An extension of the KdV hierarchy arising from Weyl algebra representations of toroidal Lie algebras

by

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Abstract

The construction of the Korteweg-de Vries (KdV) hierarchy of partial differential equations using the affine Lie algebra $\hat{\mathfrak{sl}}_2$ illustrates a remarkable connection between Lie algebras and partial differential equations. Recently Billig has extended this construction in two ways, first by using toroidal Lie algebras instead of affine algebras, and second by considering representations in a Weyl algebra. Here we combine these two methods to obtain a doubly extended hierarchy of partial differential equations.

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1 Introduction

It has been observed that waves traveling in shallow water interact in a non-linear fashion (see [17]). Large waves travel faster than small waves, and as waves cross they experience a phase shift. The differential equation which determines this interaction is the Korteweg-de Vries (KdV) equation (see [13]):

$$4u_t = 6uu_x + u_{xxx}.$$

The general solution to this equation is very difficult to find. One important family of solutions are the so called soliton solutions, which consist of a number of essentially isolated pulses. Solutions consisting of a single pulse, known as 1-solitons, have the form

$$u = c \frac{\partial^2}{\partial x^2} \log(1 + ke^\omega),$$

where $\omega = ax + bt$ is a linear combination of x and t , and c is constant (for the KdV equation it turns out that $a = 2z, b = 2z^3$ for some z and $c = 2$). Solutions with N pulses, known as N -soliton solutions, have the form

$$u = c \frac{\partial^2}{\partial x^2} \log(1 + k_1 e^{\omega_1} + \dots + k_N e^{\omega_N} + \text{higher order terms}),$$

where each of the higher order terms consists of a constant times the exponential of a sum of ω_i 's (see the statement of theorem 6). A solution of this form for any equation is known as a soliton-type solution.

It is natural to wonder whether N different 1 soliton solutions to a given equation can combine to form an N -soliton solution. It turns out that any two 1-soliton

solutions do always combine to give a 2-soliton solution, but in general for $N > 2$ this does not happen. However, for many interesting equations, including those studied here, there are large classes of 1-soliton solutions any N of which can combine to give an N -soliton solution.

The methods we use date back to work of Sato [18] as well as Date, Jimbo, Kashiwara and Miwa [6] in the early 1980's, in which they discovered an elegant connection between Lie algebras and partial differential equations. In particular, the Lie algebra $gl(\infty)$ acts on the solutions of the Kadomtsev-Petviashvili (KP) hierarchy of partial differential equations. This action allows non-trivial solutions of the hierarchy to be obtained by letting the Lie algebra act on trivial solutions. In this way, certain large classes of solutions can be found, including N -soliton solutions.

Using similar methods, Kac and Wakimoto [11] later constructed hierarchies of partial differential equations, along with N -soliton solutions, using vertex algebra representations for general affine Lie algebras (see [10]). In particular, for the case of the affine Lie algebra of type $A_1^{(1)}$, the hierarchy obtained by their construction contains the KdV equation. This is now known as the KdV hierarchy.

Affine Lie algebras can be constructed from finite dimensional Lie algebras in the following way: First one constructs the loop algebra over the finite dimensional Lie algebra, consisting of all Laurent polynomials with coefficients in the Lie-algebra. Then one takes a (1-dimensional) universal central extension of the loop algebra. Finally one adjoins an outer derivation to obtain the full affine Lie algebra. It is

natural to generalize this construction by looking at Laurent polynomials in several variables instead of just one. This can be done, resulting in toroidal Lie algebras. However, for all cases with $n \geq 2$ variables, the universal central extension is infinite dimensional, as is the space of outer derivations which must be adjoined. There is also some additional freedom since this algebra can then be twisted by some 2-cocycle chosen from a 2-dimensional space. We will also need to consider representation theory for these Lie algebras, which has proven to be considerably more difficult than in the affine case.

The study of representations of toroidal Lie algebras began with a paper of Moody, Rao and Yokonuma [16], and was further developed in [2, 1, 14, 15]. For the purposes of this paper, we only use a representation on the subalgebra not including the derivations. We also have representations for the derivations, but this does not give a representation of the full toroidal Lie algebra as the brackets of the representations of different derivations do not all work out correctly. It should be noted that Billig [5] has recently constructed a vertex operator representation for the full toroidal Lie algebras, at least for some of the possible non-trivial 2-cocycles.

In [2] and [3], Billig demonstrates how to extend the methods of Kac and Wakimoto to give hierarchies related to toroidal Lie algebras. In particular, Billig constructs the hierarchy associated with $\mathfrak{sl}_2(\mathbb{C}[t_0, t_0^{-1}, t_1, t_1^{-1}])$, which properly contains the KdV hierarchy. As well, in [4], Billig demonstrates how to extend the KdV-hierarchy by again considering the affine Lie algebra $\hat{\mathfrak{sl}}_2$, but looking at its represen-

tation as a Weyl algebra. The new hierarchy obtained in this way contains variables of ‘negative degree’ which are not visible in the standard construction. In this paper, we combine these two methods to develop a hierarchy which properly contains both of these extensions. The methods displayed work for any toroidal Lie algebra, although we focus mostly on the case $\mathfrak{sl}_2(\mathbb{C}[t_0, t_0^{-1}, t_1, t_1^{-1}])$.

The idea of using representation theory of various Lie algebras to study partial differential equations has resulted in the construction of other hierarchies as well. Notably, Ikeda and Takasaki [8] have recently constructed hierarchies which extend various reduced versions of the KP-hierarchy. In particular, one of these extensions contains Bogoyavlensky’s 2+1-dimensional equation.

This paper will be organized as follows. In the next section, we give a brief introduction to toroidal Lie algebras. Section 3 contains a description of the representations which we will need, following the methods of Billig in [2] and [3]. We use section 4 to present some preliminary results used later on, and also to explain in general terms how we construct our hierarchy of partial differential equations. Sections 5 and 6 contain the necessary calculations to find a generating series for the hierarchy associated with $\mathfrak{sl}_2(\mathbb{C}[t_0, t_0^{-1}, t_1, t_1^{-1}])$. In section 7 we present some of the equations we obtain, then we explicitly give the N -soliton solutions to these equations in section 8. Once we know the solutions, certain symmetries in our equations become apparent which we discuss in section 9. During much of the paper our equations are in the form of Hirota polynomials (see section 4), so in section 10 we briefly discuss the classical

forms of these same equations. Finally, in the appendix we give a list of equations of small degree which appear in our hierarchy (in Hirota form).

2 Toroidal Lie algebras

The basis for this work lies with toroidal Lie algebras and their representation. So before moving on we must first define these algebras and set up some notation. For the most part, we are following the conventions from [2]. We will be working with the so called principal realization of the toroidal Lie algebras, in which we construct Laurent series in N variables over our underlying finite-dimensional Lie algebra, with one variable singled out and treated differently. There is another approach known as the homogeneous realization in which all the variables originally look symmetric, although there will still end up being a ‘special’ direction, and the final Lie algebras obtained are isomorphic.

Let $\dot{\mathfrak{g}}$ be a simply laced (type ADE) finite-dimensional simple Lie algebra with root system $\dot{\Delta}$, and fix a non-degenerate symmetric invariant form, $(\cdot|\cdot)$ on $\dot{\mathfrak{g}}$ (for example the Killing form). It can easily be seen that $(\cdot|\cdot)$ is non-degenerate on any Cartan subalgebra of $\dot{\mathfrak{g}}$. In fact for any root $\alpha \in \dot{\Delta}$, the form $(\cdot|\cdot)$ pairs the root space $\dot{\mathfrak{g}}^\alpha$ non-degenerately with the root space $\dot{\mathfrak{g}}^{-\alpha}$. Restricting this form to some Cartan subalgebra $\dot{h}$ then induces a natural map $\pi : \dot{h} \rightarrow \dot{h}^*$ by $\pi(u)(v) = (u|v)$. We also get a bilinear form on $\dot{h}^*$ by letting $(x|y) = (\pi^{-1}(x)|\pi^{-1}(y))$. Since $\dot{\mathfrak{g}}$ is simply laced, we can normalize both of these forms by taking $(\alpha|\alpha) = 2$ for $\alpha \in \dot{\Delta}$.

Define a ‘height function’ ht on $\dot{h}^*$ by $\text{ht}(\sum_{j=1}^l k_j \alpha_j) = \sum_{j=1}^l k_j$, where $\alpha_1, \dots, \alpha_\ell$

are the simple roots. Let $\rho \in \dot{h}^*$ be such that $(\rho|\alpha) = \text{ht}(\alpha)$ for $\alpha \in \dot{\Delta}$. We consider the principal $\mathbb{Z}_h$ -grading of $\dot{\mathfrak{g}}$,

$$\dot{\mathfrak{g}} = \sum_{j \in \mathbb{Z}_h} \dot{\mathfrak{g}}_j,$$

where h is the Coxeter number of $\dot{\mathfrak{g}}$ and $\dot{\mathfrak{g}}_j$ is the direct sum of root spaces $\dot{\mathfrak{g}}^\alpha$ with $\text{ht}(\alpha) = j \pmod{h}$. It is well known that $\dot{\mathfrak{g}}$ contains a Cartan subalgebra $\dot{s}$ which has a basis $\{T_1, \dots, T_\ell\}$ such that $T_i \in \dot{\mathfrak{g}}_{m_i}$ for $i = 1, \dots, \ell$, where $1 \leq m_1 \leq m_2 \leq \dots \leq m_\ell = h-1$ are called the exponents of $\dot{\mathfrak{g}}$. Since $(T_i|T_j) \neq 0$ implies $i+j = 0 \pmod{h}$ and $(\cdot|\cdot)$ is non-degenerate on $\dot{s}$, the basis can be normalized so that $(T_i|T_{\ell+1-j}) = h\delta_{ij}$.

Now consider the root system of $\dot{\mathfrak{g}}$ with respect to the Cartan subalgebra $\dot{s}$, which we shall denote $\dot{\Delta}_s$. For $\alpha \in \dot{\Delta}_s$, fix a root element $A^\alpha \in \dot{\mathfrak{g}}^\alpha$, and decompose A^α with respect to the principal grading of $\dot{\mathfrak{g}}$:

$$A^\alpha = \sum_{j \in \mathbb{Z}_h} A_j^\alpha, \quad A_j^\alpha \in \dot{\mathfrak{g}}_j.$$

Define the constants λ_i by $\lambda_i^\alpha = \alpha(T_i)$, so that $[T_i, A^\alpha] = \lambda_i^\alpha A^\alpha$.

The construction of the $n+1$ -toroidal Lie algebra over $\dot{\mathfrak{g}}$ then proceeds similarly to the construction of the principal realization of the untwisted affine Kac-Moody algebra. First, let $\mathfrak{g}$ be the universal central extension of

$$\sum_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \mathbb{C}[t_1^\pm, \dots, t_n^\pm].$$

The full toroidal Lie algebra is then obtained by adjoining some derivations to this central extension (possibly twisted by some 2-cocycle). Certain vertex operator representations of $\mathfrak{g}$ were studied in [2]. In this paper it is convenient to consider the

central extension of the larger Lie algebra

$$\tilde{\mathfrak{g}} = \sum_{j \in \mathbb{Z}} \dot{\mathfrak{g}}_j \otimes s^j \mathbb{C}[\mathbb{R}^n],$$

where we replace the algebra of Laurent polynomials $\mathbb{C}[t_1^\pm, \dots, t_n^\pm]$, which is the group algebra of $\mathbb{Z}^n$, with the group algebra of $\mathbb{R}^n$. The algebra $\mathbb{C}[\mathbb{R}^n]$ has a basis of monomials $\mathbf{t}^{\mathbf{r}} = t_1^{r_1} \cdots t_n^{r_n}$ for $\mathbf{r} \in \mathbb{R}^n$, and multiplication in $\mathbb{C}[\mathbb{R}^n]$ is defined by $\mathbf{t}^{\mathbf{r}} \cdot \mathbf{t}^{\mathbf{m}} = \mathbf{t}^{\mathbf{r}+\mathbf{m}}$. All the results from [2] remain true for this algebra.

The following description of the central extension of $\tilde{\mathfrak{g}}$ is based on a general result of Kassel [12]. Let $\dot{\mathcal{K}}$ be an $(n+1)$ -dimensional space with the basis $\{K_0, K_1, \dots, K_n\}$. Consider the space

$$\tilde{\mathcal{K}} = \dot{\mathcal{K}} \otimes \mathbb{C}[s^h, s^{-h}] \otimes \mathbb{C}[\mathbb{R}^n]$$

and its subspace $d\tilde{\mathcal{K}}$ spanned by the elements

$$r_0 K_0 \otimes s^{r_0 h} \mathbf{t}^{\mathbf{r}} + r_1 K_1 \otimes s^{r_1 h} \mathbf{t}^{\mathbf{r}} + \dots + r_n K_n \otimes s^{r_n h} \mathbf{t}^{\mathbf{r}}, \quad \text{where}$$

$$\mathbf{r} = (r_1, r_2, \dots, r_n).$$

It can then be shown that $\mathcal{K} = \tilde{\mathcal{K}}/d\tilde{\mathcal{K}}$ is the universal central extension of $\tilde{\mathfrak{g}}$. The Lie bracket in this central extension $\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \mathcal{K}$ is given by:

$$[g_1 \otimes f_1(s, \mathbf{t}), g_2 \otimes f_2(s, \mathbf{t})] = [g_1, g_2] \otimes f_1 f_2 + (g_1 | g_2) \left\{ \frac{1}{h} s \frac{\partial f_1}{\partial s} f_2 K_0 + \sum_{p=1}^n t_p \frac{\partial f_1}{\partial t_p} f_2 K_p \right\}$$

and $[\mathfrak{g}, \mathcal{K}] = 0$. From now on we will omit the tensor product sign when writing the elements of $\mathfrak{g}$.

The operators $d_i = t_i \frac{\partial}{\partial t_i}$ and $d_s = s \frac{\partial}{\partial s}$ act naturally as derivations of the Lie algebra $\tilde{\mathfrak{g}}$. Thus

$$\mathcal{D} = \left\{ \sum_{p=1}^n f_p(s, t_1, \dots, t_n) d_p + f_0(s, p_1, \dots, p_n) d_s \mid f_0, \dots, f_n \in \mathbb{C}[s^\pm, t_1^\pm, \dots, t_n^\pm] \right\}$$

is a space of derivations on $\tilde{\mathfrak{g}}$. This action can then be lifted in a unique way to the Lie algebra $\mathfrak{g}$. We denote the lifting of d_a by D_a . Its action on $\mathcal{K}$ is given by:

$$f_1 D_a(f_2 K_b) = f_1 d_a(f_2) K_b + \delta_{ab} \left(\sum_{p=1}^n f_2 d_p(f_1) K_p + f_2 d_s(f_1) K_s \right).$$

In order to construct a full toroidal Lie algebra, we would have to include these derivations. However, as we have mentioned, the representations used in this paper do not behave well with respect to some of these derivations (see e.g. [2]). For this reason, we treat $\mathfrak{g}$ as our basic Lie algebra, and handle the derivations separately. In general, it is probably better to include them in the algebra. The representation theory in this case is still being developed.

