FASTER MIP SOLUTIONS VIA NEW NODE SELECTION RULES

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Abstract

When a branch and bound method is used to solve a linear mixed integer program (MIP), the order in which the nodes of the branch and bound tree are explored significantly affects how quickly the MIP is solved. When choosing a node to solve, it is desirable to either choose an ancestor of the MIP optimal node or to choose a node that is not superfluous to solving a MIP to optimality.

In this thesis, three new backtracking node selection and two backtrack triggering methods are proposed. A modification to the best-projection method is proposed that removes the requirement of finding an incumbent solution in order for this method to be used. This first proposed method is used as both a backtracking node selection method and as a method of backtrack triggering. A second proposed backtracking node selection method balances the goals of MIP feasibility and MIP optimality by using probability distributions to model these measures based on all nodes solved thus far. A third proposed backtracking node selection method determines when the cost of using a backtracking node selection method outweighs its benefit, in which case it is abandoned in favor of simple backtracking. Finally, a proposed backtrack triggering method estimates the MIP optimal objective value using the objective values of nodes at the estimated depth of a MIP optimal node.

Empirical results show that these proposed methods outperform the state of the art by a large margin with respect to speeding up the solution process of many MIP instances.

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List of Abbreviations

ALLT Trigger backtracking after every node solution.....	78
ANST Active Node Search Threshold.....	8
BDRT0 Branching variable selection method by Driebeek [14] and Tomlin [38]	73
BEES Best-estimate node selection	77
BESF Best-first node selection.....	77
BFBE BEES interleaved with BESF.....	77
BFIRST Branch on the first candidate variable.....	73
BLAST Branch on the last candidate variable	73
BMFRAC Branch on the most fractional candidate variable.....	73
BREF Breadth-first node selection.....	77
CUTS root node cuts.....	75
DBPA Default best-projection aspiration	78
DEBP Default best-projection node selection	77
DEPF Depth-first node selection.....	77
DIST Distribution node selection	78

LEXA Linear feasibility depth extrapolation aspiration	78
LP Linear Program	1
MIP Mixed Integer Program	1
MOBP Modified best-projection node selection	78
MPAS Modified best-projection aspiration	78
NOAN Do not use ANST	78
NOCUTS Do not use root node cuts	75
NONA Non-aspiration backtrack triggering	78
PCAS Pseudo-cost (best-estimate) aspiration	78
PERA Use optimal objective value to trigger backtracking	31

List of Symbols

z	The objective value of an LP.....	3
z_{inc}	The objective value of an incumbent solution.....	3
z^*	The MIP optimal objective value.....	4
x_j	A MIP variable.....	10
L_u	The list of unexplored nodes.....	13
s_i	The sum of infeasibilities.....	18
f_j	The fractional part of a variable.....	18
$\tilde{z}^*$	An aspiration value.....	23
c	The number of candidate variables.....	34
$z^{min}(c)$	The minimum objective over all nodes with c candidate variables....	34
c_{min}	The minimum number of candidate variables over all nodes.....	37
σ_z	The standard deviation of z	37
σ_c	The standard deviation of c	37
d_i	Depth of node i	48
$\tilde{d}^*$	An estimate of depth of an optimal node.....	53

r_{MI}	(time to solve a MIP MI) / (best time to solve MI)	67
<i>FAIL</i>	The number of unsolved MIP instances	68
<i>TOTTM</i>	The total time to solve a set of MIP instances	68
<i>MRATE</i>	The mean ratio to best time to solve a set of MIP instances	69
G_o	The provable optimality gap	70
<i>AVGRANK</i>	The average rank	71
<i>NFIRSTS</i>	The number of times that a method is ranked first	71
<i>NINC</i>	The number of times that no incumbent solution is found	71
<i>Cols</i>	The number of variables	72
<i>Real</i>	The number of real variables	72
<i>Int</i>	The number of non-binary integer variables	72
<i>Bin</i>	The number of binary variables	72
<i>Rows</i>	The number of constraints	72
<i>Ineq</i>	The number of inequality constraints	72
<i>Eq</i>	The number of equality constraints	72
<i>NZ</i>	The number of non-zero constraint and objective function coefficients ...	72
<i>TR</i>	Time rank	81
<i>RR</i>	Ratio rank	81
<i>SR</i>	Sum rank	81
<i>R</i>	Rank	81

Chapter 1

Introduction

Linear Mixed Integer Programs (MIPs) are used to model many problems which involve optimization such as spacecraft trajectory planning [36], hardware/software codesign [31], and scheduling [23].

A MIP, described in Section 2.1, is a *Linear Program (LP)* in which one or more of the variables are constrained to be integer valued. The most commonly used method of solving MIPs to proven optimality is called *branch and bound*, described in Section 2.2. A difficulty with branch and bound is that it can take an impractical amount of time to solve many MIPs even when using the most sophisticated computers.

The chosen *node selection* method is one aspect of branch and bound that greatly affects how quickly it can solve MIPs by determining the manner in which a branch and bound tree is explored.

To demonstrate how node selection can affect the performance of the branch and bound method, the following *MIP instance MIPEX* is solved using the branch and bound method with different node selection methods and the resulting branch and bound trees are shown.

$$\begin{array}{ll}
 & \text{MIPEX} \\
 \hline
 \min & z = x + 3y \\
 \text{st.} & 4x + 5y \leq 41 \\
 & 13x - 8y \geq 12 \\
 & 7x + 33y \geq 96 \\
 & 0 \leq x \leq 10 \\
 & 0 \leq y \leq 10 \\
 & x, y \in \mathbb{Z}
 \end{array}$$

A solution of *MIPEX* is a point (x, y) that: yields the minimum value of z , satisfies all of the linear constraints, and has integer values for both x and y . The branch and bound method starts by solving *MIPEX* without the integer constraints on x and y , called the *LP-relaxation*. *MIPEX* without the integer constraints is the *root node* of the branch and bound tree. The solution of the root node is

$$\begin{array}{l}
 z = 9.6 \\
 x = 2.4 \\
 y = 2.4.
 \end{array}$$

This solution is *LP feasible* since all of the linear constraints are satisfied. The value of y in the solution to the root node is fractional so two child nodes are created by changing the bounds on y so that its value is forced away from this fractional value. In one child node, the upper bound on the value of y is changed to $0 \leq y \leq 2$, called the *down branch*. In the other child node, the lower bound on the value of y is changed to $3 \leq y \leq 10$, called the *up branch*. In this case, y is the *branching variable*. It should be noted that x can be chosen as the branching variable instead of y and that this will result in a different branch and bound tree. At this point a node selection method is employed to choose which of these child nodes will be solved next. If the solution of a node is LP feasible and has integer values for both x and y then this node is *MIP feasible*. The process of solving a node, choosing

a branching variable, creating child nodes, and selecting the next unsolved node to solve is repeated until (i) a MIP feasible node is found that has an *objective value* z that is less than or equal to the value of z for all of the nodes that have at least one unsolved child node, or (ii) there are no unsolved nodes left in the tree. The MIP feasible node with the smallest objective value over all MIP feasible nodes is the MIP optimum. The optimal solution of *MIPEX* is

$$z = 11$$

$$x = 5$$

$$y = 2.$$

There are many possible node selection methods that can be used to solve *MIPEX*, each of which may result in a different branch and bound tree. Two possible node selection methods that can be used to solve *MIPEX* are described below. These methods are chosen arbitrarily to demonstrate how using different node selection methods can affect the branch and bound tree.

DF – BF – ASP = 11: *Depth-first* with *best-first backtracking* node selection and aspiration value of 11 chooses the next node to solve according to the following rules. If the most recently solved node is LP feasible but not MIP feasible, then choose a child of this node if $z \leq 11$ for this node. Backtracking occurs if the most recently solved node

- is not LP feasible, or
- is MIP feasible, or
- has an objective value that is greater than or equal to the smallest objective value z_{inc} over all MIP feasible solutions found so far, or
- has an LP-relaxation objective value that is greater than 11.

Best-first backtracking chooses a child of the node that has the smallest value of z .

DF – DF: Depth-first with depth-first backtracking node selection chooses the next node to solve according to the following rules. If the most recently solved node is LP feasible but not MIP feasible, then choose a child of this node. If the most recently solved node is a *leaf node*, meaning that it is either MIP feasible, not LP feasible, or has a value of $z \geq z_{inc}$, then backtrack by choosing a child of the most recently solved LP feasible node.

If an LP feasible node has two unsolved child nodes then choose the up branch. For example, the root node child that corresponds with the bounds on y being $3 \leq y \leq 10$ is solved before its sibling.

The aspiration value used in $DF - BF - ASP = 11$ is based on an *a priori* estimate of the MIP optimal objective value z^* to demonstrate how such knowledge can be used to avoid the exploration of *superfluous nodes*. Superfluous nodes are nodes for which $z > z^*$. It is not necessary to explore these nodes in order to prove that a MIP feasible solution is optimal. Knowledge of the value of z^* for a given MIP instance is usually not available until the MIP instance has been solved, so its value must be estimated in order for it to be used to avoid the exploration of superfluous nodes.

Figures 1.1 and 1.2 show the solved nodes in the branch and bound trees when using the $DF - BF - ASP = 11$ and the $DF - DF$ node selection methods respectively to solve *MIPEX*, along with the order in which each node is solved.

During the branch and bound process of solving *MIPEX* using $DF - BF - ASP = 11$, the second node that is solved has a value of $z > 11$ so backtracking is triggered. Best-first backtracking then chooses *node2* which is an ancestor of the MIP optimal node. Depth-first node selection then proceeds to choose *node3*, the remaining ancestor of the MIP optimal node, followed by *node4*, the MIP optimal node. At this point *node2* and *node3* both have an unsolved child with $z < z_{inc}$ that are each chosen in turn using best-first backtracking.

During the branch and bound process of solving *MIPEX* using $DF - DF$, the third node that is solved, *node2*, is MIP feasible so this method requires less effort to

find a MIP feasible solution than $DF - BF - ASP = 11$, which solves 5 nodes before finding a MIP feasible solution. Depth-first backtracking then chooses *node3* to be solved next. *Node3* is a leaf, so depth-first backtracking then chooses *node4* which is an ancestor of the MIP optimal node. Depth-first node selection then proceeds to choose *node5*, the remaining ancestor of the MIP optimal node, followed by *node6*, the MIP optimal node. At this point *node5* and *node4* both have an unsolved child which are each chosen in turn using depth-first backtracking.

In this example, solving *MIPEX* using $DF - BF - ASP = 11$ resulted in over 25% fewer solved nodes than when using $DF - DF$. Solving *MIPEX* using $DF - DF$, on the other hand, resulted in a MIP feasible solution being found in over 65% fewer solved nodes. Therefore, if the goal is to find a MIP feasible solution of *MIPEX* as quickly as possible, then $DF - DF$ should be used rather than $DF - BF - ASP = 11$. If the goal is to solve *MIPEX* as quickly as possible, then $DF - BF - ASP = 11$ should be used rather than $DF - DF$.

In order to quickly solve a MIP, node selection methods try to order the exploration of nodes in such a way that the MIP optimal node is quickly found by exploring the ancestors of the MIP optimal node before all of the other nodes. Whether a node is an ancestor of a MIP optimal node is generally unknown, therefore node selection methods employ heuristic rules in order to quickly find a MIP optimal solution. A commonly used framework for node selection is depth-first with backtracking. This framework uses depth-first node selection except when it is determined that backtracking is necessary and a different node selection method is employed. Two aspects of depth-first with backtracking are (i) the backtracking node selection and (ii) back-track triggering methods. Backtracking node selection methods rank all available nodes according to some criteria and choose the next node to solve accordingly. Back-track triggering methods classify the most recently solved node as either desirable or undesirable. If this node is undesirable then a backtracking node selection method is used to choose the next node to solve. Backtracking node selection is also used if the most recently solved node is a leaf.

Figure 1.1: Branch and bound tree of MIPEX when it is solved using $DF - BF - ASP = 11$.

Figure 1.2: Branch and bound tree of MIPEX when it is solved using $DF - DF$.

In this thesis, three backtracking node selection and two backtrack triggering methods are proposed.

A modification to the best-projection method is proposed in Section 5.1 that estimates the objective value of the best MIP feasible solution attainable from any node using nodes that are not MIP feasible, thus eliminating the requirement of finding an incumbent solution before computing an estimate. This method can be used for backtracking node selection and for backtrack triggering.

A backtracking node selection method is proposed in Section 5.2 that balances the goals of reaching MIP feasibility and MIP optimality by using probability distributions to model these measures based on data from all nodes solved so far. This method dynamically adjusts the weights of the feasibility and optimality goals so that neither can dominate the other for too long, and to favour choosing nodes with lower objective values and lower MIP infeasibility.

The active node search threshold (ANST) proposed in Section 5.4 determines when the cost of using a backtracking node selection method outweighs its benefit. If the cost of a method does outweigh its benefit, then a simpler node selection method is used instead.

A backtrack triggering method is proposed in Section 5.3 that estimates the value of z^* based on the objective values of nodes at the estimated depth of a MIP optimal node. The depth of a MIP optimal node is estimated using linear extrapolation of MIP infeasibility along each path of the branch and bound tree.

Empirical results of the experiments are presented in Chapter 7. These results show that the ANST, the modified best-projection method, and the distribution backtracking node selection method proposed in this work greatly speed up branch and bound for solving MIPs. The distribution node selection method outperforms all of the other node selection methods with respect to either finding good MIP feasible solutions or reducing the provable optimality gap for very difficult MIPs. The modified best-projection and the feasibility depth extrapolation backtrack triggering methods speed up the MIP solution process when used with either the modified best-projection

or the distribution backtracking node selection methods.

Chapter 2

A Review of Mixed Integer Programming

2.1 Definition of Mixed Integer Program

Linear Mixed Integer Programs all conform to the following generic formulation. The objective function is defined as

$$\text{minimize } z = \sum_{j \in I} o_j x_j + \sum_{i \in C} o_i x_i$$

The constraints are defined as

$$\sum_{j \in I} a_{kj} x_j + \sum_{i \in C} a_{ki} x_i \{ \leq, =, \geq \} v_k, \quad k = 1, \dots, k_{\max}$$

$$l_j \leq x_j \leq u_j \quad \forall j \in I \cup C$$

$$x_j \in \mathbb{Z} \quad \forall j \in I$$

$$x_i \in \mathbb{R} \quad \forall i \in C$$

z : the objective value

x_j : a variable

o_j : the objective value coefficient for x_j

a_{kj} : the constraint coefficient for x_j in constraint k

v_k : the bound on constraint k

l_j : the lower bound on the value of x_j

u_j : the upper bound on the value of x_j

I : If $j \in I$ then in order for a solution to be MIP feasible, the value of x_j is required to be an integer.

C : If $j \in C$ then the value of x_j can be any Real number.

Any maximization problem can be transformed into a minimization problem by multiplying the objective function by -1 .

2.2 The Branch and Bound Method

Many methods for solving MIPs employ a branch and bound methodology that involves relaxing the integer constraints to get a lower bound on the solution. This means solving the problem as if it is a purely continuous linear program (LP), commonly called the *initial LP-relaxation*. If all variables that are constrained to be integer valued in the MIP are integer valued in the solution to the initial LP-relaxation, then this solution is the MIP optimum. Otherwise, the integer constraints are incorporated into the LP formulation by recursively subdividing the feasible region and solving each subdivision until a MIP feasible solution is found. The LP-relaxations are commonly solved using the *simplex method* since the simplex solution information of any LP-relaxation can be used to speed up the solution of LP-relaxations with small differences in formulation.

The most commonly used method for subdividing the feasible region to solve MIPs proceeds by selecting a *candidate variable*, x_j , and then creating two subproblems

which differ from the initial LP-relaxation only in the bounds on this variable. A candidate variable of an LP-relaxation is a variable that is constrained to be integer valued (referred to as an *integer variable*) in the MIP but has a fractional value in the solution to the LP-relaxation. In one subproblem the lower bound of x_j is set to the nearest integer value that is greater than the LP-relaxation solution value of x_j . In the other subproblem the upper bound is set to the nearest integer that is less than the value of x_j . These are referred to as the *up* and *down branches* respectively. This is called *branching* on variable x_j . One of the subproblems is then chosen to be solved. If the subproblem solution is LP feasible but not MIP feasible then a candidate variable is chosen and branched on as before.

This process is repeated to create a binary tree where the *root node* is the initial LP-relaxation and each child node represents a subproblem created by branching on a chosen variable. This method of branching ensures that the feasible region of a node excludes the solution to the parent node and therefore the value of z at a node, commonly referred to as the *lower bound* of a node, will be greater than or equal to the lower bound of its parent. At an intermediate point in the MIP solution, the MIP feasible solution with the currently lowest value of z is commonly called the *incumbent solution* and its value of z is the upper bound of the MIP objective value. This means that a node need not be explored if its lower bound is greater than that of the incumbent solution. An incumbent solution is optimal when all nodes with a lower bound less than that of the incumbent have been explored. The branch and bound solution method is summarized in Algorithm 1.

Algorithm 1 (Branch and Bound)*Input:*

MIP instance. Incumbent solution, $X_{inc} = \phi$.

Objective value of incumbent solution, $z(X_{inc}) = \infty$. List of unexplored nodes, $L_u = \phi$.

Procedure:

1. Add the initial LP-relaxation to L_u as the root node. *currentNode* = root node.
2. Solve the LP-relaxation of the root node.
 - If the LP-relaxation is infeasible then exit with MIP infeasible outcome.
 - If the LP-relaxation is integer-feasible then exit with relaxation solution as optimum.
3. Choose a candidate variable in *currentNode* for branching.
4. Create two child nodes from *currentNode* by branching on the selected variable and add these new nodes to L_u . Remove *currentNode* from L_u .
5. Choose a node from L_u for exploration and label it *currentNode*.
6. Solve the LP-relaxation for *currentNode*.
 - If LP-relaxation is infeasible or it is feasible with a lower bound that is greater than $z(X_{inc})$ then discard *currentNode* and go to Step 8.
 - If LP-relaxation is MIP feasible then:
 - If LP-relaxation objective function value is less than $z(X_{inc})$ then replace X_{inc} with this solution, else discard *currentNode*.
 - Go to Step 8
7. Go to Step 3
8. If list of unexplored nodes is empty then:
 - 7.1 If $X_{inc} = \phi$, then exit with infeasible outcome.
 - 7.2 Optimum is X_{inc} : exit with optimal outcome.
9. Go to Step 5

2.2.1 Branching Variable Selection

If the solution to the initial LP-relaxation has more than one candidate variable then selecting which one to branch on may have a significant impact on the performance of the branch and bound method. Therefore, it is important to make a good choice. A prevalent paradigm in branching variable selection methods is to pick the variable which will lead to the largest change in the value of z between the current node and one of its children [26]. Towards this end, various methods have been developed to estimate the impact that branching on a given variable has on the value of z .

Two simple methods for branching variable selection are to

- choose the candidate variable that corresponds to the largest objective coefficient value, or
- choose the candidate variable that is furthest from having an integer value.

A very computationally expensive way to measure the impact of branching variable choice on the value of z is to try branching on each candidate variable and choosing the subproblem with the largest value of z . To reduce the computational cost of this strategy only a promising subset of candidate variables are branched on and the solution process of the subproblems is interrupted after a given number of simplex iterations [26, 6]. This method is referred to as *strong branching*.

An observation made by Bènichou *et al.* [7] is that branching on a given variable had a similar effect on the change in the value of z at different nodes. This observation led to what is called *pseudo-cost branching* in which the effect that branching on a variable at a given node has on the value of z is used to estimate this effect at a different node.

Driebeek [14] proposed a method, which was later improved upon by Tomlin [38], in which the minimum change in objective value between a parent and child node when branching on a given candidate variable is computed using its dual cost and a constraint coefficient in the *simplex tableau* for the candidate variable.

A class of methods developed by Patel and Chinneck [33] focus on choosing the branching variable that has the largest impact on the active constraints at the LP-relaxation solution for the current node. The goal of these methods is to decrease the amount of time required to find a MIP feasible solution. These methods involve analyzing the values of a_{ij} corresponding to candidate variables and the active constraints. Active constraints are those at which the current solution satisfies

$$\sum_{j \in I} a_{kj} x_j + \sum_{i \in C} a_{ki} x_i = v_k.$$

2.2.2 Node Selection

If there are two or more unexplored (*active*) nodes then the method of choosing which node to explore next can have a significant impact on the performance of the branch and bound method. When selecting nodes to explore, the goal is to choose nodes that are ancestors of a MIP optimal node. No method has been discovered, yet, that accurately determines whether or not an unexplored node is an ancestor of a MIP optimal node. Many node selection heuristic methods have been proposed to achieve this goal, or that use some other strategy of developing the branch and bound tree to achieve a different goal.

When choosing a node to explore, it is most efficient to avoid choosing one whose lower bound is greater than the (unknown) MIP optimal objective value, z^* . Such a node is commonly called a *superfluous node*. Since the LP-relaxation of an unexplored node has not been solved yet, the lower bound of each unexplored node is initially set equal to that of its parent node. This is also the case for any other LP solution characteristics of a node such as the number of candidate variables.

Many implementations of the branch and bound method use an estimate of the value of z^* to avoid the exploration of superfluous nodes. An estimate of z^* that is used for the purpose of avoiding the exploration of superfluous nodes is referred to as an *aspiration value*. The aspiration value is typically set by the user before the branch and bound process begins.

The state of the art in heuristic node selection methods is covered in depth in Chapter 3.

