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Turbulence Modeling for Curved Wall Jets under Adverse Pressure Gradient

by

Palanunt Pajayakrit
M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements
for the degree of

Doctor of Philosophy

in

Aerospace Engineering

Ottawa-Carleton Institute for Mechanical
and Aerospace Engineering

Department of Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario
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**Turbulence Modeling for Curved Wall Jets
under Adverse Pressure Gradient**

submitted by

Palanunt Pajayakrit, M.Sc.

in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

Chairman, Department of Mechanical and Aerospace Engineering

Thesis Supervisor

External Examiner

Carleton University

September 1997

The ability of four turbulence models to predict curved wall jets under adverse pressure gradient was assessed. An experiment on a nominally two-dimensional curved wall jet over a cylinder was also conducted to study an existence of three-dimensional flow structures that may affect overall flow development.

The turbulence models assessed were the mixing length model of Baldwin and Lomax (1978), the k - ϵ model of Dash et al. (1983), the k - ω model of Wilcox (1988a), and the multiscale model of Wilcox (1988b). None of the models were found to be satisfactory for prediction of curved wall jets, although the k - ϵ model appeared to be the least deficient.

Attempts were made to adjust the constants in the turbulence models to improve their accuracy of prediction of wall jet development. Tuning the k - ω model and the multiscale model had only limited success. The modified models gave much better predictions than the original models; however, the modifications were not universal, and the calculated results for the curved wall jets were still relatively poor.

The results of the experiment on a wall jet over a circular cylinder demonstrated that steady streamwise vortices existed in the outer layer of curved turbulent wall jets. The vortices were organized into counter-rotating pairs similar to Görtler vortices found

in laminar boundary layers over a concave surface. The effect of streamwise vortices on momentum mixing was significant, but not dominant. These vortices were triggered by slight irregularities of the upper slot lip. The sensitivity of the flow to these irregularities was such that vortex-free wall jet flows are unlikely in practical applications. Successful attempts were made to intentionally generate these vortices or alter their patterns using flow disrupters attached to the top of the upper slot lip.

Original contributions in this thesis include:

- The methodology for deriving tuning models to adjust turbulence model constants:
- The experimental demonstration of streamwise vortices in the outer layer of a wall jet over a convex wall; and the effects
- The description of of such vortices.

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List of Symbols

English symbols

b. jet slot opening height

C_f . friction coefficient $= \frac{\tau_w}{\frac{1}{2}\rho U_c^2}$ for boundary layer flows.

$= \frac{\tau_w}{\frac{1}{2}\rho U_m^2}$ for wall jet flows.

C_p . pressure coefficient $= \frac{p - P_\infty}{\frac{1}{2}\rho U_\infty^2}$.

e. lower partition energy in the multiscale model.

G. Görtler number $= \left(\frac{U_\infty \delta}{\nu} \right) \left(\frac{\delta}{R} \right)^{1/2}$.

h. curvature parameter $= 1 + \frac{y}{R}$.

H. shape factor $= \frac{\delta^*}{\theta}$

k. turbulent kinetic energy $= \frac{\overline{u'^2}}{2} + \frac{\overline{v'^2}}{2} + \frac{\overline{w'^2}}{2}$

l. length scale of turbulent eddy

p. static pressure

p_0 . total pressure

P_c . plenum chamber pressure

q. dynamic pressure $= \frac{p_0 - p}{\frac{1}{2}\rho U_\infty^2}$

- R . radius of curvature.
- Re . Reynolds number $= \frac{U_{\infty} L}{\nu}$.
- x . streamwise coordinate.
- y . normal coordinate.
- z . spanwise coordinate.
- y^+ . inner wall coordinate.
- y_j . wall jet half width.
- y_{zss} . location of zero-shear-stress point.
- u . streamwise velocity.
- u^+ . inner wall velocity $= \frac{u}{u_{\tau}}$
- u_p . potential flow velocity.
- u_{τ} . friction velocity $= \sqrt{\frac{\tau_w}{\rho}}$
- U_{∞} . boundary layer, or wall jet freestream edge velocity.
- U_m . wall jet maximum velocity.
- U_{pw} . wall potential velocity.
- U_{Δ} . wall jet maximum excess velocity $= U_m - U_e$.
- U_{∞} . remote freestream velocity.
- v . normal velocity.
- w . spanwise velocity.

Greek Symbols

- δ . boundary layer thickness.
- δ^* . displacement thickness.
- ε . dissipation.
- η . wall jet similarity variable $= \frac{y}{y_{1/2}}$.
- κ . von Karman constant.
- ν . kinematic viscosity.
- ν_t . turbulent eddy viscosity $= C_\mu \frac{k^2}{\varepsilon}$ for the k- ε model.
 $= \frac{k}{\omega}$ for the k- ω and multiscale models.
- θ . momentum thickness.
- ρ . density.
- σ_u . streamwise stress deviator $= k - \frac{2}{3} \overline{u^2}$
- σ_v . normal stress deviator $= k - \frac{2}{3} \overline{v^2}$
- τ . Reynolds shear stress $= -\rho \overline{u'v'}$
- ω . specific dissipation $= \frac{\varepsilon}{k}$

Abbreviation

- BL. Baldwin-Lomax model.
- ERS. equivalent Reynolds stress.
- KE. k- ε model.
- KW. k- ω model.
- KW1. modified k- ω model.

MS. multiscale model.

MS1. modified multiscale model.

MS2. modified multiscale model with constant $(k-e)/k$.

MS3. modified multiscale model with algebraic $(k-e)/k$.

MS4. modified multiscale model with differential $(k-e)/k$.

RS. Reynolds stress.

1.1 Scope of the Thesis

This thesis describes a study of wall jet flows using both numerical simulation and experiment. The main focus is on assessment of turbulence models for wall jets in adverse pressure gradient and with streamline curvature.

1.2 Wall Jet Flows

A wall jet is a rather peculiar flow. It has characteristics of both wall-bounded flow and free shear flow. The particular flow at the center of this study, the curved wall jet under adverse pressure gradient, is not only a severe challenge for numerical prediction, but is also a flow with many practical applications, including the following:

- Wide Angle Diffusers Diffuser flows are subject to adverse pressure gradient and the angle of divergence (Figure 1-1) must be limited to prevent stall. Wall jets can be used to delay separation and increase the angle of divergence, therefore, reduce the diffuser length.