3 Representation theory for toroidal Lie algebras

We are now ready to describe the representation of $\mathfrak{g}$ constructed in [3]. The representation space is the Fock space,

$$F = \mathbb{C}[\mathbb{R}^n] \otimes \mathbb{C}[x_{b_i}, u_{pi}, v_{pi}]_{i \in \mathbb{N}}^{p=1, \dots, n},$$

where $\{b_i\}$ is defined by $b_{i+jl} = m_i + jh$. We will define a representation φ of $\mathfrak{g}$ on F by giving generating series whose coefficients are the images of elements of $\mathfrak{g}$ under

φ . The various series involved are known as vertex operators. They are related to the study of vertex operator algebras, a fairly new field of mathematics which is closely tied to mathematical physics and string theory. It is by working with these vertex algebras that Billig has recently been able to improve the representation theory for toroidal Lie algebras (see [5]). It turns out that the space F should be enlarged in order to obtain a representation of $\mathfrak{g} \oplus \mathcal{D}$. However, for our purposes it suffices to work with the following correspondence of fields:

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \varphi(s^{jh} t^{\mathbf{r}} K_0) z^{-jh} &= K_0(z, r), \text{ where} \\ K_0(z, r) &= \mathbf{q}^{\mathbf{r}} \exp \left(\sum_{p=1}^n r_p \sum_{j \geq 1} z^{jh} u_{pj} \right) \exp \left(- \sum_{p=1}^n r_p \sum_{j \geq 1} \frac{z^{-jh}}{j} \frac{\partial}{\partial v_{pj}} \right). \\ \sum_{j \in \mathbb{Z}} \varphi(s^{jh} \mathbf{t}^{\mathbf{r}} K_p) z^{-jh} &= K_p(z, \mathbf{r}) = K_p(z) K_0(z, \mathbf{r}), \text{ where} \\ K_p(z) &= \sum_{i \geq 1} i z^{ih} u_{pi} + \sum_{i \geq 1} z^{-ih} \frac{\partial}{\partial v_{pi}}. \\ \sum_{j \in \mathbb{Z}} \varphi(T_i s^{m_i+jh} t^{\mathbf{r}}) z^{-m_i-jh} &= T_i(z, \mathbf{r}) = T_i(z) K_0(z, \mathbf{r}), \quad i = 1, \dots, \ell, \text{ where} \\ T_i(z) &= \sum_{j \geq 0} (jh - m_i) z^{jh-m_i} x_{jh-m_i} + \sum_{j \geq 0} z^{-jh-m_i} \frac{\partial}{\partial x_{jh+m_i}}. \\ \sum_{j \in \mathbb{Z}} \varphi(A_j^{\alpha} s^j \mathbf{t}^{\mathbf{r}}) z^{-j} &= A^{\alpha}(z, \mathbf{r}) = A^{\alpha}(z) K_0(z, \mathbf{r}), \alpha \in \dot{\Delta}_s, \text{ where} \\ A^{\alpha}(z) &= -\frac{\rho(A_0^{\alpha})}{h} \exp \left(\sum_{i \geq 1} \lambda_i^{\alpha} z^{b_i} x_{b_i} \right) \exp \left(- \sum_{i \geq 1} \lambda_{\ell+1-i}^{\alpha} \frac{z^{-b_i}}{b_i} \frac{\partial}{\partial x_{b_i}} \right). \end{aligned}$$

It should be mentioned that each coefficient of $K_0(z, r)$ is actually an infinite sum. So essentially every element in $\mathfrak{g}$ is represented by an infinite sum of differential operators on F . However, for any $g \in \mathfrak{g}$, and any fixed element of F , all but finitely

many of the terms in $\varphi(g)(x)$ will turn out to be zero. To see this, consider the grading of the space of differential operators on F defined by setting

$$\deg \frac{\partial}{\partial v_{pi}} = i, \text{ and } \deg \frac{\partial}{\partial x_i} = i.$$

Then for any fixed $x \in F$, all homogeneous operators of sufficiently high degree will send x to zero. It is not hard to see that each of the coefficients in our series will consist of a sum of homogeneous terms whose degrees tend to infinity. Hence, each of these infinite sums will become finite as soon as it is applied to some element of F , and so this representation is well defined.

We also want to represent the derivations of $\mathfrak{g}$ as operators on F . This is again done by giving generating series, but before stating these we need to discuss the operation of normal ordering.

Consider the algebra of differential operators $\text{diff}(y_1, y_2, \dots)$ acting on the space $\mathbb{C}[y_1, y_2, \dots]$,

$$\text{diff}(y_1, y_2, \dots) = \left\{ \sum_{\mathbf{n} \in \mathcal{A}} f_{\mathbf{n}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}} \right)^{\mathbf{n}} \mid f_{\mathbf{n}}(y) \in \mathbb{C}[y_1, y_2, \dots] \right\},$$

where $\mathcal{A}$ is the set of sequences $\mathbf{n} = (n_1, n_2, \dots)$ with $n_i \in \mathbb{Z}_+$ with only finitely many nonzero terms. We will be using the notation $\mathbf{y} = (y_1, y_2, \dots)$ and $\left(\frac{\partial}{\partial \mathbf{y}} \right)^{\mathbf{n}} = \left(\frac{\partial}{\partial y_1} \right)^{n_1} \left(\frac{\partial}{\partial y_2} \right)^{n_2} \dots$. The space $\text{diff}(y_1, y_2, \dots)$ has the structure of an associative algebra with respect to composition of operators. However, this algebra is not commutative since $\frac{\partial}{\partial y_i}$ and y_i do not commute. The normally ordered product $: \cdots :$ is a

new commutative product on $\text{diff}(y_1, y_2, \dots)$ defined by

$$: \left(\sum_{\mathbf{n} \in \mathcal{A}} f_{\mathbf{n}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}} \right)^{\mathbf{n}} \right) \left(\sum_{\mathbf{m} \in \mathcal{A}} g_{\mathbf{m}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}} \right)^{\mathbf{m}} \right) := \left(\sum_{\mathbf{n} + \mathbf{m} \in \mathcal{A}} f_{\mathbf{n}}(\mathbf{y}) g_{\mathbf{m}}(\mathbf{y}) \left(\frac{\partial}{\partial \mathbf{y}} \right)^{\mathbf{n} + \mathbf{m}} \right).$$

With this tool available, we can now give the generating series for the outer derivations of $\mathfrak{g}$. Consider the following operators on F :

$$D_p(z, \mathbf{r}) =: D_p(z) K_0(z, \mathbf{r}) :, \quad p = 1, \dots, n, \quad \text{where}$$

$$D_p(z) = \sum_{i \geq 1} i z^{ih} v_{pi} + q_p \frac{\partial}{\partial q_p} + \sum_{i \geq 1} z^{-ih} \frac{\partial}{\partial u_{pi}}, \quad \text{and}$$

$$D_s(z, \mathbf{r}) =: D_s(z) K_0(z, \mathbf{r}) :, \quad \text{where}$$

$$D_s(z) = -\frac{1}{2} \sum_{i=1}^{\ell} : T_i(z) T_{\ell+1-i}(z) : -h \sum_{p=1}^n : D_p(z) K_p(z) : .$$

Expanding the generating series $D_p(z, \mathbf{r})$ and $D_s(z, \mathbf{r})$, we obtain operators $s^{jh} \mathbf{t}^{\mathbf{r}} D_p$ and $s^{jh} \mathbf{t}^{\mathbf{r}} D_s$ by setting

$$D_p(z, \mathbf{r}) = \sum_{j \in \mathbb{Z}} s^{jh} \mathbf{t}^{\mathbf{r}} D_p z^{-jh}, \quad D_s(z, \mathbf{r}) = \sum_{j \in \mathbb{Z}} s^{jh} \mathbf{t}^{\mathbf{r}} D_s z^{-jh}.$$

The adjoint actions of $s^{jh} \mathbf{t}^{\mathbf{r}} D_p$ and $s^{jh} \mathbf{t}^{\mathbf{r}} D_s$ on $\varphi(\mathfrak{g})$ will then correspond to the outer derivations of $\mathfrak{g}$ described in the previous section. We summarize this by stating the main results of (Theorem 5 and Proposition 8) of [2]:

Theorem 1. (a) *The preceding formulas define a representation of the Lie algebra $\mathfrak{g}$ on the Fock space F .*

(b) *The operators $s^{roh} \mathbf{t}^{\mathbf{r}} D_p$ and $s^{roh} \mathbf{t}^{\mathbf{r}} D_s$ as defined above act on $\varphi(\mathfrak{g})$ as derivations:*

$$[s^{roh} \mathbf{t}^{\mathbf{r}} D_p, \varphi(A_j^{\alpha} s^j \mathbf{t}^{\mathbf{m}})] = m_p \varphi(A_j^{\alpha} s^{roh+j} \mathbf{t}^{\mathbf{r}+\mathbf{m}}),$$

$$[s^{r_0 h} \mathbf{t}^{\mathbf{r}} D_s, \varphi(A_j^\alpha s^j \mathbf{t}^{\mathbf{m}})] = j \varphi(A_j^\alpha s^{r_0 h + j} \mathbf{t}^{\mathbf{r} + \mathbf{m}}),$$

$$[s^{r_0 h} \mathbf{t}^{\mathbf{r}} D_p, \varphi(T_i s^j \mathbf{t}^{\mathbf{m}})] = m_p \varphi(T_i s^{r_0 h + j} \mathbf{t}^{\mathbf{r} + \mathbf{m}}),$$

$$[s^{r_0 h} \mathbf{t}^{\mathbf{r}} D_s, \varphi(T_i s^j \mathbf{t}^{\mathbf{m}})] = j \varphi(T_i s^{r_0 h + j} \mathbf{t}^{\mathbf{r} + \mathbf{m}}).$$

Although the operators $s^{r_0 h} \mathbf{t}^{\mathbf{r}} D_p$ and $s^{r_0 h} \mathbf{t}^{\mathbf{r}} D_s$ act on $\varphi(\mathfrak{g})$ as derivations on $\mathbb{C}[s^h, s^{-h}] \otimes \mathbb{C}[\mathbb{R}^n]$, the algebra they generate together with $\varphi(\mathfrak{g})$ is not isomorphic to the semi-direct product of $\mathfrak{g}$ with $\mathcal{D} = \text{Der}(\mathbb{C}[s^h, s^{-h}] \otimes \mathbb{C}[\mathbb{R}^n])$. A direct computation shows that the span of $s^{r_0 h} \mathbf{t}^{\mathbf{r}} D_p$ and $s^{r_0 h} \mathbf{t}^{\mathbf{r}} D_s$ is not closed under the Lie bracket. The commutators contain extra terms which commute with $\varphi(\mathfrak{g})$. So as we have said we do not really have a representation of the full algebra $\mathfrak{g} \oplus \mathcal{D}$. However, it will retain enough properties of a representation of this algebra for our purposes.

Once we have this representation, the main tool we need to construct our hierarchy of partial differential equations will be some generalized Casimir operators. In general, the Casimir element of a Lie algebra is an element of the universal enveloping associative algebra which commutes with the image of the Lie algebra. Here we would like an operator on $F \otimes F$ which commutes with the image of $\mathfrak{g}$ under the representation φ . There is a standard way of constructing a Casimir element for any Lie algebra $\mathfrak{k}$ possessing a non-degenerate invariant bilinear form, and in general if there is a representation of $\mathfrak{k}$ on F , then this Casimir maps naturally to an operator on $F \otimes F$. There are two difficulties with this construction. First, our representation does not extend to the full toroidal Lie algebra $\mathfrak{g} \oplus \mathcal{D}$, and second $\mathfrak{g} \oplus \mathcal{D}$ does not possess the required form. However, there is a slightly smaller algebra which does.

Define the subspace $\mathcal{D}_{\text{div}}$ of $\mathcal{D}$ by

$$\mathcal{D}_{\text{div}} = \left\{ f_s(s, \mathbf{t})D_s + f_1(s, \mathbf{t})D_1 + \dots + f_n(s, \mathbf{t})D_n \mid \frac{1}{h}s\frac{\partial f_s}{\partial s} + t_1\frac{\partial f_1}{\partial t_1} + \dots + t_n\frac{\partial f_n}{\partial t_n} = 0 \right\}.$$

The expression $\frac{1}{h}s\frac{\partial f_s}{\partial s} + t_1\frac{\partial f_1}{\partial t_1} + \dots + t_n\frac{\partial f_n}{\partial t_n} = 0$ is actually the divergence of the vector field in coordinates $(x_0, \dots, x_n)$ on a torus, where $s = e^{\frac{ix_0}{h}}$, and $t_j = e^{ix_j}$, so $\mathcal{D}_{\text{div}}$ can be considered as the divergence free part of $\mathcal{D}$. Then there is a natural invariant bilinear form defined on $\mathfrak{g} \oplus \mathcal{D}_{\text{div}}$ defined by setting

$$(s^{r_0}\mathbf{t}^{\mathbf{r}}g_1 | s^{m_0}\mathbf{t}^{\mathbf{m}}g_2) = \delta_{r_0, -m_0}\delta_{\mathbf{r}, -\mathbf{m}}(g_1 | g_2)$$

where $(g_1 | g_2)$ is the Killing form on $\mathfrak{g}$,

$$\left(\sum_{p=0}^n a_p s^{r_0}\mathbf{t}^{\mathbf{r}} D_p | s^{m_0}\mathbf{t}^{\mathbf{m}} K_q \right) = \delta_{r_0, -m_0}\delta_{\mathbf{r}, -\mathbf{m}} a_q,$$

$$\text{and } (\mathfrak{g} | \mathcal{K} \oplus \mathcal{D}_{\text{div}}) = 0,$$

and extending linearly. This formula is not well defined if we try to apply it to the $\mathfrak{g} \oplus \mathcal{D}$ since $\sum_{q=0}^n r_q s^{r_0 h} \mathbf{t}^{\mathbf{r}} K_q = 0$ in $\mathcal{K}$, and hence must be in the kernel of the form. $\mathcal{D}_{\text{div}}$ is exactly the subspace of $\mathcal{D}$ where this holds. Verifying that the form is invariant and non-degenerate poses no significant problems.

Using this form, it is possible to construct some ‘generalized Casimir operators’ which commute with $\varphi(\mathfrak{g})$. Details for this construction can be found in the appendix of [3]. The resulting operators are introduced by the following generating series:

$$\Omega(z) = \sum_{j \in \mathbb{Z}} \Omega_j z^{-jh} =$$

$$\begin{aligned}
&=: \left\{ \frac{1}{h} \sum_{i=1}^{\ell} T_i(z) \otimes T_{\ell+1-i}(z) + \sum_{\alpha \in \Delta} \frac{1}{(A^\alpha | A^{-\alpha})} A^\alpha(z) \otimes A^\alpha(z) - \frac{\ell(h+1)}{12h} \right. \\
&\quad \left. + \frac{1}{h} D_s(z) \otimes 1 + \frac{1}{h} 1 \otimes D_s(z) + \sum_{p=1}^n D_p(z) \otimes K_p(z) + \sum_{p=1}^n K_p(z) \otimes D_p(z) \right\} \times \\
&\quad \times \sum_{\mathbf{r} \in \mathbb{R}^n} K_0(z, \mathbf{r}) \otimes K_0(z, -\mathbf{r}) : .
\end{aligned}$$

Both $\mathfrak{g}$ and its module F are $\mathbb{Z} \times \mathbb{R}^n$ graded. Thus the $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ module $F \otimes F$ is graded by $(\mathbb{Z} \times \mathbb{R}^n) \times (\mathbb{Z} \times \mathbb{R}^n)$. Consider the completion $\overline{F \otimes F}$ of $F \otimes F$ with respect to this grading. Then the operators $\Omega_k = \text{Res}(z^{-k-1} \Omega(z))$ act from $F \otimes F$ to $\overline{F \otimes F}$. The following result, also proven in the appendix of [3], gives us what we need:

Proposition 1. *The operators $\Omega_k : F \otimes F \rightarrow \overline{F \otimes F}$ commute with the action of $\mathfrak{g}$.*

4 Some background results

This work combines methods of extending the KdV hierarchy developed by Billig in [3] and [4]. The main idea in [4] is to convert the representation of $\mathfrak{g}$ as differential operators into a homomorphism from $\mathfrak{g}$ to the Weyl algebra. Not surprisingly, this is important here as well.

Differential operators on the space $\mathbb{C}[y_1, y_2, y_3, \dots]$ naturally form an associative algebra with generators $q_i = y_i$ (as a multiplication operator) and $p_i = \frac{\partial}{\partial y_i}$. The commutation relations in this algebra are given by $p_i q_j - q_j p_i = \delta_{ij} \cdot 1$, and $p_i p_j - p_j p_i = q_i q_j - q_j q_i = 0$. Often it is convenient to generalize slightly by allowing $p_i q_j - q_j p_i = c_i \delta_{ij} \cdot 1$ where the c_i are fixed constants. This is known as the Weyl

algebra, and will be denoted by W . We can clearly see that W has Poincaré-Birkhoff-Witt decomposition $W = \mathbb{C}[q_1, q_2, q_3, \dots] \otimes \mathbb{C}[p_1, p_2, p_3, \dots]$. Since both $\mathbb{C}[q_1, q_2, \dots]$ and $\mathbb{C}[p_1, p_2, \dots]$ are commutative, there is a linear bijection π between W and the polynomial algebra $\mathbb{C}[y_{-1}, y_{-2}, \dots] \otimes \mathbb{C}[y_1, y_2, \dots]$ by setting $\pi(q_i) = y_i$ and $\pi(p_i) = y_{-i}$. The image of a differential operator under this map is known as the symbol of the operator.

Although the Weyl algebra is a non-commutative, the partial differentiation operators $\frac{\partial}{\partial p_i}$ and $\frac{\partial}{\partial q_i}$ remain well defined. Furthermore, notice that for $f \in W$, $\pi\left(\frac{\partial}{\partial q_i}f\right) = \frac{\partial}{\partial y_i}\pi(f)$ and $\pi\left(\frac{\partial}{\partial p_i}f\right) = \frac{\partial}{\partial y_{-i}}\pi(f)$. So the operation of taking symbols commutes with partial differentiation. Observe that the commutation relation of the Weyl algebra are equivalent to: For each $\rho \in W$,

$$q_i\rho = c_i\frac{\partial}{\partial p_i}\rho + \rho q_i, \quad \text{and} \quad \rho p_i = c_i\frac{\partial}{\partial q_i}\rho + p_i\rho.$$

We make use of this to create partial differential equations from expressions in the Weyl algebra. The following result is also useful:

Lemma 2. *Let p, q be generators of the Weyl algebra satisfying $pq - qp = c \cdot 1$. Then for $\tau = \tau(p, q)$ we have*

$$\exp(zp)\tau = \left(\exp\left(cz\frac{\partial}{\partial q}\right)\tau\right)\exp(zp)$$

and

$$\tau\exp(zq) = \exp(zq)\left(\exp\left(cz\frac{\partial}{\partial p}\right)\tau\right).$$

For a proof, see [4].

