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RECEPTION OF DIGITAL SIGNALS
OVER RANDOMLY TIME VARIANT
DISPERSIVE CHANNELS

by

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ABSTRACT

In pulse amplitude modulation systems a set of message parameters θ_i is encoded by forming a signal $\sum \theta_i s_i(t)$ where the signal, or pulse, shapes $s_i(t)$ are known at the receiver. To facilitate the demodulation process these signals are usually chosen to be orthogonal or, at least, to have low cross-correlation properties.

The distortion of the pulses during transmission through a band-limited channel modifies, and usually increases, their cross-correlation causing intersymbol interference at the receiver. The received signal is usually also subject to an additive random disturbance, or noise. When the response of the channel is not exactly known, or if it is randomly time variant, the problems of intersymbol interference are much worse. Even in the absence of such interference the estimation of the message parameters by the receiver is rendered much more difficult by the random nature of the channel.

This thesis obtains a canonical form for the linear receiver which is optimum, in the sense that it forms the

minimum mean square error estimate of each message parameter, for any communication system of the type described. The receiver consists of a bank of filters, the outputs of which are sampled and fed to a weighting network which forms the set of estimates. The filters are shown to be an extension of the concept of the matched filter in that each one is matched to the expected value of the received signal corresponding to a single transmitted pulse in a "noise" environment that includes the random process generated by the passage of the transmitted signals through the random medium together with the additive noise.

A special case of the general signalling scheme is sequential signalling in which the pulses $s_i(t)$ all have the same shape but are shifted in time by multiples of a basic signalling interval. When such a system is used, and the random channel is also wide sense stationary, the optimum receiver described above can be reduced to a single filter whose output is sampled at the signalling interval and fed to a time invariant, tapped delay line. In this case the receiver is similar in form to the one obtained by George⁽¹⁾ for the same signalling system when the channel response is exactly known.

When there is no intersymbol interference it is shown that the filters are closely related to a special case of the work of Kailath⁽⁷⁾ as applied to a comparable, though different communication system.

The problem of receiver design is reduced to the solution of an integral equation for each filter response and the solution of a set of simultaneous algebraic equations for the weighting network. A general expression is obtained for the estimation error and it is shown that the error approaches a non-zero limit as the transmitter power is increased.

Detailed results are obtained for a class of systems in which there is no intersymbol interference. The emphasis of this investigation is on the characterisation, and the effects of, randomness of communication channels. In particular the dependence of the estimation error on the space and time correlation properties of the channel is discussed.

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LIST OF SYMBOLS

The most frequently used symbols are listed below. Except for symbols used in a general way, the number of the equation defining the symbol is given in brackets.

I Characterising functions for time variant media

$a(t, \alpha)$	weighting function (1.6)
$T(t, f)$	transfer function (2.5)
$v(\lambda, \alpha)$	Doppler-delay function (2.7)
$V(\lambda, f)$	bifrequency function (2.8)

II Covariance functions for random media

$R_a(t, r; \alpha, \sigma)$	(2.13)
$R_T(t, r; f, u)$	(2.14)
$R_V(\beta, \lambda; \alpha, \sigma)$	(2.15)
$R_V(\beta, \lambda; f, u)$	(2.16)

For wide sense stationary media

$P_a(\tau; \alpha, \sigma)$	(2.20)
$P_T(\tau; f, u)$	(2.21)
$S_V(\lambda; \alpha, \sigma)$	(2.22), (2.23)
$S_V(\lambda; f, u)$	(2.23), (2.25)

For wide sense stationary, uncorrelated scattering media

$P_a(\tau; v)$	(2.56)
$P_T(\tau; \beta)$	(2.61), (2.62)

LIST OF SYMBOLS (Cont'd)

$S_V(\lambda; \alpha)$ (2.57), (2.58)

$S_V(\lambda; \beta)$ (2.59), (2.60)

III Second order measures of signals

$R_y(t, \tau), P_y(\tau, t)$ time autocovariance functions (2.26), (2.27)

$R_Y(f, u), P_Y(f, \beta)$ frequency autocovariance function (2.28), (2.29)

$S_y(f, t)$ time dependent spectral density (2.30)

$\chi_y(\tau, \beta)$ ambiguity function (2.33)

See also Section 2.6.1 for a notation convention.

IV Miscellaneous

θ_i message parameter

$\hat{\theta}_i$ estimate of θ_i

e_{mi} estimation error of the optimum receiver in estimating θ_i (1.2)

a_j^i weighting factor applied to the output of the filter $h_j(t)$ in estimating θ_i

E_t energy of the transmitted pulse (4.6)

E_r energy of the received pulse (4.7)

N_o spectral density of stationary, white, additive noise (1.9)

g^2 E_r/E_t for a WSS medium (4.12)

γ^2 total variance of a WSS random medium (4.16)

η g^2/γ^2 quality factor of a random medium (4.23)

ν E_r/N_o signal to noise ratio (4.22)

λ ν/η (4.24)

ρ effective signal to noise ratio (4.33)

Chapter 1

INTRODUCTION

A frequently occurring problem in communications is that of transmitting the values of a set of message parameters θ_i from one point to another. These parameters might be samples arising from the measurement of an analogue signal, or they might be numbers generated by a computer, or in general they might come from any one of many possible sources. The basic problem is to reproduce the parameters at a distance with fidelity and the general form of a communication system for this purpose is shown in Figure 1.1. In this thesis we shall not be concerned with the source of the θ_i , we shall assume only that they are random variables with known statistics. It will also be assumed that the number of variables to be transmitted in a given time is known.

The first thing to be done with the message is to form it into a signal, or waveform, suitable for transmission over the channel and the piece of equipment used to accomplish this is usually called the message encoder. There are many ways in which the message may be encoded but we shall con-

Figure 1.1 The Communication System

sider only the linear, analogue method called pulse amplitude modulation. In this scheme the encoder has available a set of waveforms or pulses $s_i(t)$, one for each of the numbers θ_i . Each pulse is multiplied by the corresponding message parameter θ_i and the resulting signals are added to form the signal

$$m(t) = \sum \theta_i s_i(t), \quad (1.1)$$

which is to be transmitted over the channel.

After encoding the signal enters the channel which we shall take to include the transmitter, the transmission medium and the demodulator as shown in Figure 1.1.

The transmitter will frequently transpose the signal to a different part of the frequency spectrum according to the nature of the transmission medium to be used. In some cases it also performs a non-linear operation on the signal, as in frequency or phase modulation systems, but in this thesis such systems will not be considered, in fact it will be assumed that the whole channel is linear.

Following the channel is the receiver which is the subject of this thesis and whose task it is to reconstruct,

as accurately as possible, the set $\{\theta_i\}$. In measuring the accuracy of this reconstruction the criterion of performance used here is the mean square estimation error for each element of the set. That is

$$e_{m_i} = E \{(\theta_i - \hat{\theta}_i)^2\} \quad (1.2)$$

where $\hat{\theta}_i$ is the estimate of θ_i and the average is taken over the ensemble of possible signals which can emerge from the channel. This ensemble involves all possible values of the set $\{\theta_i\}$ and all possible random disturbances introduced by the channel. The receivers discussed will be optimum in the sense that they achieve the minimum possible value of the performance criterion for the class of linear channels, subject to the constraint that the receiver be linear. That is to say the receiver forms the linear, minimum mean square estimate of the set.

1.1 SEQUENTIAL SIGNALLING.

A very common example of the type of encoding specified in Eqn. 1.1 is the one in which the signals $s_i(t)$ are all the same shape but shifted in time so that

$$s_i(t) = s_o(t-iT) = s(t-iT) \quad (1.3)$$

where $s(t)$ is taken as the basic shape and T is the signalling interval. The corresponding version of Eqn. 1.1 is then

$$m(t) = \sum \theta_i s(t-iT) \quad (1.4)$$

Usually in a system of this type $s(t)$ is a pulse which has most of its energy concentrated in an interval T seconds long. In this way the signals do not overlap appreciably at the input of the channel.

A closely allied encoding scheme is the multiplexing system in which a finite number, say M , of basic waveforms are used in each interval. Each of the M waveforms is multiplied by one of the θ_i and transmitted in each interval. To assist in separating the M simultaneous pulses at the receiver the pulse shapes are normally chosen to have low cross-correlation or perhaps to be orthogonal.

1.2 MESSAGE STATISTICS.

It will be assumed for convenience throughout the work described in this thesis that the numbers have been biased and scaled so that they all have zero mean and unit

variance. If the message was not originally in this form it is a simple matter to introduce the inverse scaling and biasing operations following the output of the optimum receiver. Although it is not strictly necessary, all the detailed exposition of the concepts introduced later will be written in terms of a message set in which the θ_i are uncorrelated with each other. This results in great clarification of the various equations and the extension to the correlated case is simple in principle though cumbersome algebraically. This extension is carried out in Appendix C.

The statistics just described are summarised symbolically by the following equations.

$$E \{ \theta_i \theta_j \} = \delta_{ij} \quad (1.5a)$$

$$E \{ \theta_i \} = 0 \quad (1.5b)$$

where δ_{ij} is the Kronecker delta.

1.3 THE COMMUNICATION CHANNEL

The basic model of a linear communication channel is shown in Figure 1.2a. It consists of a transmission medium characterised by a weighting function $a(t, \alpha)$ defined as

(a)

(b)

Figure 1.2 Basic Channel Model

$$a(t, \alpha) = \text{response of the medium at time } t \text{ due to} \\ \text{an impulse applied at time } t - \alpha, \quad (1.6)$$

followed by a source of additive noise. The output of the medium in response to an input signal $s(t)$ is then

$$z(t) = \int a(t, \alpha) s(t - \alpha) d\alpha \quad (1.7)$$

Since it has already been stated that the channel is taken to include the transmitter and certain parts of the receiver a word of explanation is necessary here regarding what is meant by the "medium". This term will include the whole channel with the exception of the source of additive noise. This is convenient because a great deal of the later discussion concerns the properties of the signal before the noise is added to it. If we wish to discuss the actual medium between the transmitter and receiver antennas we shall refer to it as the physical medium.

It will sometimes be convenient to assume that the parts of the transmitter and receiver included in the channel have no distorting effect on the signal since lack of this assumption causes added complexities in the discussion of random media (see Sections 2.11.1 and 2.12.2). This is not,

however, a necessary assumption and for much of the work it will in any case make no difference.

The additive noise, as is usual, in work of this kind, will be taken to be a zero mean, real valued random process with autocovariance function

$$R_n(t,r) = E \{n(t)n(r)\} \quad (1.8)$$

where the symbol E represents averaging over the ensemble of possible sample functions of the noise. While it is not necessary to assume stationary noise in the later theoretical results, it will normally be assumed in detailed results for the sake of simplicity. Stationarity of course implies that the autocovariance function is a function only of $(t-r)$. Often it will be assumed that the noise is also white with spectral density N_0 so that

$$R_n(t,r) = N_0 \delta(t-r) \quad (1.9)$$

1.4 THE MEDIUM

The vast bulk of past work in the theory of communication has concerned itself with media whose weighting function

is exactly known. Since no physical quantity can ever be measured exactly what is really meant is that the analyst has available an estimate of the weighting function so good that the error of the estimate is negligible. We shall assume that an estimate of the weighting function is available but that the error is not negligible; the error must then be regarded as a random process. The estimate of the weighting function will be denoted by $\bar{a}(t, \alpha)$ and the actual value of the weighting function can then be written

$$a(t, \alpha) = \bar{a}(t, \alpha) + \tilde{a}(t, \alpha) \quad (1.10)$$

where $\tilde{a}(t, \alpha)$ is a sample function of a random process. Of course it could well be that the weighting function of the channel is indeed generated by some random process of nature which has a very precisely known mean value. In such a case the estimate $\bar{a}(t, \alpha)$ would be that known value. It will also be assumed that the covariance function¹ of the process of which $\tilde{a}(t, \alpha)$ is a sample function is known or, equivalently, that it too has been estimated. This covariance function will

¹ It should be clearly understood that all covariance functions used in this work are defined in the true statistical sense as ensemble averages, that is they are the result of averaging over the ensemble of possible sample functions of the random process for which they are defined. This is in contrast to the common practice, when dealing with time invariant systems, of using time averages.

be denoted by

$$R_a(t, r; \alpha, \sigma) = E \{ \tilde{a}(t, \alpha) \tilde{a}(r, \sigma) \} \quad (1.11)$$

With the weighting function split into two parts as in Eqn. 1.10 the output of the medium in response to an input signal $s(t)$ can be written, from Eqn. 1.7,

$$z(t) = \bar{z}(t) + \tilde{z}(t) \quad (1.12)$$

where

$$\bar{z}(t) = \int \bar{a}(t, \alpha) s(\alpha) d\alpha \quad (1.13)$$

and

$$\tilde{z}(t) = \int \tilde{a}(t, \alpha) s(\alpha) d\alpha \quad (1.14)$$

This suggests splitting the medium into two parts as shown in Figure 1.2b. There is a random branch of the medium having the random weighting function $\tilde{a}(t, \alpha)$ and a known, or deterministic, branch having the weighting function $\bar{a}(t, \alpha)$. The outputs of the two branches are added to form the total signal $z(t)$ as in Eqn. 1.12 before the noise signal is added.

1.4.1 The deterministic branch

Since the weighting function of this branch of the medium is a deterministic function all the theory of exactly known channels applies to it. Clearly the output function

$\bar{z}(t)$ of the branch is a deterministic function when the input signal $s(t)$ is deterministic and in fact it is the ensemble average of that random process $z(t)$ which is the output of the total medium.

In general the available estimate of the medium weighting function may be a function of time either because the medium has some deterministic trend in time, or simply because our ability to estimate the medium is a function of time. In a wide variety of cases, however, the estimate will not be time dependent and then the function $\bar{a}(t, \alpha)$ is a function only of the delay variable. In such cases we shall write

$$\bar{a}(t, \alpha) = \bar{a}(\alpha) \quad (1.15)$$

The special property of a time invariant medium is that application of a signal of fixed shape at different times produces responses which have the same shape. More precisely application of the signal of Eqn. 1.4 in the sequential signalling system would result in an output of the form

$$\sum \theta_i \bar{z}(t - iT) \quad (1.16)$$

where $\bar{z}(t)$ is defined in Eqn. 1.13.

1.4.2 The random branch

Since the weighting function $\tilde{a}(t, \alpha)$ of this branch is a sample function of a zero mean random process the same is true of the output signal $\tilde{z}(t)$ in response to a deterministic signal $s(t)$. The theory of random media is not so generally known as that of deterministic media and it is reviewed in some detail in Chapter 2. There are however, two important subclasses of random media which need to be discussed here and this is done in the following two sections.

1.4.3 Wide sense stationary media

Two ensemble averages, the expected value of the weighting function $\bar{a}(t, \alpha)$ and the covariance function $R_a(t, r; \alpha, \sigma)$ have been used to describe the random medium. These two functions are in general functions of time and delay variables. If $\bar{a}(t, \alpha)$ is not time dependent, as in Eqn. 1.15, and if $R_a(t, r; \alpha, \sigma)$ can be written in terms only of a time shift variable and the delay variables, the medium is said to be wide sense stationary. The statement with regard to $R_a(t, r; \alpha, \sigma)$ amounts to being able to write

$$R_a(t, r; \alpha, \sigma) = P_a(r-t; \alpha, \sigma) \quad (1.17)$$

This is a very important class of media and has been fairly extensively discussed in the literature. (See Chapter 2 for references.)

If the complete channel is to be described as wide sense stationary then the additive noise process $n(t)$ must also have a stationary covariance function, that is,

$$R_n(t, r) = P_n(r - t) \quad (1.18)$$

1.4.4 The randomly selected, time invariant channel

If the medium is known not to be time variant so that the weighting functions of the two branches can be written

$$\bar{a}(t, \alpha) = \bar{a}(\alpha)$$

as in Eqn. 1.15 and

$$\tilde{a}(t, \alpha) = \tilde{a}(\alpha) \quad (1.19)$$

the medium is said to be randomly selected. If the additive noise process is stationary the channel is said to be a randomly selected, time invariant channel. Such a channel is, a fortiori, wide sense stationary. In this thesis the simplification necessary to specialise most of the results to this case is rather trivial and will not be discussed at length.

A very obvious example of the occurrence of a channel of this type is in switched route communication systems such as the telephone network.

1.5 INTERSYMBOL INTERFERENCE

When pulse signals are passed through a communication channel they will in general suffer some distortion due to the bandlimiting of the channel or to the inband imperfections of the channel response. In particular the pulses will be "smeared" in time, that is to say the energy of the output pulse will be distributed over a greater period of time than that of the input pulse. This effect is known as time dispersion.

In a signalling scheme such as that described in Section 1.1 a large number of pulses is transmitted, each of them occupying a time interval essentially discrete from the others at the transmitter. The time dispersion of the received pulses may then cause the energy of one pulse to spread into the time interval devoted to others thus causing interference with the process of deciphering the information contained in those pulses at the receiver. This is the effect

known as intersymbol interference. When a number of pulses is transmitted simultaneously as in the multiplexing system also described in Section 1.1, the distortion of the pulses may destroy the low cross-correlation properties of the pulses so that again the reception process becomes more difficult. This effect adds to the difficulty already caused by time dispersion in the sequential multiplexing scheme described. If the signalling interval were reduced, or the degree of multiplexing increased, in order to increase the information rate of the system, the intersymbol interference effects would clearly increase in severity until they would eventually prevent the information rate being increased further unless special steps are taken.

All that has been said so far applies equally well to any channel, deterministic or random, time variant or not. However, if the channel is not known exactly the effects are more severe since the form of interference is also not known. It will be one of the purposes of this thesis to examine the additional intersymbol interference problems introduced by a random channel. Before proceeding to the discussion of this problem, previous work in the field of intersymbol interference and in the field of random channels will be reviewed.

1.6 PREVIOUS WORK ON INTERSYMBOL INTERFERENCE

Previous work on the problem of intersymbol interference (for instance references 1,2,3,4,5,17,18) has been confined to the sequential signalling scheme described in Section 1.1. With one exception (14) the work has also been confined to systems operating over as known, time invariant channels. For such channels the received signal (see Eqn. 1.15) is of the form

$$w(t) = \sum \theta_i z(t-iT) + n(t) \quad (1.20)$$

An excellent discussion of work in the field up to 1966 is given by Coll⁽³⁾ who also gives a very complete bibliography. Some of the more important of these works are reviewed below.

1.6.1 Equalisation receivers

For many years attempts to minimise the effects of intersymbol interference were based on creating a channel with ideal non-distorting transmission characteristics⁽¹⁷⁾. This was achieved by measuring the response of the existing channel and inserting a filter with the inverse response before the receiver. The trouble with such systems is that

they invariably increase the power of the additive noise reaching the receiver since they basically increase the bandwidth of the receiver. Even when the additive noise was originally negligible, attempts to increase the information rate using such techniques eventually become limited by additive noise. It was the realisation of this that led to the later work, described in the next section, which allows for the effect of additive noise.

1.6.2 Optimum receivers for noisy, known channels

Before discussing the optimum receiver for interfering symbols when the channel is exactly known it is desirable to mention the optimum receiver when the symbols do not interfere; the isolated pulse problem. It is well known that the optimum receiver in this case is the matched filter⁽¹⁸⁾. Treating the estimation of one of the parameters, say θ_0 , the estimate is of the form

$$\theta_0 = a_0 \int h(-t)w(t)dt \quad (1.21)$$

where $w(t)$ is the received signal and the filter weighting function $h(t)$ is the solution of the integral equation

$$\bar{z}(t) = \int R_n(t,r)h(-r)dr \quad (1.22)$$

If the noise is stationary and white, so that its autocovariance function is given by Eqn. 1.9, the solution of this equation is

$$h(t) = \bar{z}(t) / N_0 \quad (1.23)$$

and the scaling constant a_0 is

$$a_0 = \frac{1}{1 + E_r/N_0} \quad (1.24)$$

where E_r is the received signal energy

$$E_r = \int \bar{z}^2(t) dt \quad (1.25)$$

In the light of later results of this thesis it will become significant that the signal to noise ratio can be expressed in the following form:

$$E_r/N_0 = \int h(-t)\bar{z}(t)dt \quad (1.26)$$

The form of the receiver implied by Eqn. 1.21 is shown in Figure 1.3. The received signal is passed through a filter having the weighting function $h(t)$, which is in general non-causal, and the output is sampled at time $t = 0$. The non-causality is allowed for in practice by allowing sufficient delay in the receiver and using a time shifted, truncated approximation of $h(t)$.

Figure 1.3 Matched Filter Receiver

When there is interference between the individual pulses of the received signal the optimum receiver has been shown by George⁽¹⁾ and Tufts⁽²⁾ to be the matched filter described above followed by a tapped delay line as illustrated in Figure 1.4. The estimate of θ_i is obtained by sampling the signal on the summing bus at $t = iT$. In general the tapped delay line is infinitely long, but in practice a good approximation is obtained by using a tapped delay line of finite length. Practical receivers of this type have been investigated by Coll⁽³⁾ and some of the implications of finite receivers with respect to the effect on the estimation error have been investigated by Kaye⁽⁵⁾.

It has been shown by George⁽⁴⁾ that the tap gains must satisfy a set of simultaneous equations which we will express in the following form for comparison with later results of this thesis:

$$a_i = 1 - \sum a_i C_i \quad (1.27)$$

$$a_n = - \sum a_i C_{i-n}, \quad n \neq 0$$

The coefficients C_{i-n} constitute the sampled autocovariance function of the transmitted pulse shape $s(t)$. That is

Figure 1.4 Receiver for Interfering Symbols

$$C_{i-n} = \frac{1}{N_0} \int \bar{z}(t-iT) \bar{z}(t-nT) dt \quad (1.28)$$

Later it will become significant that these coefficients can be expressed in terms of the filter weighting function in the following manner:

$$C_{i-n} = \int h(t-iT) \bar{z}(t-nT) dt \quad (1.29)$$

In the present case, of course, the equivalence is trivial.

1.6.3 Optimum receiver for the randomly selected channel

The randomly selected, time invariant channel has been defined in Section 1.4.4. A recent note by Gonsalves and Tufts⁽¹⁴⁾ discussed briefly the problem of intersymbol interference in such channels for sequential signalling. Their work stated an optimisation equation to be solved for the receiver but, except for a very restricted example, did not obtain the form of the receiver. This problem is a special case of the Section 3.23 of this thesis which deals with sequential signalling over wide sense stationary media. Consequently the canonical form of the receiver obtained in that section includes the case of Gonsalves and Tufts.

1.7 PREVIOUS WORK ON COMMUNICATION THROUGH RANDOM MEDIA

Previous work (20,21,22) on the problem of communication through random media has been directed towards systems using finite alphabets of code words, that is to the problem of hypothesis testing at the receiver and has largely been confined to the isolated pulse, non-interfering case. This work has been admirably interpreted and extended by Kailath (6,7).

In a communication system of the kind mentioned above the transmitted signal is selected from a set, or alphabet, of known signals which correspond to the words of some code. The task of the receiver is to decide which signal was transmitted and it is optimised to minimise some error cost function, usually just the probability of error. This is clearly quite different from the signalling scheme used in this thesis where a set of signals of known shape but random amplitude is transmitted and the task of the receiver is to estimate the amplitude in such a way as to minimise the mean square error. In addition the work in this field has ignored the problem of intersymbol interference except for some very general comments in an article by

Kailath⁽⁶⁾. In spite of the great differences between the two problems there are some very interesting similarities between the results obtained in this thesis and Kailath's results (see Section 4.8) and it is therefore worthwhile to review his results here.

We suppose first that information is to be transmitted by selecting one of an alphabet of M signals $s_m(t)$ which are known at the receiver. Kailath observed that the basic function of any receiver in such a system, regardless of the cost function used, is to compute the set of conditional likelihood functions $p(p|m)$. These are the conditional probabilities that the signal $p(t)$ will arrive at the receiver given that the m th signal was transmitted. When the simplest cost function, probability of error, is used the receiver makes the decision corresponding to the largest of the likelihood functions.

