Nonlinear Aeroelastic Modeling of a Flexible Wing and Comparison with Experiments

by

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Abstract

A wind tunnel experimental investigation of limit cycle oscillations (LCO) of a uniform flexible airfoil with root pitch motion occurring at transitional Reynolds number regime ($4.5 \times 10^4 \leq Re_c \leq 1.3 \times 10^5$) is presented. Depending on the initial conditions, two stable limit cycle regimes are observed: small and large amplitude LCO. The origin of the large amplitude LCO is determined to be coalescence flutter for which the exponential growth of the amplitude is limited by the stall at large angles of attack. On the other hand, the small amplitude LCO are attributed to laminar boundary layer separation related to transitional Reynolds number aerodynamics. In addition, the nonlinear equations of motion for a cantilever is developed considering chordwise and flapwise bending, torsion and base rotation. The nonlinearities arise from two main sources: the structural flexibility and the coupling of the bending and torsional motions with the base rotation. Furthermore, a linear inviscid aerodynamic model is considered using both quasisteady and unsteady forcing terms. Despite the linear aerodynamics approximations, a parallel to the experiments can be drawn with the numerical simulations presented for small and large amplitude LCO at the vicinity of the linear flutter speed, with a dominant cause factor being a coalescence flutter. Overall, the structural nonlinearities play significant role in LCO for both the experimental and numerical investigations.

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Table of Contents

Abstract	ii
Acknowledgments	iii
Table of Contents	iv
List of Tables	viii
List of Tables	viii
List of Figures	ix
List of Figures	ix
Nomenclature	xiv
1 Introduction	2
1.1 Statement of the problem	2
1.2 Literature review	3
1.3 Scope of the thesis and outline	5
2 Experimental investigation	8
2.1 Introduction	8
2.2 Experimental set up	9
2.3 Free vibration tests	15
2.3.1 System with aluminum stringer: bending-torsion free vibration tests . . .	16

2.3.2	System with Steel Stringer: bending-torsion-rotation free vibration tests	24
2.4	Wing aeroelastic dynamics: Limit Cycle Oscillations	27
2.4.1	Wing dynamics	28
2.4.2	Effect of elastic axis position	35
2.4.3	Effect of structural stiffness	36
2.4.4	Summary of findings	39
3	Nonlinear equations of motion for the elastic bending-torsion of uniform airfoils	
	with root-pitch motion	41
3.1	Introduction	41
3.2	Coordinate systems and transformations	42
3.3	Strain-displacement relation	45
3.4	Kinematics	48
3.5	Hamilton's principle	48
3.5.1	Strain energy contributions	49
3.5.2	Kinetic energy contributions	51
3.5.3	Generalized nonconservative forces contributions	55
3.6	Nonlinear bending-bending-torsion equations of motion	55
4	Structural equations: numerical modeling and solution	60
4.1	Introduction	60
4.2	Finite dimensional approximation of the equations of motion	60
4.3	Linear structural model	64
4.3.1	Free vibration tests	65
4.3.2	Coupled and uncoupled bending-torsional free vibration tests of the cantilever wing	65
4.3.3	Coupled bending-torsion free vibration tests of the wing with root-pitch rotation	71
4.4	Nonlinear structural model	76

4.4.1	Nonlinear coupled bending-bending-torsion with root-pitch rotation free vibration tests	77
5	Aerodynamic modeling and aeroelastic solution	84
5.1	Introduction	84
5.2	Quasisteady aerodynamics	87
5.2.1	Linear aeroelastic system with quasisteady aerodynamic	95
5.2.2	Nonlinear aeroelastic system with quasisteady aerodynamic	96
5.3	Unsteady aerodynamics	106
5.3.1	Linear aeroelastic system with unsteady aerodynamic	109
5.3.2	Nonlinear aeroelastic system with unsteady aerodynamic	110
6	Concluding Remarks	116
6.1	Summary of Findings	116
6.1.1	Experimental Findings	116
6.1.2	Numerical Modeling	117
6.2	Future Research	118
	List of References	120
	Appendix A Details of the derivation of the equation of motion	124
A.1	Deformed wing coordinate system transformation	124
	Appendix B Basis shape functions and the Galerkin method	127
	Appendix C Nondimensionalization and finite dimensional approximation of the equations of motion	131
C.1	Nondimensional nonlinear bending-bending-torsion equations of motion	131
C.2	Finite dimensional approximation of the structural equations of motion	132
C.3	Linear system structural matrices	139
C.4	Nonlinear system structural matrices	140
C.5	Finite dimensional approximation of aerodynamic terms	143

C.5.1	Quasisteady aerodynamics	143
C.5.2	Unsteady aerodynamics	144
Appendix D	Wing and base properties	146

List of Tables

2.1	Properties of the wing stringers	14
4.1	First five natural frequencies for the cantilever beam	66
4.2	Comparison between numerical and experimental frequencies	76
B.1	Five first solutions for the flexural characteristic equations and frequency ratios . .	129
B.2	Five first solutions for the torsional characteristic equations and amplitude ratios .	130
D.1	Baseline wing and base properties with steel stringer	146

List of Figures

Figure 2.1	Wing model on top of the base shaft before assembly with the wind tunnel section	10
Figure 2.2	Open view of the wing model shows constructive details	11
Figure 2.3	Side view of the test final assembly in the wind tunnel section	12
Figure 2.4	Bottom view of wind tunnel section and pulley-spring system	12
Figure 2.5	Schematic view of the experimental setup and instrumentation. (Figure removed due to copyrighting issue)	13
Figure 2.6	Detailed view of the strain gauges	14
Figure 2.7	Beam only free vibration test	17
Figure 2.8	Beam bending pushover test	18
Figure 2.9	Free vibration test time and frequency response with initial beam tip displacement of 2 cm	18
Figure 2.10	Damped Frequencies and Damping ratio peak to peak with respect to time and amplitude for initial beam tip displacement approximately of 2 cm	20
Figure 2.11	Damping ratio peak to peak with respect to time for initial beam tip displacement approximately of 2 cm	21
Figure 2.12	Natural frequencies peak to peak for several initial conditions	21
Figure 2.13	Amplitude Free decay and exponential fit damping ratio for several initial conditions	22
Figure 2.14	Damping and natural frequencies peak to peak with respect to time for several initial conditions	23
Figure 2.15	PSD analysis of the bending free vibration showing the higher harmonics .	23

Figure 2.16 Natural frequencies with respect to bending signal for several Wind tunnel conditions	25
Figure 2.17 Natural frequencies with respect to bending signal for wind tunnel conditions	25
Figure 2.18 Bending pitch free vibration test time response	26
Figure 2.19 Bending pitch free vibration test PSD	26
Figure 2.20 PSD of the wing free vibration test	27
Figure 2.21 Video frames of LAO motion	28
Figure 2.22 LCO time history for smoothly reduction of airspeed from $V = 9.8$ to 8.7 m/s	30
Figure 2.23 LCO time history for abruptly variation of airspeed from $V = 6$ to 10 m/s and back to 0 m/s	30
Figure 2.24 SAO LCO time response $V = 7.8$ m/s	31
Figure 2.25 SAO LCO phase space plot $V = 7.8$ m/s	31
Figure 2.26 SAO LCO PSD analysis $V = 7.8$ m/s	32
Figure 2.27 SAO LCO time response $V = 4.3$ m/s	33
Figure 2.28 SAO LCO phase space plot $V = 4.3$ m/s	33
Figure 2.29 SAO LCO PSD analysis $V = 4.3$ m/s	34
Figure 2.30 LCO summary pitch axis at 25% of the chord	34
Figure 2.31 LCO summary pitch axis at 25% of the chord	35
Figure 2.32 LCO summary pitch axis at 25% of the chord	35
Figure 2.33 LAO LCO time response $V = 7.9$ m/s pitch axis at 35% of the chord	36
Figure 2.34 LAO LCO PSD analysis $V = 7.9$ m/s pitch axis at 35% of the chord	37
Figure 2.35 LCO summary pitch axis at 35% of the chord	37
Figure 2.36 LCO summary pitch axis at 25% of the chord with the stiffer spring	38
Figure 2.37 LCO summary pitch axis at 35% of the chord with the stiffer spring	39
Figure 3.1 Wing view with undeformed coordinate systems	43
Figure 3.2 Cross section view of the root pitch rigid rotation (adapted from [1]). Here CG , PC , EA , AC denote center of gravity, pitch center, elastic axis, aerodynamic center, respectively.	43

Figure 3.3	Elastic displacements relative to the pitch base (adapted from [1])	44
Figure 3.4	Cross-section coordinates after deformation (Note: y', z' are not in the same plane as y, z . Their projections are shown in the y, z plane)	44
Figure 3.5	Deformed and undeformed position vector (adapted from [1])	46
Figure 4.1	Bending basis shape functions convergence test: initial $\bar{w} = 0.1$	66
Figure 4.2	Torsion basis shape functions convergence test: initial $\phi = 5^\circ$	67
Figure 4.3	Uncoupled bending free vibration time trace: initial $\bar{w} = 0.1$	68
Figure 4.4	Uncoupled bending free vibration PSD: initial $\bar{w} = 0.1$	68
Figure 4.5	Uncoupled torsion free vibration time trace: initial $\phi = 5^\circ$	69
Figure 4.6	Uncoupled torsion free vibration PSD: initial $\phi = 5^\circ$	70
Figure 4.7	Coupled bending-torsion free vibration time trace: initial $\bar{w} = 0.1$	71
Figure 4.8	Coupled bending-torsion free vibration PSD: initial $\bar{w} = 0.1$	71
Figure 4.9	Coupled bending-torsion free vibration time trace: initial $\phi = 5^\circ$	72
Figure 4.10	Coupled bending-torsion free vibration PSD: initial $\phi = 5^\circ$	72
Figure 4.11	Coupled bending-torsion with root-pitch rotation free vibration time traces: initial $\bar{w} = 0.1$	73
Figure 4.12	Coupled bending-torsion with root-pitch rotation free vibration PSD: ini- tial $\bar{w} = 0.1$	74
Figure 4.13	Coupled bending-torsion with root-pitch rotation free vibration time trace: initial $\phi = 5^\circ$	74
Figure 4.14	Coupled bending-torsion with root-pitch rotation free vibration PSD: ini- tial $\phi = 5^\circ$	75
Figure 4.15	Coupled bending-torsion with root-pitch rotation free vibration time trace: initial $\theta = 5^\circ$	75
Figure 4.16	Coupled bending-torsion with root-pitch rotation free vibration PSD: ini- tial $\theta = 5^\circ$	76
Figure 4.17	Nonlinear structural model mode shapes convergence test	77
Figure 4.18	Nonlinear coupled bending-bending-torsion free vibration time traces for different initial conditions	78

Figure 4.19 Nonlinear coupled bending-bending-torsion free vibration PSD: initial $\bar{w}=0.5$	79
Figure 4.20 Nonlinear coupled bending-bending-torsion free vibration time traces for different initial conditions	80
Figure 4.21 Nonlinear coupled bending-bending-torsion free vibration PSD: initial $\phi = 15^\circ$	81
Figure 4.22 Nonlinear coupled bending-bending-torsion free vibration time traces for different initial conditions	82
Figure 4.23 Nonlinear coupled bending-bending-torsion free vibration PSD: initial $\theta = 30^\circ$	83
Figure 5.1 Wing airfoil section view with important geometric parameters	86
Figure 5.2 Aerodynamic loading in the wing section	86
Figure 5.3 Modal frequency and decay plots of the linear aeroelastic problem with quasisteady aerodynamics with and without mode tracking; one basis function per state	94
Figure 5.4 Modal frequency and decay plots of the linear aeroelastic problem with unsteady aerodynamics with and without mode tracking; four basis functions per state	96
Figure 5.5 Modal frequency and decay plots of the linear aeroelastic problem with quasisteady aerodynamics; two basis functions shapes per state	97
Figure 5.6 Zoomed modal frequency and decay plots of the linear aeroelastic problem with quasisteady aerodynamics; two basis functions per state	97
Figure 5.7 Time traces of the linear aeroelastic system with quasisteady aerodynamics at $U_\infty = 0.95V_F$: initial condition $\bar{w} = 0.1$ and $\theta = 15^\circ$	98
Figure 5.8 Time traces of the linear aeroelastic system with quasisteady aerodynamics at $U_\infty = 1.05V_F$: initial condition $\bar{w} = 0.1$ and $\theta = 15^\circ$	99
Figure 5.9 Nonlinear quasisteady aeroelastic model: basis functions convergence test in time trace. Initial condition $\theta = 5^\circ$ and airspeed $1.00V_F$	100

Figure 5.10	Nonlinear quasisteady aeroelastic model: basis functions convergence test in phase space plot. Initial condition $\theta = 5^\circ$ and airspeed $1.00V_F$	100
Figure 5.11	Time traces comparison between linear and nonlinear aeroelastic system with quasisteady aerodynamics. Initial conditions: $\bar{w} = 0.1, \theta = 15^\circ$	101
Figure 5.12	Time traces of LCO of the nonlinear aeroelastic system with quasisteady aerodynamics. Initial conditions: $\theta = 10^\circ$	103
Figure 5.13	Phase space plots of LCO for the nonlinear aeroelastic system with quasisteady aerodynamics for several airspeeds. Initial conditions: $\theta = 10^\circ$	104
Figure 5.14	Nonlinear aeroelastic response PSD at airspeed $1.01V_F$ Initial conditions: $\theta_0 = 10^\circ$	105
Figure 5.15	Summary of LCO for quasisteady aerodynamics: initial $\theta = 10^\circ$	105
Figure 5.16	Time traces of LCO of the nonlinear aeroelastic system with quasisteady aerodynamics. Initial conditions: $\theta = 10^\circ$	106
Figure 5.17	Modal frequency and decay plots of the linear aeroelastic problem with unsteady aerodynamics	110
Figure 5.18	Modal frequency and decay plots of the linear aeroelastic problem with unsteady aerodynamics	111
Figure 5.19	Time traces comparison between linear and nonlinear aeroelastic system with unsteady aerodynamics. Initial conditions: $\bar{w} = 0.1$ and $\theta = 15^\circ$	112
Figure 5.20	Time response LCO of the nonlinear aeroelastic system with unsteady aerodynamics. Initial condition: $\theta = 5^\circ$	113
Figure 5.21	Phase space plots of the nonlinear aeroelastic system with unsteady aerodynamics. Initial condition: $\theta = 5^\circ$	114
Figure 5.22	Nonlinear aeroelastic response PSD at airspeed $1.05V_F$ Initial conditions: $\theta_0 = 5^\circ$	114
Figure 5.23	Summary of LCO for unsteady aerodynamics: initial $\theta = 5^\circ$	115
Figure A.1	Deformation and Euler (adapted from [1])	125
Figure B.1	Three first flexural and torsional basis functions	128

Nomenclature

$[A]$	aeroelastic system matrix
α_{eff}	effective angle of attack
a_h	distance between the half-chord and the elastic axis
b	wing semi chord
$[C]$	damping matrix
C_L	lift coefficient
c	wing chord
EI_y	flapwise rigidity
EI_z	chorwise rigidity
f_i	damped frequency
f_d	natural frequency
GJ	torsional rigidity
I_s	wing inertia about EA
I_B	base inertial about CM
$[K]$	stiffness matrix
K_θ	pitch stiffness
λ	eigenvalues vector
L	wing span
$L(x, t)$	lift per unit length

M_{ac}	aerodynamic moment with respect to the aerodynamic center
M_∞	freestream mach number
$[M]$	mass matrix
ω_d	damped frequency
ω_i	natural frequency
ϕ	torsion angle
$\{q\}$	state vector
$\vec{r}_0$	undeformed position vector
$\vec{r}_1$	deformed position vector
T_i	peak-to-peak period
θ	root-pitch rotation
U_∞	freestream airspeed
v	chordwise bending
V_F	linear flutter speed
w	flapwise bending
X	eigenvectors
y_ϕ	distance between the aerodynamic center and the elastic axis
ζ	damping ratio

Chapter 1: Introduction

1.1 Statement of the problem

Aeroelasticity is a multidisciplinary field that involves the aerodynamics, elasticity, and dynamics and their interactions. The classical linear theories were able to predict aeroelastic instability, i.e. flutter, through which the stable region of the aircraft's flight envelope may be defined. Nevertheless, some aircraft can experience another type of aeroelastic instabilities that cannot be described by the linear theories; such instabilities can also occur in regions of the flight envelope that were previously considered as stable using linear analysis. The advance in the technology of high-performance military aircraft and high aspect ratio UAV lead to highly flexible structures which require a nonlinear aeroelastic analysis. A typical nonlinear phenomenon observed in aircraft is Limit Cycle Oscillations (LCO), which is the object of the current investigation.

LCO, in the context of aeroelasticity, is characterized by sustained periodic oscillations for a given flight condition. The LCO is typically exhibited through a combined bending and torsional oscillation of the wing and its amplitude depends on the nature of the nonlinearity and flight conditions. One important application of LCO is in energy harvesting and wind turbine blade flutter.

One of the main concerns about the LCO is that they may occur at flight conditions below the flutter speed predicted by linear theory. The LCO phenomena is not yet fully understood and hence is difficult to predict accurately. Even when the amplitude of LCO are below the structural failure limits, its presence affects aircraft performance and airframe fatigue life. Depending on the amplitude and frequency involved, the LCO-induced motion may result in the pilot's inability to read cockpit displays or accomplish critical tasks related to the flight mission [2]. Finally, from

an economical point of view, extensive and costly aircraft flight tests for LCO clearance.

There is extensive documentation of flight tests with nonlinear aeroelastic effects. LCO has been observed on several fighter aircraft, such as F-16 [3, 4, 5] and F-5, F/A-18 [6]. Based on the phenomena observations in flight tests, Denegri [5] distinguishes three types of response: typical LCO, nontypical LCO, and classic flutter. For typical LCO, an increment in Mach number at constant altitude does not lead to a significant variation in the LCO amplitude. For nontypical LCO, the amplitude of oscillation first increases and then decreases with the Mach number changes.

Finally, in the classic flutter LCO, the increment of the LCO amplitude with the change in Mach number is so rapid that the safety of the aircraft is in jeopardy. Military specifications [7] have long emphasized 3% total damping as the minimum value acceptable within the speed limit envelope of military airplanes.

It has long known [5] that the addition of external stores to the aircraft affects the stability of the system [8]. Aircraft with different store configurations have persistently presented LCO. Bunton and Denegri [2] have described the characteristics of LCO observed in the F-16 and F/A-18 fighters with different store configurations. As noted by these authors, the linear analysis fails to predict the onset of LCO consistently. Therefore, the nonlinear analysis is necessary to predict LCO and their characteristics.

1.2 Literature review

The nonlinear aeroelasticity has been the subject of extensive investigation [9]. This section provides a brief literature review of aeroelasticity relevant to LCO.

In modeling LCO, either or both structural and aerodynamic models must be considered as nonlinear. Stated differently, LCO occurs due to the nonlinear interaction between the structure and the aerodynamic loads. Usually, only one nonlinear mechanism is assumed to be primarily responsible for the LCO; although the source of nonlinearity that is the dominant contributor to the phenomenon is not always known. Note that the presence of nonlinearity limits LCO amplitude, preventing catastrophic flutter and loss of the air vehicle.

In the context of nonlinear system theory [10], LCO is one of the simplest types of behaviour. The nonlinear aeroelastic response behaviours, however, are not limited to LCO alone. Other common possible behaviours include super and sub harmonic resonances, entrainment, etc. However, the detailed knowledge required to describe these behaviours are often unavailable. In addition, the LCO provides a general description for several nonlinear aeroelastic behaviours.

The physical sources of nonlinearities that lead to LCO have been identified by several authors [11] by using theoretical models and experimental tests in wind tunnel or flight tests. The nonlinearities may arise either from the aerodynamics or structural properties of the system. The most common source of aerodynamic nonlinearities is flow separation. This aerodynamic phenomenon can cause a nonlinear relationship between the structural motion and the aerodynamic pressure acting on the structure. Cunningham and Meijer [12] and Meijer and Cunningham [13] have shown that nonlinear aerodynamic forces originated by shock induced trailing edge separation can be a dominant mechanism in the LCO. Detailed discussions of these aerodynamic nonlinearities issues are available, for instance, in [14, 15, 16].

The aerodynamic nonlinearities are difficult to model theoretically and also to investigate empirically. Roughen et al. [17] have compared results of several linear and nonlinear aerodynamic models. The correlations between theory and experiment were not consistently achieved for all the different tests conditions.

The structural nonlinearities are also important for LCO. Free play and geometric nonlinearities are typical structural nonlinearities. Nonlinear damping forces in a structure may also cause the nonlinear effects. The impacts of nonlinear damping on flutter and LCO have not been yet fully understood, although dry friction damping has received some attention [18].

Dowell and Tang [19] investigated the influence of the free play nonlinearities on the dynamic behaviour of the aeroelastic system. Dowell found good correlation between theory and wind tunnel tests by using a model with free play nonlinearity in stiffness and an unsteady aerodynamic model.

O'Neil et al. [20] used a model with cubic nonlinearities in structural stiffness, and the results showed that LCO amplitudes increased with increasing the free stream airspeed and less time was required to reach the LCO steady state. On the other hand, the stability boundary was

shown to be insensitive to the variation in initial conditions.

Several authors [21, 22, 23] investigated wings with very high aspect ratio and both structural and aerodynamic nonlinearities. For such wings, the geometric nonlinearities, associated with large deflections in torsional and bending motions, and the aerodynamic nonlinearities, related to flow separation, have shown to be significant. Dowell and Tang [19] have shown that the wing stall causes the hysteretic LCO response between the small and large amplitude oscillations. Dowell and Tang [19] also obtained good correlation of theory and experiments by using a semi-empirical stall model.

Kim and Strganac [24] used geometrically nonlinear structural model developed by Crespo da Silva and Glynn [25] to investigate LCO induced by store kinematics. The model contains structural coupling terms and cubic nonlinearities due to the large deflections and store inertia, and nonlinear aerodynamics were introduced through a cubic stall model. Kim and Strganac [24] have found that depending on the airspeed the system can reach an LCO or the nominal static equilibrium.

Even though all the physical mechanisms responsible for LCO have not yet been fully well-known, substantial advance has been made in modeling nonlinear aeroelastic phenomena. The correlation between theoretical models and experiments is further advanced for nonlinear structural mechanism than for nonlinear aerodynamic effects. Even though, more precise and computationally efficient aerodynamic models are becoming available for better addressing these nonlinear aeroelastic phenomena; therefore the research know-how in this field is moving forward.

1.3 Scope of the thesis and outline

Based on the literature review and recent studies by the author's co-supervisor [26, 27, 28, 29] the follow research avenues are pursued in this thesis.

First, this investigation aims to move forward the experimental studies of LCO. A rectangular cantilever wing model mounted on top of a rotating base (with linear rotational stiffness) was used wind tunnel facility at Royal Military College, Kingston, Ontario. In addition to that,

the theoretical modeling of the structure and the aerodynamics of the system is carried out to compare the wind tunnel results. The experimental results were also compared to the previous 2D tests [29].

The mathematical formulation of a nonlinear structural model of the wing with a base rotation was developed by using the Hodges and Dowell [1] beam theory with geometric nonlinearities involving in-plane and out-of-plane bending, torsion, and root-pitch motion. Although the in-plane bending motion is negligible in traditional aspect ratio wings, it has to be considered in high aspect ratio ones. This model extends the Hodges and Dowell [1] formulation by including inertial and geometric nonlinearities due to the wing's base rotation. The structural and aeroelastic nonlinear partial differential equations (PDEs) are converted into a coupled set of ordinary differential equations (ODEs) by applying Galerkin method.

For simplicity, the aerodynamic model considered in this thesis assumes that the air flow is inviscid around the wing and rely upon the potential flow theory [30]. The steady and unsteady aerodynamic models are based on thin airfoil theory [30] and Wagner's function [31], respectively. In order to properly address the nonlinear aerodynamics (viscous effects, flow separation, large deflections, etc.), the full Navier-Stokes equations should be solved numerically using a Computational Fluid Dynamics (CFD) methods [31, 32, 33, 34]. The linear aerodynamic methods are chosen in the present work despite large deflections, as the primary goal is to focus on the structural nonlinearities. Also, a complex aerodynamic model would be too costly regarding computational time and is beyond the scope of this work.

Based on the above discussions, the following aspects are investigated in the thesis:

- Experimental investigation of LCO of a flexible cantilever wing with a base rotation having a linear elastic restraint.
- Development of a nonlinear structural model that includes in-plane and out-of-plane bending, torsion and root-pitch motion of the wing by adding geometric nonlinearities.
- Using this nonlinear structural model, the aeroelastic analysis is performed by using linear quasisteady and unsteady aerodynamic models at low subsonic flow.

- Verification of the time integration scheme utilized for the nonlinear aeroelastic analysis computation.

The layout of this thesis contains six chapters including this introduction. The chapter 2 contains the results of the experimental tests of a cantilever wing with base rotation conducted in the Royal Military College wind tunnel by the author. Particular attention is given to free vibration tests in still air and aeroelastic test with the focus in LCO. A detailed derivation of the nonlinear structural equations of motion of the system is provided in chapter 3 and the nonlinear structural effects due to the base rotation are briefly discussed. The chapter 4 presents comparative numerical tests between the Euler-Bernoulli linear beam theory and the formulation developed on chapter 3. The chapter 5 describes the linear quasisteady and unsteady aerodynamic models and the nonlinear aeroelastic analysis of the system. The chapter 6 concludes the thesis with remarks on experimental and numerical findings and future work suggestions.

Chapter 2: Experimental investigation

2.1 Introduction

This chapter presents the experimental investigation of a flexible cantilever wing model with a root-pitch motion that was tested at the Royal Military College (RMC) wind tunnel facility. The objective of the tests was to observe and evaluate the nonlinear aeroelastic behavior of the wing concerning LCO response, for both small and large amplitude oscillations. Since one of the purposes of this work is to investigate the effects of structural nonlinearities, the experimental program will also provide an insight into the structural nonlinearities of the wing in free vibration test. Wing dynamics in the self-sustained oscillation is also summarized in this chapter, and an investigation is done to evaluate the effect of the root-pitch stiffness and the position of the pitch axis in the system behaviour.

