VARIATIONS OF ZOMBIES AND SURVIVOR IN SIMPLE POLYGONS

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A thesis submitted to the Faculty of Graduate and Postdoctoral Affairs in partial fulfillment of the requirements for the degree of

Master
in
M.C.S. Computer Science

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ABSTRACT

We study the pursuit-evasion game of a zombie and a survivor in simple polygons. The game is played in a graph where a zombie has one objective: catch a survivor by occupying the vertex of the survivor. On its turn, a zombie can only move on the first edge of a *shortest* path to the survivor's location. A survivor may choose to move to any vertex adjacent to its current vertex; the objective of the survivor is to survive as long as possible. Both players take turns and have complete information of the graph and each others' position. We consider two variants of the zombie's path in P: the case where the zombie travels on the *minimum link path*, where an edge/link of the path has weight one; and the case where the zombie travels on the *geodesic path*, where an edge of the path has the Euclidean distance of its embedded endpoints. We restrict the game to one zombie and one survivor and show that the geodesic path variant wins on any simple polygon, and the minimum link path variant wins on any spiral polygon.

ACKNOWLEDGMENTS

Many thanks to my supervisors Prosenjit Bose, Jean-Lou De Carufel and Thomas C. Shermer as this thesis would not have been possible without their help, dedication and careful guidance.

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ACRONYMS

VP Visibility Polygon

BFS Breadth-First Search

Part I

ZOMBIES AND SURVIVOR

1 INTRODUCTION

Pursuit evasion games have a rich history involving mathematical interests and their applications in search and rescue, surveillance, defence, and mobile robotics. In a pursuit evasion game, one player an "evader" attempts to avoid capture by the "pursuers" as all players move in some domain. There are many variations in which the domain is discrete and when the game is continuous.

The game of Zombies and Survivors is one such pursuit evasion game played on a graph and is a variant of the game of cops and robbers. Initially, the cops choose vertices of a graph G as their starting positions, followed by the robbers choosing vertices of G as their starting positions. The game consists of rounds wherein a round; first, the cops take their turn, followed by the robbers' turn. A turn for a cop or a robber consists of a cop or robber moving to a vertex adjacent to its current location, or remaining on its current location. In one round, cops play their turn before any robbers may play their turn. If a cop occupies a vertex of a robber, that robber is caught and no longer in the game. Once all players have completed their turn, the next round commences. In a game of cops and robbers, each player has complete information about the graph and the other players' location. Cops, prior to their turn can communicate their intentions with other cops, and robbers can communicate their intentions with other robbers prior to their turn.

If all robbers are caught in G, then the cops win. If the cops win with only one cop, G is called a cop-win graph. If a robber evades capture indefinitely in G, the cops lose. Some graphs require more cops than other graphs in-order for cops to win, for example: cycles with four or more vertices require two cops, while paths only require one. We call the minimum number of cops required to win for a graph G to be the *cop number*, denoted $c(G)$ [7]. While for a class of graphs, the graph in the class with the largest cop number is the cop number of the entire class of graphs.

Compared to cops and robbers, the game of zombies and survivors on a graph G is a restricted version of the cops and robbers game. In this game, zombies play the role of cops, and survivors play the role of robbers. The difference is that zombies are restricted in their movement; on its turn, a zombie must move to a vertex that is adjacent to their current position, which lies on a *shortest path* to the nearest survivor position. A shortest path between two points is a path of minimum total weight between those two points. The movement of a survivor is the same as that of a robber: on a survivor's turn it may move to an adjacent vertex, or remain at its current vertex. Akin to the cop number, the *zombie number* is the minimum number of zombies required to win on a graph G, denoted as $z(G)$. While for a class of graphs, the graph in the class with the largest zombie number is the zombie number of the entire class of graphs. If $z(G) = 1$, then G is a zombie-win graph.

Let us take some natural number k. If k zombies can win on G, then G is also winnable with k cops since the k cops can follow the same moves as the k zombies. Whereas, if we have a graph winnable with k cops, k zombies can not always replicate the strategy of k cops. From the prior two statements, the cop number is a lower bound on the zombie number $c(G) \leqslant z(G)$ [14].

In our game of zombies and survivors, we play in a simple polygon P. On a player's turn, a player moves along a segment lying entirely in P, ending its turn on the other endpoint of said segment. If a player wants to get from some point $s \in P$ to some point $t \in P$, they must follow a path in P. Informally, a path in P is a series of connected line segments in P (a polygonal chain); in addition the *shortest path*, is a path from s to t, such that compared to any other path from s to t the total sum of the lengths of the segments of the path are minimum. A zombie may only choose the initial segment/edge of the shortest path.

We consider two variations of a shortest path in P: the *geodesic path*, and the *minimum link path*. The geodesic path is a shortest path in P, where any segment in P is weighted with the Euclidean distance of its endpoints. Whereas, a minimum link path is a shortest path in P, where any segment in P has weight one, a *link*.

Our Results. In this thesis we consider the game of only one zombie, and one survivor in a simple polygon P. Our first variation requires that the zombie follows a geodesic path to the survivor. We show a strategy that follows the geodesic path and only requires one zombie in order to win. The number of rounds is limited to $O(r)$, where r are the reflex vertices of P. The time to calculate the zombies position for all rounds takes $O(n + r \log r)$, where n is the number of vertices in P. These results mainly stem from the results proved from a game of cops and robber in Lubiw et al. [17].

Our second variation requires that the zombie follows a minimum link path to the survivor. For the family of any simple polygon, to date this problem remains unsolved. We show that for any spiral polygon, the number of rounds required is $O(n)$. Furthermore, the time to calculate the zombies position for all rounds takes $O(n)$ time. Lastly, we show some strategies that fail on simple polygons.

2 RELATED WORKS

2.1 COPS AND ROBBERS

A rich comprehensive book on the subject of cops and robbers has been written by Bonato and Nowakowski [7]. This section will outline the key topics pertaining to multiple cops and one robber in a connected graph G. By proving the former, it also proves that multiple cops and multiple robbers have the same cop number in G as multiple cops and one robber. It can be done by focusing and catching one robber at a time. If there does not exist a path between two vertices in G, then the described property is not true as the graph is *disconnected*.

2.1.1 *Dismantlability and Cop-Win Trees*

Given a graph G, the *closed neighbourhood* of a vertex u denoted as $N[u]$, is the set of all adjacent vertices of u in graph G including u itself. A vertex u is said to be a *corner* of a distinct vertex v if the closed neighbourhood of u is a subset of the closed neighbourhood of a vertex $v : N[u] \subseteq N[v]$. In addition, u is said to be *dominated* by v. If the robber moves to vertex u, then the optimal placement for the cop would be vertex v as this would corner the robber; any move the robber makes, the cop is also able to make.

A function f that maps the vertices of a graph G to vertices of a graph H is a *graph homomorphism* from G to H if for every edge (x, y) in G, $(f(x), f(y))$ is an edge of H (edge preserving). A *retract* of a graph G is a subgraph H of G where a homomorphism f exists such that for all vertices z in H, $f(z) = z$. Let us consider a graph homomorphism from G to H where G contains a corner vertex u that is dominated by some vertex v in G. If x is a vertex in G, then for all vertices in G, $f(x)$ maps accordingly:

$$f(x) = \begin{cases} v & \text{if } u = x \\ x & \text{otherwise} \end{cases} \tag{1}$$

The function f is a homomorphism from graphs G to H where H is a subgraph of G, and H is a retract of G. With this, a *shadow strategy* is devised in a game with one cop and one robber. In this strategy, we assume a winning strategy in graph G. The cop in G starts its turn on vertex s in G, and moves to $t \in N[s]$. The cop in H starts on some vertex $f(s)$, and moves to vertex $f(t)$; essentially playing in the 'shadow' of the cop in G. With this strategy Berarducci and Intrigila prove that $c(H) \leqslant c(G)$ [6].

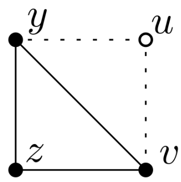


Figure 1: A transformation following Equation 1

Let G be a graph where every vertex has an edge that connects a vertex onto itself. Then $G = G_1$ is a reflexive graph, and $f_1(G_1) = G_2$ be a homomorphism from G_1 to G_2, such that f_1 is a homomorphism following the transformation seen in Equation 1, where $u_1 = u$ is a corner in G_1. More generally $G_i = f_{i-1}(G_{i-1})$, such that f_i is a homomorphism following the transformation seen in Equation 1, where $u_i = u$ is a corner in G_i. A dismantling is a sequence of homomorphisms $f_1, f_2 ..., f_{n-2}, f_{n-1}$ such that the composition $F_n = f_{n-1} \circ f_{n-2} \circ ... \circ f_2 \circ f_1$ results in $F_n(G) = K_1$ a single vertex. Note that not all graphs have a dismantling. An interesting fact of dismantlable graphs is that *a graph is cop-win if and only if it is dismantlable*. This fact was proved by Nowakowski and Winkler [21], and independently by Quilliot [22].

A *cop-win ordering* is a sequence $\mathcal{O} = u_n, u_{n-1}, u_{n-2}, ..., u_2, u_1$ of vertices from the dismantling where u_n is the vertex which dominates corner u_{n-1} found in the mapping of f_{n-1}. The cop starts on $u_n = F_n(r_1)$, a shadow of the robber's position r_1 on round 1. Then the sequence of moves $F_{n-1}(r_2), F_{n-2}(r_3), ..., F_3(r_{n-2}), F_2(r_{n-1})$ results in $N[r_{n-1}] \subseteq N[F_2(r_{n-1})]$, a cornering of the robber. Clarke and Nowakowski proved that at any round $F_{n-i+1}(r_i)$ and $F_{n-i}(r_{i+1})$ are either equal, or adjacent in G [12]; resulting in any dismantlable graph being a cop-win graph.