When both the additive and multiplicative random processes of the channel are Gaussian the receiver takes the form of a bank of estimator-correlators, which form the likelihood functions, followed by processing which depends on the particular cost function chosen. The m th likelihood function computer takes the form shown in Figure 1.5. The received

Figure 1.5 mth Branch of the Kallath Receiver

waveform is

$$p(t) = z(t) + n(t) \quad (1.30)$$

where

$$z(t) = \int a(t, \alpha) s_k(\alpha) d\alpha \quad (1.31)$$

assuming that the k th signal was transmitted. The m th branch of the receiver first forms the maximum likelihood estimate of the random part $\tilde{z}(t)$ of the received signal on the assumption that the m th signal was transmitted. This estimate is the output of a time variant estimating filter with impulse response $l^{(m)}(t, r)$. (This impulse response is defined as the output of the filter at time t due to an impulse applied at time r .) The estimate is then correlated with the received signal $p(t)$.

In parallel with this operation the received signal is also correlated with a waveform $\bar{z}^{(m)}(t)$ which is the signal which would be the output of the deterministic branch of the medium if the m th signal were transmitted. The result of this correlation, weighted with a factor of two, is combined with the output of the estimator-correlator to form the decision statistic. Thus the receiver uses two estimates of the received signal $p(t)$: the a priori estimate and an

estimate based on the received signal, both estimates being conditional on the hypothesis m . The result is a combination of linear and quadratic terms in forming the decision statistic.

When there are just two signals in the alphabet, and when these are antipodal signals such that

$$s^{(1)}(t) = -s^{(2)}(t) \quad (1.32)$$

the two decision statistics are subtracted from one another and the sign of the difference dictates the decision of the receiver. In this case the quadratic terms in the two receiver branches are identical (basically because $[s^{(1)}(t)]^2 = [s^{(2)}(t)]^2$) and therefore cancel out on subtraction. Similarly the two filters $\ell^{(1)}(t)$ and $\ell^{(2)}(t)$ are identical.

The decision statistic d is then of the form

$$d = \int q(t)p(t)dt \quad (1.33)$$

which is the result of correlating the received signal with the waveform

$$q(t) = \bar{z}^{(1)}(t) - \int \ell^{(1)}(t,r)\bar{z}^{(1)}(r)dr \quad (1.34)$$

The form of this receiver is shown in Figure 1.6. The correlator waveform $q(t)$ is obtained by passing $\bar{z}^{(1)}(t)$ through the time variant filter $\ell^{(1)}(t,r)$.

1.8 STRUCTURE OF THE THESIS

In this chapter we have introduced the communication problem which is the subject of this thesis and have discussed the most important previous contributions in associated fields. Since the theory of random media is not well known a detailed discussion of it is presented in Chapter 2 and this will serve as a general reference to be used freely throughout the rest of the thesis.

The major theoretical development of the thesis is the solution of the problem of communication by pulse amplitude modulation over random media and this is contained in Chapter 3. A canonical form of the optimum receiver for any communication system of the type described in the present chapter is developed and a very simple special form is presented for the problem of sequential signalling over wide sense stationary media.

Figure 1.6 Kailath Receiver for Antipodal Signals

Obtaining the exact specification of the optimum receiver derived in Chapter 3 requires solution of an integral equation and in general this involves the use of numerical techniques. There are, however, some cases in which analytical solutions are possible and these are discussed in Chapter 5.

Chapters 4, 6, and 7 are devoted to a discussion of the effects of the randomness of the channel. In order to do this the results of Chapter 3 are specialised in Chapter 4 to the non-interfering, or isolated pulse, case. This chapter also includes a discussion of those aspects of the behaviour of the communication system which do not require specification of a particular channel or signal.

In Chapters 6 and 7 channels of a particular type are introduced to allow a detailed investigation of the effect of variation of some basic parameters of random channels. Chapter 6 discusses the effect of the relative bandwidths of the signal and of the random process of the channel. Chapter 7 uses the results of Chapter 6 to discuss the performance of discrete multipath channels. The object of these two chapters is to build up an intuitive appreciation of the effects of random media and to this end very

simple channel structures are assumed. The result is that the performance of the channels can be estimated with a minimum of effort and the interaction of different parameters can therefore be visualised.

Chapter 2

ANALYSIS OF RANDOM MEDIA

This chapter is a review of the theory available for the analysis of randomly time variant media. Most of the material has been published previously, notably by Bello⁽¹⁰⁾, Kailath⁽⁸⁾ and Daly⁽⁹⁾. An attempt has been made here to present only as much of the theory as is necessary for an understanding of the action of the most frequently occurring types of channels and to present it in as understandable a way as possible. Of course the material is also presented from the particular point of view of the major work of this thesis. Wherever detailed proofs are available in the references only results will be stated here.

The analysis of the action of correlated scattering media does not appear to have been presented in readily accessible form before, since Bello's work has been largely concerned with uncorrelated scattering media, while Daly restricts himself to wide sense stationary, uncorrelated scattering media. (Definitions of these terms will be included at the appropriate point of the text.) The reference

by Kailath deals with the time variant, rather than the random, aspects.

2.1 THE MODEL OF THE MEDIUM

The basic model has already been introduced in Section 1.3 and 1.4 but for completeness some of the basic equations are restated here. (The model, shown in Figure 1.2, consists of a random medium followed by a source of additive noise. In this chapter, however, we shall be concerned only with the medium.)

The medium, it will be remembered, consists of two branches, a deterministic branch and a random branch, each described by a time variant weighting function. The total weighting function is expressed as the sum of these as in Eqn. 1.10.

$$a(t, \alpha) = \bar{a}(t, \alpha) + \tilde{a}(t, \alpha) \quad (2.1)$$

In response to an input signal $s(t)$ the output of the medium is also expressed as the sum of deterministic and random parts.

$$z(t) = \bar{z}(t) + \tilde{z}(t) \quad (2.2)$$

where

$$\bar{z}(t) = \int \bar{a}(t, \alpha) s(\alpha) d\alpha \quad (2.3)$$

and

$$\tilde{z}(t) = \int \tilde{a}(t, \alpha) s(\alpha) d\alpha \quad (2.4)$$

2.2 FUNCTIONS CHARACTERISING A TIME VARIANT MEDIUM

In this section we shall be concerned with the characterisation of any time variant medium, whether deterministic or random. Thus each of the functions defined may either be deterministic or a sample function of a random process. One characterising function, the weighting function, has already been defined. There are three others which will be used later according to which is most convenient in each application.

The time variant transfer function is defined in terms of the response of the medium to a sinusoid:

$$T(t, f) = \frac{\text{response of the medium to } e^{j2\pi ft}}{e^{j2\pi ft}} \quad (2.5)$$

This is the parallel of the normal transfer function of a time invariant channel, which is also defined in terms of the response to a sinusoidal driving function. As in the time

invariant case there is a Fourier transform relationship between this and the weighting function:

$$T(t, f) = \int a(t, \alpha) e^{-j2\pi f \alpha} d\alpha \quad (2.6)$$

By transforming the weighting function with respect to the time rather than the delay variable the Doppler-delay function

$$v(\lambda, \alpha) = \int a(t, \alpha) e^{-j2\pi \lambda t} dt \quad (2.7)$$

is obtained. This represents the output of the medium with a Doppler shift of λ from the input frequency, at a delay of α . This clearly has no counterpart in the theory of time invariant media.

Finally there is a bifrequency function obtained by transforming the weighting function with respect to both variables.

$$V(\lambda, f) = \iint a(t, \alpha) e^{-j2\pi f \alpha} e^{-j2\pi \lambda t} d\alpha dt \quad (2.8)$$

This function gives the output of the medium with Doppler shift λ in response to a sinusoidal input at frequency f . Again this has no counterpart for a time invariant channel.

The transform relationships between these functions can be conveniently expressed by means of the diagram in Figure 2.1 in which each connecting line represents a Fourier transform.

When any of these functions are used to describe a medium having deterministic and random branches they will each be expressed as the sum of deterministic and random parts as with $a(t, \alpha)$ in Eqn. 2.1.

2.3 INPUT-OUTPUT RELATIONSHIPS

Using the functions defined in the previous section a number of expressions relating the input and output time functions $s(t)$ and $z(t)$ to their respective transforms $S(f)$ and $Z(f)$ can be written. The most useful of these are indicated below.

$$z(t) = \int a(t, \alpha) s(t - \alpha) d\alpha \quad (2.9)$$

$$z(t) = \int T(t, f) S(f) e^{j2\pi ft} df \quad (2.10)$$

$$z(t) = \iint v(\lambda, \alpha) s(t - \alpha) e^{j2\pi \lambda t} d\lambda d\alpha \quad (2.11)$$

$$Z(f) = \int V(f - u, u) S(u) du \quad (2.12)$$

The essential difficulty of dealing with time variant media is emphasised by Eqns. 2.9 and 2.12. In the time

Figure 2.1 Transform Relationships for Medium Characterising Functions

invariant case Eqn. 2.9 would be a convolution which would transform to a simple product in the equivalent of Eqn.

2.12. In the time variant case there is a convolution and a product in each equation.

2.4 COVARIANCE FUNCTIONS FOR THE RANDOM BRANCH

Since, for the random branch of the medium, all the functions defined in Section 2.2 are sample functions of random processes, the most complete way of describing their properties would be to specify their probability density functions of various orders. However, since mean square error is used as the performance criterion for the communication systems discussed in this thesis it will turn out that only the second order moments, or covariance functions, of these density functions will be required. Accordingly a covariance function is defined below for each of the functions defined in Section 2.2. In these expressions the * notation denotes the complex conjugate. This is necessary in the case of frequency domain functions but not strictly for the time domain functions, since in this thesis they are all real valued. However use of the conjugate renders the development of certain Fourier transform relationships easier and so it will be retained.

$$R_a(t, r; \alpha, \sigma) = E \{ \tilde{a}^*(t, \alpha) \tilde{a}(r, \sigma) \} \quad (2.13)$$

$$R_T(t, r; f, u) = E \{ \tilde{T}^*(t, f) \tilde{T}(r, u) \} \quad (2.14)$$

$$R_v(\beta, \lambda; \alpha, \sigma) = E \{ \tilde{v}^*(\beta, \alpha) \tilde{v}(\lambda, \sigma) \} \quad (2.15)$$

$$R_v(\beta, \lambda; f, u) = E \{ \tilde{v}^*(\beta, f) \tilde{v}(\lambda, u) \} \quad (2.16)$$

There are a number of double Fourier transform relationships between these functions and the most important for our purposes are:

$$R_T(t, r; f, u) = \iint R_a(t, r; \alpha, \sigma) e^{j2\pi(\alpha f - \sigma u)} d\alpha d\sigma \quad (2.17)$$

$$R_v(\beta, \lambda; \alpha, \sigma) = \iint R_a(t, r; \alpha, \sigma) e^{j2\pi(\beta t - \lambda r)} dt dr \quad (2.18)$$

The others are indicated diagrammatically in Figure 2.2.

2.5 WIDE SENSE STATIONARY MEDIA

The definition of a wide sense stationary medium has been given in Section 1.4.3. It requires that

$$\bar{a}(t, \alpha) = \bar{a}(\alpha) \quad (2.19)$$

and

$$R_a(t, r; \alpha, \sigma) = P_a(r - t; \alpha, \sigma) \quad (2.20)$$

From here on this section is concerned only with the random branch, as the deterministic branch is so well understood.

Figure 2.2 Transform Relationships for any Medium

Figure 2.3 Transform Relationships for Wide Sense Stationary Media

The assertion that a medium is wide sense stationary may represent the true state of affairs or it may result from an observation that the statistics of the medium are approximately stationary over periods of time which are long compared to the signalling interval. In the latter case the covariance functions are the second moments of those variations of the medium which occur in short time intervals while long term drifts are considered separately. Such a medium is strictly described as quasi wide sense stationary. Since we shall be concerned here only with combatting the short term effects, we shall not distinguish between the two cases.

For wide sense stationary media it is convenient to introduce a new set of covariance functions defined in a similar way to $P_a(\tau; \alpha, \sigma)$ in Eqn. 2.20, where $\tau = (r-t)$ is a time shift variable. There is a second function directly involving the time shift variable:

$$R_T(t, r; f, u) = P_T(\tau; f, u) \quad (2.21)$$

The wide sense stationary property implies singular behaviour of the other two covariance functions as may be seen from

the following argument. From Eqn. 2.18

$$\begin{aligned}
 R_V(\beta, \lambda; \alpha, \sigma) &= \iint P_a(\tau; \alpha, \sigma) e^{j2\pi(\beta-\lambda)\tau} e^{-j2\pi\lambda\tau} d\tau d\tau \\
 &= \delta(\lambda-\beta) \int P_a(\tau; \alpha, \sigma) e^{-j2\pi\lambda\tau} d\tau \\
 &= \delta(\lambda-\beta) S_V(\lambda; \alpha, \sigma)
 \end{aligned} \tag{2.22}$$

where we define

$$S_V(\lambda; \alpha, \sigma) = \int P_a(\tau; \alpha, \sigma) e^{-j2\pi\lambda\tau} d\tau \tag{2.23}$$

Thus the output of a wide sense stationary medium is uncorrelated at different Doppler shifts. However, the outputs at the same shift caused by the action of the medium at different delays may be correlated. The function $S_V(\lambda; \alpha, \sigma)$ is called the Doppler-delay profile of and is the spectral density of Doppler shift components with the same delay, caused by each input frequency. It is closely related to the scattering function defined in Section 2.11.

A similar singular property follows for $R_V(\lambda, \beta; f, u)$.

$$R_V(\lambda, \beta; f, u) = \delta(\lambda-\beta) S_V(\lambda; f, u) \tag{2.24}$$

where we define

$$S_V(\lambda; f, u) = \iint S_V(\lambda; \alpha, \sigma) e^{j2\pi(\alpha f - u\sigma)} d\alpha d\sigma \quad (2.25)$$

The transform relationships between this new set of covariance functions are shown in Figure 2.3.

All discussion of the random branch of the medium in the remainder of this chapter will be restricted to the wide sense stationary case, since the detailed examples in the thesis involve only media of this type. Before continuing the discussion of media, however, the following three sections define some measures of signals which will be needed later.

2.6 MEASURES OF THE INPUT AND OUTPUT FUNCTIONS

In order to discuss the effect of a random medium on signals passing through it, it is necessary to define some measures of the input and output signals. These measures are defined in the same way for deterministic and random signals but in the deterministic case the statistical expectation operator is meaningless and can be disregarded. With this understanding the definitions will be made in terms of

an arbitrary, finite energy signal $y(t)$. If the input signal to any physical medium has finite energy the output signal will also have finite energy so that the existence of Fourier transforms is assured in all cases with which we shall be concerned.

First, there are two autocovariance functions defined on the time function $y(t)$ and its Fourier transform $Y(f)$. The time domain function is

$$R_y(t, r) = E \{y^*(t)y(r)\} \quad (2.26)$$

together with an equivalent definition involving the time shift parameter $\tau = r - t$.

$$P_y(\tau, t) = E \{y^*(t)y(t+\tau)\} \quad (2.27)$$

Similarly there are two alternative forms of the frequency autocovariance function.

$$R_Y(f, u) = E \{Y^*(u)Y(f)\} \quad (2.28)$$

$$P_Y(f, \beta) = E \{Y^*(f-\beta)Y(f)\} \quad (2.29)$$

$P_Y(f, \beta)$ is a measure of the coherency between the various frequency components of the signal when regarded as a function of β and its width in this variable will be referred to as the co-

herent bandwidth of the signal. $P_Y(f,0)$ is a spectral density function.

A very important function is the time dependent spectral density function

$$S_y(f,t) = E \{ \int y^*(t)y(t+\tau)e^{-j2\pi f\tau} d\tau \} \quad (2.30a)$$

$$= \int P_y(\tau,t)e^{-j2\pi f\tau} d\tau \quad (2.30b)$$

(The interchangeability of the order of integration and averaging is assured by the finite energy constraint.) The importance of this function is that it gives a measure of the distribution of the energy of the signal or process in terms of time and frequency. By integrating over the frequency variable, the time variation of the signal power is obtained:

$$E \{ y^2(t) \} = P_y(0,t) = \int S_y(f,t) df \quad (2.31)$$

Integration with respect to time gives the frequency distribution:

$$E \{ |Y(f)|^2 \} = P_Y(f,0) = \int S_y(f,t) dt \quad (2.32)$$

Of somewhat less importance in the context of the present work, but still of interest is the ambiguity function:

$$\chi_y(\tau, \beta) = E \{ \int y^*(t) y(t+\tau) e^{-j2\pi\beta t} dt \} \quad (2.33a)$$

$$= \int P_y(\tau, t) e^{-j2\pi\beta t} dt \quad (2.33b)$$

The Fourier transform relationships between these functions are indicated in Figure 2.4.

2.6.1 Notation convention¹

We shall frequently be concerned with output signals from random media, such as $z(t)$ in Eqn. 2.1, which can be expressed as the sum of a random part $\tilde{z}(t)$, and a deterministic part $\bar{z}(t)$. When writing the second order measures of such signals the following notation will be adopted for simplicity:

- 1) Terms written in the form $R_z(t, r)$, $P_z(\tau, t)$, $S_z(f, t)$, et cetera, are defined in terms of the random part of the signal $\tilde{z}(t)$. Thus

$$P_z(\tau, t) = E \{ \tilde{z}^*(t) \tilde{z}(t+\tau) \}$$

¹ After this section the reader will have sufficient information to follow the main theoretical development of this thesis, contained in Chapter 3. The remainder of this chapter contains material which is used in some of the detailed developments given in subsequent chapters.

Figure 2.4 Transform Relationships for the Signal Measures

- 2) Similar functions defined on the deterministic part of the signal will be clearly identified by a notation of the form $S_{\bar{z}}(f,t)$.

2.7 MEAN SQUARE DISPERSIONS

The channel and signal covariance functions defined in the two previous sections provide very detailed information about the nature of the processes they describe. In addition to these it is convenient to have available some simpler measures of the width or spread of the functions. A very useful measure of this kind is the second order central moment, or mean square dispersion, of a function. This is a fairly well known method of measurement and is defined precisely in Appendix D. In general the mean square dispersion of a function, say $x(t)$, will be denoted by a symbol of the form $m_t(x)$. Thus the subscript indicates the variable with respect to which the dispersion is defined while the term in parentheses is the name of the function. Measures which refer to the medium covariance functions will have the symbol "a" in parentheses.

The dispersions most used in this thesis are listed in the two following sections.

2.8 SIGNAL DISPERSIONS

The two most important dispersion measures for a signal, say $y(t)$, are the time dispersion

$$m_t(y) = \text{mean square dispersion of } P_y(0,t) \quad (2.34)$$

and the bandwidth defined as

$$m_f(y) = \text{mean square dispersion of } P_y(f,0). \quad (2.35)$$

The reason for the special names given to these measures is clear from Eqns. 2.31 and 2.32 which define the functions whose widths are being measured.

2.9 THE DELAY PROFILE AND DELAY SPREAD OF THE MEDIUM

In the case of a time invariant medium the behaviour of the medium as a function of delay is expressed by the impulse response. For a random medium this is a random process so that some substitute measurement must be devised and a very convenient one is the delay profile, which is really a mean square impulse response. It is defined as the expected value of the instantaneous power of the output process, in response to a delta function input, which is just the auto-

covariance function of the output at zero time shift. From Eqns. 2.9 and 2.27, for a delta function input signal,

$$\begin{aligned}
 E \{z^2(t)\} &= P_z(0,t) \\
 &= \iint a^*(t,\sigma)a(t,\nu)\delta^*(t-\sigma)\delta(t-\nu)d\sigma d\nu \\
 &= P_a(0;t,t) = \int S_v(\lambda;t,t)d\lambda \quad (2.36)
 \end{aligned}$$

Thus the delay profile is the integral of the Doppler-delay profile with respect to the Doppler shift variable.

The delay spread can now be defined as

$$m_\alpha(\alpha) = \text{mean square dispersion of } P(0;\alpha,\alpha) \quad (2.37)$$

2.10 WSS CORRELATED SCATTERING MEDIA

The discussion here has already been specialised to wide sense stationary media. Most previous work has also been specialised to the wide sense stationary, uncorrelated scattering (WSSUS) case, which implies that the autocovariance function $P_a(\tau;\alpha,\sigma)$, defined in Eqn. 2.20, can be expressed as

$$P_a(\tau;\alpha,\sigma) = P_a(\tau;\alpha)\delta(\alpha-\sigma) \quad (2.38)$$

Although Bello⁽¹⁰⁾ and other authors have found that many random media can be represented this way, it is felt that

correlated scattering media do warrant some investigation. This is especially so when one considers the meaning of "random" to include uncertainties of estimation. This is because there may be some a priori estimate of the range of the spectrum in which the response of the medium will occur which implies correlated scattering as shown in the next section.

2.10.1 The bandwidth of the medium

The frequency response of a random medium is of course a random process. However, as in the case of the impulse response, it is possible to discuss the frequency response in a mean square sense.

The spectral density of the output process in response to a sinusoidal input at frequency f_1 (a delta function in the frequency domain) is given by Eqn. 2.12.

$$\begin{aligned}
 E \{Z^*(f)Z(f)\} &= P_Z(f,0) \\
 &= E \{ \iint V^*(f-u,u) V(f-y,y) \delta^*(u-f_1) \delta(y-f_1) du dy \} \\
 &= S_V(f-f_1; f_1, f_1) \quad (2.39a) \\
 &= \iint S_V(f-f_1; \alpha, \alpha+\mu) e^{-j2\pi\mu f_1} d\alpha d\mu \quad (2.39b)
 \end{aligned}$$

Eqn. 2.39a shows that there are components of the output spectral density at frequencies other than the input frequency. This is the mark of a time variant medium and these frequencies are the result of Doppler shift.

To assess the ability of the medium to respond to different input frequencies, it is convenient to consider the output power at zero Doppler shift due to each input frequency. This function will be called the mean square frequency response and, from Eqn. 2.39b, is given by

$$S_v(0;f,f) = \int e^{-j2\pi\mu f} \int S_v(0;\alpha,\alpha+\mu) d\alpha d\mu \quad (2.40)$$

Clearly the bandwidth is determined by the correlation of the scattering action at different delays. This is easily understood when it is realised that this correlation is a measure of how rapidly the weighting function varies as a function of delay. For instance if it is a slow, smooth function it will have a high correlation and a small bandwidth. If the medium includes parts of the transmitter or receiver which are bandlimited to the extent that they cause distortion of the signal, the medium is certainly correlated scattering.

If the medium approximates uncorrelated scattering, that is if $S_V(0; \alpha, \alpha + \mu)$ is a very narrow function of μ , then $S_V(0; f, f)$ is almost constant over a wide range of f indicating that there is no reason to expect the frequency response to be concentrated in any particular part of the spectrum.

Having obtained a mean square frequency response, a bandwidth can also be defined in a mean square sense and this is done as follows:

$$m_f(\alpha) = \text{mean square dispersion of } S_V(0; f, f) \quad (2.41)$$

2.10.2 Doppler profile and Doppler spread

The introduction of Doppler shifts by a time variant medium has already been mentioned in the previous section and Eqn. 2.39a shows that, in general, the Doppler profile should be specified for each input frequency. In channel models where the autocovariance of the random process of the medium at different delays has the same shape (though possibly different amplitude) this is not necessary. From Figure 2.5 we have

$$S_V(\lambda; f, f) = \int P_T(\tau; f, f) e^{-j2\pi\lambda\tau} d\tau \quad (2.42)$$

Figure 2.5 Transform Relationships for WSSUS Media

so that when $P_T(\tau;f,f)$ is a separable function of τ and f $S_V(\lambda;f,f)$ is also a separable function of λ and f . In such a case the Doppler profile is simply defined as $S_V(\lambda;0,0)$. This will be a sufficient definition in many other cases which approximate the condition of separability.

The mean square Doppler spread is defined as

$$m_\lambda(\tilde{a}) = \text{mean square dispersion of } S_V(\lambda;0,0) \quad (2.43)$$

2.10.3 The action of the medium

With the various definitions of the preceding sections we are now in a position to investigate the effect of a random medium on a signal passing through it. Both the bandwidth and time dispersion of a signal are very easily obtained from the time dependent spectral density defined in Eqn. 2.30 and the action of the medium can consequently be expressed by its effect on this function.