This study is a continuation of the investigation initiated by Poirel *et al.* [26, 27, 28, 29]. At first, the authors conducted experimental tests of a free-mounted rigid NACA0012 in pitch only, i.e. one dimensional, in the transitional range of Reynolds numbers ($Re \approx 10^5$). The authors provide an extensive discussion of complex nonlinear viscous aerodynamic phenomena in the low-to-moderate Reynolds number region; therefore it will not be repeated here. Small amplitude LCO in pitch only were observed and attributed to laminar boundary layer separation.

In the following phase, the investigation of Poirel and Mendes [29] was extended to two degrees of freedom by allowing pitch and heave motion of the same airfoil. Two types of LCO were observed, depending on the initial conditions and set of wing configuration: one with small amplitude oscillations and the other with large amplitude oscillations. The origin of the large amplitude oscillations was determined to be coalescence flutter, for which the amplitude

exponential growth is limited by flow separation at high angles of attack. Subsequent testing also exhibited large amplitude oscillations due to stall flutter. The small amplitude oscillations, in the other hand, were found to have the same laminar separation flutter type as the pitch only case [28]. In addition to that, the release of the heave motion did not present significant effect in the pitch oscillations in comparison with the single degree of freedom of the system. Therefore, the present investigation extends the previous experiments from one and two degrees of freedom to an infinite number of them, given the fact the wing is flexible in bending and torsion and has an additional rigid body rotation degree of freedom given by the base.

2.2 Experimental set up

Test apparatus

The test assembly apparatus is composed of a flexible cantilever wing model mounted on top of a rotational base and the RMC wind tunnel section. Fig. 2.1 shows the test specimen before the final assembly within the Wind tunnel section. As seen in the figure, the wing is mounted on top of a circular base shaft that allows root-pitch rigid body rotation in addition to the bending and torsion flexibility of the cantilever wing. The base enables the wing to experience relatively high angles of attack that are observed in the large amplitude LCO.

The wing model span is $s = 0.45m$ and its chord is $c = 0.15m$, thus, giving an aspect ratio of $AR = 3$. Fig. 2.2 shows some construction details of the model opened in half. The wing was designed for easy assembly in half for access to the stringer and the sensors inside the model. The model was built at RMC machine shop: the external shell was made of fibreglass-epoxy, the ribs were made using a 3D printer, and the core was filled with foam. A slender metallic beam was used as the model stringer and also the connector between the wing and the rotational base.

The chosen wing profile is the standard NACA0012 due to its well know characteristics. An abundance of published data exists for this profile at high angles of attack, high Reynolds number, and aspect ratios in steady conditions. Some work has been done in the airfoil unsteady behaviour, especially in laminar flow separation; however, low Reynolds number 3D unsteady

Figure 2.1: Wing model on top of the base shaft before assembly with the wind tunnel section

effects is a domain to be fully explored [26].

The test assembly was set up in such a way that the specimen was mounted in the vertical position to isolate its motion from the effect of gravity and to easily instrument the test apparatus. As shown in Fig. 2.3, the base support structure was fixed underneath the wind tunnel section in a way that only the wing model was exposed to the airflow in the final assembly. It is also possible to see in the figure that a thin string is attached to the tip of the model to remotely introduce some perturbation in the system in some of the tests.

Fig. 2.4 shows the bottom view of the wind tunnel section. The rotational base can spin about a fixed axis that is connected to a circular pulley. The pulley converts the wing rotational motion into spring elongation. The spring system attached to the pulley is responsible for the rigid restoring torque in the rigid body rotation motion. The springs were set up in a way they are always extended and, in the static equilibrium position, they set the system at zero angle of

Figure 2.2: Open view of the wing model shows constructive details

attack.

Instrumentation

A schematic view of the experimental setup is shown in Fig. 2.5. Regarding the instrumentation for the tests, the wing is equipped with accelerometers, strain gauges, and a potentiometer. The signals from the sensors were recorded via a National Instruments PCB-4803-c05 signal conditioner and then sent to a LABVIEW data acquisition system, where the signals were graphically visualized and recorded for each of the tests. The sampling rate was set at a nominal value of $1kHz$.

A set of six accelerometers were attached to the wing, see Fig. 2.2, two were located near the root, two in the mid-span and two near the tip of the stringer, to capture the different levels of bending motion along the span. The two accelerometers at each location were attached from

Figure 2.3: Side view of the test final assembly in the wind tunnel section

Figure 2.4: Bottom view of wind tunnel section and pulley-spring system

an equal distance from the center of the beam in order to capture the torsion motion effect.

The set of bending and torsion strain gauges attached to the stringer are shown in detail in Fig. 2.6. Two bending strain gauges were attached to each side of the stringer, near the root and parallel to the span since the strain in bending is maximum at this location and the principal

Figure 2.5: Schematic view of the experimental setup and instrumentation. (Figure removed due to copyrighting issue)

strain is in this direction. Other two torsion strain gauges were attached to the stringer near the root with opposite angles of 45 degrees with respect to the span because the torsion strain is maximum at this location and the principal strains are in these directions.

Underneath the wind tunnel section, see Fig. 2.4, there is a potentiometer installed to measure the rotation angle of the wing. The potentiometer voltage was calibrated to set the neutral position of pitch angle as zero.

Test specimens

Two different beams were used as the stringer of the wing model. The first one was an aluminum beam that was instrumented with the strain gauges and accelerometers. The second stringer was a steel beam instrumented only with accelerometers at the tip. Table 2.1 shows a summary of the nominal properties of both beams.

Figure 2.6: Detailed view of the strain gauges

For the first set of tests, the aluminum stringer was used for the experiments but due to a yielding of the specimen during one of the tests the aluminum beam had to be replaced by the steel one. The loss of the first stringer did not compromise the investigation of LCO.

A summary of the wing with steel string and rotational base structural and geometric properties is given in Appendix D.

Wind tunnel

The experiments were performed in a closed circuit low-speed wind tunnel at Royal Military College. The flow velocity is controlled by the fan speed and it ranges from 4 to 60 m/s, which in terms of Re_c is between 4×10^4 and 6×10^5 . The test section used in the tests measures

Table 2.1: Properties of the wing stringers

Property	Stringer 1	Stringer 2
Material	Aluminum alloy 5052	Steel AISI 01
Length	18"(45.72 cm)	18"(45.72 cm)
Thickness	1/64"(0.04 cm)	1/64" (0.04 cm)
Density	2.63g/cm ³	7.8734g/cm ³
Width	2" (5.08 cm)	2" (5.08 cm)
Young Modulus	70GPa	200Gpa

0.76 m×1.08 m, the section's height corresponded to 1.67 of the wing span. The gap between the wing root and the section was 5 mm, which is equivalent to one percent of the wingspan.

Test types

Two main types of tests were performed with the specimens: structural free vibration tests of the wing in still air and wing aeroelastic dynamics investigation tests. The free vibration tests were conducted to characterize and quantify the dynamics of the structure and its properties. At first, only the stringers were tested in free vibration, then the assembled wing with the tunnel off and last with airflow. Based on this test, the rigid rotation stiffness of the system K_θ and the structural damping were determined empirically.

The wing aeroelastic dynamics investigation tests were done by running the wing tunnel for different airspeeds and wing configurations and observing the behaviour of the system. The experiments were conducted to track the LCO. The LCO data sets were recorded in the steady state region with at least 40 cycles of oscillation.

Both types of test were conducted in different days to evaluate the repeatability of the observed dynamics. The temperature and local atmospheric pressure at the laboratory were also recorded. The baseline room temperature at the laboratory was 21.5°C with a variation of $\pm 0.6^\circ\text{C}$. The baseline atmospheric pressure at the lab was 1013 mbar with variation of -9 mbar to +2 mbar.

2.3 Free vibration tests

For the experimental free vibration tests, a peak-to-peak analysis of the time response signal was done to estimate the frequency and damping for each cycle of the oscillations. The peaks and troughs of the signal were identified first and then the decay and frequency of each cycle were calculated. Considering a peak at time t_i , the period of the cycle ahead of the current time can be calculated as:

$$T_i = t_{i+2} - t_i \quad (2.1)$$

where T_i is the period of the current cycle. Note that each period overlaps with the previous and next cycle.

With the period of the current cycle, it is possible to calculate the damped angular frequency ω_{d_i} and damped frequency f_{d_i} :

$$\omega_{d_i} = \frac{2\pi}{T_i} \quad f_{d_i} = \frac{\omega_{d_i}}{2\pi} \quad (2.2)$$

The damping ratio ζ_i was also calculated peak-to-peak by comparing the logarithmic decay of the signal value x_i at the beginning and end of the current cycle i as:

$$\zeta_i = \frac{1}{2\pi} \log \frac{|x_i|}{|x_{i+2}|} \quad (2.3)$$

Note that Eq. (2.3) assumes some values of damping ratio.

With the damping ratio, the natural angular frequency ω_i and the natural frequency of the cycle f_i can be calculated as

$$\omega_i = \frac{\omega_{d_i}}{\sqrt{1 - \zeta^2}} \quad f_i = \frac{\omega_i}{2\pi} \quad (2.4)$$

The subsequent sections present the experimental results of the free vibration tests.

2.3.1 System with aluminum stringer: bending-torsion free vibration tests

Beam free vibration test

The first experimental test was done using only the cantilever beam stringer, without the wing, as shown in Fig. 2.7. Even though the base was designed to allow root-pitch rotation, for this preliminary test the base rotation was restricted. Two different types of tests were conducted with the beam. The first type was a pushover static test to observe the behavior of the signal given by the sensors to the input of a known displacement in bending and torsion. The second type was a dynamic free vibration test in still air where a known displacement in bending and a known rotation in torsion were imposed to the tip of beam and then released.

Figure 2.7: Beam only free vibration test

Static pushover test

Fig. 2.8a shows an example of a bending pushover test where the beam tip was pushed one centimeter in one direction, pushed back to the non-deformed position and then pushed back one centimeter in the opposite direction. The same process was repeated with a two centimeters displacement. The voltage signal given by the bending strain gauges shows that for this range of tip displacement, the signal is linearly dependent on the tip displacement, as it can be seen in Fig. 2.8b. This observation is important to give confidence that any eventual nonlinearities of the response are not from the sensor.

(a) Beam only Pushover static test

(b) Beam static calibration

Figure 2.8: Beam bending pushover test

Cantilever beam free vibration test

The free vibration tests of the aluminum stringer were done restricting the base rotation, i.e., only allowing motion in bending and torsion. The tip of the beam was deflected and then released for several initial deflections. The free vibration of the beam was measured by the bending strain gauges, which had their signal converted in tip displacement based on the linear fit done in the pushover test. Fig. 2.9 shows a free vibration test of the cantilever beam that was released from an initial deflection of approximately 2cm. For these tests, the accurate values of initial displacement of the beam tip were not that important because we are mainly concerned about the frequency and damping of the structure cycle by cycle.

(a) Free vibration test time response with peak identification

(b) PSD of the time response

Figure 2.9: Free vibration test time and frequency response with initial beam tip displacement of 2 cm

The Fig. 2.9a shows all the local minimums and maximums of the signal were identified as peaks and the evolution of the amplitude was used to calculate the frequency and damping at each cycle.

The Power Spectral Density (PSD) of the signals decays were calculated using the first 2^{16} data points are used to perform the fast Fourier transforms which give a frequency resolution of 0.0153Hz . Fig. 2.9b shows the PSD of the signal, indicating a clear dominant frequency of 4.578Hz , which corresponds to the first natural frequency in bending of the beam. Considering the frequency resolution, then $f_b = 4.563\text{--}4.593\text{Hz}$. The second dominant frequency is shown in the PSD of the signal, of 17.34Hz , indicates the first natural frequency in torsion of the beam. The first bending and torsion natural frequencies found in the experimental test correspond to the analytical solution of the Euler-Bernoulli beam developed in chapter 4. The PSD of the signal is very useful to give an "average" of the main frequencies of the signal, although the peak-to-peak analysis showed that the frequencies slightly vary from cycle to cycle.

Fig. 2.10 shows the peak-to-peak frequencies and damping ratios of the time response showed in Fig. 2.9a. The frequency of the cycles are slightly higher than the mean natural bending frequency indicated by the PSD because the cycle by cycle analysis does not account for the contribution of the coupled torsion vibration.

A closer look of the damping ratio is shown in Fig. 2.11. Its initial time value at the beginning of the test is about 0.01 which decreases with time to about 0.0045 near the end of the test. The damping ratio seems to be larger when the bending deflections are higher. Which may be an indication of either nonlinear structural damping mechanism or the presence of a free play gap that is accentuated at higher deflections. In order to investigate if the deflection magnitudes can influence the frequency or damping the free vibration tests were compared for several different initial tip displacements.

The free vibration tests were done for initial tip displacements of $\pm 0.5\text{cm}$, 1cm and $\pm 2\text{cm}$. Fig. 2.12 presents the peak-to-peak natural frequency and damping ratio time histories for different free vibration tests. Once again, the frequencies did not present significant variation in time with different initial conditions. The damping ratio, shown in the top plot of Fig. 2.12, presents

Figure 2.10: Damped Frequencies and Damping ratio peak to peak with respect to time and amplitude for initial beam tip displacement approximately of 2 cm

higher initial damping ratios for larger initial displacements, most likely indicating structural nonlinearity. For all cases, the damping ratio seems to reach the same steady value of 0.0045, when the bending displacements are smaller.

The comparative amplitude decays and the averaged damping ratio given by the exponential fits are shown in Fig. 2.13. For small amplitude deflections, the mean damping ratio is about 0.0047, while for the large deflections the mean damping increases with amplitude and it should not be considered as constant. Is important to notice that the exponential fit presented higher quadratic errors and residuals for the higher initial displacement decays, which indicates a significant nonlinear decay. The nonlinear structural behavior could be due to some external effect like dry friction in the rotating pulleys or some free play between the wing and the rotating shaft. The peak-by-peak damping ratio also could vary due to other modes contribution in the large deflection region, turning in hard to calculate the real damping rate for the bending mode.

Figure 2.11: Damping ratio peak to peak with respect to time for initial beam tip displacement approximately of 2 cm

Figure 2.12: Natural frequencies peak to peak for several initial conditions

Wing model free vibration test

Similar free vibration tests were conducted for the wing model with the aluminum stringer. Fig. 2.14 shows the peak to peak damping ratio and natural frequency with respect to time for free vibration tests with initial tip displacement equal to $\pm 1\text{cm}$, $\pm 2\text{cm}$ and $\pm 3\text{cm}$. Differently

Figure 2.13: Amplitude Free decay and exponential fit damping ratio for several initial conditions

from the beam-only case, the damping ratio presented either a relatively constant value around 0.025. However, the uncertainty increases significantly in the small bending deflections region. This uncertainty in the small deflections region is most likely due to two possible effects: the measurement noise which starts being more relevant compared to smaller bending deflections or dry friction between the beam and the wing.

The wing natural frequencies are not constant as it was for the beam only case. They initially presented initial lower value (about 4.2 Hz) than the beam bending frequency. The frequencies then increase with time towards the natural frequency of the beam (4.578 Hz) near the end of the test. As for the damping ratio, the natural frequency also presented a high uncertainty at the end of the test for the same reasons stated before.

The PSD analysis of the signals for the first 6 seconds of the test is shown in Fig. 2.15. It gives a better insight of the frequencies behaviour. The free vibration responses presented four dominant frequencies: the first bending frequency and its correspondent second and third super harmonics, and the first torsion frequency. The presence of the super harmonics in the free

Figure 2.14: Damping and natural frequencies peak to peak with respect to time for several initial conditions

vibration tests may indicate that some nonlinear internal resonance is happening in the system. The main frequency in bending for the wing model indicated by the PSD graphs is 4.2 Hz, slightly smaller than the beam one. The torsion frequency changed from 17.34 Hz of the beam to 17.64 Hz for the wing model.

Figure 2.15: PSD analysis of the bending free vibration showing the higher harmonics

Wing model free vibration test: aerodynamic damping

Another set of free vibration tests was conducted to investigate if the vibrations are inherent to the wind tunnel operation could affect the specimen or the sensors. The free vibrations tests were done first with the wind tunnel all off, then only its rotor on and last with the wind tunnel fan on but no airflow. Fig. 2.16 shows the bending signals for each of the cases and Fig. 2.17 shows the natural frequency and damping with respect to time and bending signal. The systems were initially deflected by about 2 cm displacement of the time for the still air cases and by an unknown deflection given by pulling the string attached to the wing tip for the cases with air-speed. Both the time response and the frequency and damping analysis of the signal for these three cases presented very similar responses, indicating that the wind tunnel fan or the rotor did not introduce any significant noise vibration on the tests.

Fig. 2.16 and 2.17 also show the time response and frequency analysis for tests that were done with wind flow. These tests cannot be called as free vibration tests in the still air, but they were performed to give some insight into the effects of the wind tunnel vibration in the system frequencies and damping. It was possible to observe that the airspeed significantly increased both the natural frequency of the system as the damping ratios, as the aerodynamic stiffness and damping act on the aeroelastic system. However, is important to note here that no significant additional vibration was introduced in the system due to the wind tunnel operation.

2.3.2 System with Steel Stringer: bending-torsion-rotation free vibration tests

Wing model free vibration test

Similar free vibration tests were conducted for the wing with the steel stringer. The main difference with respect to the previous test is that the base shaft rotational motion was allowed in order to observe the bending, torsion and root-pitch motion in the free vibration tests in air still. The system stiffness in rigid body rotation K_θ depends on the spring-pulley system underneath the wind tunnel section. The system's stiffness in rigid body rotation was initially unknown and calculated based on its inertia and natural frequency in root-pitch vibration.

Figure 2.16: Natural frequencies with respect to bending signal for several Wind tunnel conditions

Figure 2.17: Natural frequencies with respect to bending signal for wind tunnel conditions

Bending-torsion-rotation free vibration test

Fig. 2.18 shows the time response in root-pitch angle and accelerometer signal of the free vibration test with bending, torsion and pitch rotation motion. The root-pitch angle time response clearly indicates the presence of multiple frequencies in the system. The PSD of the pitch

signal, shown in Fig. 2.19, reveals two dominant frequencies in bending ($f = 4.761Hz$) and rigid pitch rotation ($f' = 2.258Hz$). In addition to the main peaks at the fundamental frequencies f and f' , weaker even and odd super and subharmonic are display in Fig. 2.19.

Figure 2.18: Bending pitch free vibration test time response

Figure 2.19: Bending pitch free vibration test PSD

Based on the rigid pitch frequency the root-pitch stiffness of the system can be calculated

as:

$$K_{\theta} = (I_s + I_B)\omega^2 = 0.71 N.m/rad \quad (2.5)$$

where I_s is the inertia of the wing and I_B is the inertia of the base, given in the material properties Appendix D, and ω is the natural angular frequency $\omega = 2\pi f'$.

It is also important to notice that with the release of the base rotation the torsion motion is significantly decreased and the torsional frequency cannot be seen in the PSD graph. The torsional frequency was obtained by another test with the restrained base, and its value is 9.095 Hz.

Figure 2.20: PSD of the wing free vibration test

2.4 Wing aeroelastic dynamics: Limit Cycle Oscillations

The aeroelastic tests were conducted for several airspeeds and wing configurations. This particular investigation aims to investigate LCO. The effects of structural stiffness and the elastic axis position on the pitch axis were also investigated. Small and large amplitude LCO were

observed with the wing with the steel stringer. The results summary and discussions about the LCO experimental tests are presented in the section .

2.4.1 Wing dynamics

The baseline results presented in this thesis are the wing with the steel beam and the pitch axis coincident with the midpoint of the beam, i.e.its elastic axis at 25% of the chord. The pitch stiffness constant, K_θ is the same as the one used in the free vibration tests. More details about the model properties are provided in Appendix D. Small amplitude oscillations (SAO) and large amplitude oscillations (LAO) were observed depending on the airspeed and test configuration.

Fig. 2.21 shows video frames of LAO, evidencing the pitch and bending motions.

Figure 2.21: Video frames of LAO motion

Fig. 2.22 and Fig. 2.23 show evidence time traces of LCO transitioning from one dynamic behaviour to the other. In Fig. 2.22, the system was in a steady state LAO when the airspeed was smoothly decreased from 9.8 to 8.7m/s. The system responded by decreasing its amplitude smoothly until about 34 seconds when the amplitude decreases abruptly as the system jumped from an LAO to an SAO regime. At this point, the airspeed was maintained constant at 8.7 m/s.

Fig. 2.23 shows a time response in which the system was in a steady state SAO and the airspeed was abruptly increased from 6 to 10 m/s; due to enormous root-pitch angles (above 80 degrees) the airflow was also quickly lowered to zero. These two examples show how perturbations, or initial conditions, can abruptly affect the dynamics of the system. The boundary between the two dynamic behaviours will be more extensively discussed later in this chapter.

In general, the SAO are initiated by some natural symmetry breaking mechanisms in the flow, such as free stream turbulence or laminar boundary layer separation [29]. For the SAO in Fig. 2.23, for example, the LCO maintains a steady small amplitude until the abrupt airspeed increment triggers the LAO.

In real aircraft, the origin of the LAO is, in general, coalescence flutter between the bending and torsion modes for which the exponentially growing amplitude is limited by flow separation at large angles of attack (AOA). In this work, the origin of the LAO is coalescence flutter between the bending and torsion modes, which will be evidenced by the linear flutter analysis conducted in chapter 5.

It was also experimentally observed that restricting the root-pitch motion prevents the LAO to happen. In fact, the wing model with the aluminum stringer and restricted root-pitch motion fluttered in the same airspeed range, deforming the beam that had to be replaced by a steel one. For the wing with steel beam, root-pitch motion was released to avoid failing another stringer; and LCO were observed in place of linear divergence.

Fig. 2.24 shows a steady state pitch time trace of a SAO observed at an airspeed of 7.8 m/s . The amplitude of the signal is not constant but it oscillates within boundaries around 6 degrees. The phase space plot of this signal, shown in Fig. 2.25, is nearly elliptical, indicating a simple harmonic motion. The PSD of the signal, shown in Fig. 2.26, has a the dominant frequency of the system and smaller peak corresponding to the root-pitch rotation frequency found in the PSD of the wing free vibration test.

Another interesting steady behavior of SAO was observed at an airspeed of 4.3 m/s, shown

Figure 2.22: LCO time history for smoothly reduction of airspeed from $V = 9.8$ to 8.7 m/s

Figure 2.23: LCO time history for abruptly variation of airspeed from $V = 6$ to 10 m/s and back to 0 m/s

Figure 2.24: SAO LCO time response $V = 7.8\text{m/s}$

Figure 2.25: SAO LCO phase space plot $V = 7.8\text{m/s}$

Figure 2.26: SAO LCO PSD analysis $V = 7.8\text{m/s}$

in Fig. 2.27. More relative variability in the root-pitch angles is observed. The phase space plot of root-pitch angle signal is shown in Fig. 2.28. The phase trajectory shows an elliptical shape with larger boundaries. The PSD of the signal, shown in Fig. 2.29, also has the second and third super harmonic frequencies in addition to the system's dominant frequency, 3.8 Hz.

The amplitude and the frequency of the LCO were measured and registered for each of the airspeeds. With the probability density function (PDF) of the signal, the amplitudes were indicated by the peaks in Fig. 2.30. From an airspeed of 3.6 to 10.7 m/s, the small amplitude LCOs were observed and the frequencies and amplitudes are summarized as 'SAO' in Fig. 2.32. The amplitude grows with airspeed the behaviour.

The frequencies and amplitudes were again found by using the PSD and PDF of the LAO signal, respectively. Fig. 2.31 shows the PDF of the signals for several airspeeds and Fig. 2.32 shows the summary of the large amplitude oscillation behavior. The angles are much higher than

Figure 2.27: SAO LCO time response $V = 4.3\text{m/s}$

Figure 2.28: SAO LCO phase space plot $V = 4.3\text{m/s}$

Figure 2.29: SAO LCO PSD analysis $V = 4.3\text{m/s}$

Figure 2.30: LCO summary pitch axis at 25% of the chord

those in the SAO regime and abruptly increase with airspeed.

Fig. 2.32 indicates that the SAO are limited between airspeed of 3.6 and 10.8 m/s; they are not sustained below 3 m/s and they can jump to the LAO for airspeeds higher than 8.9 m/s. Another interesting observation in the diagram is that the frequencies of the LAO regime start from the same value of the SAO but it decreases with airspeed.

Figure 2.31: LCO summary pitch axis at 25% of the chord

Figure 2.32: LCO summary pitch axis at 25% of the chord

2.4.2 Effect of elastic axis position

The wing pitch axis was moved along the chord to observe how the LCO behaves the pitch and the elastic axes no longer coincide. The pitch axis was moved to 35% of chord and LCO were once again observed. The main difference between time LCO for pitch axis at 35% of the chord is that the frequency and amplitude of the system does not vary significantly with airspeed.

The Fig. 2.33 shows a typical time response for the system with the pitch axis at 35% of the chord at $V = 7.9\text{m/s}$ and Fig. 2.34 show its PSD. The signal has a main frequency close to the rigid body rotation frequency and a second peak corresponds to the second superharmonic.

Figure 2.33: LAO LCO time response $V = 7.9\text{m/s}$ pitch axis at 35% of the chord

Fig. 2.35 shows the summary of the LCO behaviour for the case of pitch axis at 35% of the chord. The angle amplitude presented a higher value, around 25 degrees, without significant variance with respect to airspeed. The main frequencies of the system correspond to the rigid body rotation frequency, also presented low variance with airspeed in comparison with the LCO shown in Fig. 2.32.

2.4.3 Effect of structural stiffness

In order to observe the LCO behaviour for different configurations of the system, another set of tests were done using a stiffer spring at the rotation of the base ($K_\theta = 1.51\text{N.m/rad}$). For the sake of comparison, the cases of pitch axis at 25% and 35% of the wing chord were

Figure 2.34: LAO LCO PSD analysis $V = 7.9\text{m/s}$ pitch axis at 35% of the chord

Figure 2.35: LCO summary pitch axis at 35% of the chord

conducted again for the system with the stiffer springs. The main difference between the new and the original system is that the rigid body rotation natural frequency changed from about 2.3 to 3.4 Hz.