A *cop-win spanning tree* is a spanning tree S of graph G satisfying the following property : for all edges (x, y) in S there exists a homomorphism f_i from the dismantling of G where $f_i(x) = y$ or $f_i(y) = x$. The cop starts at the root u_n of S and follows the robber down a branch. If the robber chooses to switch a branch, the cop can also switch to a position on the robber's branch above the robber. Eventually, this leads the robber into a leaf/corner, which the cop guards. These properties are proved from Clarke in Lemmas 2.1.2, 2.1.3; and Corollary 2.1.1 [11].

2.1.2 *Cop Number on Planar graphs*

Let graph H be a subgraph of G. H is *isometric* to G if the distance between any pair of vertices in H is the same as the distance between the same pair of vertices in G [11]. An isometric path occurs when H is a path and is said to be *guarded* when the robber cannot occupy a vertex of H without getting caught on the next turn. It has been proven that after a finite number of rounds, one cop can guard the path H. Aigner and Fromme used this property to prove that any planar graph is a three cop-win graph [3]. Their method involves two cops guarding two distinct isometric paths, where eventually the robber is contained in a subgraph of the graph. The third cop splits this subgraph with a new isometric path, and induction is applied. Quilliot proved that planar graphs require at most three cops, thus this bound is tight [23].

Later Clarke proves that outerplanar graphs need at most two cops in order to win [11]. Where an *outerplanar graph* is a planar graph for which all vertices belong to the outer face. The method involves a progressive walk along the outer face.

2.1.3 *Girth and Minimum Degree of a Graph*

An interesting property of cops and robber comes from the girth of a graph G. The *girth* of G is the length of the shortest cycle in G, where each edge of G has a weight of one. Aigner and Fromme proved that if a graph has a girth of at least 5, then the cop number must be at least the minimum degree of G [3]. With this property, Barid and Bonato proved that graphs with arbitrarily large cop numbers exist [4].

2.2 ZOMBIES AND SURVIVOR

In this section, we cover topics pertaining to zombies and survivor.

2.2.1 *Simple Properties and Zombie Trees*

One of the most critical properties in a game of multiple zombies and multiple survivors is that the cop number is bounded by the zombie number $c(G) \leqslant z(G)$. This property comes from the observation that any cop can replicate a zombie's moves, and if the zombies' strategy results in the zombies winning on G, then the same number of cops can also win on G. This property was observed by Fitzpatrick et al. [14].

Other bounds stem from looking at specific graph classes with multiple zombies and one survivor. The zombie number equals the cop number if the game is played on a tree $z(T) = c(T) = 1$. On a cycle with more than four vertices, both versions are equal if zombies play on antipodal vertices $z(C) = c(C) = 2$. Similarly, on a complete bipartite graph of partition size m and n where zombies play on the dominating set $z(K_{mn}) = c(K_{mn}) = 2$. The complete graph of size m where $z(K_m) = c(K_m) = 1$. All these properties were proved by Fitzpatrick et al. [14]. Later Faubert proved that with certain zombie placements, any cycle with one chord C_{mn} the zombie number is $z(C_{mn}) = c(C_{mn}) = 2$ [13]. It is noted that a cycle with one chord is a cycle C_m, and a cycle C_n with one edge in common.

Akin to cops and robber, the zombie number of a graph is bounded by the domination number of a graph; a domination number is the size of the smallest dominating set of a graph. In the dominating set, each vertex of the graph is adjacent to at least one vertex of the dominating set. If the zombies choose the dominating set as their initial position, the survivor will be captured in the next round. This strategy is seen in Fitzpatrick et al. [14].

In Section 2.1.1 we saw that a cop-win spanning tree is a tree on which the cop plays in, to win in the original graph. This spanning tree is directly related to a cop-win ordering, as the edges of the spanning tree are related to the homomorphisms defining a dismantling of the original graph. A *breadth-first search (BFS)* is a search algorithm that satisfies a key property: it must search all the nodes of its current depth before it can search for nodes at the next depth level [20]. A *BFS ordering* is a sequence of vertices in the order that a breadth-first search first discoverers them. Likewise, a *BFS tree* is created with the following property: if our current vertex is p, and we find an adjacent vertex q which has not been discovered, we add (p, q) as an edge to the tree. There can be multiple different BFS orderings in one such graph. If one such ordering results in a BFS tree equal to the cop-win spanning tree, then such a graph is also a zombie-win graph, as proved by Fitzpatrick et al. [14].

A graph H is a subgraph of G. Then H is isometric to G if the distance between any pair of vertices in H is the same as the distance of the same pair of vertices in G [11]. In comparison, an *isometric cycle* is a cycle which is isometric with respect to G. A graph is said to be *bridged* provided that all isometric cycles of G have length three. It was proven by Fitzpatrick et al. that bridged graphs are zombie-win since there exists a cop-win spanning tree with a breadth-first search ordering [10, 14].

Determining the following: is $z(G) \leqslant c$, where c is some positive integer, and G is a simple undirected graph. It was proven to be NP-hard through a reduction of the dominating set problem on G by Keramatipour and Bahrak [16].

2.2.2 *Cartesian Product*

In graph theory, the Cartesian product $G\square H$ of graphs G and H is a graph such that the vertex set of $G\square H$ is the Cartesian product of $V(G) \times V(H)$. We denote a vertex of $G\square H$ as (v_i, u_j) where $v_i \in G$ and $u_j \in H$. The vertices (v_i, u_j) and (v_i', u_j') are distinct and adjacent in $G\square H$ if and only if either $u_j = u_j'$ and (v_i, v_i') is an edge in G; or $v_i = v_i'$ and (u_j, u_j') is an edge in H.

The topic of the zombie number of Cartesian products was first brought up by Fitzpatrick et al. Fitzpatrick et al. proposed that $z(G\square H) \leqslant z(G) + z(H)$ is a upperbound [14]. It was later proved by Keramatipour and Bahrak that $z(G\square H) \leqslant z(G) + z(H)$ is indeed an upper bound [5, 16]; with this, the upper bound on the zombie number of some Cartesian products can be improved.

We may apply this upper bound on multiple graphs. For example, the product of $z(P_n\square C_m) \leqslant 3$, where P_n is a path of length n, and C_m is a cycle of length m [16]. In addition, $z(P_2\square C_n) = 3$ is tight when C_n is odd as proven by Fitzpatrick et al. [14].

Furthermore, $z(G\square T) \leqslant z(G) + 1$, where G is any graph, and T is a tree [16]. Lastly, $z(G\square H) \leqslant z(G) + 1$ as proven by Fitzpatrick et al. [14]. G is any graph, and H is any graph of size m where there exists $v \in H$ with v having degree $m - 1$. This also applies to $z(G\square K_m) \leqslant z(G) + 1$ where K_m is the complete graph on m vertices.

2.2.3 *Hypercube*

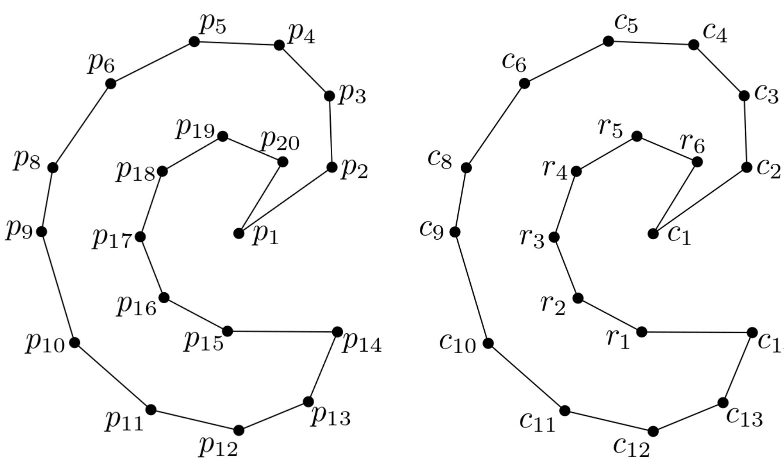
A hypercube is a generalization of a 3-cube to n dimensions, where Q_1 is a line segment, Q_2 is a square, and Q_3 is a cube. The n-th dimension can be thought of as a recursive function of the Cartesian product : $Q_n = Q_{n-1}\square Q_1$. The zombie number of Q_1 is $1 = z(P_2) = z(Q_1)$, and the zombie number of Q_2 and Q_3 is two which can be found by looking at the dominating number. Note that the total number of vertices is exponential compared to the dimensions where Q_n contains 2^n vertices.

Initially, Fitzpatrick et al. proved a lower bound on the zombie number of Q_n, where one would need at least $\lceil \frac{2n}{3} \rceil$ zombies [14]. A noteworthy property of the hypercube is that it can be defined as the product of smaller hyper-cubes $Q_{m+k} = Q_m \Box Q_k$. Through this property, Keramatipour and Bahrak proved that an upper bound on Q_n is $\lceil \frac{2n}{3} \rceil$ from the fact that the zombie number of the Cartesian product is the sum of its operands ($z(G \Box H) \leqslant z(G) + z(H)$)[16]. Thus the bound on hypercubes is tight.

3 PRELIMINARIES

A *polygon* P in the Euclidean plane $\mathbb{R}^2$ is a sequence of points $P = p_1, p_2, ..., p_n$ in $\mathbb{R}^2$ together with n line segments $p_i p_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $p_n p_1$. The points of P are referred to as the *vertices* of P, and the segments are referred to as the *edges* of P. A polygon is *simple* if no point in the plane belongs to more than two edges of P, and the points which belong to precisely two edges are vertices of P [24]. In this thesis we assume that the vertices of $P = p_1, p_2, ..., p_n$ are given in counterclockwise order along the boundary, where the *interior* of the polygon lies to the left of the directed edge $p_i p_{i+1}$. A point is *exterior* of P if it is not on the boundary or the interior of P.

