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**Weak Convergence for Empirical and
Quantile Processes of Associated Sequences with
Applications to Reliability and Economics**

by

HAO YU, B.SC., M.SC.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute for Graduate Studies and Research
in Mathematics and Statistics

Carleton University

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**Weak Convergence for Empirical and Quantile Processes of Associated
Sequences with Applications to Reliability and Economics**

submitted by

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in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

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Carleton University

1993

To my mother Zhao Shao-Ying and
the memory of my father Yu Hong-Quan

Abstract

This dissertation is devoted to develop weak convergence for empirical and quantile processes of associated random variables, a kind of dependency widely encountered in practice. With the help of an extension of Hoeffding's equality, we establish a way for estimating the covariance structures for empirical functions of associated sequences in terms of covariances of the original random variables. Consequently, we study and obtain the weak convergence, as well as the weighted weak convergence of empirical processes of stationary associated sequences under the initial restrictions on covariance structures of the original random variables. Based on these results, a new method is developed for establishing the weak and weighted weak convergence for quantile processes. We apply these results to statistical theory of reliability and life testing, and econometrics, and obtain a unified asymptotic theory for empirical mean residual life, total time on test, Lorenz, and concentration processes of stationary associated sequences. We also prove the weak convergence for empirical percentile residual life processes of stationary associated sequences without, and under random censorship. Finally, we propose a circular block procedure to bootstrap these processes of complicated covariance structures and show that this procedure provides strong consistent estimators for sample means and empirical processes of stationary mixing sequences, another and often used structure of dependence for sequences.

Chapter 1 is an introductory chapter in which we discuss problem of dependence arising in the theory and practice of reliability. We introduce informally the positive dependence notion of association. The frequently used processes in reliability theory and econometrics are also described.

Chapter 2 is primarily concerned with the definition of association, its background and its properties. An extension of Hoeffding's equality is also established here. Known limit theorems as well as some open problems on association are discussed.

Chapter 3 deals with approximations for empirical processes of associated sequences. A method for estimating the covariance structures for empirical functions of associated sequences in terms of covariances of the original random variables is developed by using the extension of Hoeffding's equality of Chapter 2. This enables us to prove a Glivenko-Cantelli lemma based on associated sequences, and a weak convergence theorem for empirical processes of stationary associated sequences of random variables.

In Chapter 4, we develop a method for obtaining weak and weighted weak convergence for uniform quantile processes, and that for the weak convergence of the normed quantile processes of stationary associated sequences from those of the uniform empirical processes.

Chapters 5 to 8 are devoted to study the problem of weak convergence for empirical mean residual life, total time on test, Lorenz and concentration processes of stationary associated sequences. Chapter 9, in turn, establishes weak convergence for empirical percentile residual life processes of stationary associated sequences without, as well as under random censorship.

In Chapter 10, we propose a circular block resampling procedure. Applica-

tions to bootstrap the sample means of stationary α -mixing, ρ -mixing and ϕ -mixing sequences are discussed, and the bootstrap of a general statistic of a stationary ρ -mixing sequence is studied by means of influence functions.

The following is a list of the author's own results which are believed to be new:

Theorem 2.3.1	Theorem 4.5.2	Theorem 6.5.2	Theorem 9.2.2
Corollary 2.3.1	Theorem 5.3.1	Theorem 7.2.1	Theorem 9.2.5
Theorem 3.2.1	Corollary 5.3.1	Theorem 7.2.2	Theorem 9.2.6
Theorem 3.3.1	Theorem 5.4.1	Theorem 7.3.1	Theorem 9.2.7
Corollary 3.3.1	Theorem 5.4.2	Corollary 7.3.1	Theorem 9.2.8
Theorem 3.4.1	Theorem 6.2.1	Theorem 7.3.2	Theorem 9.2.9
Theorem 4.2.1	Theorem 6.3.1	Corollary 7.3.2	Theorem 9.3.5
Corollary 4.2.1	Corollary 6.3.1	Theorem 8.2.1	Theorem 9.3.6
Theorem 4.3.1	Theorem 6.4.1	Theorem 8.3.1	Theorem 9.3.7
Theorem 4.4.1	Theorem 6.4.2	Corollary 8.3.1	Theorem 9.3.8
Theorem 4.5.1	Theorem 6.5.1	Theorem 9.2.1	Theorem 9.3.9

The next list of results is joint work with Dr. Qiman Shao:

Theorem 10.2.1	Theorem 10.3.1	Theorem 10.3.4
Theorem 10.2.2	Theorem 10.3.2	Theorem 10.4.1
Theorem 10.2.3	Theorem 10.3.3	Theorem 10.5.1

Acknowledgements

Although this thesis bears the name of only one author, it is the result of the efforts of many. To them all, I am grateful. I hope each and all will accept this message as an expression of my warmest thanks.

First and foremost I thank my supervisor, Professor Miklós Csörgő, for his invaluable guidance in writing this thesis and for his constructive criticism of several versions of my manuscript. I thank him for his continuous guidance, encouragement and for giving me many wonderful ideas during my stay at Carleton University as a Ph.D. student.

I express my appreciation of the collaboration with Dr. Qiman Shao during his stay at Carleton University, where he held an NSERC Canada Scientific Exchange Award, visiting Miklós Csörgő.

I thank Dr. André R. Dabrowski for sharing his forthcoming papers with me and for interesting discussions.

The overall guidance in my studies of my Advisory Committee (Professors Miklós Csörgő, Donald Dawson, André R. Dabrowski) is gratefully acknowledged.

My research and studies at Carleton University were supported by a Scholarship from the Faculty of Graduate Studies and Research, a David and Rachel Epstein Scholarship for 1988/89, the Dewan Chand and Ratna Devi Marwah Memorial Scholarship for 1992/93 at the Department of Mathematics and Statistics and the Department of Economics, and an NSERC Canada grant of Miklós Csörgő.

Lastly, but not least, I am grateful to my wife, Shu-Qian Fu, for her understanding support to persevere in my studies.

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Chapter 1

Introduction

1.1 Independence versus dependence.

The assumption of independence is customary in probability and statistics. It is this concept more than anything else which has given probability and statistics a life of their own, distinct from other branches of mathematical analysis. For example, the study of asymptotics or limit theory, one of the fundamental branches of probability and statistics, is mostly under the independence assumption, especially under the independent identically distributed (i.i.d.) assumption. During the last four decades many profound results of limit theory have been achieved in several areas of interest for independent random variables. One of these areas is the approximation of empirical and quantile processes (cf., e.g., Csörgő and Révész (1981), Csörgő (1983), Shorack and Wellner (1986), and references in these works) which have many applications in statistics and other fields. Concerning applications in the statistical theory of reliability and life testing, and in econometrics, Csörgő, Csörgő and Horváth (1986) developed a unified asymptotic theory for empirical mean residual life, total time on test, Lorenz, and concentration processes of i.i.d.

random variables. A similar asymptotic theory for percentile residual life processes of i.i.d. random variables was obtained by Csörgő and Csörgő (1987), and by Chung (1989) under random censorship.

However, in some cases, the independence assumption seems to be unrealistic. For example, when considering a system reliability of two elements, we normally assume that their life distributions, say X and Y , are independent, i.e., that the series system reliability of them is

$$P\{X \geq x, Y \geq y\} = P\{X \geq x\} \cdot P\{Y \geq y\}$$

for all $x \geq 0$ and $y \geq 0$. Before they are used, this assumption is all right. But once these two elements are connected in this series way, the independence assumption ignores the possible impact of the components on each other and that of the outside environment on them. In other words, the life distributions of X and Y will be affected by each other and outside world, and they will be no longer independent from practical point of view. If this impact is small and can be neglected, then independence is still a reasonable assumption. When this impact cannot be ignored, we have to assume that X and Y are dependent. Certainly, there are other factors which may contribute to this dependence. Though difficulties may arise due to the lack of information concerning the nature of dependence structure and the possible increase in computational complexity, we cannot afford neglecting the dependence of X and Y when their independence is unrealistic.

In reliability and other fields, people are particularly interested in the dependence structure of having

$$P\{X \leq x, Y \leq y\} \geq P\{X \leq x\} \cdot P\{Y \leq y\}$$

for all $x \geq 0$ and $y \geq 0$. This structure is called positive dependence. It describes the notion that large values of Y tend to be associated with large values of X and

small values of Y with small values of X . In the above example this means that the series system reliability is increased, while the independence assumption leads to underestimation of the system reliability.

1.2 Positive dependence and association.

In the literature, as early as in 1966, Lehmann studied systematically the nature of asymptotic independence via limit theorems for positively dependent random variables. He defined positive quadrant dependent (**PQD**) random variables X and Y as

$$(1.2.1) \quad P\{X \leq x, Y \leq y\} \geq P\{X \leq x\} \cdot P\{Y \leq y\}$$

for all $x, y \in \mathbf{R}$. Since the **PQD** is too large a class of positive dependence, it is difficult to study the asymptotic properties of **PQD**. We want to define a smaller class of positive dependence appropriate to some situations, such as in reliability, and also having nice asymptotic properties. Note that

$$(1.2.2) \quad \text{Cov}(f(X), g(Y)) \geq 0$$

for all real non-decreasing functions f and g such that $f(X)$ and $g(Y)$ have finite variances will imply (1.2.1) (in fact, they are equivalent). A stronger requirement will be

$$(1.2.3) \quad \text{Cov}(f(X, Y), g(X, Y)) \geq 0$$

for all f and g non-decreasing in each argument. This, namely the notion of association, will be seen to be a very useful definition. All the details of association are given in Chapter 2.

The purpose of this dissertation is to develop a theory of weak convergence for empirical and quantile processes of associated random variables, and to obtain an asymptotic theory for their processes in reliability and econometrics.

1.3 Processes in reliability and econometrics.

We now give a brief introduction of processes involved in reliability and econometrics. We follow the definitions of Csörgő, Csörgő and Horváth (1986), where there are also further references given concerning the historical development of the notion of these processes. We assume throughout that the underlying non-degenerate life distribution function F , $F(0) = 0$, is continuous. The quantile function is defined as

$$(1.3.1) \quad Q(y) = F^{-1}(y), \quad 0 \leq y \leq 1.$$

Here and throughout, for a non-decreasing right-continuous function $k(x)$ on the line, the right-continuous inverse k^{-1} is defined as

$$(1.3.2) \quad k^{-1}(y) = \inf\{x : k(x) > y\}.$$

To avoid confusion, a function, deterministic or random, without discontinuities of the second kind, will always be defined to be right-continuous. We also use the convention

$$\int_a^b = \int_{[a,b)}, \quad a < b,$$

for all occurring Lebesgue-Stieltjes integrals. Throughout $F(0) = 0$ will be assumed, except in Chapters 2, 3 and 4, where we mainly study association and weak convergence for empirical and quantile processes of associated random variables.

Let $X_1, \dots, X_n$ be random variables with a common distribution function F , a random sample of n observations (not necessarily independent) on X . Here, and also in the sequel, X denotes a generic random variable with distribution function F .

The first and easiest kind of process in reliability is the mean residual life process. It is defined as

$$(1.3.3) \quad z_n(x) = n^{1/2}(M_n(x) - M_F(x)), \quad 0 \leq x < \infty,$$

where

$$\begin{aligned}
 (1.3.4) \quad M_F(x) &= E(X - x | X > x) \\
 &= \frac{1}{1 - F(x)} \int_x^\infty (1 - F(t)) dt
 \end{aligned}$$

is the mean residual life function at age x , and M_n is the empirical counterpart of M_F defined by

$$M_n(x) = M_{F_n}(x) = \frac{1}{1 - F_n(x)} \int_x^\infty (1 - F_n(t)) dt,$$

where F_n is the empirical distribution function of $X_1, \dots, X_n$. The strong consistency of M_{F_n} to M_F and the weak convergence of z_n for associated sequences are given in Chapter 4.

One of the most important processes in reliability is that of total time on test which was first discussed by Marshall and Proschan (1965) in connection with estimation problems for distributions with a monotone failure rate. Introduce

$$W_{k:n} = (n + 1 - k)(X_{k:n} - X_{k-1:n}), \quad k = 1, \dots, n,$$

with $X_{0:n} \equiv 0$, where $X_{1:n} \leq \dots \leq X_{n:n}$ is the ordered sample. According to Barlow and Proschan (1981), the total time on test up to the k th order statistic, $T(X_{k:n})$, is defined by

$$T(X_{k:n}) = \sum_{i=1}^k W_{i:n} \quad \text{for } k = 1, \dots, n.$$

We define the n th total time on test function as

$$\begin{aligned}
 (1.3.5) \quad H_n^{-1}(u) &= \frac{1}{n} T(X_{[nu]+1:n}) \\
 &= \frac{1}{n} \sum_{i=1}^{[nu]+1} W_{i:n} \\
 &= \frac{1}{n} \sum_{i=1}^{[nu]} X_{i:n} + \left(1 - \frac{[nu]}{n}\right) X_{[nu]+1:n}
 \end{aligned}$$

for $0 \leq u < 1$ and

$$(1.3.6) \quad H_n^{-1}(1) = \lim_{u \uparrow 1} H_n^{-1}(u) = \frac{1}{n} \sum_{i=1}^n X_{i:n} = \bar{X}_n,$$

where $[\cdot]$ is the integer part function. We define the theoretical counterpart of H_n^{-1} , the total time on test transform of F , by

$$(1.3.7) \quad \begin{aligned} H_F^{-1}(u) &= \int_0^u (1-y) dQ(y) + t_F \\ &= (1-u)Q(u) + \int_0^u Q(y) dy, \quad 0 \leq u \leq 1, \end{aligned}$$

where

$$t_F = \sup\{t : F(t) = 0\}$$

is the lower endpoint of the support of F . If the quantile function $Q = F^{-1}$ is continuous, then we have

$$(1.3.8) \quad H_F^{-1}(u) - t_F = \int_0^{Q(u)} (1-F(y)) dy, \quad 0 \leq u \leq 1.$$

Clearly, $H_F^{-1}(u) \leq H_F^{-1}(1)$ for all $u \in [0, 1]$, and hence it is a finite function on the whole interval $[0, 1]$ if and only if

$$(1.3.9) \quad \mu = EX = H_F^{-1}(1) - t_F < \infty.$$

Now we define the total time on test empirical process by

$$(1.3.10) \quad t_n(u) = n^{1/2}(H_n^{-1}(u) - H_F^{-1}(u)), \quad 0 \leq u \leq 1$$

and its scaled version as

$$(1.3.11) \quad s_n(u) = n^{1/2}(D_n^{-1}(u) - D_F^{-1}(u)), \quad 0 \leq u \leq 1,$$

where

$$D_F^{-1}(u) = \frac{1}{\mu} H_F^{-1}(u)$$

and

$$D_n^{-1}(u) = \frac{1}{X_n} H_n^{-1}(u).$$

Chapter 6 is devoted to study the strong consistency of H_n^{-1} to H_F^{-1} and the weak convergence for the (scaled) total time on test empirical processes of associated sequences.

The Lorenz curve of F is defined by

$$(1.3.12) \quad L_F(u) = \frac{1}{\mu} \int_0^u Q(y) dy, \quad 0 \leq u \leq 1.$$

We have the relation

$$(1.3.13) \quad H_F^{-1}(u) - t_F = (1 - u)Q(u) + \mu L_F(u), \quad 0 \leq u \leq 1.$$

In econometrics, $L_F(u)$ is commonly interpreted as the fraction of total income that the holders of the lowest u th fraction of incomes processes. The empirical counterpart of L_F is

$$(1.3.14) \quad L_n(u) = \begin{cases} \frac{1}{X_n} \frac{1}{n} \sum_{i=1}^{[nu]+1} X_{i:n} & , \quad 0 \leq u < 1, \\ 1 & , \quad u = 1. \end{cases}$$

The scaled empirical Lorenz process is defined as

$$(1.3.15) \quad \ell_n(u) = n^{1/2}(L_n(u) - L_F(u)), \quad 0 \leq u \leq 1.$$

The strong consistency of L_n to L_F and the weak convergence for the empirical Lorenz processes of associated sequences are given in Chapter 7.

In Chapter 8, we study the strong consistency of L_n^{-1} to L_F^{-1} and the weak convergence for the concentration process of associated sequences defined by

$$(1.3.16) \quad c_n(u) = n^{1/2}(L_n^{-1}(u) - L_F^{-1}(u)), \quad 0 \leq u \leq 1.$$

In Chapter 9, we study convergence problems for percentile residual life processes which have potential advantages over the more frequently used mean residual life processes in reliability. Let $0 < p_0 < 1$ be any fixed number. Then the $(1 - p_0)$ -percentile residual lifetime is defined as

$$(1.3.17) \quad R^{(p_0)}(t) = Q(1 - p_0(1 - F(t))) - t, \quad t > 0.$$

This definition was introduced by Haines and Singpurwalla (1974) and $R^{(p_0)}(t)$ can be interpreted as the $(1 - p_0)$ -percentile additional time to failure, given no failure by time t . The natural empirical counterpart of $R^{(p_0)}$, the sample $(1 - p_0)$ -percentile residual life at $t > 0$ is

$$(1.3.18) \quad R_n^{(p_0)}(t) = Q_n(1 - p_0(1 - F_n(t))) - t, \quad 0 < p_0 < 1,$$

where Q_n is the sample quantile function of $X_1, \dots, X_n$. Assume that F has a density function $f = F'$ that is positive over the support $(0, b)$ of F , and define the empirical $(1 - p_0)$ -percentile life process $r^{(p_0)}(t)$ by

$$(1.3.19) \quad \begin{aligned} r^{(p_0)}(t) &= n^{1/2} f(Q(1 - p_0(1 - F(t)))) \cdot \{R_n^{(p_0)}(t) - R^{(p_0)}(t)\} \\ &= n^{1/2} f(R^{(p_0)}(t) + t) \cdot \{R_n^{(p_0)}(t) - R^{(p_0)}(t)\}, \quad t > 0. \end{aligned}$$

We will consider $r^{(p_0)}(t)$ as a function of $t > 0$, or of $p_0, 0 < p_0 < 1$ as a stochastic process and study their convergence problems without random censorship and under random censorship from the right respectively.

The background and definitions of these processes above are well documented in Csörgő, Csörgő and Horváth (1986), Csörgő and Csörgő (1987), Chung (1989), and in their references. Since the development of the weak convergence of these processes heavily depends on weak and weighted weak convergence for empirical and quantile processes of associated sequences, Chapter 3 is devoted to studying the

latter problems. Chapter 4 gives a method for obtaining the weak and weighted weak convergence for quantile processes of associated sequences from those of empirical processes.

1.4 Bootstrap and mixing dependence.

From Chapter 3 to Chapter 9 we will see that there are many processes which involve very complicated covariance structures, whether they are the processes we are interested in or their limiting Gaussian processes. Consequently, it is difficult to estimate distributions of functionals of these processes. To overcome this problem, in Chapter 10 we turn our attention to the bootstrap methodology. Since the original bootstrap works only for independent observations, we propose a circular block bootstrap to estimate distributions of functionals of these processes of complicated covariance structures. We, however, confirm only the strong consistencies for the sample means and sample empirical processes of stationary mixing sequences, another and often used structure of dependence for sequences. Due to technical difficulties, we are unable to prove similar results for associated sequences at the present time. However, we will see that mixing dependence is a fairly general structure of dependence, and that our procedure still provides a simple Monte Carlo simulation determined by the given associated observations. Hence, this makes it possible to find applications of these processes of complicated covariance structure, such as to set confidence bands for example.

Chapter 2

Association

This chapter is primarily concerned with the definition of association, its background and its properties. An extension of Hoeffding's equality is presented, which plays a uniquely important role in the proofs later on in the sequel. Known limit theorems as well as some open problems on association are discussed.

2.1 Definitions.

In order to generalize the notion of association as in (1.2.3) to cover multivariate random variables, we use the following definition.

DEFINITION 2.1.1 (Esary, Proschan and Walkup, 1967) A random vector $\mathbf{X} = (X_1, \dots, X_n)$ is said to be associated if

$$\text{Cov}(f(X_1, \dots, X_n), g(X_1, \dots, X_n)) \geq 0$$

for all nondecreasing in each argument functions $f, g : \mathbf{R}^n \rightarrow \mathbf{R}$ for which $Ef(\mathbf{X})$, $Eg(\mathbf{X})$ and $Ef(\mathbf{X})g(\mathbf{X})$ exist. An infinite family of random variables is associated if every finite subfamily is associated.

This definition was introduced by Esary, Proschan and Walkup (1967), mainly for the sake of applications in reliability and statistics. It is of interest to note that

there are several similar definitions around in different areas. The basic concept of association first was defined by Harris (1960) in the context of percolation models. Independently in 1971, Fortuin, Kastelyn and Ginibre established a so-called FKG inequality to define a dependence structure in studying the Ising models of statistical mechanics. It was found later that this notion was actually coincided with that of association. In statistical mechanics the term "association" is usually not used but rather variables are said to satisfy the FKG inequalities.

In what follows, a function defined on $\mathbf{R}^n \rightarrow \mathbf{R}$ will be said to be increasing (decreasing) if it is nondecreasing (nonincreasing) in each of its arguments.

There are several stronger concepts than association in the literature. We only present two of them. These are of interest on their own in probability and statistics.

DEFINITION 2.1.2 (Barlow and Proschan, 1981) A random vector $\mathbf{X} = (X_1, \dots, X_n)$ is said to be stochastically increasing in $\mathbf{Y}$ if for every increasing function $f : \mathbf{R}^n \rightarrow \mathbf{R}$, $E(f(\mathbf{X})|\mathbf{Y} = \mathbf{y})$ is increasing in $\mathbf{y}$. Further, $X_1, \dots, X_n$ are said to be conditionally increasing in sequence (CIS) if X_i is stochastically increasing in $X_1, \dots, X_{i-1}$ for every $i = 2, \dots, n$.

DEFINITION 2.1.3 (Karlin and Rinott, 1980ab) A real-valued function of two variables, $f(x, y)$, is totally positive of order two ($\mathbf{TP}_2$) if

$$f(x_1, y_1)f(x_2, y_2) \geq f(x_1, y_2)f(x_2, y_1)$$

for all $x_1 < x_2$ and $y_1 < y_2$. Suppose that a random vector $\mathbf{X} = (X_1, \dots, X_n)$ has a joint density function $h(x_1, \dots, x_n)$. Then $\mathbf{X}$ (or h) is said to be multivariate totally positive of order 2 ($\mathbf{MTP}_2$) if for every choice of $n - 2$ fixed variables, h as a function of the remaining two is $\mathbf{TP}_2$.

A concept weaker than association, but still similar, was defined by Burton, Dabrowski and Dehling (1986).

DEFINITION 2.1.4 (Burton, Dabrowski and Dehling, 1986) A random vector $\mathbf{X} = (X_1, \dots, X_n)$ is said to be weakly associated if whenever π is a permutation of $\{1, 2, \dots, n\}$, $1 \leq k < n$, and $f: \mathbf{R}^k \rightarrow \mathbf{R}, g: \mathbf{R}^{n-k} \rightarrow \mathbf{R}$ are increasing, then

$$\text{Cov}(f(X_{\pi(1)}, \dots, X_{\pi(k)}), g(X_{\pi(k+1)}, \dots, X_{\pi(n)})) \geq 0$$

whenever the covariance is defined. An infinite family of random variables is weakly associated if every finite subfamily is weakly associated.

A much weaker concept than association is defined analogously to PQD in the binary case (see (1.2.1)).

DEFINITION 2.1.5 (Joag-Dev, 1983) A random vector $\mathbf{X} = (X_1, \dots, X_n)$ is said to be strongly positive orthant dependent (**SPOD**) if for every I , an arbitrary proper subset of the index set $1, 2, \dots, n$, $\bar{I}$ its complement, and $\mathbf{x} = (x_1, \dots, x_n)$, a vector constants, the following three conditions hold.

$$P\{X_i \geq x_i, i \in I \cup \bar{I}\} \geq P\{X_i \geq x_i, i \in I\}P\{X_j \geq x_j, j \in \bar{I}\},$$

$$P\{X_i \leq x_i, i \in I \cup \bar{I}\} \geq P\{X_i \leq x_i, i \in I\}P\{X_j \leq x_j, j \in \bar{I}\}$$

and

$$P\{X_i \geq x_i, i \in I, X_j \leq x_j, j \in \bar{I}\} \leq P\{X_i \geq x_i, i \in I\}P\{X_j \leq x_j, j \in \bar{I}\}.$$

THEOREM 2.1.1 *We have*

$$\text{MTP}_2 \implies \text{CIS} \implies \text{Association} \implies \text{Weak association} \implies \text{SPOD}.$$

A part of this proof can be found in Theorem 4.7 and Theorem 4.14 of Barlow and Proschan (1981). For applications of **CIS** and **MPT**₂, we refer to Barlow and Proschan (1981), Karlin and Rinott (1980), Glaz and Johnson (1984), Sarkar

and Smith (1986), and Lefevre and Milhaud (1990). In the binary case, Lehmann (1966) defined **CIS** as positive regression dependence, and **MPT₂** as positive likelihood ratio dependence. Although weak association defines a larger class of positive dependence than association, it inherits many useful properties from association. Indeed, almost all results of association can be extended to the case of weak association without difficulty.

2.2 Properties of association. Hoeffding's equality.

We first state some important properties of association and give Hoeffding's equality without proofs. The details can be found in the papers Esary, Proschan and Walkup (1967), and Lehmann (1966). Five important and frequently used properties of association are as follows:

- (P₁) *Any subset of associated random variables is associated,*
- (P₂) *If two sets of associated random variables are independent of one another, then their union is a set of associated random variables,*
- (P₃) *The set consisting of a single random variable is associated,*
- (P₄) *Nondecreasing functions of associated random variables are associated,*
- (P₅) *If $T_1^{(k)}, \dots, T_n^{(k)}$ are associated for each k , and $\mathbf{T}^{(k)} \rightarrow \mathbf{T}$ in distribution, then $T_1, \dots, T_n$ are associated.*

Due to these properties, association is a very useful positive dependence structure in reliability theory. This is why in reliability dependence is frequently modelled in terms of association (cf. Barlow and Proschan (1981), Section 2 of Chapter 2).

PROPOSITION 2.2.1 (Höeffding, 1940) *If the covariance of X and Y exists, then*

$$\text{Cov}(X, Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (P\{X \leq x, Y \leq y\} - P\{X \leq x\} \cdot P\{Y \leq y\}) dx dy.$$

From the Höeffding's equality, it is easy to derive the following result.

THEOREM 2.2.1 (Lehmann, 1966) *If X and Y have a finite covariance and are PQD, then they are independent if and only if $\text{Cov}(X, Y) = 0$.*

To see an analogue result for association, from the definition of association and property P_4 of association, we can easily obtain that association implies **SPOD**. Thus, we have the following result, which gives an estimation of joint distributions in terms of their covariances.

THEOREM 2.2.2 *Suppose that $X_1, \dots, X_n$ are associated random variables. Then, for any real numbers $x_1, \dots, x_n$, we have*

$$0 \leq P(A_1 \cdots A_n) - P(A_1)P(A_2 \cdots A_n) \leq \sum_{i=2}^n \Delta_{1i},$$

where $A_i = \{X_i \leq x_i\}$, $i = 1, \dots, n$, and for $i, j = 1, \dots, n$,

$$(2.2.1) \quad \Delta_{ij} = P\{X_i \leq x_i, X_j \leq x_j\} - P\{X_i \leq x_i\}P\{X_j \leq x_j\}.$$

Proof. This theorem is a special case of a result of Lebowitz (1972). It also follows easily by the proof of Theorem 1 of Joag-Dev (1983). Simply let $U(t_2, \dots, t_n) = \sum_{i=2}^n I(t_i \leq x_i) - \prod_{i=2}^n I(t_i \leq x_i)$ and $V(t_1) = I(t_1 \leq x_1)$, where $I(A)$ is the indicator function, i.e., $I(A) = 1$ if $A \neq \emptyset$, otherwise 0. Then

$$\text{Cov}(U(X_2, \dots, X_n), V(X_1)) \geq 0,$$

since U and V are both decreasing functions of $t_1, \dots, t_n$. ■

From Theorems 2.2.1 and 2.2.2, we get immediately the following result by induction.

COROLLARY 2.2.1 *If $X_1, \dots, X_n$ are associated and uncorrelated, then $X_1, \dots, X_n$ are mutually independent.*

An elegant proof of this result was obtained by Lebowitz (1972) and also was shown in Theorem 1 and Corollary of Joag-Dev (1983). Corollary 2.2.1, in turn, shows that the covariance structure of associated random variables controls the nature of their approximate independence. Hence all assumptions concerning the convergence of associated sequences should be given in terms of their covariance structure.

2.3 An Extension of Hoeffding's equality.

Hoeffding's equality alone is not enough to study the approximate independence of associated random variables. Most of the time we have to estimate the covariance of some functions (not necessarily increasing or decreasing) of associated random variables. Moreover, estimating the covariances or moments of multivariate random variables becomes unavoidable when studying empirical processes. Hence, we should find a suitable extension of Hoeffding's equality.

The following theorem is an extension of Hoeffding's equality. In the case $k = 1$, the result was mentioned by Newman (1984) in his (4.10), without proof. Since we will use this result often in the sequel and its proof is not very long, we give a proof in this section. For other, different generalizations of the Hoeffding equality, we refer to Jogdeo (1968) and Block and Fang (1988).

THEOREM 2.3.1 *For any positive integer k , let $f_i, i = 1, \dots, 2k$ be absolutely continuous functions in any finite interval of R^1 . Then, for any random variables $Y_1, \dots, Y_{2k}$, we have*

$$(2.3.1) \quad K(f_i(Y_i), i = 1, \dots, 2k)$$

$$= \int_{R^{2k}} \left(\prod_{i=1}^{2k} f'_i(x_i) \right) K(I[Y_i \leq x_i], i = 1, \dots, 2k) dx_1 \cdots dx_{2k},$$

if the right hand side of the equation is absolutely integrable, where

$$(2.3.2) \quad K(\xi_i, i = 1, \dots, 2k) = \sum_{\text{card}\{\pi\} \leq k} (-1)^{\text{card}\{\pi\}} E\left(\prod_{j \in \pi} \xi_j\right) E\left(\prod_{j \notin \pi} \xi_j\right),$$

and the sum is taken over all the subsets $\pi \subseteq \{1, 2, \dots, 2k\}$ with $\text{card}\{\pi\}$ standing for the number of integers in π .

Proof. Let $(X_1, \dots, X_{2k})$ be an independent copy of $(Y_1, \dots, Y_{2k})$. Then, by some calculations,

$$(2.3.3) \quad \text{left hand side of (2.3.1)} = \frac{1}{2} E \prod_{i=1}^{2k} (f_i(Y_i) - f_i(X_i)).$$

It is easy to check that for $i = 1, \dots, 2k$

$$f_i(Y_i) - f_i(X_i) = \int_{X_i}^{Y_i} f'_i(x_i) dx_i = \int_{R^1} f'_i(x_i) (I(X_i \leq x_i) - I(Y_i \leq x_i)) dx_i.$$

Thus, by (2.3.3), we obtain

$$\begin{aligned} K(f_i(Y_i), i = 1, \dots, 2k) &= \frac{1}{2} E \prod_{i=1}^{2k} \int_{R^1} f'_i(x_i) (I(X_i \leq x_i) - I(Y_i \leq x_i)) dx_i \\ &= \int_{R^{2k}} \left(\prod_{i=1}^{2k} f'_i(x_i) \right) K(I[Y_i \leq x_i], i = 1, \dots, 2k) dx_1 \cdots dx_{2k}. \end{aligned}$$

In the last equality the expectation can be moved inside the integral, due to assuming absolute integrability. This completes the proof. ■

REMARK 2.3.1 When $k = 1$, Theorem 2.3.1 states that

$$(2.3.4) \quad \text{Cov}(f_1(Y_1), f_2(Y_2)) = \int_{R^2} f'_1(x_1) f'_2(x_2) \Delta_{12} dx_1 dx_2,$$

where $\Delta_{12} = K(I[Y_1 \leq x_1], I[Y_2 \leq x_2]) = \text{Cov}(I[Y_1 \leq x_1], I[Y_2 \leq x_2])$. Hence, Hoeffding's equality is a special case of (2.3.4). (2.3.4), in turn, shows that the

absolute continuity assumption in Theorem 2.3.1 cannot be relaxed. For example, let $f_i(x_i) = I(x_i \leq 0)$, $i = 1, 2$. Then, the right hand side of (2.3.4) is zero, while the left hand side of (2.3.4) is normally not zero in the dependent case. This is the reason why we cannot generalize the Hoeffding equality to cover the case of step functions.

Now we give an application of Theorem 2.3.1.

COROLLARY 2.3.1 *Let X and Y be PQD. Then for any $\alpha \geq 1, \beta \geq 1$ and $p, q, r > 1$ with $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1$, we have*

$$|\text{Cov}(|X|^\alpha, |Y|^\beta)| \leq 2^{1-\frac{1}{r}} \alpha \beta (E|X|^{(\alpha-\frac{1}{p})p})^{1/p} \cdot (E|Y|^{(\beta-\frac{1}{q})q})^{1/q} \cdot (\text{Cov}(X, Y))^{1/r}.$$

Proof. Since $|x|^\alpha$ is an absolutely continuous function in any finite interval of $\mathbf{R}$ for any $\alpha \geq 1$, by Theorem 2.3.1, we have

$$\text{Cov}(|X|^\alpha, |Y|^\beta) = \alpha \beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x|^{\alpha-1} \text{Sgn}(x) \cdot |y|^{\beta-1} \text{Sgn}(y) \cdot \Delta(x, y) dx dy,$$

where $\text{Sgn}(x) = 1$ if $x > 0$; $= 0$ if $x = 0$; $= -1$ if $x < 0$, and

$$\Delta(x, y) = P\{X \leq x, Y \leq y\} - P\{X \leq x\}P\{Y \leq y\}.$$

Then, by Hölder's inequality and Theorem 2.3.1,

$$\begin{aligned} |\text{Cov}(|X|^\alpha, |Y|^\beta)| &\leq \alpha \beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |x|^{\alpha-1} |y|^{\beta-1} \cdot \Delta(x, y) dx dy, \\ &\leq \alpha \beta \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|x|^{\alpha-1} |y|^{\beta-1})^{\frac{pq}{p+q}} \cdot \Delta(x, y) dx dy \right\}^{\frac{p+q}{pq}} (\text{Cov}(X, Y))^{1/r}. \end{aligned}$$

By applying Theorem 2.3.1 and Hölder's inequality again, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|x|^{\alpha-1} |y|^{\beta-1})^{\frac{pq}{p+q}} \cdot \Delta(x, y) dx dy \\ &= ((\alpha-1) \frac{pq}{p+q} + 1)^{-1} ((\beta-1) \frac{pq}{p+q} + 1)^{-1} \\ &\quad \text{Cov}(|X|^{(\alpha-1) \frac{pq}{p+q} + 1} \text{Sgn}(X), |Y|^{(\beta-1) \frac{pq}{p+q} + 1} \text{Sgn}(Y)) \\ &\leq 2 \left(E|X|^{((\alpha-1) \frac{pq}{p+q} + 1) \frac{p+q}{p}} \right)^{\frac{p}{p+q}} \left(E|Y|^{((\beta-1) \frac{pq}{p+q} + 1) \frac{p+q}{q}} \right)^{\frac{q}{p+q}}. \end{aligned}$$

Thus, our proof is complete. ■

Using Corollary 2.3.1, we can also simplify the proof of Theorem 1 of Birkel (1988a). The details are omitted.

2.4 Limit theorems.

In this section, we discuss some known theorems for associated sequences of random variables. Under some covariance restrictions a number of limit theorems have been proved for associated sequences, such as SLLN, CLT or invariance principles, LIL or functional LIL and Berry-Esséen inequalities. In the following, $\{X_n, n \geq 1\}$ is a sequence of random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ with $EX_n = 0$, $EX_n^2 < \infty$ for $n \geq 1$. For any $n \geq 1$, let $S_n = \sum_{i=1}^n X_i$, $\sigma_n^2 = ES_n^2$. We consider only one dimensional random variables. When our results cover also random vectors or vector indexed sequences, we mention only their simple cases.

Birkel (1989) obtained a strong law of large numbers (SLLN) for association under a condition close to that of Kolmogorov's classical SLLN for independent random variables.

THEOREM 2.4.1 (Birkel, 1989) *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geq 1$. Assume that*

$$\sum_{n=1}^{\infty} n^{-2} \text{Cov}(X_n, S_n) < \infty.$$

Then

$$n^{-1}S_n \longrightarrow 0 \text{ a.s..}$$

A central limit theorem (CLT) for association was proved by Newman (1980), and later an invariance principle for association was obtained by Newman and Wright (1981). We only give the invariance principle here since it implies the CLT.

THEOREM 2.4.2 (Newman and Wright, 1981) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Assume that*

$$0 < \sigma^2 = \text{Cov}(X_1, X_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty.$$

For each $n \geq 1$, define the stochastic process

$$W_n(t) = \sigma_n^{-1} S_{[nt]}, \quad 0 \leq t \leq 1,$$

with $S_0 = 0$. Then

$$W_n(\cdot) \xrightarrow{\mathcal{D}} W(\cdot) \text{ in } D[0, 1],$$

where W is the standard Wiener process.

REMARK 2.4.1 $D = D[0, 1]$ is the usual D space on $[0, 1]$, i.e., the space of functions on $[0, 1]$ that are right-continuous and have left-hand limits, with Skorokhod J_1 topology. Also $C[0, 1]$ is the space of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$ (cf. Chapter 3 of Billingsley (1968)). By saying *stationary* here and also in the sequel, we mean that the joint distribution of $X_{i+1}, \dots, X_{i+m}$ does not depend on i for any fixed positive integer m .

There are several extensions of Theorem 2.4.2. Cox and Grimmett (1984) weakened the assumption of stationarity and replaced it by certain conditions on the moments of associated random variables. Using the coefficient

$$u(n) = \sup_{k \geq 1} \sum_{j: |j-k| \geq n} \text{Cov}(X_j, X_k), \quad \text{for } n \geq 0,$$

they obtained the following **CLT**.

THEOREM 2.4.3 (Cox and Grimmett, 1984) *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geq 1$. Assume that*

$$u(0) < \infty, \quad u(n) \longrightarrow 0 \text{ as } n \rightarrow \infty,$$

and

$$\inf_{n \geq 1} \text{Var}(X_n) > 0, \quad \sup_{n \geq 1} E|X_n|^3 < \infty.$$

Then

$$\sigma_n^{-1} S_n \xrightarrow{\mathcal{D}} N(0, 1),$$

where $N(0, 1)$ is the standard normal random variable.

The third moment condition can be dropped. We have the following result in which A and B are necessary conditions.

THEOREM 2.4.4 (Yu, 1985ab) *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geq 1$. Assume that*

(A) $\sigma_n^2 = nh(n)$, where $h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$ is slowly varying,

(B) $\sigma_n^{-2} ES_{[nt]}(S_n - S_{[nt]}) \rightarrow 0$ for every $t \in (0, 1)$,

and $\{\sigma_n^{-2}(S_{k+n} - S_k)^2 : k \geq 0, n \geq 1\}$ is uniformly integrable. Then

$$W_n(\cdot) \xrightarrow{\mathcal{D}} W(\cdot) \text{ in } D[0, 1].$$

COROLLARY 2.4.1 (Yu, 1985ab) *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geq 1$. Assume that A holds with $\inf_{n \geq 1} h(n) > 0$, $\{X_n^2, n \geq 1\}$ is uniformly integrable and*

$$u(0) < \infty, \quad u(n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then

$$W_n(\cdot) \xrightarrow{\mathcal{D}} W(\cdot) \text{ in } D[0, 1].$$

Birkel (1987) proved a more general result by replacing the condition of uniform integrability by the Lindeberg condition.

THEOREM 2.4.5 (Birkel, 1987) Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geq 1$. Assume that A holds with $\inf_{n \geq 1} h(n) > 0$,

$$\sigma_n^{-2} \sum_{i=1}^n EX_i^2 I\{|X_i| \geq \varepsilon \sigma_n\} \rightarrow 0 \text{ for any } \varepsilon > 0$$

and

$$u(0) < \infty, \quad u(n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Then

$$W_n(\cdot) \xrightarrow{\mathcal{D}} W(\cdot) \text{ in } D[0, 1].$$

For the law of the iterated logarithm (LIL) or functional law of the iterated logarithm (FLIL) for association, we have the following results.

THEOREM 2.4.6 (Yu 1985a, 1986) Let $\{X_n, n \geq 1\}$ be a sequence of stationary associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Assume that

$$(C) \quad EX_1 X_n = O(n^{-1}(\log n)^{-(4+a)}), \text{ for all } n > 2 \text{ and some } a > 0$$

and

$$\sup_{-\infty < x < \infty} |P\{\sigma_n^{-1} S_n \leq x\} - \Phi(x)| = O((\log n)^{-(1+b)}),$$

for all $n > 2$ and $b > 0$, where Φ is the standard normal distribution function. Then

$$\limsup_{n \rightarrow \infty} (2\sigma_n^2 \log \log \sigma_n^2)^{-1/2} S_n = 1 \text{ a.s.}$$

COROLLARY 2.4.2 (Yu 1985a, 1986) Let $\{X_n, n \geq 1\}$ be a sequence of stationary associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Assume that (C) holds and

$$E|X_1|^{2+\delta} < \infty \text{ for some } \delta \in (0, 1].$$

Then

$$\limsup_{n \rightarrow \infty} (2\sigma_n^2 \log \log \sigma_n^2)^{-1/2} S_n = 1 \text{ a.s.}$$

Independently of Yu (1985a, 1986), Dabrowski (1985) obtained a **FLIL** for associated sequences. Later, Dabrowski and Dehling (1988) generalized it for weakly associated sequences. We give the latter result.

THEOREM 2.4.7 (Dabrowski and Dehling, 1988) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary weakly associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Assume that*

$$0 < \sigma^2 = \text{Cov}(X_1, X_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty,$$

$$\sigma^2 - n^{-1}\sigma_n^2 = O(n^{-\epsilon}) \text{ for some } \epsilon > 0$$

and

$$E|X_1|^{2+\delta} < \infty \text{ for some } \delta > 0.$$

Define random variables $\phi_n(t)$ taking values in $C[0, 1]$ in the following way:

$$\phi_n(t) = \begin{cases} (2n \log \log n)^{-1/2} S_k & \text{if } t = k/n, \\ \text{linear} & \text{in between.} \end{cases}$$

Then the set $\{\phi_n(\cdot), n \geq 1\}$ is almost surely relatively compact with the set of limit points

$$K_\sigma = \left\{ \sigma f : f \in C[0, 1] \text{ is absolutely continuous and } \int_0^1 \|f'\|^2 dt \leq 1 \right\}.$$

Weakening the assumption on moments, Shao (1989) obtained the following result.

THEOREM 2.4.8 (Shao, 1989) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary associated random variables with $EX_1 = 0$ and $EX_1^2 < \infty$. Assume that*

$$EX_1^2 (\log(1 + |X_1|))^{1+\delta} < \infty \text{ for some } \delta > 0$$

and

$$EX_1 X_n = O(n^{-\nu}) \text{ for some } \nu > 1 + 1/\delta.$$

Then

$$\limsup_{n \rightarrow \infty} (2\sigma_n^2 \log \log \sigma_n^2)^{-1/2} S_n = 1 \quad \text{a.s.}$$

Wood (1983) obtained the first Berry-Esséen inequalities for association. His proof, however, was in error, and the Berry-Esséen rate of $O(n^{-1/5})$ was far from the optimal rate $O(n^{-1/2})$. Finally Birkel (1988b) obtained the best possible rate for association. The following is his result.

THEOREM 2.4.9 (Birkel, 1988b) *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geq 1$. Assume that*

$$u(n) = O(\exp(-\lambda n)) \quad \text{for some } \lambda > 0,$$

and

$$\inf_{n \geq 1} n^{-1} \sigma_n^2 > 0, \quad \sup_{n \geq 1} E|X_n|^3 < \infty.$$

Then there exists a constant B , not depending on n , such that for all $n \geq 1$, we have

$$\sup_{-\infty < x < \infty} |P\{\sigma_n^{-1} S_n \leq x\} - \Phi(x)| \leq B n^{-1/2} \log^2 n.$$

On replacing the last assumption by

$$\sup_{n \geq 1} E|X_n|^{3+\delta} < \infty \quad \text{for some } \delta > 0,$$

then we have

$$\sup_{-\infty < x < \infty} |P\{\sigma_n^{-1} S_n \leq x\} - \Phi(x)| \leq B n^{-1/2} \log n.$$

2.5 Open problems.

So far, we quoted a number of limit theorems for association. However, the important problem of almost sure invariance principle (**ASIP**), or strong invariance

principle is still open. It is well known that invariance principles as well as **FLIL** and other asymptotic fluctuation results can be derived from **ASIP** for partial sums of a sequence (cf. Theorems A-E in Section 1 of Philipp and Stout, 1975). There are a lot of **ASIP** results available for mixing sequences, but not for association. Unfortunately, association is extremely hard to deal with. For example, we cannot get good estimates for conditional expectations of associated sequences. Also, the Skorohod embedding technique cannot be used for association. To tackle this problem, Yu (1987) tried to use the quantile transform method developed by Csörgő and Révész (1975ab) and later refined by Komlós, Major and Tusnády (1975ab). Generally speaking, this method is restricted to independent random variables. It however also is well suited for associated random variables since they are **PQD**, and Theorem 2.3.1 can be used to get good estimates on the covariances of the new random variables. Indeed, under an adequate decay rate (power rate) of covariances, the partial sum of an associated sequence can be approached almost surely by the partial sum of an associated sequence having normal marginal distributions. Unfortunately, we are unable to show that the latter partial sum can be approximated by a time transformed standard Wiener process under the same decay rate. Hence, the **ASIP** for association under a power decay rate of covariances is not solved yet. The only known **ASIP** result we are aware of is the following one.

THEOREM 2.5.1 (Yu, 1987) *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for $n \geq 1$. Assume that*

$$\sup_{k \geq 1} EX_k X_{n+k} = O(\exp(-an^b)) \text{ for some } a, b > 0,$$

and

$$\inf_{n \geq 1} EX_n^2 > 0, \quad \sup_{n \geq 1} E|X_n|^{2+\delta} < \infty \text{ for some } \delta > 0.$$

Then, without changing its distribution, we can redefine the sequence $\{X_n, n \geq 1\}$ on a richer probability space together with a standard Wiener process $W(t)$ such that

$$S_{[t]} - W(\sigma_{[t]}^2) = O(t^{1/2-\nu}) \text{ a.s.}$$

for some small $\nu > 0$.

It is worthy of mentioning that Lindqvist (1988) extended the notion of association to partially ordered spaces and there are some applications of association found in the areas of random measures and point processes (cf., e.g., Burton and Waymire (1985), Burton and Franzosa (1990) and Evans (1990)). As we know, limit theorems for association should require at least the existence of second moments, since covariance structure is used to control the approximate independence for association. However, in some cases, the second moments do not exist, especially so in case of random measures. Thus, a challenging problem that should be answered is to find a way for defining the covariance structure of associated sequences when second moments do not exist. Only after this problem will be answered can we proceed to study the CLT for association in such situations.

Recently Dabrowski and Jakubowski (1993) studied the problem of stable limits for associated random variables. They state conditions which guarantee that partial sums of a stationary sequence of associated random variables, when properly normalized, converge in distribution to a stable, non-Gaussian limit. Since the second moments of this sequence do not exist, they use the following way to define the corresponding covariance structure. Let

$$\Delta_{(X,Y)}(x,y) = P\{X \leq x, Y \leq y\} - P\{X \leq x\}P\{Y \leq y\}.$$

Then, fix $A > 0$, and define

$$I_p^A(X,Y) = \sup_{a \geq A} a^{p-2} \int_{-a}^a \int_{-a}^a \Delta_{(X,Y)}(x,y) dx dy.$$

DEFINITION 2.5.1 (Dabrowski and Jakubowski, 1993) Let $\{X_n, n \geq 1\}$ be an arbitrary strictly stationary sequence and let $\{Y_n, n \geq 1\}$ be a jointly strictly p -stable strictly stationary sequence. We say that $\{X_n\}$ belongs to the domain of strict normal attraction of $\{Y_n\}$ (and write $\{X_n\} \in \mathcal{D}_{s,n}(\{Y_n\})$), if for each N , the joint distribution of $\mathbf{Z}_N = (X_1, \dots, X_N)$ belongs to the domain of strict normal attraction of $\mathcal{L}(\mathbf{W}_N) = \mathcal{L}((Y_1, \dots, Y_N))$, that is

$$\frac{\mathbf{Z}_{n,1} + \dots + \mathbf{Z}_{n,n}}{n^{1/p}} \xrightarrow{\mathcal{D}} (Y_1, \dots, Y_N), \text{ as } n \rightarrow \infty,$$

where $\mathbf{Z}_{n,1}, \dots, \mathbf{Z}_{n,n}$ are independent copies of $(X_1, \dots, X_N)$.

THEOREM 2.5.2 (Dabrowski and Jakubowski, 1993) Let $\{X_n, n \geq 1\}$ be stationary and associated. If $\{X_n\} \in \mathcal{D}_{s,n}(\{Y_n\})$, where $\{Y_n, n \geq 1\}$ is jointly strictly p -stable, $0 < p < 2$, and if for some $A > 0$

$$\sum_{k=2}^{\infty} I_p^A(X_1, X_k) < \infty,$$

then there exists a strictly p -stable distribution μ_{∞} such that both

$$\frac{X_1 + \dots + X_n}{n^{1/p}} \xrightarrow{\mathcal{D}} \mu_{\infty},$$

and

$$\frac{Y_1 + \dots + Y_n}{n^{1/p}} \xrightarrow{\mathcal{D}} \mu_{\infty}.$$

THEOREM 2.5.3 (Dabrowski and Jakubowski, 1993) Let $\{X_n, n \geq 1\}$ be stationary, associated and such that for some $A \geq 0$ and $0 < p < 1$,

$$\sum_{k=2}^{\infty} I_p^A(X_1, X_k) < \infty.$$

If for each m

$$\mathcal{L}(S_m) \in \mathcal{D}_{s,n}(\mu_m),$$

then there exists a strictly p -stable distribution μ_∞ such that

$$\mu_m^{1/m} \xrightarrow{\mathcal{D}} \mu_\infty, \quad \text{as } m \rightarrow \infty,$$

and

$$\frac{S_n}{n^{1/p}} \xrightarrow{\mathcal{D}} \mu_\infty.$$

Note that Theorem 2.5.3 holds true only for $0 < p < 1$.

Chapter 3

Empirical Processes

In this chapter, we deal with approximations of empirical processes. Without loss of generality we assume that our basic sequence $\{X_n, n \geq 0\}$ is defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$. With the help of our extension of Hoeffding's equality, we develop a method for estimating the covariance structures for empirical functions of associated sequences in terms of covariances of the original random variables. Based on these estimations, we obtain a Glivenko-Cantelli lemma for associated sequences and prove weak convergence results for empirical processes of stationary associated sequences, all under conditions on covariances of the original random variables.

3.1 Empirical processes. Definitions.

Let $\{X_n, n \geq 0\}$ be a sequence of random variables with common distribution function F . For each $n \geq 1$ put $S_n = \sum_{i=1}^n X_i$ and $S_0 = 0$. Then the empirical distribution function F_n of $X_1, \dots, X_n$ is defined as

$$(3.1.1) \quad F_n(x) = \frac{1}{n} \sum_{i=1}^n I(X_i \leq x), \quad x \in \mathbf{R}.$$

The n th empirical process β_n is defined by

$$(3.1.2) \quad \beta_n(x) = n^{1/2}(F(x) - F_n(x)), \quad x \in \mathbf{R}.$$

Define the quantile function $Q(y)$, $0 \leq y \leq 1$, by

$$(3.1.3) \quad Q(y) = F^{-1}(y) = \inf\{x : F(x) > y\}, \quad 0 \leq y < 1, \quad Q(1) = Q(1-),$$

i.e., $Q(y)$ is the right continuous inverse of the right continuously defined F (see (1.3.2)). When F is continuous, Q satisfies

$$(3.1.4) \quad Q(y) = F^{-1}(y) = \inf\{x : F(x) = y\}, \quad F(Q(y)) = y \in [0, 1].$$

In what follows we always assume that F is continuous.