The summary of the LCO frequencies and amplitudes for the case of the pitch axis at 25% of the chord is show in Fig. 2.36. The range of airspeeds for which the LCOs were observed changed from about 3.6 to 9.6 m/s in the system with the original spring to between 4.8 and 6.3 m/s. The range of airspeeds for the systems is not only smaller but the type of dynamic behaviour is also different, for the system with the stiffer spring the large amplitude oscillation behaviour presented a bigger range than the small amplitude one.

Figure 2.36: LCO summary pitch axis at 25% of the chord with the stiffer spring

Finally, a set of tests using the stiffer spring system and the pitch axis located at 35% of the wing chord was done and the results are summarized in Fig. 2.37. In comparison with the original system with pitch axis at 35%, the root-pitch angles of the system with the stiffer spring presented an SAO behaviour. The frequencies presented lower values than the ones of the system free vibration tests and not a lot of variability with the airspeed range.

Figure 2.37: LCO summary pitch axis at 35% of the chord with the stiffer spring

2.4.4 Summary of findings

The results of the experimental program show that a flexible wing mounted on an elastically restrained base is capable of achieving stable limit cycle oscillations (LCO). This nonlinear phenomenon occurs mainly due to two possible sources of nonlinearities: the structure and the aerodynamics. The wing exhibited some degree of structural nonlinearity in the free vibration tests, evidenced by the damping and frequency dependency on the states displacements. The damping nonlinearity could be attributed to dry friction between the wing and the beam. Furthermore, an evidence of structural nonlinear restoring forces is found in the power spectral density (PSD) of the LCO time traces, in which superharmonic frequencies are present.

The aerodynamic nonlinearities arise from two main causes. Firstly due to a high angle of attack, which induces flow separation, and due to the complex viscous phenomena of the flow around the wing in the low-to-moderate Reynolds number for which the tests were conducted. At this range of Reynolds number, the relationship between the lift coefficient and the angle of attacks is highly nonlinear well below the stall angle [26].

Two types of LCO have been observed and described as small and large amplitude, respectively, with both oscillating around zero degrees pitch angle and dominated by the pitch mode frequency. Furthermore, the small amplitude oscillations have a strong dependency on the airspeed, whereas the large amplitude limit cycle has minimal airspeed dependency. For the same airspeed, the initiation of either one of the two oscillations depends on the initial conditions, which indicates strong nonlinear effect.

The large amplitude limit cycle can be triggered by either coalescence flutter or stall flutter, the former limit the amplitude growth causing the LCO and the latter triggers the amplitude growth to reach the LCO steady state regime. The large amplitude oscillations of the considered wing originate from coalescence flutter instability, as predicted by inviscid unsteady thin airfoil theory to occur between bending and pitch modes.

The small amplitude oscillations are related to transitional Reynolds number aerodynamics. It is hypothesized by Poirel et al. [26] that the small amplitude LCO is originated from a nonlinear coupling between the wing pitch and the viscous flow effects. It is speculated that the flow viscosity at the low-to-moderate Reynolds number can induce a laminar trailing edge separation and a laminar separation bubble, which have a dominant role in the small amplitude nonlinearities. It is suggested that the small amplitude oscillations are somehow particularly initiated by laminar trailing edge separation.

The small amplitude oscillations herein observed are the same type as those previously reported in pitch only [26] and pitch-heave rigid model [29]. The laminar separation type of flutter is not significantly affected by the additional flexible degrees of freedom. Besides, the large amplitude oscillations are originated from a coalescence flutter that occurs between the pitch and bending modes for the flexible wing and between pitch and heave for the rigid model.

Chapter 3: Nonlinear equations of motion for the elastic bending-torsion of uniform airfoils with root-pitch motion

3.1 Introduction

The derivation of the equations of motion for a cantilever beam follows the method used by Hodges and Dowell [1], extended to consider root pitch rotation. For simplicity, the elastic energy is described by Euler-Bernoulli theory and the warping effects are omitted. Hamilton's principle is used in this chapter to derive the equations of motion and boundary conditions. Coordinate system transformations between undeformed and deformed frames and the strain-displacements are defined to apply this principle. Then the potential energy and kinetic energy of the system are obtained.

Consider a system composed of an inextensible cantilever beam, with flapwise and torsional degrees of freedom, mounted on a rotating base. The beam is subjected to aerodynamic loads that are assumed to be transmitted from the airfoil. The system does not include the influence of chordwise bending or warping. These simplifications are justified because the focus of the thesis is to analyze the effect of the root-pitch motion for nonlinear aeroelastic interaction in the cantilever wing inducing LCO. Thus, assumes that nonlinear interactions betww

To avoid additional complications of the equations of motion, relatively small terms (high-order) are neglected. This is conducted in a systematic and consistent way by using an ordering scheme, introducing $\epsilon \ll 1$ as a bookkeeping non-dimensional parameter to denote the smallness of terms. Unless otherwise specified, the ordering scheme is applied in such a way that terms of order higher than three are neglected.

3.2 Coordinate systems and transformations

In the present analysis, several coordinate systems will be used for the derivation of the equations of motion. The wing is initially straight, rectangular and symmetric of length L and mass per unit length m and the undeformed coordinate systems are shown in Fig. 3.1 and 3.2. The orthogonal coordinate system (X, Y, Z) is inertially fixed at frame N . The orthogonal axes (X, y_p, z_p) are fixed in the reference frame E , which can rotate with respect to N by the rigid base rotation angle θ about the pitch axis PA . The plane containing Y, y_p, Z, z_p is called reference plane. The orthogonal coordinate system (x, y, z) contains the x axis lying on the elastic axis of the undeformed beam and it also fixed in the reference frame E .

Fig. 3.3 shows the deformation of the wing that can be described by the displacements at each cross section of the elastic axis u, v, w , in the x, y, z directions, respectively. A point originally located in the undeformed elastic axis at $(x, 0, 0)$ in the (x, y, z) coordinate system is displaced to $(x + u, v, w)$ after the deformation.

The wing cross section after the deformation is shown in Fig. 3.4. The principal axes (η, ζ) of the cross section are initially aligned with (y, z) before deformation. When the wing is deformed, the shear center of a cross section located at x is displaced by v in y direction and w in z direction. The twist angle of the cross section changes from zero to ϕ about the x' axis, shown in Fig. 3.4. The orthogonal system (x', y', z') is associated to the deformed wing.

Several coordinate transformations are used in this analysis. The most simple one is from (X, Y, Z) to (X, y_p, z_p) for the rigid body rotation of the wing base with respect to the pitch axis

Figure 3.1: Wing view with undeformed coordinate systems

Figure 3.2: Cross section view of the root pitch rigid rotation (adapted from [1]). Here CG , PC , EA , AC denote center of gravity, pitch center, elastic axis, aerodynamic center, respectively.

Figure 3.3: Elastic displacements relative to the pitch base (adapted from [1])

Figure 3.4: Cross-section coordinates after deformation (Note: y', z' are not in the same plane as y, z . Their projections are shown in the y, z plane)

[1].

$$\begin{aligned}
 \vec{i}' &= (1 - \frac{1}{2}v'^2 - \frac{1}{2}w'^2)\vec{i} + v'\vec{j} + w'\vec{k} \\
 \vec{j}' &= -(v'\cos\phi + w'\sin\phi)\vec{i} + (1 - \frac{1}{2}v'^2)\cos\phi\vec{j} + (1 - \frac{1}{2}w'^2)\sin\phi\vec{k} \\
 \vec{k}' &= (v'\sin\phi - w'\cos\phi)\vec{i} - (1 - \frac{1}{2}v'^2)\sin\phi\vec{j} + (1 - \frac{1}{2}w'^2)\cos\phi\vec{k}
 \end{aligned} \tag{3.1}$$

The transformation between the deformed axes $(x', y', z') = (\epsilon, \eta, \zeta)$ and the undeformed axes (x, y, z) is represented by matrix $[T]$ and its derivation in terms of displacements is shown in Appendix A.

3.3 Strain-displacement relation

Fig. 3.5 shows the point P is initially located at an arbitrary position $\vec{r}_0$, at a distance x from the root with respect to the undeformed local coordinate system (x, y, z) . The point P moves to $\vec{r}_1$ after deformation. The deformed and undeformed positions can be given in terms of the undeformed system coordinates by using the transformation matrix $[T]$

$$\vec{r}_1 = [\vec{i} \quad \vec{j} \quad \vec{k}] \left[\begin{array}{c} x \\ v \\ w \end{array} \right] + [T]^T \left[\begin{array}{c} 0 \\ \eta \\ \zeta \end{array} \right], \tag{3.2}$$

where $[T]$ is the transformation matrix, given in the appendix A, that relates the deformed axes $(\vec{i}', \vec{j}', \vec{k}')$ and the undeformed axes $(\vec{i}, \vec{j}, \vec{k})$.

Figure 3.5: Deformed and undeformed position vector (adapted from [1])

Before deformation,

$$\vec{r}_0 = \vec{r}_1|_{w=\phi=0} = [\vec{i} \ \vec{j} \ \vec{k}] \left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} \right\} + [T]^T|_{w=\phi=0} \left\{ \begin{bmatrix} 0 \\ \eta_0 \\ \zeta_0 \end{bmatrix} \right\} = [\vec{i} \ \vec{j} \ \vec{k}] \left\{ \begin{bmatrix} x \\ \eta_0 \\ \zeta_0 \end{bmatrix} \right\}, \quad (3.3)$$

where

$$\eta_0 = \eta|_{w=\phi=0} \quad \zeta_0 = \zeta|_{w=\phi=0}. \quad (3.4)$$

The Green's strain tensor ε_{ij} is given in terms of $\vec{r}_1$ and $\vec{r}_0$ as

$$d\vec{r}_1 \cdot d\vec{r}_1 - d\vec{r}_0 \cdot d\vec{r}_0 = 2 \begin{bmatrix} dr & d\eta & d\zeta \end{bmatrix} [\varepsilon_{ij}] \begin{bmatrix} dr \\ d\eta \\ d\zeta \end{bmatrix}, \quad (3.5)$$

where dr , $d\eta$, and $d\zeta$ are the increments along the deformed axes x' , y' , and z' , respectively.

The variation of the position vector is given by:

$$d\vec{r}_0 = [\vec{i} \quad \vec{j} \quad \vec{k}] \begin{Bmatrix} \frac{dx}{dr} dr \\ \frac{d\eta_0}{d\eta} d\eta \\ \frac{d\zeta_0}{d\zeta} d\zeta \end{Bmatrix}, \quad (3.6)$$

$$d\vec{r}_1 = [\vec{i}' \quad \vec{j}' \quad \vec{k}'] [T] \begin{Bmatrix} \frac{dx}{dr} \\ \frac{dv}{dr} \\ \frac{dw}{dr} \end{Bmatrix} + \begin{bmatrix} 0 & -\omega_k & \omega_j \\ \omega_k & 0 & -\omega_i \\ -\omega_j & \omega_i & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ \eta \\ \zeta \end{Bmatrix} dr + \begin{Bmatrix} 0 \\ d\eta \\ d\zeta \end{Bmatrix}, \quad (3.7)$$

and

$$[T][T]^{T'} = -[T]'[T]^T = \begin{bmatrix} 0 & -\omega_k & \omega_j \\ \omega_k & 0 & -\omega_i \\ -\omega_j & \omega_i & 0 \end{bmatrix}. \quad (3.8)$$

The explicit expressions the angular velocities $\omega_i, \omega_j, \omega_k$ are given in the Appendix A.

Eq. (3.6) and Eq. (3.7) can be substituted into Eq.(3.5) to solve for the components of the strain tensor ε_{ij} . Assuming the beam is slender, $\sigma_{22} = \sigma_{33} = \sigma_{23} = 0$, where σ_{ij} is the stress tensor, and Hooke's law gives $\varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11}$, where ν is Poisson's ratio. By reducing the strain components to second order, it can be shown ([1]) that $\eta = \eta_0$ and $\zeta = \zeta_0$ and the elements of the strain tensor are

$$\begin{aligned} \varepsilon_{11} &= -v''(\eta \cos \phi - \zeta \sin \phi) - w''(\eta \sin \phi + \zeta \cos \phi) \\ \varepsilon_{12} &= -\frac{1}{2}\zeta\phi' \\ \varepsilon_{13} &= \frac{1}{2}\eta\phi'. \end{aligned} \quad (3.9)$$

The relationship between the Green's strain tensor and engineering strain can be written as:

$$\begin{aligned}
\varepsilon_{xx} = \varepsilon_{11} &= -v''(\eta \cos \phi - \zeta \sin \phi) - w''(\eta \sin \phi + \zeta \cos \phi) \\
\varepsilon_{x\eta} = 2\varepsilon_{12} &= -\zeta \phi' \\
\varepsilon_{x\zeta} = 2\varepsilon_{13} &= \eta \phi'.
\end{aligned} \tag{3.10}$$

3.4 Kinematics

Using the transformation matrix $[T]$, the position of an arbitrary point of the beam after deformation, with respect to the rotating frame, can be given by $\vec{r} = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}$, where

$$\begin{aligned}
x_1 &= x - v'(\eta \cos \phi - \zeta \sin \phi) - w'(\eta \sin \phi + \zeta \cos \phi) = x - v'(y_1 - v) - w'(z_1 - w) \\
y_1 &= v + \eta \cos \phi - \zeta \sin \phi = v + (y_1 - v) \\
z_1 &= w + \eta \sin \phi + \zeta \cos \phi = w + (z_1 - w).
\end{aligned} \tag{3.11}$$

In order to find the velocity $\vec{V}$ of the point P with respect to the inertial frame, the rigid body rotation with respect to the base must also be included as

$$\vec{V} = \dot{\vec{r}} + \dot{\theta} \vec{I} \times \vec{r}. \tag{3.12}$$

Therefore, the total velocity is given by

$$\vec{V} = \dot{x}_1 \vec{i} + (\dot{y}_1 - \dot{\theta} z_1) \vec{j} + (\dot{z}_1 + \dot{\theta} y_1) \vec{k}. \tag{3.13}$$

In order to find the kinetic energy of the wing, the square of the velocity will be calculated as

$$\vec{V} \cdot \vec{V} = \dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2 + 2\dot{\theta}(\dot{y}_1 z_1 - \dot{z}_1 y_1) + \dot{\theta}^2(y_1^2 + z_1^2). \tag{3.14}$$

3.5 Hamilton's principle

The equations of motion and the boundary conditions for a uniform flexible wing with rigid base rotation are obtained by the extended Hamilton's principle that can be expressed as:

$$\int_{t_1}^{t_2} (\delta(U - T) + \delta W) dt = 0, \quad (3.15)$$

where U is the strain energy of the system, T is the kinetic energy, and δW is the virtual work done by the external forces.

Due to the rotational stiffness and inertia of the base, the kinetic and strain energy functions can be divided in two parts, one corresponding to the wing and another to the base, which are expressed with subscripts W and B respectively.

3.5.1 Strain energy contributions

The expression of the strain energy of the beam in terms of engineering stresses and strains is

$$U_W = \int_0^L \iint_A (\sigma_{xx} \varepsilon_{xx} + \sigma_{x\eta} \varepsilon_{x\eta} + \sigma_{x\zeta} \varepsilon_{x\zeta}) d\eta d\zeta dx, \quad (3.16)$$

where

$$\begin{aligned} \sigma_{xx} &= E \varepsilon_{xx} \\ \sigma_{x\eta} &= G \varepsilon_{x\eta} \\ \sigma_{x\zeta} &= G \varepsilon_{x\zeta}. \end{aligned} \quad (3.17)$$

The variation of the strain energy is

$$\delta U_W = \int_0^L \iint_A (\sigma_{xx} \delta \varepsilon_{xx} + \sigma_{x\eta} \delta \varepsilon_{x\eta} + \sigma_{x\zeta} \delta \varepsilon_{x\zeta}) d\eta d\zeta dx \quad (3.18)$$

where

$$\begin{aligned} \delta \varepsilon_{xx} &= -(\eta \sin \phi + \zeta \cos \phi)(\delta w'' - v'' \delta \phi) - (\eta \cos \phi - \zeta \sin \phi)(\delta v'' + w'' \delta \phi) \\ \delta \varepsilon_{x\eta} &= -\zeta \delta \phi' \\ \delta \varepsilon_{x\zeta} &= \eta \delta \phi'. \end{aligned} \quad (3.19)$$

Substituting the terms of Eq. (3.17) and Eq. (3.19) into Eq. (3.18) and regrouping them

in terms of shear stress S'_x and bending moments $M_{y'}$ and $M_{z'}$ about y' and z' , respectively, the strain energy variation can be rewritten as it is done in [1]

$$\delta U_W = \int_0^L \{S_{x'} \delta \phi' + (M_{z'} \cos \phi + M_{y'} \sin \phi)(\delta v'' + w'' \delta \phi) + (M_{z'} \sin \phi - M_{y'} \cos \phi)(\delta w'' - v'' \delta \phi)\} dx, \quad (3.20)$$

where the stress resultants and moments are defined as

$$\begin{aligned} S_{x'} &= \iint_A (\eta \sigma_{x\zeta} - \zeta \sigma_{x\eta}) d\eta d\zeta = GJ\phi' \\ M_{y'} &= \iint_A (\zeta \sigma_{xx}) d\eta d\zeta = EI_{y'}(v'' \sin \phi - w'' \cos \phi) \\ M_{z'} &= - \iint_A (\eta \sigma_{xx}) d\eta d\zeta = EI_{z'}(v'' \cos \phi + w'' \sin \phi). \end{aligned} \quad (3.21)$$

The section integrals used in (3.21) are defined as

$$\begin{aligned} J &\equiv \iint_A (\eta^2 + \zeta^2) d\eta d\zeta \\ I_{y'} &= \iint_A (\zeta^2) d\eta d\zeta \\ I_{z'} &= \iint_A (\eta^2) d\eta d\zeta, \end{aligned} \quad (3.22)$$

where J is the polar moment of inertia, and $I_{y'}$ and $I_{z'}$ are the flapwise and chordwise moments of inertia, respectively.

Integrating the strain energy by parts yields

$$\delta U_W = \int_0^L \{\bar{Y}_v \delta v + \bar{Y}_w \delta w + \bar{Y}_\phi \delta \phi\} dx + b(U_W), \quad (3.23)$$

where

$$\begin{aligned}
 \bar{Y}_v &= [M_{z'} \cos \phi + M_{y'} \sin \phi]'' = [(EI_{z'} - EI_{y'})w'' \sin \phi \cos \phi + (EI_{z'} \cos^2 \phi + EI_{y'} \sin^2 \phi)v'']'' \\
 \bar{Y}_w &= [M_{z'} \sin \phi - M_{y'} \cos \phi]'' = [(EI_{z'} - EI_{y'})v'' \sin \phi \cos \phi + (EI_{z'} \sin^2 \phi + EI_{y'} \cos^2 \phi)w'']'' \\
 \bar{Y}_\phi &= S'_{x'} - v''[M_{z'} \sin \phi + M_{y'} \cos \phi] + w''[M_{z'} \cos \phi + M_{y'} \sin \phi] \\
 &= -GJ\phi'' + (EI_{z'} - EI_{y'})[(w''^2 - v''^2) \sin \phi \cos \phi + v''w'' \cos 2\phi],
 \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
 b(U_W) &= [M_{z'} \cos \phi + M_{y'} \sin \phi] \delta v'|_0^L - [M_{z'} \cos \phi + M_{y'} \sin \phi]' \delta v|_0^L \\
 &\quad + [M_{z'} \sin \phi - M_{y'} \cos \phi] \delta w'|_0^L - [M_{z'} \sin \phi - M_{y'} \cos \phi]' \delta w|_0^L + S_{x'} \delta \phi|_0^L,
 \end{aligned} \tag{3.25}$$

where $b(U_W)$ are the boundary conditions of the beam.

The strain energy associated to the rotational stiffness of the base also needs to be taken into account and this energy can be written in terms of the torsional stiffness of the base K_θ as

$$U_B = \frac{K_\theta \theta^2}{2} \tag{3.26}$$

The variation of the strain energy of the spring is

$$\delta U_B = K_\theta \theta \delta \theta \tag{3.27}$$

3.5.2 Kinetic energy contributions

The expression of the kinetic energy of the beam can be given as

$$T_W = \frac{1}{2} \int_0^L \iint_A \rho (\vec{V} \cdot \vec{V}) d\eta d\zeta dx. \tag{3.28}$$

The variation of the kinetic energy is given by:

$$\delta T_W = \int_0^L \iint_A \rho (\vec{V} \cdot \delta \vec{V}) d\eta d\zeta dx, \tag{3.29}$$

where, from Eq. (3.13),

$$\delta \vec{V} = \delta \dot{x}_1 \vec{i} + (\delta \dot{y}_1 - z_1 \delta \dot{\theta} - \dot{\theta} \delta z_1) \vec{j} + (\delta \dot{z}_1 + y_1 \delta \dot{\theta} + \dot{\theta} \delta y_1) \vec{k}. \quad (3.30)$$

Thus, the integrand of Eq.(3.29) becomes

$$\begin{aligned} \vec{V} \cdot \delta \vec{V} = & \dot{x}_1 \delta \dot{x}_1 + (\dot{y}_1 - \dot{\theta} z_1) \delta \dot{y}_1 + (\dot{z}_1 + \dot{\theta} y_1) \delta \dot{z}_1 + (\dot{\theta} \dot{z}_1 + \dot{\theta}^2 y_1) \delta y_1 + (-\dot{\theta} \dot{y}_1 + \dot{\theta}^2 z_1) \delta z_1 \\ & + (-\dot{y}_1 z_1 + \dot{\theta} z_1^2 + \dot{z}_1 y_1 + \dot{\theta} y_1^2) \delta \dot{\theta} \end{aligned} \quad (3.31)$$

Eq. (3.31) can be integrated in time between two arbitrary instants t_1 and t_2 and the initial and final values (e.g., $\dot{y}_1 \delta \dot{y}_1|_{t_1}^{t_2}$) are taken as zero. Therefore, integrating the terms of (3.31) by parts in the expression of the kinetic energy one has

$$\begin{aligned} \delta T_W = & \frac{1}{2} \int_0^L \iint_A [-\ddot{x}_1 \delta x_1 + (-\ddot{y}_1 + \ddot{\theta} z_1 + 2\dot{\theta} \dot{z}_1 + \dot{\theta}^2 y_1) \delta y_1 + (-\ddot{z}_1 - \ddot{\theta} y_1 - 2\dot{\theta} \dot{y}_1 + \dot{\theta}^2 z_1) \delta z_1 \\ & + (\ddot{y}_1 z_1 - \ddot{\theta} z_1^2 - 2\dot{\theta} z_1 \dot{z}_1 - \ddot{z}_1 y_1 - \ddot{\theta} y_1^2 - 2\dot{\theta} y_1 \dot{y}_1) \delta \theta] d\eta d\zeta dx, \end{aligned} \quad (3.32)$$

where

$$\begin{aligned} \dot{x}_1 &= -(\dot{v}' - \dot{\phi} w')(y_1 - v) - (\dot{w}' - \dot{\phi} v')(z_1 - w) \\ \dot{y}_1 &= \dot{v} - \dot{\phi}(z_1 - w) \\ \dot{z}_1 &= \dot{w} + \dot{\phi}(y_1 - v) \\ \ddot{x}_1 &\cong -(\ddot{v}' + \ddot{\phi} w' + 2\dot{\phi} \dot{w}')(y_1 - v) - (\ddot{w}' - \ddot{\phi} v' - 2\dot{\phi} \dot{v}')(z_1 - w) \\ \ddot{y}_1 &\cong \ddot{v} - \ddot{\phi}(z_1 - w) \\ \ddot{z}_1 &\cong \ddot{w} + \ddot{\phi} y_1 \\ \delta x_1 &= -(\delta v' + w' \delta \phi)(y_1 - v) - (\delta w' - v' \delta \phi)(z_1 - w) \\ \delta y_1 &= \delta v - (z_1 - w) \delta \phi \\ \delta z_1 &= \delta w + (y_1 - v) \delta \phi. \end{aligned} \quad (3.33)$$

These equations have been approximated consistently with the ordering scheme [1].

