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METHODS OF NON-LINEAR ANALYSIS
WITH APPLICATIONS TO
MULTIPLE ACCESS SATELLITE COMMUNICATIONS

by

ROBERT G. LYONS, B.Sc., M.Sc.

A thesis submitted to the Faculty of
Graduate Studies in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

Faculty of Engineering,
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July 1971

The undersigned recommend to the Faculty of
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"METHODS OF NON-LINEAR ANALYSIS WITH APPLICATIONS
TO MULTIPLE ACCESS SATELLITE COMMUNICATIONS",

submitted by Robert G. Lyons, B.Sc., M.Sc. in partial
fulfillment of the requirements for the degree of
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ABSTRACT

Due to the power limitation on board a satellite, travelling wave tube amplifiers presently employed are operated close to saturation. The resulting nonlinear gain characteristic, coupled with an amplitude controlled phase shift through the tube, generate intermodulation product interference when several FM carriers are inputted to the device. In addition, the carriers are not necessarily equally amplified, and there is a change in their relative power levels. To evaluate the performance of a frequency division multiple access (FDMA) satellite communications system, it is necessary to analyze and allow for nonlinear signal suppression and intermodulation interference.

The thesis presents a new method of computing the line spectrum and statistical properties of intermodulation noise, and the powers of the output carriers. The solution is partly an analytical one, partly a simulation algorithm; it is totally general in that it applies to any bandpass nonlinearity with arbitrary multi-carrier and gaussian noise input. The case of a single sine wave and noise undergoing amplitude limiting and/or AM to PM transfer is dealt with separately and in considerable detail.

Approaches heretofore taken in analyzing the effect of a bandpass nonlinearity have involved either representing the output in a Fourier series or suitably expanding its autocorrelation function. In either case, signal amplitudes and intermodulation product spectra can be obtained by locating the appropriate terms in the series. The complexity of these methods grows exponentially with the number of independent inputs, and approximations are usually made in implementing them. The new method developed here

is based on a stochastic decomposition of the output, and complexity grows only linearly with the number of input signals. The i 'th signal in the output is simply defined to be the expected value of the output given the i 'th signal at the input. The noise component of the output must complete the phasor polygon whose sides are the resultant output and the desired signals. The signal amplitudes and phase shifts are evaluated from analytical expressions; the first and second order statistics of the noise are then computed by simulation.

While principally concerned with analyzing the properties of RF noise out of a bandpass nonlinearity, the thesis also examines the related problem of signal modulation and detection. The phenomenon of signal-intermodulation noise coherence is given a rigorous explanation, and its effect on signal detectability is discussed. A detailed application of the theory to the bandpass hard limiter is presented; other examples considered are the piecewise linear limiter and linear and quadratic AM/PM devices.

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LIST OF SYMBOLS

A	amplitude of the desired signal at the input to a bandpass non-linearity
A'	amplitude of the desired signal at the output from a bandpass non-linearity
A_k	amplitude of the k'th interfering signal at the input to a bandpass non-linearity
A'_k	amplitude of the k'th interfering signal at the output from a bandpass non-linearity
$\alpha(t), \alpha$	the RF phase difference between two independent signals, a random variable uniformly distributed on $[0, 2\pi)$
$\alpha(P_d)$	the suppression or change in desired signal power to total power through a bandpass non-linearity
b_o	the output power backoff in a saturating bandpass amplitude non-linearity
β	the phase shift incurred on each of several constant-amplitude, angle-modulated signals comprising the input to a device exhibiting AM to PM conversion
C	the relative change in amplitude on each of several signals passed through an AM-PM device ($0 \leq C \leq 1$)
$\delta(z)$	the Dirac delta function evaluated at z

$\delta(t), \delta$ the phase difference between the resultant of a constant amplitude signal and noise, and the signal; a stationary random variable with probability density function $f_{\delta}(y)$

Δ_k the RF phase difference between the k'th signal and the resultant input

$\varepsilon(t)$ the instantaneous phase error in a received PM or FM signal

$E(t), E$ the envelope of the composite input to a bandpass non-linearity, a stationary random variable with density $f_E(r)$

$\text{erfc}(z)$ the complimentary error function of z , in this case defined by

$$\text{erfc}(z) = \frac{2}{\pi} \int_z^{\infty} e^{-t^2} dt$$

$E\{Q\}$ the expected value of the random variable Q

$E(z)$ the complete elliptic integral of the second kind

$f_1(x)$ transfer function of an amplitude to amplitude non-linearity

$f_2(x)$ transfer function of an amplitude to phase non-linearity

$\vec{f}(x)$ the general bandpass non-linearity $(f_1(x), f_2(x))$. exhibiting both AM-AM and AM-PM transfer

$f_Q(q)$	probability density function of the random variable Q evaluated at q
${}_2F_1(a,b;c;z)$	the gaussian hypergeometric function
${}_1F_1(a,b;z)$	the confluent hypergeometric function
$g(t)$	a baseband pulse which amplitude modulates the desired signal
$g(x)$	the transfer function of a non-linearity, either AM-AM or AM-PM
$\gamma(t), \gamma$	the phase of the noise component of the non-linearity output relative to the signal component, in general, a random variable not uniformly distributed on $[0, 2\pi)$
$I[z]$	the imaginary part of the complex number z
$\vec{i}(t), i$	the intermodulation product interference from a bandpass non-linear device
$I(t)$	the totality of interference (including interfering signals and noise) at the input to a non-linearity
$I'(t)$	the total interference at the output from a non-linearity
$I_n(z)$	modified Bessel function of the first kind of order n evaluated at z
$J_n(z)$	Bessel function of order n evaluated at z
K_p	AM-PM conversion coefficient
$K(z)$	the complete elliptic integral of the first kind

$N(\vec{\xi}, \Lambda)$	a multivariate normal random variable with mean $\vec{\xi}$ and covariance matrix Λ .
$n(t)$	additive band-limited gaussian noise of power (statistical) σ^2 at the input to a non-linearity
$n_u(t)$	gaussian noise of power σ_u^2 on the up path to the satellite
$n_d(t)$	gaussian noise of power σ_d^2 on the down path from the satellite
$n'(t)$	the noise component of the output from a non-linearity whose input consists of a signal and additive gaussian noise
$n_1'(t), n_2'(t)$	baseband components of $n'(t)$ or $I'(t)$ in and out of phase with the desired signal
N_1', N_2'	$E\{n_1'^2\}$ and $E\{n_2'^2\}$, the powers of the in and out of phase noises out of a non-linearity
P	input power of desired signal
P'	output power of desired signal
P_k	input power of k'th interfering signal
P_k'	output power of k'th interfering signal
P_o	total power out of the non-linear device
P_s	total power out of a hard limiting satellite repeater
ρ	signal-to-noise power ratio at the input to a non-linearity
ρ_i	signal-to-total interference power ratio at the input to a non-linearity

ρ_o	signal-to-total interference power at the output from a non-linearity
$R_q(\tau)$	autocorrelation function of the process $q(t)$
$\rho_q(\tau)$	correlation coefficient, $R_q(\tau)/R_q(0)$, of the process $q(t)$
$R[z]$	the real part of the complex number z
$R(t), R$	the envelope of the total interference at the input to a non-linear device, a random variable with density $f_R(r)$
R'	a random variable denoting the envelope of the noise out of a non-linearity whose input consists of a sine wave and gaussian noise
SNR	signal-to-noise power ratio
SIR	signal-to-interference power ratio
S_o	total output signal power
$S_q(f)$	power spectrum of the process $q(t)$
$S_D(t)$	desired signal at the input to a non-linear device
$S'_D(t)$	desired signal at the output from a non-linear device
$S_k(t)$	k 'th interfering signal at the input to a non-linear device
$S'_k(t)$	k 'th interfering signal at the output from a non-linear device
σ^2	the power (statistical) of additive gaussian noise

$\theta_D(t)$	angle modulation on desired signal
$\theta_k(t)$	angle modulation on k'th interfering signal
$T\{g(t)\}$	Fourier Transform of the finite energy waveform $g(t)$
ω_D	angular frequency of desired carrier
ω_k	angular frequency of k'th interfering carrier
$x(t)$	composite input to non-linearity
$y(t)$	composite output from non-linearity

ERRATA FOR LYONS' PH.D. THESIS

<u>PAGE</u>		<u>CHANGE</u>
v	last line, first paragraph	...initiating <u>and</u> nurturing...
20	first line	$x \leq -1$
64		footnote refers to E_s and E_m in first sentence
68	middle of page	[4,p.22]
68	middle of page	Neumann
140	fifth last line	$\cos^{-1}(\quad)$
150	equation 7.5.7	$-2A'n_1'$
151	seventh line	$[0,2\pi)$
165	second line	Blachman's eq. (7.5.17)
168 & 169		abscissa should be labelled (f/f_n)
170	last line	should read $\frac{E\{n_2^2\}}{A^2}$
174	fifth last line	$\approx E\{\quad\}$
183	equation 7.6.2	$\frac{e^{-\rho}}{2\sigma^2} \left[\quad + \quad \right]$
220	equation 8.1.2	$-g(E(t)))$ ↑ <i>missing in original</i>

221 equation 8.1.5

$$A' = \left[\begin{array}{c} \\ \end{array} \right]^2 + \left[\begin{array}{c} \\ \end{array} \right]^2 \Bigg]^{\frac{1}{2}} A$$

227 equation 8.4.2

$$e^{-r^2/2}$$

268 last line

$$\sum_{k=1}^2$$

278 sixth line

$$[0, 2\pi)$$

CHAPTER 1

INTRODUCTION

The thesis reviews known methods of analyzing channel nonlinearities with multiple inputs, and presents a new unified approach given the title "The Stochastic Method". Conceptually, this approach focuses attention on a particular input RF carrier statistically impressing its phase characteristic on the composite output from a bandpass nonlinear device. The component of the output which behaves randomly with respect to all the individual input signal RF phases consists of the intermodulation products generated in the device. Specifically, the stochastic method applied to analyzing the effect of AM to AM and AM to PM nonlinearities on a sum of constant amplitude angle modulated signals and noise leads to evaluation of:

- (1) the output signal amplitudes and constant phase shifts,
- (2) the first order statistics (i.e. envelope of phase probability density functions) of the net output noise process,
- (3) the power spectrum of the intermodulation product noise.

A detailed presentation of the theory of the stochastic method, the principal theoretical contribution of the thesis, is given in 4.3. The method is applied, both as a concept and as an analytical tool, in chapters 5 to 11; its merits (and limitations) are summarized in chapter 12. So, the primary aim of the thesis is to explain, by means of general discussion, analytical development, and examples, a simple new approach to analyzing a bandpass nonlinearity with a multi-carrier input.

As a secondary and somewhat independent goal, the thesis briefly describes multiple access satellite communications. Equations developed through the stochastic approach are applied in studying the theoretical performance of a frequency division multiple access satellite system. Effects of the satellite's nonlinear gain and amplitude dependent phase shift characteristics are analyzed. The literature is extensively surveyed and important results are incorporated into the presentation of the new work; the thesis is very much organized as a tutorial. Particular attention is given to the effects of up and down path thermal noise on single access analog and digital communications. The application of the stochastic method is not limited to the satellite channel; however, since this is perhaps the most important nonlinear RF channel presently under study, it serves as a timely and useful example.

Having stated the technical goals of the thesis, we discuss briefly the motivations which led the author to pursue the work in a particular order and in a particular way. An explanation of the philosophy and format of the presentation is included. Involvement with problems of communication through an active satellite repeater* is the author's primary motivation for a theoretical study of the effects of nonlinear elements. The travelling wave tube high frequency amplifiers presently employed in communications satellites exhibit the amplitude limiting and AM/PM characteristics of the general band-pass nonlinearity analyzed in the thesis. As the satellite represents a large and irretrievable expenditure in the communications network,

* see Vita

it is important to predict and optimize the system performance beforehand. In satellite communications, trial and error cannot economically replace accurate analysis.

On investigation, one quickly discovers that the state of the art is deficient in handling the problems posed when analog or digital information is passed through bandpass nonlinear devices. In particular:

- (1) Existing methods of nonlinear analysis are tedious even to describe, and seem to be far removed from the communications problem. The equations themselves are complicated and give no insight whatsoever.
- (2) It follows that there is no simple unified approach which can give solutions; considerable experience is required to obtain a deep understanding of, and a computational facility with, existing methods.
- (3) Some questions cannot reasonably be answered by applying existing techniques even on a powerful computer. The presence of additive noise which is not gaussian or independent of a desired signal especially complicates the course of existing analytical methods.

The research work given here helps to solve the difficulties posed by (1), (2), and (3). As such, the thesis is a contribution to both nonlinear systems analysis and the satellite communications problem.

Because the thesis work is so wide ranging in scope, each chapter is somewhat self-contained, and includes new and known results under a particular topic. References are also given chapter by chapter. Generally speaking, the text proceeds in tutorial form from the simple to the complex, defining new terms and concepts as they appear. Chapters 2 and 3 discuss satellite communications; chapter 4 summarizes existing methods of nonlinear analysis, and introduces the new approach. Chapters 5 to 10 deal with a host of communications problems arising from the amplitude limiting and AM/PM characteristics of the repeater. Chapter 11 describes a computer program which simulates a bandpass nonlinearity and makes use of the general equations previously derived. The a priori discussion given here has its a posteriori counterpart in the final chapter of the thesis.

CHAPTER 2

THE SATELLITE REPEATER

2.1 Discussion

Communications satellites presently in orbit demonstrate that a single satellite repeater can provide many communication channels between widely separated points. With satellite orbits at medium altitudes or above (higher than 2,000 miles), the repeater is visible over large portions of the earth's surface and hence is potentially able to interconnect a large number of far-flung points. A synchronous orbit (on equatorial plane with one day period) gives the satellite a fixed position relative to the rotating earth; therefore, tracking is unnecessary and the communications facility is present twenty-four hours a day. Ideally it would be desirable to permit any station in view of the synchronous satellite to communicate with one or more other stations at will and with as little interaction as possible with other users of the repeater. The theoretical difficulties involved in providing a system with this characteristic constitutes the multiple-access problem considered here.

The size and weight limits on all satellites presently in orbit, and those to be put into orbit in the near future, impose severe limits on the amount of power generated on board the satellite. Therefore, satellite repeaters use travelling-wave

tube (TWT) amplifiers, which are the most efficient and reliable devices for wide-band microwave power amplification yet developed. To make maximum use of the dc power consumed by the TWT, it must be operated near the point at which maximum signal output power is obtained — i.e. near saturation. But when a TWT is operated in saturation, it exhibits undesirable properties which include a non-uniform gain - vs. - frequency characteristic and phase distortion. These properties can cause crosstalk among signals sharing the repeater. Therefore, analysis is required to determine an optimum TWT operating point — one which will be near, but not in, saturation.

Each user station transmits a narrow band signal to the satellite with two kinds of modulation. The first kind is multiple-access modulation which is designed to accomplish two main purposes:

- (1) it must make possible separate reception of each transmitted message by any receiving station possessing the requisite demodulator;
- and (2) it must interfere as little as possible with the transmissions of other user stations in passing through the satellite repeater.

The second kind is message modulation, which conveys the message information. In the following development, the RF signal with multiple-access modulation but without message modulation is referred to as a carrier. The allocation of repeater capacity given to a multiple access carrier with its message modulation is referred to as a channel. Two stations communicating in one direction comprise a link.

Domestic satellite systems can be expected to handle a large amount of traffic compared with that carried by transoceanic systems. For this reason, the presently allocated frequency bands at 4 and 6 GHz are not well suited to domestic use. Frequencies above 10 GHz are not as heavily loaded as the lower frequency bands, but they are subject to propagation difficulties which generate new problems.

In our analysis we shall not be concerned with propagation impairments, and carrier frequencies are totally arbitrary. The multiple access satellite channel is modelled simply by:

- (1) additive gaussian noise on the up link;
- (2) a bandpass transponder with nonlinear
memoryless and frequency independent
gain and phase shift characteristics,
- and (3) additive gaussian noise on the down link.

For the most part we shall consider constant amplitude angle modulated signals. Effects of group delay (resulting from non-ideal phase versus frequency characteristics) and a frequency translation in the satellite are ignored.*

References in this introductory discussion have been taken from Schwartz, Aein, and Kaiser [1] and Tillotson [2].

2.2 The Travelling Wave Tube

To facilitate an understanding of its behaviour, a brief and elementary physical description of the travelling wave tube (TWT) is now given.

In the TWT an electron beam generated by an electron gun traverses the cylindrical axis of an evacuated glass tube about 30 cm. long and about 25 mm. wide. The beam, which travels through the centre of the tube, is enclosed by a spiral wire coil (the helix). The coil and the electron collector are both given a positive bias. In order to keep the beam focussed, the tube is surrounded by an axial magnetic field; this focussing field, which is produced by an external magnetic coil, is also known as the beam directing field.

* Frequency translation is required to ensure isolation between the satellite's transmitted and received signals (i.e. to prevent feedback). A single satellite antenna receives and re-radiates the signals.

The high frequency signal that is to be amplified is fed in at the beginning of the helix; it becomes increasingly amplified in its passage along the helix and is then uncoupled at the end. The continuous and progressive interaction of the fields of the electron beam and the signal leads to a bunching of the beam electrons. These bunched electrons form electric fields and consequently induce electric fields on the helix — these induced fields are 90° out of phase with the initially present field. The resultant field present on the helix appears greater in comparison to the original field, and thus exhibits the amplification effect. Essentially, there is a transfer of energy from the electrons to the signal field. Under correct operating conditions, the total energy of the electrons decreases along the beam, and the total energy of the high frequency signal increases along the helix.

Advantages of this type of tube are the great bandwidth and the low noise factor. This makes it possible to amplify weak high frequency signals that would otherwise be drowned in noise. In general the amplitude of the wave increases uniformly along the helix up to some maximum value at which the electron beam can give no further energy to the signal (i.e. saturation). Also, the instantaneous output from the helix has experienced a delay which is partly dependent on the corresponding instantaneous amplitude of the input. Hence the TWT introduces a phase lag or differential delay which is a memoryless function of input amplitude. This phenomenon is known as AM to PM conversion.

The travelling wave tube is described in much more detail in most texts on electromagnetic device. It has been in use since its invention in 1942. Recently, TWT's have found considerable use in communication satellite applications, and renewed effort has been undertaken in improving the efficiency and capability of these tubes. Early satellites were severely power limited, and their output TWT's were operated with multiple carriers in the vicinity of saturation. For such operation, with the distortion due to AM/PM negligible in comparison with the effect of amplitude limiting, the TWT could be reasonably modelled as a hard limiter. As the capacity of satellites increased, systems became backed off from saturation to reduce the level of intermodulation products. At back off, the effect of AM to PM conversion is significant, and the simple hard limiter model totally inappropriate.

Bretting [3] has discussed in some detail the performance of a modern travelling wave tube of European origin. Wescott [4] has examined multiple FM carriers passed through a TWT operating near saturation. Foster and Kunz [5] discuss intermodulation and cross-modulation in TWT's; Dionne [6] and Giarolo [7] analyze harmonic generation and multiple-signal operation respectively. Recent papers on the nonlinear phase shift in travelling wave tubes have been given by Berman and Mahle [8] and Falconer [9].

The former presents a theoretical model which allows calculation of the multicarrier performance of the TWT. Falconer is concerned with measuring the nonlinear phase distortion under dynamic conditions.

2.3 Modelling the Communication Link

In order to facilitate analysis of the travelling wave tube, Berman and Mahle [6] have assumed that the AM to PM conversion effect and the amplitude limiting effect occur independently in the tube. From the theory of the TWT, it can be concluded that the amplification nonlinearity originates in the last part of the electron beam-helix wave rather than at the input connector. In contrast, as a first approximation, the phase shift takes place evenly distributed over the whole helix length. It therefore seems reasonable to connect both effects in series, giving rise to the two-lump physical model of the TWT depicted in figure 1.

Physical Model of TWT

Figure 1

While figure 1 may represent only an approximation to the physical operation of the TWT, as an analytical model it can specify exactly the TWT transfer characteristic: If the bandpass input to the TWT (or to any bandpass nonlinearity) is expressed in the form

$$x(t) = E(t) e^{j(\omega t + \phi(t))},$$

the instantaneous output can be written as*

$$y(t) = f_1(E(t)) e^{-j f_2(E(t))} e^{j(\omega t + \phi(t))} \quad (2.3.1)$$

Here the bandpass (i.e. two dimensional) transfer characteristic of the device is defined as:

$$\vec{f}(x) = (f_1(x), f_2(x)) \quad (2.3.2)$$

where $f_1(x)$ and $f_2(x)$ are the amplitude-amplitude and phase shift-amplitude transfer functions respectively. So while the TWT and other bandpass nonlinearities may physically only be modelled by many AM/AM and AM/PM blocks, there are equivalent two lump models (reference figure 1) which give the same input-output characteristics. The fact that the two nonlinear effects do not occur sequentially and independently

* This constitutes a definition of a bandpass nonlinear device.

does not preclude the use of figure 1 as an exact analytical representation of the physical device.

Now, if the TWT input consists of a sum of constant amplitude angle modulated sinusoids, the output can be decomposed into components due to the presence only of individual signals at the input, and components which have resulted from the interaction of signals. The first group of terms retains the angle modulation present on the input signals; each signal has experienced a slightly different degree of effective amplification and a constant phase shift in the TWT. The second group of terms has been called intermodulation product interference. This interference is ordinarily uncorrelated in amplitude and phase with the angle modulation in the signals; that component of it located in band with signals constitutes noise and limits performance.

Without specifying the form of message modulation on the signals, we write the input to the repeater as

$$x(t) = \sum_{i=1}^m S_i(t) + n_u(t) \quad (2.3.3)$$

Here, $S_i(t)$ is the i 'th signal accessing the satellite and $n_u(t)$ is up path bandlimited gaussian noise. In figure 2, a model of the multiple access satellite communications channel is presented. $n_d(t)$ is additive gaussian noise on the down link.

Figure 2

MULTIPLE ACCESS SATELLITE CHANNEL

The zonal filter in the repeater is assumed to remove harmonics generated in the TWT, but to pass without distortion the fundamental frequency terms of the signals and IM products. Equation 2.3.1, in fact, is an expression for only the first harmonic of the output from a nonlinear device. However, when referring to a bandpass nonlinearity, we assume the inclusion of a post zonal filter in order that equation 2.3.1 apply directly.

The output from the analog channel of figure 2 consists of the signals (possibly phase shifted and amplitude distorted), $\{S_i'(t)\}$, and the totality of noise and interference, $n'(t)$.

In sections 2.4 and 2.5 we briefly discuss separately the nature and effects of a nonlinear gain characteristic and AM/PM conversion.

2.4 Nonlinear Amplification

As has been pointed out, amplification in the satellite takes place near the point of saturation. Such operation is a result of the basic power limitation in the satellite; the TWT output power can only increase to a certain point as the input level is raised. The amplitude transfer characteristic of the transponder* is defined by the nonlinear memoryless

* A transponder is a single multiple access channel through the satellite; it consists of cascaded TWT's and associated electronics. A satellite may include many transponders (usually 6-12) occupying different frequency bands and operating approximately independently. They do, however, share common antennas and multiplexing (for frequency translation) circuitry in the satellite.

function $f_1(x)$. If the input amplitude is given by $x(t)$, the output amplitude is then

$$y(t) = f_1(x(t)) \quad (2.4.1)$$

Equation 2.4.1 can be simply regarded as relating the output and input from a baseband memoryless nonlinearity. Davenport and Root [10], Middleton [11], Blachman [12], and others have presented methods of analyzing the effects of such devices.

In this section we do not attempt to survey known methods of nonlinear analysis, or to outline our own approach. Such a development will be given in chapter 4. We here confine the discussion to a qualitative or conceptual description of the nonlinear process. Such a description, too often omitted, can facilitate a deeper understanding of rather tedious mathematical developments.

The nonlinear device defined by equation 2.4.1 is discussed briefly under the following headings:

- (1) the probability density function
- (2) loss of information
- (3) operating point and the linear model
- (4) the power spectrum
- (5) the intermodulation process.

(1) The Probability Density Function

The input, $x(t)$, is a stationary random process with probability density function $f_X(x)$. A is defined to be the smallest subset of the real line which satisfies the relation:

$$f_X(x) = 0, x \notin A \quad (2.4.2)$$

Here A is the range of the random variable X or the support of the function $f_X(x)$.

Now, $y(t)$ can be regarded as a random variable defined by the mapping

$$f_1: X \rightarrow Y \quad (2.4.3)$$

Equation 2.4.3 implies that sample values of the random variable X are mapped deterministically onto sample values of the random variable Y . The domain of the function f_1 is A ; the range is denoted by $f_1(A)$.

The sets $\{X \leq x\}$ and $\{Y \leq x\}$ are events that occur with probabilities

$$F_X(x) = \int_{-\infty}^x f_X(z) dz$$

and

$$F_Y(x) = \int_B f_X(z) dz$$

where

$$B = \{z | f_1(z) \leq x\}$$

Since, in general,

$$f_1(z) \neq z$$

for a nonlinear device, we can conclude that

$$F_X(x) \neq F_Y(x), \text{ for all } x.$$

Thus, a nonlinear device alters the distribution function of its input. Papoulis [13] and others have shown that the output probability density function is expressible in terms of the input probability density function as follows:

$$f_Y(y) = \frac{f_X(x_1)}{|f'_1(x_1)|} + \dots + \frac{f_X(x_n)}{|f'_1(x_n)|} \quad (2.4.4)$$

where $y = f_1(x_1) = \dots = f_1(x_n)$ for given y .

The first and second moments of Y are given by

$$E\{Y\} = \int_{-\infty}^{\infty} f_1(x) f_X(x) dx \quad (2.4.5)$$

and

$$E\{Y^2\} = \int_{-\infty}^{\infty} f_1^2(x) f_X(x) dx \quad (2.4.6)$$

respectively. These represent the DC level and total power in the output.

In particular, we note that if $x(t)$ is a gaussian process distributed according to

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}},$$

and f_1 has an inverse f_1^{-1} , the output has the density

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(f_1^{-1}(y))^2}{2\sigma^2}} \bigg/ |f_1'(f_1^{-1}(y))| \quad (2.4.7)$$

The fact that a nonlinear device with a gaussian input gives a nongaussian output constitutes the principal difficulty in analyzing the ultimate effect of $n_u(t)$ (reference figure 2) on system performance.

(2) Loss of Information

If $f_1^{-1}(x)$ does not exist, the output cannot uniquely specify the input and there is a loss of information. For example, if

$$f_1(x) = \begin{cases} -1 & x \leq -1 \\ x & -1 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (2.4.8)$$

(i.e. the piecewise linear limiter), it is observed that the input signal can be reconstructed only in the range $|x| < 1$. Hence saturation does represent an inherent loss of information about the input.

Even if f_1^{-1} exists, the signal out of a non-linearity may be more susceptible to noise degradation than was the input. For example, if $x(t)$ were an optimally quantized PCM signal, the nonlinearity would requantize in a non-optimum fashion. Near the point of saturation, inputs widely separated in value are squeezed into outputs narrowly spaced and difficult to distinguish.

Blachman [14] has considered the amount of information that a specific value y of a random variable Y gives about another random variable X . His presentation is from an information theory standpoint. Good [15] and Sobolev and Telepnev [16] are concerned with the information in a clipped waveform.

(3) Operating Point and the Linear Model

In order to demonstrate the meaning and application of the term "operating point", we consider an

input consisting of the sum of two constant amplitude sinusoids; i.e.,

$$x(t) = A_1 \cos w_1 t + A_2 \cos w_2 t, \quad A_1 \neq A_2.$$

Then, at any time t , the resultant input amplitude lies in the interval $(A_1 - A_2, A_1 + A_2)$, and this defines the operating region for the output; i.e., $[f_1(A_1 - A_2), f_1(A_1 + A_2)]$. The operating point is defined as the RMS value of the output; i.e.,

$$y_0 = \sqrt{\langle f_1^2(x(t)) \rangle} \quad (\langle \cdot \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \cdot dt) \quad (2.4.9)$$

that could also result from a single input sinusoid of amplitude

$$x_0 = f_1^{-1}(y_0) = \sqrt{A_1^2 + A_2^2}$$

The equivalent single carrier operating point (equivalent in terms of identical input powers) is given by $(\sqrt{A_1^2 + A_2^2}, f_1(\sqrt{A_1^2 + A_2^2}))$, which is slightly offset from the actual (multicarrier) operating point (x_0, y_0) .

The term backoff gives a measure of how far from saturation the device is operating. Physically, saturation in the TWT represents a state at which

the electron beam can give no further energy to the signal (reference 2.2). Mathematically the point of saturation (x_s, y_s) is defined by (sup = supremum)

$$\begin{cases} y_s = f_1(x_s) = \sup\{f_1(x)\} \\ \text{and} \\ x_s \leq x | y_s = f_1(x). \end{cases} \quad (2.4.10)$$

In other words, x_s is the minimum amplitude of a single carrier which can drive the output to its maximum (limiting) amplitude y_s . The mathematical representation is somewhat artificial and arbitrary since an increasing input may continue to produce infinitesimal increases in the output level. However, armed with our definition 2.4.10, we define output backoff (in dB) by the equation

$$b_o = 20 \log_{10} \left(\frac{y_s}{y_o} \right) \quad (2.4.11)$$

If, for example,

$$y_s = \sqrt{2} y_o ,$$

the operating point is 3 dB away from saturation, and we speak of a 3 dB backoff at the output. Input backoff similarly is defined by

$$b_i = 20 \log_{10} \left(\frac{x_s}{x_o} \right) \quad (2.4.12)$$

Output backoff is generally the preferred specification since y_s is a more easily determined and less arbitrary point than x_s .

The linear model makes the assumption that $f_1(x)$ is linear with slope $\frac{df_1(x_o)}{dx}$ in its operating region. Based on the linear model, the output can be written as the first two terms of a Taylor series expansion of $f_1(x)$ about the point (x_o, y_o) :

$$y(x) \approx y_o + \frac{df_1(x_o)}{dx} (x - x_o) \quad (2.4.13)$$

As with small signal linear transistor models, such an assumption is reasonable only for inputs that take small excursions from the operating point. The form is useful in analyzing a constant amplitude signal and low level noise through a nonlinear device. In this case excursions from the signal operating point result from an additive random process with a second moment much smaller than the signal power.

Figure 3 portrays the operating region, operating point, backoff and the linear model for the two sinusoid input case.

Figure 3

(4) The Power Spectrum

When $x(t)$ is a gaussian process with autocorrelation function $R_x(\tau)$, the output autocorrelation function, $R_y(\tau)$, can rather easily be expressed in the form

$$R_y(\tau) = g(R_x(\tau)) \quad (2.4.14)$$

Baum [17] has given a systematic and comprehensive tutorial paper on determining the autocorrelation function of the output from a nonlinear device for gaussian inputs. The basic approaches have been categorized under:

- (i) The Direct Method
- (ii) The Transform Method
- (iii) Price's Theorem.

The first two methods are described in Davenport and Root [10]. The third, which relates $R_y(\tau)$ and $R_x(\tau)$ by a differential equation, was first reported by Price [18].

Equation 2.4.14 reduces to a simple result in the case of the hard limiter and the erf limiter. These are given by:*

$$(1) \quad f_1(x) = \text{sgn}(x)$$

$$R_y(\tau) = \frac{2}{\pi} \sin^{-1} \left(\frac{R_x(\tau)}{R_x(0)} \right),$$

* These results will be used in succeeding development.

and

$$(2) \quad f_1(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2L^2}} dt$$

$$R_y(\tau) = \frac{L^2}{2\pi} \sin^{-1} \left[\frac{R_x(\tau)}{R_x(0) (1 + L^2)} \right]$$

respectively for zero mean noise at the input.

(5) The Intermodulation Process

If the input to a nonlinear device consists of a sum of signals, the instantaneous output contains components dependent on the individual signals, and components which result from the interaction of two or more of the signals. The second group of terms are called the intermodulation products.

To visualize the process, we consider a simple square law device,

$$f_1(x) = x^2 ,$$

with two signals as input; i.e.,

$$x(t) = s_1(t) + s_2(t).$$

The instantaneous output from the device is immediately

$$y(t) = s_1^2(t) + s_2^2(t) + 2 s_1(t) s_2(t) \quad (2.4.15)$$

The first term in 2.4.14 is the component of the output due only to $s_1(t)$. Its presence or absence is totally independent of the presence or absence of $s_2(t)$ at the input, and it is often referred to as the $s_1 \times s_1$ term. Similarly $s_2^2(t)$ is the second signal or $s_2 \times s_2$ term. The third component of the output, $2 s_1(t) s_2(t)$, depends on the presence of both signals at the input, and is called an intermodulation (IM) product. If $s_1^2(t)$ and $s_2^2(t)$ were the only terms present in the output, and if they were non-interfering (i.e. in some way orthogonal), except for the sign ambiguity in taking square roots, $s_1(t)$ and $s_2(t)$ could be recovered undistorted from $y(t)$. However, the IM ($s_1 \times s_2$) term, which depends on the presence of both signals at the input, and is not generally orthogonal to the $s_1 \times s_1$ and $s_2 \times s_2$ terms, makes perfect detection of the signal components in $y(t)$ impossible.

This is precisely the problem faced in frequency division multiple access communications through a nonlinear device — the original signals, while separated in frequency, ultimately impose distortion (co-channel interference) on one another through the nonlinear process*. In our case, $f_1(x)$ is a convex or limiting function (as opposed to the square law device, which is concave). Input

* A nonlinear device, unlike a linear one, can generate frequencies not present at the input.

signals must in fact share the total available power out of the device, and there is an amplitude (only) dependence of the $s_1 \times s_1$ term on the other signal inputs. Such a dependence is not seen in equation 2.4.15.

Equation 2.4.14 represents instantaneous $s_1 \times s_1$, $s_2 \times s_2$, and $s_1 \times s_2$ components out of a square law nonlinearity; the autocorrelation function of the output can be written in the similar form:

$$R_y(\tau) = R_{s_1} \times s_1(\tau) + R_{s_2} \times s_2(\tau) + R_{s_1} \times s_2(\tau) \quad (2.4.16)$$

with

$$R_{s_i} \times s_i(\tau) = E\{s_i^2(t) s_i^2(t + \tau)\}$$

and

$$R_{s_1} \times s_2(\tau) = 2 R_{s_1}(0) R_{s_2}(0) + 4 R_{s_1}(\tau) R_{s_2}(\tau)$$

if the input processes are zero mean, stationary and independent. From an expression like 2.4.16, we can find the power spectra of the signal and IM product terms out of any nonlinearity. This is the usual approach to nonlinear analysis with multiple inputs.

Sunde [17, chapter 8] has written a rather comprehensive description of intermodulation distortion in nonlinear channels. His presentation serves as a reference to this and succeeding discussions of intermodulation in the travelling wave tube.

2.5 AM/PM Conversion

As mentioned previously in section 2.2, in a travelling wave tube a change in amplitude of the input signal causes a variation in phase shift. That is, amplitude variation or modulation will be converted to phase modulation. This unwanted phase modulation creates distortion in the FM carrier spectrum, producing intermodulation noise in a similar fashion to that resulting from a nonlinear amplitude characteristic.

Postponing analytical development to chapter 8, we here briefly discuss the basic cause of AM/PM distortion in travelling-wave tubes, and describe its characterization and effects. References are taken from Benton et al [20], Wilson [21], Granlund [22], and Berman and Mahle [6].

For negligibly small values of input power, the phase velocity everywhere along the TWT helix is a constant, and the total phase shift from input to output is the circuit phase constant, $B = \frac{\omega}{v}$, times the length of the helix, d . Bd is normally in the order of thousands of degrees. At higher

input powers, the circuit phase velocity is not constant with distance; it decreases in the output region of the tube. Then the phase shift through the TWT is given by the integral of Bdz , where z is the displacement along the tube. Since v is reduced in the high power region, B is larger, which causes a larger total phase shift. This difference between total phase shift through the travelling-wave tube for different input power levels will cause a phase modulation to appear at the output if a pure amplitude modulation is impressed at the input.

In fact, when the input level is varied from well below saturation to saturation, the change in phase shift may exceed a radian. In the case of a multichannel input signal, this phase shift variation is applied to each of the signals being amplified. This AM/PM conversion distorts the phase modulation already present on the signals.

The envelope of the multichannel input signal varies at a rate much greater than the radio frequency (RF) bandwidths of the individual ground station signals; it varies at least as rapidly as the total bandwidth of the composite signal amplified by the TWT. The corresponding rapid variation of phase, no matter how small, spreads sidebands of each ground station signal over the composite satellite bandwidth and beyond. If the amplitude and hence TWT phase shift varied only slowly, the effect of AM/PM conversion would be simply to add disturbances to the individual baseband

signals without affecting the pre-detection signal-to-distortion ratios.

The AM/PM conversion of an individual tube is characterized by the AM/PM conversion coefficient, K_p , generally a number in degrees per decibel. (it should be noted that K_p can be obtained from the $f_2(x)$ — reference 2.3.1 — curve by differentiation). Since K_p varies considerably with the input power, calculations based on a constant value are restricted to small envelope fluctuations. In the multicarrier case, then, it is more reasonable to use the phase transfer function, $f_2(x)$, which links relative phase shift to instantaneous input power. An $f_2(x)$ curve can be obtained by a static measurement with a single carrier. However, in some tubes a slight difference exists between the static and dynamic phase transfer characteristics, which is not the case for the gain transfer characteristic. This is a thermal phenomenon which must be taken into account in a transfer characteristic measurement since the thermal time constant is generally large compared to the fluctuation of the instantaneous power; therefore, the input signal will see the dynamic characteristic.

The curve of relative phase shift versus input power can be approximated in many ways, for instance by a power

series. However, many higher order terms must be considered, should this series approximation be reasonably accurate for input power exceeding saturation.

An equation of the form

$$\phi = k_1(1 - e^{-k_2 P_{IN}}) + k_3 P_{IN} \quad (2.5.1)$$

where k_1 , k_2 , k_3 are tube constants, P_{IN} is the normalized input power, and ϕ is the relative phase shift, has been found [6] to give reasonably good agreement over the whole range, especially for a very small input power and above saturation. The coefficients k_1 , k_2 , and k_3 can be obtained by a least squares procedure on measured data. Berman and Mahle [6] have analyzed the effect of a phase nonlinearity given by 2.5.1 on a sum of p sine waves.

It has been suggested that AM/PM conversion may cause partly intelligible crosstalk between different ground station signals. So may the planned soft limiting of the multichannel signal. Both are envelope-controlled nonlinearities that cause odd-order distortion products, generated by certain pairs of signals, to fall on top of the signal to which a ground station receiver is tuned. Each distortion product is itself an FM signal. Third-order products, for example, are frequency-modulated by the difference of twice one of the baseband waveforms and the baseband waveform of

the other. But an FM receiver, in responding to the sum of a desired signal and a distortion product, does not deliver a baseband output that is the sum of the baseband waveforms of signal and distortion product. The receiver output is the sum of the baseband of the stronger signal and a beat whose frequency is the sum of the centre frequency difference between signal and distortion product, and a term proportional to the difference of the two baseband voltages. In order for this beat to be intelligible, the following conditions must be satisfied:

- (1) There must be beats from only a few sources
(so this one will be recognizable).
- (2) The centre frequency difference between the two signals causing the distortion must be simply harmonically related to the centre frequency difference between one of them and the desired signal.
- (3) All three signals must have low modulation indices producing total peak phase deviations less than one or two radians, depending on the degree of intelligibility of interest.

A simple gain slope producing FM to AM transfer, followed by a device which exhibits AM/PM conversion will result in

intelligible crosstalk. The gain slope produces AM coherent with the FM on the carriers, and the AM in turn is converted to PM on each of the carriers present in the TWT. Phase modulation present on one carrier will then appear, to some extent, in each of the carriers at the output of the transponder. Chapman and Millard [23] have given a detailed analysis of intelligible crosstalk together with experimental data for a travelling-wave tube.

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CHAPTER 3

MULTIPLE ACCESS COMMUNICATIONS

3.1 Frequency Division Multiple Access (FDMA)

Satellite communication design is complicated by the need for the satellite to relay signals from a number of ground stations simultaneously. This need for the satellite to provide multiple access for many signals requires that the signals be distinguishable by the ground stations as they emerge from the satellite. The signals should be orthogonal*, or nearly so. A frequency-division multiple access system is one in which orthogonality is achieved by assigning a different frequency band to the signal from each ground station. The satellite of the frequency-division multiple access system must amplify and deliver to a single antenna the sum of a number of signals in adjacent frequency bands; this composite signal, as received, has a large peak-to-RMS ratio.