- Film Cooling A wall jet is used as a cooling film over metal surfaces that are subject to high temperature gas flow. In some cases the film cooling of a jet engine turbine blade involves a curved wall jet under adverse pressure gradient.

- Multi-Element Airfoils Airfoils with slats and slotted flaps are used in high lift applications. The wall-wake flows near the gaps of the slats and flaps (Figure 1-2) share many similarities with wall jets. These gaps are usually located in or near the adverse pressure gradient region on the upper surface of the airfoil, and at high slat/flap deflection angle the curvature of the flow is high.

- Circulation Control Airfoils Circulation control airfoils employ tangential blowing near a blunt trailing edge to control the location of the rear dividing streamline (Figure 1-3), thus the circulation and the lift. This method of direct lift control, independent of angle of attack, is used for directional control in the NOTAR (no-tail-rotor) helicopter concept, and is also potentially useful for low vibration main rotor application. The flow over the trailing edge of the circulation control airfoil is that of a curved wall jet under adverse pressure gradient.

1.3 Turbulence Modelling

Turbulence is one of the most complex fluid motions. Although the governing equations, i.e. the Navier-Stokes equations, are known, their complexity is such that the solutions are not attainable within the practical limits of current computational resources. Turbulence modelling is the art of creating mathematical models that are both physically realistic and computationally reasonable. Preferably these models should rest on sound physics but, being an art rather than an exact science, these models are judged more on their results than their foundations.

Figure 1-1 Wall jet is used to delay separation along diffuser wall.

Figure 1-2 Wall jet like flows are found near the gaps of multi-element airfoils

Figure 1-3 Flow over blunt trailing edge of a circulation control airfoil.

Even in the most simplified form of the turbulent boundary layer equations, called the Reynolds averaged, parabolized Navier-Stokes equations, there are more unknowns than governing equations. Finding additional equations to close this system is called the "closure problem" and is the main objective of turbulence modelling.

The general approach to turbulence modelling is to deduce a functional form of a system of equations usually based on observations and dimensional analysis. These equations contain a number of constants which can be evaluated by calibration against experimental data. Once all constants are known, the models can be used to predict other flows. A good model should be universal, i.e. it should be able to predict a wide range of flows with the same set of constants.

1.4 Three-Dimensional Flow Structures in Curved Wall Jets

Most experiments on wall jets are intended to produce two-dimensional flows. Curved wall jets over convex surfaces are known to be prone to developing secondary flow involving three-dimensional structures in the flow and extreme care has to be taken to maintain true two-dimensionality. It was suspected that in most practical cases of convex curved wall jets, three dimensional flow structures are present and affect the development of the overall flow. If these effects are substantial and are not accounted for in their own right, that could lead to erroneous conclusions in an assessment of turbulence models. The existence of these three-dimensional structures in nominally two-dimensional convex curved wall jets has never been demonstrated experimentally. An

experiment was carried out and did indeed find such flow structures. The data constitute the first documented evidence of these structures.

1.5 Outline of Thesis Contents

An introduction to turbulence modelling and a review of previous studies on wall jets is presented in Chapter 2. Chapter 3 describes the numerical turbulence modelling work. It outlines the main features of the computer code developed for this study. The selected turbulence models and the flows that were tested are identified. It also discusses the modification efforts to improve the model prediction for wall jet flows. Chapter 4 describes the experiment on curved wall jet over a cylinder. The apparatus and instrumentation are described and the experimental results are presented. Conclusions and recommendations are provided in Chapters 5 and 6 respectively.

The review of previous work on wall jets was conducted in two general areas, experiments on wall jets and numerical prediction of wall jet flows.

2.1 Wall Jet Notation

Figure 2-1 shows a typical velocity profile of a curved wall jet and a surface-normal co-ordinate system. Further definitions of terms are as follows:

b = jet exit slot height.

R = wall radius of curvature.

U_e = edge velocity.

U_j = jet velocity at exit.

U_m = maximum velocity in a profile.

U_λ = maximum excess velocity = $U_m - U_e$.

x = streamwise distance for an arbitrary reference.

y_m = vertical displacement of the maximum velocity point.

$y_{1/2}$ = vertical displacement of the jet half width, where $u - U_e = 1/2 U_\lambda$.

Figure 2-1 Notation for a curved wall jet.

2.2 Experiments on Wall Jets

The first study on wall jet flows was reported by Förthman in 1934. This study was on a wall jet over a flat plate in still air. Although an interest in curved wall jets had appeared since 1800, it was not until the early 1960's that a detailed study of a curved wall jet was conducted. Since then there are more than 200 papers published on wall jet flows. There are also several reviews of literature, such as Newman's (1961) and Wille and Fernholz's (1965). Most recently Launder and Rodi (1981) reviewed the available wall jet experiments in preparation for the 1980/81 Stanford Conference on Complex Turbulent Shear Flows. Their paper is very useful in putting all the experimental results into perspective.

Table 2-1 and 2-2 list some representative experiments on plane and curved wall jets, including the following:

- Plane wall jets in still air This flow is the most thoroughly studied and the experimental data can be considered the most reliable.

- Plane self-similar wall jets in a moving stream Similarity analysis shows that plane wall jets are self-similar when $\frac{U_c}{U_m}$ is constant (Irwin, 1973). The plane self-similar wall jets are achieved experimentally by adjusting the external pressure gradient to maintain constant $\frac{U_c}{U_m}$ with downstream distance. Self-similar wall jets grow linearly with downstream distance, or $\frac{dy_{1,2}}{dx}$ is constant.

- Plane wall jets in a uniform stream This flow is subject to zero pressure gradient and $\frac{U_e}{U_m}$ increases with downstream distance. The parameter $\frac{U_e}{U_i}$ in Table 2-1 indicates the initial strength of the wall jet at the slot exit.

- Curved self-similar wall jets over a log-spiral Curved wall jets in still air are self-similar when $\frac{v_{12}}{R}$ is constant. This can be achieved over a log-spiral surface where $\frac{x}{R}$ is constant (Guitton, 1970).

- Curved wall jets over a cylinder in still air The growth rate of these wall jets is spectacular, and it is very difficult to maintain two-dimensionality when slot-height to cylinder radius, $\frac{h}{R}$, is appreciable. (Launder and Rodi, 1981).

- Curved wall jets under adverse pressure gradient This flow results when a curved wall jet is issued into a moving stream. Only limited data are available.