Chapter 3

State of the Art in Heuristic Node Selection Methods

3.1 Basic Node Selection Methods

The best-first, or *best-bound*, node selection method avoids the exploration of superfluous nodes by choosing the unexplored node whose lower bound is the lowest of all the unexplored nodes. The best-first method causes the tree to be searched in such a way that nodes that are closer in the tree to the root node are more likely to be chosen (because the lower bounds are generally worse deeper in the tree). Since MIP feasible solutions usually occur at nodes that are far from the root node, the best-first search method is not a good choice for quickly finding a MIP feasible solution. This is also true of the *breadth-first* node selection method, which chooses the earliest created active node.

If the goal is to find a MIP feasible solution as quickly as possible then it is probably better to use depth-first node selection. After a node is solved and a branching variable is chosen, depth-first chooses one of that node's children to solve next. If both children are unsolved, then some other criteria is used to determine which child to explore first. Some common methods for choosing a branch direction are to choose the up branch, the down branch, the branch of the nearest integer value for the

branching variable, or the branch that forces the value of the branching variable away from this variable's value at the root node [28]. If the last solved node is either LP infeasible, MIP feasible, or worse than the incumbent solution, then the last created active node is explored.

In depth-first node selection the most recently created unexplored node is explored first, so the difference in formulation between successive nodes solved is usually very small. Best-first node selection, as well as most other methods, cause the tree to be searched in such a way that the formulations of successive nodes solved are likely to be much different. A consequence of this is that the average number of simplex iterations to solve each node will be significantly higher using best-first versus depth-first node selection [6, 12, 26]. This assumes that the simplex basis of only the most recently solved node is available which, due to memory limitations of the computers on which they are executed, is true for most MIP solvers. Another consequence of using the best-first search is that the number of unexplored nodes will likely increase much more quickly than if depth-first search is used and, therefore, is more likely to become too large for the amount of available memory. Some MIP solvers, such as SCIP [4], use a predefined threshold of memory usage to ensure that the size of the branch and bound tree does not exceed the available memory by switching to depth-first node selection if this threshold is exceeded.

Another node selection method that is used to find incumbent solutions quickly is the *most-feasible* method. This method chooses the node with the smallest sum of integer infeasibilities, s_i . The value of s_i is computed using Equation 3.2 where

$$f_j = x_j - \lfloor x_j \rfloor \quad (3.1)$$

$$s_i = \sum_{j \in I} \min\{f_j, 1 - f_j\} \quad (3.2)$$

Many node selection schemes have been proposed to combine the benefits of best-first and depth-first methods. One such scheme uses depth-first search until a MIP

feasible solution is found and then switches back and forth between best-first and depth-first strategies [6].

Backtracking node selection uses depth-first search until the current node is either: MIP feasible, LP infeasible, has a lower bound that is greater than the value of z of the incumbent solution, or is considered undesirable to explore, at which point backtracking occurs. When backtracking occurs, a backtracking node selection method is used to select the next node to explore. Typically either the best-first, *best-projection*, or *best-estimate* criterion is used to select the next node to be explored [26].

3.2 Estimate based Node Selection

Knowing the value of z of the best possible MIP feasible solution attainable at a descendent of a given node would be useful in selecting which node to explore. Methods which estimate this value, called *estimate based methods*, include the best-projection method [19, 29] and the best-estimate method [7, 15]. In these methods the active node with the smallest such estimate is explored next.

3.2.1 Best-Projection Node Selection

The best-projection method [19, 29] uses the change in the value of z between the root node and the incumbent solution as well as the change in the sum of integer infeasibilities between the root node and the node for which the estimate is being computed. The best projection estimate at node i is computed using Equation 3.3 where z_i is the lower bound of node i , z_{inc} is the value of z for the incumbent solution, z_0 is the lower bound of the root node, s_0 (see Equation 3.2) is the sum of integer infeasibilities at the root node, and s_i is the sum of integer infeasibilities at node i .

$$E_i = z_i + \left(\frac{z_{inc} - z_0}{s_0} \right) s_i \quad (3.3)$$

The rationale for this method is that the change in the value of s_i between a

parent and child node is proportional to the change in the value of z between these nodes. This influence is approximated using the incumbent solution and is used to determine how much the value of z_i will have to change in order for the value of s_i to be reduced to 0 if this node is explored. If no incumbent solution has been found, then a backtracking node selection method that emphasizes finding MIP feasible solutions quickly, such as the depth-first method or the most-feasible method, is first used until an initial incumbent is found.

3.2.2 Best-Estimate Node Selection

The best-estimate method [7, 15] uses pseudo-cost estimates. Each integer variable has two pseudo-cost values associated with it, one for the up branch, P_j^U , and the other for the down branch, P_j^D . When a variable is branched on, the change in the lower bound between the parent node and the down child node is $\Delta z_j^D = z_j^D - z_j^P$ where z_j^D is the lower bound for the down child node and z_j^P is the lower bound for the parent node. Likewise for the up child node $\Delta z_j^U = z_j^U - z_j^P$. The pseudo-cost value for the down branch of x_j is

$$P_j^D = \Delta z_j^D / f_j$$

and similarly for the up branch

$$P_j^U = \Delta z_j^U / (1 - f_j).$$

The value of f_j is defined in Equation 3.1.

Updating the pseudo-costs for x_j is typically done by averaging the values from every instance that x_j was branched on. The estimate of the best integer feasible solution attainable from a node i is calculated using Equation 3.4.

$$E_i = z_i + \sum_{j \in I} \min\{P_j^D f_j, P_j^U (1 - f_j)\} \quad (3.4)$$

If there is no up pseudo-cost estimate for a candidate variable then the up pseudo-cost estimate for this variable is set to the average of the up pseudo-costs of all of the variables with at least one up pseudo-cost estimate. If there are no pseudo-cost estimates for all other variables then set the up pseudo-cost to 1. This is also done for the down pseudo-cost [3].

Forrest *et al.* [15] proposed a method that uses best-estimate node selection until a MIP feasible solution is found and then chooses nodes according to the percentage error criterion computed for a node i using Equation 3.5.

$$PE_i = 100 \times \frac{E_i - z_{inc}}{z_{inc} - z_i} \quad (3.5)$$

The percentage error method chooses the node with the smallest value of PE_i . The idea behind this method is to determine how much error there is in the estimate E_i in order for the current incumbent solution to not be optimal. If, for all nodes, $E_i < z_{inc}$, then $PE_i < 0$ and nodes with larger z_i are favored. If $E_i \geq z_{inc}$ for all nodes, then nodes with smaller z_i are favored.

Achterberg [3] proposed interleaving best-estimate with best-first node selection by performing best-first node selection once for every b_{fq} best-estimate backtracking node selections. It is suggested that $b_{fq} = 9$ should be used.

Linderoth and Savelsbergh [26] considered the pseudo-cost estimates to be optimistic in that they assume that rounding the value of each candidate variable such that the value of E_i is minimized will result in feasible solutions. They propose an alternative to Equation 3.4 in which there is a probability, q_j , associated to each candidate variable indicating the likelihood that the variable will be rounded to the nearest integer in an integer feasible solution. The *adjusted pseudo-cost* is computed using Equation 3.6.

$$\begin{aligned}
E_i = z_i &+ \sum_{j \in I: f_j \leq 0.5} f_j P_j^D q_j + (1 - f_j) P_j^U (1 - q_j) \\
&+ \sum_{j \in I: f_j > 0.5} f_j P_j^D (1 - q_j) + (1 - f_j) P_j^U q_j
\end{aligned} \tag{3.6}$$

The value of each q_j is computed by performing an LP-based successive rounding heuristic algorithm starting at an LP-relaxation solution $\mathbf{x}$ and recording how often the value of each integer infeasible variable was rounded to an integer value whose difference is greater than 0.5 from its initial value, to create a MIP feasible solution $\hat{\mathbf{x}}$. This flip percentage, ζ , is

$$\zeta = \frac{|\{j \in I: |\hat{x}_j - x_j| > 0.5\}|}{|I|}.$$

The value of each q_j is then calculated as

$$q_j = 1 - \zeta.$$

The value of q_j is modified to reflect the likelihood that a variable that is very close to integer valued will be rounded to the nearest integer value in a MIP feasible solution. Another modification to q_j is based on the frequency at which x_j takes on a given value over all of the MIP feasible solutions.

3.3 Genetic Programming and Node Selection

Kostikas and Fragakis [21, 22] have experimented with using genetic programming (GP) to create customized node selection rules for a given MIP instance during the branch and bound solution process of this instance. The method proceeds by running branch and bound using a standard node selection method until a set of criteria are met. GP is then run using some characteristics of the current branch and bound tree

as a training set, and a predefined function set, to evolve a node selection rule that is then used for the remainder of the branch and bound process. Node characteristics such as depth in the tree and the number of candidate variables are examples of the branch and bound tree characteristics used in the training set. The function set contains several arithmetic and logical operations such as “add”, “subtract” and “or”.

3.4 Triggering Backtracking

If an estimate $\tilde{z}^*$ of the MIP optimal objective value for a problem is computed early during the branch and bound process then backtracking node selection can be initiated when the lower bound of a node exceeds this estimate. A value that is used to trigger backtracking in this way is called an *aspiration value*. The purpose of using an aspiration value is to avoid exploring superfluous nodes while exploring the tree more efficiently than when using best-first node selection. The aspiration value is typically set by the user before the beginning of the branch and bound process.

It should be noted that active nodes with $z > \tilde{z}^*$ are not removed from the tree because it is not guaranteed that $\tilde{z}^* \geq z^*$ using heuristic methods of estimating z^* , so all of these active nodes are eligible to be explored. If $\tilde{z}^* > z^*$ and $\tilde{z}^*$ is not too large, then hopeless dives will be stopped and the amount of effort required to solve a MIP will likely be smaller than if no aspiration value is used. If $\tilde{z}^*$ is too large, then backtracking is triggered only when a leaf node is solved so the amount of effort required to solve a MIP is greater than not using an aspiration value due to the cost of estimating z^* . If $\tilde{z}^*$ is too small, then backtracking is triggered after each node is solved. For some MIP instances and backtracking node selection methods, triggering backtracking after every node is solved increases the amount of effort required to solve these instances relative to not using an aspiration method, whereas the opposite is true for others.

Various methods have been proposed to compute $\tilde{z}^*$ including best-projection estimates [26] as well as some estimation methods based on pseudo-costs [17, 26].

The idea of using an estimate of the objective value of the best integer feasible solution attainable from a given node as an aspiration value was first proposed by Gauthier and Ribière [17]. They used a pseudo-cost based estimate similar to the one in the best-estimate node selection method described in Section 3.2.2. Linderoth and Savelsbergh [26] revisited the idea by experimenting with using the best-projection and pseudo-cost estimates as well as proposing a modification to the pseudo-cost method.

If E_i is the estimate of the best integer feasible solution attainable from node i , then the estimate $\tilde{z}^*$ of z^* is taken as the smallest E_i over all of the active nodes in the tree as shown in Equation 3.7.

$$\tilde{z}^* = \min_{i \in L_u} \{E_i\} \quad (3.7)$$

Gauthier and Ribière [17] proposed calculating an upper bound, Q , for z using $\tilde{z}^*$ as follows:

$$Q = \tilde{z}^* + \rho(\tilde{z}^* - z_0)$$

where ρ is a parameter which is usually set to 0.5.

Many backtrack triggering methods that do not involve computing an estimate of z^* have also been implemented in many of the currently available MIP solvers.

A backtrack triggering method used in *CPLEX* [6] has a parameter whose value affects how frequently backtracking occurs [6]. Backtracking occurs when the difference between the objective value of the most recently solved node and the current lower bound on the MIP optimal objective value z_{lb} exceeds the difference between the incumbent objective value and z_{lb} multiplied by this parameter.

Xpress-MP [6] has a method that triggers backtracking after every solved node until a user-defined number of backtracks have occurred. Once enough backtracks have been performed, the backtracking trigger is changed so that backtracking occurs when the most recently solved node is a leaf.

The idea of switching between triggering backtracking after every node is solved

and triggering backtracking after a leaf node is solved is also used in *ABACUS* [1] and *Bcp* [35]. These methods are described in Section 3.5.

The *SCIP* [4] MIP solver triggers backtracking if at least one of the following conditions is satisfied:

- the number of nodes solved during the current dive exceeds a minimum value that is set to 0.1 times the maximum depth of all solved nodes, and $\gamma_i > \gamma_{max}$ where γ_i is computed using Equation 3.8.

$$\gamma_i = \frac{z_i - z_{lb}}{z_{inc} - z_{lb}} \quad (3.8)$$

- the number of nodes solved during the current dive exceeds a maximum value that is set to 0.5 times the maximum depth of all solved nodes.

Symphony [34] has three options available to the user for determining when to backtrack from a dive. The first option is to not backtrack until the last solved node has no child nodes. The second option is to backtrack if the difference between the average of the objective values of the best k solved nodes and the objective value of the most recently solved node is greater than a threshold that is computed using a user-defined parameter. The last option is to backtrack if the difference between the average of the lower bounds of the best k solved nodes and the objective value of the most recently solved node is greater than a given fraction of the gap between this average and the upper bound of the MIP optimal objective value.

3.5 Solver Specific Heuristic Node Selection Methods

In addition to the standard methods, some solvers have their own customized heuristic backtracking node selection and backtrack triggering methods. Some solvers also have seldom used heuristic node selection methods not included in the previous sections.

In some solvers such as CPLEX and Xpress-MP, the details of some of these methods are not given in the literature.

CPLEX [6] has a custom backtracking node selection method based on pseudo-costs as well as a method that probes both children of the most recently solved node.

Xpress-MP [6] uses best-first backtracking node selection for a given number of times, and then the backtracking node selection method used is changed to a method chosen by the user.

SCIP [4] has two custom backtracking node selection methods. One method orders nodes according to a weighted sum of the best-estimate value of E_i (Equation 3.4) and the lower bound of a node. The other uses depth-first node selection and periodically selects a node according to the best-first method. A user-defined parameter determines how often best-first search is used.

Cbc [16] has a node selection method that switches between depth-first, least number of candidate variables first, best-first, and a weighted sum of the objective value with the number of candidate variables. The choice of method depends on a user-defined parameter value as well as the number of MIP feasible solutions found, the smallest lower bound over all of the active nodes, and the number of nodes in the tree.

MINTO [25] has a hybrid method that combines the depth-first and best-estimate methods.

Lp_solve [25] has a two-phase method that switches between depth-first and breadth-first node selection.

Symphony [34] uses depth-first dives at random intervals that are controlled by a user-defined parameter that determines the approximate frequency of these dives.

Lindo [6] has a node selection option that chooses the node with the worst objective value, *worst-first*. Another option alternates between the depth-first, best-first, worst-first, and best-estimate node selection methods.

ABACUS [1] has a two-phase node selection method that uses depth-first backtracking until a MIP feasible solution is found, then it uses best-first node selection

with backtracking triggered after every node is solved.

Bcp [35] uses best-first backtracking node selection with backtracking triggered after every node is solved except if the number of candidate variables of the most recently solved node is small or this node is determined to be “close” to the best node according to a user-defined parameter value. If this exception occurs, then backtracking is triggered when the most recently solved node is a leaf.

BonsaiG [18] has an option for ordering node selection according to the weighted sum of the best-projection estimate with the sum of infeasibilities defined by Equation 3.2. Backtracking is triggered after every node is solved.

3.6 State of the Art in MIP Software

This section summarizes the available software for solving MIPs, and the node selection methods that are provided in each solver. The available MIP solvers are those examined in [25, 30, 6]. These solvers are shown in Table 3.1 along with the corresponding options available for the node selection methods described below.

DF: Depth-first (Section 3.1).

BR: Breadth-first (Section 3.1).

BP: Best-projection (Section 3.2.1).

BF: Best-first (Section 3.1).

BE: Best-estimate (Section 3.2.2).

BT: Custom heuristic backtrack triggering methods (Sections 3.4 and 3.5)

UA: User defined fixed aspiration value (set to infinity by default)

OC: Other custom heuristic node selection methods (Section 3.5)

Table 3.1: MIP solver software and available standard backtracking node selection methods.

Solver	DF	BR	BP	BF	BE	BT	UA	OC
CPLEX 11.0 [6]	x			x	x	x	x	x
Xpress-MP [6]	x			x	x	x	x	
SCIP [4]	x			x	x	x		x
Cbc [16]	x			x	x			x
MINTO [37]	x		x	x	x			x
GLPK [27]	x	x	x	x				
lp_solve [20]	x			x			x	x
Symphony [34]	x	x		x		x	x	x
Lindo [6]	x			x	x			x
ABACUS [1]	x	x		x				x
Bcp [35]	x			x		x		
BonsaiG [18]			x					x

Chapter 4

The Node Selection Problem

The goal of this thesis is to develop new node selection methods that reduce the amount of effort required to solve mixed-integer programs. This is a desirable outcome because some MIP instances require an impractical amount of time to solve. Towards this end, this work develops methods to order the nodes during the node selection process in branch and bound so as to quickly find good incumbent solutions and to avoid the exploration of superfluous nodes. Backtracking node selection is used as the framework for the development of node selection methods in this thesis. Specifically, the focus of this work is on developing node selection methods for use during backtracking, and methods of determining when to backtrack.

4.1 Performance of the State of the Art

4.1.1 Backtracking Node Selection

There are no documented empirical studies that show that any one of the state of the art backtracking node selection methods described in Chapter 3 greatly outperforms all of the other methods. The empirical study presented in [26] shows that, over all of the tested MIP instances, the average ranking of the best node selection method is 5.42 out of the 13 tested node selection methods. The empirical study presented in [22]

shows that, over all of the tested MIP instances, the average ranking of the best node selection method is 2.46 out of the 6 tested node selection methods. The empirical study presented in [3] shows that the best 4 out of the 7 tested node selection methods are within 4% of each other with respect to a time based performance measure.

The survey of the state of the art in node selection presented in Chapter 3 shows that none of the currently available node selection methods exploit correlation or distribution characteristics of the branch and bound tree. There is also no documented consideration given to the cost versus the benefit of using any of the state of the art backtracking node selection methods. Therefore, there is an opportunity to exploit these ideas to develop better backtracking node selection methods.

4.1.2 Backtrack Triggering

The survey of the state of the art MIP software presented in Section 3.6 shows that no MIP software uses a method for estimating the MIP optimal objective value for triggering backtracking. This survey also shows that all of the backtrack triggering methods available in the state of the art MIP software rely on user set parameter values. Another deficiency is that there is very little documentation giving details of how these methods work and there are no documented empirical results that demonstrate that these methods work well. Some MIP software allows the user to set the aspiration value, which indicates that such a value may be useful in reducing the amount of effort required to solve a MIP; however, an empirical study presented in [26] shows that the state of art methods for estimating the optimal objective value do not significantly reduce the amount of effort required to solve most MIPs.

A proof-of-concept experiment was performed to demonstrate the potential value of using a good aspiration value on the performance of branch and bound. The experiment demonstrates how using *a priori* knowledge of the MIP optimal objective value to trigger backtracking results in a reduction of the effort required to solve MIPs relative to the state of the art backtrack triggering methods. In this experiment each MIP instance is solved using NONA backtrack triggering (see Page 78 for the

definition of NONA) to determine its optimal objective value. Each MIP instance is then solved using its optimal objective value to trigger backtracking (PERA). Finally, each MIP instance is solved 2 more times, each time using the following state of the art backtrack triggering methods: DBPA and PCAS (see Page 78 for the definitions of DBPA and PCAS).

The details of the experimental setup are given in Chapter 6. The branch and bound options used in this experiment are given below.

- Branching variable selection is performed using the method by Driebeek [14] and Tomlin [38]. This is the best method according to the experiment described in Section 6.5.1.
- Backtracking node selection is performed using the best-projection method (the default method in GLPK) described in Section 3.2.1.
- Gomory’s mixed integer cuts are used at the root node. This is the best option according to the experiment described in Section 6.5.2.

In this experiment the solution process for each MIP instance is halted after 1 hour. The 50 MIP instances used for this experiment are listed in Table A.2, excluding instances that are (i) not solved within 1 hour using NONA, or (ii) MIP feasible at the root node.

Table 4.1 shows that using an aspiration value that is equal to the MIP optimal objective value reduced *TOTTM* (see Page 68 for the definition of *TOTTM*) by 15% relative to the second best aspiration method. It also reduced the value of *MRATE* (see Page 69 for the definition of *MRATE*) by 48% relative to the second best aspiration method. This result demonstrates that it is worthwhile developing better ways of estimating the MIP optimal objective value for use in triggering backtracking. The performance profile in Figure 4.1 shows that using a perfect aspiration value solves most of the MIP instances more quickly than the state of the art aspiration methods.

Table 4.1: Performance of perfect aspiration value versus the state of the art methods. The best values in each of the TOTTM and MRATE columns are in boldface. (see Page 68 for the definition of *FAIL*)

Contender	FAIL	TOTTM	MRATE
NONA	0	5670.57	1.52
PERA	0	4758.84	1.01
DBPA	0	5487.83	1.5
PCAS	0	5727.81	1.59

Figure 4.1: A performance profile comparing the NONA, PERA, DBPA, and PCAS options for aspiration method.

Chapter 5

New Methods for Backtrack Triggering and Node Selection

The main contributions of this work on improving the branch and bound method of solving MIPs are:

- the modified best-projection backtracking node selection and aspiration methods;
- the distribution backtracking node selection method;
- the feasibility depth extrapolation aspiration method;
- and the active node search threshold.

This chapter presents the ideas and observations that underlie these methods, and their algorithmic details.

5.1 Modified Best Projection Node Selection and Aspiration Method

A modification to the best-projection method is proposed that can estimate the objective value of the best MIP feasible solution attainable from any node without using an incumbent solution. It is important to be able to compute an estimate without requiring an incumbent solution because an incumbent solution can be very difficult to find in some MIP instances. The modified best-projection method is used as a backtracking node selection method as well as a backtrack triggering method.

