conditions in finite element solution in uniaxial bar test for examining work hardening rules.
- Figure 6.6: Stress-strain curve and unexpanding yield surface. Results of uniaxial bar test with load reversal and perfectly plastic material, $E_p = 0$.
- Figure 6.7: Stress-strain curve and expanding yield surface. Results of uniaxial bar test with load reversal and isotropic work hardening material, $\beta = 1$.
- Figure 6.8: Stress-strain curve and yield surface translation. Results of uniaxial bar test with load reversal and kinematic work hardening material, $\beta = 0$.

- Figure 6.9:** Thin circular disk with symmetrical temperature distribution $T = r$.
- Figure 6.10:** Finite element mesh and boundary conditions. 10° Sector of thin circular disk.
- Figure 6.11:** Radial stress comparison between analytical and finite element solutions. Thin circular disk.
- Figure 6.12:** Circumferential stress comparison between analytical and finite element solutions. Thin circular disk.
- Figure 6.13:** Displacement comparison between analytical and finite element solutions. Thin circular disk.
- Figure 6.14:** Finite element mesh for 10° sector of thick cylinder
- Figure 6.15:** Temperature gradient along radius of thick cylinder
- Figure 6.16:** Comparison of radial stress values. Results of elasto-plastic analysis of thick cylinder with various pressure loading
- Figure 6.17:** Comparison of circumferential stress values. Results of elasto-plastic analysis of thick cylinder with various pressure loading
- Figure 6.18:** Comparison of radial stress values. Results of thermo-elastic analysis of thick cylinder with various pressure loading and temperature gradient
- Figure 6.19:** Comparison of circumferential stress values. Results of thermo-elastic analysis of thick cylinder with various pressure loading and temperature gradient
- Figure 6.20:** Comparison of radial stress values. Results of elasto-plastic analysis of thick cylinder with various pressure loading and temperature gradient
- Figure 6.21:** Width dimension of infinitely long thin strip
- Figure 6.22:** Temperature distribution across half width of thin strip
- Figure 6.23:** Finite element mesh and boundary conditions. Thermo-elasto-plastic analysis of thin strip
- Figure 6.24:** Comparison of axial stress (σ_{11}) distribution. Elasto-plastic analysis of thin strip, isothermal material properties, Case I
- Figure 6.25:** Comparison of axial stress (σ_{11}) distribution. Elasto-plastic analysis of thin strip, nonisothermal material properties, Case II
- Figure 6.26:** Comparison of axial stress (σ_{11}) distribution. Thermo-elastic analysis of thin strip, isothermal material properties, Case III
- Figure 6.27:** Comparison of axial stress (σ_{11}) distribution. Elasto-plastic analysis of thin strip, nonisothermal material properties, Case IV
- Figure 6.28:** Comparison of plastic strain (ϵ^p) distribution. Elasto-plastic analysis of thin strip, isothermal material properties, Case III

- Figure 6.29:** Comparison of plastic strain (ϵ^p) distribution. Elasto-plastic analysis of thin strip, nonisothermal material properties, Case IV
- Figure 7.1:** Geometry of edge welded bar
- Figure 7.2:** Double ellipsoidal Gaussian distributed power density model
- Figure 7.3:** Geometry, finite element mesh and boundary conditions for transient heat cross-sectional heat transfer analysis
- Figure 7.4:** Time-temperature history for steel bars. Results of heat transfer analysis
- Figure 7.5:** Spreading of HAZ in steel bars. Results of heat transfer analysis
- Figure 7.6:** Geometry of bar slice for 3-D stress analysis
- Figure 7.7:** Mapping and interpolation of nodal temperature due to subdivision of time steps
- Figure 7.8:** Time-temperature history of node in heat transfer and mapped history for node in 3-D bar slice
- Figure 7.9:** Comparison between mapped and unmapped temperature histories
- Figure 7.10:** Finite element mesh for 3-D bar slice for stress analysis
- Figure 7.11:** History of thermal loads for 3-D stress analysis for carbon steel bar
- Figure 7.12:** Transient history of longitudinal stress, σ_{11} , along bar depth. Results of 3-D stress analysis
- Figure 7.13:** Transient history of effective plastic strain, $\bar{\epsilon}_p$, along bar depth. Results of 3-D stress analysis
- Figure 7.14:** Geometry, boundary conditions and finite element mesh for elastic 2-D beam deflection analysis
- Figure 7.15:** Camber of steel bars. Results of 2-D beam deflection analysis
- Figure 7.16:** Comparison between finite element and experimental results of vertical deflection of edge welded steel bars
- Figure B.1:** Cantilever with constant shear tip load. Mesh and boundary conditions for 8-noded quadrilateral element
- Figure B.2:** Nodal and Gauss point shear stress, σ_{12} , along beam diagonal for different integration schemes in cantilever with constant shear load. Beam diagonal is shown as dashed line
- Figure B.3:** Cantilever beam with uniform load. Finite element mesh and boundary conditions
- Figure B.4:** Gauss point and least squared nodal shear stress, σ_{12} , along cantilever beam diagonal

Figure B.5: Cantilever beam with parabolic shear tip load. Boundary conditions for one 12-noded quadrilateral element

Figure B.6: Deep beam with uniform load. Boundary conditions for one 17-nodes quadrilateral element

Figure B.7: Order of Gauss points for various integration schemes

List of Tables

- Table 4.1: Two and three dimensional finite elements for stress analysis
- Table 6.1: Comparison of Newton-Raphson iterative schemes with and without line search
- Table 6.2: Results of uniaxial bar test. Perfectly plastic material with loading and unloading.
- Table 6.3: Results of uniaxial bar test. Isotropic work hardening material with loading and unloading.
- Table 6.4: Results of uniaxial bar test. Kinematic work hardening material with loading and unloading.
- Table 6.5: Stresses and displacements by theoretical solution. Thin circular disk with radial temperature distribution and isothermal properties.
- Table 6.6: Stresses, displacements and % error between theoretical and finite element solutions. Thin circular disk with radial temperature distribution and isothermal properties.
- Table 6.7: Radial stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of thick cylinder, Case I
- Table 6.8: Circumferential stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of thick cylinder, Case I
- Table 6.9: Stress values and % difference in finite element and theoretical solutions. Thermo-elastic analysis of thick cylinder, Case II
- Table 6.10: Radial stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of thick cylinder, Case III
- Table 6.11: Axial stress values and % difference in finite element and theoretical solutions. Thermo-Elastic analysis of thin flat, Case I and II
- Table 6.12: Axial stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of thin flat, Case III and IV
- Table 6.13: Plastic strain values in finite element and theoretical solutions. Elasto-plastic analysis of thin flat, Case III and IV
- Table 7.1: Typical time-temperature values of two node in HAZ for stainless steel bar
- Table 7.2: Transient history of longitudinal stress, σ_{11} , along bar depth. Results of 3-D stress analysis
- Table 7.3: Transient history of effective plastic strain, ϵ_p , along bar depth. Results of 3-D stress analysis

Table 7.4: Vertical deflection of unwelded bar edge. Finite element results of edge welded steel bars

Table 7.5: Vertical deflection of unwelded bar edge. Experimental results of edge welded steel bars

Table A.1: Temperature dependent material properties for 0.23% carbon steel

Table A.2: Temperature dependent material properties for 304 stainless steel

Table A.3: Temperature dependent material properties for fine grain steel

Table B.1: Gauss point shear stress, σ_{12} , in cantilever beam with constant shear tip load. Results of finite element analysis

Table B.2: Weighted least square nodal shear stress, σ_{12} , in cantilever beam with constant shear tip load. Results of finite element analysis using one 8-noded quadrilateral element

Table B.3: Least square and weighted least square nodal shear stress, σ_{12} , in cantilever beam with constant shear tip load. Results of finite element analysis using four 8-noded quadrilateral elements

Table B.4: Gauss point and weighted least square nodal stress components in cantilever beam with parabolic shear tip load. Results of finite element analysis using one 12-noded quadrilateral element

Table B.5: Gauss point and weighted least square nodal stress components in deep beam with uniform load. Results of theoretical and finite element analysis using one 17-noded quadrilateral element

List of Symbols

- α Coefficient of thermal expansion
- α Instantaneous coefficient of thermal expansion
- α_i Center of yield surface in stress space
- β Work hardening coefficient
- $\delta\Phi$ Virtual Φ or variation in Φ , where Φ is any function
- δ_{ij} Kronecker delta
- $\Delta\Phi$ Increment of Φ
- ϵ_{ij} Green-Lagrangian strain tensor
- $\epsilon_{ij}^L, \epsilon_{ij}^{NL}$ Linear and nonlinear Green-Lagrangian strain tensor
- ϵ_{ij}^P Plastic strain tensor
- ϵ_{ij}^T Thermal strain tensor
- $\bar{\epsilon}$ Effective plastic stress
- ϵ_{ij} Almansi strain tensor
- ϵ Hemispherical emissivity
- $\epsilon^{(k)}$ Inner product of residual force and iterative displacement for k^{th} line search step
- η_w Weld arc efficiency
- $\eta^{(i,k)}$ Step length parameter for iteration number i and line step k
- θ° Angle between contact point unit normal and trial stress unit normal to yield surface
- Θ Time integration scheme parameter
- θ Constant used for calculating number of stress subincrements N_{step}
- λ Lamé's constant
- $d\lambda$ Scalar function for proportionality in plastic potential function
- μ Lamé's constant
- $d\mu$ Scalar to satisfy condition that stress point remains on yield surface
- ν Poisson's ratio
- ξ_i Coordinates of point in local (element) coordinate system
- ρ Density of material
- σ_{ij} 2nd Piola-Kirchhoff stress tensor
- τ_{ij} Cauchy stress tensor
- σ_{ij}^T Thermal stress tensor
- σ_y Uniaxial yield stress

- σ Effective stress
- σ_y Effective yield stress
- σ_r Stefan- Boltzmann constant for radiation
- $\sum_i \Phi_i$ Sum of i to e components of function Φ
 - ϕ Plastic potential function
 - Φ Any function
- φ_d Displacement convergence criteria tolerance
- φ_e Energy convergence criteria tolerance
- φ_f Force convergence criteria tolerance
- ψ, ψ_i Residual force and residual force vector, external minus internal force
- Ψ Line search tolerance
- ω Weight of Gaussian integration sampling point
- b Body force per unit mass
- B Strain displacement matrix
- B^L, B^{NL} Linear and nonlinear strain displacement matrix
 - c Scaling parameter for kinematic work hardening
 - c_p Volumetric specific heat
 - C_{ij} Green deformation tensor
 - C Global capacitance matrix
- D_{ijmn} Elastic material constitutive tensor
- D'_{ijmn} Elasto plastic material constitutive tensor
- D Elastic material constitutive matrix
 - e Column matrix of Green-Lagrangian strain components
- E Elastic or Young's modulus
- E_p Hardening modulus
- E^t Tangent modulus
- f^S Traction force
- f^B Body force
- f^P Point force
- f^I Initial force
- f_{ga} Correction factor for rate of heat generation
- $f(\sigma_{ij})$ Function whose state variable is stress tensor
 - f^B Column matrix of nodal body forces of element
 - f^I Column matrix of nodal initial forces of element
 - f^S Column matrix of nodal forces of element

- F Yield function
- $\mathbf{F}$ Global matrix of internal forces
- h^c, h^r Heat transfer coefficient for convection and radiation
- h^{cr} Combined heat transfer coefficient
- $J_{k,M}$ Deformation gradient
- J_2' Second invariant of deviatoric stress
- $\mathbf{J}$ Jacobian operator matrix
- k Work Hardening function
- k_{z_i, k_n} Thermal conductivity in z_i and surface normal direction
- $\mathbf{k}$ Matrix of material thermal conductivity
- $\mathbf{K}$ Global stiffness matrix
- $\mathbf{K}^L, \mathbf{K}^{NL}$ Global linear and nonlinear stiffness matrix
- N Shape functions or interpolation polynomials
- N_{step} Number of subincrements in yield stress projection
- q_{z_i} Heat flow conducted per unit area in z_i direction
- Q Rate of heat generation
- $\hat{Q}$ Rate of heat generation used in finite element analysis
- R Total forces on body
- $\mathbf{R}$ Global matrix of external forces
- S Surface bounding volume V
- S_{ij} Deviatoric stress tensor
- $\mathbf{S}$ Matrix of 2nd Piola-Kirchhoff stress components
- $\mathbf{S}^d$ Matrix of deviatoric stress components
- t Time
- T Temperature
- $\mathbf{T}$ Global column matrix for nodal temperature
- u_i Displacement vector
- $\mathbf{u}$ Global column matrix of nodal displacements
- $\mathbf{u}^e$ Column matrix of nodal displacements for element
- V Volume or domain of body
- v Volume of element
- v_i Velocity of point
- z_i Coordinates of point in global reference system

Chapter 1

Problem Identification

1.1 Introduction

Fusion welding is the method of choice for assembling most large metal structures. Such structures are: ships, bridges, nuclear reactors, pipelines, trains, and cars. Safety and economy are important issues. In simple terms, a fusion weld is produced by moving an intense localized heat source along the joint. The composition, energy, and position of the weld must be carefully controlled to achieve the desired weld quality. Often filler metal is added. This complex process which mimics the entire steel making process in a volume of roughly one cubic centimeters has largely been developed by experiment ; i.e., trial and error. The process is so complex that, to date, mathematical modelling has contributed little to modern welding technology.

In particular, the heat source generates a transient temperature cycle that has important consequences. It modifies the microstructure by recrystallization, grain growth and phase transformations such as ferrite to austenite to martensite. The microstructure controls the thermal and mechanical properties of the weldment. The transient temperature cycle causes thermal expansion, stress and strain. These usually cause plastic deformation in the weld neighbourhood and result in residual stress and strain ; i.e., when the weldment cools a stress remains and the structure is distorted from its original shape.

Distortion is the bane of fabricators because it increases their costs and delays production. It is managed by a variety of techniques including balanced welds, rigid jiggling, and preweld distortion. Indeed flame straightening and flame bending use localized heating to distort structures into desirable shapes. In most cases, this is done empirically.

Residual stresses affect the in-service performance of welded structures. In

particular, low temperature brittle fracture, fatigue, stress corrosion cracking, and buckling can be significantly aggravated by residual stresses in weldments. In many structures, thermal stress relief is not feasible; when it is, it is expensive.

The goal of this thesis was to develop and demonstrate a rigorous methodology for thermo-plastic analysis of the residual stress and distortion in real welds in real structures. A high priority was given to building upon the fundamental laws of continuum mechanics. Ad hoc empirical procedures were discouraged. The complexity of the process can only be managed by computerizing the model. Nonlinear transient finite element analysis provided an ideal numerical method for solving the equations of continuum mechanics for the complex geometries of welded structures, the temperature-dependent and history-dependent constitutive equations, and the complex boundary conditions associated with welding arcs. The software should be extensible to grow as new knowledge becomes available. The software design should be capable of shielding the users from procedures and data structures that need not concern them.

This thesis is unique in that it is part of a larger project that attempts to develop a general purpose finite element analysis code specifically for the analysis of welds and welded structures. It draws upon continuum mechanics, finite element analysis, computer science and welding technology.

To demonstrate the capability of the system, the distortion due to edge welds on rectangular bars of AISI 1020 structural steel and AISI 304 austenitic stainless steel are analysed and compared with experiment. To a large extent this resolves the mystery of why stainless steel weldments distort roughly three times as much structural steel weldments.

1.2 Review of past work in weld analysis

In 1835 Duhamel derived an equation for elasticity problems including the effect of temperature. If the temperatures are high then most problems require a two-dimensional (2D) or three-dimensional (3D) thermal stress and strain analysis of a nonlinear material with temperature-dependent properties. The investigations of inelastic response due to the effects of temperature began in the 1960's. The development of the isothermal plasticity solutions have been given some attention since then. In contrast, very little attention has been given to nonisothermal inelastic

behaviour. In the few instances in which such behaviour was modelled, the material properties were temperature-independent.

Most existing knowledge of stresses and deformation due to welding is based on weld experiments. Attempts were made to formulate an analytical method to predict stresses and deformations, but these depended on highly simplifying assumptions. The considerable progress in predicting residual stress and strain in recent years is due to the use of finite element analysis. The outstanding contributions to the thermo-mechanical analysis of the welding process are reviewed below.

A broad account of the theoretical techniques for calculating elastic and inelastic thermal stresses in a solid object can be found in the texts by Boley and Weiner [1] and Burgreen [2].

Okerblom [3] presented a simplified approach to predict weld distortion. For example, the longitudinal contraction due to a single weld pass was calculated by considering only the sequential deformation of a transverse strip. The material was assumed to have linear elastic-perfectly plastic stress-strain behaviour. The yield stress was assumed to be constant up to a certain temperature and zero above that. Other material parameters were considered temperature-independent. The equations for deflection were derived using the compatibility condition and stress equilibrium in the longitudinal direction. Measured deformation of steel bars were quoted in support of the derived relationship. However, the theoretical proof cannot be demonstrated.

Hogan [4] measured camber in stainless steel and mild steel due to edge welding. The results were rationalized in terms of the thermo-mechanical model suggested by Okerblom. The measured deflections in the stainless steel bar were much greater than those predicted by Okerblom's theory, which predicted similar camber for stainless steel and mild steel. This is completely contrary to experimental work and fabricator's experience. Some of the terms neglected by Okerblom were added by Hogan, however these modifications did little to improve the agreement between the experimental and the predicted values.

The paper by Hibbitt and Marcal [6] describing simulation of gas-metal arc welding, represented the initial step in the development of numerical techniques to calculate both temperatures and residual stresses in welds using the finite element

method. The finite element model for the thermal analysis was developed by uncoupling the thermal and mechanical aspects of the problem. The energy equation included convection and radiation boundary conditions. The thermal properties were taken as temperature-dependent. The time integration of the model used a Crank-Nicholson operator. The phase change and latent heat phenomena was considered by an increased specific heat at solid-liquid interfaces. Modelling of the heat input was mentioned as the most critical parameter in a welding analysis. However in the example solved later, a very simplistic model of the flux input was considered. The deposition of flux and filler material was incorporated by adding several elements per step.

The mechanical response was based on small strain, large displacement theory of incremental finite element analysis. The stress-strain relation included dependence of the stress on temperature variation of material properties. The paper did not include any constitutive equations for plasticity. Also, without a detailed explanation of the temperature-dependence of the yield criterion it is not possible to judge the rigor of the formulation. They agreed that the stress accumulation leads to degradation of the solution accuracy and that it should be calculated in summed strain increments.

Several numerical examples were given, including a finite element solution of a circular bead on plate weld on two high strength steel plates. The heat input was modelled as a three stage step function. The fusion zone was predicted with fair accuracy using an axi-symmetric model thermal analysis. It was noted that the heat input rate had a significant effect on the peak temperature.

The predicted residual stress was not in complete agreement with the experimental results in one of the plates. In the other case mentioned, agreement could be obtained only in the peak stress value. It was observed that creep was useful in modelling the stress relief process. Time-independent plasticity theory with temperature-dependent properties was demonstrated as an appropriate approach in solving the weld problem.

Friedman [7] proposed a model for calculating temperatures, stresses and distortions due to welding processes. Friedman first reviewed the analytical models used in predicting transient temperatures in the welding process. Detailed studies of heat sources moving at constant speed on the surface of a weld plate of infinite

extent were made. He pointed out several limitations of point source or line source models.

Noting the advantages of using the finite element method in transient heat conduction analysis, the method was used to find the transient temperature distribution for a butt weld in a relatively thin plate. A radially symmetric normal distribution of heat flux on the weld metal surface was used to model the heat input of the welding arc. Based on the quasi-stationary condition, a 2-D cross-sectional analysis was done using isoparametric elements. Temperature-dependent properties were used and the transient temperature distribution was found in incremental time steps.

Using the transient temperature distribution the thermo-mechanical response was calculated with 2-D finite elements. In contrast to earlier work, a plane stress analysis was done using fewer steps in the stress analysis than in the heat flow analysis. The meshes used in the analyses differ but the technique for mapping the temperature was not reported. The material was postulated to behave as an isotropic elasto-plastic strain hardening continuum. The mechanical material properties were temperature-dependent. A simplistic approach was taken in calculating the incremental plastic strain. The crucial details of stress or strain rate (increment) definition were not reported. The author provided a useful discussion on the reduction of the plastic strain hardening due to annealing above the recrystallization temperature. The results of the analysis showed maximum residual plastic strains on the interface of the molten or fusion zone and heat affected zone (HAZ). The longitudinal residual stress was greatest in the weld metal and heat affected zone. The author claimed the results to be in agreement, qualitatively, with the results of Hibbitt and Marcal [6]. As discussed earlier, the results of reference [6] were not claimed to be in agreement with experimental results.

Metals can exhibit phase changes during cooling which are sensitive to the cooling rate. Argyris *et al* [8] formulated a thermo elastic-visco plastic constitutive model to accommodate this rate dependency. They used this model in calculating residual stresses and deformation due to welding in a rectangular steel plate. The governing equations of the heat flow analysis included mechanical coupling. The thermal driving forces were composed of convective and radiative boundary conditions. Time integration was performed using the fully implicit backward algorithm.

This algorithm was predicted to be more stable.

Thermo-elastic behaviour was described by defining the problem in terms of stress and strain rates and properly accounting for the temperature dependence of the elastic material matrix. The definition of thermal strain rate was shown in more detail than in previously mentioned studies ; e.g., α depends on temperature. Later, the general temperature dependence of the elastic material was restricted to Young's modulus since Poisson's ratio was taken to be temperature-independent. A condition which approximated the critical time step was given in terms of the change in elastic modulus.

Visco-plastic behaviour strain hardening was used. The von Mises yield condition was used to determine the beginning of viscous flow. The associated flow rule was used to define the visco-plastic strain rate. Essentially, the inelastic strain term depended on two material parameters, uniaxial yield stress and fluidity. This simple form of flow mechanism did not incorporate any form of work hardening, Bauschinger effect etc.

Thermal analysis of the welded plate was divided into two parts. Two dimensional thermal analysis of the half plate (an in-plane analysis, uncoupled) was done by prescribing a high temperature over a few nodes near the electrode. The effects of weld deposition and radiation were neglected. Temperature-dependent properties for mild steel were used. This thermal analysis was uncoupled. The results of an in-plane heat flow analysis were used to obtain boundary conditions for the transient temperature distribution in the plate cross-section. This was accomplished by prescribing the temperature history (obtained from planar analysis) of a single node in the weld pool. The isotherms shown for the cross-section suggest the heat is not modelled as single point source. The cross sectional analysis was said to be coupled.

Using a thermo elastic-viscoplastic formulation for the weld problem, it was shown that the mixed model of hyperelastic-viscoplastic behaviour is better. That is, the strain rate depends not only on the total strain rate but also on the material constitutive matrix rate and the thermal expansion coefficient rate.

Results of the thermo-elastic-visco plastic analysis of the weld were presented for various time steps. They were compared to experimental data in which the scatter was large. The results showed that the thermo mechanical coupling had little

influence. The vertical deflection was oscillatory in some regions which was thought to be due to the coarse mesh. Residual displacements were larger in transverse direction than in the through-thickness direction. Residual longitudinal stress was almost uniform (tension) at the plate surface. The phase transformation seemed to change the stress history but its effect dampened gradually. If compared at the steady state conditions, the effect was very small.

In comparison to the time-independent formulations, the visco plastic formulation in the weld analysis used by Argyris *et al* did not show major differences in theoretical and computational behaviour.

Andersson [9] has published analytical (using the finite element method) results of a transient temperature and stress distribution for a butt welded steel plate. In the uncoupled thermal model, the latent heat effects were considered by modifying the heat capacity of the material. The heat source was modelled as a uniform flux deposited in three steps. The temperature distribution in the plate cross-section was obtained using 2-D triangular elements. The computed temperatures were shown to be in very good agreement with the experimental temperature measurements. It should be noted that all comparisons shown were outside the HAZ rather far from the arc. The stress distribution was calculated using a 2-D generalized plane strain condition. The von Mises' yield condition with isotropic hardening was the material stress-strain relation. Only the final stress rate formula was given in terms of the deviatoric stress rate. The phase transformation effect was incorporated in the temperature-dependent material properties with dilation (coefficient of thermal expansion) curves. Other material parameters were also taken as temperature-dependent.

The residual stress values were much higher than the measured values. The validity of the plain strain assumption and exclusion of Bauschinger's effect were stated as possible causes of the discrepancy. In the final conclusions it was suggested that the complex loading history should be given more importance and the full three-dimensional state of stress and strain should be considered.

Ueda and Nakacho [10] extended the theory of thermo-elasto-plastic analysis by considering the effects of temperature dependence and plastic history dependence of the material properties. They introduced a combined model of isotropic and kinematic work hardening. This work was similar to that of Allen and Haisler [11].

Unfortunately in both works, the presented equations are hard to follow for two reasons. Some of the steps (especially the incremental forms and the total differential of the equations) seem to violate mathematical rules (at least no explanation was given). Only the final form of each of the crucial equations was given. Hence it was not possible to judge the validity of the equations.

Ueda's work accounts for the temperature dependence of the yield criterion and temperature dependence of the material properties. The reference presented the basic finite element equations involved in the thermo-elasto-plastic analysis, the details of the numerical algorithm were not presented.

1.3 Scope of thesis

The aim of this thesis is to establish appropriate stress and strain measures for modelling thermo-mechanical behaviour. The equations of continuum mechanics are to be formulated in these measures. The temperature dependence of the material properties and all the involved functions will be taken into account as completely as possible. The plasticity aspects of the equations will be based on the thermal effects. The equations will take into account the temperature-dependent work hardening properties of the materials. The equations for calculating the displacement, stress and strain rates will be derived.

A finite element code for computing displacement, stress and strain in a weld problem which can model material as well as the geometrical nonlinearity will be developed on the basis of the established equations. A procedure to accommodate the incremental load (time) steps and iterative solution of the nonlinear equations will be developed. Techniques to aid the iterative solution will be explored.

Using several ideal test cases, the validity of the software will be demonstrated. Finally an actual weld case will be analysed using the software and the results will be compared with experimental data.

Chapter 2

Development of Fundamental Equations

In developing the basic equations for thermo-elasto-plastic stress analysis, the principles of mechanics of a continuous medium are used. For this purpose extensive reference is made to the text by Malvern [12].

2.1 Continuum

The mechanics of a continuous medium are concerned with the stresses in solids, liquids as well as gases and the deformation or flow of these materials. The continuous medium implies that all mathematical functions and their derivatives are continuous (except at the interior points separating regions of continuity). The continuum theory disregards the molecular nature of the matter. Use of the continuum concepts requires a unifying observational knowledge to form a working theory. To define the nature of the medium further assumptions such as homogeneity and isotropy of the material are required. It will be assumed that the matter of an elastic solid body is homogeneous and isotropic. The assumption of isotropy can be relaxed without increasing the mathematical complexity.

2.2 Body and forces

Consider a body of a given mass instantaneously occupying volume V bounded by surface S and acted upon by external forces. The external forces acting at any instant on a body can be of two kinds. Body forces f^B act on the elements of volume or mass inside the body. The traction forces f^S are contact forces acting on the body at its bounding surface.

The body force per unit mass acting on an infinitesimal volume element dV of the body is denoted by vector b . The body force on the volume dV is given by $\rho b dV$. The vector sum of the body forces f^B is given by the integral over the volume.

$$\int_V f^B dV = \int_V \rho b dV \quad (2.1)$$

The surface force per unit area exerted on an element dS of surface is the vector f^S . The vector sum of the surface forces will be:

$$\int_S f^S dS \quad (2.2)$$

2.3 Kinematics of continuum

There are several descriptions of the motion of a body in common use. In the continuum where stress is studied, referential (Lagrangian) or spatial (Eulerian) descriptions are commonly used. In the referential description (Lagrangian) of the medium, at a given time t , the independent variables are the particle position coordinates x_i in a arbitrarily chosen reference configuration. In the Eulerian or spatial description, the position of the particle at a given time is studied with reference to a region of space.

The kinematics of thermo-elasto-plastic deformation can be expressed more conveniently and easily in the Lagrangian description. If the reference state is taken as the initial configuration at time $t = 0$ with base temperature 0T , the formulation is called the Total Lagrangian formulation. In the Updated Lagrangian formulation the reference configuration is continuously updated. For example in a finite strain formulation, the variables at time $t + \Delta t$ refer to the last deformed configuration at time t [8,13,14, 15,16,17] or time $t + \frac{\Delta t}{2}$ [18,19]. Formulations which use rate-independent variables and do not include finite strain generally adopt the Total Lagrangian reference configuration. There are also instances of using the Total Lagrangian configuration for finite strain formulations [13,20,21,22].

In solid mechanics the concept of elasticity is often used. In elasticity, the body will return to its original undeformed state when unloaded. Here it is natural to adopt the Total Lagrangian description. From the numerical or programing point of view, the choice of the Total Lagrangian or Updated Lagrangian description poses no difficulty. The book keeping and calculation of increments of certain variables is difficult in the Updated Lagrangian configuration. It has been pointed out in previous works [23,24], that using the rate of a quantity solves the book keeping problems. Using proper definitions of the tensors, it is possible to use the same basic equations for both formulations. In the present work the variables and their

Figure 2.1: Motion of body in Total Lagrangian reference system.

rates will be presented in the Total Lagrangian configuration. The place occupied by the body at time $t = 0$ is taken to be the reference configuration. At $t = 0$, the body is assumed as strain free but initial stresses may exist. There is no apparent difficulty in adopting the Updated Lagrangian configuration if all the variables in an equation are tagged with the time at which they are evaluated.

The motion of a body in a stationary ($t = 0$) Cartesian coordinate system is shown in Figure 2.1. The point p in the body, at time $t = 0$ has coordinates ${}^0x_i, i = 1, 2, 3$ and coordinates ${}^t x_i, i = 1, 2, 3$ at time t . The notation for the displacement is ${}^t u_i, i = 1, 2, 3$. The top left prefix shows the time at which a variable is evaluated. The bottom left prefix shows the reference configuration. From now onwards a missing top and bottom prefixes should be taken as t and 0 respectively. Also, the position of point p at time t is:

$${}^t x_i = {}^0 x_i + {}^t u_i \quad (2.3)$$

2.4 Strain and deformation

The strain tensor ϵ_{ij} used in the present work is the Green-Lagrangian strain tensor which was introduced by Green (1891) and St. Venant (1844). The rectangular Cartesian components of the Green-Lagrangian tensor are given as:

$${}^t \epsilon_{ij} = \frac{1}{2} \left[\frac{\partial^t x_k}{\partial^0 x_i} \frac{\partial^t x_k}{\partial^0 x_j} - \delta_{ij} \right] \quad (2.4)$$

or

$${}^t \epsilon_{ij} = \frac{1}{2} [{}^t C_{ij} - \delta_{ij}] ; {}^t C_{ij} = \frac{\partial^t x_k}{\partial^0 x_i} \frac{\partial^t x_k}{\partial^0 x_j} \quad (2.5)$$

where C_{ij} is called the Green-deformation tensor. It is a symmetric tensor. The deformation tensor can be expressed in terms of the deformation gradient, ${}^t J_{k,M}$, given as

$${}^t J_{k,M} = \frac{\partial^t x_k}{\partial^0 x_M} \quad (2.6)$$

In the right hand side (r.h.s.) of Equa. 2.4-6, the missing bottom prefix should be taken as zero, which means that the Green deformation tensor and the deformation gradient are evaluated in the undeformed configuration. If the deformation

Figure 2.2: Stress vectors.

tensor, C_{ij} , and the gradient, J_{kM} , refer to the current configuration ; i.e., time t , then the strain will be the Almansi strain tensor, ϵ_{ij} (for details see [12]):

$$\epsilon_{ij} = \frac{1}{2} \left[\delta_{ij} - \frac{\partial^t x_k}{\partial^t x_i} \frac{\partial^t x_k}{\partial^t x_j} \right] \quad (2.7)$$

Using Equa. 2.3 and Equa. 2.4, the Green-Lagrangian strain tensor can be written as:

$$\epsilon_{ij} = \frac{1}{2} \left[\frac{\partial u_i}{\partial^0 x_j} + \frac{\partial u_j}{\partial^0 x_i} + \frac{\partial u_k}{\partial^0 x_i} \frac{\partial u_k}{\partial^0 x_j} \right] \quad (2.8)$$

For the convenience of numerical procedures, the strain tensor is rewritten as:

$$\epsilon_{ij} = \epsilon_{ij}^L + \epsilon_{ij}^{NL} \quad (2.9)$$

where ϵ_{ij}^L is the small strain tensor containing linear terms and ϵ_{ij}^{NL} is the nonlinear strain tensor containing quadratic terms.