Let $U_n = F(X_n)$ for each $n \geq 0$. Then each U_n has the uniform-[0, 1] distribution. Let the uniform empirical distribution function $E_n(y)$, $0 \leq y \leq 1$, of $U_1, \dots, U_n$ be defined as

$$(3.1.5) \quad E_n(y) = F_n(Q(y)) = \frac{1}{n} \sum_{i=1}^n I(U_i \leq y), \quad 0 \leq y \leq 1,$$

and the n th uniform empirical process $\{\alpha_n(y); 0 \leq y \leq 1\}$ by

$$(3.1.6) \quad \{\alpha_n(y); 0 \leq y \leq 1\} = \{n^{1/2}(y - E_n(y)); 0 \leq y \leq 1\}, \quad n \geq 1.$$

Thus, in terms of $U_i = F(X_i)$, $i = 1, \dots, n$, we have

$$(3.1.7) \quad \{\beta_n(Q(y)); 0 \leq y \leq 1\} \stackrel{\mathcal{D}}{=} \{\alpha_n(y); 0 \leq y \leq 1\}, \quad n \geq 1.$$

This implies that the study of weak convergence of the process $\beta_n(\cdot)$ can be carried out by studying the process $\alpha_n(\cdot)$ and using the sequence $\{U_n, n \geq 0\}$ instead of $\{X_n, n \geq 0\}$.

3.2 A Glivenko-Cantelli lemma for association.

In this section, we obtain a Glivenko-Cantelli lemma for association under a condition on the covariances of the original random variables. Through the proof of Theorem 3.2.1, we demonstrate a procedure on how to transfer the covariance of step functions of associated random variables to that of the original variables with the help of Theorem 2.3.1.

THEOREM 3.2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables having the same distribution function $F(x)$, $x \in \mathbf{R}$. If $F(x)$ is continuous and*

$$(3.2.1) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, S_{n-1}) < \infty,$$

then we have, as $n \rightarrow \infty$,

$$(3.2.2) \quad \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \rightarrow 0 \quad \text{a.s..}$$

Proof. Let x be a fixed continuity point of F . For any $\varepsilon > 0$, choose $\delta > 0$ such that

$$F(x + \delta) - F(x - \delta) < \varepsilon,$$

and define the function $h_x(s)$ on $(-\infty, \infty)$ as

$$(3.2.3) \quad h_x(s) = \begin{cases} 1 & \text{if } s \leq x \\ 1 - \frac{s-x}{\delta} & \text{if } x < s \leq x + \delta \\ 0 & \text{if } s > x. \end{cases}$$

Let $h_-(s) = h_{x-\delta}(s)$ and $h_+(s) = h_x(s)$. Then, obviously, $h_-(X_1) \leq I(X_1 \leq x) \leq h_+(X_1)$ and $E(h_+(X_1) - h_-(X_1)) \leq \varepsilon$. Noting that h_- and h_+ are nonincreasing absolutely continuous functions on $(-\infty, \infty)$, by Theorem 2.3.1, (3.2.1) and

Theorem 2.4.1. we have

$$\frac{1}{n} \sum_{i=1}^n h_-(X_i) \rightarrow Eh_-(X_1), \quad \text{a.s.},$$

$$\frac{1}{n} \sum_{i=1}^n h_+(X_i) \rightarrow Eh_+(X_1), \quad \text{a.s.}.$$

Thus almost surely for each $\varepsilon > 0$,

$$F(x) - \varepsilon \leq Eh_-(X_1) \leq \liminf F_n(x) \leq \limsup F_n(x) \leq Eh_+(X_1) \leq F(x) + \varepsilon.$$

Hence we obtain that

$$F_n(x) \rightarrow F(x) \quad \text{a.s.}$$

The rest of the proof is almost the same as in the i.i.d case (cf., e.g., Chung (1974)).

The proof of Theorem 3.2.1 is now complete. $\blacksquare$

REMARK 3.2.1 Assuming also stationarity for the associated sequence $\{X_n, n \geq 1\}$, the condition (3.2.1) can be weakened to

$$(3.2.4) \quad \frac{1}{n} \sum_{i=1}^n \text{Cov}(X_n, X_i) \rightarrow 0.$$

This can be proved by the fact that $\{X_n, n \geq 1\}$ in this case is ergodic (cf. Theorem 7 of Newman (1984)). On the other hand, (3.2.1) or (3.2.4) cannot be weakened any further, as seen by the following example.

Let X be any random variable with $EX = 0$ and $EX^2 = 1$. Then, by (P_3) and (P_4) of Association, $\{X_n = X, n \geq 1\}$ is a sequence of (stationary) associated random variables having the same distribution F of X . It is easy to check that (3.2.1) and (3.2.4) are not satisfied and neither is (3.2.2).

Theorem 3.2.1 does not cover the case of F discrete. From the proof one can only obtain, for any $x \in \mathbf{R}$, $F(x-) \leq \liminf_{n \rightarrow \infty} F_n(x-)$, a.s.. So the Glivenko-Cantelli lemma remains open for this case.

3.3 Weak convergence for empirical processes of stationary associated sequences.

In this section, we establish weak convergence for empirical processes of stationary associated sequences. We recall that Newman (1984) concluded the asymptotic finite dimensional distributions of an empirical process based on associated sequences of random variables under the additional assumption that the thus generated empirical process should have, uniformly in $\mathbf{R} \times \mathbf{R}$, a finite covariance (cf. the first Remark of page 137 of Newman, 1984). Clearly, it is not easy to verify such an assumption from the assumed covariance structure of the original random variables, and then one would also want to prove tightness as well. As we know from Theorem 2.2.1, the covariance structure of associated random variables controls the nature of their approximate independence (assuming two moments). Hence all the assumptions for the convergence of empirical processes of associated sequences should be given in terms of the covariance structure of the original variables. This is desirable not only from the theoretical, but also from practical point of view, since it is the covariance structure of the original variables that is usually given.

Our aim in this chapter is to prove a weak convergence for empirical processes of associated sequences. The most outstanding issue in weak convergence for empirical processes of associated sequences is to verify tightness. A basic approach is to estimate the covariances (moments) of empirical processes. Since the empirical process is defined in terms of step functions of the original variables, the main question is how to estimate the covariance of step functions of the original variables via their originally given covariance. One big obstacle to establishing this estimation is that we cannot apply Hoeffding's equality, which, in principle, would be the most plausible way of linking these two kinds of covariances. Motivated by Newman's

(1984) treatment of characteristic functions of associated variables, we are to see now that this question can be solved via the generalized Hoeffding's equality covering absolutely continuous functions of random variables, which is given as Theorem 2.3.1 of Chapter 2. The remaining questions then are how to choose appropriate absolutely continuous functions to approximate step functions. This procedure is established in Subsection 3.3.2.

The main results are stated in Subsection 3.3.1. Subsection 3.3.2 is devoted to prove a weak convergence for empirical processes of stationary associated sequences.

3.3.1 Results.

THEOREM 3.3.1 *Let $\{U_n, n \geq 0\}$ be a stationary associated sequence of uniform-[0, 1] random variables. Assume that there exists a positive constant ν such that*

$$(3.3.1) \quad \sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(U_0, U_n) < \infty.$$

Then

$$(3.3.2) \quad \alpha_n(\cdot) \xrightarrow{\mathcal{D}} B(\cdot) \text{ in } D[0, 1],$$

where α_n is defined by (3.1.6) and B is a zero-mean Gaussian process on $[0, 1]$ with covariance defined by

$$(3.3.3) \quad \begin{aligned} EB(s)B(t) &= s \wedge t - st \\ &+ \sum_{k=1}^{\infty} \{ \text{Cov}(I(U_0 \leq s), I(U_k \leq t)) + \text{Cov}(I(U_0 \leq t), I(U_k \leq s)) \}. \end{aligned}$$

This series converges absolutely for all $0 \leq s, t \leq 1$ and $P\{B(\cdot) \in C[0, 1]\} = 1$.

REMARK 3.3.1 Newman (1980) proved a CLT for associated random variables as in Theorem 3.3.1 under the condition

$$\sum_{n=1}^{\infty} \text{Cov}(U_0, U_n) < \infty.$$

Our theorem requires a faster decay rate of covariance than Newman's, due to having to prove tightness of α_n . We note also that the series in (3.3.3) is non-negative by association. Hence, the Gaussian process B has a stronger tendency to hang together than a standard Brownian bridge process does.

COROLLARY 3.3.1 *Let $\{X_n, n \geq 0\}$ be a stationary associated sequence of random variables. Suppose that X_0 has a bounded density function f and there exists a positive constant ν such that*

$$(3.3.4) \quad \sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(X_0, X_n) < \infty.$$

Then

$$\beta_n(Q(\cdot)) \xrightarrow{\mathcal{D}} B(\cdot) \text{ in } D[0, 1],$$

where β_n is defined by (3.1.2) and B is the same Gaussian process as in Theorem 3.3.1 with $P\{B(\cdot) \in C[0, 1]\} = 1$.

Proof. By Theorem 3.3.1 and (3.1.7), we just need to show that (3.3.4) implies (3.3.1). This can be done by applying Theorem 2.3.1. ■

3.3.2 Proof of Theorem 3.3.1.

In this subsection we always assume that $\{U_n, n \geq 0\}$ is a stationary associated sequence of random variables. Some of the technical details of our proof are presented as lemmas in the back of this subsection for convenient reference in the proof. As mentioned before, the most difficult part of proving Theorem 3.3.1 is to show tightness. To do this we use the same approach as Billingsley (1968) did in proving his Theorem 22.1. Namely, we have to get an estimation of

$$E \left\{ \sum_{i=1}^n (I[s < U_i \leq t] - (t - s)) \right\}^4$$

for any fixed $0 \leq s < t \leq 1$. If, however, we directly expand the above expression such as Billingsley (1968) did, then all the terms gained this way are not of the form required by Theorem 2.3.1. In other words, we cannot directly apply Theorem 2.3.1. Thus, the following symmetrization of $\{U_n, n \geq 0\}$ is essential in our proof.

Let $\{V_n, n \geq 0\}$ be an independent copy of $\{U_n, n \geq 0\}$. Then by Jensen's inequality and stationarity,

$$\begin{aligned}
 (3.3.5) \quad & E \left\{ \sum_{i=1}^n (I[s < U_i \leq t] - (t-s)) \right\}^4 \\
 &= E \left\{ E \left(\sum_{i=1}^n (I[s < U_i \leq t] - I[s < V_i \leq t]) \mid U_i, i = 1, \dots, 2k \right) \right\}^4 \\
 &\leq E \left\{ \sum_{i=1}^n (I[s < U_i \leq t] - I[s < V_i \leq t]) \right\}^4 \\
 &\leq 4!n \sum_{\substack{i,j,k \geq 0 \\ i+j+k \leq n}} |EK(I[s < U_0 \leq t], I[s < U_i \leq t], I[s < U_{i+j} \leq t], I[s < U_{i+j+k} \leq t])| \\
 &= 4!n \sum_{\substack{i,j,k \geq 0 \\ i+j+k \leq n}} |I_{ijk}|,
 \end{aligned}$$

where the function K is defined by (2.3.2).

As to how to estimate I_{ijk} in terms of the original covariances is shown by Lemmas 3.3.1 to 3.3.3. From now on, C_1 stands for a generic positive constant, independent of s, t and n . It may, however, take different values in each appearance. By Lemma 3.3.3, we get

$$E \left| \sum_{i=1}^n (I(s < U_i \leq t) - (t-s)) \right|^4 \leq C_1 n^2 (n^{-\frac{1}{2}-\nu_1} + (t-s)^{6/5})$$

and, consequently,

$$E|\alpha_n(t) - \alpha_n(s)|^4 \leq C_1 (n^{-\frac{1}{2}-\nu_1} + (t-s)^{6/5}).$$

For any $0 < \varepsilon < 1$, choose $r_n = \varepsilon/n^{1/2}$. If $r_n \leq t-s$, we have

$$(3.3.6) \quad E|\alpha_n(t) - \alpha_n(s)|^4 \leq \frac{C_1}{\varepsilon^{1+2\nu_1}} (t-s)^{1+2\nu_1}.$$

Consider now the random variables

$$\alpha_n(s + ir_n) - \alpha_n(s + (i-1)r_n), \quad i = 1, \dots, m,$$

where m is a positive integer. By (3.3.6) and Theorem 12.2 of Billingsley (p.94, 1968), for any $\lambda > 0$,

$$P\{\max_{i \leq m} |\alpha_n(s + ir_n) - \alpha_n(s)| \geq \lambda\} \leq \frac{C_1}{\varepsilon^{1+2\nu_1} \lambda^4} (mr_n)^{1+2\nu_1}.$$

For any $\eta > 0$, choose $\delta > 0$ such that $2^{1+2\nu_1} C_1 \delta^{2\nu_1} \varepsilon^{-(5+2\nu_1)} < \eta$ and $m_n = [\delta r_n^{-1}]$. Then, if n is large enough, $m_n \geq 1$ and

$$r_n m_n \leq \delta < (m_n + 1)r_n \leq 2m_n r_n \leq 2\delta.$$

Hence, applying (22.18) of Billingsley (1968), we have

$$\begin{aligned} P\left\{\sup_{s \leq t \leq s+\delta} |\alpha_n(t) - \alpha_n(s)| \geq 4\varepsilon\right\} &\leq P\left\{\sup_{s \leq t \leq s+(m_n+1)r_n} |\alpha_n(t) - \alpha_n(s)| \geq 4\varepsilon\right\} \\ &\leq \frac{C_1}{\varepsilon^{5+2\nu_1}} ((m_n + 1)r_n)^{1+2\nu_1} < \eta\delta. \end{aligned}$$

This proves that $\{\alpha_n(t), 0 \leq t \leq 1\}$ is tight. To prove Theorem 3.3.1, it suffices to show that for any $0 < t_1 < \dots < t_k < 1$

$$(\alpha_n(t_1), \dots, \alpha_n(t_k)) \xrightarrow{\mathcal{D}} (B(t_1), \dots, B(t_k)).$$

This can be proved directly from the CLT for weakly associated random vectors of Burton, Dabrowski and Dehling (1986). Lemma 3.3.4 shows their covariance condition is satisfied. Now the proof of Theorem 3.3.1 is complete. ■

Lemmas 3.3.1 to 3.3.3 are used to estimate the I_{jk} of (3.3.5) in terms of the original covariances. The general approach is as follows. By constructing an absolutely continuous function to approximate a step function, the I_{jk} terms of (3.3.5) can be linked to the function K inside the integral of (2.3.1) via Theorem 2.3.1. Using this

decomposition of the function K , I_{ijk} can be estimated by the original covariances via Theorem 2.3.1 again (using (2.3.4)). Note that by the definition of the function K , the Δ_{ij} in Theorem 2.2.2 can be written as

$$\Delta_{ij} = K(I[A_i], I[A_j]) = \text{Cov}(I[A_i], I[A_j]).$$

The following lemma is to decompose $K(I[A_1], I[A_2], I[A_3], I[A_4])$ into parts consisting of Δ_{ij} .

LEMMA 3.3.1 *Suppose that $Y_1, \dots, Y_4$ are associated random variables. Then, for any real numbers $x_1, \dots, x_4$, we have*

$$(3.3.7) \quad |K_1| \leq 4 \sum_{i=2}^4 \Delta_{1i},$$

$$(3.3.8) \quad K_1 = 2\Delta_{12}\Delta_{34} + 5\theta(\Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24}),$$

where $K_1 = K(I[A_1], I[A_2], I[A_3], I[A_4])$ and $|\theta| = |\theta(x_1, x_2, x_3, x_4)| \leq 1$.

Proof. For notational simplicity, we let

$$A_{1234} = A_1 A_2 A_3 A_4, \quad A_{ijk} = A_i A_j A_k, \quad A_{ij} = A_i A_j.$$

Then

$$\begin{aligned} K_1 = & -(P(A_1)P(A_{234}) + P(A_2)P(A_{134}) + P(A_3)P(A_{124}) + P(A_4)P(A_{123})) \\ & P\{A_{1234}\} + (P(A_{12})P(A_{34}) + P(A_{13})P(A_{24}) + P(A_{14})P(A_{23})). \end{aligned}$$

By Theorem 2.2.2,

$$(3.3.9) \quad 0 \leq P(A_{1234}) - P(A_1)P(A_{234}) \leq \sum_{i=2}^4 \Delta_{1i}.$$

If $P(A_2)P(A_{134}) > P(A_{12})P(A_{34})$, then by Theorem 2.2.2

$$\begin{aligned} & |P(A_2)P(A_{134}) - P(A_{12})P(A_{34})| \\ & \leq P(A_2)P(A_{134}) - P(A_1)P(A_2)P(A_{34}) \\ & \leq P(A_{134}) - P(A_1)P(A_{34}) \\ & \leq \sum_{i=3}^4 \Delta_{1i}, \end{aligned}$$

where we used the fact that $P(A_{12}) \geq P(A_1)P(A_2)$.

If $P(A_2)P(A_{134}) < P(A_{12})P(A_{34})$, then

$$|P(A_2)P(A_{134}) - P(A_{12})P(A_{34})| \leq P(A_{12})P(A_{34}) - P(A_1)P(A_2)P(A_{34}) \leq \Delta_{12},$$

since $P(A_{134}) \geq P(A_1)P(A_{34})$. Hence

$$|P(A_2)P(A_{134}) - P(A_{12})P(A_{34})| \leq \sum_{i=2}^4 \Delta_{1i}.$$

Similarly, we have

$$|P(A_3)P(A_{124}) - P(A_{13})P(A_{24})| \leq \sum_{i=2}^4 \Delta_{1i}$$

and

$$|P(A_4)P(A_{123}) - P(A_{23})P(A_{14})| \leq \sum_{i=2}^4 \Delta_{1i}.$$

Thus, these three inequalities together with (3.3.9), imply (3.3.7).

Let

$$U(t_1, t_2) = I(t_1 \leq x_1) + I(t_2 \leq x_2) - I(t_1 \leq x_1)I(t_2 \leq x_2)$$

and

$$V(t_3, t_4) = I(t_3 \leq x_3) + I(t_4 \leq x_4) + I(t_3 \leq x_3)I(t_4 \leq x_4).$$

Then U and V are non-increasing. Hence

$$\text{Cov}(U(Y_1, Y_2), V(Y_3, Y_4)) \geq 0.$$

Rearranging the above inequality, by Theorem 2.2.2 we have

$$\begin{aligned}
0 &\leq P(A_{1234}) - P(A_{12})P(A_{34}) \\
&\leq \Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24} \\
&\quad + (P(A_{12})P(A_3) - P(A_{123})) + (P(A_{12})P(A_4) - P(A_{124})) \\
&\quad + (P(A_{134}) - P(A_1)P(A_{34})) + (P(A_{234}) - P(A_2)P(A_{34})) \\
&\leq 2(\Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24}),
\end{aligned}$$

since $P(A_{12})P(A_3) - P(A_{123}) \leq 0$ and $P(A_{12})P(A_4) - P(A_{124}) \leq 0$. Thus, by Theorem 2.2.2, we obtain

$$\begin{aligned}
|K_1(x_1, x_2, x_3, x_4) - 2\Delta_{12}\Delta_{34}| &\leq (P(A_{1234}) - P(A_{12})P(A_{34})) \\
&\quad + (P(A_{13})P(A_{24}) - \prod_{i=1}^4 P(A_i)) + (P(A_{23})P(A_{14}) - \prod_{i=1}^4 P(A_i)) \\
&\quad + (P(A_1)P(A_{234}) - P(A_1)P(A_2)P(A_{34})) \\
&\quad + (P(A_2)P(A_{134}) - P(A_1)P(A_2)P(A_{34})) \\
&\quad + (P(A_3)P(A_{124}) - P(A_{12})P(A_3)P(A_4)) \\
&\quad + (P(A_4)P(A_{123}) - P(A_{12})P(A_3)P(A_4)) \\
&\leq 5(\Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{24}),
\end{aligned}$$

which shows that (3.3.8) is true. $\blacksquare$

To use Theorem 2.3.1, we have to define an absolutely continuous function h to approximate the step function $I[s \leq u \leq t]$. Let h be a function defined on

$(-\infty, \infty)$ such that it is symmetric about $u = (s + t)/2$ and

$$h(u) = \begin{cases} 0 & \text{if } u < s - a \\ 1 + \frac{u - s}{a} & \text{if } s - a \leq u < s \\ 1 & \text{if } s < u \leq (s + t)/2. \end{cases}$$

with $0 < a \leq 1$. Then $h(u)$ is absolutely continuous, and we have also

$$(3.3.10) \quad 0 \leq h(u) - I[s < u \leq t] \leq I(s - a < u < s) + I(t < u < t + a),$$

$$(3.3.11) \quad |h'(u)| = \begin{cases} a^{-1} & \text{if } u \in (s - a, s) \text{ or } (t, t + a) \\ 0 & \text{if } u \in (0, s - a) \text{ or } (s, t) \text{ or } (t + a, 1). \end{cases}$$

With $z_n = I[s < U_n \leq t] - (t - s)$, we state and prove estimates for the I_{jk} terms of (3.3.5).

LEMMA 3.3.2 *We have*

$$(3.3.12) \quad |I_{jk}| \leq 64a + 16a^{-2} \{ \text{Cov}(U_0, U_i) + \text{Cov}(U_0, U_{i+j}) + \text{Cov}(U_0, U_{i+j+k}) \},$$

as well as

$$(3.3.13) \quad |I_{jk}| \leq 80a + 2|\text{Cov}(z_0, z_i)| |\text{Cov}(z_{i+j}, z_{i+j+k})| \\ + 20a^{-2} (\text{Cov}(U_0, U_{i+j}) + \text{Cov}(U_0, U_{i+j+k}) + \text{Cov}(U_i, U_{i+j}) + \text{Cov}(U_i, U_{i+j+k})).$$

Proof. Let

$$J_{ijk} = EK(h(U_0), h(U_i), h(U_{i+j}), h(U_{i+j+k})).$$

Then, noting that $0 \leq h(u) \leq 1$ and $0 \leq I[s < u \leq t] \leq 1$ for all u , by (3.3.10) we immediately obtain that

$$(3.3.14) \quad |I_{jk} - J_{ijk}| \leq 64a.$$

Now (3.3.12) follows easily from (3.3.7), (3.3.11), (3.3.14) and Theorem 2.3.1. By (3.3.8), (3.3.11), Theorem 2.3.1, we have

$$J_{i,j,k} = 2\text{Cov}(h(U_0), h(U_i))\text{Cov}(h(U_{i+j}), h(U_{i+j+k})) \\ + 20\theta_1 a^{-2}(\text{Cov}(U_0, U_{i+j}) + \text{Cov}(U_0, U_{i+j+k}) + \text{Cov}(U_i, U_{i+j}) + \text{Cov}(U_i, U_{i+j+k})),$$

with $|\theta_1| \leq 1$. But, by (3.3.11),

$$|\text{Cov}(h(U_0), h(U_i))\text{Cov}(h(U_{i+j}), h(U_{i+j+k})) - \text{Cov}(z_0, z_i)\text{Cov}(z_{i+j}, z_{i+j+k})| \leq 16a.$$

This, together with (3.3.14), proves (3.3.13). ■

LEMMA 3.3.3 *Under the condition (3.3.1), we have for all $n \geq 1$*

$$(3.3.15) \quad E \left\{ \sum_{i=1}^n (I[s < U_i \leq t] - (t-s)) \right\}^4 \leq C_1 n^2 (n^{-\frac{1}{2}-\nu_1} + (t-s)^{6/5})$$

with $\nu_1 = \min(\nu/3, 1/5)$.

Proof. We split the summation of (3.3.5) into three parts.

$$\sum_{\substack{i,j,k \geq 0 \\ i+j+k \leq n}} = \sum_{\substack{i,j,k \geq 0, i+j+k \leq n \\ i = \max(i,j,k)}} + \sum_{\substack{i,j,k \geq 0, i+j+k \leq n \\ j = \max(i,j,k)}} + \sum_{\substack{i,j,k \geq 0, i+j+k \leq n \\ k = \max(i,j,k)}} \\ = S_{(1)} + S_{(2)} + S_{(3)}.$$

For $1 \leq i \leq n$ let $a = \text{Cov}^{\frac{1}{3}}(U_0, U_i)$. By stationarity, (3.3.1), (3.3.12) and Hölder inequality, we have

$$S_{(1)} = S_{(1), i=0} + S_{(1), i \geq 1} \\ \leq 1 + C_1 \sum_{i=1}^n (i+1)^2 \text{Cov}^{\frac{1}{3}}(U_0, U_i) \\ \leq C_1 n^{\frac{1}{2}-\nu_1}.$$

Similarly, $S_{(3)} \leq C_1 n^{\frac{1}{2}-\nu_1}$. Now we estimate $S_{(2)}$. Let $a = \text{Cov}^{\frac{1}{3}}(U_0, U_1)$. By stationarity, (3.3.1) and (3.3.13), we get

$$\begin{aligned} S_{(2)} &= S_{(2), j=0} + S_{(2), j \geq 1} \\ &\leq 1 + C_1 \sum_{j=1}^n (j+1)^2 \text{Cov}^{\frac{1}{3}}(U_0, U_j) + C_1 \sum_{j=1}^n \sum_{i,k=0}^j |E z_0 z_i| |E z_{i+j} z_{i+j+k}| \\ &\leq C_1 n^{\frac{1}{2}-\nu_1} + C_1 \sum_{j=1}^n \left(\sum_{i=0}^j |E z_0 z_i| \right)^2. \end{aligned}$$

For $1 \leq i \leq n$ we choose $a = \text{Cov}^{\frac{1}{3}}(U_0, U_i)$. Then by (3.3.1), (3.3.11) and Theorem 2.3.1,

$$|E z_0 z_i| \leq |\text{Cov}(h(U_0), h(U_i))| + 8a \leq a^{-2} \text{Cov}(U_0, U_i) + 8a \leq 9 \text{Cov}^{\frac{1}{3}}(U_0, U_i).$$

Since $E z_n^2 \leq t - s$, by the Cauchy-Schwarz inequality, we have $|E z_0 z_i| \leq t - s$, and all these give us

$$|E z_0 z_i| = |E z_0 z_i|^{3/5} |E z_0 z_i|^{2/5} \leq C_1 (t - s)^{3/5} \text{Cov}^{\frac{2}{15}}(U_0, U_i).$$

Hence, by (3.3.1) and the Hölder inequality again,

$$\sum_{i=0}^{\infty} |E z_0 z_i| \leq (t - s) + C_1 (t - s)^{3/5} \sum_{i=1}^{\infty} \text{Cov}^{\frac{2}{15}}(U_0, U_i) \leq C_1 (t - s)^{3/5}$$

and, consequently,

$$\sum_{(2)} \leq C_1 n (n^{-\frac{1}{2}-\nu_1} + (t - s)^{6/5}).$$

This completes the proof. $\blacksquare$

REMARK 3.3.2 From the above proof one can find that the best possible upper bound in (3.3.15) for association is

$$E \left\{ \sum_{i=1}^n (I[s < U_i \leq t] - (t - s)) \right\}^4 \leq C_1 n^2 (b_n + (t - s)^{2\nu_2}),$$

where

$$b_n = \begin{cases} n^{-\frac{1}{2}-\frac{\nu}{3}} & \text{if } 0 < \nu < 3/2 \\ \log n/n & \text{if } \nu = 3/2 \\ 1/n & \text{if } \nu > 3/2 \end{cases}$$

and $0 < \nu < (9 + 2\nu)(15 + 2\nu)^{-1}$.

LEMMA 3.3.4 *Let $\{U_n, n \geq 0\}$ be a stationary associated sequence of random variables. If there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^2 \log^{2+\nu}(n+1) \text{Cov}(U_0, U_n) < \infty,$$

then the series in (3.3.3) converges.

Proof. Note that the covariances in (3.3.3) involve step functions. Hence, for estimating them, we need to find appropriate forms of h_x defined in (3.2.3). For each $n \geq 1$ we choose $\delta = \text{Cov}^{\frac{1}{3}}(U_0, U_n)$ in (3.2.3) and change x to s or t . Then, by Theorem 2.3.1,

$$\begin{aligned} 0 &\leq \text{Cov}(I(U_0 \leq s), I(U_n \leq t)) = EI(U_0 \leq s)I(U_n \leq t) - st \\ &\leq Eh_s(U_0)h_t(U_n) - st \\ &\leq Eh_s(U_0)Eh_t(U_n) + \delta^{-2}\text{Cov}(U_0, U_n) - st \\ &\leq 3\delta + \delta^{-2}\text{Cov}(U_0, U_n) \\ &\leq 4\text{Cov}^{\frac{1}{3}}(U_0, U_n). \end{aligned}$$

Similarly, for all $n \geq 1$,

$$0 \leq \text{Cov}(I(U_0 \leq t), I(U_n \leq s)) \leq 4\text{Cov}^{\frac{1}{3}}(U_0, U_n).$$

By the Hölder inequality, this completes the proof. $\blacksquare$

3.4 Weighted weak convergence for empirical processes of stationary associated sequences.

In Section 3.3, we obtained a weak convergence for empirical processes of associated sequences. However, this result alone is not strong enough to study the asymptotic distribution of some statistics of interest which are based on ordered observations. One has to consider the problem of proving weak convergence of empirical processes in metrics stronger than the usual Prohorov and sup-norm metrics, namely, we have to deal with weak convergence in weighted sup-norm metrics. For example, in Chapter 5, we consider the (normalized) mean residual life process (5.1.6),

$$(1 - F_n(Q(t)))z_n(Q(t)) = - \int_t^1 \alpha_n(y) dQ(y) + M_F(Q(t))\alpha_n(t), \quad 0 \leq t \leq 1.$$

The function $M_F(Q(t))$ in the above second term, which is nonnegative and right-continuous on $[0, 1)$, is such that we have

$$M_F(Q(t)) = -Q(t) + \frac{1}{1-t} \int_t^1 Q(y) dy.$$

by $\int_0^1 (1-u) dQ(u) < \infty$ and by integrating by parts. Clearly, when t is near 1, we cannot determine whether $M_F(Q(t))$ is monotone or finite since $Q(t)$ is usually going to ∞ as $t \rightarrow 1$. This means that the weak convergence of $M_F(Q(t))\alpha_n(t)$ cannot be just simply derived from Theorem 3.3.1 by using the continuous mapping theorem. In general, the study of weighted weak convergence for empirical processes is necessary not only for such a task, but it provides also a way for further studies of quantile processes as well. In this section we study this kind of weak convergence for empirical processes of associated sequences.

THEOREM 3.4.1 *Let l be a nonnegative and right-continuous function on $(0, 1)$ such that there exist two possibly degenerate intervals $(0, a)$ and $(b, 1)$, $0 < a <$*

$b < 1$ so that l is nonincreasing on $(0, a)$ and nondecreasing on $(b, 1)$, $\sup\{l(t) : a \leq t \leq b\} < \infty$ and

$$(3.4.1) \quad \int_0^1 l^p(t) dt < \infty,$$

where

$$(3.4.2) \quad p = \begin{cases} (15 + 2\nu)(3 + 2\nu)^{-1} & \text{if } 0 < \nu < 1.5 \\ > 3 & \text{if } \nu = 1.5 \\ 3 & \text{if } \nu > 1.5. \end{cases}$$

Then under the assumptions of Theorem 3.3.1, we have

$$(3.4.3) \quad l(\cdot)\alpha_n(\cdot) \xrightarrow{\mathcal{D}} l(\cdot)B(\cdot) \text{ in } D[0, 1].$$

Proof. By Theorem 3.3.1 and Theorem 4.2 of Billingsley (1968), in order to prove (3.4.3), it is sufficient to prove that for any $\varepsilon > 0$

$$(3.4.4) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{0 < t \leq \theta} |l(t)\alpha_n(t)| \geq \varepsilon \right\} = 0,$$

$$(3.4.5) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{1-\theta \leq t < 1} |l(t)\alpha_n(t)| \geq \varepsilon \right\} = 0,$$

$$(3.4.6) \quad \lim_{\theta \rightarrow 0} P\left\{ \sup_{0 < t \leq \theta} |l(t)B(t)| \geq \varepsilon \right\} = 0,$$

and

$$(3.4.7) \quad \lim_{\theta \rightarrow 0} P\left\{ \sup_{1-\theta \leq t < 1} |l(t)B(t)| \geq \varepsilon \right\} = 0.$$

Let $q(t) = (l(t))^{-1}$ and $1/2 \leq \lambda \leq 3/4$. Then

$$\begin{aligned} P\left\{ \sup_{0 < t \leq \theta} (|l(t)\alpha_n(t)|) \geq \varepsilon \right\} &\leq \sum_{j=0}^{\infty} P\left\{ \sup_{\theta\lambda^{j+1} < t \leq \theta\lambda^j} |\alpha_n(t)| \geq \varepsilon q(\theta\lambda^{j+1}) \right\} \\ &\leq \sum_{j=0}^{\infty} P\left\{ |\alpha_n(\theta\lambda^{j+1})| \geq \frac{\varepsilon}{2} q(\theta\lambda^{j+1}) \right\} \\ &\quad + \sum_{j=0}^{\infty} P\left\{ \sup_{\theta\lambda^{j+1} < t \leq \theta\lambda^j} |\alpha_n(t) - \alpha_n(\theta\lambda^{j+1})| \geq \frac{\varepsilon}{2} q(\theta\lambda^{j+1}) \right\} \\ &= \sum_{j=0}^{\infty} A_j + \sum_{j=0}^{\infty} B_j. \end{aligned}$$

In the following, C is a positive constant, independent of s, t and n , and may take different values in each appearance.

For fixed j let $\varepsilon_0 = 8^{-1}\varepsilon q(\theta\lambda^{j+1})$ and $\delta = \theta\lambda^j(1 - \lambda)$. Choose $r_n = \varepsilon_0 n^{-1/2}$. Then, if $r_n \leq |t - s|$, by Remark 3.3.2,

$$E|\alpha_n(t) - \alpha_n(s)|^4 \leq C \left\{ |t - s|^{p_1} \varepsilon_0^{-p_1} + |t - s|^{2\nu_1} \right\},$$

where

$$p_1 = \begin{cases} 1 + 2\nu/3 & \text{if } 0 < \nu < 1.5 \\ < 2 & \text{if } \nu = 1.5 \\ 2 & \text{if } \nu > 1.5 \end{cases}$$

and $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$. Thus, applying Theorem 12.2 of Billingsley (1968), we get

$$(3.4.8) \quad \begin{aligned} P\{\max_{i \leq m} |\alpha_n(\theta\lambda^{j+1} + ir_n) - \alpha_n(\theta\lambda^{j+1})| \geq \varepsilon_0\} \\ \leq C \left\{ (mr_n)^{p_1} \varepsilon_0^{-(4+p_1)} + (mr_n)^{2\nu_1} \varepsilon_0^{-4} \right\}. \end{aligned}$$

Let $m = [\delta r_n^{-1}]$. If $\delta \geq r_n$, then $m \geq 1$ and

$$r_n m \leq \delta < (m + 1)r_n \leq 2r_n m \leq 2\delta.$$

Hence by (22.18) of Billingsley (1968) and (3.4.8),

$$\begin{aligned} B_j &\leq P\left\{ \sup_{\theta\lambda^{j+1} \leq t \leq \theta\lambda^{j+1} + (m+1)r_n} |\alpha_n(t) - \alpha_n(\theta\lambda^{j+1})| \geq 4\varepsilon_0 \right\} \\ &\leq C \left\{ ((m+1)r_n)^{p_1} \varepsilon_0^{-(4+p_1)} + ((m+1)r_n)^{2\nu_1} \varepsilon_0^{-4} \right\} \\ &\leq C \left\{ q^{-(4+p_1)} (\theta\lambda^{j+1}) \cdot (\theta\lambda^{j+1})^{p_1} + q^{-4} (\theta\lambda^{j+1}) \cdot (\theta\lambda^{j+1})^{2\nu_1} \right\}. \end{aligned}$$

If $\delta < r_n$, i.e., $\delta n^{1/2} < \varepsilon_0$, then by (22.17) of Billingsley (1968) and Chebyshev's inequality,

$$B_j \leq P\{|\alpha_n(\theta\lambda^j) - \alpha_n(\theta\lambda^{j+1})| \geq 3\varepsilon_0\}$$

$$\leq (3\varepsilon_0)^{-2} E(\alpha_n(\theta\lambda^j) - \alpha_n(\theta\lambda^{j+1}))^2.$$

On the other hand, by some calculations and stationarity, we can get for all $0 \leq s, t \leq 1$

$$E(\alpha_n(t) - \alpha_n(s))^2 \leq Ez_0^2 + 2 \sum_{i=1}^{n-1} |Ez_0 z_i|$$

and

$$E(B(t) - B(s))^2 \leq Ez_0^2 + 2 \sum_{i=1}^{\infty} |Ez_0 z_i|,$$

where $z_n = g_t^*(U_n) - g_s^*(U_n)$. Thus, similarly to the proof of Lemma 3.3.3, we have by (3.3.1)

$$(3.4.9) \quad E(\alpha_n(t) - \alpha_n(s))^2 \leq C|t - s|^{\nu_1}$$

and

$$(3.4.10) \quad E(B(t) - B(s))^2 \leq C|t - s|^{\nu_1}.$$

Hence, when $\delta n^{1/2} < \varepsilon_0$,

$$B_j \leq Cq^{-2}(\theta\lambda^{j+1}) \cdot (\theta\lambda^{j+1})^{\nu_1}.$$

Also, we have

$$A_j \leq Cq^{-2}(\theta\lambda^{j+1})E(\alpha_n(\theta\lambda^{j+1}))^2 \leq Cq^{-2}(\theta\lambda^{j+1}) \cdot (\theta\lambda^{j+1})^{\nu_1}.$$

Choose $p = (4 + p_1)p_1^{-1}$. Then by (3.4.1) we have for all $j \geq 0$ and all small θ ,

$$\left\{ q^{-p}(\theta\lambda^{j+1}) \cdot (\theta\lambda^{j+1}) \right\}^{p_1} \leq q^{-p}(\theta\lambda^{j+1}) \cdot (\theta\lambda^{j+1}) < 1.$$

For the chosen p_1 we can let ν_1 be close enough to $(9 + 2\nu)(15 + 2\nu)^{-1}$ so that

$$(3.4.11) \quad p_2 = \{(4 + p_1)\nu_1 - 2p_1\}(2 + p_1)^{-1} > 0 \text{ for all } \nu > 0.$$

Thus at $t = \theta\lambda^{j+1}$

$$q^{-2}(t) \cdot t^{\nu_1} > q^{-(4+p_1)}(t) \cdot t^{p_1}$$

implies

$$q^{-2}(t) \cdot t^{\nu_1} < t^{p_2}.$$

All these give us

$$\begin{aligned} P\left\{\sup_{0 < t \leq \theta} (|l(t)\alpha_n(t)|) \geq \varepsilon\right\} &\leq C \left\{ \sum_{j=0}^{\infty} (\theta \lambda^{j+1})^{p_2} + \sum_{j=0}^{\infty} q^{-p}(\theta \lambda^{j+1}) \cdot (\theta \lambda^{j+1}) \right\} \\ &\leq C \left\{ \theta^{p_2} + \int_0^{\theta} l^p(t) dt \right\}, \end{aligned}$$

which proves (3.4.4). Similarly we can prove (3.4.5).

From (3.4.10) we have

$$E(B(t) - B(s))^4 \leq C E^2(B(t) - B(s))^2 \leq C |t - s|^{2\nu_1}$$

for all $0 \leq s, t \leq 1$. Applying Theorem 12.2 of Billingsley (1968) directly, we can immediately get that (3.4.6) and (3.4.7) are true. This completes our proof. ■

REMARK 3.4.1 From the above proof, it is obvious that one can relax the condition (3.4.1) to $q(t) = q(1 - t) = t^{1/p}(\log(1/t))^{(p-1)(1+\mu)(4p)^{-1}}, 0 < t \leq a$ for some $\mu > 0$ and $0 < a \leq 1/2$.

REMARK 3.4.2 Theorem 3.4.1 was inspired by a version of a result of Pyke and Shorack (1968), presented as Lemma 2.4 in Csörgő, Csörgő and Horváth (1986).

Chapter 4

Quantile Processes

In this chapter, we develop a method of obtaining weak and weighted weak convergence for the uniform quantile processes, as well as for the weak convergence of the normed quantile processes of stationary associated sequences from those of the uniform empirical processes.

4.1 Quantile processes. Definitions.

In Section 3.1 of Chapter 2, we have defined basic settings for (uniform) empirical functions and (uniform) empirical processes, where the quantile function Q is defined as well. We introduce now the sample quantile function.

Let $\{X_n, n \geq 0\}$ be a sequence of random variables with common distribution function F and $X_{1:n} \leq \dots \leq X_{n:n}$ be the order statistics of $X_1, \dots, X_n$. Then the n th sample quantile function is defined as

$$\begin{aligned} (4.1.1) \quad Q_n(y) = F_n^{-1}(y) &= \inf\{x : F_n(x) > y\} \\ &= \begin{cases} X_{k:n}, & \frac{k-1}{n} \leq y < \frac{k}{n}, k = 1, \dots, n, \\ X_{n:n}, & y = 1, \end{cases} \end{aligned}$$

i.e., Q_n is the right continuous inverse of F_n (see (1.3.2)), and define the n th quantile process $\{q_n(y); 0 \leq y \leq 1\}$ is defined by

$$(4.1.2) \quad \begin{aligned} \{q_n(y); 0 \leq y \leq 1\} &= \{n^{1/2}(Q_n(y) - Q(y)); 0 \leq y \leq 1\} \\ &= \{n^{1/2}(F_n^{-1}(y) - F^{-1}(y)); 0 \leq y \leq 1\}. \quad n = 1, 2, \dots \end{aligned}$$

Let $U_n = F(X_n)$ be as in Section 3.1 of Chapter 3 with F continuous. Let the n th sample uniform quantile function $U_n(y), 0 \leq y \leq 1$, of $U_1, \dots, U_n$ be defined as

$$(4.1.3) \quad \begin{aligned} U_n(y) = E_n^{-1}(y) &= \inf\{x : E_n(x) > y\} \\ &= \begin{cases} U_{k:n}, & \frac{k-1}{n} \leq y < \frac{k}{n}, k = 1, \dots, n, \\ U_{n:n}, & y = 1, \end{cases} \end{aligned}$$

where $U_{1:n} \leq \dots \leq U_{n:n}$ is the order statistics of $U_1, \dots, U_n$, and the n th uniform quantile process $\{u_n(y); 0 \leq y \leq 1\}$ by

$$(4.1.4) \quad \{u_n(y); 0 \leq y \leq 1\} = \{n^{1/2}(U_n(y) - y); 0 \leq y \leq 1\}. \quad n \geq 1.$$

Since $U_{k:n} = F(X_{k:n})$, $k = 1, \dots, n$, we have

$$(4.1.5) \quad U_n(y) = F(Q_n(y)), \quad 0 \leq y \leq 1.$$

Now in terms of $U_{k:n} = F(X_{k:n})$, $k = 1, 2, \dots, n$, we can work with the uniform quantile process $\{u_n(y); 0 \leq y \leq 1\}$. However, unlike the case of $\beta_n(Q(y)) = \alpha_n(y)$, we no longer get such a simple relationship between the quantile processes $u_n(y)$ and $q_n(y)$. Indeed, if F is an absolutely continuous distribution function (with respect to Lebesgue measure) with a strictly positive density function $f(x) = F'(x)$ for $x \in \mathbb{R}$, then we have for $y \in (0, 1)$, $(k-1)/n \leq y < k/n$, $k = 1, 2, \dots, n$,

$$(4.1.6) \quad q_n(y) = n^{1/2}(Q_n(y) - Q(y)) = n^{1/2}(X_{k:n} - F^{-1}(y))$$

$$\begin{aligned}
&= n^{1/2}(F^{-1}(F(X_{k:n})) - F^{-1}(y)) = n^{1/2}(F^{-1}(U_{k:n}) - F^{-1}(y)) \\
&= n^{1/2}(F^{-1}(U_n(y)) - F^{-1}(y)) \\
&= \frac{1}{f(Q(\theta_{y,n}))} u_n(y),
\end{aligned}$$

where $U_n(y) \wedge y < \theta_{y,n} < U_n(y) \vee y$, $y \in (0, 1)$, $n = 1, 2, \dots$. This leads us to define the normed quantile process $\{\rho_n(y); 0 < y < 1, n = 1, 2, \dots\}$ as

$$\begin{aligned}
(4.1.7) \quad &\{\rho_n(y); 0 < y < 1, n = 1, 2, \dots\} \\
&= \{f(Q(y))q_n(y); 0 < y < 1, n = 1, 2, \dots\} \\
&= \{n^{1/2}f(F^{-1}(y))(F_n^{-1}(y) - F^{-1}(y)); 0 < y < 1, n = 1, 2, \dots\}
\end{aligned}$$

In the light of (4.1.6) above,

$$(4.1.8) \quad \rho_n(y) = \frac{f(Q(y))}{f(Q(\theta_{y,n}))} u_n(y), \quad y \in (0, 1).$$

The function $f(Q(y)) = f(F^{-1}(y))$ is called the *density-quantile* function, and $Q'(y) = 1/f(Q(y))$ the *quantile-density* function by Parzen (1979). Note that

$$(4.1.9) \quad u_n\left(\frac{k}{n}-\right) = \alpha_n(U_{k:n}), \quad k = 1, 2, \dots, n.$$

$$(4.1.10) \quad u_n\left(\frac{k-1}{n}\right) = \alpha_n(U_{k:n}-), \quad k = 1, 2, \dots, n$$

and

$$(4.1.11) \quad u_n(1) = \alpha_n(U_{n:n}).$$

Thus, we have that for each $\omega \in \Omega$

$$(4.1.12) \quad \sup_{0 \leq y \leq 1} |\alpha_n(y)| = \sup_{0 \leq y \leq 1} |u_n(y)|.$$

4.2 Strong consistency of sample quantile functions

By Theorem 3.2.1 and (4.1.12), we have the following result.

THEOREM 4.2.1 *Let $\{U_n, n \geq 1\}$ be a sequence of associated uniform-[0, 1] random variables. If*

$$(4.2.1) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(U_n, \sum_{i=1}^{n-1} U_i) < \infty,$$

then we have, as $n \rightarrow \infty$,

$$(4.2.2) \quad \sup_{0 \leq y \leq 1} |U_n(y) - y| \longrightarrow 0 \quad \text{a.s.}$$

Next by (4.1.6) and Theorem 2.3.1, we have the following corollary of Theorem 4.2.1.

COROLLARY 4.2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with common distribution function F . Assume that $f = F'$ is bounded, $\inf_{0 \leq y \leq 1} f(Q(y)) > 0$, and*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{i=1}^{n-1} X_i) < \infty.$$

Then

$$\sup_{0 \leq y \leq 1} |Q_n(y) - Q(y)| \longrightarrow 0 \quad \text{a.s.}$$

4.3 Weak convergence for uniform quantile processes of stationary associated sequences.

To get the weak convergence for the uniform quantile processes of stationary associated sequences, we can simply apply Theorem 3.3.1 and Theorem 1 of Vervaat (1972). Since we want to demonstrate how to get such convergence from that of the

uniform empirical processes and because the proof is very easy, we give the detailed proof of the following theorem.

THEOREM 4.3.1 *Let $\{U_n, n \geq 0\}$ be a sequence of stationary associated uniform-[0, 1] random variables. Assume that there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(U_0, U_n) < \infty.$$

Then

$$(4.3.1) \quad u_n(\cdot) \xrightarrow{\mathcal{D}} B(\cdot) \text{ in } D[0, 1],$$

where u_n is defined by (4.1.4) and B is the same Gaussian process as the one defined in Theorem 3.3.1 with $P\{B(\cdot) \in C[0, 1]\} = 1$.

Proof. We have

$$(4.3.2) \quad \begin{aligned} u_n(y) &= n^{1/2}(U_n(y) - E_n(U_n(y))) + n^{1/2}(E_n(U_n(y)) - y) \\ &= \alpha_n(U_n(y)) + n^{1/2}(E_n(U_n(y)) - y). \end{aligned}$$

It is well known that

$$(4.3.3) \quad 0 \leq E_n(U_n(y)) - y \leq 1/n, \quad y \in [0, 1].$$

Hence, by (4.1.12), (4.3.2) and (4.3.3), for any $\varepsilon > 0$ and $K > 0$, if n is large enough, we have

$$\begin{aligned} &P\left\{\sup_{0 \leq y \leq 1} |u_n(y) - \alpha_n(y)| \geq 2\varepsilon\right\} \\ &\leq P\left\{\sup_{0 \leq y \leq 1} |\alpha_n(U_n(y)) - \alpha_n(y)| \geq \varepsilon\right\} \\ &\leq P\left\{\sup_{0 \leq y \leq 1} |\alpha_n(U_n(y)) - \alpha_n(y)| \geq \varepsilon, |U_n(y) - y| < Kn^{-1/2}\right\} \\ &\quad + P\left\{\sup_{0 \leq y \leq 1} |u_n(y)| \geq K\right\} \\ &\leq P\{w(\alpha_n, Kn^{-1/2}) \geq \varepsilon\} + P\left\{\sup_{0 \leq y \leq 1} |\alpha_n(y)| \geq K\right\}, \end{aligned}$$

where w is Lévy's modulus of continuity:

$$(4.3.4) \quad w(f, \delta) = \sup_{|s-t| < \delta} |f(s) - f(t)|, \quad 0 < \delta \leq 1.$$

Thus, by Theorem 3.3.1, we obtain

$$\limsup_{n \rightarrow \infty} P\left\{ \sup_{0 \leq y \leq 1} |u_n(y) - \alpha_n(y)| \geq 2\varepsilon \right\} \leq P\left\{ \sup_{0 \leq y \leq 1} |B(y)| \geq K \right\}.$$

Let $K \rightarrow \infty$. Then we have

$$(4.3.5) \quad \sup_{0 \leq y \leq 1} |u_n(y) - \alpha_n(y)| \xrightarrow{P} 0.$$

Thus our theorem is proved. ■

4.4 Weighted weak convergence for uniform quantile processes of stationary associated sequences.

As we saw in Section 4.3, the weak convergence of the uniform quantile processes can be obtained from that of the uniform empirical processes. Once again, this result alone is not enough to study the asymptotic distribution of some statistics of interest, based on ordered observations. As a matter of fact, in Chapter 6, we need the weighted weak convergence for the uniform quantile processes of stationary associated sequences to obtain the weak convergence for total time on test processes. To consider the weighted weak convergence of the uniform quantile processes, we have to deal with the distributions of order statistics $U'_{1:n}, \dots, U'_{n:n}$. As we know, it is very difficult to estimate such distributions when observations are not independent. Section 4.3 shows a way of avoiding directly dealing with the order statistics $U_{1:n}, \dots, U_{n:n}$. This motivates us to exploit the result of weighted weak convergence for the uniform empirical processes of stationary associated sequences proved in Section 3.4. Thus, we are able to prove the corresponding result for the uniform quantile

processes with the help of results already proved for uniform empirical processes. This approach will be used again and again in the rest of this thesis.

Let $q(y) > 0$ be a continuous function on $(0, 1)$ which is nondecreasing on $(0, a)$ and is nonincreasing on $(b, 1)$ for some $0 < a < b < 1$ and $\inf\{q(y); a \leq y \leq b\} > 0$. We collect these functions into the set $\mathbf{Q}^*$.