- Pulsating wall jets It was recently demonstrated that oscillatory blowing can delay separation from a symmetrical airfoil much more effectively than the steady blowing used traditionally for this purpose. It therefore increases the lift and reduces the drag generated by airfoils at angles of incidence and flap deflections at which the flow would otherwise be separated (Seifert, Darabi and Wygnanski, 1996).

Flows	Reference	$\frac{U_e}{U_m}$	$\frac{U_e}{U_j}$	$\frac{dy_{1/2}}{dx}$	$\frac{y_m}{y_{1/2}}$	Notes
Plane Wall Jets in still air	Förthmann (1934)			.082	.17	Runs at several slot Reynolds numbers.
	Bradshaw and Gee (1960)			.071		
	Patel (1962)			.071	.16	
	Tailland (1967)			.076	.155	
				.074	.14	
				.073	.131	
	Guittou (1970)			.071	.15	
	Wilson (1970)			.076	.13	
Plane self-similar wall jets in a moving stream	Gartshore and Newman (1969)	.33		.046	.217	
		.5		.04	.3	
		.77		.033	.63	
Plane wall jets in a uniform stream	Irwin (1973)	.38		.044	.214	
	Bradshaw and Gee (1960)		.18			
	Kacker and Whitelaw (1971)		.38			
	Kruka and Eskinazi (1964)		.06 to .49			
	Gartshore and Newman (1969)		.2 to .49			

Table 2-1 Representative experiments on plane wall jets.

Flows	Reference	$\frac{b}{R}$	$\frac{x}{R}$	$\frac{dy_{1/2}}{dx}$	$\frac{y_m}{y_{1/2}}$	Notes
Curved self-similar wall jets over a log spiral	Giles, Hayes and Sawyer (1966)		-2/3 -1/3 1/3 2/3 1			
	Guitton (1970)		2/3 1	.155 .275		
Curved wall jets over a cylinder in still air	Fekete (1963)	.007 to .053				
	Wilson (1970)	.06				
	Alcaraz (1977)	.003				
Curved wall jets under adverse pressure gradient	Kind (1967)					Trailing edge of a cc airfoil.
	Kind, Gooden and Dvorak (1979)					Wall jet over a cylinder.

Table 2-2 Representative experiments on curved wall jets.

2.3 Experimental Results on Wall Jet Characteristics

2.3.1 Law of the Wall

Experiments show that the universal log-law as given by Patel (1965).

$$u^+ = 5.5 \log_{10}(y^+) + 5.45 . \quad (2-1)$$

generally fits the data for self-similar wall jets, but does not always apply to the non-self-similar flows (Launder and Rodi, 1981).

The universal log-law best fits plane self-similar wall jets in a moving stream (Irwin, 1973). For plane wall jets in still air, the y^+ range over which the log-law applies reduces with downstream distance. When $x/b > 139$, the universal log-law range reduces to a point, or disappears. For the curved self-similar wall jets on log spirals, the universal log-law only applies for $x/R < 1$ (Guitton, 1970). In conventional boundary layer, the range of validity of the universal log-law is also reduced by strong adverse gradient (Patel, 1965).

In those cases where the universal log-law does not apply, the Clauser plot does show a straight line region, but the slopes and intercepts are different from the universal log-law. Various researchers recommend various expressions for this straight line, but none of them is universal. The lack of universality makes it difficult to accurately determine the skin friction coefficient.

2.3.2 Normal Pressure Gradient

It is a well known fact that a wall jet over a convex surface will create a low pressure region, with respect to the freestream, over the part of the surface where it is attached. The pressure difference is the reaction to the centrifugal force acting on the wall jet fluid. Therefore, there will be a pressure gradient in the normal direction in curved wall jet flows (Kind, Gooden and Dvorak, 1979), as shown in Figure 2-2.

Figure 2-2 Pressure variation normal to the wall in curved wall jet (Kind, et al., 1979).

Figure 2-3 Typical wall jet velocity and shear stress profiles.

2.3.3 Reynolds Shear Stress

A generic Reynolds shear stress distribution across a wall jet, as well as a generic mean velocity profile, is shown in Figure 2-3. Through most of the shear layer, Reynolds shear stress has the opposite sign to the wall shear stress. The location of the zero shear stress point and the velocity maximum do not coincide. Adverse pressure gradient drives the zero-shear-stress point away from the wall. Increasing streamline curvature (reducing radius of curvature) drives the zero-shear-stress point toward the wall (Launder and Rodi, 1981).

2.3.4 Turbulence Intensity

Figure 2-4 shows that turbulent kinetic energy (k) increases with increasing x/R . Figure 2-5 shows that for curved wall jets, the normal fluctuation, $\overline{v'^2}$, increases significantly with respect to the streamwise fluctuation, $\overline{u'^2}$ (Guitton, 1970).

2.4 Wall Jet Flow Prediction

Analysis of wall jet flows was first considered by Glauert (1956) using a similarity approach. He divided the flow into an inner and outer part on either side of the maximum velocity and treated the two regions separately. In a loose sense, this is still a common practice today. The pace of development of analytical methods picked up in late 1960's and throughout 1970's, along with experimental studies.

Figure 2-4

Effects of curvature on k (Guitton, 1970).

Figure 2-5

Effects of curvature on the ratio of streamwise fluctuation $\bar{u}^2$ to normal fluctuation $\bar{v}^2$ (Guitton, 1970).

2.4.1 Similarity Analysis

Following Glauert's approach, Narayan and Narasimha (1973) applied similarity analysis to plane turbulent wall jets. They provided correlations, based on data from various experiments, to predict integral parameters for incompressible and compressible plane wall jets, with and without effects of adverse pressure gradient. Recently Zhou and Wygnanski (1993) revisited the similarity method and found more appropriate non-dimensional velocity and length scale parameters for plane wall jets.

2.4.2 Integral Method

Early methods for predicting general wall jets, both self-similar and non self-similar, employed integral methods (Gartshore and Newman, 1969; Spalding, 1964; Kind, 1967, 1971). In 1968, Nash and Hicks used an integral method to predict wall jet test cases for the AFOSR-IFP-Stanford Conference on the Computation of Turbulent Boundary Layers (Kline, Morkovin, Sovran and Cockrell, 1968). The results were mixed, and Kline (1969) concluded that the calculation of wall jets, including the effects of heat transfer and longitudinal curvature, was not a complete success. A more recent study (Kind, Gooden and Dvorak, 1979) points out that an assumed velocity profile, as needed with an integral method, makes it inflexible and unable to cope with rapidly changing profile shape near separation, over highly curved surface, or when the jet strength is not enough to maintain the characteristic wall jet velocity profile. However, an integral method is highly efficient

numerically, and it is quite attractive for an application where iteration is required.