In classical linearized small strain theory, the ϵ_{ij} is taken as ϵ_{ij}^L . This means that, if the derivatives of the displacement are small compared to unity, their product can be neglected. Hence ϵ_{ij}^{NL} is neglected in small strain formulations.

The components of the strain tensor is given as follows:

$$\epsilon_{ij}^L = \frac{1}{2} \left[\frac{\partial u_i}{\partial^0 x_j} + \frac{\partial u_j}{\partial^0 x_i} \right] \quad (2.10)$$

e.g.

$$\epsilon_{11}^L = \frac{\partial u_1}{\partial^0 x_1} \quad ; \quad \epsilon_{12} = \frac{1}{2} \left(\frac{\partial u_1}{\partial^0 x_2} + \frac{\partial u_2}{\partial^0 x_1} \right)$$

and

$$\epsilon_{ij}^{NL} = \frac{1}{2} \left[\frac{\partial u_k}{\partial^0 x_i} \frac{\partial u_k}{\partial^0 x_j} \right] \quad (2.11)$$

e.g.

$$\epsilon_{11}^{NL} = \frac{1}{2} \left[\left(\frac{\partial u_1}{\partial^0 x_1} \right)^2 + \left(\frac{\partial u_2}{\partial^0 x_1} \right)^2 + \left(\frac{\partial u_3}{\partial^0 x_1} \right)^2 \right]$$

$$\epsilon_{12}^{NL} = \frac{1}{2} \left[\frac{\partial u_1}{\partial^0 x_1} \frac{\partial u_1}{\partial^0 x_2} + \frac{\partial u_2}{\partial^0 x_1} \frac{\partial u_2}{\partial^0 x_2} + \frac{\partial u_3}{\partial^0 x_1} \frac{\partial u_3}{\partial^0 x_2} \right]$$

2.5 Stress

A vector of surface force per unit area acting on a small element is called a stress vector. The stress vectors on the three planes perpendicular to the coordinate axis,

Figure 2.2, constitute the nine rectangular components of the second order stress tensor, σ_{ij} .

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (2.12)$$

The stress tensor σ_{ij} is symmetric, hence only six of the nine components are necessary to represent the stress on the element.

$$\sigma_{12} = \sigma_{21} \quad \sigma_{13} = \sigma_{31} \quad \sigma_{23} = \sigma_{32} \quad (2.13)$$

The equation of motion or equilibrium must be satisfied in the deformed coordinates. If the element is described in the current coordinates, the stress tensor is called the Cauchy stress tensor, τ_{ij} . The Cauchy stress tensor is most appropriate for stating the equilibrium of the body. As mentioned earlier, the present formulation refers all quantities to the initial or undeformed configuration in the Total Lagrangian formulation. To be consistent, all the tensors appearing in an equation should be referred to the same configuration. Hence, one needs another stress tensor which is defined as force per unit area in the undeformed coordinates. Later, the total stress will be defined as the accumulation of incremental stress. Calculating an increment of Cauchy stress is pointless, particularly for finite strain. Hence another stress tensor the 2nd Piola-Kirchhoff is used in the present formulation. The 2nd Piola-Kirchhoff stress tensor, σ_{ij} , is very appropriate for this purpose. It is possible to express the 2nd Piola-Kirchhoff stresses in terms of the Cauchy stress or *vice-versa*:

$$\tau_{ij} = \frac{{}^t\rho}{{}^0\rho} J_{i,M} \sigma_{MN} J_{N,j} \quad (2.14)$$

and

$$\sigma_{ij} = \frac{{}^0\rho}{{}^t\rho} J_{i,M} \tau_{MN} J_{N,j} \quad (2.15)$$

where ${}^0\rho$ and ${}^t\rho$ are the densities at initial time 0 and current time t respectively.

When the displacement and its gradients are small the 2nd Piola-Kirchhoff stress is approximately equal to the Cauchy stress. Since $\frac{{}^0\rho}{{}^t\rho} \simeq 1$, $J \simeq 1$;

$$\sigma_{ij} \simeq \tau_{ij} \quad (2.16)$$

2.6 Equation of motion and equilibrium

Using the conservation of momentum principle, the Cauchy equation of motion at each point of an arbitrary volume V is given as:

$$\frac{\partial_t \sigma_{ij}}{\partial_t x_i} + \rho b_i = \rho \frac{dv_i}{dt} \quad (2.17)$$

where ρ is density, b_i is body force per unit mass and v_i is velocity of the point. In the special case of static equilibrium of the medium, the acceleration dv_i/dt is zero. Defining f_i^B to be the body force per unit volume, $f_i^B = \rho b_i$, gives

$$\frac{\partial_t \sigma_{ij}}{\partial_t x_i} + \rho f_i^B = 0 \quad (2.18)$$

These are three first order partial differential equations of equilibrium at a point in the medium. The overall equilibrium of the body in deformed coordinates may be expressed by means of the virtual work principle. A body in a certain equilibrium configuration is given an infinitesimal virtual displacement δu_i from the equilibrium configuration. It is assumed that the three functions δu_i have continuous first partial derivatives on those parts of the boundary surface where the actual displacement, u_i , is prescribed. δR is the virtual work of the external surface tractions f_i^t and body forces f_i^B (per unit volume). If the external forces are assumed to remain unchanged during the virtual displacement,

$$\delta^t R = \int_V f_i^B \delta u_i d_i V + \int_S f_i^t \delta u_i dS \quad (2.19)$$

The equilibrium of a body is given by equating the virtual external work with virtual internal work.

$$\int_V {}^t r_{ij} \frac{\partial(\delta u_i)}{\partial_t x_j} d_i V = \delta^t R$$

using Equa. 2.7 and Equa. 2.20 the above equations becomes

$$\int_V {}^t r_{ij} \delta^t \epsilon_{ij} d_i V = \int_V f_i^B \delta u_i d_i V + \int_S f_i^t \delta u_i dS \quad (2.20)$$

The strain associated with the infinitesimal virtual displacements are infinitesimal. In this case, using ${}^0 \rho dV = {}^t \rho dV$ Equa. 2.20 can be written as [25],

$$\int_V {}^t \sigma_{ij} \delta^t \epsilon_{ij} d_0 V = \int_V f_i^B \delta u_i d_0 V + \int_S f_i^t \delta u_i dS \quad (2.21)$$

It can be observed that the six independent unknown stresses σ_{ij} can not be determined by the three equations of motion *Equa. 2.19*, even if all forces are known in *Equa. 2.20*. The constitutive equations defining the nature of the material will provide additional equations. Defining a problem using the equilibrium and constitutive equations and solving these for the unknowns, requires additional boundary conditions.

2.7 Constitutive equations

Different materials produce different reactions to the applied loads. The equations characterizing the material are constitutive equations (generally known as stress-strain relations). They describe the macroscopic behaviour resulting from the internal constitution of the material. The equations are formulated to describe the ideal material response and to approximate the physical observations over a suitable range.

In an ideal elastic body the stress depends upon deformation (strain) of the body from the natural state. The relation between stress and strain may be nonlinear, but a one-to-one relationship between stress and strain is assumed. If the deformation is accompanied by a change in temperature then the equations may contain coefficients that depend on the temperature. In most applications the change in the temperature is independent of the change in strain or stress [1]. Hence in the present formulation thermal and mechanical coupling is not used.

Almost all materials (metals) used in structures and machine parts exhibit a linear relationship between stress and strain up to certain limit, and a nonlinear relationship thereafter. The nonlinear relationship (plasticity) will be given in the next chapter. The linear relationship gives the classical elastic constitutive equations, often called the generalized Hooke's law. In these equations the thermal effects are neglected ; i.e., isothermal elastic constitutive or generalized Hooke's equations are:

$$\sigma_{ij} = D_{ijmn} \epsilon_{mn} \quad (2.22)$$

where D_{ijmn} is a fourth order tensor, called the Material tensor.

As mentioned earlier, both σ_{ij} and ϵ_{mn} are symmetric and have only six independent components. Hence the above equation has only six independent equations and 36 constants in the material tensor D_{ijmn} instead of nine equations and 91 con-

stants. For hyperelastic (or Green elastic) materials (i.e. elastic material for which a strain energy function exists), the 36 constants are reduced to 21 independent constants since D_{ijmn} is symmetric. Finally for an isotropic material, the isotropic Hooke's law is

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \quad (2.23)$$

In Equa. 2.23, λ and μ (or G , shear modulus) are Lamé's constants which depend only on two material properties; namely, the elastic modulus E and Poisson's ratio ν , as follows:

$$\mu = G = \frac{E}{2(1+\nu)}; \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \quad (2.24)$$

Hence D_{ijmn} has only two independent elastic constants.

When a constrained body is subjected to a temperature change from the reference state, it may develop stress without strain as would a totally constrained plate subjected to a temperature change. On other hand, if the plate is free and the temperature is uniformly raised or lowered, the plate will expand or contract but no stress will be developed in the plate. In most practical cases the temperature changes nonuniformly over the body. This temperature change causes stress. In a welding process, a small region of the structure is heated to high temperatures. Hence temperature gradients are the major forces causing the stresses (thermal stresses). The generalized Duhamel-Neumann form of Hooke's law gives the following constitutive relationship:

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} - \frac{E\alpha}{(1-2\nu)} (T - T_0) \delta_{ij} \quad (2.25)$$

where α is the linear coefficient of thermal expansion. Only $i = j$ terms of the stress tensor will have a contribution from the thermal stress. For $i = j$ Equa. 2.24 becomes

$$\sigma_{ii} = (3\lambda + 2\mu) \epsilon_{ii}$$

Noting that $3\lambda + 2\mu = E/(1-2\nu)$, one can write

$$\sigma_{ii} = (3\lambda + 2\mu) (\epsilon_{ii} - \alpha(T - T_0))$$

Hence Equa. 2.23 is stated as:

$$\sigma_{ij} = D_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^T) \quad (2.26)$$

where ϵ_{ij}^T , the thermal strain tensor, is defined as:

$$\epsilon_{ij}^T = \alpha(T - T_0) \delta_{ij} \quad (2.27)$$

and thermal stress as:

$$\sigma_{ij}^T = (3\lambda + 2\mu) \alpha (T - T^0) \delta_{ij} \quad (2.28)$$

Usually $\frac{E\alpha}{(1-2\nu)}$ term is assumed to be independent of ϵ_{ij} . In the isothermal case, $D_{ij,ma}$ is considered independent of the temperature change ($T - T^0$). The magnitude of the change in a material property can be substantial, particularly at high temperatures. In the welding process the fusion zone is heated up to, and beyond, the melting point temperature and then cooled to room temperature. Therefore temperature dependence of $D_{ij,ma}$ must be considered in an thermo-elastic or an thermo-elasto-plastic analysis. The time derivative or rate form of the constitutive equations is required in an incremental solution. The total derivative of the Equa. 2.27 is

$$\dot{\sigma}_{ij} = D_{ij,ma} (\dot{\epsilon}_{ma} - \dot{\epsilon}_{ij}^T) + \dot{D}_{ij,ma} (\epsilon_{ij} - \epsilon_{ij}^T) \quad (2.29)$$

where

$$\dot{\epsilon}_{ij}^T = \alpha \dot{T} \delta_{ij} + \dot{\alpha} (T - T^0) \delta_{ij} \quad (2.30)$$

and

$$\dot{D}_{ij,ma} \approx \frac{dD_{ij,ma}}{dT} \dot{T} \quad (2.31)$$

In taking temperature-dependent material properties, one must properly define the coefficient of thermal expansion α . In various material handbooks, two types of coefficient of thermal expansion, $\alpha(T)$ and $\bar{\alpha}(T)$ are reported. $\bar{\alpha}$ is the instantaneous coefficient of thermal expansion at temperature T . In metallurgical literature (and used exclusively in the present work) it is customary to use another definition of the coefficient of thermal expansion :

$$\alpha = \frac{1}{T - T^0} \int_{T^0}^T \bar{\alpha} dT \quad (2.32)$$

where T^0 ($= T^0$ here) is the reference temperature. The definition of thermal strain in Equa. 2.31 is only true if α is used. If $\bar{\alpha}$ is used, the thermal strain is defined as:

$$\epsilon_{ij}^T = \delta_{ij} \int_{T^0}^T \bar{\alpha} dT$$

2.8 Compatibility Equations

The uniqueness principle states that the exact solution will be given when a single-valued displacement field satisfies the conditions of continuity and equilibrium.

The compatibility condition imposes the requirement that the displacement components be single valued continuous functions. For an assumed strain distribution, the six strain displacement equations may over determine the three displacement components. In order to integrate six equations, St. Venant's compatibility equations must be satisfied by the strain distribution. However for a given displacement, evaluating the strains offers no difficulty. In the present formulation the displacement functions are included as degrees of freedom and hence the compatibility equations are not needed to solve the unknowns. The three displacements, six stresses and six strains can be computed using three equilibrium equations, six stress-strain equations and six strain displacement equations. In a formulation where stresses are primarily unknowns, six compatibility equation will be needed to solve the problem (see [12, 26]).

2.9 Boundary conditions

In structural mechanics the boundary conditions are either displacement or traction boundary conditions. In the displacement boundary conditions, three components of u_i , are prescribed on the boundary or part of the boundary or at a point. Similar statements can be made for the three traction components f_i^t . For mixed boundary conditions either u_i or f_i^t can be prescribed at a point or on a boundary, but not both. Prescribing point boundary conditions may not be meaningful in a mathematical sense. However in engineering and particularly in numerical formulation it is used frequently. The r.h.s. of the equilibrium, Equa. 2.21, should be modified to accommodate the point forces, $f_i(k)$, and initial stress loads, f_i^t , due to existing stresses at time $t = 0$ in the body. Also, the thermal loads are considered as initial (thermal) stresses. The r.h.s. of Equa. 2.21 is written as:

$$\delta R = \int_V f_i^B \delta u_i dV + \int_S f_i^S \delta u_i dS + \int_V f_i^t \delta u_i dV + \sum_k f_i^P(k) \delta u_i^t \quad (2.33)$$

where $f_i^P(k)$ are components of the point or concentrated forces at point k .

Chapter 3

Nonlinear Equations

3.1 Nonlinearity

In the development of the fundamental equations, the relations between stress, strain, displacement and other terms have been stated. The boundary conditions were stated for displacements and tractions. The constitutive equations assume that the relation between the stress and strain is linear. However, in many real situations the stress-strain relations may be nonlinear in a time dependent or time-independent way. The change in temperature during loading may change some of the parameters in the constitutive equation. It has already been stated that the strain-displacement relations have quadratic (non linear) terms. It is possible that the displacement may alter the distribution or magnitude of loads or the distribution itself. During loading, displacement boundary conditions may change (a gap may open or close). The interdependence of terms in the fundamental equations and boundary conditions obviously make the equations nonlinear.

When phenomena such as plasticity, creep or other complex constitutive relations occur, the stress-strain relation is path dependent. A given state of stress can be produced by many different strain histories. This type of nonlinearity is commonly referred to as material nonlinearity. The essential computational problem of material nonlinearity is that the equilibrium equation must be written using material properties that depend on the strain history, but the strain is not known in advance. In addition, the solution must follow the actual path.

In practical problems involving high temperatures, such as welding, material often exhibits nonlinear behaviour. At high temperatures material parameters (E, ν, σ_y) usually change considerably. Nonuniform temperature distribution with high temperatures usually produces plastic strain. In a large class of solid mechanics problems neither stress nor strain significantly affects the change in temperature ; i.e., the effect of plastic work on temperature is negligible. Thus such nonlinearity

will not be modelled in the present work ; i.e., the heat flow equation is not coupled to the equation of motion. Since the transient heat flow equations are solved first, the temperature will be known when the temperature-dependent constitutive equations are needed to solve the equations of motion. Thus the change of temperature makes the relations temperature-dependent but does not increase the nonlinearity of the constitutive equations.

It is important to see that material nonlinearity does not necessarily mean that the equation of equilibrium must be rewritten for the linear case. By a trial-and-error process (iterations) the material parameters can be adjusted so that the constitutive law and equilibrium equation are satisfied.

Large displacements, rotations and strains introduce geometric nonlinearity. In this case the undeformed and deformed geometry differ significantly. The equilibrium equations must be written with respect to the deformed geometry, which is not known in advance. Hence, the equilibrium equations if they are written for the linear case must be modified. It has been observed that stress and strain measures are selected depending on whether the strains are large or small in the large displacement formulation. For the large rotation, large strain and large displacement case ; e.g., in rolling of sheet metal, it is preferred that the fundamental equations be stated in terms of the Jaumann stress rate and velocity strain or in terms of Cauchy stress and Almansi strain using the Updated Lagrangian reference configuration Equa. 2.21.

For the large displacement, large rotation and small strain case (finite displacement principle) the Total Lagrangian configuration, 2nd Piola-Kirchhoff stress tensor and Green-Lagrangian strain tensor are used in writing the equilibrium equations Equa. 2.21. The finite displacement principle is widely used in elastoplastic formulations for several reasons. In structural engineering, large strain cases seldom arise. Most of the incremental plasticity theories are valid only for an infinitesimal increment in plastic strain and requires incremental calculation of the strain and stress. For the present, the assumptions of classical plasticity for the finite displacement case are applicable. A temperature cycle involving high temperatures sometime causes considerable displacement. This phenomenon has been particularly observed in welding large and thick plates and in some experimental welding [4], but usually large strain is not observed in welding problems. Hence,

large strain and high strain rate are not considered in the present work.

When a body at high temperature is subjected to a continuous force or to a force for a considerable time, then the creep strains should be considered. In the welding process the temperatures are high, but the duration for which they remain high, at any point, is comparatively short. With the exception of the thermal load no other mechanical force is normally applied. Unlike the hot metal rolling process, only a small portion of the metal is heated at any given instant of the time. Hence the visco-plastic strain and the creep strains are assumed insignificant and are not modelled here. The visco plastic and creep strains could be included in an approximate manner. It has been shown that they do not alter the equations significantly [11,27,28].

The present work confines the geometric nonlinearity to that caused by large displacements, large rotations and small strain. In the material nonlinearity, plastic strain and temperature-dependence of material properties are considered.

3.2 Solution of nonlinear equations

The basic equation to be satisfied is Equa. 2.21, which expresses the equilibrium and compatibility requirements of the body at time $t + \Delta t$. As discussed earlier, the body can undergo large displacements and strain, and the constitutive equations can be nonlinear. Hence the equation cannot be solved directly. An approximate solution can be obtained by linearizing the integral equation and performing incremental integrations. It is proposed that the linearized incremental integrals should be solved iteratively. Hence the equilibrium equation is assumed to be satisfied at t . An approximate solution at $t + \Delta t$ is obtained by solving the linearized incremental equations using assumed constitutive equations ; e.g., using relations from time t . The basic assumption behind the linearization is that the motion between time t and $t + \Delta t$ is linear. Hence the rate of change of the displacement and strain is constant over the time increment Δt . In the incremental formulation all the variables are referred to a previously known equilibrium configuration.

Using the virtual work principle, an infinitesimal virtual displacement, $\delta \bar{u}_i$, is imagined to take place in an infinitesimal time interval δt , using equilibrium equation

Equa. 2.21

$$\int_V {}^t\sigma_{ij} \delta \dot{\epsilon}_{ij} dV = \int_V f_i^B \delta \dot{u}_i dV + \int_V f_i^I \delta \dot{u}_i dV + \int_S f_i^S \delta \dot{u}_i dS + \sum_k f_i^P(k) \delta \dot{u}_i \quad (3.1)$$

where $\delta \dot{\epsilon}_{ij} = \delta \epsilon_{ij} \delta t$ is the virtual rate of deformation.

As pointed out earlier the Total Lagrangian formulation is used. The stress and strain tensors refer to the same reference configuration (undeformed at time $t = 0$). In this way the appropriate variables at time $t + \Delta t$ can be obtained by adding their values at time t and their incremental values during the time interval Δt . Using the assumption of constant strain rate during the time interval Δt , the incremental stress, strain and displacement respectively can be written as:

$${}^{t+\Delta t}\sigma_{ij} = {}^t\sigma_{ij} + \Delta^t\sigma_{ij}; \quad \dot{\sigma}_{ij} = \frac{\Delta^t\sigma_{ij}}{\Delta t} \quad (3.2)$$

Strain

$${}^{t+\Delta t}\epsilon_{ij} = {}^t\epsilon_{ij} + \Delta^t\epsilon_{ij}^L + \Delta^t\epsilon_{ij}^{NL}; \quad \dot{\epsilon}_{ij} = \frac{\Delta^t\epsilon_{ij}}{\Delta t} \quad (3.3)$$

$${}^{t+\Delta t}u_i = {}^tu_i + \Delta^tu_i; \quad \dot{u}_i = \frac{\Delta^tu_i}{\Delta t} \quad (3.4)$$

The incremental linearized equation of equilibrium in the Total Lagrangian configuration using Equa. 3.1, Equa. 2.22 and the above equations is

$$\int_V {}^tD_{ijmn} \Delta^t\epsilon_{mn} \delta \dot{\epsilon}_{ij} dV + \int_V {}^t\sigma_{ij} \delta \dot{\epsilon}_{ij}^{NL} dV = \delta \Delta^t R \quad (3.5)$$

where

$$\delta \Delta^t R = \int_V \Delta^t f_i^B \delta \dot{u}_i dV + \int_V \Delta^t f_i^I \delta \dot{u}_i dV + \int_S \Delta^t f_i^S \delta \dot{u}_i dS + \sigma_k \Delta^t f_i^P \delta \dot{u}_i^P \quad (3.6)$$

The above equation approximates $\Delta^t\sigma_{ij} \approx {}^tD_{ijmn} \Delta^t\epsilon_{mn}$ or $\Delta^t\sigma_{ij} \approx {}^0D_{ijmn} \Delta^t\epsilon_{mn}$, these will lead to a Newton-Raphson or a modified Newton-Raphson iterative solution respectively. The virtual strains are considered infinitesimal. The external virtual work δR actually depends on the surface area and volume of the body in the current configuration. It is assumed that loading is independent of deformation. By keeping the time or loading steps small, the external work can be approximated using the surface area of the undeformed equilibrium state.

The incremental equations are solved (iteratively) to calculate the displacement rate, $\dot{u}_i$. Next, the incremental strain and incremental stress can be approximated using the strain-displacement relation and linear or nonlinear constitutive equations.

The internal work is calculated using the approximate ${}^{t+\Delta t}\sigma_{ij}$ and ${}^{t+\Delta t}\epsilon_{ij}$. The difference between the internal and external force is a residual or out-of-balance force, ψ .

$$\psi = \delta^{t+\Delta t} R - \int_V {}^{t+\Delta t}\sigma_{ij} \delta^{t+\Delta t}\epsilon_{ij} dV \quad (3.7)$$

The residuals are the error introduced by linearization of the equilibrium equation. The aim is to reduce the out-of-balance work; i.e., to reduce ψ to a tolerable limit. An iterative process is used for this purpose. In finite element analysis, some form of Newton-Raphson or quasi-Newton method is used as an iteration scheme. In the present work the Newton-Raphson method is used with variations. The calculated iterative displacement for i^{th} iteration found using the $\psi^{(i-1)}$ residual force is denoted as $u_i^{(i)}$.

Sometimes $u_i^{(i)}$ is accelerated and/or line search is performed. This aspect is discussed in the next section. For the next iteration, $i + 1$, the residual force will be calculated using Equa. 3.7. The linearized equations Equa. 3.5 are repeatedly solved for $i = 1, 2, 3, \dots$ until the solution converges or satisfies a specific convergence criterion (discussed later); i.e.,

$$\psi = \delta^{t+\Delta t} R - \int_V {}^{t+\Delta t}\sigma_{ij} \delta^{t+\Delta t}\epsilon_{ij} dV \simeq 0 \quad (3.8)$$

The ${}^{t+\Delta t}R$ is assumed to be independent of u_i , but the rest of the terms in the above equation depend on u_i . A Taylor series expansion of $\psi(u_i)$ about the solution of u_i (neglecting higher order terms) finally leads to a incremental equilibrium equation for the k^{th} iteration at time $t + \Delta t$ [29].

$$\int_V {}^tD_{ijmn} \Delta^t \epsilon_{mn}^{(i)} \delta \epsilon_{ij} dV + \int_V {}^t\sigma_{ij} \delta \epsilon_{ij}^{NL(i)} dV = \psi^{(i-1)} \quad (3.9)$$

where

$$\psi^{(i-1)} = \delta^{t+\Delta t} R - \int_V {}^{t+\Delta t}\sigma_{ij}^{(i-1)} \delta^{t+\Delta t}\epsilon_{ij}^{(i-1)} dV \quad (3.10)$$

The first iteration ($i = 1$) of each increment of load or time is special in the sense that the terms involving $i - 1$ are not obvious. This will be made clear in the next chapter. Depending on the choice of a convergence criterion and tolerance limit, the residual forces $\psi^{(i)}$ left at the end of time step $t + \Delta t$ may or may not be negligible. To maintain the equilibrium of the process at all stages, the residual forces, ψ , that remain at time t are added into tR .

3.3 Faster iteration using line search or step-length parameter

In a series of papers, Crisfield has recently presented a powerful and efficient methodology for solving nonlinear structural problems. In mathematical terms they are continuation methods capable of solving problems with strong nonlinearities and passing extremum or limit points [30 - 35].

Until recently, the modified Newton-Raphson method has been the dominant procedure for solving nonlinear finite element equations. In general, either the modified Newton-Raphson method or the Newton-Raphson method is used. Irons and Elsawaf [36] have shown significant advantages in applying line search to the standard modified Newton-Raphson method. Bathe *et al* [29] have studied Aitken's acceleration scheme to speed convergence of modified Newton-Raphson iterative method and have avoided divergence. Aitken's accelerator is not very suitable for complex material nonlinearity. Crisfield [32] and Hallquist have noted that line searches (with very few extra residual calculations) can significantly improve the performance of the modified Newton-Raphson method.

Crisfield [34] compared different algorithms for line search and acceleration. He concluded that a single step or multiple step length corrector with the Newton-Raphson or modified Newton-Raphson iterative scheme gives the best overall performance. However he pointed out that the multi-step length algorithms did not always improve the performance of the original Newton-Raphson method but did improve reliability. He noted that the line searches are more useful with modified Newton-Raphson, secant-Newton and quasi-Newton procedures.

Hence, the line search algorithm based on secant-type interpolations (or extrapolations) as proposed by Crisfield [34,35] is adopted in the present work and described in the following section.

3.4 Line searches

During an iteration step the total displacement $u_i^{(i)}$ is updated using Equa. 3.4. In the line search technique, Equa. 3.4 is rewritten as:

$$u_i^{(i+1)} = u_i^{(i-1)} + \eta^{(i,k)} \Delta u_i^{(i)} \quad (3.11)$$

The total displacement is updated using the previous total displacement, current iterative displacement and a scalar step length parameter $\eta^{(i,k)}$. The superscripts

(i,k) stand for the line steps ($k = 0, 1, 2, 3, \dots$) in the current iteration (i). The iterative vector $\Delta^t u_i^{(i)}$ may be found using one of a number of techniques such as the Newton-Raphson, modified Newton-Raphson, BFGS or secant-Newton. If a line search is not be used (k is limited to 1) then $\eta^{(i,k)}$ is taken as unity. In some cases it will be advantageous to apply a line search to find the optimum value of the scalar $\eta^{(i,k)}$.

The optimum value of $\eta^{(i,k)}$ is the value that makes the the inner product of the out-of-balance force vector $\psi_i^{(i)}$ and $\eta^{(i,k)} \Delta^t u_i^{(i)}$ zero. The inner product is denoted by $\epsilon^{(k)}$.

$$\epsilon^{(k)} = \psi_i^{(i)} \eta^{(i,k)} \Delta^t u_i^{(i)} = 0 \quad (3.12)$$

For a given $\Delta^t u_i^{(i)}$, satisfying the above equation involves a one-dimensional search for $\eta^{(i,k)}$. Each trial value of $\eta^{(i,k)}$ involves a calculation of the out-of-balance force vector ψ_i . This is done by finding the incremental strain from the updated displacement $^t u_i^{(i)}$. Next the total stress is calculated using elastic or thermo-elasto-plastic equations. Finally ψ_i , using Equa. 3.10, is calculated. In the present work the residual load is defined as external minus internal force. Crisfield defines residual load to be the internal force minus external force, hence he has the opposite signs for $\epsilon^{(k)}$.

In the search for the optimum $\eta^{(i,k)}$, first $\epsilon^{(0)}$ is calculated using $\Delta^t u_i^{(i)}$, which in turn was calculated using the residual load at the end of iteration ($i-1$), denoted as $\psi_i^{(i-1,k=0)}$.

$$\epsilon^{(0)} = \Delta^t u_i^{(i)} \psi_i^{(i-1,0)} \quad (3.13)$$

For the first line search step ($k=1$) (or in case of no line search) $\eta^{(i,1)} = 1.0$, $\psi_i^{(i,1)}$ is calculated as explained earlier. Now $\epsilon^{(1)}$ or in general $\epsilon^{(k)}$ can be found as:

$$\epsilon^{(k)} = \Delta^t u_i^{(i)} \psi_i^{(i,k)} \quad (3.14)$$

If the condition $\epsilon^{(k)} = 0$ is not satisfied, then $\eta^{(i,k-1)}$ was not optimum. A new trial value for $\eta^{(i,k+1)}$ is found by linear interpolation using $\epsilon^{(0)}$, $\epsilon^{(k)}$ and $\eta^{(i,k)}$. Next $\eta^{(i,k+1)}$ is used to find the next trial $^t u_i^{(i,k+1)}$ and so on. $\epsilon^{(k)} = 0$ can not be satisfied exactly. Instead it is practical to satisfy

$$\frac{\epsilon^{(k)}}{\epsilon^{(0)}} \leq \Phi^* \quad (3.15)$$

where Φ^* is a line search tolerance.

$$\begin{aligned} \eta &\geq 0 & \text{if } e(0) > 0 \\ \eta &< 0 & \text{if } e(0) < 0 \end{aligned}$$

Figure 3.1: Interpolation and extrapolation techniques in line search.

Figure 3.2: Young's Modulus E vs. temperature for various steels

Figure 3.3: Poisson's ratio (ν) vs. temperature for various steels

Figure 3.4: Linear coefficient of thermal expansion (α) vs. temperature for various steels

Figure 3.5: Uniaxial yield stress (σ_y) vs. temperature for various steels

Figure 3.6: Hardening parameter (E_p) temperature for various steels

The simplest interpolation procedure to find $\eta^{(i,k+1)}$ is to interpolate (or extrapolate) between the estimates of $\epsilon^{(k)}$ and $\epsilon^{(0)}$

$$\frac{\eta^{(i,k+1)}}{\eta^{(i,k)}} = \frac{\epsilon^{(0)}}{\epsilon^{(0)} - \epsilon^{(k)}} \quad (3.16)$$

Better interpolation may be achieved using bracketing procedures. In particular, one may interpolate (extrapolate) between the first negative value of $\epsilon^{(k)}/\epsilon^{(0)}$ and the nearest (with smaller $\eta^{(i,k)}$) $\epsilon^{(k)}/\epsilon^{(0)}$ that is positive. If $\eta^{(i,k+1)}$ involves extrapolation, it can be dangerous, so a maximum ratio of ten is stipulated. These interpolation and extrapolation techniques are shown in Figure 3.1. If $\epsilon^{(0)}$ is negative, the displacement, $\Delta^i u_i^{(i)}$, is going downhill in an energy sense. To find stable equilibrium, $\eta^{(i,1)} = -1.0$ instead of $\eta^{(i,1)} = 1.0$ is taken. A value of $\Psi^* = 0.4$ and limiting the number of line search steps to four or six is suggested by Crisfield and is confirmed in the present work.