Choose $q^* \in \mathbf{Q}^*$ so that

$$(4.4.1) \quad q^*(t) = q^*(1-t) = t^{1/p}(\log(1/t))^{(p-1)(1+\mu)(4p)^{-1}}, \quad 0 < t \leq a$$

for some $\mu > 0$ and $0 < a \leq 1/2$, where p is defined by (3.4.2).

Let

$$(4.4.2) \quad k_n = \max\{k; \frac{k}{n^{1/2}q^*(k/n)} \leq 1 \text{ and } k \text{ is an integer.}\}.$$

From the definitions of q^* and k_n , it is easy to verify that, when n is large enough,

$$(4.4.3) \quad k_n \geq 1, \quad \frac{1}{2} < \frac{k_n}{n^{1/2}q^*(k_n/n)} \leq 1.$$

By (4.4.1), we can also derive from above that

$$(4.4.4) \quad \frac{k_n}{n} \leq n^{-\frac{p}{2(p-1)}}(\log n)^{\frac{1+\mu}{4}}$$

and

$$(4.4.5) \quad \frac{1}{2} \leq \frac{t}{q^*(t)}n^{1/2}, \quad \text{for } \frac{k_n}{n} \leq t \leq a,$$

since $t/q^*(t)$ is increasing on $(0, a]$.

THEOREM 4.4.1 *Let l be a nonnegative and continuous function defined as in Theorem 3.4.1. Then, under the assumptions of Theorem 3.3.1, we have*

$$(4.4.6) \quad l(\cdot)\tilde{u}_n(\cdot) \xrightarrow{\mathcal{D}} l(\cdot)B(\cdot) \text{ in } D[0, 1],$$

where

$$\tilde{u}_n(t) = u_n(t) \cdot I\left(\frac{k_n}{n} \leq t < 1 - \frac{k_n}{n}\right).$$

Proof. By Theorem 3.3.1, Theorem 4.2 of Billingsley (1968) and (3.4.6)-(3.4.7), in order to prove (4.4.6), it is sufficient to prove that for any $\varepsilon > 0$

$$(4.4.7) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{\frac{k_n}{n} \leq t \leq \theta} |l(t)u_n(t)| \geq \varepsilon \right\} = 0$$

and

$$(4.4.8) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{1-\theta \leq t < 1-\frac{k_n}{n}} |l(t)u_n(t)| \geq \varepsilon \right\} = 0.$$

Let $q(t) = (l(t))^{-1}$. Then $q \in \mathbf{Q}^*$. For $\frac{i}{n} \leq t < \frac{i+1}{n} \leq a$, by (4.1.3),

$$\begin{aligned} \frac{|u_n(t)|}{q(t)} &\leq \frac{n^{1/2}|U_{i+1:n} - \frac{i}{n}|}{q(i/n)} \vee \frac{n^{1/2}|U_{i+1:n} - \frac{i+1}{n}|}{q(i/n)} \\ &\leq \frac{|\alpha_n(U_{i+1:n-})|}{q(E_n(U_{i+1:n-}))} + \frac{1}{n^{1/2}q(i/n)}. \end{aligned}$$

The above inequality with (4.4.3) yields

$$(4.4.9) \quad \sup_{\frac{k_n}{n} \leq t \leq \theta} \frac{|u_n(t)|}{q(t)} \leq \sup_{U_{k_n:n} \leq t < U_{k_n+1:n}} \frac{|\alpha_n(t)|}{q(E_n(t))} + \frac{1}{n^{1/2}q(1/n)}.$$

where $\frac{k_n-1}{n} \leq \theta < \frac{k_n}{n}$. On the other hand, since $E_n(t) \geq \frac{k_n}{n}$ for $t \geq U_{k_n:n}$ and q is nondecreasing on $(0, a]$,

$$\sup_{U_{k_n:n} \leq t < U_{k_n+1:n}} \frac{|\alpha_n(t)|}{q(E_n(t))} \leq \sup_{0 < t \leq \frac{k_n}{n}} \frac{|\alpha_n(t)|}{q(t)} + \sup_{\frac{k_n}{n} \leq t < U_{k_n+1:n}} \frac{|\alpha_n(t)|}{q(E_n(t))}.$$

This gives us

$$\begin{aligned} (4.4.10) \quad &P\left\{ \sup_{U_{k_n:n} \leq t < U_{k_n+1:n}} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{2} \right\} \\ &\leq P\left\{ \sup_{0 < t \leq \frac{k_n}{n}} \frac{|\alpha_n(t)|}{q(t)} \geq \frac{\varepsilon}{4} \right\} + P\left\{ \sup_{\frac{k_n}{n} \leq t < U_{k_n+1:n}} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{4} \right\}. \end{aligned}$$

By (4.4.5),

$$\sup_{\frac{k_n}{n} \leq t \leq 2\theta} \frac{|\alpha_n(t)|}{q^*(t)} \leq \frac{1}{4}$$

implies

$$E_n(t) \geq t - \frac{q^*(t)}{4n^{1/2}} \geq \frac{t}{2} \quad \text{for } \frac{k_n}{n} \leq t \leq 2\theta.$$

Also $|\alpha_n(U_{\hat{k}_n:n})| \leq K$ implies $U_{\hat{k}_n:n} \leq \frac{\hat{k}_n - 1}{n} + \frac{K}{n^{1/2}}$. Hence, for any $K > 0$, when $n \geq \frac{K^2}{\theta^2}$, we have

$$\begin{aligned} (4.4.11) \quad & P\left\{ \sup_{\frac{k_n}{n} \leq t < U_{\hat{k}_n:n}} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{4} \right\} \\ & \leq P\left\{ \sup_{0 \leq t \leq 1} |\alpha_n(t)| > K \right\} + P\left\{ \sup_{\frac{k_n}{n} \leq t \leq \theta + \frac{K}{n^{1/2}}} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{4} \right\} \\ & \leq P\left\{ \sup_{0 \leq t \leq 1} |\alpha_n(t)| > K \right\} + P\left\{ \sup_{0 < t \leq 2\theta} \frac{|\alpha_n(t)|}{q^*(t)} \geq \frac{1}{4} \right\} \\ & \quad + P\left\{ \sup_{0 < t \leq 2\theta} \frac{|\alpha_n(t)|}{q(t/2)} \geq \frac{\varepsilon}{4} \right\}. \end{aligned}$$

From the assumption (3.4.1) on the function $q = (l(\cdot))^{-1}$, it is easy to get that $n^{1/2}q(1/n) \rightarrow \infty$ as $n \rightarrow \infty$. Thus, by (3.2.4), (4.4.4), (4.4.9)-(4.4.11), Remark 3.4.1 and Theorem 3.3.1, we obtain

$$\limsup_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{\frac{k_n}{n} \leq t \leq \theta} |l(t)u_n(t)| \geq \varepsilon \right\} \leq P\left\{ \sup_{0 \leq t \leq 1} |B(t)| \geq K \right\}$$

for each $K > 0$. Therefore (4.4.7) is true by letting $K \rightarrow \infty$.

The proof of (4.4.8) is slightly different from that of (4.4.7). For $1 - a \leq \frac{i}{n} \leq t < \frac{i+1}{n}$, by (4.1.3),

$$\begin{aligned} \frac{|u_n(t)|}{q(t)} & \leq \frac{n^{1/2}|U_{i+1:n} - \frac{i}{n}|}{q((i+1)/n)} \vee \frac{n^{1/2}|U_{i+1:n} - \frac{i+1}{n}|}{q((i+1)/n)} \\ & \leq \frac{|\alpha_n(U_{i+1:n})|}{q(E_n(U_{i+1:n}))} + \frac{1}{n^{1/2}q((i+1)/n)}. \end{aligned}$$

Thus, an analogue of (4.4.9) is

$$(4.4.12) \quad \sup_{1-\theta \leq t < 1 - \frac{k_n}{n}} \frac{|u_n(t)|}{q(t)} \leq \sup_{U_{k_n:n} \leq t < U_{n-k_n:n}} \frac{|\alpha_n(t)|}{q(E_n(t))} + \frac{1}{n^{1/2}q(1-1/n)}.$$

where $\frac{\bar{k}_n - 1}{n} \leq 1 - \theta < \frac{\bar{k}_n}{n}$, and an analogue of (4.4.10) is

$$(4.4.13) \quad P\left\{ \sup_{U_{k_n/n} \leq t < U_{\bar{k}_n/n} \atop q(E_n(t))} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{2} \right\} \\ \leq P\left\{ \sup_{1 - \frac{k_n}{n} \leq t < 1} \frac{|\alpha_n(t)|}{q(t)} \geq \frac{\varepsilon}{4} \right\} + P\left\{ \sup_{U_{k_n/n} \leq t < 1 - \frac{k_n}{n} \atop q(E_n(t))} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{4} \right\}.$$

By (4.4.1) and (4.4.5),

$$\sup_{1 - 2\theta \leq t \leq 1 - \frac{k_n}{n}} \frac{|\alpha_n(t)|}{q^*(t)} \leq \frac{1}{4}$$

implies

$$E_n(t) \leq t + \frac{q^*(t)}{4n^{1/2}} \leq 1 - \frac{1-t}{2} \quad \text{for } 1 - 2\theta \leq t \leq 1 - \frac{k_n}{n}.$$

Again, for any $K > 0$ and $n \geq \frac{K^2}{\theta^2}$, an analogue of (4.4.11) is

$$(4.4.14) \quad P\left\{ \sup_{U_{k_n/n} \leq t < 1 - \frac{k_n}{n} \atop q(E_n(t))} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{4} \right\} \\ \leq P\left\{ \sup_{0 \leq t \leq 1} |\alpha_n(t)| > K \right\} + P\left\{ \sup_{1 - \theta - \frac{K}{n^{1/2}} \leq t < 1 - \frac{k_n}{n} \atop q(E_n(t))} \frac{|\alpha_n(t)|}{q(E_n(t))} \geq \frac{\varepsilon}{4} \right\} \\ \leq P\left\{ \sup_{0 \leq t \leq 1} |\alpha_n(t)| > K \right\} + P\left\{ \sup_{1 - 2\theta \leq t < 1} \frac{|\alpha_n(t)|}{q^*(t)} \geq \frac{1}{4} \right\} \\ + P\left\{ \sup_{1 - 2\theta \leq t < 1} \frac{|\alpha_n(t)|}{q(1 - (1-t)/2)} \geq \frac{\varepsilon}{4} \right\}.$$

Thus, (4.4.8) follows by (3.4.5), (4.4.12)-(4.4.14), Remark 3.4.1 and Theorem 3.3.1.

This completes our proof. $\blacksquare$

REMARK 4.4.1 By (3.4.4)-(3.4.5) and Remark 3.4.1, one can replace the function $l(\cdot)$ in (4.4.7) and (4.4.8) by the function $(q^*(\cdot))^{-1}$.

4.5 Weak convergence for normed quantile processes of stationary associated sequences.

In Section 4.3, we have obtained the weak convergence of $u_n(\cdot)$. Thus, just as in Csörgő and Révész (1978), to get the weak convergence of the normed quantile process $\rho_n(\cdot)$, it is sufficient to show that ρ_n and u_n are close in probability. Indeed, Theorems 4.5.1 and 4.5.2 constitute an extension of Theorem 3 of Csörgő and Révész (1978) to stationary associated random variables.

THEOREM 4.5.1 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables with common continuous distribution function F which is also twice differentiable on (a_1, b_1) , where*

$$a_1 = \sup\{x : F(x) = 0\}, \quad b_1 = \inf\{x : F(x) = 1\}, \quad -\infty \leq a_1 < b_1 \leq +\infty,$$

and $F'(x) = f(x) > 0$ on (a_1, b_1) . Assume that for some $\nu > 0$

$$\sum_{n=1}^{\infty} n^{\frac{13}{2}+\nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

and for some $\gamma_1 > 0$

$$\sup_{a_1 < x < b_1} F(x)(1-F(x)) \frac{|f'(x)|}{f^2(x)} = \sup_{0 < y < 1} y(1-y) \frac{|f'(Q(y))|}{f^2(Q(y))} \leq \gamma_1.$$

Then we have

$$(4.5.1) \quad \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} |\rho_n(y) - u_n(y)| \xrightarrow{P} 0,$$

where ρ_n is defined in (4.1.7) and k_n in (4.4.2).

Proof. By (4.1.8), we can write

$$\rho_n(y) = u_n(y) + u_n(y)\varepsilon_n(y), \quad 0 < y < 1.$$

where

$$\varepsilon_n(y) = \frac{f(Q(y))}{f(Q(\theta_{y,n}))} - 1, \quad U_n(y) \wedge y < \theta_{y,n} < U_n(y) \vee y, \quad 0 < y < 1.$$

Hence (4.5.1) follows if we can show that for any $\varepsilon > 0$

$$(4.5.2) \quad \lim_{n \rightarrow \infty} P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} |\varepsilon_n(y)| > \varepsilon \right\} = 0$$

Let $\lambda = 1 + \varepsilon$. Then, by our assumptions and Lemma 1.4.1 in Csörgő (1983) (cf. Lemma 1 of Csörgő and Révész, 1978), we have

$$\begin{aligned} (4.5.3) \quad & P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \varepsilon_n(y) > \lambda - 1 \right\} \leq P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{f(Q(y))}{f(Q(\theta_{y,n}))} \geq \lambda \right\} \\ & \leq P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \left(\frac{U_n(y) \vee y}{U_n(y) \wedge y} \cdot \frac{1 - (U_n(y) \wedge y)}{1 - (U_n(y) \vee y)} \right)^{\gamma_1} \right\} \\ & \leq ([\gamma_1] + 1) \left(P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{U_n(y)}{y} \geq \lambda_1 \right\} + P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{1 - U_n(y)}{1 - y} \geq \lambda_1 \right\} \right) \\ & \quad + ([\gamma_1] + 1) \left(P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{y}{U_n(y)} \geq \lambda_1 \right\} + P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{1 - y}{1 - U_n(y)} \geq \lambda_1 \right\} \right) \\ & := ([\gamma_1] + 1)(I_1 + I_2 + I_3 + I_4), \end{aligned}$$

where $\lambda_1 = \lambda^{\frac{1}{2([\gamma_1] + 1)}} > 1$. Let $\eta = \frac{\lambda_1 - 1}{4} \wedge \frac{1}{8}$. Then, for any $\theta > 0$, by (4.4.5),

$$\begin{aligned} I_1 &= P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{U_n(y) - y}{y} \geq 4\eta \right\} \\ &\leq P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{n^{1/2}y} \geq 2\eta \right\} + P\left\{ \sup_{\theta \leq y < 1 - \frac{k_n}{n}} \frac{|u_n(y)|}{y} \geq 2\eta \right\} \\ &\leq P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q^*(y)} \geq \eta \right\} + P\left\{ \sup_{0 \leq y \leq 1} |u_n(y)| \geq 2\theta\eta n^{1/2} \right\}. \end{aligned}$$

Thus, by (4.1.12), (4.4.7), Remark 4.4.1 and Theorem 3.3.1, we prove that

$$(4.5.4) \quad \lim_{n \rightarrow \infty} I_1 = 0.$$

Similarly, by (4.4.5),

$$I_2 \leq P\left\{\sup_{1-\theta \leq y < 1 - \frac{k_n}{n}} \frac{|u_n(y)|}{q^*(y)} \geq \eta\right\} + P\left\{\sup_{0 \leq y \leq 1} |u_n(y)| \geq 2\theta\eta n^{1/2}\right\},$$

which implies

$$(4.5.5) \quad \lim_{n \rightarrow \infty} I_2 = 0.$$

For I_3 , we have

$$I_3 = P\left\{\sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{y - U_n(y)}{U_n(y)} \geq 4\eta\right\} \leq P\left\{\sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{|u_n(y)|}{n^{1/2}U_n(y)} \geq 4\eta\right\}.$$

On the other hand, if $\frac{|u_n(y)|}{n^{1/2}U_n(y)} \geq 4\eta$ for some $\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}$, then

$$\begin{aligned} |u_n(y)| &\geq 4\eta n^{1/2}(U_n(y) - y) + 4\eta n^{1/2}y \\ &\geq -4\eta |u_n(y)| + 4\eta n^{1/2} \\ &\geq -\frac{1}{2}|u_n(y)| + 4\eta n^{1/2}, \end{aligned}$$

which implies

$$|u_n(y)| \geq \frac{8\eta n^{1/2}y}{3} > 2\eta n^{1/2}y,$$

or equivalently

$$\frac{|u_n(y)|}{n^{1/2}y} > 2\eta.$$

Hence

$$I_3 \leq P\left\{\sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{|u_n(y)|}{n^{1/2}y} > 2\eta\right\}.$$

This proves

$$(4.5.6) \quad \lim_{n \rightarrow \infty} I_3 = 0.$$

Similarly,

$$I_4 \leq P\left\{\sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \frac{|u_n(y)|}{n^{1/2}(1-y)} > 2\eta\right\},$$

which implies

$$(4.5.7) \quad \lim_{n \rightarrow \infty} I_4 = 0.$$

By (4.5.3)-(4.5.7), it follows that

$$\lim_{n \rightarrow \infty} P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \varepsilon_n(y) > \varepsilon \right\} = 0.$$

To finish proving (4.5.2), we need to show that

$$(4.5.8) \quad \lim_{n \rightarrow \infty} P\left\{ \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} (-\varepsilon_n(y)) > \varepsilon \right\} = 0.$$

Note that

$$\sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \{(-\varepsilon_n(y)) > \varepsilon\} = \sup_{\frac{k_n}{n} \leq y < 1 - \frac{k_n}{n}} \left\{ \frac{f(Q(\theta_{y,n}))}{f(Q(y))} > \frac{1}{1 - \varepsilon} \right\}.$$

Hence we can prove (4.5.8) arguing as in (4.5.3), just by letting $\lambda = \frac{1}{1 - \varepsilon}$ instead of $\lambda = 1 + \varepsilon$. This proves our theorem. ■

In order to include the two end points when taking sup in (4.5.1), additional assumptions are needed. First we give the following lemma.

LEMMA 4.5.1 *Let $\{U_n, n \geq 0\}$ be a sequence of associated uniform-[0, 1] random variables. Then we have*

$$(4.5.9) \quad \lim_{n \rightarrow \infty} n(\log n)U_{1:n} \stackrel{P}{=} \infty$$

and

$$(4.5.10) \quad \lim_{n \rightarrow \infty} n(\log n)(1 - U_{n:n}) \stackrel{P}{=} \infty.$$

Proof. For any $M > 0$,

$$\begin{aligned} P\left\{U_{1:n} < \frac{M}{n \log n}\right\} &= 1 - P\left\{U_1 \geq \frac{M}{n \log n}, \dots, U_n \geq \frac{M}{n \log n}\right\} \\ &\leq 1 - \left(1 - \frac{M}{n \log n}\right)^n \\ &\leq \frac{M}{\log n}. \end{aligned}$$

This implies (4.5.9). (4.5.10) follows similarly. ■

THEOREM 4.5.2 *If, in addition to the assumptions of Theorem 4.5.1, we also assume that one of the following conditions holds*

- (a) $A \wedge B > 0$, where $A := \limsup_{x \downarrow a_1} f(x) < \infty$, $B := \limsup_{x \uparrow b_1} f(x) < \infty$;
 (b) if $A = 0$ (resp. $B = 0$), when $\gamma_1 < \frac{p-1}{p-2}$ with p defined by (3.4.2), then f is nondecreasing (resp. nonincreasing) on an interval to the right of a_1 (resp. to the left of b_1); when $\gamma_1 \geq \frac{p-1}{p-2}$, then $J(1/\alpha, \infty) < \infty$ for some $\alpha < 1/p$ (resp. $J(\infty, 1/\beta) < \infty$ for some $\beta < 1/p$), where

$$(4.5.11) \quad J(1/\alpha, 1/\beta) = \sup_{0 < y < 1} \frac{y^\alpha (1-y)^\beta}{f(Q(y))},$$

then

$$(4.5.12) \quad \sup_{0 < y < 1} |\rho_n(y) - u_n(y)| \xrightarrow{P} 0,$$

Proof. By Theorem 4.5.1, (4.5.12) follows if we can show that

$$(4.5.13) \quad \sup_{0 < y \leq \frac{k_n}{n}} |\rho_n(y)| \xrightarrow{P} 0,$$

$$(4.5.14) \quad \sup_{1 - \frac{k_n}{n} \leq y < 1} |\rho_n(y)| \xrightarrow{P} 0$$

and

$$(4.5.15) \quad \lim_{n \rightarrow \infty} \sup_{0 < y \leq \frac{k_n}{n}} |u_n(y)| = \lim_{n \rightarrow \infty} \sup_{1 - \frac{k_n}{n} \leq y < 1} |u_n(y)| = 0.$$

The proof of (4.5.15) follows immediately by applying (4.3.5), (4.4.4) and (3.4.4)-(3.4.5) with $l(t) \equiv 1$. Now we prove (4.5.13). For (4.5.14) a similar argument holds.

If $A > 0$, then for $0 < y \leq \frac{k_n}{n}$, by (4.1.8)

$$\rho_n(y) = \frac{f(Q(y))}{f(Q(\theta_{y,n}))} u_n(y), \quad |y - \theta_{y,n}| \leq n^{-1/2} |u_n(y)|.$$

Hence by (4.4.1) and (4.5.15),

$$\sup_{0 < y \leq \frac{k_n}{n}} |y - \theta_{y,n}| \xrightarrow{P} 0$$

which implies

$$\sup_{0 < y \leq \frac{k_n}{n}} \frac{f(Q(y))}{f(Q(\theta_{y,n}))} = O_P(1).$$

Thus, by (4.5.15),

$$\sup_{0 < y \leq \frac{k_n}{n}} |\rho_n(y)| = O_P \left(\sup_{0 < y \leq \frac{k_n}{n}} |u_n(y)| \right) \xrightarrow{P} 0,$$

and (4.5.13) is proved in this case.

If $A = 0$ and $\gamma_1 < \frac{p-1}{p-2}$, then for $\frac{i-1}{n} \leq y < \frac{i}{n}$ and $U_{i:n} \geq y$,

$$(4.5.16) \quad |\rho_n(y)| = n^{1/2} \int_y^{U_{i:n}} \frac{f(Q(y))}{f(Q(u))} du \leq n^{1/2}(U_{i:n} - y) = u_n(y),$$

since f is nondecreasing and for $U_{i:n} < y$, by Lemma 1.4.1 in Csörgö (1983),

$$(4.5.17) \quad |\rho_n(y)| = n^{1/2} \int_y^{U_{i:n}} \frac{f(Q(y))}{f(Q(u))} du \leq \begin{cases} \frac{2^{\gamma_1}}{1-\gamma_1} n^{1/2} y & \text{if } \gamma_1 < 1, \\ 2n^{1/2} y \log \frac{y}{U_{i:n}} & \text{if } \gamma_1 = 1, \\ \frac{2^{\gamma_1}}{\gamma_1-1} n^{1/2} y^{\gamma_1} U_{i:n}^{-(\gamma_1-1)} & \text{if } \gamma_1 > 1. \end{cases}$$

Hence (4.5.13) follows by applying (4.5.15) to (4.5.16) and Lemma 4.5.1 and (4.4.4) to (4.5.17).

If $A = 0$ and $\gamma_1 \geq \frac{p-1}{p-2}$, then using $J(1/\alpha, \infty) < \infty$, we have for $\frac{i-1}{n} \leq y < \frac{i}{n}$

$$\begin{aligned} |\rho_n(y)| &= n^{1/2} \left| \int_y^{U_{i:n}} \frac{f(Q(y))}{f(Q(u))} du \right| \\ &= O \left(n^{1/2} \left| \int_0^{U_{i:n} \vee y} \frac{1}{u^\alpha} du \right| \right) \\ &= O \left(n^{1/2} (U_{i:n} \vee y)^{1-\alpha} \right). \end{aligned}$$

In addition, for $0 < y \leq \frac{k_n}{n}$,

$$U_{i:n} \leq n^{-1/2} \sup_{0 < y \leq \frac{k_n}{n}} |u_n(y)| + y.$$

Hence

$$|\rho_n(y)| = O \left(\sup_{0 < y \leq \frac{1}{n}} |u_n(y)| \right) + O \left(n^{1/2} y^{1-\alpha} \right)$$

which proves (4.5.13) by (4.5.15) and (4.4.4). ■

Chapter 5

Integrals of Empirical Processes

This chapter establishes weak convergence for the integrals of empirical processes of associated sequences, with the help of *Demimartingales*, in Section 5.3. This result is then applied to obtain weak convergence for mean residual life processes of associated sequences in Section 5.4.

5.1 Definitions.

We define the integral Δ_n of the uniform empirical process α_n by

$$(5.1.1) \quad \Delta_n(u) = \int_0^u \alpha_n(v) dQ(v), \quad 0 \leq u \leq 1,$$

and its limit counterpart of Δ by

$$(5.1.2) \quad \Delta(u) = \int_0^u B(v) dQ(v), \quad 0 \leq u \leq 1.$$

We recall that for defining a mean residual process, we assume that $X_0 \geq 0$, i.e., $F(x) = 0$ for $x \leq 0$. As in Chapter 1, the mean residual life function at age x is defined by

$$(5.1.3) \quad M_F(x) = E(X_0 - x | X_0 > x) = \frac{1}{1 - F(x)} \int_x^\infty (1 - F(t)) dt,$$

and the empirical counterpart of M_F is

$$(5.1.4) \quad M_n(x) = M_{F_n}(x) = \frac{1}{1 - F_n(x)} \int_x^\infty (1 - F_n(t)) dt.$$

Then mean residual life process defined in (1.3.3) can be written as

$$(5.1.5) \quad \begin{aligned} z_n(x) &= n^{1/2}(M_n(x) - M_F(x)), \\ &= -(1 - F_n(x))^{-1} \int_{F(x)}^1 \alpha_n(y) dQ(y) \\ &\quad + (1 - F_n(x))^{-1} M_F(x) n^{1/2}(F_n(x) - F(x)), \quad 0 \leq x < \infty \end{aligned}$$

and, consequently,

$$(5.1.6) \quad \begin{aligned} z_n(Q(t)) &= -(1 - F_n(Q(t)))^{-1} \int_t^1 \alpha_n(y) dQ(y) \\ &\quad + (1 - F_n(Q(t)))^{-1} M_F(Q(t)) \alpha_n(t), \quad 0 \leq t \leq 1. \end{aligned}$$

5.2 Demimartingales.

We now give the definition of a demimartingale which was introduced by Newman and Wright (1982).

DEFINITION 5.2.1 (Newman and Wright, 1982) $\{S_n, n \geq 1\} \in L^1(\Omega, \mathcal{F}, P)$ is called a demi-(sub)martingale if whenever f is a (nonnegative) coordinatewise nondecreasing function, it follows that

$$Ef(S_1, \dots, S_n) \cdot (S_{n+1} - S_n) \geq 0, \text{ for all } n \geq 1.$$

More generally, Wood (1983) defined a demimartingale process as follows.

DEFINITION 5.2.2 (Wood, 1983) Let $\{S(t) : t \in [0, T]\}$, $0 < T < \infty$ be a stochastic process in L^1 . We say that $S(\cdot)$ is a demi-(sub)martingale if for any k and $\{t_n : n = 0, \dots, k\} \subset [0, T]$ with $0 = t_0 < t_1 < \dots < t_k = T$, $\{S(t_n) : n = 0, 1, \dots, k\}$ is a demi-(sub)martingale.

The following proposition is an immediate consequence of the definition of association.

PROPOSITION 5.2.1 *Suppose that $\{Y_n, n \geq 1\}$ is a sequence of L^1 , mean zero, associated random variables and $S_n = Y_1 + \dots + Y_n$ ($S_0 = 0$). Then $\{S_n, n \geq 0\}$ is a demimartingale.*

We present the following lemma which can be easily derived from Theorem 3 of Newman and Wright (1982).

LEMMA 5.2.1 *If $\{S(t) : 0 \leq t \leq T\}$ is a separable L^1 -demimartingale process, then for any $0 \leq \lambda_1 < \lambda_2$,*

$$P\left\{\sup_{0 \leq t \leq T} |S(t)| \geq \lambda_2\right\} \leq (\lambda_2 - \lambda_1)^{-1} E|S(T)| I(|S(T)| \geq \lambda_1).$$

5.3 Weak convergence for integrals of uniform empirical processes of stationary associated sequences.

In what follows we will use fact that

$$(5.3.1) \quad 0 \leq E\alpha_n(s)\alpha_n(t) \leq EB(s)B(t) = \sigma_{st}$$

for all $n \geq 1$, $0 \leq s, t \leq 1$, and $\sigma_t^2 = \sigma_{tt}$. This can be shown easily from the positive covariance property of association.

THEOREM 5.3.1 *Let $\{U_n, n \geq 1\}$ be a sequence of associated uniform-[0, 1] random variables. Then, under the assumptions of Theorem 3.3.1, if $\int_0^1 \sigma_u dQ(u) < \infty$,*

$$(5.3.2) \quad \Delta_n(\cdot) \xrightarrow{\mathcal{D}} \Delta(\cdot) \quad \text{in } D[0, 1].$$

Proof. By the Schwarz's inequality and (5.3.1), for $0 \leq u \leq 1$,

$$\begin{aligned} E\Delta_n^2(u) &= \int_0^u \int_0^u E\alpha_n(s)\alpha_n(t) dQ(s) dQ(t) \\ &\leq \int_0^u \int_0^u EB(s)B(t) dQ(s) dQ(t) \\ &= E\Delta^2(u) \leq \left(\int_0^u \sigma_v dQ(v)\right)^2 \\ &\leq \left(\int_0^1 \sigma_v dQ(v)\right)^2. \end{aligned}$$

This shows that $\{\Delta_n(u), n \geq 1\} \subset L^2$ and $\Delta(u) \in L^2$ for $0 \leq u \leq 1$.

For any $0 < \varepsilon < 1$, let

$$S(t) = \int_{1-\varepsilon}^t \alpha_n(u) dQ(u), \quad 1 - \varepsilon \leq t \leq 1.$$

Since $\alpha_n(u)$ is a nonincreasing function of $U_1, \dots, U_n$ for each u , $S(s)$ and $S(t) - S(s)$ are nonincreasing functions of $U_1, \dots, U_n$ for $1 - \varepsilon \leq s \leq t \leq 1$. Thus, by the property P_4 of association, it is easy to conclude that $\{S(t) : 1 - \varepsilon \leq t \leq 1\}$ is a L^2 -demimartingale process. For any $\lambda > 0$, let $\lambda_1 = 2^{-1}\lambda$ and $\lambda_2 = \lambda$. Then, by Lemma 5.2.1, we have

$$\begin{aligned} P\left\{\sup_{1-\varepsilon \leq t \leq 1} |S(t)| > \lambda\right\} &\leq 2\lambda^{-1}E|S(1)|I(|S(1)| \geq 2^{-1}\lambda) \\ &\leq 4\lambda^{-2}ES^2(1) \\ &\leq 4\lambda^{-2}\left(\int_{1-\varepsilon}^1 \sigma_u dQ(u)\right)^2. \end{aligned}$$

Therefore, for any $\lambda > 0$,

$$(5.3.3) \quad \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{\sup_{1-\varepsilon \leq t \leq 1} \left|\int_{1-\varepsilon}^t \alpha_n(u) dQ(u)\right| > \lambda\right\} = 0.$$

It is also easy to prove that

$$(5.3.4) \quad \lim_{\varepsilon \rightarrow 0} P\left\{\sup_{1-\varepsilon \leq t \leq 1} \left|\int_{1-\varepsilon}^t B(u) dQ(u)\right| > \lambda\right\} = 0$$

since $\{\int_{1-\varepsilon}^t B(u) dQ(u) : 1-\varepsilon \leq t \leq 1\}$ is also a L^2 -demimartingale process (cf. Pitt (1982)). Hence our theorem follows from Theorem 4.2 of Billingsley (1968). ■

Let

$$(5.3.5) \quad J^*(r) = \int_0^1 (1-u)^{1/r} dQ(u) = \int_0^\infty (1-F(x))^{1/r} dx \quad \text{for } r \geq 2.$$

REMARK 5.3.1 For the parameters ν_1 and p_1 (and then p) chosen in the proof of Theorem 3.4.1, it is not difficult to find that $J^*(2/\nu_1) < \infty$ implies $\int_0^1 \sigma_u dQ(u) < \infty$ by (3.4.9), and $E|X_0|^p < \infty$ implies $J^*(2/\nu_1) < \infty$ by (3.4.11).

Theorem 5.3.1 and Remark 5.3.1 will give us the following corollary.

COROLLARY 5.3.1 *Let $\{U_n, n \geq 1\}$ be a sequence of associated uniform-[0, 1] random variables. Then, under the assumptions of Theorem 3.3.1, if $J^*(2/\nu_1) < \infty$ for some $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$,*

$$\Delta_n(\cdot) \xrightarrow{\mathcal{D}} \Delta(\cdot) \quad \text{in } D[0, 1].$$

5.4 Convergence for mean residual life processes of associated sequences

In this section, we assume again that $X_0 \geq 0$, i.e., $F(x) = 0$ for $x \leq 0$. Let $T_F = \inf\{t : F(t) = 1\}$ and define the Gaussian process $Z(\cdot)$ by

$$\begin{aligned} Z(x) = & -(1-F(x))^{-1} \int_{F(x)}^1 B(y) dQ(y) \\ & + (1-F(x))^{-1} M_F(x) B(F(x)), \quad 0 \leq x < T_F, \end{aligned}$$

First we give the strong consistency of the empirical mean residual life function M_n .

THEOREM 5.4.1 *Let $\{X_n, n \geq 0\}$ be a sequence of associated random variables having the same distribution F . If F is continuous, $T < T_F$ and*

$$(5.4.1) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{i=1}^n X_i) < \infty,$$

then

$$(5.4.2) \quad \sup_{0 \leq x \leq T} |M_n(x) - M_F(x)| \xrightarrow{\text{a.s.}} 0.$$

Proof. Theorem 5.4.1 follows from Theorem 3.2.1 if we can show that

$$(5.4.3) \quad I_n = \sup_{0 \leq x < \infty} \left| \int_0^x (F_n(u) - F(u)) du \right| \xrightarrow{\text{a.s.}} 0.$$

By (5.4.1), we arrive at

$$EX_0 = \int_0^{\infty} (1 - F(u)) du < \infty.$$

Hence, for $\varepsilon > 0$ arbitrarily small, we can choose $\beta > 0$ so large that

$$I^{(1)}(\beta) = \int_{\beta}^{\infty} (1 - F(u)) du < \varepsilon/2.$$

Then

$$I_n \leq I^{(1)}(\beta) + I_n^{(2)}(\beta) + I_n^{(3)}(\beta),$$

where

$$I_n^{(2)}(\beta) = n^{-1} \sum_{i=1}^n \int_{\beta}^{\infty} I(X_i > u) du = n^{-1} \sum_{i=1}^n Y_i,$$

and

$$\begin{aligned} I_n^{(3)}(\beta) &= \sup_{0 \leq x \leq \beta} \left| \int_0^x (F_n(u) - F(u)) du \right| \\ &\leq \beta \sup_{0 \leq x < \infty} |F_n(x) - F(x)| \xrightarrow{\text{a.s.}} 0 \end{aligned}$$

by (5.4.1) and Theorem 3.2.1.

Note that $\int_{\beta}^{\infty} I(X_0 > u) du$ is an absolutely continuous and nondecreasing function of X_0 with $EY_0 = \int_{\beta}^{\infty} (1 - F(u)) du$. Hence $\{Y_n, n \geq 0\}$ is a sequence of associated random variables by the property P_4 of association and

$$\text{Cov}(Y_i, Y_j) \leq \text{Cov}(X_i, X_j) \text{ for all } i, j = 1, 2, \dots,$$

by Theorem 2.3.1. This shows that by (5.4.1)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(Y_n, \sum_{i=1}^n Y_i) < \infty.$$

Thus, applying Theorem 2.4.1, we get

$$I_n^{(2)}(\beta) \xrightarrow{\text{a.s.}} EY_0 = I^{(1)}.$$

Therefore $\limsup_{n \rightarrow \infty} I_n \leq \varepsilon$, a.s. for all small ε and this proves our theorem. $\blacksquare$

THEOREM 5.4.2 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables. Suppose that X_0 has a continuous distribution function F and there exists a positive constant ν such that*

$$(5.4.4) \quad \sum_{n=1}^{\infty} n^{\frac{13}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

Then,

(I) *if $J^*(2/\nu_1) < \infty$ and $T < T_F$ with J^* defined by (5.3.5) and $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$, we have*

$$z_n(\cdot) \xrightarrow{\mathcal{D}} Z(\cdot) \text{ in } D[0, T].$$

(II) *if $E|X_0|^p < \infty$, we have*

$$(1 - F_n(Q(\cdot))) \cdot z_n(Q(\cdot)) \xrightarrow{\mathcal{D}} (1 - F(Q(\cdot))) \cdot Z(Q(\cdot)) \text{ in } D[0, 1],$$

where p is defined by (3.4.2).

Proof. Part (I) follows from Theorem 3.2.1, Theorem 5.3.1, Remark 5.3.1, (5.1.5) and the continuity of $B(\cdot)$. Part (II) follows from Theorem 3.2.1, Theorem 5.3.1, Remark 5.3.1 and (5.1.6) except we have to verify that $E|X_0|^p < \infty$ implies

$$(5.4.5) \quad \int_0^1 (l(t))^p dt < \infty,$$

where $l(t) = M_F(Q(t)) + Q(t) = (1-t)^{-1} \int_t^1 Q(y) dy$. It is easy to check that $l(t)$ is a nondecreasing function on $(0, 1)$ (cf. Csörgő, Csörgő and Horváth (1986), p.40) and by Hardy's inequality (cf. Hardy, Littlewood and Pólya (1959), p.240), we have

$$\int_0^1 \left(\frac{1}{t} \int_0^t Q(1-y) dy \right)^p dt \leq \left(\frac{p}{p-1} \right)^p \int_0^1 Q^p(y) dy < \infty.$$

This shows that (5.4.5) is true. Hence our proof is now complete. ■

Chapter 6

Total Time on Test Processes

This chapter is devoted to prove strong consistency for the total time on test function of associated sequences, as well as weak convergence for total time on test processes of stationary associated sequences. The scaled version of total time on test processes and total time on test from the first failure of associated sequences are also discussed.

6.1 Definitions.

We recall the definition of the total time on test processes from Chapter 1. Let $X_1, \dots, X_n$ be any random variables with a common distribution function F . Introduce

$$W_{k:n} = (n + 1 - k)(X_{k:n} - X_{k-1:n}), \quad k = 1, \dots, n,$$

with $X_{0:n} \equiv 0$, where $X_{1:n} \leq \dots \leq X_{n:n}$ is the ordered sample. Then, the total time on test up to the k th order statistic, $T(X_{k:n})$, is defined by

$$T(X_{k:n}) = \sum_{i=1}^k W_{i:n} \quad \text{for } k = 1, \dots, n.$$

We define the n th total time on test function as

$$(6.1.1) \quad H_n^{-1}(t) = \frac{1}{n} T(X_{[nu]+1:n})$$

$$\begin{aligned}
&= \frac{1}{n} \sum_{i=1}^{[nu]+1} W_{i:n} \\
&= \frac{1}{n} \sum_{i=1}^{[nu]} X_{i:n} + \left(1 - \frac{[nu]}{n}\right) X_{[nu]+1:n}
\end{aligned}$$

for $0 \leq u < 1$ and

$$(6.1.2) \quad H_n^{-1}(1) = \lim_{u \uparrow 1} H_n^{-1}(u) = \frac{1}{n} \sum_{i=1}^n X_{i:n} = \bar{X}_n.$$

The theoretical counterpart of H_n^{-1} , the total time on test transform of F , is defined by

$$\begin{aligned}
(6.1.3) \quad H_F^{-1}(u) &= \int_0^u (1-y) dQ(y) + t_F \\
&= (1-u)Q(u) + \int_0^u Q(y) dy, \quad 0 \leq u \leq 1.
\end{aligned}$$

where

$$t_F = \sup\{t : F(t) = 0\}$$

is the lower endpoint of the support of F . In Sections 6.2, 6.3 and 6.4, we assume that $t_F = 0$ and in Section 6.6, we allow that $t_F > -\infty$.

Now we define the total time on test empirical process t_n by

$$(6.1.4) \quad t_n(u) = n^{1/2}(H_n^{-1}(u) - H_F^{-1}(u)), \quad 0 \leq u \leq 1$$

and its scaled version as

$$(6.1.5) \quad s_n(u) = n^{1/2}(D_n^{-1}(u) - D_F^{-1}(u)), \quad 0 \leq u \leq 1.$$

where

$$\mu = EX_1, \quad D_F^{-1}(u) = \frac{1}{\mu} H_F^{-1}(u), \quad D_n^{-1}(u) = \frac{1}{\bar{X}_n} H_n^{-1}(u).$$

To link H_n^{-1} with the uniform empirical and quantile functions, we have the following important conclusion, as given in (6.1) of Csörgö, Csörgö and Horváth (1986):

$$(6.1.6) \quad P\left\{\sup_{0 \leq y \leq 1} |H_n^{-1}(y) - \int_0^{U_n(y)} (1 - E_n(u)) dQ(u)| = 0\right\} = 1$$

for each n .

In the case of $t_F > -\infty$, we define the n th total time on test function from the first failure on as

$$N_n^1(u) = \begin{cases} 0 & , \quad 0 \leq u < 1/n, \\ \frac{1}{n} \sum_{k=2}^{[nu]+1} W_{k:n} & , \quad 1/n \leq u < 1 \\ \frac{1}{n} \sum_{k=2}^n W_{k:n} & , \quad u = 1, \end{cases}$$

the corresponding theoretical function by

$$N_F^1(u) = H_F^{-1}(u) - t_F = \int_0^u (1-y) dQ(y), \quad 0 \leq u \leq 1,$$

and let

$$(6.1.7) \quad \eta_n^1(u) = n^{1/2} \{N_n^1(u) - N_F^1(u)\}, \quad 0 \leq u \leq 1,$$

be the corresponding normalized process.

6.2 Strong consistency for the total time on test functions of associated sequences.

THEOREM 6.2.1 *Let $\{X_n, n \geq 0\}$ be a sequence of associated random variables with common continuous distribution function F . Assume that $Q = F^{-1}$ is continuous on $[0, 1)$ and*

$$(6.2.1) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{k=1}^n X_k) < \infty.$$

Then we have

$$(6.2.2) \quad \sup_{0 \leq y \leq 1} |H_n^{-1}(y) - H_F^{-1}(y)| \xrightarrow{\text{a.s.}} 0.$$

Proof. By (6.1.3) and (6.1.6), we have

$$(6.2.3) \quad \sup_{0 \leq y \leq 1} |H_n^{-1}(y) - H_F^{-1}(y)|$$

$$\begin{aligned}
&\leq \sup_{0 \leq y \leq 1} \left| \int_0^{U_n(y)} (1 - E_n(u)) dQ(u) - \int_0^{U_n(y)} (1 - u) dQ(u) \right| \\
&\quad + \sup_{0 \leq y \leq 1} \left| \int_0^{U_n(y)} (1 - u) dQ(u) - \int_0^y (1 - u) dQ(u) \right| \\
&\leq \sup_{0 \leq x < \infty} \left| \int_0^x (F_n(u) - F(u)) du \right| + \sup_{0 \leq y \leq 1} |H_F^{-1}(U_n(y)) - H_F^{-1}(y)|.
\end{aligned}$$

Applying (4.1.12) and Theorem 3.2.1, we obtain

$$\begin{aligned}
\sup_{0 \leq y \leq 1} |U_n(y) - y| &= \sup_{0 \leq y \leq 1} |y - E_n(y)| \\
&= \sup_{0 \leq x < \infty} |F_n(x) - F(x)| \xrightarrow{\text{a.s.}} 0.
\end{aligned}$$

Hence

$$\sup_{0 \leq y \leq 1} |H_F^{-1}(U_n(y)) - H_F^{-1}(y)| \xrightarrow{\text{a.s.}} 0$$

since the function H_F^{-1} is uniformly continuous on $[0, 1]$ and thus we can prove (6.2.2) by (6.2.3) and (5.4.3). ■

6.3 Weak convergence for total time on test processes of stationary associated sequences.

DEFINITION 6.3.1 A function $h > 0$ is said to be quasi-nonincreasing on $(0, a]$ for some $0 < a \leq 1/2$ if

$$c(h) = \inf_{0 < t \leq a} \left\{ \inf_{0 < s \leq t} \frac{h(s)}{h(t)} \right\} > 0.$$

Using this definition, for the p defined by (3.4.2), we can define a subset of $\mathbf{Q}^*$ as

$$(6.3.1) \quad \mathbf{Q}^{*p} = \{q \in \mathbf{Q}^* : q(t) = q(1-t) = t^{1/p}h(t) \text{ on } (0, a]\},$$

where q is quasi-nonincreasing on $(0, a]$ for some $0 < a \leq 1/2$ and

$$\int_0^1 q^{-p}(t) dt < \infty.$$

THEOREM 6.3.1 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables. Suppose that the density function $f = F'$ of X_0 is continuous and positive on the open support of F . If there exists a positive constant ν such that*

$$(6.3.2) \quad \sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(F(X_0), F(X_n)) < \infty,$$

$$(6.3.3) \quad J = \sup_{0 < u < 1} \frac{q(u)(1-u)}{f(Q(u))} < \infty$$

for some function $q \in \mathbf{Q}^p$ and $J^*(2/\nu_1) < \infty$ for some $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$, then

$$(6.3.4) \quad t_n(\cdot) \xrightarrow{\mathcal{D}} T(\cdot) \text{ in } D[0, 1],$$

where $\{T(y), 0 \leq y \leq 1\}$ is the Gaussian process defined by

$$(6.3.5) \quad T(y) = \int_0^y B(u) dQ(u) + \frac{1-y}{f(Q(y))} B(y), \quad 0 \leq y \leq 1.$$

Proof. By (6.1.6),

$$(6.3.6) \quad \begin{aligned} t_n(y) &= \int_0^{U_n(y)} \alpha_n(u) dQ(u) + n^{1/2} \{H_F^{-1}(U_n(y)) - H_F^{-1}(y)\} \\ &:= \{\Delta_n(U_n(y)) - \Delta_n(y)\} + \Delta_n(y) + I_n(y), \quad 0 \leq y \leq 1, \end{aligned}$$

where Δ_n is defined as in (5.1.1). On the other hand, $\Delta_n(\cdot) \in C[0, 1]$ since Q is continuous. Hence, for any $K > 0$ and $\varepsilon > 0$,

$$\begin{aligned} &P\left\{\sup_{0 \leq y \leq 1} |\Delta_n(U_n(y)) - \Delta_n(y)| > \varepsilon\right\} \\ &\leq P\left\{\sup_{0 \leq y \leq 1} |u_n(y)| > K\right\} + P\{w(\Delta_n, Kn^{-1/2}) \geq \varepsilon\}. \end{aligned}$$

Thus,

$$\limsup_{n \rightarrow \infty} P\left\{\sup_{0 \leq y \leq 1} |\Delta_n(U_n(y)) - \Delta_n(y)| > \varepsilon\right\} \leq P\left\{\sup_{0 \leq y \leq 1} |B(y)| > K\right\}$$

follows by (4.1.12), Theorem 3.3.1, Corollary 5.3.1 and Theorem 8.2 of Billingsley (1968). Letting $K \rightarrow \infty$, we get

$$(6.3.7) \quad \sup_{0 \leq y \leq 1} |\Delta_n(U_n(y)) - \Delta_n(y)| \xrightarrow{P} 0.$$

Applying a one term Taylor expansion and the fact that $\frac{dH_F^{-1}(y)}{dy} = \frac{1-y}{f(Q(y))}$, we obtain

$$(6.3.8) \quad \begin{aligned} I_n(y) &= \frac{1 - \tau_n(y)}{f(Q(\tau_n(y)))} u_n(y) \\ &= \left(\frac{1 - \tau_n(y)}{f(Q(\tau_n(y)))} - \frac{1 - y}{f(Q(y))} \right) u_n(y) + \frac{1 - y}{f(Q(y))} u_n(y), \end{aligned}$$

where, $y \wedge U_n(y) \leq \tau_n(y) \leq y \vee U_n(y)$. The latter relation implies that $\tau_n(y)$ converges a.s. uniformly to y on $[0, 1]$, since

$$\sup_{0 \leq y \leq 1} |U_n(y) - y| = \sup_{0 \leq y \leq 1} |y - E_n(y)| \xrightarrow{\text{a.s.}} 0$$

by (4.1.12), (6.3.2) and Theorem 3.2.1. For any $0 < \theta < 1/4$, since the compound function $f(Q(\cdot))$ is uniformly continuous on $[\theta, 1 - \theta]$ by assumption, and since $\sup_{0 \leq y \leq 1} |u_n(y)| = \sup_{0 \leq y \leq 1} |\zeta_n(y)|$ has a limit distribution, we have

$$\sup_{\theta \leq y \leq 1-\theta} \left| \frac{1 - \tau_n(y)}{f(Q(\tau_n(y)))} - \frac{1 - y}{f(Q(y))} \right| \sup_{0 \leq y \leq 1} |u_n(y)| \xrightarrow{\text{a.s.}} 0.$$

Hence, by (6.3.6)-(6.3.8), (4.3.5), Corollary 5.3.1 and Theorem 4.2 of Billingsley (1968), our theorem follows if we can show that for any $\varepsilon > 0$,

$$(6.3.9) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{0 < y \leq \theta} |I_n(y)| \geq \varepsilon \right\} = 0.$$

$$(6.3.10) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{1-\theta \leq y < 1} |I_n(y)| \geq \varepsilon \right\} = 0,$$

$$(6.3.11) \quad \lim_{\theta \rightarrow 0} P\left\{ \sup_{0 < y \leq \theta} \left| \frac{1 - y}{f(Q(y))} B(y) \right| \geq \varepsilon \right\} = 0,$$

and

$$(6.3.12) \quad \lim_{\theta \rightarrow 0} P\left\{ \sup_{1-\theta \leq y < 1} \left| \frac{1-y}{f(Q(y))} B(y) \right| \geq \varepsilon \right\} = 0.$$

(3.4.6)-(3.4.7) and (6.3.3) already implies (6.3.11) and (6.3.12). Therefore we only need to prove (6.3.9) and (6.3.10).