It will be convenient to express our equations in the form of Hirota bilinear equations, so we now review some definitions and basic theorems concerning Hirota bilinear differentiation. If f, g are functions which depend on a variable x , then let

$$H_x f(x) \circ g(x) = \frac{\partial}{\partial \tilde{x}} (f(x + \tilde{x})g(x - \tilde{x})) \Big|_{\tilde{\mathbf{x}}=\mathbf{0}}.$$

This is called Hirota bilinear differentiation of $f \circ g$. More generally, if f, g depend of several variables, say $x_1, \dots, x_k$ and P is a polynomial in $H_{x_1}, \dots, H_{x_k}$, then we let

$$P(H_{x_1} \dots H_{x_k}) f \circ g = P \left(\frac{\partial}{\partial \tilde{x}_1}, \dots, \frac{\partial}{\partial \tilde{x}_k} \right) f(x_1 + \tilde{x}_1, \dots, x_k + \tilde{x}_k) g(x_1 - \tilde{x}_1, \dots, x_k - \tilde{x}_k) \Big|_{\tilde{\mathbf{x}}=\mathbf{0}}.$$

The polynomial P is known as a Hirota polynomial.

Using techniques as in [10], the following observation is crucial to our calculations:

Lemma 3. *For any Hirota polynomial R and any τ ,*

$$\begin{aligned} & R \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots \right) \tau(\tilde{\mathbf{y}} + \mathbf{y}) \tau(\tilde{\mathbf{y}} - \mathbf{y}) = \\ &= R \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \right) \tau(\tilde{\mathbf{y}} + (\mathbf{x} + \mathbf{y})) \tau(\tilde{\mathbf{y}} - (\mathbf{x} + \mathbf{y})) \Big|_{\mathbf{x}=\mathbf{0}} = \\ &= R \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \right) \exp \left(\sum_{i \geq 1} y_i \frac{\partial}{\partial x_i} \right) \tau(\tilde{\mathbf{y}} + \mathbf{x}) \tau(\tilde{\mathbf{y}} - \mathbf{x}) \Big|_{\mathbf{x}=\mathbf{0}} = \\ &= R(H_{\tilde{y}_1}, H_{\tilde{y}_2}, \dots) \exp \left(\sum_{i \geq 1} y_i H_{\tilde{y}_i} \right) \tau(\tilde{\mathbf{y}}) \circ \tau(\tilde{\mathbf{y}}). \end{aligned}$$

Note that by the Leibniz rule, for any function $\tau(x)$ we have $H_x \tau \circ \tau = \tau_x \tau - \tau \tau_x = 0$. In fact, extending this idea we see that whenever $P(x_1, \dots, x_k) = -P(-x_1, \dots, -x_k)$, it will follow that $P(H_1, \dots, H_k) \tau \circ \tau = 0$ regardless of τ . So if P has terms of odd

degree, these will not contribute to $Pf \circ f$ for any f , and hence for our purposes such terms should be ignored.

We now have all the tools we will need to construct our hierarchies of partial differential equations. The method will proceed as follows (for a more detailed explanation in the case $\dot{\mathfrak{g}} = sl_2(\mathbb{C})$ see the next 2 sections). Fix some finite dimensional Lie algebra $\dot{\mathfrak{g}}$. For some n , construct the toroidal algebra $\mathfrak{g}$ as in section 2, and consider its representation on the appropriate Fock space F . Let $\Omega(z)$ be the generating series for the generalized Casimir operators as in section 3, and look at the expression $\Omega(z)\tau \otimes \tau - \tau \otimes \tau\Omega(z) = 0$. Because Ω_k commutes with the action of $\mathfrak{g}$ for all k , any element of $\mathfrak{g}$ is a solution to this expression. Since $\Omega(z)$ is acting on $F \otimes F$, introduce two sets of variables, v' and v'' , to denote the first and second copy of F . This removes the $\otimes$ from the expression. Next, instead of considering Ω_k as a differential operator, think of it as an element in the Weyl algebra, where for each variable v_i , we define the variable $v_{-i} = \frac{\partial}{\partial v_i}$. We then have commutation relations $v_{-i}v_i - v_iv_{-i} = 1$ (it is actually necessary to make some changes of variables here as well, the details should be clear from the case presented in the next few sections). Then using these commutation relations, shift all negative variables to the right of $\Omega(z)$ and all positive variables to the left. This results in introducing derivatives with respect to both the positive and negative variables. Because the operation of taking symbols commutes with partial differentiation, we can then take the symbol of this generating series to get a hierarchy of partial differential equations. Finally using the

fact that $\mathfrak{g}$ commutes with Ω it is possible to construct soliton type solutions to these equations.

5 The case of $\mathfrak{sl}_2(\mathbb{C})$

We will now specialize to the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $n = 1$, and explicitly construct our hierarchy of partial differential equations along with soliton type solutions. The same method works for any simply laced finite dimensional simple complex Lie algebra, and for any n .

The Coxeter number of $\mathfrak{sl}_2(\mathbb{C})$ is $h = 2$, and its only exponent is $m_1 = 1$. The Fock space in this case is

$$F = \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[x_1, x_2, x_3, \dots] \otimes \mathbb{C}[u_1, u_2, \dots] \otimes \mathbb{C}[v_1, v_2, \dots]$$

where $\mathbb{C}[\mathbb{R}]$ is the group algebra of $(\mathbb{R}, +)$ with basis $\{q^r | r \in \mathbb{R}\}$. By substituting this data into the series from section 3, we see that the action of $\mathfrak{g}$ on F is defined by:

$$\begin{aligned} K_0(z, r) &= q^r \exp \left(r \sum_{i \geq 1} z^{2i} u_i \right) \exp \left(-r \sum_{i \geq 1} \frac{z^{-2i}}{i} \frac{\partial}{\partial v_i} \right), \\ K_1(z) &= \sum_{i \geq 1} i z^{2i} u_i + \sum_{i \geq 1} z^{-2i} \frac{\partial}{\partial v_i}, \\ T(z) &= \sum_{j \in \mathbb{N}_{\text{odd}}} j z^j x_j + \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \frac{\partial}{\partial x_j} \\ A^{\pm \alpha}(z) &= \frac{1}{2} \exp \left(\pm 2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(\mp 2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} \right), \\ T(z, r) &= T(z) K_0(z, r), \quad A^{\pm \alpha}(z, r) = A^{\pm \alpha}(z) K_0(z, r), \quad K_1(z, r) = K_1(z) K_0(z, r). \end{aligned}$$

For the positive root α of $\mathfrak{sl}_2(\mathbb{C})$, we will denote $A^\alpha(z, r)$ simply by $A(z, r)$. Then $A^{-\alpha}(z, r) = A(-z, r)$.

The derivations of $\mathfrak{g}$ are represented by:

$$D_1(z) = \sum_{i \geq 1} i z^{2i} v_i + q \frac{\partial}{\partial q} + \sum_{i \geq 1} z^{-2i} \frac{\partial}{\partial u_i},$$

$$D_s(z) = -\frac{1}{2} : T(z)T(z) : - 2 : D_1(z)K_1(z) :,$$

$$D_1(z, r) =: D_1(z)K_0(z, r) :, \quad D_s(z, r) =: D_s(z)K_0(z, r) :.$$

The generalized Casimir generating series can then be written as:

$$\Omega(z) = \sum_{j \in \mathbb{Z}} \Omega_j z^{-2j} =: \left\{ \frac{1}{4} A(z) \otimes A(-z) + \frac{1}{4} A(-z) \otimes A(z) - \frac{1}{8} - \frac{1}{4} : (T(z) \otimes 1 - 1 \otimes T(z))^2 : \right. \\ \left. - : (D_1(z) \otimes 1 - 1 \otimes D_1(z))(K_1(z) \otimes 1 - 1 \otimes K_1(z)) : \right\} \times \sum_{r \in \mathbb{R}} K_0(z, r) \otimes K_0(z, -r) :.$$

We will be considering the equation

$$\Omega(z)\tau \otimes \tau - \tau \otimes \tau\Omega(z) = 0.$$

Since $\Omega(z)$ commutes with the action of $\mathfrak{g}$ on $F \otimes F$, we will be able to use $\mathfrak{g}$ to find solutions to this expression, and hence to partial differential equations we construct from it.

Recall that the tensor square of a polynomial algebra F is again a polynomial algebra in twice as many variables. We will denote the variables in the first copy of F by q', x'_j, u'_i, v'_i and those in the second copy of F by q'', x''_j, u''_i, v''_i . Then we obtain the following representation for $\Omega(z)$:

$$\begin{aligned}
\Omega(z) = & \left\{ \frac{1}{16} \exp \left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j (x'_j - x''_j) \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} \left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j} \right) \right) \right. \\
& + \frac{1}{16} \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j (x'_j - x''_j) \right) \exp \left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} \left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j} \right) \right) - \frac{1}{8} \\
& - \frac{1}{4} : \left(\sum_{j \in \mathbb{N}_{\text{odd}}} j z^j (x'_j - x''_j) + \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j} \right) \right)^2 : \\
& - : \left(\sum_{i \geq 1} i z^{2i} (v'_i - v''_i) + \left(q' \frac{\partial}{\partial q'} - q'' \frac{\partial}{\partial q''} \right) + \sum_{i \geq 1} z^{-2i} \left(\frac{\partial}{\partial u'_i} - \frac{\partial}{\partial u''_i} \right) \right) \\
& \times \left(\sum_{i \geq 1} i z^{2i} (u'_i - u''_i) + \sum_{i \geq 1} z^{-2i} \left(\frac{\partial}{\partial v'_i} - \frac{\partial}{\partial v''_i} \right) \right) : \} \\
& \times \sum_{r \in \mathbb{R}} \left(\frac{q'}{q''} \right)^r \exp \left(r \sum_{i \geq 1} z^{2i} (u'_i - u''_i) \right) \exp \left(-r \sum_{i \geq 1} \frac{z^{-2i}}{i} \left(\frac{\partial}{\partial v'_i} - \frac{\partial}{\partial v''_i} \right) \right) : .
\end{aligned}$$

To put this expression in the form of Hirota differential equations, we need to make a change of variables as in [3]. However, unlike in [3], we want to work in a Weyl algebra. So, we define:

$$\begin{aligned}
\hat{w} &= \frac{1}{2} \ln \left(\frac{q'}{q''} \right), \tilde{w} = \frac{1}{2} \ln(q' q''), \hat{w}_- = \frac{-1}{2} \left(q' \frac{\partial}{\partial q'} - q'' \frac{\partial}{\partial q''} \right), \tilde{w}_- = \frac{-1}{2} \left(\frac{1}{q'} \frac{\partial}{\partial q'} + \frac{1}{q''} \frac{\partial}{\partial q''} \right), \\
\hat{x}_j &= \frac{1}{2} (x'_j - x''_j), \tilde{x}_j = \frac{1}{2} (x'_j + x''_j), \hat{x}_{-j} = \frac{-1}{2j} \left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j} \right), \tilde{x}_{-j} = \frac{-1}{2j} \left(\frac{\partial}{\partial x'_j} + \frac{\partial}{\partial x''_j} \right), \\
\hat{u}_i &= \frac{1}{2} (u'_i - u''_i), \tilde{u}_i = \frac{1}{2} (u'_i + u''_i), \hat{u}_{-i} = \frac{-1}{2i} \left(\frac{\partial}{\partial u'_i} - \frac{\partial}{\partial u''_i} \right), \tilde{u}_{-i} = -\frac{-1}{2i} \left(\frac{\partial}{\partial u'_i} + \frac{\partial}{\partial u''_i} \right), \\
\hat{v}_i &= \frac{1}{2} (v'_i - v''_i), \tilde{v}_i = \frac{1}{2} (v'_i + v''_i), \hat{v}_{-i} = \frac{-1}{2i} \left(\frac{\partial}{\partial v'_i} - \frac{\partial}{\partial v''_i} \right), \tilde{v}_{-i} = \frac{-1}{2i} \left(\frac{\partial}{\partial v'_i} + \frac{\partial}{\partial v''_i} \right).
\end{aligned}$$

Then $\{\hat{x}_{\pm j}, \hat{u}_{\pm i}, \hat{v}_{\pm i}, \hat{w}_{\pm}, \tilde{x}_{\pm j}, \tilde{u}_{\pm i}, \tilde{v}_{\pm i}, \tilde{w}, \tilde{w}_- | i \in \mathbb{N}, j \in \mathbb{N}_{\text{odd}}\}$ generates a Weyl algebra with commutation relations

$$\hat{w}_- \hat{w} - \hat{w} \hat{w}_- = \frac{-1}{2} \cdot 1 \quad \tilde{w}_- \tilde{w} - \tilde{w} \tilde{w}_- = \frac{-1}{2} \cdot 1,$$

$$\begin{aligned}
\hat{x}_{-j}\hat{x}_j - \hat{x}_j\hat{x}_{-j} &= \frac{-1}{2j} \cdot 1, & \tilde{x}_{-j}\tilde{x}_j - \tilde{x}_j\tilde{x}_{-j} &= \frac{-1}{2j} \cdot 1, \\
\hat{u}_{-i}\hat{u}_i - \hat{u}_i\hat{u}_{-i} &= \frac{-1}{2i} \cdot 1, & \tilde{u}_{-i}\tilde{u}_i - \tilde{u}_i\tilde{u}_{-i} &= \frac{-1}{2i} \cdot 1, \\
\hat{v}_{-i}\hat{v}_i - \hat{v}_i\hat{v}_{-i} &= \frac{-1}{2i} \cdot 1, & \tilde{v}_{-i}\tilde{v}_i - \tilde{v}_i\tilde{v}_{-i} &= \frac{-1}{2i} \cdot 1.
\end{aligned}$$

We have rescaled the ‘negative’ variables here so that they have the same coefficients as the ‘positive’ variables in $\Omega(z)$. This has been done in order to exhibit symmetry between the positive and negative variables. It will also simplify the expressions for both our equations and their solutions. Strictly speaking, the variables $\tilde{x}_{\pm j}$, $\tilde{u}_{\pm i}$, $\tilde{v}_{\pm i}$, $\tilde{w}_{\pm}$ in the Weyl algebra should be adorned with both a $\tilde{\cdot}$ and a $\hat{\cdot}$. For these variables we will use the same notation when working in the Weyl algebra or after taking symbols. Our meaning should be clear from context.

Since we have changed the variables in $\Omega(z)$, we should also change variables in τ so that they remain compatible. To this end, we define T by

$$\begin{aligned}
T(w) &= e^w, & T(w_-) &= e_-^w, \\
T(x_j) &= x_j, & T(x_{-j}) &= -jx_{-j} \quad \text{for } j \in \mathbb{N}_{\text{odd}}, \\
T(u_i) &= u_i, & T(u_{-i}) &= -iu_{-i} \quad \text{for } i \in \mathbb{N}, \\
T(v_i) &= v_i, & T(v_{-i}) &= -iv_{-i} \quad \text{for } i \in \mathbb{N},
\end{aligned}$$

and let $\dot{\tau}$ represent the composition of τ with the change of variables T . Then

$$\tau(q', \mathbf{x}', \mathbf{u}', \mathbf{v}') = \tau(T(\tilde{w} + \hat{w}, \tilde{\mathbf{x}} + \hat{\mathbf{x}}, \tilde{\mathbf{u}} + \hat{\mathbf{u}}, \tilde{\mathbf{v}} + \hat{\mathbf{v}})) = \dot{\tau}(\tilde{w} + \hat{w}, \tilde{\mathbf{x}} + \hat{\mathbf{x}}, \tilde{\mathbf{u}} + \hat{\mathbf{u}}, \tilde{\mathbf{v}} + \hat{\mathbf{v}})$$

and

$$\tau(q'', \mathbf{x}'', \mathbf{u}'', \mathbf{v}'') = \tau(T(\tilde{w} - \hat{w}, \tilde{\mathbf{x}} - \hat{\mathbf{x}}, \tilde{\mathbf{u}} - \hat{\mathbf{u}}, \tilde{\mathbf{v}} - \hat{\mathbf{v}})) = \dot{\tau}(\tilde{w} - \hat{w}, \tilde{\mathbf{x}} - \hat{\mathbf{x}}, \tilde{\mathbf{u}} - \hat{\mathbf{u}}, \tilde{\mathbf{v}} - \hat{\mathbf{v}}).$$

With these definitions, the equation

$$0 = \Omega(z)\tau(q', \mathbf{x}', \mathbf{u}', \mathbf{v}') \otimes \tau(q'', \mathbf{x}'', \mathbf{u}'', \mathbf{v}'') - \tau(q', \mathbf{x}', \mathbf{u}', \mathbf{v}') \otimes \tau(q'', \mathbf{x}'', \mathbf{u}'', \mathbf{v}'')\Omega(z)$$

can be written as:

$$\begin{aligned}
0 = & \\
& : \left\{ \frac{1}{16} \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j \hat{x}_j \right) \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \hat{x}_{-j} \right) \right. \\
& + \frac{1}{16} \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \hat{x}_j \right) \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \hat{x}_{-j} \right) - \frac{1}{8} \\
& \left. - \frac{1}{4} : \left(\sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j \hat{x}_j - \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} \hat{x}_{-j} \right)^2 : \right. \\
& - : \left(\sum_{i \geq 1} 2iz^{2i} \hat{v}_i - \hat{w}_- - \sum_{i \geq 1} 2iz^{-2i} \hat{u}_{-i} \right) \left(\sum_{i \geq 1} 2iz^{2i} \hat{u}_i - \sum_{i \geq 1} 2iz^{-2i} \hat{v}_{-i} \right) : \} \times \\
& \times \sum_{r \in \mathbb{R}} e^{2rw} \exp \left(2r \sum_{i \geq 1} z^{2i} \hat{u}_i \right) \exp \left(2r \sum_{i \geq 1} z^{-2i} \hat{v}_{-i} \right) : \\
& \dot{\tau}(\tilde{w} + \hat{w}, \tilde{\mathbf{x}} + \hat{\mathbf{x}}, \tilde{\mathbf{u}} + \hat{\mathbf{u}}, \tilde{\mathbf{v}} + \hat{\mathbf{v}}) \dot{\tau}(\tilde{w} - \hat{w}, \tilde{\mathbf{x}} - \hat{\mathbf{x}}, \tilde{\mathbf{u}} - \hat{\mathbf{u}}, \tilde{\mathbf{v}} - \hat{\mathbf{v}}) \\
& - \dot{\tau}(\tilde{w} + \hat{w}, \tilde{\mathbf{x}} + \hat{\mathbf{x}}, \tilde{\mathbf{u}} + \hat{\mathbf{u}}, \tilde{\mathbf{v}} + \hat{\mathbf{v}}) \dot{\tau}(\tilde{w} - \hat{w}, \tilde{\mathbf{x}} - \hat{\mathbf{x}}, \tilde{\mathbf{u}} - \hat{\mathbf{u}}, \tilde{\mathbf{v}} - \hat{\mathbf{v}}) \Omega(z),
\end{aligned}$$

where the expression for second term is similar to the first.