In FDMA, the repeater bandwidth is divided into a number of nonoverlapping frequency bands which constitute access channels. Each link is assigned an access channel. For the wideband hard-limiting repeater, the arrangement of channels is not set by the repeater; a particular arrangement is effected by specifying the

* A group of additive signals are orthogonal if any one can theoretically be completely recovered or detected from the resultant without distortion. Mathematically, the projection of any signal, expressed as a vector in a Hilbert space, on another is $\vec{0}$. We then speak of the signals sharing the common channel as being non-interfering.

spectrum of the signal from each transmitting user station. Assignment of distinct frequency bands (multiple access modulation) as a means of separating radio signals is a conventional communication practice. Certain complications arise, however, when the signals are mixed in the nonlinear repeater [1, 14]. When several signals are simultaneously present at the repeater input, they interact on passing through the repeater and suffer some impairments such as intermodulation noise and intelligible crosstalk (reference 2.4, 2.5). The intermodulation noise may be out-of-band noise, which only subtracts from the output power available to the desired signals, or it may be within the band of a desired signal. To allow the channels to be spaced so that, effectively, all intermodulation noise falls outside the signal bands, much of the repeater bandwidth must go unused [10].

Whenever two or more independent signals pass through a hard-limiting repeater, the power ratio of any two component signals at the output will generally differ from the power ratio of the same two components at the input, the change favouring the strongest signal at the expense of the weaker one. The worst case of this suppression involves two constant-envelope signals, one greater than the other by a factor of four or more. Then the small signal output power is reduced by another factor of four (6 dB) below its relative power at the repeater input. If the larger of the

two signals can be characterized as a gaussian process, then the small signal is reduced by only 1 dB. When many constant-envelope signals arrive simultaneously at the repeater input with nearly equal power, the effect is like the gaussian signal case, so that any one of them would be suppressed about 1 dB. If the up-link powers are not kept equal at the repeater input, the repeater output power does not divide equally, and some stations will receive more power than they require, others less.

A practical advantage of FDMA is the compatibility of the user station modulation equipment with much existing hardware. For analog FM message modulation, only FM transmitters and receivers are required, and the inputs and outputs of the links appear in analog form.

Frequency division multiple access is the method of multiple access modulation considered in the body of the thesis. Other schemes are described briefly in section 3.2.

3.2 Other Multiple Access Schemes

Following the notation of Schwartz et al [1], the multiple-access modulation techniques are grouped into four categories: frequency-division multiple access (FDMA); constant-envelope, spread-spectrum multiple access (SSMA); time-division multiple access (TDMA); and pulse-address multiple access (PAMA).

The FDMA technique was discussed in 3.1. In this section we describe the other multiple access methods separately, and compare them with FDMA.

(1) Spread Spectrum Techniques

In SSMA, each multiple-access carrier signal usually occupies the entire repeater bandwidth. An access channel consists of a carrier waveform characterized by a distinct wide-band phase or frequency modulation. Wide-band phase or frequency coding are used to address the carrier for the desired receiver. The message-modulation bandwidth is small relative to the total bandwidth and is inserted into the carrier as an additional phase or frequency variation. The multiple-access and message modulations can be either analog or digital.

A spread spectrum technique achieves both multiple and random access, and at the same time makes efficient use of the entire repeater bandwidth. A distinct pseudo-noise (PN) carrier is assigned to each user. An active user modulates his message onto his PN carrier and transmits it through the

satellite repeater to the receiving terminals. Each receiving station employs a phase coherent correlator capable of locking onto any of the transmitted signals while rejecting the others.

Computing co-channel interference out of the nonlinear repeater in this case can only be achieved by a first order analysis of the intermodulation products. We essentially must determine the projections of the total interference on each of the orthogonal signals. The IM power spectrum is of little value in analyzing distortion.

Anderson and Wintz [16] have done an analysis of a Spread-Spectrum Multiple-Access System with a hard limiter. Ward [17] has discussed acquisition of pseudo-noise signals by sequential estimation.

(2) Time-Division Multiple Access

In TDMA, the transmissions in different links do not overlap in the repeater. Each link is assigned exclusive use of the repeater during specified time slots. In this manner, the signals pass through the repeater with virtually no interaction. An access channel in TDMA designates a particular

sequence of time slots. The simplest such sequence would provide that the time slots of a channel occur periodically at a definite repetition frequency which may be termed the frame repetition rate of the system. All other channels in this arrangement would have the same frame rate but different times of occurrence, and possibly different slot durations.

If one signal emitted by the satellite is designated as a time reference, or satellite clock, the signals can be kept separated in time and can be kept in a specified sequence by maintaining specified time intervals relative to the satellite clock. This procedure does not require a given transmitting station to consider the path delay of any other station. The problem of synchronization [18, 24] however, represents a fundamental limitation in a TDMA system.

TDMA can theoretically provide the highest channel capacity because, in theory, the only inefficiency arises from the guard times between adjacent slots. No intermodulation products are developed within the satellite transponder;

therefore, the TWT can be operated at saturation without the 1.1 dB power loss occurring in FDMA, also, up-link power co-ordination is not critical. The frequency plan for TDMA is simplified since all earth stations can use the same transmit and receive frequencies*. The capacity of each earth station can be changed rapidly and easily by the reassignment of time slots and/or burst lengths to meet traffic variations, thus providing demand assignment capability. However, TDMA requires that analog message signals such as voice be transmitted in bursts or pulses, and this can necessitate analog-to-digital conversion. In most cases this conversion introduces inefficiencies, and the 'efficiency' of TDMA may, in effect, be lost. Also, equipment for high-speed high-capacity operation is expensive and not immediately available.

Takada and Doi [25] have documented a PCM TDMA Communication System (SMAX). Kutami et al [26] discuss channel control in such a system, and Inoue [27] proposes an acquisition scheme.

* Since each user requires the entire channel bandwidth to transmit the compressed information, there is no net spectrum saving over FDMA (guard bands in time simply replace guard bands in frequency).

(3) Pulse Address Multiple Access

PAMA techniques are many and varied, giving rise to systems with diverse properties. A technique can be devised with almost any single desirable property.

A pulse-address carrier waveform is a series of pulses within specific frequency bands. The pulses and the intervals between the pulses in a given carrier waveform constitute a distinct pattern, with the amount of time during which a pulse is present much less than the amount of time no pulse is present. Each waveform can be specified by dividing the time-frequency plane into a matrix. The time axis is divided into frames of length T , each of which is divided into N slots of length t_s . The frequency spectrum is divided into numbered bands of width B . B must be at least as great as the bandwidth of a pulse. An access channel specifies a distinct pattern of pulses in time and frequency, and is given by a sequence of numbers denoting the bands in successive time slots. If there is time slot synchronization without frame synchronization, a user could fill a satellite time frame with any cyclic shift of his basic pattern. When frame

synchronization is acquired at both transmitter and receiver, this redundancy in a particular access channel can be utilized to send message information. Using pulse-position modulation (PPM), for example, the pattern 2,0,0,3,0,1,.....,3,4 could convey a "mark" and the pattern 4,2,0,0,3,0,1,.....,3 could convey a "space". There are many other possible forms of message modulation, not all of which require network timing.*

A link between two users is established in the following manner. Each receiver is "tuned" to respond to a waveform with a particular pattern. When the waveform is acquired, a confirmation is transmitted on another waveform selected for answering. PAMA systems are of interest because they allow a link between user pairs to be established in this convenient manner (i.e. like dialling a telephone). They provide a large number of waveforms to be permanently assigned to individual users so that the wave shapes themselves serve as addresses. Because of the convenient access they provide, some PAMA systems are known as random-access discrete-address (RADA) systems [28].

* For example, with binary on-off signalling, frame synchronization is not necessary. Even without time slot synchronization (implying a non-orthogonal multiple access system), with the given cyclic shift redundancy, Schwartz et al [1] have stated there is no more than a 3 dB degradation from an orthogonal system.

PAMA systems take so many different forms that it is difficult to list advantages and disadvantages valid for all of them. One disadvantage they all seem to share is low data rate. TDMA is a special kind of pulse technique in which each time slot is assigned exclusively to one access channel. This is possible only with network timing, and it limits the number of access channels that can be assigned on a permanent basis. The relatively low data rates attained by PAMA techniques arises from two causes: interference among signals transmitted to the satellite and unused portions of the time frequency plane. A PAMA system which can serve as an example has been analyzed by Reiffen [29].

The wide-band limiting repeater permits a wide choice of modulation systems suitable for multiple access. Characteristics, advantages, and disadvantages of each of the four classes of techniques have been discussed by Schwartz et al [15]. We note that Blasbalg [30] has compared pseudo-noise and conventional modulation for multiple-access satellite communications. Jelinek [31] has analyzed the performances of TDMA, FDMA, and SSMA for double access to an active satellite. For constant bandwidth digital communication, he has concluded that for sufficiently

high SNR's, SSMA signalling is the most efficient. Wittman [32] has presented an analysis of the PAMA technique which takes into account the simultaneous effects of RF phase cancellation*and up-link and down-link noise. Baghdady [33] has studied the essential technical problem of spectrum sharing. The multiple-access problem has been the subject of much attention at the Institute for Defense Analyses. Invaluable contributions were provided by the IDA Multiple-Access Summer Study of July, 1964. The papers written at the study are available ([2], [29]). Werth [34] has reviewed the concept of demand assignment and presents the technical details of a PCM FDMA Demand Assignment (SPADE) System developed by Comsat Laboratories on behalf of Intelsat.

3.3 The Important Parameters in FDMA

In designing and using an FDMA satellite communications network one must consider:

- (a) method of accessing a satellite channel,
- (b) frequency band allocation,
- (c) performance of ground station equipment,
- (d) preassigned TWT operating point,
- (e) linear channel distortion,
- (f) effects of nonlinearities,
- (g) adjacent channel interference,

* Wittman has considered a lightly loaded non-orthogonal PAMA system in which a time-frequency cell might be occupied by more than one user.

(h) satellite transponder leakage,

(i) stochastic channel parameters.

In this section we discuss briefly each of these topics, at the same time defining parameters pertaining to an overall system specification.

(a) Preassigned access or demand access methods can be employed to control the division of satellite channel capacity among subscribers. In a demand access scheme a user must locate an available frequency band at message time. After the message is complete the band becomes available to someone else. Such a scheme might require a ground control station (operating like a telephone exchange) to handle and monitor the continuous allocation of carrier frequencies. Alternatively, the individual user could determine an open channel by means of an energy per signal bandwidth detector, or by consulting a common signalling channel [34].* There may be an optimum utilization of bands for given traffic which minimizes the level of resulting in-band IM products. This principle, advantageous when the total band is unfilled, could be implemented in a demand access scheme. In terms of bandwidth use, demand access is a more efficient access method; however, it requires more sophisticated terminal equipment and operation.

* The SPADE system uses a demand-assignment signalling and switching unit for self-assignment of channels based on continually updated channel-allocation status data provided via the common signalling channel.

In a preassigned access scheme, a subscriber receives on a permanent basis a particular frequency band. Since he will not transmit continuously, the satellite will be operating below capacity most of the time. However, a tuneable transmitter is not required, and the communication facility is always available without the necessity of the accessing procedure. In a thin route widely dispersed network the efficiencies of a demand access scheme make it preferable, although in very remote areas the overriding requirement of simplicity might dictate a fixed access/party line solution. For a heavy route link between heavily populated centres (i.e. east-west trunk lines), pre-assigned access is perfectly adequate. In this case the demand accessing is handled at the voice multiplexing stage, while the satellite channel itself is used almost continuously.

(b) In choosing a frequency band for the FDMA system one must consider:

- (i) traffic (i.e. bandwidth) requirement,
- (ii) existing use made of the frequency spectrum,
- (iii) cost and performance of equipment versus frequency, and
- (iv) channel characteristics versus frequency.

Of course, the FDMA signals should not interfere with microwave transmissions and other communications facilities. Given that this condition is satisfied, the choice is dependent on (i), (iii) and (iv). At frequencies above 10 GHz, bandwidths can be chosen

to satisfy any conceivable traffic requirement. However, the antenna tolerances decrease and the transmitted signals experience more air attenuation at these higher frequencies.

(c) Ground station design/antenna engineering and satellite control are required to implement a satellite communications system. There must be an on-board control facility to maintain the correct satellite attitude and geostationary orbit*. Antennas are required on the ground and in the satellite to transmit and receive the electromagnetic energy which constitutes the desired signals. Satellite antenna and control engineering encompass a vast technology in themselves, and a general discussion of the problems encountered in these areas is not undertaken. However, there is one aspect of the antenna specification that pertains to the FDMA communications system model of figure 2; so, a comment on this is pertinent.

The up and down path carrier to thermal noise power ratios (CNR's) in our model are intrinsically determined by the gain to noise temperature ratios (G/T's) of the satellite and ground receive equipment. The G/T is a figure of merit for an antenna; it is a measure of how cleanly the antenna

* The purpose of satellite control is to maintain a "target" (i.e. the satellite antenna) which remains fixed in position and direction (attitude). To accomplish this, periodic corrections to the orbit and orientation of the satellite are required as neither are perfectly stable. Although it has been designed carefully for dynamic stability, the satellite antenna may tend to rotate away from its critical direction. Due to the changing gravitational pull of the sun and moon, and solar torque, there will also be a slow precession away from synchronous orbit.

receives signals. With T in $^{\circ}$ Kelvin, K = Boltzmann's constant, we recall that a 1Ω resistor is a source of white gaussian noise with single sided spectral density

$$N_o = KT \frac{\text{watts}}{\text{Hz}} ,$$

or power in a bandwidth B of

$$N = KTB \text{ watts}$$

Specifying the antenna noise temperature and a signal bandwidth, then, is equivalent to specifying the power of the in-band additive gaussian noise present. In fact, in a system description, all sources of thermal noise on the up and down paths are included in the effective noise temperatures of the satellite and ground receive antennas respectively. Noise on the satellite channel generally results from:

- (1) resistive elements in the antennas, waveguides, and R-L-C circuits on the ground and in the satellite, and also noisy active devices (power supplies, etc.),
- (2) galactic (the earth, sun, moon, and stars) electromagnetic energy falling directly into the antenna beam path,
- (3) energy from the earth and sky coupled into the ground antenna through its sidelobes - this increases as the angle of antenna elevation decreases.

All sources of noise up to the satellite amplifiers are referred to the noise temperature of the satellite antenna, and constitute "up path thermal noise". Succeeding noise sources contribute to the "down path thermal noise", and are included in the T of the receive antenna.

The gain of an antenna (either transmitting or receiving) is a measure of its beam directing or focussing capability. It is defined by the equation

$$G = \frac{\text{peak power on axis received by antenna}}{\text{power received by a hypothetical isotropic antenna}}$$

An isotropic antenna is one with a spherical pattern (i.e. no directional orientation whatsoever). Knowing the transmitted power P , and the gain G_t of the ground transmit antenna, the flux density of the beam received at the satellite can be calculated:

$$\phi = \frac{P G_t}{4 R_u^2}$$

Here R_u is the distance from the transmit antenna to the satellite. The gain of the ground receive antenna, G_r , can be used to compute the received signal power given the power radiated from the satellite and the satellite antenna gain.

Avoiding detailed calculations, it is evident that the G/T of an antenna is a measure of its performance. From the receiver G/T 's, up and down path CNR's for particular FWT

operating points (i.e. transmitted power levels) and signal bandwidths can be calculated. Henceforth, in the model of figure 2 it is assumed that the up and down path thermal noise powers and the powers of the signals received at the satellite are given.

(d) In an FDMA communications system there must be coordination of transmitted powers for two reasons:

- (1) so that each user gets his allotted share of effective amplification in the satellite (recall the power sharing in a limiting non-linearity),
- (2) so that the TWT is loaded to a pre-assigned operating point which gives optimum system performance with respect to intermodulation product interference and up and down path thermal noise (reference 6.1).

(e) Equalization of the transmitted FM signals may be required to compensate for group delay distortion imposed on the signals by bandpass filtering in the satellite. Constant gain/linear phase shift frequency characteristics may not be attainable under the weight and complexity (circuits cannot be serviced) constraint imposed on the satellite transponder.

In any case, if distortionless communication can be achieved with simple circuits in the satellite plus equalization on the ground, this is a more economical solution than having ideal satellite filters.

Equalization must be done at the transmitter rather than the receiver. Otherwise the input signals to the nonlinear satellite TWT would have linear amplitude distortion (i.e. amplitude of signal varies with its instantaneous frequency according to the gain vs frequency of previous filters); this AM would result in further amplitude and phase distortion which couldn't be equalized at the receiver. With transmitter equalization, however, the input to the nonlinearity consists of the desired constant amplitude signals, and this problem is avoided*.

(f) The nonlinear effect of the satellite TWT on a multiple FM signal input is considered in detail in the thesis. The AM/AM and AM/PM nonlinearities introduce intermodulation products which at best only subtract from the output power available to the desired signals, and at worst constitute noise in band with the signals. In addition, the relative signal

* Equalization at the receiver might still be required for the zonal filter following the TWT. In general, to avoid nonlinear effects on the equalization, pre-TWT elements are equalized at the transmitter, post-TWT elements are equalized at the receiver.

powers change through the nonlinearity, the change generally favouring the strong signal at the expense of the weak one (there is one notable exception to this — Section 5.4).

(g) Co-channel interference is the result of a lack of perfect orthogonality among the signals sharing the multiple access channel. In the case of FDMA, a user transmits a small proportion of energy outside his allotted frequency band*, and imposes co-channel interference on another signal. To reduce co-channel interference, guard bands may be inserted between the signal bands; however, this lowers the total capacity of the channel. In FDMA system design, one must therefore consider the trade-off between maximizing capacity and minimizing adjacent channel interference. Wachs [35] and Wong [36] present interesting analyses of the effects of adjacent channel interference in a multicarrier system.

(h) As has been pointed out (see footnote on p.), there may be several transponders in a satellite. A decision as to the particular number of transponders and bandwidth spacing between them is partly influenced by an undesired effect called transponder leakage. If the transponders operated independently of one another, we could consider each FDMA network (with its own transponder) separately. However,

* A constant amplitude FM signal cannot be perfectly band-limited.

since they make common use of:

- (1) the satellite antenna,
- (2) preamplifiers (non-saturating), and
- (3) frequency translating circuitry,

the FDMA networks themselves are not orthogonal, and there is some interaction between them. Any input to a particular transponder's TWT will have an undesired component resulting from the presence of the adjacent FDMA system, and this is transponder leakage. Leakage often results from the composite transmitted signal being fed back into the input.* Since the transmitted signal might be 100 dB or more stronger than the received signal, only a little feedback is required to degrade the system significantly.

(i) In considering the hardware of a communications satellite, questions such as system reliability, redundancy, lifetime, and backup must be considered. The satellite, unlike the ground stations, cannot be serviced. Regarding only the communications equipment in the satellite, the FDMA systems will continue to function as long as each transponder operates. A failure may disrupt the entire repeating capability of the satellite, or eliminate only a single transponder. Parameters changing with time can degrade performance. A reliability analysis must be undertaken to determine expectations and confidence limits for the lifetime performance of the repeater.

* The signal out of the satellite is frequency translated away from the input band. Theoretically, then, any output component coupled back to the input is removed by the satellite prefilters. Nonlinearities, which can shift energy to new frequencies, and non-brickwall prefiltering, mean that in practice there will be feedback.

In addition to being sensitive to the satellite hardware elements, the performance of an FDMA network at a given time is also dependent on geophysical conditions (season, weather, etc.). A severe rain storm, for instance, may attenuate considerably signals transmitted from a ground antenna. In conclusion, it is important to remember that the FDMA channel is itself stochastic.

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CHAPTER 4

METHODS OF NONLINEAR ANALYSIS

In this chapter, methods of analyzing the effect of nonlinear devices on multi-carrier inputs are presented. These are described as Fourier series, autocorrelation function and stochastic methods. The stochastic method constitutes the basis for original work developed in the thesis.

4.1 Fourier Series Method

In the Fourier series approach, the nonlinear transfer function is represented by an odd periodic Fourier series expansion. Then, for a multi-carrier input, the output can be written as a double Fourier series. 'Diagonal' terms constitute signal components of the output, the remainder consists of intermodulation product terms.

Following the notation of Campanella [1], the signal entering the amplitude nonlinearity is given by

$$e_s = \sum_{j=1}^n A_j \sin(\omega_j t + \theta_j) \quad (4.1.1)$$

The normalized amplitude transfer function of the device is

$$\frac{e_o}{E_m} = g(x) \quad (4.1.2)$$

where

$$x = \frac{e_s}{E_s}.$$

Here, e_s and e_o are the instantaneous input and output amplitudes; E_s and E_m are input and output amplitudes at saturation. The transfer function can be represented by the odd periodic Fourier series expansion:

$$\frac{e_o}{E_m} = \sum_{\ell=1}^{\infty} b_{\ell} \sin\left(\frac{\ell\pi}{4} x\right) \quad (4.1.3)$$

In this expression the fundamental period of the series is 8.

The coefficients b_{ℓ} are given by the relation

$$b_{\ell} = \frac{1}{4} \int_{-4}^4 g(x) \sin\left(\frac{\ell\pi}{4} x\right) dx \quad (4.1.4)$$

For typical TWT's used in satellite communications, Berman [2] has suggested that five coefficients are sufficient to represent the amplitude characteristic. Values obtained by Berman are:

$$b_1 = 1, \quad b_2 = .183, \quad b_3 = .222, \quad b_4 = 0, \quad b_5 = .100$$

Substituting the composite multichannel signal, given by 4.1.1, into 4.1.3 results in the following expression for the instantaneous output of the TWT:

* As defined in 2.4.10, these are somewhat arbitrary.

$$\frac{e_o}{E_m} = \sum_{\ell=1}^{\infty} b_{\ell} \sin \left[\frac{\ell \pi}{4} \left(\sum_{j=1}^n \frac{A_j}{E_s} \sin (\omega_j t + \phi_j) \right) \right] \quad (4.1.5)$$

Using the identity [3, p. 321]

$$e^{j \xi \sin \gamma} = \sum_{n=-\infty}^{\infty} J_n (\xi) e^{jn\gamma} \quad (4.1.6)$$

the relation for the instantaneous TWT output becomes

$$\frac{e_o}{E_m} = \sum_{\ell=1}^{\infty} b_{\ell} \left[\sum_{(n,j)} J_n \left(\frac{\ell \pi}{4} \frac{A_j}{E_s} \right) \sin (\omega_k t + \phi_k) \right] \quad (4.1.7)$$

where

$$(n,j) = (a,1); (\beta,2); (\gamma,3); \dots$$

$$\omega_k = a\omega_1 + \beta\omega_2 + \gamma\omega_3 + \dots \quad (4.1.7a)$$

$$\theta_k = a\theta_1 + \beta\theta_2 + \gamma\theta_3 + \dots \quad (4.1.7b)$$

In this expression, a two dimensional index (n,j) is used. As an example, for carrier $j=1$, the value of n is some integer a ; for $j=2$, it is some integer β . These integers are the harmonics of carrier frequencies that take part in the generation of the component at frequency ω_k . For this reason, the values $a, \beta, \gamma \dots$ etc. are called the harmonic numbers. Each may take on either positive or negative integer values. The distortion component that results falls at frequency ω_k which is defined in equation (4.1.7a).

The instantaneous phase of the distortion component is given by ϕ_k of equation (4.1.7b). This is the result of the products of the carriers involved. The spectrum shape of such components can be obtained by convolution of the power spectra involved, or by the Fourier transformation of the product of the autocorrelations. Because of the constraints resulting from the fact that the channels are located at frequencies much greater than the separation among adjacent channels, only odd order interactions will produce disturbing in-band distortion components. Third order products predominate, all higher order terms making little additional contribution. Third order products are of two types. Type I results from harmonic number combinations in which all values of n are zero except three, these being 1, 1, -1. Type II results from harmonic number combinations in which two are non-zero, these being 2 and -1 respectively.

Campanella [1] has developed relations that permit the computation, in closed form, of the intermodulation distortion power to carrier power ratio, I/C , as a function of output back-off ratio for an FDMA PSK system. The resulting relations are exact but are rather cumbersome to apply.

An example in which both the Fourier series method and some probability theory are applied is now presented. We derive an expression for the suppression factor (reference 3.1) when a constant amplitude signal in the presence of other constant amplitude signals and stationary gaussian noise is passed through a hard limiting repeater. The result, developed by Lyons [5], will be used in Chapter 5.

The suppression factor on a particular signal is a measure of the change in proportion of the total power that signal has experienced when passed through a nonlinearity. It is defined by the equation

$$\alpha = \frac{\text{Output signal power}}{\text{Total output power}} / \frac{\text{Input signal power}}{\text{Total input power}} \quad (4.1.8)$$

The input consists of a desired signal,

$$s_D(t) = \sqrt{2P} \cos (\omega_D t + \theta_D(t)),$$

and interference

$$I(t) = \sum_{i=1}^m \sqrt{2P_i} \cos (\omega_i t + \theta_i(t)) + n(t)$$

Here $n(t)$ is gaussian noise of power (statistical) σ^2 .

Now, the limiter transfer function can be described by

$$\operatorname{sgn}(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin xu}{u} du \quad (4.1.9),$$

and applying this formula, we can write the instantaneous output from the hard limiter in the form

$$\begin{aligned} y(t) &= \operatorname{sgn}(x(t)) \\ &= \frac{2}{\pi} \int_0^{\infty} \operatorname{Im} \left\{ e^{ju \sqrt{2P} \cos(\omega_D t + \phi_D(t)) + I(t)} \right\} \frac{du}{u} \end{aligned}$$

From Watson [4, p. 22 ,

$$e^{j b \cos t} = \sum_{n=0}^{\infty} j^n \epsilon_n J_n(b) \cos nt,$$

where $\{\epsilon_n\}$ are the Newmann numbers defined by

$$\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0 \end{cases}$$

Therefore, the unfiltered output from the hard limiter can be written as

$$y(t) = \frac{2}{\pi} \int_0^{\infty} \operatorname{Im} \left\{ e^{juI(t)} \sum_{n=0}^{\infty} j^n \epsilon_n J_n(\sqrt{2P} u) \cos(n(\omega_D t + \phi_D(t))) \right\} \frac{du}{u}$$

The constant amplitude first harmonic component of $y(t)$ having the phase characteristic of $S_D(t)$ is the desired signal out of the bandpass limiter. This is given by

$$S'_D(t) = \left\langle \frac{2}{\pi} \int_0^\infty \cos(uI(t)) \cdot 2J_1(\sqrt{2P} u) \frac{du}{u} \right\rangle \cos(\omega_D t + \theta_D(t))$$

Making an ergodic assumption on $I(t)$, we write

$$S'_D(t) = E \left\{ \frac{2}{\pi} \int_0^\infty \cos(uI) \cdot 2J_1(\sqrt{2P} u) \frac{du}{u} \right\} \cos(\omega_D t + \theta_D(t))$$

(4.1.10),

where I is here regarded as a random variable defined by

$$I = n + \sum_{k=1}^m \sqrt{2P_k} \cos \alpha_k$$

with $n \sim N(0, \sigma^2)$ and $\{\alpha_k\}$ uniform and independent.

Now,

$$\begin{aligned} & E \left\{ \int_0^\infty \cos(uI) \frac{J_1(\sqrt{2P} u)}{u} du \right\} \\ &= \int_0^\infty \operatorname{Re} \left[E \left\{ e^{jIu} \right\} \right] \frac{J_1(\sqrt{2P} u)}{u} du, \end{aligned}$$

and from the mutual independence of signal and noise terms,

$$E \left\{ e^{jIu} \right\} = E \left\{ e^{jnu} \right\} \prod_{k=1}^m E \left\{ e^{j \sqrt{2P_k} \cos \alpha_k u} \right\}$$

The characteristic function of a gaussian random variable with zero mean and variance σ^2 is given by

$$E \left\{ e^{jnu} \right\} = e^{-\frac{1}{2} \sigma^2 u^2}$$

Also,

$$E \left\{ e^{j \sqrt{2P_k} \cos \alpha_k u} \right\} = \frac{1}{\pi} \int_0^\pi e^{j \sqrt{2P_k} \cos \alpha_k u} d\alpha_k$$

$$= J_0(\sqrt{2P_k} u)$$

Hence, the characteristic function of the interference process reduces to

$$E \left\{ e^{jIu} \right\} = e^{-\frac{1}{2} \sigma^2 u^2} \prod_{k=1}^m J_0(\sqrt{2P_k} u)$$

Substituting into 4.1.10 gives the result

$$S_D'(t) = \left[\frac{4}{\pi} \int_0^\infty \frac{J_1(\sqrt{2P} u)}{u} \prod_{k=1}^m J_0(\sqrt{2P_k} u) e^{-\frac{1}{2} \sigma^2 u^2} du \right]$$

$$\cos(\omega_D t + \phi_D(t))$$

Shaft [9, chapter 3] has shown that the zonal filter removes $1 - \frac{8}{\pi^2}$ of the total power out of the hard limiter. Normalizing for a resultant amplitude of unity out of the repeater gives the expression

$$S'_D(t) = \left[\int_0^\infty \frac{J_1(\sqrt{2P} u)}{u} \prod_{k=1}^m J_0(\sqrt{2P_k} u) e^{-\frac{1}{2} \sum_{k=1}^m P_k u^2} du \right] \cos(\omega_D t + \phi_D(t)) \quad (4.1.11)$$

The desired suppression factor, then, is written as

$$\alpha = \left(1 + \frac{\sigma^2 + \sum_{k=1}^m P_k}{P} \right) \left[\int_0^\infty \frac{J_1(\sqrt{2P} u)}{u} \prod_{k=1}^m J_0(\sqrt{2P_k} u) e^{-\frac{1}{2} \sum_{k=1}^m P_k u^2} du \right]^2 \quad (4.1.12)$$

The use of the Fourier series approach in analyzing intermodulation and suppression effects in an amplitude nonlinearity has been demonstrated. Berman and Mahle [6, chapter 2] have also employed a Fourier series approach to analyze the effect of AM to PM conversion on the sum of p carriers. Omitting a detailed development, they derive the equation

$$V_{out} = \sum_{j=1}^p S_j \prod_{m=1}^\infty \prod_{\ell=1}^{p-1} \prod_{k=\ell+1}^p J_{n_{m\ell k}}(a_{m\ell k}) \cos \left[\omega_j + \sum_{m=1}^\infty \sum_{\ell=1}^{p-1} \sum_{k=\ell+1}^p n_{m\ell k} (\omega_\ell - \omega_k) \right] t \quad (4.1.13)$$

for the output of the AM/PM characteristic specified by equation 2.5.1 Here S_i are the signal amplitudes and

$$a_{m\ell k} = 2K_1 e^{-K_2 \sum_{i=1}^P S_i^2} I_m(2K_2 S_\ell S_k) + \begin{cases} 2K_3 S_\ell S_k & m = 1 \\ 0 & m \neq 1 \end{cases}$$

Equation 4.1.13 can be used to determine the intermodulation noise power density. It bears resemblance in form to equation 4.1.7, a Fourier series expression for the output from an amplitude nonlinearity.

4.2 Autocorrelation Function Method

The Fourier series method outlined in 4.1 represents the instantaneous output from a nonlinear device as a function of the instantaneous input. Generating the intermodulation product line spectrum is performed by simply plotting $1/2$ the squares of the Fourier coefficients vs their corresponding IM frequencies. Phase relations among the IM products and the various signals are also evident from a Fourier expansion.

The autocorrelation or second moment function approach regards the input and output of a nonlinearity as stationary random processes. The output process contains spectral components due to the presence of individual signals at the input ($S_i \times S_i$ or signal components), and components dependent on combinations

of signals ($S_i \times S_j$ or intermodulation product components). By suitably expanding the autocorrelation function of the total output and Fourier transforming each term, the power spectra of the output signals and the IM products can be obtained. This result provides signal suppression factors and signal to distortion* power ratios.

If the input to the baseband nonlinear device $g(x)$ (assumed here to be an odd function) is a sine wave plus gaussian noise,

$$x(t) = \sqrt{2P} \cos(\omega_c t) + n(t),$$

Rice [6] and Middleton [7] have shown that the autocorrelation function of the output can be expressed as

$$R_y(\tau) = \sum_{m=0}^{\infty} \sum_{\substack{k=0 \\ (m+k \text{ odd})}}^{\infty} \frac{\epsilon_m h_{mk}^2}{k!} (R_n(\tau))^k \cos m\omega_c \tau \quad (4.2.1)$$

where

$$\begin{aligned} \epsilon_0 &= 1 \\ \epsilon_m &= 2 \quad (m \geq 1), \end{aligned}$$

$R_n(\tau)$ is the autocorrelation function of the input noise, and h_{mk} is a number defined by $(f = T[g])$

$$h_{mk} = \frac{j^{m+k}}{2\pi} \int_{-\infty}^{\infty} f(ju) e^{-\frac{\sigma^2 u^2}{2}} u^k J_m(u\sqrt{2P}) du \quad (4.2.2)$$

* Distortion is defined to be intermodulation product interference lying in band with a signal.

Equation 4.2.1 can be expanded as follows:

$$\begin{aligned}
 R_y(\tau) = & 2 \sum_{\substack{m=1 \\ (m \text{ odd})}}^{\infty} h_{m0}^2 \cos m\omega_c \tau + \sum_{\substack{k=1 \\ (k \text{ odd})}}^{\infty} \frac{h_{0k}^2}{k!} (R_n(\tau))^k \\
 & + 2 \sum_{\substack{m=1 \\ (m+k \text{ odd})}}^{\infty} \sum_{k=1}^{\infty} \frac{h_{mk}^2}{k!} (R_n(\tau))^k \cos m\omega_c \tau
 \end{aligned} \tag{4.2.3}$$

The first set of terms in 4.2.3 is periodic and consists of the signal output terms representing the interaction of the input signal with itself (s x s terms). The remaining terms are the limiter output noise terms. The second set represents the interaction of the input noise with itself (n x n terms), while the last set represents the interaction of the input signal and noise (s x n terms).

When the input consists of ℓ signals and noise, the autocorrelation function of the output can be written as an $\ell + 1$ - fold sum

$$\begin{aligned}
 R_y(\tau) = & \sum_{k=0}^{\infty} \frac{(R_n(\tau))^k}{k!} \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots \sum_{m_\ell=0}^{\infty} \epsilon_{m_1} \epsilon_{m_2} \cdots \epsilon_{m_\ell} \\
 & h_{km_1 m_2 \dots m_\ell}^2 R_x(m_1) R_x(m_2) \dots R_x(m_\ell),
 \end{aligned} \tag{4.2.4}$$

where $R_x(m_i)$ is the input autocorrelation function of the normalized i -th signal whose carrier frequency has been multiplied by m_i . Shaft [9, chapter 3] has explained and made extensive use of this representation in examining the effect of limiting several signals and noise. Sevy [8, chapter 5] employs 4.2.1 for analyzing cases of a sinusoidal, gaussian, or biphase angle modulated signal and noise.

Blachman [8] has defined $s \times s$, $s \times n$, and $n \times n$ output components in a simple manner agreeing with intuition. Basically, he states that the spectral characteristic of a particular signal in the output is unaffected, except possibly in total area, (i.e. power) by the presence of other signals in the input. The power spectrum of the intermodulation process is obtained by subtracting the spectra of the signal (i.e. $S_i \times S_i$) components from the total output power spectrum.

For example, if the input to a hard limiter* is a zero mean gaussian process

$$x(t) = s(t) + n(t),$$

with
$$R_x(\tau) = R_s(\tau) + R_n(\tau),$$

* Here we are not referring to the bandpass limiter (which includes a post zonal filter - see 2.3.1); in 4.2.5, $R_y(\tau)$ includes the harmonic components of the output. This is also the case for 4.2.3 and 4.2.4.

the output autocorrelation function ([9], also equation (1), p. 25) is

$$R_y(\tau) = \frac{2}{\pi} \sin^{-1} \left(\frac{R_s(\tau) + R_n(\tau)}{R_s(o) + R_n(o)} \right) \quad (4.2.5)$$

Here we have only a single term, depending on both $R_s(\tau)$ and $R_n(\tau)$. Expanding the result in powers of $R_s(\tau) + R_n(\tau)$ by means of the Binomial Theorem, we then have a double power series whose terms can be separated into four types [7, p. 249]: dc, s x s, s x n, n x n.

Since $\sin^{-1}(0) = 0$, there is no constant term and hence no dc component. Combining all the terms that involve only $R_s(\tau)$, the s x s and n x n parts of $R_x(\tau)$ reduce to

$$R_{s \times s}(\tau) = \frac{2}{\pi} \sin^{-1} \left(\frac{R_s(\tau)}{R_s(o) + R_n(o)} \right)$$

and
$$R_{n \times n}(\tau) = \frac{2}{\pi} \sin^{-1} \left(\frac{R_n(\tau)}{R_s(o) + R_n(o)} \right)$$

The s x n term, from 4.2.5 and the above, is necessarily

$$R_{s \times n}(\tau) = \frac{2}{\pi} \left[\sin^{-1} \left(\frac{R_s(\tau) + R_n(\tau)}{R_s(o) + R_n(o)} \right) - \sin^{-1} \left(\frac{R_s(\tau)}{R_s(o) + R_n(o)} \right) - \sin^{-1} \left(\frac{R_n(\tau)}{R_s(o) + R_n(o)} \right) \right] \quad (4.2.6)$$

We note that $R_{s \times s}(\tau)$ and $R_{n \times n}(\tau)$ are the autocorrelation functions of $s(t)$ and $n(t)$ when passed through the nonlinearities $(1 + R_n(o)/R_s(o))^{1/2} \text{sgn}(x)$ and $(1 + R_s(o)/R_n(o))^{-1/2} \text{sgn}(x)$ respectively. The power, but not the spectral characteristic, of the $s \times s$ process is affected by the presence of $n(t)$, and conversely.

Blachman's approach is useful and provides a clear meaning. The 'back door' method of obtaining $R_{s \times n}(\tau)$ (4.2.6) is similar to the new IM product analysis described in section 4.3. In the stochastic approach, the instantaneous IM phasor closes the polygon with sides the resultant output and the signal components from the nonlinearity. This avoids the necessity of a direct attack on individual intermodulation product terms.

4.3 Stochastic Method †

The limitations and complexity in applying and relating the methods of 4.1 and 4.2 to communications through bandpass nonlinearities is considered strong motivation for the development of a clearer approach. The stochastic method, hopefully demonstrated throughout the thesis as being conceptually simple, totally general, and easily implemented, leads to the derivation of original results (chapters 7 - 11).

† This method is applicable only to bandpass nonlinearities; it is implicitly assumed that there is no DC component in the output.

We suppose that the input to a general bandpass non-linearity is given by a deterministic component (the desired signal),

$$S_D(t) = \sqrt{2P} \cos(\omega_D t + \theta_D(t)) \quad (4.3.1),$$

plus a random component (the interference),

$$I(t) = R(t) \cos(\omega_0 t + p(t)) \quad (4.3.2)$$

$I(t)$ is a stationary and band-limited process, but otherwise remains unspecified.

The deterministic part of the bandpass nonlinearity output is defined as the expected value of the output over the statistics of the random input component (given by 4.3.2). The remaining output, the random component, bears no information about $\theta_D(t)$, the modulating waveform under consideration. The random-deterministic point of view at input and output is basic to the stochastic method; it leads to a definition of signal and noise components from any nonlinearity consistent with previous definitions*.

The complete input to the nonlinear device $\vec{f} = (f_1, f_2)$ is given by

* The problem of defining signal and noise components out of nonlinear devices is a rather classical one. Generally, any component of the output dependent on the presence of noise at the input is regarded totally as noise.

$$\begin{aligned}
 x(t) &= S_D(t) + I(t) \\
 &= E(t) \cos(\omega t + \theta(t));
 \end{aligned}$$

the output is then (reference 2.2).

$$y(t) = f_1(E(t)) \cos(\omega t + \theta(t) - f_2(E(t))) \quad (4.3.3)$$

From 4.3.1 and 4.3.2, the instantaneous input envelope can be written as:

$$E(t) = \sqrt{R^2(t) - 2\sqrt{2P} R(t) \cos \alpha(t) + 2P} \quad (4.3.4)$$

$$\text{where } \alpha(t) = \pi - \left[(\omega_D - \omega_0)t + \theta_D(t) - p(t) \right]$$

Substituting 4.3.4 into 4.3.3 gives the following complex representation of the output process as a function of the random variable R (the input interference envelope), and α (the phase of the interference relative to the desired signal);

$$\begin{aligned}
 \vec{y} &= f_1 \left(\sqrt{R^2 - 2\sqrt{2P} R \cos \alpha + 2P} \right) \times \\
 &\left\{ \cos \left[\tan^{-1} \left(\frac{R \sin \alpha}{\sqrt{2P} - R \cos \alpha} \right) - f_2 \left(\sqrt{R^2 - 2\sqrt{2P} R \cos \alpha + 2P} \right) \right] \right. \\
 &\left. + j \sin \left[\tan^{-1} \left(\frac{R \sin \alpha}{\sqrt{2P} - R \cos \alpha} \right) - f_2 \left(\sqrt{R^2 - 2\sqrt{2P} R \cos \alpha + 2P} \right) \right] \right\}
 \end{aligned} \quad (4.3.5)$$

The first term in 4.3.5 is the component of the output in phase with the input desired signal, the second term is out of phase.

