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SOME HANKEL TRANSFORMATIONS OF
GENERALIZED FUNCTIONS

by

LAKSHMI SHANKER DUBE

A thesis submitted to the Faculty of Graduate Studies in partial
fulfillment of the requirements for the degree of

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Ottawa, Ontario
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ABSTRACT

In Chapter 1, a classical inversion theorem for the Hankel transformation due to Schwartz [16] has been extended to a class of generalized functions. For this, the corresponding Hankel transformation for the elements of the generalized function space has been defined and an inversion theorem for the transform, interpreting the limiting operation in the distributional sense, has been established. It has been shown that the generalized Hankel transform $F(y)$ under consideration is a differentiable function of y for all $y > 0$. Asymptotic orders of $F(y)$ for large and small y are also obtained. A structure formula for the elements of the Hankel transformable generalized function space has also been established. Finally, we have applied our inversion theorem in solving a distributional differential equation.

In Chapter 2, an inversion theorem for the Hankel potential transformation due to Cholewinski and Haimo [3] has been extended to certain generalized function space. Some related results such as the infinite differentiability and boundedness of the generalized Hankel potential transform have also been proved.

In Chapter 3, a Fourier-Bessel series representation of some generalized functions has been investigated. To be more specific, a finite Hankel transformation of certain generalized functions has been defined and an inversion theorem, as an extension to generalized functions of a classical inversion formula [19, p. 591] has been proved.

(ii)

An operation transform formula has also been obtained and applied in solving a distributional differential equation.

(iii)

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INTRODUCTION

Schwartz [17(i), Vol. 2] extended the classical Fourier transformation to distributions of slow growth. Later, in 1952, Schwartz [17(ii)] also extended the Laplace transformation to a class of generalized functions. Due to widespread applicability as a powerful tool in solving certain distributional ordinary and partial differential equations, the extensions to generalized functions of Fourier transformation including several other integral transformations have become an active area of research for the last twenty years. In the 1960's, A.H. Zemanian extended the Laplace, Mellin, Hankel, K, Weierstrass and convolution transformations to certain classes of generalized functions. The references to these works are given in the Bibliography of the present dissertation. The work of Koh and Zemanian [10] on Hankel transformation of generalized functions and those of Pandey and Zemanian [11] Pandey [12] and [13] on some convolution transformations of generalized functions are worth mentioning.

The repeated extension of the same integral transformation to generalized functions should create no confusion. In fact, in such cases, either the approach of extension in one is different from the other or the latter extension is presented for a different generalized function space possibly larger than the previous one. Mostly, in all the work in this area, the inversion theorems corresponding to the generalized integral transformations have been of paramount importance.

In this thesis, we have studied some Hankel transformations of generalized functions. The whole thesis is divided into three chapters, and altogether three different problems, one in each chapter, have been solved.

Throughout the thesis, we follow the notation and terminology of Zemanian [21]. In Chapters 1 and 2, I denotes the one dimensional open interval $(0, \infty)$ while in Chapter 3, I stands for one dimensional open interval $(0, 1)$. The letters x , y and t have been used as real variables in I . By smooth function, we mean a function that possesses continuous derivatives of all orders at every point of the domain. The symbol $\langle f, \phi \rangle$ denotes the number assigned to some element ϕ in a testing function space by a member f of its dual space. If f is a generalized function on I , the notation $f(x)$ is used merely to indicate that the testing functions on which f is defined possess x as their independent variable. The k^{th} derivative of an ordinary or generalized function $f(x)$ is denoted by $D_x^k f(x)$.

$\mathcal{D}(I)$ denotes the space of smooth functions that have compact support on I . The topology of $\mathcal{D}(I)$ is that which makes its dual the space $\mathcal{D}'(I)$ of Schwartz's distribution [16, Vol. I, p. 65]. $\mathcal{E}(I)$ denotes the space of smooth functions on I . Its dual $\mathcal{E}'(I)$ is the space of distributions with compact support in I . [21, p.37].

Zemanian [22] has extended the Hankel transformation to certain generalized functions through a generalization of Parseval's equation. Later, Koh and Zemanian [10] gave an alternative approach to define the

generalized Hankel transformation which is more natural and explicit, and proved an inversion formula for the transform.

In Chapter 1, we have extended a Hankel inversion theorem due to Schwartz [16] to generalized functions. Our definition of generalized Hankel transform is as natural and explicit as that of Koh and Zemanian [10]. Moreover, our Hankel transform has some distinct advantages over theirs. We have given an example of a class of regular generalized functions in Section 7 which are Hankel transformable in our sense but not in the sense of Koh and Zemanian [10].

The classical inversion theorem due to Schwartz [16] which we have extended to generalized functions is given as follows.

Let ν be a fixed real number exceeding $-\frac{1}{2}$ and let L consist of all measurable functions defined on $0 < x < \infty$ such that

$$\int_0^{\infty} |f(x)| dm(x) < \infty$$

where

$$dm(x) = [2^{\nu} \Gamma(\nu+1)]^{-1} x^{-2\nu-1} dx$$

Let $g(x) = 2^{\nu} \Gamma(\nu+1) x^{-\nu} J_{\nu}(x)$ be defined for all $x > 0$, where $J_{\nu}(x)$ is the Bessel function of the first kind of order ν .

Theorem (Inversion). Let $f \in L$ and $\int_0^1 |f(y)y^{v+\frac{1}{2}}| dy < \infty$. If f is of bounded variation in a neighbourhood of x then,

$$\lim_{\lambda \rightarrow \infty} \int_0^\lambda \mathcal{J}(xu) dm(u) \int_0^\infty f(y) \mathcal{J}(uy) dm(y) = \frac{1}{2} \{f(x+0) + f(x-0)\}.$$

[16]

In order to extend the above theorem, we have defined testing function spaces $G_{\alpha, \delta}^v(I)$ and $H_{\alpha, \delta}^v(I)$ (see Section 2 of Chapter 1) for $v > -\frac{1}{2}$, $0 < \alpha \leq v + \frac{1}{2}$ and $\delta \geq 0$. The spaces $G_{\alpha, \delta}^v(I)$ and $H_{\alpha, \delta}^v(I)$ are related as follows.

A testing function $\phi(x) \in G_{\alpha, \delta}^v(I)$ iff $\frac{\phi(x)}{m'(x)} \in H_{\alpha, \delta}^v(I)$ where $m'(x) = [2^v \Gamma(v+1)]^{-1} x^{2v+1}$.

The dual spaces of $G_{\alpha, \delta}^v(I)$ and $H_{\alpha, \delta}^v(I)$ are denoted by $G_{\alpha, \delta}^{v'}(I)$ and $H_{\alpha, \delta}^{v'}(I)$ respectively. For $f \in H_{\alpha, \delta}^{v'}(I)$, a generalized function $m'f \in G_{\alpha, \delta}^{v'}(I)$ is defined as

$$\langle m'(x)f(x), \phi(x) \rangle = \langle f(x), m'(x)\phi(x) \rangle$$

for all $\phi(x) \in G_{\alpha, \delta}^v(I)$.

One can easily see that $f \in H_{\alpha, \delta}^{v'}(I)$ iff $m'f \in G_{\alpha, \delta}^{v'}(I)$.

The Generalized Hankel Transformation. We have defined the generalized Hankel transformation of any member of $f \in H_{\alpha, \delta}^{v'}(I)$ by the relation:

$$(1) \quad F(y) = \langle m'(x)f(x), \mathcal{J}(xy) \rangle = \langle f(x), m'(x)\mathcal{J}(xy) \rangle$$

for all $y > 0$, where

$$m'(x) = [2^{\nu} \Gamma(\nu+1)]^{-1} x^{2\nu+1}$$

and

$$J(xy) = 2^{\nu} \Gamma(\nu+1) (xy)^{-\nu} J_{\nu}(xy).$$

It has been checked that for each $y > 0$, $J(xy) \in G_{\alpha, \delta}^{\nu}(I)$ so that $m'(x) J(xy) \in H_{\alpha, \delta}^{\nu}(I)$.

We have proved the following result which gives the differentiability of $F(y)$.

Theorem. For real $y > 0$, let $F(y)$ be the Hankel transformation of $f \in H_{\alpha, \delta}^{\nu, \prime}(I)$ as defined by (1), then $F(y)$ is differentiable and

$$(2) \quad \frac{d}{dy} F(y) = \langle m'(x) f(x), \frac{\partial}{\partial y} J(xy) \rangle$$

for all $y > 0$.

We have also obtained asymptotic orders of $F(y)$ for small and large y as follows.

Theorem. Let $F(y)$ be the Hankel transform of f as defined by (1). Then, there exists a non-negative r such that

$$|F(y)| = O[y^{-\alpha}], \quad y \rightarrow 0^+$$

and

$$|F(y)| = O[y^{2r-\alpha}], \quad y \rightarrow \infty, \quad \text{where } 0 < \alpha \leq \nu + \frac{1}{2}.$$

The major result of this Chapter as an inversion theorem for our generalized Hankel transform is stated as follows.

Theorem. Let $f \in H_{\alpha, \delta}^{\nu, \prime}(I)$ where $0 < \alpha \leq \nu + \frac{1}{2}$ and $\delta \geq 0$, and let $F(y)$ be the Hankel transform of f as defined by (1). Then, for each $\phi \in \mathfrak{D}(I)$ one has

$$(3) \quad \langle m'(x) \int_0^N F(y) \mathcal{J}(xy) dm(y), \phi(x) \rangle \rightarrow \langle m'f, \phi \rangle$$

as $N \rightarrow \infty$, or equivalently,

$$\langle \int_0^N F(y) \mathcal{J}(xy) dm(y), \phi(x) \rangle \rightarrow \langle f, \phi \rangle$$

as $N \rightarrow \infty$.

A uniqueness theorem for the generalized Hankel transformation has also been proved (see Chapter 1, Theorem 3.4).

We have also established a structure formula for the elements of $H_{\alpha, \delta}^{\nu, \prime}(I)$ which is stated as follows.

Theorem. Let f be an arbitrary element of $H_{\alpha, \delta}^{\nu, \prime}(I)$ and ϕ be an arbitrary element of $\mathfrak{D}(I)$. Then, there exist bounded measurable functions $g_i(x)$ defined for $x > 0$ and polynomials $Q_i(1/x)$ in $1/x$ for $i = 0, 1, 2, \dots, 2k+1$ such that

$$(4) \quad \langle f, \phi \rangle = \left\langle \sum_{i=0}^{2k+1} (-1)^i D^{i+1} \int_a^x g_i(x) x^{\alpha-2\nu} Q_i(1/x) dx, \phi(x) \right\rangle$$

where k is some appropriate non-negative integer and a is a positive number.

Next, in Section 5 of this Chapter, some further enlargements of the Hankel transformable generalized function space have been discussed. An open problem has also been specified.

In Section 7, a few interesting examples, comparing our generalized function space $H_{\alpha, \delta}^{\nu'}(I)$ with those of Koh and Zemanian [10] and Zemanian [22] have been presented.

Finally, in Section 8 of this Chapter, we have used our generalized Hankel transformation and the corresponding inversion theorem to find a complete solution to a distributional differential equation. The operation transform formula given by (8.3) could also be used in solving some distributional boundary value problems formulated in cylindrical co-ordinates.

In Chapter 2, a Hankel potential transform of a generalized function has been defined and an inversion theorem for the Hankel potential transform given by Cholewinski and Haimo [3] has been extended to a class of generalized functions.

The Hankel potential transform of a function f (in classical sense) is defined by the following convergent integral:

$$(5) \quad F(x) = \int_0^{\infty} \frac{t}{(x^2+t^2)^{\nu+1}} f(t) dm(t)$$

where

$$dm(t) = [2^{\nu-\frac{1}{2}} \Gamma(\nu+\frac{1}{2})]^{-1} t^{2\nu} dt, \quad \nu > 0 \text{ and } x \neq 0.$$

A function $f(x)$ belongs to L for $x \in [0, R]$ $0 < R \leq \infty$, if

$$\int_0^R |f(x)| dm(x) < \infty.$$

Cholewinski and Haimo [3] proved the following inversion formula for the transform (5).

Theorem. Let $f(t)$ belong to L for $t \in [0, R]$ for every positive R , and let the integral

$$F(x) = \int_0^{\infty} \frac{t}{(x^2+t^2)^{\nu+1}} f(t) dm(t)$$

converge for some $x \neq 0$. Then,

$$(6) \quad \lim_{n \rightarrow \infty} L_{n,x}[F(x)] = f(x)$$

at all points x of the Lebesgue set for the function $f(x)$, where the operator $L_{n,x}$ is the same as defined by (1.3) in Chapter 2.

In Section 2 of this Chapter, we have defined the testing function spaces $\mathcal{T}_{\alpha,\beta}(I)$ and $\mathcal{P}_{\alpha,\beta}(I)$ for any pair of real numbers α, β such that $0 < \alpha \leq 2\nu+1$ and $0 \leq \beta < 1$, where $\nu > 0$ is a fixed real number.

The spaces $\mathcal{T}_{\alpha,\beta}(I)$ and $\mathcal{P}_{\alpha,\beta}(I)$ have the property that a testing function $\phi(x) \in \mathcal{P}_{\alpha,\beta}(I)$ iff $\frac{\phi(x)}{m'(x)} \in \mathcal{T}_{\alpha,\beta}(I)$, where

$$m'(x) = [2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})]^{-1}x^{2\nu}$$

The dual spaces of $\mathcal{T}_{\alpha,\beta}(I)$ and $\mathcal{P}_{\alpha,\beta}(I)$ are denoted by $\mathcal{T}'_{\alpha,\beta}(I)$ and $\mathcal{P}'_{\alpha,\beta}(I)$ respectively. For any $f \in \mathcal{P}'_{\alpha,\beta}(I)$, a generalized function $m'f \in \mathcal{T}'_{\alpha,\beta}(I)$ is defined as:

$$\langle m'(x)f(x), \phi(x) \rangle = \langle f(x), m'(x)\phi(x) \rangle$$

for all $\phi(x) \in \mathcal{T}_{\alpha,\beta}(I)$.

It has been checked that $f \in \mathcal{P}'_{\alpha,\beta}(I)$ iff $m'f \in \mathcal{T}'_{\alpha,\beta}(I)$.

The Generalized Hankel Potential Transformation. The generalized Hankel potential transformation of an arbitrary $f \in \mathcal{P}'_{\alpha,\beta}(I)$ is defined by the relation:

$$(7) \quad F(x) = \langle m'(t)f(t), \frac{t}{(x^2+t^2)^{\nu+1}} \rangle \equiv \langle f(t), m'(t) \frac{t}{(x^2+t^2)^{\nu+1}} \rangle$$

for all $x > 0$.

It is shown in Section 3 of this Chapter that $\frac{t}{(x^2+t^2)^{\nu+1}} \in \mathcal{T}_{\alpha,\beta}(I)$ for each $x>0$ so that $m'(t) \frac{t}{(x^2+t^2)^{\nu+1}} \in \mathcal{P}_{\alpha,\beta}(I)$.

We have proved the following main results concerning the Hankel potential transform $F(x)$.

Theorem (Infinite differentiability of $F(x)$). Let $f \in \mathcal{P}'_{\alpha,\beta}(I)$ and $F(x)$ be the Hankel potential transform as defined by (7). Then, $F^{(n)}(x)$ exists and

$$F^{(n)}(x) = \langle m'(t)f(t), \frac{\partial^n}{\partial x^n} \frac{t}{(x^2+t^2)^{\nu+1}} \rangle$$

for all $x>0$ and $n = 1, 2, 3, \dots$.

Theorem (Boundedness of $F(x)$). Let $f \in \mathcal{P}'_{\alpha,\beta}(I)$ and let $F(x)$ be the Hankel transform of f as defined by (7). Then,

$$|F(x)| \leq k_\nu \left[A + \frac{1}{x^{2(\nu+1)}} \right]$$

where A and k_ν are some constants.

Theorem (Inversion Theorem). Let $f \in \mathcal{P}'_{\alpha,\beta}(I)$ and let $F(x)$ be the Hankel potential transform of f as defined by (7). Then, for each $\phi \in \mathcal{D}(I)$,

$$\langle m'(x) L_{n,x} F(x), \phi(x) \rangle \rightarrow \langle m'f, \phi \rangle$$

as $n \rightarrow \infty$, or equivalently,

$$\langle L_{n,x} F(x), \phi(x) \rangle \rightarrow \langle f, \phi \rangle$$

as $n \rightarrow \infty$, where the operator $L_{n,x}$ is the same as defined by (1.3) in Chapter 2 and the differentiation in $L_{n,x}$ is understood in distributional sense.

An example has also been presented to illustrate the above theorem.

In Chapter 3, a Fourier-Bessel series [19, p. 576] representation of generalized functions has been investigated. Zemanian [24] has developed a procedure for expanding a generalized function into a series of Fourier type and in particular into that of the Fourier-Bessel type using some Hilbert space techniques. We have developed an alternative approach to this subject and under the usual techniques of generalized integral transformations, a classical inversion theorem [19, p. 591] for the finite Hankel transformation has been extended to a class of generalized functions giving rise to a Fourier-Bessel series representation of a generalized function in a distributional sense.

The theorem which we have extended to generalized functions is stated as follows.

Theorem. Let $f(t)$ be a function defined arbitrarily in the interval $(0,1)$ and let $\int_0^1 t^{1/2} f(t) dt$ exist and (if it is an improper integral) let it be absolutely convergent. Let

$$a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 t f(t) J_{\nu}(j_m t) dt, \quad m = 1, 2, 3, \dots$$

where $J_{\nu}(z)$, for any complex z , is the Bessel function of first kind of order $\nu \geq -1/2$ and j_m , $m = 1, 2, 3, \dots$ are positive zeros of $J_{\nu}(z)$ in ascending order. (Notice that for $\nu \geq -1/2$, all the zeros of $J_{\nu}(z)$ are real)

Let x be any internal point of an interval (a, b) such that $0 < a < b < 1$ and that $f(t)$ is of bounded variation in (a, b) . Then, the series

$$\sum_{m=1}^{\infty} a_m J_{\nu}(j_m x)$$

is convergent and its sum is $\frac{1}{2}\{f(x+0) + f(x-0)\}$ and is $f(x)$ if the function f is continuous at the point x .

In order to extend the above theorem to generalized functions, we have defined in Section 2 of Chapter 3, the testing function spaces $U_{\alpha, \nu}(I)$ and $V_{\alpha, \nu}(I)$ for any pair of real numbers α and ν such that $\alpha \geq \frac{1}{2}$ and $\nu \geq -\frac{1}{2}$. The spaces $U_{\alpha, \nu}(I)$ and $V_{\alpha, \nu}(I)$ are such that a testing function $\phi(x) \in V_{\alpha, \nu}(I)$ iff $\frac{\phi(x)}{x} \in U_{\alpha, \nu}(I)$.

The dual spaces of $U_{\alpha, \nu}(I)$ and $V_{\alpha, \nu}(I)$ are denoted by $U'_{\alpha, \nu}(I)$ and $V'_{\alpha, \nu}(I)$ respectively.

For a generalized function $f \in V'_{\alpha, \nu}(I)$ another generalized function $xf \in U'_{\alpha, \nu}(I)$ is defined by the relation

$$\langle xf(x), \phi(x) \rangle = \langle f(x), x\phi(x) \rangle$$

for all $\phi(x) \in U_{\alpha, \nu}(I)$.

This definition has a sense in view of the fact that

$\phi \in U_{\alpha, \nu}(I)$ iff $x \phi \in V_{\alpha, \nu}(I)$.

The Generalized Finite Hankel Transformation. For an arbitrary $f \in V'_{\alpha, \nu}(I)$ we have defined the finite Hankel transformation of f by the relation

$$(8) \quad F(j_m) = \langle xf(x), J_\nu(j_m x) \rangle \equiv \langle f(x), xJ_\nu(j_m x) \rangle$$

for each $m = 1, 2, 3, \dots$ where $J_\nu(z)$ and j_m ($m = 1, 2, 3, \dots$) are the same as defined on page 12.

It has been shown that $J_\nu(j_m x)$ belongs to the space $U_{\alpha, \nu}(I)$ so that the relation (8) has a sense.

The following inversion theorem as a main result of this Chapter has been proved.

Theorem. Let f be an arbitrary generalized function in the space $V'_{\alpha, \nu}(I)$ where $\nu > -\frac{1}{2}$ and $\alpha > \frac{1}{2}$, and let $F(j_m)$ be the finite Hankel transform of f for each $m = 1, 2, 3, \dots$ as defined by (8). Then, for each $\phi(x) \in \mathcal{X}(I)$ one has

$$\langle 2x \sum_{m=1}^N \frac{F(j_m)}{J_{\nu+1}^2(j_m)} J_\nu(j_m x), \phi(x) \rangle \rightarrow \langle tf(t), \phi(t) \rangle$$

as $N \rightarrow \infty$.

In the end of this Chapter, an operation transform formula has been obtained and is applied in solving a distributional differential equation with the help of the above inversion theorem. This transform technique could also be used in solving some boundary value problems in cylindrical co-ordinates.

CHAPTER 1

HANKEL TRANSFORMATION OF A CLASS OF
GENERALIZED FUNCTIONS

1. Introduction. The classical Hankel transformation of a function $f(x)$ defined over the interval $(0, \infty)$ is given by

$$(1.1) \quad F(y) = H_\nu f = \int_0^\infty f(x) \sqrt{xy} J_\nu(xy) dx$$

where $0 < y < \infty$, ν is a real number, and $J_\nu(x)$ is the Bessel function of first kind and order ν .

The factor $\sqrt{xy} J_\nu(xy)$ is called the kernel of the transformation.

A standard result concerning (1.1) is the following inversion theorem [19, p. 456].

Theorem 1.1. If $f(x) \in L_1(0, \infty)$, the space of all (equivalence classes of) functions that are Lebesgue integrable on $0 < x < \infty$, if $f(x)$ is of bounded variation in a neighbourhood of point x , and if $F(y)$ is defined by (1.1), then for $\nu > -\frac{1}{2}$

$$(1.2) \quad \int_0^\infty F(y) \sqrt{xy} J_\nu(xy) dy = H_\nu^{-1} F = \frac{1}{2} \{f(x+0) + f(x-0)\}$$

Another standard definition of the Hankel transformation, which is a little different from (1.1) is the following.

Let $f(x)$ be defined over $(0, \infty)$. Then for each $y \in (0, \infty)$, the Hankel transformation of $f(x)$ is defined as

$$(1.3) \quad F(y) = \int_0^{\infty} f(x) J_{\nu}(xy) x dx$$

A similar inversion theorem corresponding to (1.3) holds subject to the condition that

$$\int_0^{\infty} f(x) \sqrt{x} dx$$

exists and is absolutely convergent.

Another result concerning (1.1) is the Parseval's Equation.

Theorem 1.2. If $f(x)$ and $G(y)$ are in $L_1(0, \infty)$, $\nu \geq -\frac{1}{2}$, $F(y) = H_{\nu}[f(x)]$, and if $g(x) = H_{\nu}^{-1}[G(y)]$, then

$$(1.4) \quad \int_0^{\infty} f(x) g(x) dx = \int_0^{\infty} F(y) G(y) dy .$$

Zemanian [22] extended the Hankel transformation to a class of generalized functions through a generalization of the Parseval's equation (1.4). The method consists of constructing a testing-function space $\mathcal{H}_{\nu}$ on $0 < x < \infty$, on which the Hankel transformation $\mathcal{H}_{\nu}$ is an automorphism when $\nu > -\frac{1}{2}$. The so-called generalized Hankel transformation

$\mathcal{H}'_\nu$ was defined on the dual space $\mathcal{H}'_\nu$ of $\mathcal{H}_\nu$ as the adjoint of $\mathcal{H}_\nu$ through the equation

$$(1.5) \quad \langle \mathcal{H}'_\nu f, \phi \rangle = \langle f, \mathcal{H}_\nu \phi \rangle$$

where $\nu \geq -\frac{1}{2}$, $f \in \mathcal{H}'_\nu$ and $\phi \in \mathcal{H}_\nu$.

The above-mentioned testing-function space $\mathcal{H}_\nu$ has been defined as the collection of all functions $\phi(x)$ defined on $0 < x < \infty$, which are complex-valued and smooth and are such that for each pair of non-negative integers m and k

$$(1.6) \quad \gamma_{m,k}^\nu(\phi) = \sup_{0 < x < \infty} |x^m (x^{-1} D)^k [x^{-\nu-\frac{1}{2}} \phi(x)]| < \infty$$

$\mathcal{H}_\nu$ is a linear space. Also, $\{\gamma_{m,k}^\nu\}_{m,k=0}^\infty$ is a separating collection of seminorms [21, p. 8]. The topology of $\mathcal{H}_\nu$ is that generated by the collection $\{\gamma_{m,k}^\nu\}_{m,k=0}^\infty$. It is shown in [22] that any $\phi(x)$ in $\mathcal{H}_\nu$ also has the property that for each non-negative integer k , $D^k \phi(x)$ is of rapid descent as $x \rightarrow \infty$ (i.e., $D^k \phi(x)$ tends to zero faster than any power of $\frac{1}{x}$ as $x \rightarrow \infty$). It is quite obvious that $\mathcal{D}(I)$ is a subspace of $\mathcal{H}_\nu$ and $\mathcal{H}_\nu$ is a subspace of $\mathcal{E}(I)$. The dual space of $\mathcal{H}_\nu$ is denoted by $\mathcal{H}'_\nu$. The properties of the spaces $\mathcal{H}_\nu$ and $\mathcal{H}'_\nu$ have been investigated to a greater length in [22].