Using Proposition 2 we now move all ‘negative’ variables to the right of $\dot{\tau} \otimes \dot{\tau}$, and all positive variables to the left. This will introduce partial derivatives in all variables. Once this has been done, taking the symbol of the expression is equivalent to taking the symbol of each variable. We denote the symbols of each variable by taking off it’s hat. After we have taken symbols, we are again in a commutative algebra, so we can put variables on either side of $\tau \otimes \tau$ without affecting anything. For convenience, we actually write all variables on the left. Once this is done, we see that

$$\Omega(z) \dot{\tau}(\tilde{w} + w, \tilde{\mathbf{x}} + \mathbf{x}, \tilde{\mathbf{u}} + \mathbf{u}, \tilde{\mathbf{v}} + \mathbf{v}) \dot{\tau}(\tilde{w} - w, \tilde{\mathbf{x}} - \mathbf{x}, \tilde{\mathbf{u}} - \mathbf{u}, \tilde{\mathbf{v}} - \mathbf{v}) -$$

$$\begin{aligned}
& \dot{\tau}(\tilde{w} + w, \tilde{\mathbf{x}} + \mathbf{x}, \tilde{\mathbf{u}} + \mathbf{u}, \tilde{\mathbf{v}} + \mathbf{v}) \dot{\tau}(\tilde{w} - w, \tilde{\mathbf{x}} - \mathbf{x}, \tilde{\mathbf{u}} - \mathbf{u}, \tilde{\mathbf{v}} - \mathbf{v}) \Omega(z) = \\
& : \left\{ \frac{1}{16} \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} \right) \right. \\
& + \frac{1}{16} \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-j}}{j} \frac{\partial}{\partial x_j} \right) - \frac{1}{8} \\
& - \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \right)^2 + \frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} - \frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \frac{\partial}{\partial x_j} \\
& - \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} \right)^2 + \frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \frac{\partial}{\partial x_j} - \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} \frac{\partial}{\partial x_j} \right)^2 \\
& - \sum_{i \geq 1} 2iz^{2i} v_i \sum_{i \geq 1} 2iz^{2i} u_i + \sum_{i \geq 1} 2iz^{2i} v_i \sum_{i \geq 1} 2iz^{-2i} v_{-i} - \sum_{i \geq 1} 2iz^{2i} v_i \sum_{i \geq 1} z^{-2i} \frac{\partial}{\partial v_i} \\
& + \sum_{i \geq 1} 2iz^{2i} u_i 2w_- - \sum_{i \geq 1} 2iz^{2i} u_i \frac{\partial}{\partial w} - 2w_- \sum_{i \geq 1} 2iz^{-2i} v_{-i} + \sum_{i \geq 1} 2iz^{-2i} v_{-i} \frac{\partial}{\partial w} - \sum_{i \geq 1} z^{-2i} \frac{\partial}{\partial v_i} \frac{\partial}{\partial w} \\
& + \sum_{i \geq 1} 2iz^{2i} u_i \sum_{i \geq 1} 2iz^{-2i} u_{-i} - \sum_{i \geq 1} 2iz^{2i} u_i \sum_{i \geq 1} z^{-2i} \frac{\partial}{\partial u_i} \\
& - \sum_{i \geq 1} 2iz^{-2i} u_{-i} \sum_{i \geq 1} 2iz^{-2i} v_{-i} + \sum_{i \geq 1} 2iz^{-2i} v_{-i} \sum_{i \geq 1} z^{-2i} \frac{\partial}{\partial u_i} \\
& + \sum_{i \geq 1} 2iz^{-2i} u_{-i} \sum_{i \geq 1} iz^{-2i} \frac{\partial}{\partial v_i} - \sum_{i \geq 1} iz^{-2i} \frac{\partial}{\partial u_i} \sum_{i \geq 1} z^{-2i} \frac{\partial}{\partial v_i} \Big\} \times \\
& \times \left\{ \sum_{r \in \mathbb{R}} e^{2rw} \exp \left(2r \sum_{i \geq 1} z^{2i} u_i \right) \exp \left(2r \sum_{i \geq 1} z^{-2i} v_{-i} \right) \exp \left(-r \sum_{i \geq 1} \frac{z^{-2i}}{i} \frac{\partial}{\partial v_i} \right) \right\} : \\
& \dot{\tau}((\tilde{w} + w, \tilde{\mathbf{x}} + \mathbf{x}, \tilde{\mathbf{u}} + \mathbf{u}, \tilde{\mathbf{v}} + \mathbf{v})) \dot{\tau}(\tilde{w} - w, \tilde{\mathbf{x}} - \mathbf{x}, \tilde{\mathbf{u}} - \mathbf{u}, \tilde{\mathbf{v}} - \mathbf{v})) \\
& - \dot{\tau}(e^{\tilde{w}+w}, \tilde{\mathbf{x}} + \mathbf{x}, \tilde{\mathbf{u}} + \mathbf{u}, \tilde{\mathbf{v}} + \mathbf{v}) \dot{\tau}(e^{\tilde{w}-w}, \tilde{\mathbf{x}} - \mathbf{x}, \tilde{\mathbf{u}} - \mathbf{u}, \tilde{\mathbf{v}} - \mathbf{v}) \Omega(z)
\end{aligned}$$

In order to turn this into a hierarchy of equations, we will need the following results.

Let $D = \overline{\mathbb{C}[\mathbb{R}]} \otimes \text{diff}[w, y_1, y_2, \dots]$, and let $\deg : \{w, y_1, y_2, \dots\} \rightarrow \mathbb{N}$ be such that $\deg(w) = 0$, $\deg$ is non-decreasing on the sequence $\{y_i\}_{i \in \mathbb{N}}$ and $\lim_{i \rightarrow \infty} \deg(y_i) = \infty$. Then D is graded by $\deg$ (as a vector space), where the degree of a monomial

$$s_r e^{rw} y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} \left(\frac{\partial}{\partial w} \right)^{b_0} \left(\frac{\partial}{\partial y_1} \right)^{b_1} \left(\frac{\partial}{\partial y_2} \right)^{b_2} \cdots \left(\frac{\partial}{\partial y_m} \right)^{b_m}$$

is defined to be $\sum_{i=1}^n b_i \deg(y_i)$. For each $K \in \mathbb{N}$ let I_K be the right ideal in D generated by the homogeneous elements of D of degree at least K . It is clear that I_K is in fact generated by these elements as a vector space, and also that $\bigcap_{k \in \mathbb{N}} I_k = (0)$. Furthermore, note that multiplication on the left or right by $\mathbb{C}[\mathbb{R}] \left(\frac{\partial}{\partial w} \right)^k$ preserves degree, and in particular leaves I_K invariant. The following is a generalization of proposition 3 in [3].

Proposition 2. *Let $P(r) = \sum_{n \geq 0} r^n P_n$, where $P_n \in \text{diff}(w, y_1, y_2, \dots)$ are differential operators which depend on $\frac{\partial}{\partial w}$ but not on w . Also, assume there is an operator $\deg$ which assigns a positive integer to each y_i such that $\lim_{i \rightarrow \infty} \deg(y_i) = \infty$. Suppose that for every $K \in \mathbb{N}$ and every $f(w, \mathbf{y}) = \sum_{i=1}^N e^{r_i w} f_i(\mathbf{y}) \in \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[y_1, y_2, \dots]$ we have $P_n f \in I_K$ for all but finitely many n (where I_K is as defined above). If $\sum_{r \in \mathbb{R}} e^{rw} P(r) g(w, \mathbf{y}) = 0$ for some $g(w, \mathbf{y}) \in \mathbb{C}[\mathbb{R}] \otimes \mathbb{C}[y_1, y_2, \dots]$, then $P \left(\varepsilon - \frac{\partial}{\partial w} \right) g(w, \mathbf{y}) \Big|_{w=0} = 0$ as a polynomial in ε .*

In order to prove this proposition, we look at the completion of the group algebra

$$\mathbb{C}[\mathbb{R}] : \overline{\mathbb{C}[\mathbb{R}]} = \prod_{r \in \mathbb{R}} \mathbb{R} e^{rw}. \text{ Consider the element } \delta(w) = \sum_{r \in \mathbb{R}} e^{rw}. \text{ For each basis element}$$

$X(w) = e^{aw}$ of $\mathbb{C}[\mathbb{R}]$ we see that $\delta(w)e^{aw} = \sum_{r \in br} e^{(r+a)w} = \delta(w) \cdot 1$. It follows that $\delta(w)$ is a formal analog of the δ -function in the sense that for any $X(w) \in \mathbb{C}[\mathbb{R}]$, $\delta(w)X(w) = \delta(w)X(0)$. Before completing the proof of proposition 2, we need the following technical result:

Lemma 4. *Let $X(w) \in \mathbb{C}[\mathbb{R}]$. Then*

$$\left[\left(\frac{\partial}{\partial w} \right)^n \delta(w) \right] X(w) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \left[\left(\frac{\partial}{\partial w} \right)^k \delta(w) \right] \left[\left(\frac{\partial}{\partial w} \right)^{n-k} X(w) \Big|_{w=0} \right].$$

Proof. Proceed by induction on n . The case $n = 0$ holds as above. To make the inductive step, use the Leibniz rule to give

$$\left[\left(\frac{\partial}{\partial w} \right)^{n+1} \delta(w) \right] X(w) = \frac{\partial}{\partial w} \left(\left[\left(\frac{\partial}{\partial w} \right)^n \delta(w) \right] X(w) \right) - \left[\left(\frac{\partial}{\partial w} \right)^n \delta(w) \right] \frac{\partial X}{\partial w}.$$

Then using the inductive assumption, and noticing that $\frac{\partial}{\partial w} \left[\left(\frac{\partial}{\partial w} \right)^{n-k} X(w) \Big|_{w=0} \right] = 0$, we see that

$$\begin{aligned} & \left[\left(\frac{\partial}{\partial w} \right)^{n+1} \delta(w) \right] X(w) = \\ &= \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \left[\left(\frac{\partial}{\partial w} \right)^{k+1} \delta(w) \right] \left[\left(\frac{\partial}{\partial w} \right)^{n-k} X(w) \Big|_{w=0} \right] \\ &+ \sum_{k=0}^n (-1)^{n-k+1} \binom{n}{k} \left[\left(\frac{\partial}{\partial w} \right)^k \delta(w) \right] \left[\left(\frac{\partial}{\partial w} \right)^{n-k+1} X(w) \Big|_{w=0} \right] = \\ &= \sum_{k=0}^n (-1)^{n+1-k} \binom{n+1}{k} \left[\left(\frac{\partial}{\partial w} \right)^k \delta(w) \right] \left[\left(\frac{\partial}{\partial w} \right)^{n+1-k} X(w) \Big|_{w=0} \right] \quad \square \end{aligned}$$

Proof of proposition 2: Fix $K \in \mathbb{N}$. We will prove that, under the assumptions of the proposition,

$$P \left(\varepsilon - \frac{\partial}{\partial w} \right) g(w, \mathbf{y}) \Big|_{w=0}$$

has coefficients of ϵ which lie in I_K . Since the intersection of all I_K is 0, by showing this for an arbitrary K , the result will follow. From our assumptions we have

$$0 = \sum_{r \in \mathbb{R}} e^{rw} P(r) g(w, \mathbf{y}) = \sum_{n \geq 0} \sum_{r \in \mathbb{R}} r^n e^{rw} P_n g(w, \mathbf{y}).$$

It follows from the definition of $\delta(w)$ that $\sum_{r \in \mathbb{R}} r^n e^{rw} = \left(\frac{\partial}{\partial w} \right)^n \delta(w)$. Using this and lemma 4,

$$\begin{aligned} 0 &= \sum_{n \geq 0} \left[\left(\frac{\partial}{\partial w} \right)^n \delta(w) \right] P_n g(w, \mathbf{y}) = \\ &= \sum_{n \geq 0} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \left[\left(\frac{\partial}{\partial w} \right)^k \delta(w) \right] \left[\left(\frac{\partial}{\partial w} \right)^{n-k} P_n g(w, \mathbf{y}) \Big|_{w=0} \right] = \\ &= \sum_{k \geq 0} \left[\left(\frac{\partial}{\partial w} \right)^k \delta(w) \right] \sum_{m=n-k \geq 0} (-1)^m \binom{m+k}{k} \left[\left(\frac{\partial}{\partial w} \right)^m P_{m+k} g(w, \mathbf{y}) \Big|_{w=0} \right]. \end{aligned}$$

In particular, this last expression is in I_K . By our assumption, $P_{m+k} \cdot g(w, \mathbf{y}) \in I_K$ for sufficiently large $m+k$. Ignoring these terms, we have a finite linear combination of $\left\{ \left(\frac{\partial}{\partial w} \right)^k \delta(w) \right\}$, which is in I_K . Using a Vandermonde determinant argument, one can see that these elements are all linearly independent under multiplication on the right by elements of D which do not depend on w . Since multiplication on the left by $\left(\frac{\partial}{\partial w} \right)^k \delta(w)$ preserves degree, and I_K just cuts off terms of large degree, they remain linearly independent as left I_K cosets. So we see that each coefficient must be in I_K . Hence for all k

$$\sum_{m \geq 0} (-1)^m \binom{m+k}{k} \left[\left(\frac{\partial}{\partial w} \right)^m P_{m+k} g(w, \mathbf{y}) \Big|_{w=0} \right] \in I_K.$$

Then the following polynomial in ϵ must have coefficients in I_K :

$$\sum_{k \geq 0} \epsilon^k \sum_{m \geq 0} (-1)^m \binom{m+k}{k} \left[\left(\frac{\partial}{\partial w} \right)^m P_{m+k} g(w, \mathbf{y}) \Big|_{w=0} \right] =$$

$$= P \left(\varepsilon - \frac{\partial}{\partial w} \right) g(w, \mathbf{y}) \Big|_{w=0}.$$

This completes the proof. $\square$

6 The generating series for our hierarchy

We will now transform our expression into Hirota form. This is done in two steps. First, consider the grading $\deg(u_i) = 2i$, $\deg(v_i) = 2i$ and $\deg(x_j) = j$. We then see that for each k , Ω_k (the coefficients of z^k in $\Omega(z)$) satisfies the conditions of proposition 2. Thus we may replace r by $\varepsilon - \frac{\partial}{\partial w}$. Second, we apply lemma 3 to this differential operator changing all partial derivatives to Hirota bilinear derivations. this will give the following expression.