This method is based on Observation 1.

Observation 1 (Node Infeasibility versus Optimality) *The approximately linear correlation between $z^{\min}(c)$ and c for many MIPs indicates that a reasonably accurate estimate of the MIP optimal objective value may be computed early in the branch and bound process. Note that when a MIP solution is completed successfully, $z^{\min}(0)$ is the MIP optimal objective value. ■*

c_i is the number of candidate variables at node i . c_0 is the number of candidate variables at the root node.

z_i is the objective value at node i . z_0 is the objective value at the root node.

$z^{\min}(c)$ is the smallest z_i over all nodes with c candidate variables:

$$z^{\min}(c) = \min_i \{z_i : c_i = c\} \quad (5.1)$$

For many MIP instances, plots of $z^{\min}(c)$ versus c show a trend that indicates the possibility of estimating the value of z^* early in the branch and bound process by simple linear extrapolation. For the *dcmulti* MIP instance, the trend can be seen in the plot of $z^{\min}(c)$ shown in Figure 5.1 for $10 \leq c \leq 30$. For the *fixnet4* MIP instance, the trend is shown in Figure 5.2 for $5 \leq c \leq 15$.

Figure 5.1: Plot of z_i and $z^{\min}(c)$ versus c_i for all of the nodes in the tree at the completion of the branch and bound process for the *dcmulti* MIP instance. Also shown is the best fit line through $z^{\min}(c)$ for $10 \leq c \leq 30$. The DEBP-NONA-NOAN node selection configuration is used (see Chapter 7 for the configuration definitions).

Figure 5.2: Plot of z_i and $z^{min}(c)$ versus c_i for all of the nodes in the tree at the completion of the branch and bound process for the *fixnet4* MIP instance. Also shown is the best fit line through $z^{min}(c)$ for $5 \leq c \leq 15$. The DEBP-NONA-NOAN node selection configuration is used.

Observation 1 reveals a potential improvement to the best-projection method described in Section 3.2.1. Recall that in the best-projection method an incumbent solution is required to compute an estimate of the degradation in objective value per unit MIP infeasibility (Equation 3.3). At any time during the branch and bound process, the incumbent solution objective value is $z^{min}(0)$ which is the minimum objective value found at a node with 0 candidate variables, i.e. a MIP feasible node. According to Observation 1, a useful estimate of the degradation in objective value per unit MIP infeasibility can be made using $z^{min}(c)$ for $c > 0$ thus allowing modified best-projection node selection to proceed without an incumbent solution.

5.1.1 Algorithm

The following are some definitions needed for this algorithm.

c_{min} is the minimum number of candidate variables over all nodes solved so far. This value is initially equal to the number of candidate variables at the root node and is updated after each node is solved.

σ_z is the standard deviation of z_i over all nodes solved so far.

σ_c is the standard deviation of c_i over all nodes solved so far.

The estimate of degradation of the objective value per unit change in MIP infeasibility, m , is computed using Equation 5.2.

$$m = \frac{z^{min}(c_{min}) - z_0}{c_0 - c_{min}} \quad (5.2)$$

The estimate of the objective value of the best MIP feasible solution attainable from a node i , $\tilde{z}_i^*$, is computed using Equation 5.3.

$$\tilde{z}_i^* = c_i * m + z_i \quad (5.3)$$

The value of $-m$ is the slope of the line through $(c_{min}, z^{min}(c_{min}))$ and (c_0, z_0) . Figure 5.3 shows an example of this line at a time in the branch and bound process

Figure 5.3: Plot of z_i and $z^{\min}(c)$ versus c_i for all of the nodes in the tree at the time in the branch and bound process when $c_{\min} = 23$ for the *dcmulti* MIP instance. Most of the points in the plot overlap because $z_i = z^{\min}(c)$ for most of the nodes at this time. Also shown is the line through $z^{\min}(55)$ (the root node) and $z^{\min}(23)$.

when $c_{\min} = 23$, $c_0 = 55$ (the number of candidate variables at the root node), $z^{\min}(c_{\min}) = 187851.53$, and $z_0 = 187022.36$ (the objective value at the root node). The slope of the line through the points $(c_{\min}, z^{\min}(c_{\min}))$ and (c_0, z_0) is $-m = -25.91$. At this time the value of $z_i = z^{\min}(c_{\min})$ for the most recently solved node i . Likewise, the value of $c_i = c_{\min}$. An estimate of the objective value of the best MIP feasible solution attainable from node i , is $\tilde{z}_i^* = 188447.5$. This estimate is the extrapolated value of $z^{\min}(0)$ using the line through (c_i, z_i) with a slope equal to $-m$.

If at least one of the following conditions is satisfied then this method does not work well.

- $c_{min} = c_0$: In this case the denominator in Equation 5.2 is 0.
- $\sigma_c < \sigma_c^{min}$: In this case most nodes have very similar values of c_i . This causes the value of m to have little effect on node selection. $\sigma_c^{min} = 3$.
- $\sigma_z < \sigma_z^{min}$: In this case there is very little change in z_i over all nodes. This causes the value of m to be almost 0 which then results in c_i having little effect on node selection. $\sigma_z^{min} = 0.001$.

The thresholds used for the last two conditions were determined empirically. Some of the MIP instances for which the modified best-projection did not work well satisfied at least one of these conditions whereas this was not the case for MIP instances for which it did work well.

The modified best-projection backtracking node selection method is summarized in Algorithm 2. This algorithm is performed each time that backtracking is triggered.

Algorithm 2 (Modified best-projection backtracking node selection)
<p><i>Input:</i></p> <p>A list of active nodes L_u in the branch and bound tree.</p> <p>$\sigma_c^{min} = 3$.</p> <p>$\sigma_z^{min} = 0.001$.</p>
<p><i>Procedure (performed each time that backtracking is triggered):</i></p> <ol style="list-style-type: none"> 1. If $c_{min} = c_0$ or $\sigma_c < \sigma_c^{min}$ or $\sigma_z < \sigma_z^{min}$ then use the best-projection backtracking node selection method described in Section 3.2.1. 2. Otherwise: <ol style="list-style-type: none"> (a) For every node i in L_u, compute $\tilde{z}_i^*$ using Equation 5.3. (b) Select the node i with the smallest value of $\tilde{z}_i^*$ as the node to be explored next.

Algorithm 3 (Modified best-projection backtrack triggering)
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<p><i>Input:</i></p> <p>The most recently solved node i.</p> <p>A list of active nodes L_u in the branch and bound tree.</p> <p>$\sigma_c^{min} = 3$.</p> <p>$\sigma_z^{min} = 0.001$.</p>
<p><i>Procedure (performed after each node is solved):</i></p> <ol style="list-style-type: none"> 1. If $c_{min} = c_0$ or $\sigma_c < \sigma_c^{min}$ or $\sigma_z < \sigma_z^{min}$ then use the best-projection aspiration method described in Section 3.4 to determine whether or not to trigger backtracking. 2. Otherwise: <ol style="list-style-type: none"> (a) For every node i in L_u, compute $\tilde{z}_i^*$ using Equation 5.3. (b) Compute $\tilde{z}^* = \min_{i \in L_u} \tilde{z}_i^*$. (c) Trigger backtracking if $z_i > \tilde{z}^*$.

The modified best-projection backtrack triggering method is summarized in Algorithm 3. This algorithm is performed after each node is solved.

5.2 Distribution Node Selection

This backtracking node selection method balances the goals of MIP feasibility and MIP optimality by using probability distributions to model these measures using all nodes solved so far in the solution process. This method weights the MIP feasibility and optimality goals dynamically so that neither can dominate the other for too long. It favours choosing nodes with lower objective values and lower MIP infeasibility.

The most commonly used measure of a node's closeness to optimality is its value of z . The measure of MIP infeasibility used in this method is the number of candidate variables c . The number of candidate variables is used instead of the sum of the fractional components of the integer variables because a candidate variable will likely become integer valued if that variable is branched on regardless of what its fractional value is. For example, suppose one node has one candidate variable with a fractional value of 0.5 whereas another node has five candidate variables each with a fractional value of 0.09. If all other integer variables are fixed for both of these nodes, then choosing the first node will likely lead to a MIP feasible node more quickly than choosing the second node, which has a lower sum of integer infeasibilities.

Observation 2 (Balancing MIP Feasibility and Optimality) *A characteristic that is typical of many branch and bound trees is that there is a negative correlation between the objective value and the number of candidate variables of a node; i.e. z_i tends to increase with depth while c_i tends to decrease with depth.* ■

A consequence of Observation 2 is that when trying to balance MIP feasibility and MIP optimality during node selection, if greater weight is being given to z_i than to c_i , then the fraction of nodes chosen with small z_i will likely be large relative to the fraction of nodes chosen with small c_i . The opposite is likely true if greater weight is given to c_i than to z_i . Note that MIP feasibility is reached when $c_i = 0$.

Since it is desirable to choose nodes with both a small z_i and a small c_i , these values need to be combined into one value in order to rank all of the active nodes. This combined value should be directly proportional to both z_i and c_i . Combining

z_i and c_i in such a way that neither value dominates the other can be difficult due to the difference between the range of z_i and the range of c_i across all of the nodes. For example, Figure 5.1 shows that the range of z_i for *dcmulti* is over 3000, whereas the range of c_i is less than 60. This problem is solved by assembling the probability distributions of z_i and c_i , respectively, as data becomes available during the solution process. Note that the range of a probability distribution function is $[0, 1]$ and that each distribution is a monotonically increasing function of its independent variable.

The choice of the distribution function that is used in this method is suggested by empirical experiments which showed that using the Normal distribution function solved MIPs more quickly than the other standard distribution functions that were tested: the Uniform and Rayleigh distributions. The Normal distribution is well suited for balancing the pursuit of MIP feasibility with optimality since (i) it requires very little extra work to compute than the Uniform distribution, (ii) it estimates the values of c_i and z_i that most frequently appear in the tree using their respective means, and (iii) it uses information on the ranges of c_i and z_i via their respective standard deviations.

The means and standard deviations used to compute the Normal distributions of z_i and c_i are defined below.

$m_z =$ the arithmetic mean of z_i over all solved nodes.

$m_c =$ the arithmetic mean of c_i over all solved nodes.

$\sigma_z =$ the biased maximum likelihood estimate of the standard deviation of z_i over all solved nodes.

$\sigma_c =$ the biased maximum likelihood estimate of the standard deviation of c_i over all solved nodes.

Let $F_Z(z) = P(Z \leq z)$ and $F_C(c) = P(C \leq c)$ be estimates of the Normal cumulative distribution functions for z_i and c_i , respectively, of all of the nodes solved so far. The value of $F_Z(z_i)$ is the estimated fraction of nodes with a value of $z \leq z_i$.

Similarly, the value of $F_C(c_i)$ is the estimated fraction of nodes with a value of $c \leq c_i$. The distribution backtracking node selection method chooses the active node with the smallest value of $B_{zc}(z_i, c_i) = F_Z(z_i) \times F_C(c_i)$ to be explored next. $B_{zc}(z_i, c_i)$ is used as the comparison criterion because its value is directly proportional to both $F_Z(z_i)$ and $F_C(c_i)$, and $B_{zc}(z_i, c_i) \leq \min(F_Z(z_i), F_C(c_i))$ (recall that $0 \leq F_Z(z_i) \leq 1$, and $0 \leq F_C(c_i) \leq 1$) which causes this node selection method to behave a little like a combination of the best-first and the most-feasible node selection methods.

The following example demonstrates how the pursuit of MIP feasibility and optimality can be balanced using Normal distributions. Suppose that during backtracking node selection there are 4 LP feasible nodes in the tree, two of which have one unsolved child each. The values of z_i and c_i for each LP feasible node are given below.

i	z_i	c_i	child
0	1	100	n1
1	100	1	
2	100	1	
3	100	1	n2

In this example, one of the candidate nodes is an unsolved child $n1$ of node 0, the other an unsolved child $n2$ of node 3. Recall that the value of z_i and c_i at an unsolved node is initially set to its parent's value of z_i and c_i respectively. The choice of which node to solve next is between node $n1$ which has a small value of z_i and a large value of c_i , and node $n2$ which has a large value of z_i and a small value of c_i . In order to achieve the goals of MIP feasibility and optimality, we would like to choose the node i that has the smallest fraction of nodes with $z \leq z_i$ on the one hand, and the smallest fraction of nodes with $c \leq c_i$ on the other hand. Estimates of these fractions are computed for our example using $F_Z(z_i)$ and $F_C(c_i)$, and are given below as well as the product of these fractions $B_{zc}(z_i, c_i)$ for each candidate node.

i	$F_Z(z_i)$	$F_C(c_i)$	$B_{zc}(z_i, c_i)$
$n1$	0.067	0.933	0.062
$n2$	0.691	0.309	0.213

The node $n1$ has the smallest value of $B_{zc}(z_i, c_i)$ so it is chosen to be solved next. Node $n1$ is likely the better choice because three of the above nodes have a large value of z_i and a small value of c_i whereas one of the above nodes has a small value of z_i and a large value of c_i . This indicates that more emphasis has been placed on the pursuit of MIP feasibility than the pursuit of optimality so the emphasis should be shifted towards the pursuit of optimality by selecting node $n1$ to be solved next.

5.2.1 Algorithm

The distribution backtracking node selection method proceeds by computing $B_{zc}(z_i, c_i)$ for each active node i . The node n that will be explored next is chosen using Equation 5.4.

$$n = \arg \min_i B_{zc}(z_i, c_i) \quad (5.4)$$

If there is very little variation in the value of z_i , or if c_i does not change significantly in proportion to the depth of the nodes in the tree, then this method does not work well since either of these measures will be given too much weight in node selection. This is summarized in the following two conditions where d is the depth of the latest solved node before backtracking node selection.

- $\sigma_c/d < \sigma_{c/d}^{min}$: In this case most nodes have very similar values of c_i over a large range of depths in the tree. $\sigma_{c/d}^{min} = 0.1$.
- $\sigma_z < \sigma_z^{min}$: In this case there is very little change in z_i over all nodes. $\sigma_z^{min} = 0.001$.

The thresholds used for these conditions were determined empirically. Some MIP instances for which distribution node selection did not work well satisfied either of these conditions whereas this was not the case for MIP instances for which it did work well.

The distribution backtracking node selection method is summarized in Algorithm 4. This algorithm is performed each time that backtracking is triggered.

Algorithm 4 (Distribution backtracking node selection)

Input:

The most recently solved node.

A list of active nodes L_u in the branch and bound tree.

$$\sigma_{c/d}^{min} = 0.1.$$

$$\sigma_z^{min} = 0.001.$$

Procedure (performed each time that backtracking is triggered):

1. If $\sigma_{c/d} < \sigma_{c/d}^{min}$ or $\sigma_z < \sigma_z^{min}$ then use the best-projection backtracking node selection method described in Section 3.2.1.
2. Otherwise:
 - (a) For every node i in L_u , compute $B_{zc}(z_i, c_i)$.
 - (b) Select the node i with the smallest value of $B_{zc}(z_i, c_i)$ as the node to be explored next.

5.3 Feasibility Depth Extrapolation Aspiration

This method of estimating the MIP optimal objective value computes its estimate using the objective values of nodes at the estimated depth of a MIP optimal node. The depth of a MIP optimal node is estimated using linear extrapolation of the number of candidate variables along the ancestors of each node in the branch and bound tree. This method is based on Observations 3, 4 and 5.

Observation 3 (MIP Optimal Depth and Objective Value) *The largest objective value over all nodes at the depth of a MIP optimal node in the branch and bound tree is greater than or equal to the MIP optimal objective value. ■*

Observation 4 (Relative Depth of MIP Optimal Node) *The depth of a MIP optimal node in the branch and bound tree tends to be closer to the root node than the depth of most other MIP feasible nodes. ■*

Observation 5 (Node Infeasibility versus Depth) *Extrapolating the rate of decrease of the number of candidate variables along a dive in the branch and bound tree provides a reasonable estimate of the depth of the first MIP feasible solution for the dive. ■*

Figure 5.4 shows an example that demonstrates Observation 3. In this example, the largest objective value over all nodes at a MIP optimal depth is less than the objective values of a significant number of nodes in the tree for the *ran10x10a* MIP instance. Therefore, if the largest objective value over all nodes at the MIP optimal depth was known and was used as an aspiration value, then the amount of effort required to solve this MIP instance would be reduced. Figure 5.4 also shows that nodes that are close to a MIP optimal depth tend to have objective values that are relatively close to the MIP optimal objective value. This suggests a method of estimating a suitable aspiration value. Finally, Figure 5.4 shows that in some instances the estimate of the depth of the optimal node does not have to be very accurate in order to get a good aspiration value.

Figure 5.4: A plot of the objective value versus depth for all nodes for the *ran10x10a* MIP instance. Also shown are lines representing the MIP optimal objective value and depth. The DIST-NONA-NOAN node selection configuration is used.

To demonstrate Observation 4, all MIP instances listed in Table A.1 were solved to optimality, using default GLPK with a 1 hour time limit, recording the depth of the optimal node as well as the depth of the closest MIP feasible node to the root node. Figure 5.5 shows that for a majority of MIP instances, the optimal node is close to the shallowest MIP feasible node in the branch and bound tree.

This tendency is because fewer changes in variable bounds between the initial LP relaxation and a MIP feasible solution usually lead to less change in objective values between the two solutions. Hence a shallower MIP feasible solution tends to have a better objective function value, so the shallowest MIP feasible solution is often the optimum. The rate of increase of MIP feasibility with depth tends to be greater for the ancestors of the optimal node than for other MIP feasible nodes.

If the depth d_i^* of the shallowest MIP feasible node attainable by exploring the descendants of node i is known, then the shallowest of these depths over all nodes in the tree is likely a MIP optimal depth. According to Observation 5, an estimate of the value of d_i^* , $\tilde{d}_i^*$, may be computed using the correlation between the number of candidate variables c_i and the depth d_i .

Figures 5.6, 5.7 and 5.8 show examples in which changes in the number of candidate variables as one progresses along a path deeper into the tree give some indication of the depth of a MIP feasible node at the end of this path. Any node in the branch and bound tree is likely more MIP feasible than its parent because branching on a candidate variable in the parent node almost always results in that variable being integer valued in the resulting child node. Thus, it is expected that MIP infeasibility should decrease as one moves along a path deeper into the tree.

For most MIP instances, plots of c_i versus d_i of all nodes on a given path show some linearity. Figures 5.9, 5.10 and 5.11 show three examples of these plots for the *10teams*, *bell5*, and *vpm2* MIP instances respectively. These plots also show that extrapolating a linear best-fit line on c_i for $0 \leq d_i < d_i^*$ can result in a good value of $\tilde{d}_i^*$. It should be noted that this pattern does not hold for all MIP instances so this method may not work well in some cases. Linear extrapolation is used to estimate

Figure 5.5: A plot of the ratio of depth of a MIP optimal node to the minimum depth over all MIP feasible nodes for a set of MIP instances. The DEBP-NONA-NOAN node selection configuration is used.

Figure 5.6: A plot of the number of candidate variables versus depth for all ancestors of a MIP optimal node for the *10teams* MIP instance.

Figure 5.7: A plot of the number of candidate variables versus depth for all ancestors of a MIP optimal node for the *bell5* MIP instance.

Figure 5.8: A plot of the number of candidate variables versus depth for all ancestors of a MIP optimal node for the *vpm2* MIP instance.

the value of $\tilde{d}_i^*$ of a given node i , i.e. the depth at which the number of candidate variables reaches zero. Figure 5.10 shows that $d_i^* = 28$ and $\tilde{d}_i^* - d_i^* = -4$ for the node at $d_i = 20$. Figure 5.11 shows that $d_i^* = 44$ and $\tilde{d}_i^* - d_i^* = -8$ for the node at $d_i = 20$. Figure 5.9, on the other hand, shows how the value of d_i^* can be much too large. In this case $d_i^* = 24$ and $\tilde{d}_i^* - d_i^* = 12$ for the node at $d_i = 20$. An estimate of the depth of the optimal solution $\tilde{d}^*$, according to Observation 4, is then taken as the smallest $\tilde{d}_i^*$ over all solved nodes in the tree.

The aspiration value $\tilde{z}^*$, according to Observation 3 is set to the maximum objective value over all solved nodes at $\tilde{d}^*$.

Figure 5.9: A plot of the number of candidate variables versus depth for the earliest 20 ancestors of a MIP optimal node for the *10teams* MIP instance along with the best fit line.

Figure 5.10: A plot of the number of candidate variables versus depth for the earliest 20 ancestors of a MIP optimal node for the *bell5* MIP instance along with the best fit line.

Figure 5.11: A plot of the number of candidate variables versus depth for the earliest 20 ancestors of a MIP optimal node for the *vpm2* MIP instance along with the best fit line.

5.3.1 Algorithm

The following are some definitions needed for this algorithm.

m_i is the slope of the least squares best fit line for (c_i, d_i) over all ancestors of node i .

b_i is the d -intercept of the least squares best fit line ($d = m_i \cdot c + b_i$) over all ancestors of node i .

S is the set of solved nodes in the current branch and bound tree that are LP feasible but not MIP feasible.

The algorithm proceeds by computing the least squares best fit line of the ancestors for each solved node with at least 20 ancestors. The condition that a node have at least 20 ancestors is imposed to ensure enough information is present to compute a reasonably accurate best fit line while still being able to compute an aspiration value early in the branch and bound process. An estimate of the depth of the first MIP feasible solution attainable from a node i is computed as

$$\tilde{d}_i^* = [b_i] \quad (5.5)$$

where $[b_i]$ represents the rounding of b_i to the nearest integer value.