Regrouping and consistently eliminating the terms according to the second order scheme, the variation of the kinetic energy of the beam becomes

$$\delta T_W = \int_0^L \{Z_v \delta v + Z_w \delta w + Z_\phi \delta \phi\} dx + Z_\theta \delta \theta, \quad (3.34)$$

where

$$\begin{aligned} Z_v &= -\ddot{v} + \ddot{\phi}(z_1 - w) + \ddot{\theta}z_1 + \dot{\theta}^2 y_1 + 2\dot{\theta}(\dot{w} + \dot{\phi}(y_1 - v)) \\ Z_w &= -\ddot{w} - \ddot{\phi}(y_1 - v) - \ddot{\theta}y_1 + \dot{\theta}^2 z_1 - 2\dot{\theta}(\dot{v} - \dot{\phi}(z_1 - w)) \\ Z_\phi &= \ddot{v}(z_1 - w) - \ddot{w}(y_1 - v) - \ddot{\phi}[(y_1 - v)^2 + (z_1 - w)^2] - \ddot{\theta}[y_1(y_1 - v) + z_1(z_1 - w)] \\ &\quad - 2\dot{\theta}[\dot{v}(y_1 - v) + \dot{w}(z_1 - w)] - \dot{\theta}^2[y_1(z_1 - w) - z_1(y_1 - v)] \\ Z_\theta &= \int_0^L \{\ddot{v}z_1 - \ddot{w}y_1 - [y_1(y_1 - v) + z_1(z_1 - w)]\ddot{\phi} - \ddot{\theta}(y_1^2 + z_1^2) - 2\dot{\theta}\phi[z_1(y_1 - v) - y_1(z_1 - w)] \\ &\quad - 2\dot{\theta}\dot{w}z_1 - 2\dot{\theta}\dot{v}y_1\} dx \end{aligned} \quad (3.35)$$

Integrating over the cross section of the beam

$$\delta T_W = \int_0^L \{\bar{Z}_v \delta v + \bar{Z}_w \delta w + \bar{Z}_\phi \delta \phi\} dx + \bar{Z}_\theta \delta \theta, \quad (3.36)$$

where

$$\begin{aligned}
\bar{Z}_v &= -m\ddot{v} - me\ddot{\phi} \sin \phi + m\ddot{\theta}(\dot{w} + e \sin \phi) + m\dot{\theta}^2(v + e \cos \phi) + 2m\dot{\theta}(\dot{w} + e \cos \phi \dot{\phi}) \\
\bar{Z}_w &= -m\ddot{w} - me\ddot{\phi} \cos \phi - m\ddot{\theta}(v + e \cos \phi) + m\dot{\theta}^2(\dot{w} + e \sin \phi) - 2m\dot{\theta}(\dot{v} + e \sin \phi \dot{\phi}) \\
\bar{Z}_\phi &= me(\ddot{v} \sin \phi - \ddot{w} \cos \phi) - mK_m^2 \ddot{\phi} - m\ddot{\theta}(K_m^2 + e\dot{w} \sin \phi) - 2me\dot{\theta}(\dot{v} \cos \phi + \dot{w} \sin \phi) \\
&\quad - me\dot{\theta}^2(v \sin \phi - w \cos \phi) \\
\bar{Z}_\theta &= \int_0^L \{m\ddot{v}(\dot{w} + e \sin \phi) - m\ddot{w}(v + e \cos \phi) - m\ddot{\phi}(K_m^2 + e\dot{w} \sin \phi) - m\dot{\theta}[(v^2 + w^2 + K_m^2 \\
&\quad + 2e(v \cos \phi + w \sin \phi))] - 2me\dot{\theta}\dot{\phi}(\dot{w} \cos \phi - v \sin \phi) - 2m\dot{\theta}\dot{w}(\dot{w} + e \sin \phi) \\
&\quad - 2m\dot{\theta}\dot{v}(v + e \cos \phi)\} dx,
\end{aligned} \tag{3.37}$$

and the sectional integrals are defined by

$$\begin{aligned}
m &\equiv \iint_A \rho d\eta d\zeta \\
me &\equiv \iint_A \rho \eta d\eta d\zeta \\
mK_{m_1} &\equiv \iint_A \rho \zeta^2 d\eta d\zeta \\
mK_{m_2} &\equiv \iint_A \rho \eta^2 d\eta d\zeta \\
mK_m^2 &\equiv \iint_A \rho (\eta^2 + \zeta^2) d\eta d\zeta \\
\iint_A \rho \zeta d\eta d\zeta &= \iint_A \rho \eta \zeta d\eta d\zeta = 0
\end{aligned} \tag{3.38}$$

The kinetic energy due to the rigid body rotation of the base can be given in terms of the inertia of the base I_B as

$$T_B = \frac{I_B \dot{\theta}^2}{2} \tag{3.39}$$

The variation of the kinetic energy of the beam is

$$\delta T_B = I_B \dot{\theta} \delta \dot{\theta} \tag{3.40}$$

The term of Eq. 3.40 will also be integrated in Eq. Hamilton. Integrating by parts between two arbitrary instants t_1 and t_2 and taking the initial and final values as zero (i.e., $\dot{\theta}\delta|_{t_1}^{t_2} = 0$) one has.

$$\delta T_B = -I_B \ddot{\theta} \delta \theta \quad (3.41)$$

3.5.3 Generalized nonconservative forces contributions

The virtual work done by the nonconservative forces may be expressed by

$$\delta W = \int_0^L \{L_v \delta v + L_w \delta w + M_\phi (\delta \phi + \delta \theta)\} dx, \quad (3.42)$$

where L_w and M_ϕ are the distributed loads that act in the z direction and the twisting moment about the elastic axis, respectively. For the scope of this thesis the nonconservative forces are the aerodynamic ones.

3.6 Nonlinear bending-bending-torsion equations of motion

According to Eq. (3.15) of the Hamilton Principle, the strain energy in (3.23) and (3.27), the kinetic energy in Eq. (3.36) and Eq. (3.41), and the work done by the nonconservative forces in Eq. (3.42) need to be added in order to obtain the total variational equation in terms of v , w , ϕ , and θ . The terms δv , δw , $\delta \phi$ and $\delta \theta$ can be dropped in the integrand for arbitrary admissible variations. For the space-dependent variational terms δw and $\delta \phi$ the integral evaluation from 0 to L can also be dropped. The equations are assembled as follows:

δv equation:

$$\begin{aligned} & (EI_{z'} - EI_{y'})(w'' \sin \phi \cos \phi)'' + EI_{z'}(v'' \cos^2 \phi)'' + EI_{y'}(v'' \sin^2 \phi)'' \\ & + m\ddot{v} + me\ddot{\phi} \sin \phi - \underline{m\ddot{\theta}(w + e \sin \phi)} - \underline{m\dot{\theta}^2(v + e \cos \phi)} - \underline{2m\dot{\theta}(\dot{w} + e \cos \phi \dot{\phi})} = L_v \end{aligned} \quad (3.43)$$

δw equation:

$$(EI_{z'} - EI_{y'})(v'' \sin \phi \cos \phi)'' + EI_{z'}(w'' \sin^2 \phi)'' + EI_{y'}(w'' \cos^2 \phi)'' + m\ddot{w} + me\ddot{\phi} \cos \phi + \underline{m\ddot{\theta}(v + e \cos \phi)} - \underline{m\dot{\theta}^2(w + e \sin \phi)} + \underline{2m\dot{\theta}(\dot{v} + e \sin \phi \dot{\phi})} = L_w \quad (3.44)$$

$\delta \phi$ equation:

$$-GJ\phi'' + (EI_{z'} - EI_{y'})[(w''^2 - v''^2) \sin \phi \cos \phi + v''w'' \cos 2\phi] + me(\ddot{v} \sin \phi + \ddot{w} \cos \phi) + mK_m^2 \ddot{\phi} + \underline{m\ddot{\theta}(K_m^2 + ew \sin \phi)} + \underline{2me\dot{\theta}(\dot{v} \cos \phi + \dot{w} \sin \phi)} + \underline{me\dot{\theta}^2(v \sin \phi - w \cos \phi)} = M_\phi \quad (3.45)$$

$\delta \theta$ equation:

$$\int_0^L (-m\ddot{v}(w + e \sin \phi) + m\ddot{w}(v + e \cos \phi) + m\ddot{\phi}(K_m^2 + ew \sin \phi) + \underline{m\ddot{\theta}[(v^2 + w^2 + K_m^2 + 2e(v \cos \phi + w \sin \phi)]} + \underline{2me\dot{\theta}\dot{\phi}(w \cos \phi - v \sin \phi)} + \underline{2m\dot{\theta}\dot{w}(w + e \sin \phi)} + \underline{2m\dot{\theta}\dot{v}(v + e \cos \phi)} + K_\theta \theta + I_B \ddot{\theta}) dx = \int_0^L M_\phi dx \quad (3.46)$$

The boundary conditions from (3.15) and (3.25) becomes

$$b(U) - b(T) = 0, \quad (3.47)$$

$$\begin{aligned} b(U) - b(T) = 0 = & -[(EI_{z'} - EI_{y'})w'' \sin \phi \cos \phi + (EI_{z'} \cos^2 \phi + EI_{y'} \sin^2 \phi)v'']' \delta v|_0^L \\ & + [(EI_{z'} - EI_{y'})w'' \sin \phi \cos \phi + (EI_{z'} \cos^2 \phi + EI_{y'} \sin^2 \phi)v''] \delta v'|_0^L \\ & - [(EI_{z'} - EI_{y'})v'' \sin \phi \cos \phi + (EI_{z'} \sin^2 \phi + EI_{y'} \cos^2 \phi)w'']' \delta w|_0^L \\ & + [(EI_{z'} - EI_{y'})v'' \sin \phi \cos \phi + (EI_{z'} \sin^2 \phi + EI_{y'} \cos^2 \phi)w''] \delta w'|_0^L \\ & + GJ\phi' \delta \phi|_0^L. \end{aligned} \quad (3.48)$$

The equations of motion contain similar terms as those found on Hodges and Dowell [1],

in addition to several linear and nonlinear terms associated to the root pitch rotation, and these terms are underlined in the Eq. (3.43) to Eq. (3.46). An extensive explanation about the nonlinear elastic terms is given in [1]. The additional nonlinear terms associated with the root-pitch motion are underlined and they will be briefly discussed to give better physical understanding of these equations. Due to the similarity between the terms from the equations of in-plane and out-of-plane bending, only the terms from equation of in-plane bending will be discussed.

The additional nonlinear terms due to the root-pitch rotation were compared to those found by Kallesoe [35], who developed nonlinear equations of motion for a rotor blade, including gravity, pitch action and rotor speed variations. The nonlinear terms of the current work show good agreement with the terms related to the pitch action in [35].

As the system is described in a moving frame, fictitious force terms will arise [35]. The first underlined term in Eq. (3.43) represents the inertial force in the y-direction associated with the root-pitch acceleration $\ddot{\theta}$. The second and third terms represent, respectively, the centrifugal and the Coriolis forces associated with the root-pitch angular velocity and the offset of the center of gravity (c.g.) in the y-direction.

The first nonlinear underlined term in Eq. (3.45) represents the angular acceleration of the local frame about the fixed one. The second and third terms respectively represent the centrifugal and Coriolis terms associated with the rotation of the local frame with respect to the reference.

The first nonlinear underlined term of Eq. (3.46) represents the nonlinear inertia of the system. The second, third, and fourth nonlinear terms in Eq. (3.46) represent the moments caused by the Coriolis force associated with the relative velocity of the wing and the rotation between the local and inertial frames. The second and third terms are the Coriolis forces associated with the angular velocity about the x-axis of the local frame with respect to the inertial one and the twisting motion and the velocity in the z-direction, respectively.

In the equations of motion and boundary conditions $\sin \phi$ and $\cos \phi$ terms could be expanded, in order to avoid complicating the equations. For this work, the twist angles are assumed to be small, therefore: $\sin \phi \approx \phi$, $\cos \phi \approx 1$ and $\phi^2 \approx 0$. The equations of motion can therefore be simplified as:

δv equation:

$$(EI_{z'} - EI_{y'})[w''''\phi + w''(\phi'' - 4\phi\phi'^2)] + EI_{z'}[v'''' - v''(2\phi\phi'' + 2\phi'^2)] + EI_{y'}[v''(2\phi\phi'' + 2\phi'^2)] + m\ddot{v} + me\ddot{\phi} - m\ddot{\theta}(w + e\phi) - m\dot{\theta}^2(v + e) - 2m\dot{\theta}(\dot{w} + e\dot{\phi}) = L_v \quad (3.49)$$

δw equation:

$$(EI_{z'} - EI_{y'})[v''''\phi + v''(\phi'' - 4\phi\phi'^2)] + EI_{z'}[w''(2\phi\phi'' + 2\phi'^2)] + EI_{y'}[w'''' - w''(2\phi\phi'' + 2\phi'^2)] + m\ddot{w} + me\ddot{\phi} + m\ddot{\theta}(v + e) - m\dot{\theta}^2(w + e\phi) + 2m\dot{\theta}(\dot{v} + e\dot{\phi}) = L_w \quad (3.50)$$

$\delta\phi$ equation:

$$-GJ\phi'' + (EI_{z'} - EI_{y'})[(w''^2 - v''^2)\phi + v''w''] + me(\ddot{v}\phi + \ddot{w}) + mK_m^2\ddot{\phi} + m\ddot{\theta}(K_m^2 + ew\phi) + 2me\dot{\theta}(\dot{v} + \dot{w}\phi) + me\dot{\theta}^2(v\phi - w) = M_\phi \quad (3.51)$$

$\delta\theta$ equation:

$$\int_0^L (-m\ddot{v}(w + e\phi) + m\ddot{w}(v + e) + m\ddot{\phi}(K_m^2 + ew\phi) + m\ddot{\theta}[(v^2 + w^2 + K_m^2 + 2e(v + w\phi)] + 2me\dot{\theta}\dot{\phi}(w - v\phi) + 2m\dot{\theta}\dot{w}(w + e\phi) + 2m\dot{\theta}\dot{v}(v + e) + K_\theta\theta + I_B\ddot{\theta})dx = \int_0^L M_\phi dx \quad (3.52)$$

Boundary conditions:

$$\begin{aligned} b(U) - b(T) = 0 = & -[(EI_{z'} - EI_{y'})(w'''\phi + w''\phi') + EI_{z'}(v''' - 2v''\phi\phi') + EI_{y'}(2v''\phi\phi')] \delta v|_0^L \\ & + [(EI_{z'} - EI_{y'})w''\phi + EI_{z'}v''] \delta v'|_0^L \\ & - [(EI_{z'} - EI_{y'})(v'''\phi + v''\phi') + EI_{z'}(2w''\phi\phi') + EI_{y'}(w''' - 2w''\phi\phi')] \delta w|_0^L \\ & + [(EI_{z'} - EI_{y'})v''\phi + EI_{y'}w''] \delta w'|_0^L + GJ\phi'\delta\phi|_0^L. \end{aligned} \quad (3.53)$$

The equations of motion developed in this chapter and presented in Eq. (3.49) to Eq. (3.52) have new terms corresponding to the root-pitch motion when compared with Hodges and Dowell [1] equations. Although, the terms involving warp effects, extension and rotation of the beam are not included here because these effects can be neglected without loss of accuracy for fix wing applications [1].

The inertial terms in Eq.(3.49) and Eq.(3.50) couple the in-plane and out-of-plane bending to the torsional motion of the wing, with the degree of coupling depending on the distance between the center of gravity and the elastic axis. The terms $-m\ddot{\theta}(w + e\phi)$ and $m\ddot{\theta}(v + e)$ in Eq.(3.49) and Eq.(3.50), respectively, shows that the root-pitch acceleration excites both the out-of-plane and in-plane bending motion. It means that the root-pitch acceleration of a out-of-plane deflected beam excites the in-plane bending motion and vice versa. Eq. (3.51) shows the coupling between the torsional motion and the root-pitch acceleration and Eq. (3.52) shows the coupling between bending deflections and the root-pitch acceleration.

For sake of simplicity, structural damping is not included in the derivation of the equations of motion. However, a linear viscous damping assumed to be proportional to the mass and stiffness is considered for the aeroelastic model simulations described in chapter 5.

Chapter 4: Structural equations: numerical modeling and solution

4.1 Introduction

This chapter presents numerical simulations of the nonlinear structural equations for flexible cantilever wing with root-pitch motion. The objective of these simulations is to observe the nonlinear structural behaviour in the free vibration responses in the same fashion that was presented in the experimental tests in chapter 2.

First, the finite dimensional representation of the equations of motion is shown using the Galerkin method. Next the linear terms of the equations of motion are isolated and the structural free vibration tests are simulated by the linear system, in order to identify the basic concepts of coupling between bending, torsion and root-pitch motion. Finally, the fully nonlinear equations of motion derived in chapter 3 are also numerically simulated in free vibration and compared to the linear results.

4.2 Finite dimensional approximation of the equations of motion

In this section, the finite dimensional approximation of the solution of the bending-bending-torsion equations of motion, Eqs. (3.49) to (3.52), is formulated. These nonlinear partial differential equations can be solved using the Galerkin method. The equations of motion can be

reduced to ordinary differential equations by approximating the states by:

$$\begin{aligned}\bar{v} &\approx \sum_{i=1}^M V_i(t) \Psi_i(\bar{x}) \\ \bar{w} &\approx \sum_{i=1}^M W_i(t) \Psi_i(\bar{x}) \\ \phi &\approx \sum_{j=1}^N \Phi_j(t) \Omega_j(\bar{x})\end{aligned}\tag{4.1}$$

where $\bar{v} = v/L$, $\bar{w} = w/L$, and $\bar{x} = x/L$. This approximation yields $2M + N$ ordinary differential equations in terms of the generalized coordinates V_i , W_i , Φ_j , and one additional equation in terms of the root-pitch angle θ . The basis functions, $\Psi_i(\bar{x})$ and $\Omega_j(\bar{x})$, are given by the exact basis shape functions [36] for bending and torsional deflections, respectively, of a uniform cantilever beam as follows

$$\begin{aligned}\Psi_i(\bar{x}) &= \cosh(\beta_i L \bar{x}) - \cos(\beta_i L \bar{x}) - \sigma_i [\sinh(\beta_i L \bar{x}) - \sin(\beta_i L \bar{x})] \\ \Omega_j(\bar{x}) &= \sin(\gamma_j L \bar{x}),\end{aligned}\tag{4.2}$$

where β_i is the i^{th} root of the pure bending characteristic equation,

$$1 + \cos(\beta_i L) \cosh(\beta_i L) = 0,\tag{4.3}$$

and σ_i is defined as

$$\sigma_i = \frac{\cosh(\beta_i L) + \cos(\beta_i L)}{\sinh(\beta_i L) + \sin(\beta_i L)},\tag{4.4}$$

γ_j is the j^{th} root of the pure torsion characteristic equation

$$\sin(\gamma_j L) = 0.\tag{4.5}$$

Considering only the structural terms at first, substituting the expanded variables and applying the Galerkin procedure, we obtain the following nonlinear-ordinary differential equations.

δV_i equations:

$$\begin{aligned}
& (\beta_z - 1) \sum_{j=1}^M \sum_{k=1}^N [C_{v1}^{ijk} W_j \Phi_k + C_{v2}^{ijk} W_j \Phi_k - 4 \sum_{l=1}^N \sum_{m=1}^N C_{v3}^{ijklm} W_j \Phi_k \Phi_l \Phi_m] + \beta_z \sum_{j=1}^M [C_{v4}^{ij} V_j \\
& - 2 \sum_{k=1}^N \sum_{l=1}^N C_{v5}^{ijkl} V_j \Phi_k \Phi_l - 2 \sum_{k=1}^N \sum_{l=1}^N C_{v6}^{ijkl} V_j \Phi_k \Phi_l] + \sum_{j=1}^M \sum_{k=1}^N \sum_{l=1}^N [2C_{v7}^{ijkl} V_j \Phi_k \Phi_l + 2C_{v8}^{ijkl} V_j \Phi_k \Phi_l] \\
& + \sum_{j=1}^M C_{v9}^{ij} \ddot{V}_j + e \sum_{k=1}^N \sum_{l=1}^N C_{v10}^{ikl} \ddot{\Phi}_k \Phi_l - \sum_{j=1}^M C_{v11}^{ij} \ddot{\theta} W_j - e \sum_{k=1}^N C_{v12}^{ik} \ddot{\theta} \Phi_k - \sum_{j=1}^M C_{v13}^{ij} \dot{\theta}^2 V_j - e C_{v14}^i \dot{\theta}^2 \\
& - 2 \sum_{j=1}^M C_{v15}^{ij} \dot{\theta} \dot{W}_j - 2e \sum_{k=1}^N C_{v16}^{ik} \dot{\theta} \dot{\Phi}_k = 0
\end{aligned} \tag{4.6}$$

δW_i equations:

$$\begin{aligned}
& (\beta_z - 1) \sum_{j=1}^M \sum_{k=1}^N [C_{w1}^{ijk} V_j \Phi_k + C_{w2}^{ijk} V_j \Phi_k - 4 \sum_{l=1}^N \sum_{m=1}^N C_{w3}^{ijklm} V_j \Phi_k \Phi_l \Phi_m] + \beta_z \sum_{j=1}^M \sum_{k=1}^N \sum_{l=1}^N [2C_{w4}^{ijkl} W_j \Phi_k \Phi_l \\
& + 2C_{w5}^{ijkl} W_j \Phi_k \Phi_l] + \sum_{j=1}^M [C_{w6}^{ij} W_j - 2 \sum_{k=1}^N \sum_{l=1}^N C_{w7}^{ijkl} W_j \Phi_k \Phi_l - 2 \sum_{k=1}^N \sum_{l=1}^N C_{w8}^{ijkl} W_j \Phi_k \Phi_l] \\
& + \sum_{j=1}^M C_{w9}^{ij} \ddot{W}_j + e \sum_{k=1}^N C_{w10}^{ik} \ddot{\Phi}_k + \sum_{j=1}^M C_{w11}^{ij} \ddot{\theta} V_j + e C_{w12}^i \ddot{\theta} - \sum_{j=1}^M C_{w13}^{ij} \dot{\theta}^2 W_j - e \sum_{k=1}^N C_{w14}^{ik} \dot{\theta}^2 \Phi_k \\
& + 2 \sum_{j=1}^M C_{w15}^{ij} \dot{\theta} \dot{W}_j + 2e \sum_{k=1}^N \sum_{l=1}^M C_{w16}^{ikl} \dot{\theta} \dot{\Phi}_k \dot{\Phi}_l = 0
\end{aligned} \tag{4.7}$$

$\delta\Phi_j$ equations:

$$\begin{aligned}
& -\beta_x \sum_{j=1}^N C_{p1}^{ij} \Phi_j + (\beta_z - 1) \sum_{k=1}^M \sum_{l=1}^M \left[\sum_{j=1}^N (C_{p2}^{ijkl} \Phi_j W_k W_l - C_{p3}^{ijkl} \Phi_j V_k V_l) + C_{p4}^{ikl} V_k W_l \right] \\
& + e \sum_{k=1}^M \left(\sum_{j=1}^N C_{p5}^{ijk} \Phi_j \ddot{V}_k + C_{p6}^{ik} \ddot{W}_k \right) + \mu^2 \sum_{j=1}^N C_{p7}^{ij} \ddot{\Phi}_j + \ddot{\theta} (\mu^2 C_{p8}^i + e \sum_{j=1}^N \sum_{k=1}^M C_{p9}^{ijk} \Phi_j W_k) + 2e\dot{\theta} \sum_{k=1}^M (C_{p10}^{ik} \dot{V}_k \\
& + \sum_{j=1}^N C_{p11}^{ijk} \Phi_j \dot{W}_k) + e\dot{\theta}^2 \sum_{k=1}^M \left(\sum_{j=1}^N C_{p12}^{ijk} \Phi_j V_k - C_{p13}^{ik} W_k \right) = 0
\end{aligned} \tag{4.8}$$

$\delta\theta$ equation:

$$\begin{aligned}
& - \sum_{j=1}^M \left(\sum_{k=1}^M C_{t1}^{jk} \ddot{V}_j W_k + \sum_{l=1}^N C_{t2}^{jl} e \ddot{V}_j \Phi_l \right) + \sum_{j=1}^M \left(\sum_{k=1}^M C_{t3}^{jk} \ddot{W}_j V_k + e C_{t4}^j \ddot{W}_j \right) + \sum_{l=1}^N (\mu^2 C_{t5}^l \ddot{\Phi}_l \\
& + e \sum_{j=1}^M \sum_{m=1}^N C_{t6}^{jlm} W_j \ddot{\Phi}_l \Phi_m) + \ddot{\theta} \left[\sum_{j=1}^M \sum_{k=1}^M (C_{t7}^{jk} V_j V_k + C_{t8}^{jk} W_j W_k) + \mu^2 \right. \\
& + 2e \sum_{j=1}^M (C_{t10}^j V_j + \sum_{l=1}^N C_{t11}^{jl} W_j \Phi_l) \left. \right] + 2e \sum_{j=1}^M \sum_{l=1}^N \dot{\theta} (C_{t12}^{jl} W_j \dot{\Phi}_l - \sum_{m=1}^N C_{t13}^{jlm} V_j \dot{\Phi}_l \Phi_m) \\
& + 2\dot{\theta} \sum_{j=1}^M \left(\sum_{k=1}^M C_{t14}^{jk} \dot{W}_j W_k + e \sum_{l=1}^N C_{t15}^{jl} \dot{W}_j \Phi_l \right) + 2\dot{\theta} \sum_{j=1}^M \left(\sum_{k=1}^M C_{t16}^{jk} \dot{V}_j V_k + C_{t17}^j \dot{V}_j e \right) + \beta_\theta \theta + \beta_B \ddot{\theta} = 0
\end{aligned} \tag{4.9}$$

where the coefficients for the equations are defined in Appendix C.

The nonlinear finite dimensional equations of motion can be approximated by splitting

the linear and nonlinear terms in a matrix form, as

$$([M]_{L_w} + [M]_{NL_w}) \begin{Bmatrix} \ddot{V}_i \\ \ddot{W}_i \\ \ddot{\Phi}_j \\ \ddot{\theta} \end{Bmatrix} + ([C]_{L_w} + [C]_{NL_w}) \begin{Bmatrix} \dot{V}_i \\ \dot{W}_i \\ \dot{\Phi}_j \\ \dot{\theta} \end{Bmatrix} + ([K]_{L_w} + [K]_{NL_w}) \begin{Bmatrix} V_i \\ W_i \\ \Phi_j \\ \theta \end{Bmatrix} = 0, \quad (4.10)$$

where the matrices $[M]_{L_w}$, $[C]_{L_w}$, $[K]_{L_w}$ are the linear structural mass, damping and stiffness matrices, respectively, and $[M]_{NL_w}$, $[C]_{NL_w}$, $[K]_{NL_w}$ are the nonlinear structural mass, damping and stiffness matrices, respectively. The coefficients of these matrices are defined in Appendix C.

4.3 Linear structural model

The ordinary differential equations with just the linear terms can be written in matrix form as

$$[M]_{L_w} \begin{Bmatrix} \ddot{V}_i \\ \ddot{W}_i \\ \ddot{\Phi}_j \\ \ddot{\theta} \end{Bmatrix} + [C]_{L_w} \begin{Bmatrix} \dot{V}_i \\ \dot{W}_i \\ \dot{\Phi}_j \\ \dot{\theta} \end{Bmatrix} + [K]_{L_w} \begin{Bmatrix} V_i \\ W_i \\ \Phi_j \\ \theta \end{Bmatrix} = 0, \quad (4.11)$$

where the matrices $[M]_{L_w}$, $[C]_{L_w}$, $[K]_{L_w}$ are defined in Appendix C.

In order to perform a time domain analysis, these equations are transformed into state space format as

$$\{\dot{q}\} = [A]\{q\}, \quad (4.12)$$

where $\{q\}$ is the state vector that will have $4M + 2N + 2$ components such as

$$q = \begin{bmatrix} V_i & W_i & \Phi_j & \theta & \dot{V}_i & \dot{W}_i & \dot{\Phi}_j & \dot{\theta} \end{bmatrix}^T, \quad (4.13)$$

and $[A]$ is the system matrix, from which the aeroelastic frequencies and damping values are obtained.