$\vec{y}$ is a random variable with mean

$$\begin{aligned}
 E\{\vec{y}\} &= \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} (y_1(r, \alpha) + j y_2(r, \alpha)) f_R(r) d\alpha dr \\
 &= E\{y_1\} + j E\{y_2\} \\
 &= \sqrt{E^2\{y_1\} + E^2\{y_2\}} \left/ \tan^{-1} \left(\frac{E\{y_2\}}{E\{y_1\}} \right) \right.
 \end{aligned} \tag{4.3.6}$$

where $y_1(R, \alpha)$ and $y_2(R, \alpha)$ are the output processes in and out of phase with $S_D(t)$, $f_R(r)$ is the probability density function of R , and α has been assumed uniformly distributed on $[0, 2\pi)$.

By definition, then, the deterministic signal out of the combined AM/AM and AM/PM nonlinearity is

$$S'_D(t) = \sqrt{2P'} \cos(\omega_D t + \theta_D(t) - \beta) \tag{4.3.7}$$

where

$$P' = \frac{1}{2} (E^2\{y_1\} + E^2\{y_2\}) \tag{4.3.8}$$

and

$$\beta = \tan^{-1} \left(\frac{E\{y_2\}}{E\{y_1\}} \right) \tag{4.3.9}$$

The random component of the output, from 4.3.3 and 4.3.7, becomes

$$\begin{aligned}
 I'(t) &= y(t) - S_D'(t) \\
 &= f_1(E(t)) \cos(\omega t + \theta(t) - f_2(E(t))) - \sqrt{2P_T} \cos(\omega_D t + \theta_D(t) - \beta) \\
 &= n_1'(t) \cos(\omega_D t + \theta_D(t) - \beta) + n_2'(t) \sin(\omega_D t + \theta_D(t) - \beta)
 \end{aligned}
 \tag{4.3.10}$$

Here $n_1'(t)$ is output interference in phase with the desired signal; $n_2'(t)$ is quadrature interference. The significance of such a representation will be demonstrated throughout the thesis.

Equations 4.3.7 and 4.3.10 define the deterministic and random components of the output. It remains to show that they are in fact the desired signal and interference out of the nonlinearity. Any part of the output that contains information about $\theta_D(t)$ is partly desired signal. With this criterion in mind, we observe from 4.3.7 that $S_D'(t)$ is angle modulated by a phase shifted replica of $\theta_D(t)$. Hence $S_D'(t)$ is an undistorted carrier of the desired information, and is defined to be totally desired signal. The important question is whether there is any recoverable information about $\theta_D(t)$ in the expression for $I'(t)$ given by 4.3.10. It will be shown that

$I'(t)$ and $e^{j(\omega_D t + \theta_D(t) - \beta)}$ are uncorrelated (but not necessarily independent) processes (see Appendix D). Therefore, although a linear processor cannot recover from $I'(t)$ an estimate of $\theta_D(t)$, a nonlinear one might.

In 4.3.10, $n_1'(t)$ and $n_2'(t)$ are zero mean baseband random processes which are stationary but not independent. If $I'(t)$ were additive gaussian noise, $n_1'(t)$ and $n_2'(t)$ would be independent equal variance gaussian random variables regardless of the choice of the orthogonal axes in the quadrature representation. In this case, as expected, $I'(t)$ could disclose no information about $\theta_D(t)$. However, here the properties of $n_1'(t)$ and $n_2'(t)$ are very dependent on a choice of axes quadrature to the desired signal out of the nonlinearity. The particular representation of 4.3.10 will be used to demonstrate that $I'(t)$, or in fact the component of $I'(t)$ resulting from intermodulation of $s_D(t)$ and $I(t)$, does contain useable information about $\theta_D(t)$.

The phase difference between $s_D'(t)$ and $I'(t)$, defined by

$$\gamma(t) = \tan^{-1} \left[\frac{n_2'(t)}{n_1'(t)} \right] \quad , \quad (4.3.11)$$

is a stationary random process with probability density function $f_Y(x)$. As $n_2'(t)$ and $n_1'(t)$ both have zero means, $f_Y(x)$ must satisfy the following equations:

$$\begin{aligned}
\int_{-\pi}^{-\pi/2} f_Y(x) dx &= P\{n'_1 < 0, n'_2 < 0\} \quad , \quad \int_{-\pi/2}^0 f_Y(x) dx = P\{n'_1 < 0, n'_2 > 0\} \quad , \\
\int_0^{\pi/2} f_Y(x) dx &= P\{n'_1 > 0, n'_2 > 0\} \quad \& \quad \int_{\pi/2}^{\pi} f_Y(x) dx = P\{n'_1 > 0, n'_2 < 0\} = \frac{1}{4} \quad .
\end{aligned}$$

Since, from 4.3.10, n'_1 and n'_2 as functions of P and ϕ (reference 4.3.4, 4.3.9) are not necessarily identically or independently distributed, γ is not generally uniformly distributed on $[-\pi, \pi)$.

As has been pointed out, $I'(t)$ consists of $s_D \times I$ and $I \times I$ components from the nonlinearity. Due to the $s_D \times I$ interaction term, $I'(t)$ and $\theta_D(t)$ are not independent; therefore $I'(t)$ can be used to estimate $\theta_D(t)$. At high signal-to-interference ratios (i.e. $I \times I$ component negligible in comparison with $s_D \times I$ component), $I'(t)$ may in fact be strongly coherent with $s_D(t)$ ($f_Y(x)$ may be peaked at angle(s) which tend to be the phase difference between the desired signal and interference out of the nonlinear device). So while the $s \times n$ component out of a nonlinear device has generally been and is here considered as noise, in some cases a decision theory oriented FM or PM demodulator could make use of it to improve performance. This principle is discussed further in section 4.4.

With $E(t)$ the resultant input envelope, the total power out of the nonlinearity is given by

$$\begin{aligned}
P_0 &= \frac{1}{2} \langle f_1^2(E(t)) \rangle \\
&= \frac{1}{2} E \{ f_1^2(E) \} \\
&= \frac{1}{2} \int_0^\infty f_1^2(r) f_E(r) dr
\end{aligned} \tag{4.3.12}$$

where $f_E(r)$, the probability density function of $E(t)$, is given by 4.3.16. The total interference power is then

$$\begin{aligned}
P_I &= P_0 - P' \\
&= \frac{1}{2} E \left\{ n_1'^2 \right\} + \frac{1}{2} E \left\{ n_2'^2 \right\}
\end{aligned} \tag{4.3.13}$$

When the interference at the input consists of the sum of m constant amplitude signals and gaussian noise

$$\text{i.e. } I(t) = \sum_{k=1}^m \sqrt{2P_k} \cos(\omega_k t + \theta_k(t)) + n(t), \tag{4.3.14}$$

Lyons [5] and others have shown that the interference and total input envelopes have the probability density functions

$$f_R(r) = r \int_0^\infty u J_0(ru) \prod_{k=1}^m J_0(\sqrt{2P_k} u) e^{-\frac{1}{2} \sigma^2 u^2} du \tag{4.3.15}$$

and

$$f_E(r) = r \int_0^\infty u J_0(ru) J_0(\sqrt{2P} u) \prod_{k=1}^m J_0(\sqrt{2P_k} u) e^{-\frac{1}{2} \sigma^2 u^2} du \quad (4.3.16)$$

respectively.

Substituting 4.3.15 into 4.3.6 will give expressions for the amplitude and phase shift (or differential delay) of $S_D'(t)$. By writing the input as a desired signal plus interference, the process can be repeated to determine the output amplitudes, $\{\sqrt{2P_i}\}$, and phase shifts, $\{\beta_i\}$, of each of the other m signals. The effective output interference process is then written as the resultant output minus the sum of the output signal components. First and second order properties of this process can be examined without the necessity of synthesizing individual IM product terms that have contributed to it.

The stochastic method, which represents the input to and output from a nonlinearity as the sum of a deterministic process (the signal) and a stationary stochastic process (the interference), has been introduced and described very briefly. In succeeding sections the method will be explained more fully and applied to solve problems. A discussion of its merits as a concept and as a tool will follow.

In resumé, the stochastic approach consists of:

- (1) writing the input to the nonlinearity as the sum of a desired signal and interference;
- (2) defining the desired signal out of the nonlinearity as the deterministic component of the output;
- (3) writing an expression for the amplitude and phase shift of the desired signal as a functional of the first order statistics of the interference;
- (4) writing the output interference process as the difference between the total output and the desired signal component.

4.4 Coherence of Intermodulation Interference

An important property of IM product interference not usually considered is its possible coherence with signal components of the output. The interference has a phase which is uncorrelated with the phase of a signal; for an amplitude nonlinearity the phase difference between the two is a symmetrically distributed random variable. However, the phase of the IM relative to a signal is not necessarily uniformly distributed.

For a concave nonlinearity, the IM product interference tends to be in phase with the larger of the two signals that have contributed to it; for a limiting nonlinearity it tends to be quadrature to the larger signal. In particular, Manasse et al [10] have pointed out that for high signal to noise ratios an envelope detector suppresses the component of noise out of phase with the signal, while a hard limiter suppresses the component of noise in phase with the signal. Although there is an SNR improvement of 3 dB, signal detectability in each case is degraded. This seemingly contradictory statement emphasizes the importance of the coherence problem considered in this section.

The stochastic method of analysis lends itself to a rather simple evaluation of the coherence properties of intermodulation interference. Recall from 4.3.11 that $\gamma(t)$ is the instantaneous phase of the total interference relative to the desired signal. As a random variable, γ is a function of R and α , the amplitude and phase of the input interference, and P , the power of the desired signal. As will be shown (e.g. 7.5-2), the probability density function of γ can be determined and it may have rather interesting properties. Computation of $f_\gamma(x)$ is in fact required to evaluate the ultimate performance of coherent/incoherent continuous wave

(CW)/digital signals transmitted through the satellite channel. For example, in CW angle modulation systems (Ref. 7.9) in phase noise can cause 180° phase reversals (clicks), but otherwise does not degrade the demodulated waveform. Quadrature noise, however, gives a continuous error in the received signal. In a binary coherent PSK system, (Ref. 7.8) in phase noise alone can cause errors; for multi-phase PSK systems, in phase noise causes flips to the opposite signal quadrant (or decision region), whereas quadrature noise introduces a rotation to adjacent quadrants. Further discussion of the implications of the in phase-out of phase interference representation on various forms of message modulation will be given.

We note that the autocorrelation function method destroys the phase information, and is of no value in analyzing the coherence problem. The Fourier method represents each IM product component, (i.e. output due to interaction of two particular signals) as a sum of terms (e.g., third order, fifth order etc. terms) each having its own distinct amplitude, frequency, and phase. The phase relation between the component and a signal is not easily obtainable from such an expression. The stochastic approach appears to be the most practical one in examining IM product coherence.

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CHAPTER 5

THE HARD LIMITING REPEATER

5.1 Introduction

The multiple-access hard limiting satellite repeater has been the subject of extensive study ([1], [2], [4]-[9], [11], [28], chapter 3; [5], chapter 4; [1]-[3]). The model chosen for analysis (figure 5) has been a reasonably accurate representation of early satellite transponders, and is mathematically tractable. However, as was pointed out in chapter 2, increases in satellite power led to bandwidth rather than power limited systems, and the optimum operating point is now backed off from saturation to reduce the level of intermodulation products (reference 3.1). The hard limiter no longer realistically models the satellite gain characteristic; and the relative effect of AM/PM conversion is more significant (and may in fact dominate the nonlinear behaviour of the TWT).

Figure 5

However, examination of the hard limiting repeater is both chronologically and analytically a first step in a study of the satellite repeater. As an extreme case, it serves as a simple example for nonlinear analysis, and demonstrates some peculiar effects of a nonlinearity. The results themselves are crude approximations of actual system performance.

The $\text{sgn}(x)$ nonlinearity has received much attention in contexts other than satellite communications. Cobb [4] has given a solution for the distribution of intervals between zero crossings of a sine wave plus random noise. Pitassi [5] derived closed form expressions for the partial derivatives with respect to the time delays of the fourth product moment

$$W(t_1, t_2, t_3) = E\{y_1(t) y_2(t + t_1) y_3(t + t_2) y_4(t + t_3)\}$$

when the $y_i(t)$ are infinitely clipped, zero mean, jointly gaussian processes. Doyle and Reed [6] have derived an approximate distribution for the envelope of a sine wave plus noise after passage through a limiter and narrow band filter. Davenport [7] has written a rather classical paper on signal to noise ratios in bandpass limiters. Sevy [8] extended his work in comparing the performance of amplitude and angle modulated signals plus noise through a hard limiter. Aein [9], [10] performed SNR and error rate calculations for coherent

hard limiting channels; Aien and Doyle [11] have considered cascaded limiters. Bello and Higgins [12] examined the effect of hard limiting on the probabilities of incoherent dismissal and false alarm at the output of an envelope detector, while Kak [13] has investigated the effect of infinite clipping on the information content of analog waveforms. Ward [14] and Wilson [15] have written papers on the effect of a hard limiter on signal spectra. Max [16] analyzes limiter output envelope fluctuations which result from the spectral overlapping of fundamental and harmonic terms (i.e. for wideband inputs). Additional bibliography is given by [8 - 10], chapter 4.

5.2 Stochastic Analysis

Equations developed in section 4.3 are here applied to the case of the hard limiter. The input is again

$$x(t) = s_D(t) + I(t)$$

where $s_D(t)$ (the desired signal) and $I(t)$ (the interference) are given by 4.3.1 and 4.3.14 respectively.

With

$$\begin{aligned} f_1(x) &= g(x) = \text{sgn}(x) \\ &= \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0 \end{cases} \end{aligned}$$

and $f_2(x) = 0,$

from 4.3.5 and 4.3.8, the output amplitude of the desired signal is

$$\sqrt{2P'} = \frac{1}{\pi} \int_0^\infty \left[\int_0^\pi \frac{(\sqrt{2P} - r \cos \alpha) d\alpha}{\sqrt{\sqrt{2P} - 2\sqrt{2P} r \cos \alpha + r^2}} \right] f_R(r) dr \quad (5.2.1)$$

From Gradshteyn and Ryzhik [17, p. 387],

$$\int_0^\pi \frac{dx}{\sqrt{1-2p \cos x + p^2}} = \begin{cases} 2 K(p) & p < 1 \\ \frac{2}{p} K\left(\frac{1}{p}\right) & p > 1 \end{cases}$$

and

$$\int_0^\pi \frac{p \cos x dx}{\sqrt{1-2p \cos x + p^2}} = \begin{cases} 2 K(p) - E(p) & p < 1 \\ 2p K\left(\frac{1}{p}\right) - E\left(\frac{1}{p}\right) & p > 1 \end{cases}$$

Here $K(z)$ and $E(z)$ are the complete elliptic integrals of the first and second kind respectively. Applying these results, equation 5.2.1 reduces to

$$\begin{aligned} \sqrt{2P'} &= \frac{2}{\pi} \left[\int_0^{\sqrt{2P}} E\left(\frac{r}{\sqrt{2P}}\right) f_R(r) dr + \int_{\sqrt{2P}}^\infty \left[E\left(\frac{\sqrt{2P}}{r}\right) - \left(1 - \frac{2P}{r^2}\right) K\left(\frac{\sqrt{2P}}{r}\right) \right] f_R(r) dr \right] \\ &= \int_0^{\sqrt{2P}} {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; \frac{r^2}{2P}\right) f_R(r) dr + \int_{\sqrt{2P}}^\infty \sqrt{\frac{P}{2r^3}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; \frac{2P}{r^2}\right) f_R(r) dr \end{aligned} \quad (5.2.2)$$

$f_R(r)$, the probability density function of the envelope of $I(t)$, is given by 4.3.15.

Equation 5.2.2, an expression for the output amplitude of the desired signal, has been derived through use of the stochastic method of analysis. A simpler result was obtained (reference 4.1.11) from a Fourier series representation of the output:

$$\sqrt{2P'} = \left[\int_0^\infty \frac{J_1(\sqrt{2P} u)}{u} \prod_{k=1}^m J_0(\sqrt{2P_k} u) e^{-\frac{1}{2} \sigma^2 u^2} du \right]^{\frac{1}{2}} \quad (5.2.3)$$

While 5.2.3 is a more tractable expression than 5.2.2, it has been developed by manipulations not applicable to the general case. Equation 5.2.2 demonstrates the result of a procedure applicable to any nonlinear device $g(x)$. For particular cases, alternative methods may give simpler results, but the stochastic method alone simply describes a unified approach.

We note that the power of the output interference quadrature to the desired signal is (reference 4.3.10):

$$\begin{aligned} N_2' &= E\{n_2'^2\} \\ &= \frac{1}{\pi} \int_0^\infty r^2 \left[\int_0^\pi \frac{\sin^2 \alpha \, d\alpha}{r^2 - 2\sqrt{2P} r \cos \alpha + 2P} \right] f_R(r) \, dr \end{aligned}$$

From a result taken from Grobner and Hofreiter [18, p. 114], the integral in brackets reduces to

$$\begin{cases} \frac{1}{4P} & \text{for } r \leq \sqrt{2P} \\ \frac{1}{2r^2} & \text{for } r > \sqrt{2P} \end{cases}$$

N_2' can now be written in the form:

$$N_2' = .5 \left(1 - \int_0^{\sqrt{2P}} \left(1 - \frac{r^2}{2P} \right) f_R(r) dr \right) \quad (5.2.4)$$

Since the total power out of the bandpass hard limiter has the constant value of .5, from 4.3.13 the in-phase interference power is simply:

$$N_1' = 1 - 2P' - N_2' \quad (5.2.5)$$

5.3 Suppression Characteristics

We now examine the nonlinear power sharing or suppression effect the limiter has on various inputs. With ρ_i the input signal to interference power, i.e.,

$$\rho_i = \frac{P}{\sigma^2 + \sum_{k=1}^m P_k}, \quad ,$$

and P' the output signal power, figure 6 presents graphs of

Figure 6

$$\begin{aligned}
 \alpha(\rho_i) &= [\text{output/input}] \text{ signal to total power} \\
 &= \left(1 + \frac{1}{\rho_i}\right) (2 P'(\rho_i))
 \end{aligned} \tag{5.3.1}$$

for several cases of interference. The results are taken from a research report by Lyons [5, chapter 4]. $\alpha(\rho_i)$ is the ratio of the output to input signal to total power as a function of the input SIR (signal to total interference power ratio) ρ_i . It is a measure of the relative power gain or loss experienced by the desired signal when passed through the limiter.

Some of the analytical results from Lyons' report are now stated without proof:

$$\begin{aligned}
 (1) \quad \lim_{\rho_i \rightarrow \infty} \alpha(\rho_i) &= 1 \\
 (2) \quad \lim_{\rho_i \rightarrow 0} \alpha(\rho_i) &= \frac{1}{4} \left[\int_0^\infty \frac{f_A(a)}{a} da \right]^2,
 \end{aligned}$$

where $f_A(a)$ is the probability density function of the amplitude $/\sqrt{2}$ of $I(t)$ under the normalization:

$$\sum_{i=1}^m P_i + \sigma^2 = 1$$

and under the assumption that the phase angles of the signals and noise are independent random variables uniform on $[0, 2\pi)$.

The above limit is finite except for interference consisting of two equal amplitude sinusoids, in which case

$$\alpha(\rho_i) \rightarrow \infty \text{ as } \rho_i^{-.203} \quad *$$

$$(3) \quad \lim_{\rho_i \rightarrow 0} \alpha(\rho_i) = \frac{1}{2} \left[\int_0^{\infty} \sum_{k=1}^m \frac{\pi}{\pi} J_0(\sqrt{2P_k} u) e^{-\frac{1}{2} \sigma^2 u^2} du \right]^2$$

$$\rho_i \rightarrow 0$$

$$(4) \quad \lim_{\rho_i \rightarrow 0} \alpha(\rho_i) \geq \frac{1}{4} (= -6 \text{ dB}),$$

$$\rho_i \rightarrow 0$$

equality if and only if the interference consists of one sinusoid.

(5) When the interference is gaussian noise alone,

$$\alpha(\rho_i) = \frac{\pi}{4} (1 + \rho_i) \left[{}_1F_1(.5, 2, -\rho_i) \right]^2$$

(6)* For 2m equal amplitude sinusoids,

$$\lim_{\rho_i \rightarrow 0} \alpha(\rho_i) > \frac{\pi}{4} ;$$

for 2m + 1 equal amplitude sinusoids,

$$\lim_{\rho_i \rightarrow 0} \alpha(\rho_i) < \frac{\pi}{4} .$$

* Observed from numerical results rather than proved analytically.

$\alpha(\rho_i)$ converges pointwise to the result given in (5) as $m \rightarrow \infty$, $P_i \rightarrow 0$. In other words, at low SIR's, a signal is suppressed more by an odd number of equal carriers than by gaussian noise of the same power; an even number gives less suppression. As the interference power becomes distributed among many carriers, its effect approaches that of gaussian noise.

(7) When the interference is a single sinusoid,

$$\alpha(\rho_i) = \begin{cases} \frac{\rho_i + 1}{4} \left[{}_2F_1(.5, .5; 2; \rho_i) \right]^2 & \text{for } \rho_i < 1 \\ \frac{8}{\pi^2} & \text{for } \rho_i = 1 \\ (1 + 1/\rho_i) \left[{}_2F_1(.5, -.5; 1; 1/\rho_i) \right]^2 & \text{for } \rho_i > 1 \end{cases}$$

(8) When the interference consists of one sinusoid and gaussian noise,

$$\lim_{\rho_i \rightarrow 0} \alpha(\rho_i) = \frac{\pi}{4\sigma^2} \left[{}_1F_1\left(\frac{1}{2}, 1, -\frac{1-\sigma^2}{\sigma^2}\right) \right]^2$$

In the above equations ${}_2F_1(a, b; c; x)$ and ${}_1F_1(a, b; x)$ are the gaussian and confluent hypergeometric functions given by

$${}_2F_1(a, b; c; x) = 1 + \frac{ab}{c} \frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \dots$$

and

$${}_1F_1(a, b; x) = 1 + \frac{a}{b} \frac{x}{1!} + \frac{a(a+1)}{b(b+1)} \frac{x^2}{2!} + \dots$$

respectively.

We observe from (7) that a low power signal sharing the limiter with a high power signal loses an additional 6 dB due to the small signal suppression effect. This is the maximum loss any signal can experience due to suppression (see result (4)). When interference consists of gaussian noise or a large number of signals whose amplitudes are of the same order, the small signal suppression loss is only about 1 dB.

A signal sharing the limiter with two larger equal amplitude signals, however, achieves a gain theoretically unbounded at low signal-to-interference ratios. This interesting phenomenon is caused by the periodic cancellation of the two large signals when as vectors they add destructively. At these times, the small signal experiences the full power gain of the repeater. This positive small signal suppression effect has been noted by Jones [6, Chapter 3] and Shaft [9, Chapter 3], and is analyzed in detail in section 5.4.

For a signal and noise, as has been suggested by Jones [6, Chapter 3], the emphasis on signal-to-noise ratio may not always be justified. SNR is important where fidelity

of transmission is of concern (i.e. for AM signals). But the fact that a limiter gives a two-fold (3 dB) SNR enhancement at high SNR's does not imply that the detectability of an angle-modulated signal is improved (this is explained in 7.9). A loss in signal detectability is always experienced with a limiter, but Manasse et al [10, Chapter 4] have shown that the loss can be made to approach a small value at low input SNR's. Davenport's [7] 3 dB SNR improvement in this case is therefore somewhat misleading in terms of system performance. It does, however, have the one advantageous effect in the satellite channel of preventing up and down path noise in phase with the signal from giving any more 180° phase reversals* at the receiver than the sum of the phase reversals incurred independently by the two noises. The bandpass hard limiter, affecting the in phase axis like a regenerative binary repeater operating at baseband, splits the noise sources and prevents them from adding constructively. For equal up and down path signal-to-noise ratios, this is mathematically expressed by the inequality

$$\operatorname{erfc}\left(\sqrt{\frac{\text{SNR}}{2}}\right) > 2 \operatorname{erfc}(\sqrt{\text{SNR}})$$

where $\operatorname{erfc}(x)$ is the complimentary error function.

* These click errors (ref. p. 213), which degrade the demodulated waveform, occur when noise in phase with the signal becomes less than π (the FM signal amplitude). Out-of-phase noise gives a continuous error in the RF phase of the received signal; the presence of the limiter has little effect on this.

5.4 Effects of Dithering

We here show that at low signal to interference ratios, adding a dither signal to the limiter input may so improve the suppression characteristic of the repeater as to overcompensate the reduction in signal to interference power at the output. To demonstrate this seemingly unlikely statement, we first present a hypothetical example and prove analytically that an improvement can be obtained by dithering. The result is then extended and justified on an heuristic basis.

The context of the example may be slightly unrealistic, but it effectively points out the possible application of the result. Recalling the power sharing effect in the hard limiter, consider a user X communicating through the repeater under the following conditions:

- (1) X and Y are the only two users of the satellite;
- (2) Y is operating at a much higher power level than X;
- (3) X is in an emergency or top priority mode at this time (e.g., "S.O.S.");
- (4) X cannot, either directly or indirectly, communicate with Y in order to monopolize the full channel capacity; he must share the satellite with Y.

Ordinarily, from figure 6, X would suffer a 6 dB loss in the limiter due to small signal suppression. The question is, under the above assumptions, is there anything X, the emergency user, can do to improve his performance (even at the overcompensated expense of Y)*. Since the down path link often represents the primary source of noise (i.e. loss of fidelity), the signal power reradiated from the satellite is an excellent measure of performance. By activating a dither signal within the satellite**, X can increase his share of the reradiated power, although Y must lose more than half his. The problem is now described mathematically.

X and Y transmit signals of the form

$$x(t) = \sqrt{2P} \cos(\omega_D t + \theta_D(t))$$

and
$$y(t) = \sqrt{2P_I} \cos(\omega_I t + \theta_I(t))$$

respectively. By assumption

$$P \ll P_I .$$

Consider now the system depicted in figure 7.

* The solution to be described here is certainly not the only possible means of giving X improved performance.

** There could be a narrow band power detector at a particular frequency in the satellite capable of integrating a low power S.O.S. tone and triggering the dither signal. In the analysis, the false alarm/incorrect dismissal problem is ignored, and in fact implementation has not really been considered.

Figure 7

$x(t)$ and $y(t)$, the small and large signals respectively, are inputs to the repeater. When switch S_d is open X has 6 dB less of the output power than he had of the input power; i.e.,

$$\frac{\langle x'^2(t) \rangle}{.5} = \frac{P}{4(P + P_I)}$$

$x'(t)$ and $y'(t)$ are outputs corresponding to the inputs $x(t)$ and $y(t)$. $n'(t)$ represents first harmonic IM product terms (having frequencies $|m\omega_D - n\omega_i|$, $|m - n| = 1$) from the hard limiter.

So, X normally loses 6 dB; consider now the effect of closing switch S_d . A dither signal, $d(t)$, which has been activated by X is added to the input of the limiter. From figure 7,

$$\begin{aligned} R(t) &= \text{envelope of } (x(t) + y(t)) \\ &\approx \sqrt{2P_I} \end{aligned}$$

and ω_0 , the angular frequency of the local oscillator, is chosen arbitrarily, but not in the passband of either of the two signals. In fact, we would likely choose ω_0 to lie outside the band of the post limiter zonal filter to avoid radiating components of the limiter output dependent on $d(t)$. $d(t)$ is added to alter the envelope distribution of the interference. Although the second moment of this interference is doubled, the fact that it can now assume values near zero means that an increase in the output signal power can occur.

With S_d closed $x(t)$ competes (for power) with interference consisting of the sum of two approximately equal amplitude sinusoids which have a uniformly distributed relative phase angle. Figure 8 depicts the probability density functions of the interference envelope when S_d is open and closed. These are given by

Figure 8

$$f_I(r) = \delta(r - \sqrt{2P_I}) \quad (\text{i.e. } I = \sqrt{2P_I} \text{ with probability 1})$$

and

$$f_{Id}(r) = \begin{cases} \frac{2}{\pi\sqrt{8P_I - r^2}} & \text{for } 0 \leq r \leq 2\sqrt{2P_I} \quad * \\ 0 & \text{elsewhere} \end{cases} \quad (5.4.1)$$

It can be shown that

$$E\{I_d^2\} = 2E\{I^2\} ;$$

i.e. adding the dither signal has reduced the input signal to interference ratio by 3 dB. Figure 6 is used to demonstrate that an increased output signal power can be obtained when S_d is closed. This will be explained as both a statistical and time process phenomenon.

Suppose the input SIR is -25 dB. From figure 6, then, the output signal power relative to unity total output power is given by

$$\begin{aligned} P' &= \rho_i + \alpha \\ &= -31 \text{ dB} \end{aligned}$$

when S_d is open. When S_d is closed, we move to the parametric curve described by '2 equal amplitude sinusoids'. The input SIR is reduced by 3 dB. The new signal power out the limiter is then

* This is a special case of equation 9.6.5, an expression for the density of the envelope of two random phase sinusoids. The subscript d henceforth denotes a dithered input.

$$\begin{aligned}
 P'_d &= P_i + 2 \\
 &= -28 + 2 \\
 &= -26 \text{ dB.}
 \end{aligned}$$

Hence,

$$P'_d = P' + 5 \text{ dB,}$$

and a 5 dB gain in output signal power has been achieved. This result is discussed first as a time domain or periodic effect.

Without dithering, the input to the limiter is the vector sum of large and small amplitude phasors denoted by $\vec{y}$ and $\vec{x}$ respectively. With $\vec{x}$ fixed, $\vec{y}$ rotates at an angular frequency $\omega_i - \omega_p$. Since the interference at the input has a constant amplitude, the mean component of the output in phase with the desired signal has a constant amplitude 6 dB down relative to the interference.

With dithering, the input becomes the vector sum of a small signal and two equal amplitude* large signals. These are denoted by $\vec{x}$, $\vec{y}$, and $\vec{d}$ respectively, and are depicted in figure 9.

* To make the amplitudes exactly theoretically equal, the envelope detector in figure 7 must remove the fluctuating (desired signal) component of the measurement. Equality is assumed in the analysis.

Figure 9

The resultant interference vector $\vec{i}$ assumes in a periodic fashion amplitudes in the interval $[0, 2\sqrt{2P_I}]$. The extremes correspond to perfectly constructive and destructive addition of the vectors $\vec{y}$ and $\vec{d}$. Given a particular input interference amplitude, R the expected value over α of the output varies in amplitude from a small value (when $R = 2\sqrt{2P_I}$) to 1 (when $R = 0$). The constant component (or expected value over R) of this has been defined in equation 4.3.6 to be the desired signal out of the repeater.

The complete limiter output is represented as

$$z(t) = r_1(t) \cos(\omega_D t + \theta_D(t)) + r_2(t) \sin(\omega_D t + \theta_D(t)),$$

and $S'_0(t)$ is defined by

$$S'_0(t) = E\{r_1(t) | R(t)\} ,$$

the expected value taken for a fixed input interference amplitude over a uniformly distributed phase angle relative to the signal.

The resultant input interference envelope, $R(t)$, varies slowly in comparison with the carrier frequencies, but more rapidly than the modulating waveforms. Recalling that the $SIR \ll 1$, $S'_0(t)$ will also be $\ll 1$ except during those moments when $R(t)$ is nearly zero. Then the desired signal benefits from the full amplification potential of the hard limiting repeater, and its RF phase characteristic is briefly imprinted on the net output. $S'_0(t)$, therefore, displays short, high energy pulses corresponding to the small signal, in fact, capturing the limiter. These pulses occur at a rate high enough to track continuously the desired information $\phi_D(t)$. Due to the phase modulation in $y(t)$ (a noisy or random process as far as X is concerned), the exact times of occurrence of $S'_0(t)$ peaks are unknown to X (but they will occur!). We can regard these peaks as having a frequency $|\omega_0 - \omega_1|$, and a phase uniformly distributed on $[0, 2\pi)$ with the phases of successive peaks correlated.

The random or fluctuating amplitude component of the output in phase with the signal depends only on the relative RF phases of $x(t)$, $y(t)$, and $d(t)$. It therefore contains no information about $\theta_D(t)$, and is regarded as interference. The desired signal out of the repeater is the constant amplitude or deterministic component given by

$$\sqrt{2P'_d} \cos(\omega_D t + \theta_D(t)),$$

$$P'_d = \frac{1}{2} \langle S'_o(t) \rangle^2 \quad (5.4.2)$$

We now compare the cases with and without dithering. The averaging in equation 5.4.2 gives more signal power when the interference envelope is zero than it takes when it is $2\sqrt{2P_I}$. The relative gain becomes increasingly profound at low SIR's. In effect, the high energy bursts depicted and described produce an output signal of more power than in the previous case; i.e., $P'_d \gg P'$.

This result can also be obtained from a probabilistic argument. Instead of looking at rotating phasors, we make use of the interference envelope densities depicted in figure 8. From the second result of Lyons' research report,

$$\lim_{\rho_i \rightarrow 0} \alpha(\rho_i) = \frac{1}{4} \left[\int_0^\infty \frac{f_I(r)}{r} dr \right]^2 \left[\int_0^\infty r^2 f_I(r) dr \right] \quad (5.4.3)$$

The output signal power for a very weak signal is then given by (ref. 5.2.2)

$$P' \approx \frac{1}{2} \alpha(\rho_i) \rho_i$$

$$\approx \frac{1}{2} \left[\int_0^{\sqrt{2P}} {}_2F_1(.5, -.5; 1; r^2/2P) f_I(r) dr + \sqrt{\frac{P}{2}} \int_{\sqrt{2P}}^\infty \frac{f_I(r)}{r} dr \right]^2 \quad (5.4.4)$$

where P is input signal power (constant), and I is the input interference envelope.

To achieve an optimum $f_I(r)$ (i.e. maximum P') for $\rho_i \downarrow 0$ and limited I , from 5.4.4 we would maximize

$$\lim_{P \downarrow 0} \int_0^{\sqrt{2P}} {}_2F_1(.5, -.5; 1; r^2/2P) f_I(r) dr$$

under the constraints that

$$\int_0^{r_0} f_I(r) dr = 1$$

$$\int_0^{r_0} r^2 f_I(r) dr = 2N_i$$

when the interference is peak amplitude limited to $r_0 \leq \sqrt{2N_i}$.

The solution is given by (* denotes optimum)

$$f_I^*(r) = \left(1 - \frac{2N_i}{r_0^2}\right) \delta(r) + \frac{2N_i}{r_0^2} \delta(r - r_0) \quad (5.4.5)$$

Due to the $1/r$ weighting in the integrand of 5.4.3, we have concentrated as much probability of having a zero interference amplitude as the constraints allow. With ideal hard limiting (physically unrealizable since there is infinite gain in the limit as the input $\rightarrow 0$), solution 5.4.5 in 5.4.4 implies that

$$\lim_{\rho_i \rightarrow 0} P' = \frac{1}{2} \left(1 - \frac{2N_i}{r_0^2}\right)^2$$

$$> 0,$$

an impossible situation (i.e. signal power out, no signal power in).

For interference constrained to consist of the sum of gaussian noise and constant amplitude sinusoids, Lyons [5] has shown that

$$\lim_{r \rightarrow 0} \frac{f_I(r)}{\sqrt{r}} = 0$$

† A continuous stochastic process cannot have the on-off statistics defined by 5.4.5.

except when I consists of two equal amplitude sinusoids, in which case

$$\sqrt{N_i} f_I(o) = \frac{\sqrt{2}}{\pi}$$

So while a non-zero probability of zero interference cannot be satisfied as long as Y is transmitting, the 1:1 condition, from 5.4.3 and the above, is the only that gives a suppression factor which increases without bound as $\rho_i \rightarrow 0$. (This phenomenon can be observed from figure 6). Hence, it is immediately concluded that for ρ_i sufficiently small, attaining the suppression factor for 1:1 interference overcompensates the corresponding increase in total interference power. An advantage can therefore be gained with dithering.

We have taken a rather singular and unrealistic example in order to demonstrate the possible use of dithering. The unavoidable presence of up path thermal noise ensures the net interference does not have a constant amplitude. More important, the interference envelope measurement includes the desired signal itself. The question is: in practical cases of interest, can a gain through dithering be achieved? In order to gain, $\alpha(\rho_i)$ must increase by more than 3 dB in order to overcompensate the 3 dB loss in ρ_i . We might suspect that dithering could be advantageous when:

- (1) ρ_i is sufficiently low
- (2) the interference is dominated by a single signal.

The following example demonstrates that considerable gain can still be achieved in more practical cases. Suppose the interference consists of two signals and gaussian noise of relative powers 50: 5 :: 1. Then the interference has an envelope density given by $f_{\hat{I}}(r)$, and depicted in figure 8.

Normally, for $\rho_i = -25$ dB, the desired signal loses 5.2 dB under this interference configuration. With dithering, assuming the envelope detector ideally matches (without delay or error due to the presence of $S_D(t)$) the interference envelope, and introduces random phase, the interference envelope density becomes that of

$$\hat{I}_d = \hat{I} \left[(1 + \cos \alpha)^2 + \sin^2 \alpha \right]^{\frac{1}{2}} \quad (5.4.6)$$

where $\hat{I}$ is distributed according to $f_{\hat{I}}(r)$ and α is uniform.

This can be written in the form

$$\begin{aligned} f_{\hat{I}_d}(r) &= \int f_{\hat{I}_d}(r|\hat{I} = \rho) f_{\hat{I}}(\rho) d\rho \\ &= \frac{2}{\pi} \int \frac{f_{\hat{I}}(\rho)}{\sqrt{2 - \frac{r^2}{2\rho^2}}} d\rho \\ &= \frac{2}{\pi \sqrt{2 - \frac{r^2}{2\bar{\rho}^2}}} \quad \text{for } 0 \leq r \leq 2\bar{\rho} \end{aligned} \quad (5.4.7)$$

We have here made use of the mean value theorem for expected values. $\bar{\rho}$ must lie in the support of $f_{\hat{\rho}}(\rho)$ depicted in figure 8. With P_I the total interference power at the input, this support is given by the interval

$$\left[\frac{(10 - \sqrt{10} - \sqrt{2})}{\sqrt{100 + 10 + 1}} \sqrt{P_I}, \frac{(10 + \sqrt{10} + \sqrt{2})}{\sqrt{100 + 10 + 1}} \sqrt{P_I} \right]$$

Equation 5.4.7 has the same form as 5.4.1; hence, we lie on the '2 equal amplitude sinusoid' parametric curve given in figure 6. If

$$\bar{\rho} = \sqrt{P_I}$$

we could immediately conclude that $\alpha = 2$ dB, as before. The lowest possible value of α would occur for

$$\bar{\rho} = \frac{(10 - \sqrt{10} - \sqrt{2})}{\sqrt{100 + 10 + 1}} \sqrt{P_I}$$

This corresponds to an input SIR with dithering of -23 dB.

From figure 6, then,

$$\alpha \geq 1.5 \text{ dB},$$

and the reradiated power of the desired signal has increased by at least

$$5.2 + 1.5 - 3 = 3.7 \text{ dB}.$$

The frequency of variation of the interference envelope is of the order of $|f_1 - f_2|$, the difference between the frequencies of the two interference signals. It has been assumed that the envelope detector can follow such variations on a carrier of frequency f_1 (the frequency of the larger of the two signals) with negligible delay. Otherwise, the assumption inherent in equation 5.4.6 is no longer valid, and we do not lie on the 1:1 interference curve which gives us the desired suppression effect. In fact, dithering can always give improvement at sufficiently low SIR's whenever the resultant interference envelope varies slowly enough to be tracked effectively.

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CHAPTER 6

SOFT AMPLITUDE LIMITING

6.1 Operating Point and Backoff

Based on the hard limiter model analyzed in Chapter 5, the TWT is always saturated and is said (reference equation 2.4.10) to be operating at 0 dB output backoff. In fact, such a model is not only physically unrealizable, it does not represent a desirable mode of operation for the higher power, bandwidth-limited satellites. By controlling the level of the inputs to the limiting TWT, we can choose an operating point which gives a maximum received carrier to intermodulation distortion plus thermal noise power ratio. Since the effects of IM product noise and gaussian noise are not identical, this may not specify a strictly optimum choice. However, assuming the criterion is reasonable, a method of determining the optimum operating point will be described.