For $q \in \mathbf{Q}^{*p}$, by definition, we have $q(t) = t^{1/p}h(t)$ on $(0, a]$ for some quasi-nonincreasing function h and $0 < a \leq 1/2$. Next, we choose

$$\delta = \frac{c(h) \cdot (p-1)}{4pJ} \varepsilon$$

and define

$$(6.3.13) \quad k_n = k_n(\delta) = \max\{k: \frac{k}{n^{1/2}q(k/n)} \leq \delta \text{ and } k \text{ is an integer.}\}.$$

Just like the k_n defined in (4.4.2), we have $k_n \geq 1$ for n large enough and $\frac{k_n}{n} \rightarrow 0$ as $n \rightarrow \infty$. By the definitions of q and h , we have also for n large enough

$$(6.3.14) \quad \frac{\delta}{2} < \frac{k_n}{n^{1/2}q(k_n/n)} \leq \delta$$

and

$$(6.3.15) \quad \frac{\delta}{2} \leq \frac{y}{c(h) \cdot q(y)} n^{1/2}, \quad \text{for } \frac{k_n}{n} \leq y \leq a.$$

By (6.3.15),

$$\sup_{\frac{k_n}{n} \leq y \leq 2\theta} \frac{|\alpha_n(y)|}{q(y)} \leq \frac{\delta \cdot c(h)}{4}$$

implies

$$E_n(y) \geq y - \frac{\delta \cdot c(h) \cdot q(y)}{4n^{1/2}} \geq \frac{y}{2} \quad \text{for } \frac{k_n}{n} \leq y \leq 2\theta.$$

This means that, along the lines of the proof of Theorem 4.4.1, we can obtain that for any $\eta > 0$,

$$(6.3.16) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y)} \geq \eta \right\} = 0.$$

On the other hand, by (6.3.3) and (6.3.14),

$$\begin{aligned}
\sup_{0 < y \leq \theta} |I_n(y)| &= \sup_{0 < y \leq \theta} n^{1/2} \left| \int_y^{U_n(y)} \frac{1-u}{f(Q(u))} du \right| \\
&\leq \sup_{0 < y < \frac{k_n}{n}} |I_n(y)| + \sup_{\frac{k_n}{n} \leq y \leq \theta} |I_n(y)| \\
&\leq J \int_0^{\frac{k_n}{n}} \frac{n^{1/2}}{q(u)} du + J \int_0^{U_{k_n:n}} \frac{n^{1/2}}{q(u)} du + J \sup_{\frac{k_n}{n} \leq y \leq \theta} n^{1/2} \left| \int_y^{U_n(y)} \frac{1}{q(u)} du \right| \\
&\leq \frac{2Jn^{1/2}}{c(h) \cdot h(\frac{k_n}{n})} \int_0^{\frac{k_n}{n}} u^{-1/p} du + J \int_{\frac{k_n}{n}}^{U_{k_n:n}} \frac{n^{1/2}}{q(u)} \cdot I\{U_{k_n:n} \geq \frac{k_n}{n}\} du \\
&\quad + J \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y)} + J \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(U_n(y))} \\
&\leq \varepsilon/2 + 2J \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y)} + J \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(U_n(y))}.
\end{aligned}$$

Let $\eta = \frac{\delta \cdot c(h)}{4} = \frac{c^2(h) \cdot (p-1)}{16pJ} \varepsilon$. Then, by (6.3.15), $\sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y)} \leq \eta$ implies

$$U_n(y) \geq y - \eta q(y) n^{-1/2} \geq y/2 \quad \text{for } \frac{k_n}{n} \leq y \leq \theta.$$

Hence

$$\begin{aligned}
P\left\{ \sup_{0 < y \leq \theta} |I_n(y)| \geq \varepsilon \right\} &\leq P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y)} \geq \frac{\varepsilon}{8J} \right\} + P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(U_n(y))} \geq \frac{\varepsilon}{4J} \right\} \\
&\leq P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y)} \geq \frac{\varepsilon}{8J} \right\} + P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y)} \geq \eta \right\} \\
&\quad + P\left\{ \sup_{\frac{k_n}{n} \leq y \leq \theta} \frac{|u_n(y)|}{q(y/2)} \geq \frac{\varepsilon}{4J} \right\}.
\end{aligned}$$

and thus (6.3.9) follows from (6.3.16). The proof of (6.3.10) can be handled in a similar way. This proves our theorem. $\blacksquare$

Remark 5.3.1 and Theorem 6.3.1 immediately imply the following corollary.

COROLLARY 6.3.1 *If the condition $J^*(2/\nu_1) < \infty$ is replaced by $E|X_0|^p < \infty$, then Theorem 6.3.1 remains true.*

REMARK 6.3.1 It is possible to replace the function $q \in \mathbf{Q}^{*p}$ by the function q^* defined by (4.4.1), simply by using Remarks 3.4.1 and 4.4.1.

6.4 Convergence for scaled version of total time on test of associated sequences.

THEOREM 6.4.1 *Let $\{X_n, n \geq 0\}$ be a sequence of associated random variables with common continuous distribution function F . Assume that $Q = F^{-1}$ is continuous on $[0, 1)$ and that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{k=1}^n X_k) < \infty.$$

Then we have

$$\sup_{0 \leq y \leq 1} |D_n^{-1}(y) - D_F^{-1}(y)| \xrightarrow{\text{a.s.}} 0.$$

Proof. It follows easily from Theorem 2.4.1 and Theorem 6.2.1. ■

For studying the weak convergence of the scaled total time on test processes, we need the following mean zero Gaussian process

$$S(y) = \mu^{-1}T(y) - \mu^{-2}H_F^{-1}(y) \cdot T(1), \quad 0 \leq y \leq 1,$$

where $T(\cdot)$ is defined by (6.3.5).

THEOREM 6.4.2 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables. Suppose that the density function $f = F'$ of X_0 is continuous and positive on the open support of F . If there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{13}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty,$$

and

$$J = \sup_{0 < u < 1} \frac{q(u)(1-u)}{f(Q(u))} < \infty$$

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for some function $q \in \mathbf{Q}^{*p}$ and $J^*(2/\nu_1) < \infty$ for some $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$, then

$$s_n(\cdot) \xrightarrow{\mathcal{D}} S(\cdot) \text{ in } D[0, 1].$$

Proof. Using (6.1.2) and the fact that $\mu = H_F^{-1}(1)$ ($t_F = 0$ is assumed), we can easily derive that

$$\begin{aligned} s_n(y) = & \mu^{-1}t_n(y) - \mu^2 H_F^{-1}(y) \cdot t_n(1) + t_n(y) \left\{ \frac{1}{H_n^{-1}(1)} - \frac{1}{H_F^{-1}(1)} \right\} \\ & + \frac{H_F^{-1}(y)}{\mu} t_n(1) \left\{ \frac{1}{H_n^{-1}(1)} - \frac{1}{H_F^{-1}(1)} \right\}. \end{aligned}$$

Thus our theorem follows from Theorem 6.3.1 if we show that

$$(6.4.1) \quad H_n^{-1}(1) \xrightarrow{P} H_F^{-1}(1).$$

By Theorem 6.3.1, we have

$$t_n(1) = n^{1/2} \{ H_n^{-1}(1) - H_F^{-1}(1) \} \xrightarrow{\mathcal{D}} T(1).$$

This immediately implies (6.4.1). ■

6.5 Convergence for total time on test from the first failure of associated sequences.

In this section, we allow for the possibility that the lower end of the support of F , t_F , is not necessarily zero, but $t_F > \infty$.

THEOREM 6.5.1 *Let $\{X_n, n \geq 0\}$ be a sequence of associated random variables with common continuous distribution function F . Assume that $Q = F^{-1}$ is continuous on $[0, 1)$ and that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{k=1}^n X_k) < \infty.$$

Then we have

$$\sup_{0 \leq y \leq 1} |N_n^1(y) - N_F^1(y)| \xrightarrow{\text{a.s.}} 0.$$

Proof. By (6.1.6) and the definition of $N_n^1(\cdot)$, we have

$$\begin{aligned} \sup_{0 \leq y \leq 1} |N_n^1(y) - N_F^1(y)| &\leq \sup_{0 \leq y \leq 1} |H_n^{-1}(y) - H_F^{-1}(y)| + |X_{1:n} - t_F| \\ &\leq \sup_{0 \leq y \leq 1} |H_n^{-1}(y) - H_F^{-1}(y)| + |Q(U_{1:n}) - Q(0)|. \end{aligned}$$

On the other hand, by (4.1.12) and Theorem 3.2.1,

$$\begin{aligned} \sup_{0 \leq y \leq 1} |U_n(y) - y| &= \sup_{0 \leq y \leq 1} |y - E_n(y)| \\ &= \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \xrightarrow{\text{a.s.}} 0 \end{aligned}$$

implies $U_{1:n} \rightarrow 0$, a.s.. Hence our theorem follows from Theorem 6.2.1 and the fact that Q is continuous at 0. ■

THEOREM 6.5.2 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables. Suppose that the density function $f = F'$ of X_0 is continuous and positive on the open support of F . If there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{11}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty,$$

and

$$J = \sup_{0 < u < 1} \frac{q(u)(1-u)}{f(Q(u))} < \infty$$

for some function $q \in \mathbf{Q}^{*p}$ and $J^*(2/\nu_1) < \infty$ for some $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$, then

$$\eta_n^1(\cdot) \xrightarrow{\mathcal{D}} T(\cdot) \text{ in } D[0, 1].$$

Proof. Comparing $\eta_n^1(\cdot)$ with $t_n(\cdot)$, we just need to show that

$$(6.5.1) \quad n^{1/2} \{Q(U_{1:n}) - Q(0)\} \xrightarrow{P} 0.$$

As seen in the proof of Theorem 6.3.1, for any $\varepsilon > 0$ and n large enough, we have

$$\begin{aligned}
 n^{1/2}\{Q(U_{1:n}) - Q(0)\} &= n^{1/2} \int_0^{U_{1:n}} dQ(u) \\
 &\leq 2J n^{1/2} \int_0^{U_{1:n}} \frac{1}{q(u)} dQ(u) \\
 &\leq 2J \int_0^{\frac{k_n}{n}} \frac{n^{1/2}}{q(u)} dQ(u) + 2J \int_{\frac{k_n}{n}}^{U_{1:n}} \frac{n^{1/2}}{q(u)} I\{U_{k_n:n} \geq \frac{k_n}{n}\} dQ(u) \\
 &\leq \varepsilon + 2J \frac{|u_n(\frac{k_n}{n})|}{q(\frac{k_n}{n})}.
 \end{aligned}$$

This proves (6.5.1) by (6.3.16). ■

Chapter 7

Empirical Lorenz Processes

In this chapter, we prove the strong consistency for (unscaled) empirical Lorenz functions of associated sequences and the weak convergence for (unscaled) empirical Lorenz processes of stationary associated sequences as well.

7.1 Definitions.

In Chapter 1, we have defined the Lorenz curve of F as

$$(7.1.1) \quad L_F(y) = \frac{1}{\mu} \int_0^y Q(u) du, \quad 0 \leq y \leq 1$$

and the empirical counterpart of L_F as

$$(7.1.2) \quad L_n(y) = \begin{cases} \frac{1}{X_n} \frac{1}{n} \sum_{i=1}^{[ny]+1} X_{i:n} & , \quad 0 \leq y < 1, \\ 1 & , \quad y = 1. \end{cases}$$

The scaled empirical Lorenz process l_n is defined by

$$(7.1.3) \quad \ell_n(y) = n^{1/2}(L_n(y) - L_F(y)), \quad 0 \leq y \leq 1.$$

Introduce the unscaled Lorenz curve as

$$(7.1.4) \quad G_F(y) = \mu L_F(y) = \int_0^y Q(u) du, \quad 0 \leq y \leq 1$$

and the unscaled empirical Lorenz curve as

$$(7.1.5) \quad G_n(y) = \bar{X}_n L_n(y) = \begin{cases} \frac{1}{n} \sum_{i=1}^{[ny]+1} X_{i,n} & , \quad 0 \leq y < 1. \\ 1 & , \quad y = 1. \end{cases}$$

The unscaled empirical Lorenz process g_n is defined by

$$(7.1.6) \quad g_n(y) = n^{1/2}(G_n(y) - G_F(y)), \quad 0 \leq y \leq 1.$$

To link G_n with the uniform empirical and quantile functions, we have the following important conclusion, given as (10.1) by Csörgő, Csörgő and Horváth (1986),

$$(7.1.7) \quad P\left\{ \sup_{0 \leq y \leq 1} |G_n(y) - \int_0^{U_n(y)} Q(u-) dE_n(u)| = 0 \right\} = 1$$

for each n .

7.2 Strong consistency for (unscaled) empirical Lorenz functions of associated sequences.

We first give the strong consistency of the unscaled empirical Lorenz function G_n to G_F .

THEOREM 7.2.1 *Let $\{X_n, n \geq 0\}$ be a sequence of associated random variables with common continuous distribution function F . Assume that $Q = F^{-1}$ is continuous on $[0, 1)$ and that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{k=1}^n X_k) < \infty.$$

Then we have

$$(7.2.1) \quad \sup_{0 \leq y \leq 1} |G_n(y) - G_F(y)| \xrightarrow{\text{a.s.}} 0.$$

Proof. By (7.1.4) and (7.1.7), we have

$$(7.2.2) \quad \sup_{0 \leq y \leq 1} |G_n(y) - G_F(y)| \leq \sup_{0 \leq y \leq 1} \left| \int_0^{U_n(y)} Q(u) dE_n(u) - \int_0^{U_n(y)} Q(u) du \right| \\ + \sup_{0 \leq y \leq 1} |G_F(U_n(y)) - G_F(y)|.$$

Applying (4.1.12) and Theorem 3.2.1, we obtain

$$(7.2.3) \quad \sup_{0 \leq y \leq 1} |U_n(y) - y| = \sup_{0 \leq y \leq 1} |y - E_n(y)| \\ = \sup_{0 \leq x < \infty} |F_n(x) - F(x)| \xrightarrow{\text{a.s.}} 0.$$

Hence the continuity of G_F implies

$$(7.2.4) \quad \sup_{0 \leq y \leq 1} |G_F(U_n(y)) - G_F(y)| \xrightarrow{\text{a.s.}} 0.$$

By assumption, $EX_0^2 < \infty$. This implies

$$\mu = EX_0 = \int_0^1 Q(u) du < \infty.$$

Hence, for any $\varepsilon > 0$, there exists a $\beta \in (0, 1)$ so that

$$\int_{1-\beta}^1 Q(u) du < \varepsilon/2.$$

Now

$$(7.2.5) \quad \sup_{0 \leq y \leq 1} \left| \int_0^{U_n(y)} Q(u) dE_n(u) - \int_0^{U_n(y)} Q(u) du \right| \\ \leq \sup_{0 \leq y < 1-\beta} \left| \int_0^{U_n(y)} Q(u) dE_n(u) - \int_0^{U_n(y)} Q(u) du \right| \\ + \sup_{1-\beta \leq y \leq 1} \left| \int_0^{U_n(y)} Q(u) dE_n(u) - \int_0^{U_n(y)} Q(u) du \right| \\ \leq 2 \sup_{0 \leq y < 1-\beta} \left| \int_0^{U_n(y)} Q(u) d(1 - E_n(u)) - \int_0^{U_n(y)} Q(u) d(1 - u) \right| \\ + \int_{1-\beta}^1 Q(u) dE_n(u) + \int_{1-\beta}^1 Q(u) d(u)$$

$$\begin{aligned}
&\leq 2 \sup_{0 \leq y < 1-\beta} \left| \int_0^y (u - E_n(u)) dQ(u) \right| + 2Q(1-\beta) \sup_{0 \leq y \leq 1} |y - E_n(y)| \\
&\quad + \int_{1-\beta}^1 Q(u) du + \frac{1}{n} \sum_{k=1}^n Q(U_k) I\{U_k > 1-\beta\} \\
&\leq 4Q(1-\beta) \sup_{0 \leq y \leq 1} |y - E_n(y)| + \frac{\varepsilon}{2} + \frac{1}{n} \sum_{k=1}^n Q(U_k) I\{U_k > 1-\beta\}.
\end{aligned}$$

By the continuity of F and Q , we have

$$\begin{aligned}
(7.2.6) \quad &\frac{1}{n} \sum_{k=1}^n Q(U_k) I\{U_k > 1-\beta\} = \frac{1}{n} \sum_{k=1}^n X_k I\{X_k > Q(1-\beta)\} \\
&= \frac{1}{n} \sum_{k=1}^n (X_k I\{X_k > Q(1-\beta)\} + Q(1-\beta) I\{X_k \leq Q(1-\beta)\}) \\
&\quad - Q(1-\beta) F_n(Q(1-\beta)) \\
&= \frac{1}{n} \sum_{k=1}^n Y_k - Q(1-\beta) F_n(Q(1-\beta)),
\end{aligned}$$

where $Y_k = X_k I\{X_k > Q(1-\beta)\} + Q(1-\beta) I\{X_k \leq Q(1-\beta)\}$, $k = 1, \dots, n$. Note that $Y_1 = X_1 I\{X_1 > Q(1-\beta)\} + Q(1-\beta) I\{X_1 \leq Q(1-\beta)\}$ is an absolutely continuous and nondecreasing function of X_1 with

$$EY_1 = Q(1-\beta)F(Q(1-\beta)) + \int_{1-\beta}^1 Q(u) du.$$

Hence $\{Y_n, n \geq 1\}$ is a sequence of associated random variables by the property P_4 of association and

$$\text{Cov}(Y_i, Y_j) \leq \text{Cov}(X_i, X_j), \quad \text{for all } i, j = 1, 2, \dots$$

by Theorem 2.3.1. This shows by assumption that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(Y_n, \sum_{k=1}^n Y_k) < \infty.$$

Thus, applying (7.2.3) and Theorem 2.4.1 to (7.2.6), we obtain

$$\frac{1}{n} \sum_{k=1}^n Q(U_k) I\{U_k > 1-\beta\} \xrightarrow{\text{a.s.}} \int_{1-\beta}^1 Q(u) du.$$

This, together with (7.2.3) and (7.2.5), shows that

$$\limsup_{n \rightarrow \infty} \sup_{0 \leq y \leq 1} \left| \int_0^{U_n(y)} Q(u) dE_n(u) - \int_0^{U_n(y)} Q(u) du \right| \leq \varepsilon \quad \text{a.s.}$$

for all $\varepsilon > 0$ and thus (7.2.1) follows by (7.2.2) and (7.2.4). This completes our proof. ■

Applying Theorem 2.4.1, we get the following strong consistency of the scaled empirical Lorenz function L_n to L_F .

THEOREM 7.2.2 *Under the assumptions of Theorem 7.2.1, we have*

$$\sup_{0 \leq y \leq 1} |L_n(y) - L_F(y)| \xrightarrow{\text{a.s.}} 0.$$

7.3 Weak convergence for (unscaled) empirical Lorenz processes of stationary associated sequences.

THEOREM 7.3.1 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables with common continuous distribution function F . Assume that there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

If Q is continuous on $[0, 1)$ and there exists $q \in \mathcal{Q}^p$ so that $Q(y) \leq (q(y))^{-1}$ for all $y \in (0, 1)$, then

$$(7.3.1) \quad g_n(\cdot) \xrightarrow{\mathcal{D}} \Delta(\cdot) \text{ in } D[0, 1],$$

where $\Delta(\cdot)$ is the zero-mean Gaussian process defined by (5.1.2).

Proof. We have by (7.1.7) and integrating by parts,

$$\begin{aligned}
 (7.3.2) \quad g_n(y) &= - \int_0^y Q(u) d\alpha_n(u) + n^{1/2} \int_y^{U_n(y)} Q(u) dE_n(u) \\
 &= \int_0^y \alpha_n(u) dQ(u) + Q(y)(\alpha_n(y) - \alpha_n(y-)) \\
 &\quad - \int_y^{U_n(y)} Q(u) d\alpha_n(u) + \{n^{1/2} \int_y^{U_n(y)} Q(u) du - Q(y)\alpha_n(y)\} \\
 &:= \Delta_n(y) + I_{1,n}(y) + I_{2,n}(y) + I_{3,n}(y),
 \end{aligned}$$

where $\Delta_n(\cdot)$ is the same integral as that of (5.1.1).

We first prove that for any $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} P\left\{ \sup_{0 < y < 1} |I_{1,n}(y)| > \varepsilon \right\} = 0,$$

via proving the more general statement

$$(7.3.3) \quad \lim_{n \rightarrow \infty} P\left\{ \sup_{0 < y < 1} \frac{|\alpha_n(y) - \alpha_n(y-)|}{q(y)} > \varepsilon \right\} = 0,$$

for $q \in Q^*$ with $\int_0^1 q^{-p}(t) dt < \infty$. It is easy to check that

$$|\alpha_n(y) - \alpha_n(y-)| \leq n^{-1/2}, \quad 0 < y < 1.$$

Hence, for any $0 < \theta < 1/2$ and n large enough,

$$\begin{aligned}
 (7.3.4) \quad &P\left\{ \sup_{0 < y < 1} \frac{|\alpha_n(y) - \alpha_n(y-)|}{q(y)} > \varepsilon \right\} \\
 &\leq P\left\{ \sup_{0 < y < \theta} \frac{|\alpha_n(y) - \alpha_n(y-)|}{q(y)} > \frac{\varepsilon}{4} \right\} + P\left\{ \sup_{1-\theta < y < 1} \frac{|\alpha_n(y) - \alpha_n(y-)|}{q(y)} > \frac{\varepsilon}{4} \right\} \\
 &\leq P\left\{ \sup_{0 < y \leq \theta} \frac{|\alpha_n(y)|}{q(y)} > \frac{\varepsilon}{8} \right\} + P\left\{ \sup_{1-\theta \leq y < 1} \frac{|\alpha_n(y)|}{q(y)} > \frac{\varepsilon}{8} \right\}.
 \end{aligned}$$

Thus, (7.3.3) follows by (3.4.4) and (3.4.5).

Next, integrating by parts again, we have

$$\begin{aligned}
 (7.3.5) \quad I_{2,n}(y) &= \{\Delta_n(U_n(y)) - \Delta_n(y)\} \\
 &\quad + \{Q(y)\alpha_n(y^*) - Q(U_n(y))\alpha_n(U_n^*(y))\}, \quad 0 < y < 1,
 \end{aligned}$$

where y^* (resp. $U_n^*(y)$) may take the values y or $y-$ (resp. $U_n(y)$ or $U_n(y-)$). Applying (7.3.3) to (7.3.5), we will get

$$(7.3.6) \quad \lim_{n \rightarrow \infty} \sup_{0 < y < 1} |I_{2,n}(y)| \stackrel{P}{\rightarrow} 0$$

if we can show that

$$(7.3.7) \quad \sup_{0 \leq y \leq 1} |\Delta_n(U_n(y)) - \Delta_n(y)| \stackrel{P}{\rightarrow} 0$$

and

$$(7.3.8) \quad \sup_{0 < y < 1} |Q(y)\alpha_n(y) - Q(U_n(y))\alpha_n(U_n(y))| \stackrel{P}{\rightarrow} 0.$$

Note that (7.3.7) is the same as (6.3.7). To apply (6.3.7), we have to have $J^*(2/\nu_1) < \infty$ for some $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$. By assumption, $Q(y) \leq (q(y))^{-1}$ will imply $E|X_0|^p < \infty$. This, by Remark 5.3.1, implies also $J^*(2/\nu_1) < \infty$. Hence, (7.3.7) is proved.

For (7.3.8), we have for any $0 < \theta < 1/2$,

$$\begin{aligned} & \sup_{0 < y < 1} |Q(y)\alpha_n(y) - Q(U_n(y))\alpha_n(U_n(y))| \\ & \leq \sup_{0 < y \leq 1-\theta} |Q(y) - Q(U_n(y))| \cdot |\alpha_n(y)| \\ & \quad + \sup_{0 < y \leq 1-\theta} |\alpha_n(y) - \alpha_n(U_n(y))| \cdot |Q(U_n(y))| \\ & \quad + \sup_{1-\theta < y < 1} |Q(y)\alpha_n(y)| + \sup_{1-\theta < y < 1} |Q(U_n(y))\alpha_n(U_n(y))|. \end{aligned}$$

Thus, by the uniform continuity of Q on $[0, 1 - \theta/2]$, Theorem 3.3.1 and (4.1.12), for any $\varepsilon > 0$, $K > 0$, and n large enough, we have

$$\begin{aligned} P\left\{ \sup_{0 < y < 1} |Q(y)\alpha_n(y) - Q(U_n(y))\alpha_n(U_n(y))| > \varepsilon \right\} & \leq P\left\{ \sup_{0 \leq y \leq 1} |u_n(y)| > K \right\} \\ & + P\{Q(1 - \theta/2) \cdot w(\alpha_n, Kn^{-1/2}) \geq \varepsilon/4\} + P\left\{ \sup_{1-2\theta \leq y < 1} |Q(y)\alpha_n(y)| \geq \varepsilon/4 \right\}. \end{aligned}$$

This proves (7.3.8) by (3.4.5), (4.1.12) and Theorem 3.3.1. Hence (7.3.6) holds true.

For the term $I_{3,n}$, we have for any $0 < \theta < 1/2$,

$$\begin{aligned} \sup_{0 < y < 1} |I_{3,n}(y)| &\leq \sup_{0 < y \leq 1-\theta} |I_{3,n}(y)| + \sup_{1-\theta < y < 1} |I_{3,n}(y)| \\ &\leq \sup_{0 < y \leq 1-\theta} \left(\sup_{y \wedge U_n(y) \leq x \leq y \vee U_n(y)} |Q(y) - Q(x)| \right) \cdot \sup_{0 \leq y \leq 1} |u_n(y)| \\ &\quad + Q(1-\theta) \cdot \sup_{0 \leq y \leq 1} |\alpha_n(y) - U_n(y)| + \sup_{1-\theta < y < 1} |Q(y)\alpha_n(y)| \\ &\quad + \sup_{1-\theta < y < 1} n^{1/2} \left| \int_y^{U_n(y)} Q(u) du \right|. \end{aligned}$$

By the uniform continuity of Q on $[0, 1 - \theta/2]$, Theorem 3.3.1, (3.4.5), (4.1.12) and (4.3.5),

$$(7.3.9) \quad \lim_{n \rightarrow \infty} \sup_{0 < y < 1} |I_{3,n}(y)| \stackrel{P}{\equiv} 0$$

follows from

$$\lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P \left\{ \sup_{1-\theta < y < 1} n^{1/2} \left| \int_y^{U_n(y)} Q(u) du \right| > \varepsilon \right\} = 0,$$

for any $\varepsilon > 0$. The above can be proved by using the same technique we exploited in proving Theorem 6.3.1, since we assume that $Q(\cdot) \leq (q(\cdot))^{-1}$ for some $q \in \mathbf{Q}^p$. Now (7.3.1) follows by (7.3.2)-(7.3.3), (7.3.6), (7.3.9) and Corollary 5.3.1. This completes our proof. ■

REMARK 7.3.1 Based on Remark 6.3.1, one can replace the function q in Theorem 7.3.1 by the function q^* defined by (4.4.1).

COROLLARY 7.3.1 *If the condition $Q(\cdot) \leq (q(\cdot))^{-1}$ for some $q \in \mathbf{Q}^p$ is replaced by*

$$(7.3.10) \quad E|X_0|^p (\log(1 + |X_0|))^{-\frac{(p-1)(1+\gamma)}{4}} < \infty, \text{ for some } \gamma > 0,$$

then Theorem 7.3.1 remains true.

Proof. By (7.3.10), we have

$$\int_0^1 Q^p(y)(\log(1 + Q(y)))^{\frac{(p-1)(1+\gamma)}{4}} dy < \infty.$$

Thus, by the monotonicity of $Q(y)$, there exists $0 < b < 1$ so that

$$Q^p(y)(\log(1 + Q(y)))^{\frac{(p-1)(1+\gamma)}{4}} \leq \frac{1}{1-y}, \quad y \in (1-b, 1).$$

This will give us

$$Q(y) \leq C(1-y)^{-\frac{1}{p}} \left(\log \frac{1}{1-y} \right)^{\frac{(p-1)(1+\gamma)}{4}}$$

for $y \in (1-b, 1)$ and some $C > 0$. Hence our corollary follows from Remark 7.3.1 and Theorem 7.3.1. ■

Concerning the weak convergence of the scaled Lorenz processes, we have the following theorem.

THEOREM 7.3.2 *Under the assumptions of Theorem 7.3.1,*

$$\ell_n(\cdot) \xrightarrow{\mathcal{D}} \Lambda(\cdot) \text{ in } D[0, 1],$$

where $\Lambda(\cdot)$ is the zero-mean Gaussian process defined by

$$\Lambda(y) = \mu^{-1} \{ \Delta(y) - L_F(y) \Delta(1) \}, \quad 0 \leq y \leq 1.$$

Proof. By some calculations,

$$\begin{aligned} \ell_n(y) &= \mu^{-1} g_n(y) - \mu^{-2} G_F(y) \cdot g_n(1) \\ &\quad + \{ g_n(y) - \frac{G_F(y)}{\mu} g_n(1) \} \cdot \left\{ \frac{1}{G_n(1)} - \frac{1}{G_F(1)} \right\}. \end{aligned}$$

Thus, our theorem follows by Theorem 7.3.1 and

$$G_n(1) \xrightarrow{P} G_F(1),$$

which, in turn, follows by

$$g_n(1) = n^{1/2}(G_n(1) - G_F(1)) \xrightarrow{\mathcal{D}} \Delta(1)$$

of Theorem 7.3.1. The proof is now complete. ■

Theorem 7.3.2 and Corollary 7.3.1 yield the following result.

COROLLARY 7.3.2 *If the condition $Q(\cdot) \leq (q(\cdot))^{-1}$ for some $q \in \mathbf{Q}^{*p}$ is replaced by (7.3.10), then Theorem 7.3.2 remains true.*

Chapter 8

Empirical Concentration Processes

In this chapter, we prove the strong consistency for empirical concentration functions of associated sequences as well as weak convergence for empirical concentration processes of stationary associated sequences.

8.1 Definitions.

In Chapter 7, we studied the strong consistency of L_n to L_F and the weak convergence for the empirical Lorenz process ℓ_n of associated sequences. Since the Lorenz function L_F is continuous and strictly increasing on $[0, 1]$, it has a well-defined continuous and strictly increasing inverse function L_F^{-1} on $[0, 1]$. We call this inverse the concentration curve pertaining to F . Naturally, the inverse empirical Lorenz function can be defined as

$$(8.1.1) \quad L_n^{-1}(y) = \inf\{x : L_n(x) > y\},$$

or in more detail as

$$L_n^{-1}(y) = \begin{cases} 0 & , \text{ if } 0 \leq y \leq (n\bar{X}_n)^{-1}X_{1:n}, \\ \frac{k-1}{n} & , \text{ if } (n\bar{X}_n)^{-1}\sum_{i=1}^{k-1}X_{i:n} \leq y < (n\bar{X}_n)^{-1}\sum_{i=1}^kX_{i:n}, \quad 2 \leq k \leq n, \\ 1 & , \text{ if } y = 1. \end{cases}$$

Introducing

$$(8.1.2) \quad C_n(y) = (\bar{X}_n)^{-1} \int_0^y Q(u-) dE_n(u),$$

we have by (7.1.5) and (7.1.7),

$$(8.1.3) \quad P\left\{ \sup_{0 \leq y \leq 1} |L_n(y) - C_n(U_n(y))| = 0 \right\} = 1,$$

while

$$(8.1.4) \quad P\left\{ \sup_{0 \leq y \leq 1} |L_n^{-1}(y) - E_n(C_n^{-1}(y))| = 0 \right\} = 1,$$

since the inverse function to the compound function $C_n(U_n(y))$ is $E_n(C_n^{-1}(y))$.

The (Goldie) concentration process is defined by

$$(8.1.5) \quad c_n(y) = n^{1/2}(L_n^{-1}(y) - L_F^{-1}(y)), \quad 0 \leq y \leq 1.$$

8.2 Strong consistency for empirical concentration functions of associated sequences.

THEOREM 8.2.1 *Let $\{X_n, n \geq 0\}$ be a sequence of associated random variables with common continuous distribution function F . Assume that $Q = F^{-1}$ is continuous on $[0, 1)$ and that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{k=1}^n X_k) < \infty.$$

Then we have

$$\sup_{0 \leq y \leq 1} |L_n^{-1}(y) - L_F^{-1}(y)| \xrightarrow{\text{a.s.}} 0.$$

Proof. By applying Theorem 7.2.2, the proof is the same as that of Theorem 13.2 of Csörgő, Csörgő and Horváth (1986). Hence the details are omitted. ■

8.3 Weak convergence for empirical concentration processes of stationary associated sequences.

For the proof of weak convergence of the empirical concentration process $c_n(\cdot)$, we need several preliminary lemmas.

LEMMA 8.3.1 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables with common continuous distribution function F . Assume that Q is continuous on $[0, 1)$ and there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{13}{2}+\nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

If $J^(2/\nu_1) < \infty$ for some $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$, then*

$$(8.3.1) \quad n^{1/2}(\bar{X}_n - \mu) \xrightarrow{\mathcal{D}} \Delta(1).$$

Proof. Since $J^*(2/\nu_1) < \infty$ implies $EX_0 = \int_0^1 Q(y) dy < \infty$, for each $n \geq 2$, there exists $\varepsilon_n > 0$ so that

$$(8.3.2) \quad 0 < \varepsilon_n \leq \frac{1}{n \log n}, \quad n^{1/2} \int_{1-\varepsilon_n}^1 Q(y) dy < \frac{1}{n}.$$

Let $\eta_n = - \int_0^{1-\varepsilon_n} Q(y) d\alpha_n(y)$. Then, by integrating by parts, we have

$$\eta_n = -Q(1 - \varepsilon_n)\alpha_n((1 - \varepsilon_n)-) + \int_0^{1-\varepsilon_n} \alpha_n(y) dQ(y).$$

Hence

$$(8.3.3) \quad \Delta_n(1) - \eta_n = Q(1 - \varepsilon_n)\alpha_n((1 - \varepsilon_n)-) + \int_{1-\varepsilon_n}^1 \alpha_n(y) dQ(y).$$

On the other hand, if $1 - U_{n:n} > \varepsilon_n$, then it is easy to find that

$$(8.3.4) \quad \eta_n = n^{1/2}(\bar{X}_n - \mu) + n^{1/2} \int_{1-\varepsilon_n}^1 Q(y) dy$$

and $\alpha_n((1 - \varepsilon_n)-) = \alpha_n(1 - \varepsilon_n)$. Thus for any $\varepsilon > 0$ and $n > 2/\varepsilon$, by (8.3.2)-(8.3.4),

$$(8.3.5) \quad P\{|\Delta_n(1) - n^{1/2}(\bar{X}_n - \mu)| > \varepsilon\} \leq P\{1 - U_{n:n} \leq \varepsilon_n\} \\ + P\{Q(1 - \varepsilon_n) \cdot \alpha_n(1 - \varepsilon_n) > \varepsilon/4\} + P\{|\Delta_n(1) - \Delta_n(1 - \varepsilon_n)| > \varepsilon/4\}.$$

Since $J^*(2/\nu_1) < \infty$ implies $Q(1 - \varepsilon_n) \cdot \varepsilon_n^{\nu_1/2} \rightarrow 0$ as $n \rightarrow \infty$ and

$$(8.3.6) \quad P\{Q(1 - \varepsilon_n) \cdot \alpha_n(1 - \varepsilon_n) > \varepsilon/4\} \leq \frac{16}{\varepsilon^2} Q^2(1 - \varepsilon_n) E \alpha_n^2(1 - \varepsilon_n) \\ \leq C Q^2(1 - \varepsilon_n) \cdot \varepsilon_n^{\nu_1} \rightarrow 0$$

by (3.4.9), we have by (5.3.3), (8.3.2) and Lemma 4.5.1,

$$\lim_{n \rightarrow \infty} P\{|\Delta_n(1) - n^{1/2}(\bar{X}_n - \mu)| > \varepsilon\} = 0.$$

This with Corollary 5.3.1 proves (8.3.1) and hence our lemma. $\blacksquare$

From Lemma 8.3.1, it is easy to derive the following result.

COROLLARY 8.3.1 *Under the assumptions of Lemma 8.3.1, we have*

$$\eta_n - n^{1/2}(\bar{X}_n - \mu) \xrightarrow{P} 0$$

and

$$\bar{X}_n \xrightarrow{P} \mu.$$

Introduce the following process

$$(8.3.7) \quad \ell_n^*(y) = n^{1/2}(C_n(y) - L_F(y)), \quad 0 \leq y \leq 1.$$

Then

$$(8.3.8) \quad \ell_n^*(y) = (\bar{X}_n)^{-1} \left\{ L_F(y) \cdot n^{1/2}(\mu - \bar{X}_n) - \int_0^y Q(u) d\alpha_n(u) \right\}, \quad 0 \leq y \leq 1.$$

Let

$$(8.3.9) \quad \tilde{\ell}_n(y) = (\bar{X}_n)^{-1} \left\{ L_F(y) \cdot \int_0^{1-\varepsilon_n} Q(u) d\alpha_n(u) - \int_0^y Q(u) d\alpha_n(u) \right\}, \quad 0 \leq y \leq 1.$$

LEMMA 8.3.2 *Under the assumptions of Lemma 8.3.1, we have for any $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} P \left\{ \sup_{0 \leq y \leq 1} \frac{|\tilde{\ell}_n(y) - \ell_n^*(y)|}{Q(y)} > \varepsilon \right\} = 0.$$

Proof. By the monotonicity of $Q(\cdot)$,

$$(8.3.10) \quad \mu \cdot L_F(y) \leq Q(y) \cdot y, \quad 0 \leq y \leq 1.$$

Hence by (8.3.8) and (8.3.9), we obtain

$$\sup_{0 \leq y \leq 1} \frac{|\tilde{\ell}_n(y) - \ell_n^*(y)|}{Q(y)} \leq \frac{1}{\mu \bar{X}_n} |\eta_n - n^{1/2}(\bar{X}_n - \mu)|.$$

Thus our lemma follows by Corollary 8.3.1. $\blacksquare$

Define the zero-mean Gaussian process $\{\Gamma^*(y), 0 \leq y \leq 1\}$ as

$$(8.3.11) \quad \Gamma^*(y) = \mu^{-1} \left\{ \int_0^y B(u) dQ(u) - B(y)Q(y) - L_F(y) \int_0^1 B(u) dQ(u) \right\}.$$

LEMMA 8.3.3 *Under the assumptions of Lemma 8.3.1, we have*

$$(8.3.12) \quad \frac{\mu}{Q(\cdot)} \ell_n^*(\cdot) + \alpha_n(\cdot) \xrightarrow{\mathcal{D}} \frac{\mu}{Q(\cdot)} \Gamma^*(\cdot) + B(\cdot) \text{ in } D[0, 1].$$

Proof. By integrating by parts, we have

$$\begin{aligned} \tilde{\ell}_n(y) &= (\bar{X}_n)^{-1} \left\{ \int_0^y \alpha_n(u) dQ(u) - \alpha_n(y)Q(y) - L_F(y) \int_0^{1-\varepsilon_n} \alpha_n(u) dQ(u) \right\} \\ &\quad + (\bar{X}_n)^{-1} Q(y)(\alpha_n(y) - \alpha(y-)) + (\bar{X}_n)^{-1} L_F(y) Q(1 - \varepsilon_n) \alpha_n((1 - \varepsilon_n)-) \\ &:= I_{1,n}(y) + I_{2,n}(y) + I_{3,n}(y). \end{aligned}$$

By applying (7.3.3), (8.3.6), (8.3.10) and Corollary 8.3.1 to $I_{2,n}$ and $I_{3,n}$, it is easy to get

$$\sup_{0 < y < 1} \frac{|I_{i,n}(y)|}{Q(y)} \xrightarrow{P} 0, \quad i = 2, 3.$$

So, by Lemma 8.3.2, the proof of (8.3.12) is reduced to showing that

$$\frac{\mu}{Q(\cdot)} I_{1,n}(\cdot) + \alpha_n(\cdot) \xrightarrow{\mathcal{D}} \frac{\mu}{Q(\cdot)} \Gamma^*(\cdot) + B(\cdot) \text{ in } D[0, 1].$$

This can be accomplished by applying Theorems 3.3.1 and 5.3.1, Corollary 8.3.1 and Theorem 4.1 of Billingsley (1968), except that we have to show that for any $\varepsilon > 0$

$$(8.3.13) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P \left\{ \sup_{0 < y \leq \theta} \frac{|I_{1,n}(y)|}{Q(y)} > \varepsilon \right\} = 0$$

and

$$(8.3.14) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P \left\{ \sup_{0 < y \leq \theta} \frac{|\Gamma^*(y)|}{Q(y)} > \varepsilon \right\} = 0.$$

Since for $0 < \theta < 1/2$,

$$\sup_{0 < y \leq \theta} \frac{|\int_0^y \alpha_n(u) dQ(u)|}{Q(y)} \leq \sup_{0 < y \leq \theta} |\alpha_n(y)|$$

and

$$\sup_{0 < y \leq \theta} \frac{|\int_0^y B(u) dQ(u)|}{Q(y)} \leq \sup_{0 < y \leq \theta} |B(y)|,$$

(8.3.13) and (8.3.14) follow by (3.4.4)-(3.4.6), (8.3.10) and Corollary 8.3.1. This completes our proof. ■

LEMMA 8.3.4 *Under the assumptions of Lemma 8.3.1, if $q \in \mathcal{Q}^*$ with $\int_0^1 q^{-p}(t) dt < \infty$, then for any $\varepsilon > 0$,*

$$\lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P \left\{ \sup_{0 < y \leq \theta} \frac{|\ell_n^*(y)|}{Q(y) \cdot q(y)} > \varepsilon \right\} = 0.$$

Proof. By (8.3.10) and integrating by parts,

$$\begin{aligned} \sup_{0 < y \leq \theta} \frac{|\ell_n^*(y)|}{Q(y) \cdot q(y)} &\leq \frac{1}{\mu \bar{X}_n} |n^{1/2}(\mu - \bar{X}_n)| \cdot \sup_{0 < y \leq \theta} \frac{y}{q(y)} \\ &\quad + \frac{1}{\bar{X}_n} \sup_{0 < y \leq \theta} \frac{|\alpha_n(y) - \alpha_n(y-)|}{q(y)} \\ &\quad + \frac{1}{\bar{X}_n} \sup_{0 < y \leq \theta} \frac{\alpha_n(y)}{q(y)} \sup_{0 < y \leq \theta} \left(\frac{Q(y) - Q(0)}{Q(y)} + 1 \right). \end{aligned}$$

Hence the lemma follows by (3.4.4), (7.3.3) and Lemma 8.3.1. ■

LEMMA 8.3.5 *Under the assumptions of Lemma 8.3.1, we have*

$$\sup_{0 < y < 1} |C_n(y) - L_F(y)| \xrightarrow{P} 0.$$

Proof. By (8.1.2) and Corollary 8.3.1, it suffices to check that

$$(8.3.15) \quad \sup_{0 < y < 1} \left| \int_0^y Q(u) dE_n(u) - \int_0^y Q(u) du \right| \xrightarrow{P} 0.$$

For any $0 < \beta < 1$, like in the proof of Theorem 7.2.1, we have

$$(8.3.16) \quad \begin{aligned} & \sup_{0 < y < 1} \left| \int_0^y Q(u) dE_n(u) - \int_0^y Q(u) du \right| \\ & \leq 4Q(1 - \beta) \sup_{0 < y < 1} |E_n(y) - y| + \int_{1-\beta}^1 Q(y) dy + \int_{1-\beta}^1 Q(y) dE_n(y). \end{aligned}$$

Let $\zeta_n = - \int_{1-\beta}^{1-\epsilon_n} Q(y) d\alpha_n(y)$, and define ϵ_n as in the proof of Lemma 8.3.1. Then, by integrating by parts, we have

$$\zeta_n = Q(1 - \beta) \cdot \alpha_n(1 - \beta) - Q(1 - \epsilon_n) \cdot \alpha_n((1 - \epsilon_n)-) + \int_{1-\beta}^{1-\epsilon_n} \alpha_n(y) dQ(y).$$

Thus, by (8.3.6), Theorems 3.3.1 and Corollary 5.3.1, it is easy to derive that

$$\zeta_n \xrightarrow{D} Q(1 - \beta) \cdot B(1 - \beta) + \int_{1-\beta}^1 B(y) dy.$$

On the other hand, $\epsilon_n < 1 - U_{n:n}$ implies

$$\zeta_n = n^{1/2} \left\{ \int_{1-\beta}^1 Q(y) dE_n(y) - \int_{1-\beta}^{1-\epsilon_n} Q(y) dy \right\}.$$

Hence, by Lemma 4.5.1,

$$\int_{1-\beta}^1 Q(y) dE_n(y) \xrightarrow{P} \int_{1-\beta}^1 Q(y) dy.$$

This and (8.3.16) imply (8.3.15) since $\int_{1-\beta}^1 Q(y) dy \rightarrow 0$ as $\beta \rightarrow 1$. The proof of our lemma is now complete. ■

LEMMA 8.3.6 *Under the assumptions of Lemma 8.3.1, we have*

$$\sup_{0 < y < 1} |C_n^{-1}(y) - L_F^{-1}(y)| \xrightarrow{P} 0.$$

Proof. By (8.1.4) and Theorem 3.2.1,

$$\sup_{0 < y < 1} |L_n^{-1}(y) - C_n^{-1}(y)| \leq \sup_{0 \leq y \leq 1} |E_n(y) - y| \xrightarrow{\text{a.s.}} 0.$$

Thus, the proof of our lemma is reduced to showing

$$(8.3.17) \quad \sup_{0 \leq y \leq 1} |L_n^{-1}(y) - L_F^{-1}(y)| \xrightarrow{P} 0.$$

Now the Lemma 13.1 of Csörgő, Csörgő and Horváth (1986) states that for each $\omega \in \Omega$,

$$\sup_{0 \leq y \leq 1} |L_n^{-1}(L_F(y)) - y| = \sup_{0 \leq y \leq 1} |L_F^{-1}(L_n(y)) - y|.$$

This implies

$$\sup_{0 \leq y \leq 1} |L_n^{-1}(y) - L_F^{-1}(y)| = \sup_{0 \leq y \leq 1} |L_F^{-1}(L_n(y)) - L_F^{-1}(L_F(y))|.$$

Since $L_F^{-1}(\cdot)$ is continuous on $[0, 1]$, (8.3.17) will follow if we have

$$\sup_{0 \leq y \leq 1} |L_n(y) - L_F(y)| \xrightarrow{P} 0.$$

This, in turn, is true since

$$\begin{aligned} \sup_{0 \leq y \leq 1} |L_n(y) - L_F(y)| &= \sup_{0 \leq y \leq 1} |C_n(U_n(y)) - L_F(y)| \\ &\leq \sup_{0 \leq y \leq 1} |C_n(U_n(y)) - L_F(U_n(y))| + \sup_{0 \leq y \leq 1} |L_F(U_n(y)) - L_F(y)| \\ &\leq \sup_{0 \leq y \leq 1} |C_n(y) - L_F(y)| + \sup_{0 \leq y \leq 1} |L_F(U_n(y)) - L_F(y)| \\ &\xrightarrow{P} 0 \end{aligned}$$

by (8.1.3), Lemma 8.3.5 and on account of having

$$\sup_{0 \leq y \leq 1} |U_n(y) - y| = \sup_{0 \leq y \leq 1} |E_n(y) - y| \xrightarrow{\text{a.s.}} 0$$

by (4.1.12) and Theorem 3.2.1. This completes our proof. $\blacksquare$

LEMMA 8.3.7 *Let $X_n, Y_n \in D[0, 1]$. If for any $\varepsilon > 0$, we have*

$$P\left\{\sup_{0 \leq u \leq 1} |X_n(u) - Y_n(u)| > \varepsilon\right\} \longrightarrow 0$$

and

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P\{w(Y_n, \delta) \geq \varepsilon\} = 0,$$

then

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P\{w(X_n, \delta) \geq \varepsilon\} = 0,$$

where w is Lévy's modulus of continuity defined as in (4.3.4).

Proof. It easily follows by the fact that

$$w(X_n, \delta) \leq 2 \sup_{0 \leq u \leq 1} |X_n(u) - Y_n(u)| + w(Y_n, \delta).$$

The details are omitted. ■

Now we can prove the following weak convergence result for empirical concentration processes of stationary associated sequences.

THEOREM 8.3.1 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables with common continuous distribution function F . Assume that Q is continuous on $[0, 1)$ and there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

If $J^(2/\nu_1) < \infty$ for some $1/2 < \nu_1 < (9 + 2\nu)(15 + 2\nu)^{-1}$ and*

$$(8.3.18) \quad \limsup_{y \rightarrow 0} \frac{Q(y)q(y)}{Q(y/\lambda)} < \infty$$

for some $\lambda > 1$ and $q \in \mathbf{Q}^$ with $\int_0^1 q^{-p}(t) dt < \infty$, then*

$$(8.3.19) \quad c_n(\cdot) \xrightarrow{\mathcal{D}} \Phi(L_F^{-1}(\cdot)) \text{ in } D[0, 1],$$

where $\{\Phi(y), 0 \leq y \leq 1\}$ is the zero-mean Gaussian process defined by

$$\begin{aligned}
 (8.3.20) \quad \Phi(y) &= -\frac{\mu}{Q(y)}\Gamma^*(y) - B(y) \\
 &= -\frac{1}{Q(y)}\left\{\int_0^y B(u) dQ(u) - B(y)Q(y)\right\} \\
 &\quad + \frac{L_F(y)}{Q(y)}\int_0^1 B(u) dQ(u) - B(y).
 \end{aligned}$$

Proof. We have by (8.1.4)

$$\begin{aligned}
 (8.3.21) \quad c_n(y) &= n^{1/2}(E_n(C_n^{-1}(y)) - L_F^{-1}(y)) \\
 &= -\alpha_n(L_F^{-1}(y)) + n^{1/2}(C_n^{-1}(y) - L_F^{-1}(y)) + \Delta_n^{(1)}(y),
 \end{aligned}$$

where

$$\Delta_n^{(1)}(y) = \alpha_n(L_F^{-1}(y)) - \alpha_n(C_F^{-1}(y)), \quad 0 \leq y \leq 1.$$

Using Lemma 8.3.6, we can easily prove that

$$(8.3.22) \quad \sup_{0 \leq y \leq 1} |\Delta_n^{(1)}(y)| \xrightarrow{P} 0.$$

For the convenience of our proof, we need to change the scale y of $c_n(y)$ so that

(8.3.21) becomes

$$(8.3.23) \quad c_n(L_F(y)) = -\alpha_n(y) + n^{1/2}(C_n^{-1}(L_F(y)) - y) + o_P(1)$$

by (8.3.22). Let $\gamma_n(y) = n^{1/2}(L_F^{-1}(C_n(y)) - y)$, $0 \leq y \leq 1$. Then the second right term of (8.3.23) becomes

$$\begin{aligned}
 (8.3.24) \quad n^{1/2}(C_n^{-1}(L_F(y)) - y) &= n^{1/2}\{L_F^{-1}(C_n(C_n^{-1}(F_F(y)))) - y\} \\
 &\quad - \gamma_n(C_n^{-1}(L_F(y))) \\
 &= -\gamma_n(y) + \Delta_n^{(2)}(y) + \Delta_n^{(3)}(y), \quad 0 \leq y \leq 1,
 \end{aligned}$$

where

$$\Delta_n^{(2)}(y) = \gamma_n(y) - \gamma_n(C_n^{-1}(L_F(y))), \quad 0 \leq y \leq 1$$

and

$$\Delta_n^{(3)}(y) = n^{1/2} \{ L_F^{-1}(C_n(C_n^{-1}(F_F(y)))) - y \}, \quad 0 \leq y \leq 1.$$

First we show that for any $\varepsilon > 0$,

$$(8.3.25) \quad \lim_{n \rightarrow \infty} P \left\{ \sup_{0 < y < 1} \left| \gamma_n(y) - \frac{\mu}{Q(y)} \ell_n^*(y) \right| > \varepsilon \right\} = 0.$$

We have for any $0 < \theta < 1/2$,

$$\begin{aligned} (8.3.26) \quad & \sup_{0 < y < 1} \left| \gamma_n(y) - \frac{\mu}{Q(y)} \ell_n^*(y) \right| \\ & \leq \sup_{0 < y \leq \theta} |\gamma_n(y)| + \sup_{0 < y \leq \theta} \frac{\mu |\ell_n^*(y)|}{Q(y)} + \sup_{\theta < y \leq 1} \left| \gamma_n(y) - \frac{\mu}{Q(y)} \ell_n^*(y) \right| \\ & := \Delta_{1,n}(\theta) + \Delta_{2,n}(\theta) + \Delta_{3,n}(\theta). \end{aligned}$$

By Lemma 8.3.4

$$(8.3.27) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P \{ \Delta_{2,n}(\theta) > \varepsilon \} = 0.$$

By one-term Taylor expansion,

$$\begin{aligned} \Delta_{3,n}(\theta) &= \sup_{\theta < y \leq 1} \left| \frac{\mu}{Q(L_F^{-1}(\tau_n(y)))} \ell_n^*(y) - \frac{\mu}{Q(y)} \ell_n^*(y) \right| \\ &\leq \sup_{\theta < y \leq 1} \mu |\ell_n^*(y)| \cdot \sup_{\theta < y \leq 1} \left| \frac{1}{Q(L_F^{-1}(\tau_n(y)))} - \frac{1}{Q(y)} \right|, \end{aligned}$$

where

$$(8.3.28) \quad C_n(y) \wedge L_F(y) \leq \tau_n(y) \leq C_n(y) \vee L_F(y).$$

In view of the fact that $Q(\theta) > 0$, Lemmas 8.3.3 and 8.3.5 imply

$$(8.3.29) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P \{ \Delta_{3,n}(\theta) > \varepsilon \} = 0.$$

Let $\theta_n = n^{-1/2}\varepsilon$ and

$$K(\theta) = \sup_{0 < y \leq \theta} \frac{Q(y)q(y)}{Q(y/\lambda)}.$$

Again by Taylor expansion,

$$\begin{aligned} (8.3.30) \quad \Delta_{1,n}(\theta) &\leq \sup_{0 < y \leq \theta_n} n^{1/2} L_F^{-1}(C_n(y)) + n^{1/2} \theta_n + \sup_{\theta_n \leq y \leq \theta} n^{1/2} |L_F^{-1}(C_n(y)) - y| \\ &\leq 2\varepsilon + n^{1/2} |L_F^{-1}(C_n(\theta_n)) - \theta_n| + \sup_{\theta_n \leq y \leq \theta} n^{1/2} |L_F^{-1}(C_n(y)) - y| \\ &\leq 2\varepsilon + 2 \sup_{\theta_n \leq y \leq \theta} n^{1/2} |L_F^{-1}(C_n(y)) - y| \\ &\leq 2\varepsilon + 2 \sup_{\theta_n \leq y \leq \theta} \frac{\mu Q(y)q(y)}{Q(L_F^{-1}(\tau_n(y)))} \cdot \sup_{0 < y \leq \theta} \frac{|\ell_n^*(y)|}{Q(y)q(y)} \end{aligned}$$

with $\tau_n(y)$ as in (8.3.28). For $\theta_n \leq y \leq \theta$, we have

$$L_F(y) = \mu^{-1} \int_0^y Q(y) dy \geq \mu^{-1} \int_{y/\lambda}^y Q(y) dy \geq \frac{\lambda-1}{\lambda\mu} Q(y/\lambda) \cdot y.$$

Hence, by letting $\eta = \frac{(\lambda-1)^2}{\lambda^2\mu K(\theta)}\varepsilon$, $\sup_{\theta_n \leq y \leq \theta} \frac{|\ell_n^*(y)|}{Q(y)q(y)} \leq \eta$ implies that

$$\begin{aligned} C_n(y) &\geq L_F(y) - \eta n^{-1/2} Q(y)q(y) \\ &\geq L_F(y) - \eta K(\theta) n^{-1/2} Q(y/\lambda) \\ &\geq L_F(y) - \eta K(\theta) n^{-1/2} \frac{\lambda\mu}{(\lambda-1)y} L_F(y) \\ &\geq L_F(y) - (1 - 1/\lambda) L_F(y) \\ &= L_F(y)/\lambda \\ &\geq L_F(y/\lambda), \quad \theta_n \leq y \leq \theta, \end{aligned}$$

where the last inequality follows by Lemma 13.6 of Csörgö, Csörgö and Horváth (1986). Therefore, by (8.3.28) and (8.3.30),

$$\begin{aligned} P\{\Delta_{1,n}(\theta) > 4\varepsilon\} &\leq P\left\{ \sup_{\theta_n \leq y \leq \theta} \frac{\mu Q(y)q(y)}{Q(L_F^{-1}(\tau_n(y)))} \cdot \sup_{0 < y \leq \theta} \frac{|\ell_n^*(y)|}{Q(y)q(y)} > \varepsilon \right\} \\ &\leq P\left\{ \sup_{0 < y \leq \theta} \frac{|\ell_n^*(y)|}{Q(y)q(y)} > \eta \right\} + P\left\{ \mu K(\theta) \cdot \sup_{\theta_n \leq y \leq \theta} \frac{|\ell_n^*(y)|}{Q(y)q(y)} > \varepsilon \right\}, \end{aligned}$$

and hence

$$(8.3.31) \quad \lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P\{\Delta_{1,n}(\theta) > \varepsilon\} = 0$$

by Lemma 8.3.4. Thus, (8.3.25) is proved by (8.3.26)-(8.3.27), (8.3.29) and (8.3.31).