4.3.1 Free vibration tests

The wing with properties described in Appendix D is modeled and implemented in Matlab according to Eq. (4.11) with no structural damping and a series of free vibration tests were simulated to compare to the analytical natural frequencies of the beam. For the time domain discretization, a fourth order explicit Runge-Kutta method was used with a time step of 0.001 seconds.

4.3.2 Coupled and uncoupled bending-torsional free vibration tests of the cantilever wing

The first vibration tests were simulated for a cantilever beam, by setting the stiffness of the root-pitch rotation K_θ to a high value to prevent any significant rotation of the wing base. The free vibration tests are induced by giving initial deflection in out-of-plane bending $\bar{w}$ or torsion ϕ .

For free vibration tests of the cantilever, only the bending and torsional deflections were shown because the pitch motion is assumed to be negligible and the in-plane-bending is decoupled from the other states when only the linear terms are considered (see Appendix C).

For a cantilever beam, the decoupled natural frequencies in bending and torsion can be expressed analytically [36] as in Eq. (4.14) and, for the wing properties described in Appendix D, the first five natural frequencies in bending and torsion are shown in Table 4.1.

$$\omega_{bending} = \left(\frac{\beta_i}{L}\right)^2 \sqrt{\frac{EI_{y'}}{m}}, \quad \omega_{torsion} = \left(\frac{(2j-1)\pi}{2L}\right) \sqrt{\frac{GJ}{mLK_m^2}} \quad (4.14)$$

Table 4.1: First five natural frequencies for the cantilever beam

Natural Frequency	$\omega_{bending}(Hz)$	$\omega_{torsion}(Hz)$
1	4.0002	9.6450
2	25.0687	28.9350
3	70.1931	48.2250
4	137.5506	67.5150
5	227.3814	86.8050

Basis functions convergence tests

A convergence test is carried out in the initial and final regimes of the free vibration to determine how many basis functions are necessary in bending and torsion to obtain converged solution. The biggest variations in the systems response are found in the coupled free vibrations tests. Fig. 4.1 shows the torsional time trace of a free vibration test in bending. In the initial regime (beginning of the simulation), shown in Fig. 4.1a, the response is accurately computed by 2 basis functions (mode shapes) in bending and torsion. In the final regime (near the end of the simulation), shown in Fig. 4.1b, at least 3 mode shapes in bending and torsion are required to accurately simulate the responses.

(a) Transient region (beginning of the test) **(b)** Steady region**Figure 4.1:** Bending basis shape functions convergence test: initial $\bar{w} = 0.1$

Fig. 4.2 shows the bending time trace of a free vibration test obtained from giving initial torsional deflection. In the initial regime, shown in Fig. 4.2a, the response solution is adequately approximated by 2 basis functions in bending and torsion. In the final regime, shown in Fig. 4.2b, 2 basis functions in bending and torsion are still adequate to accurately capture the

system response.

Figure 4.2: Torsion basis shape functions convergence test: initial $\phi = 5^\circ$

Through this convergence study it is concluded that the numerical simulation of the linear system can be accurately conducted by 3 basis functions for the in-plane and out-of-plane bending and 2 basis functions for torsion.

Uncoupled bending free vibration test

By considering the center of gravity and the elastic axis at the same location (i.e. $e = 0$), in Fig. 3.2, the out-of-plane bending and torsional motion are decoupled from each other, which is the same assumption made by the analytical description in Eq. (4.14). Applying an initial bending displacement of $W_1 = 0.1$ in the first basis function and $W_j = 0.01$ in the following four basis functions (modes) to artificially excite the higher modes. The tip deflection is shown in Fig. 4.3 and the power spectral density (PSD) of the signal (more precisely, the magnitude of the Fourier transform) is shown in Fig. 4.4.

In Fig. 4.3, the response has a predominant harmonic component from the initial perturbation of the first mode along with the high-frequency components arising from higher modes. The PSD is shown in Fig. 4.4, it has the five first natural frequencies in bending that correspond to the numerical values of frequencies in Table 4.1.

Figure 4.3: Uncoupled bending free vibration time trace: initial $\bar{w} = 0.1$

Figure 4.4: Uncoupled bending free vibration PSD: initial $\bar{w} = 0.1$

Uncoupled torsion free vibration test

Similar to the free-vibration test in bending, an initial torsion angle of 5 degrees at the tip of the beam is applied to the first torsional mode Φ_1 and the four following torsional modes are initially excited with 0.5 degrees. The beam tip rotation is shown in Fig. 4.5 and the corresponding PSD of the signal shown in Fig. 4.6.

Figure 4.5: Uncoupled torsion free vibration time trace: initial $\phi = 5^\circ$

Similar to the previous test, the tip rotation exhibits dominant harmonic component with high frequency components arising from the higher modes, as shown in Fig. 4.5, and the corresponding PSD in Fig. 4.6 displaying the five frequencies in torsion that correspond to the torsional frequencies are shown in 4.1.

Coupled bending-torsion free vibration test

Setting the distance $e = 0.18L$ between the center of gravity and the elastic axis, the bending and torsional motions show an inertial coupling effect. The time trace and its PSD are shown

Figure 4.6: Uncoupled torsion free vibration PSD: initial $\phi = 5^\circ$

in Fig. 4.7 and Fig. 4.8, respectively. The inertial coupling effect is observed in the torsional angle time trace that is excited by the bending motion and in the PSD of the responses highlighted by frequencies close to the uncoupled bending and torsion motion.

For $e = 0.18L$ and an initial tip rotation of 5 degrees, the coupled torsion free vibration test is performed. The time traces and its PSD are shown in Fig. 4.9 and Fig. 4.10, respectively. The coupling effect is observed in the time trace of the bending signal and the energies close to the natural bending and torsional motion are observed in the corresponding PSD. In terms of amplitude, the influence of the torsion in the bending motion is less apparent in contrast to the inverse case. For 5-degree initial torsional rotation, only less than $0.01L$ bending amplitude is observed on Fig. 4.9.

Figure 4.7: Coupled bending-torsion free vibration time trace: initial $\bar{w} = 0.1$

Figure 4.8: Coupled bending-torsion free vibration PSD: initial $\bar{w} = 0.1$

4.3.3 Coupled bending-torsion free vibration tests of the wing with root-pitch rotation

All the free vibration tests were conducted so far with the assumption that the wing behaves as a cantilever beam, i.e. no rotation is allowed at the root. To investigate the effect of root-pitch motion with elastic restraint in the linear equations of motion in Eq. (4.11), a set of similar free

Figure 4.9: Coupled bending-torsion free vibration time trace: initial $\phi = 5^\circ$

Figure 4.10: Coupled bending-torsion free vibration PSD: initial $\phi = 5^\circ$

vibration tests are performed. The root-pitch stiffness of the system was configured to be equal to 0.8 N.m/degree.

The initial tip displacement in out-of-plane bending was set to $0.1L$ and the time step set as 0.0005 seconds. The time trace of the four degrees of freedom normalized by the beam length at the tip namely $\bar{w}$, $\bar{v}$, ϕ , and θ are shown in Fig. 4.11. For the system with fixed base and

with the root-pitch motion for comparison and the corresponding PSDs of the traces are shown in Fig. 4.12.

For both fixed base and root-pitch rotation configurations, the in-plane bending $\bar{v}$ has no response because, in the linear model, this state is decoupled from the other three states. The out-of-plane bending $\bar{w}$ shows very similar responses for the two different configurations, except with a time delay that increases with time of the system with root motion in comparison with the one with the fixed base. The torsional angle ϕ of the system with root-pitch motion displays a time delay and lower magnitude in comparison to the cantilever system due to transferred energy from the torsion to the root-pitch motion through inertial coupling. As the consequence of this energy transfer, the root-pitch motion θ shows an oscillatory response perhaps including the influence of other states.

The PSD of root-pitch response θ of the system with pitch base rotation displays energies at the decoupled frequencies in bending and torsion and also at a lower frequency that corresponds closely to the natural frequency of rigid body rotation of the base with the elastic restraint.

Figure 4.11: Coupled bending-torsion with root-pitch rotation free vibration time traces: initial $\bar{w} = 0.1$

Similarly, the torsion free vibration test is conducted for both configurations. The time

Figure 4.12: Coupled bending-torsion with root-pitch rotation free vibration PSD: initial $\bar{\omega} = 0.1$

traces and the corresponding PSD comparison are shown in Fig. 4.13 and Fig. 4.14, respectively. The out-of-plane bending of the system with base rotation presented a time delay and smaller amplitudes when compared to the system with the fixed root, indicating the transfer of energy between the bending root-pitch motions. The torsional responses of the two configurations display similar harmonic behaviours although the system with base rotation displays a time delay with respect to the fixed base system. Finally, the pitch rotational response is influenced by its natural frequency and its coupling with the bending and torsional motions.

Figure 4.13: Coupled bending-torsion with root-pitch rotation free vibration time trace: initial $\phi = 5^\circ$

Figure 4.14: Coupled bending-torsion with root-pitch rotation free vibration PSD: initial $\phi = 5^\circ$

The third test to evaluate the influence of the pitch rotation is conducted by giving the system an initial root-pitch rotation of 5 degrees, as shown in Fig. 4.15 and Fig. 4.16. The pitch rotation induces the out-of-plane bending and the torsional response of the wing.

Figure 4.15: Coupled bending-torsion with root-pitch rotation free vibration time trace: initial $\theta = 5^\circ$

4.2 shows the comparison between numerical and experimental frequencies.

Figure 4.16: Coupled bending-torsion with root-pitch rotation free vibration PSD: initial $\theta = 5^\circ$

Table 4.2: Comparison between numerical and experimental frequencies

Natural Frequency	Analytical	Numerical	Experimental
$f_{bending}(Hz)$	4.0002	4.029	4.212
$f_{torsion}(Hz)$	9.6450	9.645	9.095
$f_{pitch}(Hz)$	–	2.564	2.258

4.4 Nonlinear structural model

The algorithm for solution of Eq. (4.10) is implemented in Matlab environment using a forth order explicit Runge-Kutta method with a time step of 0.0005 seconds. The wing properties are described in Appendix D is modeled and implemented initially with no linear structural damping. A series of free vibration tests were simulated in order to investigate the nonlinear effects into the vibration of the system.

The converge test for the nonlinear structural model, shown in Fig. 4.17 indicates that two basis functions in bending and torsion are necessary to achieve convergence of the time trace solution.

(a) Bending basis functions convergence test

(b) Torsion basis functions convergence test

Figure 4.17: Nonlinear structural model mode shapes convergence test

4.4.1 Nonlinear coupled bending-bending-torsion with root-pitch rotation free vibration tests

In order to investigate the influence of the nonlinearities in the system, a series of free vibration tests are again conducted, focusing on the effect of increasing the initial displacements in the time traces.

The nonlinear coupled bending-bending-torsion with root-pitch rotation free vibration test responses with normalized initial tip displacement equal to 0.001, 0.1, 0.3 and 0.5 are shown in Fig. 4.18a, Fig. 4.18b, Fig. 4.18c and Fig. 4.18d, respectively. For the initial tip displacements 0.001 and 0.1, the linear and nonlinear responses are very close. For larger initial tip displacements 0.3 and 0.5, nonlinear effects become more significant and the nonlinear time traces begin to differ from the linear response.

Fig. 4.19 shows the PSD of the free vibration test in bending with initial tip deflection 0.5. The PSD plots show superharmonic response in bending ($3f$ and $5f$ denoted in Fig. 4.19 in bending PSD) and torsion ($2f$ denoted in Fig. 4.19 in torsional PSD) indicating nonlinearity effects.

The free vibration tests of the nonlinear coupled bending-bending-torsion system with root-pitch rotation with initial tip torsional angles ϕ equal to 5° , 15° and 30° are shown in Fig. 4.20a, Fig. 4.20b and Fig. 4.20c, respectively. For the small initial angle 5° , the linear and nonlinear responses are very close. As the initial displacements becomes larger the nonlinear effects become more significant; and the nonlinear time responses begin to differ from the linear

Figure 4.18: Nonlinear coupled bending-bending-torsion free vibration time traces for different initial conditions

response.

Fig. 4.21 shows the PSD of the free vibration test in bending with initial tip torsional rotation 15° . The PSD plots show super harmonic frequencies in bending ($3f$ denoted in Fig. 4.21 in bending PSD) indicating nonlinearity effects.

For the free vibration tests of the nonlinear coupled bending-bending-torsion system with root-pitch rotation is studied with initial pitch rotation θ equal to 5° , 30° and 60° with corresponding time traces shown shown in Fig. 4.22a, Fig. 4.22b and Fig. 4.22c, respectively. For the small initial rotation of 5° , the linear and nonlinear responses are very close. For larger initial

Figure 4.19: Nonlinear coupled bending-bending-torsion free vibration PSD: initial $\bar{w} = 0.5$

pitch rotations the nonlinear effects become significant as evidences in the time traces of the linear and nonlinear responses.

Fig. 4.23 shows the PSD of the free vibration test in bending with initial root-pitch rotation 30° . The PSD plots show super harmonic frequencies in torsion ($2f'$ denoted in Fig. 4.23 in torsional PSD) indicating nonlinearity effects.

The simulations of the free vibration tests of the nonlinear bending-bending-torsion equations of motion developed in chapter 3 confirm that the nonlinear effects are expressive for large deflections or angles. Additionally, the coupling effects between the root-pitch motion and bending and torsional motion is also observed through the time traces of the tests.

Figure 4.20: Nonlinear coupled bending-bending-torsion free vibration time traces for different initial conditions

Figure 4.21: Nonlinear coupled bending-bending-torsion free vibration PSD: initial $\phi = 15^\circ$

Figure 4.22: Nonlinear coupled bending-bending-torsion free vibration time traces for different initial conditions

Figure 4.23: Nonlinear coupled bending-bending-torsion free vibration PSD: initial $\theta = 30^\circ$

Chapter 5: Aerodynamic modeling and aeroelastic solution

5.1 Introduction

A realistic dynamic behaviour representation of a flexible wing with root-pitch motion with elastic restraint would require a nonlinear 3D structural and aerodynamic model, which takes into account the 3D flow effects as well the large deflections and angles of attack. A direct analysis of this fully nonlinear aeroelastic model of the wing, however, poses substantial theoretical, analytical and computational challenges [30]. These complexities arise in the development of a nonlinear aerodynamic model analytical, semi-empirical or numerical (high fidelity CFD). Furthermore, a relatively large computational time is required to obtain an accurate numerical solution of the problem.

Several nonlinear aerodynamic models have been developed in literature as powerful CFD tools. However, the scope of this thesis is primarily to do an extensive investigation of the structural nonlinearities in the aeroelastic system. Consequently, simpler quasisteady and unsteady aerodynamic models were chosen.

In this chapter, the nonlinear aeroelastic analysis of a flexible wing with root-pitch motion with elastic restraint is simplified by using linear aerodynamic models. The objective of these models is to capture the essential aeroelastic effects without introducing theoretical and computational complexities in the formulation and numerical solution. Consequently, the linear aerodynamic formulation is used to introduce the basic physics of the coalescence flutter phenomenon. The derivation of the wing aeroelastic dynamics detailed in this chapter for both,

quasisteady and unsteady aerodynamic model is taken from Fung [30].

The dynamics of a flexible wing model are derived based on the following assumptions that justify the linear aerodynamic analysis:

- the wing section airfoil is thin
- the structural vibration is limited to relatively small amplitudes
- the Reynolds number is high, representative of inviscid flow
- the Mach number is restricted to the low subsonic regime

These assumptions simplify the theoretical treatment of the aerodynamic problem and permit its linear representation. The derivation of the aerodynamic loading is based on thin airfoil theory and potential flow theory. Note that symmetry is not required as the camber of the wing airfoil does not affect the stability of the linear aeroelastic system [30].

Regarding the small amplitude assumption, this condition is evidently violated considering the large amplitude oscillations observed in the experimental investigation. Hence, the LCO cannot be modeled correctly, but the (linear) coalescence flutter behaviour can be simulated with sufficient accuracy. Also regarding the high Reynolds assumption, this is also violated since the Reynolds number is about 10^5 for the experimental tests. Accordingly, it is expected that the flutter speed prediction will show some discrepancy with the experiments.

Fig. 5.1 shows an airfoil section of the wing going to bending, torsion and rigid body rotation; that is chordwise rigid (no warping). The chord length of the airfoil is denoted by c and the half-chord by b . The center of gravity (CG), aerodynamic center (AC) and elastic axis (EA) locations are also illustrated in this figure. The aerodynamic center is the point where the aerodynamic moment remains constant with respect to the angle of attack (defined for small angles and high Reynolds Number). For subsonic flow, this point is at the quarter-chord point of the airfoil leading edge. The shear center is assumed to coincide with the elastic axis of the airfoil, and the root pitch axis is assumed to coincide with the elastic axis. Therefore the elastic axis, and in effect the shear center, serves as both the rotational and the out-of-plane bending axis.

Figure 5.1: Wing airfoil section view with important geometric parameters

Additionally in Fig. 5.1, important distance parameters are noted like e is the distance between the elastic axis and the center of gravity, previously present in the structural equation of motion (it is positive for CG aft of EA). The a_h is the distance between the half-chord point and the elastic axis, normalized by the half chord b (it is positive for EA aft of half chord). The distance by_ϕ represents the distance between the elastic axis and the aerodynamic center (it is positive for EA aft of AC).

Fig. 5.2 shows the lift $L(x, t)$ and moment $M_{ac}(x, t)$, positive nose up, acting on a wing section aerodynamic center. To simplify the analysis, the aerodynamic loading are calculated with respect to the elastic axis, therefore the equivalent moment can be calculated as:

$$M_{ea} = by_\phi L + M_{ac}. \quad (5.1)$$

Figure 5.2: Aerodynamic loading in the wing section

Additionally as shown in Fig. 5.2, the geometric angle of attack is assumed to be equal to

the sum of the rigid body root pitch θ and the torsion angle at the section ϕ .

The aerodynamic lift and moment are included in the nonlinear equations of motion presented in Eq. (3.43) to Eq. (3.46) as the forcing terms on the right side of the equations. In the next sections, a brief derivation of linear quasisteady and unsteady aerodynamic models is provided, and the aeroelastic models are simulated using both the linear and nonlinear structural model.

5.2 Quasisteady aerodynamics

This sections provides a brief derivation of the quasisteady aerodynamic loading. The equations illustrated in this section are adapted from references [30], [23] and [37]. The general quasisteady aerodynamic forcing terms consider the effects of the wing motion that alters the angle of attack of the sections, i.e. $\dot{w}$, $\dot{\theta}$, $\dot{\phi}$, without considering any memory effects. Effectively, the aerodynamic lift per unit length of span can be calculated as:

$$L(x, t) = \rho_{\infty} U_{\infty}^2 b C_L, \quad (5.2)$$

where ρ_{∞} and U_{∞} are the free stream air density and airspeed respectively, C_L is the lift coefficient:

$$C_L(x, t) = C_{L_{\alpha}} \alpha_{eff}(x, t), \quad (5.3)$$

and where $C_{L_{\alpha}}$ is the $C_L - \alpha$ curve slope, i.e. $\frac{\partial C_L}{\partial \alpha}$, of three-dimensional wing.

When linear, two-dimensional airfoil, and incompressible aerodynamics are assumed, $C_{L_{\alpha}}$ is approximately 2π . Later in this section the conversion factor between 2D and 3D of $C_{L_{\alpha}}$ is discussed.

The effective angle of attack α_{eff} is composed of two parts : the angle of incidence α_i , that is assumed to be equal to the sum of the root-pitch angle and the torsion angle at the section

$$\alpha_i(x, t) = \theta(t) + \phi(x, t), \quad (5.4)$$

and a part due to bending, torsion and pitch motion, that leads to the effective angle of attack (adapted from [30])

$$\alpha_{eff}(x, t) = \theta + \phi + \frac{\dot{w}}{U_\infty} + \frac{b(\frac{1}{2} - a_h)(\dot{\theta} + \dot{\phi})}{U_\infty}. \quad (5.5)$$

It leads to the following expression for the lift force:

$$L(x, t) = \rho_\infty U_\infty^2 b C_{L_\alpha} \left(\theta + \phi + \frac{\dot{w}}{U_\infty} + \frac{b(\frac{1}{2} - a_h)(\dot{\theta} + \dot{\phi})}{U_\infty} \right). \quad (5.6)$$

The quasisteady aerodynamic moment, per unit length of span, about the elastic axis can be calculated by substituting Eq. (5.6) in Eq. (5.1). The distance by_ϕ between the elastic axis and the aerodynamic center can be expressed as $b(\frac{1}{2} + a_h)$, therefore:

$$M_{ea}(x, t) = b^2(\frac{1}{2} + a_h)\rho_\infty U_\infty^2 C_{L_\alpha} \left(\theta + \phi + \frac{\dot{w}}{U_\infty} + \frac{b(\frac{1}{2} - a_h)(\dot{\theta} + \dot{\phi})}{U_\infty} \right) + M_{ac} \quad (5.7)$$

where M_{ac} is derived from the thin airfoil theory as (adapted from [30]):

$$M_{ac} = -\frac{\rho_\infty U_\infty b^3 C_{L_\alpha}}{4}(\dot{\theta} + \dot{\phi}). \quad (5.8)$$

Finally, M_{ea} can be expressed by:

$$M_{ea}(x, t) = b^2(\frac{1}{2} + a_h)\rho_\infty U_\infty^2 C_{L_\alpha} \left(\theta + \phi + \frac{\dot{w}}{U_\infty} + \frac{b(\frac{1}{2} - a_h)(\dot{\theta} + \dot{\phi})}{U_\infty} \right) - \frac{\rho_\infty U_\infty b^3 C_{L_\alpha}}{4}(\dot{\theta} + \dot{\phi}). \quad (5.9)$$

The added mass effect, as function of the acceleration terms, is included in this quasisteady aerodynamic model. Typically, the added mass is not included in quasisteady aerodynamics, however, its inclusion does not invalidate the quasisteady model assumption. This inclusion here is for the sake of comparison of adding only the circulatory components in the unsteady model. The lift and moment for the quasisteady aerodynamics can be summarized as:

$$\begin{aligned}
L(x, t) = & \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L_{\alpha}} \left[\frac{b^2}{U_{\infty}^2} \ddot{w} - \frac{b^3 a_h}{U_{\infty}^2} (\ddot{\theta} + \ddot{\phi}) + \frac{2b}{U_{\infty}} \dot{w} + \frac{2b^2(\frac{1}{2} - a_h)}{U_{\infty}} (\dot{\theta} + \dot{\phi}) \right. \\
& \left. + 2b(\theta + \phi) \right] \\
M_{ea}(x, t) = & \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L_{\alpha}} \left[-\frac{b^3 a_h}{U_{\infty}^2} \ddot{w} + \frac{b^4(\frac{1}{8} + a_h^2)}{U_{\infty}^2} (\ddot{\theta} + \ddot{\phi}) + \frac{2b^2(\frac{1}{2} + a_h)}{U_{\infty}} \dot{w} \right. \\
& \left. + \frac{2b^3(\frac{1}{2} - a_h)a_h}{U_{\infty}} (\dot{\theta} + \dot{\phi}) + 2b^2(\frac{1}{2} + a_h)(\theta + \phi) \right]
\end{aligned} \tag{5.10}$$

Mach correction factor

In subsonic flow, the compressibility effects can be often negligible for very low Mach numbers. As the airspeed increases and the Mach number gets higher than 0.3, the compressibility effects become more significant and therefore need to be accounted for. In order to include compressibility effects in the aeroelastic analysis the Prandtl-Glauert Mach Correction Factor [38] is applied to the lift curve slope coefficient as:

$$C_{L_{\alpha}} = \frac{\bar{C}_{L_{\alpha}}}{\sqrt{1 - M_{\infty}^2}}, \tag{5.11}$$

where $\bar{C}_{L_{\alpha}}$ is the lift curve slope at $M_{\infty} = 0$ and M_{∞} is the free flow Mach number. Note that the free stream Mach number for our experiment is very low in the order of 0.03.

3D aerodynamic lift reduction factor

The dynamics of the three-dimensional wing has to account for the 3D aerodynamic flow due to tip vortex effects, that reduce the lift towards the tip of the wing. A simple lift reduction factor is assumed in order to include the 3D aerodynamic effects. The lift coefficient distribution along the span is assumed to have an elliptical shape as:

$$\bar{C}_{L_{\alpha}} = \bar{C}_{L_{\alpha}}^{2D} \sqrt{1 - \bar{x}^2}, \tag{5.12}$$

where $\bar{C}_{L\alpha}^{2D}$ represents the 2D lift curve slope and $\bar{x}$ is the location along the span normalized by L .

Finite dimensional approximation of the equations of motion

For the aerodynamic load calculations, the states of the system is approximate as

$$\begin{aligned}\bar{w} &= \sum_{i=1}^M W_i(t) \Psi_i(\bar{x}) \\ \phi &= \sum_{j=1}^N \Phi_j(t) \Omega_j(\bar{x}).\end{aligned}\tag{5.13}$$

The nonlinear aeroelastic finite dimensional equations of motion can be given in a matrix form, as

$$\begin{aligned} & \left([M]_{L_w} + [M]_{NL_w} \right) \begin{Bmatrix} \ddot{V}_i \\ \ddot{W}_i \\ \ddot{\Phi}_j \\ \ddot{\theta} \end{Bmatrix} + \left([C]_{L_w} + [C]_{NL_w} \right) \begin{Bmatrix} \dot{V}_i \\ \dot{W}_i \\ \dot{\Phi}_j \\ \dot{\theta} \end{Bmatrix} \\ & + \left([K]_{L_w} + [K]_{NL_w} \right) \begin{Bmatrix} V_i \\ W_i \\ \Phi_j \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ L_i \\ M_{ea_j} \\ \int_0^1 M_{ea} d\bar{x} \end{Bmatrix}, \end{aligned}\tag{5.14}$$

where $i = \{1, \dots, M\}$, $j = \{1, \dots, N\}$ and

$$L_i = \frac{1}{2} \rho_\infty U_\infty^2 C_{L_\alpha} \left[\frac{2b}{U_\infty} \sqrt{\frac{L^4}{mEI_{y'}}} \sum_{j=1}^M C_{\Psi\Psi}^{ij} \dot{W}_j + \frac{2b^2(1-a_h)}{U_\infty} (\dot{\theta} + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \dot{\Phi}_k) \sqrt{\frac{L^2}{mEI_y}} + 2b(\theta + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \Phi_k) \frac{L^3}{EI_y} \right] \quad (5.15)$$

$$M_{ea_j} = \frac{1}{2} \rho_\infty U_\infty^2 C_{L_\alpha} \left[\frac{2b^2(\frac{1}{2} + a_h)}{U_\infty} \sqrt{\frac{L^2}{mEI_y}} \sum_{k=1}^M C_{\Omega\Psi}^{ik} \dot{W}_k + \frac{2b^3(\frac{1}{2} - a_h)a_h}{U_\infty} (\dot{\theta} + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \Phi_k) \sqrt{\frac{1}{mEI_y}} + 2b^2(\frac{1}{2} + a_h)(\theta + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \Phi_j) \frac{L^2}{EI_y} \right]. \quad (5.16)$$

A more extensive derivation of the aeroelastic equations of motion and its parameters is shown in Appendix C.