Given two segments $p_i p_{i+1}$ and $p_{i+1} p_{i+2}$ in P. An *internal angle* is the angle formed by rotating $p_i p_{i+1}$ about p_{i+1} clockwise towards $p_{i+1} p_{i+2}$. The *external angle* is the angle formed by rotating $p_i p_{i+1}$ about p_{i+1} counterclockwise towards $p_{i+1} p_{i+2}$. If the internal angle is greater than π, then the corresponding vertex is referred to as a *reflex vertex*; otherwise, the vertex is *convex*. A *spiral polygon* is a simple polygon P whose sequence of vertices can be decomposed into a sequence $r_1, r_2, ..., r_k$ of reflex vertices followed by a sequence $c_1, c_2, ..., c_j$ of convex vertices [18].



(a) Labelled Simple Polygon. (b) Labelled Spiral Polygon.

Figure 2: Polygon Labels.

Furthermore, we say two points p_i and p_j are *visible* if the segment $p_i p_j$ contains no points of the polygon's exterior. Note that the segment may touch the polygon's boundary at one or more points. In addition, we define the visibility polygon $VP(p_j) = \{p_i \in P \mid p_j \textit{ is visible to } p_i\}$ of point p_j as the set of all points visible to p_j [19]. A segment is *weakly visible* to some point $p \in P$ if there exists a point on the segment visible to p.

A *chord* is a segment that does not intersect the exterior nor the boundary of the polygon except at the segment's endpoints which are points on the polygon's boundary. A *diagonal* is a chord whose endpoints are vertices of the polygon.

A *path* of length $n-1$ is a sequence of n vertices $v_1, ..., v_i, ..., v_n$ in a graph G. Furthermore, for $1 \leqslant i < n$, $v_i v_{i+1}$ is an edge in G and an edge of the path. A *path vertex* is a vertex in a path's sequence. We say v_i is an *inner vertex* of the path if v_i is in the range of $1 < i < n$; if not then v_i is one of the two endpoints of the path. A path is *open* if we only include the edges of the path and inner vertices.

In a simple polygon a path vertex is a point in P, and an edge of a path is a segment in P whose endpoints must be visible to each other. An open segment is an open path of length one. We consider two types of paths in P : the *link path*, where each edge of the path has weight one; and the *Euclidean path*, where each edge of the path has the Euclidean weight of its endpoints.

In a simple polygon a vertex of a path is a point in the interior of P, and an edge of a path is a segment in P between two path vertices which are visible to each other. We consider two types of paths in P : the *link path*, where each edge of the path has weight one; and the *Euclidean path*, where each edge of the path has the Euclidean weight of its endpoints.

The *minimum link path* is a link path between two points s and t in P, such that the number of links is minimum compared to any other link path from s to t. The number of links on the minimum link path from s to t is the *link distance*. The *geodesic path* from vertices s to t is an Euclidean path such that compared to any other Euclidean path from s to t, the geodesic path has the total minimum edge weight.

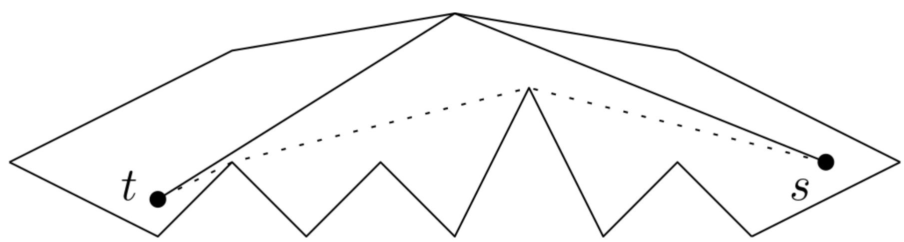


Figure 3: The geodesic path vs the minimum link path (dotted vs solid).

A *metric space* is a non-empty set combined with a metric on the set, where a *metric* is a function that defines a concept of distance between any two members of a set. Formally given a set X, a function $d : X \times X \to \mathbb{R}$ is a metric on X if d satisfies the following properties [1]:

1. for all $p, q \in X$, $d(p, q) \geqslant 0$ (positive)
2. for $d(p, q) = 0 \iff p = q$ (identity of indiscernibles)
3. $d(p, q) = d(q, p)$ (symmetry)
4. for all s in X $d(p, q) \leqslant d(p, s) + d(s, q)$ (triangle inequality)

Lemma 1. *The length of the geodesic path between two vertices* p *and* q *in* P *is a metric on the vertex set of the points in* P*.*

Proof. Let us consider the points in P as the set for a metric space X. Our distance function $d(p, q)$ is defined as the length of the geodesic path from p to q in P, where $p, q \in X$. We must prove the above four properties to prove a valid metric space.

Firstly, we prove that for all $p, q \in X$, $d(p, q) \geqslant 0$. This is obvious as the Euclidean distance between any two points is always positive, and any geodesic path is the sum of the Euclidean distances of its edges. Thus this sum will also be positive.

Secondly, $d(p, q) = 0 \iff p = q$. Thus if our path from p to q contains any other edge to a vertex that is not p or q, that edge must have a weight greater than zero. Therefore the only viable path is to stay on the same vertex, thus $p = q$. In which case, the Euclidean distance must be zero.

Thirdly, $d(p, q) = d(q, p)$. Let us state that if given an edge/segment of a path, the distance is equal independent of the order of its endpoints as the Euclidean distance is symmetric. As a consequence, if we are given a path from p to q denoted as $\pi(p, q)$, then traveling along $\pi(p, q)$ in reverse from q to p results in a path of equal length. We denote the reverse path as $\pi(q, p)$, and denote its path length as $|\pi(q, p)|$.

Let $|\pi(p, q)| = d(p, q)$, and suppose for contradiction that $d(p, q) \neq d(q, p)$. However, as seen prior $|\pi(p, q)| = |\pi(q, p)|$. Therefore we have a contradiction since if $d(q, p) < |\pi(p, q)|$, then $\pi(p, q)$ is not the shortest path; and if $d(q, p) > |\pi(p, q)|$, then $d(q, p)$ is not the length of the geodesic path. Thus property three holds.

Lastly, for all s in X, $d(p,q) \leqslant d(p,s) + d(s,q)$. We consider the case when $d(p,q) < d(p,s) + d(s,q)$ as its trivial when equal. Let us assume that $|\pi(p,q)| = d(p,q)$ is the geodesic path, and suppose for contradiction that $d(p,q) > d(p,s) + d(s,q)$. This implies that there is a path from p to s, and from s to q which is shorter than $\pi(p,q)$. Thus a contradiction that $\pi(p,q)$ is the geodesic path. $\square$

Therefore geodesic path in P is a metric space whose metric set is the points in P, and the distance function as the geodesic path between any two points in P. A similar analysis and conclusion can be applied to P with the minimum link path as the distance function.

Given a point $z \in P$, we define the windows of the visibility polygon VP(z) of P from z as the set of all boundary segments of VP(z) that does not belong to the boundary of P; a *window* is one such boundary segment. It can be seen that the points of P, which are link distance two from z, are the points of $P \backslash VP(z)$ that are weakly visible to some window of VP(z). Note that the closest point of a window to z is a reflex vertex of the polygon P; we call this vertex the *base vertex*, or the *base* of the window.

We say a chord *separates* two points in P if and only if the chord splits the polygon into two sub-polygons, and each point lies in a different sub-polygon. Given the positions of the survivor s_i and the zombie z_i at the beginning of round i, either the survivor is in VP(z_i), or there is some window separating z_i from s_i. We denote this window W_{i+1} and call it the *survivor window*. Denote the base of W_{i+1} as b_{i+1}, and the other endpoint of the window as t_{i+1}. Furthermore, let f_{i+1} be the boundary point after which the ray $\overrightarrow{b_{i+1}z_i}$ first leaves the polygon P. The *survivor window* W_{i+1} splits the polygon into two pieces. We call the one which contains s_i the *survivor pocket*.

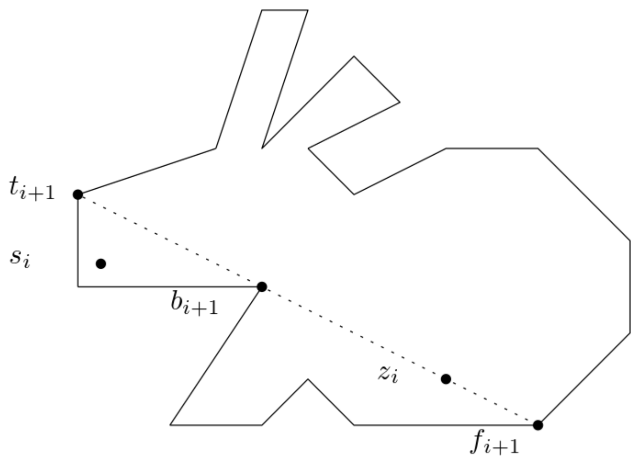


Figure 4: An example of a survivor window

4 GEODESIC PATHS IN SIMPLE POLYGONS

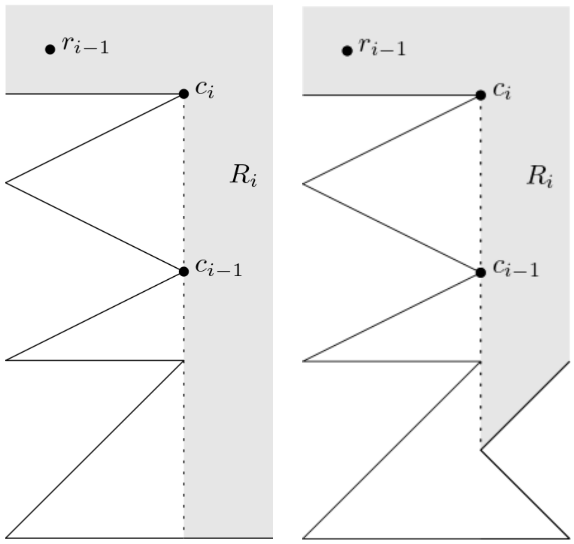
Recall that the Euclidean path is a path in P for which any edge of the path has weight equal to the Euclidean distance determined by the edge's endpoints. Furthermore, the geodesic path between two points is an Euclidean path with the minimum sum of edge weights, compared to any other Euclidean path with equal endpoints. Moving across an edge of a path costs a player one turn. In this section, we show that one zombie is sufficient in a one survivor game when following the geodesic path. The main result stems from the work of Lubiw et al. in the game of cop and robber in a simple polygon [17].