The estimate of the optimal solution depth, $\tilde{d}^*$, is computed as

$$\tilde{d}^* = \min_{\{i \in S: d_i \geq 20, \tilde{d}_i^* > 0\}} \tilde{d}_i^* \quad (5.6)$$

The aspiration value $\tilde{z}^*$, according to Observation 3, is the maximum value of z_i over all solved nodes with $d_i = \tilde{d}^*$.

$$\tilde{z}^* = \max_{i \in S: d_i = \tilde{d}^*} z_i \quad (5.7)$$

The condition that $\tilde{d}_i^* > 0$ is imposed in Equation 5.6 because nonpositive values would result in unreasonable values of $\tilde{d}^*$. A value of $\tilde{d}^* = 0$ is unreasonable since the

root node is not MIP feasible. A value of $\tilde{d}^* < 0$ is unreasonable since there are no nodes with $d_i < 0$.

The feasibility depth extrapolation backtrack triggering method is summarized in Algorithm 5. This algorithm is performed after each node is solved.

Algorithm 5 (Feasibility depth extrapolation backtrack triggering)

Input:

The most recently solved node i .

The set S of solved nodes that are LP feasible but not MIP feasible in the current branch and bound tree.

The threshold on the minimum node depth $d^{min} = 20$.

The current value of $\tilde{d}^*$.

The current value of $\tilde{z}^*$.

Procedure (performed after each node is solved):

1. If $d_i = 0$ then the current node is the root node so set $\tilde{d}^* = \infty$ and $\tilde{z}^* = \infty$.
2. If $d_i \geq d^{min}$:
 - (a) Compute $\tilde{d}_i^*$ using Equation 5.5.
 - (b) If $\tilde{d}_i^* > 0$ and $\tilde{d}_i^* < \tilde{d}^*$ then set $\tilde{d}^* = \tilde{d}_i^*$;
3. If $\tilde{d}^* < \infty$ then update the value of $\tilde{z}^*$ using Equation 5.7.
4. Trigger backtracking if $z_i > \tilde{z}^*$.

5.4 Active Node Search Threshold

Some backtracking node selection methods can be computationally expensive. The active node search threshold (ANST) estimates whether or not the cost of using a backtracking node selection method outweighs its benefit. If the cost of a method does outweigh its benefit, then a simpler node selection method should be used instead. This method is based on Observation 6.

Observation 6 (Cost of Node Selection Methods) *For some MIPs, the amount of time required to perform a given backtracking node selection method can become a significant fraction of the overall amount of time required to solve it. This occurs when the number of active nodes in the tree becomes very large. ■*

The MIP instance *mas76*, for example, is solved in 20609 seconds, requiring 3186117 simplex iterations and 1177063 nodes, using best-projection backtracking node selection. The same MIP instance is solved in 1306 seconds, requiring 3314480 simplex iterations and 1229699 nodes using depth-first node selection. This MIP is solved in significantly less time but requiring more simplex iterations and nodes using depth-first versus best-projection.

To verify that the extra time needed to solve MIPs such as *mas76* using best-projection versus depth-first backtracking is due to the node selection method, the maximum ratio, R_t , of total time spent performing all of the best-projection backtracking node selections to the total time spent performing all of the other branch and bound operations was recorded for the experiments reported in this thesis. For a small set of MIP instances, the amount of effort needed to solve each MIP instance is shown in Table 5.1 along with the corresponding value of R_t . The value of R_t for MIPs that are solved in less time but more simplex iterations using depth-first versus best-projection are much larger than the values of R_t for the remaining MIPs.

For the *mas76*, *pk1*, and *ran10x10c* MIP instances, the cost of using best-projection node selection outweighs the benefit because this method searches the entire set of active nodes and the number of active nodes in the tree becomes very large. The

cost of using a *simple* backtracking node selection method does not increase with the number of active nodes in the tree. In GLPK [27], the depth-first backtracking node selection method is simple and requires very little computation because it chooses the active node that was last created and resides at the end of the list of active nodes. A backtracking node selection method that searches through the entire set of active nodes is not simple because the amount of effort required to search through a set increases as the number of nodes in the set increases. Examples of backtracking node selection methods that are not simple include the best-projection, best-first, and best-estimate methods.

Table 5.1: Comparing the solution time, number of nodes, and number of simplex iterations using best-projection (DEBP-NONA-NOAN) vs. depth-first (DEPF-NONA-NOAN) node selection. Also shown is R_t . The best values with respect to time in each of the rows are in bold as are the best values with respect to itns and nodes.

MIP	depth-first			best-projection		
	time (s)	itns	nodes	time (s)	itns	nodes
bal8x12	4.51	5375	1483	1.73	2592	376
10teams	1985.79	1250128	14258	1076.87	711067	6043
fiber	1434.93	276720	37298	617.71	129490	15673
l152lav	706.34	113407	8168	515.35	82776	5333
blend2	679.12	1649444	316634	40.27	49393	18186
pipeX	2.32	14695	5701	1.39	9936	3007
misc07	4171.5	1464603	131477	2937.81	1058824	93303
pp08a	1932.69	515883	71946	1527.03	417930	53975
bienst1	2673.93	4107520	34256	2530.14	3971433	32621
mas76	1306.61	3314480	1229699	20609.65	3186117	1177063
pk1	4058.23	9778734	1965503	12623.15	4329434	820924
ran10x10c	930.96	1214630	566587	3095.58	907230	428535
						Max R_t
						0.0185
						0.00194
						0.0060
						0.0025
						0.0202
						0.0215
						0.0142
						0.0146
						0.0020
						0.8790
						0.7904
						0.6552

An alternative to putting the last created active node at the end of the list is used by Achterberg [3, 4] in the implementation of SCIP for solving MIPs and constraint integer programs. In SCIP, each node is placed in a list according to its priority with respect to the node selection criteria. A drawback to this method is that the relative priority of nodes already in the list may change depending on the node selection criteria used. In the best-estimate node selection method described in Section 3.2.2, for example, the pseudo-cost estimates are updated after every node is solved. So to ensure that the nodes in the list have the correct order, the best-estimate computation of each node in the list affected by the change in pseudo-cost estimates should be performed again and then all of the active nodes should be sorted.

In SCIP, the amount of computation required to place each node in the list according to its node selection priority increases with the number of nodes already in the list since a search of these nodes needs to be performed every time a new node is placed in the list. Therefore it is possible for the cost of node selection methods of this type to become very large. This issue is noted by Achterberg [3].

The cost of a backtracking node selection method is mostly affected by the number of active nodes in the tree, the amount of computation required per active node during the backtracking search, and by how frequently backtracking node-selection is performed. If an aspiration backtrack triggering method is used then backtracking frequency is affected by the aspiration value. Therefore, if the cost of backtracking node selection is perceived to outweigh its benefit for a given MIP, and an aspiration method is being used, then it may be better to first reduce the backtracking frequency by increasing or removing the aspiration value to see if backtracking continues to cost too much. If the backtracking node selection method continues to cost too much, then it should be changed to a simple method. This is one way in which an aspiration value that is too small may be detected.

5.4.1 Algorithm

The following are some definitions needed for this algorithm.

t_{NS} is the total amount of time spent performing all backtracking node selections.

t_{BB} is the total amount of time spent performing all of the operations in the branch and bound method.

q is the threshold that determines when to switch to a simple node selection method.

$$q = 0.1$$

dq is an increment that is used to determine the effect of the currently used aspiration backtrack triggering method on the cost of backtracking node selection. $dq = 0.01$

The active node search threshold algorithm proceeds by computing the value of R_t , using Equation 5.8, after each time that backtracking node selection is performed.

$$R_t = \frac{t_{NS}}{t_{BB} - t_{NS}} \quad (5.8)$$

If the value of $R_t \geq q$ then either backtracking is being performed too frequently because of an aspiration backtrack triggering method, or the cost of backtracking outweighs its benefit. If no aspiration backtrack triggering method is currently being used, then the current backtracking node selection method is too costly, and depth-first backtracking node selection replaces the current node selection method. If an aspiration backtrack triggering method is currently being used, then the use of the aspiration value is discontinued, the current backtracking node selection method continues, and the value of q is set to the current value of $R_t + dq$ so that now $R_t < q$. If the value of R_t continues to increase and exceeds this new value of q then the current backtracking node selection method is too costly and the depth-first backtracking node selection method is then substituted.

Depth-first is used as the simple backtracking node selection method because of its low cost and its good efficiency in exploring the branch and bound tree as discussed in Section 2.2.2.

Table 5.1 shows that setting $q = 0.1$ allows a given node selection method to continue to be used when it is not too costly but quickly forces a change to the depth-first method when its cost becomes too great. Although the values of R_t for a small number of MIP instances were used to determine the value of q , the chosen value of $q = 0.1$ is over 4 times larger than the value of R_t that is closest to $q = 0.1$. The value of dq is set to a small value that, if an aspiration value is used, is added to R_t to make a new threshold that allows the current backtracking node selection method to continue to be used without an aspiration value. The value of dq is set to 10% of the value of q to allow the current backtracking node selection method to continue to be used a sufficient number of times without an aspiration backtrack triggering method so that the backtracking node selection method is not mistakenly determined to be too costly.

The active node search threshold (ANST) method is summarized in Algorithm 6.

Algorithm 6 (Active Node Search Threshold)
Input:

A MIP instance using a backtracking node selection method other than depth-first.
 The total amount of time spent performing all backtracking node selections, $t_{NS} = 0$.
 The total amount of time spent performing branch and bound, $t_{BB} = 0$.
 The threshold value, $q = 0.1$.
 The threshold increment value, $dq = 0.01$.

Procedure:

1. Perform branch and bound until backtracking is triggered.
2. Perform backtracking node selection.
3. If depth-first backtracking node selection is being used then go to Step 1.
4. Add the time required to perform Steps 1 and 2 to t_{BB} . Add the time required to perform Step 2 to t_{NS} .
5. Compute the ratio of time spent backtracking to the time spent on branch and bound, R_t , using Equation 5.8.
6. If $R_t < q$ then go to Step 1.
7. If no aspiration value is in use then
 - (a) change backtracking node selection method to depth-first.
 - (b) Go to Step 1.
8. $q = R_t + dq$.
9. Stop using aspiration value.
10. Go to Step 1.

Chapter 6

Experimental Setup

This chapter presents details of the hardware, the software, the MIP solver parameters, the MIP instances, and the metrics that are used in the experiments that evaluate the performance of various configurations of backtracking node selection methods, aspiration methods, and active node search thresholds.

6.1 Hardware

The following hardware and software was used in all experiments.

CPU: Intel Core 2 6600 @ 2.4 GHz

RAM: 4 GB

OS: Linux 2.6.18

Although the CPU has two processing cores, each MIP instance is solved using only 1 core.

6.2 MIP Solver

The GLPK 4.9 [27] MIP solver was used in all experiments because:

- it is a free open source software that has excellent documentation;
- it uses the depth-first with backtracking framework;
- it is easy to add new backtracking node selection methods, aspiration methods, and active node search thresholds to it;
- as shown in Table 3.1, it has most of the standard node selection methods;
- and it is capable of solving reasonably difficult MIP instances within a reasonable time limit [25].

The GLPK solver has many parameters that do not affect the branch and bound solution method. Unless otherwise stated, all of these parameters are set to their default values except for those described below.

LPX_K_TMLIM: The time limit of the MIP solver in seconds.

LPX_K_MSGLEV: The type of messages the solver outputs to the console.

For all experiments the value of LPX_K_MSGLEV is set to 1 so that messages to the console are limited to error messages in order to reduce the corresponding overhead and thus reduce the solution time for all MIP instances.

6.3 Performance Measures

Solution time t_{MI} is the most important measure of the amount of effort required to solve a MIP instance MI and is frequently used to evaluate the performance of a branch and bound method [3, 8, 9, 2, 26]. The number of nodes solved and the number of simplex iterations are also commonly used to measure the amount of effort required to solve a MIP instance [3, 8, 14, 33, 2]. Solution time is used for the experiments in this thesis because the time to solve a MIP instance includes the overhead computing time that is not reflected in the number of nodes and the

number of simplex iterations required to solve this MIP instance. Section 5.4 shows an example of how the time spent performing node selection may become large relative to that spent performing the other aspects of branch and bound. Table 5.1 shows that although best-projection node selection solves the *mas76*, *pk1* and *ran10x10c* MIP instances in fewer nodes and simplex iterations than depth-first node selection, these MIP instances are solved in significantly less time with depth-first versus best-projection node selection. Therefore the number of nodes and simplex iterations do not always accurately reflect the amount of effort required to solve a MIP instance.

In order to evaluate the relative performance of a set of branch and bound methods, a set of many MIP instances should be solved using each method because the heuristic nature of these methods does not guarantee that the relative performance of each method will be consistent for every possible MIP instance. A difficulty in using a set of many MIP instances to determine the relative performance of each method lies in how to combine the solution times for all of the MIP instances into a single performance measure.

A simple method of combining the solution times of all of the MIP instances is simply to compute their total using equation 6.1.

$$TOTTM = \sum_{MI} t_{MI} \quad (6.1)$$

A drawback of using *TOTTM* is that its value can be dominated by a small number of very difficult MIP instances. The impact of a particularly difficult MIP instance on the overall performance of a method can be eliminated by using the best solution time t_{MIB} for *MI* over all of the contending methods to normalize t_{MI} using Equation 6.2.

$$r_{MI} = t_{MI}/t_{MIB} \quad (6.2)$$

For a given method, $r_{MI} - 1$ is the percentage of extra time required to solve *MI* relative to the best method for solving *MI*. The geometric mean defined in Definition 1 is used to combine the r_{MI} for every *MI* into a single measure *MRATE*. The

geometric mean is preferable to the arithmetic mean because the geometric mean is less sensitive to outlying values of r_{MI} that may occur with large sets of MIP instances. For example, the arithmetic means of each of the sets $A = 1, 2, 3, 4, 5$ and $B = 1, 1, 1, 1, 11$ are both equal to 3 whereas the geometric mean of each set is 2.61 and 1.62 respectively. The outlying member in B , 11, has less impact on the geometric mean than on the arithmetic mean.

Definition 1 (Geometric Mean) *Let $r_1, \dots, r_N$ be a set of real numbers with values greater than 0. The geometric mean of this set is computed using Equation 6.3.*

$$\bar{r} = \sqrt[N]{\prod_{j=1}^N r_j} \quad (6.3)$$

If a time limit is imposed on the branch and bound method, then not all MIP instances will be solved in an experiment. The number of MIP instances *FAIL* that are not solved to optimality is a commonly used performance measure for a branch and bound method [26, 9].

The details of computing *FAIL*, *TOTTM* and *MRATE* for a set of MIP instances are given below.

FAIL: The number of MIP instances used in an experiment that are not solved to optimality within the time limit. If a MIP instance is not solved using any contending method, then this MIP instance is excluded from *FAIL*. Methods with a smaller value of *FAIL* are better.

TOTTM: The total amount of time required to solve the set of MIP instances. Any MIP instance that is not solved using any contending method is excluded from this total. If a MIP instance is not solved using a given contending method but is solved using at least one other contender, then the time limit is added to this total for this instance. *TOTTM* is, therefore, a lower bound on the total time required to solve the set of MIP instances if a method fails on at least one MIP

instance (it is exact if the method solves all instances). Methods with a smaller value of $TOTTM$ are better.

MRATE: The geometric mean, see Definition 1, of t_{MI}/t_{MIB} over all solved MIP instances for a given contending method. t_{MI} is the time required to solve a MIP instance MI . t_{MIB} is the best time to solve MIP instance MI over all contending methods. Any MIP instance that is not solved using any contending method is excluded from this mean. If a MIP instance is not solved using a given contending method but is solved using at least one other contender, then $t_{MI} =$ the time limit. $MRATE$ is, therefore, a lower bound if a method fails on at least one MIP instance (it is exact if the method solves all instances). If t_{MI} is smaller than 10 seconds for every contending method then MI is excluded from $MRATE$ to reduce the effects of timing error on this measure. Methods with a smaller value of $MRATE$ are better.

Another method of combining the solution times of several MIP instances in order to evaluate the relative performance of contending methods is to use a *performance profile* [13]. A performance profile for a contending method is a plot that displays a point representing each MI that is solved. For a given MIP instance $MI1$ and contending method, the x-axis represents the value of r_{MI1} and the y-axis represents the fraction f_{MI1} of MIP instances in the set that have a value of $r_{MI} \leq r_{MI1}$ for a particular method. Figure 7.10 shows an example of a performance profile. In this example, $r_{MI} \leq 1.5$ for a little more than 80% of MIP instances for the MOBP-MPAS-ANST contender. In general, contending methods with points located nearest the top-left of the performance profile are considered to be best. By looking for points where $r_{MI} = 1$, the performance profile shows the fraction of MIP instances that a contending method performs best. The largest value of f_{MI} over all of the MIP instances for a contending method is the fraction of MIP instances that are solved using this method. If t_{MI} is smaller than 10 seconds for every contending method then MI is excluded to reduce the effects of timing error on the performance profile.

Commonly used performance measures for unsolved MIP instances are the best incumbent solution found and proven optimality gap at the termination of the branch and bound method [26, 9]. In order to determine how the competitor methods performed on the more difficult MIP instances, the following scheme is used to determine the relative ranking of two methods for a given MIP instance. Ties are allowed.

- If both methods find a MIP feasible solution then the better method is the one that finds the solution with the best value of the objective function.
- If both methods find equally good MIP feasible solutions then the method with the smaller provable optimality gap (See Definition 2) is better.
- A method is better if it finds a MIP feasible solution whereas the other method does not.

Definition 2 (Provable optimality gap) *The provable optimality gap G_o represents the proximity of the upper and lower bounds on the MIP optimal objective value. The value of G_o is computed as follows.*

$$G_o = \frac{|z_{inc} - z_g^{min}|}{|z_{inc}| + \epsilon}$$

where

z_g^{min} = the minimum objective value over all active nodes;

z_{inc} = the objective value of the incumbent solution;

$\epsilon = 10^{-9}$ (used to avoid dividing by 0)

A MIP is solved if $G_o = 0$. ■

The following measures are based on the above ranking scheme.

AVGRANK is the average rank of a given method over all MIP instances that are not solved using any contending method. MIP instances for which no incumbent solution is found by any contending method are also excluded. Methods with a smaller value of *AVGRANK* are better.

NFIRSTS is the number of MIP instances for which a given method is ranked first. MIP instances that are solved by at least 1 contending method are excluded. MIP instances for which no incumbent solution is found by any contending method are also excluded. Methods with a larger value of *NFIRSTS* are better.

NINC is the number of MIP instances for which a given method did not find an incumbent solution. MIP instances that are solved by at least 1 contending method are excluded. MIP instances for which no incumbent solution is found by any contending method are also excluded. Methods with a smaller value of *NINC* are better.

6.4 MIP Instances

As is mentioned in Section 6.3, in order to evaluate the performance of branch and bound methods, a sufficiently large set of MIP instances should be solved using these methods. The set of MIP instances used in the experiments can have a large impact on the results. The set of MIP instances used in the experiments to determine the best node selection configuration is chosen such that MIP instances with a wide variety of MIP formulation characteristics are included because there are no restrictions placed on the new node selection methods proposed in Chapter 5 with respect to MIP formulation characteristics.

The experiments presented in this thesis use all 116 MIP instances from MIPLIB (MIPLIB 2003 [5], MIPLIB 3.0 [11], and MIPLIB 2.0 [10]) and all 156 MIP instances from CORAL [24], for a total of 272 MIP instances. The MIP instances taken from these sources are problems from industry, theoretical problems, or problems submitted

Table 6.1: The minimum, average, and maximum values of the given formulation characteristics.

	Min	Avg	Max
<i>Cols</i>	9	10251.27	236376
<i>Real</i>	0	2906.45	218806
<i>Int</i>	0	491.05	67732
<i>Bin</i>	0	6853.77	236376
<i>Rows</i>	4	7897.77	318335
<i>Ineq</i>	0	6640.47	318334
<i>Eq</i>	0	1256.3	73559
<i>NZ</i>	54	104453.55	3946449

to the NEOS server. These MIP instances are commonly used in testing branch and bound methods [3, 25]. A complete list of these MIP instances is shown in Table A.1 along with the formulation characteristics described below. Table 6.1 shows the minimum, average, and maximum values of these formulation characteristics.

Cols: The number of variables.

Real: The number of real variables.

Int: The number of non-binary integer variables.

Bin: The number of binary variables.

Rows: The number of constraints.

Ineq: The number of inequality constraints.

Eq: The number of equality constraints.

NZ: The number of non-zero constraint and objective function coefficients.

6.5 Branch and Bound Parameters

Three parameters that affect the performance of the branch and bound method are described below.

- The backtracking node selection method used (GLPK parameter: LPX_K_BTRACK).
- The branching variable selection method used (GLPK parameter: LPX_K_BRANCH).
- The root node heuristics used (GLPK parameter: LPX_K_USECUTS).

6.5.1 Branching Variable Selection Parameter

When performing the experiments to determine the best node selection configuration to use when solving MIPs, the branching variable selection method used should be the best available. To determine the best available branching variable selection method, the set of 116 MIP instances shown in Table A.2 was solved four times, each time using a different contending method with the same settings for all other parameters. The contending branching variable selection methods are described below.

BFIRST: Branch on the first candidate in the list of variables.

BLAST: Branch on the last candidate in the list of variables.

BDRTO: Branch on the candidate variable selected using the method by Driebeek [14] and Tomlin [38] described in Section 2.2.1.

BMFRAC: Branch on the candidate variable with the fractional value closest to 0.5.