3.5 Constitutive equations for thermo-plasticity

The constitutive equations for ideally elastic and thermo-elastic material were given earlier. The deformation of metal beyond the elastic limit is known as plasticity. The one-to-one relationship between the stress and the strain is lost. The material behaviour can be complex when a wide range of temperatures and deformations are considered. Even for a single class of materials such as ductile materials it is difficult to devise a general constitutive law. Several different idealizations are proposed in literature, needless to say, none account completely for the phenomena observed experimentally. A history and account of these idealizations are given elsewhere [37, 38]. The isothermal nonlinear constitutive equations for ductile material have been proposed and investigated by many researchers [39,40,41]. Almost all the relations are due to early work by Tresca, Saint Venant, Levy, von Mises, Prandtl, Reuss, Hencky, Nadai and others. The nonisothermal plastic stress-strain relation was not explored until recently. In the 1960's (Landau [42], Kammash [43]) plastic relations, were proposed in which the uniaxial yield stress was temperature-dependent. More recently Hallquist [19] and Haisler [44] have proposed a generalization of existing plastic relations by including the temperature-dependent material properties. Ueda [10] has proposed a modified work hardening rule.

During a welding cycle a localized region of the metal is heated above the melting point and cooled to room temperature. The material properties such as E , ν , α and σ_y ,

change substantially during a heating and cooling cycle as given in *Appendix A* (see *Figures 3.2-6*). The plastic stress-strain relations must be based on non isothermal material parameters. Most metals work harden. Experimental investigations for determining work hardening parameters are complex and costly, especially if thermo-plasticity is included. From the few experimental studies which have been done [45][46], it can be concluded that work hardening not only exists but may not be restricted to isotropic work hardening. Hence the nonisothermal plastic constitutive relation should include work hardening properties.

In the following sections a nonisothermal plastic stress-strain relation is proposed which is selected to reflect the plastic behaviour of metals over a temperature cycle. Experimental data related to the metal plasticity are incorporated by using a hypothetical uniaxial curve. As is done in any development of plasticity theory, a yield function (or yield condition) and work hardening rules are proposed. During the development of the equations small strain rates are assumed. Another fundamental assumption of the plasticity theory is that the total strain ϵ_{ij} can be decomposed into elastic and plastic components ϵ_{ij}^p . The influence of the stress path on plastic deformation is given by Mendelson [47], Malvern [12] and Desai [48]. The stress-path-dependent history requires that the plastic strains should be calculated incrementally throughout the loading history and should be accumulated to get the total plastic strains.

3.6 Temperature-dependent yield function —

In a plastic state, an infinitesimal change in σ_{ij} will cause a permanent change in strain. The elastic range is defined as the set of elastic stress states in which $\dot{\sigma}_{ij}$ is not accompanied by a permanent change in the strain. The surface bounding the elastic range defines the yield function. For a given state of stress and temperature the existence of a yield function F which satisfies the yield conditions is assumed. The function F defines a yield surface in nine-dimensional stress space with the state variables of stress f and the hardening parameter k . The yield condition that F has to satisfy is

$$F(\sigma_{ij}, \alpha_{ij}, k) = 0 \quad (3.17)$$

$$F = f(\sigma_{ij}, \alpha_{ij}) - k^2 \left(\int \bar{\epsilon}_p, T \right) = 0$$

Figure 3.7: Yield surface and stress vectors in stress space

$$f(\sigma_{ij}^*) - k^2 \left(\int \epsilon_p, T \right) = 0 \quad (3.18)$$

where

$$\sigma_{ij}^* = \sigma_{ij} - \alpha_{ij} \quad (3.19)$$

The function F (see Figure 3.6) is stated in terms of f and k ; i.e., depends on the state of stress σ_{ij} , the position (center) of the yield surface α_{ij} , effective plastic strain ϵ_p , and temperature T at a point.

The necessary and sufficient condition for a state to be elastic is $F \leq 0$ and $dF < 0$. It is a necessary and sufficient condition for a state to be plastic when both $F = 0$ and $dF > 0$. Consider the time rate of Equa. 3.17

$$\dot{F} = \frac{\partial f}{\partial \sigma_{ij}^*} \dot{\sigma}_{ij}^* - 2k \frac{\partial k}{\partial \epsilon_p} \dot{\epsilon}_p - 2k \frac{\partial k}{\partial T} \dot{T} = 0$$

or

$$\dot{F} = \frac{\partial f}{\partial \sigma_{ij}^*} \dot{\sigma}_{ij}^* - \frac{\partial f}{\partial \sigma_{ij}^*} \dot{\sigma}_{ij} - 2k \frac{\partial k}{\partial \epsilon_p} \dot{\epsilon}_p - 2k \frac{\partial k}{\partial T} \dot{T} = 0 \quad (3.20)$$

$F < 0$ can only be an elastic state since $F \leq 0$ and hence $F + dF$ must be less than zero. A change from a plastic to an elastic state unaccompanied by plastic strain (the plastic strain rate is zero) is called *unloading*. The criterion for unloading is

$$\frac{\partial f}{\partial \sigma_{ij}^*} \dot{\sigma}_{ij}^* - 2k \frac{\partial k}{\partial T} \dot{T} < 0, \quad F = 0 \quad \text{during unloading} \quad (3.21)$$

When a change in σ_{ij} from one plastic state to another plastic state is not accompanied by a change in plastic strain, the process is called *neutral loading*. This means that the trajectory of the stress point or loading path always moves in, and hence tangential to the loading surface with zero plastic strain. Neutral loading takes place when

$$\frac{\partial f}{\partial \sigma_{ij}^*} \dot{\sigma}_{ij}^* - 2k \frac{\partial k}{\partial T} \dot{T} = 0, \quad F = 0 \quad \text{during neutral loading} \quad (3.22)$$

A change from one plastic state to another accompanied by plastic strain constitutes *loading* and occurs when

$$\frac{\partial f}{\partial \sigma_{ij}^*} \dot{\sigma}_{ij}^* - 2k \frac{\partial k}{\partial T} \dot{T} > 0, \quad F = 0 \quad \text{during loading} \quad (3.23)$$

This shows that, starting from a state of stress on the yield surface, *loading*, *unloading* and *neutral loading* take place according to whether the stress increment

vector (isothermal) is directed *outward*, *inward* or *along* the tangent to the yield surface respectively. These loading conditions are also known as the conditions of continuity and consistency.

3.7 Particular yield function F , von Mises

In the case of metal and strain rate-independent theories of continuum plasticity, specific yield functions, for f and k are given by von Mises' (1913) and Tresca. Both functions assume that the influence of the hydrostatic stress on the yield function is negligible. Some original work supporting this fact for metals can be found in the text by Hill [49], Malvern [12] and others [50,51]. For metals, the experimental results suggest the von Mises' yield criteria is more accurate. von Mises' theory assumes that yielding begins when the second invariant of the deviatoric stress reaches a certain value. Actually, the value is the distortional energy. von Mises' yield criterion is a sphere in deviatoric stress space, a 9-dimensional space, described by the following equation.

$$F = J_2' - k^2 = \frac{1}{2} S_{ij}^* S_{ij}^* - k^2 = 0 \quad (3.24)$$

where J_2' is the second invariant of the deviatoric stress S_{ij}^* , k is a material constant (assumed to depend on $\bar{\epsilon}_p$ and T) to be determined from experiments. The deviatoric stress is

$$S_{ij}^* = \sigma_{ij}^* - \frac{1}{3} \sigma_{mm}^* \delta_{ij} \quad (3.25)$$

The above equation can be rewritten explicitly in terms of the stress tensor σ_{ij}^* ;

$$\frac{1}{6} \left[(\sigma_{11}^* - \sigma_{22}^*)^2 + (\sigma_{22}^* - \sigma_{33}^*)^2 + (\sigma_{33}^* - \sigma_{11}^*)^2 \right] + (\sigma_{23}^*)^2 + (\sigma_{31}^*)^2 + (\sigma_{12}^*)^2 - k^2 = 0 \quad (3.26)$$

It is convenient to define an *equivalent* or an *effective stress*, $\bar{\sigma}$, and *equivalent* or *effective plastic strain*, $\bar{\epsilon}_p$, as:

$$\bar{\sigma} = \sqrt{\frac{3}{2} S_{ij}^* S_{ij}^*} \quad (3.27)$$

$$\bar{\epsilon}_p = \sqrt{\frac{2}{3} \epsilon_{ij}^p \epsilon_{ij}^p} \quad (3.28)$$

3.8 Plastic potential function, Prandtl-Reuss equations

When the state of stress reaches the yield condition, plastic deformation, which is called plastic flow, takes place. The direction of the plastic strain rate $\dot{\epsilon}_{ij}^p$ is defined

Figure 3.8: The uniaxial engineering stress-strain curve for typical metal

Figure 3.9: Idealized uniaxial true stress-strain curve for metal plasticity

by an assumed plastic potential function ϕ to which the incremental plastic strain vector is orthogonal.

$$\dot{\epsilon}_{ij}^p = d\lambda \frac{\partial \phi}{\partial \sigma_{ij}^*} \quad (3.29)$$

where $d\lambda$ is a non-negative scalar function for proportionality. $d\lambda$ can vary throughout the loading history. The flow rule provides the needed equations to solve the additional unknowns, $\dot{\epsilon}_{ij}^p$.

In metal plasticity, the yield function F (or $f(\sigma_{ij}^*)$) is taken as the plastic potential function. This is called the *associative flow rule*.

$$\dot{\epsilon}_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}^*} \quad (3.30)$$

An elastic, plastic generalization proposed by Prandtl (1924) for two dimensions, and by Reuss (1930) for three dimensions, assumes that the increment in the total strain can be decomposed into elastic and plastic components. It also assumes that the plastic strain increment is proportional to the deviatoric stress tensor at any instant of loading ; i.e.,

$$\dot{\epsilon}_{ij}^p = d\lambda S_{ij}^* \quad (3.31)$$

3.9 Uniaxial stress-strain curve

Many concepts in plasticity are generalizations of the simple stress state observed in a uniaxial tension test. The stress-strain diagram of a typical elasto-plastic material in simple tension is shown in Figure 3.3 . The stress σ is a monotonically increasing function of the strain ϵ ; i.e., the material is work-hardening) from O to H after which it falls off until fracture occurs. From O to the proportional limit A the material is linearly elastic (deformation is reversible). However the elastic limit generally extends beyond A up to B. For loading above B the deformation is irreversible so unloading from C to zero stress would leave a (permanent) plastic strain. Re-loading from a point E proceeds along EF to subsequent yielding at G, whereupon further loading follows the path GH. The HI portion is generally ignored. In 1928 Prandtl idealized the stress-strain (see Figure 3.4) curve under following assumptions.

- 1) The slope of EFG at E is taken to be same as the slope at O.

Rigid Perfectly Plastic

Elastic Perfectly Plastic

Rigid Work Hardening

Elastic Work Hardening

Figure 3.10: Types of most frequently used idealized stress-strain curves in metal plasticity

Figure 3.11: Idealized uniaxial stress-strain curve in loading and unloading

- 2) The hysteresis loop is assumed to be uniquely determined by point E.
- 3) The re-loading path is assumed to pass through C and proceeds thereafter from C to H as if unloading never occurred.

These assumptions are taken to imply the hysteresis loss is zero if unloading occurs after the loading. This curve is sometimes further idealized. Idealized stress-strain curves that are frequently used are illustrated in Figure 3.5.

Each point in the continuum is assumed to have its own idealized stress-strain curve or curves in nonisothermal loading. The effective stress at a point in the continuum is assumed to equal the uniaxial stress of the idealized curve. A similar assumption is made for the effective plastic strain. The experimental data needed to construct the idealized stress-strain curve is readily available at room temperature. However the availability of data over a range of temperature is rare. Some data from the published literature and private communications are given in Appendix A.

To check the yield condition at a point it is required that the k of Equa. 3.24 should be related to experimental data. The stress-strain curve at temperature T is used as follows. Denote the effective stress at a point by $\bar{\sigma}$, which is taken equal to the uniaxial stress. However the uniaxial stress (hence the effective stress at a point) beyond the elastic limit will be called the effective yield stress $\bar{\sigma}_y$.

In a uniaxial stress case, σ_{ij} components are zero except for σ_{11} . Also using Equa. 3.27, $\bar{\sigma} = \sigma_{11}$. In this case the yield condition Equa. 3.24 becomes

$$\bar{\sigma} = \sqrt{3}k \quad (3.32)$$

Also $\epsilon'_{22} = \epsilon'_{33} = 1/2\epsilon'_{11}$ since the plastic strain is assumed to be incompressible. Using Equa. 3.28, $\epsilon_p = \epsilon'_{11}$. The plastic potential equation, Equa. 3.31, after finding S_{ij} for uniaxial stress components is:

$$d\lambda = \frac{3}{2} \frac{\dot{\epsilon}_p}{\bar{\sigma}_y} \quad (3.33)$$

The assumption of the linear hardening stress-strain relation in the idealized curve gives:

$$\bar{\sigma}_y = \sigma_y(T) + E^p(T)\epsilon_p \quad (3.34)$$

where $\sigma_y(T)$ is initial yield strength, $E(T)$ is the elastic or Young's modulus (elastic slope) and $E^p(T)$ is the plastic hardening modulus at temperature T .

$$E_p(T) = \frac{\dot{\bar{\sigma}}}{\dot{\epsilon}_p} = \frac{E'(T)E(T)}{E(T) - E'(T)} \quad (3.35)$$

$E'(T)$ is the tangential modulus at temperature T found from the the uniaxial stress-strain curve. The relation between k and σ_y is given by Equa. 3.32.

3.10 Work hardening parameter k

As a result of the plastic flow, work hardening or strain hardening takes place in many materials. The meaning of the hardening in simple tension is illustrated in Figure 3.6. The initial elastic stress limit was σ_y , but after work hardening it increases to $\bar{\sigma}_y$, according to Equa. 3.34. Two hypotheses are defined in the literature, one depends on the the total plastic work (Hill [49]), the other on the effective plastic strain as a measure of hardening. The second hypothesis is used here. In the literature ([48]) the term 'work hardening' is only used for the total plastic work hypothesis, here the term is used for hardening in the broader sense. A single parameter, k , describes the size of the yield surface; e.g., for the von Mises' yield function it is the radius of the circle in the π^p plane of principle stress space (Haigh-Westergaard stress space). In deviatoric stress space, it is the radius of a hypersphere in a nine-space. In deviatoric principal stress space, σ_1 , it is a sphere and can be relatively easily visualized. If a material does not work harden then k does not change with plastic deformation as $\bar{\sigma}_y = \sigma_y$ in Equa. 3.34. When work hardening does occur, it can be of three types; namely, isotropic, kinematic or a combination of isotropic and kinematic. In an isotropic work hardening material, the radius of the sphere changes but not its shape or the location of its centre. In a kinematic work hardening material, the yield function does not change its shape, size or orientation but translates in stress space. In the case of combined work hardening, the yield function simultaneously translates and changes its radius. Most metal behaviour is closer to combined hardening. Here a fraction (β) of hardening is said to be isotropic while a fraction $(1 - \beta)$ of hardening is said to be kinematic. For pure isotropic or kinematic work hardening β is 1 or 0 respectively, Figure 3.6. Likewise for isotropic hardening Equa. 3.34 is modified to

$$\bar{\sigma}_y = \sigma_y(T) + \beta E'(T) \epsilon_p \quad (3.36)$$

The parameter β governs the amount of isotropic hardening and hence can be related to the Bauschinger effect observed in many metals. Experimentally, metals have been observed to change yield surface shape during hardening. This phenomena is not modelled in the present work.

3.10.1 Kinematic work hardening rule, Prager's rule

There are several rules which can be used for kinematic hardening. Prager [52] proposed a simple theory in which the yield surface does not change its shape or size but merely translates in stress space in the direction of the normal to the yield surface at the active stress point ; i.e., in the direction of the plastic strain increment. If α_{ij} are the coordinates of the center of the yield surface, the change is:

$$\dot{\alpha}_{ij} = c \dot{\epsilon}_{ij}^p \quad (3.37)$$

where scaling parameter, c , is a constant for linear hardening. Hodge (1957)* pointed out that the concept of kinematic hardening (as a logical generalization of the tension test) must be applied in nine-dimensional stress space. Sheild and Ziegler [53, 54] indicated that Prager's hardening rule is not invariant with respect to reductions in dimensions of the stress space. For example, although the yield surface in nine-dimensional stress space moves in the direction of the exterior normal, this is not necessarily the case in every subspace. Ziegler proposed that the center of the yield surface translates in the direction of the vector connecting the current center of the yield surface and the stress point, given by $\sigma_{ij} - \alpha_{ij}$.

$$\dot{\alpha}_{ij} = (\sigma_{ij} - \alpha_{ij}) d\mu \quad (3.38)$$

The scalar $d\mu$ must satisfy the condition that the stress point remains on the yield surface during the plastic flow ; i.e.,

$$d\mu = \frac{\frac{\partial f}{\partial \sigma_{mn}} \dot{\sigma}_{mn}}{(\sigma_{kl} - \alpha_{kl}) \frac{\partial f}{\partial \sigma_{kl}}} \quad (3.39)$$

This modification allows the flow rule to remain valid in all subspaces but has a serious drawback that it does not relate $\dot{\alpha}_{ij}$ to the plastic strain. Experimental data is needed to establish the relationship between α_{ij} and ϵ_{ij}^p for each material exactly as the yield stress and work hardening rate are now measured for each material. It should also be noted that for the von Mises' yield condition, Prager's rule and Ziegler's modification both specify the same direction for translation of the center. The normal to the surface at the active stress point and the vector joining the center and stress point are identical.

In the foregoing development a combined isotropic and kinematic hardening is proposed through a parameter β . This was mainly done because the kinematic

hardening does not allow the expansion of the yield surface while many materials are better represented by combined hardening. Mroz [55] and Iwan [56] proposed a hardening rule which considers a combination of both kinematic and isotropic hardening. A detailed explanation of this rule is also given in [48]. This rule is often referred as the nested surfaces model. Application of the rule in low cycle fatigue is attempted by Kalev[57]. Krieg [58] has suggested a two surface plasticity theory.

To take advantage of the specific relation between α_{ij} and ϵ_{ij}^p and its simplicity, Prager's rule is used here in the development of the thermo-plastic relations in nine-dimensional stress space. The $\dot{\alpha}_{ij}$ in the case of Prager's kinematic rule can be given as (or defining c of Equa. 3.37):

$$\dot{\alpha}_{ij} = (1 - \beta) \frac{2}{3} E_p(T) \dot{\epsilon}_{ij}^p \quad (3.40)$$

3.11 Determining D_{ijmn}^p and $d\lambda$

The increment in stress is given by:

$$\dot{\sigma}_{ij} = D_{ijmn} (\dot{\epsilon}_{mn} - \dot{\epsilon}_{mn}^p - \dot{\epsilon}_{mn}^T) + \dot{D}_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^p - \epsilon_{mn}^T) \quad (3.41)$$

Using the flow rule Equa. 3.30:

$$\dot{\sigma}_{ij} = D_{ijmn} (\dot{\epsilon}_{mn} - \dot{\epsilon}_{mn}^T) - D_{ijmn} \frac{\partial f}{\partial \sigma_{mn}^p} d\lambda + \dot{D}_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^p - \epsilon_{mn}^T) \quad (3.42)$$

Multiplying above equation by $\frac{\partial f}{\partial \sigma_{ij}^p}$ and using Equa. 3.20 gives:

$$\begin{aligned} \frac{\partial f}{\partial \sigma_{ij}^p} \dot{\sigma}_{ij} &= \frac{\partial f}{\partial \sigma_{ij}^p} D_{ijmn} (\dot{\epsilon}_{mn} - \dot{\epsilon}_{mn}^T) - \frac{\partial f}{\partial \sigma_{ij}^p} D_{ijmn} \frac{\partial f}{\partial \sigma_{mn}^p} d\lambda + \frac{\partial f}{\partial \sigma_{ij}^p} \dot{D}_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^p - \epsilon_{mn}^T) \\ &= \frac{\partial f}{\partial \sigma_{ij}^p} \dot{\sigma}_{ij} + 2k \frac{\partial k}{\partial \epsilon_p} \dot{\epsilon}_p + 2k \frac{\partial k}{\partial T} \dot{T} \end{aligned} \quad (3.43)$$

Some of the terms in r.h.s. of the above equation can be rewritten (using the uniaxial stress-strain idealization) as follows.

Using Equa. 3.32, Equa. 3.34 and the chain rule $\frac{\partial k}{\partial \epsilon_p} = \frac{\partial k}{\partial \bar{\sigma}_y} \frac{\partial \bar{\sigma}_y}{\partial \bar{\epsilon}_p}$ gives:

$$2k \frac{\partial k}{\partial \epsilon_p} \dot{\epsilon}_p = 2 \frac{\bar{\sigma}_y}{\sqrt{3}} \frac{\partial k}{\partial \bar{\sigma}_y} \frac{\partial \bar{\sigma}_y}{\partial \bar{\epsilon}_p} \dot{\epsilon}_p$$

$$2k \frac{\partial k}{\partial \epsilon_p} \dot{\epsilon}_p = \frac{2}{3} \bar{\sigma}_y \frac{\partial \bar{\sigma}_y}{\partial \bar{\epsilon}_p} \frac{2}{3} \bar{\sigma}_y d\lambda = \frac{4}{9} \bar{\sigma}_y \bar{\sigma}_y \beta E_p(T) d\lambda \quad (3.44)$$

Similarly Equa. 3.32 and $\frac{\partial k}{\partial T} = \frac{\partial k}{\partial \bar{\sigma}_y} \frac{\partial \bar{\sigma}_y}{\partial T}$ gives

$$2k \frac{\partial k}{\partial T} \dot{T} = \frac{2}{3} \bar{\sigma}_y \left[\frac{\partial \sigma_y}{\partial T} \dot{T} + \beta \bar{\epsilon}_y \frac{\partial E_y}{\partial T} \dot{T} \right] \quad (3.45)$$

Also, using Equa. 3.30, Equa. 3.40

$$\dot{\alpha}_{ij} = (1 - \beta) \frac{2}{3} E_y(T) d\lambda \frac{\partial f}{\partial \sigma_{ij}^*}$$

The $\frac{\partial f}{\partial \sigma_{ij}^*} \dot{\alpha}_{ij}$ term is calculated by multiplying the r.h.s. of the above equation by $\frac{\partial f}{\partial \sigma_{ij}^*}$. The Prandtl-Reuss generalization, Equa. 3.31 gives $\frac{\partial f}{\partial \sigma_{ij}^*} = S_{ij}^*$. Using the definition of $\bar{\sigma}$ ($\bar{\sigma}_y$ on yielding) gives:

$$\frac{\partial f}{\partial \sigma_{ij}^*} \dot{\alpha}_{ij} = \frac{4}{9} (1 - \beta) \sigma_y \bar{\sigma}_y E_y(T) d\lambda \quad (3.46)$$

Now the r.h.s. of Equa. 3.43 is

$$\begin{aligned} & \frac{\partial f}{\partial \sigma_{ij}^*} \dot{\alpha}_{ij} + 2k \frac{\partial k}{\partial \bar{\epsilon}_y} \dot{\epsilon}_y + 2K \frac{\partial k}{\partial T} \dot{T} \\ &= \frac{4}{9} \sigma_y \bar{\sigma}_y E_y(T) d\lambda + \frac{2}{3} \bar{\sigma}_y \left[\frac{\partial \sigma_y}{\partial T} \dot{T} + \beta \bar{\epsilon}_y \frac{\partial E_y}{\partial T} \dot{T} \right] \end{aligned}$$

Hence Equa. 3.42 is

$$\begin{aligned} & \frac{\partial f}{\partial \sigma_{rs}^*} D_{rsmn} \frac{\partial f}{\partial \sigma_{ij}^*} d\lambda + \frac{4}{9} \sigma_y \bar{\sigma}_y E_y(T) d\lambda \\ &= \frac{\partial f}{\partial \sigma_{ij}^*} D_{ijmn} (\dot{\epsilon}_{mn} - \dot{\epsilon}_{mn}^T) + \frac{\partial f}{\partial \sigma_{ij}^*} \dot{D}_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^P - \epsilon_{mn}^T) - \frac{2}{3} \bar{\sigma}_y \left[\frac{\partial \sigma_y}{\partial T} \dot{T} + \beta \bar{\epsilon}_y \frac{\partial E_y}{\partial T} \dot{T} \right] \\ d\lambda &= \frac{\frac{\partial f}{\partial \sigma_{ij}^*} D_{ijmn} (\dot{\epsilon}_{mn} - \dot{\epsilon}_{mn}^T) + \frac{\partial f}{\partial \sigma_{ij}^*} \dot{D}_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^P - \epsilon_{mn}^T) - \frac{2}{3} \bar{\sigma}_y \left[\frac{\partial \sigma_y}{\partial T} \dot{T} + \beta \bar{\epsilon}_y \frac{\partial E_y}{\partial T} \dot{T} \right]}{\frac{\partial f}{\partial \sigma_{rs}^*} D_{rsin} \frac{\partial f}{\partial \sigma_{ij}^*} + \frac{4}{9} \sigma_y \bar{\sigma}_y E_y(T)} \quad (3.47) \end{aligned}$$

The Prandtl-Reuss generalization for plastic potential function gives $\frac{\partial f}{\partial \sigma_{ij}^*} = S_{ij}^*$, hence:

$$d\lambda = \frac{S_{ij}^* D_{ijmn} (\dot{\epsilon}_{mn} - \dot{\epsilon}_{mn}^T) + S_{ij}^* \dot{D}_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^P - \epsilon_{mn}^T) - \frac{2}{3} \bar{\sigma}_y \left[\frac{\partial \sigma_y}{\partial T} \dot{T} + \beta \bar{\epsilon}_y \frac{\partial E_y}{\partial T} \dot{T} \right]}{S_{rs}^* D_{rsin} S_{ij}^* + \frac{4}{9} \sigma_y \bar{\sigma}_y E_y(T)} \quad (3.48)$$

and Equa. 3.41 becomes

$$\dot{\sigma}_{ij} = D_{ijkl} (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^T) + \dot{D}_{ijkl} (\epsilon_{kl} - \epsilon_{kl}^P - \epsilon_{kl}^T) - D_{ijkl} S_{kl}^* d\lambda \quad (3.49)$$

The elasto-plastic material tensor, D_{ijmn}^p , can be found by substituting the value of $d\lambda$ from Equa. 3.48 into above the equation and separating the terms involving $(\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^T)$. Thus

$$\dot{\sigma}_{ij} = D_{ij,kl}^p (\dot{\epsilon}_{kl} - \dot{\epsilon}_{kl}^T) + \dot{D}_{ij,kl} (\epsilon_{kl} - \epsilon_{kl}^p - \epsilon_{kl}^T) - \frac{S_{ij}^* \dot{D}_{ijmn} (\epsilon_{mn} - \epsilon_{mn}^p - \epsilon_{mn}^T) - \frac{2}{3} \bar{\sigma}_y \left[\frac{\partial \sigma_y}{\partial T} \dot{T} + \beta \epsilon_y \frac{\partial E_p}{\partial T} \dot{T} \right]}{S_{rs}^* D_{rstu} S_{tu}^* + \frac{4}{3} \bar{\sigma}_y \bar{\sigma}_y E_p(T)} \quad (3.50)$$

where

$$D_{ij,kl}^p = D_{ij,kl} - \frac{S_{ij}^* D_{ijmn} D_{stkl} S_{kl}^*}{S_{rs}^* D_{rstu} S_{tu}^* + \frac{4}{3} \bar{\sigma}_y \bar{\sigma}_y E_p(T)} \quad (3.51)$$

Chapter 4

Application of Finite Element Method

The application of the finite element method to heat flow analysis and to stress analysis will be presented separately. In this chapter the finite element formulation of the stress analysis equations (based on equations of Chapter 1-2) is given. The heat transfer equations and their finite element formulation will be given in Chapter 5. The basic concepts related to the elements, integrals and matrices are introduced briefly here. The details can be found in [25, 59-65]. These references have been used extensively in the following development.

4.1 Finite element method

Introduced in the aerospace industry in the 1950's, finite element analysis has rapidly become a very popular technique for computer solution of complex problems in physics and engineering. Mathematically it is a numerical procedure for solving differential equations. The earliest developments in the finite element method were in the field of structural engineering. In 1965 (Zienkiewicz, Vigser, Wilson) first showed the application of the method to heat conduction problems. Today, the finite element method has become the most widely used numerical technique for solving stress and heat flow problems in mechanics.

The fundamental concept of the finite element method is that a continuous function such as the displacement or temperature field, can be approximated by a discrete model composed of a set of piecewise continuous functions (polynomials) defined over a finite number of subdomains called elements. The elements have a fixed number of nodes. At the nodes the piecewise (coordinate) functions are defined using the values of continuous functions (degrees of freedom). The elements are connected at common nodal points and collectively give the shape of the domain.

In each element of the real structure, a distributed stress field is represented by forces at the element nodes. Correspondingly, the displacements of these nodes (degrees of freedom) are employed in the characterization of the displaced state of the

Figure 5.4: Thermal conductivity (k) vs. temperature for various steels

Figure 4.2: Types of elements used in present work.

element. A variety of elements are available. Each is distinguished by one or more of the following: the polynomial with which it approximate the forces/displacements (linear, cubic, quadratic, etc.); the number of dimensions for which polynomial is assumed (one, two or three dimensional); the shape of the element (triangular, quadratic bar, solid brick, tetrahedral, etc). The choice of the element type is made by keeping in mind that if the element size is reduced, the simplified form of the stress/displacement distribution it represents, should approach the actual variation of stress or displacement in that area [26]. When a displacement-based finite element formulation is used, only C^0 continuity of the nodal displacements and forces is guaranteed over the domain. The continuity and compatibility of the stress between adjacent elements depend on the choice of the element types and the problem, but they are usually C^{-1} continuous.

For most engineering problems, finite elements can precisely represent the actual geometry. If not, the geometry can be easily approximated using elements, without introducing substantial errors in the solution. The material properties are specified for the each element. Hence a body composed of several materials can be easily modelled. The major advantage of using the finite element method is flexibility in specifying the boundary conditions. They can be approximated easily as they are identified at the individual nodal points or with the individual elements. The mixed boundary conditions mentioned earlier can also be handled easily.

4.2 Elements

A list of two-dimensional and three-dimensional elements used in the present work is given in Table 4.1 and Figure 4.1. For most of problems solved in the present work, isoparametric elements are used since they are very versatile [62]. The following theory is written for isoparametric elements but the concept is valid for other types of elements. The interpolation of the element coordinates and element displacements using the same interpolation functions characterizes the isoparametric finite element formulation. The principle idea is to specify the relationship between the displacements (or coordinates) of any point within the element in terms the nodal point displacements using the shape functions (interpolation or basis polynomials) defined in a local element coordinate system. A local coordinate system (ξ) is

Element type	ASGARD type number		
	Elastic	Elas. plas.	Th. elas. plas.
3-noded triangle	62	30	126
6-noded triangle	63	31	127
4-noded quadrilateral	64	32	131
7-noded quadrilateral	65	33	132
8-noded quadrilateral	66	34	133
9-noded quadrilateral	67	35	134
12-noded quadrilateral	68	36	135
17-noded quadrilateral	69	29	137
8-noded brick	72	37	139
20-noded brick	73	38	140
8-20 variable noded brick	75	39	142

Table 4.1: Two and three dimensional finite elements for stress analysis

defined such that a point in the element has local coordinates, ξ_i , such that:

$$-1 < \xi_i \leq 1$$

regardless of the actual size of the element. The sides of element can be curved. The element can have a fixed or variable number of nodes, Figure 4.2. Once the shape functions N^n ($n = 1, 2, \dots, e$ where e is total number nodes) for the element are specified, the displacement u_i of a point in the global reference coordinate system is given as:

$$u_i = \sum_{n=1}^e N^n u_i^n \quad (4.1)$$

A coordinate of a point in the global reference system is:

$$x_i = \sum_{n=1}^e N^n x_i^n \quad (4.2)$$

Note that N^n are given in terms of ξ_i , i.e., they are functions of local coordinates.