First, we denote π as a geodesic path from p to q in P. By playing on the geodesic path in P we obtain the property that all inner vertices of π are vertices of P and all inner vertices of π are reflex.

In a one cop and one robber game played in P Lubiw et al. show that the strategy of following the first edge of the geodesic path starting from the robber's position and ending at the cop's position, results in a winning strategy for one cop [17]. If the zombie follows the cop's move in a one zombie and one survivor game, then the game also results in a zombie-win in P, since the cop follows a geodesic path to the robber.

We briefly summarize the cop's strategy for a one cop and one robber game in a simple polygon shown in Lubiw et al. [17]. In this game the cop restricts the robber to an ever decreasing *active region*. Suppose on the ith round the cop moves from c_{i-1} to c_i, and the geodesic path to the robber's position r_{i-1} is a left turn from c_i. If we shoot a ray from c_i through c_{i-1} such that the ray stops when either it first leaves the polygon, or it intersects a boundary left of the ray. We call the subpolygon of P split by this ray containing r_{i-1} to be defined as the active region R_i; the region which the survivor is restricted to. Furthermore, if the shortest path goes right at c_i, we similarly stop the ray when it intersects the boundary right of the ray. Figure 5 shows the active region, and Figure 7 shows a more complete game.

It is proved that the robber's move from r_{i-1} to r_i cannot leave R_i, and that at each iteration this region reduces in area such that $R_i \subset R_{i-1}$, where $R_{i-1} \backslash R_i$ differs by at least one vertex of P. It can be proved by induction on n the number of vertices in P, where the path length is bounded by n. In the strategy proposed by Lubiw et al., the cop always takes the first edge of the shortest path/geodesic path from its current position to the robber as seen in Theorem 5 [17]. Since the cop always takes the first edge of the geodesic path, the zombie may copy the cop's move and catch the survivor in at most n rounds. We show that this can be done in r rounds, where r is the number of reflex vertices of P, and this bound can is extendable to the game of cops and survivor in a polygon.



(a) boundary right of the ray.

(b) Boundary left of the ray.

Figure 5: An active region where the next move is a left turn to the robber.

Given a geodesic path π from z to s in P, we impose one such property: in the sequence of path vertices of π, no three consecutive vertices are colinear. Say if vertices a, b, c are colinear, then one may shortcut from a to c instead where $|ac| = |ab| + |bc|$. Thus this assumption can be made without loss of generality.

Lemma 2. *The inner vertices of the geodesic path π from z to s in P are vertices of P.*

Proof. Suppose for contradiction that the inner vertices of π are not vertices of P. Then there exists an empty ε-disc centred at c, an inner vertex of π. The disc is empty in the sense that it contains no vertices of P or path vertices of π other than c. Let l and r be points from the intersection of π and the boundary of the ε-disc. Notice that l, c, and r cannot be colinear by the property that no three consecutive vertices of π are colinear. Thus one may shortcut c by travelling from l to r instead. Since the ε-disc is empty, l is visible to r. Thus we have a contradiction that π is the geodesic path. □

Lemma 3. *The inner vertices of the geodesic path π from z to s in P are reflex vertices of P.*

Proof. Suppose for contradiction that the inner vertices of π are convex. Then there exists an empty ε-disc centred at c, an inner vertex of π. The disc is empty in the sense that it contains no vertices of P or path vertices of π other than c. Let l and r be points from the intersection of π and the boundary of ε-disc. Notice that l, c, and r cannot be colinear by the property that no three consecutive vertices of π are colinear. Also notice that the intersection of the ε-disc with P containing l, c and r is convex. Since the intersection is empty, r is visible to l, thus a shortcut. Thus we have a contradiction that π is the geodesic path.

□

Theorem 1. *After $r+1$ moves, the zombie catches the survivor.*

Proof. As defined prior, the active region R_i at iteration i contains the robber's position r_{i-1} from iteration $i-1$. The cop c_{i-1} moves at the beginning of iteration i to c_i; as a result a new active region R_i is created. From this definition we obtain a couple of properties from the strategy proved in Theorem 5 [17] : the cop always takes the first edge of the geodesic path; the previous cop position c_{i-1} is no longer in the current active region R_i; $R_i \subseteq R_{i-1}$ and differs by at least one vertex of P's boundary, specifically the cops position at c_{i-1}; and the robber's move from r_{i-1} to r_i cannot leave R_i.

By Lemma 3, the inner vertices of any geodesic path in P are reflex vertices of P. Since the cop always takes the first edge of the geodesic path, the next path vertex is either a reflex vertex of P, or the robber's position. If the robber has not been caught at iteration i, then any cop position from $c_2, ..., c_i$ are reflex vertices of P.

The set of all positions are the interior points of P, and the complement of the active region $\overline{R_i}$ are interior points of P which r_i cannot move to. As seen prior $R_i \subseteq R_{i-1}$, and observe the complement $\overline{R_i} \supseteq \overline{R_{i-1}}$. Notice that $c_{i-1} \notin R_i$ thus must be in $\overline{R_i}$, and that $c_{i-2} \notin R_{i-1}$ thus must be in $\overline{R_{i-1}}$. Thus at each iteration $\overline{R_i}$ is growing, and always contains the previous cop positions (which are reflex vertices of P). Since there are r reflex vertices in P, the number of path vertices must be at most $r+2$, resulting in at most $r+1$ edges, or $r+1$ iterations.

Since the cop always takes an edge of the shortest path, the zombie is always able to copy the cop's moves in at most $r+1$ iterations. □

Consider the input. Our input consists of a simple polygon P whose total vertices n are not always linear in the reflex vertices r.

Now suppose we have a transformation from P to P', which keeps the position of the reflex vertices of P, and transforms the input such that the total vertices of P' are linear in the reflex vertices of P, or $O(r)$. Furthermore, the transformation maintains $P \subseteq P'$, and maintains that P' is a simple polygon. Informally, this transformation replaces a large convex chain in P with a smaller convex chain containing the larger chain. Aichholzer et al. prove that this transformation is indeed possible; they prove that if given two points in P, an algorithm that computes the geodesic path in P has the same geodesic path computed with the same algorithm in P' [2]. The transformation from P to P' takes $O(n)$ in the number of vertices of P. Thus any geodesic path algorithm using input P can transform its input into P' taking $O(n)$ time while still calculating the geodesic path in P. This is immediately applicable when computing a geodesic path from the zombie to the survivor at the beginning of an iteration.

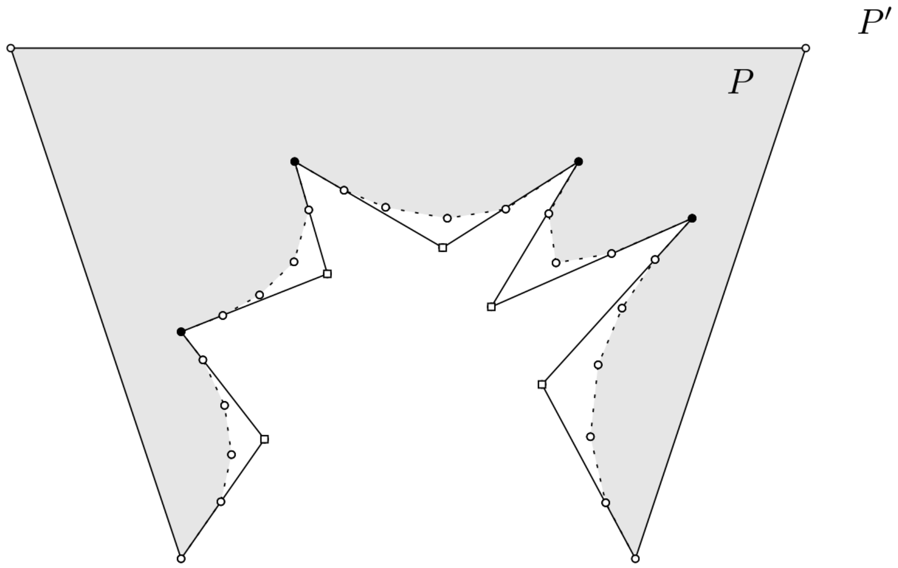


Figure 6: Transformation of P to P' where P is the shaded region.