It was a natural question to ask whether one could define a generalized Hankel transform directly as the application of a generalized function to the kernel function, i.e., as

$$(1.7) \quad (\mathcal{H}_\nu f)(y) = \langle f(x), \sqrt{xy} J_\nu(xy) \rangle$$

In order that (1.7) has a sense, it is required that $\sqrt{xy} J_\nu(xy)$ as a function of x should belong to the testing function space of the generalized functions, which is not true in case of $\mathcal{H}'_\nu$. Thus (1.7) did not possess a sense for every $f \in \mathcal{H}'_\nu$. However, restricting only to those members of $\mathcal{H}'_\nu$ which are in $\mathcal{E}'(I)$ (as $\mathcal{E}'(I)$ is a subspace of $\mathcal{H}'_\nu$), Zemanian [22] defined the generalized Hankel transform of f as

$$(1.8) \quad (\mathcal{H}_\nu f)(y) = F(y) = \langle f(x), \sqrt{xy} J_\nu(xy) \rangle$$

for all $f \in \mathcal{E}'(I)$, $\nu \geq -\frac{1}{2}$ and $0 < y < \infty$.

It was also proved that $F(y)$ in (1.8) is a smooth function of y on $0 < y < \infty$.

An inversion formula concerning (1.8) has also been established in [22] as follows:

Theorem 1.3. Let $f \in \mathcal{E}'(I)$ and $\nu \geq -\frac{1}{2}$. Let $F(y)$ be the generalized Hankel transformation of f as defined by (1.8). Then, in the sense of convergence in $\mathcal{H}'_\nu$,

$$f(x) = \lim_{Y \rightarrow \infty} \int_0^Y F(y) \sqrt{xy} J_\nu(xy) dy .$$

i.e.

$$\langle f(x), \phi(x) \rangle = \lim_{Y \rightarrow \infty} \langle \int_0^Y F(y) J_\nu(xy) dy, \phi(x) \rangle$$

for each $\phi(x) \in \mathcal{F}$.

Further attempt was made by Koh and Zemanian [10] to give more general conditions under which the relation (1.7) defining the generalized Hankel transformation is meaningful, and an inversion theorem similar to Theorem 1.3 was established for a larger space of generalized functions. They [10] constructed a testing function space $\mathcal{J}_{\nu, a}$ as follows.

Let a denote a positive real number and ν any real number. Then for each pair of a and ν , $\mathcal{J}_{\nu, a}$ is defined as the collections of functions $\phi(x)$ which are defined and smooth on $0 < x < \infty$ and for which

$$(1.9) \quad \tau_k^{\nu, a}(\phi) = \sup_{0 < x < \infty} | e^{-ax} x^{-\nu-\frac{1}{2}} S_\nu^k(\phi(x)) | < \infty$$

for each $k=0, 1, 2, \dots$, where the operator S_ν^k is defined as:

$$S_\nu^k = (x^{-\nu-\frac{1}{2}} D x^{2\nu+1} D x^{-\nu-\frac{1}{2}})^k; D = \frac{d}{dx}, k = 0, 1, 2, \dots$$

$\mathcal{J}_{\nu, a}$ is a linear space. The topology of $\mathcal{J}_{\nu, a}$ is that generated by the countable multinorm [21, p. 7], $\left\{ \tau_k^{\nu, a} \right\}_{k=0}^\infty$.

The space $\mathcal{J}_{v,a}$ has been shown to be a Fréchet space [21, p. 12]. A countable union space [21, p. 15] from the spaces $\mathcal{J}_{v,a}$ has been defined as $\mathcal{J}_v(\sigma) = \bigcup_{n=1}^{\infty} \mathcal{J}_{v,a_n}$, where $\{a_n\}_{n=1}^{\infty}$ is a monotonically increasing sequence of positive numbers tending to σ ($\sigma=+\infty$ is allowed). The space $\mathcal{J}_v(\sigma)$ has also been shown to be a complete space [21, p. 12]. The dual spaces of $\mathcal{J}_{v,a}$ and $\mathcal{J}_v(\sigma)$ are denoted respectively by $\mathcal{J}'_{v,a}$ and $\mathcal{J}'_v(\sigma)$.

A generalized complex Hankel transformation of order v of any member $f \in \mathcal{J}'_v(\sigma)$ is defined as

$$(1.10) \quad F(y) = (\mathcal{H}_v f)(y) = \langle f(x), \sqrt{xy} J_v(xy) \rangle$$

where y is a complex number belonging to a strip $\Omega_\sigma = \{y: |\operatorname{Im} y| < \sigma, y \neq 0 \text{ or a negative number}\}$.

$F(y)$, as defined by (1.10) is proved to be analytic for each y in the region Ω_σ .

An inversion formula for the generalized Hankel transform (1.10) with y restricted to the positive real axis is established and stated as:

Theorem 1.4. Let $f \in \mathcal{J}'_v(\sigma)$ and $F(y)$ be the generalized Hankel transform of f as defined by (1.10) for each $y \in (0, \infty)$. Let $v > -\frac{1}{2}$. Then, in the sense of convergence in $\mathcal{D}'(I)$,

$$(1.11) \quad f(x) = \lim_{r \rightarrow \infty} \int_0^r F(y) \sqrt{xy} J_\nu(xy) dy .$$

[10]

Now, our main aim in this chapter is to define a generalized Hankel transformation similar to the one defined by (1.7) and to prove an inversion formula for a sufficiently large generalized function space. We have provided an example showing that there are many regular generalized functions in our space $H_{\alpha, \delta}^{\nu'}(I)$ which are not contained in the space $\mathcal{J}_{\nu, a}'$. On the other hand, it is shown that all the regular generalized functions in the space $\mathcal{J}_{\nu, a}'$ for $\nu > -\frac{1}{2}$, which are determined by note (ix) in [10] are contained in $H_{\alpha, \delta}^{\nu'}(I)$ (defined in the next section). It is seen in [10] that the space $\mathcal{J}_{\nu, a}'$ is a proper subspace of the space $\mathcal{H}_\nu'$, considered by Zemanian [22]. But there is no such containment between $H_{\alpha, \delta}^{\nu'}(I)$ and $\mathcal{H}_\nu'$. To this effect, examples in Section 7 are presented showing that none of the generalized function spaces $H_{\alpha, \delta}^{\nu'}(I)$ and $\mathcal{H}_\nu'$ is properly contained in the other, and that they have a non-empty intersection.

We have proved several related results including a structure formula for the members of $H_{\alpha, \delta}^{\nu'}(I)$ in sequel. In the end, we have applied our inversion theorem in solving a distributional differential equation.

We design our generalized Hankel transform and its inversion through an extension to a class of generalized functions of a classical inversion theorem for Hankel transform due to Schwartz [16]. The inversion theorem of Schwartz is given below.

Theorem 1.5. Let ν be a fixed real number exceeding $-\frac{1}{2}$ and let L consist of all measurable functions defined on $0 < x < \infty$ such that

$$\|f\| = \int_0^{\infty} |f(x)| dm(x) < \infty$$

where

$$(1.12) \quad dm(x) = [2^{\nu} \Gamma(\nu+1)]^{-1} x^{2\nu+1} dx,$$

Let

$$(1.13) \quad g(x) = 2^{\nu} \Gamma(\nu+1) x^{-\nu} J_{\nu}(x),$$

where J_{ν} is the Bessel function of the first kind of order ν .

Suppose $f \in L$ and $\int_0^1 |f(x) x^{\nu+\frac{1}{2}}| dx < \infty$. If f is of bounded variation in a neighbourhood of x , then,

$$(1.14) \quad \lim_{\lambda \rightarrow \infty} \int_0^{\lambda} g(xu) dm(u) \int_0^{\infty} f(y) g(uy) dm(y) = \frac{1}{2} [f(x+0) + f(x-0)].$$

Notice that this result of Schwartz is different from that due to Hankel [19, p. 456]. In fact, the function spaces for which the two results are valid are different with some non-empty intersection, and for a subspace of this non-empty intersection one inversion formula can be derived from the other.

2. In order to define our generalized Hankel transformation and to extend the inversion formula of Schwartz [17] to generalized functions, we need first to specify our generalized function space.

Let x denote a real one dimensional variable in the open interval $I=(0,\infty)$. Let $\nu > -\frac{1}{2}$ be a fixed real number. Let α and δ be real numbers such that $0 < \alpha \leq \nu + \frac{1}{2}$ and $\delta \geq 0$. Let $\xi(x)$ be an infinitely differentiable function defined over $(0,\infty)$, satisfying $\xi(x) > 0$ for all $x > 0$ and

$$\xi(x) = \begin{cases} x^{\nu+\frac{1}{2}+\delta}, & 0 < x < 1 \\ x^{\alpha-2}, & x \geq 1 \end{cases}.$$

The testing function space $G_{\alpha,\delta}^{\nu}(I)$ and its dual $G_{\alpha,\delta}^{\nu'}(I)$. Let ν , α and δ be fixed real numbers such that $\nu > -\frac{1}{2}$, $0 < \alpha \leq \nu + \frac{1}{2}$ and $\delta \geq 0$. For fixed ν , α and δ , the space $G_{\alpha,\delta}^{\nu}(I)$ is defined as the collection of all functions $\phi(x)$ defined on $0 < x < \infty$, which are complex-valued and smooth and are such that

$$(2.1) \quad \gamma_k(\phi) = \sup_{0 < x < \infty} |\xi(x) \Delta_x^k \phi(x)| < \infty$$

for $k=0, 1, 2, \dots$, where $\xi(x)$ is the same as defined above and the operator Δ_x is defined by

$$\Delta_x = D_x^2 + \frac{2\nu+1}{x} D_x; \quad D_x = \frac{d}{dx}.$$

$G_{\alpha,\delta}^v(I)$ is a linear space under the addition of functions and their multiplication by complex numbers. Each γ_k is clearly a seminorm on $G_{\alpha,\delta}^v(I)$, and γ_0 is a norm. Consequently, $\{\gamma_k\}_{k=0}^\infty$ is a separating collection of seminorms [21, p. 7], i.e., a multinorm on $G_{\alpha,\delta}^v(I)$. We assign to $G_{\alpha,\delta}^v(I)$ the topology generated by $\{\gamma_k\}_{k=0}^\infty$, thereby making it a countably multinormed space. A sequence $\{\phi_n\}_{n=1}^\infty$ is said to converge in $G_{\alpha,\delta}^v(I)$ to a function ϕ if $\gamma_k(\phi_n - \phi) \rightarrow 0$ as $n \rightarrow \infty$ for each $k=0, 1, 2, \dots$. A sequence $\{\phi_n\}_{n=1}^\infty$ is said to be a Cauchy sequence in $G_{\alpha,\delta}^v(I)$ if $\gamma_k(\phi_m - \phi_n) \rightarrow 0$ as m and n both tend to ∞ independently of each other for each $k=0, 1, 2, \dots$. It can be readily seen that $G_{\alpha,\delta}^v(I)$ is a locally convex, sequentially complete [21, p. 12], Hausdorff topological vector space.

Let the dual space of $G_{\alpha,\delta}^v(I)$, i.e. the collection of all continuous linear functionals on $G_{\alpha,\delta}^v(I)$ be denoted by $G_{\alpha,\delta}^{v'}(I)$. Under the usual definition of equality, addition and multiplication, $G_{\alpha,\delta}^{v'}(I)$ is also a linear space. We assign to $G_{\alpha,\delta}^{v'}(I)$ the weak topology generated by the multinorm $\{\eta_\phi(f)\}_\phi$, where $\eta_\phi(f) = |\langle f, \phi \rangle|$ and ϕ varies through $G_{\alpha,\delta}^v(I)$. It follows from [21, Th. 1.8-3] that $G_{\alpha,\delta}^{v'}(I)$ is also complete.

It can be easily checked that $\mathfrak{D}(I)$ is a subspace of $G_{\alpha,\delta}^v(I)$ and the topology of $\mathfrak{D}(I)$ is stronger than that induced on it by $G_{\alpha,\delta}^v(I)$. Hence the restriction of any $f \in G_{\alpha,\delta}^{v'}(I)$ to $\mathfrak{D}(I)$ is in $\mathfrak{D}'(I)$.

It can also be seen that $G_{\alpha,\delta}^v(I)$ is a subspace of $\mathcal{E}(I)$. More-

over, it is dense in $\mathcal{E}(I)$ as $\mathcal{A}(I)$ is dense in $\mathcal{E}(I)$. The topology of $G_{\alpha,\delta}^v(I)$ is stronger than that induced on it by $\mathcal{E}(I)$. Hence, $\mathcal{E}'(I)$ can be identified with a subspace of $G_{\alpha,\delta}^{v'}(I)$.

The testing-function space $H_{\alpha,\delta}^v(I)$ and its dual $H_{\alpha,\delta}^{v'}(I)$. A complex-valued smooth function $\phi(x)$ defined on $0 < x < \infty$ is said to belong to $H_{\alpha,\delta}^v(I)$ iff $\phi(x)/m'(x)$ belongs to the space $G_{\alpha,\delta}^v(I)$, where

$$(2.2) \quad m'(x) = \frac{x^{2v+1}}{2^v \Gamma(v+1)}.$$

The topology on $H_{\alpha,\delta}^v(I)$ is defined by means of the separating collection of seminorms [21, p. 7], $\{\beta_k\}_{k=0}^{\infty}$, where

$$(2.3) \quad \beta_k(\phi) = \gamma_k \left(\frac{\phi(x)}{m'(x)} \right), \quad k=0, 1, 2, \dots$$

for all $\phi(x) \in H_{\alpha,\delta}^v(I)$.

The convergence and completeness in $H_{\alpha,\delta}^v(I)$ are defined in a way similar to that defined in $G_{\alpha,\delta}^v(I)$. The space $H_{\alpha,\delta}^v(I)$ is also a locally convex, sequentially complete Hausdorff topological vector space. The dual space of $H_{\alpha,\delta}^v(I)$ is denoted by $H_{\alpha,\delta}^{v'}(I)$. The space $H_{\alpha,\delta}^{v'}(I)$ is also equipped with the usual weak topology. $H_{\alpha,\delta}^{v'}(I)$ is a linear space, under the usual definition, moreover it is complete.

The space $\mathcal{D}(I)$ is a subspace of $H_{\alpha,\delta}^v(I)$, and the topology of $\mathcal{D}(I)$ is stronger than that induced on it by $H_{\alpha,\delta}^v(I)$. Hence the restriction of any $f \in H_{\alpha,\delta}^{v'}(I)$ to $\mathcal{D}(I)$ is in $\mathcal{D}'(I)$.

The space $H_{\alpha,\delta}^v(I)$ is also a dense subspace of $\mathcal{E}(I)$ and the topology of $H_{\alpha,\delta}^v(I)$ is stronger than that induced on it by $\mathcal{E}(I)$. Hence $\mathcal{E}'(I)$ can be identified as a subspace of $H_{\alpha,\delta}^{v'}(I)$.

The elements of $G_{\alpha,\delta}^{v'}(I)$ and $H_{\alpha,\delta}^{v'}(I)$ will be called generalized functions or distributions.

For $f \in H_{\alpha,\delta}^{v'}(I)$, we define a generalized function $m'f \in G_{\alpha,\delta}^{v'}(I)$ as

$$(2.4) \quad \langle m'(x)f(x), \phi(x) \rangle = \langle f(x), m'(x)\phi(x) \rangle$$

for all $\phi \in G_{\alpha,\delta}^v(I)$, where $m'(x) = x^{2v+1}/2^v \Gamma(v+1)$.

The relation (2.4) has a sense as $\phi(x) \in G_{\alpha,\delta}^v(I)$ iff $m'(x)\phi(x) \in H_{\alpha,\delta}^v(I)$.

It can be readily seen that $f \in H_{\alpha,\delta}^{v'}(I)$ iff $m'f \in G_{\alpha,\delta}^{v'}(I)$.

Now we prove the following result.

Theorem 2.1. Let $f(x)$ be locally integrable on $0 < x < \infty$ such that $\int_0^\infty \frac{|f(x)|}{\xi(x)} dm(x) < \infty$, where $\xi(x)$ is the same function as defined by

(2.1). Then, f generates a regular generalized function in $H_{\alpha,\delta}^{v'}(I)$ defined by

$$(2.5) \quad \langle f, \phi \rangle = \int_0^\infty f(x)\phi(x)dx$$

for all $\phi \in H_{\alpha, \delta}^{\nu}(I)$.

Proof: Rewrite (2.5) as

$$\langle f, \phi \rangle = \int_0^{\infty} \frac{m'(x)f(x)}{\xi(x)} \xi(x) \frac{\phi(x)}{m'(x)} dx,$$

where $m'(x)$ is the same as defined by (2.2). Then,

$$|\langle f, \phi \rangle| \leq \int_0^{\infty} \frac{|f(x)|}{\xi(x)} \left| \xi(x) \frac{\phi(x)}{m'(x)} \right| dm(x)$$

$$\leq \gamma_0 \left[\frac{\phi(x)}{m'(x)} \right] \int_0^{\infty} \frac{|f(x)|}{\xi(x)} dm(x)$$

$$\text{i.e.} \quad \leq \beta_0[\phi(x)] \int_0^{\infty} \frac{|f(x)|}{\xi(x)} dm(x),$$

which clearly shows that (2.5) truly defines a functional on $H_{\alpha, \delta}^{\nu}(I)$.

This functional is clearly a linear one. The continuity of f on $H_{\alpha, \delta}^{\nu}(I)$

can also be easily proved by the help of the above inequality. Hence

$f \in H_{\alpha, \delta}^{\nu'}(I)$.

Corollary 2.1. Let $\nu + \frac{1}{2} > \alpha > 2$. Then the regular generalized function

space generated by the elements of the function space for which the

inversion theorem 1.5 of Schwartz is valid, is contained in the general-

ized function space $H_{\alpha, 0}^{\nu'}(I)$.

Proof: The proof follows easily in view of the preceding theorem.

3. The generalized Hankel transformation. Let $\nu > -\frac{1}{2}$, $0 < \alpha \leq \nu + \frac{1}{2}$ and $\delta \geq 0$.

We define the generalized Hankel transformation of $f \in H_{\alpha, \delta}^{\nu, \prime}(I)$ by the relation

$$(3.1) \quad F(y) = (\mathcal{H}_\nu f)(y) = \langle m'(x)f(x), g(xy) \rangle$$

where the parameter y is a real number such that $y > 0$ and $x \in (0, \infty)$;

$m'(x)$ and $g(xy)$ are given as

$$m'(x) = [2^\nu \Gamma(\nu+1)]^{-1} x^{2\nu+1}$$

$$g(xy) = 2^\nu \Gamma(\nu+1) (xy)^{-\nu} J_\nu(xy) \quad .$$

Since $\Delta_x^k g(xy) = (-1)^k y^{2k} g(xy)$ for each $k=0, 1, 2, \dots$, one can easily show that for each fixed $y > 0$, $g(xy) \in G_{\alpha, \delta}^\nu(I)$. The relation (3.1) is therefore meaningful.

It can be readily checked that $\frac{\partial^k}{\partial y^k} g(xy) \in G_{\alpha, \delta}^\nu(I)$ for each $y > 0$ and for $k=1$ and 2 . Now, we prove the following results.

Theorem 3.1 (Differentiability of $F(y)$). For real $y > 0$, let $F(y)$ be defined by (3.1), then $F(y)$ is differentiable and

$$(3.2) \quad \frac{dF(y)}{dy} = \langle m'(x)f(x), \frac{\partial}{\partial y} g(xy) \rangle$$

for all $y > 0$.

Remark: We do not know whether the second order derivative of $F(y)$ exists in general. Perhaps one can find an example of a generalized function in $H_{\alpha, \delta}^V(I)$ whose Hankel transform $F(y)$ does not possess second order derivative at least at one point $y > 0$.

Proof: Let Δy be an arbitrary increment in y such that $y > |\Delta y|$. Without any loss of generality, let us restrict Δy to $0 < |\Delta y| \leq 1$. Now,

$$(3.3) \quad \frac{F(y+\Delta y) - F(y)}{\Delta y} = \langle m'(x)f(x), \frac{\mathcal{J}\{x(y+\Delta y)\} - \mathcal{J}(xy)}{\Delta y} \rangle.$$

If we could show that as $\Delta y \rightarrow 0$, the testing function in the righthand side of (3.3) converges in $G_{\alpha, \delta}^V(I)$ to $\frac{\partial}{\partial y} \mathcal{J}(xy)$, then our result would follow from the continuity of $m'(x)f(x)$. Denote the expression

$$\left[\frac{\mathcal{J}\{x(y+\Delta y)\} - \mathcal{J}(xy)}{\Delta y} - \frac{\partial}{\partial y} \mathcal{J}(xy) \right] \text{ by } \theta_{\Delta y}(x).$$

We wish to show that the function $\theta_{\Delta y}(x)$ converges in $G_{\alpha, \delta}^V(I)$ to zero as $\Delta y \rightarrow 0$. Now

$$\xi(x) \Delta_x^k \theta_{\Delta y}(x)$$

$$= \xi(x) (-1)^k \left[\frac{(y+\Delta y)^{2k} g(x(y+\Delta y)) - y^{2k} g(xy)}{\Delta y} - \frac{\partial}{\partial y} (y^{2k} g(xy)) \right]$$

$$= (-1)^k \xi(x) \left[\frac{1}{\Delta y} \int_y^{y+\Delta y} \left\{ \frac{\partial}{\partial u} (u^{2k} g(xu)) - \frac{\partial}{\partial y} (y^{2k} g(xy)) \right\} du \right]$$

$$= (-1)^k \xi(x) \left[\frac{1}{\Delta y} \int_y^{y+\Delta y} \left(\left\{ \frac{\partial}{\partial t} (t^{2k} g(xt)) \right\}_{t=u} - \left\{ \frac{\partial}{\partial t} (t^{2k} g(xt)) \right\}_{t=y} \right) du \right]$$

$$= (-1)^k \xi(x) \left[\frac{1}{\Delta y} \int_y^{y+\Delta y} \int_y^u \frac{\partial^2}{\partial t^2} (t^{2k} g(xt)) dt du \right]$$

$$= (-1)^k 2^v \Gamma(v+1) \xi(x) x^{-v} \frac{1}{\Delta y} \left[\int_y^{y+\Delta y} \int_y^u \{ (2k-v)(2k-v-1) \right.$$

$$\cdot t^{2k-v-2} J_v(xt) + 2(2k-v) t^{2k-v-1} J_v^{(1)}(xt) x + t^{2k-v} J_v^{(2)}(xt)$$

$$\left. \cdot x^2 \} dt du \right]$$

$$= I_1 + I_2 + I_3$$

where

$$I_1 = (-1)^k 2^v \Gamma(v+1) (2k-v) (2k-v-1) \frac{1}{\Delta y} \int_y^{y+\Delta y} \int_y^u \{ \xi(x) (xt)^{-v} J_v(xt) \} t^{2k-2} dt du$$

and we have similar other expressions for I_2 and I_3 . Now, in view of the asymptotic orders of Bessel's functions and their derivatives, it can be seen that for $m=0, 1$, and 2 , the expression $|\xi(x) x^{-v+m} J_v^{(m)}(xt)|$ is bounded uniformly for all $x>0$ and $y-|\Delta y| \leq t \leq y+|\Delta y|$. Let $B_m, m=0, 1$ and 2 be the corresponding bounds. Then,

$$|I_1| \leq 2^v \Gamma(v+1) (2k-v) (2k-v-1) B_0 \frac{1}{|\Delta y|} \int_y^{y+\Delta y} \int_y^u t^{2k-2} dt du$$

Thus, for $\Delta y > 0$

$$|I_1| \leq 2^v \Gamma(v+1) (2k-v) (2k-v-1) B_0 \frac{(y+\Delta y)^{2k}}{y^2} \Delta y \rightarrow 0 \text{ as } \Delta y \rightarrow 0^+.$$

Similarly, for $\Delta y < 0$,

$$|I_1| \leq 2^v \Gamma(v+1) (2k-v) (2k-v-1) B_0 \frac{(y+|\Delta y|)^{2k}}{(y-|\Delta y|)^2} |\Delta y|$$

which tends to zero as $\Delta y \rightarrow 0^-$.