The shape functions for isoparametric elements and the other elements listed in Table 4.1 are given in the literature cited earlier. The highest ξ_i term in the shape functions characterize the order of the displacement field which an element can approximate. The highest order term is also important in selecting the numerical integration scheme and the scheme for projecting nodal stresses Appendix B:

The matrices which follows are defined for later use for an element of e nodes. The column matrix of element nodal displacements is

$$u^e = [u_1^1, u_1^2, u_1^3, \dots, u_1^e]^T \quad (4.3)$$

where u_i^n are the displacements of the node, n ; for a three dimensional element $i = 1, 2, 3$. Hence, for a twenty-noded brick element u^e is a column matrix of sixty (20×3) entries. Similarly, the nodal force vector for an element is

$$f = [f_1^1, f_1^2, f_1^3, \dots, f_1^e]^T \quad (4.4)$$

The shape function matrix is

$$N = [\tilde{N}^1, \tilde{N}^2, \tilde{N}^3, \dots, \tilde{N}^e] \quad (4.5)$$

where

$$\tilde{N}^n = \begin{bmatrix} N^n & 0 & 0 \\ 0 & N^n & 0 \\ 0 & 0 & N^n \end{bmatrix}$$

Note that N has only 6 nonzero components which are shape functions at each node of the element.

4.3 Strain-displacement transformation matrix, B

The Green-Lagrangian strain definition *Equa. 2.4* involves the derivatives of displacements with respect to coordinates in the global reference system. Hence to calculate strains at point within an element, the displacement u_i are first found. The derivatives of u_i involve derivatives of the shape functions in local coordinate system ξ_i . Hence the Cartesian derivatives have to be related to local derivatives. Using the chain rule, a Jacobian operator can be defined which provides the relationship.

$$\frac{\partial}{\partial x_i} = J^{-1} \frac{\partial}{\partial \xi_i}$$

where J is the Jacobian operator. In matrix form J is

$$J = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_2} \\ \frac{\partial x_1}{\partial \xi_3} & \frac{\partial x_2}{\partial \xi_3} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix} \quad (4.6)$$

Another relationship from calculus is that the element volume differential dv can be written in terms of the local coordinates ξ_i using J .

$$dv = \det J \, d\xi_1 \, d\xi_2 \, d\xi_3 \quad (4.7)$$

The linear and nonlinear strain terms of the Green-Lagrangian strain tensor are given in *Equa. 2.10-11*. Using the symmetry of the strain tensor ϵ_{ij} , it can be expressed as a strain vector of six components. In matrix form, the strain vector at a point can be given by:

$$e = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{12}, 2\epsilon_{23}, 2\epsilon_{31}]^T \quad (4.8)$$

The $\frac{\partial u_i}{\partial x_j}$ terms at a point in the element can be evaluated from the nodal displacements, shape functions and coordinates using *Equa. 4.1* and *Equa. 4.6*. In matrix form a strain-deformation transformation matrix B is constructed such that

$$e = B u^e \quad (4.9)$$

To suit the linear and nonlinear strain definition

$$\mathbf{B} = \mathbf{B}^L + \mathbf{B}^{NL} \quad (4.10)$$

$\mathbf{B}^L$ for 2-D and 3-D elements can be found in any standard reference on the finite element method. $\mathbf{B}^{NL}$ components have derivatives of updated displacements or updated coordinates in addition to the derivatives of the nodal shape functions. Hence the definition of $\mathbf{B}^{NL}$ must include the time at which the derivatives are evaluated. References [25] and [66] give definitions of $\mathbf{B}^{NL}$ if the derivatives are evaluated at time t or $t + \Delta t$. In [19], Hallquist defined the derivatives at time $t + \frac{\Delta t}{2}$.

4.4 Linearized finite element equations of motion

In finite elements, the integrals (over total domain) in the linearized equations of motion are replaced by the sum over the volume and areas of each element. Using Equa. 3.5, the linearized incremental equations for a time increment from t to $t + \Delta t$, are

$$\begin{aligned} \sum_{N=1}^E \int_V {}^t D_{ijmn} \Delta^t \epsilon_{mn} \delta \epsilon_{ij} dv + \sum_{N=1}^E \int_V {}^t \sigma_{ij} \delta \epsilon_{ij}^{NL} = \\ \sum_{N=1}^E \int_V \Delta^t f_i^b \delta u_i^t dv + \sum_{N=1}^E \int_V \Delta^t f_i^c \delta u_i^t dv + \sum_{N=1}^E \int_S \Delta^t f_i^t \delta u_i^t dS + f_i^p(k) \dot{u}_i^k \end{aligned}$$

The displacement within an element is assumed to be given by Equa. 4.1. Using the strain displacement transformation matrices and the element matrices, the interpolation function matrices for the above equations are written as:

$$\delta \dot{\mathbf{u}} \left(\left[\sum_{N=1}^E \int_V \mathbf{B}^T {}^t \mathbf{D} \mathbf{B} dv + \sum_{N=1}^E \int_V \mathbf{B}^{NL T} {}^t \mathbf{S}^{NL} \mathbf{B}^{NL} dv \right] \Delta^t \mathbf{u} = [\Delta^t \mathbf{R}] \right) = 0$$

where $\mathbf{D}$ is the constitutive or material matrix, ${}^t \mathbf{S}^{NL}$ is the stress matrix [bathe80] and $\Delta^t \mathbf{R}$ is the incremental nodal force and reaction matrix (load vector).

$\mathbf{D}$ is a square matrix whose components are given by Equa. 2.24-25 for thermo-elastic analysis. For the thermo-elasto-plastic case, the components of ${}^t D_{ijmn}$ are given by Equa. 3.51.

Using Equa. 2.33 the load vector is given as:

$$\Delta^t \mathbf{R} = \sum_{N=1}^E \mathbf{r}^b + \sum_{N=1}^E \mathbf{r}^c + \sum_{N=1}^E \mathbf{r}^s + \Delta^t \mathbf{r}^p \quad (4.11)$$

where the f^B , f^I and f^S are body, thermal (initial) and surface nodal load matrices for an element respectively. The details of these matrices are given in the next section. The $\Delta^t f^N$ is a matrix of nodal point forces.

Since the virtual displacement is arbitrary, the linearized incremental equations of motion are written in matrix form as:

$$[{}^tK^L + {}^tK^{NL}] \Delta^t u = \Delta^t R$$

or global equation

$$K^{(i)} u^{(i)} = \Delta^t R^{(i)} \quad (4.12)$$

where i is the iteration number and K is the global stiffness matrix. It is assembled by summing linear and nonlinear element stiffness matrices K^{L^e} and K^{NL^e} which are defined as follows.

$$K^{L^e} = \int_v {}^0 B^{L^e T} {}^t D {}^0 B^{L^e} dv \quad (4.13)$$

$$K^{NL^e} = \int_v {}^t B^{NL^e T} {}^t S {}^t B^{NL^e} dv \quad (4.14)$$

In an iterative scheme for solving Equa. 4.12, the load matrix, $\Delta^t R$, can be either an externally applied load or a residual load. The global incremental force matrix $\Delta^t R^{(i)}$ for $i = 1$ is the incremental external load which is the given as:

$$\Delta^t R^{(1)} = \Delta^t R \quad (4.15)$$

For $i > 1$, it is the residual load defined as:

$$\Delta^t R^{(i)} = {}^{t+\Delta t} R - {}^{t+\Delta t} F^{(i)} \quad (4.16)$$

where ${}^{t+\Delta t} R = {}^t R + \Delta^t R$. The global matrix of nodal point forces equivalent to the element stresses at time $t + \Delta t$ and iteration i is

$${}^{t+\Delta t} F^{(i)} = \sum_{N=1}^N \int_v {}^{t+\Delta t} B^{(i-1)T} {}^{t+\Delta t} S^{(i-1)} dv \quad (4.17)$$

The global matrix of iterative displacement, $u^{(i)}$, is summed in the following way to give the total displacement, ${}^{t+\Delta t} u^{(i)}$.

$${}^{t+\Delta t} u^{(i)} = {}^{t+\Delta t} u^{(i-1)} + u^{(i)} \quad (4.18)$$

It is generally better to retain ${}^t\mathbf{u}$; i.e., retain the total displacements at the last equilibrium position instead of contaminating them with the iterative displacement of the current increment. This will require one more global displacement matrix called the incremental displacement matrix which is nothing but the sum of the iterative displacements. In this view, the iterative displacement matrix for iteration i is denoted as $\mathbf{u}^{(i)}$. The incremental displacement matrix at the end of iteration i during the Δt time increment is denoted as $\Delta^t \mathbf{u}^{(i)}$.

$$\Delta^t \mathbf{u}^{(i)} = \Delta^t \mathbf{u}^{(i-1)} + \eta^{(i,k)} \mathbf{u}^{(i)} \quad (4.19)$$

where $\eta^{(i,k)}$ is a scalar step length parameter used in the line search Equa. 3.11. The total displacement at the end of time $t + \Delta t$ now can be given as:

$${}^{t+\Delta t} \mathbf{u} = {}^t \mathbf{u} + \Delta^t \mathbf{u}^{(i)} \quad (4.20)$$

4.5 Element load matrices

The externally applied body forces and traction forces on the domain were defined earlier. In finite element analysis these forces are called generalized forces and can be specified at the nodes. The traction forces, body forces and thermal loads (in terms of thermal stress) can only be specified on the individual elements. The equivalent nodal forces for these element loads can be calculated as shown in the following sections.

4.5.1 Body forces, $\mathbf{f}^b$

Let $\mathbf{f}_i$ be a column matrix of i body force components ($i = 1, 2$ for 2-D and $i = 1, 2, 3$ for 3-D elements) acting at every point in the element. The equivalent nodal force matrix can be calculated using shape functions and Equa. 2.1

$$\mathbf{f}^b = \int_v \mathbf{N} \mathbf{f}_i dv \quad (4.21)$$

4.5.2 Initial stress and temperature loads

A typical step in any thermal stress analysis is to calculate the initial stress and equivalent nodal force ("equivalent thermal nodal load") from the body temperatures. First, the temperature T at any point within the element can be calculated

from the nodal temperatures T^n using the shape functions,

$$T = \sum_{n=1}^i N^n T^n$$

Next, using definition of thermal stress Equa. 2.28 (or in general using initial stress σ'_{ij}) and B , the thermal nodal load matrix (or load vector due to initial stress) for an element can be given as:

$$f^T = \int_v B^T S' dv \quad (4.22)$$

4.5.3 Surface traction forces

Pressure forces or distributed forces with varying intensity are common in engineering mechanics. For a solid (three-dimensional) element these forces are applied on the surface (or face). For two-dimensional elements they are prescribed on the element edges. If p_i^n is the intensity of a traction force at the nodes on a surface (or edge) then the intensity f_i at any point on the surface (or edge) can be evaluated using the surface or edge shape functions N^s . Finally, the equivalent nodal force matrix due to the surface loads can be given as:

$$f^s = \int_s N^s T f_i ds \quad (4.23)$$

The surface shape functions N^s are calculated from the same shape functions used in displacement interpolation. When the local coordinates of a face (surface) are substituted in the shape functions of a 20-noded brick, the shape functions become that of an 8-noded 2-D isoparametric element. This will be the N^s for 20-node brick element. Similarly the shape functions for edge loads in 2-D elements can be associated with those of one-dimensional bar elements.

4.6 Numerical integration

One of the advantages of using the finite element method is that the integrals are evaluated over an element volume v and not over the total domain V . For convenience one may use different element local coordinate system for evaluating the integral. The volume differential dv in the global coordinates can be extended over the local coordinate system. Using Equa. 4.7:

$$dv = \det J d\xi_1 d\xi_2 d\xi_3 \quad (4.24)$$

Numerical integration of a function stated in local coordinate system can be done by any of several methods. Gauss-Legendre quadrature (or simply Gaussian) has proved most useful in finite element work [62]. The Gaussian integration process is used exclusively in the present work. Let a function $\Phi(x)$ be integrated over a length x . The basic assumption in the Gaussian integration rule is that:

$$\int_a^b \Phi(x) dx = \sum_{i=1}^n \omega^i \Phi(x^i) \quad (4.25)$$

where ω^i and x^i are the weight and coordinates of the sampling points i (Gauss points). The Gaussian integration rule needs n sampling points to exactly integrate a polynomial of order $(2n - 1)$. More details about Gauss rules, sampling points and weights can be found in [60,61, 25]. The Gauss point coordinates and weights for various Gaussian integration schemes for 2-D and 3-D elements are given in Appendix C. The choice of a scheme or order of integration depends on the order of $\Phi(x)$ terms. Generally in finite elements the highest order term will be found in calculations of $B^T D B$ and $N^T N$.

An integral over an element volume, $\int_v \Phi(x_i) dv$, using Equa. 4.7 and Gaussian rule, can be numerically integrated as:

$$\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 \Phi(\xi_1, \xi_2, \xi_3) \det J d\xi_1 d\xi_2 d\xi_3 = \sum_i \sum_j \sum_k \omega^i \omega^j \omega^k \Phi(\xi_1^i, \xi_2^j, \xi_3^k) \det J \quad (4.26)$$

Similarly, the surface and the length integrals can be evaluated numerically.

4.7 Convergence criterion

In an incremental iterative scheme, the global equation, Equa. 4.12 is iteratively solved. At the end of each iteration, a convergence criterion should be used to test whether residual and displacement adjustments are smaller than a pre-selected tolerance level. When that is achieved, the iteration loop for the current increment can be terminated.

In a displacement configuration one is seeking equilibrium between the internal and the external forces. Therefore it is desirable that the criterion used to check the convergence takes into account the displacements and/or forces. In this context three criteria are found to be useful. All criteria involve the use of some type of vector norm. There are, again, three types of vector norms.

1. Infinity Norm

$$\|\Phi\|_{\infty} = \max_i \|\Phi_i\|$$

2. One or Taxi Cab Norm

$$\|\Phi\|_1 = \sum_{i=1}^m \|\Phi_i\|$$

3. Two or Euclidean Norm

$$\|\Phi\|_2 = \left(\sum_{i=1}^m \|\Phi_i\|^2 \right)^{1/2}$$

In the above, Φ is a vector of length m .

It is observed that, in most of the problems, if a criterion shows convergence by the Euclidean norm then it will also do the same if the Taxi cab norm is used. In the following three criteria the Taxi cab norm will be used because it involves fewer numerical calculations.

4.7.1 Displacement criterion

In the displacement norm, the displacements of a current iteration are compared with the total displacements at the end of the first iteration:

$$\frac{\|\Delta^t \mathbf{u}_i^{(1)}\|_1}{\|\mathbf{t} + \Delta^t \mathbf{u}_i^{(1)}\|_1} \leq \varphi_d \quad (4.27)$$

In Equa. 4.27, φ_d is the tolerance for the displacement convergence scheme.

It should be noted that for $t = 0$ and $i = 1$; $\Delta^t \mathbf{u}_i^{(1)} = \mathbf{t} + \Delta^t \mathbf{u}_i^{(1)}$. Therefore, for the first iteration of the first increment this criterion may not be useful.

4.7.2 Force criterion

In the force convergence criterion, the out-of-balance forces ; i.e., residual forces are monitored.

$$\frac{\|\mathbf{t} + \Delta^t \mathbf{R}^{(1)} - \mathbf{t} + \Delta^t \mathbf{F}^{(1)}\|_1}{\|\mathbf{t} + \Delta^t \mathbf{R}^{(1)}\|_1} \leq \varphi_f \quad (4.28)$$

The preset force criterion tolerance is φ_f .

4.7.3 Energy criterion

In this criterion, the amount of work done by the residual forces is used to check

the convergence.

$$\frac{\Delta^t \mathbf{u}_i^{(1)T} ({}^{t+\Delta t} \mathbf{R}^{(1)} - {}^{t+\Delta t} \mathbf{F}^{(1)})}{\Delta^t \mathbf{u}_i^{(1)T} ({}^{t+\Delta t} \mathbf{R}^{(1)} - {}^{t+\Delta t} \mathbf{F}^{(1)})} \leq \varphi_e \quad (4.29)$$

The pre-selected energy convergence criterion tolerance is φ_e .

The tolerance values (φ_d , φ_f and φ_e) affect the accuracy of the solution. If they are too stringent then the solution may require more iterations without providing a substantial gain in solution accuracy. If the problem definition (in terms of the mesh, boundary conditions etc.) is not accurate then setting a tighter tolerance may divert a solution which was converging initially. On the other hand, a coarser tolerance may give a converged solution although the answers may be wrong and may leave a substantial residual load. In the latter case the solution may diverge in subsequent increments of the loads since the energy balance was not obtained in the previous load steps.

The tolerance should be selected on the basis of the nature of problem, the desired or expected accuracy, and most of all from the experience of solving relevant test cases. It is generally recommended that $\varphi_e < \varphi_f < \varphi_d$.

4.8 Stress increment calculations

The stress tensor, σ_{ij} , a symmetrical tensor, can be expressed by a column matrix.

$$\mathbf{S} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}] \quad (4.30)$$

In an incremental and iterative formulation the global equation Equa. 4.12 is solved at each iteration; i.e., $\mathbf{u}^{(i)}$ is calculated. The incremental displacement $\Delta^t \mathbf{u}^{(i)}$ is obtained using Equa. 4.12. The increment in strain $\Delta^t \mathbf{e}$ can be calculated in a manner similar to Equa. 4.9.

To calculate the actual stress increment $\Delta^t \mathbf{S}$, first, a trial stress increment $\Delta^t \tilde{\mathbf{S}}$ and a trial total stress ${}^{t+\Delta t} \tilde{\mathbf{S}}$ are calculated assuming a purely elastic (or thermo-elastic) state. Appropriately, Equa. 2.29 and Equa. 3.2 are used. The trial elastic deviatoric stress state, ${}^{t+\Delta t} \tilde{\mathbf{S}}^d$ is calculated using Equa. 3.19 and Equa. 3.25. Ultimately a trial yield function $\tilde{F}$ and its time rate $\dot{\tilde{F}}$ are evaluated (Equa. 3.17 and Equa. 20). If $\tilde{F} < 0$ or $\tilde{F} = 0$ and $\dot{\tilde{F}} \leq 0$ the material is in an elastic state. The

Figure 4.3: Incremental stress in elasto-plastic medium.

Figure 4.2: Types of elements used in present work.

the papers published on FEM, 7115 citations in 1976, are directed toward stress analysis. The stress field actually used by the analyst is not the discontinuous FEM stress field, $\mathbf{S} = \mathbf{DBu}$, but some projection that is C^0 smooth to facilitate contoured displays. The literature on such projections appears to be limited to perhaps five papers [94]. It is curious that such an important topic has received so little attention. In Appendix B, a new and more general projection methodology is presented.

- Chapter 5

The Heat Transfer Analysis

5.1 Heat transfer equations

To derive the governing classical heat transfer equation a three-dimensional body (domain V) is considered as shown in Figure 5.1. Assuming the material obeys Fourier's law of heat conduction

$$q_{x_1} = -k_{x_1} \frac{\partial T}{\partial x_1} \quad (5.1)$$

where q_{x_1} , q_{x_2} , and q_{x_3} are the heat flow conducted per unit area, T is the body temperature and k_{x_1} , k_{x_2} and k_{x_3} are the thermal conductivities in the x_1 , x_2 and x_3 directions respectively.

The temperature, $T(x_i, t)$, as a function of spatial coordinates (x_i) and time, t , satisfies the following parabolic differential equation, called the heat equation, at every point in the domain, V .

$$\frac{\partial}{\partial x_1} k_{x_1} \frac{\partial T}{\partial x_1} + \frac{\partial}{\partial x_2} k_{x_2} \frac{\partial T}{\partial x_2} + \frac{\partial}{\partial x_3} k_{x_3} \frac{\partial T}{\partial x_3} + Q = c_p \dot{T} \quad (5.2)$$

The rate of heat generated (source or sink) per unit volume is $Q(x_i, t)$. The volumetric specific heat or the heat capacity per unit volume is c_p .

On the boundary of V either the essential or natural boundary conditions must be satisfied. The essential boundary condition can be defined as:

$$T(x_i, t) = T^{e1}(x_i, t) \text{ on the boundary } S_1 \quad (5.3)$$

The natural boundary condition can be defined as

$$k_{x_n} \frac{\partial T^{e2}}{\partial n} + q^{e2} + h^c (T^{e2} - T^\infty) + \sigma^r \epsilon (T^{e2^4} - T^\infty^4) = 0 \quad (5.4)$$

on the boundary S_2 . In Equa. 5.4 :

q^{e2} = a prescribed flux or heat flow input on the S_2 surface of the body;

Figure 5.1: Body subjected to heat transfer

- h^c = heat transfer coefficient for convection;
 σ^r = Stefan - Boltzmann constant;
 ϵ = hemispherical emissivity $0 < \epsilon \leq 1$;
 T^∞ = the ambient temperature for convection and/or radiation.

In addition, the initial condition must be specified for $(x_i) \in V$

$$T(x_i, 0) = {}^0T(x_i) \quad (5.5)$$

If the partial differential Equa. 5.2, the boundary conditions Equa. 5.3 and Equa. 5.4, and the initial condition Equa. 5.5 are consistent, the problem is well posed and a unique solution exists.

In the foregoing statement of equations it is assumed that the material particles of the body are at rest ; i.e., heat conduction in solids is considered. The other assumption is that the heat transfer analysis is not coupled to the stress analysis. In other words, the heat generated by stress and strain is negligible. The phase changes and latent heat effects are incorporated approximately through the specific heat.

If k_x , or c_p are functions of T , Equa. 5.2 is nonlinear. Also, if radiation is included in the boundary conditions or if the convective heat transfer coefficient is temperature-dependent, then the boundary conditions Equa. 5.3-4 are nonlinear.

5.2 Finite element formulation for transient heat flow

The basis of the finite element method is analogous to that employed in stress analysis. The piecewise polynomial approximation of the temperature within each element is given by

$$T = \sum_{n=1}^N N^n T^n \quad (5.6)$$

where N^n are basis or shape functions. Physically $T_i(t)$ are the nodal values of the temperature at time t . Mathematically they are undetermined coefficients and if they are orthogonal, they can be considered as generalized Fourier coefficients. The analyst specifies the basis functions N^n by creating a mesh which specifies the type and position of each element.

The temperature gradient at any point (x_i, t) can be computed directly from Equa. 5.6:

$$\frac{\partial T}{\partial x_i} = \frac{\partial N^n}{\partial x_i} T^n \quad (5.7)$$

Figure 5.2: Linear time marching basis function

The question is how to evaluate T . Galerkin's weighted residual method is the most convenient and general of the methods for solving Equa. 5.2 using N^n as weighting terms. If Equa. 5.6 is substituted into Equa. 5.2, a residual or error term must be added. If this were not true Equa. 5.6 would be the exact solution. Indeed when Equa. 5.6 is the exact solution, there is no finite element method. The details of applying Galerkin's method in developing finite element equations for transient heat flow are given by Chakravarti [69], Beresford [70] and Moore [71] and are not repeated here. The final finite element matrix form of Equa. 5.2 is given as

$$\mathbf{K} \mathbf{T}(t) + \mathbf{C} \dot{\mathbf{T}} = \mathbf{R} \quad (5.8)$$

The stiffness matrix $\mathbf{K}$, capacitance matrix $\mathbf{C}$ and loading vector $\mathbf{R}$ are defined as follows:

$$\mathbf{K} = \sum_{N=1}^E \int_V \mathbf{B}^T \mathbf{k} \mathbf{B} dv + \sum_{N=1}^E \int_s (h^c + h^r) \mathbf{N}^T \mathbf{N} ds \quad (5.9)$$

$$\mathbf{C} = \sum_{N=1}^E \int_V c_p \mathbf{N}^T \mathbf{N} dv \quad (5.10)$$

$$\mathbf{R} = - \sum_{N=1}^E \int_V \mathbf{Q} \mathbf{N} dv + \sum_{N=1}^E \int_s \mathbf{q}^r \mathbf{N} ds + \sum_{N=1}^E \int_s (h^c + h^r) {}^0 \mathbf{T} \mathbf{N} ds \quad (5.11)$$

This set of simultaneous equations is solved for the nodal temperatures $\mathbf{T}$. Usually some form of Newton-Raphson method together with Gaussian elimination and back-substitution would be employed to solve these above equations.

5.3 Time discretization

So far, the finite element equation formulation of the heat conduction problem has been presented in the space-time domain, but the formulation has not been explicitly written for the time domain. This was in order to reduce the number of simultaneous variables. Since the final equation results in an identical expression, the time-marching formulation has been included separately.

For the time-marching equation formulation, consider a typical time element of duration Δt with nodal values t and $t + \Delta t$. Linear basis functions can be written

Figure 5.3: Specific heat c_p vs. temperature for various steels

Figure 5.4: Thermal conductivity (k) vs. temperature for various steels

in terms of the local variable, ξ , (since only first order derivatives will be required) and are shown in Figure 5.2.

$$N_1 = 1 - \frac{t}{\Delta t} = 1 - \xi \quad (5.12)$$

and

$$N_2 = \frac{t}{\Delta t} = \xi \quad (5.13)$$

(Note: By definition $\xi = t/\Delta t$, therefore, $0 \leq \xi \leq 1$). The linear interpolated temperatures and their time derivatives at time $t + \Theta \Delta t$ are given by

$$\begin{aligned} {}^{t+\Theta\Delta t}T &= N_1 {}^tT + N_2 {}^{t+\Delta t}T \\ &= (1 - \xi) {}^tT + \xi {}^{t+\Delta t}T \end{aligned} \quad (5.14)$$

and

$$\frac{\partial {}^tT}{\partial t} = \frac{1}{\Delta t} [{}^{t+\Delta t}T - {}^tT] \quad (5.15)$$

Once again Galerkin's the weighted residual method is applied to Equa. 5.8 to form the transient finite element equation. Finally:

$$\left[\frac{1}{\Delta t} C + \Theta K \right] {}^{t+\Delta t}T + \left[(1 - \Theta) K - \frac{1}{\Delta t} C \right] {}^tT + (1 - \Theta) {}^{t+\Delta t}R + \Theta {}^tR = 0 \quad (5.16)$$

The value for Θ varies between $0 \leq \Theta \leq 1$. Its value determines the time integration scheme for the set of equations produced when the domain is discretized into elements. It is important for the analyst to recognize the effect of interaction between the integration scheme (explicit, $\Theta = 0$; Crank-Nicholson, $\Theta = 0.5$; Galerkin, $\Theta = 2/3$; and implicit, $\Theta = 1.0$) and any prescribed thermal loads, fluxes or power densities. The prescribed thermal loads, fluxes, and power densities are all converted to an equivalent nodal thermal load specified at the beginning of each time step and another specified at the end of each time step. The time integration scheme uses Θ to linearly interpolate for the applied thermal load at the Θ point. Any convective or radiative loads are computed at the Θ temperature and added to the applied thermal load. This load is considered to be applied for the full time increment.

5.4 Nonlinear material properties

Over the past forty years most thermal analyses of welds have assumed that the thermal properties were constant. In fact Rosenthal's analysis cannot be extended

for nonlinear properties because the final transformation he applies is only valid for a linear equation. The thermal conductivity and specific heat of two steels and one aluminium are shown in *Figure 5.3-4* and given in *Appendix A*. The error caused by assuming a temperature-independent thermal conductivity and specific heat are substantial [71].

5.5 Convection and radiation

Given the heat transfer coefficient, h^c , convective heat transfer is easily included in the finite element method analysis. The usual equation, which may be taken as the definition of the heat transfer coefficient is $q = h^c(T^s - T^\infty)$. The value of heat transfer coefficient, h^c , must be measured experimentally or taken from handbooks which are based, in turn, on experiment. For the welding hot rolled steel plates, Vinokurov [72] suggests the following equation for a combined heat transfer coefficient, h^{cr} . This equation includes both radiative and convective effects.

$$h^{cr} = h^c + \sigma' \epsilon = 24.1 \times 10^{-4} \epsilon T^{1.61} \quad (5.17)$$

Using a linearized form of the Stefan-Boltzmann equation, Equa. 5.4 can be written as

$$k_n \frac{\partial T^{s2}}{\partial n} + q^{s2} + h^{cr}(T^{s2} - T^\infty) = 0 \quad (5.18)$$

Convection and radiation have little effect near the heat source where heat flow is conduction dominated. Far from the heat source, heat flow can be convection dominated if the workpiece and jigging are so large that the temperature rise in the workpiece is negligible.

In radiative heat transfer the emissivity must be measured experimentally. In exceptional situations, it may be desirable to take into account view factors. The analysis of an electron beam cavity is an example. In a laser beam plume, interaction between radiation and the plume should be considered. Neither view factors nor participating media are considered in the present work.

Chapter 6

Code Verification Cases and Examples

6.1 Code development

A software system based on theory presented in the previous chapters. was developed. This software was an integral part of a larger system called **ASGARD**. The data structures and algorithms to solve nonlinear equations with transient loads are complex. Detailed descriptions of the above would be too long to include here and brief descriptions would not be useful. The extensive use of comments in the computer code provides an excellent account of the procedures used in implementing the numerical methods. Substantial time and effort was devoted to the development of the code. Hence it is appropriate to list the major contributions of the present work to **ASGARD**.

- Procedures for handling time or load increments and iterative solutions were added to the code.
- Iron's [60] frontal solver was modified to handle the nonlinear global equations.
- Software was written for calculating the load, stiffness, strain-displacement and constitutive matrices for the elements listed in Table 4.1.
- An algorithm to calculate residual loads, perform line searches and check the convergence was developed.
- Procedures were developed for text, binary and graphic input and output of data.
- User documents were prepared for the elastic, elasto-plastic and thermo-elasto-plastic finite element analysis using **ASGARD**.
- **ASGARD** was successfully implemented on the Honeywell CP-6, Apollo, Vax VMS and FPS-164 computer systems.

Iterative Schemes and Line Search Comparison									
	Case I♣			Case II◇			Case II♣		
Increments t	1	2	3	1	2	3	1	2	3
Iterations $\sum i$	1	48	44	1	3	3	1	1	1
Line searches $\sum k$	0	0	0	0	4	4	0	0	0

$\clubsuit$ 279 seconds of computer time.

$\diamond$ 4 seconds of computer time.

$\clubsuit$ 27 seconds of computer time.

Table 6.1: Comparison of Newton-Raphson iterative schemes with and without line search

6.2 Forms of Newton-Raphson iterative schemes and line search

A simple example, the uniaxial tensile test, was chosen to compare three Newton-Raphson schemes and the usefulness of the line search. A rectangular bar was loaded with a prescribed tensile force in two increments. During the first load increment the stress state was elastic so only one iteration was needed and no line search was necessary. In the second increment, the state of stress becomes plastic and several iterations were needed to satisfy the equilibrium of forces. Three cases considered are outlined below.

- Case I** A modified Newton-Raphson iterative scheme was used to satisfy the equilibrium.
- Case II** A modified Newton-Raphson with the line search was employed.
- Case III** A Newton-Raphson scheme was used in the third trial.

6.2.1 Results

The results for the uniaxial tensile test are given in Table 6.1. They clearly show that the line search involves considerably fewer iterations with the small penalty of repeating the out-of-balance force calculation. The results of this simple test promote Newton-Raphson as the best iterative scheme. In the simple tensile test, Newton-Raphson was expected to perform the best since recalculation of the stiffness results in an exact stiffness. However this is an impossible and dangerous task for more complex problems.