$$\begin{aligned} \Omega(z) \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = & \\ & \left\{ \frac{1}{16} \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} \right) \right. \\ & + \frac{1}{16} \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} \right) - \frac{1}{8} \\ & - \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \right)^2 + \frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} - \frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} \\ & - \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} \right)^2 + \frac{1}{4} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} - \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} \right)^2 \\ & - \sum_{i \geq 1} 2i z^{2i} v_i \sum_{i \geq 1} 2i z^{2i} u_i + \sum_{i \geq 1} 2i z^{2i} v_i \sum_{i \geq 1} 2i z^{-2i} v_{-i} - 2w_- \sum_{i \geq 1} 2i z^{2i} u_i \\ & \left. - \sum_{i \geq 1} 2i z^{2i} u_i H_{\tilde{w}} - 2w_- \sum_{i \geq 1} 2i z^{-2i} v_{-i} + \sum_{i \geq 1} 2i z^{-2i} v_{-i} H_{\tilde{w}} \right\} \end{aligned}$$

$$\begin{aligned}
& +2w_- \sum_{i \geq 1} z^{-2i} H_{\tilde{v}_i} - \sum_{i \geq 1} z^{-2i} H_{\tilde{v}_i} H_{\tilde{w}} \\
& + \sum_{i \geq 1} 2iz^{2i} u_i \sum_{i \geq 1} 2iz^{-2i} u_{-i} - \sum_{i \geq 1} 2iz^{2i} u_i \sum_{i \geq 1} z^{-2i} H_{\tilde{u}_i} \\
& - \sum_{i \geq 1} 2iz^{-2i} u_{-i} \sum_{i \geq 1} 2iz^{-2i} v_{-i} + \sum_{i \geq 1} 2iz^{-2i} v_{-i} \sum_{i \geq 1} z^{-2i} H_{\tilde{u}_i} \\
& + \sum_{i \geq 1} 2iz^{-2i} u_{-i} \sum_{i \geq 1} z^{-2i} H_{\tilde{v}_i} - \sum_{i \geq 1} z^{-2i} H_{\tilde{u}_i} \sum_{i \geq 1} z^{-2i} H_{\tilde{v}_i} \Big\} \times \\
& \exp \left((\epsilon - H_{\tilde{w}}) \sum_{i \geq 1} z^{2i} u_i \right) \exp \left((\epsilon - H_{\tilde{w}}) \sum_{i \geq 1} z^{-2i} v_{-i} \right) \exp \left(-\frac{1}{2} (\epsilon - H_{\tilde{w}}) \sum_{i \geq 1} \frac{z^{-2i}}{i} H_{\tilde{v}_i} \right) \\
& \exp \left(\sum_{j \in \mathbb{Z}_{\text{odd}}} x_j H_{\tilde{x}_j} \right) \exp \left(\sum_{i \neq 0} u_i H_{\tilde{u}_i} \right) \exp \left(\sum_{i \neq 0} v_i H_{\tilde{v}_i} \right) \\
& \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \circ \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}).
\end{aligned}$$

The expression for $\dot{\tau} \otimes \dot{\tau} \Omega(z)$ is similar. We will eventually construct solutions to these equations by using the fact that $A(r, z)$ commutes with $\Omega(z)$. Notice that $A(r, z)$ depends on u_i only for $i \geq 1$, and depends on v_i only for $i \leq 0$. So, if these are the only solutions we are interested in, we may make the reduction $H_{\tilde{u}_{-i}} = H_{\tilde{v}_i} = 0$ for all $i \geq 1$. It should be noted that the variables u_{-i} and v_i come from derivations of the algebra $\mathfrak{g}$, which we already know to be troublesome, so discarding them is not unreasonable. After this reduction, the expression $\Omega(z) \dot{\tau} \circ \dot{\tau} - \dot{\tau} \circ \dot{\tau} \Omega(z)$ simplifies to give

$$\begin{aligned}
& \left\{ \frac{1}{16} \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} \right) \right. \\
& \left. + \frac{1}{16} \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{16} \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j H_{\tilde{x}_{-j}} \right) \\
& -\frac{1}{16} \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j \right) \exp \left(-4 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} x_{-j} \right) \exp \left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j H_{\tilde{x}_{-j}} \right) \\
& -\frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \sum_{j \in \mathbb{N}_{\text{odd}}} z^j H_{\tilde{x}_{-j}} + \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} z^j H_{x_{-j}} \right)^2 \\
& -\frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^j x_j \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} + \frac{1}{2} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} \sum_{j \in \mathbb{N}_{\text{odd}}} z^j H_{\tilde{x}_{-j}} \\
& + \frac{1}{4} \sum_{j \in \mathbb{N}_{\text{odd}}} 2j z^{-j} x_{-j} \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} - \frac{1}{4} \left(\sum_{j \in \mathbb{N}_{\text{odd}}} z^{-j} H_{\tilde{x}_j} \right)^2 \\
& - \sum_{i \geq 1} z^{2i} H_{\tilde{v}_{-i}} \sum_{i \geq 1} 2i z^{2i} u_i + \sum_{i \geq 1} z^{2i} H_{\tilde{v}_{-i}} \sum_{i \geq 1} 2i z^{-2i} v_{-i} \\
& - \sum_{i \geq 1} 2i z^{2i} u_i \sum_{i \geq 1} z^{-2i} H_{\tilde{u}_i} + \sum_{i \geq 1} z^{-2i} H_{\tilde{u}_i} \sum_{i \geq 1} 2i z^{-2i} v_{-i} \\
& - \sum_{i \geq 1} 2i z^{2i} u_i H_{\tilde{w}} + \sum_{i \geq 1} z^{-2i} 2i v_{-i} H_{\tilde{w}} \} \\
& \exp \left((\epsilon - H_{\tilde{w}}) \sum_{i \geq 1} z^{2i} u_i \right) \exp \left((\epsilon - H_{\tilde{w}}) \sum_{i \geq 1} z^{-2i} v_{-i} \right) \\
& \exp \left(\sum_{j \in \mathbb{N}_{\text{odd}}} x_j H_{\tilde{x}_j} \right) \exp \left(\sum_{i \geq 1} u_i H_{\tilde{u}_i} \right) \exp \left(\sum_{i \geq 1} v_{-i} H_{\tilde{v}_{-i}} \right) \\
& \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \circ \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}).
\end{aligned}$$

It is interesting to consider the last two terms in this sum. Notice that in expanding the products, we may first multiply these by

$$\exp \left(-H_{\tilde{w}} \sum_{i \geq 1} z^{2i} u_i \right) \exp \left(-H_{\tilde{w}} \sum_{i \geq 1} z^{-2i} v_{-i} \right),$$

and then by the other exponentials. We then have a factor of

$$\left(-H_{\tilde{w}} \sum_{i \geq 1} 2iz^{2i}u_i + H_{\tilde{w}} \sum_{i \geq 1} 2iz^{-2i}v_{-i} \right) \exp \left(-H_{\tilde{w}} \sum_{i \geq 1} z^{2i}u_i - H_{\tilde{w}} \sum_{i \geq 1} z^{-2i}v_{-i} \right).$$

But this is exactly the expression for

$$-z \frac{\partial}{\partial z} \exp \left(-H_{\tilde{w}} \sum_{i \geq 1} z^{2i}u_i - H_{\tilde{w}} \sum_{i \geq 1} z^{-2i}v_{-i} \right),$$

which clearly has coefficient 0 at z^0 , so we see that these two terms do not contribute to the coefficient at z^0 .

We also note that our expression is in fact a power series in variables $x_{\pm j}$, $u_{\pm i}$, $v_{\pm i}$, z , z^{-1} , ϵ with coefficients being Hirota polynomials in $H_{\tilde{x}_{\pm j}}$, $H_{\tilde{u}_{\pm i}}$, $H_{\tilde{v}_{\pm i}}$, $H_{\tilde{w}}$. This can be seen by considering the two terms $\dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \Omega(z)$ and $\Omega(z) \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \dot{\tau}(e^{\tilde{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ separately. By degree considerations, each of these is a power series in the appropriate variables, and hence so is their difference. Furthermore, it is easy to see that only even powers of z will occur. We get our equations by setting each coefficient in this power series to zero.

7 Extracting equations

At this point it simplifies things to make a minor change of notation. There are no x variables of even index, so we can let $x_0 = \epsilon$, $x_{2n} = u_n$ and $x_{-2n} = v_{-n}$ without causing confusion. We will also let $H_0 = H_{\tilde{w}}$, and $H_i = H_{\tilde{x}_i}$ for $i \neq 0$. We say x_i has degree i and z has degree -1 , and we extend this grading to monomials in these variables in the usual way by $\deg(m_1 m_2) = \deg(m_1) + \deg(m_2)$. Similarly, we set the

degree of H_i to be i and extend to monomials in the H_i . Notice that in the generating series for our hierarchy, x_i always appears multiplied by z^i and H_i always appears multiplied by z^{-i} . So the coefficient of each monomial m in the variables $z, \mathbf{x}$ will be a homogeneous polynomial in the H_i 's of degree $\deg(m)$. We will only explicitly calculate the coefficients of monomials $m = z^{2j} x_{i_1} \cdots x_{i_k}$ where

$$\sum_{\substack{1 \leq s \leq k \\ i_s \geq 0}} i_s \leq 5 \quad \text{and} \quad 2j + \sum_{\substack{1 \leq s \leq k \\ i_s \leq 0}} i_s \leq 5.$$

First consider the case of a monomial with the power of z equal to zero and no nonzero powers of x_i for $i < 0$. We may reduce our generating series by setting $x_j = 0$ for $j < 0$ and $v_{-i} = 0$ for $i \in \mathbb{N}$. Notice that this completely removes that part of the generating series which came from $\tau \otimes \tau \Omega(z)$, and in fact what remains is simply $\Omega(z)(\tau \otimes \tau) = 0$, where $\Omega(z)$ is now acting on $\tau \otimes \tau$ as a differential operator, not an element of the Weyl algebra. This is exactly the expression studied in [3]. So these coefficients give exactly the extension of the KdV hierarchy displayed in that paper. By further restricting our attention to monomials that depend only on the x_j 's, and not the u_i 's or ε , the KdV hierarchy is recovered.

All the equations in our hierarchy have the form $P(\dots H_{-1}, H_0, H_1, \dots) \tau \circ \tau = 0$ for some Hirota polynomial P . For a fixed τ , the polynomials P such that $P\tau \circ \tau = 0$ form a linear subspace of $\mathbb{C}[\dots H_{-1}, H_0, H_1, \dots]$. So we are really interested in finding the subspace spanned by the Hirota polynomials in our hierarchy, as the solutions we find will be solutions for any P in this space.

The following basis for such P of (positive) degree at most 5 is given in [3] (the coefficient at which they can be found is listed in brackets):

$$\begin{aligned}
(x_1 u_1) & H_0 H_1^3 + 2H_0 H_3 - 6H_1 H_2 \\
(x_1 x_3) & H_1^4 - 4H_1 H_3 \\
(u_1^2) & H_0^2 H_1^4 - 4H_0^2 H_1 H_3 + 48H_0 H_4 - 48H_2^2 \\
(x_1 u_2) & H_0 H_1^5 + 20H_0 H_1^2 H_3 + 24H_0 H_5 - 120H_1 H_4 \\
(\varepsilon x_1 u_1^2) & H_0 H_1^5 + 20H_0 H_1^2 H_3 + 24H_0 H_5 - \frac{40}{3}H_1^3 H_2 - \frac{80}{3}H_2 H_3 - 40H_1 H_4 \\
(u_1 x_3) & H_0 H_1^5 + 5H_0 H_1^2 H_3 + 24H_0 H_5 - 5H_1^3 H_2 - 40H_2 H_3.
\end{aligned}$$

Notice that the coefficient of $x_1 x_3$ above gives exactly the KdV equation. Before describing the new equations we find in the extended hierarchy, we first discuss their solutions. By considering natural symmetries in the solutions, we see symmetries in our equations. This reduces the number of truly different equations which must be considered.

8 Soliton solutions

The same methods as in [10] allow us to generate soliton solutions to our Hierarchy of equations. This is based on the following fact, which is proven in [3] using results from [7]:

Proposition 3. *The operator $(1 + \lambda A(z, r)) \otimes (1 + \lambda A(z, r))$ commutes with $\Omega(z)$.*

The following is then an immediate consequence of this proposition:

Corollary 5. *If τ is a solution of $\Omega(z)\tau \otimes \tau - \tau \otimes \tau\Omega(z) = 0$, then so is $\rho = (1 + \lambda A(z, r))\tau$. Hence $\dot{\rho}$, the composition of ρ with the change of variables T defined in section 5, is a solution to the above hierarchy of Hirota equations.*

Proof.

$$\begin{aligned} & \Omega(z)(1 + \lambda A(z, r))\tau \otimes (1 + \lambda A(z, r))\tau - (1 + \lambda A(z, r))\tau \otimes (1 + \lambda A(z, r))\tau \Omega(z) \\ &= ((1 + \lambda A(z, r)) \otimes (1 + \lambda A(z, r)))(\Omega(z)\tau \otimes \tau - \tau \otimes \tau\Omega(z)) = 0 \quad \square \end{aligned}$$

Applying this corollary to the trivial solution $1 \otimes 1$ essentially gives us 1-soliton solutions to our hierarchy. To get N -soliton solutions, we simply act on the trivial solution several times with different λ , z and r . The result is as follows.

Theorem 6. *For $\lambda_1, \dots, \lambda_N, z_1, \dots, z_N, r_1, \dots, r_N \in \mathbb{R}$ the function*

$$\begin{aligned} & \tau(w, \dots, x_{-1}, x_1, x_3, \dots, \dots, u_{-2}, u_{-1}, u_1, u_2, \dots) = \\ &= \sum_{\substack{0 \leq k \leq N \\ 1 \leq i_1 < \dots < i_k \leq N}} \lambda_{i_1} \dots \lambda_{i_k} \prod_{1 \leq \mu \leq \nu \leq k} \left(\frac{z_{i_\mu} - z_{i_\nu}}{z_{i_\mu} + z_{i_\nu}} \right)^2 \times \\ & \times \exp \left(\sum_{m=1}^k r_{i_m} w + 2 \sum_{j \in \mathbb{Z}_{\text{odd}}} \sum_{m=1}^k z_{i_m}^j x_j + \sum_{j \in \mathbb{N}} \sum_{m=1}^k r_{i_m} z_{i_m}^{2j} u_j + \sum_{j \in \mathbb{N}} \sum_{m=1}^k r_{i_m} z_{i_m}^{-2j} v_{-j} \right) \end{aligned}$$

is a solution of our hierarchy.

Proof. Clearly $1 \otimes 1$ commutes with $\Omega(z)$. Corollary 5 tells us that we can multiply on the left by $(1 + \lambda A(z, r)) \otimes (1 + \lambda A(z, r))$ and the resulting operator still commutes with $\Omega(z)$. Applying this repeatedly, and letting,

$$\tau = (1 + 2\lambda_1 A(z_1, r_1)) \dots (1 + 2\lambda_N A(z_N, r_N))1 =$$

$$= \sum_{\substack{0 \leq k \leq N \\ 1 \leq i_1 < \dots < i_k \leq N}} \lambda_{i_1} \dots \lambda_{i_k} 2^k A(z_{i_1}, r_{i_1}) \dots A(z_{i_k}, r_{i_k}),$$

we see that $\tau \otimes \tau$ commutes with $\Omega(z)$. Once we move to the Weyl algebra, this means that $\dot{\tau}$, the composition of τ with T , is a solution to our equations. Denote the composition of $A(z, r)$ with T by $\dot{A}(z, r)$

The following identity follows from Proposition 3.4.1 of [7]:

$$\dot{A}(z_{i_1}, r_{i_1}) \dots \dot{A}(z_{i_k}, r_{i_k}) = \prod_{1 \leq \mu < \nu \leq k} \left(\frac{z_{i_\mu} - z_{i_\nu}}{z_{i_\mu} + z_{i_\nu}} \right)^2 : \dot{A}(z_{i_1}, r_{i_1}) \dots \dot{A}(z_{i_k}, r_{i_k}) : .$$

It follows immediately from our definitions of normally ordered products and of the symbol of an operator that the symbol of a normally ordered product of operators is equal to the product of the symbols of those operators. So this identity allows us to calculate the image $\bar{A}$ of $\dot{A}$ in the symbol algebra. We can now see that

$$\begin{aligned} \dot{\tau} = & \sum_{\substack{0 \leq k \leq N \\ 1 \leq i_1 < \dots < i_k \leq N}} \lambda_{i_1} \dots \lambda_{i_k} 2^k \times \\ & \times \prod_{1 \leq \mu < \nu \leq k} \left(\frac{z_{i_\mu} - z_{i_\nu}}{z_{i_\mu} + z_{i_\nu}} \right)^2 \bar{A}(z_{i_1}, r_{i_1}) \dots \bar{A}(z_{i_k}, r_{i_k}) \end{aligned}$$

is a solution to our hierarchy, where $\bar{A}(z, r)$ is the symbol of the composition of $A(z, r)$ with T . Now recall our previous expression for $A(z, r)$:

$$\begin{aligned} A(z, r) = & \frac{1}{2} \exp \left(r w + 2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j + r \sum_{i \geq 1} z^{2i} u_i \right) \exp \left(-2 \sum_{j \in \mathbb{N}_{\text{odd}}} \frac{z^{-2j}}{j} \frac{\partial}{\partial x_j} - r \sum_{i \geq 1} \frac{z^{-2i}}{i} \frac{\partial}{\partial v_i} \right). \end{aligned}$$

Transferring $A(z, r)$ to the Weyl algebra, applying the change of variables T , and taking symbols, we get

$$\bar{A}(z, r) = \frac{1}{2} \exp \left(rw + 2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^j x_j + r \sum_{i \geq 1} z^{2i} u_i \right) \exp \left(2 \sum_{j \in \mathbb{N}_{\text{odd}}} z^{-2j} x_{-j} + r \sum_{i \geq 1} z^{-2i} v_{-i} \right).$$

It then follows that

$$\begin{aligned} \dot{\tau} = & \sum_{\substack{0 \leq k \leq N \\ 1 \leq i_1 < \dots < i_k \leq N}} \lambda_{i_1} \dots \lambda_{i_k} \prod_{1 \leq \mu \leq \nu \leq k} \left(\frac{z_{i_\mu} - z_{i_\nu}}{z_{i_\mu} + z_{i_\nu}} \right)^2 \exp \left(\sum_{m=1}^k r_{i_m} w \right. \\ & \left. + 2 \sum_{j \in \mathbb{Z}_{\text{odd}}} \sum_{m=1}^k z_{i_m}^j x_j + 2 \sum_{j \in \mathbb{N}_{\text{even}}} \sum_{m=1}^k r_{i_m} z_{i_m}^j x_j + 2 \sum_{j \in \mathbb{N}_{\text{even}}} \sum_{m=1}^k r_{i_m} z_{i_m}^j x_j \right) \end{aligned}$$

is a solution to our hierarchy. $\square$

As in the previous section, we can set $x_{2i} = u_i$, $x_{-2i} = v_{-i}$ and $x_0 = w$, and the solutions become

$$\begin{aligned} & \tau(w, \dots, x_{-1}, x_1, x_3, \dots, u_{-2}, u_{-1}, u_1, u_2, \dots) = \\ & = \sum_{\substack{0 \leq k \leq N \\ 1 \leq i_1 < \dots < i_k \leq N}} \lambda_{i_1} \dots \lambda_{i_k} \prod_{1 \leq \mu \leq \nu \leq k} \left(\frac{z_{i_\mu} - z_{i_\nu}}{z_{i_\mu} + z_{i_\nu}} \right)^2 \times \\ & \times \exp \left(\sum_{m=1}^k \sum_{j \in \mathbb{Z}_{\text{even}}} r_{i_m} z_{i_m}^j x_j + 2 \sum_{m=1}^k \sum_{j \in \mathbb{Z}_{\text{odd}}} z_{i_m}^j x_j \right). \end{aligned}$$

This is considerably simpler, and agrees with the form of the equations given in the appendix.