The expression for the time dependent spectral density of the output as given by Eqn. 2.30b is first expanded by using Eqn. 2.9.

$$\begin{aligned}
S_z(f, t) &= \int P_z(\tau, t) e^{-j2\pi f\tau} d\tau \\
&= \int E \left\{ \iiint s(\rho+\sigma) s^*(\sigma) a^*(t, t-\sigma) \right. \\
&\quad \left. \cdot a(t+\tau, t+\tau-\rho-\sigma) e^{-j2\pi f\tau} d\rho d\sigma \right\} d\tau
\end{aligned}$$

After an interchange of the order of integration and averaging this is

$$S_z(f, t) = \iint S^*(\sigma) S(\rho+\sigma) \int P_a(\tau; t-\sigma, t-\sigma-\rho+\tau) e^{-j2\pi f\tau} d\tau d\rho d\sigma \quad (2.44)$$

At this stage the analysis becomes much clearer in concept if it is assumed that the correlation of the scattering action of the medium at different delays is a function only of the difference in delay. The concepts which follow are still valid if this is not the case but they are difficult to visualise and describe. The assumption amounts to saying that

$$P_a(\tau; \alpha, \alpha+\mu) = P_a''(\tau; \alpha) \psi(\mu) \quad (2.45)$$

Corresponding to this a new function

$$S_v''(\lambda; \alpha) = \int P_a''(\tau; \alpha) e^{-j2\pi\lambda\tau} d\tau \quad (2.46)$$

is defined. Then we have

$$S_v(\lambda; \alpha, \alpha+\mu) = S_v''(\lambda; \alpha) \psi(\mu) \quad (2.47)$$

and, following Eqn. 2.25,

$$S_V(\lambda; f-\beta, f) = S_V''(\lambda; \beta) \Psi(f) \quad (2.48)$$

where

$$S_V''(\lambda; \beta) = \int S_V''(\lambda; \alpha) e^{-j2\pi\alpha\beta} d\alpha \quad (2.49)$$

and

$$\Psi(f) = \int \psi(\mu) e^{-j2\pi f\mu} d\mu \quad (2.50)$$

The result of these definitions is that $\psi(\mu)$ represents the correlation of scatterers with delay difference μ while its transform is the mean square frequency response of the medium. $S_V''(\lambda; \alpha)$ is the equivalent of the scattering function of a wide sense stationary, uncorrelated scattering medium and will be discussed further in Section 2.11.

With the assumption of Eqn. 2.45, Eqn. 2.44 can be written

$$S_z(f, t) = \iiint s^*(\sigma) s(\rho + \sigma) \int P_a''(\tau; t - \sigma) \psi(\tau - \rho) e^{-j2\pi f\tau} d\tau d\rho d\sigma$$

By using the fact that a product becomes a convolution transformation, together with the time shift theorem for the transform of $\psi(\tau - \rho)$ we obtain

$$S_z(f, t) = \iiint s^*(\sigma) s(\rho + \sigma) e^{-j2\pi\rho\lambda} \cdot \int S_V''(f - \lambda; t - \sigma) \Psi(\lambda) d\lambda d\rho d\sigma$$

Carrying out the integration with respect to ρ , using the definition (Eqn. 2.30) of the spectral density function of the signal, gives the final form for $S_z(f, t)$.

$$S_z(f, t) = \iint [S_s(\lambda, \sigma) \Psi(\lambda)] S_v''(f - \lambda; t - \sigma) d\lambda d\sigma \quad (2.51)$$

This equation gives the time dependent spectral density of the output process in terms of the same function of the signal, the scattering function of the medium and the frequency response of the medium

The term in parentheses in Eqn. 2.51 represents essentially a filtering, or bandlimiting, action by which the spectrum of the signal is modified by the mean square frequency response of the channel. Thus we might speak of a modified input signal with time dependent spectral density

$$S_e(f, t) = S_z(f, t) \Psi(f) \quad (2.52)$$

(Unfortunately this does not correspond to the action of a physical filter.) With this modified input signal Eqn. 2.51 can be written

$$S_z(f, t) = \iint S_e(f, t) S_v''(f - \lambda; t - \sigma) d\lambda d\sigma \quad (2.53)$$

The action of the medium is now clear. The input spectral density function is first modified according to the "filtering" equation, Eqn. 2.52, and the output spectral density function is then given by a double convolution in both the delay and frequency variables. This latter action implies smoothing and stretching of the modified signal spectral den-

sity function in both variables. The bandlimited frequency response tends to restrict the output bandwidth while the Doppler effect tends to increase it.

It is well known, and is shown in Appendix D, that the mean square dispersion of the convolution of two functions is the sum of their individual dispersions. Using this result the time dispersion and bandwidths of the output process of a correlated scattering medium are easily determined.

In the time variable, $S_z(f, t)$ is a simple convolution of $S_z(f, t)$ and $S_v''(f, t)$. Thus using the definitions in Eqns. 2.34 and 2.37 we obtain

$$m_t(\tilde{z}) = m_t(s) + m_\alpha(\tilde{a}) \quad (2.54)$$

In the frequency variable things are not quite so simple since there is a combination of multiplication and convolution. There are three basic situations which can arise:

- a) The medium bandwidth is much greater than the signal bandwidth. In this case the output bandwidth neglecting Doppler effects, is very slightly less than the input bandwidth.

- b) The medium bandwidth is much less than the signal bandwidth. In this case the output bandwidth is very slightly less than the medium bandwidth, neglecting Doppler effects.
- c) The two bandwidths are approximately equal. In this case the output bandwidth is less than either the medium or the signal bandwidth. The well known result that passing a signal with a first order roll-off spectrum through a filter of similar shape and equal bandwidth produces an output with half the bandwidth, may be used as an approximation of the output bandwidth in the present case.

To all the results indicated above the Doppler spread must be added by virtue of the convolution in Eqn. 2.51. The following expression can thus be written for the output bandwidth.

$$m_{\lambda}(\check{a}) + \frac{1}{2} \inf(m_f(\check{a}), m_f(s)) \leq m_f(\check{z}) \leq m_{\lambda}(\check{a}) + \inf(m_f(\check{a}), m_f(s))^1 \quad (2.55)$$

This is not intended to be a result of mathematical rigour but rather an engineering approximation to be used with discretion.

¹ The function $\inf(x,y)$ takes on the value of the smaller of the two arguments x and y .

2.11 WSS UNCORRELATED SCATTERING MEDIA

The theory of wide sense stationary, uncorrelated scattering (WSSUS) media has been discussed at length by Bello⁽¹⁰⁾ and Daly⁽⁹⁾. It is much simpler than that of correlated scattering media and will be reviewed only very briefly here.

The lack of correlation between scatterers is expressed by defining the time autocovariance function to be

$$P_a(\tau; \alpha, \nu) = P_a(\tau; \alpha) \delta(\alpha - \nu) \quad (2.56)$$

$P_a(\tau; \alpha)$ is clearly comparable to the function $P_a''(\tau; \alpha)$ defined in Eqn. 2.45 for the separable correlated channel.

Transforming $P_a(\tau; \alpha, \nu)$ with respect to τ gives

$$S_v(\lambda; \alpha, \nu) = S_v(\lambda; \alpha) \delta(\alpha - \nu) \quad (2.57)$$

where

$$S_v(\lambda; \alpha) = \int P_a(\tau; \alpha) e^{-j2\pi\lambda\tau} d\tau \quad (2.58)$$

This is a very important function called the scattering function. It is the Doppler-delay profile of the medium and indicates the scattered energy density as a function of Doppler shift and delay.

Transformation of $S_V(\lambda; \alpha, \nu)$ according to Eqn. 2.25 gives

$$S_V(\lambda; f-\beta, f) = S_V(\lambda; \beta) \quad (2.59)$$

where

$$S_V(\lambda; \beta) = \int S_V(\lambda; \alpha) e^{-j2\pi\beta\alpha} d\alpha \quad (2.60)$$

The independence of f here results from the delta function in Eqn. 2.57. Referring to Eqn. 2.48 for correlated media Eqn. 2.59 shows that $\Psi(f)$ is a constant function in the uncorrelated case. A corresponding independence of f occurs in the time-frequency autocovariance function $P_T(\tau; \beta)$ defined by

$$R_T(t, t+\tau; f-\beta, f) = P_T(\tau; \beta) \quad (2.61)$$

where

$$P_T(\tau; \beta) = \int P_a(\tau; \alpha) e^{-j2\pi\beta\alpha} d\alpha \quad (2.62)$$

The Fourier transform relationships between the functions just defined are summarised in Figure 2.5.

2.11.1 The action of the medium

The effect of a WSSUS medium on the time variant spectral density function of a signal passing through it is

obtainable immediately from Eqn. 2.51 using the fact that $\Psi(f)$ is a constant function. Hence

$$S_z(f, t) = \iint S_s(\lambda, \sigma) S_v(f - \lambda; t - \sigma) d\sigma d\lambda \quad (2.63)$$

and the output time variant spectral density is the result of a double convolution of the input spectral density with the scattering function. As a result of this, Eqn. 2.54 for the output time dispersion holds while the output bandwidth is just the sum of the input bandwidth and the Doppler spread.

That is

$$m_f(\tilde{z}) = m_\lambda(\tilde{a}) + m_f(s) \quad (2.64)$$

A number of other very simple relations between input and output covariance functions follow from the properties described above. These are fully derived in Daly⁽⁹⁾ and are listed below.

$$P_z(\tau, t) = \int P_a(\tau; \alpha) P_s(\tau, t - \alpha) d\alpha \quad (2.65)$$

$$\chi_z(\tau, \beta) = P_T(\tau; \beta) \chi_s(\tau, \beta) \quad (2.66)$$

$$P_z(f, \beta) = \int P_v(\lambda; \beta) P_s(f - \lambda, \beta) d\lambda \quad (2.67)$$

They are very elegantly summarised, together with all the other transform relationships affecting the medium, in Figure 2.6 which was first published by Daly⁽⁹⁾.

Figure 2.6 Input-Output Relationships for WSSUS Media

2.11.2 Inclusion of the transmitter and receiver

It has already been pointed out in Section 2.10.1 that inclusion of bandlimited parts of the transmitter and receiver in the channel means that the "medium" is correlated scattering. If the actual physical medium is uncorrelated this involves a great increase in complexity of the theory. It is therefore desirable to plan the model in such a way that as many as possible of the bandlimiting effects are represented in the signal shape $s(t)$.

2.12 THE DETERMINISTIC BRANCH

To complete the study of the total channel model of Figure 1.2 it is necessary to include a review of the effect of the deterministic branch of the medium from the same point of view as was adopted for the random branch. Specifically we shall require the time dispersion and bandwidth of the output.

In the detailed discussion of the random branch it was assumed that the medium was wide sense stationary, or approximately so. The weighting function of the deterministic branch is also independent of time and is denoted by $\bar{a}(\alpha)$ as

in Eqn. 1.15. It is then a simple matter to show that

$$S_{\bar{z}}(f, t) = \int S_s(f, \sigma) S_{\bar{a}}(f, t - \sigma) d\sigma \quad (2.68)$$

where

$$S_{\bar{a}}(f, t) = \int \bar{a}^*(t) \bar{a}(t + \tau) e^{-j2\pi f \tau} d\tau \quad (2.69)$$

which is a time dependent spectral density function defined on the deterministic weighting function.

From Eqn. 2.68 it is clear that the output time dispersion is

$$m_t(\bar{z}) = m_t(\bar{a}) + m_t(s) \quad (2.70)$$

where $m_t(\bar{a})$ is defined, in the usual way, as the mean square dispersion of $S_{\bar{a}}(f, t)$ with respect to t . By the same arguments as led to Eqn. 2.55 the output bandwidth is bounded by

$$\frac{1}{2} \inf(m_f(\bar{a}), m_f(s)) \leq m_f(\bar{z}) \leq \inf(m_f(\bar{a}), m_f(s)) \quad (2.71)$$

since there is no Doppler shift in this case.

Chapter 3

THE OPTIMUM RECEIVER

The problem of communication by pulse amplitude modulation over a randomly time variant, dispersive channel has been described in Chapter 1 and is restated briefly below. The remainder of this chapter is devoted to obtaining the optimum linear receiver for such communication systems.

3.1 THE COMMUNICATION PROBLEM

The communication system is shown in Figure 1.1. A message consisting of a set of random variables $\{\theta_i\}$ is to be transmitted in the form of a signal

$$m(t) = \sum \theta_i s_i(t) \quad (3.1)$$

where the pulse shapes $s_i(t)$ are known to the receiver. In general the sets $\{\theta_i\}$ and $\{s_i(t)\}$ may be infinite but if this is the case the waveforms $s_i(t)$ must be so distributed in time as to restrict the transmitter power to some arbitrary but finite value.

In general the message variables may have arbitrary statistics but in this chapter and in the work to follow it will be assumed that they are uncorrelated, have zero mean and unit variance. That is

$$E \{ \theta_i \theta_j \} = \delta_{ij} \quad (3.2a)$$

and

$$E \{ \theta_i \} = 0 \quad (3.2b)$$

The theory presented in this chapter is extended to the case of correlated variables with unequal variances in Appendix C.

The channel model, shown in Figure 1.2, consists of a random medium represented by separate deterministic and random branches followed by a source of zero mean additive noise with covariance function

$$R_n(t, r) = E \{ n(t) n(r) \} \quad (3.3)$$

In response to the signal $m(t)$ the channel output is

$$w(t) = \sum \theta_i z_i(t) + n(t) \quad (3.4)$$

The signal $z_i(t)$ is the total response of the medium to the signal $s_i(t)$ and may therefore be expressed as the sum of deterministic and random parts

$$z_i(t) = \bar{z}_i(t) + \tilde{z}_i(t) \quad (3.5)$$

after the manner of Eqn. 1.12. Each of these signals may be related to the corresponding transmitted signal $s_i(t)$ using any of the input-output relationships listed in Eqns. 2.9 to 2.12. For instance

$$\tilde{z}_i(t) = \int \tilde{a}(t, \alpha) s(\alpha) d\alpha$$

The receiver is required to form the linear, minimum mean square estimate of each of the random variables θ_i . Thus it must be optimum in the sense that it minimises each of the estimation errors

$$e_{m_i} = E \{(\theta_i - \hat{\theta}_i)^2\} \quad (3.6)$$

where $\hat{\theta}_i$ is the estimate of θ_i . This is subject to the constraint that each estimate is to be a linear function of the data available at the receiver, that is, the received signal $w(t)$. The receiver is not constrained to be causal.

3.2 AUTOVARIANCE OF THE CHANNEL OUTPUT

The received signal $w(t)$ defined in Eqn. 3.4 is a random process resulting from the action generated by three independent ensembles, those of the message, the random medium and the additive noise. The autocovariance of this signal,

defined as in Eqn. 2.26, will be required later and is derived here. Because of the independence of the three ensembles, the averaging process may be carried out separately for each one. Since all the time functions are real, the complex conjugate sign in Eqn. 2.26 will be dropped. With these stipulations the autocovariance can be written

$$\begin{aligned}
 R_w(t,r) &= \sum_i \sum_j E \{ \theta_i \theta_j \} E \{ z_i(t) z_j(r) \} + E \{ n(t) n(r) \} \\
 &= \sum_i [R_{z_i}(t,r) + \bar{z}_i(t) \bar{z}_i(r)] + R_n(t,r) \quad (3.7)
 \end{aligned}$$

where

$$R_{z_i}(t,r) = E \{ \tilde{z}_i(t) \tilde{z}_i(r) \} \quad (3.8)$$

The elimination of one of the summations results from the property specified in Eqn. 3.2a. For convenience the simplified notations

$$R_y(t,r) = \sum_i R_{z_i}(t,r) \quad (3.9)$$

and

$$R_x(t,r) = R_y(t,r) + \sum_i \bar{z}_i(t) \bar{z}_i(r) \quad (3.10)$$

are adopted. $R_w(t,r)$ then becomes

$$R_w(t,r) = R_x(t,r) + R_n(t,r) \quad (3.11)$$

3.3 THE OPTIMISATION EQUATION

Since the error of each estimate $\hat{\theta}_j$ is to be minimised separately, no loss of generality is involved in concentrating attention on the estimation of one of them, say $\hat{\theta}_1$. The estimate $\hat{\theta}_1$ is constrained to be a linear function of the received signal and so can be represented with complete generality as the output of a linear filter sampled at some instant in time as shown in Figure 3.1a. Because this filter is not constrained to be causal this instant can conveniently be chosen to be $t = 0$. If the impulse response of the filter is $h^1(t)$ and we define

$$k^1(t) = h^1(-t) \quad (3.12)$$

and the estimate can be expressed as

$$\hat{\theta}_1 = \int_{-\infty}^{\infty} k^1(t)w(t)dt \quad (3.13)$$

Although in principle the limits of integration in Eqn. 3.13 are infinite they can always be replaced with finite, though arbitrarily large limits in practice. This follows from the fact that, in any practical system, the transmitter waveforms will be essentially time limited. By this we mean that they can be approximated arbitrarily

Figure 3.1 Optimum Receiver for Estimating θ_i

closely, in the integral square sense, by time limited functions which are square integrable. The same argument applies to the received waveforms $z_n(t)$ since they are the finite energy responses of a physical channel to a finite energy stimulus. In any practical receiver the range of integration will in any case be finite. This assures the existence of the integral in Eqn. 3.13. Even in the case of infinite limits the integral exists with probability one since the function $k^i(t)$ will be chosen to minimise the error defined in Eqn. 3.6 which must be less than the value of the variance of θ_i which is unity. With this understanding the limits of integration will not be marked hereafter.

By the principle of orthogonality (see for instance reference 11) the linear, minimum mean square estimate of a random variable is orthogonal to the data on which the estimate is based, in this case the received signal $w(t)$.

Symbolically this is expressed by the equation

$$E \{(\theta_i - \hat{\theta}_i)w(t)\} = 0 \quad (3.14)$$

for all t , or equivalently

$$E \{\theta_i w(t)\} = E \{\hat{\theta}_i w(t)\} \quad (3.15)$$

Substitution of Eqn. 3.4 for $w(t)$ in the left hand side of Eqn. 3.15 together with consideration of the properties of Eqn. 3.2 shows that

$$E \{ \hat{\theta}_i w(t) \} = \bar{z}_i(t) \quad (3.16)$$

The same procedure for the right hand side of Eqn. 3.15 gives

$$E \{ \hat{\theta}_i w(t) \} = \int R_w(t,r) k^i(r) dr \quad (3.17)$$

Finally substitution of these results in Eqn. 3.15 gives the optimisation equation

$$\bar{z}_i(t) = \int R_w(t,r) k^i(r) dr \quad (3.18)$$

the solution of which defines the optimum filter $k^i(-t)$.

If the additive noise is white, with spectral density N_0 so that

$$R_w(t,r) = R_x(t,r) + N_0 \delta(t-r) \quad (3.19)$$

Eqn. 3.18 becomes

$$N_0 k^i(t) = \bar{z}_i(t) - \int R_x(t,r) k^i(r) dr \quad (3.20)$$

It will be assumed from now on that this is the case since it can always be arranged by using a prewhitening filter before the receiver.

3.3.1 Mathematical nature of the optimisation equation

The function $R_x(t, r)$ is the autocovariance function of the output of the random medium in response to the transmitted signal. Since the transmitter power is bounded the power represented by $R_x(t, t)$ must also be bounded and $R_x(t, r)$ is therefore square integrable over any finite range. With the understanding introduced above that the limits of integration can always be replaced with finite limits with arbitrarily small error, Eqn. 3.20 becomes a Fredholm equation of the second kind with a square integrable kernel. The function $\bar{z}_1(t)$ is also square integrable and consequently the existence of a solution is assured provided -1 is not an eigenvalue of the kernel. The fact that $R_x(t, r)$ is an autocovariance function and is therefore non-negative definite precludes the existence of a negative eigenvalue. (An excellent discussion of integral equations of this type is given by Smithies⁽¹²⁾.)

3.4 SOLUTION OF THE OPTIMISATION EQUATION

For any particular problem Eqn. 3.20 can be solved to give the impulse response of the optimum receiver, usually this must be done using numerical techniques on a digital

computer. Solutions obtained in this manner, however, are not very interesting or helpful in the sense of contributing to the understanding of the principles on which the receiver operates. Furthermore, the solution for one type of channel and signal does not give any information regarding the solution to different, though closely related channels or signals. The purpose of this chapter is to obtain a canonical form of the receiver which applies to all linear channels. The physical principles underlying the operation of the receiver will be clearly demonstrated by this form.

The key to obtaining a meaningful general solution of Eqn. 3.20 lies in formulating the receiver in a somewhat different form from the one adopted in Eqn. 3.13. This is done by specifying that the filter $h^i(t)$ in Figure 3.1a may consist of a bank of filters $h_n^i(t)$ as shown in Figure 3.1b. The number of filters in this bank is allowed to be as large as necessary for the solution of the equation in the manner to be described. Each of the filters is followed by a constant scaling factor a_n^i as shown. This allows complete freedom to determine the shape of each function $h_n^i(t)$ without worrying about its amplitude. The amplitude can then be

determined later by solving for the a_n^i . The outputs of the bank of filter-multipliers are summed on a summing bus which is sampled at $t = 0$ for the estimate of θ_i . The response of the original filter is now given by

$$h^i(t) = \sum_n a_n^i h_n^i(t) \quad (3.21)$$

and again a set of inverse time responses

$$k_n^i(t) = h_n^i(-t) \quad (3.22)$$

is defined. The estimate of θ_i is given by

$$\hat{\theta}_i = \sum_n a_n^i \int k_n^i(t) w(t) dt \quad (3.23)$$

Each of the two forms postulated for the receiver is a completely general expression for a linear filter so that no loss of generality has been incurred by the change just introduced.

Substituting the new forms defined in Eqns. 3.21 and 3.22 in Eqn. 3.20 give the following rather cumbersome equation

$$\begin{aligned} N_o \sum_n a_n^i k_n^i(t) &= \bar{z}_i(t) - \sum_n a_n^i \int R_y(t, r) k_n^i(r) dr \\ &\quad - \sum_n \sum_j a_n^i \bar{z}_j(t) \int \bar{z}_j(r) k_n^i(r) dr \end{aligned} \quad (3.24)$$

The first major simplification of this equation is made by realising that, once the $k_n^i(t)$ are determined, the definite integral in the last term is just a constant

$$C_{nj}^i = \int \bar{z}_j(r) k_n^i(r) dr \quad (3.25)$$

Using the set of unknown constants C_{nj}^i , Eqn. 3.24 can be rewritten and after some rearrangement,

$$\begin{aligned} \bar{z}_i(t) = N_o \sum_n a_n^i k_n^i(t) + \sum_n a_n^i \int R_y(t, r) k_n^i(r) dr \\ + \sum_n \sum_j a_n^i C_{nj}^i \bar{z}_j(t) \end{aligned} \quad (3.26)$$

Finally Eqn. 3.26 can be reduced to a set of integral equations and a set of algebraic equations by defining each $k_n^i(t)$ to be the solution of an integral equation of the form

$$N_o k_n^i(t) = \bar{z}_n(t) - \int R_y(t, r) k_n^i(r) dr \quad (3.27)$$

There is a point of great significance to be noted from Eqn. 3.27. The solution for $k_n^i(t)$ is in no way dependent on the value of the superscript i , that is on the particular parameter θ_i being estimated. This is shown by the fact that the superscript appears only on the unknown function $k_n^i(t)$ for which the equation is to be solved. The implication of this

is that when the optimum receiver for the estimation of any other parameter θ_j is considered, the same equation will appear and its solution will be the same. Accordingly the superscript will be dropped so that

$$k_n^i(t) = k_n(t) \quad (3.28)$$

for all i . The immediate corollary of this result is that the constants C_{nj} are also independent of i and will be defined by

$$C_{nj} = \int \bar{z}_j(r) k_n(r) dr = C_{nj}^i \quad (3.29)$$

for all i .

We can now use Eqn. 3.27 to substitute for the integral in Eqn. 3.26 with the following result:

$$\bar{z}_i(t) = \sum_n a_n^i \bar{z}_n(t) + \sum_n \sum_j a_n^i C_{nj} \bar{z}_j(t) \quad (3.30)$$

Equating coefficients of each of the time functions yields the following set of simultaneous algebraic equations for the a_n^i .

$$\begin{aligned} a_i^i &= 1 - \sum_n a_n^i C_{ni} \\ a_j^i &= - \sum_n a_n^i C_{nj}, \quad j \neq i \end{aligned} \quad (3.31)$$

The similarity of Eqn. 3.31 to the special form given in Eqn. 1.27 of the equations obtained by George⁽⁴⁾ is already striking. When Eqn. 1.29 is used as the definition of the C_{nj} for the George receiver these constants have identical forms, though different values, in the two cases. The correspondence with the George receiver will be dealt with at greater length in Section 3.9 which covers the special case most closely corresponding to George's work. Although in general the set of equations for the a_n^i may be infinitely large, the work of Coll⁽³⁾ and Kaye⁽⁵⁾ in connection with practical realisations of the closely allied George receiver shows that in practice the set of a_n^i may be approximated closely by a finite set. In this case there is no difficulty in solving for the a_n^i .