These methods are used because they are the only methods available in GLPK. A commonly used state of the art branching variable selection method that uses pseudo-costs is not implemented in GLPK and is thus excluded from the list of contending methods.

Table 6.2: Performance of contending branching variable selection methods. The best values in each of the FAIL, TOTTM and MRATE columns are in bold.

Contender	FAIL	TOTTM	MRATE
BFIRST	8	33543.63	2.75
BLAST	11	46272.01	3.93
BDRTO	4	24966.18	2.09
BMFRAC	8	41363.03	3.49

The values of the GLPK parameters not involving branching variable selection are set as follows.

- The best-projection backtracking node selection method (Section 3.2.1) is used. This is the default method in GLPK (LPX_K_BTRACK = 2).
- Do not use Gomory's mixed integer cuts at the root node. This is the default setting in GLPK (LPX_K_USECUTS = 0).
- A time limit of 1 hour is used (LPX_K_TMLIM = 3600 seconds).

The results of this experiment are summarized in Table 6.2. 64 MIP instances were solved by at least 1 contending method. The BDRTO method is clearly the best of the contending methods and therefore is used in all subsequent node selection experiments. Note that BDRTO is the default method in GLPK. The complete results of using BFIRST, BLAST, BDRTO and BMFRAC in this experiment are shown in Appendix B.1.

Table 6.3: Performance of root node cuts vs no cuts. The best values in each of the FAIL, TOTTM and MRATE columns are in bold.

Contender	FAIL	TOTTM	MRATE
NOCUTS	5	28567.19	5.85
CUTS	3	16740.15	5.08

6.5.2 Root Node Cuts

This experiment determined whether it is generally better to include or exclude root node cuts. The contending methods are (i) root node cuts, CUTS , and (ii) no cuts, NOCUTS .

The 116 MIP instances used for this experiment are shown in Table A.2. In this experiment, the values of the GLPK parameters not involving root node cuts are the same values as those used in the experiment described in Section 6.5.1.

Table 6.3 shows the performance of both contenders. 65 MIP instances were solved by at least 1 contending method. Using root node cuts is clearly better than not using cuts and therefore root node cuts are used in the contending node selection methods experiments. The complete results of using default GLPK with root node cuts are shown in Appendix B.2.

6.5.3 Node Selection Parameters

The experiment that determined the best node selection configuration to use when solving MIPs is described in Chapter 7, as well as the contending node selection configurations used in this experiment. There are state of the art contending node selection methods that are not available in GLPK, so these methods were implemented and added to GLPK for this experiment. These methods are listed below.

- The best-estimate backtracking node selection method (Section 3.2.2).
- The best-projection aspiration backtrack triggering method (Sections 3.2.1 and 3.4).
- The best-estimate pseudo-cost aspiration backtrack triggering method (Sections 3.2.2 and 3.4).
- The hybrid best-estimate and best-first backtracking node selection method (Section 3.2.2).
- The method that triggers backtracking after every node is solved.

Chapter 7

Experimental Results

The purpose of the experiments in this section is to test all configurations of backtracking node selection, backtrack triggering, and active node search threshold methods to determine which is the best configuration to use for solving MIP instances. All configurations are tested to assess how well each option for each component of a configuration performs in conjunction with various options for the other components.

The options for backtracking node selection method, backtrack triggering method and active node search threshold are described below.

- Backtracking node selection methods:
 - State of the art methods that are available in GLPK.
 - DEPF**: Depth-first (Section 3.1).
 - BREF**: Breadth-first (Section 3.1).
 - DEBP**: Default best-projection (Default GLPK, Section 3.2.1).
 - BESF**: Best-first (Section 3.1).
 - State of the art methods that were added to GLPK for this experiment.
 - BEES**: Best-estimate (Section 3.2.2).
 - BFBE**: BEES interleaved with BESF (Section 3.2.2).
 - New methods that are proposed in this thesis.

DIST: Distribution (Method proposed in Section 5.2).

MOBP: Modified best-projection (Method proposed in Section 5.1).

- Backtrack triggering methods:

- State of the art methods that are available in GLPK.

NONA: Non-aspiration backtracking: trigger backtracking only at leaf nodes (Default GLPK).

- State of the art methods that were added to GLPK for this experiment.

ALLT: Perform backtracking node selection after every node solution.

DBPA: Default best-projection aspiration (Section 3.4).

PCAS: Pseudo-cost (best-estimate) aspiration (Section 3.4).

- New methods that are proposed in this thesis.

LEXA: Linear feasibility depth extrapolation aspiration (Proposed in Section 5.3).

MPAS: Modified best-projection aspiration (Proposed in Section 5.1).

- Active node search threshold (Proposed in Section 5.4) options:

NOAN: Do not use ANST (Default GLPK).

ANST: Use ANST.

Each state of the art method that is not listed above is excluded from this experiment because (i) there are no documented empirical studies in which the excluded method outperforms all of the above methods, (ii) the principal behavior of the excluded method relies on user defined parameters, or (iii) the details required to properly implement the excluded method are not documented.

One example of a node selection configuration is DEBP-NONA-NOAN. In this configuration best-projection backtracking node selection is used whenever a leaf node is reached, and the ANST method is not used. This is the default configuration in

GLPK. All of the competitor configurations in this experiment are listed in Table 7.1. This list excludes any configuration in which depth-first backtracking is combined with any backtrack triggering method other than NONA because depth-first backtracking will choose a child of the most recently solved node if the most recently solved node is not a leaf. Also excluded from this list is any configuration in which ANST is combined with either depth-first or breadth-first node selection since depth-first is the simple node selection method used in ANST and since breadth-first is usually no more costly than depth-first.

The list of competitor configurations is quite large; there are 79 configurations. Due to the impractical amount of time required to evaluate the performance of every configuration on a sufficiently large set of MIP instances, this experiment is carried out in two stages. The purpose of the first stage is to reduce the list of competitor configurations to a small subset of the most promising configurations. The set of MIP instances used in stage 1 is composed such that the performance of all of the competitor configurations can be evaluated within a reasonable amount of time, i.e. these are the models that tend to get solved within a shorter timeframe. The details and results of stage 1 of this experiment are presented in Section 7.1. The purpose of the second stage is to evaluate the performance of the most promising configurations determined in stage 1 on a larger set of MIP instances. The details and results of stage 2 of this experiment are presented in Section 7.2.

For both stages of this experiment, if the amount of memory used when solving a MIP instance exceeds 1.5 gigabytes then the depth-first backtracking node selection method without an aspiration backtrack triggering method is used until the memory usage drops below 1 gigabytes. This method is commonly used to avoid using the extremely slow swap space of memory [4]. This threshold was never exceeded in these experiments.

Table 7.1: List of all competitor node selection configurations. The configurations that are available in the original GLPK solver are in boldface.

BEES-ALLT-ANST	BFBE-DBPA-NOAN	DEPF-NONA-NOAN
BEES-ALLT-NOAN	BFBE-LEXA-ANST	DIST-ALLT-ANST
BEES-DBPA-ANST	BFBE-LEXA-NOAN	DIST-ALLT-NOAN
BEES-DBPA-NOAN	BFBE-MPAS-ANST	DIST-DBPA-ANST
BEES-LEXA-ANST	BFBE-MPAS-NOAN	DIST-DBPA-NOAN
BEES-LEXA-NOAN	BFBE-NONA-ANST	DIST-LEXA-ANST
BEES-MPAS-ANST	BFBE-NONA-NOAN	DIST-LEXA-NOAN
BEES-MPAS-NOAN	BFBE-PCAS-ANST	DIST-MPAS-ANST
BEES-NONA-ANST	BFBE-PCAS-NOAN	DIST-MPAS-NOAN
BEES-NONA-NOAN	BREF-ALLT-NOAN	DIST-NONA-ANST
BEES-PCAS-ANST	BREF-DBPA-NOAN	DIST-NONA-NOAN
BEES-PCAS-NOAN	BREF-LEXA-NOAN	DIST-PCAS-ANST
BESF-ALLT-ANST	BREF-MPAS-NOAN	DIST-PCAS-NOAN
BESF-ALLT-NOAN	BREF-NONA-NOAN	MOBP-ALLT-ANST
BESF-DBPA-ANST	BREF-PCAS-NOAN	MOBP-ALLT-NOAN
BESF-DBPA-NOAN	DEBP-ALLT-ANST	MOBP-DBPA-ANST
BESF-LEXA-ANST	DEBP-ALLT-NOAN	MOBP-DBPA-NOAN
BESF-LEXA-NOAN	DEBP-DBPA-ANST	MOBP-LEXA-ANST
BESF-MPAS-ANST	DEBP-DBPA-NOAN	MOBP-LEXA-NOAN
BESF-MPAS-NOAN	DEBP-LEXA-ANST	MOBP-MPAS-ANST
BESF-NONA-ANST	DEBP-LEXA-NOAN	MOBP-MPAS-NOAN
BESF-NONA-NOAN	DEBP-MPAS-ANST	MOBP-NONA-ANST
BESF-PCAS-ANST	DEBP-MPAS-NOAN	MOBP-NONA-NOAN
BESF-PCAS-NOAN	DEBP-NONA-ANST	MOBP-PCAS-ANST
BFBE-ALLT-ANST	DEBP-NONA-NOAN	MOBP-PCAS-NOAN
BFBE-ALLT-NOAN	DEBP-PCAS-ANST	
BFBE-DBPA-ANST	DEBP-PCAS-NOAN	

7.1 Identification of Promising Configurations

The purpose of this experiment is to test all configurations of backtracking node selection, backtrack triggering, and active node search threshold methods on a small set of MIP instances in order to determine which are the most promising configurations to test on a larger set of MIP instances.

The time limit for this experiment is 30 minutes per MIP solution. The MIP instances used in this experiment are those in Table A.1 that were solved within 30 minutes using the default GLPK node selection configuration, DEBP-NONA-NOAN. There are 79 of these MIP instances and they are shown in Table A.3. Choosing the set of MIP instances in this manner biases the results to favor DEBP-NONA-NOAN by removing any MIP instances that are not solved within the time limit using DEBP-NONA-NOAN even though these MIP instances may be solved within the time limit using a different configuration. This experiment required nearly 3 weeks of computation time to complete.

In order to determine how each competitor configuration ranks compared to the others, a method is needed that incorporates the three performance measures, TOTTM, MRATE and FAIL, described in Section 6.3 into a single measure. Towards this end, the following definitions are used to rank each competitor configuration.

TR: Time rank is the rank of a competitor configuration according to TOTTM.

RR: Ratio rank is the rank of a competitor configuration according to MRATE.

SR: Sum rank is the measure used to determine the overall rank of each competitor configuration. SR is computed using Equation 7.1. Configurations with smaller values of SR are better.

R: The rank of each competitor configuration according to SR.

$$SR = FAIL + TR + RR \quad (7.1)$$

Equation 7.1 is used to rank the competitor configurations because it is desirable for a competitor configuration to have the smallest value of $FAIL$, TR , and RR . Addition is used to prevent one of these measures from dominating the others. Table 7.5 shows that the largest difference either between TR and R , or between RR and R , is 11 for BFBE-PCAS-NOAN which has $R = 43$, $TR = 52$, and $RR = 32$. The average of this difference over all configurations is 3.2. $FAIL$ is added to SR instead of the ranking according to $FAIL$ because (i) $FAIL$ is integer valued, and (ii) by adding the $FAIL$ rank to SR , the difference between the values of $FAIL$ for two configurations is lost if this difference is greater than the difference in their ranks. In this experiment, the ranking of each configuration according to $FAIL$ is equal to $FAIL + 1$ for all but the worst 3 configurations, therefore either method of computing SR would give similar results.

Table 7.2 shows $MRATE$ and RR as well as the variance $MVAR$ in r_{MI} (see Page 67 for definition) for each configuration. RR is 19 for the best state of the art configuration, and its value of $MRATE$ that is 16% worse than that of the smallest value of $MRATE$. This table shows that the range in $MVAR$ is quite large, the smallest value being 0.52 and the largest being 250679, and that the smaller values of $MVAR$ tend to correspond to smaller values of $MRATE$. Note that the smallest $MVAR$ corresponds to the smallest $MRATE$, and that both of these are associated with DIST-ALLT-NOAN, indicating that this is a very high quality method. Figure 7.1 shows a plot of $MRATE$, and the 10th and 90th percentiles for r_{MI} for each configuration. This plot also shows that smaller values of $MRATE$ tend to correspond to smaller values of both percentiles, especially the 90th percentile.

Table 7.2: Performance of each configuration in Table 7.1 with respect to $MRATE$. Also shown is the variance of r_{MI} over all MIP instances for each configuration. The configurations that are available in the original GLPK solver are in boldface.

RR	Configuration	MRATE	MVAR
1	DIST-ALLT-NOAN	1.28927	0.52
Continued on next page			

Table 7.2 – continued from previous page

RR	Configuration	MRATE	MVAR
2	MOBP-MPAS-ANST	1.35222	50.37
3	MOBP-ALLT-NOAN	1.35982	24.63
4	MOBP-MPAS-NOAN	1.36108	50.88
5	MOBP-ALLT-ANST	1.36519	32.72
6	MOBP-PCAS-ANST	1.37487	50.21
7	MOBP-LEXA-ANST	1.38146	61.11
8	MOBP-PCAS-NOAN	1.38447	49.42
9	MOBP-DBPA-ANST	1.38864	63.11
10	DIST-MPAS-ANST	1.39187	113.8
11	MOBP-LEXA-NOAN	1.41064	60.94
12	MOBP-DBPA-NOAN	1.41183	62.67
13	MOBP-NONA-ANST	1.41603	64.79
14	MOBP-NONA-NOAN	1.42075	62.46
15	BFBE-ALLT-ANST	1.43168	0.97
16	BFBE-MPAS-ANST	1.45506	99.28
17	DIST-MPAS-NOAN	1.46461	105.76
18	DIST-DBPA-NOAN	1.47475	69.85
19	BFBE-ALLT-NOAN	1.53420	1.53
20	BFBE-PCAS-ANST	1.55149	108.2
21	DIST-DBPA-ANST	1.55960	78.2
22	DIST-LEXA-NOAN	1.56728	76.09
23	BFBE-DBPA-ANST	1.56836	23.77
24	BFBE-LEXA-ANST	1.59054	102.7
25	DIST-ALLT-ANST	1.59373	170.65
26	DIST-PCAS-NOAN	1.60639	195.85
27	DIST-NONA-ANST	1.61207	74.7
28	BFBE-NONA-ANST	1.61824	66.24
29	BEES-PCAS-ANST	1.62567	112.87
30	DEBP-DBPA-NOAN	1.63204	85.86
31	DIST-PCAS-ANST	1.64611	201.61
32	BEES-MPAS-ANST	1.64937	117.82
33	DEBP-DBPA-ANST	1.65889	75
34	DEBP-PCAS-ANST	1.67976	41.47
35	BFBE-MPAS-NOAN	1.68251	380.18
36	DEBP-ALLT-ANST	1.68649	13.53
37	BFBE-LEXA-NOAN	1.69066	313.12
38	DEBP-LEXA-NOAN	1.70648	90.84
39	DIST-NONA-NOAN	1.70819	76.69
40	DIST-LEXA-ANST	1.71448	99.53
41	BFBE-NONA-NOAN	1.72498	169.42
42	DEBP-NONA-NOAN	1.73127	92.05
43	BFBE-PCAS-NOAN	1.73483	910
44	BEES-DBPA-ANST	1.74104	116.75
45	BEES-LEXA-ANST	1.75071	116.44
46	DEBP-NONA-ANST	1.75084	75.54

Continued on next page

Table 7.2 – continued from previous page

RR	Configuration	MRATE	MVAR
47	BFBE-DBPA-NOAN	1.75157	190.93
48	BEES-ALLT-ANST	1.75423	155.33
49	DEBP-PCAS-NOAN	1.75796	156
50	BEES-NONA-ANST	1.76326	115.47
51	DEBP-LEXA-ANST	1.76442	117.23
52	BEES-PCAS-NOAN	1.76983	112.92
53	BEES-MPAS-NOAN	1.77120	115.09
54	BESF-PCAS-ANST	1.88164	86.18
55	DEBP-MPAS-NOAN	1.89579	215.62
56	BESF-LEXA-ANST	1.89786	79.79
57	DEBP-MPAS-ANST	1.89901	387.24
58	DEBP-ALLT-NOAN	1.91466	2609.09
59	BEES-DBPA-NOAN	1.91629	322.56
60	BEES-LEXA-NOAN	1.91848	315.19
61	BESF-LEXA-NOAN	1.92596	170.11
62	BEES-ALLT-NOAN	1.92709	328.18
63	BESF-DBPA-ANST	1.95607	193.56
64	BESF-NONA-NOAN	1.96045	170.92
65	BESF-PCAS-NOAN	1.96116	162.79
66	BESF-NONA-ANST	1.96341	85.02
67	BESF-DBPA-NOAN	1.96601	186.24
68	BESF-MPAS-ANST	1.99262	370.14
69	BESF-MPAS-NOAN	2.04399	165.5
70	BEES-NONA-NOAN	2.06815	354.12
71	BREF-LEXA-NOAN	2.07558	551.76
72	BREF-DBPA-NOAN	2.09071	566.65
73	BREF-NONA-NOAN	2.10032	566.16
74	BESF-ALLT-ANST	2.18260	310.81
75	BESF-ALLT-NOAN	2.33030	557.86
76	BREF-PCAS-NOAN	2.39173	540.84
77	DEPF-NONA-NOAN	2.71091	250679.22
78	BREF-MPAS-NOAN	3.00261	3670.75
79	BREF-ALLT-NOAN	4.44186	3921.76

Table 7.4 shows the ranking of each configuration according to *TOTTM*. *TR* is 21 for the best state of the art configuration, and its value of *TOTTM* is 11.7% worse than that of the smallest value of *TOTTM*.

Table 7.5 shows the performance of each competitor configuration along with their ranks. Standard node selection methods such as depth-first (DEPF-NONA-NOAN), best-first (BESF-NONA-NOAN), and breadth-first (BREF-NONA-NOAN) ranked

Figure 7.1: A plot showing MRATE, r_{p10} , and r_{p90} for each configuration. r_{p10} and r_{p90} are the 10th and 90th percentiles, respectively, of r_{MI} (see Page 67 for definition) over all MIP instances for a given configuration. Each x-axis value of RR uniquely corresponds to a configuration as is shown in Table 7.2.

76, 64, and 71 respectively. The value of FAIL for DEBP-NONA-NOAN is 0 because the set of MIP instances used in this experiment excludes any MIP instance that was not solved by DEBP-NONA-NOAN within the time limit. The top 21 ranked competitor configurations involve at least one method out of those newly proposed in Sections 5.1 through 5.4. The top ranked competitor configuration, MOBP-MPAS-ANST, consists only of methods proposed in Sections 5.1 and 5.4. Each of the new methods proposed in Sections 5.1 through 5.4 are used at least once in the top 7 ranked competitor configurations. For this reason, these top 7 ranked configurations will be tested in the final node selection experiment described in Section 7.2. The best state of the art configuration, DEBP-DBPA-NOAN, is ranked 22 whereas the default GLPK configuration, DEBP-NONA-NOAN, is ranked 39. These two configurations are also tested in the final node selection experiment to provide a comparison to the best state of the art and to the baseline GLPK.

Analysis of variance (ANOVA) tests on r_{MI} show that there is statistical significance in the differences between the best ranked state of the art configuration

Table 7.3: The p-values of the analysis of variance (ANOVA) tests of *MRATE* between the best ranked state of the art configuration DEBP-DBPA-NOAN and each of the top seven ranked configurations. Smaller p-values mean that the difference between the values of *MRATE* for considered methods is more significant.

DEBP-DBPA-NOAN vs.	p-value
DIST-ALLT-NOAN	0.05
DIST-MPAS-ANST	0.11
MOBP-LEXA-ANST	0.1
MOBP-PCAS-NOAN	0.12
MOBP-PCAS-ANST	0.1
MOBP-MPAS-NOAN	0.06
MOBP-MPAS-ANST	0.05

DEBP-DBPA-NOAN and that of each of the top 7 ranked configurations with respect to *MRATE*. The p-values of these tests are shown in Table 7.3. ANOVA tests also show that there is very little statistical significance in the differences among the top 7 methods with respect to *MRATE*.

To determine the overall effect of using ANST versus not using it, the arithmetic means of each of the performance measures ($\overline{TOTTM}$, $\overline{MRATE}$ and $\overline{FAIL}$) are computed over all configurations with ANST and compared to the corresponding means taken over all configurations with NOAN. All configurations with BREF and DEPF are excluded from the means since neither of these methods is combined with ANST in this experiment. Table 7.6 shows that on average, ANST outperforms NOAN for all three of these measures. With respect to $\overline{FAIL}$, ANST improves performance by about 22.5%. With respect to $\overline{TOTTM}$, ANST improves performance by about 5.1%. With respect to $\overline{MRATE}$, ANST improves performance by about 1.7%.

Table 7.7 shows the values of $\overline{TOTTM}$, $\overline{MRATE}$ and $\overline{FAIL}$ over all of the configurations, separating them according to backtrack triggering method. Figures 7.2, 7.3, and 7.4 show that the performance with respect to solving MIPs quickly of any

Table 7.4: Performance of each configuration in Table 7.1 with respect to TOTTM. The configurations that are available in the original GLPK solver are in boldface.