9 Equations Revisited

Consider the solutions given at the end of section 8. If we perform the change of variables $r_k \leftarrow r_k \cdot z_k^{2m}$ and for $j \in \mathbb{Z}_{\text{even}}, x_j \leftarrow x_{j+2m}$, the expression remains unchanged. Since our solutions hold for all r_k , we see that we may relabel all our variables by shifting the even ones, and still have a solution to our hierarchy. Equivalently, we can relabel our equations by

$$H_j \leftarrow H_j \text{ for odd } j$$

$$H_j \leftarrow H_{j+2m} \text{ for even } j$$

and we will preserve all our N -Soliton solutions.

Similarly, using the fact that

$$\left(\frac{z^{-1} - w^{-1}}{z^{-1} + w^{-1}} \right)^2 = \left(\frac{z - w}{z + w} \right)^2,$$

we see that we may perform the change of variable $z_k \leftarrow z_k^{-1}$ and for all $j, x_j \leftarrow x_{-j}$, and leave our solutions unchanged. So, as before, we can relabel the equations by for all $i, H_i \leftarrow H_{-i}$ and our N -soliton solutions are preserved.

Furthermore, there are some ‘reductive’ changes of variables we can apply to our equations and still preserve our N -soliton solutions, although due to eliminating certain variables, the transformed equations will have a less rich solution set. In this sense these are non-invertible transformations, and the solutions to the resulting equations are somewhat degenerate.

These transformations come from setting $r_k = z_k^t$ for some $t \in \mathbb{Z}_{\text{odd}}$. Then for

each even m , the coefficient of x_m in our solution is equal to the coefficient of x_{m+k} . Naively following the above method, we would relabel our equations by $H_i \leftarrow H_{i+t}$ for $i \in \mathbb{Z}_{\text{even}}$, and expect our solutions to still hold. However, there is a potential problem here, in that H_{i+t} may already be present in our Hirota polynomial, and so we are not really just changing variables. To see that this does not in fact cause a problem, we use the following standard fact about Hirota polynomial equations.

Lemma 7. *For any Hirota polynomial P and any constants $a_1 \dots a_k, b_1 \dots b_k$,*

$$\begin{aligned} & P(H_{x_1}, \dots, H_{x_k}) e^{a_1 x_1 + \dots + a_k x_k} \circ e^{b_1 x_1 + \dots + b_k x_k} \\ &= P(a_1 - b_1, \dots, a_k - b_k) e^{(a_1 + b_1)x_1 + \dots + (a_k + b_k)x_k}. \end{aligned}$$

Now, using this fact, along with the bilinearity of Hirota differentiation, we see that to ensure our solutions still hold we need only check various polynomial relations of the form $\sum P(c_{i_1}, \dots, c_{i_k}) = 0$, where P is the Hirota polynomial being considered, and the c_{i_j} are linear combinations of the coefficients of x_j in the different exponentials whose sum forms our solution. So if setting the coefficients of x_i equal to the coefficients of x_j gives a solution to these polynomial identities, then by setting $H_i = H_j$, our solutions still hold. By this argument, our solutions do still work for equations obtained from our hierarchy through the reductive transformations described above.

The equations we obtain do contain a lot of redundancy, both from linear dependence and from the internal symmetries of the equations. Using the computer program Maple 7 (Waterloo Maple Inc.), we have removed much of this redundancy

to find a set of equations such that all equations in the hierarchy coming from coefficients with both positive degree and negative degree at most 5 can be obtained by first for each equation taking all equations obtained by internal symmetry, then taking linear combinations of all those equations, and which is reasonably minimal subject to this condition. For details and the list itself, see the appendix.

10 Final comments about our equations and solutions

The solutions we have obtained look like exponentials, not the soliton waves we claim to be studying. The soliton behavior is uncovered by performing the change of variables $v = \frac{\partial^2}{\partial x_i^2} \log(u)$ for some i . If we try to write our equations in terms of v , the first and second integrals of v with respect to x_i will appear. In some instances, the equation can then be transformed into a partial differential equation for v , but in general this does not occur. So in some sense our solutions cannot be considered to be soliton solution in a natural way. In order to make this distinction, functions u such that $\frac{\partial^2}{\partial x^2} \log(u)$ are sometimes called presolitons. So to be precise we only have presoliton solutions to our equations.

Up to this point, all of our equations have been left in Hirota form. Now we will briefly consider their classical forms. In general, transforming an equation in Hirota form to classical form can be quite messy, although for some cases it works out nicely. For instance, consider the KdV equation, $(H_1^4 - 4H_1H_3)\tau(x_1, x_3) \circ \tau(x_1, x_3) = 0$. We

may set $x_1 = x$, $x_3 = t$, and $v = 2\log(\tau)$, so that

$$0 = (H_1^4 - 4H_1H_3)e^{\frac{v}{2}} \circ e^{\frac{v}{2}} = (3v_{xx}^2 + u_{xxxx} - 4u_{xt})e^v.$$

Taking out the factor of e^v , differentiating again by x , and letting $w = u_{xx}$, we get

$$6ww_x + w_{xxx} - 4w_t = 0,$$

which is the classic form of the KdV equation. Notice that $w = 2\frac{\partial^2}{\partial x^2}\log(\tau)$, so in this case we do end up with bonafide soliton solutions.

Another example which has a reasonably nice classical form comes from the equation $(H_0H_1^3 + 2H_0H_3 - 6H_1H_2)\tau \circ \tau = 0$, which is equivalent to the third equation in our list, and can also be found in [3]. By letting $x_0 = y$, $x_1 = x$, $x_3 = z$, and $x_2 = t$, then setting $u = \frac{\partial}{\partial x}\ln\tau$, it transforms to the form

$$u_{yt} = \frac{\partial}{\partial x}(u_{xxy} + u_xu_y + u_x).$$

For the most part, the new equations we have obtained in this extension do not seem to have nice classical forms. For instance, equation 7 on our list,

$$-H_{-2}H_0H_1^4 + 4H_{-2}H_0H_1H_3 - 24H_{-2}H_4 + 24H_0H_2 = 0,$$

is apparently new. Setting $x_{-2} = x$, $x_0 = y$, $x_1 = z$, $x_2 = w$, $x_3 = s$, and $x_4 = t$, and letting $u = \ln\tau$ the equation transforms to

$$\begin{aligned} & -12u_{xyz}u_{zz} - 8u_{xzz}u_{yz} - 8u_{xz}u_{yzz} - 12u_{xy}u_{zz}^2 - 2u_{xy}u_{zzz} - 48u_{xz}u_{yz}u_{zz} \\ & -u_{xyz}u_{zzz} + 8u_{xy}u_{zs} + 8u_{xs}u_{yz} + 4u_{xyzs} + 8u_{ys}u_{xz} - 24u_{xt} + 24u_{yw} = 0, \end{aligned}$$

which does not seem to simplify further.

11 Appendix

The following list describes all equations of small degree in our hierarchy. We list Hirota polynomials P in variables $\dots H_{-1}, H_0, H_1 \dots$, and the differential equations are obtained by setting $P_{\tau \circ \tau} = 0$. The Hirota form of every equation in our hierarchy coming from a coefficient which is of degree at most five in both the positive and the negative variables can be obtained from the following list by using the internal symmetries discussed in section 9, and by taking linear combinations. Furthermore, this list is minimal in the sense that for any $M < 65$, the polynomial $M + 1$ cannot be obtained from polynomials 1 through M , subject to the further condition that no symmetries used in constructing polynomial M involve variables with indices greater than 5 or less than -5. The use of this ‘ordered minimality’ rather than looking for a truly minimal set helps with calculations, and also, because we order the polynomials by the number of terms, ensures that as many of the ‘simple’ equations as possible remain. Cutting off the degree at ± 5 is arbitrary, so a truly minimal description would be of limited value anyway. This list is only intended to present the equations of small degree in the hierarchy in as accessible a form as possible. It appears that the list is at least very close to being truly minimal, although there is some possibility that the list could be shorted by reordering, or by considering further symmetries including variables of higher degree, as the following example shows:

Consider the following 3 polynomials (not in the hierarchy!!)

$$H_{-3}H_4 + H_0H_1$$

$$H_{-3}H_4 + H_{-2}H_1$$

$$H_2H_1 - H_0H_1.$$

The first and second polynomials can transform to

$$H_{-3}H_6 + H_2H_1 \quad \text{and} \quad H_{-3}H_6 + H_0H_1$$

whose difference is then the third polynomial. However, as H_6 appears in these transformed equations, they would not be considered, and so this dependency would be missed.

For each polynomial in the list, we include the coefficient of the generating series at which it can be found. We have multiplied each equation by a constant to force all coefficients to be integers.

1	x_1x_3	$H_1^4 - 4H_1H_3$
2	$x_{-3}x_3$	$H_{-3}H_1^3 - H_{-1}^3H_3$
3	x_1u_1	$H_0H_1^3 + 2H_0H_3 - 6H_1H_2$
4	u_1^2	$48H_2^2 - 48H_0H_4 - H_0^2H_1^4 + 4H_0^2H_1H_3$
5	x_1u_2	$120H_1H_4 - H_0H_1^5 - H_0H_1^2H_3 - 24H_0H_5$
6	$v_{-2}u_2$	$H_{-4}H_0H_1^4 - 4H_{-4}H_0H_1H_3 + 4H_{-3}H_{-1}H_0H_4 - H_{-1}^4H_0H_4$
7	$v_{-1}u_2$	$H_{-2}H_0H_1^4 - 4H_{-2}H_0H_1H_3 + 24H_{-2}H_4 - 24H_0H_2$
8	$x_{-5}x_5$	$H_{-5}H_1^5 + 5H_{-5}H_1^2H_3 - 5H_{-3}H_{-1}^2H_5 - H_{-1}^5H_5$
9	$x_{-1}^3\varepsilon x_1u_1$	$2H_{-1}^3H_3 + H_{-1}^3H_1^3 + 48H_{-1}H_1 - 18H_{-1}^2H_1^2$
10	u_1x_3	$40H_2H_3 + 5H_1^3H_2 - H_0H_1^5 - 24H_0H_5 - 5H_0H_1^2H_3$
11	$x_{-3}u_2$	$8H_{-3}H_4 - 8H_{-1}^3H_4 - 4H_{-3}H_0H_1H_3 + H_{-3}H_0H_1^4 + 24H_0H_1$
12	$x_{-1}u_2$	$24H_{-1}H_4 - 4H_{-1}H_0H_1H_3 + H_{-1}H_0H_1^4 + 8H_0H_3 - 8H_0H_1^3$
13	$v_{-2}x_5$	$72H_{-4}H_5 + 8H_{-4}H_1^5 + 40H_{-4}H_1^2H_3 + 20H_{-3}H_{-1}H_0H_5$ $- 5H_{-1}^4H_0H_5 - 360H_0H_1$
14	$x_{-3}x_1^2u_1$	$2H_{-3}H_1^2H_2 + H_{-1}^3H_1^2H_2 - H_{-1}^2H_1H_2 + 8H_{-1}H_2$ $- 2H_{-1}H_0H_1^2 + 8H_0H_1$
15	$v_{-1}^2u_1^2$	$48H_{-2}^2H_0H_4 + H_{-2}^2H_0^2H_1^4 - 4H_{-2}^2H_0^2H_1H_3 - 48H_{-4}H_0H_2^2$ $+ 4H_{-3}H_{-1}H_0^2H_2^2 - H_{-1}^4H_0^2H_2^2$
16	$v_{-1}^2\varepsilon u_2$	$H_{-2}H_1^4 - 4H_{-2}^2H_1H_3 - 48H_{-2}H_2 + 48H_{-4}H_4$ $- 8H_{-3}H_{-1}H_0H_4 + 2H_{-1}^4H_0H_4$
17	$v_{-1}x_{-1}x_1u_1$	$2H_{-2}H_{-1}H_0H_3 + H_{-2}H_{-1}H_0H_1^3 - 6H_{-2}H_0H_1^2 - 2H_{-3}H_0H_1H_2$ $- H_{-1}^3H_0H_1H_2 + 6H_{-1}^2H_0H_2$
18	$v_{-1}x_{-1}\varepsilon x_1^5$	$224H_{-1}H_1^3 - 128H_{-1}H_3 + 4H_{-3}H_1^5 + H_{-1}^3H_1^5$ $- 384H_1^2 - 30H_{-1}^2H_1^4$
19	$x_{-1}^4\varepsilon u_2$	$288H_{-1}^2H_1^2 + 32H_{-1}^3H_3 - 32H_{-1}^3H_1^3 - 4H_{-1}^4H_1H_3$ $H_{-1}^4H_1^4 - 768H_{-1}H_1$
20	$x_{-1}^2u_1^2$	$-48H_{-1}^2H_2^2 + H_{-1}^2H_0^2H_1^4 - 4H_{-1}^2H_0^2H_1H_3 + 48H_{-1}^2H_0H_4$ $+ 16H_{-1}H_0^2H_3 - 16H_{-1}H_0^2H_1^3 + H_0^2H_1^2$
21	$v_{-1}x_{-1}^2u_2$	$H_{-2}H_{-1}H_0H_1^4 - 4H_{-2}H_{-1}^2H_0H_1H_3 + 24H_{-2}H_{-1}^2H_4$ $+ 16H_{-2}H_{-1}H_0H_3 - 16H_{-2}H_{-1}H_0H_{-1}H_1^3 + 48H_{-2}H_0H_1^2$ $- 24H_{-1}^2H_0H_2$