Next we show that for any $\varepsilon > 0$,

$$(8.3.32) \quad \lim_{n \rightarrow \infty} P\left\{\sup_{0 < y < 1} |\Delta_n^{(i)}(y)| > \varepsilon\right\} = 0, \quad i = 2, 3.$$

For any $\delta > 0$ and $i = 2$, it is easy to get that

$$\begin{aligned} P\left\{\sup_{0 < y < 1} |\Delta_n^{(2)}(y)| > \varepsilon\right\} &\leq P\left\{\sup_{0 < y < 1} |C_n^{-1}(y) - L_F^{-1}(y)| \geq \delta\right\} \\ &\quad + P\{w(\gamma_n, \delta) > \varepsilon\} \end{aligned}$$

and for $i = 3$, by (13.24) of Csörgő, Csörgő and Horváth (1986),

$$\begin{aligned} P\left\{\sup_{0 < y < 1} |\Delta_n^{(3)}(y)| > \varepsilon\right\} &\leq P\left\{\max_{1 \leq k \leq n} \left\{\gamma_n\left(\frac{k}{n}\right) - \gamma_n\left(\frac{k}{n} - \right)\right\} > \varepsilon\right\} \\ &\leq P\{w(\gamma_n, \delta) > \varepsilon\}. \end{aligned}$$

In general, we have for any $\delta > 0$ and $i = 2, 3$,

$$(8.3.33) \quad \begin{aligned} P\left\{\sup_{0 < y < 1} |\Delta_n^{(i)}(y)| > \varepsilon\right\} &\leq P\left\{\sup_{0 < y < 1} |C_n^{-1}(y) - L_F^{-1}(y)| \geq \delta\right\} \\ &\quad + P\{w(\gamma_n, \delta) > \varepsilon\}. \end{aligned}$$

For $0 \leq y \leq 1$, let

$$\hat{\ell}_n(y) = (\bar{X}_n)^{-1} \left\{ \int_0^y \alpha_n(u) dQ(u) - \alpha_n(y)Q(y) - L_F(y) \int_0^{1-\varepsilon_n} \alpha_n(u) dQ(u) \right\}.$$

Then, from the proof of Lemma 8.3.3, it is not difficult to find that

$$(8.3.34) \quad \lim_{n \rightarrow \infty} P\left\{\sup_{0 < y < 1} \left| \frac{\mu}{Q(y)} \bar{\ell}_n(y) - \frac{\mu}{Q(y)} \hat{\ell}_n(y) \right| > \varepsilon\right\} = 0.$$

Now it is easy to prove that

$$\lim_{\delta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{w\left(\frac{\mu}{Q(\cdot)} \hat{\ell}_n(\cdot), \delta\right) > \varepsilon\right\} = 0$$

simply by applying Theorems 3.3.1 and 5.3.1, Theorem 8.2 of Billingsley (1968) and (8.3.10). Hence, (8.3.32) follows by (8.3.25), (8.3.33)-(8.3.34), Lemmas 8.3.2, 8.3.6 and 8.3.7.

Finally, by Lemma 8.3.3 and (8.3.25),

$$\alpha_n(\cdot) + \gamma_n(\cdot) \xrightarrow{\mathcal{D}} \frac{\mu}{Q(\cdot)} \Gamma^*(\cdot) + B(\cdot) = -\Phi(\cdot) \text{ in } D[0, 1]$$

and hence

$$c_n(L_F(\cdot)) \xrightarrow{\mathcal{D}} \Phi(\cdot) \text{ in } D[0, 1]$$

by (8.3.23)-(8.3.24) and (8.3.32). Theorem 8.3.1 is now proved. ■

Chapter 9

Percentile Residual Life Processes

9.1 Definitions.

Let F be a life distribution function with support $(0, b_1)$, where $b_1 = \inf\{t > 0 : F(t) = 1\} \leq \infty$, and $\{X_n, n \geq 1\}$ be a sequence of random variables with common distribution function F . Let $0 < p_0 < 1$ be any fixed number. Then the $(1 - p_0)$ -percentile residual lifetime is defined as

$$(9.1.1) \quad R^{(p_0)}(t) = Q(1 - p_0[1 - F(t)]) - t, \quad t > 0.$$

The natural empirical counterpart of $R^{(p_0)}$, the sample $(1 - p_0)$ -percentile residual life at $t > 0$ is

$$(9.1.2) \quad R_n^{(p_0)}(t) = Q_n(1 - p_0[1 - F_n(t)]) - t, \quad 0 < p_0 < 1.$$

Assume that F has a density function $f = F'$ that is positive over the support $(0, b_1)$ of F . Then the empirical $(1 - p_0)$ -percentile life process $r_n^{(p_0)}(t)$ is defined by

$$(9.1.3) \quad \begin{aligned} r_n^{(p_0)}(t) &= n^{1/2} f(Q(1 - p_0[1 - F(t)])) \{R_n^{(p_0)}(t) - R^{(p_0)}(t)\} \\ &= n^{1/2} f(R^{(p_0)}(t) + t) \cdot \{R_n^{(p_0)}(t) - R^{(p_0)}(t)\}, \quad t > 0. \end{aligned}$$

The above notions are good only for studying percentile life processes in the absence of random censorship. Under the random censorship model from right, we have to set new definitions of sequences and percentile lifetime functions.

Let F^0 be a life distribution function with support $(0, b_1)$, where $b_1 = \inf\{t > 0 : F^0(t) = 1\} \leq \infty$, and $X_1^0, \dots, X_n^0$ be associated random variables with common distribution function F^0 . Then the corresponding quantile function is defined as

$$(9.1.4) \quad Q(y) = \inf\{t > 0 : F^0(t) > y\}, \quad 0 < y < 1, \quad Q(0) = 0, \quad Q(1) = b_1.$$

Let $Y_1, \dots, Y_n$ be i.i.d. random variables with distribution function H . Assume that the two sequences $\{X_i^0\}$ and $\{Y_i\}$ are independent. In the random censorship model from right, the X_i^0 may be censored on the right by the Y_i , so that one observes only the pairs (X_i, δ_i) , $i = 1, \dots, n$, where

$$(9.1.5) \quad X_i = X_i^0 \wedge Y_i, \quad \delta_i = I\{X_i^0 \leq Y_i\}, \quad i = 1, \dots, n.$$

Thus, by the properties P_2 and P_4 of association, $X_1, \dots, X_n$ are associated random variables with distribution function F given by

$$(9.1.6) \quad 1 - F(t) = (1 - F^0(t))(1 - H(t)), \quad -\infty < t < \infty.$$

Let the subdistribution function of the uncensored observations be defined as

$$(9.1.7) \quad \tilde{F}(t) = P\{X_1 \leq t \text{ and } \delta_1 = 1\} = \int_{-\infty}^t (1 - H(s)) dF^0(s).$$

To estimate F^0 in the above model, the product-limit (PL) estimator (cf. Kaplan and Meier (1958)) F_n^0 has been widely used and is defined by

$$(9.1.8) \quad 1 - F_n^0(t) = \begin{cases} \prod_{X_i \leq t, 1 \leq i \leq n} \left(\frac{n - R_i}{n - R_i + 1} \right)^{\delta_i}, & \text{if } t \leq X_{n:n}, \\ 0, & \text{if } t > X_{n:n}, \end{cases}$$

where R_i is the rank of $(X_i, 1 - \delta_i)$ in the lexicographic ordering of $\{(X_i, 1 - \delta_i), i = 1, \dots, n\}$.

The counterpart of the PL estimator F_n^0 of F^0 for Q is the PL-quantile function Q_n defined by

$$(9.1.9) \quad Q_n(y) = \inf\{t : F_n^0(t) > y\}, \quad 0 < y < 1.$$

F_n^0 and Q_n have been generally accepted as substitutes for the empirical distribution function and quantile function, respectively, in the random censorship model from the right.

An analogue of the empirical process β_n defined in Chapter 3 is the PL-process defined as

$$(9.1.10) \quad \beta_n(x) = n^{1/2}(F^0(x) - F_n^0(x)), \quad x \in \mathbb{R}.$$

Again here and in the sequel, for convenience, we let

$$U_i^0 = F^0(X_i^0), \quad V_i = F^0(Y_i), \quad U_i = F^0(X_i), \quad i = 1, \dots, n.$$

Then, by the property of P_4 of association, $U_1^0, \dots, U_n^0$ are associated with uniform-[0, 1] distribution, $V_1, \dots, V_n$ are i.i.d. with the distribution function $H^*(t) = H(Q(t))$, and $U_1, \dots, U_n$ are associated with the distribution function $F^*(t)$ given by

$$(9.1.11) \quad 1 - F^*(t) = (1 - t)(1 - H^*(t)), \quad 0 \leq t \leq 1.$$

We also have

$$(9.1.12) \quad \delta_i = I\{X_i^0 \leq Y_i\} = I\{U_i^0 \leq V_i\}, \quad i = 1, \dots, n,$$

and the subdistribution function in this case is defined as

$$(9.1.13) \quad \tilde{F}^*(t) = P\{U_1 \leq t \text{ and } \delta_1 = 1\} = \int_0^t (1 - H^*(s)) ds.$$

Let F_n^{*0} be the PL-estimator based on $\{(U_i, \delta_i), i = 1, \dots, n\}$, i.e.,

$$(9.1.14) \quad 1 - F_n^{*0}(t) = \begin{cases} \prod_{U_i \leq t, 1 \leq i \leq n} \left(\frac{n - R_i}{n - R_i + 1} \right)^{\delta_i}, & \text{if } t \leq U_{n:n}, \\ 0, & \text{if } t > U_{n:n}, \end{cases}$$

where R_i is the rank of $(U_i, 1 - \delta_i)$ in the lexicographic ordering of $\{(U_i, 1 - \delta_i), i = 1, \dots, n\}$. Clearly, by (9.1.12),

$$(9.1.15) \quad F_n^{*0}(t) = F_n^0(Q(t)), \quad 0 \leq t \leq 1.$$

Now define the uniform PL-process by

$$(9.1.16) \quad \alpha_n(t) = n^{1/2}(t - F_n^{*0}(t)), \quad 0 \leq t \leq 1.$$

Define U_n^{*0} , the inverse of F_n^{*0} , by

$$(9.1.17) \quad U_n^{*0}(y) = \inf\{t : F_n^{*0}(t) > y\}, \quad 0 < y < 1$$

and the uniform PL-quantile process u_n by

$$(9.1.18) \quad u_n(y) = n^{1/2}(U_n^{*0}(y) - y), \quad 0 < y < 1.$$

Define the empirical distribution function of $\{U_i, i = 1, \dots, n\}$ by

$$(9.1.19) \quad F_n^*(t) = \frac{1}{n} \sum_{i=1}^n I(U_i \leq t), \quad 0 \leq t \leq 1$$

and the corresponding empirical process $\{\alpha_n^*(t); 0 \leq t \leq 1\}$ by

$$(9.1.20) \quad \{\alpha_n^*(t); 0 \leq t \leq 1\} = \{n^{1/2}(F^*(t) - F_n^*(t)); 0 \leq t \leq 1\}, \quad n \geq 1.$$

Define the sub-empirical distribution function of the uncensored observations by

$$(9.1.21) \quad \tilde{F}_n^*(t) = \frac{1}{n} \sum_{i=1}^n I(U_i \leq t, \delta_i = 1) = \frac{1}{n} \sum_{i=1}^n I(U_i^0 \leq t \wedge V_i), \quad 0 \leq t \leq 1,$$

and the corresponding empirical process $\{\tilde{\alpha}_n(t); 0 \leq t \leq 1\}$ by

$$(9.1.22) \quad \{\tilde{\alpha}_n(t); 0 \leq t \leq 1\} = \{n^{1/2}(\tilde{F}_n^*(t) - \tilde{F}^*(t)); 0 \leq t \leq 1\}, \quad n \geq 1.$$

Let $0 < p_0 < 1$ be any fixed number. Then the $(1 - p_0)$ -percentile residual lifetime under random censorship from the right is defined as

$$(9.1.23) \quad R^{(p_0)}(t) = Q(1 - p_0[1 - F^0(t)]) - t, \quad t > 0$$

and the empirical counterpart of $R^{(p_0)}(t)$ as

$$(9.1.24) \quad R_n^{(p_0)}(t) = Q_n(1 - p_0[1 - F_n^0(t)]) - t, \quad 0 < p_0 < 1.$$

Assume that F^0 has a density function $f^0 = F^{0'}$ that is positive over the support $(0, b_1)$ of F^0 . Then the empirical $(1 - p_0)$ -percentile life process $r_n^{(p_0)}(t)$ is defined by

$$(9.1.25) \quad \begin{aligned} r_n^{(p_0)}(t) &= n^{1/2} f^0(Q(1 - p_0[1 - F^0(t)])) \{R_n^{(p_0)}(t) - R^{(p_0)}(t)\} \\ &= n^{1/2} f^0(R^{(p_0)}(t) + t) \cdot \{R_n^{(p_0)}(t) - R^{(p_0)}(t)\}, \quad t > 0. \end{aligned}$$

9.2 Estimation of percentile residual life without random censorship.

This section is divided into three parts. In the first subsection, we show that $R_n^{(p_0)}(t)$ is a pointwise and strongly consistent estimator of $R^{(p_0)}(t)$ and $r_n^{(p_0)}(t)$ converges in distribution to a normal random variable for fixed p_0 and t . The second subsection derives confidence intervals for $R^{(p_0)}(t)$ for fixed p_0 and t . In the third subsection we view $R_n^{(p_0)}(t)$ as a stochastic process of p_0 or t , and derive the corresponding confidence bands.

9.2.1 Pointwise and strong consistency of $R_n^{(p_0)}(t)$ to $R^{(p_0)}(t)$ and the asymptotic distribution of $r_n^{(p_0)}$.

By (4.1.6), Theorem 3.2.1 and Corollary 4.2.1, it is easy to derive the following result.

THEOREM 9.2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of associated random variables with common distribution function F . Let $t > 0$ and $0 < p_0 < 1$ be fixed, and assume that $f = F'$ is bounded, the density-quantile function $f(Q(y))$ is positive and continuous at $1 - p_0(1 - F(t))$, and that*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{Cov}(X_n, \sum_{i=1}^{n-1} X_i) < \infty.$$

Then

$$R_n^{(p_0)}(t) \xrightarrow{\text{a.s.}} R^{(p_0)}(t).$$

THEOREM 9.2.2 *Let $\{X_n, n \geq 0\}$ be a sequence of associated random variables with common continuous distribution function F . Let $t > 0$ and $0 < p_0 < 1$ be fixed, and assume that $f = F'$ exists, the density-quantile function $f(Q(y))$ is positive and continuous at $1 - p_0(1 - F(t))$, and there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

Then

$$r_n^{(p_0)}(t) \xrightarrow{\mathcal{D}} B(1 - p_0[1 - F(t)]) - p_0 B(F(t)) = N(0, \sigma_{F(t), p_0}^2),$$

where

$$\begin{aligned} (9.2.1) \quad \sigma_{F(t), p_0}^2 &= E\{B(1 - p_0[1 - F(t)]) - p_0 B(F(t))\}^2 \\ &= p_0(1 - p_0)(1 - F(t)) + 2 \sum_{k=1}^{\infty} \text{Cov}(h_{F(t), p_0}(U_0), h_{F(t), p_0}(U_k)) \end{aligned}$$

and

$$h_{F, p}(x) = I(x \leq 1 - p(1 - F)) - pI(x \leq F).$$

Proof. By (3.1.6), (4.1.3), (4.1.5) and one-term Taylor expansion,

$$\begin{aligned}
 (9.2.2) \quad r_n^{(p_0)}(t) &= n^{1/2} f(Q(1 - p_0[1 - F(t)])) \{Q(U_n(1 - p_0[1 - F_n(t)])) \\
 &\quad - Q(1 - p_0[1 - F(t)])\} \\
 &= \{u_n(1 - p_0[1 - F_n(t)]) - p_0 \alpha_n(F(t))\} \frac{f(Q(1 - p_0[1 - F(t)]))}{f(Q(\theta_n(t)))},
 \end{aligned}$$

where

$$\begin{aligned}
 &U_n(1 - p_0[1 - F_n(t)]) \wedge (1 - p_0[1 - F(t)]) \\
 &\leq \theta_n(t) \leq U_n(1 - p_0[1 - F_n(t)]) \vee (1 - p_0[1 - F(t)]).
 \end{aligned}$$

By (4.1.2) and Theorem 3.2.1,

$$\sup_{0 \leq t \leq b_1} |F_n(t) - F(t)| = \sup_{0 \leq t \leq 1} |E_n(t) - t| = \sup_{0 \leq t \leq 1} |U_n(t) - t| \xrightarrow{\text{a.s.}} 0,$$

which implies that

$$U_n(1 - p_0[1 - F_n(t)]) \xrightarrow{\text{a.s.}} 1 - p_0(1 - F(t)).$$

Thus, $\theta \xrightarrow{\text{a.s.}} 1 - p_0[1 - F(t)]$, which implies also

$$\frac{f(Q(1 - p_0[1 - F(t)]))}{f(Q(\theta))} \xrightarrow{\text{a.s.}} 1$$

since $f(Q(t))$ is continuous at $1 - p_0[1 - F(t)]$. Finally,

$$r_n^{(p_0)}(t) \xrightarrow{\mathcal{D}} B(1 - p_0[1 - F(t)]) - p_0 B(F(t))$$

follows by Theorem 3.3.1 and (4.3.5). This also completes our proof. ■

9.2.2 Confidence intervals for $R^{(p_0)}(t)$ for fixed p_0 and t .

Let Φ be the standard normal distribution function and for a given $0 < \alpha < 1$ let $\lambda = \lambda(\alpha) > 0$ be the solution of the equation $\Phi(\lambda) = 1 - \alpha/2$. Assume that for

fixed $t > 0$ and $0 < p_0 < 1$, $\sigma_n^{(p_0)}(t)$ is a weak estimator of $\sigma_{F(t), p_0}$, i.e.,

$$(9.2.3) \quad \sigma_n^{(p_0)}(t) \xrightarrow{P} \sigma_{F(t), p_0}.$$

In case of i.i.d. sampling, it is easy to find such an estimator, such as

$$\sigma_n^{(p_0)}(t) = (p_0(1 - p_0)[1 - F_n(t)])^{1/2},$$

for example. For estimating $\sigma_{F(t), p_0}$ of (9.2.1), however, there is no such an immediate estimator available. One can use the circle block bootstrap method of Chapter 10 to find an estimator. We are not going to discuss how to find such an estimator here.

THEOREM 9.2.3 *Let $t > 0$ and $0 < p_0 < 1$ be fixed. If f_n is any sequence of density estimators for f that is uniformly consistent in a neighborhood of $Q(1 - p_0[1 - F(t)])$, then under the assumptions of Theorem 9.2.2,*

$$\begin{aligned} \lim_{n \rightarrow \infty} P \left\{ -\frac{\lambda \sigma_n^{(p_0)}(t)}{n^{1/2} f_n(Q_n(1 - p_0[1 - F_n(t)]))} + R_n^{(p_0)}(t) \leq R^{(p_0)}(t) \leq R_n^{(p_0)}(t) \right. \\ \left. + \frac{\lambda \sigma_n^{(p_0)}(t)}{n^{1/2} f_n(Q_n(1 - p_0[1 - F_n(t)]))} \right\} \\ = 1 - \alpha. \end{aligned}$$

Proof. This theorem can be easily proved by Theorem 9.2.2 and our assumption (9.2.3). The details are omitted. ■

One can avoid the problem of estimating the density-quantile function $f(Q(\cdot))$, via the following result.

THEOREM 9.2.4 *Let $t > 0$ and $0 < p_0 < 1$ be fixed. Then, under the assumptions of Theorem 9.2.2, we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} P \left\{ Q_n(1 - p_0[1 - F_n(t)]) - \frac{\lambda}{n^{1/2}} \sigma_n^{(p_0)}(t) - t \leq R^{(p_0)}(t) \right. \\ \left. \leq Q_n(1 - p_0[1 - F_n(t)]) + \frac{\lambda}{n^{1/2}} \sigma_n^{(p_0)}(t) - t \right\} \\ = 1 - \alpha. \end{aligned}$$

Proof. The proof is exactly the same as that of Theorem 3 of Csörgö and Csörgö (1987). Hence it is omitted. ■

9.2.3 Weak convergence of $r_n^{(p_0)}(t)$ as a stochastic process of p_0 or t and confidence bands.

In this subsection, we will view $r_n^{(p_0)}(t)$ as a stochastic process of p_0 or t and consider the weak convergence of $r_n^{(p_0)}(t)$ in a proper D space.

Let

$$v_n^{(p_0)}(t) = u_n(1 - p_0[1 - F_n(t)]) - p_0\alpha_n(F(t)), \quad 0 < t < b_1.$$

Then

$$v_n^{(p_0)}(Q(t)) = u_n(1 - p_0[1 - E_n(t)]) - p_0\alpha_n(t), \quad 0 < t < 1.$$

LEMMA 9.2.1 *Let q^* be the function defined by (4.4.1) and k_n by (4.4.2). Then, for any $\varepsilon > 0$ and fixed $0 < p_0 < 1$, under the assumptions of Theorem 3.3.1, we have*

$$(9.2.4) \quad \lim_{\theta \rightarrow 0} \limsup_{n \rightarrow \infty} P\left\{ \sup_{1-\theta \leq t \leq 1 - \frac{k_n}{n}} \frac{|v_n^{(p_0)}(Q(t))|}{q^*(1-t)} \geq \varepsilon \right\} = 0.$$

Proof. We have for $0 < \varepsilon < 1/8$

$$(9.2.5) \quad \begin{aligned} P\left\{ \sup_{1-\theta \leq t \leq 1 - \frac{k_n}{n}} \frac{|v_n^{(p_0)}(Q(t))|}{q^*(1-t)} \geq 4\varepsilon \right\} &\leq P\left\{ \sup_{1-\theta \leq t < 1} \frac{|\alpha_n(t)|}{q^*(1-t)} \geq 2\varepsilon \right\} \\ &+ P\left\{ \sup_{1-\theta \leq t \leq 1 - \frac{k_n}{n}} \frac{|u_n(1 - p_0[1 - E_n(t)])|}{q^*(1-t)} \geq 2\varepsilon \right\} \end{aligned}$$

From

$$\sup_{1-\theta \leq t \leq 1 - \frac{k_n}{n}} \frac{|\alpha_n(t)|}{q^*(1-t)} \leq 1/4$$

and (4.4.5), we can derive that

$$|E_n(t) - t| \leq \frac{q^*(1-t)}{4n^{1/2}} \leq \frac{1-t}{2}, \quad 1-\theta \leq t \leq 1 - \frac{k_n}{n},$$

which implies that $1 - E_n(t) \leq 3(1 - t)/2$. Hence we have

$$\sup_{1-\theta \leq t \leq 1-\frac{k_n}{n}} \frac{q^*(p_0[1 - E_n(t)])}{q^*(1 - t)} \leq 2.$$

At the same time, for $1 - \theta \leq t \leq 1 - k_n/n$,

$$1 - 2\theta \leq 1 - \frac{3p_0(1 - t)}{2} \leq 1 - p_0[1 - E_n(t)] \leq 1 - \frac{p_0(1 - t)}{2} \leq 1 - \frac{p_0 k_n}{2n}.$$

Finally, we arrive at

$$\begin{aligned} & \sup_{1-\theta \leq t \leq 1-\frac{k_n}{n}} \frac{|u_n(1 - p_0[1 - E_n(t)])|}{q^*(1 - t)} \\ & \leq \sup_{1-\theta \leq t \leq 1-\frac{k_n}{n}} \frac{|u_n(1 - p_0[1 - E_n(t)])|}{q^*(p_0[1 - E_n(t)])} \sup_{1-\theta \leq t \leq 1-\frac{k_n}{n}} \frac{q^*(p_0[1 - E_n(t)])}{q^*(1 - t)} \\ & \leq 2 \sup_{1-2\theta \leq t \leq 1-\frac{p_0 k_n}{2n}} \frac{|u_n(t)|}{q^*(1 - t)}, \end{aligned}$$

and thus by (9.2.5)

$$\begin{aligned} & P\left\{ \sup_{1-\theta \leq t \leq 1-\frac{k_n}{n}} \frac{|v_n^{(p_0)}(Q(t))|}{q^*(1 - t)} \geq 4\epsilon \right\} \\ & \leq 2P\left\{ \sup_{1-\theta \leq t < 1} \frac{|\alpha_n(t)|}{q^*(1 - t)} \geq 2\epsilon \right\} + P\left\{ \sup_{1-2\theta \leq t \leq 1-\frac{p_0 k_n}{2n}} \frac{|u_n(t)|}{q^*(1 - t)} \geq \epsilon \right\}. \end{aligned}$$

This proves (9.2.4) by Remark 3.4.1 and (4.4.8). ■

Let $\{G^{(p_0)}(t); 0 < t < b_1, 0 < p_0 < 1\}$ be the zero-mean Gaussian process defined by

$$G^{(p_0)}(t) = B(1 - p_0[1 - F(t)]) - p_0 B(F(t)).$$

THEOREM 9.2.5 Assume that $0 < p_0 < 1$ is fixed. Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables with common continuous distribution function F which is also twice differentiable on $(Q(1 - p_0), b_1)$ and $F'(x) = f(x) > 0$ on $(Q(1 - p_0), b_1)$. Assume that for some $\nu > 0$

$$\sum_{n=1}^{\infty} n^{\frac{11}{2}+\nu} \text{Cov}(F(X_0), F(X_n)) < \infty,$$

and for some $\gamma_1 > 0$

$$\sup_{1-p_0 < y < 1} y(1-y) \frac{|f'(Q(y))|}{f^2(Q(y))} \leq \gamma_1.$$

Then we have

$$(9.2.6) \quad \hat{r}_n^{(p_0)}(Q(\cdot)) \xrightarrow{\mathcal{D}} G^{(p_0)}(Q(\cdot)) \text{ in } D[0, 1],$$

where

$$\hat{r}_n^{(p_0)}(Q(t)) = r_n^{(p_0)}(Q(t))I(0 < t < 1 - \frac{k_n}{n})$$

and k_n is defined by (4.4.2).

Proof. By (9.2.2)

$$r_n^{(p_0)}(t) = v_n^{(p_0)}(t) + v_n^{(p_0)}(t) \varepsilon_n(t), \quad 0 < t < b_1,$$

where

$$\varepsilon_n(t) = \frac{f(Q(1 - p_0[1 - F(t)]))}{f(Q(\theta_n(t)))} - 1, \quad 0 < t < b_1.$$

Hence, by Theorem 3.3.1 and (4.3.5), in order to prove (9.2.6), it suffices to show that for any $\varepsilon > 0$

$$(9.2.7) \quad \lim_{n \rightarrow \infty} P\left\{ \sup_{0 < t < 1 - \frac{k_n}{n}} |\varepsilon_n(Q(t))| > \varepsilon \right\} = 0.$$

Let $\lambda = 1 + \varepsilon$. Note that $v_n^{(p_0)}(Q(t)) = n^{1/2}(U_n(1 - p_0[1 - E_n(t)]) - (1 - p_0[1 - t]))$.

Using an argument similar to that of (4.5.4), by our assumptions and Lemma 1.4.1 in Csörgő (1983), we have

$$\begin{aligned} (9.2.8) \quad P\left\{ \sup_{0 < t < 1 - \frac{k_n}{n}} \varepsilon_n(t) > \lambda - 1 \right\} &\leq P\left\{ \sup_{0 < t < 1 - \frac{k_n}{n}} \frac{f(Q(1 - p_0[1 - t]))}{f(Q(\theta_n(Q(t))))} \geq \lambda \right\} \\ &\leq ([\gamma_1] + 1)P\left\{ \sup_{0 < t < 1 - \frac{k_n}{n}} \frac{v_n^{(p_0)}(Q(t))}{(1 - p_0[1 - t])} \geq 4n^{1/2}\eta \right\} \\ &\quad + ([\gamma_1] + 1)P\left\{ \sup_{0 < t < 1 - \frac{k_n}{n}} \frac{-v_n^{(p_0)}(Q(t))}{p_0[1 - t]} \geq 4n^{1/2}\eta \right\} \end{aligned}$$

$$\begin{aligned}
& +([\gamma_1] + 1)P\left\{\sup_{0 < t < 1 - \frac{h_n}{n}} \frac{-v_n^{(p_0)}(Q(t))}{U_n(1 - p_0[1 - E_n(t)])} \geq 4n^{1/2}\eta\right\} \\
& +([\gamma_1] + 1)P\left\{\sup_{0 < t < 1 - \frac{h_n}{n}} \frac{v_n^{(p_0)}(Q(t))}{(1 - U_n(1 - p_0[1 - E_n(t)]))} \geq 4n^{1/2}\eta\right\} \\
& := ([\gamma_1] + 1)(I_1 + I_2 + I_3 + I_4),
\end{aligned}$$

where $\lambda_1 = \lambda^{\frac{1}{\lambda(\gamma_1+1)}} > 1$ and $\eta = \frac{\lambda_1 - 1}{4} \wedge \frac{1}{8}$. Since $1 - p_0[1 - t] > 1 - p_0 > 0$ for $0 < t < 1$, it is easy to show that

$$(9.2.9) \quad \lim_{n \rightarrow \infty} I_1 = 0.$$

For small $\theta > 0$, by (4.4.5), we have

$$I_2 \leq P\left\{\sup_{0 < t \leq 1 - \theta} |v_n^{(p_0)}(t)| \geq 4n^{1/2}p_0\theta\eta\right\} + P\left\{\sup_{1 - \theta < t < 1 - \frac{h_n}{n}} \frac{|v_n^{(p_0)}(t)|}{q^*(1 - t)} \geq 2p_0\eta\right\}.$$

Hence, by Lemma 9.2.1,

$$(9.2.10) \quad \lim_{n \rightarrow \infty} I_2 = 0.$$

To deal with I_3 and I_4 , we use an argument similar to that in the proof of Theorem 4.5.1, and obtain easily that

$$(9.2.11) \quad \lim_{n \rightarrow \infty} I_k = 0, \quad k = 3, 4.$$

By (9.2.8)-(9.2.11), it follows that

$$\lim_{n \rightarrow \infty} P\left\{\sup_{0 < t < 1 - \frac{h_n}{n}} \varepsilon_n(Q(t)) > \varepsilon\right\} = 0.$$

Similarly, we can show that

$$\lim_{n \rightarrow \infty} P\left\{\sup_{0 < t < 1 - \frac{h_n}{n}} (-\varepsilon_n(Q(t))) > \varepsilon\right\} = 0.$$

This proves (9.2.7) and hence our theorem. $\blacksquare$

THEOREM 9.2.6 *If, in addition to the assumptions of Theorem 9.2.5, we also assume that one of the following conditions holds*

- (a) $B > 0$, where $B := \limsup_{x \uparrow b_1} f(x) < \infty$,
- (b) if $B = 0$, when $\gamma_1 < \frac{p-1}{p-2}$ with p defined by (3.4.2), then f is nonincreasing on an interval to the left of b_1 ; when $\gamma_1 \geq \frac{p-1}{p-2}$, then $J(\infty, 1/\beta) < \infty$ for some $\beta < 1/p$, where $J(1/\alpha, 1/\beta)$ is defined by (4.5.11),

then

$$(9.2.12) \quad r_n^{(p_0)}(Q(\cdot)) \xrightarrow{\mathcal{D}} G^{(p_0)}(Q(\cdot)) \text{ in } D[0, 1],$$

Proof. From the proof of Theorem 9.2.5, it is easy to see that (9.2.12) follows if we can show that for any $\varepsilon > 0$

$$(9.2.13) \quad \lim_{n \rightarrow \infty} P\left\{ \sup_{1 - \frac{k_n}{n} \leq t < 1} |r_n^{(p_0)}(Q(t))| > \varepsilon \right\} = 0.$$

If $B > 0$, then, by (4.5.15)

$$\sup_{1 - \frac{k_n}{n} \leq t < 1} |1 - p_0[1 - t] - \theta_n(t)| \leq n^{-1/2} \sup_{1 - \frac{k_n}{n} \leq t < 1} |v_n^{(p_0)}(Q(t))| \xrightarrow{P} 0,$$

which implies that

$$\sup_{1 - \frac{k_n}{n} \leq t < 1} \frac{f(Q(1 - p_0[1 - F(t)]))}{f(Q(\theta_n(t)))} = O_P(1).$$

Thus, by (9.2.2),

$$\sup_{1 - \frac{k_n}{n} \leq t < 1} |r_n^{(p_0)}(Q(t))| = O_P \left(\sup_{1 - \frac{k_n}{n} \leq t < 1} |v_n^{(p_0)}(Q(t))| \right) \xrightarrow{P} 0$$

and (9.2.13) is proved in this case.

The proof in the case that $B = 0$ is much like that of Theorem 9.2.5, by using the following facts:

$$r_n^{(p_0)}(Q(t)) = n^{1/2} \int_{1 - p_0[1 - t]}^{U_n(1 - p_0[1 - E_n(t)])} \frac{f(Q(1 - p_0[1 - t]))}{f(Q(u))} du, \quad 1 - \frac{k_n}{n} \leq t < 1,$$

$$U_n(1 - p_0[1 - E_n(t)]) \leq U_{n:n}, \quad 0 < t < 1,$$

$$1 - U_n(1 - p_0[1 - E_n(t)]) \leq n^{-1/2} \sup_{1 - \frac{k_n}{n} \leq t < 1} |v_n^{(p_0)}(Q(t))| + p_0[1 - t], \quad 1 - \frac{k_n}{n} \leq t < 1.$$

The details are omitted. ■

THEOREM 9.2.7 Assume that $t > 0$ is fixed. Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables with common continuous distribution function F which is also twice differentiable on (t, b_1) and $F'(x) = f(x) > 0$ on (t, b_1) . Assume that for some $\nu > 0$

$$\sum_{n=1}^{\infty} n^{\frac{1}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty$$

and for some $\gamma_1 > 0$

$$\sup_{F(t) < y < 1} y(1-y) \frac{|f'(Q(y))|}{f^2(Q(y))} \leq \gamma_1.$$

Then we have

$$\hat{r}_n^{(\cdot)}(Q(t)) \xrightarrow{\mathcal{D}} G^{(\cdot)}(Q(t)) \text{ in } D[0, 1],$$

where

$$\hat{r}_n^{(p_0)}(Q(t)) = r_n^{(p_0)}(Q(t))I\left(\frac{k_n}{n} \leq t < 1\right).$$

THEOREM 9.2.8 If, in addition to the assumptions of Theorem 9.2.7, we also assume that one of the following conditions holds

- (a) $B > 0$, where $B := \limsup_{x \uparrow b_1} f(x) < \infty$,
- (b) if $B = 0$, when $\gamma_1 < \frac{p-1}{p-2}$ with p defined by (3.4.2), then f is nonincreasing on an interval to the left of b_1 ; when $\gamma_1 \geq \frac{p-1}{p-2}$, then $J(\infty, 1/\beta) < \infty$ for some $\beta < 1/p$,

then

$$r_n^{(\cdot)}(Q(t)) \xrightarrow{\mathcal{D}} G^{(\cdot)}(Q(t)) \text{ in } D[0, 1].$$

The proofs of Theorems 9.2.7 and 9.2.8 are similar to those of Theorems 9.2.5 and 9.2.6. So we omit them here.

To construct the corresponding confidence bands for $R^{(p_0)}(t)$, we need to change the definition of the right-continuous inverse function of a right-continuous function (see (1.3.2)), so that we can implement a method developed by Song (1987).

For a non-decreasing right-continuous function $K(x)$ on the line, the left-continuous inverse K^{-1} is defined as

$$K^{-1}(y) = \inf\{x : K(x) \geq y\}, \quad 0 < y < 1.$$

Using this definition, we redefine the Q , Q_n and U_n respectively. Note that (3.1.4)-(3.1.5) and (4.1.5) are still true since F is assumed to be continuous. For any distribution function K on the real line and any $0 < t < 1$ we have (cf., e.g., Csörgő (1986), or pages 5-8 in Shorack and Wellner (1986))

$$(9.2.14) \quad K(x) \geq t \text{ if and only if } K^{-1}(t) \leq x,$$

$$(9.2.15) \quad K(x) < t \text{ if and only if } K^{-1}(t) > x.$$

For simplicity, we introduce the event

$$\begin{aligned} B_n^{(p_0)}(c; t) &= \left\{ Q_n(1 - p_0[1 - F_n(t)] - \frac{c}{n^{1/2}}) - t \leq R^{(p_0)}(t) \right. \\ &\quad \left. < Q_n(1 - p_0[1 - F_n(t)] + \frac{c}{n^{1/2}}) - t \right\}, \end{aligned}$$

where c is a positive number.

THEOREM 9.2.9 *Let $\{X_n, n \geq 0\}$ be a sequence of stationary associated random variables with common continuous distribution function F on $(0, b_1)$. Assume that for some $\nu > 0$*

$$\sum_{n=1}^{\infty} n^{\frac{13}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

Then for fixed $0 < p_0 < 1$ we have

$$(9.2.16) \quad \lim_{n \rightarrow \infty} P\{B_n^{(p_0)}(c; t); 0 \leq t \leq Q_n(1 - \varepsilon_n)\} \\ = P\left\{\sup_{0 \leq t \leq 1} |B(1 - p_0[1 - t]) - p_0 B(t)| \leq c\right\},$$

where $\varepsilon_n \downarrow 0$ and $n^{1/2}\varepsilon_n \rightarrow \infty$.

Proof. By (3.1.4)-(3.1.5) and (4.1.5), we have

$$(9.2.17) \quad \{B_n^{(p_0)}(c; t); 0 \leq t \leq Q_n(1 - \varepsilon_n)\} \\ = \{B_n^{(p_0)}(c; Q(t)); 0 \leq t \leq U_n(1 - \varepsilon_n)\} \\ = \{Q_n(1 - p_0[1 - E_n(t)] - \frac{c}{n^{1/2}}) \leq Q(1 - p_0[1 - t]) \\ < Q_n(1 - p_0[1 - E_n(t)] + \frac{c}{n^{1/2}}); 0 \leq t \leq U_n(1 - \varepsilon_n)\}.$$

We have by (3.1.5) and (9.2.15)

$$(9.2.18) \quad Q(1 - p_0[1 - t]) < Q_n(1 - p_0[1 - E_n(t)] + cn^{-1/2}) \quad \text{if and only if}$$

$$E_n(1 - p_0[1 - t]) < 1 - p_0[1 - E_n(t)] + cn^{-1/2},$$

provided that

$$1 - p_0[1 - E_n(t)] + cn^{-1/2} < 1 \quad \text{for } 0 \leq t \leq U_n(1 - \varepsilon_n).$$

This can be done since

$$E_n(U_n(t)) < t + \frac{1}{n} \quad \text{for } 0 \leq t \leq U_n(1 - \varepsilon_n),$$

$$1 - p_0[1 - E_n(t)] + cn^{-1/2} < 1 - n^{-1/2}(p_0 n^{1/2} \varepsilon_n - c - 1) < 1$$

as $n \rightarrow \infty$. On the other hand, it is easy to derive by (9.2.14) that we have

$$(9.2.19) \quad Q_n(1 - p_0[1 - E_n(t)] - \frac{c}{n^{1/2}}) \leq Q(1 - p_0[1 - t]) \quad \text{if and only if}$$

$$1 - p_0[1 - E_n(t)] - \frac{c}{n^{1/2}} \leq E_n(1 - p_0[1 - t]),$$

since

$$1 - p_0[1 - E_n(t)] - cn^{-1/2} \geq 1 - p_0 - cn^{-1/2} > 0$$

as $n \rightarrow \infty$. Now (9.2.18) and (9.2.19) yield as $n \rightarrow \infty$

$$\begin{aligned} & P\left\{ \sup_{0 \leq t \leq U_n(1-\epsilon_n)} n^{1/2} |1 - p_0[1 - E_n(t)] - E_n(1 - p_0[1 - t])| < c \right\} \\ & \leq P\{B_n^{(p_0)}(c; Q(t)); 0 \leq t \leq U_n(1 - \epsilon_n)\} \\ & \leq P\left\{ \sup_{0 \leq t \leq U_n(1-\epsilon_n)} n^{1/2} |1 - p_0[1 - E_n(t)] - E_n(1 - p_0[1 - t])| \leq c \right\}. \end{aligned}$$

Hence (9.2.16) follows by (9.2.17) and Theorem 3.3.1. $\blacksquare$

THEOREM 9.2.10 *Under the assumptions of Theorem 9.2.9, we have for fixed $t > 0$*

$$\begin{aligned} & \lim_{n \rightarrow \infty} P\{B_n^{(p_0)}(c; t); \epsilon_n \leq p_0 \leq 1\} \\ & = P\left\{ \sup_{0 \leq p_0 \leq 1} |B(1 - p_0[1 - F(t)]) - p_0 B(F(t))| \leq c \right\}, \end{aligned}$$

where $\epsilon_n \downarrow 0$ and $n^{1/2}\epsilon_n \rightarrow \infty$.

THEOREM 9.2.11 *Under the assumptions of Theorem 9.2.9, we have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} P\{B_n^{(p_0)}(c; t); Q_n(\epsilon_n) \leq t \leq Q_n(1 - \epsilon_n^{1/2}), \epsilon_n^{1/2} \leq p_0 \leq 1\} \\ & = P\left\{ \sup_{0 \leq t, p_0 \leq 1} |B(1 - p_0[1 - t]) - p_0 B(t)| \leq c \right\}, \end{aligned}$$

where $\epsilon_n \downarrow 0$ and $n^{1/2}\epsilon_n \rightarrow \infty$.

9.3 Estimation of percentile residual life with random censorship from the right.

In this section, we utilize all notations defined from (9.1.4) to (9.1.25). $\{U_n^0, n \geq 0\}$ is assumed to be a sequence of stationary associated random variables.

9.3.1 Weak convergence of $\alpha_n^*(\cdot)$ in $D[0, 1]$

In this subsection, we establish weak convergence of α_n^* of (9.1.20). Since $U_1, \dots, U_n$ are no longer $[0, 1]$ -uniform distributed random variables, the approach developed in proving Theorem 3.3.1 cannot be applied directly. Fortunately, the difference is only minor and we can still use a similar approach to show the tightness of α_n^* . First we need several lemmas.

LEMMA 9.3.1 *We have for any $i, j \geq 1$*

$$0 \leq \text{Cov}(U_i, U_j) \leq \text{Cov}(U_i^0, U_j^0).$$

Proof. Since $\{U_i^0, U_j^0\}$ and $\{V_i, V_j\}$ are independent, we have

$$\begin{aligned} EU_i U_j &= E[E(U_i U_j | V_i, V_j)] \\ &= \int_0^1 \int_0^1 E(U_i^0 \wedge v_i)(U_j^0 \wedge v_j) dH^*(v_i) dH^*(v_j) \\ &\leq \text{Cov}(U_i^0, U_j^0) + \int_0^1 \int_0^1 E(U_i^0 \wedge v_i) E(U_j^0 \wedge v_j) dH^*(v_i) dH^*(v_j) \\ &= \text{Cov}(U_i^0, U_j^0) + EU_i EU_j. \end{aligned}$$

The inequality in above follows by the fact that

$$\text{Cov}(U_i^0 \wedge v_i, U_j^0 \wedge v_j) \leq \text{Cov}(U_i^0, U_j^0),$$

which can be easily derived from Theorem 2.3.1. This proves our lemma. ■

Let $\{B^*(t), 0 \leq t \leq 1\}$ be a mean-zero Gaussian process specified by

$$(9.3.1) \quad EB^*(s)B^*(t) = F^*(s) \wedge F^*(t) - F^*(s)F^*(t) \\ + \sum_{k=1}^{\infty} \{\text{Cov}(I(U_0 \leq s), I(U_k \leq t)) + \text{Cov}(I(U_0 \leq t), I(U_k \leq s))\}.$$

LEMMA 9.3.2 *If there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^2 (\log(n+1))^{2+\nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then the series in (9.3.1) converges and for any $0 < t_1 < \dots < t_k < 1$

$$(\alpha_n^*(t_1), \dots, \alpha_n^*(t_k)) \xrightarrow{\mathcal{D}} (B^*(t_1), \dots, B^*(t_k)).$$

Proof. As we see, $\{U_n, n \geq 0\}$ is a sequence of stationary associated random variables. Hence, the proof is basically the same as that of Lemma 3.3.4 except that we have to verify that for any $0 < s < t < 1$

$$(9.3.2) \quad \sum_{j=1}^{\infty} \text{Cov}(I(U_0 \leq s), I(U_j \leq t)) < \infty.$$

Note that we cannot directly apply Lemma 9.3.1 to prove (9.3.2) since, as we mentioned before, U_0 and U_j are no longer $[0, 1]$ -uniform distributed random variables and hence the proof of Lemma 3.3.4 does not work for them. One way to solve this problem is to remove V_i from U_i ($i = 0, j$) so that Lemma 3.3.4 can be applied to U_0^0 and U_j^0 instead. For doing this, we use the technique of conditional probabilities. It is easy to check that

$$EI(U_0^0 \wedge v_0 \leq s)I(U_j^0 \wedge v_j \leq t) = EI(U_0^0 \wedge v_0 \leq s)EI(U_j^0 \wedge v_j \leq t)$$

for $0 < v_0 \leq s$ or $0 < v_j \leq t$. Hence we have

$$(9.3.3) \quad EI(U_0 \leq s)I(U_j \leq t) = E[E(I(U_0 \leq s)I(U_j \leq t)|V_0, V_j)]$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 EI(U_0^0 \wedge v_0 \leq s) I(U_j^0 \wedge v_j \leq t) dH^*(v_0) dH^*(v_j) \\
&= \iint_{A^c} EI(U_0^0 \wedge v_0 \leq s) EI(U_j^0 \wedge v_j \leq t) dH^*(v_0) dH^*(v_j) \\
&\quad + \iint_A EI(U_0^0 \leq s) I(U_j^0 \leq t) dH^*(v_0) dH^*(v_j),
\end{aligned}$$

where $A = (s, 1] \times (t, 1]$. Thus, (9.3.2) follows by Lemma 3.3.4 and (9.3.3). ■

Based on Lemma 9.3.2, the weak convergence of α_n^* can be obtained by proving its tightness. For doing this, we follow an approach similar to that developed in Subsection 3.3.2. Basically, we try to use the same notations whenever possible so that for those lemmas whose proofs are similar, they will be skipped. Otherwise, we will give the detailed proofs.

By Jensen's inequality and stationarity,

$$\begin{aligned}
&E \left\{ \sum_{i=1}^n (I[s < U_i \leq t] - (F^*(t) - F^*(s))) \right\}^4 \\
&\leq 4!n \sum_{\substack{i,j,k \geq 0 \\ i+j+k \leq n}} |EK(I[s < U_0 \leq t], I[s < U_i \leq t], I[s < U_{i+j} \leq t], I[s < U_{i+j+k} \leq t])| \\
&= 4!n \sum_{\substack{i,j,k \geq 0 \\ i+j+k \leq n}} |I_{ijk}|,
\end{aligned}$$

where the function K is defined by (2.3.2).