Hence, $|I_1| \rightarrow 0$ as $\Delta y \rightarrow 0$.

Similarly, we can show that for fixed $y > 0$, $|I_2|$ and $|I_3|$ both go to zero uniformly for all $x > 0$ as $\Delta y \rightarrow 0$. Hence, for each $k = 0, 1, 2, \dots$ the expression $|\xi(x) \Delta_x^{(k)}(\theta_{\Delta y}(x))| \rightarrow 0$ uniformly for all $x \in (0, \infty)$ as $\Delta y \rightarrow 0$. This completes the proof of the theorem.

Theorem 3.2. For each fixed $\delta \geq 0$, $0 < \alpha \leq \nu + \frac{1}{2}$ and $f \in H_{\alpha, \delta}^{\nu'}(I)$, let $F(y)$ be the Hankel transform of f as defined by $F(y) = \langle m'(x)f(x), g(xy) \rangle$, then there exists a non-negative integer r satisfying

$$|F(y)| = O[y^{-\alpha}], \quad y \rightarrow 0^+, \text{ and}$$

$$|F(y)| = O[y^{2r-\alpha}], \quad y \rightarrow \infty.$$

Proof. In view of a general result [21, Th. 1.8.1], there exist a constant $C > 0$ and a non-negative integer r such that

$$|\langle m'(x)f(x), g(xy) \rangle| = |F(y)| \leq C \max_{0 \leq k \leq r} \gamma_k[g(xy)]$$

$$= C \max_{0 \leq k \leq r} \sup_{0 < x < \infty} |\xi(x) \Delta_x^k[g(xy)]|$$

$$= C \max_{0 \leq k \leq r} \sup_{0 < x < \infty} |\xi(x) (-1)^k y^{2k} g(xy)|$$

$$= C 2^{\nu} \Gamma(\nu+1) \max_{0 \leq k \leq r} \sup_{0 < x < \infty} |\xi(x) y^{2k} (xy)^{-\nu} J_{\nu}(xy)|$$

Here $\xi(x)$ is the same as defined in Section 2.

Now we can choose a positive constant M such that

$$|\xi(x)| \leq Mx^\alpha \text{ for all } x > 0.$$

Therefore,

$$|F(y)| \leq C 2^\nu \Gamma(\nu+1) M \max_{0 \leq k \leq r} \sup_{0 < x < \infty} |(xy)^{\alpha-\nu} J_\nu(xy)| y^{2k-\alpha}.$$

By using asymptotic orders of $J_\nu(x)$, it can be seen that

$|(xy)^{\alpha-\nu} J_\nu(xy)|$ is bounded for all x, y in the interval $(0, \infty)$. Taking B as its bound, we get

$$|F(y)| \leq C 2^\nu \Gamma(\nu+1) M B \max_{0 \leq k \leq r} y^{2k-\alpha}.$$

Hence

$$|F(y)| = O(y^{-\alpha}) \text{ as } y \rightarrow 0^+$$

and

$$|F(y)| = O(y^{2r-\alpha}) \text{ as } y \rightarrow \infty.$$

This completes the proof of Theorem 3.2.

Before giving the main theorem, we prove the following necessary lemmas.

Lemma 3.1. Let $\nu > \frac{1}{2}$, $0 < \alpha \leq \nu + \frac{1}{2}$ and $\delta \geq 0$. Let $x, t > 0$. Then, for fixed $x > 0$,

$$(3.4) \quad \int_0^\eta g(yt)g(xy)dm(y) \rightarrow 0 \text{ in } G_{\alpha, \delta}^\nu(I)$$

as $\eta \rightarrow 0^+$ where

$$g(xy) = 2^\nu \Gamma(\nu+1)(xy)^{-\nu} J_\nu(xy)$$

and

$$dm(y) = [2^\nu \Gamma(\nu+1)]^{-1} y^{2\nu+1} dy = m'(y) dy.$$

Proof. To prove (3.4), we need show that for a fixed $x > 0$,

$$(3.5) \quad \xi(t) \Delta_t^k \int_0^\eta g(yt)g(xy)dm(y) \rightarrow 0$$

as $\eta \rightarrow 0^+$ uniformly for all $t > 0$ and for $k=0, 1, 2, \dots$

Since the order of differentiation and integration in (3.5) is interchangeable in view of [18, p. 59], therefore for any non-negative integer k

$$\Delta_t^k \int_0^\eta g(yt)g(xy)dm(y) = \int_0^\eta (-1)^k y^{2k} g(yt)g(xy)dm(y).$$

Then

$$(3.6) \quad \left| \xi(t) \Delta_t^k \int_0^\eta g(yt) g'(xy) dm(y) \right| \leq \xi(t) \int_0^\eta |y^{2k} g(yt) g(xy)| dm(y) .$$

By the definition of $\xi(t)$ in Section 2, we can find a positive constant M such that $\xi(t) \leq M t^\alpha$ for all $t > 0$. Notice that $\xi(t)$ is a positive function.

Hence the righthand side expression of (3.6) is bounded by

$$(3.7) \quad M[2^\nu \Gamma(\nu+1)] \int_0^\eta |(ty)^\alpha g'(ty)| \cdot |(xy)^\alpha g(xy)| x^{-\alpha} y^{2\nu+1-2\alpha+2k} dy .$$

Since $0 < \alpha \leq \nu + \frac{1}{2}$ the quantity $|(ty)^\alpha g'(ty)|$ is uniformly bounded for all $t, y > 0$ in view of the asymptotic orders of Bessel function. Similarly, $|(xy)^\alpha g(xy)|$ is uniformly bounded for all $x, y > 0$. Therefore, there is an obvious constant k_ν such that the expression (3.7) is bounded by

$$(3.8) \quad k_\nu x^{-\alpha} \int_0^\eta y^{2(\nu+\frac{1}{2}-\alpha)+2k} dy$$

which obviously tends to zero as $\eta \rightarrow 0^+$ for any fixed $x > 0$ and for any non-negative integer k . Notice that in (3.8), $2(\nu+\frac{1}{2}-\alpha)+2k \geq 0$ for any non-negative integer k as $0 < \alpha \leq \nu + \frac{1}{2}$.

This completes the proof of Lemma 3.1.

Lemma 3.2. Let $f \in H_{\alpha, \delta}^{v, \prime}(I)$ and let $0 < \alpha \leq v + \frac{1}{2}$ and $\delta \geq 0$. Then for a fixed $x > 0$ and for a fixed integer $N > 0$ one has

$$(3.9) \quad \int_0^N \langle m'(t)f(t) \mathcal{J}(yt) \rangle \mathcal{J}(xy) dm(y) = \langle m'(t)f(t), \int_0^N \mathcal{J}(yt) \mathcal{J}(xy) dm(y) \rangle$$

where $m'(t)$ and $\mathcal{J}(ty)$ are the same as defined in the preceding lemma.

Proof. In view of Theorems 3.1 and 3.2, the integral in the lefthand side of (3.9) exists. It can be shown quite readily that for fixed $x > 0$ the integral appearing in the righthand side of (3.9) belongs to $G_{\alpha, \delta}^v(I)$ and so the righthand side expression of (3.9) exists (is meaningful).

To justify (3.9), we first establish that for $\eta > 0$,

$$(3.10) \quad \int_{\eta}^N \langle m'(t)f(t) \mathcal{J}(yt) \rangle \mathcal{J}(xy) dm(y) = \langle m'(t)f(t), \int_{\eta}^N \mathcal{J}(yt) \mathcal{J}(xy) dm(y) \rangle.$$

Then, using Lemma 3.1 and letting $\eta \rightarrow 0^+$ in (3.10), the result (3.9) would follow.

We therefore need to justify (3.10). This can be proved by following the technique of Riemann sums. Since the method of the proof is very similar to that used in [11, Th. 2], further detail is omitted.

Now, let us denote the integral $\int_0^N \mathcal{J}(yt) \mathcal{J}(xy) dm(y)$ by the symbol $G_N(t, x)$. It is a fact that

$$(3.10) \quad G_N(t, x) = \frac{(xt)^{-\nu} N [x J_{\nu+1}(Nx) J_{\nu}(Nt) - t J_{\nu+1}(Nt) J_{\nu}(Nx)]}{(x^2 - t^2)}.$$

[19, p. 134]

Lemma 3.3. Let a and b be real numbers such that $b > a > 0$. Then,

$$\lim_{N \rightarrow \infty} \int_a^b G_N(t, x) dm(x) = \begin{cases} 1, & t \in (a, b) \\ \frac{1}{2}, & t = a, t = b \\ 0, & t \notin [a, b] \end{cases}.$$

Proof. The result follows quite readily in view of the Hankel's inversion formula [9, pp. 96-97]. This lemma can be proved still more readily by using Theorem 1.5 (Schwartz's inversion theorem [17]). By taking a function $f(y)$ defined on $0 < y < \infty$ as

$$f(y) = \begin{cases} 1, & a < y < b \\ 0, & \text{elsewhere} \end{cases}$$

and changing the order of integration in (1.14) by Fubini's theorem, the lemma follows.

Lemma 3.4. Let $\phi(x) \in \mathcal{D}(I)$ with support contained in $[a, b]$ $b > a > 0$. Then for $0 < \alpha \leq \nu + \frac{1}{2}$ and $\delta \geq 0$, the integral

$$(3.11) \quad \int_a^b G_N(t, x) \phi(x) dm(x) \rightarrow \phi(t) \text{ in } G_{\alpha, \delta}^{\nu}(I) \text{ as } N \rightarrow \infty.$$

Proof. It can be easily seen that $\Delta_t G_N(t, x) = \Delta_x G_N(t, x)$. Hence

$$\begin{aligned} \Delta_t \int_a^b G_N(t, x) \phi(x) dm(x) &= \int_a^b \Delta_x G_N(t, x) \phi(x) dm(x) \\ &= \int_a^b G_N(t, x) [\Delta_x \phi(x)] dm(x) . \end{aligned}$$

(by integration by parts)

Operating by Δ_t successively, it can be seen in view of Lemma 3.3 that

$$\lim_{N \rightarrow \infty} \left[\Delta_t^k \int_a^b G_N(t, x) \phi(x) dm(x) - \phi(t) \right] = \lim_{N \rightarrow \infty} \int_a^b G_N(t, x) [\phi_k(x) - \phi_k(t)] dm(x)$$

where

$$\Delta_x^k \phi(x) \equiv \phi_k(x) \in \mathcal{D}(I) \text{ for } k=0, 1, 2, \dots$$

Now, our problem is reduced to proving the following

$$(3.12) \quad \xi(t) \int_a^b G_N(t, x) [\psi(x) - \psi(t)] dm(x) \rightarrow 0 \text{ as } N \rightarrow \infty$$

uniformly for all $t > 0$ where $\psi(x) \in \mathcal{D}(I)$ with support contained in $[a, b]$.

Thus, substituting the expression for $G_N(t, x)$ from (3.10) and denoting the expression (3.12) by I_N , we get $I_N = I_1 - I_2$ where

$$I_1 = N[2^v \Gamma(v+1)]^{-1} \xi(t) t^{-v} J_v(Nt) \int_a^b x^{v+2} \frac{\psi(x) - \psi(t)}{x^2 - t^2} J_{v+1}(Nx) dx$$

and

$$I_2 = N[2^v \Gamma(v+1)]^{-1} \xi(t) t^{-v+1} J_{v+1}(Nt) \int_a^b x^{v+1} \frac{\psi(x) - \psi(t)}{x^2 - t^2} J_v(Nx) dx$$

Since for $t > b$, $\psi(t) = 0$, therefore making a little rearrangement in the term N , we rewrite I_2 as

$$I_2 = [2^v \Gamma(v+1)]^{-1} \xi(t) t^{-v+\frac{1}{2}} \sqrt{Nt} J_{v+1}(Nt) \int_a^b x^{v+\frac{1}{2}} \frac{\psi(x)}{x^2 - t^2} \sqrt{Nx} J_v(Nx) dx$$

Now, it is a fact that $\sqrt{Nt} J_{v+1}(Nt)$ is uniformly bounded for all $N > 0$ and $t > 0$; similarly, $\sqrt{Nx} J_v(Nx)$ is also uniformly bounded for all $N > 0$ and $x > 0$. Let A and B be the respective bounds so that

$$|\sqrt{Nt} J_{v+1}(Nt)| \leq A$$

and

$$|\sqrt{Nx} J_v(Nx)| \leq B$$

Also, for all $t > 0$ we can find a constant $K' > 0$ such that $|\xi(t)| \leq K' t^{\alpha-2}$. Moreover, for $t > b$, the inequality $|x^2 - t^2| > t^2 - b^2$ holds

when $a \leq x \leq b$. Hence,

$$|I_2| \leq [2^{\nu} \Gamma(\nu+1)]^{-1} K'_{ABM} \frac{t^{\alpha-(\nu+\frac{1}{2})-1}}{t^2-b^2} \int_a^b x^{\nu+\frac{1}{2}} dx.$$

$$(3.13) \text{ i.e. } \leq [(\nu+\frac{3}{2}) 2^{\nu} \Gamma(\nu+1)]^{-1} \left(\frac{b^{\nu+\frac{3}{2}}}{2} - \frac{a^{\nu+\frac{3}{2}}}{2} \right) K'_{ABM} \frac{t^{\alpha-(\nu+\frac{1}{2})-1}}{t^2-b^2}$$

where $M = \sup_{a \leq x \leq b} |\psi(x)|$.

Since $0 < \alpha \leq \nu + \frac{1}{2}$, for a given arbitrary $\epsilon > 0$, we can get a positive integer $N_1 > b$ sufficiently large such that the absolute value of the expression (3.13) is less than ϵ , i.e.

$$(3.14) \quad |I_2| < \epsilon \quad \text{for all } t > N_1 > b.$$

Next, we consider separately the remaining three ranges of t , namely $a \leq t \leq b$, $0 < t < a$ and $b < t \leq N_1$ and prove the convergence of I_2 to zero as $N \rightarrow \infty$, in each of the intervals uniformly.

Now, for the case $a \leq t \leq b$, first we establish that

$$(3.15) \quad \int_a^b x^{\nu+1} \frac{\psi(x) - \psi(t)}{x^2 - t^2} J_{\nu}(Nx) dx = o\left(\frac{1}{\sqrt{N}}\right) \text{ as } N \rightarrow \infty$$

where the order $o(1/\sqrt{N})$ is independent of t . We prove this by following the techniques used in the analogue of Riemann-Lebesgue Lemma [19, p. 457].

Define a function $f(x,t)$ of x and t in the square region $\{a \leq x \leq b; a \leq t \leq b\}$ as follows:

$$f(x,t) = \begin{cases} \frac{\psi(x) - \psi(t)}{x^2 - t^2}, & x \neq t \\ \frac{\psi'(x)}{2x}, & x = t \end{cases}$$

Obviously $f(x,t)$ is uniformly continuous in the domain $\{a \leq x \leq b; a \leq t \leq b\}$. Replacing the expression

$$\frac{\psi(x) - \psi(t)}{x^2 - t^2}$$

by the function $f(x,t)$ in the integral in (3.15), the value of the integral is unchanged. Hence we need show that

$$(3.16) \quad \int_a^b x^{\nu+1/2} f(x,t) J_\nu(Nx) \sqrt{x} \, dx = o\left(\frac{1}{\sqrt{N}}\right), \text{ as } N \rightarrow \infty$$

independently of t , $a \leq t \leq b$.

Let $\sup_{\substack{a \leq x \leq b \\ a \leq t \leq b}} |x^{\nu+1/2} f(x,t)| = K$. Clearly such a supremum

exists. Divide the range of integration (a,b) into n equal intervals

$a=x_0, x_1, x_2, \dots, x_{n-1}, x_n=b$. Let

$$\sup_{x_{m-1} \leq x \leq x_m} |x^{v+\frac{1}{2}} f(x, t)| = U_m$$

$$x_{m-1} \leq x \leq x_m$$

$$a \leq t \leq b$$

and

$$\inf_{x_{m-1} \leq x \leq x_m} |x^{v+\frac{1}{2}} f(x, t)| = L_m$$

$$x_{m-1} \leq x \leq x_m$$

$$a \leq t \leq b$$

for each $m=1, 2, \dots, n$.

Now, in view of uniform continuity of $x^{v+\frac{1}{2}} f(x, t)$, ($v \geq -\frac{1}{2}$) in the region $\{a \leq x \leq b; a \leq t \leq b\}$, for a given arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|x^{v+\frac{1}{2}} f(x, t) - x'^{v+\frac{1}{2}} f(x', t)| < \frac{\varepsilon}{b-a}$$

whenever $|x-x'| < \delta$ and $a \leq t \leq b$, for every pair of points (x, t) and (x', t) in the region.

Choose n large enough so that the norm of the subdivision $a=x_0, x_1, x_2, \dots, x_n=b$, is less than δ . Then

$$U_m - L_m < \frac{\varepsilon}{b-a}$$

for each $m=1, 2, \dots, n$, and therefore

$$(3.17) \quad \sum_{m=1}^n (U_m - L_m)(x_m - x_{m-1}) < \varepsilon.$$

Now write $f(x, t)x^{v+\frac{1}{2}}$ as

$$f(x, t)x^{v+\frac{1}{2}} = f(x_{m-1}, t)x_{m-1}^{v+\frac{1}{2}} + w_m(x, t)$$

so that when x lies in the m^{th} subinterval, $|w_m(x, t)| \leq U_m - L_m$ for each $m=1, 2, \dots, n$.

Now for $v \geq -\frac{1}{2}$, both of the functions of x ,

$$\sqrt{x} J_v(x), \text{ and } \int_0^x \sqrt{t} J_v(t) dt$$

are bounded when $x \geq 0$, even though the integral $\int_0^\infty \sqrt{t} J_v(t) dt$ is not convergent. Let A and B be the upper bounds of the moduli of these functions respectively when $a \leq x \leq b$. Then it is clear that

$$\begin{aligned}
 & \left| \int_a^b x^{\nu+\frac{1}{2}} f(x, t) J_\nu(Nx) \sqrt{x} \, dx \right| \\
 &= \left| \sum_{m=1}^n x_{m-1}^{\nu+\frac{1}{2}} f(x_{m-1}, t) \int_{x_{m-1}}^{x_m} J_\nu(Nx) \sqrt{x} \, dx + \sum_{m=1}^n \int_{x_{m-1}}^{x_m} w_m(x, t) J_\nu(Nx) \sqrt{x} \, dx \right| \\
 &\leq \sum_{m=1}^n \left| \frac{x_{m-1}^{\nu+\frac{1}{2}} f(x_{m-1}, t)}{N^{3/2}} \int_{Nx_{m-1}}^{Nx_m} J_\nu(x) \sqrt{x} \, dx \right| + \sum_{m=1}^n \int_{x_{m-1}}^{x_m} |w_m(x, t)| \frac{A}{\sqrt{N}} \, dx \\
 &\leq \frac{2B}{N^{3/2}} \sum_{m=1}^n \left| x_{m-1}^{\nu+\frac{1}{2}} f(x_{m-1}, t) \right| + \frac{A\epsilon}{\sqrt{N}} \\
 &\leq \frac{2BnK}{N^{3/2}} + \frac{A\epsilon}{\sqrt{N}} .
 \end{aligned}$$

By taking N sufficiently large (n remaining fixed after ϵ has been chosen) the expression $2BnK/N^{3/2}$ can be made less than $A\epsilon/\sqrt{N}$ so that

$$\left| \int_a^b x^{\nu+\frac{1}{2}} f(x, t) J_\nu(Nx) \sqrt{x} \, dx \right| < \frac{2A\epsilon}{\sqrt{N}} .$$

Let the upper bound of the expression $[2^{\nu}\Gamma(\nu+1)]^{-1} \xi(t)t^{\nu+\frac{1}{2}}$
 $\sqrt{Nt} J_{\nu+1}(Nt)$ for all $a \leq t \leq b$ be C which clearly exists. Then,

$$|I_2| < \frac{2AC}{\sqrt{N}} \varepsilon (b-a) \text{ for } a \leq t \leq b, \text{ i.e.}$$

$$(3.18) \quad |I_2| \rightarrow 0 \text{ as } N \rightarrow \infty \text{ uniformly for } a \leq t \leq b.$$

Using similar techniques, we can show that

$$(3.19) \quad |I_2| \rightarrow 0 \text{ as } N \rightarrow \infty \text{ uniformly for } 0 < t \leq a$$

and

$$(3.20) \quad |I_2| \rightarrow 0 \text{ as } N \rightarrow \infty \text{ uniformly for } b \leq t \leq N_1.$$

Combining (3.14), (3.18), (3.19) and (3.20), we conclude
that

$$(3.21) \quad |I_2| \rightarrow 0 \text{ as } N \rightarrow \infty \text{ uniformly for all } t > 0.$$

Using the similar arguments, it can be shown that

$$(3.22) \quad |I_1| \rightarrow 0 \text{ as } N \rightarrow \infty \text{ uniformly for all } t > 0.$$

Combining (3.21) and (3.22) we conclude that

$|I_N| \rightarrow 0$ as $N \rightarrow \infty$ uniformly for all $t > 0$, where I_N denotes the expression (3.12). This completes the proof of Lemma 3.4.

Inversion theorem: Now we prove the inversion theorem.

Theorem 3.3. Let $f \in H_{\alpha, \delta}^{v, 1}(I)$ where $0 < \alpha \leq v + \frac{1}{2}$, $v > -\frac{1}{2}$ and $\delta \geq 0$, and let $F(y)$ be the Hankel transform of f defined by (3.1). Then for each $\phi \in \mathcal{D}(I)$ one has

$$(3.23) \quad \langle m'(x) \int_0^N F(y) \mathcal{J}(xy) dm(y), \phi(x) \rangle \rightarrow \langle m'f, \phi \rangle$$

as $N \rightarrow \infty$. $\mathcal{J}(xy)$ and $m'(x)$ are the same as defined in Lemma 3.1.

Proof. Suppose that the support of $\phi(x)$ is contained in the interval (a, b) , $b > a > 0$. Now the result (3.23) will be proved by justifying the steps in the following manipulations.

$$(3.24) \quad \langle m'(x) \int_0^N F(y) \mathcal{J}(xy) dm(y), \phi(x) \rangle = \int_a^b \int_0^N F(y) \mathcal{J}(xy) dm(y) \phi(x) dm(x)$$

$$(3.24)' \quad = \int_a^b \phi(x) dm(x) \int_0^N \langle m'(t) f(t), \mathcal{J}(yt) \rangle \mathcal{J}(xy) dm(y)$$

$$(3.25) \quad = \int_a^b \langle m'(t) f(t), \int_0^N \mathcal{J}(yt) \mathcal{J}(xy) dm(y) \rangle \phi(x) dm(x)$$

$$(3.25)' \quad = \int_a^b \langle m'(t)f(t), G_N(t,x) \rangle \phi(x) dm(x)$$

$$(3.26) \quad = \langle m'(t)f(t), \int_a^b G_N(t,x) \phi(x) dm(x) \rangle$$

$$(3.27) \quad \rightarrow \langle m'f, \phi \rangle \text{ as } N \rightarrow \infty$$

The step (3.24) is obvious in view of Theorems 3.1 and 3.2. That the expression (3.24)' equals the expression (3.25) follows from Lemma 3.2. By following the technique very similar to that used in proving Lemma 3.2 one can show that the expression (3.25)' equals the expression (3.26). The expression (3.26) goes to that in (3.27) as $N \rightarrow \infty$ by Lemma 3.4. This completes the proof of Theorem 3.3.

Theorem 3.4. (Uniqueness Theorem) Let f and g belong to the space $H_{\alpha, \delta}^{V'}(I)$. Let for each $y > 0$, $F(y)$ and $G(y)$ be the generalized Hankel transformations of f and g respectively as defined by (3.1). If $F(y) = G(y)$ for each $y > 0$ then $f = g$ in the sense of equality in $\mathcal{S}'(I)$.

Proof. By Theorem 3.3, we have for any $\phi \in \mathcal{S}(I)$,

$$(3.28) \quad \langle m'(f-g), \phi \rangle = \lim_{N \rightarrow \infty} \int_a^b \int_0^N [F(y) - G(y)] g(xy) dm(y) \phi(x) dm(x)$$

where support of $\phi(x)$ is contained in an interval $[a,b]$ and $m'(x)$ and $\mathcal{J}(xy)$ are the same as defined in (3.1). But the righthand side of (3.28) equals zero as $F(y)=G(y)$ for all $y>0$. Hence for any $\phi \in \mathfrak{D}(I)$

$$\langle m'(f-g), \phi \rangle = \langle f-g, m'\phi \rangle = 0$$

and therefore, $f-g=0$ i.e. $f=g$ in $\mathfrak{D}'(I)$. This completes the proof of Theorem 3.4.