The linear structural damping $[C]_{L_w}$ is represented by a viscous and proportional Rayleigh damping. The proportionality is taken with respect to the linear mass and stiffness matrices as

$$[C]_{L_w} = \alpha_0 [M]_{L_w} + \alpha_1 [K]_{L_w}. \quad (5.17)$$

The parameters α_0 and α_1 are taken to attain the prescribed damping ratios ϵ_i and ϵ_j of two structural modes. Based on ϵ_i and ϵ_j , α_0 and α_1 can be obtained as

$$\begin{bmatrix} \frac{1}{2\omega_i} & \frac{\omega_i}{2} \\ \frac{1}{2\omega_j} & \frac{\omega_j}{2} \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \end{Bmatrix} = \begin{Bmatrix} \epsilon_i \\ \epsilon_j \end{Bmatrix}, \quad (5.18)$$

where ω_i and ω_j are the natural frequencies of the two structural modes used to formulate damping.

The two chosen modes to model damping in this work are the two coalescing modes, out-of-plane bending and root-pitch rotation. When greater number of modes is involved in the system the Rayleigh damping can take other forms. However, since coalescence flutter primarily depends on the two coalescing modes, a two modal damping approximation is reasonable.

Eigenvalue analysis of the linear quasisteady aeroelastic system

This section presents the eigenvalue analysis of the linear aeroelastic system and its interpretation of eigenvalues and eigenvectors in terms of the properties of the system. Assuming the system state as

$$\begin{Bmatrix} v \\ w \\ \phi \\ \theta \end{Bmatrix} = X e^{\lambda t}, \quad (5.19)$$

the linear aeroelastic system can be written as

$$\left[\lambda^2 M_{ae} + \lambda C_{ae} + K_{ae} \right] X e^{\lambda t} = \{0\}, \quad (5.20)$$

which, for a non-trivial solution, leads to a quadratic eigenvalue problem. The solution of Eq. (5.20) is a set of eigenvalues λ and associated eigenvectors X .

Using the state space representation, the quadratic eigenvalue problem can be transformed into the standard eigenvalue problem defined as:

$$[A]\{Y\} = \lambda Y, \quad (5.21)$$

where

$$[A] = \begin{bmatrix} 0 & I \\ -M_{ae}^{-1}K_{ae} & -M_{ae}^{-1}C_{ae} \end{bmatrix}, \quad (5.22)$$

$$\text{and } \{Y\} = \begin{Bmatrix} X \\ \dot{X} \end{Bmatrix}.$$

The eigenvalues of an aeroelastic system normally appear in complex conjugate pairs:

$$\lambda_i = -\beta_i \pm j(\omega_{d_i}), \quad (5.23)$$

where β_i is the aeroelastic modal decay (damping) rates and ω_{d_i} is the aeroelastic modal frequency of the i^{th} mode.

The number of roots in the eigenvalue analysis depends on the number of modes chosen from each state in the spatial discretization method. In the present work, a convergence test of the linear flutter speed has shown that two mode shapes in each bending and two in torsion are enough to reach convergence of the flutter speed. Therefore, the solution involves $2M + N + 1$ complex conjugate root pairs, each pair representing the damping and frequency of an aeroelastic mode.

Mode tracking

In aeroelastic analysis, the air-on vibratory frequencies and modes are ordered by their air-off frequencies (free vibration). As the airspeed changes, the vibratory frequencies will change and mode crossing can occur. Mathematically, mode crossing occurs due to abrupt numerical changes in the aeroelastic system state matrix with the airspeed increment. The failure in tracking and correcting these mode crossings can cause misinterpretation of aeroelastic phenomena. Without such corrections, the factors leading to flutter and divergence may be not properly understood and, therefore, attempts to make engineering decisions to improve performance will be misguided and likely to be unsuccessful.

Fig. 5.3a shows an example of a simple aeroelastic analysis plot of the discussed wing of this work using one basis function for each state and quasisteady aerodynamics. In this figure, damping represents the structural damping that would be required to stabilize the system. Hence, positive damping means the system is unstable, i.e., the amplitudes of one or more states exponentially grow. If two modal frequencies coalesce in the vicinity of the point where instability starts, the energy exchange between these two aeroelastic modes leads to the so-called coalescence

flutter. Although the flutter speed is commonly indicated by the damping, the primary mechanism for flutter is in fact the frequency coalescence. The damping variation is important to determine if the flutter phenomenon occurs in a mild, moderate or explosive manner. Applying a mode tracking methodology in the eigenvalue analysis, shown in Fig. 5.3b, the graph provides a clear description of the system instability.

(a) Without mode tracking

(b) With mode tracking

Figure 5.3: Modal frequency and decay plots of the linear aeroelastic problem with quasisteady aerodynamics with and without mode tracking; one basis function per state

The author of this work, in partnership with the Aerodynamics Lab of the National Research Council of Canada [39], successfully developed a capability to establish a reliable and automated post-processing module for correctly visualizing the aeroelastic modes in flutter analysis that can be expressed regarding complex conjugates.

The mode tracking methodology was developed based on the work of [40] and relies on the scalar product of two complex eigenvectors to build the modal correlation matrix. The elements

of this matrix can be expressed by

$$C_{ij} = \frac{|\{z_R^{(k-1)}\}_i^T \{z_R^{(k)}\}_j|}{|\{z_R^{(k-1)}\}_i| |\{z_R^{(k)}\}_j|}, \quad (5.24)$$

where $\{z_R^{(k)}\}_j$ is the right eigenvector correspondent to the j^{th} mode at the k^{th} airspeed.

The advantage of this method is that is mathematically simple; it does not require the use of the aeroelastic matrix of the system, neither its left eigenvector. Also, this method was found to be the most robust mode tracking method for p-k flutter analysis [41] .

To demonstrate the use of this capability for more complex problems, the eigenvalue analysis of the wing is conducted, using the unsteady aerodynamic model and four basis functions for each state. The unsteady model will be extensively discussed in the upcoming sections. Fig. 5.4a shows many mode crossings occur before and after the flutter speed, making it even more difficult to understand the complex interaction among the modes. Applying the mode tracking methodology, the mode crossings are eliminated in Fig. 5.4b.

5.2.1 Linear aeroelastic system with quasisteady aerodynamic

Eigenvalue analysis

Fig. 5.5 illustrates the eigenvalue analysis of the linear aeroelastic system with quasisteady aerodynamic using two basis functions for each state. The wing parameters are defined in Appendix D. The graph shows the aeroelastic modal frequencies and decay rates against airspeed. At $V_F = 13.28$ m/s the decay rate of the root-pitch dominated mode becomes positive, leading to instability. Recall that this is defined as the structural damping required for stabilizing the system. To get a better physical insight about the flutter type, the zoomed plot is showed in Fig. 5.6. On the left graph, the frequencies of the first mode in bending and the root-pitch mode coalesce, indicating coalescence flutter. On the right side of the graph, it is important to notice that, after the flutter speed, another instability point appears for the first mode in torsion at 21.2 m/s. For the linear problem, the flutter speed is sufficient to define the stability margin of the system.

(a) Without mode tracking

(b) With mode tracking

Figure 5.4: Modal frequency and decay plots of the linear aeroelastic problem with unsteady aerodynamics with and without mode tracking; four basis functions per state

Fig. 5.7 shows the time trace of the system states for an airspeed of $0.95V_F$. The initial conditions were $\bar{w}=0.1$, normalized by L , and $\theta = 15^\circ$. As expected from linear theory, the system is stable and therefore the amplitudes decay with time. Furthermore, Fig. 5.8 shows the response of the system for an airspeed of $1.05V_F$. The amplitudes grow with time, indicating the instability of the system. The in-plane bending showed no disturbance in time response because the state is fully decoupled from the others for the linear structural model.

5.2.2 Nonlinear aeroelastic system with quasisteady aerodynamic

Introducing the structural nonlinearities developed in chapter 3, the quasisteady aeroelastic system does not possess a distinct instability point demarcating the stable and unstable regions by the flutter speed. Instead, there is a transition region in the vicinity of the flutter speed in

Figure 5.5: Modal frequency and decay plots of the linear aeroelastic problem with quasisteady aerodynamics; two basis functions shapes per state

Figure 5.6: Zoomed modal frequency and decay plots of the linear aeroelastic problem with quasisteady aerodynamics; two basis functions shapes per state

Figure 5.7: Time traces of the linear aeroelastic system with quasisteady aerodynamics at $U_\infty = 0.95V_F$: initial condition $\bar{w} = 0.1$ and $\theta = 15^\circ$

which the system response reaches a steady regime in which the amplitudes neither decay nor grow, known as the limit cycle oscillation (LCO).

The convergence test for the nonlinear aeroelastic model with quasisteady aerodynamics is shown in Fig. 5.9 for an LCO encountered at airspeed equal to the linear flutter speed. The convergence test indicates that two basis functions in each bending and torsion are necessary to achieve convergence of the time trace. A phase space plot of the convergence test is shown in Fig. 5.10 as well.

A comparison between the linear and nonlinear system time responses for several airspeeds is shown Fig. 5.11. The initial conditions were $\bar{w} = 0.1$ and $\theta = 15^\circ$. Two basis functions were used for each state. For airspeeds well below from the flutter speed the linear and nonlinear time responses are similar, as seen in Fig. 5.11a and Fig. 5.11b for airspeeds $0.01V_F$ and $0.5V_F$ respectively. In this airspeed range, the system is stable for both linear and nonlinear systems and the displacements are not large enough to induce significant nonlinear effects. The nonlinear

Figure 5.8: Time traces of the linear aeroelastic system with quasisteady aerodynamics at $U_\infty = 1.05V_F$: initial condition $\bar{w} = 0.1$ and $\theta = 15^\circ$

coupling effect between the in-plane bending $\bar{v}$ and the other states is the main difference between the linear and nonlinear responses in this airspeed range below the flutter speed.

As the system approaches the instability point, the linear and nonlinear systems predominantly oscillate in the modal frequency of the root-pitch angle, which is the dominant mode in the flutter phenomena. Fig. 5.11c and Fig. 5.11d show the time traces near the linear flutter speed at $0.95V_F$ and $0.99V_F$ respectively. In this range of airspeeds, the linear system shows smaller decay than the nonlinear one because the linear system is at the vicinity of the instability point. At airspeed $0.99V_F$, the amplitudes of the linear system are larger than those of the nonlinear one.

The time traces in the post flutter region are shown in Fig. 5.11e and Fig. 5.12a for airspeeds $1.01V_F$ and $1.05V_F$ respectively. As expected, the linear response grows more strongly for increasing airspeeds. The nonlinear response, however, reaches a self-sustained oscillation, i.e., the amplitude neither decay nor grow but stay within boundaries.

Figure 5.9: Nonlinear quasisteady aeroelastic model: basis functions convergence test in time trace. Initial condition $\theta = 5^\circ$ and airspeed $1.00V_F$

Figure 5.10: Nonlinear quasisteady aeroelastic model: basis functions convergence test in phase space plot. Initial condition $\theta = 5^\circ$ and airspeed $1.00V_F$

Figure 5.11: Time traces comparison between linear and nonlinear aeroelastic system with quasisteady aerodynamics. Initial conditions: $\bar{w} = 0.1$, $\theta = 15^\circ$

The time traces of the nonlinear structural response of the aeroelastic system with quasisteady aerodynamics are shown in Fig. 5.12. Simulations of the first 30 seconds with airspeeds

equal to $1.01V_F$, $1.05V_F$ and $1.15V_F$ are shown in Fig. 5.12a, Fig. 5.12c and Fig. 5.12e respectively. Thirty seconds of simulation were chosen to guarantee that the system is no longer in a transient region. In addition, the steady regime of the responses is shown in Fig. 5.12b, Fig. 5.12d and Fig. 5.12f.

Significant differences in the time traces are found in the steady regime in Fig. 5.12. The time traces for airspeeds $1.01V_F$ and $1.05V_F$ have relatively small pitch rotation than the one for airspeed $1.15V_F$. For $1.01V_F$ and $1.05V_F$, the two coalescing modes, out-of-plane bending and root-pitch rotation, show a behaviour similar to harmonic motion with one dominant frequency. The in-plane bending and torsion time traces, however, show more than one dominant frequency and non-harmonic motion. For $1.15V_F$, the nonlinear time trace does not diverge, as the linear response does. Instead, for airspeeds higher than $1.05V_F$ the nonlinear system reaches a self-sustained oscillation with relatively high root-pitch angle, in the order of 20° , which is beyond the scope of the linear aerodynamic model.

The quasisteady aeroelastic system presents LCO in the vicinity of the flutter point for an airspeed between $1.01V_F$ and $1.05V_F$. By increasing the airspeed beyond $1.05V_F$, the system reaches a large amplitude oscillation concerning root-pitch angle which is not adequately modeled by using a linear aerodynamic model.

Phase plots of the nonlinear time traces in the steady regime are shown in Fig. 5.13 for several airspeeds. The small amplitude LCO observed for airspeeds $1.01V_F$, $1.03V_F$ and $1.05V_F$ are shown in Fig. 5.13a, Fig. 5.13b and Fig. 5.13c respectively. The phase space plot for $1.01V_F$ presents well-defined elliptical shapes in both bending and root-pitch modes, indicating that the linear structural dynamics is predominant for these modes. The torsion and in-plane bending phase space plots show different shapes and inner loops, indicating a nonlinear behaviour. As the airspeed increases in the small amplitude oscillation range (from $1.01V_F$ to $1.05V_F$), the nonlinearities start being more significant altering the shape of the phase space plots.

Fig. 5.13d, Fig. 5.13e and Fig. 5.13f show the phase space plots of the large amplitude LCO observed for airspeeds $1.10V_F$, $1.15V_F$ and $1.19V_F$ respectively. The phase space plots of the large amplitude LCO shows that the amplitudes do not vary significantly and they show similar

Figure 5.12: Time traces of LCO of the nonlinear aeroelastic system with quasisteady aerodynamics. Initial conditions: $\theta = 10^\circ$

nonlinear behavior indicated by the phase space plots shape.

The Fig. 5.14 shows the PSD of the small amplitude LCO for airspeed $1.01 V_F$ showed in Fig. 5.12a. The PSD of the pitch time trace shows the odd superharmonic frequencies ($3f$, $5f$ and

Figure 5.13: Phase space plots of LCO for the nonlinear aeroelastic system with quasisteady aerodynamics for several airspeeds. Initial conditions: $\theta = 10^\circ$

7f denoted in pitch PSD) indicating nonlinear effects.

Fig. 5.15 shows the summary of LCO amplitudes for quasisteady aerodynamics. Two types of behaviours were found: one with small amplitude (between 1.01 and 1.07 V_F) and another

Figure 5.14: Nonlinear aeroelastic response PSD at airspeed $1.01V_F$ Initial conditions: $\theta_0 = 10^\circ$

with large amplitude (between 1.08 and $1.19V_F$). Fig. 5.16 shows the time traces of the small and large amplitude oscillations for 3 basis functions per state.

Figure 5.15: Summary of LCO for quasisteady aerodynamics: initial $\theta = 10^\circ$

Figure 5.16: Time traces of LCO of the nonlinear aeroelastic system with quasisteady aerodynamics. Initial conditions: $\theta = 10^\circ$

5.3 Unsteady aerodynamics

In general, unsteady aerodynamic modeling includes the additional considerations of added inertia terms and the circulatory effects due to the vortex shedding. In this study, however, the added inertia has already been included in the quasisteady aerodynamics. Thus, only the unsteady circulatory terms still need to be accounted for the unsteady aerodynamics. These circulatory effects account for the lag originating from the starting vortices traveling downstream [30]. In the time domain approach, they can be modeled using Wagner's function.

Wagner's method introduces unsteady effects by adding an additional degree of freedom z . This variable carries the contribution of the lag effects of the starting vortices. As the vortex travels downstream, its influence on the wing diminishes with the distance to the wing. For incompressible flow, Wagner's function approximates the lag effect due to vortex shedding. A two term approximation of this function as developed by Jones [31] is given:

$$\Phi(\tau) = 1 - 0.165e^{-0.0455\tau} - 0.335e^{-0.3\tau}, \quad (5.25)$$

where τ is the dimensionless time parameter $\tau = \frac{U_\infty t}{b}$, U_∞ is the free stream airspeed and b is the half-chord length.

The Wagner's function can be written in the dimensional form

$$\Phi(t) = 1 - 0.165e^{\frac{-0.0455U_\infty}{b}t} - 0.335e^{\frac{-0.3U_\infty}{b}t}, \quad (5.26)$$

where $b_1 = 0.0455\frac{U_\infty}{b}$, $b_2 = 0.3\frac{U_\infty}{b}$, $A_1 = 0.165$, and $A_2 = 0.335$.

For a sudden increase in downwash (or angle of attack) the circulatory component of the lift can be written as [42]

$$L_c(x, t) = \rho_\infty U_\infty b C_{L_\alpha} \omega(\alpha_{eff}) \Phi(t), \quad (5.27)$$

where $\omega(\alpha_{eff})$ is the downwash, which is the vertical component of the airspeed due to the effective angle of attack (AOA) of the wing chord. For a small AOA assumption, the downwash term is given by $\omega(\alpha_{eff}) = U_\infty \theta_{eff}$.

Using the thin airfoil theory for continuous arbitrary motion, the downwash is obtained from the $\frac{3}{4}c$ position (adapted from [42]):

$$\omega_{\frac{3}{4}c} = U_\infty \alpha_{eff} = U_\infty \left(\theta + \phi + \frac{\dot{w}}{U_\infty} + \frac{b(\frac{1}{2} - a_h)(\dot{\theta} + \dot{\phi})}{U_\infty} \right). \quad (5.28)$$

Therefore, when the unsteady term is considered, an additional differential equation needs to be included to the unsteady aeroelastic model [42]

$$\ddot{z}(t) + \dot{z}(t)(b_1 + b_2) + z(t)b_1b_2 = \omega_{\frac{3}{4}c}. \quad (5.29)$$

The unsteady equations of lift, moment and downwash are summarized as follows (adapted from [42]):

$$\begin{aligned} L(x, t) = & \frac{1}{2} \rho_\infty U_\infty^2 C_{L_\alpha} \left[\frac{b^2}{U_\infty^2} \ddot{w} - \frac{b^3 a_h}{U_\infty^2} (\ddot{\theta} + \ddot{\phi}) + \frac{b}{U_\infty} \dot{w} + \frac{b^2(\frac{1}{2} - a_h)}{U_\infty} (\dot{\theta} + \dot{\phi}) \right. \\ & \left. + \frac{2b(A_1 b_1 + A_2 b_2)}{U_\infty} \dot{z} + b(\theta + \phi) + \frac{2b(b_1 b_2 (A_1 + A_2))}{U_\infty} z \right] \end{aligned} \quad (5.30)$$

$$\begin{aligned}
M_{ea}(x, t) = & \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L\alpha} \left[\frac{b^3 a_h}{U_{\infty}^2} \ddot{w} - \frac{b^4 (\frac{1}{8} + a_h^2)}{U_{\infty}^2} (\ddot{\theta} + \ddot{\phi}) + \frac{b^2 (\frac{1}{2} + a_h)}{U_{\infty}} \dot{w} \right. \\
& + \frac{b(b(\frac{1}{2} - a_h))^2}{U_{\infty}} (\dot{\theta} + \dot{\phi}) + \frac{2b^2 (\frac{1}{2} + a_h)(A_1 b_1 + A_2 b_2)}{U_{\infty}} \dot{z} \\
& \left. + b^2 (\frac{1}{2} + a_h)(\theta + \phi) + \frac{2b^2 (\frac{1}{2} + a_h)(b_1 b_2 (A_1 + A_2))}{U_{\infty}} z \right]
\end{aligned} \tag{5.31}$$

$$\omega_{\frac{3c}{4}}(t) = U_{\infty}(\theta + \phi) + \dot{w} + b\left(\frac{1}{2} - a_h\right)(\dot{\theta} + \dot{\phi}). \tag{5.32}$$

Finite dimensional approximation of the equations of motion

The finite dimensional approximation of the unsteady aerodynamic loads considers the same state approximation by basis functions as the quasisteady model. However, the unsteady model requires that the additional degree of freedom z also be approximated by basis functions as:

$$z = \sum_{i=1}^M \iota_i(t) \Psi_i(\bar{x}) + \sum_{j=1}^N \zeta_j(t) \Omega_j(\bar{x}). \tag{5.33}$$

Considering only the aerodynamic equations, substituting the expanded variables and applying the Galerkin procedure, we obtain the following non-dimensional linear ordinary-differential equations. The lift and moment for the unsteady aerodynamics are given as:

$$\begin{aligned}
L_i = & \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L\alpha} \left[\frac{b^2}{m U_{\infty}^2} \sum_{j=1}^M C_{\Psi\Psi}^{ij} \ddot{W}_j - \frac{b^3 a_h}{m L U_{\infty}^2} (\ddot{\theta} + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \ddot{\Phi}_k) \right. \\
& + \frac{b}{U_{\infty}} \sqrt{\frac{L^4}{m E I_y}} \sum_{j=1}^M C_{\Psi\Psi}^{ij} \dot{W}_j + \frac{b^2 (\frac{1}{2} - a_h)}{U_{\infty}} \sqrt{\frac{L^2}{m E I_y}} (\dot{\theta} + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \dot{\Phi}_k) \\
& + \frac{2b(A_1 b_1 + A_2 b_2)}{U_{\infty}} \sqrt{\frac{L^2}{m E I_y}} \sum_{k=1}^N C_{\Psi\Omega}^{ik} \dot{\zeta}_k + b(\theta + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \Phi_k) \\
& \left. + \frac{2b(b_1 b_2 (A_1 + A_2))}{U_{\infty}} \frac{L^3}{E I_y} \sum_{k=1}^N C_{\Psi\Omega}^{ik} \zeta_k \right]
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
M_{ea_j} = & \frac{1}{2} \rho_\infty U_\infty^2 C_{L_\alpha} \left[\frac{b^3 a_h}{m L U_\infty^2} \sum_{k=1}^M C_{\Omega\Psi}^{ik} \ddot{W}_k - \frac{b^4 (\frac{1}{8} + a_h^2)}{m L^2 U_\infty^2} (\ddot{\theta} + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \ddot{\Phi}_j) \right. \\
& + \frac{b^2 (\frac{1}{2} + a_h)}{U_\infty} \sqrt{\frac{L^2}{m E I_y}} \sum_{k=1}^M C_{\Omega\Psi}^{ik} \dot{W}_k \\
& + \frac{b (b (\frac{1}{2} - a_h))^2}{U_\infty} (\dot{\theta} + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \dot{\Phi}_j) \sqrt{\frac{1}{m E I_y}} \\
& + \frac{2 b^2 (\frac{1}{2} + a_h) (A_1 b_1 + A_2 b_2)}{U_\infty} \sqrt{\frac{1}{m E I_y}} \sum_{j=1}^N C_{\Omega\Omega}^{ij} \dot{\zeta}_j \\
& + b^2 (\frac{1}{2} + a_h) (\theta + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \Phi_j) \frac{L^2}{E I_y} \\
& \left. + \frac{2 b^2 (\frac{1}{2} + a_h) (b_1 b_2 (A_1 + A_2))}{U_\infty} \frac{L^2}{E I_y} \sum_{j=1}^N C_{\Omega\Omega}^{ij} \zeta_j \right] \quad (5.35)
\end{aligned}$$

$$\omega_{\frac{3c}{4}j} = U_\infty (\theta + \sum_{k=1}^N C_{\Omega}^k \Phi_k) + \sqrt{\frac{E I_y}{m L^2}} \sum_{j=1}^M C_{\Psi}^j \dot{W}_j + b (\frac{1}{2} - a_h) \sqrt{\frac{E I_y}{m L^4}} (\dot{\theta} + \sum_{k=1}^N C_{\Omega}^k \dot{\Phi}_k) \quad (5.36)$$

A more extensive derivation of the aeroelastic equations of motion and its parameters is shown in Appendix C.

5.3.1 Linear aeroelastic system with unsteady aerodynamic

Eigenvalue analysis

The eigenvalue analysis of the linear aeroelastic system with unsteady aerodynamic was performed using two mode shapes for each state. The wing parameters are defined in Appendix D. Fig. 5.17 shows the modal frequency and decay rate changes with airspeed. The first instability point occurs at $V_F = 13.46 \text{ m/s}$ dominated by the root-pitch mode. In order to get a better physical insight into the flutter type, the zoomed plot is shown in Fig. 5.18. On the left plot, the frequencies of the first mode in out-of-plane bending and pitch modes coalesce, indicating coalescence flutter. On the right plot, notice that another instability point appears for the first mode in torsion at 22.77 m/s . Flutter occurs at slightly higher speed than the case with quasisteady aerodynamics.

For the linear unsteady system, the flutter speed is sufficient to determine the stability. The

system is stable before the flutter speed and unstable after the flutter speed.