Figure 10 presents plots of carrier-to-up path noise, carrier-to-IM, and carrier-to-down path power ratios versus equivalent single carrier output backoff in a multicarrier situation [1]. Defining $(C/N)_R$ to be the carrier-to-noise power at the desired receiver, we wish to maximize

$$\left(\frac{C}{N} \right)_R = \left(\left[\hat{\alpha} \left(\frac{C}{N} \right)_u \right]^{-1} + \left(\frac{C}{N} \right)_I^{-1} + \left(\frac{C}{N} \right)_d^{-1} \right)^{-1} \quad (6.1.1)$$

with respect to the output backoff b_o .* Here $(C/N)_u$, $(C/N)_d$, and $(C/N)_I$ are the up path thermal, down path thermal, and intermodulation product desired carrier-to-noise power ratios. $\hat{\alpha}$ is the change in (C/N) when the signals and up path noise are passed through the TWT. All of these are functions of the output backoff, b_o which is determined by the transmitted power levels. With

$$q = \left(\frac{C}{N} \right)^{-1},$$

equation 6.1.1 is rewritten as

$$\left(\frac{C}{N} \right)_R = \left(\frac{q_u(b_o)}{\hat{\alpha}(b_o)} + q_d(b_o) + q_I(b_o) \right)^{-1} \quad (6.1.2)$$

Maximizing this with respect to b_o leads to the equation

$$\begin{aligned} \frac{d \left(\frac{C}{N} \right)_R}{db_o} &= - \left(\frac{C}{N} \right)_R^2 \left[- \frac{q_u(b_o) \hat{\alpha}'(b_o)}{\hat{\alpha}^2(b_o)} + \frac{q_u'(b_o)}{\hat{\alpha}(b_o)} + q_d'(b_o) + q_I'(b_o) \right] \\ &= 0 \end{aligned}$$

or

$$\hat{\alpha}^2(b_o^*) (q_d'(b_o^*) + q_I'(b_o^*)) + \hat{\alpha}(b_o^*) q_u'(b_o^*) - \hat{\alpha}'(b_o^*) q_u(b_o^*) = 0 \quad (6.1.3)$$

* It is here assumed that the powers of the transmitted signals are in fixed proportion; the only variable is the constant of proportionality which defines their absolute values and determines the TWT backoff. If the signals are of equal amplitude, there is one TWT operating point which is optimum for each signal -- in 6.1.1 we are considering a maximization of the $(C/N)_R$ of a particular desired signal.

at the optimum backoff point $b_o = b_o^{*\dagger}$. Assuming the suppression effect is insignificant in the vicinity of b_o^* (i.e. $\hat{\alpha} \approx 1$), equation 6.1.3 reduces to

$$q_d'(b_o^*) + q_I'(b_o^*) + q_u'(b_o^*) = 0 \quad (6.1.4)$$

Graphically, this is equivalent to determining the minimum point of $q_d(b_o) + q_I(b_o) + q(b_o)$.

At low transmitted power levels (i.e. large backoff), the effect of intermodulation is negligible, and the system is limited by up and down path thermal noise. At high power levels the intermodulation process becomes the primary source of distortion. The optimum operating point will be located somewhere between the two extremes of complete saturation and perfectly linear operation.

Intermodulation in the TWT is a result of both a non-linear amplitude characteristic and AM to PM conversion. In figure 11 their effects are separated, and the C/I due to each, and the resultant C/I, are plotted [2] versus input backoff. The C/I due to the amplitude nonlinearity increases approximately linearly from 0 dB backoff, whereas the AM/PM gives a minimum C/I at about -6 dB backoff.* Figure 12

† This assumes that $(C/N)_R$ in fact assumes a maximum at some non-extreme value of backoff. The validity of this assumption is demonstrated in the succeeding paragraph.

* This is for a particular TWT and multicarrier input, and cannot be interpreted as an exact general result.

Figure 11

Figure 12

Y 12 10 8 6 4 2 0 -2 -4 -6 -8 -10 -12 -14 -16 -18 -20 -22 -24 -26 -28 -30

[1, Chapter 2] demonstrates that the total IM/SIGNAL power ratio at a particular backoff is a function of the number of carriers present. It also depends on their relative powers. For equal carriers there is a 3 - 4 dB difference from 4 to an infinite number of carriers for input backoff greater than 5 dB.

The effects described by figures 11 and 12 have been pointed out as characteristics important to an understanding of multicarrier operation through existing satellite transponders. In the next two sections the piecewise linear and error function soft amplitude nonlinearities are considered.

6.2 The Piecewise Linear Characteristic

The piecewise linear amplitude characteristic described by

$$g(x) = \begin{cases} -1 & x \leq -x_0 \\ x/x_0 & -x_0 < x < x_0 \\ 1 & x \geq x_0 \end{cases} \quad (6.2.1)$$

has been analyzed by Cahn [5, Chapter 3], Van Vleck [9, chapter 4] and others. In this section we apply the results of 4.3 to analyze its effect on a multiple input given by

$$x(t) = S_D(t) + I(t) \quad (6.2.2)$$

where $S_D(t)$ and $I(t)$ are given by 4.3.1 and 4.3.9. The special case of

$$\begin{aligned} I(t) &= n(t) \\ &= \text{gaussian noise} \end{aligned}$$

(i.e. $m = 0$) is considered in 7.6.

From 4.3.5, the amplitude of the desired signal out of the piecewise linear characteristic is given by

$$\begin{aligned} \sqrt{2P^*} &= \frac{1}{\pi} \int_0^\pi \left[\int_{L_\alpha} \left(\frac{r}{x_0} \right) \left(\frac{\sqrt{2P} - r \cos \alpha}{\sqrt{r^2 - 2\sqrt{2P} r \cos \alpha + 2P}} \right) f_R(r) dr \right. \\ &\quad \left. + \int_{C_{L_\alpha}} 1 \left(\frac{\sqrt{2P} - r \cos \alpha}{\sqrt{r^2 - 2\sqrt{2P} r \cos \alpha + 2P}} \right) f_R(r) dr \right] d\alpha \end{aligned} \quad (6.2.2)$$

where L_α denotes the region of linear operation for values of R (see figure 13).

$$L_\alpha = \begin{cases} \left(\max \left(\sqrt{2P} \cos \alpha - \sqrt{x_0^2 - 2P \sin^2 \alpha}, 0 \right), \right. \\ \quad \left. \sqrt{2P} \cos \alpha + \sqrt{x_0^2 - 2P \sin^2 \alpha} \right) \text{ for } \sin \alpha < \frac{x_0}{\sqrt{2P}} \\ \phi \text{ (the empty set) for } \sin \alpha \geq \frac{x_0}{\sqrt{2P}} \end{cases} \quad (6.2.3)$$

C_{L_α} is the complement of L_α on the interval $[0, \infty)$ and is the region of saturation. $f_R(r)$, the density of the envelope of $I(t)$, is given by 4.3.10.

Figure 13

The total power out of the nonlinearity, from 4.3.12 is given by

$$P_o = \frac{1}{2} \left[\int_0^{x_0} \left(\frac{r}{x_0} \right)^2 \cdot f_E(r) dr + \int_{x_0}^{\infty} 1 \cdot f_E(r) dr \right] \quad (6.2.4)$$

$f_E(r)$ is the envelope density of the resultant input, and is given by 4.3.16.

The suppression or change in desired signal to total power ratio is then simply

$$\alpha = \frac{P'}{P_0} \cdot \frac{P + \sum_{k=1}^m P_k + \sigma^2}{P}$$

$$= \frac{P'}{P_0} \left(1 + \frac{1}{\rho_i}\right) \quad (6.2.5)$$

where ρ_i is the input signal to interference power ratio.

6.3 The Error Function Characteristic

The error function characteristic as described by (ref. (2), p. 26)

$$g(x) = \frac{1}{K} \frac{1}{\sqrt{2\pi}L} \int_0^x e^{-\frac{t^2}{2L^2}} dt \quad (6.3.1)$$

lends itself to simple analysis. It has been studied by Baum [3], Galejs [4], Lee [13, chapter 3], and Kaye [5]. The curve described by 6.3.1 limits the input to values between plus and minus $1/2K$. The special case of extreme clipping is obtained as L approaches zero.

We now consider the input to be gaussian noise with power σ^2 . The input probability density function is then given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}} \quad (6.3.2)$$

The output density can easily be established by application of the well-known formula for a monotonic function $y(x)$ (2.4.4)

$$f_Y(y) = \frac{f_X(x)}{\left| \frac{dy}{dx} \right|}$$

Using the derivative of 6.3.1, we obtain the result

$$f_Y(x) = K\sqrt{a} e^{-\frac{x^2}{2\sigma^2} (1-a)} \quad (6.3.3)$$

where $a = \frac{L}{\sigma}$

Since x and y are related by 6.3.1, equation 6.3.3 also gives $f_Y(y)$ as an implicit function of y .

Baum [3] has demonstrated the result given by (2), p. 26 . For $a \rightarrow 0$ (extreme clipping), one obtains the arcsin result (1), p. 25 . When $a = 1$ (constant output probability density distribution), the correlation coefficient of the output as a function of the input correlation coefficient can be written as

$$\rho_Y(\tau) = \frac{6}{\pi} \sin^{-1} \left(\frac{\rho_X(\tau)}{2} \right)$$

The power spectrum of the output in general is given by

$$S_y(\omega) = \frac{4}{\sin^{-1}\left(\frac{1}{1+a}\right)} \int_0^{\infty} \sin\left(\frac{\rho_x(\tau)}{1+a}\right) \cos \omega\tau \, d\tau \quad (6.3.4)$$

The percentage harmonic content in the output is a monotonic decreasing function of a , or is far less for smoothly limited noise compared to extremely clipped noise.

Lee [13, chapter 3] considers multichannel FDM signals passing through an erf limiter given by equation 6.3.1. The signal in the channel under consideration is represented by a sine wave; invoking the central limit theorem, the remaining signals are represented by a gaussian noise. Lee has shown that the output signal power to crosstalk power at the center channel output for a gaussian input spectrum is then given by

$$\frac{S_o}{N_c} = \left[\sum_{v=1}^{\infty} \frac{[1.2.3.....(2v-1)]^2}{2^{2v}! (v+1)! \sqrt{2v+1} (1+a^2)^{2v}} \right]^{-1} \quad (6.3.5)$$

Since the crosstalk power is a maximum in the centre channel, equation 6.3.5 serves as a lower bound for S_o/N_c as a function of a (the "limiter rms-to-input noise rms ratio"). We note from 6.3.5 that S_o/N_c is an increasing function of a , and is a minimum for extreme clipping ($a = 0$).

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CHAPTER 7

EFFECTS OF AMPLITUDE NON-LINEARITY ON ONE SIGNAL AND NOISE

7.1 Signal and Noise Components of Output

The stochastic approach developed in section 4.3 is here applied to the case of a signal and noise* passed through a bandpass amplitude non-linearity $g(x)$. The input and output are

$$\begin{aligned} x(t) &= s(t) + n(t) \\ &= A \cos(\omega_0 t + \theta(t)) + R(t) \cos(\omega_0 t + \phi(t)) \\ &= E(t) \cos(\omega_0 t + \phi(t)) \end{aligned} \quad (7.1.1)$$

and

$$y(t) = g(E(t)) \cos(\omega_0 t + \phi(t)) \quad (7.1.2)$$

respectively. $n(t)$ is stationary band-limited gaussian noise of power (statistical) σ^2 . From 4.3.5, with

$$f_2(x) = 0 ,$$

the output can be written in quadrature form as a function of $R(t)$, the envelope of the gaussian interference, and

$$\alpha(t) = \phi(t) - \theta(t) ,$$

its phase relative to the signal:

$$y(t) = g(\sqrt{R^2(t) - 2AR(t)\cos\alpha(t) + A^2}) \left[\cos\left(\tan^{-1}\left(\frac{R(t)\sin\alpha(t)}{A - R(t)\cos\alpha(t)}\right)\right) \cos(\omega_0 t + \theta(t)) + \sin\left(\tan^{-1}\left(\frac{R(t)\sin\alpha(t)}{A - R(t)\cos\alpha(t)}\right)\right) \sin(\omega_0 t + \theta(t)) \right]$$

* as would be the case in a TDMA satellite communications system

$$= y_1(R(t), \alpha(t)) \cos(\omega_0 t + \theta(t)) + y_2(R(t), \alpha(t)) \sin(\omega_0 t + \theta(t)) \quad (7.1.4)$$

Here $R(t)$ has the Rayleigh envelope density function

$$f_R(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}} & \text{for } r \geq 0 \\ 0 & \text{for } r < 0 \end{cases} \quad (7.1.4)$$

and $\alpha(t)$ is assumed uniformly distributed. In 7.1.3, then, $y_1(R(t), \alpha(t))$ and $y_2(R(t), \alpha(t))$ are processes having means

$$E\{y_1(R, \alpha)\} = A' = \frac{1}{\pi\sigma^2} \int_0^\infty r e^{-\frac{r^2}{2\sigma^2}} \left[\int_0^\pi g(\sqrt{r^2 - 2A\cos\alpha + A^2}) ; \frac{(A - r\cos\alpha) d\alpha}{\sqrt{r^2 - 2A\cos\alpha + A^2}} \right] dr \quad (7.1.5)$$

and

$$E\{y_2(R, \alpha)\} = 0$$

respectively. From 4.3.7, 4.3.8, and 4.3.9, the signal or deterministic part of the output is therefore given by

$$\begin{aligned} s'(t) &= (E^2\{y_1\} + E^2\{y_2\})^{\frac{1}{2}} \cos\left(\omega_0 t + \theta(t) - \tan^{-1}\left(\frac{E\{y_2\}}{E\{y_1\}}\right)\right) \\ &= A' \cos(\omega_0 t + \theta(t)) \end{aligned} \quad (7.1.6)$$

Following the formulation of equation 4.3.10, the random or noise component is

$$\begin{aligned} n'(t) &= y(t) - s'(t) \\ &= g(E(t)) \cos(\omega_0 t + \phi(t)) - A' \cos(\omega_0 t + \theta(t)) \\ &= n_1'(t) \cos(\omega_0 t + \theta(t)) + n_2'(t) \sin(\omega_0 t + \theta(t)) \end{aligned} \quad (7.1.7)$$

$n_1'(t)$ and $n_2'(t)$ are instantaneous projections of the output noise process on axes in phase and out of with the signal. Defining

$$\begin{aligned}\delta(t) &= \phi(t) - \theta(t) \\ &= \tan^{-1} \left(\frac{R(t) \sin \alpha(t)}{A - R(t) \cos \alpha(t)} \right) \\ &= \text{the phase difference between } s(t) + n(t) \text{ and } s(t) \quad (7.1.8),\end{aligned}$$

7.1.6 and the expansion of 7.1.3 imply that

$$n_1'(t) = g(E(t)) \cos(\delta(t)) - A' \quad (7.1.9)$$

and

$$n_2'(t) = g(E(t)) \sin(\delta(t)) \quad (7.1.10)$$

7.2 Probability Density Functions of Quadrature Noise Components

From 7.1.9 and 7.1.10, the in and out of phase noise components from the non-linearity are distributed as

$$\begin{aligned}n_1' &= g(E) \cos \delta - A' \\ &= g(\sqrt{R^2 - 2AR \cos \alpha + A^2}) \left(\frac{A - R \cos \alpha}{\sqrt{R^2 - 2AR \cos \alpha + A^2}} \right) - A' \quad (7.2.1)\end{aligned}$$

and

$$\begin{aligned}n_2' &= g(E) \sin \delta \\ &= g(\sqrt{R^2 - 2AR \cos \alpha + A^2}) \left(\frac{R \sin \alpha}{\sqrt{R^2 - 2AR \cos \alpha + A^2}} \right) \quad (7.2.2)\end{aligned}$$

The variables included in these equations are defined by the input/output phasor diagram of figure 14. Here

$$A' = E\{g(E)\cos\delta\} = \text{output signal amplitude}$$

is given by 7.1.5, R is distributed according to 7.1.4, and α is uniform on $[0, 2\pi)$.

Figure 14

Davenport and Root [10, chapter 2, p.167] and Lucky, Salz, and Weldon [1, p.249] have stated that δ , a random variable defined by equation 7.1.8, has the distribution

$$f_{\delta}(y) = \begin{cases} \frac{1}{2\pi} e^{-\rho} \left(1 + \sqrt{\rho} \cos y e^{\rho \cos^2 y} \operatorname{erfc}(\sqrt{\rho} \cos y) \right) & \text{for } -\pi \leq y < \pi \\ 0 & \text{elsewhere} \end{cases} \quad (7.2.3)$$

ρ is the input signal-to-noise ratio (SNR) and erfc is the complementary error function defined by

$$\operatorname{erfc}(z) = \sqrt{\frac{2}{\pi}} \int_z^{\infty} e^{-t^2} dt$$

Then it can be shown that:

$$\cos \delta \sim \frac{2 f_{\delta}(\cos^{-1} z)}{\sqrt{1-z^2}} \quad (7.2.4)$$

$$\sin \delta \sim \frac{f_{\delta}(\sin_1^{-1}|z|) + f_{\delta}(\sin_2^{-1}|z|)}{\sqrt{1-z^2}} \quad (7.2.5)$$

Here $\sim$ means "is distributed as" and $\sin_i^{-1}(x)$ denotes the angle taken from the i 'th quadrant.

We now examine the first representations of n_1' and n_2' given in 7.2.1 and 7.2.2. Davenport and Root [10, chapter 2, p. 166] have shown that E , the envelope of a sine wave and noise, has the probability function

$$f_E(r) = \begin{cases} \frac{r}{\sigma^2} e^{-\frac{r^2+A^2}{2\sigma^2}} I_0\left(\frac{Ar}{\sigma^2}\right) & \text{for } r \geq 0 \\ 0 & \text{for } r < 0 \end{cases} \quad (7.2.6)$$

where $I_0(z)$, the zero order modified Bessel function of the first kind, is given by

$$\begin{aligned} I_0(z) &= \frac{1}{\pi} \int_0^{\pi} e^{z \cos \theta} d\theta \\ &= \sum_{v=0}^{\infty} \frac{(z^2)^v}{(v!)^2} \end{aligned}$$

If E and δ were independent random variables, their joint density would be the product of 7.2.6 and 7.2.3, and the statistical properties of $n_1' = g(E)\cos\delta - A'$ and $n_2' = g(E)\sin\delta$ could be immediately examined. In fact, E and δ are not independent; however, it can be shown (see Appendix A) that

$$f_{E|\delta}(r|y) = \begin{cases} \left(\frac{\pi}{2} \sec^2 y \sigma^2\right)^{-\frac{1}{2}} \exp\left(-\frac{(r - A \sec y)^2}{2 \sec^2 y \sigma^2}\right) \left[\operatorname{erfc}\left(\operatorname{sgn}(|y| - \frac{\pi}{2}) \sqrt{\rho}\right)\right]^{-1} & \text{for } r \geq 0 \\ 0 & \text{for } r < 0 \end{cases} \quad (7.2.7)$$

Hence, the joint density of E and δ is

$$\begin{aligned} f_{E,\delta}(r,y) &= f_{E|\delta}(r|y) f_{\delta}(y) \\ &= \left(\frac{\pi}{2} \sec^2 y \sigma^2\right)^{-\frac{1}{2}} \exp\left(-\frac{(r - A \sec y)^2}{2 \sec^2 y \sigma^2}\right) \left[\operatorname{erfc}\left(\operatorname{sgn}(|y| - \frac{\pi}{2}) \sqrt{\rho}\right)\right]^{-1} \\ &\quad \frac{1}{2\pi} \left(1 + \sqrt{\pi\rho} \cos y e^{\rho \cos^2 y} \operatorname{erfc}(\sqrt{\rho} \cos y)\right) \end{aligned} \quad (7.2.8)$$

In succeeding analysis both the (R, α) and (E, δ) expressions for n'_1 and n'_2 in 7.2.1 and 7.2.2 are used. Each represents the quadrature noise processes as functions of two random variables; in one case the independent envelope and phase of a gaussian process, in the second the dependent envelope and phase error of a sine wave and noise process. Based on the (E, δ) representation, we now proceed to derive expressions for the densities of n'_1 and n'_2 .

From 7.2.2 the distribution function of the out of phase noise is written as

$$P\{g(E) \sin \delta \leq z\} = P\{E \leq g^{-1}(z \csc \delta)\}$$

if $\delta \leq 0$, and $g(E)$ is a monotonic increasing function of E and therefore has an inverse g^{-1} . Noting that n'_2 is symmetrically distributed about 0, its density is then

$$\begin{aligned}
f_{n_2'}(z) &= \frac{d}{dz} \int_0^\pi \left[\int_0^{g^{-1}(z \csc y)} f_{E,\delta}(r,y) dr \right] dy \\
&= \int_0^\pi \left. \frac{dg^{-1}(s)}{ds} \right|_{s=z \csc y} f_{E,\delta}(g^{-1}(z \csc y), y) \csc y dy \quad (7.2.9)
\end{aligned}$$

where $f_{E,\delta}(r,y)$ is given by 7.2.8.

Following a similar procedure, the density of the in phase noise reduces to

$$f_{n_1'}(z) = \int_0^\pi \left. \frac{dg^{-1}(s)}{ds} \right|_{s=(z+A') \csc y} f_{E,\delta}(g^{-1}((z+A') \csc y), y) \csc y dy \quad (7.2.10)$$

where it is assumed that

$$g^{-1}(s) = \frac{dg^{-1}(s)}{ds} = 0 \quad \text{for } s < 0$$

7.3 Noise Envelope and Phase Densities

The envelope and phase (relative to the signal) of the noise component of the output are defined by

$$R'(t) = \sqrt{n_1'^2(t) + n_2'^2(t)}$$

and

$$\gamma(t) = \tan^{-1}(n_2'(t)/n_1'(t))$$

As random variables they are written as

$$R' = \sqrt{n_1'^2 + n_2'^2} \quad (7.3.1)$$

and

$$\gamma = \tan^{-1}(n_2'/n_1') \quad (7.3.2)$$

where n_1' and n_2' are the in and out of phase noise processes discussed in 7.2. In this section derivations of the densities of R' and γ are presented.

Equations 7.3.1 and 7.3.2 represent R' and γ as functions of two random variables which are not independently distributed. From trigonometric relations evident in figure 14, they can also be written in the form

$$R' = \sqrt{g^2(E) - 2g(E)A'\cos\delta + A'^2} \quad (7.3.3)$$

and

$$\gamma = \cot^{-1} \left(\frac{g(E)\cos\delta - A'}{g(E)\sin\delta} \right) \quad (7.3.4)$$

Now, from 7.3.3, the distribution function of R' is

$$\begin{aligned} P\{R' \leq r\} &= P\left\{|\delta| \leq \cos^{-1} \frac{A'^2 + g^2(E) - r^2}{2g(E)A'}\right\} \\ &= 2 \int_0^\infty \left[\int_0^{\cos^{-1} \left(\frac{A'^2 + g^2(e) - r^2}{2g(e)A'} \right)} f_{E,\delta}(e,y) dy \right] de \end{aligned}$$

where $f_{E,\delta}(r,y)$, the joint density of E and δ , is given by 7.2.8. From the above expression, the probability density function of the noise envelope is written as

$$\begin{aligned}
 f_{P'}(r) &= \frac{dP\{P' < r\}}{dr} \\
 &= \frac{4r}{A'} \int_S \frac{f_{E,\delta} \left(e, \cos^{-1} \frac{A'^2 + g^2(e) - r^2}{2g(e)A'} \right)}{\sqrt{4A'^2 g^2(e) - (A'^2 + g^2(e) - r^2)^2}} de \quad (7.3.5)
 \end{aligned}$$

where S is defined to be the set of all values of e satisfying

$$g(e) \geq \frac{|A'^2 + g^2(e) - r^2|}{2A'}$$

From equation 7.3.4, for $0 \leq \gamma < \pi$,

$$\begin{aligned}
 P\{\gamma \leq x\} &= P\{\cot \gamma \geq \cot x\} \\
 &= P\left\{ \frac{g(E)\cos\delta - A'}{g(E)\sin\delta} \geq \cot x \right\} \\
 &= P\{g(E) \geq A'(\cos\delta - \cot x \sin\delta)^{-1}\} \\
 &= 2 \int_0^\pi \left[\int_0^\infty f_{E,\delta}(r, y) dr \right. \\
 &\quad \left. g^{-1}(A'(\cos y - \cot x \sin y)^{-1}) \right] dy \quad (7.3.6)
 \end{aligned}$$

if g is a monotonic increasing function of E and hence has an inverse g^{-1} . Since $f_\gamma(x) = f_\gamma(-x)$, the phase density of the output noise can be written as the derivative of 7.3.6; i.e.,

$$f_\gamma(x) = \begin{cases} \frac{2\csc^2 x}{A'} \int_0^\pi u^2(x, y) \frac{dg^{-1}(u(x, y))}{du} f_{E,\delta}(r, g^{-1}(u(x, y))) dy & \text{for } -\pi \leq x < \pi \\ 0 & \text{elsewhere} \end{cases} \quad (7.3.7)$$

where

$$u(x,y) = A'(\cos y - \cot x \sin y)^{-1}$$

7.3.5 and 7.3.7 are rather cumbersome equations for the envelope and phase densities of the non-gaussian noise out of the bandpass non-linear device given by $g(x)$. In the case of the hard limiter, they will be shown to reduce to simple expressions (reference 7.5-2). Apart from this example, no further development or application of the results is undertaken.

7.4 Second Moments of Noise Components

The effect of noise on communication is largely dependent on its power or second moment. Based on results developed in 7.2, we here determine the second moments of the in and out of phase noise components. *

The power of the out of phase noise is

$$\begin{aligned} N_2' &= \langle n_2'^2(t) \rangle = E\{n_2'^2\} \\ &= E\{g^2(E)\sin^2\delta\} \\ &= \int_0^\infty \int_{-\pi}^\pi f_{E,\delta}(r,y) g^2(r) \sin^2 y \, dy dr \end{aligned}$$

* The important question: "How is this power distributed in the frequency spectrum?" remains unanswered. The spectral densities of the in and out of phase noises from a bandpass hard limiter are evaluated in 7.5-5, but a general and tractable result appears unattainable. The autocorrelation method (reference 4.2.3) gives an expression for the power spectrum of the total noise process out of any non-linearity; however the in and out of phase components cannot be separated.

Applying 7.2.7, N'_2 can be written as

$$N'_2 = \int_{-\pi}^{\pi} \left[\int_0^{\infty} \frac{2g^2(r) e^{-\frac{(r-A\cos y)^2}{2\sigma^2}}}{\sqrt{2\pi} \operatorname{erfc}(-\sqrt{\rho}\cos y) \sigma} dr \right] \sin^2 y f_{\delta}(y) dy \quad (7.4.1)$$

with $f_{\delta}(y)$ given by 7.2.3. From 7.2.1, the in phase noise power is

$$\begin{aligned} N'_1 &= E\{n_1'^2\} \\ &= E\{g^2(E)\cos^2\delta\} - 2A'E\{g(E)\cos\delta\} + A'^2 \end{aligned} \quad (7.4.2)$$

In 7.1.5 the amplitude of the signal out of the non-linear device has been defined to be

$$A' = E\{g(E)\cos\delta\}$$

With $\cos^2\delta = 1 - \sin^2\delta$,
equation 7.4.2 reduces to

$$N'_1 = 2(P_0 - \frac{1}{2}A'^2 - \frac{1}{2}N'_2) \quad (7.4.3)$$

where, from 4.3.12 and 7.2.6, the total output power is

$$\begin{aligned} P_0 &= \frac{1}{2}E\{g^2(E)\} \\ &= \frac{e^{-\rho}}{2} \int_0^{\infty} r g^2(\sigma r) e^{-\frac{r^2}{2}} I_0(\sqrt{2\rho}r) dr \end{aligned} \quad (7.4.4)$$

Equation 7.4.3 is an expression for twice the power of the noise in phase with the signal.* Note that N'_1 is simply

* N'_1 and N'_2 are the second moments of the noise modulation on carriers² in and out of phase with the transmitted signal $s(t)$.

2 (total power - signal power - out of phase noise power) at the output. The total output power, $\frac{1}{2}\langle g^2(E(t)) \rangle$, was obtained in 7.4.4 by integrating $\frac{1}{2}g^2(E)$ over the density of the input envelope E .

Knowing the total output power P_o and the output signal power $\frac{1}{2}A'^2$, the output signal-to-noise ratio can be written:

$$\rho_o = \frac{A'^2}{2P_o - A'^2} \quad (7.4.5)$$

The suppression of the signal, or change in signal-to-total power (reference 5.3.1) is then

$$\alpha = \left(1 + \frac{2\sigma^2}{A^2}\right) \frac{A'^2}{2P_o} \quad (7.4.6)$$

7.5 The Hard Limiter

A detailed presentation of the multiple-access hard limiting repeater was given in chapter 5. We now apply the results of 7.1-7.4 in the particular case of an input consisting of one signal plus noise. The in-phase/out-of-phase noise densities and their second moments, and envelope/phase densities reduce to simple forms derived here. Applications of the results to TDMA satellite communications problems are given in 7.7 and 7.8.

7.5-1 Quadrature Noise Densities:

From 7.1.1, with the additive gaussian noise written in quadrature form, the input to the limiter is

$$x(t) = (A + n_1(t)) \cos(\omega_0 t + \theta(t)) + n_2(t) \sin(\omega_0 t + \theta(t)) \quad (7.5.1)$$

The output is then

$$\begin{aligned} y(t) &= \frac{x(t)}{\text{env}(x(t))} \quad (\text{env}=\text{envelope}) \\ &= \left(\frac{A + n_1(t)}{\sqrt{(A+n_1(t))^2 + n_2^2(t)}} \right) \cos(\omega_0 t + \theta(t)) \\ &\quad + \left(\frac{n_2(t)}{\sqrt{(A+n_1(t))^2 + n_2^2(t)}} \right) \sin(\omega_0 t + \theta(t)) \end{aligned} \quad (7.5.2)$$

The signal (deterministic) component of the output is

$$\begin{aligned} E\{y(t)\} &= E \left\{ \frac{A + n_1(t)}{\sqrt{(A+n_1(t))^2 + n_2^2(t)}} \right\} \cos(\omega_0 t + \theta(t)) \\ &= \frac{1}{2\pi\sigma^2} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(A+u) e^{-\frac{u^2+v^2}{2\sigma^2}}}{\sqrt{(A+u)^2 + v^2}} du dv \right] \cos(\omega_0 t + \theta(t)) \\ &= A' \cos(\omega_0 t + \theta(t)) \end{aligned}$$

We have here integrated over the independent gaussian random variables $n_1(t)$ and $n_2(t)$.^{*} Alternatively, from 5.2.2 and

7.1.4, the amplitude of the signal can be written as a function of the input SNR as

* $n_1(t)$ and $n_2(t)$ are not necessarily independent processes. The possible correlation between $n_1(t)$ and $n_2(t+\tau)$ ($\tau \neq 0$) is irrelevant here, but is an important consideration when second order statistics are desired (see 7.5-5).

$$A'(\rho) = \int_0^{\sqrt{2\rho}} {}_2F_1(.5, -.5; 1; \frac{r^2}{2\rho}) re^{-\frac{r^2}{2}} dr + \int_{\sqrt{2\rho}}^{\infty} (\frac{\sqrt{2\rho}}{2r}) {}_2F_1(.5, .5; 2; \frac{2\rho}{r^2}) re^{-\frac{r^2}{2}} dr \quad (7.5.3)$$

Simpler expressions for $A'(\rho)$ are given by 7.7.3.

The noise (random) output component is now written in the quadrature form:

$$\begin{aligned} n'(t) &= y(t) - s'(t) \\ &= \left(\frac{A + n_1(t)}{\sqrt{(A+n_1(t))^2 + n_2^2(t)}} - A' \right) \cos(\omega_0 t + \theta(t)) \\ &\quad + \left(\frac{n_2(t)}{\sqrt{(A+n_1(t))^2 + n_2^2(t)}} \right) \sin(\omega_0 t + \theta(t)) \\ &= n_1'(t) \cos(\omega_0 t + \theta(t)) + n_2'(t) \sin(\omega_0 t + \theta(t)) \quad (7.5.4) \end{aligned}$$

From 7.2.1 to 7.2.5 inclusive, with the limiter transfer function described by

$$g(E) = 1,$$

the in and out of phase noise processes have probability density functions

$$f_{n_1'}(z) = \begin{cases} \frac{2 f_0(\cos^{-1}(z+A'))}{\sqrt{1 - (z+A')^2}} & \text{for } |z+A'| \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad (7.5.5)$$

and

$$f_{n_2'}(z) = \begin{cases} \frac{f_{\delta}(\sin_1^{-1}|z|) + f_{\delta}(\sin_2^{-1}|z|)}{\sqrt{1-z^2}} & \text{for } |z| \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad (7.5.6)$$

Here, $f_{\delta}(y)$, the probability density function of the phase deviation incurred on a sinusoid by additive gaussian noise, is given by equation 7.2.3. We note that $n_1'(t)$ and $n_2'(t)$ are uncorrelated but not independent random variables; in fact, given one, the other can assume only two values. This coupling is not generally the case for quadrature noise components out of a bandpass non-linearity. It is also observed that $n_1'(t)$ and $n_2'(t)$ take values in the intervals $[-1-A', 1-A']$ and $[-1, 1]$ respectively; at the extreme values the densities have poles of order $\frac{1}{2}$.

Computer programs were written to implement equations 7.5.5 and 7.5.6, and plots of $f_{n_1'}(z)$ and $f_{n_2'}(z)$ for input signal-to-noise ratios of .25, 1, and 4 are presented in figures 15 and 16. Further discussion follows in 7.5-7.

Figure 15

Figure 16

7.5-2 Envelope and Phase Densities:

A transformation from cartesian to polar co-ordinates is performed and probability density functions of the noise envelope and phase (relative to the signal) are derived. With n_1' and n_2' (random variables) denoting the in and out of phase noise processes respectively, the envelope of $n'(t)$ is written as

$$\begin{aligned} R' &= \sqrt{n_1'^2 + n_2'^2} \\ &= \sqrt{1 - A'^2 + 2A'n_1'} \end{aligned} \quad (7.5.7)$$

since

$$n_2' = \pm \sqrt{1 - (n_1' + A')^2} \quad (7.5.8)$$

from simple trigonometry.

Equation 7.5.7 is a monotonic transformation described by

$$r'(y) = \sqrt{1 - A'^2 - 2A'y}$$

$$\text{or} \quad y(r) = \frac{(1 - A'^2) - r^2}{2A'}$$

From 2.4.4, then, the density of R' is

$$\begin{aligned} f_{R'}(r) &= f_{n_1'}(y) \left| \frac{dy}{dr} \right| \\ &= \begin{cases} \frac{r}{A'} f_{n_1'}\left(\frac{1 - A'^2 - r^2}{2A'}\right) & \text{for } r \geq 0 \\ 0 & \text{for } r < 0 \end{cases} \end{aligned} \quad (7.5.9)$$

The phase of the output noise is a random variable defined by

$$\begin{aligned}\gamma &= \cot^{-1}(n'_1/n'_2) \\ &= \cot^{-1}\left(\frac{n'_1}{\sqrt{1-(A'+n'_1)^2}}\right)\end{aligned}\quad (7.5.10)$$

With

$$f_Y(x) = \frac{f_{n'_1}(x)}{\left|\frac{dy}{dx}\right|},$$

the density of γ on $0, \pi$ is

$$f_Y(x) = \begin{cases} \sqrt{1-(A'+y)^2} \left(\frac{1-A'^2-2A'y}{1-A'^2-A'y} \right) f_{n'_1}(y) & \text{for } 0 \leq x < \pi \\ 0 & \text{elsewhere} \end{cases}\quad (7.5.11)$$

where

$$\begin{aligned}y &= \left[-A' + (-1)^i \sqrt{A'^2 + (1-A'^2)\sec^2 x} \right] \cos^2 x \\ i &= \begin{cases} 0 & \text{for } 0 \leq x < \pi/2 \\ 1 & \text{for } \pi/2 \leq x < \pi \end{cases}\end{aligned}$$

We now show that 7.5.9 and 7.5.11 can be obtained by applying the general results given by 7.3.5 and 7.3.7. With

$$g(E) = 1,$$

equation 7.2.9 implies that

$$f_{E,\delta} \left(e, \cos^{-1} \left(\frac{A'^2 + 1 - r^2}{2A'} \right) \right) = f_{E|\delta}(e|q) f_{\delta}(q)$$

$$\text{where } q = \cos^{-1} \left(\frac{A'^2 + 1 - r^2}{2A'} \right)$$

From 7.5.5,

$$f_{\delta}(q) = \frac{\sqrt{1 - \left(\frac{A'^2 + 1 - r^2}{2A'} \right)^2}}{2} f_{n_1'} \left(\frac{1 - A'^2 - r^2}{2A'} \right),$$

and 7.3.5 can be written in the form

$$f_{R'}(r) = \frac{r}{A'} f_{n_1'} \left(\frac{1 - A'^2 - r^2}{2A'} \right) \int_S f_{E|\delta}(e|q) de \quad (7.5.12)$$

Here S is the set defined by those values of e for which

$$2A' \geq A'^2 + 1 - r^2 \quad (\text{i.e. } |r| \leq A' + 1)$$

Hence, for $0 \leq r \leq A' + 1$, S is the entire real line; for $r > A' + 1$, it is the empty set. Recalling that for any value of q

$$\int_{-\infty}^{\infty} f_{E|\delta}(e|q) de = 1,$$

equation 7.5.12 therefore reduces to the result given by 7.5.9.

In the case of the hard limiter, $g(E)$ does not have a unique inverse, and equation 7.3.7 cannot be applied directly to determine $f_Y(x)$. From 7.3.6, however, with

$$g(E) = 1 ,$$

$$\cos(\delta) = A' + n_1' ,$$

$$\text{and } \sin(\delta) = n_2' = \sqrt{1 - (A' + n_1')^2} \quad (\text{for } \delta \geq 0) ,$$

for $0 \leq x < \pi$,

$$\begin{aligned} P\{\gamma \leq x\} &= P\{A'(A' + n_1' - \cot x \sqrt{1 - (A' + n_1')^2})^{-1} \leq 1\} \\ &= P\{n_1' \geq \cot x \sqrt{1 - (A' + n_1')^2}\} \\ &= P\{\sec^2 x n_1'^2 + 2A'n_1' - (1 - A'^2) \geq 0\} \\ &= P\{n_1' \geq y\} \end{aligned}$$

where y is given in equation 7.5.11. Hence,

$$f_Y(x) = \frac{dP\{\gamma \leq x\}}{dx} = f_{n_1'}(y) \left| \frac{dx}{dy} \right|^{-1} \quad (7.5.13)$$

$$\text{with } \frac{dx}{dy} = \frac{d}{dy} \cot^{-1} \left(\frac{y}{\sqrt{1 - (A' + y)^2}} \right)$$

Equation 7.5.13 then reduces to the formula given by 7.5.11.

$f_R(r)$ and $f_Y(x)$ for input signal-to-noise ratios of .25,

1, and 4 are plotted in figures 17 and 18. As the SNR increases

Figure 18

the range of possible values for R' , $[1-A', 1+A']$, increases (although its second moment diminishes). The phase density of $n'(t)$ becomes peaked at $\gamma = \pi/2$ (i.e. along the out of phase axis) as the SNR increases. Further explanation and detailing of these effects appears in 7.5-7.

7.5-3 Second Moment Analysis:

Expressions for the output signal-to-noise ratio and second moments of the in and out of phase noise processes are now derived. From 7.1.5 the amplitude of the signal out of the limiter is

$$\begin{aligned}
 A' &= \frac{1}{\pi\sigma^2} \int_0^\infty r e^{-\frac{r^2}{2\sigma^2}} \left[\int_0^\pi \frac{A - r \cos\alpha}{\sqrt{r^2 - 2A r \cos\alpha + A^2}} d\alpha \right] dr \\
 &= \frac{1}{\pi} \int_0^\infty r e^{-\frac{r^2}{2}} \left[\int_0^\pi \frac{1 - \frac{r}{\sqrt{2\rho}} \cos\alpha}{\sqrt{1 - \sqrt{\rho/2} r \cos\alpha + r^2/2\rho}} d\alpha \right] dr
 \end{aligned}
 \tag{7.5.1-}$$

where $\rho = \frac{A^2}{2\sigma^2}$ = input signal-to-noise ratio.