3.5 INTERPRETATION OF THE INTEGRAL EQUATION

Eqn. 3.27 defining the individual filters in the receiver is best understood from an engineering point of view by expressing it as

$$\bar{z}_n(t) = \int [R_y(t,r) + N_o \delta(t-r)] k_n(r) dr \quad (3.32)$$

This is a well known form, introduced in this thesis in Eqn. 1.22, defining a filter $h_n(t)$ matched to the deterministic signal $\bar{z}_n(t)$ in a noise environment characterised by the autocovariance function within the square brackets. This noise is the total random signal at the output of the channel due to the additive noise and the interaction of the channel with all the transmitted signals $\theta_i s_i(t)$.

The above argument shows that the receiver contains a bank of filters each of which is matched, in the special sense defined by Eqn. 3.32, to the deterministic received signal corresponding to a transmitted signal $s_n(t)$. The purpose of these filters is to compensate for the effects of the random multiplicative and additive noise. If there is intersymbol interference between the received pulses $z_n(t)$ then its effect is compensated by the network of constant multipliers a_n^i , as is shown in the next section.

3.6 COMPENSATION FOR INTERSYMBOL INTERFERENCE

We shall show that the purpose of the weighting network is to compensate for intersymbol interference by showing that it is degenerate when there is no intersymbol interference.

If the deterministic and random received signals due to the transmitted signal $\theta_n s_n(t)$ occupy a time interval which does not overlap with the corresponding time interval for the received signals due to any of the other transmitted signals, then we say that there is no intersymbol interference. To be specific let the time interval occupied by $z_n(t)$ be $t_{n-1} < t \leq t_n$ and let the region of the (t, r) plane for which $R_{z_n}(t, r)$ is non-zero be $t_{n-1} < t, r \leq t_n$. These intervals and regions for different values of n are to be disjoint. In this case Eqn. 3.27 reduces to

$$N_0 k_n(t) = \bar{z}_n(t) - \int R_{z_n}(t, r) k_n(r) dr \quad (3.33)$$

which is identical to the result which would be obtained if only one pulse, $\theta_n s_n(t)$, were transmitted. It is clear that the non-zero range of $k_n(t)$ is also contained within the interval $t_{n-1} < t \leq t_n$ and for this reason

$$C_{nj} = 0, \quad j \neq n \quad (3.34)$$

Substituting this result in Eqn. 3.31 shows that

$$a_j^i = 0, \quad j \neq i \quad (3.35)$$

Thus in the isolated pulse case the weighting network reduces to a single multiplier at the output of each filter and the

estimate of θ_i depends only on the output of the i th filter.

$$\hat{\theta}_i = a_i^i \int k_i(t)w(t)dt \quad (3.36)$$

A similar result arises in a different way when the various signals do not overlap in frequency as is discussed in Section 3.7.1.

3.7 FREQUENCY DOMAIN DEFINITION OF THE RECEIVER

So far the receiver has been described in terms of its impulse response. It is equally possible to define it in terms of its frequency response. The derivation of the optimisation follows parallel lines to the time domain definition and will not be repeated.

The frequency response of the n th filter is the solution of the integral equation

$$N_o K_n(f) = \bar{Z}_n(f) - \int R_Y(f,u)K_n(u)du \quad (3.37)$$

which is of exactly the same form as the time domain equation. $R_Y(f,u)$ is defined, in a similar manner to $R_Y(t,r)$, as the sum

$$R_Y(f,u) = \sum R_{Z_i}(f,u) \quad (3.38)$$

The constants C_{nj} can be defined, using Parseval's theorem, as

$$C_{nj} = \int \bar{Z}_j^*(f) K_n(f) df \quad (3.39)$$

and the equations for the a_n^i are the same as in the time domain.

3.7.1 Intersymbol interference

In Section 3.6 a situation was described in which no intersymbol interference occurs because the received energy due to individual transmitted pulses occupied discrete time intervals. A parallel case is defined in the frequency domain when the received energy due to different transmitted pulses occupies discrete frequency bands, as for instance in a frequency division multiplexing system. In the same way that the delay dispersion of the channel can cause the received signals to overlap in time at the receiver when they do not do so at the transmitter, Doppler spread can cause overlapping of the frequency bands of received signals when the transmitted signals occupy discrete bands.

3.8 REVIEW OF THE RECEIVER STRUCTURE

We have shown that the optimum receiver consists of two main sections, a bank of filters and a weighting network. Each of the filters is matched to one of the deterministic pulse shapes $\bar{z}_n(t)$ in the extended sense of matching which includes the output of the random branch of the channel together with the additive noise. With this interpretation of the filters it is hardly surprising that the same set of filters is required for estimating each of the θ_i .

The output of each of the filters is fed to the weighting or compensation network which provides a separate set of weights a_n^i for the estimation of each of the θ_i . The purpose of this network is to compensate for intersymbol interference.

While the receiver structure described might be extremely large and impractical in some situations it should be remembered that it is a completely general result covering all situations within the class defined. In any particular example it may be possible to achieve major simplifications without significant loss.

All the filters in the bank have been allowed to be non-causal in the derivation. In practice, of course, this property would be approximated by causal filters with appropriate delay. By the same token it is not necessary that all the filters be sampled at the same time or that all the estimates be obtained at the same time, matters of this kind must be decided according to the properties of the particular signalling scheme used. An example of a situation in which a very great simplification is achievable is given in Section 3.10.

3.9 PERFORMANCE OF THE RECEIVER

The mean square error of each estimate $\hat{\theta}_i$ obtained by the optimum receiver is quite easily found. By definition it is

$$e_{m_i} = E \{ (\theta_i - \hat{\theta}_i)^2 \} \quad (3.40)$$

and this can be written

$$e_{m_i} = E \{ (\theta_i - \hat{\theta}_i) \theta_i \} - E \{ (\theta_i - \hat{\theta}_i) \hat{\theta}_i \} \quad (3.41)$$

But $\hat{\theta}_i$ is a linear function of the received signal and the error $(\theta_i - \hat{\theta}_i)$ is orthogonal to the received signal by virtue

of the optimisation procedure (see Eqn. 3.14); the second term of Eqn. 3.41 is therefore zero. Expanding the first term in terms of the estimate, from Eqn. 3.23, gives

$$e_{m_i} = E \{ (\theta_i - \sum a_n^i \int k_n(t) w(t) dt) \theta_i \}$$

and by using Eqn. 3.4 for $w(t)$, together with Eqn. 3.2 for the correlation properties of the $\{\theta_i\}$ ensemble,

$$e_{m_i} = 1 - \sum a_n^i c_{ni} \quad (3.42)$$

This equation enables the error to be determined once the solution for the receiver has been found.

It is an interesting property of the receiver that the expression for e_{m_i} coincides with that for a_i^i (see Eqn. 3.31) so that

$$e_{m_i} = a_i^i \quad (3.43)$$

This is sufficient to determine the range of values which can be taken by a_i^i . The mean square error is non-negative by definition and is also less than the variance of θ_i so that

$$0 \leq a_i^i \leq 1 \quad (3.44)$$

3.10 SEQUENTIAL SIGNALLING OVER A WSS CHANNEL

A very common form of communication system was described in Section 1.1 in which the set $\{s_i(t)\}$ consists of a sequence of pulses of the same shape transmitted at intervals of T seconds. Thus

$$s_i(t) = s(t-iT) \quad (3.45)$$

In general this does not introduce any simplification in the structure of the receiver, but if also the transmission medium is wide sense stationary a very great simplification results.

A wide sense stationary medium was defined in Section 1.4.3 as one in which the first and second order statistics are not functions of time. Thus from Eqn. 1.17

$$R_a(t, r; \sigma, \nu) = P_a(r-t; \sigma, \nu) \quad (3.46)$$

and the weighting function of the deterministic branch is also time invariant and is denoted $\bar{a}(\alpha)$. The deterministic output pulses are then given by

$$\bar{z}_n(t) = \bar{z}(t-nT) \quad (3.47)$$

where

$$\bar{z}(t) = \int \bar{a}(t-\alpha)s(\alpha)d\alpha \quad (3.48)$$

From Eqns. 2.9 and 2.13 the autocovariance function $R_{z_j}(t, r)$ defined in Eqn. 3.8 is given in this case by

$$\begin{aligned} R_{z_j}(t, r) &= \iint P_a(r-t; \sigma, v) s(t-jT-\sigma) s(r-jT-v) d\sigma dv \\ &= R_z(t-jT, r-jT) \end{aligned} \quad (3.49)$$

and the total autocovariance function is

$$R_y(t, r) = \sum R_z(t-jT, r-jT) \quad (3.50)$$

$$= R_y(t-nT, r-nT) \quad (3.51)$$

for any n . Hence the autocovariance function of the output of the random channel is periodic. Expressing these properties in terms of the alternative notation for the autocovariance function, as in Eqn. 2.27, we have

$$P_y(\tau, t) = \sum P_y(\tau, t-jT) \quad (3.52)$$

and this has a similar periodic property

$$P_y(\tau, t) = P_y(\tau, t-nT) \quad (3.53)$$

for any n .

3.10.1 The optimum receiver

Substituting the form for $\bar{z}_n(t)$ from Eqn. 3.47 into the integral equation (Eqn. 3.27) defining the n th filter

characteristic gives

$$k_n(t) = \bar{z}(t-nT) - \int R_y(t,r)k_n(r)dr$$

and using the periodic property of Eqn. 3.41

$$k_n(t) = \bar{z}(t-nT) - \int R_y(t,r)k_n(r)dr \quad (3.54)$$

But the equation defining the filter with index zero is

$$k_o(t-nT) = \bar{z}(t-nT) - \int R_y(t-nT,r-nT)k_o(r-nT)dr \quad (3.55)$$

where a time shift nT has been introduced. Now comparing equations 3.54 and 3.55 shows that

$$k_n(t) = k_o(t-nT) \quad (3.56)$$

for all n . Remembering (Eqn. 3.22) that the $k_n(t)$ are the inverse-time responses of the filters, the impulse response of the n th filter is

$$h_n(t) = h_o(t-nT) \quad (3.57)$$

where

$$h_o(t) = k_o(-t) \quad (3.58)$$

so that the receiver consists of a bank of filters of identical characteristics, except for a time shift, and the output

of each of these filters is sampled at $t = 0$. The output of the n th filter at $t = 0$ is

$$\begin{aligned} v_n &= \int h_n(-\alpha)w(\alpha)d\alpha \\ &= \int h_0(-\alpha-nT)w(\alpha)d\alpha \end{aligned} \quad (3.59)$$

But this is just the output of the filter $h_0(t)$ at $t = -nT$. Consequently the receiver can be reduced to a single filter $h_0(t)$ the output of which is sampled at intervals of T seconds, the signalling interval, provided provision is made for storing the values of the samples. This storage can conveniently be provided by using a tapped delay line as shown in Figure 3.2a.

So far the bank of filters has been reduced to a single filter but a completely different set of weights a_n^i may still be required for the estimation of each θ_j from the information stored in the delay line. Fortunately this turns out to be unnecessary as is shown by the following argument.

3.10.2 Properties of the estimation weights

First it is a general property of the constants C_{nj} , not dependent on the wide sense stationarity of the

(a)

(b)

Figure 3.2 Optimum Receiver for Sequential Signalling Over Wide-Sense Stationary Media

channel or the use of sequential signalling, that

$$C_{nj} = C_{jn} \quad (3.60)$$

This can be seen by using Eqn. 3.27 for $\bar{z}_j(t)$ in the formula Eqn. 3.29 for C_{nj} to give

$$C_{nj} = \int k_n(t)k_j(t)dt + \iint k_n(t)R_y(t,r)k_j(r)dr$$

But $R_y(t,r)$, being an autocovariance function is symmetric in t and r so that the whole equation is symmetric and Eqn. 3.60 is proved.

Next a property peculiar to the particular channel and signalling system of this section is that

$$C_{nj} = D_{n-j} = D_{j-n} \quad (3.61)$$

where

$$D_{n-j} = \int k_o(r-nT)\bar{z}(r-jT)dr \quad (3.62)$$

The proof is trivial since equation 3.62 is just the definition of C_{nj} with the particular signals and filter characteristics developed in Section 3.10. The second part of Eqn. 3.61 follows from the symmetric property of the C_{nj} .

Finally the property which will be used for simplification of the receiver is that

$$a_{j+n}^j = a_n^0 \quad (3.63)$$

for all n, j . The equations corresponding to Eqn. 3.31 for the weights for estimating θ_j and θ_0 are, respectively

$$a_j^j = 1 - \sum_i a_i^j D_{i-j} \quad (3.64a)$$

$$a_n^j = - \sum_i a_i^j D_{i-n}, \quad i \neq n \quad (3.64b)$$

and

$$a_0^0 = 1 - \sum_i a_i^0 D_i \quad (3.65a)$$

$$a_n^0 = - \sum_i a_i^0 D_{i-n}, \quad i \neq n \quad (3.65b)$$

Now assuming that Eqn. 3.63 is correct and using it to substitute in Eqn. 3.65b gives

$$a_j^j = 1 - \sum_i a_{j+i}^j D_i$$

which with a change in variable is

$$a_j^j = 1 - \sum_k a_k^j D_{k-j}$$

which is in agreement with Eqn. 3.64a. Similarly substituting from Eqn. 3.63 in Eqn. 3.65b and changing variables yields Eqn. 3.64b. The result of Eqn. 3.63 is therefore true.

3.10.3 The receiver structure

The property expressed in Eqn. 3.63 for the estimation weights shows that only one set of weights need be used if the weighting multipliers are used as tap gains on the tapped delay line of Figure 3.2a. This arrangement is shown in Figure 3.2b. The estimate of θ_j is then obtained by sampling the summing bus at $t = jT$.

The receiver is now identical in form to the receiver derived by George for the time invariant, exactly known channel although, of course, the filters and weights all have different values.

In general the tapped delay line must be of infinite dimensions but in practice, as in the finite version of the George receiver discussed by Coll⁽³⁾, a good approximation can be obtained with one of finite length.

3.10.4 Sequential signalling with multiplexing

If, instead of just one basic waveform $s(t)$ as used in this section so far, there was a set of say M basic waveforms each of which was modulated and transmitted at inter-

vals of T seconds, as in a multiplexing scheme, the receiver would be a trivial extension of the form derived. The receiver would consist of M filters each feeding a tapped delay line in the same way as in the $M = 1$ case discussed above. The integral equation defining the filters would then take account of the noise contributions due to all the multiplex signals and the tap weights would take account of the intersymbol interference not only of the shifted signals in each multiplex group but also that due to signals from the other multiplex groups.

3.10.5 The frequency domain equation

The periodic nature of the noise autocorrelation function demonstrated in Eqn. 3.51 leads to some interesting effects on the filter frequency response given by Eqn. 3.37.

Using the alternative form of the frequency autocovariance function, as in Eqn. 2.29, leads to the following form of Eqn. 3.37.

$$K_n(f) = \bar{Z}_n(f) - \int P_Y(f, \beta) K_n(f - \beta) d\beta \quad (3.67)$$

in which, according to Eqn. 3.38

$$P_Y(f, \beta) = \sum P_{Z_i}(f, \beta) \quad (3.68)$$

Now, from Eqn. 2.12, each component of $P_{Z_i}(f, \beta)$ can be written in terms of the channel and signal functions:

$$\begin{aligned} P_{Z_i}(f, \beta) &= E \{ \tilde{Z}_i^*(f - \beta) \tilde{Z}_i(f) \} \\ &= E \{ \iiint V^*(f - \beta - \nu, \nu) S_i^*(\nu) V(f - \sigma, \sigma) S_i(\sigma) d\nu d\sigma \} \\ &= \iint S_i^*(\nu) S_i(\sigma) R_V(f - \beta - \nu, f - \sigma; \nu, \sigma) d\nu d\sigma \end{aligned} \quad (3.69)$$

This is for the general channel, in the WSS case use of Eqn. 2.24 yields

$$P_{Z_i}(f, \beta) = \int S_i^*(\nu) S_i(\beta + \nu) S_V(f - \beta - \nu; \nu, \beta + \nu) d\nu \quad (3.70)$$

Now using the particular form, Eqn. 3.45 for the signal waveforms and using the time shift theorem for their transform gives

$$\begin{aligned} P_{Z_i}(f, \beta) &= \int S_o^*(\nu) e^{j2\pi\nu iT} S_o(\beta + \nu) e^{-j2\pi(\beta + \nu) iT} \\ &\quad \cdot S_V(f - \beta - \nu; \nu, \beta + \nu) d\nu \\ &= e^{-j2\pi\beta iT} \int S_o^*(\nu) S_o(\beta + \nu) S_V(f - \beta - \nu; \nu, \beta + \nu) d\nu \\ &= e^{-j2\pi\beta iT} P_{Z_o}(f, \beta) \end{aligned} \quad (3.71)$$

Summing these terms to form $P_Y(f, \beta)$

$$\begin{aligned} P_Y(f, \beta) &= \frac{1}{T} P_{Z_0}(f, \beta) \int \delta(\beta - k/T) \\ &= \frac{1}{T} \int P_{Z_0}(f, k/T) \delta(\beta - k/T) \end{aligned} \quad (3.72)$$

This is the frequency domain equivalent of Eqn. 3.51. It has the effect of transforming the integral equation Eqn. 3.67 for the filter frequency responses into the following algebraic equation:

$$K_n(f) = \bar{Z}_n(f) - \int K_n(f - k/T) P_{Z_0}(f, k/T). \quad (3.73)$$

3.11 SUMMARY

In this chapter a completely general solution has been obtained for the problem of signalling by pulse amplitude modulation over a randomly time variant, dispersive channel. The optimum receiver has been shown to be a bank of filters followed by a compensation network. The filters are matched in a special sense to the individual signalling waveforms and are used to combat the combined effects of multiplicative and additive noise. The compensation network exists for the purpose of combatting the effects of intersymbol interference and corresponds to the similar network obtained by previous workers in the case of the known, time invariant channel.

In the special case of sequential signalling over a wide sense stationary channel the receiver has been shown to be a single filter followed by a time invariant tapped delay line.

When there is no intersymbol interference the compensation network is degenerate and consists of a single constant multiplier at the output of each filter.

Chapter 4

THE RECEIVER FOR ISOLATED PULSES

In Chapter 3 it was shown that the two types of random disturbance interfering with the estimation of θ_i are dealt with in distinct parts of the receiver. The disturbance due to the multiplicative and additive noise of the channel is minimised by the bank of filters while the intersymbol interference is compensated by a weighting network acting on the sampled outputs of the filters.

This chapter and those following concentrate on the degradation of performance caused by the random multiplicative disturbances in the channel. In order to do this without confusing the situation with intersymbol interference effects, the isolated pulse problem is considered. The estimate of the i th message parameter is given in this case by Eqn. 3.36 as

$$\hat{\theta}_i = a_i^i \int k_i(t)w(t)dt$$

Since we shall focus attention on the estimation of only one parameter, all subscripts and superscripts will be dropped

for greater clarity except for the multiplying constant a_i^i which will be written a_o . Thus the transmitted waveform is denoted by $s(t)$ and the resulting deterministic received waveform is

$$\bar{z}(t) = \int \bar{a}(t, \alpha) s(t - \alpha) d\alpha \quad (4.1)$$

4.1 THE INTEGRAL EQUATION AND VECTOR SPACE NOTATION

The autocovariance function of the output of the random medium will be the part of Eqn. 3.9 due to a single pulse and from Eqns. 2.9 and 2.13 this is

$$\begin{aligned} R_y(t, r) &= R_z(t, r) \\ &= \iint R_a(t, r; \alpha, \sigma) s(t - \alpha) s(r - \sigma) d\alpha d\sigma \end{aligned} \quad (4.2)$$

With these definitions the integral equation defining the single filter in this receiver is Eqn. 3.33:

$$N_o k(t) = \bar{z}(t) - \int R_z(t, r) k(r) dr \quad (4.3)$$

To facilitate some of the discussions in this and later chapters it is convenient to define a vector space L^2 of functions square integrable in the Lebesgue sense. All the time functions involved have already been shown to satisfy this requirement. Time functions will be denoted in this space by an appropriate single letter, for instance $k(t)$ will

be denoted simply by k . The usual inner product

$$(x, y) = \int x(t)y(t)dt \quad (4.4)$$

is defined and leads to the definition of a norm

$$||x|| = (x, x)^{\frac{1}{2}} \quad (4.5)$$

Using this norm the energies of the transmitted and deterministic received signals can be defined respectively as

$$E_t = ||s||^2 = (s, s) \quad (4.6)$$

and

$$E_r = ||\bar{z}||^2 = (\bar{z}, \bar{z}) \quad (4.7)$$

The quantity E_r will usually be referred to simply as the received energy.

The integral transform of a function with respect to the kernel $R_z(t, r)$ will be denoted by a linear operator R_z with norm

$$||R_z|| = \{\iint |R_z(t, r)|^2 dt dr\}^{\frac{1}{2}}. \quad (4.8)$$

With these definitions the integral equation can be written

$$N_o k = \bar{z} - R_z k \quad (4.9)$$

Wherever necessary it will be understood that all time functions are essentially time limited in the sense defined in Section 3.3. The kernel $R_z(t,r)$ will then also be non-zero only in some square $a \leq t, r \leq b$ of the (t,r) plane with the assumption that the channel also has essentially time limited memory. Thus limits of integration will always be (a,b) . With these assumptions the vector space and its operators are well defined and unique solutions always exist for integral equations of the form of Eqn. 4.3.

4.2 NORMALISATION OF THE INTEGRAL EQUATION

In order to investigate the behaviour of the solutions of Eqn. 4.8 as various parameters of the channel are varied it is necessary first to identify the most significant parameters of a randomly time variant channel. This can most conveniently be done by normalising the integral equation in such a way that desired conditions can be easily formulated in a physically meaningful way. That is the object of this section.

In order to be able to select and discuss a variety of different transmitted and deterministic received signal shapes, these functions will be expressed as scaled unit energy pulses.

Thus $s(t)$ and $\bar{z}(t)$ will be written

$$s(t) = \sqrt{E_t} s'(t) \quad (4.10)$$

and

$$\bar{z}(t) = \sqrt{E_r} z'(t) \quad (4.11)$$

where $s'(t)$ and $z'(t)$ are the unit energy pulses.

Whereas in the case of an exactly known channel it is usually unimportant, from the theoretical standpoint, to know the attenuation of the channel, it turns out to be of great significance in the case of a random channel. The ratio of the transmitted and (deterministic) received energies is therefore defined as

$$g^2 = E_r / E_t \quad (4.12)$$

and has the value

$$g^2 = \int [\bar{a}(t, \alpha) s'(t - \alpha) d\alpha]^2 dt \quad (4.13)$$

In the case of the random branch of the medium it is again desirable to be able to consider different shapes of the covariance functions separately from their amplitudes. The normalisation should therefore be of the form

$$R_a(t, r; \nu, \sigma) = \gamma^2 R'_a(t, r; \nu, \sigma) \quad (4.14)$$

where γ^2 is an amplitude factor. The actual choice of this factor seems best made separately for each class of channels. The examples worked out in later chapters involve the class of wide sense stationary channels and for these a good choice is to choose γ^2 to be the total variance of the medium integrated over all delay values. Since the covariance function is written, according to Eqn. 2.20, in terms of a time shift variable, Eqn. 4.14 becomes

$$P_a(\tau; \nu, \sigma) = \gamma^2 P'_a(\tau; \nu, \sigma) \quad (4.15)$$

where

$$\gamma^2 = \int P_a(0; \nu, \nu) d\nu \quad (4.16)$$

With both the transmitted signal and the channel covariance function normalised, the autocovariance of the output of the random medium is automatically normalised since

$$R_z(t, r) = \gamma^2 E_t R'_z(t, r) = \frac{\gamma^2}{g^2} E_r R'_z(t, r) \quad (4.17)$$

where, from Eqn. 4.2,

$$R'_z(t, r) = \iint R'_a(t, r; \nu, \sigma) s'(t-\nu) s'(r-\sigma) d\nu d\sigma \quad (4.18)$$

The integral equation Eqn. 4.3 in terms of the normalised functions is then

$$k(t) = \frac{\sqrt{E_r}}{N_0} z'(t) - \frac{\gamma^2 E_r}{g^2 N_0} \int R'_z(t,r) k(r) dr \quad (4.19)$$

The appropriate normalisation for the inverse-time impulse response is then clearly

$$k(t) = \frac{\sqrt{E_r}}{N_0} k'(t) \quad (4.20)$$

While this last normalisation may seem rather odd at first sight, reference to Eqn. 1.23 shows that the normalisation factor here is precisely the scaling factor defining the amplitude of a matched filter used for parameter estimation in the case of an exactly known channel.