TR	Configuration	TOTTM	TR	Configuration	TOTTM
1	MOBP-PCAS-ANST	16812.65	41	BFBE-MPAS-NOAN	21122.74
2	MOBP-PCAS-NOAN	16861.29	42	BEES-PCAS-ANST	21775.88
3	MOBP-MPAS-ANST	17115.24	43	BFBE-NONA-NOAN	21827.77
4	MOBP-MPAS-NOAN	17393.7	44	BFBE-ALLT-NOAN	22000.64
5	BFBE-MPAS-ANST	17669.74	45	BEES-ALLT-ANST	22031.86
6	DIST-MPAS-ANST	17695.79	46	BEES-MPAS-ANST	22169.74
7	DIST-ALLT-NOAN	17776.13	47	BFBE-DBPA-NOAN	22257.84
8	MOBP-LEXA-ANST	17906.25	48	BEES-DBPA-ANST	22365.05
9	MOBP-NONA-NOAN	17970.21	49	BEES-LEXA-ANST	22514.53
10	MOBP-ALLT-ANST	18045.45	50	BEES-NONA-ANST	22520.3
11	MOBP-DBPA-ANST	18075.98	51	DEBP-MPAS-ANST	22643.89
12	MOBP-DBPA-NOAN	18143.23	52	BFBE-PCAS-NOAN	22976.51
13	MOBP-LEXA-NOAN	18183.25	53	DEBP-ALLT-NOAN	23087.6
14	MOBP-ALLT-NOAN	18201.74	54	DEBP-MPAS-NOAN	23710.79
15	MOBP-NONA-ANST	18279.89	55	BESF-PCAS-ANST	23855.92
16	DIST-NONA-ANST	18290.62	56	BESF-LEXA-ANST	24022.76
17	DIST-DBPA-ANST	18455.03	57	BESF-LEXA-NOAN	24096.97
18	DIST-MPAS-NOAN	18577.81	58	BESF-NONA-ANST	24156.89
19	BFBE-LEXA-ANST	18621.14	59	BEES-MPAS-NOAN	24303.55
20	DIST-DBPA-NOAN	18986.03	60	BESF-DBPA-ANST	24345.53
21	DEBP-DBPA-NOAN	19042.51	61	BEES-LEXA-NOAN	24393.86
22	DIST-LEXA-NOAN	19091.69	62	BESF-NONA-NOAN	24472.94
23	DIST-LEXA-ANST	19439	63	BEES-DBPA-NOAN	24534.95
24	DEBP-DBPA-ANST	19455.25	64	BESF-PCAS-NOAN	24559.31
25	BFBE-DBPA-ANST	19548.95	65	BESF-DBPA-NOAN	24595.95
26	DEBP-ALLT-ANST	19553.35	66	BEES-PCAS-NOAN	24649.17
27	BFBE-NONA-ANST	19619.03	67	BESF-MPAS-ANST	24657.48
28	DIST-NONA-NOAN	19869.09	68	BEES-ALLT-NOAN	25029.99
29	DEBP-NONA-ANST	19870.05	69	BEES-NONA-NOAN	25343.76
30	DIST-ALLT-ANST	19963.48	70	BESF-MPAS-NOAN	25414.45
31	BFBE-ALLT-ANST	19967.51	71	BREF-NONA-NOAN	25816.53
32	BFBE-PCAS-ANST	19992.11	72	BREF-LEXA-NOAN	25855.29
33	DIST-PCAS-ANST	20190.27	73	BREF-DBPA-NOAN	26079.39
34	DEBP-LEXA-NOAN	20240.77	74	BESF-ALLT-ANST	27829
35	DEBP-PCAS-ANST	20390.92	75	BESF-ALLT-NOAN	28713.73
36	DEBP-NONA-NOAN	20584.63	76	DEPF-NONA-NOAN	30570.64
37	DIST-PCAS-NOAN	20650.23	77	BREF-PCAS-NOAN	32525.43
38	DEBP-LEXA-ANST	20832.68	78	BREF-MPAS-NOAN	36409.78
39	DEBP-PCAS-NOAN	20900.44	79	BREF-ALLT-NOAN	45973.67
40	BFBE-LEXA-NOAN	21077.97			

Table 7.5: Performance of each configuration in Table 7.1 and their rankings. The configurations that are available in the original GLPK solver are in boldface. The smallest values of *FAIL*, *TR*, and *RR* are also in boldface.

R	Configuration	FAIL	TR	RR	R	Configuration	FAIL	TR	RR
1	MOBP-MPAS-ANST	1	3	2	41	DEBP-PCAS-ANST	3	35	48
2	MOBP-MPAS-NOAN	1	4	3	42	DEBP-LEXA-ANST	3	38	46
3	MOBP-PCAS-ANST	1	1	7	43	BFBE-PCAS-NOAN	5	52	32
4	DIST-ALLT-NOAN	2	7	1	44	BFBE-DBPA-NOAN	3	47	40
5	DIST-MPAS-ANST	1	6	4	45	BEES-ALLT-ANST	2	45	45
5	MOBP-PCAS-NOAN	1	2	8	45	BEES-MPAS-ANST	3	46	43
7	MOBP-LEXA-ANST	2	8	6	45	BEES-PCAS-ANST	3	42	47
8	MOBP-DBPA-ANST	2	11	5	45	DEBP-PCAS-NOAN	3	39	50
9	BFBE-MPAS-ANST	1	5	16	49	BEES-DBPA-ANST	3	48	42
10	MOBP-DBPA-NOAN	2	12	9	50	BEES-NONA-ANST	3	50	49
10	MOBP-NONA-NOAN	2	9	12	51	BEES-LEXA-ANST	4	49	51
12	MOBP-LEXA-NOAN	2	13	10	52	DEBP-MPAS-ANST	3	51	52
13	MOBP-ALLT-ANST	2	10	14	53	DEBP-ALLT-NOAN	4	53	53
14	MOBP-NONA-ANST	2	15	13	54	DEBP-MPAS-NOAN	3	54	55
15	DIST-MPAS-NOAN	2	18	11	55	BESF-PCAS-ANST	5	55	57
15	MOBP-ALLT-NOAN	2	14	15	56	BESF-LEXA-NOAN	6	57	56
17	BFBE-LEXA-ANST	1	19	19	57	BESF-LEXA-ANST	5	56	59
17	DIST-DBPA-NOAN	2	20	17	58	BEES-DBPA-NOAN	5	63	54
17	DIST-NONA-ANST	1	16	22	59	BEES-MPAS-NOAN	5	59	61
20	DIST-DBPA-ANST	1	17	26	60	BESF-DBPA-ANST	6	60	60
21	DIST-LEXA-NOAN	2	22	21	60	BESF-NONA-ANST	6	58	62
22	DEBP-DBPA-NOAN	0	21	27	62	BESF-DBPA-NOAN	6	65	58
23	BFBE-ALLT-ANST	2	31	18	63	BEES-LEXA-NOAN	5	61	66
24	BFBE-PCAS-ANST	2	32	20	64	BESF-NONA-NOAN	6	62	65
25	BFBE-DBPA-ANST	1	25	30	64	BESF-PCAS-NOAN	6	64	63
25	BFBE-NONA-ANST	1	27	28	66	BESF-MPAS-ANST	6	67	64
27	DEBP-DBPA-ANST	1	24	35	67	BEES-PCAS-NOAN	4	66	69
28	DIST-LEXA-ANST	1	23	37	68	BEES-ALLT-NOAN	4	68	68
29	DEBP-ALLT-ANST	2	26	34	69	BESF-MPAS-NOAN	6	70	67
30	DIST-NONA-NOAN	2	28	33	70	BEES-NONA-NOAN	5	69	73
30	DIST-PCAS-NOAN	3	37	23	71	BREF-LEXA-NOAN	6	72	70
32	DIST-PCAS-ANST	2	33	31	71	BREF-NONA-NOAN	6	71	71
33	BFBE-LEXA-NOAN	2	40	25	73	BREF-DBPA-NOAN	6	73	72
34	BFBE-MPAS-NOAN	3	41	24	74	BESF-ALLT-ANST	8	74	74
34	DIST-ALLT-ANST	2	30	36	75	BESF-ALLT-NOAN	8	75	75
36	DEBP-NONA-ANST	1	29	41	76	DEPF-NONA-NOAN	7	76	76
37	DEBP-LEXA-NOAN	3	34	38	77	BREF-PCAS-NOAN	10	77	77
38	BFBE-ALLT-NOAN	3	44	29	78	BREF-MPAS-NOAN	13	78	78
39	DEBP-NONA-NOAN	0	36	44	79	BREF-ALLT-NOAN	19	79	79
40	BFBE-NONA-NOAN	2	43	39					

Table 7.6: Average performance of configurations with NOAN vs ANST. Configurations with BREF and DEPF are excluded. The best values of *FAIL*, *TOTTM*, and *MRATE* are in boldface.

	<i>FAIL</i>	<i>TOTTM</i>	<i>MRATE</i>
NOAN	3.33	21684.53	1.72
ANST	2.58	20574.42	1.69

Table 7.7: Average performance of each backtrack triggering method. DEPF-NONA-NOAN is excluded. The best values of *FAIL*, *TOTTM*, and *MRATE* are in boldface.

Aspiration	<i>FAIL</i>	<i>TOTTM</i>	<i>MRATE</i>
DBPA	2.92	21221.98	1.7
LEXA	3.23	21252.01	1.73
NONA	2.85	21432.44	1.76
PCAS	3.69	22010.78	1.8
MPAS	3.69	22221.9	1.76
ALLT	4.62	23705.7	1.94

given backtrack triggering method varies greatly depending on the choice of backtracking node selection method. For example, although MOBP-MPAS-ANST is the best ranked configuration, Table 7.7 shows that, on the average, MPAS is inferior to DBPA. Similarly, MOBP-LEXA-ANST ranks better than the highest ranked configuration with DBPA but LEXA is inferior to DBPA on average. Table 7.8 shows that the LEXA backtrack triggering method outperforms the state of the art backtrack triggering methods when either the BREF or BFBE backtracking node selection method is used. It also shows that the MPAS backtrack triggering method outperforms the state of the art backtrack triggering methods when either the BREF, DIST or MOBP node selection method is used.

Figure 7.5 shows that the average, minimum, and maximum rankings for MOBP and DIST are each less than the respective values for all of the state of the art

Figure 7.2: A plot showing the average, minimum, and maximum value of *FAIL* for each of the following backtrack triggering methods: method 1 = NONA, method 2 = ALLT, method 3 = DBPA, method 4 = PCAS, method 5 = LEXA, method 6 = MPAS.

Figure 7.3: A plot showing the average, minimum, and maximum value of *TOTTM* for each of the following backtrack triggering methods: method 1 = NONA, method 2 = ALLT, method 3 = DBPA, method 4 = PCAS, method 5 = LEXA, method 6 = MPAS.

Figure 7.4: A plot showing the average, minimum, and maximum value of *MRATE* for each of the following backtrack triggering methods: method 1 = NONA, method 2 = ALLT, method 3 = DBPA, method 4 = PCAS, method 5 = LEXA, method 6 = MPAS.

backtracking node selection methods. Figures 7.6, 7.7, and 7.8 show that the best backtracking node selection method has the smallest range of *FAIL*, *TOTTM*, and *MRATE*. The ranges of *FAIL*, *TOTTM*, and *MRATE* for the backtracking node selection methods tends to be smaller than for backtrack triggering methods.

The complete results of this experiment are given in Appendix B.3.

Table 7.8: Rank of configurations with the same backtracking node selection option.

R	Method 1	R	Method 2	R	Method 3
1	MOBP-MPAS-ANST	4	DIST-ALLT-NOAN	9	BFBE-MPAS-ANST
2	MOBP-MPAS-NOAN	5	DIST-MPAS-ANST	17	BFBE-LEXA-ANST
3	MOBP-PCAS-ANST	15	DIST-MPAS-NOAN	23	BFBE-ALLT-ANST
5	MOBP-PCAS-NOAN	17	DIST-DBPA-NOAN	24	BFBE-PCAS-ANST
7	MOBP-LEXA-ANST	17	DIST-NONA-ANST	25	BFBE-DBPA-ANST
8	MOBP-DBPA-ANST	20	DIST-DBPA-ANST	25	BFBE-NONA-ANST
10	MOBP-DBPA-NOAN	21	DIST-LEXA-NOAN	33	BFBE-LEXA-NOAN
10	MOBP-NONA-NOAN	28	DIST-LEXA-ANST	34	BFBE-MPAS-NOAN
12	MOBP-LEXA-NOAN	30	DIST-NONA-NOAN	38	BFBE-ALLT-NOAN
13	MOBP-ALLT-ANST	30	DIST-PCAS-NOAN	40	BFBE-NONA-NOAN
14	MOBP-NONA-ANST	32	DIST-PCAS-ANST	43	BFBE-PCAS-NOAN
15	MOBP-ALLT-NOAN	34	DIST-ALLT-ANST	44	BFBE-DBPA-NOAN
R	Method 4	R	Method 5	R	Method 6
22	DEBP-DBPA-NOAN	45	BEES-ALLT-ANST	55	BESF-PCAS-ANST
27	DEBP-DBPA-ANST	45	BEES-MPAS-ANST	56	BESF-LEXA-NOAN
29	DEBP-ALLT-ANST	45	BEES-PCAS-ANST	57	BESF-LEXA-ANST
36	DEBP-NONA-ANST	49	BEES-DBPA-ANST	60	BESF-DBPA-ANST
37	DEBP-LEXA-NOAN	50	BEES-NONA-ANST	60	BESF-NONA-ANST
39	DEBP-NONA-NOAN	51	BEES-LEXA-ANST	62	BESF-DBPA-NOAN
41	DEBP-PCAS-ANST	58	BEES-DBPA-NOAN	64	BESF-NONA-NOAN
42	DEBP-LEXA-ANST	59	BEES-MPAS-NOAN	64	BESF-PCAS-NOAN
45	DEBP-PCAS-NOAN	63	BEES-LEXA-NOAN	66	BESF-MPAS-ANST
52	DEBP-MPAS-ANST	67	BEES-PCAS-NOAN	69	BESF-MPAS-NOAN
53	DEBP-ALLT-NOAN	68	BEES-ALLT-NOAN	74	BESF-ALLT-ANST
54	DEBP-MPAS-NOAN	70	BEES-NONA-NOAN	75	BESF-ALLT-NOAN
R	Method 7				
71	BREF-LEXA-NOAN				
71	BREF-NONA-NOAN				
73	BREF-DBPA-NOAN				
77	BREF-PCAS-NOAN				
78	BREF-MPAS-NOAN				
79	BREF-ALLT-NOAN

Figure 7.5: A plot showing the average, minimum, and maximum ranking for each of the backtracking node selection methods shown in Table 7.8; method 1: MOBP, method 2: DIST, method 3: BFBE, method 4: DEBP, method 5: BEES, method 6: BESF, method 7: BREF.

Figure 7.6: A plot showing the average, minimum, and maximum value of *FAIL* for each of the backtracking node selection methods shown in Table 7.8; method 1: MOBP, method 2: DIST, method 3: BFBE, method 4: DEBP, method 5: BEES, method 6: BESF, method 7: BREF.

Figure 7.7: A plot showing the average, minimum, and maximum value of $TOTTM$ for each of the backtracking node selection methods shown in Table 7.8; method 1: MOBP, method 2: DIST, method 3: BFBE, method 4: DEBP, method 5: BEES, method 6: BESF, method 7: BREF.

Figure 7.8: A plot showing the average, minimum, and maximum value of $MRATE$ for each of the backtracking node selection methods shown in Table 7.8; method 1: MOBP, method 2: DIST, method 3: BFBE, method 4: DEBP, method 5: BEES, method 6: BESF, method 7: BREF.

Table 7.9: List of promising node selection configurations for the stage 2 experiment along with their rank R determined in Section 7.1 and why they are included in this experiment.

R	Configuration	Description
39	DEBP-NONA-NOAN	Default GLPK: included as a baseline for GLPK
22	DEBP-DBPA-NOAN	Top ranked state of the art configuration
1	MOBP-MPAS-ANST	Top ranked proposed configuration
2	MOBP-MPAS-NOAN	Second ranked proposed configuration
3	MOBP-PCAS-ANST	Third ranked proposed configuration
4	DIST-ALLT-NOAN	Fourth ranked proposed configuration
5	DIST-MPAS-ANST	Fifth ranked proposed configuration
5	MOBP-PCAS-NOAN	Fifth ranked proposed configuration
7	MOBP-LEXA-ANST	Highest ranked proposed configuration with LEXA

7.2 Testing of Promising Configurations

The purpose of this experiment is to perform a more thorough test to determine which of the most promising configurations found in Section 7.1 is best for solving MIPs. Towards this end, both the time limit for this experiment and the number of MIP instances are larger than those used in Section 7.1.

The competitor configurations for this experiment are the top 7 ranked configurations from Section 7.1 as well as the top ranked state of the art and GLPK baseline configurations. Seven of the top ranked configurations are used because each of the new methods proposed in this thesis is used in at least one of these configurations. These configurations are listed in Table 7.9.

The set of MIP instances used in this experiment is listed in Table A.1. This list includes the MIP instances from the experiment in Section 7.1. The time limit for each MIP solution in this experiment is 1 hour. This experiment required over 9 weeks of computation time to complete.

- 272 MIP instances are used in this experiment.
- 109 MIP instances were solved to optimality using at least 1 of the competitor configurations. These instances are listed in Table A.4.

- 130 MIP instances were not solved to optimality using any configuration, but at least one configuration found a feasible solution within the time limit. These MIP instances are listed in Table A.5.
- 33 MIP instances were not solved and no MIP feasible solution was found using any configuration. These MIP instances are listed in Table A.6.

Table 7.10 shows the performance of each competitor configuration. All but one of the configurations proposed in this thesis have fewer unsolved MIPs than the state of the art configurations. The configurations with the fewest unsolved MIPs are DIST-MPAS-ANST, MOBP-LEXA-ANST, MOBP-PCAS-ANST and MOBP-MPAS-ANST. These configurations all use ANST whereas all of the others do not, which indicates that using ANST increases the chances of solving difficult MIPs within a stated time limit.

Over all MIP instances, *TOTTM* for the worst proposed configuration, MOBP-PCAS-NOAN, is 7.8% less than the best state of the art configuration, DEBP-DBPA-NOAN. In the case of the best proposed configuration, MOBP-MPAS-ANST, *TOTTM* is 20.7% less than DEBP-DBPA-NOAN. With respect to *MRATE*, the worst proposed configuration, MOBP-PCAS-NOAN, is 18.7% better than the best state of the art configuration, DEBP-DBPA-NOAN. The best proposed configuration, MOBP-MPAS-ANST, has a value of *MRATE* that is 24.2% better than DEBP-DBPA-NOAN. The *TOTTM* for the second best proposed method, DIST-MPAS-ANST, is 0.001% worse than MOBP-MPAS-ANST and the *MRATE* for DIST-MPAS-ANST is 3.1% worse than MOBP-MPAS-ANST which indicates that the performance of DIST-MPAS-ANST for solving MIPs is very similar to MOBP-MPAS-ANST.

To investigate the effect of any bias introduced by including the results from the first experiment in the second experiment, we recalculated the results with those models removed. The bias is negligible. Over all MIP instances that were not used in the experiment in Section 7.1, *TOTTM* for the worst proposed configuration, MOBP-PCAS-NOAN, is 6.9% less than the best state of the art configuration,

DEBP-DBPA-NOAN. In the case of the best proposed configuration, MOBP-MPAS-ANST, $TOTTM$ is 22% less than DEBP-DBPA-NOAN. With respect to $MRATE$, the worst proposed configuration, MOBP-PCAS-NOAN, is 23.5% better than the best state of the art configuration, DEBP-DBPA-NOAN. The best proposed configuration, MOBP-MPAS-ANST, has a value of $MRATE$ that is 35.2% better than DEBP-DBPA-NOAN.

ANOVA tests on r_{MI} show that there is statistical significance in the differences between DEBP-DBPA-NOAN and most of the proposed configurations with respect to $MRATE$. The p-values of these tests are shown in Table 7.11. The MOBP-PCAS-NOAN configuration is not significantly different from DEBP-DBPA-NOAN, but it is also the worst ranked proposed configuration.

Table 7.10: The Performance of each configuration in Table 7.9: (a) over all MIP instances; (b) excluding MIP instances used in the experiment in Section 7.1. The smallest values of *FAIL*, *TOTM*, *MRATE*, *AVGRANK*, and *NINC* are in boldface. The largest value of *NFIRSTS* is also in boldface. Also shown is the variance of *MRATE*, *MVAR*, and the variance of *AVGRANK*, *RVAR*.

Configuration	Solved MIPs				Unsolved MIPs			
	FAIL	TOTM	MRATE	MVAR	AVGRANK	RVAR	NFIRSTS	NINC
MOBP-MPAS-ANST	6	86653.72	1.32825	179.24	4.54	3.49	8	75
DIST-MPAS-ANST	6	86654.38	1.37045	176.78	2.89	4.08	34	21
MOBP-PCAS-ANST	6	88560.12	1.34409	22.83	4.34	3.88	11	72
MOBP-LEXA-ANST	6	91509.11	1.39778	177.01	4.18	4.05	17	57
DIST-ALLT-NOAN	15	97340.26	1.34277	161.26	2.48	4.27	67	17
MOBP-MPAS-NOAN	13	99798.67	1.42522	185.78	4.62	3.51	7	81
MOBP-PCAS-NOAN	12	100752.39	1.49822	21.99	4.6	4.19	9	82
DEBP-DBPA-NOAN	14	109239.58	1.75282	1729.37	3.74	4.89	21	15
DEBP-NONA-NOAN	15	109579.06	1.78324	1729.55	3.47	4.68	25	15

		(a)		(b)	
MOBP-MPAS-ANST	6	71701.63	1.32	362.65	
DIST-MPAS-ANST	6	72487.06	1.41	361.79	
MOBP-PCAS-ANST	6	73780.67	1.35	17.95	
MOBP-LEXA-ANST	6	76864.76	1.48	361.4	
DIST-ALLT-NOAN	15	82927.2	1.61	360.57	
MOBP-MPAS-NOAN	13	84230.99	1.60	361.12	
MOBP-PCAS-NOAN	12	85611.83	1.56	18.28	
DEBP-DBPA-NOAN	14	91985.95	2.04	3814.98	
DEBP-NONA-NOAN	15	93186.08	2.08	3810.44	

Table 7.11: The p-values of the ANOVA tests of *MRATE* between the best state of the art configuration and each of the proposed configurations.