Development of integral methods is still active (Roberts, 1987).

2.4.3 Algebraic Eddy Viscosity Models

Superseding integral methods are the algebraic eddy viscosity, or mixing length, turbulence models. Actually, the first mixing length model for wall jet flows was introduced by Spalding and Patankar (1967) at about the same time as the integral method. This model specified a simple ramp function as the distribution of the mixing length. Initially the model was successful in predicting the growth rate of a plane wall jet in still air. Later it was found that the constants in the model had to be altered significantly to make the model perform satisfactorily for other types of wall jets, such as the radial wall jet or wall jet over a cone (Sharma, 1972). This model is now virtually abandoned, but the concept of the mixing length, in a slightly modified form, is still in routine use as a model for the inner layer of the wall jet.

Another significant algebraic eddy viscosity model is that of Dvorak (1973). The eddy viscosity function is evaluated from a number of empirical expressions involving integral parameters, δ^+ , θ , H , C_f and the intermittency distribution. It also considers history effects by using a rate equation for the eddy viscosity.

$$\delta^+ \frac{dv_{t,\max}}{dx} = (constant) (v_{t,i} - v_{t,\max}) \quad (2-2)$$

where $v_{t,\max}$ = maximum eddy viscosity at that streamwise station, and

$\nu_{t,i}$ = local eddy viscosity.

Therefore, in a strict sense, this model should be called a one-equation model. However, in terms of computational intensity, it is more similar to the algebraic eddy viscosity models than to other one-equation models.

Dvorak's model was successful in predicting the plane wall jet in uniform freestream and plane wall jet under adverse pressure gradient cases (Dvorak, 1973), as shown in Figures 2-6 and 2-7. For the case of curved wall jet under adverse pressure gradient, the prediction was less satisfactory, especially at stations close to separation (Kind, Gooden and Dvorak 1979, as shown in Figure 2-8. This model went on to form the basis of a prediction method for flow over multi-element airfoils (Dvorak and Woodward, 1974) and circulation control airfoils (Dvorak and Kind, 1979; Dvorak and Choi, 1983) with considerable success.

2.4.4 k- ϵ Models

The k- ϵ turbulence model was first introduced by Harlow and Nagayama (1968) and soon began to replace other less sophisticated models. At the 1980/1981 AFOSR-HTTM Stanford Conference on Complex Turbulent Flows (Kline, Cantwell and Lilley, 1982), three groups of modelers submitted their calculations for the case of a plane self-similar wall jet, and two groups for the case of a curved wall jet over a logarithmic spiral. All of the predictions were based on the k- ϵ model.

(a)

(b)

Figure 2-6 Prediction of plane wall jet under uniform freestream with Dvorak's (1973) algebraic eddy viscosity model.

(a)

(b)

Figure 2-7 Prediction of plane wall jet under adverse pressure gradient with Dvorak's (1973) algebraic eddy viscosity model.

Figure 2-8 Prediction of curved wall jet over a cylinder under adverse pressure gradient (Kind, Gooden, Dvorak, 1979).

— — — — — Dvorak's (1973) (algebraic eddy viscosity) model
 - - - - - Kind's (1971) (integral method) model

Launder and Rodi (1983) reviewed the accuracy of the k - ϵ model for wall jet calculations and noted that the model overpredicted the rate of growth of the plane wall jet in still air by 30% with respect to the experimental data. This error diminished as U_e/U_m increased (Figure 2-9). For the case of a curved wall jet over a logarithmic spiral, the k - ϵ model, without modifications to its constants, seriously underestimated the effects of curvature (Figure 2-10).

The most sophisticated k - ϵ model to date for wall jet applications is that of Dash et al. (1983). The model employs a two layer approach, with the near-wall layer (up to $y^+ = 50$) being modelled with the van Driest mixing length formula, and the outer layer with the k - ϵ model. Curvature correction factors are applied to both the mixing length and the k - ϵ models. This model still overpredicts the growth rate of the plane wall jet in still air by 22%, but the predictions of the velocity decay, and of the streamwise and normal velocity profiles are good (Figure 2-11). The predictions for the plane wall jet in a uniform freestream are also good (Figure 2-12). For the case of a curved wall jet over a cylinder in still air, when the recommended curvature correction factor is used, the prediction is fairly good (Figure 2-13).

Figure 2-9 Prediction of plane wall jet in still air and self-preserving plane wall jet (Launder and Rodi, 1983).

Figure 2-10 Prediction of curved wall jet over a logarithmic spiral in still air (Launder and Rodi, 1983).

Figure 2-11 Prediction of plane wall jet in still air by Dash ($k-\epsilon$) model (Dash, 1983).

Figure 2-12 Prediction of plane wall jet under uniform freestream by Dash (k-ε) model (Dash, et al., 1983).

Figure 2-13 Prediction of curved wall jet over a cylinder in still air by Dash (k - ϵ) model (Dash, et al., 1983).

Note Recommended curvature correction is labelled "Penn State Correction $C_0 = 0.16$ "

2.5 General Remarks

Only finite-difference or finite-volume type methods have the flexibility required for predicting the development of wall jets and wall-jet-like flows. The prediction of such methods can only be as good as the turbulence models that they use, and turbulence models for this type of flow are not well developed. The main aim of this thesis is to make a contribution towards improving this situation.

Chapter 3

ASSESSMENT AND MODIFICATION OF TURBULENCE MODELS FOR WALL JETS

This chapter describes an attempt to predict the development of several wall jet flows using selected turbulence models. The objective was to find a model which could accurately predict a wide range of wall jet flows, including the effects of pressure gradient and streamline curvature. As discussed in Chapter 2, wall jet flows have relatively complex velocity profiles, particularly when an external flow is present and the profiles have both a velocity maximum and a velocity minimum. These flows are thus similar to the type of flow that occurs downstream of the slots in multi-element high-lift airfoils. The present objective was thus seen as a first step in efforts to develop turbulence models suitable for use in a wide range of high lift applications.

3.1 Basic Approach

3.1.1 Model Selection

Turbulence calculations can be ranked according to the potential range of applicability, from narrow to wide (Kline et al. 1982) as:

- integral models.