Alternate expressions for $A'(\rho)$ are given by 7.5.3 and 7.7.3. Since the total power out of the hard limiting repeater is .5, the output noise power and signal-to-noise ratio are:

$$N' = \frac{1}{2} (1 - A'^2(\rho)) \quad (7.5.15)$$

$$\text{and} \quad \rho_o = \frac{A'^2(\rho)}{1 - A'^2(\rho)} \quad (7.5.16)$$

A computer program implementing equation 7.5.15 was written. In figure 19 we have plotted ρ_o/ρ , the change in signal-to-noise ratio, versus ρ . At high SNR's it is observed there is a 3 dB SNR improvement, at low SNR's about a 1 dB degradation. Blachman [12, chapter 2, p. 86] has examined the effect of a limiter on SNR for a constant envelope signal and gaussian noise. He has shown that

$$\rho_o = \frac{\pi\rho \left[I_0\left(\frac{\rho}{2}\right) + I_1\left(\frac{\rho}{2}\right) \right]^2}{4e^\rho - \pi\rho \left[I_0\left(\frac{\rho}{2}\right) + I_1\left(\frac{\rho}{2}\right) \right]^2} \quad (7.5.17)$$

is an expression for the output SNR as a function of the input SNR. It is noteworthy that there is pointwise agreement between the numerical results given by 7.5.16 and 7.5.17.

The output noise power N' can be divided into the powers of noise components in phase with and quadrature to the desired signal. From 5.2.4 (hard limiter with multiple input), or 7.4.2 (general non-linearity with signal and noise input), the out of phase noise power is

Figure 19

$$\begin{aligned}
N'_2 &= E\{n_2'^2\} = E\{\sin^2\delta\} \\
&= \frac{1}{2} - \frac{1}{2\sigma^2} \int_0^A r \left(1 - \frac{r^2}{A^2}\right) e^{-\frac{r^2}{2\sigma^2}} dr \\
&= \frac{1 - e^{-\rho}}{2\rho}
\end{aligned} \tag{7.5.18}$$

after performing the integration. The in phase noise power can then (reference 7.4.3) be written as

$$N'_1 = E\{n_1'^2\} = 1 - A'^2 - \left(\frac{1 - e^{-\rho}}{2\rho}\right) \tag{7.5.19}$$

$N'_1(\rho)$ and $N'_2(\rho)$ are plotted in figure 20. For

$\rho \gg 1$, it is observed that

$$N'_2 \gg N'_1,$$

and in fact the limiter has virtually removed the in phase noise and given a resulting 3 dB improvement in signal-to-noise ratio.

Figure 20

7.5-4 Price - Mehler Methods:

The problem of the computation of the moments of non-zero mean circularly complex gaussian noise has been treated by McGee [2]. We here make use of some of his results to derive alternate expressions for the signal amplitude and quadrature noise powers out of the bandpass hard limiter.

From 7.5.2, the input to the limiter is represented as a bivariate gaussian random variable given by

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N \left(\begin{pmatrix} A \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right) \quad (7.5.20)$$

In complex notation this is written as

$$x = x_1 + jx_2 \quad (7.5.21)$$

Here x_1 (the in phase process) has mean A (the signal amplitude) and variance σ^2 . x_2 , the quadrature noise, has zero mean and variance σ^2 , and is independent of x_1 .† The output from the limiter can then be written in the form (* denotes complex conjugate)

† Hence the locus of $p(x_1, x_2) = c$ is a circle on the x_1 - x_2 plane centered at $(A, 0)$. In general, for equal variances gaussian variates, the surface of constant probability is an ellipsoid; the joint distribution is circular only when the variates are uncorrelated.

$$y = \frac{x}{|x|} = \frac{x}{(xx^*)^{1/2}} = x^{1/2} x^{*-1/2}$$

y is a complex number having the phase characteristic of x , but unity amplitude. The mean of y has the real value

$$\mu_y = E\{x^{1/2} x^{*-1/2}\}, \quad (7.5.22)$$

and the variance of its real (i.e. in phase) component is

$$\begin{aligned} \sigma_i^2 &= E\{(\tfrac{1}{2}(y+y^*))^2\} - \mu_y^2 \\ &= \frac{1}{2} + \frac{1}{2} E\{xx^{*-1}\} - \mu_y^2 \end{aligned} \quad (7.5.23)$$

McGee has shown that if x is a circularly complex gaussian process:

$$E\{x^m x^{*n}\} = (-j\sigma)^{n+m} H_{n,m}(j\frac{\bar{x}}{\sigma}, j\frac{\bar{x}^*}{\sigma}) \quad (7.5.24)$$

where $\bar{x} = E\{x_1 + jx_2\} = A$,

in our case, and $H_{n,m}(z, z^*)$ is the univariate complex Hermite function defined by

$$H_{n,m}(z, z^*) = \frac{1}{2\pi} e^{\frac{1}{2}zz^*} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (j\zeta^*)^n (j\zeta)^m e^{-j \frac{(\zeta^*z + z^*\zeta - \zeta\zeta^*)}{2}} du dv$$

with $\zeta = u + jv$,

and $n-m = \text{an integer.}$

By changing to polar co-ordinates and using the infinite integral formulas in Watson [4, chapter 4], the following representation can also be derived:

$$H_{n,m}(z, z^*) = \begin{cases} \frac{n!}{(n-m)!} (z^*)^{n-m} (-2)^m {}_1F_1(-m, n-m+1; \frac{1}{2}zz^*) \\ \text{for } n-m \geq 0, n \geq 0 \\ \frac{m!}{(n-m)!} (z)^{m-n} (-2)^n {}_1F_1(-n, m-n+1; \frac{1}{2}zz^*) \\ \text{for } m-n \geq 0, m \geq 0 \end{cases} \quad (7.5.25)$$

${}_1F_1(a, b; x)$ is the confluent hypergeometric function defined on page

From equation 7.5.24 and the above, then, 7.5.22 and 7.5.23 reduce to

$$\begin{aligned} \mu_y &= H_{-\frac{1}{2}, \frac{1}{2}}(j \frac{A}{\sigma}, j \frac{A}{\sigma}) \\ &= \sqrt{\frac{\pi}{4} \rho} {}_1F_1(.5, 2; -\rho) \end{aligned} \quad (7.5.26)$$

$$\begin{aligned} \text{and } \sigma_i^2 &= \frac{1}{2} + \frac{1}{2} H_{-1, 1}(j \frac{A}{\sigma}, j \frac{A}{\sigma}) - \mu_y^2 \\ &= \frac{1}{2} + \frac{\rho}{4} {}_1F_1(1, 3; -\rho) - \mu_y^2 \end{aligned}$$

respectively. Now

$$\begin{aligned}
{}_1F_1(2,3;\rho) &= 2 \frac{d}{d\rho} {}_1F_1(1,2;\rho) \\
&= 2 \frac{d}{d\rho} \left(\frac{e^\rho - 1}{\rho} \right) \\
&= 2 \left(\frac{e^\rho}{\rho} - \frac{e^\rho - 1}{\rho^2} \right)
\end{aligned}$$

and

$${}_1F_1(1,3;-\rho) = {}_1F_1(2,3;\rho) e^{-\rho}$$

Therefore,

$$\sigma_i^2 = 1 - \mu_y^2 - \frac{1 - e^{-\rho}}{2\rho} \quad (7.5.27)$$

The signal component of the limiter output is defined to be the expected value of $y(t)$ (reference equation 4.3.3). Hence its amplitude is given by

$$A' = \sqrt{\mu_y \mu_y^*} = \mu_y$$

From 5.3.1 and 7.5.26, then, the signal suppression factor is

$$\begin{aligned}
\alpha &= \left(1 + \frac{1}{\rho} \right) A'^2 \\
&= \frac{\pi}{4} (1 + \rho) \left[{}_1F_1(.5,2;-\rho) \right]^2 \quad (7.5.28)
\end{aligned}$$

This verifies Lyons' result (5), p. 99, and can also be shown to be consistent with Blachman's SNR equation 7.5.15.

σ_i^2 in equation 7.5.27 is precisely the in phase noise power defined by

$$\begin{aligned} N'_1 &= E\{\cos^2\delta\} - E^2\{\cos\delta\} \\ &= 1 - A'^2 - E\{\sin^2\delta\} \end{aligned}$$

The out of phase noise power is then

$$\begin{aligned} N'_2 &= E\{\sin^2\delta\} \\ &= \frac{1 - e^{-\rho}}{2\rho} \quad , \end{aligned}$$

thus verifying equation 7.5.18.

In conclusion, a Price-Mehler method can be used to compute moments of a non-linear function of a circularly complex gaussian process with a non-zero mean. The approach can be regarded as an extension of techniques used for baseband signals (reference 4, 5, 11, chapter 2). Further references are given by [3] to [9] inclusive.

7.5-5 Spectral Analysis:

We here develop expressions for the autocorrelation functions of the quadrature noise components from a hard limiter. If the signal is an unmodulated carrier and $n(t)$ has a spectral density which is symmetric about ω_0 *, the autocorrelation function of the out of phase noise is written as

$$\begin{aligned}
 R_{n_2'}(\tau) &= E\{n_2'(t+\tau) n_2'(t)\} \\
 &= E \left\{ \frac{n_2(t+\tau) n_2(t)}{\sqrt{(A^2 - 2An_1(t+\tau) + n_1^2(t+\tau) + n_2^2(t+\tau)) (A^2 - 2An_1(t) + n_1^2(t) + n_2^2(t))}} \right\} \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{i=1}^2 \left(\frac{y_i}{\sqrt{A^2 - 2Ax_i + x_i^2 + y_i^2}} \right) e^{-\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2r_n(\tau)(x_1x_2 + y_1y_2)}{2(1-r_n^2(\tau))\sigma^4}} \\
 &\quad \frac{1}{4\pi^2 (1-r_n^2(\tau))\sigma^4} dx_1 dx_2 dy_1 dy_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^2 \left(\frac{(1-r_n^2(\tau)) y_i}{\sqrt{\rho^2 - 2\rho\sqrt{1-r_n^2(\tau)}\rho x_i + (1-r_n^2(\tau))(x_i^2 + y_i^2)}} \right) e^{r_n(\tau)(x_1x_2 + y_1y_2)} \right] \\
 &\quad \frac{1}{\pi^2} e^{-\frac{x_1^2 + x_2^2 + y_1^2 + y_2^2}{\sigma^2}} dx_1 dx_2 dy_1 dy_2 \quad (7.5.29)
 \end{aligned}$$

where $r_n(\tau)$ is the correlation coefficient of the baseband equivalent of $n(t)$. We have here integrated over the variables

* This is a necessary and sufficient condition [13, chapter 2, equation 10-9²] for $n_1(t)$ and $n_2(t)$ to be uncorrelated and hence independent processes. For a modulated carrier and/or a non-symmetric noise spectrum, the cross-correlation function $E\{n_1(t+\tau)n_2(t)\} = -E\{n_2(t+\tau)n_1(t)\}$ is non-zero and must be computed in order to write the joint density of the four variates in 7.5.30, also the baseband equivalent correlation function is undefined [10, chapter 2, p. 162].

$$n_1(t), n_1(t+\tau), n_2(t), n_2(t+\tau) \sim N \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1-r_n^2(\tau) & 0 & 0 \\ 1-r_n^2(\tau) & 1 & 0 & 0 \\ 0 & 0 & 1 & 1-r_n^2(\tau) \\ 0 & 0 & 1-r_n^2(\tau) & 1 \end{bmatrix} \sigma^2 \right) \quad (7.5.30)$$

Following a similar procedure, the autocorrelation function of the in phase noise is written as

$$\begin{aligned} R_{n_1}(\tau) &= E\{n_1'(t+\tau)n_1'(t)\} \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\prod_{i=1}^2 \left(\frac{\sqrt{1-r_n^2(\tau)}\sqrt{\rho} - (1-r_n^2(\tau))x_i}{\sqrt{\rho-2\sqrt{1-r_n^2(\tau)}\sqrt{\rho}x_i + (1-r_n^2(\tau))(x_i^2+y_i^2)}} \right) e^{r_n(\tau)(x_1x_2+y_1y_2)} \right] \\ &\quad \frac{1}{\pi^2} e^{-x_1^2+x_2^2+y_1^2+y_2^2} dx_1 dx_2 dy_1 dy_2 = A'^2 \end{aligned} \quad (7.5.31)$$

Computer programs were written to evaluate $R_{n_1}(\tau)$ and $R_{n_2}(\tau)$ and their Fourier transforms, the spectra of the quadrature noises. Four-dimensional 10 and 16 point Hermite quadrature * formulas respectively were employed to perform the multiple integrations in 7.5.29 and 7.5.31.

* Note that equations 7.5.29 and 7.5.31 are in the desired form for the application of Hermite quadrature [10,p.924].

Figure 21

Figure 22

Figures 21 and 22 present numerical results for ideal band-limited noise *; i.e.,

$$r_n(\tau) = \frac{\sin(2\pi f_n \tau)}{2\pi f_n \tau} \quad (7.5.32)$$

In the $\rho = 5$ case, note that the $n_1'(t)$ and $n_2'(t)$ spectral density curves are approaching the triangular and rectangular shapes to be predicted by the high SNR asymptotic forms 7.5.38 and 7.5.40. Observe also the rapid decrease in the power of the in phase noise as ρ increases.

7.5-6 Asymptotic Behaviour:

Limiting results as $\rho \rightarrow \infty$ & 0 are now examined. First -1 dB and 3 dB changes in SNR respectively are demonstrated.

Recalling that for $\rho \gg 1$, δ is, with a high degree of probability, a very small angle, we write

$$\begin{aligned} E\{n'^2(t)\} &= \frac{1}{2} (1 - E^2\{\cos(\delta)\}) \\ &\approx \frac{1}{2} E\{\delta^2\} \\ &\approx \frac{n_2^2}{2A^2} \end{aligned}$$

* The output autocorrelation functions were sampled at $(8f_n)^{-1}$ intervals (thereby allowing for an output noise bandwidth 4 times the input noise bandwidth). 64 samples of each were generated - the accuracy of the transformed (i.e. spectral) samples is limited by the numerical integration and the truncation of the input correlation function at .0125.

Hence the output SNR is

$$\begin{aligned}
 \rho_o &= \frac{.5 - E\{n'^2(t)\}}{E\{n'^2(t)\}} \\
 &\approx \frac{A^2}{E\{n_2'^2\}} \\
 &= 20
 \end{aligned} \tag{7.5.33}$$

or there is a 3.01 dB improvement.

For $\rho \ll 1$, the output signal amplitude is

$$\begin{aligned}
 A' &= E\{\cos(\delta)\} \\
 &= \frac{1}{\pi} \int_0^\infty \int_0^\pi \frac{(1 - \frac{r}{\sqrt{2\rho}} \cos\alpha) e^{-\frac{r^2}{2}} d\alpha dr}{\sqrt{(1 - \frac{r}{\sqrt{2\rho}} \cos\alpha)^2 + (\frac{r}{\sqrt{2\rho}} \sin\alpha)^2}} \\
 &= \frac{1}{\pi} \int_0^\infty \int_0^\pi \frac{\text{sgn}(1 - \frac{r}{\sqrt{2\rho}} \cos\alpha) e^{-\frac{r^2}{2}} d\alpha dr}{\sqrt{1 + (\frac{r \sin\alpha}{r \cos\alpha - \sqrt{2\rho}})^2}} \\
 &\approx \frac{1}{\pi} \int_0^\infty \left[\int_{\pi/2}^\pi - \int_0^{\pi/2} \right] \frac{r e^{-\frac{r^2}{2}} d\alpha dr}{\sqrt{1 + \tan^2\alpha (1 - \frac{\sqrt{2\rho}}{r \cos\alpha})^{-2}}} \\
 &\approx \frac{1}{\pi} \int_0^\infty \int_0^\pi \frac{-\cos\alpha r e^{-\frac{r^2}{2}} d\alpha dr}{\sqrt{1 + \frac{\sqrt{2\rho}}{r} \tan\alpha \sin\alpha}}
 \end{aligned}$$

where we have here used

$$\left(1 - \frac{\sqrt{2\rho}}{r \cos \alpha}\right)^{-2} \approx 1 + \frac{2\sqrt{2\rho}}{r \cos \alpha}$$

and $\sec^2 \alpha = \tan^2 \alpha + 1$

With

$$\left(1 + \frac{2\sqrt{2\rho}}{r} \tan \alpha \sin \alpha\right)^{-\frac{1}{2}} \approx 1 - \frac{\sqrt{2\rho}}{r} \tan \alpha \sin \alpha,$$

our result becomes

$$\begin{aligned} A' &\approx \frac{1}{\pi} \int_0^\infty \int_0^\pi -\cos \alpha \left(1 - \frac{\sqrt{2\rho}}{r} \tan \alpha \sin \alpha\right) r e^{-\frac{r^2}{2}} d\alpha dr \\ &= \frac{\sqrt{\pi\rho}}{2} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\frac{r^2}{2}} dr \right] \left[\frac{2}{\pi} \int_0^\pi \sin^2 \alpha d\alpha \right] \\ &= \frac{\sqrt{\pi\rho}}{2} \end{aligned}$$

The output SNR is then

$$\rho_o = \frac{A'^2}{1 - A'^2} \approx \frac{\pi}{4} \rho, \quad (7.5.34)$$

or there is a 1.05 degradation in signal-to-noise ratio for a weak signal sharing the hard limiter with gaussian noise.

Equations 7.5.33 and 7.5.34 are depicted in the ρ_o/c graph of figure 19. They may be readily obtained from Blackman's expression 7.5.15, and were first demonstrated by Davenport [7, chapter 5]. We again point out that the 3 dB high SNR gain is achieved by the removal (in terms of its second moment) of noise which is in phase with the signal. The SNR gain, however, does not improve the detectability of angle modulation in the signal (see 7.7 and 7.8), and therefore is somewhat misleading.

We now state without proof that δ has asymptotic distributions:

$$\begin{aligned} f_{\delta}(y) &\approx \frac{1}{\sqrt{\pi}} e^{-\rho \tan^2 y} \sqrt{\rho} \sec^2 y \quad \text{for } |y| \leq \pi/2 \\ &\approx \sqrt{\frac{\rho}{\pi}} e^{-\rho y^2} \quad (\text{i.e. } N(0, 1/2\rho)) \end{aligned} \quad (7.5.35)$$

for $\rho \gg 1$, and

$$f_{\delta}(y) \approx \frac{1}{2\pi} (1 + \sqrt{\rho} \cos y) \quad \text{for } -\pi \leq y < \pi \quad (7.5.36)$$

for $\rho \ll 1$. These lead to asymptotic forms (reference 7.5.5, 7.5.6) for the densities of the in and out of phase noise. As $\rho \rightarrow 0$, it is observed that

$$f_{n_1'}(z), f_{n_2'}(z) \rightarrow \frac{1}{\pi\sqrt{1-z^2}} \quad \text{on } -1 \leq z \leq 1 \quad (7.5.37)$$

Feller [11, p. 29] and others have shown that the projections of a constant amplitude, random phase two-dimensional vector on the co-ordinate axes have the above probability density function. The constant amplitude is produced by the hard limiter; in our case the phase is only randomly (i.e. uniformly) distributed in the limit as $\rho \rightarrow 0$. Equation 7.5.37 is obtained by letting $f_\delta(y) \rightarrow 1/2\pi$ and $A' \rightarrow 0$ in 7.5.5 and 7.5.6.

Finally, high SNR limiting forms are developed for $R_{n_2'}(\tau)$ and $R_{n_1'}(\tau)$. For $\rho \gg 1$,

$$\begin{aligned} R_{n_2'}(\tau) &= E\{\sin(\delta(t+\tau)) \sin(\delta(t))\} \\ &= E\left\{ \frac{n_2(t+\tau) n_2(t)}{A^2} \right\} \\ &= \frac{r_n(\tau)}{2\rho} \end{aligned} \quad (7.5.38)$$

where $r_n(\tau)$ is the correlation function of the baseband equivalent of the input band-limited gaussian noise (see footnote on p. 166). Further,

$$\begin{aligned}
 R_{n_1}(\tau) &= E\{(\cos(\delta(t+\tau)) - A') (\cos(\delta(t)) - A')\} \\
 &= E\{\cos(\delta(t+\tau)) \cos(\delta(t))\} - A'^2
 \end{aligned}$$

from

$$E\{\cos(\delta(t+\tau))\} = E\{\cos(\delta(t))\} = A'.$$

With a high probability δ is a small angle. Therefore,

$$\begin{aligned}
 \cos(\delta) &\approx \sqrt{1 - n_2^2/A^2} \quad \text{and} \\
 \cos(\delta(t+\tau))\cos(\delta(t)) &\approx \sqrt{1 - \frac{n_2^2(t+\tau)+n_2^2(t)}{A^2} + \frac{n_2^2(t+\tau)n_2^2(t)}{A^4}} \\
 &\approx 1 - \frac{n_2^2(t+\tau)+n_2^2(t)}{2A^2} - \frac{n_2^4(t+\tau)-2n_2^2(t+\tau)n_2^2(t)+n_2^4(t)}{8A^4}
 \end{aligned}$$

Using

$$E\{n_2^2(t+\tau)\} = E\{n_2^2(t)\} = \sigma^2$$

and

$$E\{n_2^4(t+\tau)\} = E\{n_2^4(t)\} = 3\sigma^4$$

for stationary gaussian noise, $R_{n_1}(\tau)$ reduces to

$$R_{n_1'}(\tau) \approx 1 - A'^2 - \frac{1}{2\rho} - \frac{3}{8\rho^2} + \frac{E\{n_2^2(t+\tau)n_2^2(t)\}}{4A^4} \quad (7.5.39)$$

Davenport and Root [10, chapter 2, p. 225] have shown that if $z(t)$ is a zero mean stationary gaussian process,

$$E\{z^2(t+\tau) z^2(t)\} = E^2\{z^2(t)\} + 2E^2\{z(t+\tau) z(t)\} \quad .$$

Applying this result 7.5.39 becomes

$$R_{n_1'}(\tau) \approx 1 - A'^2 - \frac{1}{2\rho} - \frac{3}{8\rho^2} + \frac{1+2r_n^2(\tau)}{16\rho^2}$$

Using

$$\lim_{\tau \rightarrow \infty} r_n(\tau) = E^2\{n(t)\}/\sigma^2 = 0 \quad \text{and}$$

$$\lim_{\tau \rightarrow \infty} R_{n_1'}(\tau) = E^2\{n_1'(t)\} = 0 \quad ,$$

this reduces to

$$R_{n_1'}(\tau) \approx \frac{r_n^2(\tau)}{8\rho^2} \quad (7.5.40)$$

Also obtained are a second order asymptotic expression for the output signal amplitude as a function of the input SNR,

$$A' = \sqrt{1 - \frac{1}{2\rho} - \frac{5}{16\rho^2}} \quad , \quad (7.5.41)$$

and an asymptotic form for the in phase noise power,

$$N'_1 = R_{n_1}(0) \approx \frac{1}{8\rho^2} \quad (7.5.42)$$

Defining $n_b(t)$ to be the low pass equivalent of $n(t)$ (which has been assumed to have a spectral density symmetric about ω_0), we conclude from 7.5.38 and 7.5.40 that for high input signal-to-noise ratios:

- (1) $n'_2(t)$ has the same spectral characteristic as $n_b(t)$,
- (2) $n'_1(t)$ has the spectral characteristic, minus the DC term, of $n_b(t)$ passed through a square law device, and
- (3) N'_2 and N'_1 diminish as ρ^{-1} and ρ^{-2} respectively - in a dB plot N'_1 decreases at twice the rate of N'_2 (this can be observed in figure 20).

7.5-7 Geometric Interpretation:

We now take the opportunity to re-examine from a geometric point of view the bandpass non-linear process and the parameters which describe it. This serves to consolidate an understanding of the definitions and results of this section, provides a simple bridge relating the

physical process to the analytical development, and introduces an approach used extensively for purposes of insight and explanation in succeeding chapters.

The input to a hard-limiting bandpass limiter consists of bandpass signal and noise components denoted in phasor (or vector) notation by $\vec{s}(t)$ and $\vec{n}(t)$ respectively. The noise envelope is given by $R(t)$ (a Rayleigh distributed random variable) and its phase relative to the signal is $\phi(t)$ (uniformly distributed on $[0, 2\pi)$). Relative to a stationary signal vector, the noise vector changes in length and direction randomly but in a continuous fashion according to its baseband correlation function. In figure 23 we have 'photographed' the phasors at time t_0 , and can now regard $\vec{n}(t_0) = (n_1, n_2)$ as a sample vector of the up-path gaussian noise process.

The output signal vector is written as

$$\vec{s}'(t_0) = A' \frac{\vec{s}(t_0)}{|\vec{s}(t_0)|} \quad (A' = E\{\cos(\delta)\})$$

The input noise sample $\vec{n}(t_0)$ uniquely determines samples of the random variables

δ = phase error on input signal resulting from additive noise

and

Figure 23

$$\begin{aligned}
\vec{n}' &= \text{stationary non-gaussian bandpass output noise} \\
&= (R', \gamma) \\
&= (n'_1, n'_2)
\end{aligned}$$

These are all depicted in figure 23. δ is the angle subtended by the resultant input (or output) vector and the input (or output) signal vector. $\vec{n}'$ is the vector difference between the total output, $(\vec{s} + \vec{n})/|\vec{s} + \vec{n}|$, and the signal component, $A' \vec{s}/|\vec{s}|$. Its amplitude/phase and quadrature representations are (R', γ) and (n'_1, n'_2) respectively. The limiter output noise, $\vec{n}'$, follows deterministically the input noise as it traces out a sample function in time. $n'(t)$ has non-gaussian statistics (reference 7.5.1 and 7.5.2) and occupies more bandwidth than the input noise. From figure 23, it is observed that

$$R' \in [1-A', 1+A'] \subseteq [0, 2] \quad \dagger$$

The support (range of possible values) of the envelope of the output noise increases with the input signal-to-noise ratio; for $\rho = 0$ (i.e. no signal at all), $A' = 0$ and $R' = 1$ with probability one. For $\rho \gg 1$, $n_2'^2 \gg n_1'^2$ and $|\gamma| \approx \pi/2$ with a high degree of probability.

$\dagger \in$ denotes 'element of' a set; $\subseteq$ means 'is a subset of'

For $\rho \ll 1$, $\frac{n'_2}{n'_1} \approx \frac{n_2}{n_1}$, implying that

$\gamma \approx$ uniformly distributed .

Hopefully, the geometric representation of figure 23 relates the definitions and results of sections 7.5-1 to 7.5-6. This type of description is applicable to more complicated non-linearities and inputs, and will be useful in the intermodulation product analysis of chapters 9 and 10.

Again, based on figure 23, a brief conceptual description of the stochastic approach to the analysis of general bandpass non-linearities is given. A narrow band input, consisting of signal and interference components, can be regarded as a rotating vector with slow variations in length (AM) and angular velocity (FM). This vector has an easily visualized instantaneous amplitude and RF phase; the output from the bandpass non-linearity similarly is a vector with an amplitude and phase shift determined by the AM/AM and AM/PM transfer characteristics of the device. The output vector can be decomposed into two vectors: one rotates in such a way as to follow the angle modulation on the signal component of the input, the second moves randomly with respect to this modulation. The first component is the signal out of the

non-linearity, the second is the interference. If the input interference consists of gaussian noise alone, the output interference is non-gaussian noise characterized as a two-dimensional random variable with statistics defined in sections 7.2 and 7.3.

7.6 The Piecewise Linear Limiter

The piecewise linear limiter (described by equation 6.2.1) has been analyzed for general inputs in section 6.2. With the input signal amplitude given by

$$\sqrt{2P} = A$$

and the interference envelope density given by

$$f_R(r) = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}},$$

equation 6.2.2 reduces to the following result for the output signal amplitude:

$$\begin{aligned} A' = \frac{1}{\pi\sigma^2} \int_0^\pi \left[\int_{L_\alpha} \left(\frac{r}{x_0} \right) \left(\frac{A - r \cos \alpha}{\sqrt{r^2 - 2Ar \cos \alpha + A^2}} \right) re^{-\frac{r^2}{2\sigma^2}} dr \right. \\ \left. + \int_{C_{L_\alpha}} \left(\frac{A - r \cos \alpha}{\sqrt{r^2 - 2Ar \cos \alpha + A^2}} \right) re^{-\frac{r^2}{2\sigma^2}} dr \right] d\alpha \end{aligned} \quad (7.6.1)$$

L_α and C_{L_α} are set functions of α defined by equation 6.2.3.

From 6.2.4 and 7.2.6, the total power out of the non-linearity is

$$P_o = \frac{e^{-\rho}}{2\sigma^2} \int_0^{x_0} \left(\frac{r^3}{x_0^2} \right) e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{Ar}{\sigma^2}\right) dr + \int_{x_0}^{\infty} r e^{-\frac{r^2}{2\sigma^2}} I_0\left(\frac{Ar}{\sigma^2}\right) dr \quad (7.6.2)$$

Substituting equations 7.6.1 and 7.6.2 into 6.2.5 gives an expression for the change in signal-to-noise ratio through the piecewise linear limiter.

From 7.2.1 and 7.2.2, the in and out of phase output noise processes are conditionally distributed as

$$n'_1 = \begin{cases} \frac{E}{x_0} \cos(\delta) - A' & \text{for } E \leq x_0 \\ \cos(\delta) - A' & \text{for } E > x_0 \end{cases} \quad (7.6.3)$$

$$\text{and } n'_2 = \begin{cases} \frac{E}{x_0} \sin(\delta) & \text{for } E \leq x_0 \\ \sin(\delta) & \text{for } E > x_0 \end{cases} \quad (7.6.4)$$

E , the input envelope, and δ , the phase distortion incurred by the additive noise, have a joint probability density function given by 7.2.8. For $0 \leq z \leq 1$, the out of phase noise has the distribution function

$$\begin{aligned}
P\{n'_2 \leq z\} &= P\{E \leq x_0 \csc(\delta)z, E \leq x_0\} + P\{\sin(\delta) \leq z, E > x_0\} \\
&= \int_0^\pi \left[\int_0^{\min\{\csc(y)z, 1\}x_0} f_{E,\delta}(r,y) dr \right] dy + \left[\int_0^{\sin^{-1}(z)} + \int_{\sin^{-1}(z)}^\pi \right] \int_{x_0}^\infty f_{E,\delta}(r,y) dr dy
\end{aligned}$$

where $\min\{a, b\}$ = the minimum of a and b .

Differentiating the above expression with respect to z gives

the following equation for the density of n'_2 :

$$\begin{aligned}
f_{n'_2}(z) &= x_0 \left[\int_{\sin^{-1}|z|}^{\pi/2} + \int_{\pi/2}^{\sin^{-1}|z|} \right] f_{E,\delta}(x_0 \csc(y)|z|, y) \csc(y) dy \\
&+ \sum_{i=1}^2 \operatorname{erfc} \left(\frac{x_0 + (-1)^i (1-z^2)^{-\frac{1}{2}} A}{\sqrt{2} \sigma} \right) \left(\operatorname{erfc} \left(\frac{(-1)^i (1-z^2)^{-\frac{1}{2}} A}{\sqrt{2} \sigma} \right) \right)^{-1} \frac{f_\delta(\sin^{-1}|z|)}{(1-z^2)^{\frac{1}{2}}}
\end{aligned}$$

(7.6.5)

where from 7.2.7 we have used

$$\int_{x_0}^\infty f_{E|\delta}(r|y) dr = \frac{1}{2} \operatorname{erfc} \left(\frac{x_0 - A \cos y}{\sqrt{2} \sigma} \right) \left(\frac{1}{2} \operatorname{erfc}(-\sqrt{2} \cos y) \right)^{-1}$$

It can then be concluded from figure 14 that n'_2 is a random variable symmetrically distributed. Hence the insertion of the absolute values around z means that 7.6.5 applies to $-1 \leq z \leq 1$, the support or full range of values of n'_2 .

Letting $x_0 \rightarrow 0$ (i.e. the hard limiting case) in 7.6.5 gives the asymptotic form

$$f_{n_2}'(z) \rightarrow \frac{f_{\delta}(\sin^{-1}_1|z|) + f_{\delta}(\sin^{-1}_2|z|)}{\sqrt{1-z^2}}, \quad |z| \leq 1,$$

the expected result from 7.5.6. In fact, the first term in 7.6.5 results from operation in the ideal linear region and the second term from operation in the region of complete saturation.

Development of an equation for $f_{n_1}'(z)$, the density of the in phase noise, follows the same procedure as that for $f_{n_2}'(z)$. It is a slightly more tedious derivation due to the presence of the A' term (see 7.6.3), and is not given here. The densities of the envelope and phase of the noise from the piecewise limiter can be theoretically determined from 7.3.5 and 7.3.6, and are also not included.

Now, from 7.4.2, the power of the out of phase noise is

$$N_2' = \int_{-\pi}^{\pi} \left[\int_0^{x_0} \left(\frac{r}{x_0} \right)^2 f_{E|\delta}(r|y) dr + \operatorname{erfc}\left(\frac{x_0 - A \cos y}{\sqrt{2} \sigma}\right) \left(\operatorname{erfc}\left(-\frac{A \cos y}{\sqrt{2} \sigma}\right) \right)^{-1} \right] \sin^2 y f_{\delta}(y) dy \quad (7.6.6)$$

Letting $x_0 \rightarrow 0$ here gives the hard limiting result of equation 7.5.18. Again, the first term in 7.6.6 results from the linear operation of the device, the second from operation in saturation. From 7.6.1, 7.6.2, and 7.6.6, equation 7.4.4 may be applied to compute N_1' , the power of the in phase noise.

Results for the piecewise limiter are functions of three parameters: A (the signal amplitude) , σ^2 (the power of the gaussian noise) , and $(x_0)^{-1}$ (the gain of the device in the linear region). For $A > x_0$, the signal is said to saturate the device, and excursions to the linear region can result from noise samples which sufficiently reduce the resultant input amplitude. For $A < x_0$, the signal operates in the linear region, and saturation can occur only with the sympathetic addition of signal and noise vectors at the input. When $A = x_0$, a singular but interesting case (since it might represent a desired operating point), there is some simplification in the equations; for example, in 7.6.1 (from 6.2.3 with $A = \sqrt{2P}$)

$$L_\alpha = \left[0, 2x_0 \cos \alpha \right] ,$$

implying that the output signal amplitude as a function of the input signal-to-noise ratio is given by

$$\begin{aligned}
A'(\rho) = \frac{1}{\pi} \int_0^\pi \left[\int_0^{2\sqrt{2\rho}\cos\alpha} \left(\frac{r}{\sqrt{2\rho}} \right) \left(\frac{\sqrt{2\rho} - r\cos\alpha}{\sqrt{r^2 - 2\sqrt{2\rho}r\cos\alpha + 2\rho}} \right) r e^{-\frac{r^2}{2}} dr \right. \\
\left. + \int_{2\sqrt{2\rho}\cos\alpha}^\infty \left(\frac{\sqrt{2\rho} - r\cos\alpha}{\sqrt{r^2 - 2\sqrt{2\rho}r\cos\alpha + 2\rho}} \right) r e^{-\frac{r^2}{2}} dr \right] d\alpha
\end{aligned}$$

At high signal-to-noise ratios there is approximately a 1.25 dB SNR improvement through the piecewise linear limiter when $A = x_0$. The hard limiter removes all noise in phase with the signal, and achieves a two-fold or 3 dB SNR gain (reference 7.5.31); the piecewise linear limiter removes only in phase noise of the same polarity as the signal, implying a gain of 1.33 or 1.25 dB.

Finally, we note that if $A = cx_0$ (c is an arbitrary constant), the output signal amplitude and noise densities become functions of the single parameter ρ (the input signal-to-noise ratio). The case of $c = 1$ was discussed in the previous paragraph.

7.7 Effects of a Fluctuating Signal Amplitude

With narrow-band amplitude modulated signals, as the instantaneous input signal-to-noise ratio to the non-linearity varies, output parameters computed for a fixed SNR follow without experiencing any transient effects.* In analyzing the case of an input given by

$$x(t) = g(t) \cos(\omega_0 t + \theta(t)) + n(t) ,$$

ρ is replaced by

$$\rho(t) = \frac{g^2(t)}{2\sigma^2} \quad (7.7.1)$$

and A by $g(t)$ in previously derived equations. Output noise distributions, spectra, etc., are now functions of time.† The second moments (or areas under the power spectra) of the in and out of phase noise components vary in an inverse way with the desired signal power into the amplitude limiter.

* The assumption here is really that the signal amplitude does not change appreciably during the time required for the RF phase difference between the signal and noise to pass through all angles in $[0, 2\pi)$ and establish an approximately uniform phase distribution. The clipping is imprinted on the RF waveform; hence, with respect to slow amplitude variations on the signal, the non-linear suppression is an instantaneous effect. Results derived from an ensemble analysis valid for the static case are therefore applicable to the dynamic case.

† Since the $s \times n$ output component is a time-varying process, $n'(t)$ itself is non-stationary.

Consider, for example, the transmission of a single pulse with the triangular frequency response

$$G(j\omega) = \begin{cases} \frac{\varepsilon_0}{c} (1 - \frac{|\omega|}{2c}) & \text{for } |\omega| \leq 2c \\ 0 & \text{elsewhere} \end{cases}$$

through a bandpass hard limiter. Then

$$g(t) = \varepsilon_0^2 \frac{\sin^2(ct)}{(ct)^2} \quad *,$$

and applying 7.7.1,

$$\rho(t) = \frac{\varepsilon_0^2}{2\sigma^2} \frac{\sin^4(ct)}{(ct)^4} \quad (7.7.2)$$

is the instantaneous input signal-to-noise ratio.

-
- * Such a pulse shape could be applied to the transmission of a band-limited binary PSK signal constrained to have no intersymbol interference at the sampling times. In this case,

$$\begin{aligned} s(t) &= \left[\sum_{-\infty}^{\infty} a_n g(t - n\pi/c) \right] \cos(\omega_0 t + \theta) \quad , \quad a_n = \pm 1 \quad , \\ &= R(t) \cos(\omega_0 t + \theta) \quad , \text{ and} \end{aligned}$$

c/π bits per second are sent. The limiter preserves the zero intersymbol interference property of $s(t)$, i.e.,

$$R'(k\pi/c) = a_k g(0)$$

where $R'(t)$ is the envelope of the desired output signal, but produces pulse shapes dependent on the particular sequence $\{a_n\}$. (Regarding the k 'th RF pulse as the desired signal and the remaining input as the interference, the k 'th output RF pulse is the signal component of the output as defined in 4.3.)

From 7.5.16 and 7.5.26, the amplitude of the output signal is

$$\begin{aligned} A'(t) &= \frac{\sqrt{\pi \rho(t)}}{2} e^{-\frac{1}{2}\rho(t)} \left[I_0\left(\frac{\rho(t)}{2}\right) + I_1\left(\frac{\rho(t)}{2}\right) \right] \\ &= \frac{\sqrt{\pi \rho(t)}}{2} {}_1F_1(.5, 2; -\rho(t)) \end{aligned} \quad (7.7.3)$$

The output pulse is then given by

$$g'(t) = A'(t) \operatorname{sgn}(g(t)) = A'(t) \quad (7.7.4)$$

$g'(t)$ vs. t was computed for

$$\rho(0) = \frac{g_0^2}{2\sigma^2} = .5, 2, \text{ and } 8,$$

and figure 24 presents corresponding plots of

$$G'(f) = T\{g'(t)\}$$

obtained from a Fast Fourier Transform routine.

We observe from figure 24 that in each case the output pulse does not have the exact triangular frequency response of the input pulse. The distortion in time results from the non-linear suppression characteristic of the limiter which suppresses more strongly low voltage points in the pulse

than high voltage points. The $\rho(0) = .5$ case displays the least distortion, as the suppression loss is nearly constant at about 1 dB (reference figure 6) throughout the pulse.

In the other cases, the 1 dB suppression loss is inevitable near the zeros of $g(t)$, but there is a gain for values of $g(t) > \sqrt{2}\sigma$; hence, more distortion occurs in the cases $\rho(0) = 2$ & 8.

$g(t) = 0$ implies that $g'(t) = 0$; therefore, $g'(t)$ retains the property of zero intersymbol interference at the sampling times. From Nyquist's first criterion [1, p. 45], then, $G'(f)$ must be obtainable from the sum of an ideal low pass filter characteristic up to $f = c/2\pi$, and a characteristic having skew-symmetry about this frequency. The triangular response obviously satisfies the condition, but the other curves in figure 24 can also be shown to satisfy this property.

Figure 24

*LIMITER DISTORTION OF A PULSE WITH
A TRIANGULAR FREQUENCY RESPONSE*

7.8 Combined Effects of Up and Down Path Thermal Noise on PSK Transmission Through a Hard Limiting Repeater

The results of sections 7.5 and 7.7 are here applied to computing the probability of error in detecting a single positive or negative polarity amplitude-modulating pulse transmitted through a hard limiting satellite channel having arbitrary up and down path noise levels. The approach is generalized to intersymbol interference free, digitally phase modulated signals, and then extended to the continuous transmission of a band-limited PSK signal.