Now we use conditional probabilities so that the proof of Theorem 3.3.1 can be applied to $U_0^0, \dots, U_n^0$ instead of $U_0, \dots, U_n$. Let $g_i(u_i^0, v_i) = I(s < u_i^0 \wedge v_i \leq t)$. Then

$$\begin{aligned}
I_{ijk} &= E[E(I_{ijk} | V_0, V_i, V_{i+j}, V_{i+j+k})] \\
&= \int_{[0,1]^4} \prod_{i=1}^4 I_{ijk}^* dH^*(v_0) dH^*(v_i) dH^*(v_{i+j}) dH^*(v_{i+j+k}),
\end{aligned}$$

where

$$I_{ijk}^* = K[g_0(U_0^0, v_0), g_i(U_i^0, v_i), g_{i+j}(U_{i+j}^0, v_{i+j}), g_{i+j+k}(U_{i+j+k}^0, v_{i+j+k})].$$

Now we can estimate $I_{ij,k}^*$ instead of $I_{ij,k}$ (the difference is that $I_{ij,k}^*$ is a function of $v_0, v_i, v_{i+j}, v_{i+j+k}$). Since the function h defined in Subsection 3.3.2 depends on s, t, a , we rewrite it as $h(u; s, t, a) = h(u)$. Using this new h function notation, we let

$$h_1(u_1^0, v_1) = h(u_1^0; s, 1, a)I(s < v_1 \leq t) + h(u_1^0; s, t, a)I(t < v_1 \leq 1).$$

Then

$$0 \leq h_1(u_1^0, v_1) - g_1(u_1^0, v_1) \leq I(s - a < u_1^0 < s) + I(t < u_1^0 < t + a)I(t < v_1 \leq 1),$$

$$Eh_1(U_1^0, v_1) = Eg_1(U_1^0, v_1) + a_1, \quad 0 \leq a_1 \leq a.$$

Based on the above notations, we give the following two lemmas without proofs.

LEMMA 9.3.3 *We have*

$$|I_{ij,k}^*| \leq 64a + 16a^{-2} \{ \text{Cov}(U_0, U_i) + \text{Cov}(U_0, U_{i+j}) + \text{Cov}(U_0, U_{i+j+k}) \},$$

$$\begin{aligned} I_{ij,k}^* &= 3K[g_0(U_0^0, v_0), g_i(U_i^0, v_i)]K[g_{i+j}(U_{i+j}^0, v_{i+j}), g_{i+j+k}(U_{i+j+k}^0, v_{i+j+k})] + 128\theta a \\ &\quad + 22\theta a^{-2}(\text{Cov}(U_0^0, U_{i+j}^0) + \text{Cov}(U_0^0, U_{i+j+k}^0) + \text{Cov}(U_i^0, U_{i+j}^0) + \text{Cov}(U_i^0, U_{i+j+k}^0)) \end{aligned}$$

LEMMA 9.3.4 *If there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{4}{3}+\nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then we have for all $n \geq 1$

$$E \left\{ \sum_{i=1}^n (I[s < U_i \leq t] - (F^*(t) - F^*(s))) \right\}^4 \leq C_1 n^2 (n^{-\frac{1}{3}-\nu_1} + (F^*(t) - F^*(s))^{6/5})$$

with $\nu_1 = \min(\nu/3, 1/5)$.

The next lemma is an extension of (22.18) of Billingsley (1968). The proof is omitted.

LEMMA 9.3.5 *We have for any $0 \leq s \leq t \leq s + p_1, p_1 > 0$*

$$|\alpha_n^*(t) - \alpha_n^*(s)| \leq |\alpha_n^*(s + p_1) - \alpha_n^*(s)| + n^{1/2}(F^*(s + p_1) - F^*(s)).$$

LEMMA 9.3.6 *Under the assumption of Lemma 9.3.4, if $F^*(t)$ is continuous on $[0, 1]$, then $\{\alpha_n^*(t), 0 \leq t \leq 1\}$ is tight on $D[0, 1]$.*

Proof. By applying Lemma 9.3.4 to $\{I(s < U_n \leq t) - (F^*(t) - F^*(s))\}$, we get

$$(9.3.4) \quad E|\alpha_n^*(t) - \alpha_n^*(s)|^4 \leq C_1(n^{-\frac{1}{2}-\nu_1} + (F^*(t) - F^*(s))^{6/5}).$$

To get rid of the function F^* in above so that we can apply Lemma 9.3.5, we partition $[0, 1]$ in such a way that for any $0 < \varepsilon < 1$, there exist $0 = r_1 < \dots < r_m < r_{m+1} = 1$ for which we have

$$(9.3.5) \quad F^*(r_{i+1}) - F^*(r_i) = n^{-1/2}\varepsilon, \quad i = 1, 2, \dots, m-1,$$

$$(9.3.6) \quad F^*(r_{m+1}) - F^*(r_m) \leq n^{-1/2}\varepsilon.$$

This is possible since F^* is assumed to be continuous on $[0, 1]$. Obviously, $\{r_i, 1 \leq i \leq m\}$ depends on ε, n and $[\varepsilon^{-1}n^{1/2}] \leq m \leq [\varepsilon^{-1}n^{1/2}] + 1$.

For any $0 < \delta < 1$ (δ will be chosen later), choose an integer m_1 such that $r_{m_1-1} \leq \delta < r_{m_1}$. Let $t_n(1) = r_{m_1}$ and $t_n(0) = 0$. Then $t_n(1) - t_n(0) \geq \delta$. By Lemma 9.3.5 and (9.3.5),

$$\sup_{t_n(0) < t \leq t_n(1)} |\alpha_n^*(t) - \alpha_n^*(t_n(0))| \leq 3 \max_{1 \leq i \leq m_1} |\alpha_n^*(t_n(0) + r_i) - \alpha_n^*(t_n(0))| + \varepsilon.$$

This implies

$$(9.3.7) \quad \begin{aligned} P\left\{ \sup_{t_n(0) < t \leq t_n(1)} |\alpha_n^*(t) - \alpha_n^*(t_n(0))| \geq 4\varepsilon \right\} \\ \leq P\left\{ \max_{1 \leq i \leq m_1} |\alpha_n^*(t_n(0) + r_i) - \alpha_n^*(t_n(0))| \geq \varepsilon \right\}. \end{aligned}$$

Consider now the random variables

$$\alpha_n^*(t_n(0) + r_i) - \alpha_n^*(t_n(0) + r_{i-1}), \quad i = 1, \dots, m_1.$$

By (9.3.4) and (9.3.5),

$$\begin{aligned} & E|\alpha_n^*(t_n(0) + r_i) - \alpha_n^*(t_n(0) + r_{i-1})|^4 \\ & \leq \frac{C_1}{\varepsilon^{1+2\nu_1}} \{F^*(t_n(0) + r_i) - F^*(t_n(0) + r_{i-1})\}^{1+2\nu_1}. \end{aligned}$$

Hence, by (9.3.7) and Theorem 12.2 of Billingsley (1968).

$$\begin{aligned} (9.3.8) \quad & P\left\{ \sup_{t_n(0) < t \leq t_n(1)} |\alpha_n^*(t) - \alpha_n^*(t_n(0))| \geq 4\varepsilon \right\} \\ & \leq \frac{C_1}{\varepsilon^{3+2\nu_1}} \{F^*(t_n(1)) - F^*(t_n(0))\}^{1+2\nu_1}. \end{aligned}$$

On the other hand, by (9.3.5),

$$\begin{aligned} (9.3.9) \quad & F^*(t_n(1)) - F^*(t_n(0)) \\ & = F^*(t_n(1)) - F^*(t_n(0) + r_{m_1-1}) + F^*(t_n(0) + r_{m_1-1}) - F^*(t_n(0)) \\ & \leq n^{-1/2}\varepsilon + \sup_{|s-t| \leq \delta} |F^*(s) - F^*(t)|. \end{aligned}$$

To choose the next interval $(t_n(1), t_n(2)]$, we can find an integer m_2 such that $r_{m_2-1} - t_n(1) \leq \delta < r_{m_2} - t_n(1)$. Let $t_n(2) = r_{m_2}$. Then $t_n(2) - t_n(1) \geq \delta$. In general, there exist integers $m_2 < m_3 < \dots < m_k \leq m$ so that $t_n(j) = r_{m_j}$, $t_n(j) - t_n(j-1) \geq \delta$, $j = 2, 3, \dots, k$, and $1 - t_n(k) \leq \delta$. Repeating the process of proving (9.3.8) and (9.3.9), we finally obtain for $j = 1, 2, \dots, k$,

$$\begin{aligned} (9.3.10) \quad & P\left\{ \sup_{t_n(j-1) < t \leq t_n(j)} |\alpha_n^*(t) - \alpha_n^*(t_n(j-1))| \geq 4\varepsilon \right\} \\ & \leq \frac{C_1}{\varepsilon^{3+2\nu_1}} \{F^*(t_n(j)) - F^*(t_n(j-1))\}^{1+2\nu_1}, \end{aligned}$$

$$(9.3.11) \quad F^*(t_n(j)) - F^*(t_n(j-1)) \leq n^{-1/2}\varepsilon + \sup_{|s-t| \leq \delta} |F^*(s) - F^*(t)|.$$

To deal with the last interval $(t_n(k), 1]$, we let $t_n(k+1) = 1$. Similarly to the proof of (9.3.8), we have

$$(9.3.12) \quad \begin{aligned} & P\left\{ \sup_{t_n(k) < t \leq r_m} |\alpha_n^*(t) - \alpha_n^*(t_n(k))| \geq 4\varepsilon \right\} \\ & \leq \frac{C_1}{\varepsilon^{5+2\nu_1}} \{F^*(r_m) - F^*(t_n(k))\}^{1+2\nu_1}. \end{aligned}$$

By Lemma 9.3.5 and (9.3.6),

$$\begin{aligned} & \sup_{t_n(k) < t \leq t_n(k+1)} |\alpha_n^*(t) - \alpha_n^*(t_n(k))| \\ & \leq \sup_{t_n(k) < t \leq r_m} |\alpha_n^*(t) - \alpha_n^*(t_n(k))| + |\alpha_n^*(1) - \alpha_n^*(r_m)| + \varepsilon. \end{aligned}$$

This, together with (9.3.4) and (9.3.12), implies that

$$(9.3.13) \quad \begin{aligned} & P\left\{ \sup_{t_n(k) < t \leq t_n(k+1)} |\alpha_n^*(t) - \alpha_n^*(t_n(k))| \geq 6\varepsilon \right\} \\ & \leq \frac{C_1}{\varepsilon^{5+2\nu_1}} \{F^*(t_n(k+1)) - F^*(t_n(k))\}^{1+2\nu_1} + \frac{C_1}{\varepsilon^4} n^{-\frac{1}{2}-\nu_1}. \end{aligned}$$

On the other hand, (9.3.11) holds true for $j = k+1$. Finally, by applying the Corollary of Theorem 8.3 of Billingsley (1968), (9.3.10)·(9.3.11) and (9.3.13) yield

$$\begin{aligned} & P\left\{ \sup_{|s-t| \leq \delta} |\alpha_n^*(t) - \alpha_n^*(s)| \geq 18\varepsilon \right\} \\ & \leq \sum_{j=1}^{k+1} P\left\{ \sup_{t_n(j) < t \leq t_n(j-1)} |\alpha_n^*(t) - \alpha_n^*(t_n(j-1))| \geq 6\varepsilon \right\} \\ & \leq \frac{C_1}{\varepsilon^{5+2\nu_1}} \left\{ n^{-1/2}\varepsilon + \sup_{|s-t| \leq \delta} |F^*(t) - F^*(s)| \right\}^{2\nu_1} + \frac{C_1}{\varepsilon^4} n^{-\frac{1}{2}-\nu_1}. \end{aligned}$$

By the continuity of F^* on $[0, 1]$, for any $\eta > 0$, there exist a $0 < \delta < 1$ and an integer N_0 such that for $n \geq N_0$

$$\frac{C_1}{\varepsilon^{5+2\nu_1}} \left\{ n^{-1/2}\varepsilon + \sup_{|s-t| \leq \delta} |F^*(t) - F^*(s)| \right\}^{2\nu_1} + \frac{C_1}{\varepsilon^4} n^{-\frac{1}{2}-\nu_1} < \eta.$$

Thus, for $n \geq N_0$, we obtain

$$P\left\{\sup_{|s-t|\leq\delta} |\alpha_n^*(t) - \alpha_n^*(s)| \geq 18\epsilon\right\} < \eta.$$

This proves tightness of α_n^* . ■

Based on Lemmas 9.3.2 and 9.3.6, we immediately get the following theorem.

THEOREM 9.3.1 *If for some $\nu > 0$ we have*

$$\sum_{n=1}^{\infty} n^{\frac{13}{2}+\nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then

$$\alpha_n^*(\cdot) \xrightarrow{\mathcal{D}} B^*(\cdot) \text{ in } D[0, 1]$$

with $P\{B^(\cdot) \in C[0, 1]\} = 1$.*

9.3.2 Weak convergence of $\tilde{\alpha}_n(\cdot)$ in $D[0, 1]$

Let $\{\tilde{B}(t), 0 \leq t \leq 1\}$ be a mean-zero Gaussian process specified by

$$\begin{aligned} (9.3.14) \quad E\tilde{B}(s)\tilde{B}(t) &= \tilde{F}^*(s) \wedge \tilde{F}^*(t) - \tilde{F}^*(s)\tilde{F}^*(t) \\ &+ \sum_{k=1}^{\infty} \text{Cov}(I(U_0^0 \leq s \wedge V_0), I(U_k^0 \leq t \wedge V_k)) \\ &+ \sum_{k=1}^{\infty} \text{Cov}(I(U_k^0 \leq s \wedge V_k), I(U_0^0 \leq t \wedge V_0)). \end{aligned}$$

LEMMA 9.3.7 *Under the assumption of Lemma 9.3.2, the two series in (9.3.14) converge and for any $0 < t_1 < \dots < t_k < 1$*

$$(\tilde{\alpha}_n(t_1), \dots, \tilde{\alpha}_n(t_k)) \xrightarrow{\mathcal{D}} (\tilde{B}(t_1), \dots, \tilde{B}(t_k)).$$

Proof. By applying Theorem 4.2 of Jogdeo (1977), it is easy to see that $\{I(U_i^0 \leq t \wedge V_i), i = 1, \dots, n\}$ are associated for a fixed $0 \leq t \leq 1$. Hence, similarly to the

proof of Lemma 9.3.2, our lemma follows if we can show that for any $0 < s, t < 1$

$$\sum_{j=1}^{\infty} \text{Cov}(I(U_0^0 \leq s \wedge V_0), I(U_j^0 \leq t \wedge V_j)) < \infty.$$

The proof is basically the same as that of Lemma 9.3.1. So it is omitted. ■

To prove tightness of $\tilde{\alpha}_n$, one can use exactly the same approach as that of Theorem 3.3.1 by the following fact

$$\begin{aligned} 0 \leq \tilde{F}^*(t) - \tilde{F}^*(s) &= EI(s \wedge V_1 < U_1^0 \leq t \wedge V_1) \\ &= \int_s^t EI(s \wedge v_1 < U_1^0 \leq t \wedge v_1) dH^*(v_1) \\ &\leq t - s, \quad 0 \leq s < t \leq 1. \end{aligned}$$

The details are omitted.

THEOREM 9.3.2 *If for some $\nu > 0$ we have*

$$\sum_{n=1}^{\infty} n^{\frac{13}{2}+\nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then

$$\tilde{\alpha}_n(\cdot) \xrightarrow{\mathcal{D}} \tilde{B}(\cdot) \text{ in } D[0, 1]$$

with $P\{\tilde{B}(\cdot) \in C[0, 1]\} = 1$.

9.3.3 Weak convergence of $(\alpha_n^*(\cdot), \tilde{\alpha}_n(\cdot))$ in $D[0, 1] \times D[0, 1]$

Let $\{(B^*(s), \tilde{B}(t)), 0 \leq s \leq 1, 0 \leq t \leq 1\}$ be a Gaussian process defined on $D[0, 1] \times D[0, 1]$ so that its respective marginal distributions are the same as that of B^* defined by (9.3.1) and $\tilde{B}$ defined by (9.3.14), and for any $0 \leq s, t \leq 1$

$$\begin{aligned} (9.3.15) \quad EB^*(s)\tilde{B}(t) &= \tilde{F}^*(s) \wedge \tilde{F}^*(t) - F^*(s)\tilde{F}^*(t) \\ &+ \sum_{k=1}^{\infty} \left\{ \text{Cov}(I(U_0 \leq s), I(U_k^0 \leq t \wedge V_k)) + \text{Cov}(I(U_k \leq s), I(U_0^0 \leq t \wedge V_0)) \right\}. \end{aligned}$$

LEMMA 9.3.8 *Under the assumption of Lemma 9.3.2. the series in (9.3.15) converges and for any $0 < s_1 < \dots < s_k < 1$ and $0 < t_1 < \dots < t_l < 1$*

$$(\alpha_n^*(s_1), \dots, \alpha_n^*(s_k), \tilde{\alpha}_n(t_1), \dots, \tilde{\alpha}_n(t_l)) \xrightarrow{\mathcal{D}} (B^*(s_1), \dots, B^*(s_k), \hat{B}(t_1), \dots, \hat{B}(t_l)).$$

Proof. For any $a_i \geq 0$, $i = 1, \dots, k$ and $b_j \geq 0$, $j = 1, \dots, l$, let

$$M(u, v) = \sum_{i=1}^k a_i I(u \wedge v \leq s_i) + \sum_{j=1}^l b_j I(u \leq t_j \wedge v).$$

Clearly, for each fixed v , $M(u, v)$ is a nonincreasing function of u . Let $\xi_i = M(U_i^0, V_i)$, $i = 1, \dots, n$. First we claim that $\{\xi_n, n \geq 0\}$ is a stationary sequence of weakly associated random variables. To see this, let π be an arbitrary permutation of $\{1, 2, \dots, n\}$, $1 \leq r < n$, and $f: \mathbf{R}^r \rightarrow \mathbf{R}, g: \mathbf{R}^{n-r} \rightarrow \mathbf{R}$ be increasing. Then

$$\begin{aligned} Ef(\xi_{\pi(1)}, \dots, \xi_{\pi(r)})g(\xi_{\pi(r+1)}, \dots, \xi_{\pi(n)}) &= E\{E[fg|V_1, \dots, V_n]\} \\ &= \int Ef(\eta_{\pi(1)}^*, \dots, \eta_{\pi(r)}^*)g(\eta_{\pi(r+1)}^*, \dots, \eta_{\pi(n)}^*)dH^*(v_1), \dots, dH^*(v_n) \\ &\geq \int Ef(\eta_{\pi(1)}^*, \dots, \eta_{\pi(r)}^*)Eg(\eta_{\pi(r+1)}^*, \dots, \eta_{\pi(n)}^*)dH^*(v_1), \dots, dH^*(v_n) \\ &= Ef(\xi_{\pi(1)}, \dots, \xi_{\pi(r)})Eg(\xi_{\pi(r+1)}, \dots, \xi_{\pi(n)}), \end{aligned}$$

where $\xi_i^* = M(U_i^0, v_i)$, $i = 1, \dots, n$ and the inequality above follows by the fact that $\eta_1^*, \dots, \eta_n^*$ are associated. This proves that $\{\xi_n, n \geq 0\}$ is a sequence of weakly associated random variables by Definition 2.1.4.

On the other hand, we have

$$\sum_{i=1}^k a_i \alpha_n^*(s_i) + \sum_{j=1}^l b_j \tilde{\alpha}_n(t_j) = n^{-1/2} \sum_{i=1}^n \{\xi_i - E\xi_i\}.$$

Hence, in order to apply Theorem 1 of Burton, Dabrowski and Dehling (1986) to prove the weak convergence of the finite dimensional distributions of $(\alpha_n^*(\cdot), \tilde{\alpha}_n(\cdot))$ to the corresponding Gaussian distribution, one needs to show that

$$\sum_{k=2}^{\infty} \text{Cov}(\xi_1, \xi_k) < \infty.$$

By Lemmas 9.3.2 and 9.3.7, it suffices to show that for any $0 < s, t < 1$ and $a > 0$

$$(9.3.16) \quad 0 \leq \text{Cov}(I(U_i^0 \wedge V_i \leq s), I(U_j^0 \leq t \wedge V_j)) \leq 2a + a^{-2} \text{Cov}(U_i^0, U_j^0).$$

The positive covariance above follows by the fact that $I(U_i^0 \wedge V_i \leq s)$ and $I(U_j^0 \leq t \wedge V_j)$ are weakly associated (the same proof as that of $\xi_1, \dots, \xi_n$ above). To show the upper bound, we have

$$\begin{aligned} & \text{Cov}(I(U_i^0 \wedge V_i \leq s), I(U_j^0 \leq t \wedge V_j)) \\ &= \text{Cov}(I(U_i^0 \wedge V_i \leq s, U_i^0 \leq V_i), I(U_j^0 \leq t \wedge V_j)) \\ & \quad + \text{Cov}(I(U_i^0 \wedge V_i \leq s, U_i^0 > V_i), I(U_j^0 \leq t \wedge V_j)) \\ &= \text{Cov}(I(U_i^0 \leq s \wedge V_i), I(U_j^0 \leq t \wedge V_j)) \\ & \quad + \text{Cov}(I(U_i^0 > V_i, V_i \leq s), I(U_j^0 \leq t \wedge V_j)) \\ &\leq \text{Cov}(I(U_i^0 \leq s \wedge V_i), I(U_j^0 \leq t \wedge V_j)), \end{aligned}$$

where the last inequality follows by the fact that $I(U_i^0 > v_i)$ and $-I(U_j^0 \leq t \wedge v_j)$ are associated, and hence

$$\begin{aligned} & \text{Cov}(I(U_i^0 > V_i, V_i \leq s), I(U_j^0 \leq t \wedge V_j)) \\ &= \int_0^s \int_0^1 \text{Cov}(I(U_i^0 > v_i), I(U_j^0 \leq t \wedge v_j)) dH^*(v_i) dH^*(v_j) \leq 0. \end{aligned}$$

Thus (9.3.16) follows by Lemma 9.3.7. From (9.3.16), one can show that the series in (9.3.15) converges. ■

THEOREM 9.3.3 *If for some $\nu > 0$ we have*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then

$$(\alpha_n^*(\cdot), \tilde{\alpha}_n(\cdot)) \xrightarrow{\mathcal{D}} (B^*(\cdot), \tilde{B}(\cdot)) \text{ in } D[0, 1] \times D[0, 1].$$

Proof. The weak convergence of the finite dimensional distributions of $(\alpha_n^*(\cdot), \hat{\alpha}_n(\cdot))$ is shown in Lemma 9.3.8. Hence it remains only to prove the tightness. Theorems 9.3.1 and 9.3.2 already imply the tightnesses of α_n^* and $\hat{\alpha}_n$ respectively. Consequently $(\alpha_n^*, \hat{\alpha}_n)$ induces a tight sequence of distributions on the product space $D[0, 1] \times D[0, 1]$. This proves our theorem. ■

9.3.4 Weak convergence of cumulative hazard process

The empirical cumulative hazard process Λ_n is defined by

$$\Lambda_n(t) = \sum_{i=1}^n \frac{I(U_i \leq t) \delta_i}{n+1-R_i}, \quad 0 \leq t \leq 1.$$

From the definitions of F_n^* and $\tilde{F}_n^*$, Λ_n may be written as

$$\Lambda_n(t) = \int_0^t (1 - F_n^*(s))^{-1} d\tilde{F}_n^*(s), \quad 0 \leq t < U_{n:n}.$$

Naturally, we expect that the limit of Λ_n will be

$$\Lambda(t) = \int_0^t (1 - F^*(s))^{-1} d\tilde{F}^*(s) = -\ln(1-t), \quad 0 \leq t < 1.$$

We define the cumulative hazard process as

$$\hat{Z}_n(t) = n^{1/2}(\Lambda_n(t) - \Lambda(t)), \quad 0 \leq t < U_{n:n}$$

Integrating by parts, we have (see (7.9) of Breslow and Crowley (1974))

$$\hat{Z}_n(t) = A_n(t) + B_n(t) + R_{1n}(t) + R_{2n}(t),$$

where

$$A_n(t) = \int_0^t \frac{\alpha_n^*(s)}{(1 - F^*(s))^2} d\tilde{F}^*(s)$$

$$B_n(t) = \hat{\alpha}_n(t)(1 - F^*(t))^{-1} - \int_0^t \frac{\hat{\alpha}_n(s)}{(1 - F^*(s))^2} dF^*(s)$$

$$R_{1n}(t) = n^{-1/2} \int_0^t \frac{\alpha_n^{*2}(s)}{(1 - F^*(s))^2(1 - F_n^*(s))} d\tilde{F}^*(s)$$

$$R_{2n}(t) = \int_0^t \frac{\alpha_n^*(t)}{(1 - F^*(s))(1 - F_n^*(s))} d(\tilde{F}_n^*(s) - \tilde{F}^*(s)).$$

THEOREM 9.3.4 *If for some $\nu > 0$ we have*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then for any $0 < p_1 < 1$ satisfying $F^(p_1) < 1$*

$$\hat{Z}_n(\cdot) \xrightarrow{\mathcal{D}} \hat{H}(\cdot) \text{ in } D[0, p_1],$$

where $\{\hat{H}(t); 0 \leq t \leq p_1\}$ is a Gaussian process defined by

$$(9.3.17) \quad \hat{H}(t) = \int_0^t \frac{B^*(s)}{(1 - F^*(s))^2} d\tilde{F}^*(s) + \tilde{B}(t)(1 - F^*(t))^{-1}$$

$$- \int_0^t \frac{\tilde{B}(s)}{(1 - F^*(s))^2} F^*(s).$$

Proof. Using the result of Theorem 9.3.3, the proof is the same as that of Theorem 4 of Breslow and Crowley (1974), except that one has to verify that

$$(9.3.18) \quad I = \sup_{0 \leq t \leq p_1} \left| \int_0^t \frac{B^*(s)}{(1 - F^*(s))^2} d(\tilde{F}_n^*(s) - \tilde{F}^*(s)) \right| \xrightarrow{P} 0.$$

For any $0 < \delta < 1$, let $t_i = i\delta$, $i = 0, 1, \dots, [p_1\delta^{-1}]$ and $t_{[p_1\delta^{-1}]+1} = p_1$. Then

$$I \leq 2 \max_i \sup_{t_i < t \leq t_{i+1}} \left| \frac{B^*(s)}{(1 - F^*(s))^2} - \frac{B^*(t_i)}{(1 - F^*(t_i))^2} \right|$$

$$+ \frac{\sup_{0 \leq t \leq 1} |B^*(t)|}{n^{1/2}(1 - F^*(p_1))^2} \sum_i |\hat{\alpha}_n(t_{i+1}) - \hat{\alpha}_n(t_i)|$$

$$\leq \frac{2}{(1 - F^*(p_1))^2} \sup_{|t-s| \leq \delta} |B^*(s) - B^*(t)|$$

$$+ \frac{4 \sup_{0 \leq t \leq 1} |B^*(t)|}{(1 - F^*(p_1))^3} \sup_{|t-s| \leq \delta} |F^*(s) - F^*(t)|$$

$$+ \frac{\sup_{0 \leq t \leq 1} |B^*(t)| \sup_{|t-s| \leq \delta} |\hat{\alpha}_n(s) - \hat{\alpha}_n(t)|}{(1 - F^*(p_1))^2 n^{1/2} \delta}.$$

Choose $\delta = n^{-1/2}$. Then (9.3.17) follows since B^* and F^* are continuous on $[0, 1]$, and $\{\tilde{\alpha}_n(t), 0 \leq t \leq 1\}$ is tight. This also completes our proof. ■

9.3.5 Weak convergence of the uniform PL-process α_n

To show the weak convergence of the uniform PL-process α_n , we need to use Lemma 1 of Breslow and Crowley (1974).

LEMMA 9.3.9 (Breslow and Crowley (1974)) For $0 < t < U_{n:n}$, we have

$$0 < -\ln(1 - F_n^{*0}(t)) - \Lambda_n(t) < \frac{F_n^*(t)}{n(1 - F_n^*(t))}, \quad \text{a.s.}$$

From Lemma 9.3.9, we can derive that for $0 < t < U_{n:n}$,

$$n^{1/2} |\exp\{\ln(1 - F_n^{*0}(t))\} - \exp\{-\Lambda_n(t)\}| \leq \frac{F_n^*(t)}{n^{1/2}(1 - F_n^*(t))}.$$

Hence, using Taylor expansion, we get

$$\begin{aligned} (9.3.19) \quad \alpha_n(t) &= n^{1/2} (\exp\{-\Lambda(t)\} - \exp\{-\Lambda_n(t)\}) \\ &\quad + n^{1/2} |\exp\{\ln(1 - F_n^{*0}(t))\} - \exp\{-\Lambda_n(t)\}| \\ &= \exp\{-\Lambda(t)\} \hat{Z}_n(t) + \frac{1}{2n^{1/2}} \exp\{-\Lambda_n^*(t)\} \hat{Z}_n^2(t) + \frac{F_n^*(t)}{n^{1/2}(1 - F_n^*(t))}, \end{aligned}$$

where $\Lambda_n^*(t)$ is between $\Lambda_n(t)$ and $\Lambda(t)$, and hence it is non-negative.

Now from Theorem 9.3.1, we can derive that $|F^*(U_{n:n}) - 1| \rightarrow 0$ in probability which implies that $\liminf_{n \rightarrow \infty} U_{n:n} > p_1$ in probability for any $0 < p_1 < 1$ such that $F^*(p_1) < 1$. This, together with (9.3.18), will give us the following theorem.

THEOREM 9.3.5 If for some $\nu > 0$ we have

$$\sum_{n=1}^{\infty} n^{\frac{1}{2} + \nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then for any $0 < p_1 < 1$ satisfying $F^*(p_1) < 1$

$$\alpha_n(\cdot) \xrightarrow{\mathcal{D}} H(\cdot) \text{ in } D[0, p_1],$$

where $\{H(t); 0 \leq t \leq p_1\}$ is a Gaussian process defined by

$$(9.3.20) \quad H(t) = (1 - t)\hat{H}(t).$$

The covariance structure of $H(\cdot)$ is calculated explicitly in Subsection 9.3.7. .

9.3.6 Weak convergence of the PL-quantile process u_n

THEOREM 9.3.6 *If for some $\nu > 0$ we have*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(U_0^0, U_n^0) < \infty,$$

then for any $0 < p_1 < 1$ satisfying $F^*(p_1) < 1$

$$u_n(\cdot) \xrightarrow{\mathcal{D}} H(\cdot) \text{ in } D[0, p_1].$$

Proof. The proof follows the lines of that of Theorem 3.1 of Aly, Csörgö and Horváth (1985). Hence here we give only a skeleton of the proof.

The PL-quantile process u_n can be written as

$$\begin{aligned} u_n(y) &= \alpha_n(y) + (\alpha(U_n^{*0}(y)) - \alpha_n(y)) + n^{1/2}(F_n^{*0}(U_n^{*0}(y)) - y) \\ &= \alpha_n(y) + I_{1n}(y) + I_{2n}(y). \end{aligned}$$

There exists a p^* such that

$$p_1 < p^* < 1, \quad F^*(p^*) < 1, \quad U_n^{*0}(p_1) \leq p^* \text{ a.s.}$$

Hence

$$\sup_{0 \leq y \leq p_1} |U_n^{*0}(y) - y| \leq \sup_{0 \leq t \leq U_n^{*0}(p_1)} |t - F_n^{*0}(t)| \rightarrow 0 \text{ a.s.}$$

This proves that

$$\sup_{0 \leq y \leq p_1} |I_{1n}(y)| \rightarrow 0 \text{ in } P.$$

Let $\Delta_t F_n^{*0}$ denote the jump of F_n^{*0} at the nearest observation to t which is uncensored and is equal or larger than t . Then

$$\limsup_{n \rightarrow \infty} \sup_{0 \leq t \leq p^*} n \Delta_t F_n^{*0} \leq \frac{2}{1 - H^*(p^*)} \text{ in } P.$$

Thus

$$\begin{aligned} \sup_{0 \leq y \leq p_1} |I_{2n}(y)| &\leq n^{-1/2} \sup_{0 \leq t \leq p_1} n \Delta_{U_n^{*0}(t)} F_n^{*0} \\ &\leq n^{-1/2} \sup_{0 \leq t \leq p^*} n \Delta_t F_n^{*0} \\ &\rightarrow 0 \text{ in } P. \end{aligned}$$

This completes our proof. ■

9.3.7 Calculation of covariance structure of $H(\cdot)$

In this subsection, we calculate the covariance structure of the Gaussian process $H(\cdot)$. Intuitively, the covariance structure of $H(\cdot)$ should consist of two parts, where the first part is the same as that in the i.i.d. case, while the second part reflects the influence of dependence. Since we assume that $\{U_n^0, n \geq 1\}$ and $\{V_n, n \geq 1\}$ are independent, the second part should only involve the dependent variables $\{U_n^0, n \geq 1\}$. The following theorem verifies our conjecture.

THEOREM 9.3.7 *For the Gaussian process $\hat{H}(\cdot)$ defined by (9.3.17), we have for $0 \leq s, t \leq p_1 < 1$*

$$\begin{aligned} (9.3.21) \quad E \hat{H}(s) \hat{H}(t) &= d^*(s \wedge t) \\ &+ (1-s)^{-1} (1-t)^{-1} \sum_{k=2}^{\infty} \{ \text{Cov}(I(U_1^0 \leq s), I(U_k^0 \leq t)) + \text{Cov}(I(U_1^0 \leq t), I(U_k^0 \leq s)) \} \end{aligned}$$

with

$$d^*(t) = \int_0^t (1 - F^*(u))^{-2} d\tilde{F}^*(u) = \int_0^t \frac{1}{(1-u)^2(1-H^*(u))} du.$$

Consequently, for the Gaussian process $H(\cdot)$ defined by (9.3.20), we have for $0 \leq s, t \leq p_1$

$$(9.3.22) \quad EH(s)H(t) = (1-s)(1-t)d^*(s \wedge t) + \sum_{k=2}^{\infty} \{ \text{Cov}(I(U_1^0 \leq s), I(U_k^0 \leq t)) + \text{Cov}(I(U_1^0 \leq t), I(U_k^0 \leq s)) \}.$$

Proof. Let $\hat{d}(t) = (1 - F^*(t))^{-1}$, $0 \leq t \leq p_1$. Then, by the definition of $\hat{H}(\cdot)$,

$$\hat{H}(t) = \int_0^t B^*(u) dd^*(u) + \hat{d}(t)\tilde{B}(t) - \int_0^t \tilde{B}(u) d\hat{d}(u).$$

Hence

$$(9.3.23) \quad E\hat{H}(s)\hat{H}(t) = \hat{d}(s)\hat{d}(t)E\tilde{B}(s)\tilde{B}(t) + \int_0^s \int_0^t EB^*(u)B^*(v) dd^*(u) dd^*(v) + \int_0^s \int_0^t E\tilde{B}(u)\tilde{B}(v) d\hat{d}(u) d\hat{d}(v) + \hat{d}(s) \int_0^t E^*B(v)\tilde{B}(s) dd^*(v) + \hat{d}(t) \int_0^s E^*B(u)\tilde{B}(t) dd^*(u) - \hat{d}(s) \int_0^t E\tilde{B}(s)\tilde{B}(v) d\hat{d}(v) - \hat{d}(t) \int_0^s E\tilde{B}(u)\tilde{B}(t) d\hat{d}(u) - \int_0^s \int_0^t E^*B(u)\tilde{B}(v) dd^*(u) d\hat{d}(v) - \int_0^t \int_0^s E^*B(u)\tilde{B}(v) dd^*(u) d\hat{d}(v).$$

It is easy to show that

$$\begin{aligned} & \int_0^s \int_0^t \text{Cov}(I(U_1 \leq u), I(U_k \leq v)) dd^*(u) dd^*(v) \\ &= \text{Cov}(d^*(s) - d^*(s \wedge U_1), d^*(t) - d^*(t \wedge U_k)) \end{aligned}$$

and

$$\begin{aligned} & \int_0^s \int_0^t \text{Cov}(I(U_1^0 \leq u \wedge V_1), I(U_k^0 \leq v \wedge V_k)) d\hat{d}(u) d\hat{d}(v) \\ &= \text{Cov}((\hat{d}(s) - \hat{d}(s \wedge U_1^0))I(U_1^0 \leq V_1), (\hat{d}(t) - \hat{d}(t \wedge U_k^0))I(U_k^0 \leq V_k)). \end{aligned}$$

By using the above two equalities together with (9.3.1), (9.3.14) and (9.3.15), we can simply rewrite (9.3.23) as

$$(9.3.24) \quad E\dot{H}(s)\dot{H}(t) = d^*(s \wedge t) + \sum_{k=2}^{\infty} \{ \text{Cov}(g_s(U_1^0, V_1), g_t(U_k^0, V_k)) + \text{Cov}(g_s(U_k^0, V_k), g_t(U_1^0, V_1)) \},$$

where

$$\begin{aligned} g_t(u, v) &= d^*(t) - d^*(t \wedge u \wedge v) + (1 - F^*(u))^{-1} I(u \leq t \wedge v) \\ &= \int_0^t I(u \wedge v \leq x) dd^*(x) + (1 - F^*(u))^{-1} I(u \leq t \wedge v). \end{aligned}$$

Comparing (9.3.21) with (9.3.24), we have to remove the independent sequence $\{V_n, n \geq 1\}$ inside of the covariance series in (9.3.24). For showing this we have

$$\begin{aligned} E g_t(U_1^0, V_1) &= E \int_0^t \frac{I(U_1 \leq u)}{(1 - F^*(u))^2} d\tilde{F}^*(u) + E \frac{I(U_1^0 \leq t \wedge V_1)}{1 - F^*(U_1^0)} \\ &= \int_0^t \frac{F^*(u)}{(1 - F^*(u))^2} d\tilde{F}^*(u) + \int_0^t \frac{E I(V_1 \geq u)}{1 - F^*(u)} du \\ &= \int_0^t (1 - F^*(u))^{-2} d\tilde{F}^*(u) + \ln(1 - t) + \int_0^t \frac{1 - H^*(u)}{(1 - u)(1 - H^*(u))} du \\ &= d^*(t). \end{aligned}$$

Thus for $k \geq 2$

$$\begin{aligned} (9.3.25) \quad & \text{Cov}(g_s(U_1^0, V_1), g_t(U_k^0, V_k)) \\ &= E \int_0^s I(U_1^0 \wedge V_1 \leq x) dd^*(x) \int_0^t I(U_k^0 \wedge V_k \leq y) dd^*(y) \\ & \quad + E \int_0^s I(U_1^0 \wedge V_1 \leq x) dd^*(x) (1 - F^*(U_k^0))^{-1} I(U_k^0 \leq t) I(U_k^0 \leq V_k) \\ & \quad + E (1 - F^*(U_1^0))^{-1} I(U_1^0 \leq s) I(U_1^0 \leq V_1) \int_0^t I(U_k^0 \wedge V_k \leq y) dd^*(y) \\ & \quad + E (1 - F^*(U_1^0))^{-1} I(U_1^0 \leq s) I(U_1^0 \leq V_1) (1 - F^*(U_k^0))^{-1} I(U_k^0 \leq s) I(U_k^0 \leq V_k) \end{aligned}$$

$$-d^*(s)d^*(t)$$

$$= I_1 + I_2 + I_3 + I_4 - d^*(s)d^*(t).$$

Let $F_{1k}^{*0}(x, y) = P\{U_1^0 \leq x, U_k^0 \leq y\}$. Then, by the assumption of independence of $\{U_n^0, n \geq 1\}$ and $\{V_n, n \geq 1\}$ and conditional expectation, we have

$$\begin{aligned}
 (9.3.26) \quad I_1 &= \int_0^s \int_0^t EI(U_1^0 \wedge V_1 \leq x) I(U_k^0 \wedge V_k \leq y) dd^*(x) dd^*(y) \\
 &= \int_0^s \int_0^t \int_0^1 \int_0^1 EI(u \wedge V_1 \leq x) EI(v \wedge V_k \leq y) dF_{1k}^{*0}(u, v) dd^*(x) dd^*(y) \\
 &= \int_0^1 \int_0^1 \left\{ \int_0^s (1 - EI(x < V_1) I(x < u)) dd^*(x) \right\} \\
 &\quad \left\{ \int_0^t (1 - EI(y < V_k) I(y < v)) dd^*(y) \right\} dF_{1k}^{*0}(u, v) \\
 &= \int_0^1 \int_0^1 (d^*(s) - \int_0^{s \wedge u} \frac{1}{(1-x)^2} dx) (d^*(t) - \int_0^{t \wedge v} \frac{1}{(1-y)^2} dy) dF_{1k}^{*0}(u, v) \\
 &= d^*(s)d^*(t) - d^*(s) \int_0^1 \frac{t \wedge v}{1 - t \wedge v} dv - d^*(t) \int_0^1 \frac{s \wedge u}{1 - s \wedge u} du \\
 &\quad + \int_0^1 \int_0^1 \frac{s \wedge u}{1 - s \wedge u} \frac{t \wedge v}{1 - t \wedge v} dF_{1k}^{*0}(u, v) \\
 &= d^*(s)d^*(t) + d^*(s) \ln(1-t) + d^*(t) \ln(1-s) \\
 &\quad + \int_s^1 \int_0^t \frac{s}{1-s} \frac{v}{1-v} dF_{1k}^{*0}(u, v) + \int_0^s \int_t^1 \frac{u}{1-u} \frac{t}{1-t} dF_{1k}^{*0}(u, v) \\
 &\quad + \int_0^s \int_0^t \frac{u}{1-u} \frac{v}{1-v} dF_{1k}^{*0}(u, v) + \frac{s}{1-s} \frac{t}{1-t} P\{U_1^0 > s, U_k^0 > t\}.
 \end{aligned}$$

We have also

$$\begin{aligned}
 (9.3.27) \quad I_2 &= \int_0^s \int_0^t \int_0^1 (1 - EI(x < V_1) I(x < u)) (1 - F^*(v))^{-1} \\
 &\quad EI(v \leq V_k) dF_{1k}^{*0}(u, v) dd^*(x) \\
 &= \int_0^1 \int_0^t (d^*(s) - \frac{s \wedge u}{1 - s \wedge u}) \frac{1}{1-v} dF_{1k}^{*0}(u, v) \\
 &= -d^*(s) \ln(1-t) - \int_0^s \int_0^t \frac{u}{(1-u)(1-v)} dF_{1k}^{*0}(u, v)
 \end{aligned}$$

$$- \int_s^1 \int_0^t \frac{s}{1-s} \frac{1}{1-v} dF_{1k}^{*0}(u, v).$$

Similarly,

$$(9.3.28) \quad I_3 = -d^*(t) \ln(1-s) - \int_0^s \int_0^t \frac{v}{(1-u)(1-v)} dF_{1k}^{*0}(u, v) \\ - \int_0^s \int_t^1 \frac{1}{1-u} \frac{t}{1-t} dF_{1k}^{*0}(u, v).$$

Finally, we have

$$(9.3.29) \quad I_4 = \int_0^s \int_0^t E \frac{I(u \leq V_1)}{1-F^*(u)} E \frac{I(v \leq V_k)}{1-F^*(v)} dF_{1k}^{*0}(u, v) \\ = \int_0^s \int_0^t \frac{1}{1-u} \frac{1}{1-v} dF_{1k}^{*0}(u, v).$$

Now (9.3.21) follows by (9.3.25)-(9.3.29). ■

9.3.8 The asymptotic distribution of $r_n^{(p_0)}$ under random censorship from the right.

By Theorems (9.3.5)-(9.3.7), it is easy to derive the following result.

THEOREM 9.3.8 *Let $t > 0$ and $0 < p_0 < 1$ be fixed such that $t \vee Q(1 - p_0(1 - F^0(t))) < b_1$, and assume that $f^0 = F^{0'}$ exists, the density-quantile function $f^0(Q(y))$ is positive and continuous at $1 - p_0(1 - F^0(t))$, and there exists a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{1}{2}+\nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

Then

$$r_n^{(p_0)}(t) \xrightarrow{\mathcal{D}} p_0 H(F^0(t)) - H(1 - p_0(1 - F^0(t))) = N(0, \sigma_{F^0(t), p_0}^2),$$

where

$$\begin{aligned}
 (9.3.30) \quad \sigma_{F^0(t), p_0}^2 &= E\{p_0 H(F^0(t)) - H(1 - p_0(1 - F^0(t)))\}^2 \\
 &= (1 - F^0(t))^2 p_0^2 \{d(Q(1 - p_0(1 - F^0(t)))) - d(t)\} \\
 &\quad + 2 \sum_{k=2}^{\infty} \text{Cov}(h_{F^0(t), p_0}(U_1^0), h_{F^0(t), p_0}(U_k^0))
 \end{aligned}$$

and

$$d(t) = \int_{-\infty}^t (1 - F(x))^{-2} d\hat{F}(x).$$

Based on Theorem 9.3.8, we are able to get the following result. Assume that for fixed $t > 0$ and $0 < p_0 < 1$

$$\sigma_n^{2(p_0)}(t) = \sigma_n^{2(p_0)}(t, X_1, \dots, X_n, \delta_1, \dots, \delta_n) \xrightarrow{P} \sigma_{F^0(t), p_0}^2.$$

THEOREM 9.3.9 *Let $t > 0$ and $0 < p_0 < 1$ be fixed such that $t \vee Q(1 - p_0(1 - F^0(t))) < b_1$, and assume that $f^0 = F^{0'}$ exists, the density-quantile function $f^0(Q(y))$ is positive and continuous at $1 - p_0(1 - F^0(t))$, and there is a positive constant ν such that*

$$\sum_{n=1}^{\infty} n^{\frac{13}{2} + \nu} \text{Cov}(F(X_0), F(X_n)) < \infty.$$

If f_n^0 is any sequence of density estimators for f^0 that is uniformly consistent in a neighborhood of $Q(1 - p_0(1 - F^0(t)))$, then

$$\begin{aligned}
 \lim_{n \rightarrow \infty} P\left\{-\frac{\lambda \sigma_n^{(p_0)}(t)}{n^{1/2} f_n^0(Q_n(1 - p_0[1 - F_n(t)]))} + R_n^{(p_0)}(t) \leq R^{(p_0)}(t) \leq R_n^{(p_0)}(t) \right. \\
 \left. + \frac{\lambda \sigma_n^{(p_0)}(t)}{n^{1/2} f_n^0(Q_n(1 - p_0[1 - F_n(t)]))}\right\} \\
 = 1 - \alpha.
 \end{aligned}$$

THEOREM 9.3.10 *Let $t > 0$ and $0 < p_0 < 1$ be fixed. Then, under the assumptions of Theorem 9.3.8, we have*

$$\begin{aligned} \lim_{n \rightarrow \infty} P\{Q_n(1 - p_0[1 - F_n^0(t)] - \frac{\lambda}{n^{1/2}}\sigma_n^{(p_0)}(t)) - t \leq R^{(p_0)}(t) \\ \leq Q_n(1 - p_0[1 - F_n^0(t)] + \frac{\lambda}{n^{1/2}}\sigma_n^{(p_0)}(t)) - t\} \\ = 1 - \alpha. \end{aligned}$$

9.3.9 Weak convergence of $r_n^{(p_0)}(t)$ as a stochastic process of t or p_0 , and confidence bands.

In this subsection, we simply present the following results without proofs since the proofs can be easily implemented from previous results.

THEOREM 9.3.11 *For a fixed $p_0 \in (0, 1)$, let $T \in (0, T_F)$ be fixed such that $Q(1 - p_0(1 - F^0(T))) < T_F$. Assume that $f^0 = F^{0'}$ is continuous and positive on $[Q(1 - p_0), Q(1 - p_0(1 - F^0(T)))]$. Then*

$$r_n^{(p_0)}(Q(\cdot)) \xrightarrow{\mathcal{D}} G^{(p_0)}(Q(\cdot)) \quad \text{on } D[0, F^0(T)],$$

where the Gaussian process $G^{(p_0)}$ is defined as

$$G^{(p_0)}(t) = -H(1 - p_0(1 - F^0(t))) + p_0 H(F^0(t)).$$

THEOREM 9.3.12 *For a fixed $t \in (0, T_F)$, let $\Gamma \in (0, 1)$ be fixed such that $Q(1 - \Gamma(1 - F^0(t))) < T_F$. Assume that $f^0 = F^{0'}$ is continuous and positive on $[t, Q(1 - \Gamma(1 - F^0(t)))]$. Then*

$$r_n^{(\cdot)}(Q(t)) \xrightarrow{\mathcal{D}} G^{(\cdot)}(Q(t)) \quad \text{on } D[\Gamma, 1],$$

Let

$$a_n(c; p_0, t) = 1 - p_0(1 - F_n^0(t)) - n^{-1/2}c.$$

$$\begin{aligned} b_n(c; p_0, t) &= 1 - p_0(1 - F_n^0(t)) + n^{-1/2}c, \\ D'_n(c; p_0, t) &= \{Q_n(a_n(c; p_0, t)) - t \leq R^{(p_0)}(t)\}, \\ D_n(c; p_0, t) &= \{R^{(p_0)}(t) \leq Q_n(b_n(c; p_0, t)) + t\}. \end{aligned}$$

THEOREM 9.3.13 *For a fixed $p_0 \in (0, 1)$, let $T \in (0, T_F)$ be fixed such that $Q(1 - p_0(1 - F^0(T))) < T_F$. Assume that F^0 is continuous and strictly increasing on $[Q(1 - p_0), Q(1 - p_0(1 - F^0(T)))]$. Then*

$$\begin{aligned} &\lim_{n \rightarrow \infty} P\{D'_n(c; p_0, t) \cap D_n(c; p_0, t) : 0 \leq t \leq T\} \\ &= P\left\{\sup_{0 \leq y \leq F^0(T)} |-H(1 - p_0(1 - y)) + p_0H(y)| \leq c\right\}. \end{aligned}$$

THEOREM 9.3.14 *For a fixed $t \in (0, T_F)$, let $\Gamma \in (0, 1)$ be fixed such that $Q(1 - \Gamma(1 - F^0(t))) < T_F$. Assume that F^0 is continuous and strictly increasing on $[t, Q(1 - \Gamma(1 - F^0(t)))]$. Then*

$$\begin{aligned} &\lim_{n \rightarrow \infty} P\{D'_n(c; p_0, t) \cap D_n(c; p_0, t) : \Gamma \leq p_0 \leq 1\} \\ &= P\left\{\sup_{\Gamma \leq p_0 \leq 1} |-H(1 - p_0(1 - F^0(t)) + p_0H(F^0(t)))| \leq c\right\}. \end{aligned}$$

THEOREM 9.3.15 *Let $T \in (0, T_F)$ and $\Gamma \in (0, 1)$ such that $Q(1 - \Gamma(1 - F^0(T))) < T_F$. Assume that F^0 is continuous and strictly increasing on $(0, T_F)$. Then for $n^{1/2}\varepsilon_n \rightarrow \infty$,*

$$\begin{aligned} &\lim_{n \rightarrow \infty} P\{D'_n(c; p_0, t) \cap D_n(c; p_0, t) : Q_n(\varepsilon_n) \leq t \leq T, \Gamma \leq p_0 \leq 1\} \\ &= P\left\{\sup_{0 \leq t \leq T, \Gamma \leq p_0 \leq 1} |-H(1 - p_0(1 - F^0(t)) + p_0H(F^0(t)))| \leq c\right\} \\ &= P\left\{\sup_{0 \leq y \leq F^0(T), \Gamma \leq p_0 \leq 1} |-H(1 - p_0(1 - y)) + p_0H(y)| \leq c\right\}. \end{aligned}$$

Chapter 10

Bootstrap

In this chapter, we propose a circular block resampling procedure to modify Künsch's moving block bootstrap. Our procedure has the special feature that the resampled data are like drawing from the empirical distribution function of dependent observations. No information is lost concerning the nature of dependency of the original observations coming from a general stationary sequence. We prove two general theorems on bootstrapping sample means for stationary sequences. We also prove a basic theorem on weak convergence for a bootstrapped empirical process of a general stationary sequence in D space. Applications to bootstrap the sample means of stationary α -mixing, ρ -mixing and ϕ -mixing sequences are discussed. In particular, an almost sure weak convergence for a bootstrapped empirical process of a stationary ρ -mixing sequence is obtained under logarithmic mixing rates. Applications to the bootstrap of a general statistic of a stationary ρ -mixing sequence are also studied by means of influence functions.