- mixing length (zero equation) models.
- 2-equation, eddy-viscosity models.
- Reynolds stress models.
- large eddy simulation, and
- full Navier-Stokes.

At the low end, the integral models are very limited in their use because the shape of the velocity profile must be assumed. As already mentioned, wall jet velocity profiles can be quite complex, particularly if an upstream boundary layer is also considered. This makes the integral models unattractive. Furthermore, previous studies have already made attempts to apply integral models to wall jets (Kind, 1967, 1971). Therefore, the present study concentrated at the level of mixing length models and higher. At the high end, the large eddy simulation and full Navier-Stokes methods are computationally prohibitive for practical flows and will remain so for many years. Thus, the present study was limited to the level of Reynolds stress transport models and below.

At the mixing-length level, the Baldwin-Lomax model was selected because it is one of the most popular models at present. For the two-equation, eddy-viscosity level, two models were chosen, the k - ϵ model and k - ω model. These were the first and second most popular models at this level at the time the work was done. The Reynolds stress models are still considered too computationally intensive in general. The model chosen at

this level for the present study was, therefore, a simplified version of the Reynolds stress model called a multiscale model.

Although all four models were tested, it was realized early in the study that there would not be enough time for modifying and retesting all of them. It was decided that the major modification effort would focus only on the $k-\omega$ and the multiscale model. The reason was that the $k-\omega$ model was demonstrated to be superior to $k-\epsilon$ model for boundary layers under adverse pressure gradient (Wilcox, 1993a). The pressure gradient was one of the major effects to be included in this study, therefore, the $k-\omega$ model was preferred. The multiscale model did not use the eddy viscosity assumption, which was known from experiments to be violated near the maximum velocity point of wall jets. Therefore, the multiscale model appeared to have the potential to improve predictions of wall jets and became one of the models to receive extra attention.

3.1.2 Algorithm for Computation of Flow Development

Since the focus of this study was on assessment of turbulence models, the calculation domain would be limited to simple geometries. Flow separation or flow reversal was not included. Thus, a relatively simple and fast algorithm for computing flow developments was appropriate. All of the flows of interest conformed with the assumption of a two-dimensional thin shear layer in which axial diffusion could be neglected. Given the freestream pressure distribution, the equations of motion would then

be parabolic. Implementing a parabolic, space-marching code for such a problem is relatively straight forward: therefore, such a code was developed by the author.

Developing a new computer code would ensure intimate familiarity with the code, facilitate the expected extensive turbulence model modification, and provide the author with valuable experience in computational fluid dynamics work.

3.2 Equations of Motion

3.2.1 Coordinate System

The body oriented curvilinear coordinate system is shown in Figure 3-1. The y axis is aligned with the local normal and is directed away from the centre of curvature. The x axis is perpendicular to the local normal and is directed along the flow direction. The radius of curvature of the surface is positive for a convex surface.

3.2.2 Simplifying Assumptions

The flows in this study are assumed to be:

- incompressible, $\rho = \text{constant}$:

- steady, $\frac{\partial}{\partial t}(\quad) = 0$;

- two-dimensional, $\vec{w} = 0$, $\frac{\partial}{\partial z}(\quad) = 0$; and

Figure 3-1 Body oriented coordinate system.

- thin, $\frac{\delta}{R} \ll 1$.

3.2.3 System of Equations

Bradshaw (1973) gives the equations of motion in surface-normal, curvilinear coordinates under the above assumptions as follows:

Continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial}{\partial y} (h \bar{v}) = 0 \quad (3-1)$$

Streamwise momentum:

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + h \bar{v} \frac{\partial \bar{u}}{\partial y} + \frac{\bar{v}}{R} \bar{u} = - \frac{\partial}{\partial x} \left(\frac{\bar{p}}{\rho} + \bar{u}^2 \right) - h \frac{\partial}{\partial y} \overline{u'v'} - \frac{\bar{v}}{R} \overline{u'v'} - h \bar{v} \frac{\partial}{\partial y} \left[\frac{\frac{\partial}{\partial y} (h \bar{u}) - \frac{\partial \bar{v}}{\partial x}}{h} \right] \quad (3-2)$$

Normal momentum:

$$\frac{\bar{u}^2}{hR} = \frac{\partial}{\partial y} \left(\frac{\bar{p}}{\rho} + \bar{v}^2 \right) \quad (3-3)$$

where $h = 1 + y/R$.

Substituting the Reynolds shear stress, $\tau = -\rho \overline{u'v'}$, and simplifying the viscous (last) term in equation (3-2) yields

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + h \bar{v} \frac{\partial \bar{u}}{\partial y} + \frac{\bar{v}}{R} \bar{u} = - \frac{\partial}{\partial x} \left(\frac{\bar{p}}{\rho} + \bar{u}^2 \right) + h \frac{\partial \tau}{\partial y} + \frac{\bar{v} \tau}{R} + \frac{\partial}{\partial y} \left(h \bar{v} \frac{\partial \bar{u}}{\partial y} \right) - \left(\frac{\bar{v}}{hR} \right) \frac{\bar{u}}{R} \quad (3-4)$$

The Reynolds shear stress, τ , is assumed to be represented by the Boussinesq eddy viscosity assumption:

$$\tau = \nu_t \left(\frac{\partial \bar{u}}{\partial y} - \frac{\bar{u}}{hR} \right) . \quad (3-5)$$

where ν_t is the turbulent eddy viscosity. Substituting the Boussinesq relation in equation (3-4) and simplifying, one gets

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + h \bar{v} \frac{\partial \bar{u}}{\partial y} - \frac{\bar{v}}{R} \bar{u} = - \frac{\partial}{\partial x} \left(\frac{\bar{p}}{\rho} + \bar{u}^2 \right) + \frac{\partial}{\partial y} \left[h (v + \nu_t) \frac{\partial \bar{u}}{\partial y} \right] - \left(\frac{v + \nu_t}{hR} + \frac{\partial \nu_t}{\partial y} \right) \frac{\bar{u}}{R} . \quad (3-6)$$

Substituting the equivalent viscosity, $\nu_e = \nu + \nu_t$, yields

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + h \bar{v} \frac{\partial \bar{u}}{\partial y} - \frac{\bar{v}}{R} \bar{u} = - \frac{\partial}{\partial x} \left(\frac{\bar{p}}{\rho} + \bar{u}^2 \right) + \frac{\partial}{\partial y} \left[h \nu_e \frac{\partial \bar{u}}{\partial y} \right] - \left(\frac{\nu_e}{hR} + \frac{\partial \nu_e}{\partial y} \right) \frac{\bar{u}}{R} . \quad (3-7)$$

Therefore, without the Boussinesq eddy viscosity assumption, the equations of motion consist of equations (3-1), (3-4) and (3-3). With the Boussinesq assumption, the set of equations becomes (3-1), (3-7) and (3-3).