4. A structure formula. In this section, we give a structure formula for the restriction of an element $f \in H_{\alpha, \delta}^{\nu'}(I)$ to $\mathfrak{D}(I)$.

Theorem 4. Let f be an arbitrary element of $H_{\alpha, \delta}^{\nu'}(I)$ and ϕ be an arbitrary element of $\mathfrak{D}(I)$. Then, there exist bounded measurable functions $g_i(x)$ defined for $x>0$ and polynomials $Q_i(1/x)$ in $1/x$ for $i=0, 1, 2, \dots, 2k+1$ such that

$$(4.1) \quad \langle f, \phi \rangle = \left\langle \sum_{i=0}^{2k+1} (-1)^i D^{i+1} \int_a^x g_i(x) x^{\alpha-2\nu} Q_i(1/x) dx, \phi(x) \right\rangle$$

where k is some appropriate non-negative integer and a is a positive number.

Here D denotes the distributional differentiation.

Proof. Let $\{\gamma_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ be the sequences of seminorms as defined in Section 2. Let f and ϕ be arbitrary elements of $H_{\alpha, \delta}^{\nu'}(I)$ and $\mathfrak{D}(I)$ respectively. By virtue of a general theorem [21, Th. 1.8.1] there exist a constant $C>0$ and a non-negative integer r such that

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq r} \beta_k(\phi)$$

i.e.

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq r} \gamma_k \left(\frac{\phi(x)}{m'(x)} \right)$$

$$= C' \max_{0 \leq k \leq r} \sup_{0 < x < \infty} \left| \xi(x) \Delta_x^k \left(\frac{\phi(x)}{x^{2\nu+1}} \right) \right|$$

$$(4.2) \quad = C'' \max_{0 \leq k \leq r} \sup_{0 < x < \infty} x^\alpha \left(D_x^2 + \frac{2\nu+1}{x} D_x \right)^k \frac{\phi(x)}{x^{2\nu+1}}$$

where $\xi(x) \leq K'x^\alpha$ for some constant $K' > 0$, $C' = 2^\nu \Gamma(\nu+1)C$ and $C'' = C'K'$.

Thus, the expression (4.2)

$$= C'' \max_{0 \leq k \leq r} \sup_{0 < x < \infty} \left| \sum_{i=0}^{2k} x^{\alpha-2\nu} P_i(1/x) \phi^{(i)}(x) \right|$$

$$= C'' \max_{0 \leq k \leq r} \sup_{0 < x < \infty} \left| \int_x^\infty \sum_{i=0}^{2k} D_x \{ x^{\alpha-2\nu} P_i(1/x) \phi^{(i)}(x) \} dx \right|$$

$$\leq C'' \max_{0 \leq k \leq r} \sup_{0 < x < \infty} \int_x^\infty \sum_{i=0}^{2k+1} |x^{\alpha-2\nu} Q_i(1/x) \phi^{(i)}(x)| dx$$

Here $P_i(1/x)$ and $Q_i(1/x)$ are obvious polynomials in $1/x$, and $\phi^{(i)}(x)$ denotes the i^{th} derivative of $\phi(x)$. Thus, we have

$$|\langle f, \phi \rangle| \leq C'' \max_{0 \leq k \leq r} \sum_{i=0}^{2k+1} \int_0^\infty |x^{\alpha-2\nu} Q_i(1/x) \phi^{(i)}(x)| dx.$$

Therefore, in view of Reisz representation theorem and Hahn Banach extension theorem, there exist bounded measurable functions $g_i(x)$ defined over $I=(0, \infty)$, $i=0, 1, 2, \dots, 2k+1$, satisfying

$$(4.3) \quad \langle f, \phi \rangle = \sum_{i=0}^{2k+1} \langle g_i(x), x^{\alpha-2\nu} Q_i(1/x) \phi^{(i)}(x) \rangle.$$

Since it can be easily verified that the function $\int_a^x g_i(x) x^{\alpha-2\nu} Q_i(1/x) dx$ generates a regular distribution in $\mathcal{D}'(I)$, therefore it follows from

(4.3) that

$$(4.4) \quad \langle f, \phi \rangle = \sum_{i=0}^{2k+1} \langle (-1)^i D^{i+1} \int_a^x g_i(x) x^{\alpha-2\nu} Q_i(1/x) dx, \phi(x) \rangle$$

where differentiation in (4.4) is in distributional sense, and a is a positive real number.

This completes the proof of Theorem 4.

5. Further enlargement of Hankel transformable space of generalized

functions. (i) In Section 2, we observe that for any two real numbers $\delta_1, \delta_2 > 0$ such that $\delta_1 \leq \delta_2$, $G_{\alpha, \delta_1}^v(I) \subseteq G_{\alpha, \delta_2}^v(I)$. Moreover, topology on $G_{\alpha, \delta_1}^v(I)$ is stronger than that induced on it by $G_{\alpha, \delta_2}^v(I)$. Consequently, the restriction of any $f \in G_{\alpha, \delta_2}^{v'}(I)$ to $G_{\alpha, \delta_1}^v(I)$ is in $G_{\alpha, \delta_1}^{v'}(I)$.

It is also observed that the space $G_{\alpha, 0}^v(I) \subset G_{\alpha, \delta}^v(I)$ for any real $\delta > 0$ and the topology of $G_{\alpha, 0}^v(I)$ is stronger than that induced on it by $G_{\alpha, \delta}^v(I)$. Therefore, the restriction of any $f \in G_{\alpha, \delta}^{v'}(I)$ to $G_{\alpha, 0}^v(I)$ is in $G_{\alpha, 0}^{v'}(I)$. Similarly for all $\delta > 0$, the restriction of any $f \in H_{\alpha, \delta}^{v'}(I)$ to $H_{\alpha, 0}^v(I)$ is in $H_{\alpha, 0}^{v'}(I)$.

Hence in Section 2, taking $\delta=0$, we obtain the largest Hankel transformable generalized function space.

(ii) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers such that $0 < \alpha_n \leq v + \frac{1}{2}$ for each $n=1, 2, 3, \dots$ and $\alpha_n \rightarrow 0^+$ as $n \rightarrow \infty$. It can be easily verified that $H_{\alpha_n, 0}^v(I) \subset H_{\alpha_{n+1}, 0}^v(I)$ and topology of $H_{\alpha_n, 0}^v(I)$ is stronger than that induced on it by $H_{\alpha_{n+1}, 0}^v(I)$ for $n=1, 2, 3, \dots$. We define a countable union space $H_0^v(I)$ as

$$H_0^v(I) = \bigcup_{n=1}^{\infty} H_{\alpha_n, 0}^v(I) . \quad [21, p. 15] .$$

The concept of convergence and Cauchy sequence in $H_0^v(I)$ is defined as in [21, p. 15]. The space $H_0^v(I)$ is a sequentially complete space [21, p. 12]. Denote the dual space of $H_0^v(I)$ by $H_0^{v'}(I)$. The Hankel transform of an element $f \in H_0^{v'}(I)$ is defined in the way similar to that defined in Section 3.

An inversion formula similar to Theorem 3.3 holds in this case too.

(iii) It is easy to check that for $\alpha > 0$, $H_{\alpha+1,\delta}^v(I) \subset H_{\alpha,\delta}^v(I)$ and topology of $H_{\alpha+1,\delta}^v(I)$ is stronger than that induced on it by $H_{\alpha,\delta}^v(I)$. Therefore, the restriction of any $f \in H_{\alpha+1,\delta}^{v'}(I)$ to $H_{\alpha,\delta}^v(I)$ is in $H_{\alpha+1,\delta}^{v'}(I)$.

One can check that when $f \in H_{\alpha+1,\delta}^{v'}(I)$ its Hankel transform as defined by $F(y) = \langle m'(x)f(x), \varphi(xy) \rangle$, exists and is continuous for all $y > 0$. The proof of the continuity of $F(y)$ goes very similar to Theorem 3.1. But the differentiability of $F(y)$ is suspected in general. Perhaps one could get some example of $f \in H_{\alpha+1,\delta}^{v'}(I)$ whose Hankel transform $F(y)$ may not be differentiable at least for some $y > 0$. We regret that we have not yet got any counter example to this effect.

The theorems analogous to Theorem 3.3 and Theorem 4 are valid in this case too.

6. An open problem. For $0 < \alpha \leq \nu + \frac{1}{2}$, $\delta \geq 0$ and $f \in H_{\alpha+2, \delta}^{\nu'}(I)$ let Hankel transform of f be defined by

$$F(y) = \langle m'(x)f(x), \mathcal{H}(xy) \rangle \quad \text{for } y > 0.$$

Can one show that

- (i) $F(y)$ is continuous or at least continuous almost everywhere?
- (ii) the inversion formula analogous to Theorem 3.3 is valid in this case?

7. Some Examples. In this section, we give some interesting examples concerning the space $H_{\alpha, \delta}^{\nu'}(I)$.

Example 7.1. We give an example of a class of regular distributions $g(x)$ which belong to our space $H_{\alpha, \delta}^{\nu'}(I)$ but do not belong to the Hankel transformable generalized function space $\mathcal{G}'_{\nu, \alpha}$ of Koh and Zemanian [10] defined in Section 1. Let $g(x)$ be a locally integrable function defined for $x > 0$ such that

$$g(x) = \begin{cases} 0[x^{-\nu - \frac{3}{2} + \eta + \delta}], & x \rightarrow 0^+ \\ 0[x^{\alpha - 2\nu - 4 - \epsilon}], & x \rightarrow \infty \end{cases}$$

where ϵ, η are some positive numbers and ν, α and δ are as defined in Section 2. Clearly, the regular generalized function generated by $g(x)$ is in $H_{\alpha, \delta}^{\nu'}(I)$.

Now, let $\psi(x)$ be an infinitely differentiable function defined over $(0, \infty)$ satisfying the following conditions:

$$\psi(x) = \begin{cases} 0, & 0 < x < 1 \\ e^{-ax}, & x \geq 1; a > 0 \end{cases}$$

It is easy to show that $\psi(x)$ belongs to the testing function space $\mathcal{J}_{v,a}$ as defined on page 19. . . But, the integral $\int_0^\infty g(x)\psi(x)dx$ does not exist, showing that $g(x)$ does not belong to the corresponding dual space $\mathcal{J}'_{v,a}$

Next, we want to show that all those regular generalized functions in $\mathcal{J}'_{v,a}$ (the generalized function space considered by Koh and Zemanian [10]) for $v > -\frac{1}{2}$, which are determined by note (ix) in [10], also belong to our generalized function space $H_{\alpha,\delta}^{v'}(I)$ for $0 < \alpha \leq v + \frac{1}{2}$ and $\delta = 0$.

By note (ix) in [10], a locally integrable function $f(x)$ on $0 < x < \infty$ generates a regular generalized function in $\mathcal{J}'_{v,a}$ if

$$(i) \quad \int_0^\infty |f(x)e^{-ax}x^{v+\frac{1}{2}}|dx < \infty$$

Let (i) be satisfied.

Now, in order to show that the locally integrable function $f(x)$ satisfying (i) generates a regular generalized function in $H_{\alpha,0}^{\nu'}(I)$, it suffices to show in view of Theorem 2.1 that

$$(ii) \quad \int_0^{\infty} \frac{|f(x)|}{\xi(x)} dm(x) < \infty$$

where $dm(x)$ and $\xi(x)$ are defined in Sections 1 and 2, respectively.

Indeed,

$$\begin{aligned} \int_0^{\infty} \frac{|f(x)|}{\xi(x)} dm(x) &= \left(\int_0^1 + \int_1^2 + \int_2^{\infty} \right) \left(\frac{|f(x)|}{\xi(x)} \right) dm(x) \\ &= [2^{\nu} \Gamma(\nu+1)]^{-1} \left[\int_0^1 \frac{|f(x)|}{x^{\nu+\frac{1}{2}}} x^{2\nu+1} dx + \int_1^2 \frac{|f(x)|}{\xi(x)} x^{2\nu+1} dx + \int_2^{\infty} \frac{|f(x)|}{x^{\alpha-2}} x^{2\nu+1} dx \right] \\ &= [2^{\nu} \Gamma(\nu+1)]^{-1} \left[\int_0^1 |f(x) e^{ax} x^{\nu+\frac{1}{2}}| e^{-ax} dx + \int_1^2 \frac{|f(x)|}{\xi(x)} x^{2\nu+1} dx \right. \\ &\quad \left. + \int_2^{\infty} |f(x) e^{ax} x^{\nu+\frac{1}{2}}| e^{-ax} x^{\nu + \frac{5}{2} - \alpha} dx \right]. \end{aligned}$$

Let $\inf_{1 \leq x \leq 2} \xi(x) = A$, which exists by the definition of $\xi(x)$ and
 $\nu + 5/2 - \alpha$
 let $\sup_{2 < x < \infty} |x| e^{-ax} = B$ which also exists as $a > 0$. Then,

$$(iii) \quad \int_0^{\infty} \frac{|f(x)|}{\xi(x)} dm(x) \leq [2^{\nu} \Gamma(\nu+1)]^{-1} \left[\int_0^1 |f(x) e^{ax} x^{\nu+1/2}| dx + \frac{1}{A} \int_1^2 |f(x)| x^{2\nu+1} dx \right. \\ \left. + B \int_2^{\infty} |f(x) e^{ax} x^{\nu+1/2}| dx \right].$$

The second integral in the right hand side of (iii) is obviously finite. The first and third integrals in the right hand side of (iii) are also finite in view of (i). Hence, (ii) is proved.

Example 7.2. We have seen that for each fixed $x > 0$, $m'(x) \mathcal{J}(xy) \in H_{\alpha, \delta}^{\nu}(I)$ where $m'(x)$ and $\mathcal{J}(xy)$ are the same as defined in Section 3. But $m'(x) \mathcal{J}(xy) \notin \mathcal{H}_{\nu}$, the testing function space considered by Zemanian [22] defined on pages 16-17. It is due to the reason that $m'(x) \mathcal{J}(xy)$ is not of rapid descent.

Example 7.3. Let $\psi(x)$ be an infinitely differentiable function defined over I satisfying the following conditions:

$$\psi(x) = \begin{cases} x^{\nu+1/2}, & 0 < x < 1; \nu \geq -1/2 \\ e^{-x}, & x \geq N \end{cases}$$

It is easy to verify that $\psi \in \mathcal{H}_{\nu}$ [22] but $\psi \notin H_{\alpha, \delta}^{\nu}(I)$.

The examples (7.2) and (7.3) together with the fact that the space $\mathfrak{H}(I)$ is contained in both the spaces $H_{\alpha,\delta}^{\nu}(I)$ and $\mathcal{H}_{\nu}$, imply that the spaces $H_{\alpha,\delta}^{\nu}(I)$ and $\mathcal{H}_{\nu}$ have non-empty intersection and none is contained in the other properly. Therefore, the generalized function spaces $H_{\alpha,\delta}^{\nu'}(I)$ and $\mathcal{H}_{\nu}'$ [22] have non-empty intersection and none is properly contained in the other.

8. An application. In this section, we will apply our inversion formula to solve certain differential equations.

We define an operator $\Delta_x^*: H_{\alpha,\delta}^{\nu'}(I) \rightarrow H_{\alpha,\delta}^{\nu}(I)$ given by the relation

$$(8.1) \quad \langle \Delta_x^* f(x), \phi(x) \rangle = \langle f(x), m'(x) \Delta_x \frac{\phi(x)}{m'(x)} \rangle$$

for all $f \in H_{\alpha,\delta}^{\nu'}(I)$ and $\phi \in H_{\alpha,\delta}^{\nu}(I)$, $0 < \alpha \leq \nu + \frac{1}{2}$, $\delta \geq 0$. Obviously the righthand side of (8.1) has a sense. Let us call Δ_x^* the adjoint of the operator $\Delta_x = D_x^2 + \frac{2\nu+1}{x} D_x$. Equivalently, the operator Δ_x^* can also be defined by the relation

$$(8.2) \quad \langle m'(x) \Delta_x^* f(x), \psi(x) \rangle = \langle m'(x) f(x), \Delta_x \psi(x) \rangle$$

for all $\psi(x) \in G_{\alpha,\delta}^{\nu}(I)$, $0 < \alpha \leq \nu + \frac{1}{2}$, $\delta \geq 0$. Since $G_{\alpha,\delta}^{\nu}(I)$ is closed with respect to the differentiation operator Δ_x , $H_{\alpha,\delta}^{\nu'}(I)$ is also closed with respect to the operator Δ_x^* . This means that for all $k=1, 2, 3, \dots$ and $\phi(x) \in G_{\alpha,\delta}^{\nu}(I)$ one will have

$$\langle m'(x) (\Delta_x^*)^k f(x), \phi(x) \rangle = \langle m'(x) f(x), \Delta_x^k \phi(x) \rangle .$$

It can be readily seen that if f is a regular distribution in $H_{\alpha, \delta}^{\nu'}(I)$ generated by a member of $\mathcal{D}(I)$, then

$$\Delta_x^* f \equiv \Delta_x f .$$

For each $k=1, 2, 3, \dots$ and $y>0$ one can show that for any $f \in H_{\alpha, \delta}^{\nu'}(I)$,

$$\langle m'(x) (\Delta_x^*)^k f(x), \mathcal{J}(xy) \rangle = (-1)^k y^{2k} \langle m'(x) f(x), \mathcal{J}(xy) \rangle .$$

i.e.

$$(8.3) \quad \mathcal{H}_y [(\Delta_x^*)^k f(x)] = (-1)^k y^{2k} \mathcal{H}_y [f(x)] .$$

Now consider the operator equation

$$(8.4) \quad P(\Delta_x^*) u = g ,$$

where $g \in H_{\alpha, \delta}^{\nu'}(I)$ and $P(x) = (x-a_1^2)(x-a_2^2) \dots (x-a_n^2)$, the a_i 's are all distinct real numbers. Our object is to find the restriction to $\mathcal{D}(I)$ of a generalized function $u \in H_{\alpha, \delta}^{\nu'}(I)$ satisfying the operator equation (8.4). Taking Hankel transform of both sides of (8.4) and using (8.3) we get

$$(8.5) \quad P(-y^2) U(y) = G(y)$$

where U and G are Hankel transforms of u and g , respectively.

Therefore, in view of our inversion formula, we can see that f , the restriction of u to $\mathcal{D}(I)$ is given by

$$(8.6) \quad f = \lim_{N \rightarrow \infty} \int_0^N \frac{G(y)}{P(-y^2)} \mathcal{J}(yx) dm(y) .$$

In fact, it is quite easy to show that the expression in the righthand side of (8.6) converges in $\mathcal{D}'(I)$ weakly.

We now proceed to show that the distribution f obtained in (8.6) is actually a solution to the distributional differential equation

$$(8.7) \quad P(\Delta_x)u = g$$

where equality is in the sense of $\mathcal{D}'(I)$ and $\Delta_x = D^2 + \frac{2\nu+1}{x} D$ and $g \in H_{\alpha, \delta}^{\nu'}(I)$. Differentiations in (8.7) have been performed in the distributional sense.

Now, one can see that

$$P(\Delta_x) \int_0^N \frac{G(y)}{P(-y^2)} \mathcal{J}(xy) dm(y) = \int_0^N G(y) \mathcal{J}(xy) dm(y) .$$

Therefore, for $\phi(x) \in \mathcal{D}(I)$ we have

$$\begin{aligned} & \langle m'(x) P(\Delta_x) \int_0^N \frac{G(y)}{P(-y^2)} \mathcal{J}(xy) dm(y), \phi(x) \rangle \\ &= \langle m'(x) \int_0^N G(y) \mathcal{J}(xy) dm(y), \phi(x) \rangle \end{aligned}$$

$\rightarrow \langle m'g, \phi \rangle$ as $N \rightarrow \infty$ in view of Theorem 3.3.

But, as in (8.6)

$$\int_0^N \frac{G(y)}{P(-y^2)} \mathcal{J}(xy) dm(y) \rightarrow f \text{ in } \mathcal{D}'(I) \text{ as } N \rightarrow \infty.$$

Therefore f satisfies the distributional differential equation (8.7).

We now want to find the complete solution in $\mathcal{D}'(I)$ of the distributional differential equation (8.7).

One can see that $\mathcal{J}(axi) = 2^\nu \Gamma(\nu+1) (axi)^{-\nu} J_\nu(axi)$ satisfies the distributional differential equation

$$(8.8) \quad (D^2 + \frac{2\nu+1}{x} D - a^2)u = 0.$$

Since $\mathcal{J}(aix)$ is smooth and positive on I , we can say that the general solution to (8.8) is of the form

$$(8.9) \quad u(x) = \mathcal{J}(aix)v(x)$$

where $v(x)$ is to be determined in $\mathfrak{D}'(I)$.

Using the substitution (8.9) in (8.8), we get the following distributional differential equation:

$$\mathcal{J}(aix)v''(x) + \frac{2v+1}{x} \mathcal{J}(aix)v'(x) + 2ai\mathcal{J}'(aix)v'(x) = 0 .$$

Solving for $v'(x)$, we get

$$v'(x) = \frac{c}{x^{2v+1}\mathcal{J}^2(aix)} , \text{ in } \mathfrak{D}'(I)$$

Therefore,

$$v(x) = c \int_1^x \frac{1}{x^{2v+1}\mathcal{J}^2(aix)} dx + d,$$

where c and d are arbitrary constants.

Hence, the general solution in $\mathfrak{D}'(I)$ to the distributional differential equation

$$P(\Delta x)u = 0$$

is given by

$$(8.10) \quad u(x) = \sum_{k=1}^n \mathcal{J}(a_k x) \left[c_k \int_1^x [t^{2\nu+1} \mathcal{J}^2(a_k t)]^{-1} dt + d_k \right]$$

where c_k and d_k are arbitrary constants.

Hence, the general solution to the distributional differential equation (8.7) is given by

$$u(x) = f + \sum_{k=1}^n \mathcal{J}(a_k x) \left[c_k \int_1^x [t^{2\nu+1} \mathcal{J}^2(a_k t)]^{-1} dt + d_k \right]$$

where f is a distribution in $\mathcal{D}'(I)$ as determined in (8.6).

Remark 8.1. In equation (8.7), let $g(x)$ be a regular distribution as taken in example 7.1. With such a choice of g , the differential equation (8.7) cannot be solved by using the transform technique of Koh and Zemanian as used in [10], whereas it is solvable by our transform technique.

Remark 8.2. Our generalized Hankel transform technique can also be applied in solving Dirichlet problems similar to that discussed by Zemanian in [22].

CHAPTER 2

"Hankel Potential Transform of a Class of Generalized Functions"

1. Introduction. In this Chapter, the Hankel potential transform of a generalized function is defined and an inversion formula is established. The limit operation in the formula is interpreted in the weak distributional sense. This work is, in fact, an extension to a class of generalized functions of an inversion formula for the Hankel potential transform (in the classical sense) given by Cholewinski and Haimo [3]. A theorem giving a bound on the Hankel potential transform of the generalized function under consideration is also proved. The inversion theorem is also illustrated by means of an example.

The notations and the terminologies used in this chapter are the same as in the previous chapter.

The Hankel potential transform of a function f is defined by the following convergent integral:

$$(1.1) \quad F(x) = \int_0^{\infty} \frac{t}{(x^2 + t^2)^{\nu+1}} f(t) dm(t)$$

where $dm(t) = \frac{1}{2^{\nu-\frac{1}{2}}\Gamma(\nu + \frac{1}{2})} t^{2\nu} dt$, $\nu > 0$ and $x \neq 0$.

A function $f(x)$ is said to belong in $L[0, \infty)$ for $x \in [0, R)$, $0 < R \leq \infty$, if $\int_0^{\infty} |f(x)| dm(x) < \infty$.

Cholewinski and Haimo proved the following inversion formula for the transform (1.1):

Let $f(t)$ belong to L for $t \in [0, R]$ for every positive R , and let the integral

$$F(x) = \int_0^\infty \frac{t}{(x^2 + t^2)^{\nu+1}} f(t) dm(t)$$

converge for some $x \neq 0$. Then

$$(1.2) \quad \lim_{n \rightarrow \infty} L_{n,x}[F(x)] = f(x),$$

at all points x of the Lebesgue set for the function $f(x)$. The operator $L_{n,x}$ is defined as follows,

$$(1.3) \quad L_{1,x}[F] = \frac{\sqrt{\pi}\Gamma(2\nu+1)}{2^{\frac{5}{2}+\nu} [\Gamma(2+\frac{\nu}{2})]^2 x^{1+2\nu}} D x^{5+2\nu} D_x^{-1} [F] \quad \text{and}$$

$$L_{n,x}[F] = \frac{(-1)^{n+1} \sqrt{\pi}\Gamma(2\nu+1)}{2^{2n+\nu+\frac{1}{2}} [\Gamma(n+\frac{\nu}{2}+1)]^2 x^{2n+2\nu-1}} D x^{4n+2\nu+1} \left[\prod_{k=1}^{n-1} D \frac{1}{x^{4k+2\nu+1}} \right] D_x^{-1} [F],$$

$n = 2, 3, \dots; D = \frac{d}{dx}.$

Our main object in this chapter is to extend the inversion formula (1.2) to certain space of generalized functions.