Figure 5.17: Modal frequency and decay plots of the linear aeroelastic problem with unsteady aerodynamics

5.3.2 Nonlinear aeroelastic system with unsteady aerodynamic

The linear and nonlinear aeroelastic systems responses with unsteady aerodynamics are compared in Fig. 5.19 for several airspeeds before and after the predicted linear flutter speed $V_F = 13.46 \text{ m/s}$. The initial conditions of the states were $\bar{w} = 0.1$ and $\theta = 15^\circ$. For low airspeeds, i.e. well below the flutter speed, the linear and nonlinear time responses are similar because the system is stable and the deflections are not large enough to induce the nonlinear effects.

Fig. 5.19c and Fig. 5.19d shows the time traces near the linear flutter speed, at $0.95V_F$ and $0.99V_F$ respectively. In this range of airspeeds, both linear and nonlinear systems are stable and the response decay with time. The linear response shows slightly higher decay rate than the nonlinear one.

Figure 5.18: Modal frequency and decay plots of the linear aeroelastic problem with unsteady aerodynamics

The time traces in the post flutter region are shown in Fig. 5.19e and Fig. 5.19f, for airspeeds $1.01V_F$ and $1.05V_F$ respectively. As expected, the linear response grows more strongly for increasing airspeed. However, the nonlinear response reaches a self-sustained oscillation with large root-pitch angles.

Even though the large amplitude LCO cannot be model by a linear aerodynamic model, the time traces of the nonlinear unsteady aeroelastic system are shown in Fig. 5.20. The transient regime to reach the LCO diminishes as the airspeed increases although the frequencies and amplitudes of the time traces do not vary significantly. The phase space plots of the time traces showed in Fig. 5.20 are presented in Fig. 5.21. The main differences between the phase space plot with the increased airspeed are found in the in-plane bending and torsion, while the out-of-plane bending and root-pitch phase space plots do not vary significantly with airspeed.

The Fig. 5.22 shows the PSD of the large amplitude LCO for airspeed $1.05V_F$ showed in Fig. 5.20c. The PSD of the pitch time trace shows the odd super harmonic frequencies ($3f$, $5f$ and

Figure 5.19: Time traces comparison between linear and nonlinear aeroelastic system with unsteady aerodynamics. Initial conditions: $\bar{w} = 0.1$ and $\theta = 15^\circ$

7f denoted in pitch PSD) indicating nonlinear effects.

Fig. 5.23 shows the summary of LCO amplitudes for quasisteady aerodynamics. A large

(a) $U_{\infty} = 1.01V_F$ (b) $U_{\infty} = 1.01V_F$ steady state (c) $U_{\infty} = 1.05V_F$ (d) $U_{\infty} = 1.05V_F$ steady state (e) $U_{\infty} = 1.10V_F$ (f) $U_{\infty} = 1.10V_F$ steady state

Figure 5.20: Time response LCO of the nonlinear aeroelastic system with unsteady aerodynamics. Initial condition: $\theta = 5^\circ$

amplitude oscillation behaviour was found.

Figure 5.21: Phase space plots of the nonlinear aeroelastic system with unsteady aerodynamics. Initial condition: $\theta = 5^\circ$

Figure 5.22: Nonlinear aeroelastic response PSD at airspeed $1.05 V_F$ Initial conditions: $\theta_0 = 5^\circ$

Figure 5.23: Summary of LCO for unsteady aerodynamics: initial $\theta = 5^\circ$

Chapter 6: Concluding Remarks

6.1 Summary of Findings

This research has provided insight into aeroelastic nonlinearity through experimental investigations of the flexible airfoil. Furthermore, theory and computational models used to evaluate the nonlinear aeroelastic behaviour of the flexible cantilever airfoil with root-pitch elastic restraint have been described. Several conclusions have been drawn which led to recommendations that can advance modeling and experimental investigations of the nonlinear aeroelastic systems.

6.1.1 Experimental Findings

The results of the experimental program show that a flexible wing mounted on an elastically restrained base is capable of achieving stable limit cycle oscillations (LCO). This nonlinear phenomenon occurs mainly due to two possible sources of nonlinearities: the structural dynamics and aerodynamics. The airfoil exhibited some degree of structural nonlinearity in the free vibration tests, evidenced by the dependence of the damping and frequency dependency on the structural displacements. The damping nonlinearity could be attributed to dry friction between the wing and the beam. Furthermore, an evidence of structural nonlinear restoring forces is found in the power spectral density (PSD) of the LCO time traces, in which superharmonic frequencies are present.

Two types of LCO have been observed and described for small and large amplitude oscillation. Both LCO involve oscillations around zero degree pitch angle and dominated by the pitch. Furthermore, the small amplitude oscillations have a strong dependence on the airspeed, whereas

the large amplitude limit cycle has minimal airspeed dependency. For the same airspeed, the initiation of either one of the two oscillations depends on the initial conditions, which indicates strong nonlinear effects.

The large amplitude oscillations of the airfoil originate from coalescence flutter instability, as predicted by inviscid unsteady thin airfoil theory (in chapter 5) due to coupling between out-of-plane bending and pitch motions. The small amplitude oscillations are related to transitional Reynolds number aerodynamics and specifically due to laminar boundary-layer separation.

6.1.2 Numerical Modeling

The development of the nonlinear equations of motion in chapter 3 has contributed to extending the Hodges and Dowell [1] equations for the flexible airfoil with base rotation. The equations demonstrate that the structural nonlinearities appear due to curvature effects for large displacements and due to the rotation of the base. The base rotation is coupled with bending and torsion.

Free vibration simulations have shown that the structural nonlinearities have a considerable effect for large displacements, evidenced by the differences in time traces compared to the linear response, and by superharmonic frequencies present in the responses.

Even though the inviscid linear aerodynamics for both quasisteady and unsteady cases have limitations with respect to the experimental system, the nonlinear elastic simulations give an insight into the effects of the structural nonlinearities in self-sustained oscillations. For the quasisteady aerodynamic model, LCO is encountered in the vicinity of the predicted flutter speed. At this airspeed range, two distinct limit cycles are observed: one with small amplitude pitch angles, for airspeed between 1 and 5 percent above the flutter speed, and the other with larger pitch angles, for airspeed beyond 5 percent above the flutter speed. For the unsteady aerodynamic model, large amplitude LCO is also encountered for the unsteady aerodynamic model for airspeed between 1 and 10 percent above the linear flutter speed, for which amplitudes do not vary significantly with the airspeed.

A few observations are noteworthy for the aeroelastic models. The linear flutter happens

due a coalesce-type instability and it occurs between the pitch and the out-of-plane bending motions, which are the same states participating in coalescence flutter in the experimental investigation. In addition, the small amplitude LCO found in the numerical simulations increase with the airspeed, which is the same trend observed in the experimental small amplitude LCO. The large amplitude LCO in the numerical simulations is not dependent on the airspeed, as also observed in the experiments. For the quasisteady aerodynamic model, the system oscillates with a dominant root-pitch motion, which is the same case noted in the experiments. Finally, superharmonics are observed numerical simulations of the LCO as in the experiments.

Quantitatively, LCO amplitudes are larger than those found in the experimental tests due to the absence of aerodynamic nonlinearity in simulations. Overall, the success of the study in developing and implementing the nonlinear equations of motion for the particular experimental setup suggests the potential of these structural models to enhance the understanding of nonlinear aeroelasticity of flexible systems.

6.2 Future Research

Some suggestions for future research are proposed based on the findings of the experimental and numerical study:

- Comprehensive bifurcation analysis of the nonlinear equations of motion using nonlinear software packages such as AUTO [43] and DsTool [44].
- Experimental investigation of LCO of a flexible cantilever airfoil with a base rotation having a linear elastic restraint with different specimens and configurations.
- Experimental study of the boundary layer tripping mechanics to observe the effect of laminar separation into the small amplitude LCO.
- Use of dynamic stall model to the system to reproduce large amplitudes LCO.
- Use of a low-to-moderate Reynolds number aerodynamic model to attempt reproducing small amplitudes LCO.

- Use of nonlinear structural damping model.

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Appendix A: Details of the derivation of the equation of motion

A.1 Deformed wing coordinate system transformation

The derivation of the coordinate system transformation matrix is adapted from Hodges and Dowell [1]. The transformation matrix $[T]$ describes the transformation between the undeformed local coordinate system (x, y, z) into the deformed coordinate system $(\epsilon, \eta, \zeta) = (x', y', z')$ and it can be expressed as

$$\begin{bmatrix} \vec{i}' \\ \vec{j}' \\ \vec{k}' \end{bmatrix} = [T] \begin{bmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{bmatrix}. \quad (\text{A.1})$$

The transformation matrix may be expressed in terms of Euler angles $\bar{\zeta}$, $\bar{\beta}$ and $\bar{\theta}$, which represent the rotation with respect to z , y and x axes respectively[1]. The Euler angles are shown in Fig. A.1.

$$[T] = \begin{bmatrix} \cos \bar{\beta} \cos \bar{\zeta} & \cos \bar{\beta} \sin \bar{\zeta} & \sin \bar{\beta} \\ -\sin \bar{\theta} \sin \bar{\beta} \cos \bar{\zeta} - \cos \bar{\theta} \sin \bar{\zeta} & \cos \bar{\theta} \cos \bar{\zeta} - \sin \bar{\zeta} \sin \bar{\beta} \cos \bar{\theta} & \cos \bar{\beta} \sin \bar{\theta} \\ -\cos \bar{\theta} \sin \bar{\beta} \cos \bar{\zeta} + \sin \bar{\theta} \sin \bar{\zeta} & -\sin \bar{\theta} \cos \bar{\zeta} - \sin \bar{\zeta} \sin \bar{\beta} \cos \bar{\theta} & \cos \bar{\beta} \cos \bar{\theta} \end{bmatrix}. \quad (\text{A.2})$$

Figure A.1: Deformation and Euler (adapted from [1])

The Euler angles at $[T]$ are taken in the order $\bar{\zeta}$, $\bar{\beta}$, $\bar{\theta}$. The transformation matrix may vary with the order of Euler angles but the expression in terms of displacements should be unique. The sine and cosine of the Euler angle $\bar{\zeta}$ and $\bar{\beta}$ can be expressed in terms of displacements as

$$[T] = \begin{bmatrix} \sqrt{1 - v'^2 - w'^2} & v' & w' \\ -\frac{[\sin \bar{\theta} w' \sqrt{1 - v'^2 - w'^2} + \cos \bar{\theta} v']}{\sqrt{1 - w'^2}} & \frac{[\cos \bar{\theta} \sqrt{1 - v'^2 - w'^2} - \sin \bar{\theta} v' w']}{\sqrt{1 - w'^2}} & \sqrt{1 - w'^2} \sin \bar{\theta} \\ -\frac{[\cos \bar{\theta} w' \sqrt{1 - v'^2 - w'^2} + \sin \bar{\theta} v']}{\sqrt{1 - w'^2}} & -\frac{[\sin \bar{\theta} \sqrt{1 - v'^2 - w'^2} + \cos \bar{\theta} v' w']}{\sqrt{1 - w'^2}} & \sqrt{1 - w'^2} \cos \bar{\theta} \end{bmatrix}. \quad (\text{A.3})$$

The Euler angle $\bar{\theta}$ can be determined by the solution of a differential equation for $[T]$. Considering a small rotation $\vec{\omega} dr$ of the undeformed local coordinate system (x, y, z) . The vector components of the rate of rotation $\vec{\omega}$ can be identified as the torsional rate $\omega_i = \dot{\phi}$, and the bending curvature $\omega_j = -\dot{\bar{\beta}} \cos \theta$ and $\omega_k = \dot{\bar{\beta}} \sin \theta$. This infinitesimal rotation can be written in

terms of the transformation matrix $[T]$ as

$$\begin{bmatrix} 0 & -\omega_k & \omega_j \\ \omega_k & 0 & -\omega_i \\ -\omega_j & \omega_i & 0 \end{bmatrix} = [T]^+ [T]^T. \quad (\text{A.4})$$

The angle $\bar{\theta}$ is simply equal to the elastic twist at the section ϕ . Therefore, the transformation matrix can be rewritten to second order as

$$[T] = \begin{bmatrix} 1 - \frac{1}{2}v'^2 - \frac{1}{2}w'^2 & v' & w' \\ -v' \cos \phi - w' \sin \phi & (1 - \frac{1}{2}v'^2) \cos \phi & (1 - \frac{1}{2}w'^2) \sin \phi \\ v' \sin \phi - w' \cos \phi & -(1 - \frac{1}{2}v'^2) \sin \phi & (1 - \frac{1}{2}w'^2) \cos \phi \end{bmatrix}. \quad (\text{A.5})$$

The expressions for the angular velocities ω_i and ω_j can be found by solving the Eq. Fig. A.4, simplified to second order one has:

$$\begin{aligned} \omega_j &= v'' \sin \phi - w'' \cos \phi \\ \omega_k &= v'' \cos \phi + w'' \sin \phi \end{aligned} \quad (\text{A.6})$$

Appendix B: Basis shape functions and the Galerkin method

The derivation of the exact basis shape functions for flexural and torsional vibration of a clamped-free beam and shaft, respectively, is detailed in [36]. These basis shape functions are the vibration mode shapes for the uncoupled, undamped and uniform cantilever beam. These basis shape functions described in equation 4.2 are shown in Fig. B.1.

Flexural basis functions

The flexural basis functions and its derivatives can be written as

$$\begin{aligned}
 \Psi_i(\bar{x}) &= \cosh(\beta_i L \bar{x}) - \cos(\beta_i L \bar{x}) - \sigma_i [\sinh(\beta_i L \bar{x}) - \sin(\beta_i L \bar{x})] \\
 \frac{1}{\beta_i} \frac{d\Psi_i(\bar{x})}{d\bar{x}} &= \Psi_i'(\bar{x}) = \sinh(\beta_i L \bar{x}) + \sin(\beta_i L \bar{x}) - \sigma_i [\cosh(\beta_i L \bar{x}) - \cos(\beta_i L \bar{x})] \\
 \frac{1}{\beta_i^2} \frac{d^2\Psi_i(\bar{x})}{d\bar{x}^2} &= \Psi_i''(\bar{x}) = \cosh(\beta_i L \bar{x}) + \cos(\beta_i L \bar{x}) - \sigma_i [\sinh(\beta_i L \bar{x}) + \sin(\beta_i L \bar{x})] \\
 \frac{1}{\beta_i^3} \frac{d^3\Psi_i(\bar{x})}{d\bar{x}^3} &= \Psi_i'''(\bar{x}) = \sinh(\beta_i L \bar{x}) - \sin(\beta_i L \bar{x}) - \sigma_i [\cosh(\beta_i L \bar{x}) + \cos(\beta_i L \bar{x})] \\
 \frac{1}{\beta_i^4} \frac{d^4\Psi_i(\bar{x})}{d\bar{x}^4} &= \Psi_i''''(\bar{x}) = \cosh(\beta_i L \bar{x}) - \cos(\beta_i L \bar{x}) - \sigma_i [\sinh(\beta_i L \bar{x}) - \sin(\beta_i L \bar{x})].
 \end{aligned} \tag{B.1}$$

The boundary values that satisfy the essential boundary conditions are

$$\begin{aligned}
 \Psi_i(0) &= \Psi_i'(0) = 0 \\
 \Psi_i''(0) &= \Psi_i'''(0) = 0.
 \end{aligned} \tag{B.2}$$

Figure B.1: Three first flexural and torsional basis functions

The five first solutions for the flexural characteristic Eq. (4.3) are shown in Table B.1 as well the values of σ_i and the frequency ratio between the first mode and the others.

Table B.1: Five first solutions for the flexural characteristic equations and frequency ratios

i	$\beta_i L$	σ_i	$\frac{\omega_i}{\omega_1} = (\frac{\beta_i}{\beta_1})^2$
1	1.87510	0.7341	1.0000
2	4.69409	1.0185	6.2669
3	7.85476	0.9992	17.5476
4	10.9955	1.0000	34.3863
5	14.1372	1.0000	56.8432

The flexural basis functions satisfy the orthogonality

$$\int_0^1 \Psi_r \Psi_s d\bar{x} = \begin{cases} 1, & r = s \\ 0, & r \neq s. \end{cases} \quad (\text{B.3})$$

Torsional basis functions

The basis functions and its derivatives of the torsional modes can be derived as

$$\begin{aligned} \Omega_j(\bar{x}) &= \sin(\gamma_j L \bar{x}) \\ \frac{1}{\gamma_j} \frac{d\Omega_j(\bar{x})}{d\bar{x}} &= \Omega'_j(\bar{x}) = \cos(\gamma_j L \bar{x}) \\ \frac{1}{\gamma_j^2} \frac{d^2\Omega_j(\bar{x})}{d\bar{x}^2} &= \Omega''_j(\bar{x}) = -\sin(\gamma_j L \bar{x}). \end{aligned} \quad (\text{B.4})$$

The boundary values that satisfy the essential boundary conditions are

$$\begin{aligned} \Omega_j(0) &= 0 \\ \Omega'_j(1) &= 0. \end{aligned} \quad (\text{B.5})$$

The solution for the torsional characteristic Eq. (4.5) is given by

$$\gamma_j L = \left(j - \frac{1}{2}\right) \pi. \quad (\text{B.6})$$

The five first values of $\gamma_j L$ and the frequency ratio between the first mode and the other ones are shown in Table B.2.

Table B.2: Five first solutions for the torsional characteristic equations and amplitude ratios

j	$\gamma_j L$	$\frac{\omega_j}{\omega_1} = \left(\frac{\gamma_j}{\gamma_1}\right) = 2j - 1$
1	1.5708	1.0
2	4.7124	3.0
3	7.8540	5.0
4	10.9956	7.0
5	14.1372	9.0

The torsional basis functions satisfy the orthogonality

$$\int_0^1 \Omega_r \Omega_s d\bar{x} = \begin{cases} \frac{1}{2}, & r = s \\ 0, & r \neq s. \end{cases} \quad (\text{B.7})$$

Appendix C: Nondimensionalization and finite dimensional approximation of the equations of motion

C.1 Nondimensional nonlinear bending-bending-torsion equations of motion

In order to get the nondimensionalized equations, the states v , w and time t are made dimensionless by introducing a characteristic length L , equal to the wing span, and a characteristic time $\sqrt{\frac{mL^4}{EI_{y'}}$. The characteristic time adopted in this study is inversely proportional to the system natural frequency. Using these characteristic terms, the system parameters are written in nondimensional form as follows.

$$\begin{aligned} \bar{v} &= \frac{v}{L}, \quad \bar{w} = \frac{w}{L}, \quad \bar{x} = \frac{x}{L}, \quad \bar{e} = \frac{e}{L}, \quad t^* = t\sqrt{\frac{EI_{y'}}{mL^4}}, \\ \beta_z &= \frac{EI_{z'}}{EI_{y'}}, \quad \beta_x = \frac{GJ}{EI_{y'}}, \quad \beta_\theta = \frac{K_\theta L}{EI_{y'}}, \quad \beta_B = \frac{I_B}{mL^3}, \\ \mu &= \frac{K_m}{L}, \quad \mu_1 = \frac{K_{m1}}{L}, \quad \mu_2 = \frac{K_{m2}}{L}, \\ ()' &= \frac{\partial}{\partial \bar{x}}, \quad \dot{()} = \frac{\partial}{\partial t^*}, \end{aligned} \tag{C.1}$$

The nonlinear equations of motion can be rewritten in the nondimensional form. For

notation purposes, the $\bar{(\)}$ superscripts are omitted. δv equation:

$$(\beta_z - 1)[w''''\phi + w''(\phi'' - 4\phi\phi'^2)] + \beta_z[v'''' - v''(2\phi\phi'' + 2\phi'^2)] + [v''(2\phi\phi'' + 2\phi'^2)] + \ddot{v} + e\ddot{\phi} - \ddot{\theta}(w + e\phi) - \dot{\theta}^2(v + e) - 2\dot{\theta}(\dot{w} + e\dot{\phi}) = 0 \quad (\text{C.2})$$

δw equation:

$$(\beta_z - 1)[v''''\phi + v''(\phi'' - 4\phi\phi'^2)] + \beta_z[w''(2\phi\phi'' + 2\phi'^2)] + [w'''' - w''(2\phi\phi'' + 2\phi'^2)] + \ddot{w} + e\ddot{\phi} + \ddot{\theta}(v + e) - \dot{\theta}^2(w + e\phi) + 2\dot{\theta}(\dot{v} + e\dot{\phi}) = 0 \quad (\text{C.3})$$

$\delta\phi$ equation:

$$-\beta_x\phi'' + (\beta_z - 1)[\phi(w''^2 - v''^2) + v''w''] + e(\phi\ddot{v} + \ddot{w}) + \mu^2\ddot{\phi} + \ddot{\theta}(\mu^2 + e\phi w) + 2e\dot{\theta}(\dot{v} + \phi\dot{w}) + e\dot{\theta}^2(\phi v - w) = 0 \quad (\text{C.4})$$

$\delta\theta$ equation:

$$\int_0^L (-\ddot{v}(w + e\phi) + \ddot{w}(v + e) + \ddot{\phi}(\mu^2 + ew\phi) + \ddot{\theta}[(v^2 + w^2 + \mu^2 + 2e(v + w\phi)] + 2e\dot{\theta}\dot{\phi}(w - v\phi) + 2\dot{\theta}\dot{w}(w + e\phi) + 2\dot{\theta}\dot{v}(v + e) + \beta_\theta\theta + \beta_B\ddot{\theta})dx = 0 \quad (\text{C.5})$$

C.2 Finite dimensional approximation of the structural equations of motion

Substituting the expanded variables and applying the Galerkin procedure we obtain finite dimensional approximation of the structural equations of motion, that are nonlinear, homogeneous, ordinary differential equations.

δV_i equations:

$$\begin{aligned}
& (\beta_z - 1) \sum_{j=1}^M \sum_{k=1}^N [C_{v1}^{ijk} W_j \Phi_k + C_{v2}^{ijk} W_j \Phi_k - 4 \sum_{l=1}^N \sum_{m=1}^N C_{v3}^{ijklm} W_j \Phi_k \Phi_l \Phi_m] + \beta_z \sum_{j=1}^M [C_{v4}^{ij} V_j \\
& - 2 \sum_{k=1}^N \sum_{l=1}^N C_{v5}^{ijkl} V_j \Phi_k \Phi_l - 2 \sum_{k=1}^N \sum_{l=1}^N C_{v6}^{ijkl} V_j \Phi_k \Phi_l] + \sum_{j=1}^M \sum_{k=1}^N \sum_{l=1}^N [2C_{v7}^{ijkl} V_j \Phi_k \Phi_l + 2C_{v8}^{ijkl} V_j \Phi_k \Phi_l] \\
& + \sum_{j=1}^M C_{v9}^{ij} \ddot{V}_j + e \sum_{k=1}^N \sum_{l=1}^N C_{v10}^{ikl} \ddot{\Phi}_k \Phi_l - \sum_{j=1}^M C_{v11}^{ij} \ddot{\theta} W_j - e \sum_{k=1}^N C_{v12}^{ik} \ddot{\theta} \Phi_k - \sum_{j=1}^M C_{v13}^{ij} \dot{\theta}^2 V_j - e C_{v14}^i \dot{\theta}^2 \\
& - 2 \sum_{j=1}^M C_{v15}^{ij} \dot{\theta} \dot{W}_j - 2e \sum_{k=1}^N C_{v16}^{ik} \dot{\theta} \dot{\Phi}_k = 0
\end{aligned} \tag{C.6}$$

δW_i equations:

$$\begin{aligned}
& (\beta_z - 1) \sum_{j=1}^M \sum_{k=1}^N [C_{w1}^{ijk} V_j \Phi_k + C_{w2}^{ijk} V_j \Phi_k - 4 \sum_{l=1}^N \sum_{m=1}^N C_{w3}^{ijklm} V_j \Phi_k \Phi_l \Phi_m] + \beta_z \sum_{j=1}^M \sum_{k=1}^N \sum_{l=1}^N [2C_{w4}^{ijkl} W_j \Phi_k \Phi_l \\
& + 2C_{w5}^{ijkl} W_j \Phi_k \Phi_l] + \sum_{j=1}^M [C_{w6}^{ij} W_j - 2 \sum_{k=1}^N \sum_{l=1}^N C_{w7}^{ijkl} W_j \Phi_k \Phi_l - 2 \sum_{k=1}^N \sum_{l=1}^N C_{w8}^{ijkl} W_j \Phi_k \Phi_l] \\
& + \sum_{j=1}^M C_{w9}^{ij} \ddot{W}_j + e \sum_{k=1}^N C_{w10}^{ik} \ddot{\Phi}_k + \sum_{j=1}^M C_{w11}^{ij} \ddot{\theta} V_j + e C_{w12}^i \ddot{\theta} - \sum_{j=1}^M C_{w13}^{ij} \dot{\theta}^2 W_j - e \sum_{k=1}^N C_{w14}^{ik} \dot{\theta}^2 \Phi_k \\
& + 2 \sum_{j=1}^M C_{w15}^{ij} \dot{\theta} \dot{V}_j + 2e \sum_{k=1}^N \sum_{l=1}^M C_{w16}^{ikl} \dot{\theta} \dot{\Phi}_k \dot{\Phi}_l = 0
\end{aligned} \tag{C.7}$$

$\delta\Phi_j$ equations:

$$\begin{aligned}
& -\beta_x \sum_{j=1}^N C_{p1}^{ij} \Phi_j + (\beta_z - 1) \sum_{k=1}^M \sum_{l=1}^M \left[\sum_{j=1}^N (C_{p2}^{ijkl} \Phi_j W_k W_l - C_{p3}^{ijkl} \Phi_j V_k V_l) + C_{p4}^{ikl} V_k W_l \right] \\
& + e \sum_{k=1}^M \left(\sum_{j=1}^N C_{p5}^{ijk} \Phi_j \ddot{V}_k + C_{p6}^{ik} \ddot{W}_k \right) + \mu^2 \sum_{j=1}^N C_{p7}^{ij} \ddot{\Phi}_j + \ddot{\theta} (\mu^2 C_{p8}^i + e \sum_{j=1}^N \sum_{k=1}^M C_{p9}^{ijk} \Phi_j W_k) + 2e\dot{\theta} \sum_{k=1}^M (C_{p10}^{ik} \dot{V}_k \\
& + \sum_{j=1}^N C_{p11}^{ijk} \Phi_j \dot{W}_k) + e\dot{\theta}^2 \sum_{k=1}^M \left(\sum_{j=1}^N C_{p12}^{ijk} \Phi_j V_k - C_{p13}^{ik} W_k \right) = 0
\end{aligned} \tag{C.8}$$