3.2.4 Non-Dimensionalization

The equations of motion are non-dimensionalized by introducing the following non-dimensional variables.

$$\begin{aligned} x^* &= \frac{x}{L} , & y^* &= \frac{y}{L} , & R^* &= \frac{R}{L} , \\ \bar{u}^* &= \frac{\bar{u}}{U_{ref}} , & \bar{v}^* &= \frac{\bar{v}}{U_{ref}} . \end{aligned}$$

$$\begin{aligned}\overline{u^{*2}} &= \frac{\overline{u^2}}{U_{ref}^2} , & \overline{v^{*2}} &= \frac{\overline{v^2}}{U_{ref}^2} , & \tau^* &= \frac{\tau}{U_{ref}^2} , \\ \overline{C_p} &= \frac{\overline{P} - P_{ref}}{\frac{1}{2} \rho U_{ref}^2} , & Re &= \frac{L U_{ref}}{\nu} , \\ v_i^* &= \frac{v_i}{L U_{ref}} , \text{ and } & v_e^* &= \frac{v + v_i}{L U_{ref}} .\end{aligned}$$

where L is a reference length, U_{ref} is a reference velocity, P_{ref} is a reference static pressure and Re is the Reynolds number. When the non-dimensional variables are substituted into the equations of motion, the system of equations becomes:

- without Boussinesq assumption,

$$\frac{\partial \overline{u^*}}{\partial x^*} - \frac{\partial}{\partial y^*}(h v^*) = 0 . \quad (3-8)$$

$$\begin{aligned}\overline{u^*} \frac{\partial \overline{u^*}}{\partial x^*} - h v^* \frac{\partial \overline{u^*}}{\partial y^*} - \frac{\overline{v^2}}{R^*} \overline{u^*} &= \frac{\partial}{\partial y^*} \left(\frac{h}{R^*} \frac{\partial \overline{u^*}}{\partial y^*} \right) - \left(\frac{1}{R^* h R^*} \right) \frac{\overline{u^*}}{R^*} \\ &+ h \frac{\partial \tau^*}{\partial y^*} + \frac{2 \tau^*}{R^*} - \frac{\partial}{\partial x^*} \left(\frac{\overline{C_p}}{2} + \overline{u^{*2}} \right) .\end{aligned} \quad (3-9)$$

$$\frac{\overline{u^{*2}}}{h R^*} = \frac{\partial}{\partial y^*} \left(\frac{\overline{C_p}}{2} + \overline{v^{*2}} \right) . \quad (3-10)$$

- with the Boussinesq assumption,

$$\frac{\partial \overline{u^*}}{\partial x^*} - \frac{\partial}{\partial y^*}(h v^*) = 0 . \quad (3-11)$$

$$\begin{aligned}\overline{u^*} \frac{\partial \overline{u^*}}{\partial x^*} - h v^* \frac{\partial \overline{u^*}}{\partial y^*} - \frac{\overline{v^2}}{R^*} \overline{u^*} &= \frac{\partial}{\partial y^*} \left(h v_e^* \frac{\partial \overline{u^*}}{\partial y^*} \right) - \left(\frac{v_e^*}{h R^*} + \frac{\partial v_e^*}{\partial y^*} \right) \frac{\overline{u^*}}{R^*} \\ &- \frac{\partial}{\partial x^*} \left(\frac{\overline{C_p}}{2} + \overline{u^{*2}} \right) , \text{ and}\end{aligned} \quad (3-12)$$

$$\frac{\overline{u^{*2}}}{h R^*} = \frac{\partial}{\partial y^*} \left(\frac{\overline{C_p}}{2} + \overline{v^{*2}} \right) . \quad (3-13)$$

3.3 Selected Turbulence Models

3.3.1 Baldwin-Lomax (BL)

This is a two-layer algebraic (zero equation) eddy viscosity model presented by Baldwin and Lomax (1978). The turbulent eddy viscosity is defined as:

$$\nu_t^* = \begin{cases} (\nu_t^*)_{outer} & y^* > y_{cross}^* \\ (\nu_t^*)_{inner} & y^* \leq y_{cross}^* \end{cases} \quad (3-14)$$

where $(\nu_t^*)_{outer}$ is the outer layer eddy viscosity,

$(\nu_t^*)_{inner}$ is the inner layer eddy viscosity, and

y_{cross}^* is the minimum y^* where $(\nu_t^*)_{inner} = (\nu_t^*)_{outer}$.

The inner eddy viscosity is calculated from the Prandl-Van Driest formulation,

$$(\nu_t^*)_{inner} = l^{*2} \Omega^* \quad (3-15)$$

The Ω^* is the vorticity which, in two-dimensional, thin-layer flow, is given by

$$\Omega^* = \frac{\tilde{c} \bar{u}^*}{\tilde{c} y^*} - \frac{\bar{u}^*}{h R^*} \quad (3-16)$$

The mixing length, l^* , contains the wall damping, as well as curvature correction factors,

$$l^* = l_0^* F_f \quad (3-17)$$

The wall damping effect is represented by

$$l_v^* = \kappa y^* \left[1 - \exp\left(-\frac{y^*}{A^*}\right) \right] . \quad (3-18)$$

The curvature effect term is suggested by Shrewsbury (1989), who applied the correction factor as suggested by Bradshaw (1973).

$$F_c = 1 - \alpha S , \quad (3-19)$$

where S is the ratio of the “extra” strain due to curvature to the primary strain, or

$$S = \frac{\frac{\bar{u}}{hR}}{\frac{\partial \bar{u}}{\partial y}} . \quad (3-20)$$

The value of the constant α is found by trial and error and may change from flow to flow.