Theorem 2. *The time to compute the zombie's position for all rounds takes* $O(n + r \log r)$ *time.*

Proof. First, the survivor and the zombie choose their points in P. For simplicity, the zombie chooses a vertex of P taking $O(1)$ time; we assume vertices are stored in a list. We transform the input from P into P' taking $O(n)$ time. At the beginning of the zombie's turn, we compute the geodesic path from the zombie's position to the survivor's position and take the first edge of the geodesic path. Guibas et al. have shown that given a triangulation of P', a Euclidean/geodesic path can be calculated in time logarithmic in the number of vertices of P' or $O(\log r)$ [15]. Note that it is necessary to preprocess P' to achieve a query time of $O(\log r)$; preprocessing takes time linear in the number of vertices of P', or $O(r)$. Furthermore, Chazelle has proved that triangulating a simple polygon P' with no holes can be done in time linear in the number of vertices of P' [9], also $O(r)$. Since the number of iterations to catch the survivor takes no longer than $r+1$ iterations, the worst case running time of calculating the zombie's moves takes $O(n+r+r+(r+1)\log r) = O(n + r\log r)$ time. □

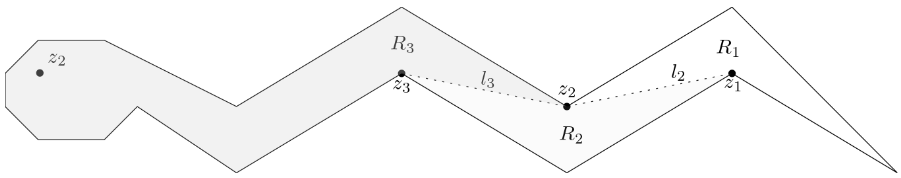


Figure 7: An example of the *active region* at round 3

5 MINIMUM LINK PATHS IN SIMPLE POLYGONS

In this section, we show a variety of strategies when the zombie follows a minimum link path in P and show how these strategies fail. One of the strategies attempts to imitate the work done in Section 4. Later we focus on the family of spiral polygons P_s, where a spiral polygon is a simple polygon whose vertices seen travelling counter-clockwise along the boundary can be decomposed into a sequence $r_1, r_2, \ldots, r_k$ of reflex vertices followed by a sequence $c_1, c_2, \ldots, c_j$, of convex vertices, where $n = k + j$. We show that a one zombie and one survivor game results in a zombie winning strategy when following the minimum link path in P_s.

5.1 GENERAL STRATEGIES IN SIMPLE POLYGONS

As seen in Chapter 4, if the zombie moves solely on the reflex vertices of P, the zombie will capture the survivor in $r + 1$ steps. However, the minimum link path in P differs from the geodesic path as the inner vertices may not all be reflex vertices of P, but may be points in the interior of P.

Let us consider the following approach. At the beginning of a zombie's turn, we look at the survivor window with base vertex b. All minimum link paths from the zombie to the survivor in P must cross this window as it separates the survivor from the zombie. We find an edge of a minimum link path whose endpoint o lies on the survivor window, and its other endpoint is the zombie's current position. Endpoint o is the closest endpoint to b than all other endpoints on the survivor window. On its next move, the zombie moves to o.

Strategy Strategy-One(P)
Input. A simple polygon P.

The zombie chooses a point z_1 in P;
Let s_1 be any point in P as the starting position of the survivor;
Let *round* = 1;
for *round* i **do**
  if z_i *is visible to* s_i **then**
    $z_{i+1} = s_i$;
    Terminate;
  else
    The zombie chooses z_{i+1} to be endpoint o, where (z_{i+1}, o) is an edge of a minimum link path where o is the closest endpoint to the base of the survivor window.
  end
  Let $s_{i+1} \in VP(s_i)$ be the next position of the survivor in P;
  Let *round*= *round* $+1$;
end

Strategy 1: Strategy-One

However, there is a counter-example to Strategy 1 seen in Figure 8. The zombie starts in position z_i, and the survivor starts in position s_i. The zombie moves to s_{i-1} since it is the only point that will decrease link distance. The survivor then moves to s_{i+1}. The same thing happens for rounds $i+1$, $i+2$, and $i+3$. Notice that the bottom half of the polygon is mirrored both horizontally and vertically to the top half of the polygon. Thus s_{i+5} would play in the mirror of s_{i-1}, and z_{i+6} would be forced to play in the mirror of z_i Thus we are back at our initial positions and have shown an infinite loop.

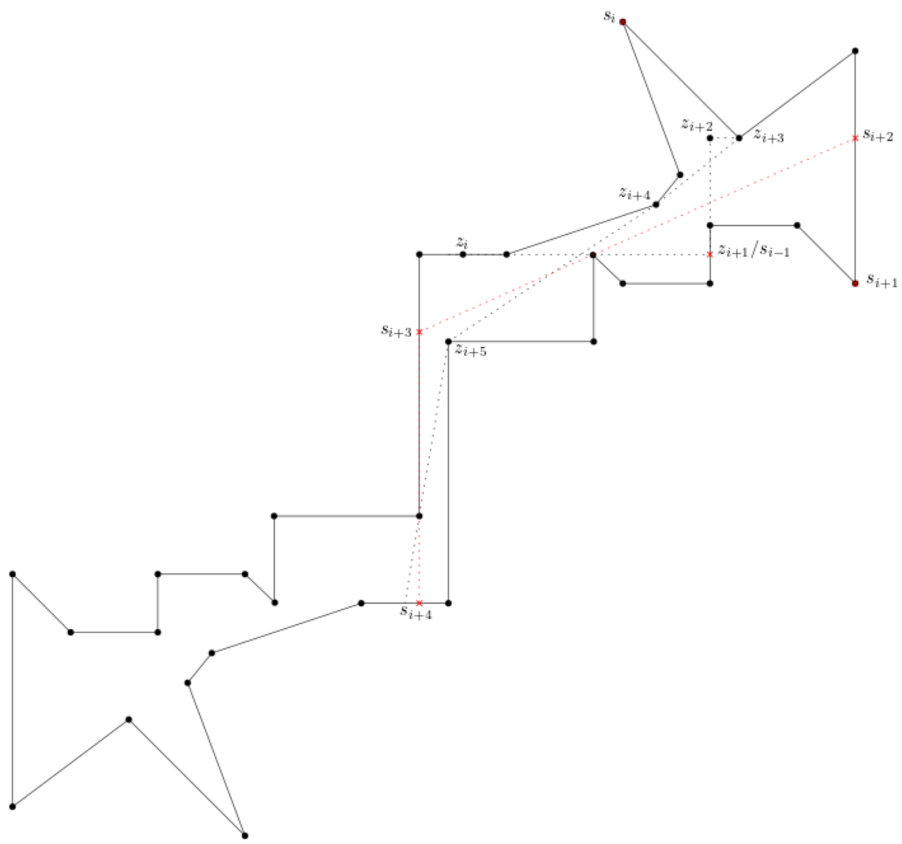


Figure 8: Counter Example to Strategy 1

As discussed prior, Strategy 1 attempts to approximate Lubiw et al.'s cop winning strategy which plays on the geodesic path, not the minimum link path [17]. As a result of the strategy, the zombie's current position borders the survivor's visibility polygon. Strategy 2 instead finds the opposite side of the visibility polygon intersecting the window on which the zombie last travelled. More formally, the previous move of the zombie was from z_{i-1} to its current position z_i. On the zombie's turn, if s_i is visible to z_i, then the survivor is caught. Else, if the segment $z_{i-1}z_i$ is weakly visible to s_i, then the next zombie move z_{i+1} is the last visible point to s_i traveling from z_i to z_{i-1}. If both propositions are false, we select endpoint o as seen in Strategy 1. Nevertheless, this strategy fails with the same setup seen in Figure 9. The zombie starts at position z_i, and the survivor starts at position s_i.

Strategy Strategy-Two(P)
Input. A simple polygon P.

The zombie chooses a point z_1 in P;
Let s_1 be any point in P as the starting position of the survivor;
Let *round* = 1;
for *round* i **do**
  if z_i *is visible to* s_i **then**
    $z_{i+1} = s_i$;
    Terminate;
  else
    if $z_i z_{i-1}$ *is weakly visible to* s_i **then**
      z_{i+1} is the the last visible point to s_i traveling from z_i to z_{i-1};
    else
      The zombie chooses z_{i+1} to be endpoint o, where (z_{i+1}, o) is an edge of a minimum link path where o is the closest endpoint to the base of the survivor window.
    end
  end
  Let $s_{i+1} \in VP(s_i)$ be the next position of the survivor in P;
  Let *round*= *round* +1;
end

Strategy 2: Strategy-Two

Both strategies get caught in the same corner of the polygon, this is seen on the survivor move from s_{i+2} to s_{i+3} in Figure 9. Furthermore, depending on the configuration the survivor evades capture with a left or a right turn depending on the placement of vertices in regions R_1 and R_2 as seen in Figure 10. By limiting our problem to only orthogonal polygons, we demonstrate a similar setup in Figure 11, where the zombie starts at z_i, and the survivor starts at s_i.

The previous examples show the zombie and survivor looping in two 'branches' of a polygon. However, there is a setup where more than two 'branches' in P exist. We show a polygon with four 'branches' that apply to both strategies seen in Figure 12.