10.1 Introduction.

It is well known that in the case when $\{X_n, n \geq 1\}$ is a sequence of i.i.d random variables with a common continuous distribution function F , we have the following

result, due to Donsker (1952),

$$(10.1.1) \quad \alpha_n(\cdot) \xrightarrow{\mathcal{D}} B^*(\cdot) \text{ in } D[0, 1],$$

where α_n is defined by (3.1.6) and B^* is a Brownian bridge, a zero-mean Gaussian process specified by $EB^*(s)B^*(t) = s \wedge t - st$.

Distribution functions of functionals of statistical interest have been extensively tabulated and used in asymptotic statistical inference on F via (10.1.1), like e.g. in the Kolmogorov-Smirnov tests and when constructing confidence bands for F . If, however, $\{X_n, n \geq 1\}$ is not an i.i.d. sequence, then invariance principles like (10.1.1) may not be of much direct help. For example, compare the weak convergence for the empirical process α_n of a stationary associated sequence of Theorem 3.3.1 to that of (10.1.1). In general, when $\{X_n, n \geq 1\}$ is sequence of stationary random variables, under some regularity conditions, we have (cf. Billingsley (1968), Deo (1973) and Philipp (1986))

$$(10.1.2) \quad \alpha_n(\cdot) \xrightarrow{\mathcal{D}} B(\cdot) \text{ in } D[0, 1],$$

where α_n is defined by (3.1.6) and B is a zero-mean Gaussian process specified by

$$(10.1.3) \quad EB(s)B(t) := s \wedge t + \sum_{k=1}^{\infty} \{ \text{Cov}(I(U_0 \leq s), I(U_k \leq t)) + \text{Cov}(I(U_0 \leq t), I(U_k \leq s)) \}.$$

Due to the covariance series in (10.1.3), it is difficult to estimate distributions of functionals of B even when the covariance structure of $\{U_n, n \geq 1\}$ is known. Hence (10.1.2) tells us very little from the statistical inference point of view, unless we can get at least some approximate distributions of B .

It is well known that Efron's (1979) bootstrap provides very good estimators in nonparametric statistical analysis. For example, Bickel and Freedman (1981) prove

that the i.i.d. bootstrap works for sample means and empirical processes. Indeed, the assumption of having i.i.d. observations has been an essential ingredient in most of the studies of its properties. Otherwise, however, the bootstrap will not provide consistent estimators (see Remark 2.1 of Singh (1981)). This is due to the fact that a bootstrap method using an i.i.d. resampling procedure cannot reflect the nature of dependence relation among the observations. To make the bootstrap suitable for dependent observations, various block resampling procedures have been introduced. Instead of drawing one sample observation at a time, a block of sample observations can be drawn according to some order, increasing the block size as the whole sample size is getting large. This procedure will reflect some aspects of dependence among the observations and some studies have already exploited this idea.

Shi and Shao (1988) proposed a block resampling procedure for m -dependent observations. In addition to the original sample of m -dependent observations, they simulated i.i.d. observations from a sample source of interest, e.g., a standard normal distribution, and then they combined these data with block samples from the original m -dependent observations (cf. Section 2 of Shi and Shao (1988)). The merit of this method will be determined not only by the original data structure, but also by the choice of resample distribution as well. For example, if observations are from a very non-normal distribution and the resample distribution is chosen to be normal, then this procedure cannot be expected to provide a good estimator, specially so, when the sample size is relatively small. Hence the choice of a resample distribution plays a crucial role in this block bootstrap methodology. In order to obtain strong consistency of their sample mean distribution Shi and Shao (1988) also required the existence of the third moment.

It seems quite natural to use all possible block data of the original observations to construct an empirical distribution directly and then choose this distribution as

a resample source to obtain the final resamples. Based on this idea, Künsch (1989) proposed a moving block bootstrap to estimate a sample mean for general stationary observations. While his bootstrapped sample mean is not an unbiased estimator for the sample mean, it can be easily corrected by slightly modifying the procedure, e.g., by using $F_n^* - E^*(F_n^*)$ instead of $F_n^* - F_n$. On the other hand, this procedure still poses problems when trying to prove strong consistency for the sample mean and empirical processes of dependent observations. In our opinion, these difficulties are due to the fact that the block data proposed by Künsch lacks symmetry. We believe that, for the sake of proving strong consistency of the sample means assuming only two moments, either the block data structure or the resample source should be modified.

In this chapter we introduce a circular block bootstrap which can take care of the problem of biasedness. Our procedure has the special feature that the resampled data are like drawing from the empirical distribution function of observations. No information is lost concerning the nature of dependency of the original observations coming from a general stationary sequence. Our approach makes it also possible to prove strong consistency of empirical processes based on stationary mixing observations. Exploiting our procedure further, we find that it works well with bootstrapped empirical processes. In particular, our procedure enables us to prove the weak convergence of finite dimensional distributions and tightness of bootstrapped empirical processes almost surely for stationary ρ -mixing sequences, and thus establish the weak convergence in D space.

This chapter is organized as follows. The circular block procedure and basic theorems for a general stationary sequence are given in Section 10.2 with proofs. In Section 10.3, we give the definitions of mixing dependence and present the results on bootstrapping the sample means of several stationary mixing sequences with proofs.

In Section 10.4, an almost sure weak convergence for a bootstrapped empirical process of a stationary ρ -mixing sequence is given with proof. Finally, Section 10.5 is to consider the bootstrap of a general statistic of a stationary ρ -mixing sequence by means of influence functions.

We have run several simulations, similar to those of Künsch (1989), and found that the outcome of these were results more or less the same as those of Künsch (1989). Hence we omit the details here.

10.2 Procedure and basic theorems with proofs.

10.2.1 Procedure and basic theorems.

First we introduce our circle block resampling procedure. For $n \geq 1$, let ℓ be a positive integer such that $1 \leq \ell \leq n$. Define

$$X_{i,n} = \begin{cases} X_i, & i = 1, \dots, n \\ X_{i-n}, & i = n+1, \dots, n+\ell-1. \end{cases}$$

Set $\mathbf{Y}_{i,n} = (X_{i,n}, \dots, X_{i+\ell-1,n})$, $i = 1, \dots, n$. Then we use $\{\mathbf{Y}_{i,n}, 1 \leq i \leq n\}$ as the resample source, sampling with replacement up to m -times. The final resampled sample will be $\mathbf{Y}_1^*, \dots, \mathbf{Y}_m^*$, which are i.i.d. with

$$(10.2.1) \quad P^*\{\mathbf{Y}_1^* = \mathbf{Y}_{i,n}\} = \frac{1}{n}, \quad i = 1, \dots, n,$$

where $\mathbf{Y}_i^* = (X_{(i-1)\ell+1}^*, \dots, X_{i\ell}^*)$, $i = 1, \dots, m$. The special choice of our block data $\{\mathbf{Y}_{i,n}, 1 \leq i \leq n\}$ enables us to get the following important result for $i = 1, \dots, m\ell$,

$$(10.2.2) \quad P^*\{X_i^* = X_j\} = \frac{1}{n}, \quad j = 1, \dots, n,$$

i.e., $\{X_i^*, 1 \leq i \leq m\ell\}$ are random variables with a common distribution function $F_n(x) = n^{-1} \sum_{i=1}^n I(X_i \leq x)$, the n th empirical distribution function of $X_1, \dots, X_n$.

However, within one block, $X_{(i-1)\ell+1}^*, \dots, X_{i\ell}^*$ are no longer independent. They contain information about the nature of dependency of the observations $X_1, \dots, X_n$.

Using the above procedure, we establish two general theorems on bootstrapping the sample mean of a general stationary sequence.

In the sequel, ℓ is assumed to be function of n . Let $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ and $S_n = \sum_{i=1}^n X_i$. Then bootstrap sample mean is defined as

$$\bar{X}_{n,m}^* = (m\ell)^{-1} \sum_{j=1}^{m\ell} X_j^*$$

and its normalized version is

$$(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n).$$

Before stating our basic theorems, we list some general assumptions on the observations $\{X_n, n \geq 1\}$ and sample mean, which we will use in the sequel.

$$(A1): (n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \left(\sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \right)^2 - E \left(\sum_{i=1}^{\ell} (X_i - \mu) \right)^2 \right\} \xrightarrow{\text{a.s.}} 0,$$

$$(A2): (n\ell)^{-1} \sum_{j=0}^{n-1} \left(\sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \right)^2 I(\{ \sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \}^2 \geq \varepsilon m\ell) \xrightarrow{\text{a.s.}} 0,$$

for any $\varepsilon > 0$ as $n, m \rightarrow \infty$,

$$(A3): n^{-1} \sum_{j=0}^{n-1} (I\{\ell^{-\frac{1}{2}} \sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \leq x\} - P\{\ell^{-\frac{1}{2}} \sum_{i=1}^{\ell} (X_i - \mu) \leq x\}) \rightarrow 0$$

almost surely for each fixed x ,

$$(B1): \text{Var}(n^{\frac{1}{2}} \bar{X}_n) \rightarrow \sigma^2 > 0,$$

$$(B2): n^{\frac{1}{2}}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2).$$

THEOREM 10.2.1 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary random variables with $\mu = EX_1$ and $\text{Var}X_1 < \infty$. Suppose that (B1), (B2), (A1) and (A2) are satisfied. Then, if $\ell \rightarrow \infty$ and $\ell n^{-1} \log^4 n \rightarrow 0$, we have*

$$\sup_x |P\{(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) \leq x | X_1, \dots, X_n\} - P\{n^{\frac{1}{2}}(\bar{X}_n - \mu) \leq x\}| \xrightarrow{\text{a.s.}} 0$$

as $n, m \rightarrow \infty$.

THEOREM 10.2.2 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary random variables with $\mu = EX_1$ and $\text{Var} X_1 < \infty$. Suppose that (B1), (B2), (A1) and (A3) are satisfied. Then, if $\ell = o(n) \rightarrow \infty$ as $n \rightarrow \infty$, we have*

$$\sup_x |P\{(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) \leq x | X_1, \dots, X_n\} - P\{n^{\frac{1}{2}}(\bar{X}_n - \mu) \leq x\}| \xrightarrow{\text{a.s.}} 0$$

as $n \rightarrow \infty$.

To bootstrap the empirical process of a stationary sequence, we need to assume that m is a function of n . The bootstrapped empirical uniform empirical process is defined by

$$(10.2.3) \quad \alpha_n^*(t) = (m\ell)^{-1}(E_n^*(t) - E_n(t)), \quad 0 \leq t \leq 1,$$

where $E_n^*(t) = (m\ell)^{1/2} \sum_{i=1}^{m\ell} I(U_i^* \leq t)$, $0 \leq t \leq 1$ and $U_i^* = F(X_i^*)$, $i = 1, \dots, m\ell$.

Let T_0 be the set of all rational numbers in $[0, 1]$. Then we have the following basic theorem on bootstrapping the empirical process of a general stationary sequence.

THEOREM 10.2.3 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary random variables with common continuous distribution function F . Assume that the covariance series in (10.1.3) is absolutely convergent and*

(C1): *for any $s, t \in T_0$,*

$$(n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \left(\sum_{i=1}^{\ell} (I(U_{i+j,n} \leq s) - s) \right) \left(\sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t) - t) \right) \right. \\ \left. - E \left(\sum_{i=1}^{\ell} (I(U_i \leq s) - s) \right) \left(\sum_{i=1}^{\ell} (I(U_i \leq t) - t) \right) \right\} \longrightarrow 0 \quad \text{a.s.}$$

as $\ell = o(n(\log n)^{-4}) \rightarrow \infty$, where $U_{i,n} = F(X_{i,n})$,

(C2): for some $q > 1$ and $t \in T_0$,

$$(m\ell)^{1-q}(n\ell)^{-1} \sum_{j=0}^{n-1} \left| \sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t) - t) \right|^{2q} \longrightarrow 0 \quad \text{a.s.}$$

as $n, m \longrightarrow \infty$,

(C3): almost surely, for any $\varepsilon, \eta > 0$, there exist $0 < \delta < 1$ and an integer N_0 such that

$$P^* \{w(\alpha_n^*, \delta) \geq \varepsilon\} \leq \eta, \quad n \geq N_0,$$

where w is Lévy's modulus of continuity defined as in (4.3.4).

Then we have

$$\alpha_n^* \xrightarrow{\mathcal{D}} B \quad \text{in } D[0, 1] \quad \text{a.s.}$$

with $P^* \{B \in C[0, 1]\} = 1$ a.s..

10.2.2 Proofs.

LEMMA 10.2.1 Let $\{\xi_n, n \geq 1\}$ be a sequence of random variables with $E\xi_n = 0$ and $\sup_{n \geq 1} E\xi_n^2 < \infty$. Assume that there is a constant $C > 0$ such that for any $n \geq 1$,

$$(10.2.4) \quad \sup_{k \geq 0} E \left(\sum_{i=k+1}^{k+n} \xi_i \right)^2 \leq Cn.$$

Then

$$(10.2.5) \quad \limsup_{n \rightarrow \infty} \frac{|\sum_{i=1}^n \xi_i|}{n^{\frac{1}{2}} \log^2 n} = 0 \quad \text{a.s..}$$

Proof. The proof is similar to that of Theorem 3.7.6 of Stout (1974) and it is omitted here. ■

Proof of Theorem 10.2.1.

Let

$$Z_{i,n}^* = \ell^{-\frac{1}{2}} \sum_{j=1}^{\ell} (X_{(i-1)\ell+j}^* - \bar{X}_n), \quad i = 1, 2, \dots, m.$$

Then $Z_{1,n}^*, \dots, Z_{m,n}^*$ are i.i.d. random variables and

$$(10.2.6) \quad (m\ell)^{\frac{1}{2}} (\bar{X}_{n,m}^* - \bar{X}_n) = m^{-\frac{1}{2}} \sum_{i=1}^m Z_{i,n}^*.$$

We have

$$\begin{aligned} E^* Z_{i,n}^{*2} &= \frac{1}{n\ell} \sum_{j=0}^{n-1} \left(\sum_{i=1}^{\ell} (X_{i+j,n} - \bar{X}_n) \right)^2 \\ &= \frac{1}{n\ell} \sum_{j=0}^{n-1} \left(\sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \right)^2 - \ell (\bar{X}_n - \mu)^2 \\ &= \frac{1}{n\ell} \sum_{j=0}^{n-1} \left\{ \left(\sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \right)^2 - E \left(\sum_{i=1}^{\ell} (X_i - \mu) \right)^2 \right\} \\ &\quad + \ell \text{Var}(\bar{X}_\ell) - \ell (\bar{X}_n - \mu)^2. \end{aligned}$$

Thus **(B1)**, **(A1)** and Lemma 10.2.1 imply that

$$(10.2.7) \quad E^* Z_{1,n}^{*2} \longrightarrow \sigma^2 \quad \text{a.s..}$$

On the other hand, for any $\varepsilon > 0$,

$$\begin{aligned} &E^* Z_{1,n}^{*2} I(Z_{1,n}^{*2} \geq \varepsilon m) \\ &= \frac{1}{n\ell} \sum_{j=0}^{n-1} \left(\sum_{i=1}^{\ell} (X_{i+j,n} - \bar{X}_n) \right)^2 I(\{ \sum_{i=1}^{\ell} (X_{i+j,n} - \bar{X}_n) \}^2 \geq \varepsilon m\ell) \\ &\leq \frac{4}{n\ell} \sum_{j=0}^{n-1} \left(\sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \right)^2 I(\{ \sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \}^2 \geq \frac{\varepsilon m\ell}{4}) + 4\ell (\bar{X}_n - \mu)^2, \end{aligned}$$

since $(a+b)^2 I((a+b)^2 \geq d) \leq 4a^2 I(4a^2 \geq d) + 4b^2 I(4b^2 \geq d)$ is true for all real numbers a, b and $d \geq 0$. Hence, **(A2)** and Lemma 10.2.1 imply that for any $\varepsilon > 0$

$$(10.2.8) \quad E^* Z_{1,n}^{*2} I(Z_{1,n}^{*2} \geq \varepsilon m) \longrightarrow 0 \quad \text{a.s..}$$

which means that $\{Z_{i,n}^*, i = 1, \dots, m, n \geq 1\}$ satisfies the Lindeberg condition. Therefore, by (10.2.6)-(10.2.8)

$$(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) \xrightarrow{D} N(0, \sigma^2) \quad \text{a.s.}$$

as $n, m \rightarrow \infty$ and our conclusion follows from (B2). ■

Proof of Theorem 10.2.2.

Let $G^{(m)}$ be the distribution function of $m^{-\frac{1}{2}} \sum_{i=1}^m (Z_i - EZ_i)$, where $Z_1, \dots, Z_m$ are i.i.d. random variables with the distribution function G . Using Mallows's metric d_2 (see Section 8 of Bickel and Freedman (1981)), we have

$$(10.2.9) \quad d_2(G^{(m)}, H^{(m)}) \leq d_2(G, H).$$

Let

$$\tilde{F}_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} I(\ell^{-\frac{1}{2}} \sum_{i=1}^{\ell} (X_{i+j,n} - \mu) \leq x).$$

Then it is easy to see that $\{\ell^{-\frac{1}{2}} \sum_{j=1}^{\ell} (X_{(i-1)\ell+j}^* - \mu)\}_{i=1}^m$ is an i.i.d. sequence with the distribution function $\tilde{F}_n$, and the distribution function of

$$(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) = m^{-\frac{1}{2}} \sum_{i=1}^m \{\ell^{-\frac{1}{2}} \sum_{j=1}^{\ell} ((X_{(i-1)\ell+j}^* - \mu) - E^*(X_{(i-1)\ell+j}^* - \mu))\}$$

is $\tilde{F}_n^{(m)}$. Now (B1) and (A1) imply that

$$(10.2.10) \quad \int x^2 d\tilde{F}_n(x) \rightarrow \sigma^2 \quad \text{a.s.}$$

and (B2) and (A3) imply that

$$(10.2.11) \quad \tilde{F}_n \xrightarrow{D} N(0, \sigma^2) \quad \text{a.s.,}$$

since (A3) is true almost surely for each fixed x and hence it is also true almost surely for all rational numbers. Applying (10.2.9), we have

$$\begin{aligned} d_2(\tilde{F}_n^{(m)}, N(0, \sigma^2)) &= d_2(\tilde{F}_n^{(m)}, N^{(m)}(0, \sigma^2)) \\ &\leq d_2(\tilde{F}_n, N(0, \sigma^2)) \rightarrow 0 \quad \text{a.s.} \end{aligned}$$

by (10.2.10), (10.2.11) and Lemma 8.3 of Bickel and Freedman (1981). This completes the proof of Theorem 10.2.2. ■

Proof of Theorem 10.2.3.

Since (C3) implies the tightness of α_n^* almost surely, by Theorem 15.4, 15.5 and problem 1 on page 136 of Billingsley (1968), Theorem 10.2.3 follows if we can show that there exists a $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 0$ and for every $\omega \notin \Omega_0$ and for any $0 < t_1 < \dots < t_k < 1$, $t_i \in T_0$, $i = 1, \dots, k$, we have

$$(10.2.12) \quad (\alpha_n^*(t_1), \dots, \alpha_n^*(t_k)) \xrightarrow{\mathcal{D}} (B(t_1), \dots, B(t_k)).$$

For any $a_i \in \mathbf{R}$, $i = 1, \dots, k$, let

$$W_j = \sum_{i=1}^k a_i I(U_j \leq t_i), \quad j = 1, \dots, n$$

and

$$W_j^* = \sum_{i=1}^k a_i I(U_j^* \leq t_i), \quad j = 1, \dots, m\ell.$$

Then

$$(10.2.13) \quad \sum_{i=1}^k a_i \alpha_n^*(t_i) = (m\ell)^{1/2} (\bar{W}_n^* - E^* \bar{W}_n^*)$$

by (10.2.1) and (10.2.2), where $\bar{W}_n^* = (m\ell)^{-1} \sum_{j=1}^{m\ell} W_j^*$. Set $\bar{W}_n = n^{-1} \sum_{j=1}^n W_j$. Then, by (10.1.3) and (C1), there exists a $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 0$ and for every $\omega \notin \Omega_0$,

$$\begin{aligned} (10.2.14) \quad nE(\bar{W}_n - E\bar{W}_n)^2 &= n^{-1} E \left\{ \sum_{i=1}^k a_i \left(\sum_{j=1}^n (I(U_j \leq t_i) - t_i) \right) \right\}^2 \\ &= n^{-1} \sum_{i_1, i_2=1}^k a_{i_1} a_{i_2} E \left(\sum_{j=1}^n (I(U_j \leq t_{i_1}) - t_{i_1}) \right) \left(\sum_{j=1}^n (I(U_j \leq t_{i_2}) - t_{i_2}) \right) \\ &\longrightarrow \sum_{i_1, i_2=1}^k a_{i_1} a_{i_2} E B(t_{i_1}) B(t_{i_2}) = E \left(\sum_{i=1}^k a_i B(t_i) \right)^2 \end{aligned}$$

as $n \rightarrow \infty$, and

$$\begin{aligned}
 (10.2.15) \quad & (n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \left(\sum_{i=1}^{\ell} (W_{i+j,n} - EW_{i+j,n}) \right)^2 - E \left(\sum_{i=1}^{\ell} (W_i - EW_i) \right)^2 \right\} \\
 &= (n\ell)^{-1} \sum_{i_1, i_2=1}^k a_{i_1} a_{i_2} \sum_{j=0}^{n-1} \left\{ \left(\sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t_{i_1}) - t_{i_1}) \right) \left(\sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t_{i_2}) - t_{i_2}) \right) \right. \\
 &\quad \left. - E \left(\sum_{i=1}^{\ell} (I(U_i \leq t_{i_1}) - t_{i_1}) \right) \left(\sum_{i=1}^{\ell} (I(U_i \leq t_{i_2}) - t_{i_2}) \right) \right\} \\
 &\rightarrow 0
 \end{aligned}$$

as $\ell = o(n(\log n)^{-4}) \rightarrow \infty$, where $W_{i,n}$ is defined the same way as $X_{i,n}$. Next, by (C2), there exists a $\Omega'_0 \in \mathcal{F}$ such that $P(\Omega'_0) = 0$ and for every $\omega \notin \Omega'_0$ and any $\varepsilon > 0$, we have

$$\begin{aligned}
 (10.2.16) \quad & (n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \sum_{i=1}^{\ell} (W_{i+j,n} - EW_{i+j,n}) \right\}^2 I \left\{ \left(\sum_{i=1}^{\ell} (W_{i+j,n} - EW_{i+j,n}) \right)^2 \geq \varepsilon m \ell \right\} \\
 &\leq (\varepsilon m \ell)^{1-q} (n\ell)^{-1} \sum_{j=0}^{n-1} \left| \sum_{i=1}^{\ell} \sum_{r=1}^k a_r (I(U_{i+j,n} \leq t_r) - t_r) \right|^{2q} \\
 &\leq k^{2q-1} (\varepsilon m \ell)^{1-q} (n\ell)^{-1} \sum_{r=1}^k |a_r|^{2q} \sum_{j=0}^{n-1} \left| \sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t_r) - t_r) \right|^{2q} \\
 &\rightarrow 0
 \end{aligned}$$

as $n, m \rightarrow \infty$, where in the last inequality, we used the C_r inequality, namely that, for any $(a_1, \dots, a_n) \in \mathbf{R}^n$ and $r \geq 1$, we have $|\sum_{i=1}^n a_i|^r \leq n^{r-1} \sum_{i=1}^n |a_i|^r$. It is not difficult to see that the Ω_0 and Ω'_0 can be chosen independently of $(t_1, \dots, t_k)$ and $(a_1, \dots, a_k)$ since T_0 is countable. Thus, the assumptions of Theorem 10.2.1 are satisfied for the stationary sequence $\{W_n, n \geq 1\}$ by (10.2.14)-(10.2.16). This, by (10.2.13), implies that

$$\sum_{i=1}^k a_i \alpha_n^*(t_i) \xrightarrow{\mathcal{D}} \sum_{i=1}^k a_i B(t_i)$$

for each point $(a_1, \dots, a_k)$ of $\mathbf{R}^k$. Hence (10.2.12) follows by the Cramér and Wold theorem. The proof of Theorem 10.2.3 now is complete. ■

10.3 Definitions of mixing dependence and the bootstrap of sample means for stationary mixing sequences.

10.3.1 Definitions of mixing dependence.

Theorems 10.2.1 and 10.2.2 enable us to establish the bootstrap of sample mean for stationary mixing sequences. We first introduce the following dependence relations.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_1$ and $\mathcal{F}_2$ two σ -algebras contained in $\mathcal{F}$. Define the following measures of dependence between $\mathcal{F}_1$ and $\mathcal{F}_2$ by

$$\rho(\mathcal{F}_1, \mathcal{F}_2) = \sup_{X \in L_2(\mathcal{F}_1), Y \in L_2(\mathcal{F}_2)} \frac{|\text{Cov}(X, Y)|}{(\text{Var} X \cdot \text{Var} Y)^{\frac{1}{2}}},$$

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} |P(A \cap B) - P(A)P(B)|$$

and

$$\phi(\mathcal{F}_1, \mathcal{F}_2) = \sup_{A \in \mathcal{F}_1, B \in \mathcal{F}_2} |P(B | A) - P(B)|.$$

Let $\{X_n, n \geq 1\}$ be a sequence of real valued random variables on $(\Omega, \mathcal{F}, P)$, $\mathcal{F}_n^m = \sigma(X_i, n \leq i \leq m)$, σ -algebras generated by the indicated random variables, and put

$$\rho(n) = \sup_{k \geq 1} \rho(\mathcal{F}_1^k, \mathcal{F}_{n+k}^\infty), \quad \alpha(n) = \sup_{k \geq 1} \alpha(\mathcal{F}_1^k, \mathcal{F}_{n+k}^\infty) \quad \text{and} \quad \phi(n) = \sup_{k \geq 1} \phi(\mathcal{F}_1^k, \mathcal{F}_{n+k}^\infty).$$

The sequence $\{X_n, n \geq 1\}$ is said to be ρ -mixing, α -mixing or ϕ -mixing, according as $\rho(n) \rightarrow 0$, $\alpha(n) \rightarrow 0$ or $\phi(n) \rightarrow 0$ as $n \rightarrow \infty$, respectively. It is well-known that

$$\alpha(n) \leq \rho(n) \leq 2\phi^{\frac{1}{2}}(n).$$

10.3.2 Results.

THEOREM 10.3.1 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary ϕ -mixing sequence of random variables with $EX_1 = \mu$ and $\text{Var}X_1 < \infty$. Assume that*

$$(10.3.1) \quad \text{Var}S_n \rightarrow \infty,$$

$$(10.3.2) \quad \sum_{n=1}^{\infty} \phi^{\frac{1}{2}}(2^n) < \infty,$$

$$(10.3.3) \quad \ell(n) = \ell(2^k) \text{ if } 2^k \leq n < 2^{k+1}, \quad k = 1, 2, \dots,$$

$$(10.3.4) \quad \ell \rightarrow \infty \text{ and } \ell n^{-1}(\log n)^{\varepsilon_0} \rightarrow 0 \text{ for some } \varepsilon_0 > 0.$$

Then, we have

$$\sup_x |P\{(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) \leq x | X_1, \dots, X_n\} - P\{n^{\frac{1}{2}}(X_n - \mu) \leq x\}| \xrightarrow{\text{a.s.}} 0$$

as $n \rightarrow \infty$.

THEOREM 10.3.2 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary α -mixing sequence of random variables with $EX_1 = \mu$ and $(E|X_1|^{2+\delta})^{1/(2+\delta)} < \infty$ for some $0 < \delta \leq \infty$. Assume that (10.3.3) and*

$$(10.3.5) \quad \alpha(n) \leq C n^{-r} \text{ for some } C > 0, \quad r > \frac{2+\delta}{\delta},$$

$$(10.3.6) \quad \ell \rightarrow \infty, \quad \ell \leq n^{1-\varepsilon_0} \text{ for some } \varepsilon_0 > 0.$$

Then $\text{Var}X_1 + 2 \sum_{i=2}^{\infty} \text{Cov}(X_1, X_i)$ converges to a constant $\sigma^2 \geq 0$. In the case of $\sigma^2 > 0$, we have

$$\sup_x |P\{(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) \leq x | X_1, \dots, X_n\} - P\{n^{\frac{1}{2}}(X_n - \mu) \leq x\}| \xrightarrow{\text{a.s.}} 0$$

as $n \rightarrow \infty$.

THEOREM 10.3.3 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary α -mixing sequence of random variables with $EX_1 = \mu$ and $(E|X_1|^{2+\delta})^{1/(2+\delta)} < \infty$ for some $0 < \delta \leq \infty$. Assume that (10.3.1), (10.3.3) and (10.3.6) are satisfied. Moreover, suppose that*

$$\alpha(n) \leq C n^{-r} \text{ for some } C > 0, r > \frac{2(2+\delta)}{\delta}.$$

Then, as $n \rightarrow \infty$,

$$\sup_x |P\{(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) \leq x | X_1, \dots, X_n\} - P\{n^{\frac{1}{2}}(\bar{X}_n - \mu) \leq x\}| \xrightarrow{\text{a.s.}} 0.$$

THEOREM 10.3.4 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary ρ -mixing sequence of random variables satisfying $EX_1 = \mu$, $\text{Var}X_1 < \infty$, (10.3.1) and (10.3.3). Assume that*

$$(10.3.7) \quad \sum_{n=1}^{\infty} \rho^{\frac{1}{2}}(2^n) < \infty,$$

$$(10.3.8) \quad \ell \rightarrow \infty, \ell n^{-1} \log^4 n \rightarrow 0.$$

Then, we have

$$\sup_x |P\{(m\ell)^{\frac{1}{2}}(\bar{X}_{n,m}^* - \bar{X}_n) \leq x | X_1, \dots, X_n\} - P\{n^{\frac{1}{2}}(\bar{X}_n - \mu) \leq x\}| \xrightarrow{\text{a.s.}} 0$$

as $n \rightarrow \infty$.

10.3.3 Proofs.

The following lemmas are standard. We state them here for easy references (see, e.g., Shao (1989c)).

LEMMA 10.3.1 (Peligrad (1982)) *Let $\{X_n, n \geq 1\}$ be a sequence of stationary ρ -mixing sequence of random variables with $EX_1^2 < \infty$. Assume that*

$$(10.3.9) \quad \sum_{n=1}^{\infty} \rho(2^n) < \infty,$$

$$(10.3.10) \quad \text{Var} S_n \rightarrow \infty.$$

Then, there is a positive constant σ such that

$$(10.3.11) \quad \frac{\text{Var} S_n}{n} \rightarrow \sigma^2.$$

LEMMA 10.3.2 (Ibragimov (1975)) Let $\{X_n, n \geq 1\}$ be a sequence of stationary ρ -mixing sequence of random variables with $EX_1 = \mu$ and $EX_1^2 < \infty$. Assume that (10.3.9) and (10.3.10) are satisfied. Then

$$n^{\frac{1}{2}}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2).$$

LEMMA 10.3.3 (Ibragimov (1962)) Let $\{X_n, n \geq 1\}$ be a sequence of stationary α -mixing sequence of random variables with $EX_1 = \mu$ and $(E|X_1|^{2+\delta})^{1/(2+\delta)} < \infty$ for some $0 < \delta \leq \infty$. Assume that

$$(10.3.12) \quad \sum_{n=1}^{\infty} \alpha^{\frac{\delta}{2+\delta}}(n) < \infty.$$

Then

$$\sigma^2 = \text{Var} X_1 + 2 \sum_{k=2}^{\infty} \text{Cov}(X_1, X_k) \text{ exists and } \frac{\text{Var} S_n}{n} \rightarrow \sigma^2.$$

If in addition $\sigma^2 > 0$, then

$$n^{\frac{1}{2}}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2).$$

LEMMA 10.3.4 (Bradley (1985)) Let $\{X_n, n \geq 1\}$ be a sequence of stationary α -mixing sequence of random variables with $EX_1 = \mu$ and $(E|X_1|^{2+\delta})^{1/(2+\delta)} < \infty$ for some $0 < \delta \leq \infty$. Assume that (10.3.10) and

$$(10.3.13) \quad \sum_{n=1}^{\infty} n \alpha^{\frac{\delta}{2+\delta}}(n) < \infty$$

are satisfied. Then (10.3.11) holds true.

LEMMA 10.3.5 (Shao (1988), Peligrad (1989)) Let $\{\xi_n, n \geq 1\}$ be a ϕ -mixing sequence of random variables. Then, for each $q \geq 2$, there exists a constant K , depending only on q and $\phi(\cdot)$, such that for each $n \geq 1$

$$(10.3.14) \quad E \max_{1 \leq i \leq n} \left| \sum_{j=1}^i \xi_j \right|^q \leq K \left\{ \max_{1 \leq i \leq n} (E \left| \sum_{j=1}^i \xi_j \right|^2)^{\frac{q}{2}} + E \max_{1 \leq i \leq n} |\xi_i|^q \right\}.$$

LEMMA 10.3.6 (Shao (1989d)) Let $\{\xi_n, n \geq 1\}$ be a ρ -mixing sequence of random variables with $E\xi_n = 0$ and $E|\xi_n|^q < \infty$ for some $q \geq 2$. Then, there exists a constant K , depending only on q and $\rho(\cdot)$, such that for each $n \geq 1$

$$(10.3.15) \quad E \left| \sum_{j=1}^n \xi_j \right|^q \leq K \left\{ n^{\frac{q}{2}} \exp(K \sum_{j=0}^{\lfloor \log_2 n \rfloor} \rho(2^j)) \max_{1 \leq i \leq n} (E\xi_i^2)^{\frac{q}{2}} + n \exp(K \sum_{j=0}^{\lfloor \log_2 n \rfloor} \rho^{\frac{2}{q}}(2^j)) \max_{1 \leq i \leq n} E|\xi_i|^q \right\}.$$

In particular, if $\sum_{n=1}^{\infty} \rho(2^n) < \infty$, then, for every $\varepsilon > 0, q \geq 2$, there exists a constant K , depending only on ε, q and $\rho(\cdot)$, such that

$$E \left| \sum_{i=1}^n \xi_i \right|^q \leq K \left\{ n^{q/2} \max_{1 \leq i \leq n} (E\xi_i^2)^{q/2} + n^{1+\varepsilon} \max_{1 \leq i \leq n} E|\xi_i|^q \right\}.$$

LEMMA 10.3.7 (Shao (1989c, 89d)) Let $\{\xi_n, n \geq 1\}$ be a ρ -mixing sequence of random variables with $E\xi_n = 0$ and $E\xi_n^2 < \infty$. Then, for each $q \geq 2$, there exists a constant K , depending only on q and $\rho(\cdot)$, such that for any $x > 0, 0 < B \leq A \leq \infty, n \geq 1$,

$$(10.3.16) \quad P \left\{ \max_{1 \leq i \leq n} \left| \sum_{j=1}^i \xi_j \right| \geq x \right\} \leq \sum_{j=1}^n P \{ |\xi_j| \geq A \} \\ + K x^{-q} \left\{ n^{\frac{q}{2}} \exp(K \sum_{j=0}^{\lfloor \log_2 n \rfloor} \rho(2^j)) \max_{1 \leq i \leq n} (E\xi_i^2 I\{|\xi_i| \leq A\})^{\frac{q}{2}} \right. \\ + n \exp(K \sum_{j=0}^{\lfloor \log_2 n \rfloor} \rho^{\frac{2}{q}}(2^j)) \log^2 n \max_{1 \leq i \leq n} E|\xi_i|^q I\{|\xi_i| \leq B\} \\ \left. + n \exp(K \sum_{j=0}^{\lfloor \log_2 n \rfloor} \rho^{\frac{2}{q}}(2^j)) \max_{1 \leq i \leq n} E|\xi_i|^q I\{B < |\xi_i| < A\} \right\}.$$

provided that

$$(10.3.17) \quad 48n \max_{i \leq n} E\xi_i^2 I\{|\xi_i| \geq B\} \leq Bx.$$

LEMMA 10.3.8 (Shao (1989c, 1991)) Let $\{\xi_i, 1 \leq i \leq n\}$ be an α -mixing sequence of random variables with $E\xi_i = 0$ and $(E|\xi_i|^s)^{1/s} \leq D_n$ for $1 \leq i \leq n$ and for some $1 < s \leq \infty$. Assume that $\alpha(i) \leq C_0 i^{-\theta}$ for some $C_0 > 1$ and $\theta > 0$. Then, there exists a constant K , depending only on C_0, θ, s , such that for any $x \geq K D_n n^{\frac{1}{2}} \log n$

$$P\{\max_{i \leq n} |\sum_{j=1}^i \xi_j| \geq x\} \leq K n \left(\frac{D_n}{x}\right)^{\frac{s(\theta+1)}{1+\theta}} \log^\theta \frac{x}{D_n}.$$

LEMMA 10.3.9 (Yokoyama (1980)) Let $\{X_n, n \geq 1\}$ be a sequence of stationary α -mixing sequence of random variables with $EX_1 = \mu$ and $(E|X_1|^{2+\delta})^{1/(2+\delta)} < \infty$ for some $0 < \delta \leq \infty$. Suppose that $2 \leq s < 2 + \delta$ and

$$\sum_{n=1}^{\infty} n^{\frac{s}{2}-1} \alpha^{\frac{2+\delta-s}{2+\delta}}(n) < \infty.$$

Then, there exists a constant K , depending only on s, δ and $\alpha(\cdot)$, such that

$$E|\sum_{i=1}^n (X_i - \mu)|^s \leq K n^{\frac{s}{2}} (E|X_1|^{2+\delta})^{\frac{s}{2+\delta}}.$$

LEMMA 10.3.10 (Shao (1988)) Let $\{X_n, n \geq 1\}$ be a sequence of stationary ϕ -mixing sequence of random variables with $EX_1 = \mu$ and $\text{Var} X_1 < \infty$. Assume that $\text{Var} S_n \rightarrow \infty$ and $\sum_n \phi^{\frac{1}{2}}(2^n) < \infty$. Then

$$\limsup_{n \rightarrow \infty} \frac{|S_n - n\mu|}{(2 \text{Var} S_n \cdot \log \log n)^{\frac{1}{2}}} = 1 \quad \text{a.s.}$$

LEMMA 10.3.11 Let X be a random variable with $EX^2 < \infty$. Then

$$\sum_{k=1}^{\infty} 2^k P\{|X| \geq 2^{\frac{k}{2}}\} < \infty$$

and

$$\sum_{k=1}^{\infty} 2^{-k(r-1)} E|X|^{2r} I\{|X| \leq 2^{\frac{k}{2}}\} < \infty \quad \text{for every } r > 1.$$

Proof. The proof is trivial and it is omitted here. ■

Proof of Theorem 10.3.1.

Lemmas 10.3.1 and 10.3.2 imply that conditions (B1) and (B2) are satisfied. We first prove that (A1) is also satisfied. Without loss of generality, assume $\mu = EX_1 = 0$. In this case (A1) is equivalent to

$$(10.3.18) \quad (n\ell)^{-1} \sum_{j=0}^{n-1} \{(\sum_{i=1}^{\ell} X_{i+j,n})^2 - E(\sum_{i=1}^{\ell} X_i)^2\} \rightarrow 0 \quad \text{a.s..}$$

It suffices to show that $\forall \varepsilon > 0$ we have

$$(10.3.19) \quad \sum_{k=1}^{\infty} P\{\max_{2^k \leq n < 2^{k+1}} |(n\ell_n)^{-1} \sum_{j=0}^{n-1} \{(\sum_{i=1}^{\ell_n} X_{i+j,n})^2 - E(\sum_{i=1}^{\ell_n} X_i)^2\}| \geq \varepsilon\} < \infty$$

by the Borel-Cantelli lemma. In terms of (10.3.3), (10.3.19) is also equivalent to

$$(10.3.20) \quad \sum_{k=1}^{\infty} P\{\max_{2^k \leq n < 2^{k+1}} |\sum_{j=0}^{n-1} \{(\sum_{i=1}^{\ell(2^k)} X_{i+j,n})^2 - E(\sum_{i=1}^{\ell(2^k)} X_i)^2\}| \geq 4\varepsilon 2^k \ell(2^k)\} < \infty$$

for every $\varepsilon > 0$. Put

$$\bar{X}_{i,k} = X_i I\{|X_i| \leq 2^{\frac{k}{2}}\}, \quad \hat{X}_{i,k} = X_i I\{|X_i| > 2^{\frac{k}{2}}\},$$

$$\bar{S}_j(n) = \sum_{i=j+1}^{j+n} \bar{X}_{i,k}, \quad S_j(n) = \sum_{i=j+1}^{j+n} X_i, \quad p_{k,n} = \lfloor \frac{n - \ell(2^k) + 1}{2\ell(2^k)} \rfloor,$$

$$\xi_{u,k} = \sum_{j=2u\ell(2^k)}^{(2u+1)\ell(2^k)-1} \{(\sum_{i=1}^{\ell(2^k)} X_{i+j})^2 - E(\sum_{i=1}^{\ell(2^k)} X_{i+j})^2\}, \quad u = 0, 1, \dots,$$

$$\eta_{u,k} = \sum_{j=(2u+1)\ell(2^k)}^{(2u+2)\ell(2^k)-1} \{(\sum_{i=1}^{\ell(2^k)} X_{i+j})^2 - E(\sum_{i=1}^{\ell(2^k)} X_{i+j})^2\}, \quad u = 0, 1, \dots.$$

The definitions of $\bar{\xi}_{u,k}$ and $\bar{\eta}_{u,k}$ correspond to those of $\xi_{u,k}$ and $\eta_{u,k}$, respectively, with $\bar{X}_{j,k}$ instead of X_j . For $2^k \leq n < 2^{k+1}$, write

$$|\sum_{j=0}^{n-1} \{(\sum_{i=1}^{\ell(2^k)} X_{i+j,n})^2 - E(\sum_{i=1}^{\ell(2^k)} X_i)^2\}|$$

$$\begin{aligned}
&\leq \left| \sum_{j=0}^{n-\ell(2^k)} \left\{ \left(\sum_{i=1}^{\ell(2^k)} X_{i+j} \right)^2 - E \left(\sum_{i=1}^{\ell(2^k)} X_i \right)^2 \right\} \right| \\
&\quad + \sum_{j=n-\ell(2^k)+1}^{n-1} \left\{ \left(\sum_{i=1}^{n-j} X_{i+j} + \sum_{i=n-j+1}^{\ell(2^k)} X_{i+j-n} \right)^2 + E \left(\sum_{i=1}^{\ell(2^k)} X_i \right)^2 \right\} \\
&\leq \left| \sum_{u=0}^{p_{k,n}} \xi_{u,k} \right| + \left| \sum_{u=0}^{p_{k,n}} \eta_{u,k} \right| \\
&\quad + \left| \sum_{j=2(p_{k,n}+1)\ell(2^k)}^{n-\ell(2^k)} \left\{ \left(\sum_{i=1}^{\ell(2^k)} X_{i+j} \right)^2 - E \left(\sum_{i=1}^{\ell(2^k)} X_{i+j} \right)^2 \right\} \right| \\
&\quad + 2 \sum_{j=n-\ell(2^k)+1}^{n-1} \left\{ \left(\sum_{i=1}^{n-j} X_{i+j} \right)^2 + \left(\sum_{i=n-j+1}^{\ell(2^k)} X_{i+j-n} \right)^2 \right\} + K \cdot \ell^2(2^k) \\
&\leq \left| \sum_{u=0}^{p_{k,n}} \xi_{u,k} \right| + \left| \sum_{u=0}^{p_{k,n}} \eta_{u,k} \right| + 6\ell(2^k) \max_{j \leq 2^{k+1}} \max_{i \leq \ell(2^k)} |S_j(i)|^2 + K \cdot \ell^2(2^k),
\end{aligned}$$

where, and in the sequel as well, K denotes a positive constant, whose value is not important and may be different from time to time.

Since $\ell^2(2^k) = o(2^k \ell(2^k))$, we obtain from the above inequality that

$$\begin{aligned}
(10.3.21) \quad &P \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{j=0}^{n-1} \left\{ \left(\sum_{i=1}^{\ell(2^k)} X_{i+j,n} \right)^2 - E \left(\sum_{i=1}^{\ell(2^k)} X_i \right)^2 \right\} \right| \geq 4\epsilon 2^k \ell(2^k) \right\} \\
&\leq P \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{u=0}^{p_{k,n}} \xi_{u,k} \right| \geq \epsilon 2^k \ell(2^k) \right\} + P \left\{ \max_{j \leq 2^{k+1}} \max_{i \leq \ell(2^k)} |S_j(i)|^2 \geq \frac{\epsilon 2^k}{6} \right\} \\
&\quad + P \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{u=0}^{p_{k,n}} \eta_{u,k} \right| \geq \epsilon 2^k \ell(2^k) \right\} \\
&\leq P \left\{ \max_{\frac{2^k}{3\ell(2^k)} \leq i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \xi_{u,k} \right| \geq \epsilon 2^k \ell(2^k) \right\} + P \left\{ \max_{j \leq 2^{k+1}} \max_{i \leq \ell(2^k)} |S_j(i)| \geq \frac{\epsilon}{3} 2^{\frac{k}{2}} \right\} \\
&\quad + P \left\{ \max_{\frac{2^k}{3\ell(2^k)} \leq i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \eta_{u,k} \right| \geq \epsilon 2^k \ell(2^k) \right\} \\
&:= I_{1,k} + I_{2,k} + I_{3,k},
\end{aligned}$$

provided that k is sufficiently large.

Using Lemma 10.3.6, we have

$$\begin{aligned}
 (10.3.22) \quad n^{-1} E(\sum_{i=1}^n \hat{X}_{i,k})^2 &\leq 2n^{-1} \{E(\sum_{i=1}^n (\hat{X}_{i,k} - E\hat{X}_{i,k}))^2 + (\sum_{i=1}^n E\hat{X}_{i,k})^2\} \\
 &\leq n^{-1} (K n E X_1^2 I\{|X_1| \geq 2^{\frac{k}{2}}\} + 2n^{-1} E^2 |X_1| I\{|X_1| \geq 2^{\frac{k}{2}}\}) \\
 &\rightarrow 0 \text{ as } k \rightarrow \infty \text{ uniformly in } 1 \leq n \leq 2^{k+2}.
 \end{aligned}$$

By (10.3.22), we obtain

$$\begin{aligned}
 (10.3.23) \quad I_{2,k} &\leq P\{\max_{j \leq 2^{k+1} + \ell(2^k)} |X_j| \geq 2^{\frac{k}{2}}\} + P\{\max_{j \leq 2^{k+1}} \max_{i \leq \ell(2^k)} |\bar{S}_j(i)| \geq \frac{\varepsilon}{3} 2^{\frac{k}{2}}\} \\
 &\leq 2^{k+2} P\{|X_1| \geq 2^{\frac{k}{2}}\} + P\{\max_{j \leq 2^{k+1}} \max_{i \leq \ell(2^k)} |\hat{S}_j(i) - E\hat{S}_j(i)| \geq \frac{\varepsilon}{6} 2^{\frac{k}{2}}\} \\
 &\leq 2^{k+2} P\{|X_1| \geq 2^{\frac{k}{2}}\} + P\{\max_{v \leq \frac{2^{k+1}}{\ell(2^k)}} \max_{i \leq \ell(2^k)} |\bar{S}_{v\ell(2^k)}(i) - E\bar{S}_{v\ell(2^k)}(i)| \geq \frac{\varepsilon}{18} 2^{\frac{k}{2}}\} \\
 &\leq 2^{k+2} P\{|X_1| \geq 2^{\frac{k}{2}}\} + \frac{2^{k+2}}{\ell(2^k)} P\{\max_{i \leq \ell(2^k)} |\bar{S}_0(i) - E\bar{S}_0(i)| \geq \frac{\varepsilon}{18} 2^{\frac{k}{2}}\}.
 \end{aligned}$$

It follows from Lemmas 10.3.5. and 10.3.6 that for $p > 2 + 1/\varepsilon_0$

$$\begin{aligned}
 (10.3.24) \quad &\sum_{k=1}^{\infty} \frac{2^k}{\ell(2^k)} P\left\{\max_{i \leq \ell(2^k)} |\bar{S}_0(i) - E\bar{S}_0(i)| \geq \frac{\varepsilon}{18} 2^{\frac{k}{2}}\right\} \\
 &\leq K \sum_{k=1}^{\infty} \frac{2^k}{\ell(2^k)} \cdot \frac{1}{(\varepsilon 2^{k/2})^p} (\ell^{p/2}(2^k) + \ell(2^k) E|X_1|^p I\{|X_1| \leq 2^{\frac{k}{2}}\}) \\
 &\leq K \left(\sum_{k=1}^{\infty} \left(\frac{\ell(2^k)}{2^k}\right)^{p-1} + \sum_{k=1}^{\infty} 2^{-k(\frac{p}{2}-1)} E|X_1|^p I\{|X_1| \leq 2^{\frac{k}{2}}\} \right) \\
 &< \infty
 \end{aligned}$$

by (10.3.4) and Lemma 10.3.11. Similarly, we have

$$(10.3.25) \quad \sum_{k=1}^{\infty} 2^k P\{|X_1| \geq 2^{\frac{k}{2}}\} < \infty.$$

This proves

$$(10.3.26) \quad \sum_{k=1}^{\infty} I_{2,k} < \infty$$

by (10.3.23), (10.3.24) and (10.3.25).