The outer eddy viscosity is calculated from

$$(v^*)_{outer} = K C_{ep} F_{wake} F_{KLEB} . \quad (3-21)$$

$$F_{KLEB} = \frac{1}{1 - 5.5 \left(C_{KLEB} \frac{y^*}{y_{max}^*} \right)^6} , \text{ and} \quad (3-22)$$

$$F_{wake} = Min \left(y_{max}^* F_{max} , \frac{C_{wake} v_{max} \bar{u}_{diff}^2}{F_{max}} \right) . \quad (3-23)$$

$\bar{u}_{diff}$ is the difference between the maximum and the minimum total non-dimensional mean velocity in the velocity profile at any x. For two-dimensional flows where there are no regions of reverse flow, such as those in this study,

$$\bar{u}_{diff} = \left(\sqrt{\bar{u}^2 + v^2} \right)_{max} . \quad (3-24)$$

The y_{max} and F_{max} are evaluated at the point in the profile where the following function obtains its maximum value:

$$F(y^*) = y^* |\Omega^*| \left(1 - \exp\left(-\frac{y^*}{A^*}\right) \right). \quad (3-25)$$

The following are model constants;

$$\kappa = 0.4, \quad K = 0.0168, \quad A^* = 26.$$

$$C_{cp} = 1.6, \quad C_{wk} = 1.0, \text{ and } C_{KLEB} = 0.3.$$

3.3.2 Dash k-ε (KE)

This model consists of two layers, a mixing length model in the near-wall region and a standard high Reynolds number k-ε model everywhere else. Dash et al. (1983) had already successfully calculated the development of plane wall jets in still air and of wall jets over a cylinder in still air with this model. The k and ε equations in two-dimensional, body-oriented, curvilinear coordinates, as given by Dash et al., are as follows.

$$\bar{u}^* \frac{\partial k^*}{\partial x^*} + h \bar{v}^* \frac{\partial k^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left(\frac{h \bar{v}^*}{\sigma_k} \frac{\partial k^*}{\partial y^*} \right) + h(P - \varepsilon^*) \quad (3-26.a)$$

$$\bar{u}^* \frac{\partial \varepsilon^*}{\partial x^*} + h \bar{v}^* \frac{\partial \varepsilon^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left(\frac{h \bar{v}^*}{\sigma_\varepsilon} \frac{\partial \varepsilon^*}{\partial y^*} \right) + h \frac{\varepsilon^*}{k^*} (C_1 P - C_2 \varepsilon^*) \quad (3-26.b)$$

$$v_i^* = C_u \frac{k^*}{\varepsilon^*} \quad (3-26.c)$$

$$P = v_i^* \left(\frac{\partial \bar{u}^*}{\partial y^*} - \frac{\bar{u}^*}{h R^*} \right)^2 \quad (3-26.d)$$

The non-dimensional k^* and ε^* are defined as:

$$k^* = \frac{k}{U_{ref}^2}, \text{ and } \varepsilon^* = \frac{\varepsilon L}{U_{ref}^3}.$$

The model constants are:

$$\begin{aligned} \sigma_k &= 1.0, & \sigma_\varepsilon &= 1.3. \\ C_2 &= 1.92, \text{ and} & C_\mu &= 0.09. \end{aligned}$$

Dash et al. considered various modifications for the curvature effects. They recommended the modification by Hah and Lakshminarayana (1980). in which C_1 is modified to account for the effects of curvature as follows.

$$C_1 = 1.43 \left(1 - C_c Ri \right) \quad (3-27)$$

Ri is a Richardson number which is defined by Launder, Pridden, and Sharma (1977) as:

$$Ri = \frac{\bar{u}}{R} \left(\frac{k}{\varepsilon} \right)^{1/2} \frac{\partial \bar{u}}{\partial y} \quad (3-28)$$

C_c is a constant for which Dash suggested a value of 0.16.

In the near wall region, the eddy viscosity is calculated using a mixing length model.

$$\nu_t^* = l^{*2} \frac{\partial \bar{u}^*}{\partial y^*} \quad (3-29)$$

The mixing length, l^* , contains a wall damping term, as well as a curvature correction term. The wall damping is given by

$$l_0^* = \kappa y^* \left[1 - \exp\left(\frac{-y^*}{A}\right) \right] . \quad (3-30)$$

where the model constants are $\kappa = 0.41$, and $A = 27$. The curvature correction is applied as follows:

$$l^* = \frac{l_0^* (1 - \alpha S)}{1 - S} , \quad (3-31)$$

where S is the ratio of the “extra” strain due to curvature to the primary strain, or

$$S = \frac{\frac{\bar{u}}{R}}{\frac{\partial \bar{u}}{\partial y^*}} . \quad (3-32)$$

α is the constant that is used to account for the effects of curvature. Dash et al.(1983) stated that it had little effect unless curvature was large. Nonetheless, they suggested values for α in the range of 5-10.

The matching point between the near wall and the outer region was set to occur at $y^* \sim 50$. Dash et al. reported that the model prediction was insensitive to the exact location of the matching point, as long as it remained in the log-law region ($20 < y^* < 100$). At the matching point, v_i calculated from both the mixing length model and the k - ϵ model are equal, and the production and dissipation terms in the k equation (3-26.a) are in equilibrium. These conditions yield boundary values for k and ϵ at the matching point as follows:

$$k_{l^*} = \frac{l_i^2 \left| \frac{\partial \bar{u}}{\partial y^*} \right|_{l^*}^2}{C_\mu^{1/2}} , \text{ and} \quad (3-33.a)$$

$$\varepsilon_{I^*} = \frac{C_{\mu}^{3/4} k_{I^*}^{3/2}}{l_{I^*}} \quad (3-33.b)$$

The subscript I^* indicates values at the matching (or interface) point. The values of k_{I^*} and ε_{I^*} then become the wall boundary values of k and ε , and the k - ε model equations are solved only at the nodes above I^* .

The boundary conditions at the edge between the shear layer and the freestream are set by solving a reduced form of the k and ε equations. At the interface with the freestream, it is assumed that $\frac{\partial}{\partial y}(\quad) = 0$ and that the production terms are zero. Under these assumptions, the k and ε equations reduce to

$$\bar{u}_e^* \frac{dk_e^*}{dx^*} = -h_e^* \varepsilon_e^*, \text{ and} \quad (3-34.a)$$

$$\bar{u}_e^* \frac{d\varepsilon_e^*}{dx^*} = -C_{\varepsilon_2} h_e^* \frac{\varepsilon_e^{*2}}{k_e^*} \quad (3-34.b)$$

where subscript e indicates values at the freestream edge of the shear layer. k_e^* and ε_e^* can be solved using Runge-Kutta integration.