The final form of the normalised equation is thus

$$k'(t) = z'(t) - \frac{\gamma^2 E_r}{g^2 N_0} \int R'_z(t,r) k'(r) dr \quad (4.21)$$

and we see that two very important channel parameters have appeared in the course of the normalisation.

The first parameter is the ratio

$$\nu = E_r / N_0 \quad (4.22)$$

of the energy of the deterministic received pulse to the spectral density of the additive noise. This corresponds exactly to the usual signal to noise ratio defined for an exactly known channel. When we refer in this work to the "signal to noise ratio" we shall mean this quantity. When we speak of increasing the signal to noise ratio we shall infer that this is done by increasing the transmitter power.

The second parameter

$$\eta = g^2/\gamma^2 \quad (4.23)$$

which has no counterpart for an exactly known channel, is a quality factor indicating the ratio of deterministic energy to scattered, or random, energy delivered by the channel in response to a transmitted pulse. It is because of the appearance of this quality factor involving the attenuation ratio g^2 of the deterministic branch of the medium that this ratio is important in the theory of random media.

The signal to noise ratio ν and the quality factor η together determine the performance of a channel for given shapes of the signal and covariance function. In particular it is their ratio

$$\lambda = \nu/\eta \quad (4.24)$$

which determines the solution of the normalised integral equation. Writing Eqn. 4.21 in terms of this ratio gives

$$k'(t) = z'(t) - \lambda \int R'_Z(t,r)k'(r) dr \quad (4.25)$$

and there is a corresponding form

$$K'(f) = Z'(f) - \lambda \int R'_Z(f,u)K'(u) du \quad (4.26)$$

in the frequency domain resulting from Eqn. 3.37. In Section 4.3 it will be shown that determination of $k'(t)$ is sufficient, together with knowledge of the signal to noise ratio ν , to determine the performance of the system. Thus although the ratio λ is sufficient to define the filter shape, the multiplying constant a_0 defining the filter amplitude and the mean square estimation error e_m require knowledge of the value of each of the parameters ν and η .

In the vector notation of Section 4.2 the normalised equation can be written

$$k' = z' - \lambda R'_Z k' \quad (4.27)$$

By introducing an identity operator I such that, for any vector x in L^2 ,

$$Ix = x \quad (4.28)$$

Eqn. 4.27 becomes

$$(I + \lambda R'_z)k' = z' \quad (4.29)$$

Since a solution is known to exist we can define an inverse operator $(I + \lambda R'_z)^{-1}$ such that

$$k' = (I + \lambda R'_z)^{-1} z' \quad (4.30)$$

4.3 THE MEAN SQUARE ESTIMATION ERROR

Eqns. 3.42 and 3.43 show that the mean square estimation error, and the value of the multiplying constant, in the isolated pulse case is

$$e_m = a_o = 1/(1 + C_{oo}) \quad (4.31)$$

where

$$C_{oo} = \int k(t)\bar{z}(t)dt \quad (4.32)$$

The error is therefore determined by the coefficient C_{oo} and, because of its special significance, this will be denoted by the symbol ρ .

In terms of the normalised time functions defined in Section 4.2,

$$\rho = \frac{E_r}{N_o} \cdot (k', z') = v \cdot (k', z') \quad (4.33)$$

and, by comparison with Eqn. 4.31,

$$e_m = 1/(1 + \rho) \quad (4.34)$$

Eqn. 4.33 shows that a solution for the normalised function $k'(t)$, together with a knowledge of the signal to noise ratio ν , determines the mean square error, as was pointed out in the previous section.

By using Eqn. 4.30 for k' , ρ can be expressed as

$$\rho = \nu \cdot (z', (I + \lambda R_z')^{-1} z') \quad (4.35)$$

and this form will be used in some later results.

4.4 THE EFFECTIVE SIGNAL TO NOISE RATIO

If the channel were exactly known the solution of Eqn. 4.25 would be

$$k'(t) = z'(t) \quad (4.36)$$

which is, of course, the well known matched filter solution for this problem. In this case the value of ρ would be, from Eqn. 4.33,

$$\rho = \nu \cdot (z', z') = \nu \quad (4.37)$$

since $z'(t)$ is a unit energy pulse. Thus in the exactly known case ρ is just the signal to noise ratio and the mean square error is

$$e_m = 1/(1 + \nu) \quad (4.38)$$

By comparison with Eqn. 4.34 it can be seen that, in this sense, ρ can be regarded as an effective signal to noise ratio for the random channel since a known channel with this value of signal to noise ratio would have the same performance. Because the presence of the random medium can only degrade performance, that is cause a larger error, Eqns. 4.34 and 4.38 yield the following inequality

$$\rho \leq \nu \quad (4.39)$$

which is useful as a check on numerical results.

4.5 PERFORMANCE OF A SIMPLE MATCHED FILTER

So far expressions have been developed for the optimum receiver and its performance and in later chapters some examples will be worked out for particular types of channels. At that stage it will be interesting to see what advantage the optimum receiver has over a simple matched filter which takes

no account of the random branch of the medium. Many existing communication systems are designed in this way at least with regard to random fluctuations on the same time scale as the signalling interval (although diversity systems and adaptive systems are used to combat the effects of longer term drifts). In this section an expression is developed for the performance of the simple matched filter, defined by Eqns. 1.23 and 1.24.

In vector notation the estimate of θ is given by

$$\hat{\theta} = a_0(k, w) \quad (4.40)$$

where a_0 is defined in Eqn. 1.24 and the received signal is

$$w = \theta (\bar{z} + \tilde{z}) + n \quad (4.41)$$

The mean square error will be denoted by e_m^f and can be expanded as follows

$$\begin{aligned} e_m^f &= E \{(\theta - \hat{\theta})^2\} \\ &= 1 - 2a_0 E \{\theta(k, w)\} + a_0^2 E \{(k, w)^2\} \end{aligned} \quad (4.42)$$

On substituting the normalised vectors and after some algebra the final form for e_m^f is

$$e_m^f = \left[\frac{1}{1+\nu} \quad 1 + \frac{\lambda\nu}{1+\nu} (z', R_z' z') \right] \quad (4.43)$$

Of main interest in this expression are its values for extreme values of the signal to noise ratio ν . For very small values of ν ,

$$e_m^f \approx 1 - \nu + \frac{\nu^2}{\eta} (z', R_z' z') \quad (4.44)$$

neglecting terms higher than second order in ν . For very large values,

$$\begin{aligned} e_m^f &\approx \frac{1}{\nu} \left[1 + \lambda (z', R_z' z') \right] \\ &\approx \frac{1}{\eta} (z', R_z' z') \end{aligned} \quad (4.45)$$

4.6 COMPARISON OF PERFORMANCE AT SMALL SNR

The performance of the optimum receiver is defined in Eqns. 4.33 and 4.34 in terms of the solution $k'(t)$ of the integral equation and the signal to noise ratio ν . Now the solution $k'(t)$ is shown by Eqn. 4.25 to depend on the ratio $\lambda = \nu/\eta$ so that λ might be quite large even when ν is small, depending on the value of η . We shall therefore assume that ν is small enough that λ is also small. In this case it is not difficult to show (see Appendix A) that Eqn. 4.30 reduces to

$$k' \approx (I - \lambda R_z') z' \quad (4.46)$$

so that

$$e_m \approx \frac{1}{1 + v[1 - \lambda(z', R'_2 z')]} \\ \approx 1 - v + \frac{v^2}{\eta} (z', R'_2 z') \quad (4.47)$$

to within second order terms in v .

Eqn. 4.47 is identical to Eqn. 4.45 so that the simple matched filter is as good as the optimum receiver for very small signal to noise ratios; in fact the two receivers are themselves identical in the limit as v tends to zero. This is hardly surprising since as the multiplicative noise is proportional to the signal energy. Thus as the signal energy is reduced to small values compared to the additive noise, the multiplicative noise is also reduced to a lower level than the additive noise and the matched filter is designed in an optimum fashion for combatting additive noise.

4.7 PERFORMANCE AT LARGE SIGNAL TO NOISE RATIO

With an exactly known channel Eqn. 4.38 shows that the estimation error can be decreased to arbitrarily small values by increasing the transmitter power and thus the signal to noise ratio. This is not the case with random

channels as is shown by Eqn. 4.35. As the transmitter power and the signal to additive noise ratio ν are increased without limit the effective signal to noise ratio ρ approaches a finite limit ρ^* , where

$$\rho^* = \eta(z', (R'_z)^{-1} z') \quad (4.48)$$

and

$$e_m \rightarrow e_m^* = \frac{1}{1 + \rho^*} \quad (4.49)$$

so that the estimation error approaches a non-zero limit.

The physical reason for this is that the power output of the random branch of the medium (the multiplicative noise) is proportional to the transmitter power. Thus as the signal to additive noise ratio increases the signal to total noise ratio approaches a limit which is proportional to the quality factor η .

It has been implicitly assumed in the above derivation that the inverse operator $(R'_z)^{-1}$ exists. The kernel $R'_z(t, r)$ will always be non-negative definite and, except in the most pathological of channels¹, will be positive definite so that the inverse will exist. Only channels for which the inverse does exist will be discussed.

¹ Basically the only channels for which this is not true are those in which the random branch is a pure frequency translating medium. A discussion of the procedures involved in such extreme cases is given by Kailath⁽¹⁵⁾.

The limiting error given by Eqn. 4.49 is an excellent way of comparing the usefulness of different channels and will be used for this purpose in the remainder of this thesis. Unfortunately the evaluation of this error involves solution of the integral equation which in most cases is extremely difficult to do analytically. Usually it is best accomplished by numerical methods using a digital computer although in some rather restricted cases approximate analytical methods, such as the use of a kernel of finite rank, are useful. A discussion of the various methods available will be found in Appendix A and in Appendix B an example using a kernel of finite rank is worked out. In Chapters 6 and 7 a class of channels is discussed for which some very good approximations for the limiting error can be obtained. Chapter 5 contains a discussion of some circumstances under which exact solutions can be obtained analytically.

4.8 RELATION TO THE WORK OF KAILATH

The work of Kailath as applied to the problem of hypothesis testing for an antipodal signalling scheme was reviewed in Section 1.7. In this section it is shown that the filter defined by Eqn. 4.3 for our parameter estimation pro-

blem corresponds exactly to the part of Kailath's receiver shown in Figure 1.6, when the signal shape used in the two systems is the same.

The decision statistic of the Kailath receiver is, according to Eqn. 1.32,

$$d = \int q(t)p(t)dt$$

where $p(t)$ is the received signal. This is achieved by correlating the received signal with the waveform $q(t)$, defined in Eqn. 1.34. In the parameter estimation problem, the estimate of θ is in the form

$$\hat{\theta} = \int k(t)w(t)dt$$

where $k(t)$ is the solution of Eqn. 4.3 and has been regarded as the inverse-time impulse response of a filter the output of which is sampled at $t = 0$. This is entirely equivalent to correlating $w(t)$ with the waveform $k(t)$ as can be seen by comparing the equations for d and $\hat{\theta}$. We shall show that, in fact, the waveforms $q(t)$ and $k(t)$ are identical.

In order to show that $k(t) = q(t)$ we must first show that $q(t)$ can be expressed in terms of the parameters of the communication problem of this chapter.

According to Eqn. 1.34 the waveform $q(t)$ is

$$q(t) = \bar{z}(t) - \int \ell(t,r)\bar{z}(r)dr \quad (4.50)$$

where the integral implies the filtering of the waveform $\bar{z}(t)$ by a time-variant filter having impulse response $\ell(t,r)$. (The superscripts of Eqn. 1.34 have been dropped for simplicity.) Now if we take the two possible transmitted signals to be $\pm s(t)$ and write the random part of the channel output $p(t)$ as

$$\tilde{p}(t) = \tilde{z}(t) + n(t) \quad (4.51)$$

the autocovariance function of $\tilde{p}(t)$ is

$$\begin{aligned} R_p(t,r) &= \iint R_a(t,r;\alpha,\sigma)s(t-\alpha)s(r-\sigma) d\alpha d\sigma \\ &\quad + N_o \delta(t-r) \\ &= R_z(t,r) + N_o \delta(t-r) \end{aligned} \quad (4.52)$$

This is also the autocovariance of the random part of the channel output when the signalling scheme of this chapter is used, as can be seen from a comparison with Eqn. 4.2. This occurs because the variance of θ is unity and this is also the absolute value of the two possible multipliers of $s(t)$ in the Kailath scheme. According to Kailath the filter $\ell(t,r)$ is

the Wiener filter⁽¹⁶⁾ which, given the signal $\tilde{p}(t)$, forms the minimum mean square estimate of $\tilde{z}(t)$. It must therefore satisfy the following equation

$$R_z(u, r) = \int R_p(u, r) \ell(r, t) dr \quad (4.53)$$

Now $R_p(t, r)$ has been expressed in terms of the parameters of the communication problem of this thesis in Eqn. 4.52 so that $q(t)$ is now related to this problem.

It remains to show that $q(t) = k(t)$ and this will be done by showing that $q(t)$ gives the integral equation for $k(t)$. After rearranging Eqn. 4.3 this is

$$\begin{aligned} \bar{z}(t) &= \int [R_z(t, r) + N_0 \delta(t-r)] k(r) dr \\ &= \int R_p(t, r) k(r) dr \end{aligned} \quad (4.54)$$

Substituting the value of $q(r)$ from Eqn. 4.50 in the right hand side of Eqn. 4.54 gives

$$\begin{aligned} \int R_p(t, r) q(r) dr &= \int R_p(t, r) \bar{z}(r) dr \\ &\quad - \iint R_p(t, r) \ell(r, u) \bar{z}(u) du dr \end{aligned}$$

Interchanging the order of integration and using the defining equation (Eqn. 4.53) for $\ell(r, t)$ gives

$$\int R_p(t, r) q(r) dr = \int R_p(t, r) \bar{z}(r) dr - \int R_z(u, r) \bar{z}(u) du$$

Since $R_z(t,r)$ is symmetric for real time functions, Eqn. 4.52 shows that this is

$$\int R_p(t,r)q(r)dr = \bar{z}(t)$$

which defines $q(t)$ in precisely the same way as the optimum waveform $k(t)$ in expressed in Eqn. 4.54. Thus we have shown that $q(t) = k(t)$.

The above argument shows that the parameter estimation receiver for the special case of an isolated pulse is identical, except for a scaling factor, to the hypothesis testing receiver of Kailath in the special case of antipodal signals. This equivalence is a very satisfying confirmation of the results of this thesis.

Chapter 5

ANALYTIC SOLUTIONS OF THE INTEGRAL EQUATION

In Chapter 3 the integral equation defining the various filters in the receiver was formulated and in Chapter 4 it was specialised to the isolated pulse problem for the purpose of focussing attention on the effects of the randomness of the channel. In this chapter we investigate some situations in which it is possible to obtain exact solutions of the integral equation by analytic means. We shall deal with the normalised form of the equation since this is most convenient for the purpose later of evaluating the estimation error.

In the work of this chapter free use will be made of the inter-relationships of the various second order measures of the random output signal defined in Section 2.6 and illustrated in Figure 2.4.

5.1 THE TIME DOMAIN EQUATION

Using the alternative form of the autocovariance function, defined in Eqn. 2.27, the normalised time domain

equation, Eqn. 3.27, can be written

$$k'(t) = z'(t) - \lambda \int P'_z(\tau, t) k'(t+\tau) d\tau \quad (5.1)$$

Now suppose that $P'_z(\tau, t)$ is a very narrow function of τ compared with $k'(\tau)$. Because of the transform relationship between $P'_z(\tau, t)$ and $S'_z(f, t)$ (see Figure 2.4) this implies that $S'_z(f, t)$ can be written approximately

$$S'_z(f, t) \approx S'_z(0, t) \quad (5.2)$$

over the band of frequencies occupied by $K'(f)$. In terms of $P'_z(\tau, t)$, Eqn. 5.2 amounts to being able to write

$$P'_z(\tau, t) \approx S'_z(0, t) \delta(\tau) \quad (5.3)$$

where the delta function is a statement of the narrowness of $P'_z(\tau, t)$ compared with $k'(\tau)$. If Eqn. 5.3 is true the integral equation has the following very simple solution.

$$k'(t) = \frac{z'(t)}{1 + \lambda S'_z(0, t)} \quad (5.4)$$

The implication of Eqn. 5.2 is that the condition for the solution amounts to requiring a large bandwidth of $S'_z(f, t)$ compared to the bandwidth of $K'(f)$. A problem is that, until the equation is solved, we do not know the band-

width of $K'(f)$ so that we cannot see in advance when the simple solution of Eqn. 5.4 will apply. Fortunately the situation is retrievable for we shall show that it is sufficient to compare the bandwidths of $Z'(f)$ and $S'_Z(f,t)$ which are both known. To do this we assume first that Eqn. 5.3, and therefore Eqn. 5.4 are valid. In this case we have

$$z'(t) = k'(t) + \lambda S'_Z(0,t)k'(t)$$

which yields, on transformation

$$Z'(f) = K'(f) + \lambda \int P_Z(0,u) K'(f-u) du \quad (5.5)$$

Since the second term on the right hand side is a convolution, it has a wider bandwidth than either of its two component functions and therefore the whole right hand side has a bandwidth greater than that of $K'(f)$. But the bandwidth of the two sides must be equal and so the bandwidth of $K'(f)$ is less than that of $Z'(f)$. This is what we set out to prove and therefore the condition for the validity of the simple solution, Eqn. 5.4 is that

$$m_f(\tilde{Z}) \gg m_f(\bar{Z}) \quad (5.7)$$

(See Eqn. 2.35 for the definition of these dispersions.)

Eqn. 5.7 states that the bandwidth of the random part of $z(t)$ must greatly exceed that of the deterministic part. In the two following sections we investigate what this means in terms of the parameters of the channel.

5.1.1 Conditions for a WSSUS channel

For a wide sense stationary, uncorrelated scattering channel the results of Eqns. 2.64 and 2.71 can be substituted in Eqn. 5.7 to give

$$m_{\lambda}(\tilde{a}) + m_f(s) \gg \inf(m_f(\bar{a}), m_f(s)) \quad (5.8)$$

as the condition for a simple solution. Because of the infimum on the right hand side two cases must be considered:

Case 1) The bandwidth of the deterministic output is limited by the bandwidth of the deterministic branch, that is

$$m_f(\bar{a}) < m_f(s)$$

Eqn. 5.8 then becomes

$$m_{\lambda}(\tilde{a}) + m_f(s) \gg m_f(\bar{a}) \quad (5.9)$$

which states that the sum of the Doppler spread and the signal bandwidth must be greater than the bandwidth of the deterministic branch.

Case 2) The deterministic output is not limited by the bandwidth of the branch, that is

$$m_f(\bar{a}) > m_f(s)$$

in which case Eqn. 5.8 is

$$m_\lambda(\tilde{a}) + m_f(s) \gg m_f(s)$$

or simply

$$m(\tilde{a}) \gg m_f(s) \quad (5.10)$$

Equation 5.10 states that the Doppler spread must be much larger than the signal bandwidth, a condition which implies very wideband random multiplicative processes.

5.1.2 Conditions for a WSS channel

If the random medium is not uncorrelated scattering, the bandwidth of the random branch of the medium complicates the expressions obtained for a WSSUS channel. The equivalent of Eqn. 5.8 is obtained by substituting the results of Eqn. 2.55 in Eqn. 5.7 to give

$$m_\lambda(\tilde{a}) + \frac{1}{2} \inf(m_f(\tilde{a}), m_f(s)) \gg \inf(m_f(\bar{a}), m_f(s)) \quad (5.11)$$

If the bandwidth of the random branch is greater than the signal bandwidth, Eqn. 5.11 reduces to Eqn. 5.8 as would be expected since, in this case, the bandwidth of the branch has no effect on the bandwidth of its output. The remaining conditions are then exactly as for the WSSUS channel.

If the bandwidth of the random medium is less than the signal bandwidth then Eqn. 5.11 becomes

$$m_{\lambda}(\tilde{\alpha}) + m_f(\tilde{\alpha}) \gg \inf(m_f(\bar{\alpha}), m_f(s)) \quad (5.12)$$

where the factor $\frac{1}{2}$ has been dropped since a large inequality is required in any case. The two cases relating to the bandwidths of the signal and the deterministic medium, as in the WSSUS case, lead to the following two conditions:

Case 1) The sum of the Doppler spread and the bandwidth of the random branch must exceed the bandwidth of the deterministic branch, that is

$$m_{\lambda}(\tilde{\alpha}) + m_f(\tilde{\alpha}) \gg m_f(\bar{\alpha}) \quad (5.13)$$

Case 2) The Doppler spread must exceed the signal bandwidth

$$m_{\lambda}(\tilde{\alpha}) \gg m_f(s)$$

which is identical to Eqn. 5.10 for the WSSUS case.

5.1.3 Summary

Essentially the conditions above state that the simple solution Eqn. 5.4 of the integral equation may result either from using a very wideband signal with a narrow band deterministic branch, or from having a very rapidly varying random branch. When the random branch is correlated the situation is changed in a rather obvious way depending on the bandwidth of the random branch.

5.2 THE FREQUENCY DOMAIN EQUATION

Writing the frequency domain equation, Eqn. 4.26, in terms of the covariance function $P_Z(f, \beta)$ gives

$$K'(f) = Z'(f) - \lambda \int P_Z'(f, \beta) K'(f - \beta) d\beta \quad (5.14)$$

This equation is of precisely the same form as Eqn. 5.1 and the condition for its reduction to an algebraic equation is similar. If $P_Z'(f, \beta)$ is a very narrow function of β compared with $K'(\beta)$ so that its transform can be written approximately

$$S_Z'(f, t) \approx S_Z'(f, 0) \quad (5.15)$$

and correspondingly

$$p'_z(f, \beta) \approx s'_z(f, 0) \delta(\beta) \quad (5.16)$$

then equation 5.14 has the solution

$$K'(f) = \frac{Z'(f)}{1 + \lambda S'_z(f, 0)} \quad (5.17)$$

The above requirement can be expressed (see Eqn. 2.34 for definitions) as

$$m_t(\tilde{z}) \gg m_t(k) \quad (5.18)$$

and again the problem is that the time dispersion of $k(t)$ is not known until the equation has been solved. By a precisely parallel argument to that used in the time domain case it can be shown that the time dispersion of $k(t)$ is less than that of $\tilde{z}(t)$ and therefore the requirement of Eqn. 5.18 can be replaced by

$$m_t(\tilde{z}) \gg m_t(\bar{z}) \quad (5.19)$$

and in fact we err conservatively by making the replacement.

For both WSS and WSSUS channels this requirement is equivalent to

$$m_t(\hat{a}) + m_t(s) \gg m_t(\bar{a}) + m_t(s) \quad (5.20)$$

which is obtained by substituting from Eqns. 2.54 and 2.71 in Eqn. 5.19. Eqn. 5.20 can be expressed as the following two conditions

$$\begin{aligned} m_t(\tilde{a}) &>> m_t(\bar{a}) \\ m_t(\tilde{a}) &> m_t(s) \end{aligned} \quad (5.21)$$

which is to say that the random branch of the medium must have a much larger delay dispersion than the deterministic branch and the delay dispersion of the random branch must be greater than the time dispersion of the transmitted signal.