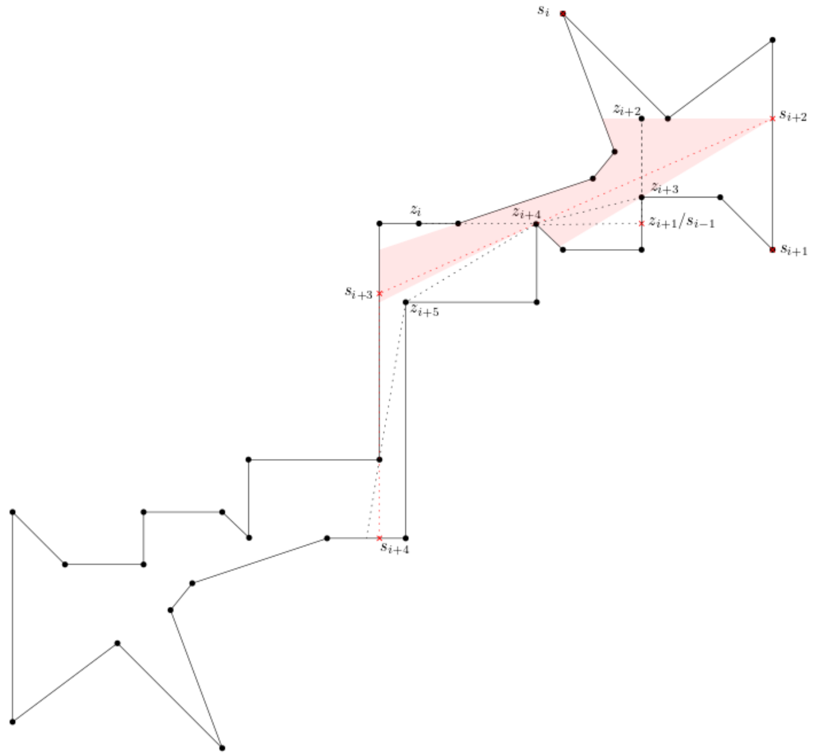
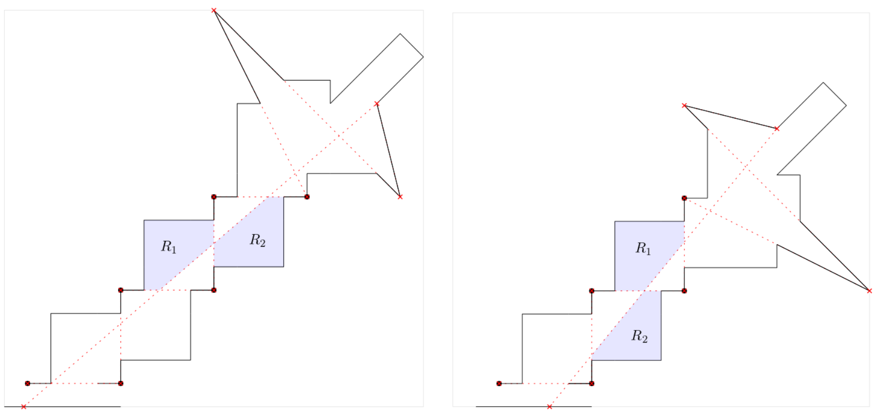


Figure 9: Counter for Strategy 2. See that z_{i+3} lies on edge of visibility polygon of s_{i+2}.



(a) Escaping through the left reflex vertex

(b) Escaping through the right reflex vertex.

Figure 10: Two examples illustrating an evasion with a left turn and a right turn.

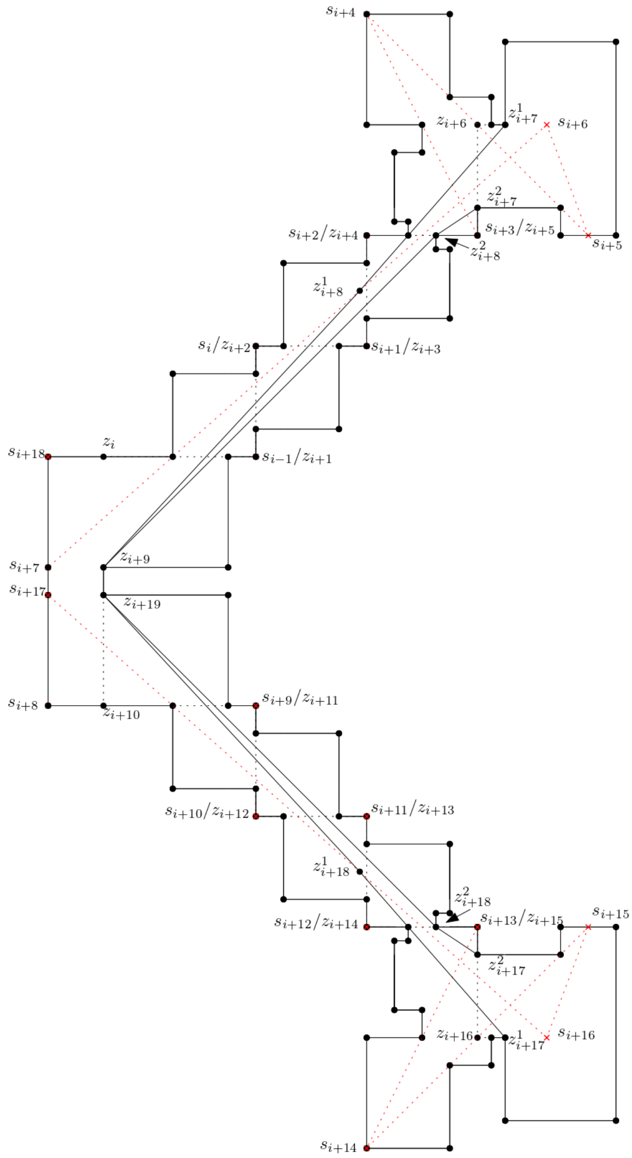


Figure 11: An orthogonal Zombie Loss on Strategies 1 and 2, where z^1 is Strategy 1, and z^2 is Strategy 2

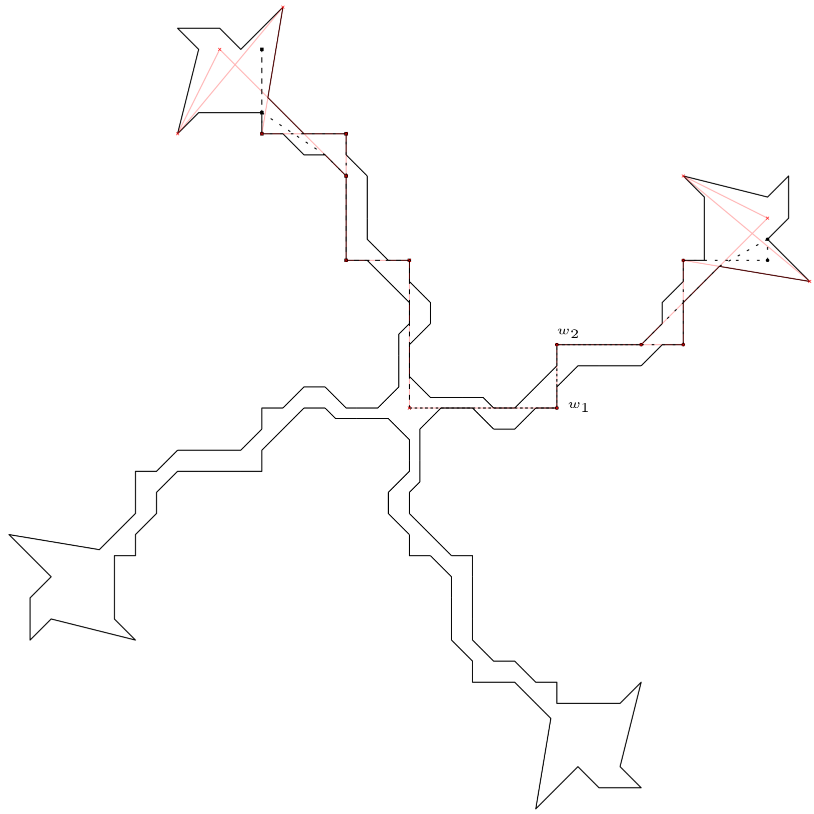


Figure 12: A multiple branch example where Strategy 1 is exhibited on the right branch, and Strategy 2 is exhibited on the upper branch. The survivor waits at w_2 until the zombie reaches w_1.

5.2 PLAYING ON A SPIRAL

Recall that a spiral polygon denoted as P_s is a simple polygon whose vertices seen when traveling counterclockwise along the boundary can be decomposed into a sequence $r_1, r_2, ..., r_k$ of reflex vertices followed by a sequence $c_1, c_2, ..., c_j$ of convex vertices, where $n = k + j$.

We outline a strategy for one zombie and one survivor, which results in a zombie-win when playing on the minimum link path in P_s. In this strategy the zombie at the beginning of its turn checks to see if the survivor is in its visibility polygon. If the survivor is in the visibility polygon, then the zombie can capture the survivor on its turn. If not, then the zombie chooses the non-base endpoint on the survivor window. This repeats until the survivor is caught.

Strategy Spiral-Strategy(P_s)
Input. A spiral polygon P_s with a sequence $r_1, r_2, ..., r_k$ of reflex vertices, and a sequence of $c_1, c_2, ..., c_j$ of convex vertices.

```
The zombie chooses z_1 to be c_1;
Let s_1 be any point in P_s as the starting position of the
   survivor ;
Let round = 1;
for round i do
    if z_i is visible to s_i then
        z_{i+1} = s_i;
        Terminate;
    else
        Let d be the link distance from z_i to s_i;
        The zombie chooses z_{i+1} to be t_{i+1}, the non-base
           endpoint of the survivor window;
    end
    Let s_{i+1} ∈ VP(s_i) be the next position of the survivor in the
       polygon;
    Let round= round +1;
end
```

Strategy 3: Spiral-Strategy, where z_i and s_i are the locations of the zombie and survivor at round i.

We show that if a single zombie follows the Spiral-Strategy then it wins against a single survivor in P_s. First we show that the point z_{i+1} chosen by the zombie always exists, and has link distance $d - 1$ to s_i, where initially z_i had link distance d to s_i. We refer to Figure 13 in reference to the following lemma.