$\delta\theta$ equation:

$$\begin{aligned}
& - \sum_{j=1}^M \left(\sum_{k=1}^M C_{t1}^{jk} \ddot{V}_j W_k + \sum_{l=1}^N C_{t2}^{jl} e \ddot{V}_j \Phi_l \right) + \sum_{j=1}^M \left(\sum_{k=1}^M C_{t3}^{jk} \ddot{W}_j V_k + e C_{t4}^j \ddot{W}_j \right) + \sum_{l=1}^N (\mu^2 C_{t5}^l \ddot{\Phi}_l \\
& + e \sum_{j=1}^M \sum_{m=1}^N C_{t6}^{jlm} W_j \ddot{\Phi}_l \Phi_m) + \ddot{\theta} \left[\sum_{j=1}^M \sum_{k=1}^M (C_{t7}^{jk} V_j V_k + C_{t8}^{jk} W_j W_k) + \mu^2 + 2e \sum_{j=1}^M (C_{t10}^j V_j + \sum_{l=1}^N C_{t11}^{jl} W_j \Phi_l) \right] \\
& + 2e \sum_{j=1}^M \sum_{l=1}^N \dot{\theta} (C_{t12}^{jl} W_j \dot{\Phi}_l - \sum_{m=1}^N C_{t13}^{jlm} V_j \dot{\Phi}_l \Phi_m) + 2\dot{\theta} \sum_{j=1}^M \left(\sum_{k=1}^M C_{t14}^{jk} \dot{W}_j W_k + e \sum_{l=1}^N C_{t15}^{jl} \dot{W}_j \Phi_l \right) \\
& + 2\dot{\theta} \sum_{j=1}^M \left(\sum_{k=1}^M C_{t16}^{jk} \dot{V}_j V_k + C_{t17}^j \dot{V}_j e \right) + \beta_\theta \dot{\theta} + \beta_B \ddot{\theta} = 0
\end{aligned} \tag{C.9}$$

where the coefficients for the bending-bending-torsion modal equations are defined by:

$$\begin{aligned}
C_{v1}^{ijk} &= \int_0^1 \Psi_i \Psi_j'''' \Omega_k d\bar{x} \\
C_{v2}^{ijk} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k'' d\bar{x} \\
C_{v3}^{ijklm} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k \Omega_l' \Omega_m' d\bar{x} \\
C_{v4}^{ij} &= \int_0^1 \Psi_i \Psi_j'''' d\bar{x} \\
C_{v5}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k \Omega_l'' d\bar{x} \\
C_{v6}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k' \Omega_l' d\bar{x} \\
C_{v7}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k \Omega_l'' d\bar{x} \\
C_{v8}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k' \Omega_l' d\bar{x} \\
C_{v9}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{v10}^{ikl} &= \int_0^1 \Psi_i \Omega_k \Omega_l d\bar{x} \\
C_{v11}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{v12}^{ik} &= \int_0^1 \Psi_i \Omega_k d\bar{x} \\
C_{v13}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{v14}^i &= \int_0^1 \Psi_i d\bar{x} \\
C_{v15}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{v16}^{ik} &= \int_0^1 \Psi_i \Omega_k d\bar{x}
\end{aligned} \tag{C.10}$$

$$\begin{aligned}
C_{w1}^{ijk} &= \int_0^1 \Psi_i \Psi_j'''' \Omega_k d\bar{x} \\
C_{w2}^{ijk} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k'' d\bar{x} \\
C_{w3}^{ijklm} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k \Omega_l' \Omega_m' d\bar{x} \\
C_{w4}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k \Omega_l'' d\bar{x} \\
C_{w5}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k' \Omega_l' d\bar{x} \\
C_{w6}^{ij} &= \int_0^1 \Psi_i \Psi_j'''' d\bar{x} \\
C_{w7}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k \Omega_l'' d\bar{x} \\
C_{w8}^{ijkl} &= \int_0^1 \Psi_i \Psi_j'' \Omega_k' \Omega_l' d\bar{x} \\
C_{w9}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{w10}^{ik} &= \int_0^1 \Psi_i \Omega_k d\bar{x} \\
C_{w11}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{w12}^i &= \int_0^1 \Psi_i d\bar{x} \\
C_{w13}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{w14}^{ik} &= \int_0^1 \Psi_i \Omega_k d\bar{x} \\
C_{w15}^{ij} &= \int_0^1 \Psi_i \Psi_j d\bar{x} \\
C_{w16}^{ikl} &= \int_0^1 \Psi_i \Omega_k \Omega_l d\bar{x}
\end{aligned} \tag{C.11}$$

$$\begin{aligned}
C_{p1}^{ij} &= \int_0^1 \Omega_i \Omega_j'' d\bar{x} \\
C_{p2}^{ijkl} &= \int_0^1 \Omega_i \Omega_j \Psi_k'' \Psi_l'' d\bar{x} \\
C_{p3}^{ijkl} &= \int_0^1 \Omega_i \Omega_j \Psi_k'' \Psi_l'' d\bar{x} \\
C_{p4}^{ikl} &= \int_0^1 \Omega_i \Psi_k'' \Psi_l'' d\bar{x} \\
C_{p5}^{ijk} &= \int_0^1 \Omega_i \Omega_j \Psi_k d\bar{x} \\
C_{p6}^{ik} &= \int_0^1 \Omega_i \Psi_k d\bar{x} \\
C_{p7}^{ij} &= \int_0^1 \Omega_i \Omega_j d\bar{x} \\
C_{p8}^i &= \int_0^1 \Omega_i d\bar{x} \\
C_{p9}^{ijk} &= \int_0^1 \Omega_i \Omega_j \Psi_k d\bar{x} \\
C_{p10}^{ijk} &= \int_0^1 \Omega_i \Omega_j \Psi_k d\bar{x} \\
C_{p11}^{ijk} &= \int_0^1 \Omega_i \Omega_j \Psi_k d\bar{x} \\
C_{p12}^{ijk} &= \int_0^1 \Omega_i \Omega_j \Psi_k d\bar{x} \\
C_{p13}^{ik} &= \int_0^1 \Omega_i \Psi_k d\bar{x}
\end{aligned} \tag{C.12}$$

$$\begin{aligned}
C_{t1}^{jk} &= \int_0^1 \Psi_j \Psi_k d\bar{x} \\
C_{t2}^{jl} &= \int_0^1 \Psi_j \Omega_l d\bar{x} \\
C_{t3}^{jk} &= \int_0^1 \Psi_j \Psi_k d\bar{x} \\
C_{t4}^j &= \int_0^1 \Psi_j d\bar{x} \\
C_{t5}^l &= \int_0^1 \Omega_l d\bar{x} \\
C_{t6}^{jlm} &= \int_0^1 \Psi_j \Omega_l \Omega_m d\bar{x} \\
C_{t7}^{jk} &= \int_0^1 \Psi_j \Psi_k d\bar{x} \\
C_{t8}^{jk} &= \int_0^1 \Psi_j \Psi_k d\bar{x} \\
C_{t9} &= \int_0^1 d\bar{x} = 1 \\
C_{t10}^j &= \int_0^1 \Psi_j d\bar{x} \\
C_{t11}^{jl} &= \int_0^1 \Psi_j \Omega_l d\bar{x} \\
C_{t12}^{jl} &= \int_0^1 \Omega_j \Psi_l d\bar{x} \\
C_{t13}^{jlm} &= \int_0^1 \Psi_j \Omega_l \Omega_m d\bar{x} \\
C_{t14}^{jk} &= \int_0^1 \Psi_j \Psi_k dx \\
C_{t15}^{jl} &= \int_0^1 \Psi_j \Omega_l d\bar{x} \\
C_{t16}^{jk} &= \int_0^1 \Psi_j \Psi_k d\bar{x} \\
C_{t17}^j &= \int_0^1 \Psi_j dx \\
C_{t18} &= \int_0^1 d\bar{x} = 1 \\
C_{t19} &= \int_0^1 d\bar{x} = 1
\end{aligned} \tag{C.13}$$

C.3 Linear system structural matrices

The matrices $[M]_{L_w}$, $[C]_{L_w}$, and $[K]_{L_w}$ of the linear ordinary differential equations are summarized as

$$[M]_{L_w} = \begin{bmatrix} \sum_{j=1}^M C_{v9}^{ij} & 0 & 0 & 0 \\ 0 & \sum_{j=1}^M C_{w9}^{ij} & e \sum_{k=1}^N C_{w10}^{ik} & e C_{w12}^i \ddot{\theta} \\ 0 & e \sum_{j=1}^M C_{p6}^{ik} & \mu^2 \sum_{j=1}^N C_{p7}^{ij} & \mu^2 C_{p8}^i \\ 0 & e \sum_{j=1}^M C_{t4}^j & \mu^2 \sum_{l=1}^N \mu^2 C_{t5}^l & \mu^2 + \beta_B \end{bmatrix} \quad (C.14)$$

$$[K]_{L_w} = \begin{bmatrix} \beta_z \sum_{j=1}^M C_{v4}^{ij} & 0 & 0 & 0 \\ 0 & \sum_{j=1}^M C_{w6}^{ij} & 0 & 0 \\ 0 & 0 & -\beta_x \sum_{j=1}^N C_{p1}^{ij} & 0 \\ 0 & 0 & 0 & \beta_\theta \end{bmatrix} \quad (C.15)$$

$$[C]_{L_w} = \alpha_0 [M]_{L_w} + \alpha_1 [K]_{L_w} \quad (C.16)$$

C.4 Nonlinear system structural matrices

The matrices $[M]_{NL_w}$, $[C]_{NL_w}$, and $[K]_{NL_w}$ of the nonlinear ordinary differential equations are summarized as.

$$[M]_{NL_w} = \begin{bmatrix} 0 & 0 & eC_{v10}\Phi_p & -C_{v11}W_i - eC_{v12}\Phi_j \\ 0 & 0 & 0 & C_{w11}V_i \\ eC_{p5}\Phi_j & 0 & 0 & eC_{p9}W_i\Phi_j \\ -C_{t1}W_i - eC_{t2}\Phi_j & C_{t3}V_i & eC_{t6}W_i\Phi_p & C_{t7}V_iV_p + C_{t8}W_iW_p + 2e(C_{t10}V_i + C_{t11}W_i\Phi_j) \end{bmatrix} \quad (C.17)$$

$$[K]_{NL_w} = \begin{bmatrix} K_{11NL_w} & K_{12NL_w} & K_{13NL_w} & K_{14NL_w} \\ K_{21NL_w} & K_{22NL_w} & K_{23NL_w} & K_{24NL_w} \\ K_{31NL_w} & K_{32NL_w} & K_{33NL_w} & K_{34NL_w} \\ K_{41NL_w} & K_{42NL_w} & K_{43NL_w} & K_{44NL_w} \end{bmatrix} \quad (C.18)$$

where

$$\begin{aligned}
K_{11_{NLw}} &= 2(-\beta_z(C_{v5} + C_{v6}) + (C_{v7} + C_{v8}))\Phi_j\Phi_p) \\
K_{12_{NLw}} &= \frac{1}{2}(\beta_z - 1)(C_{v1} + C_{v2} - 4C_{v3}\Phi_p\Phi_q)\Phi_j \\
K_{13_{NLw}} &= \frac{1}{2}(\beta_z - 1)(C_{v1} + C_{v2} - 4C_{v3}\Phi_p\Phi_q)W_i \\
K_{14_{NLw}} &= 0 \\
K_{21_{NLw}} &= \frac{1}{2}(\beta_z - 1)(C_{w1} + C_{w2} - 4C_{w3}\Phi_p\Phi_q)\Phi_j \\
K_{22_{NLw}} &= 2(\beta_z(C_{w4} + C_{w5}) - (C_{w7} + C_{w8}))\Phi_j\Phi_p) \\
K_{23_{NLw}} &= \frac{1}{2}(\beta_z - 1)(C_{w1} + C_{w2} - 4C_{w3}\Phi_p\Phi_q)V_i \\
K_{24_{NLw}} &= 0 \\
K_{31_{NLw}} &= \frac{1}{2}(\beta_z - 1)(C_{p4}W_i) \\
K_{32_{NLw}} &= \frac{1}{2}(\beta_z - 1)(C_{p4}V_i) \\
K_{33_{NLw}} &= (\beta_z - 1)(C_{p2}W_iW_p - C_{p3}V_iV_q) \\
K_{34_{NLw}} &= 0 \\
K_{41_{NLw}} &= 0 \\
K_{42_{NLw}} &= 0 \\
K_{43_{NLw}} &= 0 \\
K_{44_{NLw}} &= 0
\end{aligned} \tag{C.19}$$

$$[C]_{NLw} = \begin{bmatrix} C_{11_{NLw}} & C_{12_{NLw}} & C_{13_{NLw}} & C_{14_{NLw}} \\ C_{21_{NLw}} & C_{22_{NLw}} & C_{23_{NLw}} & C_{24_{NLw}} \\ C_{31_{NLw}} & C_{32_{NLw}} & C_{33_{NLw}} & C_{34_{NLw}} \\ C_{41_{NLw}} & C_{42_{NLw}} & C_{43_{NLw}} & C_{44_{NLw}} \end{bmatrix} \tag{C.20}$$

where

$$\begin{aligned}
C_{11_{NLw}} &= 0 \\
C_{12_{NLw}} &= -C_{v15}\dot{\theta} \\
C_{13_{NLw}} &= -eC_{v16}\dot{\theta} \\
C_{14_{NLw}} &= -C_{v15}\dot{W}_i - eC_{v16}\dot{\Phi}_j - \dot{\theta}(C_{v13}V_i + eC_{v14}) \\
C_{21_{NLw}} &= C_{w15}\dot{\theta} \\
C_{22_{NLw}} &= 0 \\
C_{23_{NLw}} &= eC_{w16}\Phi_j\dot{\theta} \\
C_{24_{NLw}} &= C_{w15}\dot{V}_i + eC_{w16}\Phi_j\dot{\Phi}_p - \dot{\theta}(C_{w13}W_i + eC_{w14}\Phi_j) \\
C_{31_{NLw}} &= e\dot{\theta}C_{p10} \\
C_{32_{NLw}} &= e\dot{\theta}C_{p11}\Phi_j \\
C_{33_{NLw}} &= 0 \\
C_{34_{NLw}} &= e(C_{p10}\dot{V}_i + C_{p11}\dot{W}_i\Phi_j) + e\dot{\theta}(C_{p12}V_i\Phi_j - C_{p13}W_i) \\
C_{41_{NLw}} &= \dot{\theta}(C_{t16}V_i + eC_{t17}) \\
C_{42_{NLw}} &= \dot{\theta}(C_{t14}W_i + eC_{t15}\Phi_j) \\
C_{43_{NLw}} &= e\dot{\theta}(C_{t12}W_i - C_{t13}V_i\Phi_p) \\
C_{44_{NLw}} &= (C_{t16}V_i + eC_{t17})\dot{V}_i + (C_{t14}W_i + eC_{t15}\Phi_j)\dot{W}_i + e(C_{t12}W_i - C_{t13}V_i\Phi_p)\dot{\Phi}_j
\end{aligned} \tag{C.21}$$

C.5 Finite dimensional approximation of aerodynamic terms

C.5.1 Quasisteady aerodynamics

C.5.1.1 Nondimensionalization

$$L(x, t) = \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L_{\alpha}} \left[\frac{2b}{U_{\infty}} \dot{w} \sqrt{\frac{L^4}{mEI_{y'}}} + \frac{2b^2(1-a_h)}{U_{\infty}} (\dot{\theta} + \dot{\phi}) \sqrt{\frac{L^2}{mEI_y}} + 2b(\theta + \phi) \frac{L^3}{EI_y} \right] \quad (C.22)$$

$$M_{ea}(x, t) = \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L_{\alpha}} \left[+ \frac{2b^2(\frac{1}{2} + a_h)}{U_{\infty}} \dot{w} \sqrt{\frac{L^2}{mEI_y}} + \frac{2b^3(\frac{1}{2} - a_h)a_h}{U_{\infty}} (\dot{\theta} + \dot{\phi}) \sqrt{\frac{1}{mEI_y}} + 2b^2(\frac{1}{2} + a_h)(\theta + \phi) \frac{L^2}{EI_y} \right] \quad (C.23)$$

C.5.1.2 Ordinary differential equations

$$L_i = \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L_{\alpha}} \left[\frac{2b}{U_{\infty}} \sqrt{\frac{L^4}{mEI_{y'}}} \sum_{j=1}^M C_{\Psi\Psi}^{ij} \dot{W}_j + \frac{2b^2(1-a_h)}{U_{\infty}} (\dot{\theta} + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \Phi_k) \sqrt{\frac{L^2}{mEI_y}} + 2b(\theta + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \Phi_k) \frac{L^3}{EI_y} \right] \quad (C.24)$$

$$M_{ea_j} = \frac{1}{2} \rho_{\infty} U_{\infty}^2 C_{L_{\alpha}} \left[\frac{2b^2(\frac{1}{2} + a_h)}{U_{\infty}} \sqrt{\frac{L^2}{mEI_y}} \sum_{k=1}^M C_{\Omega\Psi}^{ik} \dot{W}_k + \frac{2b^3(\frac{1}{2} - a_h)a_h}{U_{\infty}} (\dot{\theta} + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \Phi_j) \sqrt{\frac{1}{mEI_y}} + 2b^2(\frac{1}{2} + a_h)(\theta + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \Phi_j) \frac{L^2}{EI_y} \right] \quad (C.25)$$

$$\begin{aligned}
C_{\Psi\Psi}^{ij} &= \int_0^1 \sqrt{1-\bar{x}} \Psi_i \Psi_j d\bar{x} \\
C_{\Psi\Omega}^{ik} &= \int_0^1 \sqrt{1-\bar{x}} \Psi_i \Omega_k d\bar{x} \\
C_{\Omega\Psi}^{ik} &= \int_0^1 \sqrt{1-\bar{x}} \Omega_i \Psi_k d\bar{x} \\
C_{\Omega\Omega}^{ij} &= \int_0^1 \sqrt{1-\bar{x}} \Omega_i \Omega_j d\bar{x}
\end{aligned} \tag{C.26}$$

C.5.2 Unsteady aerodynamics

C.5.2.1 Nondimensionalization

$$\begin{aligned}
L(x, t) = & \frac{1}{2} \rho_\infty U_\infty^2 C_{L\alpha} \left[\frac{b^2}{m U_\infty^2} \ddot{w} - \frac{b^3 a_h}{m L U_\infty^2} (\ddot{\theta} + \ddot{\phi}) + \frac{b}{U_\infty} \dot{w} \sqrt{\frac{L^4}{m E I_y}} + \frac{b^2 (\frac{1}{2} - a_h)}{U_\infty} (\dot{\theta} + \dot{\phi}) \sqrt{\frac{L^2}{m E I_y}} \right. \\
& \left. + \frac{2b(A_1 b_1 + A_2 b_2)}{U_\infty} \dot{z} \sqrt{\frac{L^2}{m E I_y}} + b(\theta + \phi) + \frac{2b(b_1 b_2 (A_1 + A_2))}{U_\infty} z \frac{L^3}{E I_y} \right]
\end{aligned} \tag{C.27}$$

$$\begin{aligned}
M_{ea}(x, t) = & \frac{1}{2} \rho_\infty U_\infty^2 C_{L\alpha} \left[\frac{b^3 a_h}{m L U_\infty^2} \ddot{w} - \frac{b^4 (\frac{1}{8} + a_h^2)}{m L^2 U_\infty^2} (\ddot{\theta} + \ddot{\phi}) + \frac{b^2 (\frac{1}{2} + a_h)}{U_\infty} \dot{w} \sqrt{\frac{L^2}{m E I_y}} \right. \\
& + \frac{b(b(\frac{1}{2} - a_h))^2}{U_\infty} (\dot{\theta} + \dot{\phi}) \sqrt{\frac{1}{m E I_y}} + \frac{2b^2 (\frac{1}{2} + a_h) (A_1 b_1 + A_2 b_2)}{U_\infty} \dot{z} \sqrt{\frac{1}{m E I_y}} \\
& \left. + b^2 (\frac{1}{2} + a_h) (\theta + \phi) \frac{L^2}{E I_y} + \frac{2b^2 (\frac{1}{2} + a_h) (b_1 b_2 (A_1 + A_2))}{U_\infty} z \frac{L^2}{E I_y} \right]
\end{aligned} \tag{C.28}$$

$$\omega_{\frac{3c}{4}}(t) = U_\infty (\theta + \phi) + \dot{w} \sqrt{\frac{E I_y}{m L^2}} + b \left(\frac{1}{2} - a_h \right) (\dot{\theta} + \dot{\phi}) \sqrt{\frac{E I_y}{m L^4}} \tag{C.29}$$

C.5.2.2 Ordinary differential equations

$$\begin{aligned}
L_i = & \frac{1}{2} \rho_\infty U_\infty^2 C_{L_\alpha} \left[\frac{b^2}{m U_\infty^2} \sum_{j=1}^M C_{\Psi\Psi}^{ij} \ddot{W}_j - \frac{b^3 a_h}{m L U_\infty^2} (\ddot{\theta} + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \ddot{\Phi}_k) + \frac{b}{U_\infty} \sqrt{\frac{L^4}{m E I_y}} \sum_{j=1}^M C_{\Psi\Psi}^{ij} \dot{W}_j \right. \\
& + \frac{b^2 (\frac{1}{2} - a_h)}{U_\infty} \sqrt{\frac{L^2}{m E I_y}} (\dot{\theta} + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \dot{\Phi}_k) + \frac{2b(A_1 b_1 + A_2 b_2)}{U_\infty} \sqrt{\frac{L^2}{m E I_y}} \sum_{k=1}^N C_{\Psi\Omega}^{ik} \dot{\zeta}_k \\
& \left. + b(\theta + \sum_{k=1}^N C_{\Psi\Omega}^{ik} \Phi_k) + \frac{2b(b_1 b_2 (A_1 + A_2))}{U_\infty} \frac{L^3}{E I_y} \sum_{k=1}^N C_{\Psi\Omega}^{ik} \zeta_k \right] \quad (C.30)
\end{aligned}$$

$$\begin{aligned}
M_{eaj} = & \frac{1}{2} \rho_\infty U_\infty^2 C_{L_\alpha} \left[\frac{b^3 a_h}{m L U_\infty^2} \sum_{k=1}^M C_{\Omega\Psi}^{ik} \ddot{W}_k - \frac{b^4 (\frac{1}{8} + a_h^2)}{m L^2 U_\infty^2} (\ddot{\theta} + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \ddot{\Phi}_j) + \frac{b^2 (\frac{1}{2} + a_h)}{U_\infty} \sqrt{\frac{L^2}{m E I_y}} \sum_{k=1}^M C_{\Omega\Psi}^{ik} \dot{W}_k \right. \\
& + \frac{b(b(\frac{1}{2} - a_h))^2}{U_\infty} (\dot{\theta} + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \dot{\Phi}_j) \sqrt{\frac{1}{m E I_y}} + \frac{2b^2 (\frac{1}{2} + a_h) (A_1 b_1 + A_2 b_2)}{U_\infty} \sqrt{\frac{1}{m E I_y}} \sum_{j=1}^N C_{\Omega\Omega}^{ij} \dot{\zeta}_j \\
& \left. + b^2 (\frac{1}{2} + a_h) (\theta + \sum_{j=1}^N C_{\Omega\Omega}^{ij} \Phi_j) \frac{L^2}{E I_y} + \frac{2b^2 (\frac{1}{2} + a_h) (b_1 b_2 (A_1 + A_2))}{U_\infty} \frac{L^2}{E I_y} \sum_{j=1}^N C_{\Omega\Omega}^{ij} \zeta_j \right] \quad (C.31)
\end{aligned}$$

$$\omega_{\frac{3c}{4j}} = U_\infty (\theta + \sum_{k=1}^N C_{\Omega}^k \Phi_k + \sqrt{\frac{E I_y}{m L^2}} \sum_{j=1}^M C_{\Psi}^j \dot{W}_j + b (\frac{1}{2} - a_h) \sqrt{\frac{E I_y}{m L^4}} (\dot{\theta} + \sum_{k=1}^N C_{\Omega}^k \dot{\Phi}_k) \quad (C.32)$$

where

$$\begin{aligned}
C_{\Psi}^j &= \int_0^1 \sqrt{1 - \bar{x}} \Psi_j d\bar{x} \\
C_{\Omega}^k &= \int_0^1 \sqrt{1 - \bar{x}} \Omega_k d\bar{x} \quad (C.33)
\end{aligned}$$

Appendix D: Wing and base properties

Table D.1: Baseline wing and base properties with steel stringer

Parameter	Symbol	Value
Mass	m	0.2893 kg/m
Wing span	L	0.4528 m
Wing chord	c	0.15 m
Semi chord	b	0.075 m
Elastic axis position	EA	0.25c
Center of gravity position	CG	0.43c
Eccentricity	e	0.18c
Gyration radius	K_m	0.0751m
Out-of-plane Bending rigidity	EI_y	$0.6215 \text{ N} \cdot \text{m}^2$
In-plane Bending rigidity	EI_z	$3.1075 \text{ N} \cdot \text{m}^2$
Torsional Rigidity	GJ	$0.43 \text{ N} \cdot \text{m}^2/\text{rad}$
Lift Coefficient gradient	$C_{L\alpha}$	$2\pi\sqrt{1-\bar{x}^2}$
Pitch stiffness	K_θ	0.8 N.m/rad
Damping ratio of bending and root-pitch motions	ϵ	0.05
Free stream air density	ρ_∞	$1.225 \text{ kg}/\text{m}^3$
Wing inertial about EA	I_s	$6.44 \times 10^{-4} \text{ m}^4$
Inertia of the base	I_B	$2.6 \times 10^{-3} \text{ m}^4$