2. The testing function space $\mathcal{T}_{\alpha,\beta}(I)$ and its dual space $\mathcal{T}'_{\alpha,\beta}(I)$:

Let $\nu > 0$ be a fixed real number and let α, β be a pair of real

numbers such that $0 < \alpha \leq 2\nu + 1$ and $0 \leq \beta < 1$. Let $\xi(t)$ be a smooth positive function defined over I satisfying

$$\begin{aligned}\xi(t) &= t^{-\beta} \quad \text{for } 0 < t < 1 \\ \text{and } \xi(t) &= t^{\alpha} \quad \text{for } t > 2.\end{aligned}$$

We define $\mathcal{T}_{\alpha,\beta}(I)$ as a collection of infinitely differentiable and complex-valued functions $\phi(t)$ defined over $I = (0, \infty)$ such that

$$\gamma_k(\phi) = \sup_{0 < t < \infty} |\xi(t) (t \frac{d}{dt})^k \phi(t)| < \infty \quad \text{for all } k = 0, 1, 2, \dots$$

Clearly $\mathcal{T}_{\alpha,\beta}(I)$ is a vector space over the field of complex numbers under the usual addition of functions and multiplication of a function by a scalar. It can be easily seen that γ_0 is a norm on $\mathcal{T}_{\alpha,\beta}(I)$ and therefore the countable collection of semi-norms $\{\gamma_k\}_{k=0}^{\infty}$ is separating [21, p. 8]. We assign to $\mathcal{T}_{\alpha,\beta}(I)$ the topology generated by $\{\gamma_k\}_{k=0}^{\infty}$, thereby making it a countably multi-normed space. A sequence $\{\phi_n(x)\}_{n=1}^{\infty}$ where each $\phi_n(x)$ belongs to $\mathcal{T}_{\alpha,\beta}(I)$, converges in $\mathcal{T}_{\alpha,\beta}(I)$ to $\phi(x)$ if $\gamma_k(\phi_n - \phi)$ tends to zero as $n \rightarrow \infty$ for each $k = 0, 1, 2, \dots$. A sequence $\{\phi_n(x)\}_{n=1}^{\infty}$ in $\mathcal{T}_{\alpha,\beta}(I)$ is said to be a Cauchy sequence if $\gamma_k(\phi_m - \phi_n)$ goes to zero as m and n both go to infinity independently of each other, for each $k = 0, 1, 2, \dots$. It can be seen that $\mathcal{T}_{\alpha,\beta}(I)$ is a Fréchet space, i.e., a complete countably multinormed space [21, p. 12]. Moreover $\mathcal{T}_{\alpha,\beta}(I)$ is a locally convex and Hausdorff topological vector space.

The dual space $\mathcal{T}'_{\alpha,\beta}(I)$ consists of all continuous linear

functionals on $\mathcal{T}_{\alpha,\beta}(I)$. It is also a linear space to which we assign the weak topology generated by the multi-norm $\{\eta_\phi(f)\}$ where $\eta_\phi(f) = |\langle f, \phi \rangle|$ and ϕ varies through $\mathcal{T}_{\alpha,\beta}(I)$.

The testing function space $\mathcal{P}_{\alpha,\beta}(I)$ and its dual space $\mathcal{P}'_{\alpha,\beta}(I)$.
For $\nu > 0$, $0 < \alpha \leq 2\nu + 1$ and $0 \leq \beta < 1$, we say that an infinitely differentiable function $\psi(x)$ defined over I is in $\mathcal{P}_{\alpha,\beta}(I)$ iff $\frac{\psi(x)}{m'(x)} \in \mathcal{T}_{\alpha,\beta}(I)$, where

$$m'(x) = \frac{1}{2^{\nu-\frac{1}{2}}\Gamma(\nu + \frac{1}{2})} x^{2\nu}.$$

The topology on $\mathcal{P}_{\alpha,\beta}(I)$ is the one, generated by the sequence of semi-norms $\{\beta_k\}_{k=0}^\infty$, where β_k is defined as:

$$\beta_k(\psi(x)) = \gamma_k\left(\frac{\psi(x)}{m'(x)}\right).$$

The concept of convergence and completeness in $\mathcal{P}_{\alpha,\beta}(I)$ is similar to that in $\mathcal{T}_{\alpha,\beta}(I)$. $\mathcal{P}_{\alpha,\beta}(I)$ is likewise a sequentially complete Hausdorff locally convex topological vector space. The dual space $\mathcal{P}'_{\alpha,\beta}(I)$ consists of all continuous linear functionals on $\mathcal{P}_{\alpha,\beta}(I)$, equipped with the usual weak topology.

The elements of $\mathcal{T}'_{\alpha,\beta}(I)$ and $\mathcal{P}'_{\alpha,\beta}(I)$ will be called generalized functions.

For any $f \in \mathcal{P}'_{\alpha,\beta}(I)$ we define a generalized function

$m'f \in \mathcal{T}'_{\alpha,\beta}(I)$ by the relation

$$\langle m'(x)f(x), \phi(x) \rangle = \langle f(x); m'(x)\phi(x) \rangle, \phi(x) \in \mathcal{T}_{\alpha,\beta}(I).$$

It can be easily seen that $f \in \mathcal{P}'_{\alpha,\beta}(I)$ iff $m'f \in \mathcal{T}'_{\alpha,\beta}(I)$. It can also be checked that the space $\mathcal{D}(I)$ is contained in $\mathcal{T}_{\alpha,\beta}(I)$, and the topology of $\mathcal{D}(I)$ is stronger than that induced on it by $\mathcal{T}_{\alpha,\beta}(I)$. Hence the restriction of any $f \in \mathcal{T}'_{\alpha,\beta}(I)$ to $\mathcal{D}(I)$ is in $\mathcal{D}'(I)$, and the weak convergence in $\mathcal{T}'_{\alpha,\beta}(I)$ implies the weak convergence in $\mathcal{D}'(I)$. Also the space $\mathcal{D}(I)$ is contained in $\mathcal{P}_{\alpha,\beta}(I)$ and the topology of $\mathcal{D}(I)$ is stronger than that induced on it by $\mathcal{P}_{\alpha,\beta}(I)$. Therefore the restriction of any $f \in \mathcal{P}'_{\alpha,\beta}(I)$ to $\mathcal{D}(I)$ is in $\mathcal{D}'(I)$.

3. The generalized Hankel potential transformation. For any $f \in \mathcal{P}'_{\alpha,\beta}(I)$, we define its Hankel potential transform as the application of $m'f$ to the kernel $\frac{t}{(x^2 + t^2)^{\nu+1}}$, i.e.,

$$(3.1) \quad F(x) = \langle m'(t)f(t), \frac{t}{(x^2 + t^2)^{\nu+1}} \rangle,$$

for all $x > 0$ and $\nu > 0$. The right hand side of (3.1) has a sense because for any $x > 0$, it can be checked that $\frac{t}{(x^2 + t^2)^{\nu+1}} \in \mathcal{T}_{\alpha,\beta}(I)$. In fact for any non-negative integer k ,

$$\left(t \frac{d}{dt}\right)^k \left[\frac{t}{(x^2 + t^2)^{\nu+1}} \right] = \sum_{m=1}^{k+1} C_{\nu,m} \frac{t^{2m-1}}{(x^2 + t^2)^{\nu+m}},$$

where $C_{\nu,m}$ are obvious constants. Now, it is easy to see that for

fixed $x > 0$, $\sup_{0 < t < \infty} \left| \xi(t) \left(t \frac{d}{dt} \right)^k \left[\frac{t}{(x^2 + t^2)^{\nu+1}} \right] \right|$ is finite.

Before giving the main generalized inversion theorem, we prove some useful results.

Theorem 3.1. Let $f(x)$ be a locally integrable function on $0 < x < \infty$ such that $\int_0^\infty \left| \frac{f(x)}{\xi(x)} \right| dm(x) < \infty$, where $dm(x)$ and $\xi(x)$ are the same as defined in Sections 1 and 2 respectively. Then $f(x)$ generates a regular generalized function in $\mathcal{P}'_{\alpha, \beta}(I)$ defined by

$$(3.2) \quad \langle f, \phi \rangle = \int_0^\infty f(x) \phi(x) dx \quad , \quad \phi \in \mathcal{P}_{\alpha, \beta}(I) \quad .$$

Proof. Multiplying and dividing the integrand in 3.2, by $m'(x)\xi(x)$ we get

$$\begin{aligned} |\langle f, \phi \rangle| &= \left| \int_0^\infty \frac{f(x)}{\xi(x)} \xi(x) \frac{\phi(x)}{m'(x)} dm(x) \right| \quad , \quad (m'(x)dx = dm(x)) \\ &\leq \gamma_0 \left(\frac{\phi(x)}{m'(x)} \right) \int_0^\infty \left| \frac{f(x)}{\xi(x)} \right| dm(x) \\ &= \beta_0(\phi) \int_0^\infty \left| \frac{f(x)}{\xi(x)} \right| dm(x) \end{aligned}$$

which shows that (3.2) truly defines a functional f in $\mathcal{P}'_{\alpha, \beta}(I)$.

f is clearly linear by definition (3.2). For continuity of f ,

let $\{\phi_\nu\}_{\nu=1}^\infty$ be a sequence which converges in $\mathcal{P}_{\alpha, \beta}(I)$ to zero,

then $\beta_0(\phi_\nu) \rightarrow 0$ so that $|\langle f, \phi_\nu \rangle| \rightarrow 0$. Thus f is continuous on

$\mathcal{P}_{\alpha,\beta}(I)$. Hence $f \in \mathcal{P}'_{\alpha,\beta}(I)$.

Corollary. Let $f(x) \in L(0, \infty)$. Then $f(x)$ generates a regular generalized function in $\mathcal{P}'_{\alpha,\beta}(I)$, defined as in the preceding theorem.

Proof. $f(x) \in L(0, \infty)$ implies (as given in Section 1) that $\int_0^\infty |f(x)| dm(x) < \infty$. Now, the corollary follows by proving the inequality

$$(3.3) \quad \int_0^\infty \left| \frac{f(x)}{\xi(x)} \right| dm(x) \leq A \int_0^\infty |f(x)| dm(x)$$

for some constant A . Indeed

$$\begin{aligned} \int_0^\infty \left| \frac{f(x)}{\xi(x)} \right| dm(x) &= \int_0^1 |x^\beta f(x)| dm(x) + \int_1^2 \left| \frac{f(x)}{\xi(x)} \right| dm(x) + \int_2^\infty \left| \frac{f(x)}{x^\alpha} \right| dm(x) \\ &\leq \int_0^1 |f(x)| dm(x) + \frac{1}{B} \int_1^2 |f(x)| dm(x) + \int_2^\infty |f(x)| dm(x) , \end{aligned}$$

where B is the infimum of $\xi(x)$ in the interval $1 \leq x \leq 2$. Here, the bound on both x^β and $\frac{1}{x^\alpha}$ in the intervals $0 < x < 1$ and $2 < x < \infty$ respectively is taken as 1 . ($\alpha > 0, \beta \geq 0$) . Therefore

$$\int_0^\infty \left| \frac{f(x)}{\xi(x)} \right| dm(x) \leq \left(2 + \frac{1}{B} \right) \int_0^\infty |f(x)| dm(x) .$$

Thus putting $2 + \frac{1}{B} = A$, we get inequality (3.3).

Remark 3.1. If $f(x)$ satisfies the conditions of Theorem 3.1 (or above corollary), then the definition (3.1), of the generalized Hankel potential transform of f can be given as

$$F(x) = \int_0^{\infty} f(t) \frac{t}{(x^2 + t^2)^{\nu+1}} dm(t) .$$

Theorem 3.2. (Infinite differentiability of $F(x)$). Let $f \in \mathcal{P}'_{\alpha, \beta}(I)$ and $F(x)$ be the Hankel potential transform of f , as defined by (3.1). Then $F^{(n)}(x)$ exists and

$$(3.3) \quad F^{(n)}(x) = \langle m'(t)f(t), \frac{\partial^n}{\partial x^n} \frac{t}{(x^2 + t^2)^{\nu+1}} \rangle$$

for all $x \in (0, \infty)$ and $n = 1, 2, 3, \dots$.

Proof. As $\frac{\partial^n}{\partial x^n} \frac{t}{(x^2 + t^2)^{\nu+1}}$ belongs to the space $\mathcal{T}_{\alpha, \beta}(I)$ for each $n = 1, 2, 3, \dots$, (easy to check), the right-hand expression in (3.3) is well defined. Now, we establish the differentiability of $F(x)$, and the equality (3.3) for $n = 1$. For $n > 1$ the theorem follows by using the technique of mathematical induction. Let Δx be an arbitrary increment in x . Assume $|\Delta x| < x$ without any loss. Then

$$(3.4) \quad \frac{F(x + \Delta x) - F(x)}{\Delta x} = \langle m'(t)f(t), \frac{\partial}{\partial x} \frac{t}{(x^2 + t^2)^{\nu+1}} \rangle = \langle m'(t)f(t), \theta_{\Delta x}(t) \rangle ,$$

where

$$\theta_{\Delta x}(t) = \left[\frac{t}{((x + \Delta x)^2 + t^2)^{\nu+1}} - \frac{t}{(x^2 + t^2)^{\nu+1}} \right] / \Delta x - \frac{\partial}{\partial x} \frac{t}{(x^2 + t^2)^{\nu+1}} .$$

We want to prove that the left hand side expression in (3.4) converges to zero as $\Delta x \rightarrow 0$. This can be done by showing that $\theta_{\Delta x}(t)$ converges in $J_{\alpha, \beta}(I)$ to zero as $\Delta x \rightarrow 0$, i.e. that

$$\xi(t) \left(t \frac{d}{dt} \right)^k \theta_{\Delta x}(t) \rightarrow 0 \quad \text{as } \Delta x \rightarrow 0 ,$$

uniformly for all $t \in (0, \infty)$ and $k = 0, 1, 2, \dots$. Now using the fact

$$\left(t \frac{d}{dt} \right)^k \frac{t}{(x^2 + t^2)^{\nu+1}} = \sum_{m=1}^{k+1} C_{\nu, m} \frac{t^{2m-1}}{(x^2 + t^2)^{\nu+m}} ,$$

where $C_{\nu, m}$ are obvious constants, and interchanging $\frac{\partial}{\partial x}$ with $\left(t \frac{d}{dt} \right)^k$ in $\theta_{\Delta x}(t)$ we get

$$(3.5) \quad \xi(t) \left(t \frac{d}{dt} \right)^k \theta_{\Delta x}(t) = \xi(t) \sum_{m=1}^{k+1} C_{\nu, m} t^{2m-1} \left[\left\{ \frac{1}{(x+\Delta x)^2 + t^2} - \frac{1}{(x^2 + t^2)} \right\} / \Delta x - \frac{\partial}{\partial x} \frac{1}{(x^2 + t^2)^{\nu+m}} \right] .$$

Now, the m^{th} term of the above finite series can be expressed as

$$C_{\nu, m} \xi(t) t^{2m-1} \frac{1}{\Delta x} \left[\int_x^{x+\Delta x} \left\{ \frac{\partial}{\partial u} \frac{1}{(u^2 + t^2)^{\nu+m}} - \frac{\partial}{\partial x} \frac{1}{(x^2 + t^2)^{\nu+m}} \right\} du \right]$$

$$\begin{aligned}
 &= C_{v,m} \xi(t) t^{2m-1} \frac{1}{\Delta x} \left[\int_x^{x+\Delta x} \left\{ \frac{\partial}{\partial v} \left(\frac{1}{(v^2+t^2)^{v+m}} \right)_{v=u} - \frac{\partial}{\partial v} \left(\frac{1}{(v^2+t^2)^{v+m}} \right)_{v=x} \right\} du \right] \\
 &= C_{v,m} \xi(t) t^{2m-1} \frac{1}{\Delta x} \left[\int_x^{x+\Delta x} \int_x^u \frac{\partial^2}{\partial v^2} \frac{1}{(v^2+t^2)^{v+m}} dv du \right] \\
 &= C_{v,m} \xi(t) t^{2m-1} \frac{1}{\Delta x} \int_x^{x+\Delta x} \int_x^u \left\{ \frac{-2(v+m)}{(v^2+t^2)^{v+m+1}} + \frac{4(v+m)(v+m+1)v^2}{(v^2+t^2)^{v+m+2}} \right\} dv du \\
 &= I_{1,m} + I_{2,m}, \text{ where}
 \end{aligned}$$

$$I_{1,m} = C'_{v,m} \xi(t) t^{2m-1} \frac{1}{\Delta x} \int_x^{x+\Delta x} \int_x^u \frac{1}{(v^2+t^2)^{v+m+1}} dv du, \text{ and}$$

$$I_{2,m} = C''_{v,m} \xi(t) t^{2m-1} \frac{1}{\Delta x} \int_x^{x+\Delta x} \int_x^u \frac{v^2}{(v^2+t^2)^{v+m+2}} dv du, \text{ with } C'_{v,m} \text{ and}$$

$C''_{v,m}$ as obvious constants. Let us consider two cases, namely when $\Delta x > 0$ and $\Delta x < 0$. Let $\Delta x > 0$. Then

$$\begin{aligned}
 |I_{2,m}| &\leq |C''_{v,m}| \xi(t) t^{2m-1} \frac{1}{\Delta x} \int_x^{x+\Delta x} \int_x^{x+\Delta x} \frac{(x+\Delta x)^2}{(x^2+t^2)^{v+m+2}} dv du \\
 &= |C''_{v,m}| \xi(t) \frac{t^{2m-1}}{(x^2+t^2)^{v+m+2}} \frac{(x+\Delta x)^2}{\Delta x} (\Delta x)^2 \\
 &= |C''_{v,m}| \xi(t) \frac{t^{2m-1}}{(x^2+t^2)^{v+m+2}} (x+\Delta x)^2 \Delta x.
 \end{aligned}$$

Now, for any fixed $x > 0$, the quantity $\frac{\xi(t) t^{2m-1}}{(x^2+t^2)^{v+m+2}}$ is bounded by some constant, say $B(x)$, depending upon x , for all $t \in (0, \infty)$.

This is due to the fact that for any fixed $x > 0$, $\frac{\xi(t)t^{2m-1}}{(x^2 + t^2)^{v+m+2}}$ is a continuous function of t which approaches zero, both when $t \rightarrow 0$ and $t \rightarrow \infty$. Thus

$$|I_{2,m}| \leq |C''_{v,m}| B(x)(x + \Delta x)^2 \Delta x,$$

which converges to zero, as $\Delta x \rightarrow 0+$. Thus $I_{2,m} \rightarrow 0$ as $\Delta x \rightarrow 0+$.

For the case $\Delta x < 0$ we have

$$\begin{aligned} |I_{2,m}| &\leq |C''_{v,m}| \xi(t) \frac{t^{2m-1}}{|\Delta x|} \int_{x-|\Delta x|}^x \int_{x-|\Delta x|}^x \frac{x^2}{((x - |\Delta x|)^2 + t^2)^{v+m+2}} dv du \\ &= |C''_{v,m}| \xi(t) t^{2m-1} \frac{x^2}{((x - |\Delta x|)^2 + t^2)^{v+m+2}} |\Delta x|. \end{aligned}$$

In view of this inequality, it can be easily proved, as in the first case, that $I_{2,m} \rightarrow 0$ as $\Delta x \rightarrow 0^-$. Combining both the cases, we conclude that $I_{2,m} \rightarrow 0$ as $\Delta x \rightarrow 0$ uniformly for all $t \in (0, \infty)$.

Proceeding in a similar manner, we can also show that $I_{1,m} \rightarrow 0$ as $\Delta x \rightarrow 0$ uniformly for all $t \in (0, \infty)$. Therefore, $(I_{1,m} + I_{2,m}) \rightarrow 0$ as $\Delta x \rightarrow 0$ concluding that the finite sum, $\sum_{m=1}^{k+1} (I_{1,m} + I_{2,m}) \rightarrow 0$ as $\Delta x \rightarrow 0$, uniformly for all $t \in (0, \infty)$. In other words, the expression in (3.5) converges to zero as $\Delta x \rightarrow 0$ uniformly for all $t \in (0, \infty)$. This completes the proof of the theorem.

Theorem 3.3. (Boundedness of $F(x)$): Let $f \in \mathcal{P}'_{\alpha, \beta}(I)$ and let

$F(x)$ be the Hankel Potential transform of f , i.e.,

$$F(x) = \langle m'(t)f(t), \frac{t}{(x^2 + t^2)^{\nu+1}} \rangle \text{ for all } x \in (0, \infty) \\ \text{and } \nu > 0.$$

Then

$$|F(x)| \leq K_\nu [A + \frac{1}{x^{2(\nu+1)}}],$$

where A and K_ν are some constants.

Proof. In view of a general result [21, Th. 1.8-1, p. 18] there exist a positive constant C and a non-negative integer r such that for any $f \in \mathcal{P}'_{\alpha, \beta}(I)$,

$$|F(x)| = |\langle m'(t)f(t), \frac{t}{(x^2 + t^2)^{\nu+1}} \rangle| \leq C \max_{0 \leq k \leq r} \gamma_k \left(\frac{t}{(x^2 + t^2)^{\nu+1}} \right) \\ = C \max_{0 \leq k \leq r} \sup_{0 < t < \infty} \left| \xi(t) \left(t \frac{d}{dt} \right)^k \frac{t}{(x^2 + t^2)^{\nu+1}} \right|.$$

By definition of $\xi(t)$ (Section 2), we can get a positive constant A such that $|\xi(t)| \leq At^\alpha$ for all $t \geq 1$. Now

$$\sup_{0 < t < \infty} \left| \xi(t) \left(t \frac{d}{dt} \right)^k \frac{1}{(x^2 + t^2)^{\nu+1}} \right| \leq \sup_{0 < t < 1} \left| \xi(t) \left(t \frac{d}{dt} \right)^k \frac{t}{(x^2 + t^2)^{\nu+1}} \right| + \\ + \sup_{1 \leq t < \infty} \left| \xi(t) \left(t \frac{d}{dt} \right)^k \frac{t}{(x^2 + t^2)^{\nu+1}} \right|$$

$$\leq \sup_{0 < t < 1} \left| t^{-\beta} \sum_{m=1}^{k+1} C_{v,m} \frac{t^{2m-1}}{(x^2+t^2)^{v+m}} \right| + \sup_{1 \leq t < \infty} \left| A t^{\alpha} \sum_{m=1}^{k+1} C_{v,m} \frac{t^{2m-1}}{(x^2+t^2)^{v+m}} \right| ,$$

($C_{v,m}$ are obvious constants)

$$\leq \sup_{0 < t < 1} \sum_{m=1}^{k+1} |C_{v,m}| t^{1-\beta} \left(\frac{t^2}{x^2+t^2} \right)^{m-1} \frac{1}{(x^2+t^2)^{v+1}} + \sup_{1 \leq t < \infty} \sum_{m=1}^{k+1} A |C_{v,m}| \frac{t^{\alpha+2m-1}}{(x^2+t^2)^{v+m}}$$

$$\leq \sum_{m=1}^{k+1} |C_{v,m}| \frac{1}{x^{2(v+1)}} + \sum_{m=1}^{k+1} A |C_{v,m}| \sup_{1 \leq t < \infty} t^{\alpha-(2v+1)} , \quad (\text{as } 0 \leq \beta < 1)$$

$$= \sum_{m=1}^{k+1} |C_{v,m}| \frac{1}{x^{2(v+1)}} + \sum_{m=1}^{k+1} A |C_{v,m}| , \quad \text{as } 0 \leq \alpha \leq 2v+1$$

$$= \left(A + \frac{1}{x^{2(v+1)}} \right) \sum_{m=1}^{k+1} |C_{v,m}| .$$

Hence

$$|F(x)| \leq C \max_{0 \leq k \leq r} \left[A + \frac{1}{x^{2(v+1)}} \right] \sum_{m=1}^{k+1} |C_{v,m}| = \left[A + \frac{1}{x^{2(v+1)}} \right] \sum_{m=1}^{r+1} C |C_{v,m}| ,$$

i.e.,

$$|F(x)| \leq K_v \left[A + \frac{1}{x^{2(v+1)}} \right] , \quad \text{where } K_v = \sum_{m=1}^{r+1} C |C_{v,m}| .$$

This completes the proof of Theorem 3.3.

4. Inversion theorem. We now give an inversion theorem for our generalized Hankel potential transformation.

Theorem 4.1. Let $f \in \mathcal{P}'_{\alpha, \beta}(I)$ and $F(x)$ be the Hankel potential transform of f as defined by (3.1). Then for each $\phi \in \mathcal{D}(I)$

$$\langle m'(x) L_{n,x}(F(x)), \phi(x) \rangle \rightarrow \langle m'(t) f(t), \phi(t) \rangle$$

as $n \rightarrow \infty$, where the operator $L_{n,x}$ is, as defined by (1.3) and the differentiation in it is understood to be in the distributional sense.