The portions of the code which deal with the residual force calculations, line search calculations, convergence checking and nonlinear stiffness calculations have been verified by these results. The results are also useful in showing the capability of the developed code to solve nonlinear equations using a modified frontal solver.

6.3 Examining work hardening rules for the case of a uniaxial bar

Material work hardening is very important in metal plasticity. In the development of the constitutive equations for plasticity various work hardening cases have been discussed and equations have been derived for a combination of isotropic and kinematic work hardening. In the software development, a combined isotropic and the kinematic hardening with Prager's rule has been coded. The work hardening rules and the loading/unloading capability of the developed software can be best

2

studied in the case of uniaxial bar under a tensile (or compressive) load. In a multi-axial case, it would be hard to explain and visualize the σ_{ij} , ϵ_{ij} & α_{ij} tensors.

A bar of a uniform cross-section was incrementally loaded and unloaded in tension as well as in compression by specifying extension and contraction of the bar length (prescribed incremental displacements). Hence the total incremental strain was known. The aim was to study the change in stress ($\dot{\sigma}_{ij}$), plastic strain ($\dot{\epsilon}_{ij}^p$) and the translation of the yield surface ($\dot{\alpha}_{ij}$). Assuming uniaxial behaviour, the incremental stress and incremental plastic strain were found for a prescribed incremental strain. In the present case, the incremental strain was equal to the prescribed incremental displacement since length was taken as unity. The material properties were assumed to be $E = 8/3$, $\nu = 0.3$, $\sigma_y = 8$ and $E_t = 2/6$. The material is assumed to follow von Mises' yield criterion.

Given a strain increment $\Delta \epsilon_{ij}$, other quantities can be calculated in a simple manner. The behaviour strictly follows the idealized stress-strain relation. In a load increment for which the final stress state is elastic and Equa. 3.17 is satisfied, the stress increment is given using Equa. . The plastic strain and yield surface translation increment for such a load increment will be zero.

$$\dot{\sigma}_{11} = E \dot{\epsilon}_{11} \quad ; \quad \dot{\epsilon}_{11}^p = 0 \quad ; \quad \dot{\alpha}_{11} = 0 \quad (6.1)$$

On the other hand if F satisfies Equa. 3.23 then $\dot{\epsilon}_{11}^p$ is non-zero. The increment in stress and plastic strain in this case is given as

$$\dot{\sigma}_{11} = E_t \dot{\epsilon}_{11} \quad ; \quad \dot{\epsilon}_{11}^p = \dot{\epsilon}_{11} E_t / E_p \quad (6.2)$$

The yield surface radius and yield surface translation coordinate α_{11} depend on the yield criterion chosen and the type of hardening assumed. The yield criterion is assumed to be von Mises'. Three cases of hardening were chosen: perfectly plastic ($E_p = 0$); isotropic work hardening ($\beta = 1$); and kinematic work hardening ($\beta = 0$). The k and $\dot{\alpha}_{ij}$ for these cases were:

$$\begin{aligned} \sqrt{3}k &= \sigma_{11} \quad \text{for } \beta = 1 \\ \sqrt{3}k &= \sigma_y \quad \text{for } \beta = 0 \end{aligned} \quad (6.3)$$

$$\frac{3}{2} \dot{\alpha}_{11} = E_p \dot{\epsilon}_{11}^p \quad \text{for } \beta = 0 \quad (6.4)$$

Figure 6.1: Iterative solution using modified Newton Raphson method

Figure 6.2: Iterative solution using modified Newton Raphson method and line search

Figure 6.3: Iterative solution using Newton Raphson method

Once the incremental quantity is calculated the total quantity is summed as follows:

$$\begin{aligned}
 \sigma_{11} &= \sigma_{11} + \dot{\sigma}_{11} \\
 \epsilon_{11} &= \epsilon_{11} + \dot{\epsilon}_{11} \\
 \epsilon_{11}^p &= \epsilon_{11}^p + \dot{\epsilon}_{11}^p \\
 \alpha_{11} &= \alpha_{11} + \dot{\alpha}_{11}
 \end{aligned} \tag{6.5}$$

6.3.1 Load specifications and finite element modelling

A solid, square, prismatic bar of unit cross-sectional area, fixed at one end and free to displace in all directions at the other end, was chosen. The bar was subjected to four increments of axial displacement at the free end, in four different time steps of $t = 1, 2, 3$ and 4 with initial time $t = 0$.

In the first increment ($t = 0, t + \Delta t = 1$) the bar had displacements prescribed such that the stress state just reached the full elastic limit. In the present case such a state is reached by prescribing $\Delta^t u_1 = \epsilon_{11} = 3$.

In the second increment ($t = 1, t + \Delta t = 2$) the bar was stretched by a prescribed displacement ($\Delta^t u_1 = \epsilon_{11} = 6$). This load step is entirely in the plastic zone of the uniaxial stress-strain curve.

In the third load increment ($t = 2, t + \Delta t = 3$) the bar was unloaded from the elasto-plastic stress state in tension to the elastic limit in compression. Depending on the chosen hardening, an elastic unloading limit was calculated in terms of the incremental displacement. Three cases are considered:

- Case I Perfectly plastic material.
- Case II Work hardening is isotropic.
- Case III Work hardening is kinematic.

During the fourth increment ($t = 3, t + \Delta t = 4$) the bar was compressed by the same amount as in the second increment ($\Delta^t u_1 = \epsilon_{11} = -6$). As in the second increment, this load step caused yielding (compressive). In Figure 6.4 the total displacement at the end of an increment or time step is shown.

In the finite element solutions for the above cases, one 20-node brick element was used. A tangential or Newton-Raphson iterative scheme was used for the steps in which the yielding occurred. The bar dimensions were taken as unity. The

Figure 6.4: Total prescribed deflection vs. time. Loading conditions for examining work hardening rules in case of uniaxial bar test.

Figure 6.5: Geometry and boundary conditions in finite element solution in uniaxial bar test for examining work hardening rules.

	Case I			
t	1	2	3	4
ϵ_{11}	3	6	-6	-6
ϵ_{11}	3	9	3	-3
σ_{11}	8	8	-8	-8
ϵ_{11}^p	0	5.25	5.25	10.5
$\sqrt{3}k$	8	8	8	8
$\frac{3}{2}\alpha_{11}$	0	0	0	0

Table 6.2: Results of uniaxial bar test. Perfectly plastic material with loading and unloading.

	Case II			
t	1	2	3	4
ϵ_{11}	3	6	-7.5	-6.0
ϵ_{11}	3	9	1.5	-4.5
σ_{11}	8	10	-10	-12
ϵ_{11}^p	0	5.25	5.25	10.5
$\sqrt{3}k$	8	10	10	12
$\frac{3}{2}\alpha_{11}$	0	0	0	0

Table 6.3: Results of uniaxial bar test. Isotropic work hardening material with loading and unloading.

	Case III			
t	1	2	3	4
ϵ_{11}	3	6	-6	-6
ϵ_{11}	3	9	3	-3
σ_{11}	8	10	-6	-8
ϵ_{11}^p	0	5.25	5.25	10.5
$\sqrt{3}k$	8	8	8	8
$\frac{3}{2}\alpha_{11}$	0	2	2	0

Table 6.4: Results of uniaxial bar test. Kinematic work hardening material with loading and unloading.

Figure 6.6: Stress-strain curve and unexpanding yield surface. Results of uniaxial bar test with load reversal and perfectly plastic material, $E_p = 0$.

Figure 6.7: Stress-strain curve and expanding yield surface. Results of uni-axial bar test with load reversal and isotropic work hardening material, $\beta = 1$.

Figure 6.8: Stress-strain curve and yield surface translation. Results of uni-axial bar test with load reversal and kinematic work hardening material, $\beta = 0$.

boundary conditions are shown in Figure 6.5. Any of the three convergence schemes (see Chapter 4.9) with a tolerance limit of zero was appropriate in this case. In the finite element solution for this case, all three schemes were checked. The tolerance in each scheme were taken as 10^{-6} . The line search was not required for this case since the Newton-Raphson scheme was used.

6.3.2 Results

An analytical solution was calculated using Equa. 6.1-5. The finite element solution was found using **ASGARD**. The stresses, strains and yield surface radius values of the two solutions, as expected, matched exactly. These values are given in Tables 6.2-4. The results are presented graphically by plotting a stress-strain curve for each of the three cases, Figures 6.6-8.

The three hardening rules have been verified from these results. The validity of the stress, strain and displacement rate calculations have been established for the elasto-plastic analysis. Also the surface (edge) load calculations and the displacement boundary condition aspects of the code have been shown to be in order.

6.4 Solid thin circular disk with radial temperature distribution

A thin circular solid disk (Figure 6.9) which had a symmetrical temperature distribution about its center; i.e., T was function of radius r only) was studied. The temperature, stresses and strains did not vary over the thickness t . The stresses σ_r, σ_θ satisfied the equilibrium equation [73,1].

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \quad (6.6)$$

Solving this equation for the plane stress condition for the freely expanding disk gives the following equations for stresses and radial displacements at radius r

$$\sigma_r = \alpha E \left(\frac{1}{R} \int_0^R T r dr - \frac{1}{r^2} \int_0^r T r dr \right) \quad (6.7)$$

$$\sigma_\theta = \alpha E \left(-T + \frac{1}{R} \int_0^R T r dr - \frac{1}{r^2} \int_0^r T r dr \right) \quad (6.8)$$

$$u_r = (1 + \nu) \frac{\alpha}{r} \int_0^r T r dr + (1 - \nu) \frac{\alpha r}{R^2} \int_0^R T r dr \quad (6.9)$$

Figure 6.9: Thin circular disk with symmetrical temperature distribution
 $T = r$.

1 6-noded triangle

10 8-noded quadrilateral elements

Figure 6.10: Finite element mesh and boundary conditions. 10° Sector of thin circular disk.

r/T	Analytical		
	σ_r	σ_θ	u_r
	10^6	10^6	10^{-3}
.00	.666667	.666667	.000000
5.00	.633333	.600000	.012750
10.00	.600000	.533333	.027667
14.50	.570000	.473333	.042944
19.00	.540000	.413333	.059977
23.50	.510000	.353333	.078764
28.00	.480000	.293333	.099307
32.50	.450000	.233333	.121604
37.00	.420000	.173333	.145657
41.50	.390000	.113333	.171464
46.00	.360000	.053333	.199027
50.50	.330000	-.006667	.228344
55.00	.300000	-.066667	.259417
59.50	.270000	-.126667	.292244
64.00	.240000	-.186667	.326827
68.50	.210000	-.246667	.363164
73.00	.180000	-.306667	.401257
77.50	.150000	-.366667	.441104
82.00	.120000	-.426667	.482707
86.50	.090000	-.486667	.526064
91.00	.060000	-.546667	.571177
95.50	.030000	-.606667	.618044
100.00	.000000	-.666667	.666667

Table 6.5: Stresses and displacements by theoretical solution. Thin circular disk with radial temperature distribution and isothermal properties.

r/T	FEM			% Error		
	σ_r 10^8	σ_θ 10^8	u_r 10^{-2}	σ_r	σ_θ	u_r
00	.667166	.666257	.000000	-.075	.061	-.000
5.00	.633494	.599863	.012747	-.025	.023	.027
10.00	.599893	.533417	.027676	.018	-.016	-.034
14.50	.569979	.473354	.042947	.004	-.004	-.007
19.00	.539992	.413344	.059978	.001	-.003	-.002
23.50	.509993	.353342	.078764	.001	-.002	-.000
28.00	.479995	.293339	.099310	.001	-.002	-.003
32.50	.449996	.233338	.121604	.001	-.002	-.000
37.00	.419997	.173337	.145659	.001	-.002	-.002
41.50	.389998	.113336	.171464	.001	-.003	-.000
46.00	.359998	.053336	.199029	-.000	-.004	-.001
50.50	.329999	-.006665	.228344	-.000	.031	-.000
55.00	.299999	-.066665	.259419	-.000	.003	-.001
59.50	.269999	-.126665	.292244	-.000	.001	-.000
64.00	.239999	-.186665	.326828	-.000	.001	-.000
68.50	.210000	-.246665	.363164	-.000	.001	-.000
73.00	.180000	-.306665	.401258	-.000	-.000	-.000
77.50	.150000	-.366666	.441103	-.000	-.000	-.000
82.00	.120000	-.426666	.482708	-.000	-.000	-.000
86.50	.090000	-.486666	.526063	-.000	-.000	-.000
91.00	.060000	-.546666	.571178	-.000	-.000	-.000
95.50	.030000	-.606666	.618043	-.000	-.000	-.000
100.00	.000000	-.666666	.666668	-.126	-.000	-.000

Table 6.6: Stresses, displacements and % error between theoretical and finite element solutions. Thin circular disk with radial temperature distribution and isothermal properties.

Figure 6.17: Comparison of circumferential stress values. Results of elasto-plastic analysis of thick cylinder with various pressure loading

Figure 6.12: Circumferential stress comparison between analytical and finite element solutions. Thin circular disk.

Figure 6.13: Displacement comparison between analytical and finite element solutions. Thin circular disk.

where σ_r is the radial stress, σ_θ is the circumferential stress and u_r is the radial displacement. R is the outer radius of the disk. Note that the modulus of elasticity E , Poisson's ratio ν and the coefficient of thermal expansion α are temperature-independent.

6.4.1 Example definition and solutions

The geometrical and material data selected for the disk were $R = 100.0$, $t = 1$, $\alpha = 1 \times 10^{-6}$, $E = 2 \times 10^{12}$ and $\nu = 0.3$. The temperature distribution was given (Figure 6.9) by

$$T = r \quad (6.10)$$

Using these data, the analytical solution and finite element solution were obtained. The analytical solution for this particular example now becomes:

$$\sigma_r = \frac{\alpha E}{3.0} (R - r) \quad (6.11)$$

$$\sigma_\theta = \frac{\alpha E}{3.0} (R - 2r) \quad (6.12)$$

$$u_r = \frac{\alpha r}{3.0} [(1 + \nu)r + (1 - \nu)R] \quad (6.13)$$

In the finite element solution, a 10° sector of the disk was analysed using ten 8-node quadrilaterals and one 6-node triangle. The mesh is shown in the Figure 6.10. The solution was obtained using the plane stress condition. Boundary conditions corresponded to the free expansion of the disk Figure 6.10

6.4.2 Results

For comparison, the analytical stress values were calculated at the nodes of the finite element mesh Table 6.5. The nodal values of the stresses and displacements from the finite element solution and percentage errors are given in Table 6.6. A graphical comparison is also shown in Figures 6.11-13.

Excellent agreement between the theoretical and the finite element solution was found. Hence the implementation of the thermo-elastic constitutive equation, Equa. 2.37, for the plane stress analysis has been verified.

Circumferential Stress Comparison◇				
P_c	R_c ♣	σ_θ^{theory}	σ_θ^{fem}	error
GPa	m	$Pa \times 10^{-6}$		%
.2119	.100	3.53125	3.51902	-0.35
.2323	.105	3.60364	3.46717	-3.79
.2514	.110	3.67956	3.67359	-0.16
.2692	.115	3.75902	3.63316	-3.35
.2858	.120	3.84200	3.84262	+0.02
.3012	.125	3.92852	3.81538	-2.88
.3157	.130	4.01856	4.00148	-0.43
.3291	.135	4.11214	4.00076	-2.71
.3416	.140	4.20925	4.21632	+0.17
.3533	.145	4.30989	4.20459	-2.44
.3641	.150	4.41406	4.39094	-0.52
.3741	.155	4.52177	4.44884	-1.61
.3833	.160	4.63300	4.60287	-0.65
.3919	.165	4.74777	4.67649	-1.50
.3997	.170	4.86606	4.83252	-0.69
.4069	.175	4.98789	4.91565	-1.45
.4134	.180	5.11325	5.07516	-0.74
.4193	.185	5.24214	5.16510	-1.47
.4246	.190	5.37456	5.32895	-0.85
.4293	.195	5.51052	5.42373	-1.57
.4335	.200	5.65000	5.54674	-1.83

◇ At R_c

♣ Theoretical plastic zone radius for pressure P_c .

Table 6.7: Circumferential stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of the thick cylinder, Case I

Figure 6.15: Temperature gradient along radius of thick cylinder

6.5 Thick cylinder subjected to internal pressure and a temperature gradient

A thick cylinder subjected to a gradually increasing pressure is a classical problem in plasticity. Solutions have been obtained with and without a temperature gradient, in the past, for example in references [74,49,75]. The formulae for calculating the radial and circumferential stress values for a thick cylinder with internal pressure and a temperature gradient are given by Mendelson [47]. The code verification for the thermo-elastic and thermo-plastic plane strain cases was done by solving the following cases for the thick cylinder problem.

- Case I Elasto-plastic solution of a cylinder subjected to a gradually increasing pressure. The temperature gradients were zero.
- Case II Thermo-elastic solution of a cylinder subjected to an internal pressure and a temperature gradient.
- Case III Elasto-plastic solution of a cylinder subjected to an internal pressure and a temperature gradient.

A cylinder of 100mm internal radius R^I and 200mm of outer radius R^O was considered. The material properties were taken as $E = 204\text{GPa}$, $\nu = 0.3$, $\alpha = 1.463 \times 10^{-5}/^\circ\text{C}$, $\sigma_y = 565\text{MPa}$ and $E_p = 0.1E$. Taking advantage of the symmetry in geometry and loading, a 10° sector was analysed using the plane strain condition. The sector was modelled by ten 8-node quadrilaterals, Figure 6.14. Thus the finite element mesh had 21 nodes along the radius. The boundary conditions were similar to that used in the disk case (see Chapter 6.4).

In Case I, the cylinder was subjected to twenty-one increments of the pressure applied as a distributed edge force. The radius of each node in the radial direction was R^C . The increment in the pressure was calculated in such a way that the total pressure P^C applied was sufficient to yield the $R^I \leq R \leq R^C$ zone. The twenty-one increments of the ΔP^C corresponded to the twenty-one R^C (radius of the 21 nodes) yield zones. The total pressure P^C at the end of twenty-one time steps is shown in Table 6.7.

In the thermo-elastic analysis (Case II) of the cylinder, a pressure of $P^C = 425.05\text{MPa}$ and a temperature gradient were applied in one time step. The temperature gradient was calculated by (see Figure 6.15)

$$T^{R^O} = \frac{T^{R^I}}{R^O - R^I} \left(\frac{R^O}{R^I} - 1 \right) \quad (6.14)$$

Figure 6.16: Comparison of radial stress values. Results of elasto-plastic analysis of thick cylinder with various pressure loading

Figure 6.17: Comparison of circumferential stress values. Results of elastoplastic analysis of thick cylinder with various pressure loading

Radial Stress Comparison $\diamond$				
P_c	$R_c \clubsuit$	σ_r^{theory}	σ_r^{fem}	error
GPa	m	$Pa \times 10^{-8}$		%
.2119	.100	-2.11875	-2.10652	-.58
.2323	.105	-2.32330	-2.31561	-.33
.2514	.110	-2.51397	-2.50714	-.27
.2692	.115	-2.69179	-2.68483	-.26
.2858	.120	-2.85767	-2.85058	-.25
.3012	.125	-3.01239	-3.00517	-.24
.3157	.130	-3.15667	-3.14930	-.23
.3291	.135	-3.29111	-3.28359	-.23
.3416	.140	-3.41628	-3.40861	-.22
.3533	.145	-3.53267	-3.52484	-.22
.3641	.150	-3.64071	-3.63272	-.22
.3741	.155	-3.74082	-3.73267	-.22
.3833	.160	-3.83335	-3.82500	-.22
.3919	.165	-3.91864	-3.91013	-.22
.3997	.170	-3.99697	-3.98825	-.22
.4069	.175	-4.06863	-4.05975	-.22
.4134	.180	-4.13387	-4.12475	-.22
.4193	.185	-4.19291	-4.18363	-.22
.4246	.190	-4.24597	-4.23644	-.22
.4293	.195	-4.29323	-4.28354	-.23
.4335	.200	-4.33489	-4.32494	-.23

$\diamond$ At the inner face.

$\clubsuit$ Theoretical plastic zone radius for pressure P_c .

Table 6.8: Radial stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of the thick cylinder, Case I

R_c <i>mm</i>	Radial Stress			Circumferential Stress		
	anal. <i>GPa</i>	fem <i>GPa</i>	error %	anal. <i>GPa</i>	fem <i>GPa</i>	error %
100	-.4250	-.4249	-0.03	0.1399	.1422	1.64
105	-.3968	-.3969	0.01	0.1929	.1917	-0.61
110	-.3689	-.3688	-0.05	0.2389	.2409	0.83
115	-.3416	-.3417	0.04	0.2789	.2781	-0.29
120	-.3150	-.3147	-0.08	0.3141	.3155	0.45
125	-.2892	-.2893	0.05	0.3452	.3446	-0.17
130	-.2643	-.2640	-0.11	0.3727	.3737	0.27
135	-.2402	-.2403	0.06	0.3972	.3968	-0.11
140	-.2170	-.2167	-0.13	0.4192	.4199	0.18
145	-.1947	-.1949	0.07	0.4389	.4386	-0.07
150	-.1733	-.1730	-0.16	0.4567	.4573	0.13
155	-.1527	-.1529	0.09	0.4728	.4725	-0.05
160	-.1329	-.1327	-0.19	0.4874	.4878	0.09
165	-.1139	-.1141	0.11	0.5007	.5005	-0.04
170	-.0957	-.0954	-0.25	0.5128	.5132	0.07
175	-.0781	-.0782	0.15	0.5239	.5238	-0.03
180	-.0613	-.0610	-0.35	0.5341	.5344	0.05
185	-.0450	-.0451	0.23	0.5435	.5434	-0.02
190	-.0294	-.0292	-0.66	0.5522	.5524	0.04
195	-.0144	-.0145	0.65	0.5602	.5601	-0.02
200	0.0000	.0018	-	0.5676	.5678	0.04

Table 6.9: Stress values and % difference in finite element and theoretical solutions. Thermo-elastic analysis of the thick cylinder, Case II

Figure 6.18: Comparison of radial stress values. Results of thermo-elastic analysis of thick cylinder with various pressure loading and temperature gradient

Figure 6.19: Comparison of circumferential stress values. Results of thermo-elastic analysis of thick cylinder with various pressure loading and temperature gradient

Radial Stress Comparison			
R_o	Anal	fem	error
m	GPa	GPa	%
100	-.467784	-.4781554	2.22
105	-.438965	-.4420749	0.71
110	-.411490	-.4097416	-0.43
115	-.385278	-.3922883	1.82
120	-.360204	-.3712763	3.07
125	-.336188	-.3454951	2.77
130	-.313152	-.3192101	1.93
135	-.291025	-.2905478	-0.16
140	-.269742	-.2619391	-2.89
145	-.249245	-.2350196	-5.71
150	-.200575	-.2083828	3.89
155	-.176038	-.1840928	4.58
160	-.152673	-.1597980	4.67
165	-.130405	-.1374851	5.43
170	-.109165	-.1151686	5.50
175	-.088887	-.0945458	6.37
180	-.069514	-.0739205	6.34
185	-.050988	-.0547580	7.39
190	-.033259	-.0355935	7.02
195	-.016277	-.0177041	8.77
200	0.000000	.0019382	-

Table 6.10: Radial stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of the thick cylinder, Case III

Figure 6.20: Comparison of radial stress values. Results of elasto-plastic analysis of thick cylinder with various pressure loading and temperature gradient

This corresponds to the steady-state temperature distribution of a cylinder whose inner and outer surfaces are kept at temperatures T^{∞} and zero respectively.

The loading in Case III was similar to that of Case II but the solution was based on the elasto-plastic (isothermal material properties) theory.

The finite element results were obtained from **ASGARD** while the theoretical results were obtained by numerical (Simpson's rule) integration of the formulae in [47].

The radial and circumferential stress values at the 21 points (nodes) in the radial direction are compared graphically in *Figures 6.16-17* for Case I at various time steps. The highest stress values of the two components in the finite element and theoretical solutions and the error between them are given in *Tables 6.7-8*. In the second case stress values at 21 nodal points are compared in *Table 6.9* and *Figures 6.18-19*. For the radial and circumferential stress values the average errors are 0.17% and 0.25% respectively. Finally, a comparison of the radial stress (average 4.8% error) is given for the Case III in *Table 6.10* and *Figures 6.20*. The agreement between solutions is evident in all three cases.

The results particularly verify the implementation of the elasto-plastic and thermo-plastic constitutive equations, and the stress rate, strain rate and incremental displacement calculations.

6.6 Thermo-Elasto-plastic stresses in a thin flat strip

Closed form solutions for thermal stresses are available for quite a few classical problems. However the theoretical solutions for thermal stresses in the case of the temperature-dependent problems are almost nonexistent. Mendelson and Spero [79] have given a general solution for the elasto-plastic thermal stresses in a strain hardening plate with temperature-dependent material properties. Using a simplified stress state they have given an iterative procedure to find the uniaxial stress and strain. Their solution for a thin flat strip with an arbitrary temperature distribution was used to verify the thermo-plastic aspects of the finite element code. The loads were applied in one-time step, hence this example fell short of verifying the incremental aspects of the code.

6.6.1 Theoretical solution for a thin flat strip

A thin infinitely long flat strip with a temperature gradient (T) across the width ($2c$) was considered. The strip was subjected to an axial force (P) and a moment (M). The strip was assumed to have only one nonzero stress component σ_{11} . The stress-strain relation for this uniaxial case [47] is

$$\epsilon_{11} = \frac{\sigma_{11}}{E} + \epsilon^p + \alpha T \quad (6.15)$$

The following nondimensional quantities are introduced:

$$\sigma_z = \frac{\sigma_{11}}{\sigma_y} ; \epsilon_z = \frac{\epsilon_{11}}{\epsilon_y} ; r = \frac{\alpha T}{\epsilon_y} ; \epsilon_z^p = \frac{\epsilon_{11}^p}{\epsilon_y}$$

$$\gamma = \frac{y}{c} ; G = \frac{t^E}{\sigma_y} ; P^* = \frac{P}{\sigma_y c h} ; M^* = \frac{M}{\sigma_y c^2 h} \quad (6.16)$$

where

- σ_y = Uniaxial yield stress at the reference temperature at time $t = 0$
- E = Modulus of elasticity at the reference temperature at time $t = 0$
- ϵ_y = σ_y / σ_y , Equivalent plastic strain corresponding to stress σ_y
- α = Coefficient of thermal expansion
- y = Distance from the centerline
- c = Half width of the strip
- P = Applied axial load
- M = Applied moment
- h = Thickness of the strip
- σ_z = Nondimensional stress
- $\epsilon_z, \epsilon_z^p, r$ = Nondimensional total, plastic and thermal strains respectively

The material quantities E, σ_y and ϵ^p are temperature-dependent. Now Equa. 6.1 can be written as

$$\epsilon_z = \frac{\sigma_z}{G} + r + \epsilon_z^p \quad (6.17)$$

The compatibility equations give

$$\frac{\partial^2 \epsilon_z}{\partial \gamma^2} = 0 \quad (6.18)$$

which gives

$$\epsilon_z = a + b\gamma \quad (6.19)$$

The above equation expresses the assumption of simple beam theory, that is, plane sections remain plane and normal to the neutral axis. To determine the constants a and b , the resultant force and moment acting on a cross-section must balance the externally applied force and moment. These conditions do not imply that boundary loads vanish. Hence the results obtained will, in general, not be valid near the ends of a finite length strip. The end effect can be ignored using St. Venant's principle or can be properly accommodated with appropriate use of sectorial symmetry. For equilibrium

$$\int_{-1}^1 \sigma_z d\gamma = \frac{P}{\delta \sigma_y c t} = P^* \quad (6.20)$$

$$\int_{-1}^1 \sigma_z \gamma d\gamma = \frac{M}{\delta \sigma_y c^2 t} = M^* \quad (6.21)$$

Combining Equa. 6.3, Equa. 6.5, Equa. 6.6 and Equa. 6.7 gives:

$$\begin{aligned} a &= A_1 \int_{-1}^1 G(r + \epsilon^p) d\gamma - A_2 \int_{-1}^1 G(r + \epsilon^p) \gamma d\gamma + A_1 P^* - A_2 M^* \\ b &= -A_2 \int_{-1}^1 G(r + \epsilon^p) d\gamma + A_3 \int_{-1}^1 G(r + \epsilon^p) \gamma d\gamma - A_2 P^* + A_3 M^* \end{aligned} \quad (6.22)$$

where

$$\begin{aligned} A_1 &= \frac{\int_{-1}^1 G \gamma^2 d\gamma}{\int_{-1}^1 G d\gamma \int_{-1}^1 G \gamma^2 d\gamma - \left(\int_{-1}^1 G \gamma d\gamma \right)^2} \\ A_2 &= \frac{\int_{-1}^1 G \gamma d\gamma}{\int_{-1}^1 G d\gamma \int_{-1}^1 G \gamma^2 d\gamma - \left(\int_{-1}^1 G \gamma d\gamma \right)^2} \\ A_3 &= \frac{\int_{-1}^1 G d\gamma}{\int_{-1}^1 G d\gamma \int_{-1}^1 G \gamma^2 d\gamma - \left(\int_{-1}^1 G \gamma d\gamma \right)^2} \end{aligned} \quad (6.23)$$

The mechanical strain ϵ is given by

$$\epsilon = \epsilon_z - r \quad (6.24)$$

Substituting the constant from Equa. 6.8 into Equa. 6.9 and using Equa. 6.10 gives:

$$\begin{aligned} \epsilon &= -r + (A_1 - A_2 \gamma) \left[\int_{-1}^1 G r d\gamma + P^* \right] \\ &\quad - (A_2 - A_3 \gamma) \left[\int_{-1}^1 G r \gamma d\gamma + M^* \right] \\ &\quad + (A_1 - A_2 \gamma) \int_{-1}^1 G \epsilon^p d\gamma - (A_2 - A_3 \gamma) \int_{-1}^1 G \epsilon^p \gamma d\gamma \end{aligned} \quad (6.25)$$

Figure 6.21: Width dimension of infinitely long thin strip

Figure 6.22: Temperature distribution across half width of thin strip

If E is independent of temperature, then $G = 1$, $A_1 = 1/2$, $A_2 = 0$ and $A_3 = 3/2$. If the temperature distribution is symmetric, $A_2 = 0$ and the integrals involving $\gamma d\gamma$ will be zero.

In Equa. 6.11 the plastic strain ratio ϵ_p^* is a nonlinear function of the mechanical strain ϵ . They are related to each other through the stress-strain curve of the material. If the material properties are temperature-dependent, each point on the strip obeys a different stress-strain curve.

Equa. 6.11 is an integral equation for determining the elasto-plastic strain during the loading of a thermo-mechanically stressed (uniaxial) strip for a work hardening material with temperature-dependent properties.

6.6.2 Example definition

A $2c$ ($= 2$) wide infinitely long strip Figure 6.21 was subjected to a quadratic temperature distribution across its width Figure 6.22.

$$T = 980.0y^2 + 20.0 \quad (6.26)$$

No other mechanical loads were applied i.e. $P = 0$ and $M = 0$. The material properties corresponded to those of stainless steel. They were obtained from AS-GARD's temperature-dependent property data base. At 20° , $E = 0.2040 \times 10^{12}$, $\sigma_y = .2840 \times 10^9$ and $\alpha = .1463 \times 10^{-4}$.

6.6.3 Numerical solution of analytical formulation

The nonlinear equation Equa. 6.11 can be solved using the method of successive approximations in the following steps:

- a) Assume ϵ_p^* to be zero every where.
- b) Calculate ϵ from Equa. 6.11. This gives the elastic solution.
- c) For each value of ϵ find ϵ_p^* from the stress-strain curve. If the material properties are temperature dependent, a different curve is used at every station.
- d) Using Equa. 6.11 calculate a better approximation to ϵ . The integrals are evaluated numerically.
- d) Repeat steps c and d until convergence is obtained. The convergence is checked by monitoring the change in $\|\Sigma\epsilon\|$. When the sum is less than some pre fixed tolerance (here $\leq 10^{-3}$) convergence is satisfied.