Figure 25 depicts the data channel we have described. $g(t)$, the transmit filter characteristic, and $h(t)$, the receive filter characteristic, are given. σ_u^2 and σ_d^2 , the statistical powers of the up and down path gaussian noises respectively, are parameters. The satellite repeater, modelled as a hard limiter plus zonal filter, has a constant output power P_s , which will be shared by signal and noise components. Consider the transmission of one of the two equally likely band-limited signals

$$s(t) = \pm g(t) \cos(\omega_0 t) ,$$

the + and - corresponding to a '1' and '0' respectively.

To this is added up path gaussian noise represented by

Figure 25

$$n_u(t) = n_{u_1}(t) \cos(\omega_0 t) + n_{u_2}(t) \sin(\omega_0 t) .$$

The desired signal out of the repeater is then

$$s'(t) = \pm \sqrt{2P_s} g'(t) \cos(\omega_0 t) \quad (7.8.1)$$

where, from 7.7.1, 7.7.3, and 7.7.4,

$g'(t)$ = baseband pulse out of the hard limiter

$$= \left[\frac{\sqrt{\pi}}{2\sqrt{2}\sigma_u} {}_1F_1(.5, 2; -\frac{g^2(t)}{2\sigma_u^2}) \right] g(t)$$

The noise component of the repeater output, $\sqrt{2P_s} n'(t)$, is non-stationary and non-gaussian. With a transmitted phase of

$$\theta(t) = 0 \text{ or } \pi ,$$

from 7.5.4, $n'(t)$ can be written in quadrature form

$$n'(t) = \pm n_1'(t) \cos(\omega_0 t) \mp n_2'(t) \sin(\omega_0 t) \quad (7.8.2)$$

Applying 7.5.5, $n_1'(t)$, the in phase noise, has a time-varying probability density function given by

$$f_{n_1}(z; t) = \begin{cases} \frac{2 f_g(\cos^{-1}(z+g'(t)))}{\sqrt{1 - (z+g'(t))^2}} & \text{for } |z+g'(t)| \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad (7.8.3)$$

$f_g(y)$, the probability density function of the phase distortion, is defined by equation 7.2.3 with ρ replaced by

$$\rho(t) = \frac{g^2(t)}{2\sigma_u^2}$$

Since $n_2'(t)$ is symmetrically distributed, the sign in front of the out of phase term in equation 7.8.2 is statistically unimportant. However, as is evident from figure 18, the direction of the signal vector to the left or right will alter the orientation of the noise distribution (reference figure 15) along the in phase axis. Hence, we must retain the $\pm$, corresponding to a transmitted '1' or '0', before the in phase component of $n'(t)$

The band-limited down path gaussian noise can be written as the sum of components in and out of phase with the desired signal re-radiated from the satellite:

$$n_d(t) = n_{d_1}(t) \cos(\omega_o t) + n_{d_2}(t) \sin(\omega_o t) \quad (7.8.4)$$

Now, coherent demodulation of the received signal removes the out of phase component of the noise. Hence the baseband noise out of the demodulator is given by

$$n(t) = \pm\sqrt{2P_s} n'_1(t) + n_{d1}(t) \quad (7.8.5)$$

$n_u(t)$ and $n_d(t)$ are independent random variables; therefore $n'_1(t)$ and $n_{d1}(t)$ are independent. n_{d1} has the gaussian density given by

$$f_{n_{d1}}(z) = \frac{1}{\sqrt{2\pi}\sigma_d} e^{-\frac{z^2}{2\sigma_d^2}} \quad (7.8.6)$$

$\pm n'_1(t)$ has the probability density function $f_{n'_1}(\pm z; t)$; this is given immediately by equation 7.8.3. The distribution of the sum of two independent random variables is the convolution of their distributions. $n(t)$ then has a probability density function given by

$$\begin{aligned} f_n(z; t) &= f_{n'_1}(\pm(2P_s)^{-\frac{1}{2}}z; t) * f_{n_{d1}}(z) \quad (* \text{ denotes convolution}) \\ &= \frac{1}{\sqrt{2\pi}\sigma_d} \int_{-\infty}^{\infty} e^{-\frac{(z-y)^2}{2\sigma_d^2}} f_{n'_1}(\pm(2P_s)^{-\frac{1}{2}}y; t) dy \quad (7.8.7) \end{aligned}$$

The output from the receive filter at the sampling time t_0 is written as

$$u(t_0) = \pm s_h + n_h \quad (\text{the subscript } h \text{ represents the particular receive filter } h(t))$$

where

$$s_h = \sqrt{2P_s} \int_{-\infty}^{t_0} h(t_0-t) g'(t) dt \quad (7.8.8)$$

$$\text{and } n_h = \int_{-\infty}^{t_0} h(t_0-t) n(t) dt \quad (7.8.9)$$

Assuming a '0' and '1' are transmitted with equal probability, and from the fact that

$$E\{n_h\} = 0 ,$$

the detector threshold is set at zero volts for minimum probability of incorrect detection. The probability of error is then given by

$$\begin{aligned} P_e &= P\{\text{'1' detected} \mid \text{'0' transmitted}\} \\ &= P\{n_h > s_h\} \\ &= \int_{s_h}^{\infty} f_{n_h}(z) dz \end{aligned} \quad (7.8.10)$$

where $f_{n_h}(z)$ is the probability density function of the noise component of the sampled output, n_h .

Since $h(t)$ and $g'(t)$ (reference 7.8.1) are known, computation of s_h from 7.8.8 is not difficult. But linear filtering of a non-gaussian process (i.e. $n(t)$) alters its distribution; hence, we do not know $f_{n_h}(z)$. In fact, the statistics of n_h depend on both the spectrum and statistics of $n(t)$. There is no simple general method for finding the probability density function of n_h .

Several approaches are available for obtaining an approximation to P_e in equation 7.8.10. Jacobs [12] assumes in a similar problem that if W and T are the signal pulse bandwidth and length respectively, $n = WT$ baseband samples of the received signal are independent. The error probability is then given by the probability that more than half the samples are in error. From the binomial distribution this is

$$P_e \approx \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} (1-p)^k p^{n-k} \quad (7.8.11)$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer function and

$$p = P\{n(t) > \sqrt{2P_s} |g'(t)|\}$$

is the probability that a sample is in error. Except when $g'(t)$ is a constant (i.e. a rectangular transmitted pulse), p is a function of time and 7.8.11 can not be used directly.

Gallager [13] has demonstrated several methods of obtaining bounds on the tails of distributions. The Chernoff bound is particularly useful, and can be applied to 7.8.10 if the moment generating function (i.e. the moments) of n_h are known. Aein [14] has used the Chernoff bound in computing the performance of hard-limiting correlators. He correctly points out that while the correlator output is asymptotically gaussian, there is no justification for computing an error probability with the use of the complimentary error function. However, it will be shown that there may be justification in expanding n_h about a gaussian distribution, and computing an approximation to the tail based on it. Additional references to this problem are given by [9, 10, 12, chapter 5] and [15, 16]. The remainder of section 7.8 should largely be regarded as conjecture.

Consider now the assumption that n_h has very nearly a gaussian distribution function. We shall attempt to apply this fact in solving equation 7.8.10. Recalling that for a transmitted '0',

$$n_h = \int_{-\infty}^{t_0} h(t_0-t) (-\sqrt{2P_s} n_1'(t) + n_{d1}(t)) dt ,$$

we note that n_h results from the linear filtering of the sum of gaussian and non-gaussian processes.

The filter and noise bandwidths are of the same order of magnitude. The central limit principle makes a deterministic linear combination (from filtering) of a random linear combination (sum of two sample functions from independent populations) tend to a gaussian distribution. If it is also assumed that the power of the non-gaussian component, $2P_s E\{n_l'^2\}$, is small in comparison with σ_d^2 , the power of the gaussian component, the near-gaussian assumption on n_h is valid. In fact, since the up path carrier-to-noise ratio is usually much greater than the down path carrier-to-noise ratio, and since the limiter removes most of the in phase noise in any case*, the assumption

$$2P_s E\{n_l'^2\} \ll \sigma_d^2$$

holds for most cases of interest.

With moments of n_h defined by

$$m_k = E\{n_h^k\}, \text{ and noting that}$$

$$m_1 = E\{n_h\} = 0,$$

* At the same time, however, it makes the in phase noise 'very' non-gaussian. Such is not the case for the out of phase noise.

we make use of the Hermite polynomial expansion of an arbitrary zero-mean, unity-variance cumulative distribution function about the standard $N(0,1)$ normal distribution [11, p. 515], and write

$$F_{n_h}(z) = \operatorname{erfc}(-z) + \sum_{k=2}^{\infty} c_k H_k(z) e^{-\frac{z^2}{2}} \quad (7.8.12)$$

where

$$c_k = \frac{1}{\sqrt{2\pi} k!} \int_{-\infty}^{\infty} H_k(z) \left[F_{n_h}(z) - \operatorname{erfc}(-z) \right] dz$$

can be written as a linear combination of $\{m_3, m_5, \dots, m_{k+1}\}$ for k even or $\{m_4, m_6, \dots, m_{k+1}\}$ for k odd.

The first correction term in 7.8.12 depends on the third moment of the noise, which is non-zero since $n_1'(t)$ is not symmetrically distributed. Both m_2 and m_3 can be computed from 7.8.8. From 7.8.12 and 7.8.9, we obtain the following two-term approximation for the probability of error:

$$P_e \approx \hat{P}_e = \frac{1}{2} \operatorname{erfc}(\sqrt{\rho}) + \frac{(2\rho-1)e^{-\rho}}{6\sqrt{2\pi}} \frac{|m_3|}{(m_2)^{1.5}} \quad (7.8.13)$$

where ρ and m_k are the signal-to-noise ratio and k 'th moment of the noise at the sampling time.

Note that for $\rho > -3$ dB, the non-gaussian 'correction' term in 7.8.13 is positive. In other words, for reasonable and equal sample time signal-to-noise ratios, $\hat{P}_e$ is greater than the error probability in detecting a signal corrupted by additive gaussian noise. On examining figure 15, probability density function plots of the limiter output in phase noise, we can heuristically justify this result. While n_1' has no DC bias, the tail of its distribution opposite in polarity to the signal voltage extends much further than the signal reinforcing tail. For a positive signal (i.e. $g(t) > 0$), this gives rise to a negative third moment. So the limiter reduces considerably the second moment of the in phase noise, but not the number of incorrect threshold crossings. This effect is moderated somewhat by the down path additive in phase noise, but still leaves a noise process which produces more errors than an equal variance gaussian noise.

If the first few correction terms in the Hermite expansion are small in comparison with the first term, this indicates that the truncated series gives a good approximation.* The argument offered, however, is this:

* Since P_e is unknown and there are no applicable convergence properties of the series, we can not really test for the accuracy of any approximation.

in any technique to approximate P_e , moments of n_h must be computed. Given these moments, it is suggested that the Hermite series here will give as rapid convergence to P_e as any other method (although no numerical investigation was undertaken). The series becomes difficult to apply at the extreme tail of the distribution (i.e. to compute accurately low error rates, many terms are required). The accuracy of the approximation increases as $2P_s E\{n_1'^2\}/\sigma_d^2$ and ρ decrease. Of course, for analog communications, where the tail of the noise distribution is not as critical, the Hermite expansion is an excellent statistical approximation.

In order to minimize P_e with respect to the choice of receive filter, a first step might be to minimize the dominant term in 7.8.13. This would call for a maximization of the sample time signal-to-noise ratio, and would be realized by the maximum SNR filter (time-varying in this case) given by [1, p. 94]

$$H_{\text{opt}}(j\omega; t) = \frac{G'(j\omega)}{S_n(\omega; t)} \quad (7.8.14)$$

Here, $G'(j\omega) = T\{g'(t)\}$

where $g'(t)$ is given by 7.8.2, and $S_n(\omega; t)$ is the time-varying power spectrum of the non-stationary noise $n(t)$. A tutorial paper on time-varying spectra is given

bu Ackroyd [17]; Lindenlaub and Chen [18] discuss the performance of matched filters on non-gaussian inputs. The filter defined by 7.8.14 is matched to the received pulse $g'(t)$, not the transmitted pulse $g(t)$. The fact that $n(t)$ is non-gaussian does not imply that the maximum signal-to-noise ratio filter is sub-optimum; however, due to the additional terms in the series expansion of P_e , analytically 7.8.14 is only a first order approximation of the optimum.

We make a few additional observations which serve to extend the applicability of the analysis presented:

(1) Consider the transmission of one of

$$s(t) = \begin{cases} g(t) \cos(\omega_0 t) & ('1') , \text{ or} \\ 0 & ('0') , \end{cases}$$

and an incoherent demodulator (i.e. envelope detector) at the receiver. In this case the noise component of the detector input would partly depend on the out of phase noise eliminated in the case of coherent demodulation. Based on the known probability density functions of n_1' , n_2' , n_{d1} , and n_{d2} (since n_1' and n_2' are not independent we may in fact require their joint density), the probability density function of the sampled process at the envelope

detector output could be computed. This information determines an optimum decision threshold, and the resulting probability of error.

(2) For a four phase or double binary* coherent PSK signal given by

$$s(t) = g(t) \cos(\omega_0 t + (2k+1)\pi/4) \quad \text{for } k = 0, 1, 2, 3,$$

or

$$s(t) = a_1 g(t) \cos(\omega_0 t) + a_2 g(t+T/2) \sin(\omega_0 t) \quad \text{for } a_i = \pm 1,$$

the outputs from the quadrature $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ demodulators again depend on noise components both in and out of phase with the transmitted signal. The essential method of analysis, however, remains unchanged from the binary case. We note that in phase noise can give a received phase opposite to the transmitted phase, while the out of phase noise can introduce a rotation to one

* In a double binary system [19] the transmitted energy is split equally between two quadrature carriers. For continuous transmission, time shifting the amplitude modulating pulse in one of the binary channels eliminates zero crossings of the RF signal envelope. This avoids crosstalk or the intermodulation (through the limiter) of the quadrature carriers at the sampling instants, and the effect of small signal suppression (up path noise shifts signal envelope zero crossings and the error is magnified by hard limiting).

of the adjacent quadrants. If the quadrants are coded 00, 01, 11, 10, these correspond to double and single bit errors respectively.

(3) In the case of continuous transmission, i.e.,

$$s(t) = \left[\sum a_n g(t-nT) \right] \cos(\omega_0 t) \quad , \quad a_n = \pm 1 \quad ,$$

for the binary case,

due to intersymbol interference* , the signal envelope becomes a stochastic process which can assume a countable number of possibilities depending on the particular sequence $\{a_n\}$ of transmitted bits. Since the channel is non-linear, we cannot use the principle of superposition in computing the output resulting from a sum of inputs overlapping in time. Any averaging over all possible sequence of bits adjacent to the one considered in order to determine

$$\bar{P}_e = E \{ P_e \mid \text{the intersymbol interference} \}$$

necessitates numerical evaluation of P_e for each possible envelope. Such computation could be extremely tedious.

* $g(t)$ is assumed to be a band-limited pulse; hence it is not time-limited. There is then an overlap of successive pulses spaced T seconds apart.

Possibly the most promising general method of evaluating error rates for continuous digital transmission through a non-linear RF channel is by computer simulation. A 'typical' message of sufficient length is chosen or generated randomly; the transmitted signal envelope is then known, and the limiter output signal envelope and instantaneous quadrature noise densities and correlation functions can be computed. Statistically typical samples of in and out of phase noise at the receiver can be obtained by:

- (1) Generating independent or white samples $\{p_{2i}\}$ and $\{q_i\}$ from populations defined by probability density functions $f_{n_1}(z/\sqrt{2P_s}; t)$ and $N(0, \sigma_d^2)$; then computing (for a given Ψ > the correlation interval)

$$p_{2i+1} = \pm \sqrt{1 - (A'(i\Psi) + p_{2i})^2},$$

the + or - occurring with equal probability*. Then

$\{r_{2i} = p_{2i} + q_{2i}\}$ and $\{r_{2i+1} = p_{2i+1} + q_{2i+1}\}$ are

sequences of typical uncorrelated samples of in and

out of phase received noise at times $i\Psi$.

* Unlike the down path band-limited gaussian noise, the in and out of phase noise components from a non-linearity are correlated. For the case of the hard limiter, given one, the other can only assume one of two values (see figure 23).

- (2) If samples of noise closely spaced enough to be significantly correlated are required, for example at $\Psi/2$ second intervals, the intermediate values can be obtained by applying the equation

$$x_c(\Psi/2) = \rho_x(\Psi/2) (x(0)+x(\Psi)) + \sqrt{1-2\rho_x^2(\Psi/2)} \hat{x} \quad (7.8.15)$$

$x_c(\Psi/2)$ is a correlated sample of the random variable x at time $\Psi/2$ (given its value at 0 and Ψ). Here $\rho_x(\tau)$ is the correlation function of x at times $\pm\tau$; $\hat{x}$ is a white sample taken from the distribution function of $x(\Psi/2)$. Equation 7.8.15 can be obtained from Anderson [20, p. 29], and is strictly applicable only to the case of gaussian noise.

The baseband noise samples $\{r_1\}$, along with the deterministic signal, can be applied in the discrete sense to the demodulation, filtering, and detection procedure at the receiver. The number of errors in the finite message can be counted, and for a sufficiently long message (i.e. one producing a sufficient number of errors), this can be used as an estimate of the probability of error. We note that the same noise samples can be used for any receiver configuration (i.e. to determine the optimum); however, any change in the message*, the transmit

filter, or the up or down path noise levels necessitates the generation of a new batch of statistically typical noise samples.

In a discussion of this complex simulation approach, the question arises as to whether the analytical results are of any practical value whatsoever. After all, the method outlined is perhaps more difficult than generating up and down path gaussian noise samples and simply applying the transfer function of linear and non-linear elements to samples of the input. This criticism certainly has some validity; but it should be noted that:

- (1) In particular cases the method given here greatly simplifies; for example, for binary coherent transmission we can generate the relevant noise samples from the single population defined by first and second order statistics $f_{n_1}(z/\sqrt{2P_s}; t) * N(0, \sigma_d^2)$ and $(P_s R_{n_1}(\tau; t) + R_{n_{d_1}}(\tau)) / (P_s N_1'(t) + \sigma_d^2)$ respectively.
- (2) The possibility of applying approximations or shortcuts is more evident in the analysis behind this simulation than in an exact simulation of the physical model.

* Some pre-correlation or redundancy on the transmitted bits may be included to reduce the probability of error [21].

- (3) The fact that the equations developed here have not been carried through to a tractable solution for AM-PSK performance through a satellite channel does not mean it is not the first step towards such a solution. In addition, further insight might be derived from the analytical work which could influence signal design and the choice of modem and detector.

7.9 Effects of Up and Down Path Noise on an Analog FM Signal

In this case a signal of the form

$$s(t) = A \cos(\omega_0 t + \theta(t))$$

is transmitted through the hard limiting satellite channel depicted in figure 23. Applying the same analysis as in 7.8, the received bandpass signal is

$$\begin{aligned} r(t) &= s'(t) + n'(t) + n_d(t) \\ &= (\sqrt{2P_s} A' + n_1(t)) \cos(\omega_0 t + \theta(t)) + n_2(t) \sin(\omega_0 t + \theta(t)) \end{aligned} \quad (7.9.1)$$

where $n_1(t) = \sqrt{2P_s} n'_1(t) + n_{d1}(t)$

and $n_2(t) = \sqrt{2P_s} n'_2(t) + n_{d2}(t)$

are the baseband noise components in and out of phase with the signal. From 7.5.26, A' , the signal amplitude out of the hard limiter, is given by

$$A' = \sqrt{\frac{\pi}{4}} \rho {}_1F_1(.5, 2; -\rho) \quad (7.9.2)$$

where $\rho = A^2/2\sigma_u^2$.

$n_1'(t)$ and $n_2'(t)$ have probability density functions given by 7.5.5 and 7.5.6. Convolving these with the gaussian distributions of $n_{d1}(t)$ and $n_{d2}(t)$ leads to the following expressions for the densities of $n_1(t)$ and $n_2(t)$:

$$f_{n_1}(z) = \frac{1}{\sqrt{2\pi}\sigma_d} \int_{-(1+A')\sqrt{2P_s}}^{(1-A')\sqrt{2P_s}} e^{-\frac{(z-y)^2}{2\sigma_d^2}} \frac{2 f_\delta(\cos^{-1}(y/\sqrt{2P_s} - A'))}{\sqrt{1 - (y/\sqrt{2P_s} - A')^2}} dy \quad (7.9.3)$$

and

$$f_{n_2}(z) = \frac{1}{\sqrt{2\pi}\sigma_d} \int_{-\sqrt{2P_s}}^{\sqrt{2P_s}} e^{-\frac{(z-y)^2}{2\sigma_d^2}} \frac{f_\delta(\sin_1^{-1}(|y|/\sqrt{2P_s})) + f_\delta(\sin_2^{-1}(|y|/\sqrt{2P_s}))}{\sqrt{1 - y^2/2P_s}} dy \quad (7.9.4)$$

A' is given by 7.9.2, and $f_\delta(y)$ by 7.2.3.

An FM receiver makes an estimate of $d\theta(t)/dt$ based on the received signal $r(t)$ given by 7.9.1. Figure 26 is a phasor diagram depicting $\epsilon(t)$, the error in the instantaneous received phase.

Figure 26

The recovered FM signal is then

$$\theta_r'(t) = \theta'(t) + \epsilon'(t) \quad (7.9.5)$$

where ' here denotes the derivative with respect to time.

It will be shown that $\epsilon'(t)$, the FM error signal, consists of a continuous noise process and an ideal impulse (click) noise process. First, without loss of generality, assume

$$n_1(\tau) > -\sqrt{2P_s} A \quad \text{for} \quad t-\eta < \tau < t+\eta$$

$$\text{and some } \eta > 0 . \quad (7.9.6)$$

From figure 26 we can write

$$\varepsilon'(t) = \frac{d}{dt} \tan^{-1} \left(\frac{n_2(t)}{\sqrt{2P_s} A' + n_1(t)} \right)$$

$$\approx n_2'(t) / \sqrt{2P_s} A' \quad (7.9.7)$$

for high signal-to-noise ratios. The error signal defined by 7.9.7 is background noise present continuously in the baseband. The second type of FM noise occurs at the discrete moments when (7.9.6 no longer holds)

$$n_1(t) = -\sqrt{2P_s} A' \quad (7.9.8)$$

Since $n_1(t)$ is a continuous random process, 7.9.8 implies that for $0 < \eta < \infty$ and some $\eta > 0$,

$$n_1(t-\eta) > -\sqrt{2P_s} A' \quad \text{and} \quad n_1(t+\eta) < -\sqrt{2P_s} A' ,$$

or

$$n_1(t-\eta) < -\sqrt{2P_s} A' \quad \text{and} \quad n_1(t+\eta) > -\sqrt{2P_s} A' ,$$

corresponding to $n_1(t)$ increasing or decreasing at time t .

In either event, $\theta_r(t)$ experiences a phase change of π radians in time 2η , η arbitrarily small. Hence,

$$\varepsilon'(t) = \pm \pi \delta(t) \quad , \quad (7.9.9)$$

defining a click. The clicks will occur with alternating sign in 7.9.9 as the noise swings above and back below the signal amplitude. The probability of the π radian reversal of the received phase is given by

$$P_c = \int_{-\infty}^{-\sqrt{2P_s} A'} f_{n_1}(z) dz \quad ; \quad (7.9.10)$$

along with the noise bandwidth, this approximately determines the average rate of occurrence of the clicks. Physically, clicks appear on the baseband signal as a crackling of short duration pulses.*

Defining the click times $t_0 < t_1 < \dots < t_n$ by

$$n_1(t_i) = -\sqrt{2P_s} A' \quad ,$$

and assuming $n_1'(t_0) < 0$, equations 7.9.7 and 7.9.9 can be combined to give the following exact expression for the noise additive to the baseband FM signal:

* The FM frequency detector, which differentiates the received phase, cannot produce the infinite bandwidth Dirac pulse defined by 7.9.9. The click energy is slightly dispersed in time, but would show up on an oscilloscope as a spike which obliterates the signal for a brief instant.

$$\varepsilon'(t) = \frac{(\sqrt{2P_s} A' + n_1(t)) n_2'(t) + n_1'(t) n_2(t)}{(\sqrt{2P_s} A' + n_1(t))^2 + n_2^2(t)} +$$

$$\pi \sum_{i=0}^n (-1)^i \delta(t-t_i) \quad (7.9.11)$$

Based on the probability density functions of $n_1(t)$ and $n_2(t)$ given by 7.9.3 and 7.9.4, and their correlation functions, the second moment and other properties of $\varepsilon'(t)$ can be approximated. Yavuz and Hess [22] have analyzed FM noise and clicks; Bozzoni et al [23] compute the probability density function of the click time or duration of $n_1(t) < -\sqrt{2P_s} A'$. Whelan [24] compares digital PSK and analog FM methods of communicating analog information, while Bargellini and Schmidt [25] stress the advantage of the digital approach in satellite communications.

We now make use of the results of this section to explain the fact that a hard limiter can give a 3 dB carrier-to-noise improvement, but does not improve the detectability of an FM signal. The carrier-to-out of phase noise power ratio remains essentially unchanged through the limiter; hence, the limiting does reduce the effect of the FM noise defined by 7.9.7. Although it does virtually eliminate (in terms of its power or second moment) the in phase noise, the limiter

does not alter the click probability (7.9.10), and hence does not improve the click noise. In fact, as was pointed out in 5.3, pre-detection limiting degrades performance* .

* The limiter-discriminator FM receiver is sub-optimum.

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CHAPTER 8

EFFECTS OF AM TO PM CONVERSION ON ONE SIGNAL AND NOISE

8.1 Signal and Noise Components of Output

As in 7.1, the stochastic approach of 4.3 is applied to the case of signal and noise passed through an AM/PM conversion device which does not alter amplitudes. With a phase shift vs input amplitude characteristic given by $g(x)$, the input and output are

$$\begin{aligned}x(t) &= s(t) + n(t) \\&= A \cos(\omega_0 t + \theta(t)) + R(t) \cos(\omega_0 t + p(t)) \\&= E(t) \cos(\omega_0 t + \phi(t))\end{aligned}\tag{8.1.1}$$

and

$$\begin{aligned}y(t) &= A \cos(\omega_0 t + \theta(t) - g(E(t))) + R(t) \cos(\omega_0 t + p(t) - g(E(t))) \\&= E(t) \cos(\omega_0 t + \phi(t) - g(E(t)))\end{aligned}\tag{8.1.2}$$

respectively. The signal component in $y(t)$ is defined to be

$$\begin{aligned}s'(t) &= A E \left[\cos(\omega_0 t + \theta(t) - g(E(t))) \right] \\&= A \left[E \left[\cos(g(E(t))) \right] \cos(\omega_0 t + \theta(t)) \right. \\&\quad \left. + E \left[\sin g(E(t)) \right] \sin(\omega_0 t + \theta(t)) \right]\end{aligned}\tag{8.1.3}$$

where $E(t)$, the net input envelope, is a random variable with probability density function $f_E(r)$ given by 7.2.6. Referring to 4.3.6 - 4.3.9, and applying 7.2.6, then, $s'(t)$ can be written as

$$s'(t) = A' \cos(\omega_0 t + \theta(t) - \beta) \quad (8.1.4)$$

where

$$\begin{aligned} A' &= \left[\left(\int_0^\infty \cos(g(r)) f_E(r) dr \right)^2 + \left(\int_0^\infty \sin(g(r)) f_E(r) dr \right)^2 \right]^{1/2} A \\ &= \left[\int_0^\infty \cos(g(r) - \beta) f_E(r) dr \right] A \end{aligned} \quad (8.1.5)$$

and

$$\beta = \tan^{-1} \left(\frac{\int_0^\infty \cos(g(r)) f_E(r) dr}{\int_0^\infty \sin(g(r)) f_E(r) dr} \right) \quad (8.1.6)$$

From 8.1.2, the noise component of the output is

$$\begin{aligned} n'(t) &= y(t) - s'(t) \\ &= E(t) \cos(\omega_0 t + \phi(t) - g(E(t))) - A' \cos(\omega_0 t + \theta(t) - \beta) \\ &= (E(t) \cos(\delta(t) + \beta - g(E(t))) - A') \cos(\omega_0 t + \theta(t) - \beta) \\ &\quad - \sin(\delta(t) + \beta - g(E(t))) \sin(\omega_0 t + \theta(t) - \beta) \\ &= n_1'(t) \cos(\omega_0 t + \theta(t) - \beta) + n_2'(t) \sin(\omega_0 t + \theta(t) - \beta) \end{aligned} \quad (8.1.7)$$

where

$$\delta(t) = \phi(t) - \theta(t)$$

was introduced in equation 7.1.8. Equation 8.1.7 is in the desired quadrature form given by 4.3.10, and defines output noise processes in and out of phase with the signal.

Figure 27 is a phasor diagram depicting the input to and output from a device exhibiting AM to PM transfer. $\vec{n}$ is here a sample of the input additive gaussian noise; it determines a resultant input which is phase-shifted δ radians from the signal, and an envelope at both input and output of E . The resultant output phasor lags the resultant input phasor by $g(E)$ radians, and is decomposed into $s \times s$, $n \times n$, and $s \times n$ components as follows: The $s \times s$ component, which retains the phase characteristic of the input signal, is a time replica of the input signal vector except for an attenuation factor C and phase lag β ; the $n \times n$ output component similarly is a delayed and attenuated version of the input noise; the $s \times n$ term in the output is then the vector difference between the resultant output and $s \times s$ and $s \times n$ components previously defined. This signal-noise intermodulation term is uncorrelated with the phase characteristics of both the $s(t)$ and $n(t)$, and makes up the loss in power experienced by both the sinusoid and gaussian noise.

$$\beta = \tan^{-1} \left[\frac{E [\sin g(E)]}{E [\cos g(E)]} \right]$$

$$C = E [\cos (g(E) - \beta)]$$

Figure 27

8.2 Probability Density Functions of Quadrature Noise Components

Recalling that E and δ are random variables with the joint density $f_{E,\delta}(r,y)$ given by 7.2.8, and A' and β are constants defined by 8.1.5 and 8.1.6 respectively, from 8.1.7 the in and out of phase noise processes are distributed as

$$n_1' = E \cos(\delta + \beta - g(E)) - A' \quad (8.2.1)$$

and

$$n_2' = E \sin(\delta + \beta - g(E)) \quad (8.2.2)$$

Their cumulative distribution functions are:

$$\begin{aligned} F_{n_1'}(z) &= P(E \cos(\delta + \beta - g(E)) \leq z + A') \\ &= \int_{S_1} \int_{(z)} f_{E,\delta}(r,y) \, dr \, dy \end{aligned} \quad (8.2.3)$$

where $S_1(z)$ is the set of all (r,y) such that

$$r \cos(y + \beta - g(r)) \leq z + A';$$

$$\text{and} \quad F_{n_2'}(z) = P\left[E \sin(\delta + \beta - g(E)) \leq z\right]$$

$$= \int_{S_2} \int_{(z)} f_{E,\delta}(r,y) \, dr \, dy \quad (8.2.4)$$

where $S_2(z)$ is defined by

$$r \sin(y + \beta - g(r)) \leq z$$

Differentiating 8.2.3 and 8.2.4 with respect to z gives expressions for the desired probability density functions. The computation is difficult, and not attempted for the general case.

From 8.2.1 and 8.2.2, the envelope

$$R' = \sqrt{E^2 - 2A' E \cos(\delta + \beta - g(E)) + A'^2} \quad (8.2.5)$$

and phase

$$\gamma = \tan^{-1} \left(\frac{E \sin(\delta + \beta - g(E))}{E \cos(\delta + \beta - g(E)) - A'} \right) \quad (8.2.6)$$

of the output noise have statistics that can similarly be computed.

8.3 Second Moments of Noise Components

From 8.2.2, the power of the out of phase noise is

$$\begin{aligned} N_2' &= E \{n_2'^2\} \\ &= \int_0^\infty \int_{-\pi}^\pi r^2 \sin^2(y + \beta - g(r)) f_{E,\delta}(r,y) dy dr \end{aligned} \quad (8.3.1)$$

The total power out of the AM/PM device remains unchanged; i.e.,

$$P_o = \frac{A^2}{2} + \sigma^2 \quad (8.3.2)$$

With $A' = A E(\cos(g(E) - \beta))$, as in 7.4.3, the in phase noise power is

$$N'_1 = 2 \left(P_o - \frac{A'^2}{2} - \frac{N_2'}{2} \right) \quad (8.3.3)$$

The output SNR and signal suppression factor are simply computed:

$$\rho_o = \frac{A'^2}{A^2 + 2\sigma^2 - A'^2} \quad (8.3.4)$$

and from 8.1.5

$$\alpha = \frac{A'^2}{A^2} = \left[\int_0^\infty \cos(g(r) - \beta) f_E(r) dr \right]^2 \quad (8.3.5)$$

Note that in 8.3.5 $\alpha \leq 1$, equality if and only if $\sigma = 0$.

Therefore AM/PM always results in a decrease in a signal's share of the total power. Referring to figure 6, this is not necessarily the case for an amplitude nonlinearity.

8.4 The Linear AM/PM Characteristic

In this section we consider as an example the case of

$$g(x) = \frac{a}{\sigma} x \quad (\text{i.e., } K_p = \frac{dg(x)}{dx} = \text{constant})^*$$

From 8.1.5, 8.1.6 and 7.2.6, the amplitude change and phase shift of the signal vs input SNR are

$$\frac{A'}{A} = e^{-\rho} \int_0^{\infty} r e^{-\frac{r^2}{2}} I_0(\sqrt{2\rho} r) \cos(ar - \beta) dr \quad (8.4.2)$$

and

$$\beta = \tan^{-1} \left[\frac{\int_0^{\infty} r e^{-\frac{r^2}{2}} I_0(\sqrt{2\rho} r) \sin(ar) dr}{\int_0^{\infty} r e^{-r^2/2} I_0(\sqrt{2\rho} r) \cos(ar) dr} \right] \quad (8.4.3)$$

A computer program was written to compute β and A'/A . In figure 28, $(A'/A)^2$, the power loss experienced by the signal, is plotted for $a = K_p \sigma = .2$ and $.4$. As a becomes greater, the severity or degrading effect of the AM/PM transfer increases. Note that in figure 28 σ is held fixed; an increase in ρ is accomplished purely by increasing the input signal power $A^2/2$.

For low input SNR's, 8.4.3 and 8.4.2 reduce [16, Chapter 5, p. 495] to the following asymptotic approximations for small a :

* As was pointed out in 2.4 (3) ("operating point and the linear model"), this equation can be used to approximate any AM/PM characteristic about its operating point.

Figure 28

$$\beta \approx \tan^{-1} \left(\sqrt{\frac{\pi}{2}} a \right) \quad (8.4.4)$$

$$\frac{A'}{A} \approx \frac{1 + \left(\frac{\pi}{2} - 1\right)a^2}{1 + \frac{\pi}{4}a^2} \approx 1 - .215a^2 \quad (8.4.5)$$

At high input SNR's [10, Chapter 2, p. 480] the density of the normalized input envelope density becomes

$$f_E(r) \approx \frac{1}{\sqrt{2\pi}} e^{-\frac{(r - \sqrt{2\rho})^2}{2}}$$

From this, 8.4.2 and 8.4.3 can be evaluated exactly [16, Chapter 5, p. 480]

$$\beta = a\sqrt{2\rho} \quad (8.4.6)$$

$$\frac{A'}{A} = e^{-\frac{a^2}{2}} \approx 1 - .5a^2 \quad (8.4.7)$$

Note that the AM/PM degradation expressed by 8.4.7 is independent of ρ for large ρ . As the input signal power increases, the mean phase shift β , but not the rms value of the differential phase shift, $\sqrt{E(g(E) - \beta)^2}$ *, increase.

Finally, from 8.4.7 and 8.4.5, the high and low SNR suppression factors for small a are

* This really determines the loss incurred by the AM/PM transfer.

$$\alpha = 20 \log_{10} \left(\frac{A'}{A} \right) = 8.6858 \log_e \left(\frac{A'}{A} \right) \approx 8.6858 \left(\frac{A'}{A} - 1 \right) =$$

$-1.8674a^2$ dB and $-4.3429a^2$ dB respectively. Hence, for a fixed additive noise power and a linear AM/PM characteristic, a high amplitude signal experiences a dB suppression loss 2.3 times greater than a low amplitude signal. This can be observed in figure 28.

8.5 Distortion with a Fluctuating Signal Envelope, Prelimiting

AM to PM conversion will distort a single amplitude modulated carrier input. Suppose, for example, a band-limited binary coherent AM-PSK signal of the form

$$s(t) = \left[\sum a_n g(t - nT) \right] \cos(\omega_0 t), \quad a_n = \pm 1 \quad (8.5.1)$$

is passed through the AM/PM device described by $g(x)$. The output is then

$$s'(t) = \left[\sum a_n g(t - nT) \right] \cos(\omega_0 t - g(\sum a_n g(t - nT))) \quad (8.5.2)$$

From 8.5.2, the signal has experienced a phase shift dependent on the transmitted symbols (a_n). Since these are unknown to the receiver, coherence has been lost, and the normal detection procedure cannot be used.

Consider now the problem of recovering the desired binary information from the noiseless waveform described

by 8.5.2. In order to recover coherence, it is necessary to pass $s'(t)$ through an inverse AM/PM device* described by

$$\hat{g}(x) = d_o - g(x) \quad (8.5.3)$$

d_o an arbitrary constant

Then the output is

$$s''(t) = \sum_n a_n g(t - nT) \cos(\omega_o t - d_o) \quad (8.5.4)$$

and we can proceed with coherent detection. Gaussian noise added to the signal $s(t)$ (i.e. up path noise) will not induce any phase distortion in $s''(t)$, and will remain purely additive at the detector. Noise added to $s'(t)$, however, will contribute both an error signal in the phase of $s''(t)$ and an additive disturbance at the detector. Assuming the TWT has an invertable gain characteristic in the operating region considered, figure 29 depicts the satellite channel and AM/PM equalizer at the receiver.

The effects of AM/PM conversion in the TWT can be reduced by prelimiting its input according to the amplitude characteristic of the tube. If the TWT is hard-limiting its input (i.e. 0 dB output backoff), prelimiting

* If any amplitude information is lost (e.g. due to a saturating amplitude nonlinearity), this cannot be done. With AM/PM compensation at the transmitter, the up-path noise rather than the down-path noise would cause a noisy signal phase reference.

Figure 29

$g(t)$ and $h(t)$ are transmit and receive filters for linear channel

$[a_i]$ and $[\hat{a}_i]$ are transmitted and received binary symbols

$n(t)$ is nongaussian and nonstationary noise

can theoretically eliminate completely the AM/PM degradation.* Of course, it is here assumed that the prelimiter (which is not required to be an amplifying device) does not itself exhibit any appreciable amplitude dependent phase shift. The prelimiting does not affect the intermodulation resulting from the nonlinear repeater gain characteristic, and the TWT in this case is modelled as an ideal bandpass hard limiter.

* Since the TWT input is of constant amplitude, the phase shift incurred by AM/PM is a constant and does not degrade the signal(s).

CHAPTER 9

EFFECTS OF AMPLITUDE NONLINEARITY ON MULTIPLE SIGNALS

9.1 Vector Approach

If the input to a limiter consists of the m constant amplitude signals $s_1(t)$, $s_2(t)$, ..., $s_m(t)$, the output can be represented as

$$o(t) = \sum_{i=1}^m s'_i(t) + i(t) \quad (9.1.1)$$

$s'_i(t)$ is the component of the output having the phase characteristic of $s_i(t)$, and is the i 'th signal out of the repeater. $i(t)$ contains the fundamental frequency part of those components of the limiter output which depend on the presence of two or more of the input signals. $i(t)$ is called the intermodulation product interference, and has been assumed to contain no useable intelligence about the individual signals.

In this section a deterministic representation of $i(t)$ as a function of the $\{s_i(t)\}$ is developed. Based on the assumption of independent and uniformly distributed phase differences among the input signals, we examine statistical properties of the IM product interference regarded as a

two-dimensional (i.e. bandpass) random process. In section 9.5 we extend these results to compute the intermodulation product line spectrum in order to isolate those spectral components in band with each signal.