$$\begin{aligned}
22 \quad v_{-1}^2 x_1 x_3 & 16H_{-2}H_1H_3 + 8H_{-2}^2H_1^4 - 48H_{-4}H_0H_1H_3 + 4H_{-3}H_{-1}H_0^2H_1H_3 \\
& - H_{-1}^4H_0^2H_1H_3 - 24H_{-1}H_0^2H_1 - 8H_{-3}H_0^2H_3 + 8H_{-1}^3H_0^2H_3 \\
23 \quad v_{-1}x_{-1}u_1^2 & 48H_{-2}H_{-1}H_0H_4 + H_{-2}H_{-1}H_0^2H_1^4 - 4H_{-2}H_{-1}H_0^2H_1H_3 \\
& + 8H_{-2}H_0^2H_3 - 8H_{-2}H_0^2H_1^3 - 16H_{-3}H_0H_2^2 - 8H_{-1}^3H_0H_2^2 \\
& + H_{-1}H_0^2H_2 \\
24 \quad v_{-1}x_{-1}x_1x_3 & 2H_{-2}H_{-1}H_1H_3 + H_{-2}H_{-1}H_1^4 - 10H_{-2}H_1^3 - 8H_{-2}H_3 \\
& - H_{-1}^3H_0H_1H_3 - 2H_{-3}H_0H_1H_3 + 6H_{-1}^2H_0H_3 + 24H_0H_1 \\
25 \quad x_{-1}^3u_2 & 64H_{-3}H_4 + 8H_{-1}^3H_4 - H_{-1}^3H_0H_1^4 + 4H_{-1}^3H_0H_1H_3 \\
& + 192H_0H_1 - 24H_{-1}^2H_0H_3 + 24H_{-1}^2H_0H_1^3 - 144H_{-1}H_0H_1^2 \\
26 \quad x_{-3}x_{-1}^2x_1^2x_3 & 60H_{-1}^4H_1H_3 - 24H_{-5}H_1^2H_3 + 24H_{-3}H_{-1}^2H_5 + H_{-3}H_{-1}^2H_1^5 \\
& - 60H_{-3}H_{-1}H_1^4 - H_{-1}^5H_1^2H_3 + 440H_{-3}H_1^3 - 440H_{-1}^3H_3 \\
27 \quad x_{-1}^2x_1u_2 & 360H_{-1}^2H_1H_4 - 960H_{-1}H_4 - 60H_{-1}^2H_0H_1^2H_3 - 3H_{-1}^2H_0H_1^5 \\
& - 72H_{-1}^2H_0H_5 + 400H_{-1}H_0H_1H_3 + 80H_{-1}H_0H_1^4 - 400H_0H_1^3 \\
& - 320H_0H_3 \\
28 \quad x_{-1}^4u_1^2 & 16H_{-1}^4H_2^2 - 256H_{-3}H_{-1}H_2^2 + 288H_{-1}^2H_0^2H_1^2 + 32H_{-1}^3H_0^2H_3 \\
& - 32H_{-1}^3H_0^2H_1^3 + 48H_{-1}^4H_0H_4 + H_{-1}^4H_0^2H_1^4 - 4H_{-1}^4H_0^2H_1H_3 \\
& - 768H_{-1}H_0^2H_1 \\
29 \quad x_{-3}x_{-1}^2\varepsilon u_1x_3 & -H_{-3}H_{-1}^2H_1^5 - 24H_{-3}H_{-1}^2H_5 - 8H_{-3}H_{-1}^2H_1^2H_3 + 20H_{-3}H_{-1}H_1^4 \\
& + 40H_{-3}H_{-1}H_1H_3 - 80H_{-3}H_1^3 + 50H_{-1}^3H_3 + 28H_{-1}^2H_1^2 \\
& - 384H_{-1}H_1 \\
30 \quad x_{-1}^5\varepsilon x_1u_2 & 4000H_{-1}^3H_1^3 + 3200H_{-1}^3H_3 + 61440H_{-1}H_1 + 60H_{-1}^5H_1^2H_3 \\
& + 3H_{-1}^5H_1^5 + 72H_{-1}^5H_5 - 1000H_{-1}^4H_1H_3 - 200H_{-1}^4H_1^4 \\
& - 28800H_{-1}^2H_1^2 \\
31 \quad v_{-2}x_1^2x_3 & 240H_{-4}H_1^2H_3 + 8H_{-4}H_1^5 + 192H_{-4}H_5 + 20H_{-3}H_{-1}H_0H_1^2H_3 \\
& - 5H_{-1}^4H_0H_1^2H_3 - 120H_{-1}H_0H_1^2 - 80H_{-3}H_0H_1H_3 \\
& + 80H_{-1}^3H_0H_1H_3 - 240H_{-1}^2H_0H_3 - 960H_0H_1 \\
32 \quad v_{-2}x_1^5 & 8H_{-4}H_1^5 + 640H_{-4}H_1^2H_3 + 3072H_{-4}H_5 - 2400H_{-1}^2H_0H_1^3 \\
& + 20H_{-3}H_{-1}H_0H_1^5 - 5H_{-1}^4H_0H_1^5 + 9600H_{-1}H_0H_1^2 \\
& - 200H_{-3}H_0H_1^4 + 200H_{-1}^3H_0H_1^4 - 15360H_0H_1 \\
33 \quad v_{-1}x_1u_1^2 & 240H_{-2}H_1H_2^2 - 160H_{-2}H_0H_2H_3 - 80H_{-2}H_0H_1^3H_2 \\
& - 240H_{-2}H_0H_1H_4 + 3H_{-2}H_0^2H_1^5 + 60H_{-2}H_0^2H_1^2H_3 \\
& + 72H_{-2}H_0^2H_5 + 360H_0^2H_1H_2 - 80H_0^3H_3 - 40H_0^3H_1^3
\end{aligned}$$

- 34 $x_{-1}x_1u_1^2$ $240H_{-1}H_1H_2^2 - 160H_{-1}H_0H_2H_3 - 80H_{-1}H_0H_1^3H_2 + 480H_0H_1^2H_2$
 $-240H_{-1}H_0H_1H_4 + 3H_{-1}H_0^2H_1^5 + 60H_{-1}H_0^2H_1^2H_3 + 72H_{-1}H_0^2H_5$
 $-200H_0^2H_1H_3 - 40H_0^2H_1^4$
- 35 $v_{-2}x_{-1}x_1u_2$ $3H_{-4}H_{-1}H_0H_1^5 + 60H_{-4}H_{-1}H_1^2H_0H_3 + 72H_{-4}H_{-1}H_0H_5$
 $-200H_{-4}H_0H_1H_3 - 40H_{-4}H_0H_1^4 - 60H_{-3}H_{-1}^2H_0H_1H_4$
 $-3H_{-1}^5H_0H_1H_4 + 200H_{-3}H_{-1}H_0H_4$
 $40H_{-1}^4H_0H_4 - 72H_{-5}H_0H_1H_4$
- 36 $x_{-3}x_1u_1^2$ $160H_{-3}H_1H_2^2 - 40H_{-1}^3H_1H_2^2 + 240H_{-1}^2H_2^2 - 160H_{-3}H_3H_0H_2$
 $-80H_{-3}H_0H_1^3H_2 + 480H_{-1}H_0H_1H_2 - 240H_{-3}H_0H_1H_4$
 $+3H_{-3}H_0^2H_1^5 + 60H_{-3}H_0^2H_1^2H_3 + 72H_{-3}H_0^2H_5 - 360H_0^2H_1^2$
- 37 $x_{-3}x_{-1}x_1u_2$ $200H_{-3}H_{-1}H_1H_4 + 40H_{-1}^4H_1H_4 - 400H_{-1}^3H_4 - 320H_{-3}H_4$
 $-3H_{-3}H_{-1}H_0H_1^5 - 60H_{-3}H_{-1}H_0H_1^2H_3 - 72H_{-3}H_{-1}H_0H_5$
 $+40H_{-3}H_0H_1^4 + 200H_{-3}H_0H_1H_3 + 360H_{-1}H_0H_1^2 - 960H_0H_1$
- 38 $x_{-1}^2u_1x_3$ $40H_{-1}^2H_2H_3 + 5H_{-1}^2H_1^3H_2 - 60H_{-1}H_1^2H_2 + 120H_1H_2$
 $-H_{-1}^2H_0H_1^5 - 24H_{-1}^2H_0H_5 - 5H_{-1}^2H_0H_1^2H_3 + 20H_{-1}H_0H_1^4$
 $+40H_{-1}H_0H_1H_3 - 40H_0H_3 - 80H_0H_1^3$
- 39 $x_{-3}x_{-1}u_1x_3$ $20H_{-3}H_{-1}H_2H_3 + 5H_{-3}H_{-1}H_1^3H_2 - 30H_{-3}H_1^2H_2 + 5H_{-1}^4H_2H_3$
 $-120H_{-1}H_2 - H_{-3}H_{-1}H_0H_1^5 - 24H_{-3}H_{-1}H_0H_5$
 $-5H_{-3}H_{-1}H_0H_1^2H_3 + 10H_{-3}H_0H_1^4 + 20H_{-3}H_0H_1H_3$
 $+30H_{-1}H_0H_1^2 - 120H_0H_1$
- 40 $v_{-2}x_1u_1^2$ $120H_{-4}H_1H_2^2 - 160H_{-4}H_0H_2H_3 - 80H_{-4}H_0H_1^3H_2 + 3H_{-4}H_0^2H_1^5$
 $+60H_{-4}H_0^2H_1^2H_3 - 240H_{-4}H_0H_1H_4 + 72H_{-4}H_0^2H_5$
 $+20H_{-3}H_{-1}H_0H_1H_2^2 - 5H_{-1}^4H_0H_1H_2^2 - 40H_{-3}H_0H_2^2$
 $+40H_{-1}^3H_0H_2^2 + 240H_{-2}H_0H_1H_2 - 120H_0^3H_1$
- 41 $v_{-1}^2u_1x_3$ $80H_{-2}^2H_2H_3 + 40H_{-2}^2H_1^3H_2 - 192H_{-2}^2H_0H_5 - 8H_{-2}^2H_0H_1^5$
 $-40H_{-2}^2H_0H_1^2H_3 - 480H_{-2}H_0H_1H_2 + 280H_{-2}H_0^2H_3$
 $+80H_{-2}H_0^2H_1^3 + 240H_{-4}H_0H_2H_3 - 20H_{-3}H_{-1}H_0^2H_2H_3$
 $+5H_{-1}^4H_0^2H_2H_3 + 120H_{-1}H_0^2H_2 - 240H_0^3H_1$
- 42 $v_{-1}^2x_1u_2$ $120H_{-2}^2H_1H_4 - 3H_{-2}^2H_0H_1^5 - 60H_{-2}^2H_0H_1^2H_3 - 72H_{-2}^2H_0H_5$
 $-240H_{-2}H_0H_1H_2 + 160H_{-2}H_0^2H_3 + 80H_{-2}H_0^2H_1^3$
 $-20H_{-3}H_{-1}H_0^2H_1H_4 + 5H_{-1}^4H_0^2H_1H_4 + 240H_{-4}H_0H_1H_4$
 $+40H_{-3}H_0^2H_4 - 40H_{-1}^3H_0^2H_4 - 120H_0^3H_1$

- 43 $v_{-1}x_{-1}x_1u_2$ $120H_{-2}H_{-1}H_1H_4 - 3H_{-2}H_{-1}H_0H_1^5 - 60H_{-2}H_{-1}H_0H_1^2H_3$
 $-72H_{-2}H_{-1}H_0H_5 + 200H_{-2}H_0H_1H_3 + 40H_{-2}H_0H_1^4$
 $+80H_{-3}H_0H_1H_4 + 40H_{-1}^3H_0H_1H_4 - 240H_{-1}^2H_0H_4$
 $-120H_{-1}H_0H_1H_2 + 80H_{-1}H_0^2H_3 + 40H_{-1}H_0^2H_1^3 - 240H_0^2H_1^3$
- 44 $x_{-3}v_{-1}u_1x_3$ $5H_{-3}H_{-2}H_1^3H_2 - 5H_{-2}H_{-1}^3H_2H_3 - 24H_{-3}H_{-2}H_0H_5$
 $-H_{-3}H_{-2}H_0H_1^5 - 5H_{-3}H_{-2}H_0H_1^2H_3 + 30H_{-2}H_{-1}H_0H_3$
 $+30H_{-2}H_0H_1^2 - 30H_{-3}H_0H_1H_2 + 24H_{-5}H_0H_2H_3$
 $+5H_{-3}H_{-1}^2H_0H_2H_3 + H_{-1}^5H_0H_2H_3 - 30H_{-1}^2H_0H_2$
 $+5H_{-3}H_0^2H_1^3 - 5H_{-1}^3H_0^2H_3$
- 45 $v_{-1}x_{-1}u_1x_3$ $10H_{-2}H_{-1}H_2H_3 + 5H_{-2}H_{-1}H_1^3H_2 - 30H_{-2}H_1^2H_2$
 $-H_{-2}H_{-1}H_0H_1^5 - 24H_{-2}H_{-1}H_0H_5 - 5H_{-2}H_{-1}H_0H_1^2H_3$
 $+10H_{-2}H_0H_1^4 + 20H_{-2}H_0H_1H_3 - 30H_{-1}H_0H_1H_2$
 $+10H_{-3}H_0H_2H_3 + 5H_{-1}^3H_0H_2H_3 + 10H_{-1}H_0^2H_3$
 $+5H_{-1}H_0^2H_1^3 - 30H_0^2H_1^2$
- 46 $v_{-1}^2\varepsilon x_1u_1^2$ $80H_{-2}^2H_2H_3 + 40H_{-2}^2H_1^3H_2 + 120H_{-2}^2H_1H_4 - 3H_{-2}^2H_0H_5$
 $-60H_{-2}^2H_0H_1^2H_3 - 72H_{-2}^2H_0H_5 - 480H_{-2}H_0H_1H_2$
 $+240H_{-2}H_0^2H_3 + 120H_{-2}H_0^2H_1^3 - 5H_{-1}^4H_0H_1H_2^2$
 $+20H_{-3}H_{-1}H_0H_1H_2^2 - 120H_{-4}H_1H_2^2 - 40H_{-3}H_0H_2^2$
 $+40H_{-1}^3H_0H_2^2 - 240H_0^3H_1$
- 47 $v_{-1}x_{-1}x_1^3u_1$ $40H_{-2}H_{-1}H_2H_3 + 20H_{-2}H_{-1}H_1^3H_2 - 120H_{-2}H_1^2H_2$
 $-50H_{-2}H_{-1}H_0H_1^2H_3 - H_{-2}H_{-1}H_0H_1^5 - 24H_{-2}H_{-1}H_0H_5$
 $+10H_{-2}H_0H_1^4 + 200H_{-2}H_0H_1H_3 + 240H_{-1}H_0H_1H_2$
 $+5H_{-1}^3H_0H_1^3H_2 + 10H_{-3}H_0H_1^3H_2 - 90H_{-1}^2H_0H_1H_2$
 $+40H_{-1}H_0^2H_3 + 20H_{-1}H_0^2H_1^3 - 120H_1^2H_0^2$
- 48 $v_{-1}^2x_{-1}x_5$ $48H_{-2}^2H_{-1}H_5 - 8H_{-2}^2H_{-1}H_1^5 - 40H_{-2}^2H_{-1}H_1^2H_3 + 80H_{-2}^2H_1^4$
 $+160H_{-2}^2H_1H_3 + 640H_{-2}H_{-1}H_0H_3 + 80H_{-2}H_{-1}H_0H_1^3$
 $-160H_{-3}H_{-2}H_0H_5 - 80H_{-2}H_{-1}^3H_0H_5 - 480H_{-2}H_0H_1^2$
 $-240H_{-4}H_{-1}H_0H_5 + 60H_{-3}H_{-1}^2H_0^2H_5 + 3H_{-1}^5H_0^2H_5$
 $-360H_{-1}H_0^2H_1 + 72H_{-5}h_0^2H_5$
- 49 $x_{-3}v_{-1}x_1u_2$ $40H_{-3}H_{-2}H_1H_4 - 40H_{-2}H_{-1}^3H_1H_4 + 240H_{-2}H_{-1}^2H_4$
 $-60H_{-3}H_{-2}H_0H_1^2H_3 - 3H_{-3}H_{-2}H_0H_1^5 - 72H_{-3}H_{-2}H_0H_5$
 $+360H_{-2}H_0H_1^2 + 192H_{-5}H_0H_1H_4 + 40H_{-3}H_{-1}^2H_0H_1H_4$
 $+8H_{-1}^5H_0H_1H_4 - 160H_{-3}H_{-1}H_0H_4 - 80H_{-1}^4H_0H_4$
 $-120H_{-3}H_0H_1H_2 + 80H_{-3}H_0^2H_3 + 40H_{-3}H_0^2H_1^2 - 240H_{-1}H_0^2H_1$