8-noded quadrilateral elements

Figure 6.23: Finite element mesh and boundary conditions. Thermo-elasto-plastic analysis of thin strip

Axial stress comparison							
Y	T	Case I			Case II		
		fem	anal.	Error	fem	anal.	Error
	$^{\circ}C$	10^9	10^9	%	10^9	10^9	%
1.000	1000.000	-1.950	-1.957	-0.39	-1.590	-1.595	-0.28
0.889	794.321	-1.336	-1.343	-0.51	-1.413	-1.399	0.93
0.778	612.839	-0.794	-0.798	-0.50	-1.003	-1.024	-2.11
0.667	455.556	-0.325	-0.324	0.25	-0.500	-0.489	2.19
0.556	322.469	0.072	0.072	-0.79	-0.013	-0.014	-9.30
0.444	213.580	0.397	0.399	-0.43	0.395	0.395	0.07
0.333	128.889	0.650	0.653	-0.59	0.716	0.716	-0.07
0.222	68.395	0.830	0.833	-0.37	0.949	0.950	-0.05
0.111	32.099	0.939	0.942	-0.36	1.092	1.093	-0.06
0.000	20.000	0.975	0.979	-0.46	1.139	1.141	-0.18

Table 6.11: Axial stress values and % difference in finite element and theoretical solutions. Thermo-elastic analysis of the long and thin flat, Case I and II

Axial stress comparison						
Y	Case III			Case IV		
	fem	anal.	Error	fem	anal.	Error
	10^9	10^9	%	10^9	10^9	%
1.000	-0.718	-0.730	-1.66	-0.235	-0.2469	-4.90
0.889	-0.656	-0.668	-1.78	-0.308	-0.3091	-0.48
0.778	-0.602	-0.601	0.15	-0.394	-0.3984	-0.97
0.667	-0.473	-0.493	-4.38	-0.488	-0.5292	-8.42
0.556	-0.075	-0.075	-0.08	-0.390	-0.3381	13.48
0.444	0.249	0.271	-8.87	0.003	-0.0321	-
0.333	0.502	0.538	-7.30	0.306	0.3431	-12.21
0.222	0.576	0.570	1.04	0.523	0.5071	3.03
0.111	0.587	0.583	0.62	0.573	0.5878	-2.30
0.000	0.591	0.587	0.70	0.578	0.5578	3.62

Table 6.12: Axial stress values and % difference in finite element and theoretical solutions. Elasto-plastic analysis of the long and thin flat, Case III and IV

Plastic strain comparison				
Y	Case III		Case IV	
	fem	anal.	fem	anal.
	10^{-2}	10^{-2}	10^{-2}	10^{-2}
1.000	0.6765	0.66958	1.5392	1.5312
0.889	0.4056	0.39599	1.0686	1.0619
0.778	0.1665	0.15989	0.6262	0.6168
0.667	0.0000	0.00000	0.2240	0.1926
0.556	0.0000	0.00000	0.0000	0.0172
0.444	0.0000	0.00252	0.0000	0.0000
0.333	0.0000	0.00381	0.0000	0.0566
0.222	0.0519	0.06655	0.0000	0.0150
0.111	0.0997	0.11404	0.0397	0.0271
0.000	0.1156	0.12993	0.0584	0.0681

Table 6.13: Plastic strain values in finite element and theoretical solutions.
Elasto-plastic analysis of the long and thin flat, Case III and IV

Figure 6.24: Comparison of axial stress (σ_{11}) distribution. Elastic analysis of thin strip, isothermal material properties, Case I

Figure 6.25: Comparison of axial stress (σ_{11}) distribution. Elastic analysis of thin strip, nonisothermal material properties, Case II

Figure 6.26: Comparison of axial stress (σ_{11}) distribution. Thermo-elastic analysis of thin strip, isothermal material properties, Case III

Figure 6.27: Comparison of axial stress (σ_{11}) distribution. Elasto-plastic analysis of thin strip, nonisothermal material properties, Case IV

Figure 6.28: Comparison of plastic strain ($\bar{\epsilon}$) distribution. Elasto-plastic analysis of thin strip, isothermal material properties, Case III

Figure 6.29: Comparison of plastic strain (ϵ^p) distribution. Elasto-plastic analysis of thin strip, nonisothermal material properties, Case IV

The integrals were evaluated numerically using Simpson's rule [80]. Ten equally spaced stations (nodes in a finite element solution) were used for the numerical integration. A Fortran program was coded to evaluate this analytical solution using successive approximations. The results of the analytical solution are given in the next section.

6.6.4 Finite element solution.

A region of length $L (= 6c)$ and width c of the infinite strip was modelled using nine 12-node quadrilaterals, Figure 6.23. A plane stress state was assumed. The thermal nodal loads were applied in one increment. The boundary conditions allowed the flat strip to expand freely in lateral and longitudinal directions, Figure 6.23. The finite element solutions were obtained for the following four cases:

- Case I Elastic analysis with isothermal material properties.
- Case II Elastic analysis with nonisothermal material properties.
- Case III Elasto-plastic analysis with isothermal material properties.
- Case IV Elasto-plastic analysis with nonisothermal material properties.

In the elasto-plastic analysis, the modified Newton-Raphson iterative scheme with line search was employed. Isotropic work hardening and von Mises' yield criteria were used. The energy norm with a 10^{-5} tolerance was chosen as the convergence scheme.

The axial stress (σ_{11}) values and the percentage difference for the four cases are given in Table 6.11 and Table 6.12. Nodal stresses are projected from the Gauss points using weighted least squares and nodal averaging. The plastic strain values in Cases I and IV at the nodes ($x = 0$) are shown in Table 6.13. The comparison of analytical and finite element solutions is also shown graphically in Figures 6.24-29.

The average percentage differences in the stress values for Cases I, II, III and IV are 0.5, 1.5, 2.6 and 5.5 respectively. The plastic strain values also follow the same trend. Considering the assumptions in the theoretical derivation, the differences in the values between the theoretical and the finite element methods are reasonable.

The results also show that the stress and strain values for the cases of isothermal and nonisothermal properties differ substantially. For example, σ_{11} and ϵ^p values are 200% and 127% different respectively. Hence a substantial error may be introduced if isothermal properties are used when the temperatures are high.

The results are also useful in verifying the parts of the code which deal with the calculations of temperature-dependent material properties and thermal loads.

Figure 7.1: Geometry of edge welded bar

Chapter 7

Computing the Distortion to Edge Welds in Steel Bars

In welding, one of the most widely used fabrication techniques, the distortion of the workpiece is an important limitation of the process. The temperature cycle associated with the welding process causes distortion, the result of residual stresses. Often these distortions exceed the tolerance limits and work is rejected or reworked to meet the specification. Previous attempts have failed to provide a general and reliable method for predicting the distortion caused by the welding process [3,81,82, 83,84]. Attempts to use finite element analysis have had limited success [6,7,8,9,85, 86,87] in this area. Accurate and reliable predictions must be done either by a full 3-D analysis or by properly projecting the problem to a stress subspace such as plane strain. To keep the computing costs acceptable, simplifying approximations can be introduced through the boundary conditions. In the longer term, improvements in the FEM formulation can reduce the computing costs by orders of magnitude and permit full 3-D analysis without assuming simplified boundary conditions. In this thesis, analysis of an edge weld on a long bar was chosen to demonstrate the accuracy, power and usefulness of this finite element formulation for the analysis of weld problems.

Hogan [4] and Wan [83,84] have conducted experiments to measure deflections for .23% mild steel (AISI 1020) and stainless steel (AISI 304) bars. In Wan's experimental work, bars of the various dimensions were welded. In some cases the welding parameters were also varied. The present case considered conditions based on Hogan's experimental work. Hogan restricted his studies to a bar of 1m length (l_B) with a rectangular cross-section of 0.05m depth (d_B) and 0.0125 m thickness (h_B), Figure 7.1. The bars were annealed at 1050°C and allowed to cool slowly to relieve any residual stresses. Both types of bars were then descaled and cleaned by sand-blasting. The bar was electron beam welded on one edge. The camber was measured. Henceforth AISI 1020 and AISI 304 steel bars will be referred to as carbon steel and stainless steel bars respectively.

Figure 7.2: Double ellipsoidal Gaussian distributed power density model

In this work the camber of stainless steel and carbon steel bars was studied using finite element method in three steps. First, the temperature field $T(x, t)$ due to the heat source was computed. Convection, temperature-dependent thermal conductivity and specific heat were included. A 2-D 'plane strain' model was analyzed. The second step was an elasto-thermo-plastic stress analysis to compute the residual stresses and plastic strain caused by the temperature field of the weld. A 3-D plane strain analysis of a thin slice perpendicular to the longitudinal axis of the bar was performed. The final deflection (camber) was calculated by a 2-D elastic analysis of the simply supported bar with a residual stress load. The three steps of the analysis are discussed in detail in the next sections.

7.1 Heat transfer analysis.

A full, three-dimensional model with a sufficiently fine mesh should model the heat flow as accurately as errors in the material properties, geometry, heat input, convection and radiation parameters permit. By repeating the analysis with various estimates of the data, error bounds for the computed temperatures could be estimated. The computing cost of a full, three-dimensional finite element heat transfer analysis of an edge weld with the existing program would be prohibitive unless certain accelerators were incorporated. This was done by considering a 2-D cross-section normal to the weld direction.

In the 2-D region the longitudinal temperature gradient due to conduction, convection or radiation is, of course, ignored. It is assumed that $\partial T / \partial x_1 = 0$ for a weld moving in the direction x_1 . Every line parallel to x_1 is an isotherm. Physically, this can be achieved with a strip heat source such as a sheet electron beam, a laser beam or a resistance welding strip electrode. The accuracy of the cross-sectional model increases as the welding speed increases, as the thermal diffusivity decreases, and as locations farther from the heat source are considered. As these conditions are relaxed, the error in the model grows with the gradient $\partial T / \partial x_1 > 0$ and more heat flows in the x_1 direction. Andersson [9] attempted to assess this error by comparing the analytic solutions for a one and two-dimensional heat flow problems. He argued that the results gave a qualitative estimate of the error that should be expected in the cross-sectional approximation to three-dimensional reality. In the present case, the cross-sectional model is quite accurate for the weld speed used in the experimental work.

Figure 7.3: Geometry, finite element mesh and boundary conditions for transient heat cross-sectional heat transfer analysis

The welding process was modelled by a moving heat source (moving electron beam) which deposits the heat (power density) on the top edge. In the finite element analysis this source was idealized as a double ellipsoidal Gaussian distributed power density [88] as shown in Figure 7.2. The width of the double ellipsoid was set to half of the thickness of the bar ($b_1 = b_2 = 0.00625 \text{ m}$). The depth ($c_1 = c_2$) was set to 0.001 m . The front and back ellipsoid lengths (a_1 & a_2) were 0.00625 m and 0.009375 m respectively. The power was shared equally by the front and back ellipsoids.

The welding parameters required to calculate the heat flux and distributed source boundary conditions are the magnitude of heat input, Q , and the weld speed v_w . The heat input Q is

$$Q = V I \eta_w$$

where V is voltage, I is amperage of the electron beam and η_w is the weld efficiency. To model Q as a double ellipsoidal Gaussian power density distribution, the dimensions of the ellipsoid and an adjusting factor f_{gd} are required. The ellipsoid dimensions are related to the weld pool size. The f_{gd} factor is used to scale the heat input, Q , so that in the finite element approximation, $\hat{Q}$ is exactly equal to Q . This is necessary to correct the error in approximating a Gaussian distribution with the piecewise polynomial finite element basis functions. In this view the above equation is re-stated as

$$\hat{Q} = Q f_{gd} \quad (7.1)$$

The following values were used for the weld parameters. All of these values were taken from actual experimental work except that the value for the efficiency was not measured but was estimated by the experimenter rather than analyst. Hence, the data have been entirely defined by the experimenter.

$$\begin{aligned} V &= 60,000 \text{ volts} \\ I &= 0.035 \text{ amps} \\ \eta_w &= 85 \% \\ v_w &= 5.08 \times 10^{-3} \text{ m/sec} \\ Q &= 1785 \text{ watts} \end{aligned}$$

7.2 Geometry and boundary condition for heat transfer analysis.

The cross-section of the bar was subdivided into forty-four 2-D elements. Thirty-

Figure 7.4: Time-temperature history for steel bars. Results of heat transfer analysis.

500 °C contours

Figure 7.5: Spreading of HAZ in steel bars. Results of heat transfer analysis

six 8-node quadrilaterals and eight 6-node triangles were arranged as shown in *Figure 7.3*. Four-point and six-point Gaussian integration schemes were employed for the 8-node quadrilateral and the 6-node triangular elements respectively.

The heat was deposited as surface loads on the elements in the HAZ in 13 time steps (heating steps). The fluxes were predetermined by using the ellipsoid power density distribution. For the first six heating steps, the time increment was 0.24606 seconds ; i.e., the front ellipsoid required six time steps to pass through the cross-section. In the next seven time steps of 0.26364 seconds, the back ellipsoid intersected the cross-section. It is assumed that there are no convective losses in those elements intersecting the ellipsoid during the heating steps. The bar cross-section was allowed to cool during the next seventeen time steps. The time increments for the cooling steps were selected automatically by the program, depending on the maximum temperature change at any node during the load step.

The temperature-dependent values of thermal conductivity and specific heat were taken from *ASGARD's* property database. The liquid- solid phase change and associated latent heat were included as an apparent increase in the specific heat value. When latent heat effects are accounted for in this way, inaccuracies may arise. This was partially eliminated by relating the heat capacity and enthalpy as suggested by Moore [71].

The heat flow across the liquid and solid interface is discontinuous in the case of pure metals and cannot be modelled in the existing finite element method. The molten zone is assumed to be stagnant and the thermal conductivity above the melting point was increased to account for heat transfer due to stirring.

7.3 Results of heat transfer analysis.

The nonlinear heat transfer analysis was done for both stainless steel and carbon steel bars using the same mesh and boundary conditions.

Typical computed time-temperature curves are given in *Figure 7.4*. To determine the size of the load steps for the stress analysis, it is important to determine the size of the heat-affected region. For both cases, 500°C contours at different time steps are shown in the *Figure 7.5*. A study of the time-temperature curves for the nodes on the outer boundary of the heat-affected zone provides some guidance for selecting the time step size for the stress analysis. In both cases the HAZ expands

Figure 7 6: Geometry of bar slice for 3-D stress analysis

for approximately 2 seconds. However, the computed heat-affected and fusion zone sizes for both materials are quite different. The HAZ in case of the carbon steel bar is much smaller than the HAZ of the stainless steel bar. The main reason for the different HAZ size is the difference in the temperature-dependent c_p and k values.

7.4 Deflection and stress analysis of the bar

To compute the transient stress and strain history in the bar during and after welding, a fully, three-dimensional displacement analysis would be preferred. This would involve a large number of 3-D elements and a nonlinear incremental iterative solution which would be expensive. One of the alternatives is to perform a 2-D stress analysis of the cross-section. However, the plasticity formulation in the plane strain subspace is complex, especially for the kinematic work hardening materials. To reduce the problem size, the plane strain approximation is attractive and it is reasonable for long bars. The temperature along the length can vary considerably at a given instant of time. For example, if 1/400 of the 1 m bar length and a time interval 0.5 seconds are used, the welded edge has a temperature difference of over 1000°C. The 2-D dimensional stress analysis will give very poor approximation in such a case.

In this analysis, the plasticity formulation in full 3-D stress space was retained and the plane strain approximation (along the length) was introduced through the boundary conditions. A rectangular slice with dimensions of full depth ($d_S = d_B$), half width ($h_S = 1/2 h_B$) and length ($l_S = 1/400 l_B$) of the actual bar was analysed (Figure 7.6).

7.5 Temperature mapping and time step subdivision

The 3-D slice was subjected to the thermal loading history obtained from the 2-D cross-sectional heat transfer analysis. This required that the temperature be mapped from the nodes in the cross-sectional plane to the nodes in this 3-D slice. Knowing the position of the nodes and the velocity of the arc (time steps of the cross-sectional analysis) allowed this to be done. The task can be complicated when the sizes of the time steps are not equal and when the positions of the nodal points in the 3-D slice and 2-D cross-sectional analysis do not coincide. In this analysis, the nodal positions were the same but the sizes of the time steps were not equal.

Figure 7.7: Mapping and interpolation of nodal temperature due to subdivision of time steps

Thus, mapping of the temperature history was carried out in the following manner.

The two coordinates (x_2^{3D}, x_3^{3D}) of a point n in the 3-D slice are the same as the coordinates (x_1^{2D}, x_2^{2D}) of n in the 2-D cross-section. In the 3-D field the third coordinate, x_3^{3D} is related to the cross-sectional analysis by

$$x_3^{3D} = x^* + v_a t^{2D}$$

where x^* is the location of the arc when $t^{2D} = 0$; and v_a is the arc velocity in x_3^{3D} direction. t^{2D} is the time for which the temperature distribution in the cross-sectional analysis is known. This shows that the cross-sectional analysis is equivalent to a steady-state 3-D analysis in which heat flow is conducted to the x_2, x_3 plane; i.e., $\partial T / \partial x_1 = 0$.

Referring to Figure 7.7, let the arc be positioned on the plane marked as A. The temperature of a point in the plane A will be same as the temperature of the point in the 2-D cross-sectional analysis at time $t^{3D} = t^{2D}$. For the points in B and C planes, the temperatures will correspond to those points in the cross-sectional analysis at time $t^{3D} = t^{2D} - \Delta x_1 / v_a$ and $t^{3D} = t^{2D} + \Delta x_1 / v_a$ respectively. The half length of the slice is Δx_1 .

Experience and the literature [7] indicate that computational difficulties can be expected when inelastic strains occur in an analysis which has rapid temperature changes (heating steps are not small enough). Generally the stress analysis must have shorter time steps than those of the heating phase of the heat transfer analysis. On the other hand, it may be possible to take time steps larger than those of the cooling phase of the heat transfer analysis. Hence the time step of the 3-D stress analysis (t^{3D}) may not coincide with any time step of the 2-D heat transfer analysis (t^{2D}).

The temperature of a node in the 3-D slice at time t^{3D} is denoted as $t^{3D}T(n3)$ (an unknown). Similarly the known temperatures of a node in the cross-sectional analysis at time t_i^{2D} and t_j^{2D} are denoted as $t_i^{2D}T(n2)$ and $t_j^{2D}T(n2)$ respectively. Let $t_i^{2D} \leq t^{3D} < t_j^{2D}$. Using the following linear interpolation $t^{3D}T(n3)$, can be approximated as

$$t^{3D}T(n3) = t_i^{2D}T(n2) + \left[t_j^{2D}T(n2) - t_i^{2D}T(n2) \right] \frac{(t^{3D} - t_i^{2D})}{(t_j^{2D} - t_i^{2D})} \quad (7.2)$$

TIME - TEMPERATURE HISTORY

3D Slice Stress Analysis, Node 59

TIME - TEMPERATURE HISTORY

2D c/s Heat Flow Analysis, Node 1

Figure 7.8: Time-temperature history of node in heat transfer and mapped history for node in 3-D bar slice

Figure 7.9: Comparison between mapped and unmapped temperature histories

Figure 7.10: Finite element mesh for 3-D bar slice for stress analysis

The molten pool expanded up to $t = 2$ seconds for both cases. The heat-affected zone continued to expand up to $t = 4.7$ seconds and up to $t = 10$ seconds for the carbon steel and stainless steel respectively. In contrast to the time steps (Δt) of 0.24606 and 0.26364 seconds chosen in the cross-sectional analysis, time steps of 0.1 seconds were chosen for the 3-D slice while the molten zone was expanding. Slightly larger steps of 0.25 seconds were chosen while the heat-affected zone was expanding. Larger steps (0.5 seconds and bigger) were chosen during the cooling period.

Figure 7.8b shows the time-temperature curve for a node ($n2, x = 0, y = 0$) in the 2-D cross-sectional analysis. In Figure 7.8a the time-temperature curve for the corresponding node ($n3$) in the 3-D analysis is shown. The comparison between the two temperature histories is shown in Figure 7.9.

7.6 Elasto-thermo-plastic stress analysis of the 3-D slice.

The goal of performing 3-D stress analysis is to obtain the residual stress distribution (stress distribution at the end of thermal load cycle) in the slice. This stress distribution is obtained with the best available thermo-plasticity constitutive data.

7.6.1 Geometry and boundary conditions for the 3-D slice.

The 3-D bar slice is subdivided into eleven 20-node isoparametric brick elements as shown in Figure 7.10. A $2 \times 2 \times 2$ Gaussian integration scheme was used for each element.

The longitudinal (u_1) displacements of the faces marked as 'A' in Figure 7.6 were constrained to zero. Since only half of the bar width (z_3) was modelled, the u_3 displacement of face 'C' was constrained to zero. The rigid body motion in the z_3 direction was removed by constraining the 3 nodes on the bottom face from moving in the y direction.

The material properties were taken as temperature-dependent. The sources of the material property data were [9,90- 93]. During the time step Δt , the material properties were evaluated at time $t + \Delta t$. It should be noted that any temperature above the melting point temperature (T^M) of the material was set to T^M . Therefore the liquid effects were ignored. The strain hardening accumulated during the welding process was completely relieved, when melting took place, by zeroing the accumulated effective plastic strain. Recrystallisation induced plastic strain relief

Time t seconds	Temperature	
	'T°C ◇	'T°C ●
.3000E+00	0.200000E+02	.200000E+02
.4000E+00	0.200000E+02	.200000E+02
.5000E+00	0.209552E+02	.200098E+02
.6000E+00	0.170034E+03	.201344E+02
.7000E+00	0.309113E+03	.202590E+02
.8000E+00	0.502633E+03	.209760E+02
.9000E+00	0.729781E+03	.220588E+02
.1000E+01	0.966626E+03	.237099E+02
.1100E+01	0.125531E+04	.283984E+02
.1200E+01	0.154400E+04	.330870E+02
.1300E+01	0.177871E+04	.434540E+02
.1400E+01	0.198996E+04	.562898E+02
.1500E+01	0.217354E+04	.724757E+02
.1600E+01	0.226776E+04	.994848E+02
.1700E+01	0.236198E+04	.126494E+03
.1800E+01	0.231385E+04	.166836E+03
.1900E+01	0.222457E+04	.211031E+03
.2000E+01	0.214058E+04	.256716E+03
.2250E+01	0.201226E+04	.393927E+03
.2500E+01	0.188285E+04	.531990E+03
.2750E+01	0.174248E+04	.648981E+03
.3000E+01	0.160130E+04	.736115E+03
.3500E+01	0.137322E+04	.826136E+03
.4000E+01	0.124225E+04	.847761E+03
.4500E+01	0.114746E+04	.842958E+03
.5000E+01	0.107589E+04	.828671E+03
.5500E+01	0.101896E+04	.811385E+03
.6000E+01	0.969986E+03	.793147E+03
.6500E+01	0.933168E+03	.775584E+03
.7000E+01	0.896689E+03	.758040E+03
.7500E+01	0.868778E+03	.742326E+03
.8000E+01	0.841502E+03	.726747E+03
.8500E+01	0.815478E+03	.711630E+03
.9000E+01	0.794805E+03	.698487E+03
.1500E+02	0.624361E+03	.578462E+03
.3000E+02	0.458318E+03	.440414E+03
.6000E+02	0.341511E+03	.334400E+03
.1000E+03	0.284107E+03	.280555E+03
.1800E+03	0.244101E+03	.242733E+03
.2500E+03	0.229032E+03	.228461E+03
.3650E+03	0.187730E+03	.187730E+03
.5883E+03	0.112638E+03	.112638E+03
.9232E+03	0.675829E+02	.675829E+02
.1369E+04	0.202749E+02	.202749E+02

◇ Temperature values for Node 1 (See Figure 7.10).

● Temperature values for Node 23 (See Figure 7.10).

Table 7.1: Typical time-temperature values of two node in HAZ for stainless steel bar

Figure 7.11 : Heating curve. History of thermal loads for 3-D stress analysis for carbon steel bar

also occurs at temperatures near the melting point temperature. In this analysis this is modelled using the following empirical formula proposed by Friedman [7].

$$\epsilon_p = f_r \epsilon_p \quad (7.3)$$

where

$$\begin{aligned} f_r &= e^{0.03(T-T^M)} & \text{if } T \leq T^M \\ f_r &= 0 & \text{for } T > T^M \end{aligned}$$

The nonlinear finite element equations are solved with a modified Newton-Raphson iterative scheme. Normally in this scheme the initial stiffness matrix, $^0K^L$, is used. For stress analysis involving transient thermal loads, the elastic stiffness matrix $^0K^L$ is re-computed at the beginning of each increment because the material properties are temperature-dependent. The plastic or tangential stiffness matrix is not used for the first iteration because it would be very wrong for a load reversal.

The energy criterion with a tolerance of 0.01 was used to check the convergence. The von Mises' yield criteria with work hardening was employed. For the carbon steel and stainless steel bars, the work hardening parameter, β , was taken as 0 and 0.8 respectively.

The bar was subjected to the incremental thermal load by prescribing a starting and ending temperature at each node in a time step. No other mechanical loads were applied. For the carbon steel bar, the thermal cycle had 34 time steps. In case of the stainless steel bar the thermal cycle was divided into 44 time steps. Temperature changes at two nodes on the boundary of the heat-affected zone are given in Table 7.1 for stainless steel and shown in Figure 7.11 for carbon steel.

7.7 Results of the 3-D slice analysis.

Keeping the same boundary conditions and respective thermal loading history, the elasto-thermo-plastic stress analysis of the 3-D slice was done for the two types of the bars. Once again the material properties for 0.23% carbon steel and the 304 stainless steel were taken from the ASGARD data base.

To perform the very large numerical calculations required by both analyses a dedicated Apollo work station was used. The tasks of formulating the theory, developing the software and running such huge problems are of equal importance. In idealizing the problem, the need to consider the computing cost and time was mentioned. Consider the following data obtained during the solution. For the carbon

Figure 7.13: Transient history of longitudinal stress, σ_{11} , along bar depth. Results of 3-D stress analysis

Figure 7.13: Transient history of effective plastic strain, ϵ_p , along bar depth.
Results of 3-D stress analysis

$-z_2$	Longitudinal stress σ_{11}			
$m \times 10^2$	GPa			
	t_{25}	t_{30}	t_{35}	t_{40}
0.025	.454	.592	.846	1.163
0.144	.358	.498	.750	1.048
0.263	.341	.490	.740	1.034
0.394	.035	.290	.592	.939
0.573	-.134	.147	.443	.721
0.764	-.514	-.483	-.198	.306
1.020	-.347	-.410	-.352	.239
1.410	-.086	-.202	-.274	.172
1.930	-.008	-.140	-.211	-.087
2.760	-.004	-.015	-.170	-.236
4.010	.000	-.001	-.002	-.225

Table 7.2: Transient history of longitudinal stress, σ_{11} , along bar depth. Results of 3D stress analysis

$-x_2$	Effective Plastic strain ϵ_p			
$m \times 10^3$	10^{-3}			
	t_{25}	t_{30}	t_{35}	t_{40}
0.025	4.290	4.490	4.870	5.450
0.144	2.630	2.970	3.400	4.070
0.263	2.080	2.510	2.930	3.610
0.394	2.540	2.730	3.030	3.600
0.573	1.930	1.930	2.160	2.690
0.764	1.020	1.040	1.040	1.390
1.020	.570	.700	.720	.860
1.410	.160	.390	.570	.570
1.930	.000	.030	.240	.320
2.760	.000	.000	.010	.230
4.010	.000	.000	.000	.220

Table 7.3: Transient history of effective plastic strain, ϵ_p , along bar depth.
Results of 3D stress analysis

steel bar, a total of 134 iterations were required over 34 load steps. It took 400 minutes of real time or 248 minutes of CPU time on an Apollo DN660. The corresponding data for the stainless steel bar were 440 minutes of real time and 300 minutes of the CPU time. The 44 load steps required a total of 160 iterations. Each bar had 420 degrees of freedom (3×140 nodes).

The goal of the stress analysis is to determine the state of stress at the end of the thermal cycle (residual stress) and to find the final deflection due to the weld. The transient stress and plastic strain history are also important since they give an account of the stress state at the various stages of the weld. In finite element analysis the optimal sampling points for stress and strain are the Gauss points. Reporting the full transient history (either graphically or numerically) is prohibitive. Hence eleven Gauss points, one from each element, along the bar depth were selected. The transient histories of the longitudinal stress (σ_{11}) and effective plastic strain (ϵ_p) are given in Figures 7.12-13 and in Table 7.2-3.

The nodal stresses were projected from the Gauss points for each element at all time steps. These stresses show that the σ_{11} stress component was dominant throughout the time history. It was also observed that σ_{22} and σ_{33} values became as large as the residual σ_{11} during the cycle. Also the shear stress components are far from zero during the transient loading history; e.g., at time $t = 0.9$ seconds, the shear stress components, σ_{23} and σ_{31} are 150 and -20 MPa respectively. It should be noted that in a 2-D plane strain case, these components would have been assumed zero. Hence the use of a full nine-dimensional stress space (3-D analysis) instead of a subspace (2-D analysis) was better, if not the best model. The heat-affected and fusion zones experience severe plastic deformation. The compressive stress in the HAZ is very high in comparison to the compressive and tensile stresses in the bar region outside HAZ.

The deflection or camber cannot be calculated directly from the results of the 3-D slice held in-plane strain. The results of the 3-D slice analysis, particularly the residual stress distribution, can be used to compute the camber or deflection as shown below.

7.8 Elastic 2-D deflection analysis of the bar

Neglecting end effects, the distribution of residual stress in the bar after a long

Figure 7.14: Geometry, boundary conditions and finite element mesh for elastic 2-D beam deflection analysis

Finite Element Results		
Distance	Displacement $\diamond$	
from midspan	Carbon	Stainless
m	mm	mm
0.500	0.0000	00.0000
0.475	0.4492	01.4191
0.450	0.8776	02.7629
0.425	1.2812	04.0360
0.400	1.6632	05.2349
0.375	2.0216	06.3618
0.350	2.3576	07.4153
0.325	2.6704	08.3963
0.300	2.9606	09.3042
0.275	3.2278	10.1395
0.250	3.4722	10.9017
0.225	3.6938	11.5913
0.200	3.8924	12.2078
0.175	4.0683	12.7518
0.150	4.2211	13.2225
0.125	4.3514	13.6209
0.099	4.4582	13.9456
0.075	4.5431	14.1987
0.050	4.6040	14.3773
0.025	4.6437	14.4856
0.000	4.6580	14.5175

$\diamond$ Displacement of the bottom edge

Table 7.4: Vertical deflection of unwelded bar edge. Finite element results of edge welded steel bars

Experimental Results		
Distance	Displacement $\diamond$	
from midspan	Carbon	Stainless
<i>m</i>	<i>mm</i>	<i>mm</i>
0.5000	0.0000	0.000
0.4572	0.9525	2.972
0.4064	1.7780	5.512
0.3556	2.5527	7.849
0.3048	3.1623	9.906
0.2540	3.7338	11.633
0.2032	4.1656	12.954
0.1524	4.4958	14.021
0.1016	4.7117	14.707
0.0508	4.8895	15.164
0.0000	4.9276	15.342

$\diamond$ Displacement of the bottom edge

Table 7.5: Vertical deflection of unwelded bar edge. Experimental results of edge welded steel bars

Carbon steel **Stainless steel**

Shown displacements are 5 times magnified

Figure 7.15: Camber of steel bars. Results of 2-D beam deflection analysis

Figure 7.16: Comparison between finite element and experimental results of vertical deflection of edge welded steel bars

period of cooling is likely to be uniform with respect to the length. In this case the residual stress distribution in the slice, which is very thin, can be prescribed throughout the beam length. Taking the residual stress as an initial stress, the behaviour of the full bar can be approximated by a 2-D analysis of the bar cross-section in the transverse direction. The boundary conditions are those of a simply supported beam and a plane stress approximation can be used.