DEBP-DBPA-NOAN vs.	p-value
DIST-ALLT-NOAN	0.07
DIST-MPAS-ANST	0.03
MOBP-LEXA-ANST	0.04
MOBP-PCAS-NOAN	0.27
MOBP-PCAS-ANST	0.02
MOBP-MPAS-NOAN	0.06
MOBP-MPAS-ANST	0.01

Figure 7.9 shows *MRATE*, r_{p10} , and r_{p90} for the configurations in Table 7.10. The difference between r_{p90} and r_{p10} is smallest for the MOBP-MPAS-ANST, DIST-MPAS-ANST, and MOBP-PCAS-ANST configurations which correspond to the three smallest values of *TOTTM*.

Figure 7.10 shows the performance profiles, described in Section 6.3, of the state of the art configurations: DEBP-NONA-NOAN and DEBP-DBPA-NOAN, and the two best proposed configurations with respect to *TOTTM*: MOBP-MPAS-ANST and DIST-MPAS-ANST. From this plot, it can be seen that for any given value of t_{MI}/t_{MIB} defined in Section 6.3, the fraction of MIP instances with a ratio less than this value for MOBP-MPAS-ANST and DIST-MPAS-ANST is greater than that for DEBP-NONA-ANST and DEBP-DBPA-ANST. For example, t_{MI}/t_{MIB} for MOBP-MPAS-ANST is less than 2 for about 85% of the MIP instances; for DIST-MPAS-ANST, this ratio is less than 2 for about 83% of MIP instances; for DEBP-DBPA-NOAN, this ratio is less than 2 for about 71% of MIP instances; for DEBP-NONA-NOAN, this ratio is less than 2 for about 69% of MIP instances. Over all of the MIP instances not solved within 10 seconds for every contending method: MOBP-MPAS-ANST performed best for solving 11% of these instances, DIST-MPAS-ANST performed best for solving 22% of these instances, DEBP-DBPA-NOAN performed best for solving 7% of these instances, and DEBP-NONA-NOAN performed best for

Figure 7.9: A plot showing MRATE, r_{p10} , and r_{p90} for the following configurations: method 1 = MOBP-MPAS-ANST, method 2 = MOBP-PCAS-ANST, method 3 = DIST-MPAS-ANST, method 4 = MOBP-LEXA-ANST, method 5: DIST-ALLT-NOAN, method 6: MOBP-MPAS-NOAN, method 7: MOBP-PCAS-NOAN, method 8: DEBP-DBPA-NOAN, method 9: DEBP-NONA-NOAN.

solving 0% of these instances. Over all of the MIP instances not solved within 10 seconds for every contending method: MOBP-MPAS-ANST and DIST-MPAS-ANST solved 92% of these instances, DEBP-DBPA-NOAN solved 81% of these instances, and DEBP-NONA-NOAN solved 79% of these instances.

130 MIP instances were not solved to optimality within the one hour time limit, but at least one configuration was able to find a feasible solution. These instances are listed in Table A.5. With respect to *AVGRANK* and *NFIRSTS*, DIST-ALLT-NOAN is the best configuration, the second best being DIST-MPAS-ANST. These results suggest that the DIST backtracking node selection method performs better than the other methods for either finding good MIP feasible solutions to the more difficult MIP instances or reducing the optimality gap of these MIP instances.

The best configurations with respect to *NINC* are DEBP-DBPA-NOAN and

Figure 7.10: The performance profile comparing the two best proposed configurations with the two state of the art methods. Each point in the plot represents the fraction of MIP instances for which the ratio of time to solve them using the given configuration to best time to solve them over all the configurations in Table 7.9 is less than or equal to its value at that point.

DEBP-NONA-NOAN because both of these configurations use the most-feasible backtracking node selection method without an aspiration value until an incumbent solution is found. The goal of the most-feasible backtracking node selection method is to quickly find a MIP feasible solution.

The results of this experiment suggest that DIST-MPAS-ANST is the best configuration for solving MIPs. It has the smallest number of unsolved MIPs (the same as 3 other configurations) and it is the only proposed configuration that outperforms the state of the art configurations with respect to all but one of the metrics: *FAIL*, *TOTTM*, *MRATE*, *AVGRANK*, and *NFIRSTS*. DIST-MPAS-ANST is only marginally worse than the only configuration, MOBP-MPAS-ANST, that performs better with respect to *TOTTM*. Similarly, DIST-MPAS-ANST is marginally worse than the best two configurations, MOBP-MPAS-ANST and MOBP-PCAS-ANST with respect to *MRATE*. Finally, DIST-MPAS-ANST performs better than both MOBP-MPAS-ANST and MOBP-PCAS-ANST with respect to *AVGRANK* and *NFIRSTS*.

The complete details and results of this experiment are given in Appendix B.4.

7.3 Problem-specific Node Selection

Different MIP solver configurations may perform exceptionally well on particular classes of MIP models. A preliminary analysis was undertaken to determine whether some classes of MIP models are particularly well solved by the new solver configurations. The MOBP-MPAS-ANST and DIST-ALLT-NOAN configurations perform exceptionally well on Fixed Charge Transportation Problems (FCTP). There may well be other special cases that are handled well by some of the other new configurations, but this has been left for future research. The results for the FCTP follow below.

The goal of a FCTP [32] is to determine what routes should be used to transport goods from supply locations to demand locations while minimizing

- the total of the costs that is associated with the amount of flow on each route, plus
- the total of the fixed costs associated with using each route.

Table 7.12 shows that MOPB-MPAS-ANST performs as well as or better than either state of the art configuration with respect to the amount of time required to solve the FCTPs. MOBP-MPAS-ANST reduced the time to solve three of the FCTPs by at least 15 minutes each, in one case reducing the time by over 45 minutes. Two of these FCTPs were not solved within the time limit using either state of the art configuration.

Table 7.13 shows that DIST-ALLT-NOAN performs as well as or better than either state of the art configuration with respect to the best incumbent solution found for all but one of the unsolved FCTPs. The best incumbent solutions found when using DIST-ALLT-NOAN are, on average, 27% less than the best incumbent solution found using either state of the art methods.

These results suggest that if the goal is to solve an FCTP to optimality, then it is probably best to use the MOBP-MPAS-ANST node selection configuration. If the goal is to find a good feasible solution to an FCTP given a limited amount of time, then it is probably best to use the DIST-ALLT-NOAN node selection configuration.

Table 7.12: The performance of two state of the art configurations versus the MOBP-MPAS-ANST configuration with respect to time for solving FCTPs. Each FCTP is solved by at least one of these configurations. Solved = 1 if a configuration solves a model within the time limit; Solved = 0 if a configuration does not solve a model within the time limit. Diff time is the difference between the time to solve a model using MOBP-MPAS-ANST and the smaller of the times to solve this model using either of the art configurations.

DEBP-NONA-NOAN		DEBP-DBPA-NOAN		MOBP-MPAS-ANST			
FCTP	Time (s)	Solved	Time (s)	Solved	Time (s)	Solved	Diff time
bal8x12	1.09	1	0.86	1	0.68	1	-0.18
gr4x6	0.1	1	0.09	1	0.08	1	0.004
ran10x10a	22.97	1	20.82	1	13.25	1	-7.56
ran10x10b	5.13	1	4.02	1	3.84	1	-0.18
ran10x12	6.45	1	6.14	1	3.76	1	-2.38
ran10x10c	2776.39	1	3063.2	1	547.74	1	-2228.65
ran10x26	3600	0	3600	0	2565.8	1	-1034.2
ran4x64	3.3	1	2.48	1	0.74	1	-1.74
ran6x43	1.37	1	1.32	1	1.22	1	-0.1
ran8x32	3600	0	3600	0	786.49	1	-2813.51

Table 7.13: The performance of two state of the art configurations versus the DIST-ALLT-NOAN configuration with respect to the best incumbent solution found for FCTPs that were not solved within the time limit. $r_{z_{inc}}$ is the ratio of the best incumbent solution found using DIST-ALLT-NOAN and the smallest of incumbent solutions found using either of the state of the art configurations.

FCTP	DEBP-NONA-NOAN	DEBP-DBPA-NOAN	DIST-ALLT-NOAN	$r_{z_{inc}}$
n3700	1800252	1800252	1408176	0.78
n3701	1867170	1915358	1404912	0.75
n3702	2012508	2012508	1398413	0.69
n3703	1924141	1924141	1400758	0.73
n3704	1966551	1966551	1450852	0.74
n3705	1953807	1953807	1440439	0.74
n3706	1755690	1755690	1403127	0.8
n3707	1823077	1823077	1383278	0.76
n3708	1852929	1852929	1408435	0.76
n3709	1944285	1944285	1442535	0.74
n370a	1989867	1989867	1507415	0.76
n370b	1893469	1893469	1505803	0.8
n370c	1830458	1830458	1435154	0.78
n370d	1830458	1830458	1435154	0.78
n370e	1843311	1843311	1420086	0.77
ran12x12	2294	2294	2294	1
ran12x21	3664	3675	3664	1
ran13x13	3252	3252	3276	1.01
ran14x18	4032	5326	3861	0.96
ran16x16	3954	5134	3891	0.98
ran17x17	1373	1373	1373	1

Chapter 8

Conclusions

1. The goal stated in Chapter 4 has been achieved: as shown in Chapters 5 and 7, node selection methods that reduce the amount of effort required to solve MIPs have been developed.
2. The correlation between the objective values and numbers of candidate variables, as well as between the depths and numbers of candidate variables of nodes in the tree, can be exploited by backtracking node selection and back-track triggering methods to reduce the amount of effort required to solve MIPs. This is shown in Sections 5.1, 5.3, 7.1 and 7.2.
3. For backtracking node selection, balancing the goals of MIP feasibility and MIP optimality using distributions of the objective values and the numbers of candidate variables of nodes in the tree reduces the amount of effort required to solve MIPs. This is shown in Sections 5.2, 7.1 and 7.2.
4. The amount of effort required to carry out a complex backtracking node selection algorithm can be large enough compared to the other branch and bound operations that it requires less effort to solve a MIP using a simpler node selection algorithm, as shown in Section 5.4.
5. The backtracking node selection method has much more influence on the amount

of effort required to solve MIPs than the backtrack triggering method. This is shown in Section 7.1.

8.1 Contributions

1. Developed novel backtracking node selection heuristics that improve the performance of the branch and bound method for solving MIPs.
2. Developed novel methods of estimating the optimal objective value that, when used as an aspiration value to trigger backtracking, improve the performance of the branch and bound method for solving MIPs.
3. Introduced the concept of using probability distributions for the purpose of ordering nodes for selection during backtracking to improve the performance of solving MIPs.
4. Improved the best-projection backtrack triggering and backtracking node selection method by developing a method of computing projections without using an incumbent solution.
5. Introduced the concept of using the correlation between the objective values and numbers of candidate variables for the purpose of triggering backtracking and ordering nodes for selection during backtracking to improve the performance of solving MIPs.
6. Introduced the concept of the cost versus benefit analysis of backtracking node selection methods for the purpose of ordering node selection to improve the performance of solving MIPs.
7. Introduced the concept of using the correlation between the numbers of candidate variables and depths of nodes to estimate the depth of MIP feasible solutions.

8. Introduced the concept of predicting the optimal objective value using the estimated depth of a MIP optimal node.
9. Performed an extensive computational study comparing 79 configurations of backtracking node selection methods, backtrack triggering methods, and active node search thresholds.

8.2 Future Research

1. The modified best-projection method presented in Section 5.1 may be improved by adaptively selecting the points (z_i, c_i) to be used to estimate the degradation in objective value per unit MIP infeasibility. The change in the gradient of $z^{min}(c)$ may be useful towards this end.
2. The distribution method presented in Section 5.2 may be improved by using distribution functions that are better suited for modelling the objective values and the numbers of candidate variables of nodes in the tree. These distribution functions may incorporate the correlation of these values. Other node characteristics, such as depth, may also be used.
3. The distribution method may be improved by combining correlation information on the paths from the root node to each active node with global information such as the averages and variances of the objective values, numbers of candidate variables, and depths of nodes.
4. The parameters of the distribution functions used in the distribution method are computed using the assumption that the objective value of a node is independent of the objective value of any other node. This is not true for nodes that are ancestors or descendants of each other so it may be better to use only nodes that are not ancestors or descendants of each other in the computation of the parameters of the distribution functions.

5. The ANST value may be improved if it is computed for each MIP instance based on some features of that instance and its branch and bound tree. Features that may be useful are the number of variables, the number of constraints, the number of active nodes, the optimality gap, the number of times backtracking has been performed, the number of MIP feasible nodes, and the number LP infeasible nodes. It may be possible to set the threshold value adaptively as the branch and bound method proceeds based on these features.
6. In Section 5.4 two methods of storing the active nodes in a branch and bound tree are discussed: the last active node created is placed at the end of the list; or each node is placed in the list by priority according to the node selection criteria. An empirical study comparing these methods has not been found in the literature. Such a comparative study should be conducted to determine which method is better.
7. The modified best-projection aspiration algorithm presented in Section 5.1.1 updates the aspiration value by searching through all of the active nodes after each node has been solved. If the number of active nodes becomes large then it may be better to reduce the frequency at which the aspiration value is updated.
8. The feasibility depth extrapolation aspiration algorithm presented in Section 5.3.1 estimates the depth of an optimal node as the smallest extrapolated depth of a MIP feasible node over all solved nodes. Therefore this estimate may become much too small without any way of increasing it. It may be better to increase this estimate using information such as the depths of all active nodes, or to use a different method of choosing the MIP optimal depth from all of the extrapolated depths of MIP feasible nodes.

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Appendix A

MIP Instances

The number of MIP instances used to evaluate the performance of a branch and bound method should be large enough to obtain a sufficiently accurate measure of performance while also being small enough so that all of these instances can be solved within a reasonable amount of time. Table A.1 shows the MIP instances used in the branch and bound experiments performed in this thesis along with some of the formulation characteristics that are described below.

Name: The name of a MIP instance.

Cols: The number of variables.

Real: The number of real variables.

Int: The number of non-binary integer variables.

Bin: The number of binary variables.

Rows: The number of constraints.

Ineq: The number of inequality constraints.

Eq: The number of equality constraints.

NZ: The number of non-zero constraint coefficients.

Table A.1: List of MIP instances taken from MIPLIB 2003, MIPLIB 3.0, MIPLIB 2.0 and CORAL with primal LP and MIP formulation characteristics.

Name	Cols	Real	Int	Bin	Rows	Ineq	Eq	NZ
22433	429	198	0	231	199	0	198	3409
23588	368	137	0	231	138	0	137	3702
10teams	2025	225	0	1800	231	95	135	14175
alc1sl	3648	3456	192	0	3313	2064	1248	11618
aflow30a	842	421	0	421	480	421	58	2512
aflow40b	2728	1364	0	1364	1443	1364	78	8147
air01	771	0	0	771	24	0	23	4986
air02	6774	0	0	6774	51	0	50	68329
air03	10757	0	0	10757	125	0	124	101785
air04	8904	0	0	8904	824	0	823	81869
air05	7195	0	0	7195	427	0	426	59316
air06	8627	0	0	8627	826	0	825	79433
aligninq	1831	1	0	1830	341	339	1	15735
arki001	1388	850	123	415	1049	1028	20	20440
atlanta-ip	48738	1965	106	46667	21733	14652	7080	288828
bal8x12	192	96	0	96	117	96	20	576
bc	1751	1268	0	483	1914	1128	785	276843
bc1	1751	1499	0	252	1914	1128	785	276843
bell3b	133	62	32	39	124	123	0	441
bell4	117	53	30	34	106	105	0	385
bell5	104	46	28	30	92	91	0	340
bienst1	505	477	0	28	577	448	128	2185
bienst2	505	470	0	35	577	448	128	2185
binkar10.1	2298	2128	0	170	1027	11	1015	6238
bk4x3	24	12	0	12	20	12	7	72
blend2	353	89	33	231	275	185	89	1497
bm23	27	0	0	27	21	20	0	505
cap6000	6000	0	0	6000	2177	2053	123	54238
cracpb1	572	0	0	572	144	54	89	4730
d10200	2000	0	1267	733	948	747	200	59637
d20200	4000	0	819	3181	1503	1302	200	193389
dano3.3	13873	13804	0	69	3203	1978	1224	79656
dano3.4	13873	13781	0	92	3203	1978	1224	79656
dano3.5	13873	13758	0	115	3203	1978	1224	79656
dano3mip	13873	13321	0	552	3203	1978	1224	79656
danooint	521	465	0	56	665	520	144	3233
dcmulti	548	473	0	75	291	212	78	1833
disctom	10000	0	0	10000	400	0	399	40000
ds	67732	0	67732	0	657	0	656	1091791
dsbmip	1886	1694	32	160	1855	1462	392	9768
egout	141	86	0	55	99	55	43	392
eilD76	1898	0	1898	0	76	0	75	21009
enigma	100	0	0	100	22	0	21	298
fast0507	63009	0	0	63009	508	507	0	472358
fiball	34219	1	258	33960	3708	478	3229	104793
fiber	1298	44	0	1254	364	0	363	4198
fixnet3	878	500	0	378	479	378	100	2631
fixnet4	878	500	0	378	479	378	100	2621
fixnet6	878	500	0	378	479	378	100	2550
flugpl	18	7	11	0	19	12	6	64
gen	870	720	6	144	781	630	150	3174
germanrr	10813	239	5286	5288	10780	10541	238	175548
gesa2	1224	816	168	240	1393	1344	48	6000
gesa2-o	1224	504	336	384	1249	1128	120	4608
glass4	322	20	0	302	397	360	36	1834

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Table A.1 – continued from previous page

Name	Cols	Real	Int	Bin	Rows	Ineq	Eq	NZ
gr4x6	48	24	0	24	35	24	10	144
gt2	188	0	164	24	30	29	0	468
haprp	1828	0	1828	0	1049	60	988	5363
harp2	2993	0	0	2993	113	39	73	8833
irp	20315	0	0	20315	40	1	38	118569
khh05250	1350	1326	0	24	102	24	77	3973
l152lav	1989	0	0	1989	98	1	96	11911
leo1	6731	1	0	6730	594	592	1	131219
leo2	11100	1	0	11099	594	592	1	219960
liu	1156	67	0	1089	2179	2178	0	10627
lp41	1086	0	0	1086	86	1	84	5763
lrn	7253	4798	0	2455	8702	7692	1009	52643
lseu	89	0	0	89	29	28	0	394
manna81	3321	0	3303	18	6481	6480	0	16281
markshare1	62	12	0	50	7	0	6	324
markshare2	74	14	0	60	8	0	7	448
mas284	151	1	0	150	69	68	0	9782
mas74	151	1	0	150	14	13	0	1857
mas76	151	1	0	150	13	12	0	1791
mcf2	521	465	0	56	665	520	144	3233
mcsched	1747	2	14	1731	2108	1906	201	8089
misc01	83	1	0	82	55	37	17	746
misc02	59	1	0	58	40	25	14	414
misc03	160	1	0	159	97	69	27	2054
misc04	4897	4867	0	30	1726	1414	311	17253
misc05	136	62	0	74	301	271	29	2946
misc06	1808	1696	0	112	821	570	250	5860
misc07	260	1	0	259	213	177	35	8620
mitre	10724	0	0	10724	2055	1671	383	49028
mkc	5325	2	0	5323	3412	3409	2	19984
mkc1	5325	2238	0	3087	3412	3409	2	19984
mod008	319	0	0	319	7	6	0	1562
mod010	2655	0	0	2655	147	1	145	13858
mod011	10958	10862	0	96	4481	113	4367	29839
mod013	96	48	0	48	63	48	14	288
modglob	422	324	0	98	292	196	95	1390
momentum1	5174	2825	0	2349	42681	42122	558	106005
momentum2	3732	1923	1	1808	24238	24017	220	353367
momentum3	13532	6933	1	6598	56823	55970	852	962826
msc98-ip	21143	853	53	20237	15851	11549	4301	100358
mzzv11	10240	0	251	9989	9500	6127	3372	134808
mzzv42z	11717	0	235	11482	10461	5958	4502	151455
n3700	10000	5000	0	5000	5151	5000	150	30000
n3701	10000	5000	0	5000	5151	5000	150	30000
n3702	10000	5000	0	5000	5151	5000	150	30000
n3703	10000	5000	0	5000	5151	5000	150	29999
n3704	10000	5000	0	5000	5151	5000	150	30000
n3705	10000	5000	0	5000	5151	5000	150	30000
n3706	10000	5000	0	5000	5151	5000	150	30000
n3707	10000	5000	0	5000	5151	5000	150	30000
n3708	10000	5000	0	5000	5151	5000	150	30000
n3709	10000	5000	0	5000	5151	5000	150	30000
n370a	10000	5000	0	5000	5151	5000	150	30000
n370b	10000	5000	0	5000	5151	5000	150	30000
n370c	10000	5000	0	5000	5151	5000	150	30000
n370d	10000	5000	0	5000	5151	5000	150	30000
n370e	10000	5000	0	5000	5151	5000	150	30000
nag	2884	1499	1385	0	5841	4465	1375	26500

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Table A.1 – continued from previous page