5.3 INTERPRETATION OF THE SOLUTIONS

In both of the cases in which a simple algebraic solution can be found for the optimum filter, the filter is defined in terms of the time variant spectral density function of the random multiplicative noise caused by the action of the random medium. In the two denominators $(1 + \lambda S'_z(f,0))$ and $(1 + \lambda S'_z(0,t))$ the quantity 1 corresponds to the normalized value of the time dependent spectral density of the additive noise. (It should be noted that the time variant spectral density of the stationary, white, additive noise is a constant normalised to one in both time and frequency variables since $S_n(f,t) = N_o$.) Thus each of the

solutions Eqn. 5.4 and Eqn. 5.17 is the result of weighting the deterministic received signal by an amount corresponding to the distribution of the total noise, additive plus multiplicative, at each point in time or frequency. When the time dispersion of the output of the random branch is much larger than that of the deterministic branch of the medium we have frequency weighting, as in Eqn. 5.17. When the bandwidth (frequency dispersion) of the output of the random branch is much greater than that of the output of the deterministic branch we have time weighting, as in Eqn. 5.4.

The weighting interpretation is a very satisfying one from the intuitive point of view. Basically it states that, when the multiplicative noise is uncorrelated with itself in either frequency (Eqn. 5.16) or time (Eqn. 5.3) the solution to the integral equation is found by weighting the deterministic received signal. When the noise is uncorrelated in frequency the weighting is carried out in the frequency domain and when it is uncorrelated in time the weighting is carried out in the time domain.

5.4 THE MEAN SQUARE ERROR

From Eqn. 4.34 the mean square estimation error is determined by the factor ρ expressed in terms of the solution of the normalised equation in Eqn. 4.33.

When an algebraic solution has been obtained in the time domain as in Eqn. 5.4 ρ is given by

$$\rho = \int \frac{z'^2(t)}{1 + \lambda S'_z(0,t)} dt \quad (5.22)$$

When the solution has been obtained in the frequency domain, use of Parseval's theorem gives the following:

$$\rho = \int \frac{Z'^*(f)Z'(f)}{1 + \lambda S'_z(f,0)} df \quad (5.23)$$

This is as far as it is possible to go in evaluating the error without specific forms for the signal and the channel covariance function. This will be done in some examples in Chapter 6.

5.5 INTERSYMBOL INTERFERENCE

So far in this chapter we have been exclusively concerned with the isolated pulse problem. Analytic solutions can

be obtained in the interfering symbol case under basically the same conditions as for the isolated pulse case.

5.5.1 The time domain equation

The integral equation for each of the filters of the receiver for interfering symbols (Eqn. 3.27) can be normalised to give a form similar to Eqn. 5.1:

$$k'_n(t) = z'_n(t) - \lambda \int P'_y(\tau, t) k'(t+\tau) d\tau \quad (5.24)$$

where

$$P'_y(\tau, t) = \sum P'_{z_i}(\tau, t)$$

If the $P'_{z_i}(\tau, t)$ satisfy the requirement of Eqn. 5.3 then Eqn. 5.24 has the solution

$$k'_n(t) = \frac{z'_n(t)}{1 + \lambda \sum S'_{z_i}(0, t)} \quad (5.25)$$

As in the isolated pulse case the deterministic signal is weighted at each point in time by the total random noise at all frequencies existing at that time.

5.5.2 The frequency domain equation

Again the basic frequency domain equation (Eqn. 3.37) can be written in a form similar to that of Eqn. 5.14 to give

$$K'_n(f) = Z'_n(f) - \lambda \int P'_Y(f, \beta) K'_n(f - \beta) d\beta \quad (5.26)$$

where

$$P'_Y(f, \beta) = \sum P'_{Z_i}(f, \beta)$$

If the $P'_{Z_i}(f, \beta)$ satisfy Eqn. 5.16 then the solution of Eqn. 5.26 is

$$K'_n(f) = \frac{Z'_n(f)}{1 + \lambda \sum S'_{Z_i}(f, 0)} \quad (5.27)$$

which is a frequency weighting expression.

There is an interesting additional effect of inter-symbol interference when the channel is wide sense stationary and sequential signalling is used. In this case the basic frequency domain equation is Eqn. 3.73:

$$K'_n(f) = Z'_n(f) - \lambda \sum K'_n(f - k/T) P'_{Z_0}(f, k/T) \quad (5.28)$$

Now if the coherent bandwidth (bandwidth as a function of β) of $P_{Z_0}(f, \beta)$ is less than $1/T$, or more precisely if

$$P_{Z_0}(f, \beta) = 0, \quad |\beta| > 1/T \quad (5.29)$$

Eqn. 5.28 has the simple solution

$$K'_n(f) = \frac{Z'_n(f)}{1 + \lambda P'_{Z_0}(f, 0)} \quad (5.30)$$

and in particular, for the pulse with subscript zero,

$$K'_0(f) = \frac{Z'_0(f)}{1 + \lambda P'_{Z_0}(f, 0)} \quad (5.31)$$

This is the same solution as would be obtained if there were no intersymbol interference and this comment also applies to $K'_n(f)$ since, from Eqn. 3.56

$$K'_n(f) = K'_0(f) e^{-j2\pi n f T} \quad (5.32)$$

The condition of Eqn. 5.29 is somewhat weaker than the requirement of Eqn. 5.18 since it implies only that the delay spread $m_t(\vec{a})$ of the random medium be greater than T regardless of the delay spread of the deterministic signal. The implication seems to be that multiplicative noise from different pulses tends to cancel out when the delay spread is large.

Although the filter defined by Eqn. 5.31 is the same as the one for no intersymbol interference this does not imply that the estimation error will be the same. The intersymbol interference will cause an increase in error. The error in this case is given by Eqn. 3.42 rather than Eqn. 4.31.

Chapter 6

PERFORMANCE OF A NON-SELECTIVE CHANNEL

This chapter investigates the performance of a very simple type of channel for a range of different Doppler profiles, and different values of the signal to noise ratio ν and the quality factor η , introduced in Chapter 4. The channel is assumed to be non-selective, that is it has zero memory and simply multiplies the signal by some random function of time. The mean value of this random time function is represented by a non-selective, deterministic branch of the channel and the zero mean random part by a zero memory random branch. Adoption of such a channel model allows us to focus attention on the effects of different correlation functions and variances of the random process. In the next chapter it will be shown that the results of analysis of a channel of this type can be used to determine the performance of a particular class of selective channels with very little increase in difficulty.

Since in this chapter the channel covariance function will not be restricted as it was in Chapter 5, it will be

necessary to solve the basic integral equation by numerical means and this of course involves choosing a particular covariance function and a particular signal. In spite of this it is felt that the results obtained give some considerable insight into the type of performance to be expected from randomly time variant channels in general.

Within the context described above it will be the purpose of this chapter to find means whereby the performance of channels may be estimated without the necessity of solving the integral equation in each case. The results of the various estimation methods will be compared to accurately computed results obtained by numerical solution of the integral equation.

The first few sections of the chapter are concerned with approximations for the limiting performance of each channel as the signal to additive noise ratio is increased. Following this is a method for approximating the performance at lower signal to noise ratios.

6.1 CHOICE OF SIGNAL

In this chapter it will always be assumed that the signal $s(t)$ is non-zero only in the interval $(-T/2, T/2)$. It can therefore be expressed as

$$s(t) = \sqrt{E_t} s'(t) p_T(t) \quad (6.1)$$

where $p_T(t)$ is a unit energy rectangular pulse

$$\begin{aligned} p_T(t) &= 1/\sqrt{T}, \quad -T/2 \leq t \leq T/2 \\ &= 0 \text{ elsewhere.} \end{aligned} \quad (6.2)$$

This is the only requirement for expressing the performance of the class of channels used in the next chapter in terms of the non-selective channel of this chapter. However most of this chapter is concerned with numerical results for the particular signal

$$s(t) = \sqrt{E_t} p_T(t) \quad (6.3)$$

which is a simple rectangular pulse.

6.2 CHOICE OF COVARIANCE FUNCTION

The channel is assumed to be wide sense stationary and, since it has zero memory, scattering of energy in the ran-

dom branch of the medium occurs at only one value of delay so that the formulae applying to the uncorrelated scattering medium can be used. The autocorrelation function of the single random process will be denoted by $\phi(\tau)$ where τ is the time shift variable. Using the normalisation of Eqn. 4.15 and the form of the covariance function for the uncorrelated scattering channel of Eqn. 2.56 gives

$$P_a(\tau; \alpha) = \gamma^2 \phi(\tau) \delta(\alpha) \quad (6.4)$$

where the delay of the channel is taken to be zero and

$$\phi(0) = 1$$

A particularly simple function having all the necessary properties of an autocorrelation function is the Gaussian function and this will be used in the examples of this chapter.

Thus

$$\phi(\tau) = e^{-\frac{1}{2} \left(\frac{\tau}{\mu} \right)^2} \quad (6.5)$$

where μ is the r.m.s. width of the autocorrelation function and will be called the correlation width:

$$\mu = \frac{\int \tau^2 \phi(\tau) d\tau}{\int \phi(\tau) d\tau} \quad (6.6)$$

In order to investigate the effect of the relative bandwidths of the signal and the random process of the channel it is desirable to introduce a dimensionless factor connecting the specification of the signal as in Eqn. 6.3 with the autocorrelation function $\phi(\tau)$. This can conveniently be done in the time domain by defining the quantity

$$\xi = \frac{T}{\mu} = \frac{\text{pulse width}}{\text{r.m.s. correlation width}} \quad (6.7)$$

which gives the following definition for $\phi(\tau)$.

$$\phi(\tau) = e^{-\frac{1}{2}(\xi\tau)^2} \quad (6.8)$$

The Fourier transform of the autocorrelation function is the spectral density of the random process and is

$$\Phi(f) = \sqrt{2\pi} \frac{T}{\xi} e^{-\frac{1}{2}\left(\frac{2\pi f T}{\xi}\right)^2} \quad (6.9)$$

which also has the Gaussian shape and is non-negative, as is required of a spectral density function.

6.3 THE INTEGRAL EQUATION

With the channel covariance function as in Eqn. 6.4 the normalised autocovariance of the output of the random

branch is given by Eqn. 4.2.

$$R'_z(t,r) = \phi(t-r)s'(t)s'(r) \quad (6.10)$$

Because of the time limitation of the signal it is clear that $R'_z(t,r)$ is non-zero only in the square $-T/2 \leq t, r \leq T/2$ of the (t,r) plane. The integral equation for the optimum filter characteristic is then

$$k'(t) = s'(t) - \lambda s'(t) \int \phi(t-r)s'(r)k'(r)dr \quad (6.11)^1$$

where the range of integration is $(-T/2, T/2)$. For the rectangular pulse this reduces to

$$k'(t) = 1/\sqrt{T} - \frac{\lambda}{T} \int \phi(t-r)k'(r)dr \quad (6.12)$$

Eqn. 6.12 shows that the optimum filter response is a function of T , λ and ξ .

6.4 THE MEAN SQUARE ERROR

To determine the mean square estimation error of the system we first find the effective signal to noise ratio ρ by

¹ With a non-selective channel the normalised transmitted pulse and the normalised received pulse are identical since they have the same shape and are both unit energy.

forming the inner product of $k'(t)$ with $z'(t) = s'(t)$ and multiplying by v according to Eqn. 4.33. The integration involved in obtaining the inner product removes the dependence on T so that the error is a function only of v , η and ξ for a given choice of signal.¹ This would be expected since the correlation width has been expressed, in Eqn. 6.7, as a function of T and ξ .

For the rectangular pulse shape of Eqn. 6.3 the estimation error has been computed by numerical solution of the integral equation for a range of values of the parameters listed above. The results will be presented in the remainder of this chapter in the course of comparing them with the results of a number of approximation methods.

6.5 CHANNELS WITH NARROWBAND RANDOM PROCESSES

In Appendix B it is shown that for values of ξ less than 3 the Gaussian autocorrelation function can be approximated by the raised cosine function

$$\phi_c(\tau) = \frac{1}{2}(1 + \cos \pi \xi \frac{\tau}{T}) \quad (6.13)$$

¹ This may not be immediately obvious. Basically it is the result of defining $\phi(\tau)$ in terms of ξ , in Eqn. 6.3, which means that any quadratic form with $\phi(t-r)$ as kernel will be proportional to T as is the spectral density in Eqn. 6.9. The inner product involves such a quadratic form. Examples of this occur in Section 6.6 and Appendix B where the error is obtained analytically.

The value of ζ can be chosen in two different ways which correspond to different criteria in approximating the Gaussian function. For equal 3dB widths of the two functions the value is

$$\zeta = \xi/2.35 \quad (6.14)$$

and for equal r.m.s. correlation widths it is

$$\zeta = \xi/3 \quad (6.15)$$

The restriction on ξ corresponds to the restriction

$$\zeta \leq 1 \quad (6.16)$$

Computed comparisons of the two methods of approximation, made on the basis of the accuracy with which the two methods approximate the estimation error, show that Eqn. 6.14 is best for small values of ξ while Eqn. 6.16 is best for values approaching the upper limit. The choice is not critical since with either approximation the error in approximating the estimation error is limited to about 5%.

Having chosen the appropriate value of ζ the effective signal to noise ratio ρ is

$$\rho = \frac{\nu(1 + 2\lambda g_2)}{1 + \lambda g_1 + \lambda^2 g_2} \quad (6.17)$$

where g_1 and g_2 are given in general by Eqn. B.18 and Eqn. B.19. For the case of rectangular pulses these factors are from Eqns. B.30 and B.31

$$g_1 = \frac{1}{4} \left(3 + \frac{1}{\pi\zeta} \sin\pi\zeta \right) \quad (6.18)$$

$$g_2 = \frac{1}{8} \left(1 + \frac{1}{\pi\zeta} \sin\pi\zeta - \frac{8}{(\pi\zeta)^2} \sin^2 \frac{\pi\zeta}{2} \right) \quad (6.19)$$

The value of the estimation error is then given by

$$e_m = \frac{1}{1 + \rho}$$

The method described above gives approximations to the system performance for any value of the parameters η , ν and ξ within the restriction on ξ and it is clear that it is a much simpler procedure than solving the integral equation by numerical methods. The quality of the results is indicated by Table 6.1. The first two columns of this table show the important channel parameters and the third column lists the value of the estimation error at signal to noise ratio $\nu = 512$. This error was obtained by numerical solution of the integral equation. In column four the value given by the cosine approximation is shown and it is clear that the approximation is very good.

TABLE 6.1 True and Approximate values of the Estimation Error for Rectangular Signals

ξ	η	True Error	Error Approximations			
			Cosine Formula	Eqn. 6.23	Eqn. 6.33	Eqn. 6.32
.0312	2	.334	.334	.333	- -	- -
.125	5.6	.155	.170	.153	- -	- -
.5	.72	.564	.566	.58	- -	- -
2	2	.232	.240	- -	- -	- -
3	1	.325	.337	- -	.4	.455
5	1	.259	- -	- -	.286	.333
8	2	.113	- -	- -	.11	.135
10	1	.172	- -	- -	.167	.2
32	2	.039	- -	- -	.030	.038
32	.72	.097	- -	- -	.08	.098

The cosine approximation gives good results at all signal to noise ratios and in general is better at low ratios. The large value of 512 was chosen for presentation because our main interest is in the limiting performance as the signal to noise ratio is increased. The value of 512 is high enough for the performance to have reached its limit in all channels ex-

cept those which can almost be described as exactly known. The value chosen also facilitates comparison with some other approximation methods which are restricted to high signal to noise ratio. It has already been shown (Section 4.6) that a simple matched filter gives almost the same result as the optimum receiver for low signal to noise ratios.

For values of ξ between 0.1 and 0.5, Eqn. 6.19 involves a small difference of two large numbers and is therefore difficult to use. However in this range the channel can be considered to be almost time invariant and a much simpler approximation is available as is shown in the next section.

6.5.1 Approximately time-invariant channels

For channels having values of ξ which are either very small or, in the time invariant case, zero Eqns. 6.18 and 6.19 together with Eqn. 6.14, show that

$$g_2 \approx 0 \quad \text{and} \quad g_1 \approx 1 \quad (6.20)$$

In this case

$$\rho \approx \frac{\nu}{1 + \lambda} \quad (6.21)$$

and for large values of the signal to noise ratio the limiting value ρ^* of ρ is

$$\rho^* = \eta \quad (6.22)$$

giving the corresponding limiting value of the error

$$e_m^* \approx \frac{1}{1 + \eta} \quad (6.23)$$

Consequently for channels of this type it is not necessary to use the cosine formula if only the limiting value of e_m is required. In Section 6.7 it will be shown that a good approximation can also be obtained at any signal to noise ratio without using the cosine formula.

In order to use the above approximation it is necessary to know what constitutes an approximately time invariant channel. The answer to this question can be obtained from Figure 6.1a which is a graph of the value of ρ against ξ for two different values of η . This shows that the approximation of Eqn. 6.22 is valid for ξ less than about 0.5. In addition the fifth column of Table 6.1 shows good agreement between the value of the estimation error obtained using this method and the accurate value. It is convenient that this is the range of ξ for which the cosine formula is difficult to use for the reason stated at the end of Section 6.5.

(a)

(b)

Figure 6.1 Effective Signal to Noise Ratio Versus ξ for Rectangular Pulses

6.6 CHANNELS WITH WIDEBAND RANDOM PROCESSES

In Section 5.1 it was shown that when the random multiplicative process is very wideband, that is when the Doppler spread is very large, the integral equation has the solution

$$k'(t) = \frac{z'(t)}{1 + \lambda S'_z(0, t)}$$

and in Section 5.4 this was used to obtain the following formula for ρ .

$$\rho = \int \frac{z'^2(t)}{1 + \lambda S'_z(0, t)} dt \quad (6.25)$$

A wideband process corresponds to large values of ξ and Eqn. 6.25 will now be used to arrive at the estimation error for rectangular pulses and a wideband process. The results are not restricted to the case of Gaussian correlation functions since, if the spectral density of the process is flat over the band of interest, the shape outside this band does not affect the results.

The basic assumption of Section 5.1 was that for all frequencies over the band of interest the following statement

is true

$$S'_z(f, t) \approx S'_z(0, t) \quad (6.26)$$

From Eqn. 2.63

$$S'_z(0, t) = \iint S'_v(-\lambda; t-\sigma) S'_s(\lambda, \sigma) d\sigma d\lambda \quad (6.27)$$

and in the present case

$$S'_v(f; \alpha) = \Phi(f) \delta(\alpha) \quad (6.28)$$

Substituting this in Eqn. 6.27 and using the assumption of a very wideband process gives

$$S'_z(0, t) = \Phi(0) \int S'_s(\lambda, t) d\lambda$$

and from Eqn. 2.31 this is

$$\begin{aligned} S'_z(0, t) &= \Phi(0) P'_s(0, t) \\ &= \Phi(0) p_T^2(t) \\ &= \Phi(0)/T, \quad -T/2 \leq t \leq T/2 \end{aligned} \quad (6.29)$$

This result shows that for rectangular pulses the quantity $S'_z(0, t)$ is not a function of t over the non-zero range of $z'(t) = s'(t)$. The integration in Eqn. 6.25 therefore applies

only to the numerator, the denominator is a constant multiplier.

From Eqn. 6.9

$$\Phi(0)/T = \sqrt{2\pi}/\xi \quad (6.30)$$

Substituting this value in Eqn. 6.29 and then in Eqn. 6.25

gives the final result

$$\rho = \frac{v}{1 + \sqrt{2\pi} \lambda/\xi} \quad (6.31)$$

This applies for any signal to noise ratio, for large signal to noise ratios the limiting value of ρ is

$$\rho \approx \frac{\eta\xi}{2.5} \quad (6.32)$$

The seventh column of Table 6.1 shows the values of e_m obtained using the approximation of Eqn. 6.32. For values of ρ less than 10 the approximation is appreciably higher than the true value but for the values 10 and 32 the approximation is good. For values greater than about 32 the numerical method used to solve the integral equation did not work well precisely because the autocorrelation function approximates a delta function in this range. There are sufficient results, however, to indicate that $\xi > 10$ is a reasonable definition of a wideband process.

The accurate value of ρ is plotted against ξ in Figure 6.1b which shows that, for values of ξ between 2 and 10, the graph can be approximated by a straight line with slope $\eta/2$. That is

$$\rho \approx \frac{\eta\xi}{2} \quad (6.33)$$

The results of this approximation are shown in column six of Figure 6.1 and the correspondence with the true value is good.

6.7 PERFORMANCE VERSUS SIGNAL TO NOISE RATIO

In the previous sections of this chapter our attention was focussed on the limiting performance of the communication system as the signal to noise ratio increased. In this section we consider the performance at lower signal to noise ratios.

To emphasise the dependence of the error on signal to noise ratio we write Eqn. 4.34 in the following way.

$$e_m = 1/(1 + \rho(v)) \quad (6.34)$$

Clearly it is only necessary to find the dependence of ρ on the signal to noise ratio and in particular to know the signal

to noise ratio necessary for $\rho(v)$ to approach its limiting value ρ^* .

From Eqn. 6.21, for an approximately time invariant channel ($\xi < 0.5$), we have

$$\rho(v) = \frac{v}{1 + v/\eta}$$

and

$$\rho^* = \eta$$

so that we can write

$$\rho(v) = \frac{v}{1 + v/\rho^*} \quad (6.35)$$

Eqns. 6.31 and 6.32 show that this same expression is valid for very wideband random processes, ($\xi > 10$). Thus for very wideband or very narrowband processes Eqn. 6.35 shows that the limiting value of $\rho(v)$, and thus of the error, is approached when $v \gg \rho^*$, say when

$$v = v^* \approx 10\rho^* \quad (6.36)$$

For intermediate values of ξ the situation is not so clear. When $\xi \leq 3$ the cosine formula of Eqn. 6.17 can be used though it is not quite so convenient as Eqn. 6.35. When

$\xi > 3$ we have an approximation (Eqn. 6.33) for the limiting value ρ^* but no approximation for lower signal to noise ratios. By computing accurate results for the value of ρ as a function of signal to noise ratio it has been found that Eqn. 6.35 is a good approximation to the behaviour at all values of ξ . An example of this is shown in Figure 6.2 and further examples will be given in Chapter 7. The curve plotted in Figure 6.2 is the accurately determined curve. For signal to noise ratios greater than about two the curve given by Eqn. 6.35 is indistinguishable, on the scale of the figure, from the true curve.

6.8 CONCLUSIONS

Although this chapter has been concerned mainly with the specific example in which the autocovariance function of the multiplicative random process is Gaussian and the signal pulse is rectangular, some of the results have strong implications with respect to other systems.

The results for wideband random processes are clearly not restricted to the Gaussian autocovariance function as has already been indicated. In judging whether a process is to be

$$\xi = 3, \quad \eta = 2, \quad F = 1$$

$$e_m^* = 0.2$$

$$\rho^* = 40$$

Figure 6.2 Estimation Error Versus Signal to Noise Ratio for Rectangular Pulses

described as wideband or not the ratio ξ , defined in Eqn. 6.7 as the ratio of the pulse width to the r.m.s. correlation width of the Gaussian function has been used as a criterion. Hence it seems most likely that the general statement

$$\frac{\text{pulse width}}{\text{r.m.s. correlation width}} > 10 \quad (6.37)$$

may be taken as the definition of a wideband random process for any autocorrelation function reasonably similar to the Gaussian function, although more computational work is necessary to confirm this.

For similar reasons one might take the statement

$$\frac{\text{pulse width}}{\text{r.m.s. correlation width}} < 0.5 \quad (6.38)$$

to be a definition of an approximately time invariant channel.

For situations between the two extremes indicated above the results of this chapter will certainly be more dependent on the particular signal and autocovariance functions used, but they nevertheless afford an appreciation of the type of behaviour to be expected of randomly time variant channels. In particular it has been shown that good approximations to the performance are available for any signal to noise ratio and for any value of the ratio ξ .