In this case the residual stress distribution in the slice is mapped as a simple function of the depth of the beam. Each stress component ; *e.g.*, σ_{11} , at all Gauss points at a particular beam depth was summed, averaged and saved as a function of the beam depth. In the 2-D plane stress analysis these functions were used to find, σ'_{ij} , the initial stress components at each Gauss point as a function of its depth coordinate.

Using symmetry, half of the beam length was analysed. As shown in Figure 7.14, forty 8-node quadrilaterals were used with simply supported boundary conditions. As mentioned, the only load applied was the initial stress.

7.9 Comparison of the bar deflections

The important results of this analysis are the bar camber or deflection which can be compared with the experimental values. Table 7.4 and Figure 7.15 show the vertical (in the depth direction) deflection of the nodal points (along the bar length) found using the finite element analysis. The difference between the predicted deflections of the bar for the different materials is particularly significant. The bottom central node has a vertical deflection of 0.047 m and 0.145 m for the carbon steel and stainless steel bars respectively. The analysis predicts that the stainless steel bar will deflect 3.1 times more than the carbon steel bar.

The experimental values measured along the edge opposite the welded edge are reported by Hogan and Wan. These values are given in Table 7.5. The comparison between the experimental and finite element solutions is shown in Figure 7.16.

The correlation between the experimental and analytical values of the bar deflection is excellent. This suggests that the approximations in the boundary conditions introduced only small errors. The results also prove that the temperature-dependent nonlinear constitutive equation and the incremental formulation are very effective in predicting residual deflections and stresses in weld problems.

In general, it can now be said that applying accelerators, such as multilevel

substructuring and adaptive FEM, will make it feasible to compute residual stresses in large complex structures without approximations such as plane strain or plane stress.

7.10 Residual stress field

Ideally the computed stress field should be compared directly to the experimentally measured stress field. Since this experimental data is not available, only indirect estimates of the accuracy of the computed stress field are feasible.

The fact that the camber is due to the real residual stress field and the camber is computed accurately is a necessary but not sufficient condition. Many different stress fields could cause the same camber.

Published values of the peak residual stress in stainless steel welds range from 200 to 400 MPa which is substantially lower than the 1100 MPa computed in this analysis. The computed value is entirely consistent with the constitutive properties that have been assumed. Specifically, stainless steel was assumed to have a yield stress of 0.242 GPa; a plastic modulus of 23×10^9 , and $1 - \beta = 0.8$. In a uniaxial tensile test with a strain of 0.06, the $\sigma_{11} = 0.242 + 0.8 \times 23 \times 10^9 = 1.1$ GPa. In Figure 7.12 and Table 7.2 the computed stress of 1163 MPa can clearly be attributed to strain hardening. If this stress is too high, then the constitutive data is incorrect. If a plastic modulus of 4×10^9 Pa had been assumed, the uniaxial tensile stress would have been reduced to 442 MPa.

In addition to the uncertainty in the constitutive data, little is known about thermo-elasto-plasticity during phase transformations. This is a deep issue that requires further research.

Another factor is that the bars were simply supported during the welding. The 3-D slice assumed the bars were held quite rigidly although not as rigidly as in plane strain. In the plane stress analysis, the bar was allowed to bend. This reduced the peak stress from 1163 MPa to 863 MPa. It is entirely possible that had the 3-D

slice allowed bending the stresses would have been even lower.

Until the residual stress is measured and a direct comparison made with the computed stresses, and the constitutive data measured directly, the accuracy of the computed stresses cannot be estimated with more confidence.

Chapter 8

Conclusions and Future Research

8.1 Conclusions

A software system has been designed, developed, tested and demonstrated for thermo-elasto-plastic analysis with the immediate goal of computing the distortion and residual stress of welds. The software system includes: an explicit presentation of the equations of thermo-plasticity for temperature dependent elastic and plastic properties; a nonlinear incremental finite element analysis formulation and code; test problems to verify the formulation and the code; and, as a demonstration problem, the analysis of camber due to edge welds in AISI 1020 and AISI 304 stainless steel bars.

The values of the distortion (camber) computed by the finite element analysis for bars of AISI 1020 steel and AISI 304 stainless steel due to edge welds is 4.7 mm and 14.5 mm respectively. This is in excellent agreement with the experimentally measured values of 4.9 mm and 15.3 mm respectively.

An isotropic hardening model for AISI 1020 steel, which was used because of the lack of data for temperature-dependent isotropic-kinematic hardening, agrees quite well with experiment despite an expected Bauschinger effect. An attractive hypothesis is that phase transformations, austenite-ferrite and solid-liquid, effectively reset the yield surface to the origin, which reduces the effect of load reversals. In the finite element analysis, the effective plastic strain was gradually relieved up to the melting point temperature and set to zero above it. Neglecting thermal loads above the melting point temperature is shown to be acceptable for computing camber and residual stress. It is not likely to be acceptable for an analysis of hot cracking which depends on strain and stress near the melting point.

A procedure has been presented for mapping data from a nonlinear transient finite element analysis of heat transfer to a thermo-elasto-plastic analysis of a clamped

bar, and then to the distortion and residual stress in a simply supported beam. The next generation of software must develop methods to automate such mappings.

It has been observed that the actual stress state may not be plane strain. The magnitude of shear stress components were far from zero in the transient temperature loading. The analysis of a 3-D slice of the beam held in plane strain was shown to approximate the actual stress state in a superior manner.

8.2 Future research

The most important future research topic is clearly the development of fully automatic finite element analysis systems. They can be considered automatic solvers for partial differential equations. The geometry, constitutive relations, initial conditions, boundary conditions and required accuracy must be defined independently of the discretization actually used. The software must create a near-optimum mesh and choose time steps automatically. The software must estimate errors and adapt the mesh and time steps to achieve the specified accuracy. The methodology and software developed in this thesis can be the code for a fully automatic finite element analysis system.

The ability to solve problems in continuum mechanics should be extended to include large displacements, large rotations and finite strains. Dynamic analysis should be emphasized in future research. Even though inertia effects are small, time is the fundamental parameter and therefore should play a fundamental role.

In the long term, the greatest limitation is the availability of constitutive models and data. Since software will allow such data to be used in design and analysis, the demand for it will grow rapidly.

Finally, the software should be tested on as many welding situations as possible to verify existing capabilities and identify any deficiencies in its performance.

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APPENDIX A

0.25% Carbon Steel Material Properties							
T	E	ν	α	σ_y	$E_T \Phi$	c_p	k
$^{\circ}\text{C}$	GPa		$/^{\circ}\text{C}$	MPa	GPa	J/m ²	W/m- $^{\circ}\text{C}$
			10^{-6}			10^6	
20	206.2	0.29	1.17	303.0	22.9	343.0	52.70
100	203.0	0.31	1.17	275.0	22.6	365.0	51.90
200	200.0	0.33	1.21	240.0	22.2	387.0	51.10
300	200.0	0.35	1.28	205.0	22.2	408.0	48.60
400	165.0	0.37	1.34	171.0	18.3	439.0	44.40
500	100.0	0.39	1.39	136.0	11.1	473.0	42.70
600	60.00	0.41	1.44	101.0	6.67	525.0	39.40
700	40.00	0.43	1.48	66.00	4.44	590.0	35.60
800	30.00	0.44	1.48	3.000	3.33	872.0	31.80
900	20.00	0.45	1.48	3.000	2.22	643.0	26.00
1000	10.00	0.48	1.48	3.000	1.11	493.0	26.40
1100	10.00	0.48	1.48	3.000	1.11	490.0	27.20
1200	10.00	0.48	1.48	3.000	1.11	490.0	28.50
1300	10.00	0.48	1.48	3.000	1.11	502.0	29.70
1400	10.00	0.48	1.48	3.000	1.11	508.0	29.70
1500	10.00	0.48	1.48	3.000	1.11	508.0	30.50
1600	10.00	0.48	1.48	3.000	1.11	508.0	105.0
1700	10.00	0.48	1.48	3.000	1.11	508.0	105.0
1800	10.00	0.48	1.48	3.000	1.11	508.0	105.0
1900	10.00	0.48	1.48	3.000	1.11	508.0	105.0
2000	10.00	0.48	1.48	3.000	1.11	508.0	105.0

Φ Given E^* corresponds to 1% E in Eqn. 4.35

Table A.1: Temperature dependent material properties for 0.25% carbon steel

304 Stainless Steel Material Properties♡							
T	E	ν	α	σ_y	E_p ◆	c_p	k
$^{\circ}C$	GPa		$/^{\circ}C$	MPa	GPa	J/m ³	W/m- $^{\circ}C$
			10^{-5}			10^4	
20	204.0	0.263	1.72	242.6	23.0	293.0	12.80
100	192.0	0.267	1.72	208.6	21.3	296.0	14.70
200	182.0	0.272	1.79	177.0	20.2	299.0	16.60
300	176.2	0.281	1.79	158.0	19.6	300.9	18.10
400	168.1	0.292	1.84	146.3	18.7	306.7	19.50
500	158.9	0.307	1.84	133.2	17.7	312.7	20.82
600	149.9	0.317	1.84	118.4	16.7	321.0	22.23
700	140.6	0.322	1.88	100.2	15.6	335.3	23.60
800	129.0	0.327	1.88	58.62	14.3	340.0	25.00
900	113.1	0.331	1.88	51.00	12.6	378.0	26.35
1000	102.5	0.331	1.88	51.00	11.4	379.0	27.75
1100	90.00	0.335	1.88	51.00	10.0	381.5	29.10
1200	75.00	0.350	2.05	51.00	8.33	387.3	30.50
1300	60.00	0.370	2.05	51.00	6.67	393.2	31.90
1400	45.00	0.390	2.05	51.00	5.00	399.4	33.25
1500	30.00	0.425	2.05	51.00	3.33	410.1	95.00
1600	15.00	0.450	2.05	51.00	1.67	410.7	95.00
1700	10.00	0.480	2.05	51.00	1.11	416.4	95.00
1800	10.00	0.480	2.05	51.00	1.11	422.4	95.00
1900	10.00	0.480	2.05	51.00	1.11	428.2	95.00
2000	10.00	0.480	2.05	51.00	1.11	428.2	95.00

♡ [90]

◆ Given E' corresponds to 1% E in Equa. 4.35

Table A.2: Temperature dependent material properties for 304 stainless steel

Fine Grain Steel Material Properties [♡]					
<i>T</i>	<i>E</i>	ν	α	σ_y	<i>E_p</i>
^{°C}	GPa		/°C	MPa	GPa
			10 ⁻³		
20	208.1	0.260	1.17	347.9	2.20
100	202.7	0.273	1.17	327.9	2.10
200	195.9	0.285	1.21	302.9	2.00
300	189.2	0.297	1.28	277.9	1.90
400	182.4	0.310	1.34	252.9	1.80
500	153.6	0.326	1.39	182.3	1.30
600	124.7	0.342	1.44	111.8	.50
700	95.90	0.357	1.48	41.20	.010
800	67.10	0.373	1.48	31.40	.010
900	38.10	0.389	1.48	21.60	.010
1000	9.400	0.405	1.48	11.80	.010
1100	7.800	0.421	1.48	9.800	.010
1200	6.300	0.437	1.48	7.900	.010
1300	4.700	0.453	1.48	5.900	.010
1400	3.100	0.468	1.48	3.900	.010
1500	1.600	0.480	1.48	2.000	.010
1600	1.600	0.480	1.48	2.000	.010
1700	1.600	0.480	1.48	2.000	.010
1800	1.600	0.480	1.48	2.000	.010
1900	1.600	0.480	1.48	2.000	.010
2000	1.600	0.480	1.48	2.000	.010

♡ [9]

Table A.3: Temperature dependent material properties for fine grain steel

Least Square Stress Projection in FEM

B.1 Introduction

Stresses are most accurately and conveniently computed at Gauss points but in post-processing most users prefer nodal values of stress. For optimal sampling points, such as 2×2 Gaussian integration with 8-noded quadrilaterals, Hinton and Campbell [94] derived a projection matrix which is useful. Because the 2×2 integration scheme contains a zero energy mode or mechanism, many users would prefer to accept the extra cost and slight loss of accuracy of a 5 or 9 point integration scheme to avoid the risk of gross error from a mechanism [95,25]. For these schemes the present work gives a method for computing the projection matrices that map the stresses from the Gauss points to the nodes in a least square sense. This method differs from Hinton and Campbell in that the projection surface is no longer required to contain the Gauss point stresses.

The problem can be stated in the following mathematical terms. Find functions $g(x)$ and $h(x)$ that approximate the function $f(x)$ and its gradient $\Delta f(x)$ respectively. In particular, $g(x)$ may be an excellent approximation to $f(x)$ while $\delta g(x)$ may be a particularly poor approximation to $\Delta f(x)$. It is for this reason that most FEM textbooks advise against computing stresses at the nodes. To the authors' knowledge, this problem has not been studied by mathematicians for the important case of piece-wise polynomial approximations.

There is no question that specifying a particular element, e.g. an 8-noded isoparametric quadrilateral, specifies the polynomial in (ξ_1, ξ_2) that defines the displacement field. The function space of the basis functions in (x, y) is less clear [96]. However, if a stress field polynomial could be chosen freely, which polynomial should be chosen? Zienkiewicz page 280 [63] argues that, "With a finite element approximation of degree p the Lu derivatives can at best be approximated by local polynomials of degree $(p-m)$ where m is the differentiation order implied by the operator L . Incomplete polynomials or other terms often add a spurious variation which still, however, in a least square sense approximates the true answer." Should the stress field polynomial be the complete polynomial of degree $(p-m)$ or should it be the incomplete polynomial obtained by differentiating the polynomial p . The present work will show and argue that the latter is better, but does not prove this assertion.

Given an element and integration scheme, and having chosen a stress field polynomial, the present work proposes to fit the stress field polynomial to the Gauss point values in a least square sense. If the number of Gauss points equals the number of terms in the polynomial, the polynomial fits the Gauss point values exactly. Hinton and Campbell [94] have done this exact fit of bilinear polynomials for 2×2 integration with quadrilaterals and trilinear polynomials for $2 \times 2 \times 2$ integration for bricks. However, when the number of Gauss points exceeds the number of terms in the polynomial, their approach fails. Our approach of fitting the polynomial to the Gauss point values in a least square sense, allows projection matrices to be derived for any element and any integration scheme. For the 2×2 and $2 \times 2 \times 2$ schemes, this reproduces Hinton and Campbell's matrices. Therefore it is a generalization of their method.

Reduced integration (RI) is used for four reasons.

- 1) With fewer integration points the cost of computing an element stiffness matrix and load vector is reduced.
- 2) The error incurred by using reduced (inexact) integration compensates for the truncation error caused by using polynomial basis functions of finite degree [64]
- 3) RI can analyze some problems with constraints in which full integration (FI) fails. These include incompressibility, beams, plates and shells. The explanation is that RI is equivalent to a penalty formulation in which the stiffness matrix associated with the penalty parameter becomes singular with RI [63,97].
- 4) The RI Gauss points permit convenient stress projections to the nodes. For example, a straight line projector is easily fitted to the two Gauss points in a three node element. A bilinear projector is conveniently fitted through the 2×2 Gauss points in 8 or 9-noded quadrilaterals.

RI is frequently not used because of the dangers of zero-energy or near zero-energy strain modes. In the presence of such modes, FEM results are often meaningless [95].

Argument one is not compelling as the saving is small. The situation with argument two is similar. The possibility of a small gain in accuracy hardly justifies the risk of zero-energy modes that destroy the solution.

Argument three is compelling in cases where a solution could not be obtained with FI. RI is used to modify the stiffness matrix and hence displacement field. The primary objective is not to compute or project the stress field.

3 3

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Present work focuses on argument four. In particular the present work gives a method and specific projection matrices for the convenient projection of stresses from Gauss points to nodes. The results are demonstrated to be just as accurate as those obtained with RI, provided the nodal displacements are the same. To enlarge on this point, RI and FI lead to different stiffness matrices and load vectors. This is the reason that argument three is compelling when constraints cause ill-conditioned stiffness matrices with FI. The present work only deals with those cases in which constraints do not cause ill-conditioning. However, it is worth mentioning that in those cases where an optimal scheme cannot be used because of the need to provide a singular stiffness matrix [63, page 288], the projectors presented here promise to provide the accuracy of optimal sampling points while retaining the singular stiffness matrix associated with the penalty parameter. Specifically the new methodology presented here renders argument four obsolete in such cases.

The question immediately arises as to what weights should be used? The present work will show by numerical tests, but not prove, that the square root of the Gauss point weights performs well. In fact, when the exact solution lies in the function space of the element and the element displacements are exact, this method produces exact stresses in all cases tested even when the Gauss point stresses are in error!

B.2 Theory

A Gaussian integration scheme with m sampling points is used in an n node element. The objective is to project the Gauss point stresses, σ^{gp} to the nodal stress, σ^{nd} ; i.e. find the matrix P such that

$$[\sigma^{nd}]_n = [P]_{n,m} [\sigma^{gp}]_m \quad (B.1)$$

where n is the number of nodes and m is the number of Gauss points.

B.2.1 Bilinear projector

A bilinear projector is defined to be the following polynomial

$$\sigma^{nd}(\xi_1, \eta) = a_1 + a_2 \xi_1 + a_3 \xi_2 + a_4 \xi_1 \xi_2 \quad (B.2)$$

is a complete linear polynomial with one quadratic term. In matrix form, the

nodal stresses are given in terms of the nodal local coordinates (ξ_1, ξ_2) by

$$\begin{bmatrix} \sigma^{nd^1} \\ \sigma^{nd^2} \\ \sigma^{nd^3} \\ \vdots \\ \sigma^{nd^n} \end{bmatrix} = \begin{bmatrix} 1 & \xi_1^1 & \xi_2^1 & \xi_1^1 \xi_2^1 \\ 1 & \xi_1^2 & \xi_2^2 & \xi_1^2 \xi_2^2 \\ 1 & \xi_1^3 & \xi_2^3 & \xi_1^3 \xi_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_1^n & \xi_2^n & \xi_1^n \xi_2^n \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \quad (B.3)$$

$$[\sigma^{nd}]_n = [B]_{n,c} [a]_c \quad (B.4)$$

The available data are the values of the stress at m Gauss points (gp) and the position of the Gauss points in local coordinates. This data can be written in the following form

$$\begin{bmatrix} 1 & \xi_1^1 & \xi_2^1 & \xi_1^1 \xi_2^1 \\ 1 & \xi_1^2 & \xi_2^2 & \xi_1^2 \xi_2^2 \\ 1 & \xi_1^3 & \xi_2^3 & \xi_1^3 \xi_2^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_1^m & \xi_2^m & \xi_1^m \xi_2^m \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \sigma^{gp^1} \\ \sigma^{gp^2} \\ \sigma^{gp^3} \\ \vdots \\ \sigma^{gp^m} \end{bmatrix} \quad (B.5)$$

or

$$[A]_{m,c} [a]_c = [b]_m \quad (B.6)$$

B.2.2 Trilinear projection for brick elements

For the brick elements (14 point Gauss rule [98]), the following polynomial is a trilinear projector.

$$\sigma^{nd} = a_1 + a_2 \xi_1 + a_3 \xi_2 + a_4 \xi_3 + a_5 \xi_1 \xi_2 + a_6 \xi_2 \xi_3 + a_7 \xi_3 \xi_1 + a_8 \xi_1 \xi_2 \xi_3 \quad (B.7)$$

In this case $c = 8$ and Equa. B.3 and Equa. B.5 will be ,

$$\begin{bmatrix} 1 & \xi_1^1 & \xi_2^1 & \xi_3^1 & \xi_1^1 \xi_2^1 & \xi_2^1 \xi_3^1 & \xi_3^1 \xi_1^1 & \xi_1^1 \xi_2^1 \xi_3^1 \\ 1 & \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_1^2 \xi_2^2 & \xi_2^2 \xi_3^2 & \xi_3^2 \xi_1^2 & \xi_1^2 \xi_2^2 \xi_3^2 \\ 1 & \xi_1^3 & \xi_2^3 & \xi_3^3 & \xi_1^3 \xi_2^3 & \xi_2^3 \xi_3^3 & \xi_3^3 \xi_1^3 & \xi_1^3 \xi_2^3 \xi_3^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_1^{14} & \xi_2^{14} & \xi_3^{14} & \xi_1^{14} \xi_2^{14} & \xi_2^{14} \xi_3^{14} & \xi_3^{14} \xi_1^{14} & \xi_1^{14} \xi_2^{14} \xi_3^{14} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_8 \end{bmatrix} = \begin{bmatrix} \sigma^{gp^1} \\ \sigma^{gp^2} \\ \sigma^{gp^3} \\ \vdots \\ \sigma^{gp^{14}} \end{bmatrix} \quad (B.8)$$

B.2.3 Quadratic projector

A quadratic projector is defined to be the following polynomial

$$\sigma^{nd} = a_1 + a_2 \xi_1 + a_3 \xi_2 + a_4 \xi_1^2 + a_5 \xi_1 \xi_2 + a_6 \xi_2^2 + a_7 \xi_1^2 \xi_2 + a_8 \xi_1 \xi_2^2 \quad (B.9)$$

It is a complete quadratic with two cubic terms.

In this case Equa. B.3 will take the following form

$$\begin{bmatrix} \sigma^{node^1} \\ \sigma^{node^2} \\ \sigma^{node^3} \\ \vdots \\ \sigma^{node^n} \end{bmatrix} = \begin{bmatrix} 1 & \xi_1^1 & \xi_2^1 & \xi_1^{1^2} & \xi_1^1 \xi_2^1 & \xi_2^{1^2} & \xi_1^{1^2} \xi_2^1 & \xi_1^1 \xi_2^{1^2} \\ 1 & \xi_1^2 & \xi_2^2 & \xi_1^{2^2} & \xi_1^2 \xi_2^2 & \xi_2^{2^2} & \xi_1^{2^2} \xi_2^2 & \xi_1^2 \xi_2^{2^2} \\ 1 & \xi_1^3 & \xi_2^3 & \xi_1^{3^2} & \xi_1^3 \xi_2^3 & \xi_2^{3^2} & \xi_1^{3^2} \xi_2^3 & \xi_1^3 \xi_2^{3^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_1^n & \xi_2^n & \xi_1^{n^2} & \xi_1^n \xi_2^n & \xi_2^{n^2} & \xi_1^{n^2} \xi_2^n & \xi_1^n \xi_2^{n^2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_8 \end{bmatrix} \quad (B.10)$$

and Equa. B.5 will be ,

$$\begin{bmatrix} 1 & \xi_1^1 & \xi_2^1 & \xi_1^{1^2} & \xi_1^1 \xi_2^1 & \xi_2^{1^2} & \xi_1^{1^2} \xi_2^1 & \xi_1^1 \xi_2^{1^2} \\ 1 & \xi_1^2 & \xi_2^2 & \xi_1^{2^2} & \xi_1^2 \xi_2^2 & \xi_2^{2^2} & \xi_1^{2^2} \xi_2^2 & \xi_1^2 \xi_2^{2^2} \\ 1 & \xi_1^3 & \xi_2^3 & \xi_1^{3^2} & \xi_1^3 \xi_2^3 & \xi_2^{3^2} & \xi_1^{3^2} \xi_2^3 & \xi_1^3 \xi_2^{3^2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_1^n & \xi_2^n & \xi_1^{n^2} & \xi_1^n \xi_2^n & \xi_2^{n^2} & \xi_1^{n^2} \xi_2^n & \xi_1^n \xi_2^{n^2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_8 \end{bmatrix} = \begin{bmatrix} \sigma^{pp^1} \\ \sigma^{pp^2} \\ \sigma^{pp^3} \\ \vdots \\ \sigma^{pp^n} \end{bmatrix} \quad (B.11)$$

B.2.4 Cubic projector

Define the following polynomial to be a cubic projector. It is a complete cubic with three quartic terms.

$$\begin{aligned} \sigma^{pd} = & a_1 + a_2 \xi_1 + a_3 \xi_2 + a_4 \xi_1^2 + a_5 \xi_1 \xi_2 + a_6 \xi_2^2 + a_7 \xi_1^3 \\ & + a_8 \xi_1^2 \xi_2 + a_9 \xi_1 \xi_2^2 + a_{10} \xi_2^3 + a_{11} \xi_1^3 \xi_2 + a_{12} \xi_1^2 \xi_2^2 + a_{13} \xi_1 \xi_2^3 \end{aligned} \quad (B.12)$$

In this case Equa. B.3 will take the following form

$$\begin{bmatrix} \sigma^{node^1} \\ \sigma^{node^2} \\ \sigma^{node^3} \\ \vdots \\ \sigma^{node^n} \end{bmatrix} = \begin{bmatrix} 1 & \xi_1^1 & \xi_2^1 & \xi_1^{1^2} & \xi_1^1 \xi_2^1 & \xi_2^{1^2} & \xi_1^{1^2} \xi_2^1 & \xi_1^1 \xi_2^{1^2} & \xi_2^{1^2} \xi_1^1 & \xi_1^{1^2} \xi_2^{1^2} & \xi_1^1 \xi_2^3 \\ 1 & \xi_1^2 & \xi_2^2 & \xi_1^{2^2} & \xi_1^2 \xi_2^2 & \xi_2^{2^2} & \xi_1^{2^2} \xi_2^2 & \xi_1^2 \xi_2^{2^2} & \xi_2^{2^2} \xi_1^2 & \xi_1^{2^2} \xi_2^{2^2} & \xi_1^2 \xi_2^3 \\ 1 & \xi_1^3 & \xi_2^3 & \xi_1^{3^2} & \xi_1^3 \xi_2^3 & \xi_2^{3^2} & \xi_1^{3^2} \xi_2^3 & \xi_1^3 \xi_2^{3^2} & \xi_2^{3^2} \xi_1^3 & \xi_1^{3^2} \xi_2^{3^2} & \xi_1^3 \xi_2^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_1^n & \xi_2^n & \xi_1^{n^2} & \xi_1^n \xi_2^n & \xi_2^{n^2} & \xi_1^{n^2} \xi_2^n & \xi_1^n \xi_2^{n^2} & \xi_2^{n^2} \xi_1^n & \xi_1^{n^2} \xi_2^{n^2} & \xi_1^n \xi_2^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{13} \end{bmatrix} \quad (B.13)$$

and Equa. B.5 will be ,

$$\begin{bmatrix} 1 & \xi_1^1 & \xi_2^1 & \xi_1^{1^2} & \xi_1^1 \xi_2^1 & \xi_2^{1^2} & \xi_1^{1^2} \xi_2^1 & \xi_1^1 \xi_2^{1^2} & \xi_2^{1^2} \xi_1^1 & \xi_1^{1^2} \xi_2^{1^2} & \xi_1^1 \xi_2^3 \\ 1 & \xi_1^2 & \xi_2^2 & \xi_1^{2^2} & \xi_1^2 \xi_2^2 & \xi_2^{2^2} & \xi_1^{2^2} \xi_2^2 & \xi_1^2 \xi_2^{2^2} & \xi_2^{2^2} \xi_1^2 & \xi_1^{2^2} \xi_2^{2^2} & \xi_1^2 \xi_2^3 \\ 1 & \xi_1^3 & \xi_2^3 & \xi_1^{3^2} & \xi_1^3 \xi_2^3 & \xi_2^{3^2} & \xi_1^{3^2} \xi_2^3 & \xi_1^3 \xi_2^{3^2} & \xi_2^{3^2} \xi_1^3 & \xi_1^{3^2} \xi_2^{3^2} & \xi_1^3 \xi_2^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_1^n & \xi_2^n & \xi_1^{n^2} & \xi_1^n \xi_2^n & \xi_2^{n^2} & \xi_1^{n^2} \xi_2^n & \xi_1^n \xi_2^{n^2} & \xi_2^{n^2} \xi_1^n & \xi_1^{n^2} \xi_2^{n^2} & \xi_1^n \xi_2^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{13} \end{bmatrix} = \begin{bmatrix} \sigma^{pp^1} \\ \sigma^{pp^2} \\ \sigma^{pp^3} \\ \vdots \\ \sigma^{pp^n} \end{bmatrix} \quad (B.14)$$

B.2.5 Least squares

The least squares solution to a set of inconsistent equations $Aa = b$ (6) of m (number of Gauss points) equations in c (number of coefficients in the polynomial) unknowns satisfies [99],

$$A^T A a = A^T b \quad (B.15)$$

If the columns of A are linearly independent; i.e. the rank of A is m , then the matrix $A^T A$ is invertible, and the unique least square solution is

$$a = [A^T A]^{-1} A^T b \quad (B.16)$$

Substitute a from (16) into (6)

$$\sigma^a = B [A^T A]^{-1} A^T b \quad (B.17)$$

Hence the least square projection matrix P_1 from (1) is

$$P = B [A^T A]^{-1} A^T \quad (B.18)$$

The simple least squares P will be denoted by P_1 (i.e. weights = 1).

B.2.6 Weighted least squares

Except for the 2×2 scheme all other Gaussian integration schemes have different weights at Gauss points. This suggests that values at each Gauss point should not be weighted equally and the stress projection should be done by weighted least squares and not simple least squares.

Start with a least square problem $Aa = b$ (6). If the m Gauss points are not equally important then various weights $w_1, w_2, \dots, w_m$ are associated with the m equations. Then Equa. B.6 becomes

$$W A a w = W b \quad (B.19)$$

W is a diagonal matrix with entries $w_1, w_2, \dots, w_m$. Similar to (15) the weighted least squares solution satisfies

$$(A^T W^T W A) a w = A^T W^T W b \quad (B.20)$$

Define $H = W^T W$ then

$$a w = [A^T H A]^{-1} A^T H b \quad (B.21)$$

Hence the weighted least squares projector matrix P for (1) is

$$P = B [A^T H A]^{-1} A^T H \quad (B.22)$$

P will be denoted by P_w or $P_{\sqrt{w}}$ or P_{w^2} . The subscript notation will be explained below.

This procedure can be applied to any Gaussian integration scheme for any projection polynomial, provided the projection polynomial is a subspace of A .

If the weighted least squares scheme is used then what weights should be selected? Should the Gauss point weights be used? The authors did not find a clear mathematical answer. Even if the Gauss point weights are used it is not clear whether they should be the weights (P_w) or the square root of weights ($P_{\sqrt{w}}$) or the square of weights (P_{w^2}). A partial answer to this problem is found by numerical experiment using different weights. The Gauss point weights and coordinates used are listed in Appendix B.

B.2.7 Stress space

An element has a displacement polynomial of degree p in (ξ_1, ξ_2) that depends on the element type. The strain and stress are the first partial derivatives ($m = 1$) of the displacement polynomial. It is approximately true to say that the element stress space is defined by a polynomial of degree $p - m = p - 1$. This is not exactly true because if the basis function in local coordinates (ξ_1, ξ_2) are polynomials of degree p , the isoparametric transformation to global coordinates (x, y) does not necessarily form polynomials of degree p . The same argument applies to the polynomial stress space of degree $p - m$. The space defined by a projector polynomial is referred to as projector stress space.