We now estimate $I_{1,k}$. From $E(\sum_{j=1}^n X_j)^2 - E(\sum_{j=1}^n X_{j,k})^2 = E(\sum_{j=1}^n \hat{X}_{j,k})^2 + 2E(\sum_{j=1}^n \bar{X}_{j,k})(\sum_{j=1}^n \hat{X}_{j,k})$, it is easy to see that

$$\begin{aligned}
 (10.3.27) \quad I_{1,k} &\leq P\left\{\max_{i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \xi_{u,k} \right| \geq \varepsilon 2^k \ell(2^k)\right\} \\
 &\leq P\left\{\max_{j \leq 2^{k+2}} |X_j| \geq 2^{\frac{k}{2}}\right\} \\
 &\quad + P\left\{\max_{i \leq \frac{2^k}{\ell(2^k)}} \left\{ \left| \sum_{u=0}^i \bar{\xi}_{u,k} \right| + i \ell(2^k) \left| E\left(\sum_{j=1}^{\ell(2^k)} X_j \right)^2 - E\left(\sum_{j=1}^{\ell(2^k)} X_{j,k} \right)^2 \right| \right\} \geq \varepsilon 2^k \ell(2^k)\right\} \\
 &\leq 2^{k+2} P\{|X_1| \geq 2^{\frac{k}{2}}\} + P\left\{\max_{i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \bar{\xi}_{u,k} \right| \geq \frac{1}{2} \varepsilon 2^k \ell(2^k)\right\}
 \end{aligned}$$

by (10.3.22) and Lemma 10.3.6, for every k sufficiently large. Clearly, $\{\xi_{u,k}, u = 0, 1, \dots\}$ is also a ϕ -mixing sequence. Hence, using Lemmas 10.3.5 and 10.3.6 again, we conclude that for $p > 2 + 2/\varepsilon_0$

$$\begin{aligned}
 (10.3.28) \quad &\sum_{k=1}^{\infty} P\left\{\max_{i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \bar{\xi}_{u,k} \right| \geq \frac{1}{2} \varepsilon 2^k \ell(2^k)\right\} \\
 &\leq K \sum_{k=1}^{\infty} (\varepsilon 2^k \ell(2^k))^{-p} \left\{ \frac{2^k}{\ell(2^k)} E|\bar{\xi}_{0,k}|^p + \left(\frac{2^k}{\ell(2^k)} E|\xi_{0,k}|^2 \right)^{\frac{p}{2}} \right\} \\
 &\leq K \sum_{k=1}^{\infty} (2^k \ell(2^k))^{-p} \left\{ \frac{2^k}{\ell(2^k)} \ell^p(2^k) E\left| \sum_{j=1}^{\ell(2^k)} \bar{X}_{j,k} \right|^{2p} + (2^k \ell(2^k)) E\left| \sum_{j=1}^{\ell(2^k)} \bar{X}_{j,k} \right|^4 \right\} \\
 &\leq K \sum_{k=1}^{\infty} (2^k \ell(2^k))^{-p} \left\{ 2^k \ell^{p-1}(2^k) \left\{ E\left| \sum_{j=1}^{\ell(2^k)} \bar{X}_{j,k} - E\bar{X}_{j,k} \right|^{2p} \right. \right. \\
 &\quad \left. \left. + (\ell(2^k) E|X_1| I\{|X_1| \leq 2^{\frac{k}{2}}\})^{2p} \right\} \right. \\
 &\quad \left. + (2^k \ell(2^k)) \left\{ E\left| \sum_{j=1}^{\ell(2^k)} \bar{X}_{j,k} - E\bar{X}_{j,k} \right|^4 + (\ell(2^k) E|X_1| I\{|X_1| \geq 2^{\frac{k}{2}}\})^4 \right\} \right\} \\
 &\leq K \sum_{k=1}^{\infty} (2^k \ell(2^k))^{-p} \left\{ 2^k \ell^{p-1}(2^k) \left(\ell^p(2^k) + \ell(2^k) E|X_1|^{2p} I\{|X_1| \leq 2^{\frac{k}{2}}\} \right) \right\}
 \end{aligned}$$

$$\begin{aligned}
& + \left(2^k \ell(2^k) (\ell^2(2^k) + \ell(2^k) E|X_1|^4 I\{|X_1| \leq 2^{\frac{k}{2}}\}) \right)^{\frac{p}{2}} \Big\} \\
\leq & K \left\{ \sum_{k=1}^{\infty} \left(\frac{\ell(2^k)}{2^k} \right)^{p-1} + \sum_{k=1}^{\infty} 2^{-k(p-1)} E|X_1|^{2p} I\{|X_1| \leq 2^{\frac{k}{2}}\} \right. \\
& \left. + \sum_{k=1}^{\infty} \left(\frac{\ell(2^k)}{2^k} \right)^{p/2} + \sum_{k=1}^{\infty} 2^{-\frac{1}{2}kp} (E|X_1|^4 I\{|X_1| \leq 2^{\frac{k}{2}}\})^{\frac{p}{2}} \right\} \\
\leq & K \left\{ \sum_{k=1}^{\infty} \left(\frac{\ell(2^k)}{2^k} \right)^{p/2} + \sum_{k=1}^{\infty} 2^{-k(p-1)} E|X_1|^{2p} I\{|X_1| \leq 2^{\frac{k}{2}}\} \right. \\
& \left. + \sum_{k=1}^{\infty} 2^{-k} E|X_1|^4 I\{|X_1| \leq 2^{\frac{k}{2}}\} \right\} \\
< & \infty
\end{aligned}$$

by (10.3.4) and Lemma 10.3.11. A combination of (10.3.27), (10.3.28) and (10.3.25) yields

$$(10.3.29) \quad \sum_{k=1}^{\infty} I_{1,k} < \infty.$$

Similarly, we have

$$(10.3.30) \quad \sum_{k=1}^{\infty} I_{3,k} < \infty.$$

This proves (10.3.20) by (10.3.21), (10.3.26), (10.3.29) and (10.3.30), and hence **(A1)** is satisfied.

We verify below that **(A3)** holds true as well. It suffices to show that $\forall \varepsilon > 0$

$$\begin{aligned}
(10.3.31) \quad \sum_{k=1}^{\infty} P \left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{j=0}^{n-1} (I(\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j,n}) < x) \right. \right. \\
\left. \left. - P\{\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_i < x\} \right| \geq \varepsilon 2^k \right\} < \infty.
\end{aligned}$$

Let

$$\zeta_{u,k} = \sum_{j=2u\ell(2^k)}^{(2u+1)\ell(2^k)-1} \left\{ I(\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j} < x) - P\{\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j} < x\} \right\},$$

$$\tau_{u,k} = \sum_{j=(2u+1)\ell(2^k)}^{(2u+2)\ell(2^k)-1} \{I(\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j} < x) - P\{\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j} < x\}\},$$

$$u = 0, 1, \dots.$$

Noting that

$$(10.3.32) \quad \left| \sum_{j=0}^{n-1} \{I(\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j,n} < x) - P\{\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_i < x\}\} \right|$$

$$\leq \left| \sum_{j=0}^{n-\ell(2^k)} \{I(\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j} < x) - P\{\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_i < x\}\} \right| + 2\ell(2^k)$$

$$\leq 4\ell(2^k) + \left| \sum_{u=0}^{p_{k,n}} \zeta_{u,k} \right| + \left| \sum_{u=0}^{p_{k,n}} \tau_{u,k} \right|$$

and that $\ell(2^k) = o(2^k)$, we deduce

$$(10.3.33) \quad P\left\{ \max_{2^k \leq n < 2^{k+1}} \left| \sum_{j=0}^{n-1} \{I(\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_{i+j,n} < x) - P\{\ell^{-\frac{1}{2}}(2^k) \sum_{i=1}^{\ell(2^k)} X_i < x\}\} \right| \geq \varepsilon 2^k \right\}$$

$$\leq P\left\{ \max_{\frac{2^k}{3\ell(2^k)} \leq i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{j=0}^i \zeta_{u,k} \right| \geq \frac{1}{3} \varepsilon 2^k \right\} + P\left\{ \max_{\frac{2^k}{3\ell(2^k)} \leq i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{j=0}^i \tau_{u,k} \right| \geq \frac{1}{3} \varepsilon 2^k \right\}$$

$$:= J_{1,k} + J_{2,k}$$

for every k sufficiently large. Since $\{\zeta_{u,k}, u = 0, 1, \dots\}$ is also a ϕ -mixing sequence, we derive from Lemmas 10.3.5 and 10.3.6 that for $p > 2 + 4/\varepsilon_0$

$$(10.3.34) \quad \sum_{k=1}^{\infty} J_{1,k} \leq K \sum_{k=1}^{\infty} (\varepsilon 2^k)^{-p} \left(\frac{2^k}{\ell(2^k)} E|\zeta_{0,k}|^p + \left(\frac{2^k}{\ell(2^k)} E\zeta_{0,k}^2 \right)^{\frac{p}{2}} \right)$$

$$\leq K \sum_{k=1}^{\infty} 2^{-pk} \left(\frac{2^k}{\ell(2^k)} \cdot \ell^p(2^k) + \left(\frac{2^k}{\ell(2^k)} \cdot \ell^2(2^k) \right)^{\frac{p}{2}} \right)$$

$$\leq K \sum_{k=1}^{\infty} \left(\frac{\ell(2^k)}{2^k} \right)^{\frac{p}{2}}$$

$$< \infty$$

by (10.3.4). Similarly, we have

$$(10.3.35) \quad \sum_{k=1}^{\infty} J_{2,k} < \infty.$$

This proves (10.3.31) by (10.3.33), (10.3.34) and (10.3.35) and hence the condition **(A3)** is satisfied. Now Theorem 10.3.1 follows from Theorem 10.2.2. ■

Proof of Theorem 10.3.2.

It follows from Lemma 10.3.3 that **(B1)** and **(B2)** are satisfied. We only need to verify that **(A1)** and **(A3)** hold true. Without loss of generality, we assume again that $\mu = EX_1 = 0$. According to the proof of Theorem 10.3.1, it suffices to show that (10.3.19) and (10.3.31) remain true. Let $p_{k,n}, \xi_{u,k}, \eta_{u,k}, \zeta_{u,k}, \tau_{u,k}, I_{1,k}, I_{2,k}, I_{3,k}, J_{1,k}$ and $J_{2,k}$ be defined as in the proof of Theorem 10.3.1. We have, by Lemma 10.3.8,

$$\begin{aligned} (10.3.36) \quad I_{2,k} &\leq P\left\{ \max_{0 \leq u \leq \frac{2^{k+1}}{\ell(2^k)}} \max_{1 \leq i \leq \ell(2^k)} |S_{u\ell(2^k)}(i)| \geq \frac{\varepsilon}{9} 2^{k/2} \right\} \\ &\leq \frac{2^{k+2}}{\ell(2^k)} P\left\{ \max_{i \leq \ell(2^k)} |S_i| \geq \frac{\varepsilon}{9} 2^{k/2} \right\} \\ &\leq K \cdot \frac{2^k}{\ell(2^k)} \cdot \ell(2^k) \cdot 2^{-\frac{k}{2} \cdot \frac{(2+\delta)(r+1)}{2+\delta+r}} \cdot \log^r(2^{k/2}) \\ &\leq K 2^{-k \cdot \frac{r\delta - (2+\delta)}{2(2+\delta+r)}} \cdot k^r. \end{aligned}$$

Hence

$$(10.3.37) \quad \sum_{k=1}^{\infty} I_{2,k} < \infty.$$

As to $I_{1,k}$, take $1 < s < 1/(1 + \frac{1}{r} - \frac{\delta}{2+\delta})$. Then

$$\sum_{n=1}^{\infty} n^{s-1} \alpha^{\frac{2+\delta-2s}{2+\delta}}(n) \leq K \sum_{n=1}^{\infty} n^{s-1} \cdot n^{-\frac{(2+\delta-2s)r}{2+\delta}}(n) < \infty$$

by condition (10.3.5). Using Lemma 10.3.9, we obtain

$$(10.3.38) \quad (E|\xi_{0,k}|^s)^{1/s} \leq \ell(2^k) (E|\sum_{i=1}^{\ell(2^k)} X_i|^{2s})^{1/2s} \leq K \ell^2(2^k).$$

Applying (10.3.38) and Lemma 10.3.8 again, we derive

$$\begin{aligned}
 (10.3.39) \quad I_{1,k} &\leq K \cdot \frac{2^k}{\ell(2^k)} \cdot \left(\frac{\ell(2^k)}{2^k} \right)^{\frac{s(r+1)}{s+r}} \cdot \log^r \left(\frac{2^k \ell(2^k)}{\ell^2(2^k)} \right) \\
 &\leq K \left(\frac{\ell(2^k)}{2^k} \right)^{\frac{(s-1)r}{s+r}} \cdot \log^r(2^k) \\
 &\leq K \cdot 2^{-\frac{ks_0(s-1)r}{s+r}} \cdot k^r
 \end{aligned}$$

by (10.3.6). Therefore

$$(10.3.40) \quad \sum_{k=1}^{\infty} I_{1,k} < \infty.$$

Similarly, one can get

$$(10.3.41) \quad \sum_{k=1}^{\infty} I_{3,k} < \infty.$$

This proves (10.3.19).

As to $J_{1,k}$, noting that $|\zeta_{u,k}| \leq \ell(2^k)$, by Lemma 10.3.8 again, we have

$$\begin{aligned}
 (10.3.42) \quad \sum_{k=1}^{\infty} J_{1,k} &\leq K \sum_{k=1}^{\infty} \frac{2^k}{\ell(2^k)} \left(\frac{\ell(2^k)}{2^k} \right)^{r+1} \cdot \log^r(2^k) \\
 &\leq K \sum_{k=1}^{\infty} 2^{-ks_0 r} \cdot k^r < \infty
 \end{aligned}$$

by (10.3.6). Similarly,

$$(10.3.43) \quad \sum_{k=1}^{\infty} J_{2,k} < \infty,$$

which together with (10.3.42) implies (10.3.31). This completes the proof of Theorem 10.3.2. ■

Proof of Theorem 10.3.3.

The proof is exactly the same as that of Theorem 10.3.2 except that Lemma 10.3.4 is used instead of Lemma 10.3.3. ■

Proof of Theorem 10.3.4.

Let $p_{k,n}, \xi_{u,k}, \eta_{u,k}, \zeta_{u,k}, \bar{\zeta}_{u,k}, I_{1,k}, I_{2,k}, I_{3,k}, J_{1,k}$ and $J_{2,k}$ be defined as in the proof of Theorem 10.3.1. We first prove (10.3.26). Take

$$A = 2^{\frac{k}{2}}, B = k^{-3} 2^{\frac{k}{2}}, x = \frac{\varepsilon}{18} 2^{\frac{k}{2}}, q = 4, n = \ell(2^k)$$

in Lemma 10.3.7. Then (10.3.8) implies (10.3.17). Using Lemma 10.3.7 yields

$$\begin{aligned} (10.3.44) \quad I_{2,k} &\leq \frac{2^{k+2}}{\ell(2^k)} P\left\{\max_{1 \leq i \leq \ell(2^k)} |S_i| \geq \frac{\varepsilon}{9} 2^{\frac{k}{2}}\right\} \\ &\leq K \frac{2^k}{\ell(2^k)} \left\{ \ell(2^k) P\{|X_1| \geq 2^{\frac{k}{2}}\} + 2^{-2k} \left(\ell(2^k) E X_1^4 I(|X_1| \leq 2^{\frac{k}{2}}) \right. \right. \\ &\quad \left. \left. + \ell^2(2^k) + \ell(2^k) \cdot \log^2 \ell(2^k) \cdot E X_1^4 I(|X_1| \leq 2^{\frac{k}{2}} k^{-3}) \right) \right\} \\ &\leq K \left\{ 2^k P\{|X_1| \geq 2^{\frac{k}{2}}\} + \frac{\ell(2^k)}{2^k} + k^{-4} + 2^{-k} E X_1^4 I(|X_1| \leq 2^{\frac{k}{2}}) \right\}. \end{aligned}$$

Hence

$$(10.3.45) \quad \sum_{k=1}^{\infty} I_{2,k} < \infty$$

by Lemma 10.3.11 and (10.3.8).

Let $1 < \theta < 2$ be such that

$$(10.3.46) \quad (1 - \frac{1}{\theta}) K_0 < \frac{\varepsilon}{4},$$

where K_0 is a positive constant which will be specified later on. Similarly to (10.3.23), one can show that

$$\begin{aligned} (10.3.47) \quad I_{1,k} &\leq 2^{k+2} P\{|X_1| \geq 2^{k/2}\} + P\left\{\max_{\frac{2^k}{3\ell(2^k)} \leq i \leq \frac{2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \bar{\xi}_{u,k} \right| \geq \frac{\varepsilon}{2} 2^k \ell(2^k)\right\} \\ &\leq 2^{k+2} P\{|X_1| \geq 2^{\frac{k}{2}}\} + \sum_{j=0}^{[\log_3 3]} P\left\{\max_{\frac{\theta^{-j-1} 2^k}{\ell(2^k)} \leq i \leq \frac{\theta^{-j} 2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \bar{\xi}_{u,k} \right| \geq \frac{\varepsilon}{2} 2^k \ell(2^k)\right\}. \end{aligned}$$

For $\frac{2^k}{\ell(2^k)} \theta^{-j-1} \leq i \leq \theta^{-j} \frac{2^k}{\ell(2^k)}$, we have

$$\sum_{u=0}^i \bar{\xi}_{u,k} \leq \sum_{u=0}^{[\frac{\theta^{-j-1} 2^k}{\ell(2^k)}]} \bar{\xi}_{u,k} + \sum_{u=[\frac{\theta^{-j-1} 2^k}{\ell(2^k)}]}^{[\frac{\theta^{-j} 2^k}{\ell(2^k)}]} \bar{\xi}_{u,k} \sum_{v=2u\ell(2^k)}^{(2u+1)\ell(2^k)-1} E \left(\sum_{i=1}^{\ell(2^k)} \bar{X}_{i+v,k} \right)^2$$

$$\begin{aligned}
&\leq \sum_{u=0}^{\lfloor \frac{\theta^{-j} 2^k}{\ell(2^k)} \rfloor} \bar{\xi}_{u,k} + (\theta^{-j} - \theta^{-j-1}) \frac{2^k}{\ell(2^k)} \cdot \ell^2(2^k) K_0 \\
&\leq \sum_{u=0}^{\lfloor \frac{\theta^{-j} 2^k}{\ell(2^k)} \rfloor} \bar{\xi}_{u,k} + \frac{\varepsilon}{4} \cdot 2^k \ell(2^k)
\end{aligned}$$

by Lemma 10.3.6 and (10.3.46). Similarly, we obtain

$$\sum_{u=0}^i \bar{\xi}_{u,k} \geq - \sum_{u=0}^{\lfloor \frac{\theta^{-j-1} 2^k}{\ell(2^k)} \rfloor} \bar{\xi}_{u,k} - \frac{\varepsilon}{4} \cdot 2^k \ell(2^k).$$

Therefore

$$\begin{aligned}
(10.3.48) \quad &P\left\{ \max_{\frac{\theta^{-j-1} 2^k}{\ell(2^k)} \leq i \leq \frac{\theta^{-j} 2^k}{\ell(2^k)}} \left| \sum_{u=0}^i \bar{\xi}_{u,k} \right| \geq \frac{\varepsilon}{2} 2^k \ell(2^k) \right\} \\
&\leq P\left\{ \left| \sum_{u=0}^{\lfloor \frac{\theta^{-j-1} 2^k}{\ell(2^k)} \rfloor} \bar{\xi}_{u,k} \right| + \left| \sum_{u=0}^{\lfloor \frac{\theta^{-j} 2^k}{\ell(2^k)} \rfloor} \bar{\xi}_{u,k} \right| \geq \frac{\varepsilon}{4} 2^k \ell(2^k) \right\} \\
&\leq K \left(2^k \ell(2^k) \right)^{-2} \cdot \frac{2^k}{\ell(2^k)} \cdot E \bar{\xi}_{0,k}^2 \\
&\leq K \left(2^k \ell(2^k) \right)^{-1} \cdot E \left| \sum_{v=1}^{\ell(2^k)} \bar{X}_{v,k} \right|^4 \\
&\leq K \left(2^k \ell(2^k) \right)^{-1} (E \left| \sum_{v=1}^{\ell(2^k)} \bar{X}_{v,k} - E \bar{X}_{v,k} \right|^4 + (\ell(2^k) E |X_1| I\{|X_1| \geq 2^{k/2}\})^4) \\
&\leq K \left(2^k \ell(2^k) \right)^{-1} \left(\frac{2^k}{\ell(2^k)} + \frac{1}{2^k} E X_1^4 I\{|X_1| \leq 2^{k/2}\} \right).
\end{aligned}$$

Using Lemma 10.3.11 and (10.3.6) again, we finally conclude

$$(10.3.49) \quad \sum_{k=1}^{\infty} I_{1,k} < \infty$$

by (10.3.47) and (10.3.48). Similarly, we have

$$(10.3.50) \quad \sum_{k=1}^{\infty} I_{3,k} < \infty.$$

Putting the above inequalities together, we see that (10.3.26) is satisfied. Along the lines of the proof of (10.3.49), we can arrive at

$$\sum_{k=1}^{\infty} J_{1,k} < \infty$$

and at

$$\sum_{k=1}^{\infty} J_{2,k} < \infty$$

as well. This completes the proof of Theorem 10.3.4. ■

10.4 Bootstrapping empirical processes for stationary ρ -mixing sequences.

Applying Theorem 10.2.3, we can also establish weak convergence for bootstrapped empirical processes of stationary ρ -mixing sequences.

THEOREM 10.4.1 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a stationary ρ -mixing sequence of random variables with common continuous distribution function F . Assume that the covariance series in (10.1.3) is absolutely convergent and*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty.$$

Then

$$\alpha_n \xrightarrow{\mathcal{D}} B \quad \text{in } D[0, 1]$$

with $P\{B \in C[0, 1]\} = 1$, and if $l = O(n^{1-\nu_1}) \rightarrow \infty$ as $n \rightarrow \infty$ and $n^{\nu_2} \leq m \leq n$ for some $0 < \nu_1, \nu_2 < 1$, we have almost surely as $n \rightarrow \infty$

$$\alpha_n^* \xrightarrow{\mathcal{D}} B \quad \text{in } D[0, 1]$$

with $P^\{B \in C[0, 1]\} = 1$ a.s..*

Proof. The already mentioned C_r inequality, $|\sum_{i=1}^n a_i|^r \leq n^{r-1} \sum_{i=1}^n |a_i|^r$ for any $(a_1, \dots, a_n) \in \mathbf{R}^n$ and $r \geq 1$, is often used in this proof.

The weak convergence for the original empirical process α_n was proved by Shao (1986). Hence, we concentrate mainly on showing weak convergence for the bootstrapped empirical process α_n^* . For doing this, the proof is divided into three steps. The first one is to prove that (C1) holds true. We split the summation in (C1) into two parts and have for fixed $s, t \in T_0$

$$\begin{aligned} J_n &= (n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \left(\sum_{i=1}^{\ell} (I(U_{i+j,n} \leq s) - s) \right) \left(\sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t) - t) \right) \right. \\ &\quad \left. - E \left(\sum_{i=1}^{\ell} (I(U_i \leq s) - s) \right) \left(\sum_{i=1}^{\ell} (I(U_i \leq t) - t) \right) \right\} \\ &= (n\ell)^{-1} \sum_{j=0}^{n-\ell} + (n\ell)^{-1} \sum_{j=n-\ell+1}^{n-1} \\ &:= J_{1n} + J_{2n}. \end{aligned}$$

Let

$$\begin{aligned} \eta_j &= \left(\sum_{i=1}^{\ell} (I(U_{i+j} \leq s) - s) \right) \left(\sum_{i=1}^{\ell} (I(U_{i+j} \leq t) - t) \right) \\ &\quad - E \left(\sum_{i=1}^{\ell} (I(U_i \leq s) - s) \right) \left(\sum_{i=1}^{\ell} (I(U_i \leq t) - t) \right) \end{aligned}$$

for $j = 0, \dots, n - \ell$. Then $E\eta_j = 0$ by stationarity, and

$$J_{1n} = (n\ell)^{-1} \left\{ \sum_{k=0}^{\lfloor \frac{n-\ell+1}{2\ell} \rfloor} \sum_{j=2k\ell}^{(2k+1)\ell-1} \eta_j + \sum_{k=0}^{\lfloor \frac{n-\ell+1}{2\ell} \rfloor} \sum_{j=(2k+1)\ell}^{2(k+1)\ell-1} \eta_j + \sum_{j=2\lfloor \frac{n-\ell+1}{2\ell} \rfloor \ell}^{n-\ell} \eta_j \right\}.$$

Notice that $\sum_{j=2k\ell}^{(2k+1)\ell-1} \eta_j \in \sigma(U_{2k\ell+1}, \dots, U_{2(k+1)\ell-1})$ for $k = 0, 1, \dots, \lfloor \frac{n-\ell+1}{2\ell} \rfloor$.

Hence, applying Lemma 10.3.6, the C_r and Hölder inequalities, we have for $q \geq 4$,

$$E|J_{1n}|^q \leq K(n\ell)^{-q} \left\{ \left(\frac{n}{\ell} \right)^{q/2} \left(E \left| \sum_{j=0}^{\ell-1} \eta_j \right|^2 \right)^{q/2} + \left(\frac{n}{\ell} \right)^{\ell} E \left| \sum_{j=0}^{\ell-1} \eta_j \right|^q \right\}$$

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$$\begin{aligned}
&\leq K(n\ell)^{-q} \left(\frac{n}{\ell}\right)^{q/2} \ell^q E|\eta_0|^q \\
&\leq K(n\ell)^{-q/2} E^{1/2} \left| \sum_{i=1}^{\ell} (I(U_i \leq s) - s) \right|^{2q} E^{1/2} \left| \sum_{i=1}^{\ell} (I(U_i \leq t) - t) \right|^{2q} \\
&\leq K\left(\frac{\ell}{n}\right)^{q/2},
\end{aligned}$$

where the constant K is independent of s, t and n , however, it may take different values in each appearance in the rest of proof. For the term J_{2n} , by Lemma 10.3.6, the C_r and Hölder inequalities,

$$\begin{aligned}
E|J_{n2}|^q &\leq 2^{q-1}(n\ell)^{-q} E \left| \sum_{j=n-\ell+1}^{n-1} \left\{ \sum_{i=1}^{\ell} (I(U_{i+j,n} \leq s) - s) \right\} \left(\sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t) - t) \right) \right|^q \\
&\quad + 2^{q-1} n^{-q} E \left| \sum_{i=1}^{\ell} (I(U_i \leq s) - s) \right|^q \left| \sum_{i=1}^{\ell} (I(U_i \leq t) - t) \right|^q \\
&\leq K n^{-q} \max_{n-\ell+1 \leq j \leq n-1} \sup_{0 \leq s \leq 1} E \left| \sum_{i=1}^{n-j} (I(U_{i+j} \leq s) - s) \right|^{2q} \\
&\quad + \sum_{i=1}^{\ell+j-n} (I(U_i \leq s) - s)^{2q} + K\left(\frac{\ell}{n}\right)^q \\
&\leq K\left(\frac{\ell}{n}\right)^q.
\end{aligned}$$

Consequently, for $q \geq 4$, we have

$$E|J_n|^q \leq K\left(\frac{\ell}{n}\right)^{q/2}.$$

Hence (C1) follows simply by the Borel-Cantelli lemma and q being large enough.

Next we prove that (C2) is true. By applying Lemma 10.3.6 and the C_r inequality again, for any $\varepsilon > 0$,

$$\begin{aligned}
&P\{(m\ell)^{1-q}(n\ell)^{-1} \sum_{j=0}^{n-1} \left| \sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t) - t) \right|^{2q} \geq \varepsilon\} \\
&\leq \varepsilon^{-1} (m\ell)^{1-q} (n\ell)^{-1} \left\{ \sum_{j=0}^{n-\ell} E \left| \sum_{i=1}^{\ell} (I(U_{i+j} \leq t) - t) \right|^{2q} \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=n-\ell+1}^{n-1} E \left| \sum_{i=1}^{\ell} (I(U_{i+j,n} \leq t) - t) \right|^{2q} \Big\} \\
& \leq K m^{1-q} \leq K n^{\nu_2(1-q)}.
\end{aligned}$$

Let q be large enough so that $\nu_2(1-q) < -1$. Then (C2) follows by the Borel-Cantelli lemma.

Finally, the most difficult part is to prove (C3). Introducing the following notations

$$(10.4.1) \quad Z_{i,n}^*(t) = \ell^{-1} \sum_{j=1}^{\ell} I(U_{(i-1)\ell+j}^* \leq t), \quad i = 1, \dots, m,$$

we have that

$$(10.4.2) \quad \alpha_n^*(t) = \left(\frac{\ell}{m}\right)^{1/2} \sum_{i=1}^m (Z_{i,n}^*(t) - E_n(t)),$$

and $\{Z_{i,n}^*(t), 1 \leq i \leq m\}$ is an i.i.d. sequence with $E^* Z_{1,n}^*(t) = E_n(t)$ for each fixed $t \in [0, 1]$. Since $\{Z_{i,n}^*(t) - Z_{i,n}^*(s), 1 \leq i \leq m\}$ is an i.i.d. sequence with $E^*\{Z_{1,n}^*(t) - Z_{1,n}^*(s)\} = E_n(t) - E_n(s)$, for $p \geq 2$ and $0 \leq s < t \leq 1$, we have by (10.4.2) and the Marcinkiewicz-Zygmund inequality

$$\begin{aligned}
(10.4.3) \quad & E^* |\alpha_n^*(t) - \alpha_n^*(s)|^p \\
& \leq K \ell^{p/2} m^{1-p/2} E^* |Z_{1,n}^*(t) - Z_{1,n}^*(s) - (E_n(t) - E_n(s))|^p \\
& \quad + K \{\ell E^* (Z_{1,n}^*(t) - Z_{1,n}^*(s) - (E_n(t) - E_n(s)))^2\}^{p/2}.
\end{aligned}$$

From (10.2.1)-(10.2.2) and (10.4.1), it is easy to get

$$\begin{aligned}
(10.4.4) \quad & \ell E^* \{Z_{1,n}^*(t) - Z_{1,n}^*(s) - (E_n(t) - E_n(s))\}^2 \\
& = (n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \sum_{i=1}^{\ell} (I(s < U_{i+j,n} \leq t) - (E_n(t) - E_n(s))) \right\}^2 \\
& = (n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \sum_{i=1}^{\ell} (I(s < U_{i+j,n} \leq t) - (t - s)) \right\}^2
\end{aligned}$$

$$\begin{aligned}
 & -\ell(E_n(t) - E_n(s) - (t - s))^2 \\
 & \leq (n\ell)^{-1} \sum_{j=0}^{n-1} \left\{ \sum_{i=1}^{\ell} (I(s < U_{i+j,n} \leq t) - (t - s)) \right\}^2 \\
 & := I_n(s, t).
 \end{aligned}$$

Now by the C_r inequality.

$$\begin{aligned}
 (10.4.5) \quad I_n(s, t) & \leq (n\ell)^{-1} \sum_{j=0}^{n-\ell} \left\{ \sum_{i=1}^{\ell} (I(s < U_{i+j} \leq t) - (t - s)) \right\}^2 \\
 & \quad + 2(n\ell)^{-1} \sum_{j=n-\ell+1}^{n-1} \left\{ \sum_{i=1}^{n-j} (I(s < U_{i+j} \leq t) - (t - s)) \right\}^2 \\
 & \quad + 2(n\ell)^{-1} \sum_{j=n-\ell+1}^{n-1} \left\{ \sum_{i=1}^{\ell+j-n} (I(s < U_i \leq t) - (t - s)) \right\}^2 \\
 & := I_{1n}(s, t) + 2I_{2n}(s, t) + 2I_{3n}(s, t).
 \end{aligned}$$

Let $\xi_i(s, t) = I(s < U_i \leq t) - (t - s)$. Then

$$(10.4.6) \quad I_{1n}(s, t) \leq (n\ell)^{-1} \sum_{j=0}^{2\ell(\lfloor \frac{n-\ell+1}{2\ell} \rfloor + 1) - 1} \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 := I'_{1n}(s, t).$$

With the notation

$$(10.4.7) \quad H_n(s, t) = I'_{1n}(s, t) - E I'_{1n}(s, t),$$

by the C_r inequality we get

$$(10.4.8) \quad |H_n(s, t)|^q \leq 2^{q-1} |I'_{1n}(s, t)|^q + 2^{q-1} |E I'_{1n}(s, t)|^q,$$

where

$$(10.4.9) \quad I'_{1n}(s, t) = (n\ell)^{-1} \sum_{k=0}^{\lfloor \frac{n-\ell+1}{2\ell} \rfloor} \sum_{j=2k\ell}^{(2k+1)\ell-1} \left\{ \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 - E \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 \right\}$$

and

$$(10.4.10) \quad I'_{1n}(s, t) = (n\ell)^{-1} \sum_{k=0}^{\lfloor \frac{n-\ell+1}{2\ell} \rfloor} \sum_{j=(2k+1)\ell}^{2(k+1)\ell-1} \left\{ \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 - E \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 \right\}.$$

For simplicity, let

$$\zeta_k = \sum_{j=2k\ell}^{(2k+1)\ell-1} \left\{ \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 - E \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 \right\}, \quad k = 0, 1, \dots, \left[\frac{n-\ell+1}{2\ell} \right].$$

Then $E\zeta_k = 0$ and $\zeta_k \in \sigma(U_{2k\ell+1}, \dots, U_{2(k+1)\ell-1})$. Hence, by applying Lemma 10.3.6 and the C_r inequality, for $q \geq 4$, we obtain

$$\begin{aligned} (10.4.11) \quad E|I_{1n}^{(1)}|^q &\leq K(n\ell)^{-q} \left\{ \left(\frac{n}{\ell} \right)^{q/2} (E\zeta_0^2)^{q/2} + \left(\frac{n}{\ell} \right)^2 E|\zeta_0|^q \right\} \\ &\leq K(n\ell)^{-q} \left\{ \left(\frac{n}{\ell} \right)^{q/2} \ell^q (E|\sum_{i=1}^{\ell} \xi_i(s, t)|^4)^{q/2} + \left(\frac{n}{\ell} \right)^2 \ell^q E|\sum_{i=1}^{\ell} \xi_i(s, t)|^{2q} \right\} \\ &\leq K \{ n^{2-q}(t-s) + \left(\frac{\ell}{n} \right)^{q/2} (t-s)^{q/2} \}, \end{aligned}$$

where in the last inequality, we used the fact that

$$(10.4.12) \quad E|\xi_1(s, t)|^p \leq t-s \quad \text{for any } p \geq 2.$$

With the same treatment for $I_{1n}^{(2)}$, we have by (10.4.8),

$$(10.4.13) \quad E|H_n(s, t)|^q \leq K \{ n^{2-q}(t-s) + \left(\frac{\ell}{n} \right)^{q/2} (t-s)^{q/2} \}.$$

On the other hand, direct applications of Lemma 10.3.6 and the C_r inequality lead to

$$(10.4.14) \quad E|I_{2n}(s, t)|^q \leq K \{ n^{2-q}(t-s) + \left(\frac{\ell}{n} \right)^q (t-s)^q \}$$

and

$$(10.4.15) \quad E|I_{3n}(s, t)|^q \leq K \{ n^{2-q}(t-s) + \left(\frac{\ell}{n} \right)^q (t-s)^q \}.$$

Consider again $I'_{1n}(s, t)$. We have

$$\begin{aligned} (10.4.16) \quad \sup_{1/n < t-s \leq 1} \frac{I'_{1n}(s, t)}{(t-s)^{1/2}} &\leq \max_{1 \leq i < n} \sup_{1/n < t-s \leq (i+1)/n} \frac{I'_{1n}(s, t)}{(t-s)^{1/2}} \\ &\leq \max_{1 \leq i < n} \max_{0 \leq k < n/i} \sup_{ki/n \leq s < (k+1)i/n} \sup_{i/n < t-s \leq (i+1)/n} \frac{I'_{1n}(s, t)}{(in^{-1})^{1/2}} \end{aligned}$$

$$\begin{aligned}
&\leq 4 \max_{1 \leq i < n} \max_{0 \leq k < n/i} \sup_{ki/n < t \leq (ki+2i+1)/n} \frac{I'_{1n}(kin^{-1}, t)}{(in^{-1})^{1/2}} \\
&:= 4G_n,
\end{aligned}$$

where the last inequality follows by the fact that for any $0 \leq r \leq s < t \leq 1$.

$$(10.4.17) \quad \left(\sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right)^2 \leq 2 \left(\sum_{i=1}^{\ell} \xi_{i+j}(r, s) \right)^2 + 2 \left(\sum_{i=1}^{\ell} \xi_{i+j}(r, t) \right)^2.$$

It is easy to get (cf. (22.17) of Billingsley (1968))

$$\left| \sum_{i=1}^{\ell} \xi_{i+j}(s, t) \right| \leq \left| \sum_{i=1}^{\ell} \xi_{i+j}(s, s+r) \right| + \ell r \quad \text{for } 0 \leq s < t \leq s+r \leq 1,$$

and hence

$$(10.4.18) \quad \sup_{s < t \leq s+r} I'_{1n}(s, t) \leq 2I'_{1n}(s, s+r) + 2\ell r^2$$

by the C_r inequality. Let $r_{in} = \ell^{-1/2} in^{-1}$, i.e., $\ell r_{in}^2 = (in^{-1})^2$. If $kin^{-1} + (j-1)r_{in} < t \leq kin^{-1} + jr_{in}$, then by (10.4.17) and (10.4.18),

$$\begin{aligned}
(10.4.19) \quad I'_{1n}(kin^{-1}, t) &\leq 2I'_{1n}(kin^{-1}, kin^{-1} + (j-1)r_{in}) \\
&\quad + 2I'_{1n}(kin^{-1} + (j-1)r_{in}, t) \\
&\leq 2I'_{1n}(kin^{-1}, kin^{-1} + (j-1)r_{in}) \\
&\quad + 4I'_{1n}(kin^{-1} + (j-1)r_{in}, kin^{-1} + jr_{in}, t) + 4\ell r_{in}^2 \\
&\leq 18 \max_j I'_{1n}(kin^{-1}, kin^{-1} + jr_{in}) + 4(in^{-1})^2,
\end{aligned}$$

where $1 \leq j \leq [(2i+1)(nr_{in})^{-1}] + 1$. Now, since by Lemma 10.3.6 and (10.4.12),

$$EI'_{1n}(kin^{-1}, kin^{-1} + jr_{in}) \leq K(jr_{in}) \leq K(in^{-1}),$$

$$\begin{aligned}
(10.4.20) \quad G_n &\leq 18 \max_{1 \leq i < n} \max_{0 \leq k < n/i} \max_{1 \leq j \leq [\frac{2i+1}{nr_{in}}] + 1} \frac{H_n(kin^{-1}, kin^{-1} + jr_{in})}{(in^{-1})^{1/2}} + K \\
&:= 18G'_n + K
\end{aligned}$$

by (10.4.7) and (10.4.19). Thus, by (10.4.13),

$$\begin{aligned}
 (10.4.21) \quad P\{G'_n \geq 1\} &\leq \sum_{i,j,k} P\{H_n(kin^{-1}, kin^{-1} + jr_{in}) \geq (in^{-1})^{1/2}\} \\
 &\leq K \sum_{i,j,k} (in^{-1})^{-q/2} \{n^{2-q}(jr_{in}) + (\frac{\ell}{n})^{q/2}(jr_{in})^{q/2}\} \\
 &\leq K \sum_{i=1}^n \ell^{1/2} (in^{-1})^{-1-q/2} \{n^{2-q}(in^{-1}) + (\frac{\ell}{n})^{q/2}(in^{-1})^{q/2}\} \\
 &\leq K \{n^{2-q/2} \ell^{1/2} + n^{1-q/2} \ell^{(1+q)/2} \log n\} \\
 &\leq Kn^{-2}
 \end{aligned}$$

if $q > 8/\nu_1$. Similarly, by (10.4.14) and (10.4.15), we have

$$P\left\{\sup_{1/n < t-s \leq 1} \frac{I_{in}(s, t)}{(t-s)^{1/2}} \geq 2\right\} \leq Kn^{-2}$$

for $q > 8/\nu_1$ and $i = 2, 3$. Checking (10.4.5), (10.4.6), (10.4.16), (10.4.20) and (10.4.21), we arrive at the following result by the Borel-Cantelli lemma,

$$(10.4.22) \quad I_n(s, t) \leq K(t-s)^{1/2} \quad \text{for all } t-s \geq n^{-1}$$

almost surely.

Let us now turn to the first term of (10.4.3). By (10.2.1)-(10.2.2) and the C_r inequality,

$$\begin{aligned}
 (10.4.23) \quad &\ell^{p/2} m^{1-p/2} E^* |Z_{1,n}^*(t) - Z_{1,n}^*(s) - (E_n(t) - E_n(s))|^p \\
 &= \ell^{-p/2} m^{1-p/2} n^{-1} \sum_{j=0}^{n-1} \left| \sum_{i=1}^{\ell} \{I(s < U_{i+j,n} \leq t) - (E_n(t) - E_n(s))\} \right|^p \\
 &\leq 2^{p-1} \ell^{-p/2} m^{1-p/2} n^{-1} \sum_{j=0}^{n-1} \left| \sum_{i=1}^{\ell} (I(s < U_{i+j,n} \leq t) - (t-s)) \right|^p \\
 &\quad + 2^{p-1} \ell^{p/2} m^{1-p/2} |E_n(t) - E_n(s) - (t-s)|^p \\
 &:= 2^{p-1} L_{1n}(s, t) + 2^{p-1} L_{2n}(s, t).
 \end{aligned}$$

Much like for $I_n(s, t)$, we obtain

$$(10.4.24) \quad \sup_{1/n < t-s \leq 1} \frac{L_{1n}(s, t)}{(t-s)^2} \\ \leq K \max_{1 \leq i < n} \max_{0 \leq k < n/i} \max_{1 \leq j \leq [\frac{2i+1}{nr_{in}}]+1} \frac{L_{1n}(kin^{-1}, kin^{-1} + jr_{in})}{(in^{-1})^2} + K.$$

This time it is not necessary to get an estimator like (10.4.13) for $L_{1n}(s, t)$. What we need is

$$(10.4.25) \quad EL_{1n}^2(s, t) \leq \frac{1}{\ell^p m^{p-2}} \left\{ \max_{0 \leq j \leq n-1} E \left| \sum_{i=1}^{\ell} (I(s < U_{i+j,n} \leq t) - (t-s)) \right|^{2p} \right\} \\ \leq K m^{2-p} (t-s),$$

by the C_r inequality, Lemma 10.3.6 and (10.4.12). Similarly to proving (10.4.21), (10.4.24) and (10.4.25) imply that almost surely

$$(10.4.26) \quad L_{1n}(s, t) \leq K(t-s)^2 \quad \text{for all } t-s \geq n^{-1}$$

if $p \geq 2 + 7/\nu_2$. In a similar way for $p \geq 2 + 7/\nu_2$ we have

$$(10.4.27) \quad L_{2n}(s, t) \leq K(t-s)^2 \quad \text{for all } t-s \geq n^{-1}$$

almost surely. Finally, putting (10.4.3), (10.4.4), (10.4.22), (10.4.23), (10.4.26) and (10.4.27) together, we can choose a proper $p \geq 8$ such that, almost surely

$$(10.4.28) \quad E^* |\alpha_n^*(t) - \alpha_n^*(s)|^p \leq K(t-s)^2 \quad \text{for all } t-s \geq n^{-1}.$$

For $s < t \leq s+r$ (cf. (22.17) of Billingsley (1968)),

$$(10.4.29) \quad |\alpha_n^*(t) - \alpha_n^*(s)| \leq |\alpha_n^*(s+r) - \alpha_n^*(s)| + (m\ell)^{1/2} (E_n(s+r) - E_n(s)) \\ \leq |\alpha_n^*(s+r) - \alpha_n^*(s)| + \left(\frac{m\ell}{n}\right)^{1/2} |\alpha_n(s+r) - \alpha_n(s)| + (m\ell)^{1/2} r.$$

Let $0 < \delta < 1$ and $0 < r < \delta$. If $i\delta + (j-1)r < t \leq i\delta + jr$, then by (10.4.29)

$$\begin{aligned}
 (10.4.30) \quad & |\alpha_n^*(t) - \alpha_n^*(i\delta)| \\
 & \leq |\alpha_n^*(i\delta + (j-1)r) - \alpha_n^*(i\delta)| + |\alpha_n^*(t) - \alpha_n^*(i\delta + (j-1)r)| \\
 & \leq 3 \max_{1 \leq j \leq [\delta/r]+1} |\alpha_n^*(i\delta + jr) - \alpha_n^*(i\delta)| + (m\ell)^{1/2}r \\
 & \quad + \left(\frac{m\ell}{n}\right)^{1/2} \max_{1 \leq j \leq [\delta/r]+1} |\alpha_n(i\delta + jr) - \alpha_n(i\delta + (j-1)r)|.
 \end{aligned}$$

Choose $r = n^{-1}$. Then

$$(10.4.31) \quad (m\ell)^{1/2}r = O(n^{-\nu_1/2}) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Let δ^{-1} be also an integer. Then for any $a > 0$ and $p > 2$,

$$\begin{aligned}
 & P\left\{\left(\frac{m\ell}{n}\right)^{1/2} \max_{0 \leq i < \delta^{-1}} \max_{1 \leq j \leq [n\delta]+1} |\alpha_n(i\delta + jr) - \alpha_n(i\delta + (j-1)r)| > a\right\} \\
 & \leq \sum_{i=0}^{\delta^{-1}-1} \sum_{j=1}^{[n\delta]+1} a^{-p} \left(\frac{m\ell}{n}\right)^{p/2} E|\alpha_n(i\delta + jr) - \alpha_n(i\delta + (j-1)r)|^p \\
 & \leq K a^{-p} n^{1-\nu_1 p/2}
 \end{aligned}$$

by Lemma 10.3.6 and (10.4.12). Thus by the Borel-Cantelli lemma, for $a = \log^{-1} n$ and $p \geq 8/\nu_1$,

$$(10.4.32) \quad \lim_{n \rightarrow \infty} \left(\frac{m\ell}{n}\right)^{1/2} \max_{0 \leq i < \delta^{-1}} \max_{1 \leq j \leq [n\delta]+1} |\alpha_n(i\delta + jr) - \alpha_n(i\delta + (j-1)r)| = 0 \text{ a.s.}$$

for each fixed δ and hence, (10.4.32) is still true for all $0 < \delta < 1$ such that δ^{-1} is an integer. This means that there exists $\Omega_0 \in \mathcal{F}$ such that $P(\Omega_0) = 0$ and for each $\omega \notin \Omega_0$ and any $0 < \varepsilon, \eta < 1$, by (10.4.30)-(10.4.32), there is an integer $N_1 = N_1(\varepsilon, \delta, \omega)$ so that when $n \geq N_1$ we have

$$(10.4.33) \quad P^*\left\{\sup_{i\delta < t \leq (i+1)\delta} |\alpha_n^*(t) - \alpha_n^*(s)| \geq 5\varepsilon\right\}$$

$$\begin{aligned}
&\leq P^*\left\{\max_{1 \leq j \leq [n\delta]+1} |\alpha_n^*(i\delta + jr) - \alpha_n^*(i\delta)| \geq \varepsilon\right\} \\
&\leq \frac{K}{\varepsilon^p} (n\delta r)^2 \leq \frac{K\delta^2}{\varepsilon^p} \quad \text{for } i = 0, 1, \dots, \delta^{-1} - 1
\end{aligned}$$

by (10.4.28) and Theorem 12.2 of Billingsley (1968). Since the constant K in (10.4.33) is independent of δ , we can choose δ so that $K\delta\varepsilon^{-p} < \eta$. From (10.4.33) it will follow that

$$P^*\left\{\sup_{i\delta < t \leq (i+1)\delta} |\alpha_n^*(t) - \alpha_n^*(i\delta)| \geq 5\varepsilon\right\} < \eta\delta, \quad \text{for } i = 0, 1, \dots, \delta^{-1} - 1.$$

This implies (C3) (see the corollary to Theorem 8.3 of Billingsley (1968)). This completes the proof of Theorem 10.4.1. ■

10.5 Applications.

For applications concerning the original stationary observations $X_1, \dots, X_n$, we have to change the bootstrapped uniform empirical process in Section 2 back to the ordinary one. For observations $X_1, \dots, X_n$ from a stationary sequence, the empirical distribution function is

$$F_n(x) = n^{-1} \sum_{i=1}^n I(X_i \leq x), \quad -\infty < x < \infty,$$

and the corresponding bootstrapped one is

$$F_n^*(x) = (m\ell)^{-1} \sum_{i=1}^{m\ell} I(X_i^* \leq x), \quad -\infty < x < \infty.$$

The bootstrapped empirical process is now

$$(m\ell)^{1/2}(F_n^*(x) - F_n(x)) = \alpha_n^*(F(x)), \quad -\infty < x < \infty.$$

Hence, under the assumptions of (10.1.2) and Theorem 10.2.3 or the assumptions of Theorem 10.4.1, we have the following result.

Suppose F is continuous. Fix α with $0 < \alpha < 1$. Choose $c_n(F_n)$ from the bootstrap distribution so that

$$P\{(m\ell)^{1/2} \sup_x |F_n^*(x) - F_n(x)| \leq c_n(F_n) | X_1, \dots, X_n\} \longrightarrow 1 - \alpha.$$

Then

$$P\{n^{1/2} \sup_x |F_n(x) - F(x)| \leq c_n(F_n)\} \longrightarrow 1 - \alpha.$$

This result gives us the $1 - \alpha$ asymptotic confidence band $F_n \pm n^{-1/2}c_n(F_n)$ for F .

Let us now consider a general statistic based on observations $X_1, \dots, X_n$ from a stationary sequence,

$$(10.5.1) \quad T_n(X_1, \dots, X_n) = T(\rho_n),$$

where ρ_n is the n th empirical measure

$$(10.5.2) \quad \rho_n = n^{-1} \sum_{i=1}^n \delta_{X_i}$$

and δ_y denotes the point mass at $y \in \mathbf{R}$.

Künsch (1989) shows that $T(\rho_n)$ covers a large range of statistics and gives many examples. Here we concentrate mainly on the asymptotic distribution of T_n with the help of influence functions (see Hampel, Ronchetti, Rousseeuw and Stahel (1986)).

Let $\{Y_n, n \geq 1\}$ be a sequence of random variables with a common distribution measure G . We assume that

(D1): the influence function

$$IF(y, G) = \lim_{\epsilon \downarrow 0} (T((1 - \epsilon)G + \epsilon\delta_y) - T(G))/\epsilon$$

exists for all $y \in \mathbf{R}$ and G in some neighborhood of F ;

(D2): for all G in some neighborhood of F ,

$$\begin{aligned} & T(n^{-1} \sum_{i=1}^n \delta_{Y_i}) \\ &= T(G) + n^{-1} \sum_{i=1}^n IF(Y_i, G) + n^{-1/2} R_n(\sup_x n^{1/2} |G_n(x) - G(x)|), \end{aligned}$$

where $G_n(x) = n^{-1} \sum_{i=1}^n I(Y_i \leq x)$ and $R_n \xrightarrow{P} 0$ if

$$\sup_x n^{1/2} |G_n(x) - G(x)| \xrightarrow{D} \sup_{0 \leq t \leq 1} |B(t)|.$$

Then we can apply Theorem 10.4.1 to get the following result.

THEOREM 10.5.1 (Shao and Yu, 1993) *Let $\{X_n, n \geq 1\}$ be a stationary ρ -mizing sequence of random variables with common continuous distribution function F . In addition to (D1) and (D2), we assume that the covariance series in (10.1.3) is absolutely convergent, $E(IF(X_1, F))^2 < \infty$ and*

$$\sum_{n=1}^{\infty} \rho(2^n) < \infty.$$

Then, if $l = O(n^{1-\nu_1}) \rightarrow \infty$ as $n \rightarrow \infty$ and $n^{\nu_2} \leq m \leq n$ for some $0 < \nu_1, \nu_2 < 1$, we have almost surely as $n \rightarrow \infty$

$$\sup_x |P\{(m\ell)^{1/2}(T_n^* - T_n) \leq x | X_1, \dots, X_n\} - P\{n^{1/2}(T_n - T(F)) \leq x\}| \rightarrow 0,$$

where

$$T_n^* = T((m\ell)^{-1} \sum_{i=1}^{m\ell} \delta_{X_i^*}).$$

Proof. By (D2), we have

$$(10.5.3) \quad n^{1/2}(T_n - T(F)) = n^{-1/2} \sum_{i=1}^n IF(X_i, F) + R_n(\sup_x n^{1/2} |F_n(x) - F(x)|)$$

and

$$(10.5.4) \quad \begin{aligned} (m\ell)^{1/2}(T_n^* - T_n) \\ = (m\ell)^{-1/2} \sum_{i=1}^{m\ell} IF(X_i^*, \rho_n) + R_n(\sup_x (m\ell)^{1/2} |F_n^*(x) - F_n(x)|). \end{aligned}$$

Since $X_1 \stackrel{D}{=} F$, $E IF(X_1, F) = 0$ by the property of influence function. Since also $X_i^* \stackrel{D}{=} \rho_n$ for $i = 1, \dots, m\ell$, $E^* IF(X_i^*, \rho_n) = 0$. For the two terms R_n in (10.5.3) and

(10.5.4), we have by Theorem 10.4.1

$$R_n(\sup_x n^{1/2}|F_n(x) - F(x)|) \xrightarrow{P} 0$$

and

$$R_n(\sup_x (m\ell)^{1/2}|F_n^*(x) - F_n(x)|) \xrightarrow{P^*} 0 \text{ a.s..}$$

Our theorem follows immediately from the results of Ibragimov (1975), Peligrad (1982) and Theorem 10.3.4. ■

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