The essence of the generalized IM product analysis can be described in geometric terms. From equation 9.1.1, at any time t , the intermodulation vector (or phasor) must satisfy the equation.

$$\vec{i} = \vec{o} - \sum_{i=1}^m \vec{s}_i' \quad (9.1.2)$$

The resultant output from the bandpass amplitude nonlinearity $g(x)$ has the RF phase of the input, and amplitude $g(\text{input amplitude})$. The i 'th desired output signal has the same phase as the i 'th input signal, and an amplitude given by

$$A_i' = E\{g(E)\cos\Delta_i\} .$$

Here Δ_i is the angle between the resultant input and the i 'th signal, and E is the input envelope; both are defined as random variables. The following simple vector equations can now be written:

$$\vec{o} = g\left(\left|\sum_{i=1}^m \vec{s}_i\right|\right) \frac{\sum_{i=1}^m \vec{s}_i}{\left|\sum_{i=1}^m \vec{s}_i\right|} \quad (9.1.3)$$

and

$$\vec{s}'_i = A'_i \frac{\vec{s}_i}{|\vec{s}_i|} \quad (9.1.4)$$

implying that

$$\vec{i} = g\left(\left|\sum_{i=1}^m \vec{s}_i\right|\right) \left[\frac{\sum_{i=1}^m \vec{s}_i}{\left|\sum_{i=1}^m \vec{s}_i\right|} \right] - \sum_{i=1}^m A'_i \frac{\vec{s}_i}{|\vec{s}_i|} \quad (9.1.5)$$

By making use of the relatively simple forms for the resultant output and the individual signal components of the output, we have indirectly constructed a representation for the intermodulation product interference. From equation 9.1.1, for given signal amplitudes, the IM is a deterministic function of the instantaneous relative phases of the input signals. In fact, the interference vector may be obtained by closing an $m+2$ sided polygon with $m+1$ sides given by $\vec{s}'_1, \vec{s}'_2, \dots, \vec{s}'_m$, and $\vec{o}$. The statistical properties of the interference can be obtained from 9.1.5 and the known distribution of the random variable

$$E = \text{input envelope} = \left| \sum_{i=1}^m \vec{s}_i \right|$$

9.2 Output Signal Amplitudes

From 9.1, the input to and output from the nonlinear device $g(x)$ are

$$\begin{aligned} x(t) &= \sum_{i=1}^m s_i(t) \\ &= \sum_{i=1}^m A_i \cos(\omega_i t + \theta_i(t)) \\ &= E(t) \cos(\omega_o t + \phi(t)) \end{aligned} \tag{9.2.1}$$

and

$$\begin{aligned} y(t) &= g(E(t)) \cos(\omega_o t + \phi(t)) \\ &= \sum_{i=1}^m A'_i \cos(\omega_i t + \theta_i(t)) + i(t) \end{aligned} \tag{9.2.2}$$

With R_j and α_j defined as random variables denoting the amplitude and phase (relative to the j 'th signal) of $\sum_{i \neq j} s_i(t)$, the output amplitude of the j 'th signal is given by

$$A'_j = E \left\{ g(\sqrt{R_j^2 - 2A_j R_j \cos \alpha_j + A_j^2}) \left(\frac{A_j - R_j \cos \alpha_j}{\sqrt{R_j^2 - 2A_j R_j \cos \alpha_j + A_j^2}} \right) \right\}$$

α_j is assumed uniformly distributed on $[0, 2\pi)$; R_j has the probability density function

$$f_{R_j}(r) = r \int_0^\infty u J_0(ru) \prod_{i \neq j} J_0(A_i u) du \quad (9.2.3)$$

We then write

$$A'_j = \frac{1}{\pi} \int_0^\infty \int_0^\pi g(\sqrt{r^2 - 2A_j r \cos \alpha + A_j^2}) \left(\frac{A_j - r \cos \alpha}{\sqrt{r^2 - 2A_j r \cos \alpha + A_j^2}} \right) f_{R_j}(r) d\alpha dr \quad (9.2.4)$$

Once again, A_j' represents the length of the constant component of a projection of the output on a phasor with a phase signature $\omega_j t + \theta_j(t)$.

9.3 Total Signal to Total IM Product Power Ratio

The total power out of the amplitude nonlinearity is

$$\begin{aligned} P_o &= \frac{1}{2} E\{g^2(E)\} \\ &= \frac{1}{2} \int_0^\infty g^2(r) f_E(r) dr \end{aligned} \quad (9.3.1)$$

where $f_E(r)$, the probability density function of the input envelope, is given by

$$f_E(r) = r \int_0^{\infty} u J_0(ru) \prod_{i=1}^m J_0(A_i u) du \quad (9.3.2)$$

The total desired signal power in the output is

$$S_o = \frac{1}{2} \sum_{j=1}^m A_j'^2 ; \quad (9.3.3)$$

here A_j' , the output amplitude of the j 'th signal, is defined by equation 9.2.4.

It immediately follows from 9.3.1 and 9.3.3 that the signal-to-interference power ratio (SIR) in $y(t)$ is

$$\begin{aligned} \frac{S}{I} &= \frac{S_o}{P_o - S_o} \\ &= \frac{\sum_{j=1}^m A_j'^2}{\int_0^{\infty} g^2(r) f_E(r) dr - \sum_{j=1}^m A_j'^2} \end{aligned} \quad (9.3.4)$$

9.4 The IM Product Power Spectrum

As was mentioned in 2.4 and 3.1, intermodulation product interference can impose distortion on the signals depending on where its energy lies in the frequency spectrum. Therefore, in analyzing its effect on the performance of an FDMA communications system, it is necessary to determine the spectral density of this interference.

From equation 9.1.5 an instantaneous representation of the total IM in the output is

$$i(t) = \sum_{k=1}^m \left(\frac{g(E(t))}{E(t)} A_k - A'_k \right) \cos(\omega_k t + \theta_k(t)) \quad (9.4.1)$$

Making use of the mutual independence of $E(t)$ and each $\theta_j(t)$ †, the autocorrelation function of $i(t)$ can be written as

$$\begin{aligned} R_i(\tau) &= E\{i(t+\tau) i(t)\} \\ &= \frac{1}{2} \left(\sum_{k=1}^m \rho_k(\tau) A_k^2 \right) R_{\frac{g(E)}{E}}(\tau) - \left(\sum_{k=1}^m \rho_k(\tau) A_k A'_k \right) \int_0^\infty \frac{g(r)}{r} f_E(r) dr \\ &\quad + \frac{1}{2} \sum_{k=1}^m \rho_k(\tau) A_k'^2 \end{aligned} \quad (9.4.2)$$

† Given $\theta_j(t)$ the RF phase differences between pairs of signals are still uniformly distributed on $[0, 2\pi)$.

after some manipulation.* In this equation,

$$\begin{aligned}
 \phi_k(\tau) &= \text{correlation function of } k\text{'th input signal} \\
 &= 2 E\{\cos(\omega_k(t+\tau) + \theta_k(t+\tau)) \cos(\omega_k t + \theta_k(t))\} \\
 &= R \left[R_{e^{j\theta_k}}(\tau) e^{j\omega_k \tau} \right], \quad (9.4.3)
 \end{aligned}$$

and $R_x(\tau)$ is defined to be the autocorrelation function of the random process $x(t)$; i.e.,

$$R_{\frac{g(E)}{E}}(\tau) = E \left\{ \frac{g(E(t+\tau)) g(E(t))}{E(t+\tau) E(t)} \right\} \quad \text{in 9.4.2, and}$$

$$\begin{aligned}
 R_{e^{j\theta_k}}(\tau) &= E \left\{ e^{j\theta_k(t+\tau)} \left[e^{j\theta_k(t)} \right]^* \right\} \\
 &= E \{ e^{j(\theta_k(t+\tau) - \theta_k(t))} \} \quad \text{in 9.4.3.}
 \end{aligned}$$

The spectral density $S_i(f)$ of the interference can be obtained by taking the Fourier transform of 9.4.2. If the k 'th angle modulated signal lies in the band $(f_k - B_k/2, f_k + B_k/2)$, the output k 'th signal-to-distortion power ratio is

* Note that when $g(E) = E$ (i.e. a linear device), $A'_k = A_k$, $R_{\frac{g(E)}{E}}(\tau) = 1$, and $\int_0^\infty \frac{g(r)}{r} f_E(r) dr = 1$. This gives the expected solution $R_i(\tau) = 0$ (i.e. no intermodulation).

$$(S/I)_k = \frac{A_k'^2}{4 \int_{f_k - B_k/2}^{f_k + B_k/2} S_i(f) df} \quad (9.4.4)$$

For unmodulated carriers, the IM product line spectrum is the Fourier transform of the periodic function

$$R_i(\tau) = \sum_{k=1}^m \left[\frac{1}{2} \left\langle \frac{E(t+\tau)E(t)}{E(t+\tau)E(t)} \right\rangle A_k^2 - \left\langle \frac{E(t)}{E(t)} \right\rangle A_k A_k' + \frac{1}{2} A_k'^2 \right] \cos(\omega_k \tau) \quad (9.4.5)$$

Here,

$$E(t) = \left| \sum_{k=1}^m A_k e^{j\omega_k t} \right| ,$$

and $\langle . \rangle$ denotes a time average over the finite period of $E(t)$.

The autocorrelation method (reference 4.2) has here been employed to obtain the line spectrum of the intermodulation product interference. Note that a Fourier series method (reference 4.1, 9.5) could also have been used.

9.5 Two Sine Waves Through a Hard Limiter

The input to the limiter consists of two angle modulated sinusoids; i.e.,

$$\begin{aligned} x(t) &= s_1(t) + s_2(t) \\ &= A_1 \cos(\omega_1 t + \theta_1(t)) + A_2 \cos(\omega_2 t + \theta_2(t)) \end{aligned}$$

We shall here write expressions for the signal and IM product components of the output, and examine properties of the IM interference process.

In figure 30, it is supposed that at time t , $s_1(t)$ and $s_2(t)$ have the phase difference

$$\pi - \phi = (\omega_2 - \omega_1) t + \theta_2(t) - \theta_1(t) \quad (9.5.1)$$

$\vec{s}_1'$, $\vec{s}_2'$, $\vec{0}$, and $\vec{i}$ have been depicted according to equations 9.1.2, 9.1.3, and 9.1.4.

First, the output signal amplitudes are evaluated. Without loss of generality, it is assumed that $A_1 > A_2$. From equation 5.2.2, with

$$r_{R_i}(r) = \delta(r - A_i) \quad (i = 1, 2) \quad ,$$

these are

$$A_1' = E\{\cos\Delta_1\} = {}_2F_1(.5, -.5; 1; A_2^2/A_1^2) \quad (9.5.2)$$

and

$$A_2' = E\{\cos\Delta_2\} = \frac{A_2}{2A_1} {}_2F_1(.5, .5; 2; A_2^2/A_1^2) \quad (9.5.3)$$

respectively.

Equations 9.5.2 and 9.5.3 are consistent with result (7), page 99. We note from 9.5.3 the -6 dB suppression of $s_2(t)$ when $A_2 \ll A_1$ (reference figure 6).

In figure 30, ϕ is a simple function of time expressed by equation 9.5.1. To compute the intermodulation products in the output, it is sufficient to write $\vec{i}$ as a function of ϕ . This can be done through trigonometry in figure 30. Representing $\vec{i}$ in quadrature components (i_1, i_2) relative to the signal $s_1(t)$, we can write

$$i_2 = \sin\Delta_1 - A_2' \sin\phi \quad (9.5.4)$$

$$i_1 = A_2' \cos\phi + \cos\Delta_1 - A_1' \quad (9.5.5)$$

where

$$\cos \Delta_1 = \frac{A_1 - A_2 \cos \phi}{\sqrt{A_1^2 - 2A_1 A_2 \cos \phi + A_2^2}} \quad \text{and}$$

$$\sin \Delta_1 = \frac{A_2 \sin \phi}{\sqrt{A_1^2 - 2A_1 A_2 \cos \phi + A_2^2}}$$

Substituting for ϕ from 9.5.1, and multiplying i_1 and i_2 by the quadrature carriers $\cos(\omega_1 t + \theta_1(t))$ and $-\sin(\omega_1 t + \theta_1(t))$, gives the following expression for the interference component of the limiter output:

$$\begin{aligned} i(t) &= \left[(A_1^2 + 2A_1 A_2 \cos((\omega_2 - \omega_1)t + \theta_2(t) - \theta_1(t)) + A_2^2)^{-\frac{1}{2}} A_1 - A_1' \right] \cos(\omega_1 t + \theta_1(t)) \\ &+ \left[(A_1^2 + 2A_1 A_2 \cos((\omega_2 - \omega_1)t + \theta_2(t) - \theta_1(t)) + A_2^2)^{-\frac{1}{2}} A_2 - A_2' \right] \cos(\omega_2 t + \theta_2(t)) \\ &= \frac{\vec{s}_1(t) + \vec{s}_2(t)}{|\vec{s}_1(t) + \vec{s}_2(t)|} - \vec{s}_1'(t) - \vec{s}_2'(t) \quad , \quad (9.5.6) \end{aligned}$$

the correct expression for the output from the bandpass hard limiter minus the desired signals.

Observe from figure 31 the locus of the IM interference vector $\vec{i}(t)$ as $s_2(t)$ completes one cycle relative to $s_1(t)$. Note that for any ϕ , $\vec{i}$ emanates from the s_2' circle and terminates on the $\vec{0}$ circle; $\vec{i}$ changes in magnitude and

Figure 31

direction with ϕ . If $s_1(t)$ and $s_2(t)$ were unmodulated carriers, $i(t)$ would be a periodic function with period $|f_1 - f_2|^{-1}$ in time, or period 2π with respect to ϕ . Accordingly, in order to compute the power spectrum of $i(t)$, we expand 9.5.6 in a Fourier series; each term in the series will represent a particular intermodulation product with a characteristic frequency. Writing the periodic function

$$f(x) = (1 + 2p \cos x + p^2)^{-\frac{1}{2}}, \quad |p| < 1,$$

as a Fourier series on $[-\pi, \pi)$ gives the expression

$$f(x) = \frac{1}{\sqrt{1+p^2}} \sum_{n=0}^{\infty} \left[\frac{\epsilon_n}{\pi} \int_0^{\pi} \frac{\cos(ny) dy}{\sqrt{1+b\cos y}} \right] \cos(nx) \quad (9.5.7)$$

$$\text{where} \quad \epsilon_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n \neq 0 \end{cases}$$

$$\text{and} \quad b = \frac{2p}{1+p^2}.$$

Using the binomial expansion

$$(1 + b\cos x)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} b^k \cos^k x,$$

and the relation [16, chapter 5, p. 374]

$$\frac{1}{\pi} \int_0^\pi \cos(nx) \cos^k x \, dx = \begin{cases} 2^{-k} \binom{k}{j} & \text{for } k-n = 2j \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

equation 9.5.7 becomes

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n \cos(nx) \\ &= \frac{1}{\sqrt{1+p^2}} \sum_{n=0}^{\infty} \epsilon_n \left(\frac{p}{2\sqrt{1+p^2}} \right)^n \left[\sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{n+2j} \binom{n+2j}{j} \left(\frac{p}{2\sqrt{1+p^2}} \right)^{2j} \right] \cos(nx) \end{aligned} \quad (9.5.8)$$

For the cases of $n = 0$ and $n = 1$, we can solve directly from 9.5.7 (see p. 93, $p < 1$) to obtain

$$c_0 = \frac{2}{\pi} K(p), \quad c_1 = -\frac{4}{\pi p} (K(p) - E(p)).$$

Hence,

$$c_0 + \frac{p}{2} c_1 = \frac{2}{\pi} E(p) = {}_2F_1(.5, -.5; 1; p^2) \quad (9.5.9)$$

$$c_1 + \frac{p}{2} c_0 = \frac{4}{\pi p} \left[E(p) + (p^2 - 1) \frac{K(p)}{4} \right] = \frac{p}{2} {}_2F_1(.5, .5; 1; p^2) \quad (9.5.10)$$

With $\alpha_i = \omega_i t + \theta_i(t)$ ($i = 1, 2$),

and recalling that $A_1 > A_2$, equation 9.5.6 is written

as

$$\begin{aligned}
 i(t) &= (\cos\alpha_1 + p\cos\alpha_2) \sum_{n=0}^{\infty} c_n \cos(n(\alpha_1 - \alpha_2)) - (A'_1 \cos\alpha_1 + A'_2 \cos\alpha_2) \\
 &= \sum_{n=0}^{\infty} \frac{c_n}{2} \left[\cos(\alpha_1 + n(\alpha_1 - \alpha_2)) + \cos(\alpha_1 - n(\alpha_1 - \alpha_2)) \right] \\
 &\quad + p \sum_{n=0}^{\infty} \frac{c_n}{2} \left[\cos(\alpha_2 + n(\alpha_1 - \alpha_2)) + \cos(\alpha_2 - n(\alpha_1 - \alpha_2)) \right] \\
 &\quad - A'_1 \cos\alpha_1 - A'_2 \cos\alpha_2
 \end{aligned} \tag{9.5.11}$$

where $p = A_2/A_1$ and c_n is given by 9.5.8. From 9.5.2 and 9.5.9, and 9.5.3 and 9.5.10, the coefficients of the $\cos\alpha_1$ and $\cos\alpha_2$ (i.e. fundamental) terms in 9.5.11 are zero.*

$i(t)$ then reduces to the desired form:

$$\begin{aligned}
 i(t) &= \sum_{n=1}^{\infty} \left[a_n \cos((n+1)(\omega_1 t + \theta_1(t)) - n(\omega_2 t + \theta_2(t))) + \right. \\
 &\quad \left. b_n \cos((n+1)(\omega_2 t + \theta_2(t)) - n(\omega_1 t + \theta_1(t))) \right]
 \end{aligned} \tag{9.5.12}$$

* Assuming that the fundamental frequency terms in the output are in fact the desired signals, equations 9.5.9 and 9.5.10 are expressions for their amplitudes.

where $a_n = c_n + pc_{n+1}$,
 $b_n = c_{n+1} + pc_n$. †

From 9.5.12, the line spectrum of the interference is given by

$$S_i(\omega) = \frac{1}{2} \sum_{n=1}^{\infty} \left[a_n^2 \delta(\omega - [(n+1)\omega_1 - n\omega_2]) + b_n^2 \delta(\omega - [(n+1)\omega_2 - n\omega_1]) \right] \quad (9.5.13)$$

a sum of n'th order intermodulation products at frequencies $(n+1)\omega_1 - n\omega_2$ and $(n+1)\omega_2 - n\omega_1$. To compute the power spectrum for modulated signals, n'th order convolutions of the input spectra are required to obtain the spectral shape of the n'th order product.

9.6 Three Sine Waves Through a Hard Limiter

Suppose the input to a hard limiter consists of a desired signal

$$s_D(t) = \sqrt{2P} \cos(\omega_D t + \theta_D(t)) ,$$

† For $p \neq 1$, $a_n \neq b_n$ as expected.

and two unwanted signals *

$$s_i(t) = \sqrt{2P_i} \cos(\omega_i t + \theta_i(t)) \quad (i = 1, 2)$$

Following the notation of section 4.3, the output from the nonlinearity is written as

$$\begin{aligned} y(t) &= s_D'(t) + n'(t) \\ &= (\sqrt{2P'} + n_1'(t)) \cos(\omega_D t + \theta_D(t)) + n_2'(t) \sin(\omega_D t + \theta_D(t)) \end{aligned} \quad (9.6.1)$$

$n_1'(t)$ and $n_2'(t)$ have been called the in and out of phase interference processes respectively. In this section we derive expressions for P' , $E\{n_2'\}$, and $E\{n_1'^2\}$, the signal and quadrature interference powers in the output.

From equation 5.2.3, the output signal power can be written as

$$P' = \frac{1}{2} \int_0^\infty \frac{J_1(\sqrt{2P} u)}{u} \prod_{k=1}^2 J_0(\sqrt{2P_k} u) du \quad (9.6.2)$$

This expression can be simplified for the cases of

- (i) $P > P_1 + P_2$,
- (ii) $P < |P_1 - P_2|$, and
- (iii) $P \ll P_1 = P_2$

* This could result from multiple access to a limiting repeater, or from adjacent channel tones in a limiter-discriminator FM receiver.

For cases (i) and (ii), the integral in 9.6.2 can be solved [16, chapter 5, p. 694] to give the result

$$P' = \begin{cases} \frac{1}{2} F_4(-.5, .5; 1, 1; P_1/P, P_2/P) & \text{for } P > P_1 + P_2 \\ \sqrt{\frac{P}{P_2}} F_4(.5, .5; 2, 1; P/P_2, P_1/P_2) & \text{for } P_2 > P + P_1 \end{cases} \quad (9.6.3)$$

It will be shown in Appendix B that when $P \ll P_1 = P_2$, *

$$P' \approx \frac{1}{2} \left[\frac{16}{3\pi^2} + \frac{\ln(2/\sqrt{\rho_i})}{\sqrt{2} \pi} \right]^2 \rho_i \quad (9.6.4)$$

F_4 is the hypergeometric function of two variables given by

$$F_4(a, b; c, d; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_m (d)_n m! n!} x^m y^n$$

where $(h)_k = h(h+1)(h+2)\dots(h+k)$.

Now, the probability density function of the envelope of the sum of two sinusoids of powers P_1 and P_2 is given by [1]

* This case was considered at great length in section 5.4.

$$f_R(r) = \begin{cases} \frac{2r}{\pi \sqrt{16P_1P_2 - (r^2 - 2P_1 - 2P_2)^2}} & \text{for } |\sqrt{P_1} - \sqrt{P_2}| \leq \frac{r}{\sqrt{2}} \leq \sqrt{P_1} + \sqrt{P_2} \\ 0 & \text{elsewhere} \end{cases} \quad (9.6.5)$$

Substituting this expression for the interference envelope density into equation 5.2.4 gives the following expression for the out of phase interference power from the bandpass hard limiter:

$$N'_2 = \begin{cases} \frac{1}{2} & \text{for } P \leq P_1 + P_2 - 2\sqrt{P_1P_2} \\ \frac{1}{2} - \frac{1}{2\pi} \int_{2(P_1+P_2-2\sqrt{P_1P_2})}^{2 \min\{P, P_1+P_2+2\sqrt{P_1P_2}\}} (1-\frac{r}{2P}) \frac{dr}{\sqrt{-r^2+4(P_1+P_2)-4(P_1-P_2)^2}} & \text{for } P > P_1 + P_2 - 2\sqrt{P_1P_2} \end{cases}$$

Evaluating the integral [16, chapter 5, pp. 81,83], this equation reduces to

$$N'_2 = \begin{cases} \frac{1}{2} & \text{for } P \leq P_1 + P_2 - 2\sqrt{P_1P_2} \\ \frac{1}{2} \left[\frac{(P_1+P_2)}{2P} \left(1 - \frac{2}{\pi} \sin^{-1} \left(\frac{P_1+P_2-P}{2\sqrt{P_1P_2}} \right) \right) - \frac{1}{\pi} \sqrt{\frac{2(P_1+P_2)}{P} - \left(1 - \frac{(P_1-P_2)^2}{P^2} \right)} \right] & \text{for } |P-P_1-P_2| < 2\sqrt{P_1P_2} \\ \frac{P_1 + P_2}{2P} & \text{for } P \geq P_1 + P_2 + 2\sqrt{P_1P_2} \end{cases} \quad (9.6.6)$$

We observe in 9.6.6 the interesting result that N'_1 assumes its maximum limiting value of .5 whenever the desired signal amplitude is less than the difference in amplitudes between the two interference signals. The fact that

$$N'_2 = E\{\sin^2 \Delta_d\} = .5 \quad (\Delta_d = \text{phase difference between resultant and desired signal})$$

does not imply that Δ_d is uniform on $[-\pi, \pi)$, although the converse is true. A uniformly distributed phase difference between $s_D(t) + I(t)$ and $s_D(t)$ occurs only in the limit as $P \rightarrow 0$. In this case $x(t)$ contains no information about the waveform $s_D(t)$. However, for values of P between 0 and $P_1 + P_2 - 2\sqrt{P_1 P_2}$, there is power tradeoff only between the in phase noise and the signal out of the repeater; $N'_2 = E\{n_2'^2\}$, but not the probability density function of n_2' , remain unchanged. In this case the in phase noise power is

$$N'_1 = .5 - 2P' \quad (9.6.7)$$

where P' is given by the second expression in 9.6.3 provided $P < P_2 - P_1$. In figure 32, N'_1 and N'_2 have been plotted versus the input carrier-to-interference power ratio

$$C_1 = \frac{P}{P_1 + P_2}$$

for the case of $P_1 = P_2$. As in figure 20, we shall note that

$$N_1^2 \gg N_2^2 \quad \text{for } \alpha_1 \gg 1.$$

Finally, for the special case of $P_2 = 0$ (i.e. the interfering sinusoid), from 9.6.6 the out of phase interference power reduces to

$$N_2^2 = \begin{cases} \frac{P}{P_1} & \text{for } P \leq P_1 \\ \frac{P_1}{2P} & \text{for } P > P_1 \end{cases} \quad (9.6.8)$$

This result can be verified directly by equation 5.2.4.

9.7 The Gaussian Assumption

From the central limit theorem [10, chapter 7, p. 252], the sum of m constant amplitude sinusoids with uniform and independent phase angles is nearly gaussian for m large. Lyons [5, chapter 4] in particular has demonstrated that

$$\lim_{m \rightarrow \infty} P_{\sum_{i=1}^m A_i \cos \theta_i}(r) = \frac{\sqrt{2} r}{\sqrt{\sum_{i=1}^m A_i^2}} \exp \left[-r^2 / \sum_{i=1}^m A_i^2 \right] \quad (9.7.1)$$

in 9.2.3 provided that the amplitudes diminish at least as fast as $m^{-1/2}$; i.e.,

$$A_i < \frac{K}{\sqrt{m}} \quad \text{for all } i \neq j \text{ and some } K < \infty.$$

Equation 9.7.1 implies that the probability density function of the envelope of the interference to the j th signal approaches pointwise that of a stationary gaussian process of equivalent power. For equal amplitude sinusoids, this convergence to the Rayleigh envelope is rapid, as can be seen in figure 33 [5, chapter 4].

Under the assumption that the interference to each signal is gaussian, the nonlinear analysis is greatly simplified, and we can in fact use many of the results of chapters 7 and 8. For example, with $f_{R_j}(r)$ given by 9.7.1 and σ^2 replaced by $\frac{1}{2} \sum_{i \neq j}^m A_i^2$, equations 9.2.4 and 9.3.1 reduce to 7.1.5 and 7.4.4 respectively.

Since the signal plus noise case has been discussed at length, it is unnecessary to restate the results obtainable under the gaussian interference assumption. It is pointed out, however, that in computing the power spectrum of the intermodulation product interference, the gaussian noise

Figure 33

representation of the sum of the accessing angle modulation signals must have a power spectrum which is the sum of the spectral densities of the signals. Campbell [2] has demonstrated that for an input to a bandpass hard limiter given by

$$x(t) = \sqrt{2P} \cos(\omega_D t + \theta_D(t)) + \sum_{k=1}^m \sqrt{2P_k} \cos(\omega_k t + \theta_k(t)) + z(t),$$

under the ordering and normalization constraints

$$P_1 \geq P_2 \geq P_3 \dots \geq P_m \quad \text{and}$$

$$\sum_{k=1}^m P_k + \sigma^2 = 1,$$

the output power P' of the desired signal satisfies the inequality

$$\left| P' - \frac{\pi}{8} P \left[{}_1F_1(.5, 2; -P) \right]^2 \right| \leq$$

$$.06 \sum_{k=1}^m P_k^2 + .707 P_1 e^{-P_1} + \frac{.707}{N} \left[\frac{2}{\pi} \sqrt{\frac{P_1}{P_N}} \left(1 + \frac{1}{8} \sqrt{\frac{P_1}{P_N}} \right) \right]^{\frac{1}{2}}$$

(2.7.1)

where N is chosen so that $\frac{P_1}{P_N} \leq 1.406$.

REFERENCES TO CHAPTER 9

- [1] W.A. Flood, "The Fading of Ionospheric Signals",
Cornell University School of Engineering,
August, 1954.
- [2] L.L. Campbell, "Approximations of an Integral Connected
with Hard-Limited Signals and Noise", Queen's
University Report No. 69-3, Department of Mathematics,
July, 1969.

CHAPTER 10

EFFECTS OF AM TO PM CONVERSION ON MULTIPLE SIGNALS

10.1 Vector Representation

Following the form and notation of section 9.1, we write the following equations for a multiple signal input to an AM/PM device having the transfer characteristic $g(x)$:

$$\vec{s}_k = \text{k'th input signal}$$

$$\begin{aligned} A_k &= \text{amplitude of k'th input signal} \\ &= |\vec{s}_k| \end{aligned}$$

$$E = \text{input envelope}$$

$$= \left| \sum_{k=1}^m \vec{s}_k \right|$$

$$\beta = \text{signal phase shift resulting from AM/PM (reference 4.3.9)}$$

$$= \tan^{-1} \left[\frac{E\{\sin(g(E))\}}{E\{\cos(g(E))\}} \right]$$

$$\begin{aligned} A'_k &= \text{output amplitude of k'th signal} \\ &= E\{\cos(g(E)-\beta)\} A_k \end{aligned}$$

$\vec{o}$ = instantaneous output

$$= \left[\sum_{k=1}^m \vec{s}_k \right] e^{-jg(|\sum_{k=1}^m \vec{s}_k|)}$$

$\vec{s}'_k$ = k'th output signal

$$= \left[\frac{A'_k}{A_k} \right] \vec{s}_k e^{-j\beta}$$

$\vec{i}$ = intermodulation product interference in output

$$\begin{aligned} &= \vec{o} - \sum_{k=1}^m \vec{s}'_k \\ &= \left[\sum_{k=1}^m \vec{s}_k \right] \left(e^{-jg(|\sum_{k=1}^m \vec{s}_k|)} - C e^{-j\beta} \right) \quad \text{where} \end{aligned}$$

$$C = \frac{A'_k}{A_k} = E\{\cos(g(E) - \beta)\} \quad (10.1.1)$$

is defined to be the signal compression factor resulting from AM to PM conversion.

10.2 S/I and the Compression Factor

An output signal from an amplitude linear, phase nonlinear device is an attenuated and delayed replica of the input signal. Analysis is simplified by the fact that

each signal experiences the same phase lag and amplitude reduction given by

$$\beta = \tan^{-1} \left[\frac{\int_0^{\infty} \sin(g(r)) f_E(r) dr}{\int_0^{\infty} \cos(g(r)) f_E(r) dr} \right] \quad (10.2.1)$$

and

$$C = \int_0^{\infty} \cos(g(r)-\beta) f_E(r) dr \quad (10.2.2)$$

respectively. Here $f_E(r)$, the density of the envelope of the sum of m incoherent sinusoids, is given by 9.3.2. C , a number between 0 and 1, is the maximum average projection of an output signal with its message and AM/PM angle modulation on a vector which contains only the message modulation and an arbitrary constant phase shift.* Intermodulation noise in the output results from the imposition of phase modulation coherent with the input envelope, a process which contains no useful information about any of the m messages.

* By setting $\frac{d}{dQ} \int_0^{\infty} \cos(g(r)-Q) f_E(r) dr = 0$, we obtain

the solution $Q = \beta$ given by 10.2.1. This maximizes the average value of $\cos(g(E)-Q)$, and heuristically is therefore the phase shift experienced by the signal. In addition, it can be shown that under solution 10.2.1-10.2.2, the residual component of the output is uncorrelated with each of the input signals; this sometimes serves as a definition of the intermodulation noise (see Appendix D).

The total signal power and the total power out of the nonlinearity can be simply written as

$$S_o = \frac{1}{2} \sum_{k=1}^m A_k'^2 = \frac{C^2}{2} \sum_{k=1}^m A_k^2$$

and
$$P_o = \frac{1}{2} \sum_{k=1}^m A_k^2 .$$

The total signal-to-intermodulation noise power ratio is then

$$\left(\frac{S}{I}\right) = \frac{S_o}{P_o - S_o} = \frac{C^2}{1 - C^2} \quad (10.2.3)$$

C , a measure of the signal degradation resulting from AM to PM conversion, is a decreasing function of

$$\begin{aligned} V &= \int_{-\infty}^{\infty} f_E(x) dg(x) \approx \left[g(\mu_E + \sigma_E) - g(\mu_E - \sigma_E) \right] / 2 \\ &\approx \frac{dg(\mu_E)}{dx} \sigma_E \end{aligned} \quad (10.2.4)$$

where $g(x)$ is assumed to be a non-decreasing function of x (consistent with the physical operation of a TWT),

and μ_E and σ_E^2 denote the mean and variance of the random variable E . V is the variation in phase shift over the operating region of the device. It largely depends on the variance of the input envelope and the AM/PM coefficient at the mean operating point. Assuming that $g(E) - \theta$ is a small angle with a high degree of probability, equation 10.2.2 reduces to

$$C \approx 1 - \frac{1}{2} \sigma_{g(E)}^2 \approx 1 - \frac{1}{2} V^2 \quad (10.2.5)$$

10.3 The IM Product Power Spectrum

The input to and output from the AM/PM nonlinearity are

$$\begin{aligned} x(t) &= \sum_{k=1}^m A_k \cos(\omega_k t + \theta_k(t)) \\ &= E(t) \cos(\omega_o t + \phi(t)) \end{aligned}$$

and

$$\begin{aligned} y(t) &= E(t) \cos(\omega_o t + \phi(t) - g(E(t))) \\ &= \sum_{k=1}^m A_k \cos(\omega_k t + \theta_k(t) - g(E(t))) \end{aligned} \quad (10.3.1)$$

The interference component of the output is given by

$$\begin{aligned}
 i(t) &= y(t) - \sum_{k=1}^m s'_k(t) \\
 &= \sum_{k=1}^m \left[(A_k \cos(g(E(t)) - \beta) - A'_k) \cos(\omega_k t + \theta_k(t) - \beta) \right. \\
 &\quad \left. + A_k \sin(g(E(t)) - \beta) \sin(\omega_k t + \theta_k(t) - \beta) \right]
 \end{aligned}
 \tag{10.3.2}$$

Its autocorrelation function reduces to the form

$$\begin{aligned}
 R_i(\tau) &= E\{i(t+\tau) i(t)\} \\
 &= \frac{1}{2} R \left[\sum_{k=1}^m (R_{e j g(E)}(\tau) A_k^2 - A_k'^2) R_{e j \theta_k}(\tau) e^{j \omega_k \tau} \right]
 \end{aligned}
 \tag{10.3.3}$$

We have here made use of the relations

$$E\{\cos(g(E) - \beta)\} = \frac{A'_k}{A_k}, \quad k = 1, 2, \dots, m \quad (\text{reference 10.1}),$$

$$\begin{aligned}
 E\{\sin(g(E) - \beta)\} &= E\{\sin(g(E))\} \cos \beta - E\{\cos(g(E))\} \sin \beta \\
 &= E\{\cos(g(E))\} (\tan \beta \cos \beta - \sin \beta) = 0,
 \end{aligned}$$

† When $g(E) = \text{a constant}$ (i.e. no differential phase shift),
 $R_{e j g(E)}(\tau) = 1$, $A'_k = E\{\cos(g(E) - \beta)\} A_k = A_k$, giving the
 expected solution $R_i(\tau) = 0$.

and have again employed the notation

$$R_x(\tau) = E\{x(t+\tau) x^*(t)\} .$$

The power spectrum $S_i(f)$ of the intermodulation product interference is the Fourier transform of 10.3.3. Applying equation 9.4.4, the signal-to-distortion power in each channel can be computed. The line spectrum of the interference is obtained by letting

$$R_{e^{j\theta_k}}(\tau) = 1$$

in 10.3.3.

10.4 Two Sine Waves Through a Quadratic AM/PM Device

If k_1 is set to 0 in equation 2.5.1, a transfer function of the form

$$g(x) = a x^2 \tag{10.4.1}$$

is obtained. Such an AM/PM characteristic may not be an accurate representation of the physical behaviour of a TWT, but it does facilitate a demonstration of Fourier analysis for a two sinusoid input given by

$$x(t) = \sum_{k=1}^m A_k \cos(\omega_k t + \theta_k(t))$$

From 10.3.1, the output from the AM/PM characteristic described by 10.4.1 is

$$y(t) = \sum_{k=1}^2 A_k \cos \left[\omega_k t + \theta_k(t) - a(A_1^2 + A_2^2 - 2A_1 A_2 \cos[(\omega_2 - \omega_1)t + \theta_2(t) - \theta_1(t)]) \right] \quad (10.4.2)$$

The noise component of the output is

$$i(t) = y(t) - \sum_{k=1}^2 A'_k \cos(\omega_k t + \theta_k(t) - \beta) \quad (10.4.3)$$

Here β is a constant given by

$$\beta = \tan^{-1} \left[\frac{\int_0^\infty \sin(ar^2) f_E(r) dr}{\int_0^\infty \cos(ar^2) f_E(r) dr} \right] \quad (10.4.4)$$

where $f_E(r)$, the envelope of the sum of two randomly phased sinusoids, is given by 9.6.4 with P_k replaced by $A_k^2/2$.

By making the substitution

$$r = \sqrt{r' + (A_1 - A_2)^2}, \quad dr = \frac{dr'}{2r},$$

and applying integral formulas taken from Gradshteyn and Ryzhik [16, chapter 5, p. 424, 3.768-7 & -9], we obtain the following closed form expression for 10.4.4:

$$\begin{aligned}\beta &= \tan^{-1} \left[\frac{\sin(a(A_1^2 + A_2^2)) J_0(2aA_1A_2)}{\cos(a(A_1^2 + A_2^2)) J_0(2aA_1A_2)} \right] \\ &= a(A_1^2 + A_2^2) \quad (10.4.5)\end{aligned}$$

if the peak-to-peak phase shift, $\Delta\phi$, satisfies the inequality

$$\Delta\phi = g(A_1 + A_2) - g(|A_1 - A_2|) = 4aA_1A_2 < \pi \quad ; \quad *$$

i.e., the signal amplitudes satisfy the relation

$$A_1A_2 < \frac{\pi}{4a} \quad . \quad (10.4.6)$$

Noting from 10.4.5 that

$$E\{\cos(g(E))\} = \cos(a(A_1^2 + A_2^2)) J_0(2aA_1A_2) = \cos\beta J_0(2aA_1A_2) ,$$

the output signal amplitudes are then given by (reference 10.1)

$$\begin{aligned}A'_k &= E\{\cos(g(E) - \beta)\} A_k \\ &= E\{\cos(g(E))\} (\cos\beta + \tan\beta \sin\beta) A_k \\ &= \frac{E\{\cos(g(E))\}}{\cos\beta} A_k \\ &= J_0(2aA_1A_2) A_k \quad (10.4.7)\end{aligned}$$

* Otherwise β could assume several values spaced π radians apart; each value would correspond to a particular region of E , and $E\{\cos(g(E) - \beta)\}$ could not be evaluated for a single value of β .

Now, from equations 10.4.2 and 10.4.3, the intermodulation product noise in the output is

$$i(t) = R \left[\sum_{k=1}^2 \left(e^{j2aA_1A_2 \cos(\alpha(t))} - J_0(2aA_1A_2) \right) A_k e^{j(\omega_k t + \theta_k(t) - a(A_1^2 + A_2^2))} \right] \quad (10.4.8)$$

where

$$\alpha(t) = (\omega_2 - \omega_1)t + \theta_2(t) - \theta_1(t) \quad (10.4.9)$$

Applying the Bessel expansion [16, chapter 5, p. 973]

$$e^{j2aA_1A_2 \cos(\alpha(t))} = \sum_{i=-\infty}^{\infty} (j)^i J_i(2aA_1A_2) e^{ji\alpha(t)}$$

in 10.4.8 gives the result

$$\begin{aligned} i(t) &= R \left[\sum_{k=1}^2 \left(2 \sum_{i=1}^{\infty} (j)^i J_i(2aA_1A_2) \cos(i\alpha(t)) \right) A_k e^{j(\omega_k t + \theta_k(t) - a(A_1^2 + A_2^2))} \right] \\ &= 2 \sum_{k=1}^2 A_k \left\{ \left[\sum_{i=1}^{\infty} (-1)^i J_{2i-1}(2aA_1A_2) \cos(2i\alpha(t)) \right] \cos(\omega_k t + \theta_k(t) - a(A_1^2 + A_2^2)) \right. \\ &\quad \left. + \left[\sum_{i=1}^{\infty} (-1)^i J_{2i-1}(2aA_1A_2) \cos((2i-1)\alpha(t)) \right] \sin(\omega_k t + \theta_k(t) - a(A_1^2 + A_2^2)) \right\} \end{aligned} \quad (10.4.10)$$

Recalling from 10.4.9 that $\alpha(t)$ is the RF phase difference between the signals, note that the amplitudes of all the intermodulation products are available from equation 10.4.10.

These can be used to compute the output signal-to-distortion ratios. We observe also that equation 10.4.10 is similar in form to 9.5.12, an expression for the interference generated by two sinusoids passed through an ideal amplitude limiter.