For many problems solved by the finite element method, analytical solutions may not exist. Judgement must be exercised by the user in selecting the element type. The stress space of the analytical solution (or real stress space of the problem) may or may not be a subset of the element stress space; i.e. $p - m$. If it is, a projector derived using the $p - m$ polynomial projects the Gauss point values to any other point within the element. There is no loss of accuracy. It is also possible to show that the weights are irrelevant in this case. However, if the true stress space is not a subset of the element stress space, then the $p - m$ polynomial finds a value which is a least square approximation of the Gauss point values. It is very important to remember

$$L=1000, \quad c=50, \quad Q=120000$$

Figure B.1: Cantilever with constant shear tip load. Mesh and boundary conditions for 8-noded quadrilateral element

GP No.	Stress Component	Values		
		2 × 2	5	3 × 3
1	σ_{zz}	-32784.6	0	-49485.5
	σ_{zy}	-1200.0	21807.7	19606.2
2	σ_{zz}	432784.6	-33956.1	-6285.5
	σ_{zy}	-1200.0	-2410.9	-19606.2
3	σ_{zz}	8784.6	33956.1	6285.5
	σ_{zy}	-1200.0	-2410.9	19606.2
4	σ_{zz}	8784.6	-8692.9	49485.5
	σ_{zy}	-1200.0	-2410.9	-19606.2
5	σ_{zz}		8692.9	0.0
	σ_{zy}		-2410.9	-19730.8
6	σ_{zz}			0.0
	σ_{zy}			-19730.8
7	σ_{zz}			27885.5
	σ_{zy}			21932.3
8	σ_{zz}			-27885.5
	σ_{zy}			21932.3
9	σ_{zz}			0.0
	σ_{zy}			21807.7

Table B.1: Gauss point shear stress, σ_{12} , in cantilever beam with constant shear tip load. Results of finite element analysis

Weights	Stress Values		
		schemes	
	2 × 2	5	3 × 3
P_1	-1200.0	2432.8	-5801.5
P_w	-1200.0	-2145.5	3643.7
$P_{\sqrt{w}}$	-1200.0	-1200.0	-1200.0

All nodes have the same stress value.

Table B.2: Weighted least square nodal shear stress, σ_{12} , in cantilever beam with constant shear tip load. Results of finite element analysis using one 8-noded quadrilateral element

Figure B.2: Nodal and Gauss point shear stress, σ_{12} , along beam diagonal for different integration schemes in cantilever with constant shear load. Beam diagonal is shown as dashed line

Figure B.3: Cantilever beam with uniform load. Finite element mesh and boundary conditions

A

The Gauss point stresses *

<i>z coord</i>	P_1	P_W	P_{W^2}	$P_{\sqrt{W}}$
0.0	-1.13612	-0.85671	-0.60583	-1.000
5.0	-1.01112	-0.73171	-0.68083	-0.875
10.0	-0.86672	-0.62712	-0.41200	-0.750
15.0	-0.72239	-0.52254	-0.34317	-0.625
20.0	-0.57785	-0.41805	-0.27458	-0.500
25.0	-0.43337	-0.31355	-0.20596	-0.375
30.0	-0.28891	-0.20903	-0.13730	-0.250
35.0	-0.14446	-0.10451	-0.06864	-0.125
40.0	-0.01946	-0.02048	-0.05635	-0.000

* *y* coordinate -1.54919

B

The projected nodal stresses

<i>z coord</i>	σ_{xy}
1.1270	-1.51632
5.0000	-1.06470
8.8729	-1.32267
11.1270	-1.11111
15.0000	-0.07268
18.8730	-0.91749
21.1270	-0.70532
25.0000	-0.04438
28.8730	-0.51167
31.1270	-0.29967
35.0000	-0.01464
38.8730	-0.10602

○ *y* coordinate -2.0.

Table B.3: Least square and weighted least square nodal shear stress, σ_{12} , in cantilever beam with constant shear tip load. Results of finite element analysis using four 8-noded quadrilateral elements

Figure B.4: Gauss point and least squared nodal shear stress, σ_{12} , along cantilever beam diagonal .

that in finite element analysis the error is minimized in the integral sense and not in the pointwise sense.

B.3 Numerical Tests and Results

Two cantilever beam problems are used to in this presentation to to assess these projection matrices. The displacements and stresses at Gauss points are calculated using the finite element program ASGARD [100]. The nodal stresses are found by local stress projection (i.e. using P_1 , P_w and $P_{\sqrt{w}}$) and smoothed by a nodal averaging technique to obtain a C^0 global stress field.

If the nodal displacements and Gauss point stresses exactly match the correct analytical solution then the projection matrix of degree $(p - m)$ gives exact nodal stresses. Problems for which exact values of stresses can be computed conveniently are used to verify the projection matrices. Also, problems in which the analytical stress space is not a subset of the element stress space are used to find appropriate weights for the weighted least squares projectors.

Generally the 2×2 , 3×3 and 5 point integration schemes are used with the quadratic quadrilateral elements (7, 8 and 9-noded). For these elements the bilinear projector is tested. The 3×3 and 4×4 schemes are used with cubic elements (12-noded). The quadratic projector is tested with this element. Finally the cubic projector is tested with the quartic (17-noded) element which generally requires the 4×4 or 5×5 integration schemes.

B.3.1 Tests using bilinear projector

A cantilever beam with a constant shearing load applied at the tip is solved using one 8-noded quadrilateral (Figure B.1). The displacements for all the degrees of freedom are prescribed to the values computed from the analytical solution (Equa. B.23). The same problem is solved using 2×2 , 5 point and 3×3 rules. The values of different components of the Gauss point stress are given in Table B.1. Gauss point values of σ^{12} (for different schemes) on a diagonal of the element are plotted in Figure B.2. The least square or weighted least squares stress values at the nodes are found using P_1 , P_w and $P_{\sqrt{w}}$ projectors. The least squared σ^{12} at the nodes is a constant value for each weighting scheme. But this value changes as the projector weight changes. For different integration schemes these values are tabulated in Table B.2.

GP or Node No.	Scheme 3×3		Scheme 4×4		Nodal Values	
			Stresses			
	σ_{xx}	σ_{xy}	σ_{xx}	σ_{xy}	σ_{xx}	σ_{xy}
1	-49485.0	-120.0	-57696.9	-465.2	-72000.0	0.0
2	-6285.5	-120.0	-4304.9	-465.2	-48000.0	0.0
3	6285.5	-120.0	4304.9	-465.2	-24000.0	0.0
4	49485.0	-120.0	57696.9	-465.2	0.0	0.0
5	0.0	-1800.0	-41540.6	-465.2	0.0	-1600.0
6	0.0	-1800.0	-20461.2	-465.2	0.0	-1600.0
7	27885.0	-120.0	-1699.6	-1591.9	0.0	0.0
8	-27885.0	-120.0	1699.6	-1591.9	24000.0	0.0
9	0.0	-1800.0	20461.2	-465.2	48000.0	0.0
10			41540.6	-465.2	12000.0	0.0
11			22779.0	-1591.9	24000.0	-1600.0
12			-22779.0	-1591.9	-24000.0	-1600.0
13			-16400.5	-1591.9		
14			-8078.2	-1591.9		
15			8078.2	-1591.9		
16			16400.5	-1591.9		

nodal analytical values of σ_{xx} and σ_{xy} are same
all Gauss points and nodes, $\sigma_{yy} = 0$.

Table B.4: Gauss point and weighted least square nodal stress components in cantilever beam with parabolic shear tip load. Results of finite element analysis using one 12-noded quadrilateral element

Figure B.5: Cantilever beam with parabolic shear tip load. Boundary conditions for one 12-noded quadrilateral element

GP or Node No.	Gauss Point Stresses			Nodal Stress Values		
	σ_{xx}	σ_{yy}	σ_{zy}	σ_{xx}	σ_{yy}	σ_{zy}
1	103.4476	-0.13792	0.3383	122.0000	0.0000	0.0000
2	14.4611	-0.13792	7.2150	114.5000	0.0000	0.0000
3	-14.4611	-9.86207	7.2150	92.0000	0.0000	0.0000
4	-103.4476	-9.86207	0.5383	54.5000	0.0000	0.0000
5	92.8918	-0.13792	2.5586	2.0000	0.0000	0.0000
6	57.5595	-0.13792	5.1946	-0.8750	-1.5625	22.5000
7	4.6452	-2.54839	24.8902	0.0000	-5.0000	30.0000
8	-4.6452	-7.45161	24.8902	0.8750	-8.4375	22.5000
9	-57.5595	-9.86207	5.1946	-2.0000	-10.0000	0.0000
10	-92.8918	-9.86207	-2.5586	-54.5000	-10.0000	0.0000
11	-39.7775	-7.45161	1.8421	-92.0000	-10.0000	0.0000
12	39.7775	-2.54839	1.8421	-114.5000	-10.0000	0.0000
13	35.5311	-2.54839	8.7859	-122.0000	-10.0000	0.0000
14	21.6607	-2.54839	17.7776	-59.1250	-8.3750	0.0000
15	-21.6607	-7.45161	17.7776	0.0000	-5.0000	0.0000
16	-35.5311	-7.45161	8.7559	59.1250	-1.5625	0.0000
17				0.0000	-5.0000	15.0000

The analytic and nodal stresses are equal.

Table B.5: Gauss point and weighted least square nodal stress components in deep beam with uniform load. Results of theoretical and finite element analysis using one 17-noded quadrilateral element

The second problem solved using four 8-noded quadrilaterals is also a cantilever beam but this time with a uniform load along the top edge (Figure B.3). At the encastered end, $\Delta x = 0$ along the edge and $\Delta y = 0$ on the neutral axis. In this problem, all the conditions of the analytical solution are not completely described by the prescribed boundary conditions. However, the solution should be satisfactory at considerable distances from the ends. The Gauss point and nodal σ^{12} along the length of the beam calculated using P_1 , P_w , P_w^2 and $P_{\sqrt{w}}$ are given in Table B.3 and plotted in Figure B.4.

B.3.2 Tests using quadratic projector

Here a cantilever beam with tip load applied as parabolic shear is solved. For this test, one 12-noded cubic quadrilateral element is used (Figure B.5). The analytical solution [73] for this problem is

$$\Delta x = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{IG} + \left(\frac{PL^2}{2EI} - \frac{Pc^2}{2IG} \right) y \quad (B.23a)$$

$$\Delta y = -\frac{\nu Px^2y}{2EI} - \frac{Px^3}{6EI} + \frac{PL^2x}{2EI} - \frac{Pl^3}{3EI} \quad (B.23b)$$

and

$$\sigma_{xx} = \frac{Pxy}{I} \quad \sigma_{yy} = 0 \quad \sigma_{xy} = -\frac{P}{2I} (c^2 - y^2) \quad (B.24)$$

All the load and displacement boundary conditions mentioned in the analytical solution [73] must be prescribed exactly (including both ends) to obtain exact stress values at all the points in the domain. This means the load at $x = L$ must be applied according to σ^{12} in Equa. B.24. Also the exact displacements must be prescribed at $x = 0$ and $x = L$ (23a, 23b). In this case, the analytical solution function space is a subspace of the element stress polynomials. The Gauss point and the nodal stresses found using a quadratic projector are given in Table B.4. Even in a pointwise comparison, the highest error found in FEM Gauss point stresses (all three components) is less than 0.001% relative to the analytical solution.

B.3.3 Tests using cubic projector

A deep beam of narrow rectangular cross section with uniform load, simply supported at the ends (Figure B.6), is solved using one 17-noded quartic quadrilateral element. Using 3×3 and 4×4 integration schemes one symmetrical half of the beam is analyzed. The analytical solution given in [73] for the stresses is

$$\sigma_{xx} = -\frac{q}{2I} \left((L^2 - x^2)y - \left(\frac{2}{3}y^3 - \frac{2}{5}c^2y \right) \right) \quad (B.25a)$$

Figure B.6: Deep beam with uniform load. Boundary conditions for one 17-nodes quadrilateral element

$$\sigma_{yy} = \frac{q}{2I} \left(-\frac{1}{3}y^3 + c^2y + \frac{2}{3}c^3 \right) \quad (B.25b)$$

$$\sigma_{zy} = \frac{q}{2I} (c^2 - y^2)x \quad (B.25c)$$

As mentioned in [73], in addition to the uniform load q at $y = +c$, the normal loads at $z = L$ must be applied (using *Equa. B.25a*). Also the boundary condition $\Delta y = 0$ at $z = L$ and $y = 0$ creates a singularity (nodal reaction) unless a distributed shear load is applied at $z = L$ (*Equa. B.25c*).

The nodal stresses are found by projecting the Gauss point stresses with the cubic $P_{\sqrt{w}}$ projector. The nodal values of the projected and analytic stresses are given in *Table B.5*. Once again the FEM Gauss point stresses are in exact agreement with analytical solution in both a least square and pointwise sense.

B.4 Discussion

In the first problem, *Figure B.1*, the analytical stress space is a polynomial of degree 2, which is not a subspace of the element stress space ($p-m = 2-1 = 1$). Hence an error is expected in stress component values at Gauss points. Since displacements are prescribed at all the degrees of freedom, the stress is simply given by $\sigma = DBu$. D , the modulus matrix, is a constant. The displacement vector, u , is entirely prescribed. Hence stress is independent of the integration scheme. For an eight noded quadrilateral the degree of B is fixed (approximately $= p-m = 1$). Therefore, in different integration schemes, the same B is simply evaluated at different points in the domain v . As a result all integration schemes use exactly the same stress space. In short, since the stress distribution for all schemes is the same, no scheme can be better or worse than any other! This is shown in *Figure B.2*, for a series of Gauss points from different schemes that lie on the diagonal of the element. The stress values all lie on the same curve as expected. The curve depends on the degree of B . This clearly indicates that there is a meaning in the values calculated at the points of all schemes. In the literature the Gauss point stresses calculated using 3×3 and 5 point schemes are sometimes called spurious or meaningless [101]. In our view this conclusion overlooks two facts. First to obtain exact stresses within the element, the analytical stress space should be a subset of the element stress space. Second and more important, error estimates should be in an integral — not a pointwise sense. Our results, particularly the results obtained using $P_{\sqrt{w}}$ projection in *Table B.2*, show that the error is the same for all three integration schemes. The

weighted least squares σ^{12} are the same at a particular node with the 2×2 , 5 point and 3×3 schemes. The results in Table B.1 also show that the σ_{xx} stresses which vary bilinearly are in exact agreement with the analytical solution Equa. B.24. This is expected because the σ_{xx} analytic solution function space is a subset of the bilinear projection function space. The validity of the bilinear projector is also proven because the projected nodal σ_{xx} stresses are in agreement with Equa. B.24.

The results of the first problem are also very useful in arriving at another important conclusion. It can be noted that the same displacements, u , is used for all three schemes and projection matrices. The analytical stresses are given by Equa. B.24. Hence the least square values derived using different schemes should be the same. However, as seen from Table B.2, only the $P_{\sqrt{w}}$ projector gives a constant value in all three schemes. Other projection matrices, differing only in weight, give different least square values with different schemes. This strongly suggests that the square root of the Gauss point weights is the appropriate weight to be used in computing the projection matrix P .

The results of the second problem (Figure B.2), are not straight forward to interpret. The Gauss points stresses calculated using different schemes are a result of not only evaluating B at different points but also the element displacements (stiffnesses) are different. In this case, the analytic stress space is not a subset of the element stress space. In the analytical solution, σ_{xx} has a quadratic term in x and a cubic term in y . The element stress space is only bilinear. The four elements in the x direction provide a piecewise linear approximation for the quadratic term in x . There is only one 8-noded element across the depth; i.e. the y direction. The σ^{12} stress has an xy^2 term in the analytical solution. Accordingly the FEM results show poor agreement for this component. It should be noted that the error is due to the selection of the element type and number of elements across the depth. The integration schemes should not be blamed for the error. If the results are interpreted correctly, 3×3 does give the proper approximate values for the σ^{12} stresses.

This interpretation differs from [63], where it is argued that, " σ_{xx} is in excellent agreement" and " σ^{12} provides an extremely poor representation" of actual stresses obtained by solving the equations of simple beam theory. First, the correct theory leads to Equa. B.25, not the equations of simple beam theory, and FEM approximates the equations from the correct theory. Second, the stress σ_{xx} is not in excellent agreement with Equa. B.25. Third, σ^{12} is in poor agreement because of inadequate

mesh discretisation in the y direction for the 8-noded quadrilaterals used. Table B.3 shows that in this case again the $P_{\sqrt{w}}$ matrix gives the correct least squared values for the σ_{xx} stress along the beam length. These least square values compare very well with the simple beam theory solution. However, since the FEM is solving Equa. B.25 not simple beam theory, this is of little significance.

The results for the 12-noded cubic element in Table B.4 and 17-noded quartic in Table B.5 show that the Gauss point and the projected nodal stresses (using $P_{\sqrt{w}}$) agree exactly with the analytical solution. The Gauss points stresses are correct because the analytical stress space is a subspace of the element stress space. The nodal stresses simply confirm the correctness of the projector polynomial of degree $p - m$ in each case. The projection matrices for $P_{\sqrt{w}}$ and various integration schemes and elements are listed in Appendix A.

In adaptive FEM, the discontinuity in the stress on element boundaries is integrated. In nonlinear FEM, residuals are usually computed at Gauss points. Both cases are based on the FEM stress, $\sigma = DBu$. Could an algorithm based on the projected stress be developed that would be more effective? This is an open question that warrants investigation.

On the boundary of the domain, the stress or traction vector must be prescribed. This fact together with equilibrium equations specify the normal components of the stress on the boundary. If this information could be incorporated into a stress projection method, the accuracy of the global stress field could be improved. To the authors' knowledge, this has not been investigated.

B.5 Conclusions

A least squares stress projection method for various integration schemes is presented for elements of different order. By numerical experiment it is shown that the bilinear, quadratic and cubic least squares projection matrices are most accurate for quadratic, cubic and quartic quadrilateral elements respectively. This in turn confirms the correctness of the least square polynomials suggested by the authors.

Should the Gauss point values be weighted and if so what weights should be used? A theoretical answer to this question is not known to the authors. However, numerical experiment suggests the square root of Gauss point weights is more accurate than unit weights in interpolating the Gauss point values by least squares.

Figure B.7: Order of Gauss points for various integration schemes

APPENDIX C

Projection Matrices.

This appendix lists projection matrices P for various element types and numerical integration schemes. These matrices are calculated for a particular order of integration points and nodal points (see Figure B.7). To minimize roundoff damage, the order of integration points are chosen in ascending order of their weight. The nodal order is consistent with the ASGAR program structure.

The P matrix for the 9-noded quadrilateral element can also be used for the 4, 7 and 8-noded elements. Similarly the P matrix for the 20-noded brick can be used for the 8-noded brick element with appropriate rows of the P matrix removed.

C.1 Weighted least square bilinear projection matrices $P_{\sqrt{w}}$.

P_1 matrix for 9 noded element and 2×2 Gaussian rule for numerical integration.

$$\begin{bmatrix} a & b & b & c \\ d & e & d & e \\ b & c & a & b \\ e & e & d & d \\ c & b & b & a \\ e & d & e & d \\ b & a & c & b \\ d & d & c & e \\ f & f & f & f \end{bmatrix}$$

Where

$$\begin{aligned} a &= 1.86602540378440 & b &= -.500000000000000 & c &= .13397459621556 \\ d &= .68301270189222 & e &= -.18301270189222 & f &= .250000000000000 \end{aligned}$$

$P_{\sqrt{w}}$ matrix for 9-noded element and 5 point Gaussian rule for numerical integration.

$$\begin{bmatrix}
 a & b & c & c & d \\
 a & e & f & e & f \\
 a & c & d & b & c \\
 a & f & f & e & e \\
 a & d & c & c & b \\
 a & f & e & f & e \\
 a & c & b & d & c \\
 a & e & e & f & f \\
 a & g & g & g & g
 \end{bmatrix}$$

Where

$$\begin{aligned}
 a &= .05000000000000 & b &= 1.79409714988390 & c &= -.47499999942045 \\
 d &= .10590284895704 & e &= .65954857523171 & f &= -.18454857523171 \\
 g &= .23750000000000
 \end{aligned}$$

$P_{\sqrt{w}}$ matrix for 9-noded element and 3×3 Gaussian rule for numerical integration.

$$\begin{bmatrix}
 a & b & c & b & d & e & e & d & f \\
 g & g & h & h & i & i & e & d & f \\
 b & a & b & c & e & d & e & d & f \\
 h & g & g & h & e & d & i & i & f \\
 c & b & a & b & e & d & d & e & f \\
 h & h & g & g & i & i & d & e & f \\
 b & c & b & a & d & e & d & e & f \\
 g & h & h & g & d & e & i & i & f \\
 j & j & j & j & i & i & i & i & f
 \end{bmatrix}$$

Where

$$\begin{aligned}
 a &= .85243672958711 & b &= -.33950617283951 & c &= .13521759140055 \\
 d &= .41034444539808 & e &= -.16343086515117 & f &= .19753086419753 \\
 g &= .25646527837380 & h &= -.10214429071948 & i &= .12345679012346 \\
 j &= .07716049382716
 \end{aligned}$$

C.2 Weighted least square quadratic projection matrices $P_{\sqrt{w}}$.

$P_{\sqrt{w}}$ matrix for 9-noded element and 3×3 Gaussian rule for numerical integration.

$$\begin{bmatrix} a & b & c & b & d & e & e & d & f \\ g & g & g & g & h & h & i & j & k \\ b & a & b & c & e & d & e & d & f \\ g & g & g & g & i & j & h & h & k \\ c & b & a & b & e & d & d & e & f \\ g & g & g & g & h & h & j & i & k \\ b & c & b & a & d & e & d & e & f \\ g & g & g & g & j & i & h & h & k \\ l & l & l & l & g & g & g & g & m \end{bmatrix}$$

Where

$$\begin{aligned} a &= 1.87829784308230 & b &= -.03086419753086 & c &= -.27335957147737 \\ d &= -.36860308785021 & e &= .49205987797366 & f &= -.79012345879012 \\ g &= .15452098765432 & h &= -.30864197530864 & i &= -.12080586634321 \\ j &= 1.17018858239260 & k &= -.04938271604938 & l &= -.07716049382716 \\ m &= .69135802469136 \end{aligned}$$

$P_{\sqrt{w}}$ matrix for 12-noded element and 3×3 Gaussian rule for numerical integration.

$$\begin{bmatrix} a & b & c & b & d & e & e & d & f \\ g & h & i & j & k & l & m & n & o \\ h & g & j & i & l & k & m & n & o \\ b & a & b & c & e & d & e & d & f \\ j & g & h & i & m & n & l & k & o \\ i & h & g & j & m & n & k & l & o \\ c & b & a & b & e & d & d & e & f \\ i & j & g & h & l & k & n & m & o \\ j & i & h & g & k & l & n & m & o \\ b & c & b & a & d & e & d & e & f \\ h & i & j & g & n & m & k & l & o \\ g & j & i & h & n & m & l & k & o \end{bmatrix}$$

Where

$a = 1.87829784308230$ $b = -.03086419753086$ $c = -.27335957147737$
 $d = -.36860308785021$ $e = .49205987797366$ $f = -.79012345679012$
 $g = .55800308721445$ $h = -.07838425965661$ $i = .07985699509008$
 $j = .16068878640558$ $k = -.41093353967147$ $l = -.12404588429684$
 $m = -.05270967253022$ $n = .99921173014339$ $o = -.13168724279835$

$P_{\sqrt{w}}$ matrix for 12-noded element and 4×4 Gaussian rule for numerical integration.

a	b	c	b	d	e	f	g	g	f	e	d	h	i	j	i
k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
l	k	n	m	p	o	v	u	t	s	r	q	x	w	z	y
b	a	b	c	e	d	e	f	g	g	f	i	h	i	j	
n	k	l	m	u	v	o	p	q	r	s	t	x	w	z	y
m	l	k	n	r	q	p	o	v	u	t	s	y	z	w	x
c	b	a	b	g	f	e	d	e	f	g	j	i	h	i	
m	n	k	l	s	t	u	v	o	p	q	r	y	z	w	x
n	m	l	k	t	s	r	q	p	o	v	u	z	y	x	w
b	c	b	a	f	g	g	f	e	d	e	i	j	i	h	
l	m	n	k	q	r	s	t	u	v	o	p	x	y	z	w
k	n	m	l	v	u	t	s	r	q	p	o	w	z	y	x

Where

$a = 1.05221895643980$ $b = .01359270435357$ $c = -.21745936189755$
 $d = .42935254689568$ $e = -.33940308563369$ $f = .31599665125616$
 $g = .14497981340777$ $h = -.26765350994448$ $i = -.35155566731654$
 $j = .00696798947635$ $k = .26349383596777$ $l = -.08271491472762$
 $m = .05509369330816$ $n = .13211104872521$ $o = .61561703916248$
 $p = .3593651616526$ $q = -.11386523734348$ $r = -.01186401849719$
 $s = .006449726235$ $t = .06345533885136$ $u = -.17332498484435$
 $v = -.07607993879$ $w = .17202629141099$ $x = .14405890562030$
 $y = -.11715703056347$ $z = -.23666491616109$

C.3 Weighted least square cubic projection matrices $P_{\sqrt{w}}$.

$P_{\sqrt{w}}$ matrix for 17-noded element and 4×4 Gaussian rule for numerical integration.

a	b	c	b	d	e	f	g	g	f	e	d	h	i	j	i
k	l	m	n	o	p	q	r	s	t	u	v	w	z	y	z
A	A	B	B	C	C	D	E	F	F	E	D	G	G	H	H
l	k	n	m	p	o	v	u	t	s	r	q	z	w	z	y
b	a	b	c	e	d	d	e	f	g	g	f	i	h	i	j
n	k	l	m	u	v	o	p	q	r	s	t	z	w	z	y
B	A	A	B	E	D	C	C	D	E	F	F	H	G	G	H
m	l	k	n	r	q	p	o	v	u	t	l	y	z	w	z
c	b	a	b	g	f	e	d	d	e	f	g	j	i	h	i
m	n	k	l	l	t	u	v	o	p	q	r	y	z	w	z
B	B	A	A	F	F	E	D	C	C	D	E	H	H	G	G
n	m	l	k	t	l	r	q	p	o	v	u	z	y	z	w
b	c	b	a	f	g	g	f	e	d	d	e	i	j	i	h
l	m	n	k	q	r	s	t	u	v	o	p	z	y	z	w
A	B	B	A	D	E	F	F	E	D	C	C	G	H	H	G
k	n	m	l	v	u	t	s	r	q	p	o	w	z	y	z
J	J	J	J	K	K	K	K	K	K	K	K	L	L	L	L

Where

a = 1.79827121654850	b = -.03626638531530	c = .27046581659075
d = -.19555213753760	e = -.03966456823829	f = .04686729891211
g = -.39498392646956	h = -1.22204732951010	i = .55709971864863
j = .27831029637087	k = .43046222243534	l = -.05322459853352
m = -.05603832432768	n = -.01781658433681	o = 1.08491218973070
p = .06463831182471	q = .13979824582232	r = .01449007960976
s = .11214199942271	t = -.15220428704020	u = .19667367338692
v = -.43961687942360	w = -.12140056011668	z = -.39241325576896
y = .03083082867569	z = .15876693863929	A = .04217572260431
B = -.08826984432224	C = .80556978533294	D = -.17509828262994
E = .21321725111576	F = .03131124618124	G = -.23171794211352
H = -.01283649095993	I = .11214199942270	J = .00852420077964
K = -.06468750000000	L = .35085079922036	

C.4 Weighted least square trilinear projection matrices for brick elements. $P_{\sqrt{w}}$.

$P_{\sqrt{w}}$ matrix for 20-noded element and $2 \times 2 \times 2$ point Gaussian rule for numerical integration.

a	b	c	a	c	a	d	c
e	e	f	f	f	f	g	g
b	a	a	c	a	c	c	d
e	f	e	f	f	g	f	g
a	c	b	a	c	d	a	c
f	f	e	e	g	g	f	f
c	a	a	b	d	c	c	a
f	e	f	e	g	f	g	f
f	e	g	f	f	e	g	f
e	f	f	g	e	f	f	g
f	g	e	f	f	g	e	f
g	f	f	e	g	f	f	e
c	a	d	c	a	b	c	a
f	f	g	g	e	e	f	f
a	c	c	d	b	a	a	c
f	g	f	g	e	f	e	f
e	d	a	c	a	c	b	a
g	g	f	f	f	f	e	e
d	c	c	a	c	a	a	b
g	f	g	f	f	e	f	e

Where

$$\begin{aligned}
 a &= -.68301270189222 & b &= 2.54903810567670 & c &= .18301270189222 \\
 d &= -.04903810567666 & e &= .93301270189222 & f &= -.25000000000000 \\
 g &= .06698729810778
 \end{aligned}$$

$P_{\sqrt{w}}$ matrix for 20-noded element and 14 point Gaussian rule for numerical integration.

a	b	b	c	b	c	c	d	e	e	e	f	f	f
g	g	h	h	h	h	i	i	j	e	e	j	f	f
b	a	c	b	c	b	d	c	f	e	e	e	f	f
h	g	h	g	i	h	i	h	f	j	e	e	j	f
c	b	b	a	d	c	e	b	f	f	e	e	e	f
h	h	g	g	i	i	h	h	j	f	e	j	e	f
b	c	a	b	c	d	b	c	e	f	e	f	e	f
g	h	g	h	h	i	h	i	e	j	e	f	j	f
g	h	h	i	g	h	h	i	e	e	j	f	f	j
h	g	i	h	h	g	i	h	f	e	j	e	f	j
i	h	h	g	i	h	h	g	f	f	j	e	e	j
h	i	g	h	h	i	g	h	e	f	j	f	e	j
b	c	c	d	a	b	b	c	e	e	f	f	f	e
h	h	i	i	g	g	h	h	j	e	f	j	f	e
c	b	d	c	b	a	c	b	f	e	f	e	f	e
i	h	i	h	h	g	h	g	f	j	f	e	j	e
d	c	e	b	c	b	b	a	f	f	f	e	e	e
i	i	h	h	h	h	g	g	j	f	f	j	e	e
c	d	b	c	b	c	a	b	e	f	f	f	e	e
h	i	h	i	g	h	g	h	e	j	f	f	j	e

Where

$a = 1.26545633224310$ $b = -.36595543480688$ $c = .01553992267482$
 $d = .12097025923689$ $e = .37534263473849$ $f = -.15373598643309$
 $g = .44975044871810$ $h = -.17520775606603$ $i = .06825509095586$
 $j = .11080332415270$

APPENDIX D

Gauss Point Weights and Coordinates

2 x 2 Gaussian Integration scheme.

No.	Weight	ξ_1	ξ_2
1	1.00000000000000	-.57735026918963	-.57735026918963
2	1.00000000000000	-.57735026918963	.57735026918963
3	1.00000000000000	.57735026918963	-.57735026918963
4	1.00000000000000	.57735026918963	.57735026918963

5 point Gaussian Integration Scheme.

No.	Weight	ξ_1	ξ_2
1	.200000000	.000000000	.000000000
2	.950000000	-.592348878	-.592348878
3	.950000000	-.592348878	.592348878
4	.950000000	.592348878	-.592348878
5	.950000000	.592348878	.592348878

3 x 3 Gaussian Integration Scheme.

No.	Weight	ξ_1	ξ_2
1	.30864197530864	-.77459666924148	-.77459666924148
2	.30864197530864	.77459666924148	-.77459666924148
3	.30864197530864	.77459666924148	.77459666924148
4	.30864197530864	.77459666924148	.77459666924148
5	.49382716049383	-.77459666924148	.00000000000000
6	.49382716049383	.77459666924148	.00000000000000
7	.49382716049383	.00000000000000	.77459666924148
8	.49382716049383	.00000000000000	-.77459666924148
9	.79012345679012	.00000000000000	.00000000000000

4 × 4 Gaussian Integration Scheme.

No.	Weight	ξ_1	ξ_2
1	.12100299328560	-.86113631159405	-.86113631159405
2	.12100299328560	.86113631159405	-.86113631159405
3	.12100299328560	.86113631159405	.86113631159405
4	.12100299328560	-.86113631159405	.86113631159405
5	.22685185185185	-.33998104358486	-.86113631159405
6	.22685185185185	.33998104358486	-.86113631159405
7	.22685185185185	.86113631159405	-.33998104358486
8	.22685185185185	.86113631159405	.33998104358486
9	.22685185185185	.33998104358486	.86113631159405
10	.22685185185185	-.33998104358486	.86113631159405
11	.22685185185185	-.86113631159405	.33998104358486
12	.22685185185185	-.86113631159405	-.33998104358486
13	.42529330301069	-.33998104358486	-.33998104358486
14	.42529330301069	.33998104358486	-.33998104358486
15	.42529330301069	.33998104358486	.33998104358486
16	.42529330301069	-.33998104358486	.33998104358486

END

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FIN