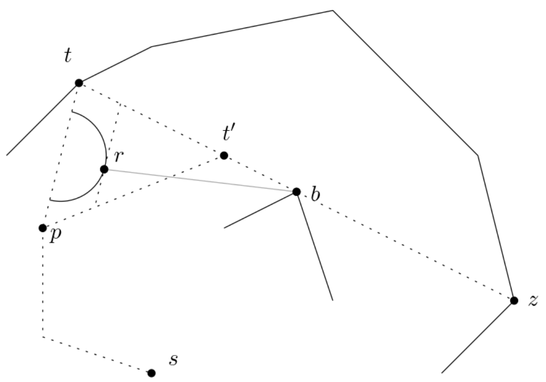


Figure 13: Lemma 4

Lemma 4. *Let* z *and* s *be the points of the zombie and survivor in a spiral polygon* P_s *with link distance* $d > 1$*. Then the non-base endpoint* t *of the survivor window of* z *has link distance* $d-1$ *from* s*.*

Proof. First we prove that there exists a point t' on the survivor window of z which separates z from s, where s is at distance $d-1$ from t'. When $d > 1$, $s \notin VP(z)$, and thus s lies in a pocket S formed by a window of z. Thus any minimum link path from z to s must intersect the window separating S from $P \backslash S$. Consider a minimum link path from z to s. When travelling on this path from z to s: we let t' be the last point of intersection along this minimum link path and the survivor window. Because t' does not lie on the (open) first link of the minimum link path, the link distance from t' to s is at most $d-1$. The link distance from t' to s is at least $d-1$ since $VP(z)$ contains it. Therefore t' must have distance $d-1$ to s.

Let p be the link path vertex after t' on the minimum link path from t' to s. To prove t has link distance $d-1$ from s, it suffices to prove that p is visible to t. We assume for the sake of contradiction that p is not visible to t. Furthermore, we assume that p is left of the directed segment $\overrightarrow{zt}$.

Consider a set S as the vertices of P_s lying inside triangle $\Delta tt'p$. The exterior angles of the convex hull vertices of S are interior angles of P_s since both tt' and $t'p$ lie entirely in the interior of P_s. Each exterior angle on the convex hull is greater than π; therefore, the hull vertices are reflex vertices of P_s. Let r be any hull vertex maximizing the orthogonal distance to segment tp. Then the line going through r parallel to tp does not intersect the hull.

Let x begin at t' and move x along segment $t'b$. Initially segment $rx = rt'$ lies entirely inside P_s. We stop moving x when the segment rx does not lie entirely inside P_s, or we reach vertex b. If we reach b

then rb is a diagonal of P_s. If not, then there must exist a reflex vertex b' on the segment rx that is not r; we define this diagonal as rb'.

Consider the polygon split by the diagonal rb or rb'. Each sub-polygon must contain a convex vertex. Furthermore, each sub-polygon has greater than three vertices since it is split by a diagonal, and there exist at least three vertices that are not colinear, else it would not be a simple polygon. Therefore if we look at either of the convex hulls of each sub-polygon. There must exist a hull vertex that is not an end-point of either diagonal; a convex vertex. Therefore we have a contradiction since walking along the boundary results in at least two contiguous reflex sequences. Therefore we have a contradiction to the definition of a spiral polygon. □

When proving the correctness of the Spiral-Strategy, we refer to the *zombie area* as a region of the polygon where the survivor cannot enter without losing. More formally, at iteration i we define the zombie area to be $R_i = R_{i-1} \cup VP(z_i)$. Where $R_0 = \{\}$ as the zombie has not chosen a vertex, and $R_1 = VP(z_1) = VP(z_1) \cup \{\}$. We show that the survivor may never enter R_i in any current or subsequent iteration; otherwise, it will result in the zombie winning. We refer to Figures 14 and 15 in reference to the following theorem.

Theorem 3. *Given a spiral polygon P_s on n vertices. When a zombie follows the Spiral-Strategy(P_s), the zombie wins in at most n iterations.*

Proof. We will show that if the survivor enters the zombie area on their turn, the next move of the zombie results in a capture. More formally we will prove that $R_{i-1} \subset R_i$, R_i has at least one more vertex than R_{i-1}, and $s_i \notin R_i$ otherwise z_{i+1} captures the survivor at s_i. We show that this is true for all $1 \leqslant i \leqslant n$.

Base Case. Consider when the zombie chooses its position at $z_1 = c_1$. Then $R_0 = \{\} \subset VP(z_1) = R_1$ differs by at least one vertex. The survivor s_1 cannot be inside $VP(z_1)$ else it will be captured.

Inductive Hypothesis. We assume that for values $i = 1, ..., m-1$ the inductive hypothesis holds : that is $R_{i-1} \subset R_i$, R_i has at least one more vertex than R_{i-1}, and $s_i \notin R_i$ otherwise z_{i+1} captures the survivor at s_i.

Inductive Step. We assume that the inductive hypothesis holds and prove for some value $i = m \leqslant n$. If s_m lies in R_{m-1} it must have crossed the chord $z_m b_m$ since $s_{m-1} \notin R_{m-1}$ by the inductive hypothesis. Thus $s_{m-1}s_m$ intersects the chord $b_m z_m$ at point x. As a result, the $VP(x) \cap R_{m-1}$ is a convex polygon by the intersection of R_{m-1}, and the restriction by vertex b_m. The zombie z_m lies inside $VP(x) \cap R_{m-1}$, and catches s_m on its turn. If s_m lies inside $R_m \backslash R_{m-1}$, then z_{m+1} captures s_m as $R_m \backslash R_{m-1} \subseteq VP(z_m)$.

We know that $R_{m-1} \subset R_m$ by two cases : If $R_m = P_s$, then $c_j \notin R_{m-1}$ thus $R_{m-1} \subset R_m$; or b_{m+1} exists and $b_{m+1} \notin R_{m-1}$ thus

$R_{m-1} \subset R_m$. Since $R_{m-1} \subset R_m$ differs by a vertex, then the total number of rounds is bounded by $m \leqslant n$.

□

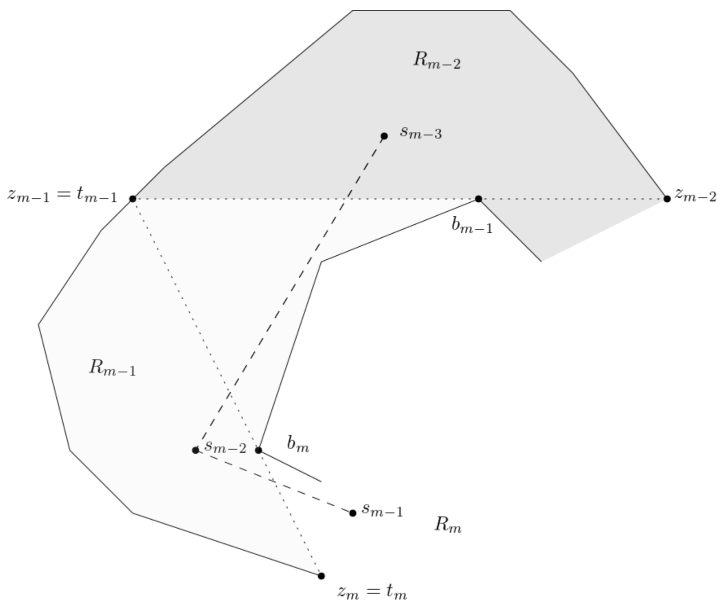


Figure 14: Theorem 3

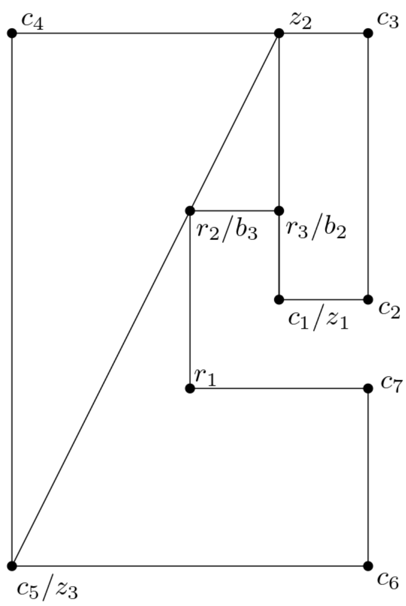


Figure 15: Case $R_m = P_s$ when $m = 3$. Case $R_m \neq P_s$ when $m = 2$.

Before we analyze the runtime, we define *winding order* and some properties associated with it. Let Δdef be a triangle with a sequence d, e, f and center c, we define a *clockwise winding order* as such. Let x be a point traveling from d to e, then segment cx rotates clockwise as x

travels to e. If cx rotates clockwise for all segments de, ef, and fd, then Δdef has a clockwise winding order. If we have a counterclockwise winding order, then cx rotates counterclockwise for all segments. A *degenerate* triangle is a triangle whose vertices are colinear. If a triangle is not degenerate and does not have a clockwise winding order, then the winding order is counterclockwise; this case is symmetric. Now suppose we have a window where the open segment ba is weakly visible to p, then the following lemma is true.

Lemma 5. *Let* b,c, *and* a *be vertices of* P, *where* c *and* a *are adjacent to* b. *If* b *is the base of a window formed from a point* $p \in P$, *then triangles* Δpab *and* Δpbc *differ in winding order, given that triangles* Δpab *and* Δpbc *are both not degenerate.*

Proof. Let a' be the last visible point when traveling from b to a. We claim that if the winding order of $\Delta pa'b$ is clockwise, then Δpbc is counterclockwise. Furthermore, observe that if $\Delta pa'b$ is clockwise, then Δpab is also clockwise.

By our assumption, $\Delta pa'b$ has a clockwise winding order. Without loss of generality, we rotate our coordinate space such that the line through bp is horizontal, and a is below the line through bp as seen in Figure 16. Since b is reflex and the base of p, c must also be below the line bp. As a result, the center of Δpbc is also below; thus, the rotation from p to b must be counterclockwise from Δpbc's center. Since Δpbc is not degenerate, its winding order must be counterclockwise. $\square$

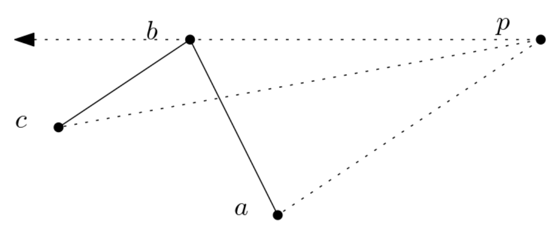


Figure 16: Winding order of Δpab is clockwise, and Winding order of Δpbc is counterclockwise.