Chapter 7

PERFORMANCE OF DISCRETE MULTIPATH CHANNELS

In this chapter the signal is again assumed to be time limited to the interval $(-T/2, T/2)$. The medium consists of a number of non-selective random paths of the type described in Chapter 6. It is assumed that, in the class of channels investigated, the responses of the various paths to the signal $s(t)$ do not overlap in time so that the response of the medium can be expressed

$$z(t) = \sum b_i(t)s(t-\alpha_i)$$

where

$$\alpha_i - \alpha_j \geq T \quad (7.1)$$

for all $i \neq j$. Each of the time functions $b_i(t)$ has a deterministic and a random part, so that

$$z(t) = \bar{z}(t) + \tilde{z}(t) \quad (7.2)$$

where

$$\bar{z}(t) = \sum \bar{b}_i(t)s(t-\alpha_i)$$

and

$$\tilde{z}(t) = \sum \tilde{b}_i(t)s(t-\alpha_i)$$

Hence both the random and the deterministic branches of the medium consist of paths which give non-overlapping responses. It is of no consequence what the precise values of the delays α_i are, provided they satisfy Eqn. 7.1. For instance, the deterministic response to the signal $s(t) = p_T(t)$ might have either of the forms shown in Figure 7.1 and still be regarded as the same channel from the point of view of the theory to be presented. With the inclusion of the additive noise the total output signal is

$$w(t) = z(t) + n(t) \quad (7.3)$$

All the random transfer functions $b_i(t)$ of the paths of the random branch are taken to have the same autocorrelation function $\phi(\tau)$ but the variance of each path may be different. The autocovariance function of the i th path can then be written $\gamma^2 \sigma_i^2 \phi(\tau)$ and the normalisation of Eqn. 4.16 dictates that

$$\sum \sigma_i^2 = 1 \quad (7.4)$$

Figure 7.1 Two Equivalent Forms of a Multipath Signal

so that the σ_i^2 indicate the relative variances of each path, while γ^2 represents the total variance of the medium. At this stage there is no restriction on the form of the autocorrelation function and the random processes of the different taps may be correlated with each other.

The proportion of the total received signal energy received over the i th path of the deterministic branch is denoted f_i^2 so that the normalised deterministic received signal may be expressed as

$$\bar{z}(t) = \sum f_i s'(t - \alpha_i) \quad (7.5)$$

where

$$\sum f_i^2 = 1 \quad (7.6)$$

In Section 7.2 we shall relate the performance of the channel described to that of an equivalent single path channel and in Section 7.3 we shall discuss the effect of correlation between the random processes in each path. In the following section a mathematical structure suited to the later discussions is set up. The analysis will be presented in detail to an arbitrary number is identical in concept but requires cumbersome algebra. To simplify the algebra further and to emphasise certain graphical concepts it will also be

assumed that the transmitted signal $s(t)$ is rectangular, so that

$$s'(t) = p_T(t) \quad (7.7)$$

7.1 ANALYSIS OF A TWO PATH CHANNEL

In analysing a two path channel we shall assume for convenience that the delays of the two paths are

$$\alpha_1 = 0 \quad \text{and} \quad \alpha_2 = T \quad (7.8)$$

so that the normalised deterministic received signal for a rectangular transmitted signal is

$$z'(t) = f_1 p_T(t) + f_2 p_T(t-T) \quad (7.9)$$

The autocovariance of the output of the random branch is

$$\begin{aligned} R_z(t, r) = & \gamma^2 \{ \sigma_1^2 \phi(t-r) p_T(t) p_T(r) \\ & + \sigma_2^2 \phi(t-r) p_T(t-T) p_T(r-T) \\ & + r \sigma_1 \sigma_2 \phi(t-r) p_T(t) p_T(r-T) \\ & + r \sigma_1 \sigma_2 \phi(t-r) p_T(t-T) p_T(r) \} \quad (7.10) \end{aligned}$$

where r is the correlation coefficient of the two random processes. The k th term of Eqn. 7.10 will be denoted by $R_k(t, r)$

so that for instance

$$R_1(t, r) = \gamma^2 \sigma_1^2 R_1'(t, r)$$

where

$$R_1'(t, r) = \phi(t-r) p_T(t) p_T(r) \quad (7.11)$$

and

$$R_3(t, r) = \gamma^2 r \sigma_1 \sigma_2 R_3'(t, r)$$

where

$$R_3'(t, r) = \phi(t-r) p_T(t) p_T(r-T) \quad (7.12)$$

Similar expressions hold for $R_2(t, r)$ and $R_4(t, r)$.

In a similar manner the two parts of the deterministic received signal in Eqn. 7.9 will be denoted by

$$z_1'(t) = f_1 p_T(t) \quad \text{and} \quad z_2'(t) = f_2 p_T(t-T) \quad (7.13)$$

and these two parts occupy discrete time intervals. We shall consider the inverse filter response $k(t)$ to be divided in the same way so that

$$k'(t) = k_1'(t) + k_2'(t) \quad (7.14)$$

where

$$k'_1(t) = 0 \text{ outside the interval } (-T/2, T/2)$$

and

$$k'_2(t) = 0 \text{ outside the interval } (T/2, 3T/2).$$

The autocovariance function $R_z(t, r)$ can be regarded as a surface plotted over the portion of the (t, r) plane illustrated in Figure 7.2. The four terms of the function occupy four discrete regions of the plane, as illustrated in the figure in which the regions are numbered with the subscript of the corresponding term. The surface has the same shape as the autocorrelation function $\phi(t-r)$ which is symmetrical about the diagonal shown. Along the borders of the four regions there are, in general, discontinuities of the surface caused by the unequal values of σ_1 and σ_2 .

Since the function $R'_z(t, r)$ is the kernel of the integral equation to be solved for $k'(t)$ it can be seen that solving for a point of $k'_1(t)$ corresponds to integrating $k'_2(t)$ along a line such as line A of the figure while solution for a point of $k'_2(t)$ involves a line such as line B.

In Chapter 4 the concept of a vector space of time functions with linear operators representing integral trans-

Figure 7.2 Regions of the Autocovariance Function

formations was introduced. Thus $k'(t)$ was represented by k' . Here we represent the discrete parts of the various functions by vectors and form them into matrices. With a corresponding representation for the integral operators we can write

$$\begin{bmatrix} k'_1 \\ k'_2 \end{bmatrix} = \begin{bmatrix} z'_1 \\ z'_2 \end{bmatrix} - \lambda \begin{bmatrix} R'_1 & R'_3 \\ R'_4 & R'_2 \end{bmatrix} \begin{bmatrix} k'_1 \\ k'_2 \end{bmatrix} \quad (7.15)$$

which is a representation of the integral equation (Eqn. 4.25) for the complete function $k'(t)$.

The vectors u_1 and u_2 will be used to represent the pulse functions $p_T(t)$ and $p_T(t-T)$ respectively so that the two parts of the deterministic received waveform can be represented by

$$z'_1 = f_1 u_1 \quad \text{and} \quad z'_2 = f_2 u_2 \quad (7.16)$$

With the above definitions the effective signal to noise ratio ρ is

$$\begin{aligned} \rho &= v(z', k') \\ &= v((z'_1, k'_1) + (z'_2, k'_2)) \\ &= \rho_1 + \rho_2 \end{aligned} \quad (7.17)$$

where

$$\rho_1 = \nu f_1(u_1, k_1')$$

and

$$\rho_2 = \nu f_2(u_2, k_2') \quad (7.18)$$

This completes the description of the problem to be solved for a two path channel. The extension of the representation to the multiple path case is obvious.

7.2 UNCORRELATED SCATTERING CHANNELS

In this section the limiting performance at high signal to noise ratios, of a channel with independent random processes in the two paths, is obtained in terms of the performance of an equivalent single path channel. The equivalence is determined by a figure of merit.

With a correlation coefficient of zero the matrix equation (Eqn. 7.15) reduces to the two independent equations:

$$\begin{aligned} k_1' &= f_1 u_1 - \lambda_1 R_1' k_1' \\ k_2' &= f_2 u_2 - \lambda_2 R_2' k_2' \end{aligned} \quad (7.19)$$

where

$$\lambda_1 = \nu \sigma_1^2 / \eta$$

and

$$\lambda_2 = v \sigma_2^2 / \eta \quad (7.20)$$

It is important that, although these two equations concern discrete regions of the (t, r) plane the two normalised kernels are related in the following way

$$R_1'(t, r) = R_2'(t-T, r-T) \quad (7.21)$$

A third equation defining a vector k_3' in either interval, say in $(-T/2, T/2)$, is now set up.

$$k_3' = u_1 - \lambda_3 R_1' k_3' \quad (7.22)$$

In this equation λ_3 is chosen to satisfy

$$\frac{1}{\lambda_3} = \frac{f_1^2}{\lambda_1} + \frac{f_2^2}{\lambda_2} \quad (7.23)$$

and a corresponding quality factor η' is defined by

$$\eta' = v / \lambda_3 = F \eta \quad (7.24a)$$

so that

$$\lambda_3 = v / F \eta \quad (7.24b)$$

where

$$F = \frac{f_1^2}{\sigma_1^2} + \frac{f_2^2}{\sigma_2^2} \quad (7.25)$$

As the signal to noise ratio is increased the solution of Eqns. 7.19 and 7.22 can be written in terms of the inverse operators in the following way

$$k_1' = \frac{f_1}{\lambda_1} (R_1')^{-1} u_1 \quad (7.26a)$$

$$k_2' = \frac{f_2}{\lambda_2} (R_2')^{-1} u_2 \quad (7.26b)$$

$$k_3' = \frac{1}{\lambda_3} (R_1')^{-1} u_1 \quad (7.27)$$

(The conditions for the existence of the inverse operators have been discussed in Section 4.7.) An effective signal to noise ratio ρ_3 is defined in terms of k_3' .

$$\rho_3 = v(u_1, k_3') = \frac{v}{\lambda_3} (u_1, (R_1')^{-1} u_1) \quad (7.28)$$

Now computing the value of ρ for the channel according to Eqns. 7.17 and 7.18 gives

$$\rho_1 = v f_1(u_1, k_1') = \frac{v}{\lambda_1} f_1^2(u_1, (R_1')^{-1} u_1)$$

$$\rho_2 = v f_2(u_2, k_2') = \frac{v}{\lambda_2} f_2^2(u_2, (R_2')^{-1} u_2)$$

$$= \frac{v}{\lambda_2} f_2^2(u_1, (R_1')^{-1} u_1) \quad \text{from Eqn. 7.21}$$

and

$$\begin{aligned}\rho &= \sqrt{\frac{f_1^2}{\lambda_1} + \frac{f_2^2}{\lambda_2}} (u_1, (R_1)^{-1} u_1) \\ &= \rho_3 \quad \text{from Eqns. 7.23 and 7.28.} \quad (7.29)\end{aligned}$$

Since the performance of a channel is completely determined by the value of ρ , Eqn. 7.29 shows that the performance of the two path channel with which we started is the same as that of a single path channel for which Eqn. 7.22 is the integral equation. This single path channel is therefore called an equivalent of our original channel. The equivalent channel has the same signal to noise ratio as the original but its quality factor is given by Eqn. 7.24a. Other parameters such as the signal shape and the autocorrelation function of the random process are the same in each channel. Eqn. 7.25 shows that the factor F is always unity for a single tap channel so that Eqn. 7.24a could be written

$$F'\eta' = F\eta \quad (7.30)$$

where $F' = 1$. The argument shows that any two channels which have the same value of $F\eta$, have the same limiting performance and so $F\eta$ will be called the figure of merit of a channel. The factor F will be called the relative figure of merit.

The extension of the above theory to an arbitrary number of paths is trivial. It is only necessary to define the relative figure of merit as

$$F = \sum f_i^2 / \sigma_i^2 \quad (7.31)$$

The problem of computing the performance of a multipath channel has now been reduced to that of computing the performance of its single path equivalent, a much simpler problem. When approximation methods, such as those of Chapter 6, are available for the single path performance they can be applied equally well to the multipath case.

7.2.1 Examples

To illustrate the accuracy and usefulness of the above methods a number of examples have been worked out for channels with random processes having a Gaussian autocorrelation function and using rectangular signals as in Chapter 6. The results for channels having narrowband random processes ($\xi = 0.5$) are shown in Table 7.1 while Table 7.2 shows results for channels having wideband processes ($\xi = 8$). In each case the signal to noise ratio is 512.

Table 7.1 Estimation Error as a Function of Figure of Merit
for Narrowband Processes ($\xi = .5$)

$F\eta$	η	F	σ_1^2	f_1^2	True Error	True Error for Equiv. Channel	Error from Cosine Formula
.72	.36	2	.5 .5 .1	.5 .1 .1	.564 .564 .566	.564	.566
2	.36 1 1 1	5.6 2 2 2	.1 .5 .5 .1	.5 .5 .1 .1	.326 .325 .325 .327	.325	.326
3.28	.36	9.1	.9	.1	.229	.229	.230
5.6	1	5.6	.1	.5	.151	.150	.150
9.1	1	9.1	.9	.1	.0984	.0983	.099

Table 7.2 Estimation Error as a Function of Figure of Merit
for Wideband Processes ($\xi = 8$)

$F\eta$	η	F	σ_1^2	f_1^2	True Error	True Error for Equiv. Channel	Approx. Error Eqn. 6.33
.72	.36	2	.5	.5	.254	.254	.268
2	.36 1 1 1	5.6 2 2 2	.1 .5 .5 .1	.5 .5 .1 .1	.114 .112 .112 .116	.112	.11
3.28	.36	9.1	.9	.1	.0734	.0731	.076
5.6	1	5.6	.1	.5	.0475	.0459	.043
9.1	1	9.1	.9	.1	.0298	.0296	.0268

For each channel the figure of merit and the various channel parameters are shown in columns one to five of the tables. Column six shows the estimation error of the optimum receiver for the channel, computed by solving the integral equation. Column seven shows the error computed in the same way for the single path equivalent channel and it is clear that the results confirm the concept of equivalence. To demonstrate the usefulness of the approximation techniques in the multipath case column eight shows the error obtained by approximation for the equivalent channel. In Table 7.1 the approximation method is the cosine formula of Eqn. 6.17 while in Table 7.2 it is the formula of Eqn. 6.33. The approximations are excellent in every case.

7.2.2 Performance versus signal to noise ratio

Curves of estimation error versus signal to noise ratio are plotted in Figure 7.3 for two quite different channels. In each case the value ν^* , suggested in Section 6.7 as the signal to noise ratio necessary to approach the limiting error, is marked and it is clear that it is also a valid criterion for multipath channels. Curves plotted on the basis of the limit-

$$\begin{aligned}\xi &= 0.5, \quad \eta = 1, \quad F = 5.6 \\ e_m^* &= 0.15 \\ v^* &= 57\end{aligned}$$

$$\begin{aligned}\xi &= 8, \quad \eta = 1, \quad F = 2 \\ e_m^* &= 0.11 \\ v^* &= 80\end{aligned}$$

Figure 7.3 Estimation Error Versus Signal to Noise Ratio for Two-Path Channels

ing error according to Eqn. 6.35 are again indistinguishable from the accurate curves for signal to noise ratios greater than about two.

7.3 CHANNELS WITH CORRELATED RANDOM PROCESSES

Insight into the effect of a non-zero correlation coefficient between the paths can be obtained from a consideration of the diagram of the (t,r) plane introduced in Section 7.1. The diagram is reproduced in Figure 7.4 and the value of $R_z(t,r)$ over regions three and four is now non-zero according to Eqn. 7.10. However the autocorrelation function $\phi(t-r)$ will decay to zero over some part of the plane outside lines parallel to the diagonal such as lines A and B. Depending on the correlation width these may or may not occur within the square $(-T/2, 3T/2)$.

If the random process is wideband lines A and B will be close together. Since solution of the integral equation involves integration along a line parallel to the r axis, the non-zero correlation coefficient will clearly have no effect on the resulting waveform outside the region of the t axis indicated. In these circumstances when the process is

Figure 7.4 Effect of Correlation Between Paths

very wideband the estimation error will be unaffected by correlation of the paths. When the process is narrowband it is not so easy to assess the effect of correlation.

In general the effect of correlation depends on the degree to which the outputs of the random branch of the medium add or subtract from each other during the integration process in the receiver. If the two paths have approximately equal quality factors f_i^2/σ_i^2 and the correlation coefficient is positive the effects of the two random processes will tend to add coherently, thus increasing the error. An example of this is the curve for Example 2 in Figure 7.5. The curve for Example 4 which is a similar channel but with a wider band random process confirms that the effect of correlation is less in such a case. By the same argument for these two examples, when the correlation coefficient is negative the effects tend to cancel out thus reducing the error. Again there is not so much cancelation when the random processes are wideband.

The filter responses shown in Figure 7.6 for Example 2 illustrate the way in which the receiver compensates for, or takes advantage of, correlation between paths. When the coefficient is positive the receiver de-emphasises the central

Example	ξ	η	f_1^2/σ_1^2	f_2^2/σ_2^2
1	0.5	1	0.2	1.8
2	0.5	1	1	1
3	8	1	0.2	1.8
4	8	1	1	1

Figure 7.5 Estimation Error Versus Correlation Coefficient for a Two-path Channel

(Vertical scale indicates relative amplitude only)

Figure 7.6 Filter Responses for Example 2

portion of the waveform where the correlation of the two processes at small time differences is large. When the coefficient is negative it emphasises this region so as to take advantage of the self-canceling effect of the processes.

When the two paths have very different quality factors f_i^2/σ_i^2 , as in Examples 1 and 3, the receiver is able to achieve some reduction in error even when the coefficient is positive, provided it is large. This is because the signal due to the poor quality path contributes little to the estimate of θ but can be used to estimate the behaviour of the other path by virtue of the correlation between them. This effect can be understood from consideration of Eqn. 7.34 when f_1 and thus z'_1 and k'_1 are small. In this case the equations for the two parts of the waveform can be written approximately

$$\begin{aligned} k'_2 &\approx f_2 u_2 - \lambda_2 R'_2 k'_2 \\ k'_1 &\approx -\lambda r \sigma_1 \sigma_2 R'_3 k'_2 \end{aligned} \quad (7.32)$$

This indicates that k'_2 is virtually the same as if there were only a single independent tap, while the value of k'_1 is determined by k'_2 and the correlation between the two output processes. The estimate of θ is then in the form

$$\hat{\theta} \propto (w, k'_2) - \lambda r \sigma_1 \sigma_2 (w, R'_3 k'_2) \quad (7.33)$$

where w represents the received signal given in Eqn. 7.3. In this expression the first term is an estimate of θ based on the signal from the second path and is subject to an error caused by the random process of the path. The second term is an estimate of this error based on the correlation between the random processes of the two paths. This effect is brought out quite clearly by the responses shown in Figures 7.7 and 7.8 for Examples 1 and 3 respectively. For large positive values of the correlation coefficient $k_1'(t)$ takes on negative values in the right hand region where there is high correlation with the process of the second path. In each example $k_2'(t)$ is substantially independent of the correlation coefficient.

7.4 COMPARISON WITH A SIMPLE MATCHED FILTER

Having obtained the performance of the optimum receiver for a number of different channels it is interesting to compare the performance of the simple matched filter, described in Section 4.5, which is suboptimum for random channels.

For the four examples used previously this comparison is made in Table 7.3. The entries in the table for each

(Vertical scale indicates relative amplitude only)

Figure 7.7 Filter Responses for Example 1

Figure 7.8 Filter Responses for Example 3

channel and correlation coefficient are

$$10 \log_{10} \frac{\text{matched filter estimation error}}{\text{optimum estimation error}}$$

Table 7.3 Comparison of the Optimum Receiver with a Matched Filter for the Examples of Figure 7.5

Example	Correlation Coefficients						
	0	-.4	-.8	-1	.4	.8	1
1	1.78	1.83	3.84	3.8	2.38	4.36	13.2
2	1.78	1.34	1.63	8.68	2.32	2.94	3.34
3	1.03	1.32	2.41	4.94	1.0	1.33	2.37
4	1.03	1.41	1.70	5.49	.894	.853	.856

Not surprisingly the optimum receiver affords the greatest improvement in performance over the matched filter when the channel has significant space and time correlation properties. These situations occur when the random processes are relatively narrowband and the correlation coefficient is large. In other cases the optimum receiver can do little more

to combat the multiplicative noise than the matched filter. In fact in these cases the optimum filter is essentially identical to the matched filter except for a multiplicative constant which reduces the scale of the estimate in order to reduce the error. This is shown clearly by the filter responses discussed earlier.

7.5 SUMMARY

In this chapter we have shown that discrete multipath channels with uncorrelated random processes may be analysed with very little increase in difficulty over the single path channel. The performance of such channels depends on a figure of merit related to the distribution of the ratio of random and deterministic energy received from each path. For channels of this type a formula has been found relating the performance to that of an equivalent single path channel.

When the random processes in the different paths are correlated no exact formula has been found for the performance. It has been shown, however, that the effect of correlation is very small if the random processes are wideband compared to the signal bandwidth. For small correlation coefficients the

effects of correlation are most marked when the individual quality factors of the separate paths are approximately equal. In general negative correlation between the paths produces a reduction in error while positive correlation increases the error.

Chapter 8

CONCLUSIONS

8.1 SUGGESTIONS FOR FURTHER WORK

8.1.1 More general channel models

It has been shown by Kailath⁽⁸⁾ and Bello⁽¹⁰⁾ that, for bandlimited signals, any randomly time variant channel can be modeled by a tapped delay line. If the signal bandwidth is W then the tap spacing must in general be $1/2W$ or less. In Chapter 7 we investigated a channel which could also be modeled by a tapped delay line but the channel was restricted so that the outputs due to the various taps, or paths, did not overlap in time. An investigation of the more general model might well use the methods described in Chapter 7 as a starting point.

The discrete multipath channel has an analogue in the frequency domain in which the channel is modeled by a number of paths each of which passes a discrete band of frequencies. Some work on models of this type has been done by

Proakis⁽²²⁾ for M-ary signalling and would seem to merit investigation for the present problem. Such models have obvious implications for time and frequency diversity systems and these might also be profitably developed.

8.1.2 Adaptive receivers

In the work of this thesis it has been assumed that an estimate of the channel impulse response and of the statistics of the error of this estimate is available. The source of the estimate has not been discussed nor does the receiver take any action to improve on this estimate. Thus the work shows what to do with an estimate once it has been obtained. The next obvious step is to include in the receiver structure some mechanism for continuing improvement of the channel estimate. A receiver for this purpose would have the basic form shown in Figure 8.1. The received signal is used by the part of the receiver with which we have been concerned to estimate the message parameters while at the same time the new section is using it to estimate the channel. The channel estimate is then fed to the message receiver either continuously or perhaps at discrete intervals of time. Such a

Figure 8.1 Basic Form of an Adaptive Receiver

structure would fit very well with the theory already presented since although the message receiver theoretically has infinite delay, in practice it must be finite.

8. SUMMARY

The major contribution of this thesis is a statement of a completely general, or canonical, form of the optimum linear receiver for the problem of signalling by pulse amplitude modulation over a randomly time variant, dispersive channel. The receiver contains a bank of filters matched in a special sense to the a priori estimates of the received signals corresponding to each transmitted pulse. The sampled outputs of these filters are fed to a weighting network which uses them to compute the estimates of the message parameters. This structure applies for all linear media whether exactly known or random. When the medium is wide sense stationary and sequential signalling is used the receiver can be reduced to a single filter followed by a tapped delay line. In this latter case the receiver has a form identical to that obtained for known media by George⁽¹⁾ and in fact for all channels the receiver has been shown to be generalisation of the form obtained by George.

When the received signals are non-interfering the weighting network is degenerate and reduces to a scaling factor applied to each filter. In this case the filters are closely related to those obtained by Kailath⁽⁷⁾ for the problem of hypothesis testing in a comparable situation.

In two subclasses of the general problem, analytic solutions for the impulse responses of the filters have been obtained. Basically these solutions apply when the output of the random medium is very wideband compared to the bandwidth of the a priori estimate of the received signal, and when the delay spread of the random output is much greater than that of the a priori estimate.

An assessment of the effect of the relative bandwidth of the signal and the random process of the channel has been achieved by analysing in detail a rather restricted channel model. This led to a general result for very wideband and very narrowband random processes. In connection with this result it was possible to define the situations in which the channel process can be regarded essentially as white noise and those in which the channel can be said to be essentially time invariant.

For channels which can be approximated by discrete multipath models some quite detailed results have been obtained and in particular the performance of such channels has been expressed in terms of equivalent non-selective channels. In this connection an assessment of the effects of correlation between the random processes at different delays was possible.

A minor contribution to the knowledge of random media has been made in the discussion and interpretation of some of the effects of correlated scattering channels on signals passing through them.

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APPENDIX A

METHODS OF SOLVING THE INTEGRAL
EQUATION

This appendix reviews some of the methods available for solving Fredholm integral equations of the second kind. Section A.2 discusses the numerical method which was used to solve the integral equations of Chapters 6 and 7 and Sections A.3 and A.4 discuss two methods which yield analytic solutions in certain restricted cases. The method of Section A.4 is used in Appendix B to develop the results for narrowband random processes with Gaussian autocorrelation functions described in Chapter 6.