Before giving the proof of the theorem, we prove a few lemmas to be used in proving the theorem.

Lemma 4.1. Let $\phi(x) \in \mathcal{D}(I)$ and $f \in \mathcal{P}'_{\alpha, \beta}(I)$. Then

$$(4.1) \quad \int_0^\infty \langle m'(t) f(t), \frac{t}{(x^2+t^2)^{\nu+1}} \rangle \phi(x) dm(x) = \langle m'(t) f(t), \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} \phi(x) dm(x) \rangle.$$

Proof. (By Riemann sum technique). $\phi(x)$ as a member of $\mathcal{D}(I)$ has a compact support in I . Then, let support of $\phi(x)$ be contained in the finite interval (a, b) ; $b > a > 0$. Thus, we have to prove that

$$(4.2) \quad \int_a^b \langle m'(t) f(t), \frac{t}{(x^2+t^2)^{\nu+1}} \rangle \phi(x) dm(x) = \langle m'(t) f(t), \int_a^b \frac{t}{(x^2+t^2)^{\nu+1}} \phi(x) dm(x) \rangle.$$

Note that, the right-hand side expression of (4.2) is well defined, as

$\int_a^b \frac{t}{(x^2 + t^2)^{\nu+1}} \phi(x) dm(x)$ belongs to $\mathcal{T}_{\alpha, \beta}(I)$ (easy to check). Now,

for each $n = 1, 2, 3, \dots$, let us partition the interval $[a, b]$

into n sub-intervals whose lengths are $\Delta_{x_{j,n}}$ ($j = 1, 2, 3, \dots, n$).

Let $\eta_{j,n}$ be an arbitrary point in the j th sub-interval ($j = 1, 2,$

$\dots, n$) and let maximum of $\Delta_{x_{j,n}}$ tend to zero as $n \rightarrow \infty$. The

Riemann sum of the integral in the left-hand side of (4.2) is given by

$$(4.3) \quad \sum_{j=1}^n \langle m'(t)f(t), \frac{t}{(\eta_{j,n}^2 + t^2)^{\nu+1}} \rangle \phi(\eta_{j,n}) m'(\eta_{j,n}) \Delta x_{j,n}.$$

The difference of the right-hand expression in (4.2) and the expression

(4.3) is given by

$$\langle m'(t)f(t), \int_a^b \frac{t}{(x^2 + t^2)^{\nu+1}} \phi(x) m'(x) dx - \sum_{j=1}^n \frac{t}{(\eta_{j,n}^2 + t^2)^{\nu+1}} \phi(\eta_{j,n}) m'(\eta_{j,n}) \Delta x_{j,n} \rangle.$$

Thus to justify (4.2), we have to show that the expression

$$(4.4) \quad \left[\int_a^b \frac{t}{(x^2 + t^2)^{\nu+1}} \phi(x) m'(x) dx - \sum_{j=1}^n \frac{t}{(\eta_{j,n}^2 + t^2)^{\nu+1}} \phi(\eta_{j,n}) m'(\eta_{j,n}) \Delta x_{j,n} \right]$$

converges to zero in $\mathcal{T}_{\alpha, \beta}(I)$ as $n \rightarrow \infty$. In other words, we have

to show that, for any non-negative integer k

$$(4.5) \quad \xi(t) \left(t \frac{d}{dt} \right)^k \left[\int_a^b \frac{t}{(x^2 + t^2)^{\nu+1}} \phi(x) m'(x) dx - \sum_{j=1}^n \frac{t}{(\eta_{j,n}^2 + t^2)^{\nu+1}} \phi(\eta_{j,n}) m'(\eta_{j,n}) \Delta x_{j,n} \right]$$

converges to zero as $n \rightarrow \infty$, uniformly for all $t \in (0, \infty)$.

As the order of differentiation and integration in (4.5) is interchangeable [18, p. 59], the expression (4.5) may be rewritten as

$$(4.6) \quad \left[\int_a^b \sum_{m=1}^{k+1} C_{v,m} \frac{t^{2m-1} \xi(t)}{(x^2+t^2)^{v+m}} \phi(x) m'(x) dx \right. \\ \left. - \sum_{j=1}^n \left(\sum_{m=1}^{k+1} C_{v,m} \frac{t^{2m-1} \xi(t)}{(\eta_{j,n}^2+t^2)^{v+m}} \phi(\eta_{j,n}) m'(\eta_{j,n}) \Delta x_{j,n} \right) \right],$$

where $C_{v,m}$ are obvious constants obtained after the differentiations. Now, let $R > 0$ be any fixed real number. The quantity under the integral sign in (4.6), being a finite sum of uniformly continuous functions of (x, t) is a uniformly continuous function on the region $(a \leq x \leq b; 0 < t \leq R)$. Consequently, for any $\epsilon > 0$, there exists a positive integer N , sufficiently large so that

$$(4.7) \quad \left| \int_a^b \sum_{m=1}^{k+1} C_{v,m} \frac{t^{2m-1} \xi(t)}{(x^2+t^2)^{v+m}} \phi(x) m'(x) dx \right. \\ \left. - \sum_{j=1}^n \left(\sum_{m=1}^{k+1} C_{v,m} \frac{t^{2m-1} \xi(t)}{(\eta_{j,n}^2+t^2)^{v+m}} \phi(\eta_{j,n}) m'(\eta_{j,n}) \Delta x_{j,n} \right) \right| < \epsilon$$

for all $n \geq N$ and for all $t \in (0, R]$.

Again, since

$$(4.8) \quad \sum_{m=1}^{k+1} C_{v,m} \frac{t^{2m-1} \xi(t)}{(x^2 + t^2)^{v+m}} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly for all x lying in any compact subset of $(0, \infty)$.

Therefore, denoting the expression in (4.8) by $\lambda(x, t)$, we can find a number $R > 0$, sufficiently large so that for a given arbitrary $\epsilon > 0$, $|\lambda(x, t)| < \epsilon$ for all $t > R$, uniformly for all x lying in any compact subset of $(0, \infty)$. Fix R this way. Now, let

$$\sup_{a \leq x \leq b} |\phi(x)m'(x)| = M.$$

Then the absolute value of the expression (4.6) is

$$\leq \int_a^b |\lambda(x, t)| |\phi(x)m'(x)| dx + \sum_{j=1}^n |\lambda(\eta_{j,n}, t)| |\phi(\eta_{j,n})m'(\eta_{j,n})|$$

$$(4.9) \quad \leq \epsilon M(b-a) + \epsilon M(b-a) = 2\epsilon M(b-a)$$

for all $t > R$. Thus, combining (4.7) and (4.9), we conclude that the absolute value of the expression in (4.6) can be made as small as we please by taking n sufficiently large, uniformly for all $t > 0$.

This completes the proof of Lemma 4.1.

Lemma 4.2. Let $\phi(x) \in \mathcal{D}(I)$. For an arbitrary $\epsilon > 0$, there exists a positive number $\eta < \frac{1}{2}$ such that for $1-\eta < x < 1+\eta$ and $t > 0$

$$\xi(t) |\phi(tx) - \phi(t)| < \epsilon,$$

where the function $\xi(t)$ is the same as defined in Section 2.

Proof. The proof of this lemma will closely follow the techniques employed in [14, Lemma 5]. Now let infimum and supremum of the support of $\phi(t)$ be r and R respectively. Let η be an arbitrary positive number less than $\frac{1}{2}$. Obviously, the interval $(1-\eta, 1+\eta)$ is contained in the interval $(\frac{1}{2}, \frac{3}{2})$. For $x \in (\frac{1}{2}, \frac{3}{2})$, the support of the quantity $\xi(t)[\phi(tx) - \phi(t)]$ is clearly contained in the interval $[\frac{2}{3}r, 2R]$. Now, for $x \in (\frac{1}{2}, \frac{3}{2})$ and $t \in [\frac{2}{3}r, R]$, tx lies in the interval $(\frac{r}{3}, 3R)$. Let $M = \sup_{\frac{r}{3} \leq t \leq 3R} \xi(t)$. Since $\phi(t)$

is uniformly continuous over the interval $[\frac{r}{3}, 3R]$, therefore for an arbitrary $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\phi(tx) - \phi(t)| < \frac{\epsilon}{M} \text{ whenever } |t - tx| < \delta.$$

Choose η , so that it satisfies

$$\eta > \frac{\delta}{3R}.$$

Then

$$\xi(t)|[\phi(tx) - \phi(t)]| < M \frac{\epsilon}{M} = \epsilon$$

uniformly for all $t \in [\frac{r}{3}, 3R]$ and $x \in (1-\eta, 1+\eta)$. But the support of $\xi(t)[\phi(tx) - \phi(t)]$ is contained in $[\frac{2}{3}r, 2R] \subset [\frac{r}{3}, 3R]$, therefore,

$$\xi(t) |\phi(tx) - \phi(t)| < \epsilon$$

uniformly for all $t > 0$ and $x \in (1-\eta, 1+\eta)$.

This completes the proof of the lemma.

Lemma 4.3. Let $\phi(x) \in \mathcal{N}(I)$ and let $\xi(t)$ be the same as defined in Section 2. Then

$$(4.10) \quad \xi(t) \frac{2\Gamma(2n+\nu+2)}{\{\Gamma(n+\frac{\nu}{2}+1)\}^2} \int_0^\infty [\phi(x) - \phi(t)] \frac{x^{2n+2\nu+2} t^{2n+1}}{(x^2+t^2)^{2n+\nu+2}} dx \rightarrow 0 ,$$

uniformly for all $t > 0$ as $n \rightarrow \infty$; $\nu > 0$.

Proof. Denoting the integral in the expression (4.10) by I , and using the substitution $x = ty$, we get

$$\begin{aligned} I &= \int_0^\infty [\phi(ty) - \phi(t)] \frac{y^{2n+2\nu+2}}{(1+y^2)^{2n+\nu+2}} dy \\ &= \int_0^\infty [\phi(tx) - \phi(t)] \frac{x^{2n+2\nu+2}}{(1+x^2)^{2n+\nu+2}} dx . \end{aligned}$$

Thus

$$(4.11) \quad |I| \leq \int_0^\infty |\phi(tx) - \phi(t)| \left(\frac{x}{1+x^2}\right)^{2n+2} \left(\frac{x^2}{1+x^2}\right)^\nu dx$$

$$(4.12) \quad \leq \int_0^\infty |\phi(tx) - \phi(t)| \left(\frac{x}{1+x^2}\right)^{2n+2} dx .$$

Now, split the integration in (4.11) into integrations on $0 < x < 1-\eta$, $1-\eta < x < 1+\eta$ and $(1+\eta, \infty)$, and represent the corresponding integrals by I_1 , I_2 and I_3 respectively. Here, η is a positive number less than $\frac{1}{2}$. Now, for an arbitrary $\epsilon > 0$, we have, in view of the preceding lemma

$$(4.13) \quad \frac{2\Gamma(2n+\nu+2)}{[\Gamma(n+\frac{\nu}{2}+1)]^2} [\xi(t)I_2] \leq \epsilon \frac{2\Gamma(2n+\nu+2)}{[\Gamma(n+\frac{\nu}{2}+1)]^2} \int_{1-\eta}^{1+\eta} \left(\frac{x}{1+x^2}\right)^{2n+2} dx.$$

Now,

$$\begin{aligned} \int_{1-\eta}^{1+\eta} \left(\frac{x}{1+x^2}\right)^{2n+2} dx &\leq \int_0^{\infty} \left(\frac{x}{1+x^2}\right)^{2n+2} dx \\ &= \int_0^{\pi/2} \sin^{2n+2}\theta \cos^{2n}\theta d\theta, \end{aligned}$$

(By the substitution $x = \tan \theta$)

$$\begin{aligned} &= \frac{1}{2} \frac{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{1}{2})}{\Gamma(2n+2)} \\ &\approx \frac{1}{2\sqrt{2\pi}} \frac{(n+\frac{1}{2})^{n+1} e^{-(n+\frac{1}{2})} (n-\frac{1}{2})^{n-\frac{1}{2}} e^{-(n-\frac{1}{2})}}{(2n+1)^{2n+\frac{3}{2}} e^{-(2n+1)}}, \text{ as } n \rightarrow \infty, \end{aligned}$$

(By Sterling's approximation formula)

$$(4.14) \quad \approx \sqrt{\pi} e^{-\frac{1}{2n+2}} \cdot \frac{1}{\sqrt{n}}, \text{ as } n \rightarrow \infty.$$

Also, we have again, in view of Sterling's approximation formula,

$$\begin{aligned} \frac{2\Gamma(2n+\nu+2)}{\{\Gamma(n+\frac{\nu}{2}+1)\}^2} &\approx \frac{2}{\sqrt{2\pi}} \frac{(2n+\nu+1)^{2n+\nu+\frac{3}{2}} e^{-(2n+\nu+1)}}{(n+\frac{\nu}{2})^{2n+\nu+1} e^{-(2n+\nu)}} , \text{ as } n \rightarrow \infty \\ (4.15) \quad &\approx \frac{1}{e\sqrt{\pi}} 2^{2n+\nu+2} \sqrt{n} , \text{ as } n \rightarrow \infty . \end{aligned}$$

Therefore, in view of (4.14) and (4.15), we get

$$\frac{2\Gamma(2n+\nu+2)}{\{\Gamma(n+\frac{\nu}{2}+1)\}^2} [\xi(t)I_2] \leq 2^\nu \epsilon , \text{ as } n \rightarrow \infty ,$$

uniformly for all $t > 0$. But ϵ is arbitrary, therefore the left-hand side expression in (4.13) goes to zero as $n \rightarrow \infty$ uniformly for all $t > 0$. Next,

$$\begin{aligned} (4.16) \quad \frac{2\Gamma(2n+\nu+2)}{\{\Gamma(n+\frac{\nu}{2}+1)\}^2} [\xi(t)I_1] &\approx \frac{1}{e\sqrt{\pi}} 2^{2n+\nu+2} \sqrt{n} \xi(t) \int_0^{1-\eta} |\phi(tx) - \phi(t)| \frac{x^{2n+2\nu+2}}{(1+x^2)^{2n+\nu+2}} dx , \\ &\text{as } n \rightarrow \infty . \end{aligned}$$

We need show that the right-hand side expression in (4.16) $\rightarrow 0$, as $n \rightarrow \infty$, uniformly for all $t > 0$. For this, first we show that, for any fixed $x \in (0, 1-\eta)$, $\xi(t) |\phi(tx) - \phi(t)|$ is uniformly bounded for all $t > 0$. In fact

$$\begin{aligned} \sup_{0 < t < \infty} \xi(t) |\phi(tx)| &\leq \sup_{0 < t < 1} \xi(t) |\phi(tx)| + \sup_{1 \leq t < \infty} \xi(t) |\phi(tx)| \\ &\leq \sup_{0 < t < 1} t^{-\beta} |\phi(tx)| + \sup_{1 \leq t < \infty} K t^\alpha |\phi(tx)| , \end{aligned}$$

where, in view of definition of $\xi(t)$, we can find a constant K such that $\xi(t) \leq Kt^\alpha$ for all $t \geq 1$ (in fact $\xi(t) = t^\alpha$ for $t > 2$). Thus,

$$\sup_{0 < t < \infty} \xi(t) |\phi(tx)| \leq \sup_{0 < t < 1} x^\beta \frac{|\phi(tx)|}{(tx)^\beta} + \sup_{1 \leq t < \infty} K \frac{1}{x^\alpha} (tx)^\alpha |\phi(tx)|.$$

Now, $x^\beta < 1$, as $x \in (0, 1-\eta)$ and $\beta \geq 0$. Also there exist appropriate constants C_1 and C_2 such that $\frac{|\phi(tx)|}{(tx)^\beta} \leq C_1$ and $(tx)^\alpha |\phi(tx)| \leq C_2$ in the range $0 < t < 1$ and $1 \leq t < \infty$ respectively. Therefore,

$$(4.17) \quad \sup_{0 < t < \infty} \xi(t) |\phi(tx)| \leq C_1 + KC_2 \frac{1}{x^\alpha}.$$

Also, there exists obviously a constant C_3 such that

$$(4.18) \quad \xi(t) |\phi(t)| \leq C_3 \text{ for all } t \in (0, \infty).$$

Combining (4.17) and (4.18) we get

$$\begin{aligned} \xi(t) |\phi(tx) - \phi(t)| &\leq (C_1 + C_3) + KC_2 \frac{1}{x^\alpha}, \text{ for all } t \in (0, \infty) \\ &\leq C + C' \frac{1}{x^\alpha}, \text{ where } C_1 + C_3 = C \text{ and } KC_2 = C'. \end{aligned}$$

Now, using the above bound in the right-hand side of (4.16) we get

as $n \rightarrow \infty$,

$$\left| \frac{2\Gamma(2n+\nu+2)}{\{\Gamma(n+\frac{\nu}{2}+1)\}^2} \xi(t) I_1 \right| \leq \frac{1}{e\sqrt{\pi}} 2^{2n+\nu+2} \sqrt{n} \left[C \int_0^{1-\eta} \frac{x^{2n+1}}{(1+x^2)^{2n+1}} \cdot \frac{x^{2\nu+1}}{(1+x^2)^{\nu+1}} dx + \right. \\ \left. C' \int_0^{1-\eta} \frac{x^{2n+1}}{(1+x^2)^{2n+1}} \cdot \frac{x^{2\nu+1-\alpha}}{(1+x^2)^{\nu+1}} dx \right]$$

$$(4.19) \quad \leq \frac{1}{e\sqrt{\pi}} 2^{2n+\nu+2} \sqrt{n} \left[(C+C') \int_0^{1-\eta} \frac{x^{2n+1}}{(1+x^2)^{2n+1}} dx \right],$$

since $x^{2\nu+1} < 1$, $\frac{1}{(1+x^2)^{\nu+1}} < 1$ and $x^{2\nu+1-\alpha} \leq 1$ for $x \in (0, 1-\eta)$; recall that $\nu > 0$, $0 < \alpha \leq 2\nu+1$ and $0 < \eta < \frac{1}{2}$.

Now, we claim that for fixed η , $0 < \eta < \frac{1}{2}$,

$$2^{2n} \sqrt{n} \int_0^{1-\eta} \left(\frac{x}{1+x^2} \right)^{2n+1} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In fact, the function $\frac{x}{1+x^2}$ is an increasing one in the interval $0 < x < 1$, and therefore,

$$2^{2n} \sqrt{n} \int_0^{1-\eta} \left(\frac{x}{1+x^2} \right)^{2n+1} dx \leq 2^{2n} \sqrt{n} \left(\frac{1-\eta}{1+(1-\eta)^2} \right)^{2n} \int_0^1 \frac{x}{1+x^2} dx \\ \leq 2^{2n} \sqrt{n} \left[\frac{1}{2} \left(1 - \frac{\eta^2}{1+(1-\eta)^2} \right) \right]^{2n} \\ = \sqrt{n} \left(1 - \frac{\eta^2}{1+(1-\eta)^2} \right)^{2n} \rightarrow 0,$$

as $n \rightarrow \infty$, by L'Hospital's rule.

Hence, the expression (4.19) $\rightarrow 0$ as $n \rightarrow \infty$, which implies that the expression (4.16) $\rightarrow 0$ as $n \rightarrow \infty$, uniformly for all $t > 0$.

Finally, consider $\frac{2\Gamma(2n+\nu+2)}{\{\Gamma(n+\frac{\nu}{2}+1)\}^2} [\xi(t)I_3]$, which is

$$(4.20) \quad \approx \frac{1}{e\sqrt{\pi}} 2^{2n+\nu+2} \sqrt{n} \xi(t) \int_{1+\eta}^{\infty} |\phi(tx) - \phi(t)| \left(\frac{x}{1+x}\right)^{2n+2} dx, \text{ as } n \rightarrow \infty.$$

We can now find the constants C_1' , C_2' and C_3 in a similar manner as in (4.17) and (4.18) so that for any $x \in (1+\eta, \infty)$

$$\xi(t) |\phi(tx) - \phi(t)| \leq C_1' x^\beta + C_2' \frac{1}{x^\alpha} + C_3, \text{ for all } t > 0.$$

Thus expression (4.20)

$$\begin{aligned} &\leq \frac{1}{e\sqrt{\pi}} 2^{2n+\nu+2} \sqrt{n} [C_1' \int_{1+\eta}^{\infty} \left(\frac{x}{1+x}\right)^{2n+2} x^\beta dx + C_2' \int_{1+\eta}^{\infty} \left(\frac{x}{1+x}\right)^{2n+2} x^{-\alpha} dx \\ &\quad + C_3 \int_{1+\eta}^{\infty} \left(\frac{x}{1+x}\right)^{2n+2} dx] \\ (4.21) \quad &\leq \frac{1}{e\sqrt{\pi}} 2^{2n+\nu+2} \sqrt{n} [C_1' \int_{1+\eta}^{\infty} \frac{x^{2n+3}}{(1+x^2)^{2n+2}} dx + (C_2' + C_3) \int_{1+\eta}^{\infty} \left(\frac{x}{1+x}\right)^{2n+2} dx], \end{aligned}$$

$$\text{since } x^\beta \leq x \text{ and } x^{-\alpha} < 1.$$

Further

$$\int_{1+\eta}^{\infty} \frac{x^{2n+3}}{(1+x^2)^{2n+2}} dx = \int_{1+\eta}^{\infty} \left(\frac{x}{1+x}\right)^{2n+1} \left(\frac{x^2}{1+x}\right) dx$$

$$\leq \int_{1+\eta}^{\infty} \left(\frac{x}{1+x^2}\right)^{2n+1} dx, \text{ as } \frac{x^2}{1+x^2} < 1.$$

Now, as $\left(\frac{x}{1+x^2}\right)$ is a decreasing function of x in the range $1+\eta < x < \infty$ therefore,

$$\begin{aligned} 2^{2n+\nu+2} \sqrt{n} \int_{1+\eta}^{\infty} \left(\frac{x}{1+x^2}\right)^{2n+1} dx &\leq 2^{2n+\nu+2} \sqrt{n} \left(\frac{1+\eta}{1+(1+\eta)^2}\right)^{2n-1} \int_0^{\infty} \left(\frac{x}{1+x^2}\right)^2 dx \\ &= \frac{\pi}{4} 2^{2n+\nu+2} \sqrt{n} \left[\frac{1}{2} \left(1 - \frac{\eta^2}{1+(1+\eta)^2}\right)\right]^{2n-1} \\ &= 2^{\nu+1} \pi \sqrt{n} \left[1 - \frac{\eta^2}{1+(1+\eta)^2}\right]^{2n-1}, \end{aligned}$$

which goes to zero as $n \rightarrow \infty$ in view of L'Hospital's rule.

Also,

$$2^{2\nu+\nu+2} \sqrt{n} \int_{1+\eta}^{\infty} \left(\frac{x}{1+x^2}\right)^{2n+2} dx \leq 2^{2n+\nu+2} \sqrt{n} \int_{1+\eta}^{\infty} \left(\frac{x}{1+x^2}\right)^{2n+1} dx \rightarrow 0$$

when $n \rightarrow \infty$ as above.

Hence the expression (4.21) and therefore the expression (4.20) goes to zero as $n \rightarrow \infty$ uniformly for all $t > 0$. In view of the convergence of the expressions (4.13), (4.16) and (4.20) respectively to zero as $n \rightarrow \infty$ uniformly for all $t > 0$, we conclude that the expression (4.10) $\rightarrow 0$ as $n \rightarrow \infty$, uniformly for all $t > 0$. This completes the proof of Lemma 4.3.