Name	Cols	Real	Int	Bin	Rows	Ineq	Eq	NZ
neos-430149	395	215	0	180	991	780	210	2910
neos-476283	11915	6327	0	5588	10016	9951	64	3946449
neos-480878	534	345	0	189	1322	1286	35	44826
neos-494568	6889	153	0	6736	2216	2091	124	122257
neos-495307	9423	0	9423	0	4	3	0	37254
neos-498623	9861	1361	0	8500	2048	1901	146	156991
neos-501453	165	18	147	0	41	26	14	685
neos-501474	206	59	147	0	266	251	14	2419
neos-503737	2850	350	0	2500	501	351	149	17200
neos-504674	844	694	0	150	1345	1094	250	3600
neos-504815	674	554	0	120	1068	867	200	2856
neos-506422	2527	2464	0	63	6812	6804	7	34083
neos-506428	42981	0	0	42981	129926	129831	94	343488
neos-512201	838	688	0	150	1338	1093	244	3568
neos-520729	91149	60441	0	30708	31179	40	31138	327996
neos-522351	1524	1284	0	240	1706	1609	96	5700
neos-525149	3640	40	0	3600	144121	144001	119	1522840
neos-530627	103	75	28	0	114	77	36	350
neos-538867	792	0	0	792	1171	1153	17	4554
neos-538916	864	0	0	864	1315	1297	17	4976
neos-544324	10080	0	0	10080	733	732	0	1767360
neos-547911	3528	0	0	3528	694	693	0	618576
neos-548047	2020	1	0	2019	3971	3970	0	26406
neos-548251	1922	992	0	930	2387	1456	930	5792
neos-551991	1730	1	0	1729	3333	3332	0	31632
neos-555001	3855	0	75	3780	3475	1270	2204	16657
neos-555298	4827	0	15	4812	2756	1336	1419	20153
neos-555343	3815	0	15	3800	3327	2599	727	16975
neos-555424	3815	0	15	3800	2677	2606	70	15675
neos-555694	4139	1320	0	2819	1949	1845	103	42443
neos-555771	4170	1338	0	2832	1979	1857	121	43255
neos-555884	3815	0	15	3800	4332	3086	1245	19075
neos-555927	1945	0	15	1930	1404	763	640	7973
neos-565672	190589	15228	0	175361	318335	318334	0	825044
neos-565815	1276	0	0	1276	15414	15372	41	124716
neos-570431	511	1	0	510	932	931	0	12042
neos-574665	740	492	64	184	3791	3335	455	16796
neos-578379	17010	0	0	17010	21703	10720	10982	101560
neos-582605	1265	865	0	400	1241	836	404	3755
neos-583731	1350	0	0	1350	1492	1471	20	5310
neos-584146	811	561	0	250	937	636	300	3056
neos-584851	445	40	0	405	662	658	3	1734
neos-584866	3674	90	0	3584	9010	9001	8	21506
neos-585192	2597	1553	0	1044	2629	1081	1547	72401
neos-585467	2116	1270	0	846	2167	901	1265	50062
neos-593853	2400	1200	0	1200	1607	1407	199	7200
neos-595904	4508	3360	0	1148	2453	2300	152	26872
neos-595905	1200	888	0	312	705	628	76	6988
neos-595925	1276	840	0	436	957	872	84	7236
neos-598183	1696	1260	0	436	993	876	116	10084
neos-603073	1696	1260	0	436	993	876	116	10084
neos-611135	6400	0	0	6400	5278	5246	31	775700
neos-611838	9954	9900	0	54	1877	1132	744	46981
neos-612125	9554	9504	0	50	1796	1051	744	45345
neos-612143	9832	9780	0	52	1843	1098	744	46475
neos-612162	9893	9840	0	53	1860	1115	744	46728
neos-619167	3452	3052	0	400	6801	6641	159	20021
neos-631164	1282	245	0	1037	407	194	212	3401

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Table A.1 – continued from previous page

Name	Cols	Real	Int	Bin	Rows	Ineq	Eq	NZ
neos-631517	1090	231	0	859	352	185	166	2974
neos-631694	3725	0	0	3725	3997	3921	75	18573
neos-631709	45150	0	0	45150	46497	46346	150	225448
neos-631710	167056	0	0	167056	169577	169276	300	834722
neos-631784	22725	0	0	22725	23997	23921	75	113323
neos-632335	12719	0	0	12719	24865	12168	12696	84663
neos-633273	11154	0	0	11154	21782	10649	11132	74074
neos-641591	18235	0	328	17907	1086	758	327	218290
neos-655508	13572	0	0	13572	13574	13457	116	53824
neos-662469	18235	0	328	17907	1086	758	327	218290
neos-686190	3660	0	60	3600	3665	3544	120	21625
neos-691058	3006	1251	0	1755	2668	2206	461	30912
neos-691073	1935	180	0	1755	2668	2206	461	29841
neos-693347	1576	171	0	1405	3193	2861	331	113538
neos-702280	7199	0	0	7199	1601	1600	0	2429081
neos-709469	224	0	0	224	470	413	56	4656
neos-717614	3049	49	2916	84	892	296	595	13526
neos-738098	9093	147	0	8946	25850	25353	496	101507
neos-775946	4710	1593	0	3117	6603	6475	127	111313
neos-777800	6400	0	0	6400	480	478	1	38400
neos-780889	182700	0	0	182700	73911	351	73559	621460
neos-785899	1320	0	0	1320	1654	1653	0	18380
neos-785912	1380	0	0	1380	1715	1714	0	17870
neos-785914	1260	0	0	1260	1591	1590	0	16430
neos-787933	236376	0	0	236376	1898	1897	0	300084
neos-791021	9448	0	0	9448	3695	970	2724	29732
neos-796608	311	192	119	0	287	95	191	873
neos-799711	41998	41088	0	910	59219	35827	23391	147165
neos-799716	30619	29709	0	910	40419	17031	23387	99929
neos-799838	20844	0	0	20844	5977	73	5903	69804
neos-801834	3220	0	0	3220	3301	3241	59	58420
neos-803219	640	580	0	60	902	431	470	3070
neos-803220	630	570	0	60	892	421	470	3030
neos-806323	1060	950	0	110	1542	691	850	5740
neos-807454	1638	0	0	1638	1623	1342	280	35323
neos-807456	1635	0	0	1635	841	0	840	6540
neos-807639	1030	950	0	80	1542	691	850	5580
neos-807705	1030	950	0	80	1542	691	850	5580
neos-808072	1702	1	0	1701	1714	1468	245	38107
neos-808214	1308	0	0	1308	641	437	203	22583
neos-810286	2915	0	0	2915	2676	2296	379	71666
neos-810326	1702	1	0	1701	1750	1468	281	39843
neos1	2112	0	0	2112	5021	4921	99	21600
neos10	23489	0	5	23484	46794	46777	16	251230
neos11	1220	320	0	900	2707	2471	235	9540
neos12	3983	847	0	3136	8318	7743	574	26229
neos13	1827	12	0	1815	20853	20852	0	253854
neos14	792	656	0	136	553	313	239	2318
neos15	792	632	0	160	553	313	239	2318
neos16	377	0	41	336	1019	1009	9	2802
neos17	535	235	0	300	487	485	1	5416
neos18	3312	0	0	3312	11403	9008	2394	24694
neos19	103789	93789	0	10000	34083	91	33991	198442
neos2	2101	1061	0	1040	1104	1060	43	7330
neos20	1165	198	111	856	2447	1693	753	7433
neos3	2747	1387	0	1360	1443	1386	56	9585
neos4	22884	5712	0	17172	38578	37441	1136	116040
neos5	63	10	0	53	64	63	0	2079

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Table A.1 – continued from previous page

Name	Cols	Real	Int	Bin	Rows	Ineq	Eq	NZ
neos6	8786	446	0	8340	1037	813	223	252169
neos7	1556	1102	30	424	1995	1525	469	5472
neos8	23228	0	4	23224	46325	46312	12	313212
neos9	81408	79309	0	2099	31601	23873	7727	264192
net12	14115	12512	0	1603	14022	13469	552	80423
newdano	505	449	0	56	577	448	128	2185
noswot	128	28	25	75	183	180	2	760
npmv07	220686	218806	0	1880	76343	52641	23701	859615
nsa	388	352	0	36	1298	1261	36	4205
nsrand-idx	6621	1	0	6620	736	735	0	223262
nug08	1632	0	1632	0	913	1	911	8304
nw04	87482	0	0	87482	37	0	36	724148
opt1217	769	1	0	768	65	16	48	1543
p0033	33	0	0	33	17	16	0	131
p0040	40	0	0	40	24	23	0	150
p0201	201	0	0	201	134	133	0	2124
p0282	282	0	0	282	242	241	0	2248
p0291	291	0	0	291	253	252	0	2321
p0548	548	0	0	548	177	176	0	2127
p2756	2756	0	0	2756	756	755	0	11103
p6b	462	0	0	462	5853	5852	0	12166
pg	2700	2600	0	100	126	25	100	7800
pg5.34	2600	2500	0	100	226	225	0	10300
pipex	48	0	0	48	26	9	16	240
pk1	86	31	0	55	46	30	15	916
pp08a	240	176	0	64	137	72	64	656
pp08aCUTS	240	176	0	64	247	182	64	1015
prod1	250	101	0	149	209	201	7	5351
prod2	301	101	0	200	212	201	10	10502
protfold	1835	0	0	1835	2113	2075	37	23611
qap10	4150	0	4150	0	1821	0	1820	20810
qiu	840	792	0	48	1193	1060	132	3744
qnet1	1541	124	129	1288	504	172	331	4746
qnet1.o	1541	124	129	1288	457	125	331	4338
ramos3	2187	0	0	2187	2188	2187	0	34992
ran10x10a	200	100	0	100	121	100	20	600
ran10x10b	200	100	0	100	121	100	20	600
ran10x10c	200	100	0	100	121	100	20	600
ran10x12	240	120	0	120	143	120	22	720
ran10x26	520	260	0	260	297	260	36	1560
ran12x12	288	144	0	144	169	144	24	864
ran12x21	504	252	0	252	286	252	33	1512
ran13x13	338	169	0	169	196	169	26	1014
ran14x18	504	252	0	252	285	252	32	1512
ran14x18.1	504	252	0	252	285	252	32	1512
ran14x18.disj-8	504	252	0	252	448	415	32	10781
ran16x16	512	256	0	256	289	256	32	1536
ran17x17	578	289	0	289	324	289	34	1734
ran4x64	512	256	0	256	325	256	68	1536
ran6x43	516	258	0	258	308	258	49	1548
ran8x32	512	256	0	256	297	256	40	1536
rd-rplusc-21	622	165	0	457	125900	125762	137	852386
rentacar	9557	9502	0	55	6804	511	6292	42019
rgn	180	80	0	100	25	4	20	540
rlp1	461	11	0	450	69	58	10	837
rlp2	451	1	0	450	69	58	10	5396
roll3000	1166	428	492	246	2296	2121	174	29387
rout	556	241	15	300	292	260	31	2432

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Table A.1 – continued from previous page

Name	Cols	Real	Int	Bin	Rows	Ineq	Eq	NZ
roy	149	99	0	50	163	153	9	501
set1al	712	472	0	240	493	252	240	1884
set1ch	712	472	0	240	493	252	240	1884
set1cl	712	472	0	240	493	252	240	1884
seymour	1372	0	0	1372	4945	4944	0	34921
seymour.disj-10	1209	0	0	1209	5109	5108	0	65913
sp97ar	14101	0	14101	0	1762	1761	0	305069
sp97ic	1662	0	944	718	2087	2086	0	68294
sp98ar	5478	0	2682	2796	4681	4680	0	237234
sp98ic	2508	0	1369	1139	2312	2311	0	140561
sp98ir	1680	0	809	871	1532	1531	0	73384
stein15	15	0	0	15	37	36	0	135
stein27	27	0	0	27	119	118	0	405
stein45	45	0	0	45	332	331	0	1079
stein9	9	0	0	9	14	13	0	54
stp3d	204880	0	0	204880	159489	131403	28085	737776
swath	6805	81	0	6724	885	380	504	34966
t1717	73885	0	0	73885	552	0	551	399574
Test3	72215	39072	25969	7174	50681	50083	597	617907
timtab1	397	226	107	64	172	0	171	957
timtab2	675	381	181	113	295	0	294	1731
tr12-30	1080	720	0	360	751	390	360	3228
vpml	378	210	0	168	235	192	42	917
vpm2	378	210	0	168	235	192	42	1085

Table A.2: List of MIP instances taken from MIPLIB 2003, MIPLIB 3.0 and MIPLIB 2.0. These MIP instances are used in the experiments described in Sections 4.1.2, 6.5.1, and 6.5.2.

10teams	dcmulti	markshare1	mzzv42z	rgn
alc1s1	disctom	markshare2	net12	roll3000
afflow30a	ds	mas284	noswot	rout
afflow40b	dsbmip	mas74	nsrand-ipx	set1al
air01	egout	mas76	nw04	set1ch
air02	enigma	misc01	opt1217	set1cl
air03	fast0507	misc02	p0033	seymour
air04	fiber	misc03	p0040	sp97ar
air05	fixnet3	misc04	p0201	stein15
air06	fixnet4	misc05	p0282	stein27
arki001	fixnet6	misc06	p0291	stein45
atlanta-ip	flugpl	misc07	p0548	stein9
bal8x12	gen	mitre	p2756	stp3d
bell3b	gesa2	mkc	pipex	swath
bell4	gesa2-o	mod008	pk1	t1717
bell5	glass4	mod010	pp08a	timtab1
bienst1	gr4x6	mod011	pp08aCUTS	timtab2
bk4x3	harp2	mod013	protfold	tr12-30
blend2	khh05250	modglob	qiu	vpm1
bm23	l152lav	momentum1	ran10x10a	vpm2
cap6000	liu	momentum2	ran10x10b	
cracpb1	lp41	momentum3	ran10x12	
dano3mip	lseu	msc98-ip	rd-rplusc-21	
danooint	manna81	mzzv11	rentacar	

Table A.3: List of MIP instances in Table A.1 which are solved within 30 minutes for the experiment in Section 7.1 using the default GLPK node selection configuration, DEBP-NONA-NOAN.

22433	enigma	misc04	neos-807639	ran10x10a
23588	fiber	misc05	neos-807705	ran10x10b
10teams	fixnet4	misc06	neos-810286	ran10x12
air06	fixnet6	misc07	nug08	ran4x64
bal8x12	flugpl	mod008	nw04	ran6x43
bell3b	gen	mod013	p0033	rentacar
bienst1	gr4x6	neos-506422	p0201	rgn
blend2	gt2	neos-584851	p0282	roy
bm23	irp	neos-709469	p0291	set1al
dano3_3	khh05250	neos-777800	p0548	stein15
dano3_4	l152lav	neos-785899	p2756	stein27
dano3_5	lseu	neos-785914	pipex	stein45
dcmulti	mas284	neos-801834	pp08a	stein9
dsbmip	misc01	neos-803219	qap10	Test3
egout	misc02	neos-803220	qnet1	vpml
eild76	misc03	neos-806323	qnet1_o	

Table A.4: List of MIP instances in Table A.1 which are solved within 60 minutes for the experiment in Section 7.2 using at least one of the node selection configurations listed in Table 7.9.

22433	fiber	neos-480878	neos-803220	qnet1_o
23588	fixnet4	neos-501453	neos-806323	ran10x10a
10teams	fixnet6	neos-503737	neos-807639	ran10x10b
air06	flugpl	neos-506422	neos-807705	ran10x10c
aligninq	gen	neos-512201	neos-810286	ran10x12
bal8x12	gr4x6	neos-522351	neos-810326	ran10x26
bcl	gt2	neos-530627	neos11	ran17x17
bell3b	irp	neos-555694	neos2	ran4x64
bell4	khh05250	neos-565815	nug08	ran6x43
bienst1	l152lav	neos-570431	nw04	ran8x32
blend2	lseu	neos-584851	p0033	rentacar
bm23	mas284	neos-612125	p0201	rgn
cap6000	mas76	neos-655508	p0282	rlp2
dano3_3	misc01	neos-686190	p0291	roy
dano3_4	misc02	neos-709469	p0548	set1al
dano3_5	misc03	neos-777800	p2756	stein15
dcmulti	misc04	neos-780889	pipex	stein27
disctom	misc05	neos-785899	pk1	stein45
dsbmip	misc06	neos-785914	pp08a	stein9
egout	misc07	neos-796608	prod1	Test3
eilD76	mod008	neos-801834	qap10	vpm1
enigma	mod013	neos-803219	qnet1	

Table A.5: List of MIP instances in Table A.1 which are not solved within 60 minutes for the experiment in Section 7.2 using any of the node selection configurations listed in Table 7.9. MIP instances for which no incumbent solution was found using any of these configurations are excluded.

alc1s1	mod011	neos-504815	neos-631694	nsrand-ipx
aflow30a	modglob	neos-520729	neos-631784	p6b
aflow40b	mzzv11	neos-525149	neos-641591	pg
air04	mzzv42z	neos-538867	neos-662469	pg5_34
air05	n3700	neos-538916	neos-691058	pp08aCUTS
bc	n3701	neos-544324	neos-775946	prod2
bell5	n3702	neos-547911	neos-787933	protfold
bienst2	n3703	neos-548047	neos-791021	qiu
binkar10_1	n3704	neos-548251	neos-799838	ran12x12
d10200	n3705	neos-551991	neos-808072	ran12x21
d20200	n3706	neos-555298	neos1	ran13x13
dano3mip	n3707	neos-555424	neos10	ran14x18
danoimt	n3708	neos-555771	neos12	ran14x18_1
ds	n3709	neos-555927	neos13	ran14x18.disj-8
fast0507	n370a	neos-574665	neos14	ran16x16
germanrr	n370b	neos-582605	neos15	rlp1
gesa2-o	n370c	neos-584146	neos17	roll3000
haprp	n370d	neos-584866	neos18	rout
liu	n370e	neos-595905	neos3	set1ch
markshare1	neos-430149	neos-598183	neos5	seymour
markshare2	neos-476283	neos-603073	neos6	seymour.disj-10
mas74	neos-494568	neos-611135	neos7	t1717
mcf2	neos-495307	neos-611838	neos9	timtab1
mcsched	neos-498623	neos-612143	newdano	timtab2
mkc	neos-501474	neos-612162	noswot	tr12-30
mkc1	neos-504674	neos-631164	nsa	vpm2

Table A.6: List of MIP instances in Table A.1 which are not solved and no incumbent solution was found within 60 minutes for the experiment in Section 7.2 using any of the node selection configurations listed in Table 7.9.

atlanta-ip	neos-565672	neos-807456
fiball	neos-578379	neos-808214
gesa2	neos-583731	neos16
momentum1	neos-585467	neos19
momentum3	neos-595904	neos20
msc98-ip	neos-595925	neos4
nag	neos-619167	npmv07
neos-506428	neos-631709	ramos3
neos-555001	neos-785912	rd-rplusc-21
neos-555343	neos-799711	sp98ic
neos-555884	neos-807454	stp3d

Appendix B

Experimental Data

B.1 Branching Variable Selection Data

This appendix shows the measurements described below for the branching variable selection methods experiment described in Section 6.5.1.

Name: The name of a MIP instance.

Time: The total time in seconds required to solve a MIP; -1 if a MIP is not solved due to solver error.

Iters: The total number of simplex iterations required to solve a MIP; -1 if a MIP is not solved due to solver error.

Nodes: The total number of branch and bound tree nodes solved; -1 if a MIP is not solved due to solver error.

Best: The last incumbent solution found; “not” if no incumbent solution is found; -1 if a MIP is not solved due to solver error.

Solved: Whether or not a MIP is solved: 0 for not solved, 1 for solved.

The tables of full results can be found at http://www.sce.carleton.ca/faculty/chinneck/students/Wojtaszek/branching-variable_selection_data.pdf.

B.2 Cuts Data

This appendix shows the measurements described in Appendix B.1 for the root node cuts experiment described in Section 6.5.2.

The tables of full results can be found at http://www.sce.carleton.ca/faculty/chinneck/students/Wojtaszek/cuts_data.pdf.

B.3 Identification of Promising Configurations Data

This appendix shows the measurements described in Appendix B.1 for the MIP instances solved in the experiment comparing all of the configurations of backtracking node selection methods, backtrack triggering methods, and active node search thresholds, described in Section 7.1. Only small MIP instances are used due to the excessive amount of time that would be required if all MIP instances were used. The MIP instances used in this experiment are those taken from Table A.1 that are solved using default GLPK within 30 minutes excluding any instances that are MIP feasible at the root node. This set of MIP instances is shown in Table A.3.

The tables of full results can be found at http://www.sce.carleton.ca/faculty/chinneck/students/Wojtaszek/identification_promising_configurations_data.pdf.

B.4 Testing of Promising Configurations Data

This appendix shows the measurements for the MIP instances solved in the experiment described in Section 7.2 comparing the proposed configurations of backtracking and aspiration node selection methods with the state of the art. These measurements are described in Appendix B.1. Another measure used in this experiment is the proveable optimality gap (see Definition 2), O_{gap} , at the termination of the branch and bound method. The MIP instances used in this experiment are those taken from Table A.1 excluding any instances that are MIP feasible at the root node.

The tables of full results can be found at http://www.sce.carleton.ca/faculty/chinneck/students/Wojtaszek/testing-promising-configurations_data.pdf.

B.5 Proof of Concept: Perfect Aspiration Value Data

This appendix shows the measurements described in Appendix B.1 for the proof of concept experiment described in Section 4.1.2.

The tables of full results can be found at http://www.sce.carleton.ca/faculty/chinneck/students/Wojtaszek/proof_of_concept_data.pdf.