A.1 GENERAL FORM OF THE EQUATION

The general form of the equation is

$$k(t) = z(t) - \lambda \int R(t,r)k(r)dr \quad (A.1)$$

in which $z(t)$ is square integrable and, in our case, $R(t,r)$ is a real, symmetric, non-negative definite kernel with finite norm. We are only interested in positive values of λ and, as

was pointed out in Chapter 3, a solution exists for all such values. In terms of the vector space L^2 , defined in Chapter 4, the equation can be written

$$k = z - \lambda Rk \quad (\text{A.2})$$

or

$$k = (I + \lambda R)^{-1} z \quad (\text{A.3})$$

A.2 REPLACEMENT BY A SET OF ALGEBRAIC EQUATIONS

The integral in Eqn. A.1 can be replaced approximately by a summation of the form

$$\int R(t,r)k(r)dr \approx \sum_{i=1}^N A_i R(t,x_i)k(x_i) \quad (\text{A.4})$$

using any integration formula, such as the trapezoidal formula or Simpson's formula. In Eqn. A.4 the A_i are the integration coefficients of the particular formula used and the x_i are the points at which the functions are sampled. The integral equation can then be replaced by a set of N simultaneous algebraic equations of the form

$$k(x_j) \approx z(x_j) - \lambda \sum_{i=1}^N A_i R(x_j, x_i) k(x_i) \quad (\text{A.5})$$

for $j = 1, 2, \dots, N$. It is a relatively simple matter to program a digital computer to solve this set of equations for the sample values of the function $k(t)$. If necessary an interpolation formula can then be used for values of t lying between the sample points x_i .

An excellent discussion of this method is given in reference 13 together with a procedure for estimating the error involved in applying an integration formula with a given grid size (that is for a given value of N). In practice it is simplest to determine the necessary fineness of the grid by solving the equation several times using increasing values of N until the solution converges.

When the kernel $R(t, r)$ has discontinuities, as in the case of the multipath problem of Chapter 7, a separate integration formula should be applied over each continuous interval. This allows a solution to be obtained for the values of $k(t)$ at the end points of each interval. The necessity for this is apparent from the discontinuities of the solutions sketched in Figures 7.7 and 7.8.

A.3 THE NEUMANN SERIES

Before introducing the Neumann series it is necessary first to introduce the concept of an iterated kernel. The second order iteration of the kernel $R(t,r)$ is denoted by the operator R^2 and is defined by

$$R^2 y = \iint R(t,s)R(s,r)y(r)drds \quad (A.6)$$

Higher order iterations are defined in a similar manner.

The Neumann series consists of an expansion of the inverse operator of Eqn. A.3 in terms of iterated kernels in the following form:

$$(I + \lambda R)^{-1} = I - \lambda R + \lambda^2 R^2 - \lambda^3 R^3 + \dots \quad (A.7)$$

It can be shown (12) that this series converges under the condition

$$|\lambda| \|R\| < 1 \quad (A.8)$$

It can also be shown that the expansion

$$k = z - \lambda R z + \lambda^2 R^2 z - \lambda^3 R^3 z + \dots \quad (A.9)$$

converges to the solution of Eqn. A.3.

Substituting the expression for λ from Eqn. 4.22 in Eqn. A.8 gives the condition

$$\nu \ll \frac{||R||}{\eta} \quad (\text{A.10})$$

which shows that the series converges only for low signal to additive noise ratios. Since this thesis concentrates on the effects of multiplicative noise, which are most pronounced at high signal to additive noise ratios, the Neumann series is not of great value here. Its only use in the thesis occurs in Section 4.6 in which the optimum performance is compared with the performance of a matched filter at low signal to noise ratios. When the signal to noise ratio ν is small enough for λ also to be small, the expansion of Eqn. A.9 can be written approximately

$$k \approx z - \lambda R z \quad (\text{A.11})$$

Although the concept of the Neumann series can be extended⁽¹³⁾ by analytic continuation to any value of λ , it requires considerably more work in numerical solutions than the method described in Section A.2.

A.4 KERNEL OF FINITE RANK

If $R(t,r)$ can be expressed in the form

$$R(t,r) = \sum_{i=1}^N B_i(t)B_i(r) \quad (A.12)$$

over its square of definition in the (t,r) plane, a closed form can be found for the resolvent kernel and the solution of Eqn. A.1 can be reduced to the solution of a set of N algebraic equations. A kernel of this form is said to be of rank N .

The time functions $B_i(t)$ are represented by the vectors B_i and the following inner products will be required.

$$\begin{aligned} z_i &= (z, B_i) \\ k_i &= (k, B_i) \\ r_{ij} &= (B_i, B_j) \end{aligned} \quad (A.13)$$

Eqn. A.1 can then be written

$$\begin{aligned} k &= z - \lambda \sum_{i=1}^N (B_i, k) B_i \\ &= z - \lambda \sum_{i=1}^N k_i B_i \end{aligned} \quad (A.14)$$

forming the inner product of this equation with B

$$k_j = z_j - \lambda \sum_{i=1}^N k_i r_{ij} \quad (A.15)$$

for $j = 1, 2, \dots, N$. This is a set of N simultaneous equations for the components k_j . Once a solution has been found for the k_j , the function $k(t)$ is given by Eqn. A.14.

A.3.1 Matrix notation

We form a set of matrices to represent arrays of the inner products defined in Section A.2. Let

$$\begin{aligned}\underline{z} &= [z_i] \\ \underline{k} &= [k_i] \\ \underline{R} &= [r_{ij}]\end{aligned}\tag{A.16}$$

The set of simultaneous equations then becomes the matrix equation

$$\begin{aligned}\underline{k} &= \underline{z} - \lambda \underline{R} \underline{k} \\ &= (\underline{I} + \lambda \underline{R})^{-1} \underline{z}\end{aligned}\tag{A.17}$$

Since $\underline{R}$ is a real, symmetric, non-negative definite matrix and $\underline{I}$ is the identity matrix of order N , the existence of the inverse is assured.

In order to find the estimation error of the optimum receiver it is necessary to determine the value of the inner

product (k, z) . This is easily done by forming the inner product of z with Eqn. A.14 to give

$$(k, z) = (z, z) - \lambda \sum_{i=1}^N k_i z_i \quad (\text{A.18})$$

which in matrix notation, is

$$(k, z) = (z, z) - \lambda \underline{z}' \underline{k} \quad (\text{A.19a})$$

$$= (z, z) - \underline{z}' (\underline{I} + \lambda \underline{R})^{-1} \underline{z} \quad (\text{A.19b})$$

where $\underline{z}'$ denotes the transpose of $\underline{z}$.

A.4.1 Approximation by a kernel of finite rank

It is only very rarely that the integral equation turns out to have a kernel of finite rank. However, it can be shown⁽¹²⁾ that any kernel of finite norm can be approximated arbitrarily closely by a kernel of finite rank. If such an approximation can be found it is possible to obtain analytic approximations to the solution of the equation using the technique described in this appendix. However, this may still be a very cumbersome algebraic problem unless the rank of the approximation is small, since it is necessary to invert a matrix of the same order as the rank. In general it is very difficult to find a sufficiently accurate approximation with low enough rank.

In Appendix B an approximation of rank three is suggested for the Gaussian autocorrelation function used in Chapter 6 for a single path channel. Detailed results are developed in the appendix using this approximation and the comparison, made in Chapter 6, with results computed by the algebraic equation method of Section A.2 shows that the approximation is excellent.

APPENDIX B

APPROXIMATION BY A KERNEL OF
FINITE RANK

Chapter 6 was concerned with a single path, non-selective channel in which the random process had an autocorrelation function

$$\phi(\tau) = e^{-\frac{1}{2}(\xi\tau)^2} \quad (\text{B.1})$$

The corresponding output autocovariance (Eqn. 6.10) is

$$R'_z(t,r) = \phi(t,r)s'(t)s'(r) \quad (\text{B.2})$$

and, since the transmitted signal $s(t)$ is restricted to be non-zero only in the range $(-T/2, T/2)$, the kernel $R'_z(t,r)$ is also non-zero in the corresponding square $-T/2 \leq t, r \leq T/2$ of the (t,r) plane.

In this appendix the Gaussian autocorrelation function is approximated by the raised cosine function

$$\phi_c(\tau) = \frac{1}{2}(1 + \cos \pi \zeta \frac{\tau}{T}) \quad (\text{B.3})$$

over the square of definition of $R'_z(t,r)$ and it will be shown that the corresponding approximation for $R'_z(t,r)$ is a kernel of finite rank. This allows an approximate solution to be found in closed form for the integral equation

$$k'(t) = s'(t) - \lambda \int R'(t,r)k'(r)dr \quad (B.4)$$

The value of ζ is chosen according to Eqn. 6.14 or Eqn. 6.15.

The Gaussian shape of Eqn. B.1 is monotonic for $\tau > 0$ but the cosine function of Eqn. B.3 is monotonic for $\tau > 0$ in the square of interest only if $\zeta \frac{T}{T} < 1$ over that square. Now the greatest possible value of $\tau = t-r$ in the square is $\tau = T$. The approximation is therefore only valid for

$$\zeta < 1 \quad (B.5)$$

B.1 ANALYSIS FOR AN ARBITRARY SIGNAL

Using the cosine approximation for $\phi(t-r)$ gives the approximation

$$R'_z(t,r) \approx R(t,r) = \phi_c(t-r)s'(t)s'(r) \quad (B.6)$$

from Eqn. B.2. This expression can now be expanded to give

$$R(t,r) = \frac{1}{2}(1 + \cos\pi \zeta \frac{t}{T} \cdot \cos\pi \zeta \frac{r}{T} \\ + \sin\pi \zeta \frac{t}{T} \sin\pi \zeta \frac{r}{T})s'(t)s'(r) \quad (B.7)$$

which shows that the approximate kernel $R(t,r)$ is a kernel of rank three by the definition of Eqn. A.12. It can be expressed as

$$R(t,r) = \sum_{i=1}^3 B_i(t)B_i(r)$$

where

$$\begin{aligned} B_1(t) &= \frac{1}{\sqrt{2}} s'(r) \\ B_2(t) &= \frac{1}{\sqrt{2}} s'(t) \cos\pi \zeta \frac{t}{T} \\ B_3(t) &= \frac{1}{\sqrt{2}} s'(t) \sin\pi \zeta \frac{t}{T} \end{aligned} \quad (B.8)$$

At this point we note two properties of the above functions.

- 1) If $s'(t)$ is an even function so are $B_1(t)$ and $B_2(t)$ while $B_3(t)$ is an odd function.
- 2) If $s'(t)$ is an odd function so are $B_1(t)$ and $B_2(t)$ while $B_3(t)$ is even.

In the remainder of this appendix it will be assumed that $s'(t)$ is either even or odd, it will not be necessary to specify

which. This restriction is not necessary in order to obtain results, but it yields some useful simplifications.

We are now in a position to follow the general procedure for a kernel of finite rank, outlined in Appendix A. First we determine the entries of the matrix $\underline{R}$ defined in Eqn. A.16.

$$r_{11} = \frac{1}{2} \int s'^2(t) dt = \frac{1}{2} \quad (\text{B.9})$$

$$r_{22} = \frac{1}{2} \int s'^2(t) \cos^2 \pi \zeta \frac{t}{T} dt \quad (\text{B.10})$$

$$r_{33} = \frac{1}{2} \int s'^2(t) \sin^2 \pi \zeta \frac{t}{T} dt \quad (\text{B.11})$$

$$r_{12} = \frac{1}{2} \int s'^2(t) \cos \pi \zeta \frac{t}{T} dt \quad (\text{B.12})$$

$$\begin{aligned} r_{13} &= \frac{1}{2} \int s'^2(t) \sin \pi \zeta \frac{t}{T} dt \\ &= 0 \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} r_{23} &= \frac{1}{2} \int s'^2(t) \cos \pi \zeta \frac{t}{T} \sin \pi \zeta \frac{t}{T} dt \\ &= 0 \end{aligned} \quad (\text{B.14})$$

The terms r_{13} and r_{23} are zero because they involve the integral of an odd function over a range symmetric about the origin. This is a result of the assumption that $s'(t)$ is either even or odd. The term r_{11} always has the value $\frac{1}{2}$ since $s'(t)$ has unit energy.

The next step is to form the matrix $(I + \lambda R)$ and invert it. We have

$$(I + \lambda R) = \left[\begin{array}{cc|c} 1 + \lambda r_{11} & , & \lambda r_{12} & , & 0 \\ & \lambda r_{12} & , & 1 + \lambda r_{22} & , & 0 \\ \hline & 0 & , & 0 & , & 1 + \lambda r_{33} \end{array} \right] \quad (\text{B.15})$$

Because of the two zero off-diagonal terms the matrix can be inverted by inverting the two partition matrices separately to give

$$(I + \lambda R)^{-1} = \left[\begin{array}{cc|c} \frac{1 + \lambda r_{22}}{d} & , & -\lambda \frac{r_{12}}{d} & , & 0 \\ & -\lambda \frac{r_{12}}{d} & , & \frac{1 + \lambda r_{11}}{d} & , & 0 \\ \hline & 0 & , & 0 & , & \frac{1}{1 + \lambda r_{33}} \end{array} \right] \quad (\text{B.16})$$

where

$$d = 1 + \lambda g_1 + \lambda^2 g_2 \quad (\text{B.17})$$

and

$$g_1 = r_{11} + r_{12} \quad (\text{B.18})$$

$$g_2 = r_{11}r_{22} - r_{12}^2 \quad (\text{B.19})$$

These two coefficients are of considerable importance and it is useful to note from the definitions that

$$1 \geq g_1 \geq \frac{1}{2} \quad (\text{B.20})$$

On investigating the matrix $\underline{z}$ we find that the whole problem reduces to second order under the assumption that $s'(t)$ is either even or odd, for, from Eqn. A.13

$$z_1 = \frac{1}{\sqrt{2}} \int s'^2(t) dt = \frac{1}{\sqrt{2}} = \sqrt{2} r_{11} \quad (\text{B.21})$$

$$z_2 = \frac{1}{\sqrt{2}} \int s'^2(t) \cos \pi \zeta \frac{t}{T} dt = \sqrt{2} r_{12} \quad (\text{B.22})$$

$$z_3 = \frac{1}{\sqrt{2}} \int s'^2(t) \sin \pi \zeta \frac{t}{T} dt = 0 \quad (\text{B.23})$$

Since all the operations to be carried out involve multiplication of the matrix $\underline{z}$ by $(I + \lambda R)^{-1}$, we have the result

$$k_3 = 0 \quad (\text{B.24})$$

B.1.1 Filter response

In order to solve for the inverse filter response $k(t)$ we must first determine the matrix $\underline{k}$, which, from Eqn. A.17 is

$$\underline{k} = (\underline{I} + \lambda \underline{R})^{-1} \underline{z}$$

Hence, after substituting the results obtained above and after a little algebra

$$k = \frac{\sqrt{2}}{d} \begin{bmatrix} \frac{1}{2} + \lambda g_2 \\ r_{12} \end{bmatrix} \quad (B.25)$$

Then from Eqn. A.14

$$k'(t) = \frac{1 - \lambda (\frac{1}{2} - g_1)}{1 + \lambda g_1 + \lambda^2 g_2} s'(t) - \frac{\lambda r_{12}}{1 + \lambda g_1 + \lambda^2 g_2} \cos \pi \zeta \frac{t}{T} s'(t) \quad (B.26)$$

B.1.2 The estimation error

To determine the estimation error we must find the value of

$$\rho = v(k', z')$$

where, in the present case $z' = s'$, and from Eqn. A.19a and from Eqn. A.18 this is

$$\begin{aligned}\rho &= v[(s', s') - \lambda \underline{z}' \underline{k}] \\ &= v[1 - \lambda \underline{z}' \underline{k}]\end{aligned}\tag{B.27}$$

Substituting the matrices $\underline{z}$ and $\underline{k}$ determined above gives the final result

$$\rho = \frac{v(1 + 2\lambda g_2)}{1 + \lambda g_1 + \lambda^2 g_2}\tag{B.28}$$

B.2 RESULTS FOR A RECTANGULAR PULSE

If the transmitted pulse is

$$\begin{aligned}s'(t) &= p_T(t) \\ &= 1/T, \quad -T/2 \leq t \leq T/2 \\ &= 0 \quad \text{elsewhere}\end{aligned}$$

then all the formulae developed in the previous section apply, for this is an even function. It is a matter of simple algebra to show that the important parameters have the following values.

$$\begin{aligned}
 r_{11} &= \frac{1}{2} \\
 r_{22} &= \frac{1}{4} \left(1 + \frac{1}{\pi \zeta} \sin \pi \zeta \right) \\
 r_{12} &= \frac{1}{\pi \zeta} \sin \frac{\pi \zeta}{2}
 \end{aligned} \tag{B.29}$$

$$g_1 = \frac{1}{4} \left(3 + \frac{1}{\pi \zeta} \sin \pi \zeta \right) \tag{B.30}$$

$$g_2 = \frac{1}{8} \left(1 + \frac{1}{\pi \zeta} \sin \pi \zeta - \frac{8}{(\pi \zeta)^2} \sin^2 \frac{\pi \zeta}{2} \right) \tag{B.31}$$

These are all that is required to determine the filter response and the estimation error from Eqns. B.26 and B.28.

B.3 QUALITY OF THE APPROXIMATION

The solutions for the filter response and the estimation error obtained in this appendix are, of course, approximations to the true values since the cosine kernel is only an approximation to the Gaussian kernel. The comparison between these results and the results of accurate numerical computation, made in Chapter 6, show that the approximation is excellent at least for rectangular pulses.

APPENDIX C

CORRELATED MESSAGE PARAMETERS

In Chapter 3 the message parameters θ_i were restricted to have unit variance and to be uncorrelated with each other (Eqn. 3.2a). This restriction is now relaxed to allow completely arbitrary statistics. The covariances of the parameters are denoted by

$$E \{ \theta_i \theta_j \} = p_{ij} \quad (C.1)$$

With these statistics the autocovariance $R_w''(t, r)$ of the received signal

$$w(t) = \sum_i \theta_i z_i(t) + n(t)$$

is

$$\begin{aligned} R_w''(t, r) &= \sum_m \sum_j p_{mj} \bar{z}_m(t) \bar{z}_j(r) \\ &\quad + \sum_m \sum_j p_{mj} Q_{mj}(t, r) + R_n(t, r) \end{aligned} \quad (C.2)$$

where

$$Q_{mj}(t, r) = E \{ \tilde{z}_m(t) \tilde{z}_j(r) \} \quad (C.3)$$

Eqns. C.2 and C.3 correspond to Eqns. 3.7 and 3.8 respectively.

Corresponding to Eqns. 3.9 and 3.10 we define the functions

$$R_y''(t, r) = \sum_m \sum_j p_{mj} Q_{mj}(t, r) \quad (C.4)$$

and

$$R_x''(t, r) = R_y''(t, r) + \sum_m \sum_j p_{mj} \bar{z}_m(t) \bar{z}_j(r) \quad (C.5)$$

Then, with white additive noise, we have

$$R_w''(t, r) = R_x''(t, r) + N_o \delta(t-r) \quad (C.6)$$

With the estimate $\hat{\theta}_i$ defined as in Eqn. 3.13 the optimisation equation corresponding to Eqn. 3.20 is

$$N_o k^i(t) = \sum_j p_{ij} \bar{z}_j(t) - \int R_x''(t, r) k^i(r) dr \quad (C.7)$$

Now we suppose the receiver to be made up of a bank of filters as before so that

$$k^i(t) = \sum_n a_n^i k_n^i(t)$$

and substitute this expression in Eqn. C.7 to obtain the equation corresponding to Eqn. 3.26

$$\begin{aligned} \sum_j p_{ij} \bar{z}_j(t) &= N_o \sum_n a_n^i k_n^i(t) + \sum_n a_n^i R_y''(t, r) k_n^i(r) dr \\ &+ \sum_m \sum_j \sum_n a_n^i c_{nj} p_{mj} \bar{z}_m(t) \end{aligned} \quad (C.8)$$

Defining the individual filter responses as in Eqn. 3.27

$$N_{on}^i k_n^i(t) = \bar{z}_n^i(t) - \int_y R''(t,r) k_n^i(r) dr \quad (C.9)$$

and substituting in Eqn. C.8 gives

$$\sum_j P_{ij} \bar{z}_j(t) = \sum_n a_n^i \bar{z}_n(t) + \sum_m \sum_j \sum_n a_n^i C_{nj} p_{mj} \bar{z}_m(t) \quad (C.10)$$

where the superscripts have been dropped on the $k_n(t)$ and the C_{nj} for the same reason as in Eqns. 3.28 and 3.29.

Finally the weights a_n^i are the solutions of the set of simultaneous equations

$$p_{ik} = a_k^i + \sum_n \sum_j a_n^i C_{nj} p_{kj} \quad (C.11)$$

for all k , obtained by equating coefficients of the time functions in Eqn. C.10. It is a simple matter to show that Eqn. C.11 reduces to Eqn. 3.31 in the special case of Chapter 3 in which

$$p_{kj} = \delta_{kj} \quad (C.12)$$

APPENDIX D

MEAN SQUARE DISPERSIONS

In this appendix we define the mean square dispersion, or second order central moment, of a function. It is convenient to do this by example and for this purpose we shall use the time dispersion of the signal $y(t)$ defined in Eqn. 2.34 as

$$m_t(y) = \text{mean square dispersion of } P_y(0,t)$$

where

$$P_y(0,t) = E \{y^2(t)\}$$

is the power distribution of $y(t)$ as a function of time.

First we form a normalised distribution function

$$p_y(t) = \frac{P_y(0,t)}{\int P_y(0,t)dt} \quad (D.1)$$

so that

$$\int p_y(t)dt = 1 \quad (D.2)$$

The first and second moments of this distribution are then defined as

$$\bar{m}_y = \int t p_y(t) dt \quad (D.3)$$

and

$$m_y^2 = \int t^2 p_y(t) dt \quad (D.4)$$

respectively. Finally the mean square dispersion is

$$m_t(y) = m_y^2 - \bar{m}_y^2 \quad (D.5)$$

Clearly the normalisation of Eqn. D.1 renders the measurement of the width or dispersion independent of the amplitude of $P_y(0,t)$. The subtraction of the first order moment in Eqn. D.5 compensates for the dependence of the second moment on the location of $P_y(0,t)$ in time.

D.1 DISPERSION OF A CONVOLUTION

In this section we suppose that we have three time functions $x(t)$, $y(t)$ and $z(t)$ whose power distributions $P_x(0,t)$, $P_y(0,t)$ and $P_z(0,t)$ are related by the convolution

$$P_z(0,t) = \int P_x(0,\tau) P_y(0,t-\tau) d\tau$$

Their normalised distributions, defined according to Eqn. D.1, are then related by the similar convolution

$$p_z(t) = \int p_x(\tau) p_y(t-\tau) d\tau \quad (D.6)$$

and we wish to determine the mean square dispersion $m_z(t)$ of $P_z(0,t)$ defined as

$$m_z(t) = m_z^2 - \bar{m}_z^2 \quad (D.7)$$

From Eqns. D.4 and D.6 the second moment of $p_z(t)$ is

$$\begin{aligned} m_z^2 &= \iint t^2 p_x(\tau) p_y(t-\tau) d\tau dt \\ &= \int p_x(\tau) \int [(t-\tau)^2 - (\tau^2 - 2t\tau)] p_y(t-\tau) dt d\tau \\ &= m_x^2 + m_y^2 + 2\bar{m}_x \bar{m}_y \end{aligned} \quad (D.8)$$

Similarly from Eqns. D.3 and D.6 the first moment of $p_z(t)$ is

$$\begin{aligned} \bar{m}_z &= \iint t p_x(\tau) p_y(t-\tau) d\tau dt \\ &= \int p_x(\tau) \int [(t-\tau) + \tau] p_y(t-\tau) dt d\tau \\ &= \bar{m}_x + \bar{m}_y \end{aligned} \quad (D.9)$$

On substituting the results of Eqns. D.8 and D.9 into Eqn. D.7 the mean square dispersion $m_z(t)$ is

$$m_z(t) = m_x(t) + m_y(t) \quad (D.10)$$

This is the desired result, used in Chapter 2 that the dispersion of a convolution is the sum of the dispersions of its component functions.