Proof of Theorem 4.1. In view of the fact that $F(x)$ is smooth (by Theorem 3.2) and that $\phi(x)$ is smooth, and has compact support in $I = (0, \infty)$, we can write

$$\langle m'(x) L_{n,x} F(x), \phi(x) \rangle = \int_0^{\infty} m'(x) L_{n,x} [F(x)] \phi(x) dx .$$

Note that, an alternative form of the operator $L_{n,x}$, which we will be using here, is given as

$$L_{n,x} = - \frac{2^{v-\frac{1}{2}} \Gamma(v+\frac{1}{2}) \Gamma(n+v+1) n!}{[\Gamma(n+\frac{v}{2}+1)]^2} \theta \prod_{k=1}^n (1 - \frac{\theta}{2k}) (1 + \frac{\theta}{2k+2v}) ,$$

where $\theta = x \frac{d}{dx}$, $\theta^2 = x \frac{d}{dx} (x \frac{d}{dx}) \dots$; $n = 1, 2, \dots$. [3, p. 320]

Now, applying the integration by parts successively, and using the fact that $\phi(x)$ has compact support in $(0, \infty)$, we get

$$\int_0^{\infty} m'(x) L_{n,x} [F(x)] \phi(x) dx = \int_0^{\infty} m'(x) F(x) [M_{n,x} \phi(x)] dx ,$$

where the operator $M_{n,x}$ is given by

$$M_{n,x} = - \frac{2^{v-\frac{1}{2}} \Gamma(v+\frac{1}{2}) \Gamma(n+v+1) n!}{[\Gamma(n+\frac{v}{2}+1)]^2} \{-\theta - (2v+1)\} \prod_{k=1}^n \{1 - \frac{1}{2k} (-\theta - (2v+1))\} \{1 + \frac{1}{2k+2v} (-\theta - (2v+1))\} ; \theta = x \frac{d}{dx} .$$

Indeed here we make use of the following fact:

$$\begin{aligned}
 \int_0^{\infty} x \frac{d}{dx} (F(x)) x^{2\nu} \phi(x) dx &= \int_0^{\infty} \frac{d}{dx} (F(x)) x^{2\nu+1} \phi(x) dx \\
 &= x^{2\nu+1} \phi(x) F(x) \Big|_0^{\infty} - \int_0^{\infty} F(x) \{ (2\nu+1) x^{2\nu} \phi(x) + \\
 &\quad x^{2\nu+1} \frac{d}{dx} \phi(x) \} dx \\
 &= 0 + \int_0^{\infty} F(x) \{ -x \frac{d}{dx} - (2\nu+1) \} \phi(x) x^{2\nu} dx ,
 \end{aligned}$$

as $\phi(x)$ vanishes at 0 and at ∞ .

Now

$$\begin{aligned}
 \int_0^{\infty} m'(x) F(x) M_{n,x} \phi(x) dx &= \int_0^{\infty} \langle m'(t) f(t), \frac{t}{(x^2+t^2)^{\nu+1}} \rangle [M_{n,x} \phi(x)] dm(x) \\
 &= \langle m'(t) f(t), \int_0^{\infty} \frac{t}{(x^2+t^2)^{\nu+1}} [M_{n,x} \phi(x)] dm(x) \rangle ,
 \end{aligned}$$

which follows in view of Lemma 4.1 with a note of the fact that

$$M_{n,x}[\phi(x)] \in \mathfrak{D}(I) .$$

Our theorem will be proved now if we could show that,

$$\int_0^{\infty} \frac{t}{(x^2+t^2)^{\nu+1}} [M_{n,x} \phi(x)] dm(x) \text{ goes to } \phi(t) \text{ in } \mathcal{T}_{\alpha,\beta}(I)$$

as $n \rightarrow \infty$.

Observe that

$$(4.22) \quad \left(t \frac{d}{dt}\right) \frac{t}{(x^2+t^2)^{\nu+1}} = \left(-x \frac{d}{dx} - (2\nu+1)\right) \frac{t}{(x^2+t^2)^{\nu+1}} .$$

We need first the justification of the steps in the following manipulations:

$$(4.23) \quad \left(t \frac{d}{dt}\right)^k \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} [M_{n,x} \phi(x)] dm(x) = \int_0^\infty \left(t \frac{d}{dt}\right)^k \frac{t}{(x^2+t^2)^{\nu+1}} [M_{n,x} \phi(x)] dm(x)$$

$$(4.24) \quad = \int_0^\infty \left(-x \frac{d}{dx} - (2\nu+1)\right)^k \frac{t}{(x^2+t^2)^{\nu+1}} [M_{n,x} \phi(x)] dm(x)$$

$$(4.25) \quad = \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} \left(x \frac{d}{dx}\right)^k [M_{n,x} \phi(x)] dm(x)$$

$$(4.26) \quad = \int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} M_{n,x} \left(x \frac{d}{dx}\right)^k \phi(x) dm(x)$$

$$(4.27) \quad = \int_0^\infty [L_{n,x} \frac{t}{(x^2+t^2)^{\nu+1}}] \left(x \frac{d}{dx}\right)^k \phi(x) dm(x)$$

$$(4.28) \quad = \frac{2\Gamma(2n+\nu+2)}{[\Gamma(n+\frac{\nu}{2}+1)]^2} \int_0^\infty \frac{x^{2n+2\nu+2} t^{2n+1}}{(x^2+t^2)^{2n+\nu+2}} \left(x \frac{d}{dx}\right)^k \phi(x) dx .$$

Note that, the differentiation under the integral sign in (4.23) is justified by a standard theorem [16, p. 59]. The step (4.24) is straightforward in view of (4.22). The step (4.25) follows by applying integration by parts successively in (4.24). The step (4.26) is true in view of the commutativity of the operators $\left(x \frac{d}{dx}\right)^k$ and $M_{n,x}$.

The integration by parts has been used again to get (4.27) from (4.26).

The step (4.28) is true in view of [3;2.4].

Also, it can be seen that

$$\frac{2\Gamma(2n+\nu+2)}{[\Gamma(n+\frac{\nu}{2}+1)]^2} \int_0^\infty \frac{x^{2n+2\nu+2} t^{2n+1}}{(x^2+t^2)^{2n+\nu+2}} dx \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

In fact this follows by taking the substitution $x = t \tan \theta$ in the integral, and using Sterling's formula. Hence, as $n \rightarrow \infty$, we have for any $k = 1, 2, 3, \dots$

$$\begin{aligned} & \xi(t) \left(t \frac{d}{dt}\right)^k \left[\int_0^\infty \frac{t}{(x^2+t^2)^{\nu+1}} [M_{n,x} \phi(x)] dm(x) - \phi(t) \right] \\ (4.29) \quad &= \xi(t) \frac{2\Gamma(2n+\nu+2)}{[\Gamma(n+\frac{\nu}{2}+1)]^2} \int_0^\infty [\phi_k(x) - \phi_k(t)] \frac{x^{2n+2\nu+2} t^{2n+1}}{(x^2+t^2)^{2n+\nu+2}} dx, \end{aligned}$$

where

$$\phi_k(x) \equiv \left(x \frac{d}{dx}\right)^k \phi(x)$$

is obviously a member of $\mathcal{N}(I)$. By Lemma 4.3, the expression (4.29) goes to zero as $n \rightarrow \infty$ uniformly for all $t > 0$.

This completes the proof of Theorem 4.1.

Remark. Let $\mathcal{E}(I)$ be the space of all complex-valued smooth functions defined over I . The topology and convergence on $\mathcal{E}(I)$ is defined

as in [21, 2.3]. The dual space $\mathcal{E}'(I)$ of the space $\mathcal{E}(I)$ is the space of distributions of compact support [21, Th. 2.3.1]. It turns out that the space $\mathcal{P}_{\alpha,\beta}(I) \subset \mathcal{E}(I)$ and the convergence of a sequence $\{\phi_v\}_{v=1}^{\infty}$ in $\mathcal{P}_{\alpha,\beta}(I)$ to ϕ implies convergence of the same sequence to ϕ in $\mathcal{E}(I)$. Therefore the restriction of any $f \in \mathcal{E}'(I)$ to $\mathcal{P}_{\alpha,\beta}(I)$ is in $\mathcal{P}'_{\alpha,\beta}(I)$.

5. Illustration of the Inversion Theorem by an example. Let us take a distribution $f(t)$ defined as

$$(5.1) \quad f(t) = \sum_{k=1}^m \delta(t - k)$$

where $\delta(t)$ is the Dirac delta functional. Obviously $f(t) \in \mathcal{E}'(I)$. The Hankel potential transform of $f(t)$ is given by

$$\begin{aligned} (5.2) \quad F(x) &= \langle m'(t) \sum_{k=1}^m \delta(t - k), \frac{t}{(x^2 + t^2)^{v+1}} \rangle \\ &= \langle \sum_{k=1}^m \delta(t - k), m'(t) \frac{t}{(x^2 + t^2)^{v+1}} \rangle \\ &= \sum_{k=1}^m m'(k) \frac{k}{(x^2 + k^2)^{v+1}}. \end{aligned}$$

In order to illustrate the inversion theorem, we need verify that, for

any $\phi(x) \in \mathfrak{D}(I)$,

$$\langle m'(x) L_{n,x}[F(x)], \phi(x) \rangle \rightarrow \langle m'(t) f(t), \phi(t) \rangle,$$

as $n \rightarrow \infty$.

Now,

$$(5.3) \quad \langle m'(x) L_{n,x}[F(x)], \phi(x) \rangle = \sum_{k=1}^m m'(k) \int_0^{\infty} L_{n,x} \left[\frac{k}{(x^2+k^2)^{v+1}} \right] \phi(x) dm(x),$$

where

$$\begin{aligned} & \int_0^{\infty} L_{n,x} \left[\frac{k}{(x^2+k^2)^{v+1}} \right] \phi(x) dm(x) \\ &= \frac{2^{v+\frac{1}{2}} \Gamma(v+\frac{1}{2}) \Gamma(2n+v+2)}{\left[\Gamma(n + \frac{v}{2} + 1) \right]^2} \int_0^{\infty} \frac{x^{2n+2} k^{2n+1}}{(x^2+k^2)^{2n+v+2}} \phi(x) dm(x) \\ & \quad \text{(By [3, (2.4)])} \\ &= \frac{2 \Gamma(2n+v+2)}{\left[\Gamma(n + \frac{v}{2} + 1) \right]^2} \int_0^{\infty} \frac{x^{2n+2v+2} k^{2n+1}}{(x^2+k^2)^{2n+v+2}} \phi(x) dx \\ (5.4) \quad &= \frac{2 \Gamma(2n+v+2)}{\left[\Gamma(n + \frac{v}{2} + 1) \right]^2} \int_0^{\infty} \left(\frac{t}{1+t^2} \right)^{2n+2} \frac{\phi(kt)}{t^{-2v} (1+t^2)^v} dt \\ & \quad \text{(By the substitution } x=kt) \end{aligned}$$

Following the techniques similar to those used in getting (3.4) from (3.3) in [3] and applying Sterling's formula, it can be readily seen that the expression (5.4) tends to $\phi(k)$ as $n \rightarrow \infty$.

Hence,

$$\sum_{k=1}^n m'(k) \int_0^{\infty} L_{n,x} \left[\frac{k}{(x^2+k^2)^{\nu+1}} \right] \phi(x) d\pi(x) \rightarrow \sum_{k=1}^n m'(k) \phi(k),$$

as $n \rightarrow \infty$.

But

$$\sum_{k=1}^n m'(k) \phi(k) = \langle m'(t) \sum_{k=1}^n \delta(t-k), \phi(t) \rangle$$

This is what we need.

CHAPTER 3

"Finite Hankel Transformation of a Class of Generalized Functions"

1. Introduction: The Fourier-Bessel expansion [19, p. 576] of an arbitrary function $f(x)$ is formally given by

$$(1.1) \quad f(x) = \sum_{m=1}^{\infty} a_m J_{\nu}(j_m x) ,$$

where $J_{\nu}(z)$ is the Bessel function of first kind of order $\nu > -\frac{1}{2}$ for any complex z ; $j_1, j_2, j_3, \dots$ are the positive zeros* of $J_{\nu}(z)$ arranged in ascending order, and the Fourier coefficients a_m are given as

$$(1.2) \quad a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 t f(t) J_{\nu}(j_m t) dt , \quad m = 1, 2, 3, \dots$$

It is also to be noted here that

$$(1.3) \quad \int_0^1 t J_{\nu}(j_m t) J_{\nu}(j_n t) dt = \begin{cases} 0 , & m \neq n \\ \frac{1}{2} J_{\nu+1}^2(j_m) , & m = n \end{cases} ,$$

which represents, for a fixed $\nu \geq -\frac{1}{2}$, the orthogonality of the set of functions $J_{\nu}(j_m t)$, $m = 1, 2, 3, \dots$ on the interval $(0, 1)$ with t as a weight function.

* For $\nu > -1$, all the zeros of $J_{\nu}(z)$ lie on the real x -axis [19, p. 483].

The integral appearing in (1.2) defines the finite Hankel Transformation of f . With a better notation, the finite Hankel Transformation of the function $f(t)$ on the interval $(0,1)$ is defined as

$$(1.4) \quad H_{\nu, m}\{f(t)\} = \int_0^1 t f(t) J_{\nu}(j_m t) dt = F_H(j_m) \quad , \quad m = 1, 2, 3, \dots$$

Note that for fixed $\nu \geq -\frac{1}{2}$, the transform $F_H(j_m)$ is a sequence of real numbers.

The inversion theorem for the transform (1.4) is given in [19, p. 591] and stated as:

Theorem 1.1: Let $f(t)$ be a function defined arbitrarily in the interval $(0,1)$; and let $\int_0^1 t^{\frac{1}{2}} f(t) dt$ exist and (if it is an improper integral) let it be absolutely convergent. Let

$$a_m = \frac{2}{J_{\nu+1}^2(j_m)} \int_0^1 t f(t) J_{\nu}(j_m t) dt \quad ,$$

where $\nu \geq -\frac{1}{2}$.

Let x be any internal point of an interval (a,b) such that $0 < a < b < 1$ and that $f(t)$ has limited total fluctuation in (a,b) ,

i.e., $f(t)$ is of bounded variation in (a,b) . Then the series

$$\sum_{m=1}^{\infty} a_m J_{\nu}(j_m x)$$

is convergent, and its sum is $\frac{1}{2} \{f(x+0) + f(x-0)\}$, and is $f(x)$ if the function f is continuous at the point x .

Zemanian [21, Ch. 9], [24] has developed a procedure for expanding a generalized function into a series of the Fourier type and in particular into that of the Fourier-Bessel type. The method involved in his work is very much related to the Hilbert space techniques, and, as Zemanian [21, Ch. 9] also has mentioned, the techniques used are of somewhat different character from the ones used in most of the works on the generalized integral transformations such as [10], [11], [22], and [21, Before Ch. 9] . Other works that discuss orthogonal series expansions involving generalized functions are by Bouix [1, Ch. 7], Barga and Schönberg [2] and Gelfand and Shilov [7, Vol. 3, Ch. 4].

In this chapter, the classical result, Theorem 1.1 is extended to a class of generalized functions. For this, a generalized function space is specified and a finite Hankel transformation of generalized functions is defined. Subsequently an inversion theorem for the transformation is established which gives rise to an expansion of the generalized function under consideration in a series of Fourier-Bessel type converging in $\mathcal{D}'(I)$.

In the end, an operation transform formula is also given and a distributional differential equation is solved.

Notice that, in the present work, we will employ mostly the techniques used in the previous chapters, and the usual ones, unlike the Zemanian's approach to this subject. This statement, at present, might seem a little vague, but will be clear with the successive development of this chapter.

2. The Testing-Function Space $u_{\alpha, \nu}(I)$ and its Dual: Let I denote

the interval $(0,1)$ and let x and t be real variables restricted to I . For each pair of real numbers ν and α such that $\nu \geq -\frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, we define the space $u_{\alpha, \nu}(I)$ of all complex-valued functions $\phi(x)$ defined over I such that $\phi(x)$ is infinitely differentiable and that

$$(2.1) \quad \gamma_k(\phi(x)) = \sup_{0 < x < 1} |x^{\alpha} \Omega_x^{(k)} \phi(x)| < \infty$$

for all $k = 0, 1, 2, \dots$, where the operator Ω_x is defined as

$$\Omega_x = \left(D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2} \right) ; \quad D_x = \frac{d}{dx} .$$

$u_{\alpha, \nu}(I)$ is a linear space. Also each γ_k is a seminorm on $u_{\alpha, \nu}(I)$ and γ_0 is a norm so that $\{\gamma_k\}_{k=0}^{\infty}$ is a separating collection of seminorms. The topology of $u_{\alpha, \nu}(I)$ is that generated by the multinorm $\{\gamma_k\}_{k=0}^{\infty}$ [21, pp. 7-8]. The concept of convergence and completeness is also given in a way similar to that in Section 2 of the previous Chapter. It can be checked that $u_{\alpha, \nu}(I)$ is a Frechet space, i.e., a complete countably mulinormed space [21, p. 12]. Let $u'_{\alpha, \nu}(I)$ denote the dual space of $u_{\alpha, \nu}(I)$. It is easy to see that the space $\mathfrak{D}(I)$ with $I = (0,1)$ is a subspace of $u_{\alpha, \nu}(I)$ and that convergence in $\mathfrak{D}(I)$ implies convergence in $u_{\alpha, \nu}(I)$. Consequently, the restriction of any $f \in u'_{\alpha, \nu}(I)$ to $\mathfrak{D}(I)$ is a member of $\mathfrak{D}'(I)$.

The Testing-Function Space $V_{\alpha, \nu}(I)$ and its dual: An infinitely

differentiable, complex-valued function $\phi(x)$ defined over I is said to belong to space $V_{\alpha, \nu}(I)$ iff $\frac{\phi(x)}{x} \in u_{\alpha, \nu}(I)$. The topology on $V_{\alpha, \nu}(I)$ is generated by the separating collection of semi-norms

$\{\beta_k\}_{k=0}^{\infty}$ where β_k 's are defined as:

$$(2.2) \quad \beta_k(\phi(x)) = \gamma_k\left(\frac{\phi(x)}{x}\right), \quad k = 0, 1, 2, \dots$$

for all $\phi(x) \in V_{\alpha, \nu}(I)$. Having defined the concept of convergence and completeness as before, it is clear that $V_{\alpha, \nu}(I)$ is also a Fréchet space. Denote the dual space of $V_{\alpha, \nu}(I)$ by $V'_{\alpha, \nu}(I)$. It follows easily that the space $\mathfrak{D}(I)$ is a subspace of $V_{\alpha, \nu}(I)$, and the convergence in $\mathfrak{D}(I)$ implies the convergence in $V_{\alpha, \nu}(I)$, so that the restriction of any $f \in V'_{\alpha, \nu}(I)$ to $\mathfrak{D}(I)$ is in $\mathfrak{D}'(I)$.

We will call the elements of $U'_{\alpha, \nu}(I)$ and $V'_{\alpha, \nu}(I)$ generalized functions.

For $f \in V'_{\alpha, \nu}(I)$, we define $x f \in U'_{\alpha, \nu}(I)$ by the relation

$$(2.3) \quad \langle x f, \phi \rangle = \langle f, x \phi \rangle, \quad \text{for all } \phi \in U_{\alpha, \nu}(I).$$

This definition has sense in view of the fact that $\phi \in U_{\alpha, \nu}(I)$ iff $x \phi \in V_{\alpha, \nu}(I)$. It can be easily verified that $f \in V'_{\alpha, \nu}(I)$ iff $x f \in U'_{\alpha, \nu}(I)$.

3. The generalized finite Hankel transformation: For an arbitrary member $f \in V'_{\alpha, \nu}(I)$, we define finite Hankel Transformation of f by the relation

$$\mathfrak{H}_{\nu, m}(f) = \langle x f(x), J_{\nu}(j_m x) \rangle = F(j_m) \quad , \quad \text{for each } m = 1, 2, 3, \dots,$$

where $j_1, j_2, \dots$ are the positive zeros of Bessel's function

$J_\nu(z)$, $\nu \geq -\frac{1}{2}$. The above relation is well defined, since

$xf \in U'_{\alpha, \nu}(I)$, and $J_\nu(j_m x) \in U_{\alpha, \nu}(I)$ for each $m = 1, 2, 3, \dots$.

Indeed, to show that $J_\nu(j_m x) \in U_{\alpha, \nu}(I)$, one has to make use of the fact

$$\left(D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2}\right)[J_\nu(j_m x)] = -j_m^2 J_\nu(j_m x).$$

Before giving the inversion theorem, we prove the following simple result:

Theorem 3.1: Let $f(x)$ be a function defined on the interval $(0, 1)$

and let $\int_0^1 |f(x)| dx$ exist. Then for $\nu \geq -\frac{1}{2}$ and $\frac{1}{2} \leq \alpha < 1$, f

generates a regular generalized function in $V'_{\alpha, \nu}(I)$, defined by,

$$(3.1) \quad \langle f, \phi \rangle = \int_0^1 f(x) \phi(x) dx,$$

for all $\phi \in V_{\alpha, \nu}(I)$.

Proof: First we show that the relation (3.1) defines a functional

on $V_{\alpha, \nu}(I)$.

Now,

$$\begin{aligned} \langle f, \phi \rangle &= \int_0^1 f(x) \phi(x) dx \\ &= \int_0^1 \frac{x f(x)}{x^\alpha} \left[\frac{x^\alpha \phi(x)}{x} \right] dx . \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle f, \phi \rangle| &\leq \beta_0(\phi(x)) \int_0^1 x^{1-\alpha} |f(x)| dx \\ (3.1)' \quad &\leq \beta_0(\phi(x)) \int_0^1 |f(x)| dx < \infty , \end{aligned}$$

which shows that (3.1) truly defines a functional f on $V_{\alpha, \nu}(I)$.

This functional is clearly linear by definition. To prove the continuity of

f , let $\{\phi_n\}_{n=1}^\infty$ be a sequence which converges in $V_{\alpha, \nu}(I)$ to zero,

then $\beta_0(\phi_n) \rightarrow 0$. Therefore in view of the inequality (3.1)'

$|\langle f, \phi_n \rangle| \rightarrow 0$ as $n \rightarrow \infty$. Hence $f \in V'_{\alpha, \nu}(I)$.

To establish the inversion formula, first we prove a few pertinent lemmas.

We will often use the expression,
$$\bigvee_{m=1}^N \frac{2 J_\nu(j_m x) J_\nu(j_m t)}{J_{\nu+1}^2(j_m)}$$

in the sequel, so let us denote it for simplicity, by the symbol

$T_N(t, x)$.

Lemma 3.1: Let $f \in V'_{\alpha, \nu}(I)$ and $\phi(x) \in \mathcal{D}(I)$. Then for any $N > 0$,

$$(3.2) \quad \int_0^1 \langle tf(t), T_N(t, x) \rangle \phi(x) dx = \langle tf(t), \int_0^1 T_N(t, x) \phi(x) dx \rangle .$$

Proof: Let the support of $\phi(x)$ be contained in the closed interval $[a, b] \subset (0, 1)$. So, in order to prove (3.2), we have to show that,

$$(3.3) \quad \int_a^b \langle tf(t), T_N(t, x) \rangle \phi(x) dx = \langle tf(t), \int_a^b T_N(t, x) \phi(x) dx \rangle .$$

Now, for each $n = 1, 2, 3, \dots$, partition the interval $[a, b]$ into n sub-intervals whose lengths are $\Delta x_{i,n}$ ($i=1, 2, \dots, n$). Let our partition be such that the maximum of $\Delta x_{i,n}$, so called the norm of the partition tends to zero as $n \rightarrow \infty$. Let $\xi_{i,n}$ be an arbitrary point in the i th sub-interval. Then the Riemann sum of the integral on left-hand side of (3.3) is given by

$$\begin{aligned} & \sum_{i=1}^n \langle tf(t), T_N(t, \xi_{i,n}) \rangle \phi(\xi_{i,n}) \xi_{i,n} \Delta x_{i,n} \\ &= \langle tf(t), \sum_{i=1}^n T_N(t, \xi_{i,n}) \phi(\xi_{i,n}) \xi_{i,n} \Delta x_{i,n} \rangle . \end{aligned}$$

To prove (3.3), we need only show that

$$(3.4) \quad \left[\int_a^b T_N(t, x) \phi(x) dx - \sum_{i=1}^n T_N(t, \xi_{i,n}) \phi(\xi_{i,n}) \xi_{i,n} \Delta x_{i,n} \right] \rightarrow 0$$

in $U_{\alpha, \nu}(I)$ as $n \rightarrow \infty$.

In fact, for any $k = 1, 2, 3, \dots$

$$\begin{aligned} \Omega_t^{(k)}(T_N(t, x)) &= (-1)^k \sum_{m=1}^N \frac{2(j_m)^{2k}}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x) J_{\nu}(j_m t) \\ &= L_{N,k}(t, x) \quad , \text{ (say) } . \end{aligned}$$

Therefore

$$\begin{aligned} & t^{\alpha} \Omega_t^{(k)} \left[\int_a^b T_N(t, x) \phi(x) dx - \sum_{i=1}^n T_N(t, \xi_{i,n}) \phi(\xi_{i,n}) \xi_{i,n} \Delta x_{i,n} \right] \\ (3.5) \quad &= t^{\alpha} \left[\int_a^b L_{N,k}(t, x) \phi(x) dx - \sum_{i=1}^n L_{N,k}(t, \xi_{i,n}) \phi(\xi_{i,n}) \xi_{i,n} \Delta x_{i,n} \right] \end{aligned}$$

Now, the integrand in (3.5) is a uniformly continuous function of (t,x) on the domain $\{0 < t \leq 1; a \leq x \leq b\}$, and t^α for $\alpha \geq \frac{1}{2}$ is bounded by 1 on $0 < t \leq 1$. Consequently, the absolute value of the expression (3.5) tends uniformly to zero on $0 < t \leq 1$ as $n \rightarrow \infty$.