$$\begin{aligned}
50 \quad v_{-1}^2 x_1^3 u_1 & 320H_{-2}^2 H_2 H_3 + 160H_{-2}^2 H_1^3 H_2 - 400H_{-2}^2 H_0 H_1^2 H_3 \\
& -8H_{-2}^2 H_0 H_1^5 - 192H_{-2}^2 H_0 H_5 - 1920H_{-2} H_0 H_1 H_2 \\
& +640H_{-2} H_0^2 H_3 + 440H_{-2} H_0^2 H_1^3 - 20H_{-3} H_{-1} H_0^2 H_1^3 H_2 \\
& +5H_{-1}^4 H_0^2 H_1^3 H_2 + 240H_{-4} H_0 H_1^3 H_2 - 960H_{-1} H_0^2 H_2 \\
& +120H_{-3} H_0^2 H_1^2 H_2 - 120H_{-1}^3 H_0^2 H_1^2 H_2 + 720H_1^2 H_0^2 H_1 H_2 \\
& -960H_0^3 H_1 \\
51 \quad v_{-1} x_{-1} \varepsilon x_1 u_1^2 & 80H_{-2} H_{-1} H_2 H_3 + 40H_{-2} H_{-1} H_1^3 H_2 - 240H_{-2} H_1^2 H_2 \\
& +120H_{-2} H_{-1} H_1 H_4 - 3H_{-2} H_{-1} H_0 H_1^5 - 60H_{-2} H_{-1} H_0 H_1^2 H_3 \\
& -72H_{-2} H_{-1} H_1 H_4 + 200H_{-2} H_0 H_1 H_3 + 40H_{-2} H_0 H_1^4 \\
& -40H_{-3} H_3 H_2^2 - 20H_{-1}^3 H_1 H_2^2 + 120H_{-1}^2 H_2^2 \\
& -120H_{-1} H_0 H_1 H_2 + 120H_{-1} H_0^2 H_3 + 60H_{-1} H_0^2 H_1^3 - 360H_0^2 H_1^2 \\
52 \quad x_{-1}^3 x_1 u_1^2 & 80H_{-1}^3 H_1 H_2^2 - 320H_{-3} H_1 H_2^2 + 960H_{-1}^2 H_2^2 - 160H_{-1}^3 H_0 H_2 H_3 \\
& -80H_{-1}^3 H_0 H_1^3 H_2 - 3840H_{-1} H_0 H_1 H_2 + 1440H_{-1}^2 H_0 H_1^2 H_2 \\
& -240H_{-1}^3 H_0 H_1 H_4 + 3H_{-1}^3 H_0^2 H_1^5 + 60H_{-1}^3 H_0^2 H_1^2 H_3 \\
& +72H_{-1}^3 H_0^2 H_5 - 2880H_0^2 H_1^2 - 600H_{-1}^2 H_0^2 H_1 H_3 \\
& -120H_{-1}^2 H_0^2 H_1^4 + 1200H_{-1} H_0^2 H_1^3 + 960H_{-1} H_0^2 H_3 \\
53 \quad v_{-1}^2 x_{-1} \varepsilon x_1 u_2 & 96H_{-2}^2 H_{-1} H_5 + 4H_{-2}^2 H_{-1} H_1^5 + 20H_{-2}^2 H_{-1} H_1^2 H_3 - 40H_{-2}^2 H_1^4 \\
& -80H_{-2}^2 H_1 H_3 + 240H_{-2} H_2 H_{-1} H_1 - 80H_{-3} H_{-2} H_2 H_3 \\
& -40H_{-2} H_{-1}^3 H_2 H_3 - 40H_{-2} H_{-1} H_0 H_3 - 80H_{-2} H_{-1} H_0 H_1^3 \\
& +480H_{-2} H_0 H_1^2 + 60H_{-3} H_{-1}^2 H_0 H_2 H_3 + 3H_{-1}^5 H_0 H_2 H_3 \\
& -120H_{-4} H_{-1} H_2 H_3 - 360H_{-1}^2 H_0 H_2 + 72H_{-5} H_0 H_2 H_3 \\
& +360H_{-1} H_0^2 H_1 - 120H_{-3} H_0^2 H_3 - 60H_{-1}^3 H_0^2 H_{-1} \\
54 \quad v_{-1}^2 x_{-1} \varepsilon u_1 x_3 & 96H_{-2}^2 H_{-1} H_5 + 4H_{-2}^2 H_{-1} H_1^5 + 20H_{-2}^2 H_{-1} H_1^2 H_3 - 40H_{-2}^2 H_1^4 \\
& -80H_{-2}^2 H_1 H_3 + 240H_{-2} H_{-1} H_1 H_2 - 80H_{-3} H_{-2} H_2 H_3 \\
& -40H_{-2} H_{-1}^3 H_2 H_3 - 40H_{-2} H_{-1} H_0 H_3 - 80H_{-2} H_{-1} H_0 H_1^3 \\
& +480H_{-2} H_0 H_1^2 + 60H_{-3} H_{-1}^2 H_0 H_2 H_3 + 3H_{-1}^5 H_0 H_2 H_3 \\
& -120H_{-4} H_{-1} H_2 H_3 - 360H_{-1}^2 H_0 H_2 + 72H_{-5} H_0 H_2 H_3 \\
& +360H_{-1} H_0^2 H_1 - 120H_{-3} H_0^2 H_3 - 60H_{-1}^3 H_0^2 H_3 \\
55 \quad v_{-1} x_{-1}^3 x_1^3 u_1 & 40H_{-2} H_{-1}^3 H_2 H_3 - 40H_{-3} H_{-2} H_1^3 H_2 - H_{-2} H_{-1}^3 H_0 H_1^5 \\
& -50H_{-2} H_{-1}^3 H_0 H_1^2 H_3 - 24H_{-2} H_{-1}^3 H_0 H_5 - 480H_{-2} H_{-1} H_0 H_1^3 \\
& +1920H_{-2} H_1^2 H_0 + 30H_{-2} H_{-1}^2 H_0 H_1^4 + 600H_{-2} H_{-1}^2 H_0 H_1 H_3 \\
& -1920H_{-2} H_{-1} H_0 H_3 + 480H_{-1}^3 H_0 H_1 H_2 + H_{-1}^5 H_0 H_1^3 H_2 \\
& +24H_{-5} H_0 H_1^3 H_2 + 50H_{-3} H_{-1}^2 H_0 H_1^3 H_2 - 1920H_{-1}^2 H_0 H_2 \\
& -600H_{-3} H_{-1} H_0 H_1^2 H_2 - 30H_{-1}^4 H_0 H_1^2 H_2 + 1920H_{-3} H_0 H_1 H_2 \\
& +40H_{-1}^3 H_0^2 H_3 - 40H_{-3} H_0^2 H_1^3
\end{aligned}$$

$$\begin{aligned}
56 \quad v_{-1}^2 x_{-1} x_1^2 x_3 \quad & 960H_{-2}H_1H_3 + 960H_{-1}H_0^2H_1 - 1280H_{-2}H_{-1}H_0H_3 \\
& + 320H_{-3}H_0^2H_3 - 3840H_{-2}H_0H_1^2 + 400H_{-1}^3H_0^2H_3 \\
& - 160H_{-3}H_{-2}H_0H_1^2H_3 - 400H_{-3}H_{-1}H_0^2H_1H_3 + 800H_{-2}H_{-1}H_0H_1^3 \\
& - 240H_{-4}H_{-1}H_0H_1^2H_3 - 192H_{-2}^2H_{-1}H_5 - 8H_{-2}^2H_{-1}H_1^5 \\
& - 160H_{-2}^2H_{-1}H_1^2H_3 + 240H_{-2}^2H_1^4 - 80H_{-1}^4H_0^2H_1H_3 \\
& - 360H_{-1}^2H_0^2H_1^2 + 60H_{-3}H_{-1}^2H_0^2H_2^2H_3 - 80H_{-2}H_{-1}^3H_0H_1^2H_3 \\
& + 960H_{-2}H_{-1}^2H_0H_1H_3 + 3H_{-1}^5H_0^2H_1^2H_3 + 72H_{-5}H_0^2H_1^2H_3 \\
57 \quad v_{-1}x_{-1}^3x_1u_2 \quad & 40H_{-2}H_{-1}^3H_1H_4 + 320H_{-3}H_{-2}H_1H_4 - 960H_{-2}H_{-1}^2H_4 \\
& + 3H_{-2}H_{-1}^3H_0H_1^5 + 60H_{-2}H_{-1}^3H_0H_1^2 + 72H_{-2}H_{-1}^3H_0H_5 \\
& - 2880H_{-2}H_0H_1^2 - 600H_{-2}H_{-1}H_0H_1H_3 - 120H_{-2}H_{-1}^2H_0H_1^4 \\
& + 1200H_{-2}H_{-1}H_0H_1^3 + 960H_{-2}H_{-1}H_0H_3 - 8H_{-1}^5H_0H_1H_4 \\
& - 192H_{-5}H_0H_1H_4 - 4000H_{-3}H_{-1}H_0H_1H_4 + 1600H_{-3}H_{-1}H_0H_4 \\
& + 80H_{-1}^4H_0H_4 + 120H_{-1}^3H_0H_1H_2 - 80H_{-1}^3H_0^2H_3 \\
& - 40H_{-1}^3H_0^2H_1^3 - 1920H_{-1}H_0^2H_1 + 720H_{-1}^2H_0^2H_1^2 \\
58 \quad x_{-1}^5x_1u_1^2 \quad & 61440H_{-1}H_0^2H_1 + 10240H_{-3}H_0H_1H_2 - 160H_{-1}^5H_0H_2H_3 \\
& + 30720H_{-1}^2H_0H_2 + 3200H_{-1}^3H_0^2H_3 + 2400H_{-1}^4H_0H_1^2H_2 \\
& - 80H_{-1}^5H_0H_1^3H_2 - 200H_{-1}^4H_0^2H_1^4 + 15360H_{-3}H_{-1}H_2^2 \\
& - 640H_{-3}H_{-1}^2H_1H_2^2 - 17920H_{-1}^3H_0H_1H_2 - 1920H_{-1}^4H_2^2 \\
& + 112H_{-1}^5H_1H_2^2 - 3072H_{-5}H_1H_2^2 + 72H_{-1}^5H_0^2H_5 \\
& + 4000H_{-1}^3H_0^2H_1^3 - 240H_{-1}^5H_0H_1H_4 + 3H_{-1}^5H_0^2H_1^5 \\
& - 1000H_{-1}^4H_0^2H_1H_3 - 28800H_{-1}^2H_0^2H_1^2 + 60H_{-1}^5H_0^2H_1^2H_3 \\
59 \quad v_{-1}x_{-1}^2x_1u_1^2 \quad & 240H_{-2}H_{-1}^2H_1H_2^2 - 960H_{-2}H_{-1}H_2^2 - 160H_{-2}H_{-1}^2H_0H_2H_3 \\
& - 80H_{-2}H_{-1}^2H_0H_1^3H_2 + 960H_{-2}H_{-1}H_0H_1^2H_2 - 1920H_{-2}H_0H_1H_2 \\
& - 240H_{-2}H_{-1}^2H_0H_1H_4 + 3H_{-2}H_{-1}^2H_0^2H_1^5 + 60H_{-2}H_{-1}^2H_0^2H_1^2H_3 \\
& + 72H_{-2}H_{-1}^2H_0^2H_5 - 400H_{-2}H_{-1}H_0^2H_1H_3 - 80H_{-2}H_{-1}H_0^2H_1^4 \\
& + 400H_{-2}H_0^2H_1^3 + 320H_{-2}H_0^2H_3 + 160H_{-1}^3H_0H_2^2 \\
& + 320H_{-3}H_0H_2^2 + 360H_{-1}^2H_0^2H_1H_2 - 1920H_{-1}H_0^2H_2 \\
& - 80H_{-1}^2H_0^3H_3 - 40H_{-1}^2H_0^3H_1^3 + 480H_{-1}H_0^3H_1^2 - 960H_0^3H_1
\end{aligned}$$

$$\begin{aligned}
60 \quad v_{-1}x_{-1}^3u_1x_3 \quad & 360H_{-1}H_0^2H_1 - 10H_{-2}H_{-1}^3H_2H_3 + 24H_{-5}H_0H_2H_3 \\
& + H_{-1}^5H_0H_2H_3 - 40H_{-3}H_0^2H_3 + 480H_{-2}H_0H_1^2 \\
& - 120H_{-1}^2H_0H_2 - 10H_{-1}^3H_0^2H_3 + 50H_{-3}H_{-1}^2H_0H_2H_3 \\
& - 30H_{-1}^3H_0H_1H_2 - 240H_{-2}H_{-1}H_0H_1^3 - 24H_{-2}H_{-1}^3H_0H_5 \\
& + 5H_{-1}^3H_0^2H_1^3 + 5H_{-2}H_{-1}^3H_1^3H_2 - 90H_{-2}H_{-1}^2H_1^2H_2 \\
& - H_{-2}H_{-1}^3H_0H_1^5 + 360H_{-2}H_{-1}H_1H_2 - 40H_{-3}H_{-2}H_2H_3 \\
& - 90H_{-1}^2H_0^2H_1^2 + 30H_{-2}H_{-1}^2H_0H_4 - 5H_{-2}H_{-1}^3H_0H_1^2H_3 \\
& + 60H_{-2}H_{-1}^2H_0H_1H_3 \\
61 \quad v_{-1}^2x_{-1}x_1u_1^2 \quad & 480H_{-2}^2H_0H_1^2H_2 + 72H_{-2}^2H_{-1}H_0H_5 + 3H_{-2}^2H_{-1}H_0^2H_1^5 \\
& + 80H_{-1}^3H_0^3H_1H_2 + 160H_{-3}H_0^3H_1H_2 - 72H_{-5}H_0^2H_1H_2^2 \\
& - 3H_{-1}^5H_0^2H_1H_2^2 - 80H_{-2}H_{-1}H_0^3H_1^3 - 160H_{-2}H_{-1}H_0^3H_3 \\
& - 480H_{-2}H_{-1}^2H_0H_2^2 + 480H_{-2}H_0^3H_1^2 - 480H_{-1}^2H_0^3H_2 \\
& + 160H_{-3}H_{-2}H_0H_1H_2^2 + 60H_{-2}^2H_{-1}H_0^2H_1^2H_3 - 240H_{-2}^2H_{-1}H_0H_1H_4 \\
& - 80H_{-2}^2H_{-1}H_0H_1^3H_2 - 60H_{-3}H_{-1}^2H_0^2H_1H_2^2 + 240H_{-4}H_{-1}H_0H_1H_2^2 \\
& - 160H_{-2}^2H_{-1}H_0H_2H_3 + 80H_{-2}H_1^3H_0H_1H_2^2 + 40H_{-1}^4H_0^2H_2^2 \\
& - 40H_{-2}^2H_0^2H_1^4 - 200H_{-2}^2H_0^2H_1H_3 + 200H_{-3}H_{-1}H_0^2H_2^2 \\
62 \quad v_{-1}x_{-1}^3\varepsilon x_1u_1^2 \quad & 2880H_{-1}H_0^2H_1 + 60H_{-1}^3H_0^2H_1^3 - 1080H_{-1}^2H_0^2H_1^2 \\
& + 800H_{-3}H_{-1}H_2^2 - 1920H_{-1}^2H_0H_2 - 96H_{-5}H_1H_2^2 \\
& - 4H_{-1}^5H_1H_2^2 + 2880H_{-2}H_0H_1^2 + 120H_{-1}^2H_0^2H_3 \\
& - 3H_{-2}H_{-1}^3H_0H_1^5 + 1920H_{-2}H_{-1}H_1H_2 - 72H_{-2}H_{-1}^3H_0H_5 \\
& + 120H_{-2}H_{-1}^3H_1H_4 + 80H_{-2}H_{-1}^3H_2H_3 + 120H_{-2}H_{-1}^2H_0H_1^4 \\
& - 960H_{-2}H_{-1}H_0H_3 - 1200H_{-2}H_{-1}H_0H_1^3 - 200H_{-3}H_{-1}^2H_1H_2^2 \\
& + 640H_{-3}H_0H_1H_2 + 200H_{-1}^3H_0H_1H_2 - 720H_{-2}H_{-1}^2H_1^2H_2 \\
& + 40H_{-2}H_{-1}^3H_1^3H_2 + 40H_{-1}^4H_2^2 - 60H_{-2}H_{-1}^3H_0H_1^2H_3 \\
& + 600H_{-2}H_{-1}^2H_0H_1H_3
\end{aligned}$$

$$\begin{aligned}
63 \quad z^2 v_{-1} x_1 u_1^2 & \quad 48H_{-2}H_0H_1H_2 - 16H_{-2}H_0^2H_3 - 8H_{-2}H_0^2H_1^3 \\
& \quad -4H_{-3}H_{-1}H_0H_1H_2^2 + H_{-1}^4H_0H_1H_2^2 + 24H_{-4}H_1H_2^2 \\
& \quad + 8H_{-3}H_0H_2^2 - 8H_{-1}^3H_0H_2^2 \\
64 \quad z^2 x_{-3} x_1 u_1^2 & \quad H_{-1}^5H_1H_2^2 + 5H_{-3}H_{-1}^2H_1H_2^2 + 24H_{-5}H_1H_2^2 \\
& \quad -20H_{-3}H_{-1}H_2^2 - 10H_{-1}^4H_2^2 - 20H_{-3}H_0H_1H_2 \\
& \quad -10H_{-1}^3H_0H_1H_2 + 60H_{-1}^2H_0H_2 - 10H_{-3}H_0^2H_3 \\
& \quad -5H_{-3}H_0^2H_1^3 + 30H_{-1}H_0^2H_1 \\
65 \quad z^2 x_{-1} u_1^2 & \quad 24H_{-5}H_1H_2^2 + 50H_{-3}H_{-1}^2H_1H_2^2 + H_{-1}^5H_1H_2^2 \\
& \quad -200H_{-3}H_{-1}H_2^2 - 10H_{-1}^4H_2^2 - 40H_{-1}^3H_0H_1H_2 \\
& \quad -80H_{-3}H_0H_1H_2 + 240H_{-1}^2H_0H_2 - 10H_{-1}^3H_0^2H_3 \\
& \quad -5H_{-1}^3H_0^2H_1^3 - 240H_{-1}H_0^2H_1 + 90H_{-1}^2H_0^2H_1^2
\end{aligned}$$

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