From 10.2.3, with the compression factor given by

$$C = J_0(2aA_1A_2) \quad (\text{under assumption 10.4.6}) ,$$

the total output signal-to-interference power is

$$(S/I) = \frac{J_0^2(2aA_1A_2)}{1 - J_0^2(2aA_1A_2)} \quad (10.4.11)$$

Since $2P_t = A_1^2 + A_2^2 \geq 2A_1A_2$, and

$J_0(x)$ is a monotonic increasing function for $0 \leq x < \pi/2$,

under the total power constraint $P_t < \pi/4a$, (S/I) satisfies the inequality

$$(S/I) \geq \frac{J_0^2(2aP_t)}{1 - J_0^2(2aP_t)} \quad (10.4.12)$$

equality if and only if $A_1 = A_2$. In figure 34, (S/I) vs A_2/A_1 has been plotted for the case of $P_t = \pi/12a$. This corresponds to a maximum change in phase shift over the operating range of 60° .

Figure 34

TWO SINUSOIDS THRU QUADRATIC AM - PM DEVICE
 WITH $20(A_1^2 + A_2^2) = \text{MAX (PEAK TO PEAK PHASE SHIFT)}$
 HELD FIXED AT $\frac{\pi}{3} = 60^\circ$

10.5 Two Sine Waves Through a Linear AM/PM Device

Consider the linear AM/PM transfer function

$$g(x) = K_p x \quad (10.5.1)$$

having as input two sinusoids of amplitudes A_1 and A_2 . The total input power is constrained to unity; i.e.,

$$A_1^2 + A_2^2 = 2 \quad (10.5.2)$$

From 9.6.5 and the above, the probability density function of the input envelope is

$$f_E(r) = \begin{cases} \frac{2r}{\pi \sqrt{4A_1^2 A_2^2 - (r^2 - 2)^2}} & \text{for } |A_2 - A_1| \leq r \leq A_2 + A_1 \\ 0 & \text{elsewhere} \end{cases} \quad (10.5.3)$$

Applying equations 10.2.3, 10.2.2, and 10.2.1, with the substitution $r^2 = r$, the output signal-to-intermodulation product interference ratio reduces to

$$(S/I) = \left[\left(\frac{1}{\pi} \int_{2(1-A_1 A_2)}^{2(1+A_1 A_2)} \frac{\cos(K_p \sqrt{r} - \beta)}{\sqrt{4A_1^2 A_2^2 - (r-2)^2}} dr \right)^{-2} - 1 \right]^{-1} \quad (10.5.4)$$

where

$$\beta = \tan^{-1} \left[\frac{\int_{2(1-A_1A_2)}^{2(1+A_1A_2)} \frac{\sin(K_p\sqrt{r})}{\sqrt{4A_1^2A_2^2 - (r-2)^2}} dr}{\int_{2(1-A_1A_2)}^{2(1+A_1A_2)} \frac{\cos(K_p\sqrt{r})}{\sqrt{4A_1^2A_2^2 - (r-2)^2}} dr} \right] \quad (10.5.5)$$

A computer program was written to implement 10.5.5 and 10.5.4, and figure 35 gives a dB plot of (S/I) vs A_2/A_1 for $K_p = .2$ & $.4$. These correspond to maximum (when $A_1=A_2$) peak-to-peak phase shifts of $.4$ and $.8$ radians respectively.

Figure 35

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CHAPTER 11

SIMULATION OF A GENERAL NON-LINEARITY HAVING A MULTI-CARRIER SIGNAL AND NOISE INPUT

11.1 Theory

We recall that the general bandpass non-linearity

$$\vec{f}(x) = (f_1(x), f_2(x))$$

is defined by the input-output relationship

$$\begin{aligned} x(t) &= E(t) \cos(\omega_0 t + \phi(t)) \\ y(t) &= f_1(E(t)) \cos(\omega_0 t + \phi(t) - f_2(E(t))) \end{aligned} \quad (11.1.1)$$

If $x(t)$ is written in the form

$$x(t) = \sum_{i=1}^m A_i \cos(\omega_i t + \theta_i(t)) + n(t) , \quad (11.1.2)$$

then applying the results obtained previously for the individual AM/AM and AM/PM non-linearities, $y(t)$ can be expressed as

$$y(t) = \sum_{i=1}^m A'_i \cos(\omega_i t + \theta_i(t) - \beta) + n'(t) \quad (11.1.3)$$

where

$$(1) \quad \beta = \tan^{-1} \left[\frac{E\{\sin(f_2(E))\}}{E\{\cos(f_2(E))\}} \right]$$

with E having the probability density function

$$f_E(r) = r \int_0^{\infty} u J_0(ur) \prod_{k=1}^m J_0(A_k u) e^{-\frac{1}{2}\sigma^2 u^2} du \quad (11.1.4)$$

$$(2) \quad A'_j = E \left\{ f_1(\sqrt{R_j'^2 - 2CA_j R_j' \cos \alpha + C^2 A_j'^2}) \left(\frac{CA_j - R_j' \cos \alpha}{\sqrt{R_j'^2 - 2CA_j R_j' \cos \alpha + C^2 A_j'^2}} \right) \right\} \quad (11.1.5)$$

where $C = E\{\cos(f_2(E) - \beta)\}$

and R'_j has the approximate probability density function

$$f_{R'_j}(r) \approx r \int_0^{\infty} u J_0(ur) \prod_{k \neq j} J_0(A_k u) e^{-(\sigma^2 + \frac{1}{2}(1-C^2)A_j'^2)u^2} du \quad (11.1.6)$$

and α is nearly uniformly distributed on $[0, 2\pi)$ and

independent of R'_j . Equation 11.1.6 is derived in Appendix C.

- (3) $n'(t)$, resulting from thermal noise at the input and intermodulation through the nonlinear device, is a stationary random process uncorrelated with the desired modulating waveforms $\{\theta_i(t)\}$ (see Appendix D).

In analyzing the communications problem, it is not sufficient to determine the amplitudes A'_i of the signals retransmitted from the satellite. It is necessary to examine

the properties, in particular the power spectrum, of the output noise component $n'(t)$. This will be accomplished from known information about the total output $y(t)$ and the signal components. For example, the total power out of the bandpass nonlinearity (f_1, f_2) is simply

$$P_o = \frac{1}{2} E\{f_1^2(E)\}$$

It can immediately be concluded that the noise power is

$$\begin{aligned} N' &= E\{n'^2\} \\ &= P_o - \frac{1}{2} \sum_{i=1}^m A_i'^2 \end{aligned}$$

The important question, "How is this noise power distributed in the frequency spectrum?", is not as easily answered. The purpose of chapter 11 is largely to extend the analytical work to a computer simulation which evaluates the power spectrum and first order statistics of $n'(t)$ for any non-linearity and input consisting of unmodulated carriers and noise.

11.2 Brief Description of Computer Programs

11.2-1 Program "NONLINEAR"

Given the individual signal and thermal noise powers into a nonlinear repeater having arbitrary AM/AM and AM/PM transfer characteristics, this program computes the output signal powers and the power of the intermodulation interference and noise. (A follow-up program "SPECTR" determines the line spectrum of the output distortion). Up to 20 signals may be present; it is implicitly assumed they have constant amplitudes and are incoherent (i.e. have uniformly and independently distributed RF phase differences). The suppression factors (changes in signal to total power) are computed for each signal. Results obtained can influence the choice of satellite TWT backoff and allocation of up-link power for a particular FDMA configuration. The pertinent equations to this program are given in section 11.1.

11.2-2 Program "SPECTR"

Given the carrier frequencies and powers at the input and output of an arbitrary bandpass nonlinearity, this program computes the line spectrum and statistics (envelope and phase densities) of the output intermodulation distortion. The program also has the facility of mathematically simulating input gaussian noise of arbitrary autocorrelation function and including it in the calculations. The output signal powers and constant phase shift β must be previously obtained from program NONLINEAR.

If the carriers are unmodulated and thermal noise is not present, the input to and output from the general bandpass non-linearity are written as the deterministic functions (reference 11.1.1 to 11.1.5):

$$\begin{aligned}
 x(t) &= \sum_{i=1}^m A_i \cos (2\pi f_i t + \theta_i) \\
 &= E(t) \cos (2\pi f_0 t + \phi(t))
 \end{aligned}
 \tag{11.2.1}$$

and

$$\begin{aligned}
 y(t) &= \sum_{i=1}^m A'_i \cos (2\pi f_i t + \theta_i - \beta) + n'(t) \\
 &= f_1(E(t)) \cos (2\pi f_0 t + \phi(t) - f_2(E(t)))
 \end{aligned}
 \tag{11.2.2}$$

It can immediately be observed that $x(t)$, $E(t)$, $\phi(t)$, $y(t)$, and $n'(t)$, are periodic functions* all having the period

$$T = [\text{greatest common factor of } \{10^k(f_1, f_2, \dots, f_m)\}]^{-1} \tag{11.2.3}$$

where k is the minimum integer chosen to make $10^k f_i = \text{integer}$ ($i=1, m$).

* Assuming the f_i can be expressed as terminating decimal numbers - otherwise the functions are nearly periodic. In this application the distinction is not important

By generating sufficiently closely spaced samples of $n'(t)$ over this period, we can compute by fast Fourier transform the line spectrum of the IM distortion. Apart from a few further complications, this basically describes the approach taken in program SPECTR.

In equation 11.2.2, $n'(t)$ is band-limited RF intermodulation noise. If it is assumed that $n'(t)$ has a bandwidth no greater than four times the total input bandwidth B_i ; i.e.,

$$B_{n'} = 4B_i, * \quad (11.2.4)$$

then $4B_i$ complex (quadrature) time samples per second over its period T are sufficient to represent the process. The total number of samples theoretically required to obtain the IM product line spectrum is then

$$N = 4TB_i \quad (11.2.5)$$

* In fact, IM products will fall at all frequencies $kf_i - \ell f_j$ where $k - \ell = 1$. Some of these frequencies will lie outside the allotted bandwidth. However, since (1) they will in general be of lower magnitude, and (2) we are principally interested in those products falling in band with the signals and only products lying $2 \times 1.5 = 3$ full bandwidths away from the signal band can cause aliasing errors in this region, the assumption is considered reasonable. (Of course, it should be remembered that out of band products will still represent interference to an adjacent transponder).

In program SPECTR, N is identically $8192 = 2^{13}$, and T is computed from 11.2.3 to a maximum of $2000/B_i$ (i.e. giving three significant figures in the equivalent IF carrier frequencies*). The sampling rate becomes

$$R = 8192/T \geq 4.096B_i \text{ samples per second,}$$

which allows for the IM bandwidth given in 11.2.4. If there is a large common factor in the carrier frequencies* (see 11.2.3), T is considerably reduced and R increases proportionately. This principle is now demonstrated by a simple example:

Suppose $f_1 = 2.4$ Hz and $f_2 = 4.7$ Hz. If the s_1 and s_2 phasors start together, they do not simultaneously return to the same positions until they have completed 24 and 47 cycles respectively. The period of $s_1(t) + s_2(t)$ is therefore 10 seconds. If, however, f_2 is replaced by 4.8 Hz, the two phasors require only 1 and 2 cycles each to coincide at their starting points. The resultant period in this case is reduced to $1/2.4$ seconds; samples taken outside this period are redundant. In the computer program it is convenient to hold the total number of

* In the simulation, the $\{f_i\}$ are in fact translated down to occupy the $(1.5B_i, 2.5B_i)$ frequency band, giving an equivalent wideband model. Under the assumption 11.2.4, the 1.5 bandwidth clearance from DC avoids reflections of low frequency IM products; the translation will therefore not affect the computed results.

samples fixed; hence, the sampling rate is increased in the second case. Finer sampling gives further protection against aliasing error, and improves the reliability of $i(t)$ envelope and phase probability density function plots also obtained from the time samples.

Gaussian noise cannot easily be adequately represented by a deterministic time function^{*}. In order to include additive noise at the input, a subroutine has been written to generate samples of a circularly complex gaussian process of arbitrary (but symmetric) spectral density. 8192 is a sufficient number of samples to reasonably^{**} represent the process. When noise is present, the coarsest sampling rate, $R = 4.096B_1$, is used so highly correlated noise samples (which may not be sufficiently representative of the gaussian ensemble) are not generated.

Note that program SPECTR alone could be used to compute the IM line spectrum out of an arbitrary non-linearity. With the output signal powers and phase shift set equal to zero, (implying that their true value is not available), SPECTR computes the total

* A sum of equal amplitude unmodulated carriers has asymptotically gaussian first order statistics, but cannot be given an arbitrary RF spectral density.

** If these samples were being used to compute error rates, many more might be required. However, the tail of the distribution is not critical for our purposes, and hence 8192 gives a sufficiently large ensemble.

output line spectrum; by simply removing the spectral lines generated at the signal frequencies, the interference spectral density is isolated. The interference envelope and phase probability density functions, however, are not obtained by this method, and IM lines falling exactly at signal frequencies are lost.

11.3 Verification of Program "NONLINEAR"

In order to test the method and accuracy of program "NONLINEAR", three simple cases were run and compared with answers obtained from analytical expressions. While the evidence is by no means conclusive, it suggests that the program is reliable and accurate. The results are now listed.

(1) Signal and Noise Through A Hard Limiter:

For an input signal-to-noise ration of -10 dB, the program predicts a signal suppression factor of -.85 dB. From equation (5), p.98, we compute a value of -.86 dB.

(2) Three Signals Through A Hard Limiter:

For two equal amplitude interferers and an input signal-to-interference ratio of 16 dB, the program predicts the positive suppression factor (reference section 5.4) .36 dB. Equation 9.6.2 gives the asymptotic value .37 dB; figure 6, representing a third independent source, gives approximately the same result.

(3) Two Signals Through A Quadratic AM/PM Characteristic:

When 2 equal sinusoids constitute the input to a quadratic AM/PM non-linearity, "NONLINEAR" computes a 7.83 dB output signal-to-IM noise power ratio for a peak-to-peak phase shift of 60° . From equation 10.4.10 (and figure 31), we obtain the result 8.25 dB. Note that the input envelope probability density function here has a pole and an abrupt cut-off at its extreme value (reference 5.4.1); the application of equation 11.1.4 to compute it at best taxes the numerical method, and may in fact not be a rigorous procedure^{*}. This should explain the larger discrepancy between answers in (3) as compared to (1) and (2).

11.4 Example: An "Arctic Model"

In Telesat's arctic model (see figure 36) it is assumed that one large FDM-FM carrier shares the repeater with 12 small FDM-FM carriers to provide telephone service between a large urban centre and 12 scattered (i.e. northern) communities. The main station multiplexes all of the south to north voice circuits on a single large wide deviation FM carrier. Each small station receives and demodulates this carrier, then demultiplexes the appropriate voice

* $\int_0^r u J_0(ur) J_0^2(cu) du$ is not Riemann integrable or even conditionally convergent for all r .

circuits. The return link from a northern community consists of a low power, narrow band FM modulated carrier. The main station receives all 12 small carriers to complete the two-way communications facility.

Figure 36

It is assumed that each one way voice circuit has the same share of the RF power, Hence, for northern communities each providing the same number of telephone channels, we conclude that each small carrier has 1/12 the power of the large carrier. In fact, due to the different G/T's (reference section 3.3) of the large and small station antennas, equal north-south, south-north transmitted power is not the condition for optimum performance. However, in the example given here this condition is assumed. The only degree of freedom in the model is the total input power to the repeater, or the constant of proportionality among the up-link powers.

While a small carrier has 1/24 of the total repeater input power, it may only retain a considerably reduced share of the total output power. The suppression factors (change in signal-to-total power) on the large carrier and on a small carrier have been plotted in figure 37. Note that the large carrier experiences a net power gain and each small carrier a net loss, the suppression effect becoming less pronounced with an increasing TWT backoff. In figure 38, the AM/PM power loss for both the large carrier and a small carrier is plotted vs input backoff. The total output signal power to inter-modulation noise power vs input backoff is presented in figure 39. The tube characteristics used in the computation,

$$f_1(x) = \sin\left(\frac{\pi}{4}x\right) + .183 \sin\left(\frac{\pi}{2}x\right) + .222 \sin\left(\frac{3\pi}{4}x\right) + .1 \sin\left(\frac{5\pi}{4}x\right) \quad (11.4.1)$$

and

$$f_2(x) = .458 (1 - e^{-4.54x^2}) + .1x^2, \quad (11.4.2)$$

have been given by Berman [2, chapter 4, 6, chapter 2] *. The computer program here does not restrict $f_1(x)$ and $f_2(x)$ to particular functional representations as do some computational methods [6, chapter 2; 1, chapter 4]. The single carrier tube characteristics given by 11.4.1 and 11.4.2, and the multi-carrier arctic model characteristics, obtained from equations 11.1(2) and 11.1(1), are plotted in figure 40. Note that the two sets of curve converge at large backoffs. Figures 41 and 42 plot the sampled probability density functions of the IM envelope and phase (relative to both the large and a small output carrier) for an input backoff of -9 dB. Observe that the interference is strongly coherent with the large carrier (reference 4.4), but has nearly a uniformly distributed phase with respect to the small carrier. The sharp spike in figure 41 indicates that the interference very nearly has a constant amplitude (one which alone is insufficient to generate clicks-reference 7.9 - into the baseband of a small FM carrier).

Finally, figures 43 and 44 are Arctic model 'intermod' plots for two frequency allocations. A 36 MHz transponder bandwidth has been assumed; the large carrier occupies 18 MHz and each small carrier occupies 1.5 MHz **. In figure 43 the large carrier is

* $x = 1$ corresponds to a saturating input voltage (i.e. 0 dB backoff).

** The assumption of equal RF bandwidth for each voice circuit, like the assumption of equal RF power, is not strictly correct.

flanked on either side by six small carriers; in figure 44 the 12 small carriers are equally spaced on the right half of the band. The input backoff in each case is -9 dB. An intermod plot can be used to determine approximately the signal-to-intermodulation distortion power ratio in each RF channel; these figures are required in evaluating the channel performances. Note that while the program does not generate an optimum FDMA configuration, it can lead to a good choice of TWT backoff (see 6.1), carrier frequencies, and the division of up-link power^{*}.

* In the Arctic model, for example, we might want to give these small carriers embedded in IM noise a little more power, and the large carrier (due to its positive suppression factor - see figure 37) a little less. Of course, all of the parameters discussed in 3.3 must be considered in a systems analysis.

Figure 41

Figure 42 (a)

Figure 42 (b)

INPUT BACKOFF = 9 dB
 OUTPUT BACKOFF = 4.52 dB
 PHASE SHIFT PER CARRIER = 10.3°
 C/I_{total} = 16.8 dB

Figure 43

INPUT BACKOFF = 9 dB
 OUTPUT BACKOFF = 4.52 dB
 PHASE SHIFT PER CARRIER = 10.3°
 (C/I)_{total} = 16.8 dB

Figure 44

11.5 An Experimental Approach Based on the Stochastic Method

The stochastic method of analyzing the effect of non-linear devices on multiple inputs also defines a simple and conceptual experimental approach to the problem. Before proceeding to the general case, we present first an example which illustrates how results previously derived mathematically can be obtained experimentally.

Suppose the input to an amplitude nonlinearity whose transfer characteristic is unknown consists of a constant amplitude FM signal and additive stationary band-limited noise. The amplitude of the output signal and properties of the in and out of phase components of the noise are to be determined. Figure 45 illustrates schematically the experimental solution to the problem. In the diagram, $n(t)$ is bandlimited gaussian noise having a statistical power σ^2 . The limiter is assumed to introduce no amplitude dependent phase shift; the output can then be demodulated coherently into components in and out of phase with the signal. A' , the amplitude of the desired signal, is simply the time average of the in phase output. $n'_1(t)$ and $n'_2(t)$, the quadrature noises, are the AC or random outputs from the demodulators.

From measurements made on the output from the experimental model, we can determine:

- (1) The output SNR as a function of input SNR.

Figure 45

- (2) The in and out of phase noise powers as functions of input SNR.
- (3) For fixed SNR, the suppression factor as a function of backoff.
- (4) For fixed A and σ , the sample probability density functions of $n'_1(t)$ and $n'_2(t)$.
- (5) For fixed A and σ , the sample densities of the envelope ($R'(t) = \sqrt{n_1'^2(t) + n_2'^2(t)}$) and phase ($\chi(t) = \tan^{-1}(n'_2(t)/n'_1(t))$) of the output noise.
- (6) For fixed A and σ , the power spectra of $n'_1(t)$ and $n'_2(t)$.

In fact, the parameters give by analytical expressions in chapter 7 can be determined by experiment.

The general nonlinearity (i.e. one exhibiting AM/PM conversion as well as amplitude limiting) with a multi-signal input can be handled in a similar fashion. If the composite bandpass input and output are written as

$$\begin{aligned}
 x(t) &= E(t) e^{j(\omega_0 t + \phi(t))} \\
 &= \sum_{i=1}^M A_i e^{j(\omega_i t + \theta_i(t))} + n(t)
 \end{aligned} \tag{11.5.1}$$

and

$$\begin{aligned}
 y(t) &= E'(t) e^{j(\omega_c t + \phi'(t))} \\
 &= \sum_{i=1}^m A_i' e^{j(\omega_i t + \theta_i(t) - \beta)} + n'(t) \quad (11.5.2)
 \end{aligned}$$

respectively, the solution consists of:

(1) Measuring

$$\beta = \cos^{-1} \left\langle \Re \left[\frac{x^*(t) y(t)}{|x(t)| |y(t)|} \right] \right\rangle / I \left[\frac{x^*(t) y(t)}{|x(t)| |y(t)|} \right]$$

the constant component of the phase shift in the output.

(2) Demodulating $y(t)$ coherently with the i 'th carrier $\cos(\omega_i t + \theta_i(t) - \beta)$. The DC component of the demodulator output is A_i' , the output amplitude of the i 'th signal. This is repeated for $i = 1, 2, \dots, m$.

(3) The noise component $y(t)$ is isolated as

$$n'(t) = y(t) - \sum_{i=1}^m A_i' \cos(\omega_i t + \theta_i(t) - \beta).$$

The power spectrum and envelope and phase statistics of $n'(t)$ can then be measured.

No laboratory verification of this experimental algorithm has been undertaken. Its advantages are again those of the stochastic method in general: i.e.,

- (1) It is simple to understand and universal in application.
- (2) It is closely tied to the communications problem. The question "What happens to the phase signature of a particular signal transmitted through a nonlinear channel?" is answered directly.
- (3) It gives more complete information on the output intermodulation noise than existing methods.
- (4) Implementation appears to be reasonably straightforward.

CHAPTER 12

CONCLUDING DISCUSSION

The presentation thusfar has consisted of qualitative discussion, general mathematical formulations and solutions, and specific examples. Under the umbrella of its title, "Methods of Nonlinear Analysis with Applications to Multiple Access Satellite Communications", the thesis touches on such wide-ranging topics as dithering, FM receivers, travelling wave tube operation, methods of satellite multiple accessing, intermodulation, AM/PM equalization, and intersymbol interference. The purpose of this chapter is to reinforce the thread of continuity which is present throughout the work, and to comment on the contribution of the thesis to communications engineering.

As was emphasized at the outset of the thesis, the theme is to demonstrate in a communications context the application of the stochastic method of analyzing a bandpass nonlinearity. Each topic covered contributes to this theme; chapters 2 and 3 establish the communications context by introducing methods of multiple access satellite communications and modelling an FDMA system; chapter 4 reviews known theoretical methods and introduces the stochastic method of analyzing the effect of nonlinear devices on multi-carrier inputs; chapter 5 applies the stochastic method to the analysis of a hard limiter, and points out some rather singular effects;

returning to the modern communications context, chapter 6 discusses the backed off satellite repeater in a band-limited FDMA system; chapters 7 to 12 apply the stochastic method to computing the signal suppression and amplitude distortion, and the noise statistics and spectral density after a general bandpass nonlinear transformation. Examples demonstrating the application of the theory are at the same time real communications situations, and the particular results obtained are themselves of interest.

The formulation and development throughout the thesis are kept as general as possible; the results can be applied to analyzing the effects of any non-linear device on the first and second order statistics of its input. Such analysis has been shown to be useful in the multiple access satellite communications problem; it also has wide potential application in evaluating the performance of non-linear receivers. The principle behind the original derivations here is basically simple: the component of the nonlinearity output which contains all, and only, information about a particular input signal is defined as the expected value of the output given that input signal. On the application of probability theory and mathematical manipulations, this definition is translated into analytical expressions for the signals out of an arbitrary nonlinear device having an arbitrary multi-carrier input.

It has been shown that the stochastic definition leads to results consistent with those obtained by other methods. The stochastic method is easily understood and implemented. It can be applied to compute the statistics as well as the power spectrum of intermodulation noise, and computation time grows only linearly with the number of input carriers. The computer implementation of the stochastic method, in fact, calculates the IM product line spectrum rather than its spectral density; this is a limitation of the program, not the stochastic method. The program is partly a numerical evaluation of integrals (to compute the signal output amplitudes and phase shifts), and partly a simulation (the output noise is sampled, these samples are used to generate the noise spectrum and statistics). The run time (23 minutes on a SIGMA 7 computer for one set of Arctic model input data - reference 11.4) is large, but it should be remembered that the program is written to give accurate results for any TWT transfer characteristic and multi-carrier input. The stochastic method does not constitute a brute force solution to the problem.

From a critical standpoint, the thesis does not give a detailed presentation of how the results obtained can be applied in computing and optimizing the ultimate performance of communications systems. The rigorous presentation of the stochastic method and a wide demonstration of its application have, to some extent, been at the expense of a follow up to specific communications problems

raised. For example, in section 7.8, a theoretical method of evaluating TDMA/PSK system performance is given, but some difficulties are left unresolved. In many ways, then, the material here is only a first step to analyzing the theoretical effects of nonlinear channels on message fidelity. However, a general approach to the problem has been given, and the most important impediment to a solution has been removed. By giving the general bandpass nonlinearity a meaningful and tractable input/output characterization, the thesis is an important tool in designing and analyzing a large class of communications systems.

APPENDIX A

DERIVATION OF THE PROBABILITY DENSITY FUNCTION OF THE ENVELOPE
OF A SINUSOID AND GAUSSIAN NOISE GIVEN THE INSTANTANEOUS
PHASE ERROR RESULTING FROM THE NOISE

Figure 46

In the above figure, the additive noise vector $\vec{n}$ consists of components n_1 and n_2 in and out of phase with the signal. Any pair (n_1, n_2) of orthogonal components of a gaussian process satisfies

$$\vec{n} = (n_1, n_2) \sim N \left((0, 0), \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} \right),$$

and from figure 46,

$$n_2 = (A + n_1) \tan \delta$$

Therefore,

$$\begin{aligned} E|\delta = y &= \left((A + n_1)^2 + n_2^2 \right)^{\frac{1}{2}} \mid \delta = y \\ &= |(A + n_1) \sec y| \begin{cases} n_1 > -A & \text{for } |y| < \frac{\pi}{2} \\ n_1 < -A & \text{for } |y| > \frac{\pi}{2} \end{cases} \end{aligned}$$

has the probability density function

$$f_{E|\delta}(r|y) = \begin{cases} \left(\frac{\pi}{2} \sec^2 y \sigma^2 \right)^{-\frac{1}{2}} \exp\left(-\frac{(r - A \sec y)^2}{2 \sec^2 y \sigma^2} \right) \left[\operatorname{erfc}\left(\operatorname{sgn}\left(|y| - \frac{\pi}{2} \right) \sqrt{\rho} \right) \right]^{-1} & \text{for } r \geq 0 \\ 0 & \text{for } r < 0 \end{cases}$$

for $|y| \neq \frac{\pi}{2}$ and

$$f_{E|\delta}\left(r \mid \frac{\pi}{2}\right) = \begin{cases} \left(\frac{\pi}{2} \sigma^2 \right)^{-\frac{1}{2}} \exp\left(-\frac{r^2}{2\sigma^2} \right) & \text{for } r \geq 0 \\ 0 & \text{for } r < 0 \end{cases}$$

APPENDIX B

DERIVATION OF AN ASYMPTOTIC EXPRESSION FOR THE OUTPUT AMPLITUDE OF A WEAK SIGNAL SHARING A HARD LIMITER WITH TWO EQUAL AMPLITUDE SINUSOIDS

Suppose the limiter input is given by

$$x(t) = \sqrt{2\rho_i} \cos(\omega_D t + \theta_D(t)) + \sum_{k=1}^2 \cos(\omega_k t + \theta_k(t))$$

where, by assumption,

$$\rho_i = \text{input signal-to-interference power ratio} \ll 1. \quad (\text{B.1})$$

From equation 5.4.1, the interference envelope has the probability density function

$$f_R(r) = \frac{2}{\pi \sqrt{4-r^2}}$$

Applying 5.2.2, then, the signal amplitude at the limiter output is

$$A' = \frac{2}{\pi} \left[\int_0^{\sqrt{2\rho_i}} {}_2F_1(.5, -.5; 1; r^2/2\rho_i) \frac{dr}{\sqrt{4-r^2}} + \right. \\ \left. \int_{\sqrt{2\rho_i}}^2 \frac{\sqrt{2\rho_i}}{2r} {}_2F_1(.5, .5; 2; 2\rho_i/r^2) \frac{dr}{\sqrt{4-r^2}} \right]$$

$$\approx \frac{2}{\pi} \left[\frac{1}{2} \int_0^{\sqrt{2\rho_i}} {}_2F_1(.5, -.5; 1; r^2/2\rho_i) dr + \frac{\sqrt{2\rho_i}}{2} \int_{\sqrt{2\rho_i}}^2 \frac{dr}{r \sqrt{4 - r^2}} \right] \quad (\text{B.2})$$

under the assumption B.1.

Making use of the integral formulas taken from Gradshteyn and Ryzhik [17, chapter 5, pp. 849, 245], equation B.2 reduces to

$$A' \approx \left[\frac{16}{3\pi^2} + \frac{\ln(2/\sqrt{\rho_i})}{\sqrt{2} \pi} \right] \sqrt{\rho_i} \quad (\text{B.3})$$

which is a low signal-to-interference asymptotic result for the output amplitude of a signal sharing a hard limiter with two equal amplitude interferers.

APPENDIX C

DERIVATION OF THE ENVELOPE PROBABILITY DENSITY FUNCTION OF THE INTERFERENCE TO THE K'th SIGNAL FROM AN AM/PM NONLINEARITY

From 11.1.2, the input to the nonlinearity is written in complex form as

$$\begin{aligned} x(t) &= E(t) e^{j(\omega_o t + \phi(t))} \\ &= s_k(t) + I_k(t) \end{aligned}$$

where

$$s_k(t) = A_k e^{j(\omega_k t + \theta_k(t))} \quad \text{and}$$

$$I_k(t) = \sum_{i \neq k} A_i e^{j(\omega_i t + \theta_i(t))} + \vec{n}(t)$$

are the k'th signal and interference to it at the input to the amplitude-to-phase transfer device $g(x)$. The output is then

$$\begin{aligned} y(t) &= E(t) e^{j(\omega_o t + \phi(t) - g(E(t)))} \\ &= CA_k e^{j(\omega_k t + \theta_k(t) - \beta)} + (e^{-jg(E(t))} - Ce^{-j\beta}) A_k e^{j(\omega_k t + \theta_k(t))} \\ &\quad + I_k(t) e^{-jg(E(t))} \\ &= s'_k(t) + I'_k(t) \end{aligned}$$

Here $I'_k(t)$, the output interference to the k 'th signal, is given by

$$I'_k(t) = \left[A_k (1 - C e^{j(g(E(t)) - \beta)}) e^{j(\omega_k t + \theta_k(t))} + I_k(t) \right] e^{-jg(E(t))}$$

It can be demonstrated (see figure 47) that $I'_k(t)$ is a process described by the complex random variable

$$I'_k = \left\{ 1 + C^2 - 2C \cos[g(\sqrt{R_k^2 - 2R_k A_k \cos \alpha + A_k^2}) - \beta] \right\}^{\frac{1}{2}} A_k \exp \left[j \left\{ \alpha + \tan^{-1} \left(\frac{\sin[g(\sqrt{R_k^2 - 2R_k A_k \cos \alpha + A_k^2}) - \beta]}{\cos[g(\sqrt{R_k^2 - 2R_k A_k \cos \alpha + A_k^2}) - \beta] - C} \right) - g(\sqrt{R_k^2 - 2R_k A_k \cos \alpha + A_k^2}) + \beta \right\} \right] + R_k \quad (C.1)$$

Here R_k and α are the independent envelope and phase of the input interference $I_k(t)$; they have the distributions

$$f_{R_k}(r) = r \int_0^\infty u J_0(ur) \prod_{i \neq k} J_0(A_i u) e^{-\frac{1}{2\sigma^2} u^2} du$$

and

$$f_\alpha(y) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \leq y < 2\pi \\ 0 & \text{elsewhere} \end{cases}$$

Figure 47

The first term in C.1 represents the $s_k \times I_k$ (i.e. intermodulation) component from the nonlinearity; the second term is the $I_k \times I_k$ component. Note that the phase of I'_k is taken relative to the $I_k \times I_k$ term, and hence $R\{I'_k\} \gg I\{I'_k\}$, or $\arg\{I'_k\} \approx 0$, with a high degree of probability. If the $s_k \times I_k$ term were additive independent gaussian noise, I'_k would have a uniformly distributed phase and an envelope density given by

$$f_{R'_k}(r) = r \int_0^\infty u J_0(ur) \prod_{i \neq k} J_0(A_i u) e^{-\frac{1}{2}(\sigma^2 + (1-C^2)A_k^2/2)u^2} du \quad (C.2)$$

In fact, the first term in C.1 is not only non-gaussian, it is dependent on the second variable R_k . Note from figure 47 that when $1-C \ll 1$ (or $g(E)-\beta$ is a very small angle with a high degree of probability), the IM product noise resulting from the presence of the k 'th signal tends to be out of phase with the k 'th signal. If the $s_k \times I_k$ and $I_k \times I_k$ terms in C.1 were independent random variables, the following expression for the two-dimensional characteristic function of I'_k would be obtained ($\xi = (\xi_1, \xi_2)$) †

$$\phi_{I'_k}(\xi) = e^{-\frac{1}{2}\sigma^2(\xi_1^2 + \xi_2^2)} \prod_{i \neq k} J_0(A_i \sqrt{\xi_1^2 + \xi_2^2}) \phi_k(\xi) \quad (C.3)$$

where

† The characteristic function of the sum of two independent random variables is the product of their characteristic functions.

$$\phi_k(\xi) = e^{j\xi_2} \int_0^\infty e^{jCA_k\{\sin(g(\rho)-\beta)\xi_1 - \cos(g(\rho)-\beta)\xi_2\}} f_E(\rho) d\rho \quad (C.4)$$

is the characteristic function of the $s_k \times I_k$ intermodulation noise and $f_E(\rho)$ is the probability density function of the input envelope. The envelope and phase densities of $I'_k(t)$ can be obtained by transforming C.3.

Computation of the statistics of $I'_k(t)$ can be done numerically by making use of C.1 and the known densities of R_k and α . Both C.2 and C.3 represent approximations; the simpler form C.2 (with uniform phase) is used in the computer program "NONLINEAR" described in chapter 11. In a typical FDMA configuration, after the AM to PM transfer, the composite interference signal competing with a desired signal for TWT power still consists mostly of randomly phased sinusoids and gaussian noise; therefore, the inaccuracy incurred by the approximation on the statistics of the added desired signal-dependent intermodulation component should not significantly affect the computed values of the signal amplitudes out of the TWT.

with R having a probability density function given by 11.1.4.

The crosscorrelation of $\vec{n}'$ and the k 'th input carrier is

$$\begin{aligned}
 & E \left\{ e^{j\theta_k} \left(\sum_{i=1}^m A_i e^{j(\theta_i - f_2(R))} + \vec{n}' e^{-jf_2(R)} - C \sum_{i=1}^m A_i e^{j(\theta_i - \beta)} \right)^* \right\} \\
 &= A_k e^{j\beta} \left[E \{ e^{j(f_2(R) - \beta)} \} - C \right] \\
 &= 0
 \end{aligned} \tag{D.1}$$

where we have here made use of the independence of θ_k and $\theta_i - f_2(R)$ ($i \neq k$). Equation D.1 implies that $\vec{n}'$ is uncorrelated with each input RF carrier.

Following the analytical model of figure 1, $\vec{y}$ is now the input to an amplitude-to-amplitude nonlinearity $f_1(x)$. With $\theta'_i = \theta_i - \beta$ uniformly distributed and independent of θ'_j ($i \neq j$), and $A'_i = CA_i$, we can write the input-output equations:

$$\begin{aligned}
 \vec{y} &= \sum_{i=1}^m A'_i e^{j\theta'_i} + \vec{n}' \\
 &= A'_k e^{j\theta'_k} + R_k e^{j\phi_k}
 \end{aligned} \tag{D.2}$$

and

$$\begin{aligned}\vec{z} &= f_1(A(R_k, \alpha_k)) \frac{(A'_k e^{j\theta'_k} + R_k e^{j\phi_k})}{A(R_k, \alpha_k)} \\ &= \sum_{i=1}^m A''_i e^{j\theta'_i} + \vec{n}''\end{aligned}$$

$$\text{Here: } \alpha_k = \pi - \theta'_k + \phi_k, \quad (D.3)$$

$$A(R_k, \alpha_k) = \sqrt{R_k^2 - 2A'_k R_k \cos \alpha_k + A'^2_k},$$

$$\text{and } A''_k = E \left\{ f_1(A(R_k, \alpha_k)) \frac{(A'_k - R_k \cos \alpha_k)}{A(R_k, \alpha_k)} \right\} \quad (D.4)$$

has been assumed (see 11.1.5) to be the amplitude of the k 'th signal out of the general bandpass nonlinearity (f_1, f_2) .

The crosscorrelation of $e^{j\theta'_k}$ and $\vec{n}''$ is then

$$\begin{aligned}& E \left\{ e^{j\theta'_k} \left(f_1(A(R_k, \alpha_k)) \frac{A'_k e^{j\theta'_k} + R_k e^{j\phi_k}}{A(R_k, \alpha_k)} - \sum_{i=1}^m A''_i e^{j\theta'_i} \right)^* \right\} \\ &= E \left\{ f_1(A(R_k, \alpha_k)) \frac{(A'_k - R_k \cos \alpha_k - j R_k \sin \alpha_k)}{A(R_k, \alpha)} \right\} - A''_k \\ &= -j E \left\{ \frac{f_1(A(R_k, \alpha_k)) R_k \sin \alpha_k}{A(R_k, \alpha)} \right\} \quad (D.5)\end{aligned}$$

on applying equations D.3 and D.4, and the fact that θ'_i ($i \neq k$) is uniformly distributed and independent of θ'_k . Since in addition it has been shown that $e^{j\theta_k}$, and hence $e^{j\theta'_k}$, is uncorrelated with $\vec{n}'$, it can be concluded from equations D.2 and D.3 that $e^{j\theta'_k}$ is uncorrelated with R_k and $e^{j\alpha_k}$. Therefore

$$E \left\{ \frac{f_1(A(R_k, \alpha_k)) R_k \sin \alpha_k}{A(R_k, \alpha_k)} \right\} = E \left\{ \frac{f_1(A(R_k, \alpha_k)) R_k \sin \alpha_k}{A(R_k, \alpha_k)} \middle| e^{j\theta'_k} \right\}$$

= component of k'th output signal
orthogonal to itself (D.6)

under the stochastic method (see 4.3) of defining an output carrier as the expected value of the nonlinearity output given the RF phase of the corresponding input carrier. Since the right hand side of equation D.6 is trivially zero, we therefore obtain in D.5 the result that $\vec{n}''(t)$, the noise component of the output, is uncorrelated with $e^{j\theta'_k}$. Since $\theta_k = \theta'_k + \beta$, and k is arbitrary ($1 \leq k \leq m$), it can be concluded that $\vec{n}''(t)$ is uncorrelated with each input carrier.

VITA

Robert G. Lyons was born in Ottawa, Ontario, on July 30, 1945. He obtained his B. Sc. (Eng.) degree at Queen's University in 1967 in the Communications option of "Engineering and Mathematics". During his undergraduate years, Mr. Lyons spent summers working at N.R.C., Forest Products Laboratories, the Defence Research Board and Northern Electric Research and Development Laboratories. In September 1968, he received an M. Sc. degree in Mathematics from Queen's University. His thesis topic was "The Effect of Hard Limiting on Signal to Interference Ratios". Mr. Lyons entered a Ph. D. program in Electrical Engineering at Carleton University in September 1968. His major field is Communications, his minor is in the Numerical Analysis field.

In May 1969, Mr. Lyons joined the Northern Electric Research and Development Labs as a summer student. He was assigned the task of studying the effect of up path noise on PSK communication through a band and amplitude limiting satellite repeater. From September 1969 to April 1970, while still pursuing Ph.D. studies, his employment at Northern Electric continued on a part-time basis. During this time, his research extended to studying the effect of a nonlinear transponder on FDMA communications.

In May 1970, Mr. Lyons joined the Communications Systems group at the Telesat Company of Canada where he has been engaged in a study of digital transmission over a satellite channel.