This completes the proof of the Lemma 3.1 .

Lemma 3.2: Let $T_N(t,x)$ denote the same expression as in the previous lemma. Let a,b be real numbers satisfying $0 < a < b < 1$. Then

$$(3.6) \quad \lim_{N \rightarrow \infty} \int_0^1 T_N(t,x) x dx = 1 , \text{ when } a < t < b .$$

Proof: By interchanging the variables x and t and putting $f(x) = 1$ in Theorem 1.1 we get

$$\lim_{N \rightarrow \infty} \int_0^1 T_N(t,x) x dx = 1 \text{ when } 0 < a < t < b < 1 .$$

Now,

$$(3.7) \quad \int_0^1 T_N(t,x) x dx = \int_0^a T_N(t,x) x dx + \int_a^b T_N(t,x) x dx + \int_b^1 T_N(t,x) x dx .$$

By virtue of the analogue of Riemann-Lebesgue Lemma [18, pp. 589-591]

for $f(x) = 1$, the first and last integral on the right hand side of (3.7) tend to zero as $N \rightarrow \infty$, when $a < t < b$. Therefore taking limit as $N \rightarrow \infty$ on both sides of (3.7) we get

$$\lim_{N \rightarrow \infty} \int_a^b T_N(t, x) x dx = 1 \quad \text{when } a < t < b.$$

Hence the lemma is proved.

Lemma 3.3: Let $\psi(x) \in \mathcal{D}(I)$; $I = (0, 1)$ and let $T_N(t, x)$ be the same as in Lemma 3.1. Then

$$t^\alpha \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx \rightarrow 0 \quad \text{as } N \rightarrow \infty,$$

uniformly for all $t \in (0, 1)$, where the support of $\psi(x)$ is contained in the interval (a, b) ; $0 < a < b < 1$.

Proof: To prove the lemma, we consider two cases. In case (i), we assume that the point t is outside the interval $[a, b]$. In case (ii), the point t is assumed to be within the interval $[a, b]$.

Case (i): For $t \notin [a, b]$, $\psi(t) = 0$. Therefore,

$$\int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx = \int_a^b T_N(t, x) \psi(x) x dx.$$

Now interchanging the variables x and t and replacing $f(x)$ by $\psi(x)$ in the analogue of Riemann-Lebesgue Lemma [19, p. 589] we infer that, for a given arbitrary $\epsilon > 0$, there exists a positive integer N_0 such that for all $N \geq N_0$,

$$\left| \int_a^b T_N(t, x) \psi(x) dx \right| \leq \frac{8 c_1^2 \epsilon}{\pi c_2^2 (2-t-b) \sqrt{t}}, \quad [19, p. 590, \text{line } 9], \quad \text{which}$$

$$\text{is again bounded by } \frac{8 c_1^2 \epsilon}{\pi c_2^2 (2-a-b) \sqrt{t}} \quad \text{or} \quad \frac{8 c_1^2 \epsilon}{\pi c_2^2 (2-1-b) \sqrt{t}}, \quad \text{according}$$

as t lies on the left or right of the interval $[a, b]$. Here

c_1 and c_2 are some constants. Now let

$$M = \max \left(\frac{8 c_1^2}{\pi c_2^2 (2-a-b)}, \frac{8 c_1^2}{\pi c_2^2 (2-1-b)} \right) = \frac{8 c_1^2}{\pi c_2^2 (1-b)}$$

Then,

$$\left| \int_a^b T_N(t, x) \psi(x) dx \right| \leq \frac{8 c_1^2 \epsilon}{\pi c_2^2 (1-b) \sqrt{t}}.$$

Hence for all $N \geq N_0$,

$$\begin{aligned}
 t^\alpha \left| \int_a^b T_N(t,x) \psi(x) x dx \right| &\leq \frac{8 c_1^2 e t^{\alpha-\frac{1}{2}}}{\pi c_2^2 (1-b)} \\
 (3.8) \qquad \qquad \qquad &\leq \frac{8 c_1^2 e}{\pi c_2^2 (1-b)} \quad , \quad \text{since } \alpha \geq \frac{1}{2}
 \end{aligned}$$

and $0 < t < 1$.

Case (ii): $0 < a \leq t \leq b < 1$: We want to show that

$$t^\alpha \int_a^b T_N(t,x) [\psi(x) - \psi(t)] x dx \rightarrow 0 \quad \text{as } N \rightarrow \infty$$

uniformly for all $t \in [a,b]$.

Let $F(t,x)(x^2-t^2) \equiv x^{-\nu} [\psi(x) - \psi(t)]$ for $0 < x < 1$ and $0 < t < 1$. Define a function $G(t,x)$ of t and x in the square domain $\{0 < t < 1; 0 < x < 1\}$ as follows:

$$G(t,x) = \begin{cases} \frac{x^{-\nu} [\psi(x) - \psi(t)]}{x^2 - t^2} & \text{for } t \neq x \\ \frac{x^{-\nu} \psi'(x)}{2x} & \text{for } t = x \end{cases} .$$

Obviously $G(t,x)$ is a continuous function of t and x in

the domain $\{0 < t < 1; 0 < x < 1\}$.

Now ,

$$\begin{aligned}
 & \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx \\
 (3.9) \quad &= \int_a^b x^{\nu+1} F(t, x) (x^2 - t^2) T_N(t, x) dx \\
 &= \int_a^b x^{\nu+1} G(t, x) (x^2 - t^2) T_N(t, x) dx .
 \end{aligned}$$

Notice that the value of the integral (3.9) remains unchanged by replacing the expression $F(t, x)(x^2 - t^2)$ by $G(t, x)(x^2 - t^2)$. In fact, for $t \neq x$ these expressions coincide while for $t = x$ they both vanish.

Now, the function $G(t, x)$ is uniformly continuous in the closed domain $\{a \leq t \leq b; a \leq x \leq b\}$. Therefore for a given $\epsilon > 0$ there exists a $\delta > 0$ such that

$$(3.10) \quad |G(t, x) - G(t, x')| < \frac{\epsilon}{b-a} , \text{ whenever } |x - x'| < \delta$$

for all $x, x' \in [a, b]$ and for all $t \in [a, b]$.

Let us divide the interval $a \leq x \leq b$ into p equal parts -

by the points $a = x_0, x_1, x_2, \dots, x_{p-1}, x_p = b$. Take p large enough so that the norm of the subdivision is less than δ , where δ is the same as determined above. Now let

$$\begin{aligned} U_m &= \sup G(t, x) \\ x_{m-1} &\leq x \leq x_m \\ a &\leq t \leq b \end{aligned}$$

$$\begin{aligned} \text{and} \quad L_m &= \inf G(t, x) \\ x_{m-1} &\leq x \leq x_m \\ a &\leq t \leq b \end{aligned}$$

for each $m = 1, 2, \dots, p$. Note that the above supremum and infimum exist, moreover they are attained in the corresponding range. Since $x_m - x_{m-1} < \delta$ for each $m = 1, 2, \dots, p$, we have, in view of (3.10)

$$U_m - L_m < \frac{\epsilon}{b-a}, \quad \text{for each } m = 1, 2, \dots, p$$

so that

$$(3.11) \quad \sum_{m=1}^p (U_m - L_m)(x_m - x_{m-1}) < \epsilon.$$

Let $G(t, x) \equiv G(t, x_{m-1}) + w_m(t, x)$, for $x_{m-1} \leq x \leq x_m$ and

$a \leq t \leq b$ so that $|w_m(t, x)| \leq U_m - L_m$, for each $m = 1, 2, \dots, p$.

Then clearly we have

$$(3.12) \quad \int_a^b x^{\nu+1} G(t, x) (x^2 - t^2) T_N(t, x) dx = \sum_{m=1}^p G(t, x_{m-1}) \int_{x_{m-1}}^{x_m} x^{\nu+1} T_N(t, x) (x^2 - t^2) dx \\ + \sum_{m=1}^p \int_{x_{m-1}}^{x_m} x^{\nu+1} T_N(t, x) (x^2 - t^2) w_m(t, x) dx .$$

The two inequalities which we shall now use are given in [19, pp. 584-585] as

$$(3.13) \quad |T_N(t, x)| \leq \frac{4 c_1^2}{\pi c_2^2 |t^2 - x^2| (2-t-x)\sqrt{tx}}, \quad (0 < t < 1, 0 < x < 1)$$

and

$$(3.14) \quad \left| \int_0^x x^{\nu+1} T_N(t, x) (x^2 - t^2) dx \right| \leq \frac{4 c_1^2 x^{\nu+1}}{\pi c_2^2 A_N (2-t-x)\sqrt{tx}}, \quad (0 < t < 1, 0 < x < 1)$$

where c_1 and c_2 are some constants and A_N is a positive number

such that $j_N < A_N < j_{N+1}$. Recall that $j_1, j_2, \dots, j_N, \dots$ are

positive zeros of $J_\nu(z)$ in the ascending order.

Let $\sup_{\substack{a \leq x \leq b \\ a \leq t \leq b}} |G(t,x)| = K$. Obviously such a supremum exists. Then

in view of the inequalities (3.13) and (3.14), (3.12) leads to

$$\begin{aligned} \left| \int_a^b x^{\nu+1} G(t,x) (x^2 - t^2) T_N(t,x) dx \right| &\leq \frac{8 c_1^2 K p}{\pi c_2^2 A_N (2-t-b)\sqrt{t}} \\ &+ \frac{4 c_1^2}{\pi c_2^2 (2-t-b)\sqrt{t}} \sum_{m=1}^p (U_m - L_m) \int_{x_{m-1}}^{x_m} dx . \\ &\leq \frac{4 c_1^2}{\pi c_2^2 (2-2b)\sqrt{t}} \left[\frac{2Kp}{A_N} + \epsilon \right] \\ &\quad \text{(by (3.11))}. \end{aligned}$$

Since p has already been fixed by the choice of ϵ (in fact ϵ fixes δ and δ fixes p), and since A_N increases with N , we can choose N_1 large enough so that $A_N > \frac{2Kp}{\epsilon}$ for all $N \geq N_1$.

Hence

$$\left| \int_a^b x^{\nu+1} G(t,x) (x^2 - t^2) T_N(t,x) dx \right| \leq \frac{4 c_1^2 \epsilon}{\pi c_2^2 (1-b)\sqrt{t}} \quad \text{for } N \geq N_1 .$$

i.e.,

$$\left| \int_a^b T_N(t,x) [\psi(x) - \psi(t)] x dx \right| \leq \frac{4 c_1^2 \epsilon}{\pi c_2^2 (1-b) \sqrt{t}} \quad \text{for } N \geq N_1 .$$

Hence

$$(3.15) \quad \left| t^\alpha \int_a^b T_N(t,x) [\psi(x) - \psi(t)] x dx \right| \leq \frac{4 c_1^2 \epsilon}{\pi c_2^2 (1-b)} t^{\alpha - \frac{1}{2}} \leq \frac{4 c_1^2 \epsilon}{\pi c_2^2 (1-b)} ,$$

for $N \geq N_1$, as $\alpha \geq \frac{1}{2}$ and $0 < a \leq t \leq b < 1$.

Now combining the relations (3.8) and (3.15) from the cases

(i) and (ii) respectively, as ϵ is arbitrary, we conclude that

$$t^\alpha \int_a^b T_N(t,x) [\psi(x) - \psi(t)] x dx \rightarrow 0 \quad \text{as } N \rightarrow \infty ,$$

uniformly for all $t \in (0,1)$. This completes the proof of the lemma.

Inversion theorem: We will now prove the following inversion theorem for the generalized finite Hankel transformation.

Theorem 3.2: Let f be an arbitrary generalized function in the space

$V'_{\alpha, \nu}(I)$ where $\nu \geq -\frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, and let $F(j_m)$ be the finite

Hankel transform of f for each $m = 1, 2, 3, \dots$ defined as

$$H_{\nu, m}(f) = \langle t f(t), J_\nu(j_m t) \rangle \equiv F(j_m) .$$

Then for each $\phi(x) \in \mathcal{D}(I)$ one has

$$(3.9) \quad \left\langle 2x \sum_{m=1}^N \frac{F(j_m)}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x), \phi(x) \right\rangle \rightarrow \langle tf(t), \phi(t) \rangle$$

as $N \rightarrow \infty$.

Here $J_{\nu}(z)$ is the Bessel's function of first kind of order ν , and j_m ($m=1,2,3,\dots$) are all the positive zeros of $J_{\nu}(z)$ in increasing order; the variables x and t range in the interval $I = (0,1)$.

Proof: As $\phi(x) \in \mathcal{D}(I)$, let us assume that the support of $\phi(x)$ is contained in the interval (a,b) , where $0 < a < b < 1$.

Now, the theorem will be proven by justifying the steps in the following manipulations.

$$(3.10) \quad \begin{aligned} & \left\langle 2x \sum_{m=1}^N \frac{F(j_m)}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x), \phi(x) \right\rangle \\ &= \int_a^b 2 \sum_{m=1}^N \frac{F(j_m)}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x) x \phi(x) dx \\ &= \int_a^b \sum_{m=1}^N 2 \frac{\langle tf(t), J_{\nu}(j_m t) \rangle}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x) x \phi(x) dx \end{aligned}$$

$$(3.11) \quad = \int_a^b \langle tf(t) , \sum_{m=1}^N \frac{2 J_{\nu}(j_m x) J_{\nu}(j_m t)}{J_{\nu+1}^2(j_m)} \rangle x \phi(x) dx$$

$$(3.12) \quad = \langle tf(t) , \int_a^b \sum_{m=1}^N \frac{2 J_{\nu}(j_m x) J_{\nu}(j_m t)}{J_{\nu+1}^2(j_m)} x \phi(x) dx \rangle$$

$$(3.13) \quad \rightarrow \langle tf(t) , \phi(t) \rangle , \text{ as } N \rightarrow \infty .$$

The step (3.10) is justified in view of the fact that the function $2x \sum_{m=1}^N \frac{F(j_m)}{J_{\nu+1}^2(j_m)} J_{\nu}(j_m x)$ is locally integrable over the interval $(0,1)$, and $\phi(x) \in \mathcal{D}(I)$. The step (3.11) follows by the linearity of the functional. That the expression (3.11) equals the expression (3.12) is proved in Lemma 3.1. To prove that the expression (3.12) converges to the expression (3.13), we need show that

$$(3.14) \quad t^{\alpha} \Omega_t^{(k)} \left[\int_a^b \sum_{m=1}^N \frac{2 J_{\nu}(j_m x) J_{\nu}(j_m t)}{J_{\nu+1}^2(j_m)} \phi(x) x dx - \phi(t) \right] \rightarrow 0$$

as $N \rightarrow \infty$, uniformly for all $t \in (0,1)$.

Recall that the operator Ω_t is defined as:

$$\Omega_t = \left(D_t^2 + \frac{1}{t} D_t - \frac{\nu^2}{t^2} \right); \quad D_t = \frac{d}{dt} \quad (\text{Section 2.})$$

As earlier, let us denote the expression $\sum_{m=1}^N \frac{2 J_\nu(j_m x) J_\nu(j_m t)}{J_{\nu+1}^2(j_m)}$ by

$T_N(t, x)$. Now, in view of the relation

$$\Omega_t[J_\nu(j_m t)] = -j_m^2 J_\nu(j_m t)$$

one can easily see that

$$(3.15) \quad \Omega_t[T_N(t, x)] = \Omega_x[T_N(t, x)]$$

Since, the order of differentiation and integration in the expression (3.14) is interchangeable in view of [18, p. 59], therefore

$$\begin{aligned} \Omega_t \int_a^b T_N(t, x) \phi(x) dx &= \int_a^b \Omega_t[T_N(t, x)] \phi(x) dx \\ &= \int_a^b \Omega_x[T_N(t, x)] \phi(x) dx, \quad (\text{By (3.15)}) \end{aligned}$$

$$= \int_a^b T_N(t, x) \Omega_x [\phi(x)] x dx, \quad (\text{By integration by parts}).$$

Now applying the same operation Ω_t successively and using the integration by parts we get

$$\Omega_t^{(k)} \int_a^b T_N(t, x) \phi(x) x dx = \int_a^b T_N(t, x) \Omega_x^{(k)} [\phi(x)] x dx,$$

for each $k = 1, 2, 3, \dots$

Hence, in the light of Lemma 3.2, we have as $N \rightarrow \infty$,

$$\begin{aligned} \Omega_t^{(k)} \left[\int_a^b T_N(t, x) \phi(x) x dx - \phi(t) \right] &= \int_a^b T_N(t, x) [\Omega_x^{(k)} \phi(x) - \Omega_t^{(k)} \phi(t)] x dx \\ &= \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx, \end{aligned}$$

where $\psi(x) = \Omega_x^{(k)} \phi(x)$, which is obviously a member of $\mathcal{D}(I)$, with support contained in (a, b) .

Hence it suffices to prove that

$$t^{\alpha} \int_a^b T_N(t, x) [\psi(x) - \psi(t)] x dx \text{ converges to zero as } N \rightarrow \infty,$$

uniformly for all $t \in (0, 1)$, which is true in view of Lemma 3.3.

This completes the proof of the theorem.

Remark. A uniqueness theorem similar to Theorem 3.4 of Chapter 1 is valid in this case too.

4. An application. We turn now to find out an operation-transform formula which can be used in solving certain types of differential equations.

Recall the operator Ω_x defined in Section 2 as

$$\Omega_x = D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2} ; \quad D_x = \frac{d}{dx} , \quad \nu \geq -\frac{1}{2} ,$$

$$(0 < x < 1) .$$

Also recall the testing function spaces $u_{\alpha, \nu}(I)$ and $v_{\alpha, \nu}(I)$ defined in the same section with a relation

$$(4.1) \quad \phi(x) \in v_{\alpha, \nu}(I) \text{ iff } \frac{\phi(x)}{x} \in u_{\alpha, \nu}(I) .$$

Let $u'_{\alpha, \nu}(I)$ and $v'_{\alpha, \nu}(I)$ be the dual spaces of $u_{\alpha, \nu}(I)$ and $v_{\alpha, \nu}(I)$ respectively.

We now define an operator $\Omega_x^*: v'_{\alpha, \nu}(I) \rightarrow v'_{\alpha, \nu}(I)$ as follows:

$$(4.2) \quad \langle \Omega_x^* f(x), \phi(x) \rangle = \langle f(x), x \Omega_x \left(\frac{\phi(x)}{x} \right) \rangle$$

for all $f \in V'_{\alpha, \nu}(I)$ and $\phi \in V_{\alpha, \nu}(I)$.

Since in view of the relation (2.1), $\gamma_k(\Omega_x^* \phi(x)) = \gamma_{k+1}(\phi(x))$, for each $k=0,1,2,\dots$ and for any $\phi(x) \in u_{\alpha, \nu}(I)$, the space $u_{\alpha, \nu}(I)$ is closed with respect to the differentiation operator Ω_x .

Hence by relation (4.1), $x \Omega_x \left(\frac{\phi(x)}{x} \right) \in V_{\alpha, \nu}(I)$, and the right hand expression of (4.2) is meaningful.

Taking note of the relation (2.3) and the fact that $f \in V'_{\alpha, \nu}(I)$ iff $xf \in u'_{\alpha, \nu}(I)$, the operator $\Omega_x^*: V'_{\alpha, \nu}(I) \rightarrow V'_{\alpha, \nu}(I)$ can equivalently be defined as

$$(4.2)' \quad \langle x \Omega_x^* f(x), \psi(x) \rangle = \langle xf(x), \Omega_x \psi(x) \rangle,$$

for all $\psi(x) \in u_{\alpha, \nu}(I)$.

The proof of equivalence of relations (4.2) and (4.2)' is quite simple and is therefore omitted.

Since the space $u_{\alpha, \nu}(I)$ is closed with respect to the

differentiation operator Ω_x , it would follow from (4.2)' that the space $V'_{\alpha, \nu}(I)$ is closed with respect to the operator Ω_x^* and for all $k = 1, 2, 3, \dots$

$$(4.3) \quad \langle x \Omega_x^{*k} f(x), \psi(x) \rangle = \langle x f(x), \Omega_x^k \psi(x) \rangle$$

for any $f \in V'_{\alpha, \nu}(I)$ and $\psi(x) \in U_{\alpha, \nu}(I)$.

Now the following theorem provides an operation transform formula.

Theorem 4.1: Let $f \in V'_{\alpha, \nu}(I)$ and $\mathcal{H}_{\nu, m}(f)$ be the finite Hankel transform of f as defined in section 3. Then for each $k = 0, 1, 2, \dots$

$$(4.4) \quad \mathcal{H}_{\nu, m}(\Omega_x^{*k} f) = (-1)^k j_m^{2k} \mathcal{H}_{\nu, m}(f), \quad m = 1, 2, \dots,$$

where $j_1, j_2, \dots$ are positive zeros of $J_\nu(x)$.

Proof: We have,

$$\mathcal{H}_{\nu, m}(\Omega_x^{*k} f) = \langle x \Omega_x^{*k} f(x), J_\nu(j_m x) \rangle$$

$$\begin{aligned}
 &= \langle xf(x), \Omega_x^{(k)} J_v(j_m x) \rangle \\
 &= \langle xf(x), (-1)^k j_m^{2k} J_v(j_m x) \rangle \\
 &= (-1)^k j_m^{2k} \langle xf(x), J_v(j_m x) \rangle \\
 &= (-1)^k j_m^{2k} H_{v,m}(f), \quad m = 1, 2, \dots
 \end{aligned}$$

This completes the proof of the theorem.

Now, consider the operator equation

$$(4.5) \quad P(\Omega_x^*)u = g,$$

where g possesses a finite Hankel transformation as defined in Section 3, and P is a polynomial satisfying $P(-j_m^2) \neq 0$ for $m = 1, 2, 3, \dots$.

We want to find the restriction to $\mathfrak{D}(I)$ of a generalized function $u \in V'_{\alpha, \nu}(I)$ satisfying the operator equation (4.5). Here I stands for the interval $(0, 1)$.

Applying finite Hankel transformation on both sides of (4.5) and using Theorem 4.1 we get

$$P(-j_m^2) U(j_m) = G(j_m), \quad m = 1, 2, \dots,$$

where $U(j_m)$ and $G(j_m)$ are finite Hankel transformations of u and g , respectively.

Now using inversion theorem 3.2, we get a solution to the operator equation (4.5) as

$$(4.6) \quad u = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{1}{P(-j_m^2)} \frac{2}{J_{v+1}^2(j_m)} G(j_m) J_v(j_m x)$$

where the limit is understood in $\mathcal{D}'(I)$.

Now consider the following distributional differential equation:

$$(4.7) \quad P(\Omega_x)u = g$$

where $g \in V'_{\alpha, v}(I)$, Ω_x is the same as defined in the beginning of this section and P is the polynomial. The differentiation in Ω_x is understood in the distributional sense.

One can verify quite easily that the distribution given by the right hand side expression in (4.6) is the restriction to $\mathcal{D}(I)$ of a generalized function in $V'_{\alpha, v}(I)$, satisfying the distributional differential equation (4.7). The proof is very similar to that used in Section 8 of Chapter 1.

Now let us take a polynomial $\tilde{P}$ defined as

$$\tilde{P}(x) = (x-a_1^2)(x-a_2^2) \dots (x-a_n^2),$$

where a_i 's are all distinct real numbers.

We want to find a complete solution in $\mathcal{D}'(I)$ to the distributional differential equation

$$(4.8) \quad \tilde{P}(\Omega_x)u = g$$

where $g \in V'_{\alpha, \nu}(I)$.

It is easy to check that $J_\nu(ax)$ for any real a satisfies the distributional differential equation

$$(4.9) \quad (D_x^2 + \frac{1}{x} D_x - \frac{\nu^2}{x^2} - a^2)u = 0.$$

Following a method similar to that employed in Section 8 of Chapter 1, the general solution to (4.9) in $\mathcal{D}'(I)$ can be given as

$$(4.10) \quad u(x) = J_\nu(ax) \left[c \int_{I_2}^x \left[t J_\nu^2(ait) \right]^{-1} dt + d \right]$$

where c and d are arbitrary constants.

Hence, in view of (4.10) and the fact that (4.6) is a solution in $\mathcal{D}'(I)$ to the distributional differential equation (4.7), the general

solution to (4.8) is given by

$$u(x) = \lim_{N \rightarrow \infty} \sum_{m=1}^N \frac{1}{\tilde{P}(\tilde{J}_m^2)} \frac{2}{J_{\nu+1}^2(J_m)} G(j_m) J_{\nu}(j_m x) +$$

$$\sum_{k=1}^n J_{\nu}(a_k x i) \left[c_k \int_{\frac{1}{2}}^x \left[t J_{\nu}^2(a_k t i) \right]^{-1} dt + d_k \right]$$

where the limit is understood in $\mathcal{D}'(I)$ and c_k and d_k are arbitrary constants.

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