Likewise, if the winding order does not differ, then b is not a base of the window. Next, we consider a similar situation in a spiral polygon.

Lemma 6. *Let* b *and* t *be a window in* P_s*, where* b *is the base and* t *is the other boundary endpoint. Let* t' *be the first intersection of the polygon boundary with ray* $\overrightarrow{tb}$*. If* a *and* d *are adjacent vertices of* t'*, then* Δtbd *and* Δtba *differ in winding order.*

Proof. We consider two cases for t' : a and d lie on the same side of the line through bt; and a and d lie on different sides of the line trough bt. Without loss of generality, we rotate the coordinate system such that the line through bt is horizontal. We refer to Figure 17 as an example of the transformation.

If a and d are on the same side, then t' must be a vertex of P_s. Since the window bt' lies inside P_s and the angle $\angle at'd > \frac{\pi}{2}$, then t' must be a reflex vertex. Since $t'b$ is a diagonal with two reflex vertices, we have a contradiction of the definition of a spiral polygon as seen in Lemma 4.

Otherwise, a and d are on different sides of the line through bt. Suppose a is above the line, and d is below. Then the center of Δtba is above the line, and Δtba has a counterclockwise winding order. The center of Δtbd is below the line, and Δtbd has a clockwise winding order. Thus the winding orders differ. The cases are symmetric when d is above the line, and a is below. □

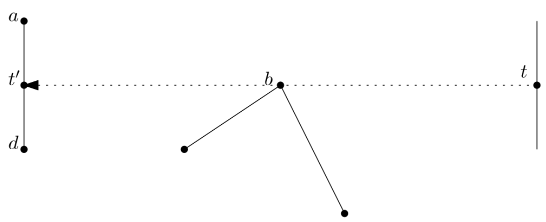


Figure 17: Setup of Lemma 6

Likewise, if the triangles do not differ in winding order, then a and d are not adjacent to t'. It is noted that if a and d are on the same side of the line through bt, then the winding order does not change in Lemma 5.

Calculating the winding order of a triangle can be checked in constant time by using the shoelace formula [8]. If we have a clockwise winding order, the triangle's area is negative, whereas a counterclockwise winding order has a positive area.

We now analyze the time complexity of an implementation of the Spiral-Strategy. The input is a spiral polygon P_s with a reflex sequence $r_1, ..., r_k$ and a convex sequence $c_1, ..., c_j$. The output is when the zombie's position lies on the survivor's position. We place the

$$\frac{1}{2}\cdot\begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix} = \frac{1}{2}\cdot\begin{vmatrix}1 & 0 & 0\\ x_1 & x_2-x_1 & x_3-x_1\\ y_1 & y_2-y_1 & y_3-y_1\end{vmatrix} = \frac{1}{2}\cdot\begin{vmatrix}x_2-x_1 & x_3-x_1\\ y_2-y_1 & y_3-y_1\end{vmatrix}$$

Figure 18: The signed area of a triangle given by the shoelace formula where point $p_i = (x_i, y_i)$

zombie z_1 onto vertex c_1, and the survivor $s_1 \in P_S$. When the zombie is positioned at z_i, we define b_{i+1} to be the base of the survivor window, and t_{i+1} to be the other endpoint of the survivor window. Therefore if the zombie is positioned at z_1 and t_2 exists, then $b_2 = r_k$. We argue that the total time complexity to compute $z_1, ..., z_m$ is $O(n)$, where m is the final round. Notice that the window $b_{i+1}t_{i+1}$ is an edge of $VP(z_i)$. If the window does not exist, then there was no survivor window, in which case the survivor is in $VP(z_i)$. If the window does exist, then the cost of calculating b_{i+1} and t_{i+1} is the same as calculating $VP(z_i)$.

Calculating b_{i+1} and t_{i+1} can be done when a sequence of triangles differ in winding order. For example, if $r_f, ..., r_u$ is the sequence of reflex vertices inclusively between b_i and b_{i+1}, and r_l is a vertex in the range of $f \geqslant l \geqslant u-1$. Then if the winding order of triangle $\Delta z_i r_l r_{l-1}$ differs from the winding order of triangle $\Delta z_i r_{l-1} r_{l-2}$, then by Lemma 5, r_{l-1} is a base of z_i. Furthermore, since r_f is b_i, then r_{l-1} must be b_{i+1}.

Similarly, if $c_b, ..., c_d$ is the sequence of convex vertices inclusively between z_i and t_{i+1}, and c_a is a vertex in the range of $b \leqslant a \leqslant d+1$. If the winding order of triangle $\Delta z_i b_{i+1} c_a$ differs from the winding order of triangle $\Delta z_i b_{i+1} c_{a+1}$, then by Lemma 6 t_{i+1} lies on segment $c_a c_{a+1}$, or t_{i+1} is a vertex of a degenerate triangle.

Notice if the winding order does not differ and we reach r_1, then all reflex vertices from $b_i, ..., r_1$ and convex vertices starting from $c_j, c_{j-1}, ...$ to z_i are visible to z_i since there is no window. Furthermore, it can be seen that computing b_{i+1}, t_{i+1}, and $VP(z_i)$ is essentially a progressive walk on the convex and reflex sequences while keeping track of the last window's vertices.

Let us walk in the counterclockwise direction starting from z_i to t_{i+1}. Along the walk, we cross $t_{i+1}b_{i+1}$ and $b_i t_{i-1}$. Eventually, we return to z_i. A triangulation of $VP(z_i)$ can be created with z_i as the common vertex for all triangles. Subsequently, the chords $t_{i+1}b_{i+1}$ and $b_i t_{i-1}$ are windows of z_i. In the case where a window does not exist we travel on the boundary visible to z_i, instead of the non-existent window. Thus, we count the number of triangles that we check to calculate the total time taken.

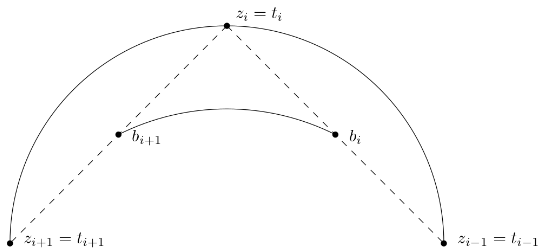


Figure 19: Subdivisions of $V(z_i)$, including the forward, center, and backward polygons.

Theorem 4. *The time to compute the zombie's position for all rounds for Spiral-Strategy takes* $O(n)$ *time, where* n *is the number of vertices in* P_s.

Proof. To analyze time complexity, consider a counting problem. If we split our visibility polygon into three disjoint polygons : the forward polygon being the chord t_it_{i+1}, and the path on the boundary from t_i to t_{i+1} not including b_i; the center polygon being the chords $b_{i+1}t_i$, t_ib_i, and the path on the boundary from b_i to b_{i+1} no including t_i; and the backwards polygon being the chord t_it_{i-1}, and the path on the boundary from t_i to t_{i-1} not including b_{i+1}. We see that for every forward polygon, there is a z_{i+1}, and z_i associated with it. The equivalent can be said for every backward polygon. Furthermore, only z_i is associated with every center polygon. Thus, in the worst case, we count each polygon twice in our progressive walk.

Suppose that we enumerate these disjoint polygons from $1, ..., \phi, ..., \psi$ where the ϕth polygon has n_ϕ triangles formed from diagonals of P_s, and at most two additional triangles from the chords. Since we check each polygon twice, then on the ϕth round we check $2 \cdot (n_\phi + 2)$ triangles. Therefore in total we check $2(n_1 + 2) + 2(n_2 + 2) + ... + 2(n_\psi + 2) = 2(n_1 + n_2 + ... + n_\psi) + 4\psi$ triangles. If there are n vertices in P_s, then an upper-bound to the number of triangles with diagonals in P_s is $n - 2$. Furthermore, we create at most two new chords every zombie iteration, thus $\psi \leqslant 2 \cdot n$. Which means, $2(n_1 + n_2 + ... + n_\psi) + 4\psi \leqslant 2(n-2) + 8n \leqslant 10n = O(n)$.

As seen previously, the calculation of t_{i+1} and b_{i+1} can be done with a progressive walk. In the worst case, we walk on a vertex twice; thus, our walk takes $2n = O(n)$ time. Therefore, the total cost of calculating all $VP(z_i)$, all b_{i+1}, and all t_{i+1} takes linear time in the number of verities of P_s in the worst case.

□

6 CONCLUSION

In this thesis, we explore two variations of zombies and survivor in a simple polygon: when a zombie follows a geodesic path in P; and when a zombie follows a minimum link path in P.

Through previous work, Lubiw et al. have shown that when the cop takes the geodesic path in a one cop and one robber game in P, the strategy results in a cop win; this also indirectly proves a cop winning strategy when the cop takes a link path in P [17]. We show that in a game of one zombie and one survivor, the zombie also has a winning strategy by following the geodesic path in P to the survivor. Furthermore, the number of iterations to capture the zombie takes $O(r)$ rounds, and the time to calculate all the zombie positions takes $O(n + r \log r)$. We note that n is the number of vertices in P, and r is the number of reflex vertices in P. These bounds can be directly applied to the game of cops and robber in P. In addition, it improves the result of Lubiw et al. by decreasing the number of rounds from $O(n)$ to $O(r)$.

We also prove that when the zombie follows the minimum link path in any spiral polygon, this also results in a zombie winning strategy for one zombie and one survivor. The maximum number of rounds required is $O(n)$. In addition, we prove that to compute all of the zombie positions, it also takes $O(n)$ time.

We explore other strategies when the zombie is restricted to the minimum link path in any simple polygon, and we show some interesting counter-examples to our proposed strategies. Our hope is that through these examples, others may garner some intuition of the complexity of link paths regarding the game of zombies and survivor in P.

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