3.3.3 Wilcox k - ω (KW)

This model was presented by Wilcox (1988a). It has an advantage of being able to be integrated all the way to the wall without various damping term which are necessary for k - ε . Wilcox (1993a) demonstrated the superiority of this model over the k - ε model for boundary layers under adverse pressure gradient and it has gained increasing popularity

since then. A two-dimensional, parabolized form of the k and ω equations in body-oriented, curvilinear coordinates is as follows (Wilcox, 1993b).

$$\bar{u}^* \frac{\partial k^*}{\partial x^*} + h \bar{v}^* \frac{\partial k^*}{\partial y^*} + C_k v_i^* \frac{\bar{u}^*}{R^*} \frac{\partial \bar{u}^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left[h \left(\frac{1}{Re} + \sigma^* v_i^* \right) \frac{\partial k^*}{\partial y^*} \right] + v_i h S^2 - \beta^* h \omega^* k^* \text{, and (3-35.a)}$$

$$\bar{u}^* \frac{\partial \omega^*}{\partial x^*} + h \bar{v}^* \frac{\partial \omega^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left[h \left(\frac{1}{Re} + \sigma^* v_i^* \right) \frac{\partial \omega^*}{\partial y^*} \right] - \alpha h S^2 - \beta^* h \omega^{*2} \text{ , (3-35.b)}$$

$$\text{where the eddy viscosity, } v_i^* = \frac{k^*}{\omega^*} \text{ , and (3-35.c)}$$

$$\text{the strain rate, } S = \frac{\partial \bar{u}^*}{\partial y^*} - \frac{\bar{u}^*}{h R^*} \text{ . (3-35.d)}$$

The third term on the left hand side of the k (3-35.a) equation is the curvature correction term as suggested by Wilcox and Chambers (1977). The non-dimensional k and ω are defined as.

$$k^* = \frac{k}{U_{ref}^2} \text{ , } \omega^* = \omega \frac{L}{U_{ref}} \text{ .}$$

and the model constants are,

$$\alpha = 5.9, \quad \beta = 3.40, \quad \beta^* = 9.100, \quad \sigma = 1.2, \quad \sigma^* = 1.2 \text{ , and } C_k = 9.2 \text{ .}$$

The boundary conditions at the edge between the shear layer and the freestream are obtained as with the k - ϵ method. That is by assuming that $\frac{\partial}{\partial y^*}$ and all production terms are zero. The k and ω equations then reduce to

$$\bar{u}_e^* \frac{dk_e^*}{dx^*} = - h_e \beta^* \omega_e^* k_e^* \text{ , and (3-36.a)}$$

$$\bar{u}_c \frac{d\omega_c}{dx} = - h_c \beta \omega_c \quad (3-36.b)$$

The solution of these reduced equations provides the boundary conditions at the freestream edge for the k and ω equations.

At the wall, the boundary conditions are

$$k_w = 0 \quad \text{and} \quad (3-37.a)$$

$$\omega_w = \frac{u_\tau^2}{\nu} \left(\frac{50}{k_R^+} \right)^2 = \left(\frac{\partial \bar{u}}{\partial y} \right)_w \left(\frac{50}{k_R^+} \right)^2 \quad (3-37.b)$$

The wall boundary condition for ω follows the suggestion by Wilcox (1993b). k_R^+ is the non-dimensional wall roughness, which Nikuradse (1933) found that $k_R^+ < 5$ for smooth walls. Wilcox also states that to apply the k - ω model equations right up to the wall, it is necessary that the grid point next to the wall be close enough to the wall such that $y^+ < 1$.

3.3.4 Wilcox Multiscale (MS)

This is a simplified Reynolds stress transport model proposed by Wilcox (1988b). The turbulent eddies are categorized into two partitions, each having a different energy level. The upper partition eddies are assumed to be essentially inviscid, and to be responsible for all production of turbulent energy by interaction with the mean flow. The lower partition eddies are assumed to be isotropic and to be responsible for all dissipation of turbulent energy into heat. These assumptions are inspired by large-eddy simulation

studies (Bardina et al., 1983). The model equations in two-dimensional, parabolized, curvilinear form are given by Wilcox (1988c).

Turbulent kinetic energy.

$$\bar{u}^* \frac{\partial k^*}{\partial x^*} + h \bar{v}^* \frac{\partial k^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left[h \left(\frac{1}{Re} + \sigma^* v_i^* \right) \frac{\partial k^*}{\partial y^*} \right] + hP - \beta^* h \omega^* k^*, \quad (3-38.a)$$

where the production term P is defined as

$$P = \tau^* \left(\frac{\partial \bar{u}^*}{\partial y^*} - \frac{\bar{u}^*}{hR^*} \right).$$

Specific dissipation rate.

$$\bar{u}^* \frac{\partial \omega^*}{\partial x^*} + h \bar{v}^* \frac{\partial \omega^*}{\partial y^*} = \frac{\partial}{\partial y^*} \left[h \left(\frac{1}{Re} + \sigma^* v_i^* \right) \frac{\partial \omega^*}{\partial y^*} \right] - \alpha \frac{\omega^*}{k^*} hP - \beta^* h \omega^* \left(\omega^* + \frac{\epsilon}{3} \Omega \right), \quad (3-38.b)$$

where the rotation, Ω , for two-dimensional flow is given by

$$\Omega = \frac{\partial \bar{u}^*}{\partial y^*} - \frac{\bar{u}^*}{hR^*}.$$

Reynolds stresses are obtained from the shear stress, τ^* , and the stress deviators,

σ_x^* and σ_y^* , which represent the amount of anisotropy of the Reynolds stresses in the streamwise and normal axes respectively. They are defined as

$$\tau^* = - \frac{\overline{u'v'}}{U_{\kappa'}^2}.$$

$$\sigma_x^* = \frac{\overline{u'^2} - \frac{2}{3}k}{U_{\kappa'}^2}, \text{ and}$$

$$\sigma_y^* = \frac{\overline{v'^2} - \frac{2}{3}k}{U_{\kappa'}^2}.$$

The differential equations for the stresses are:

- shear stress.

$$\begin{aligned} \overline{u}^* \frac{\partial \tau^*}{\partial x^*} + h \overline{v}^* \frac{\partial \tau^*}{\partial y^*} = & \left[-\hat{\beta} \sigma^* + (1 - \hat{\alpha}) \sigma_i^* + \frac{2}{3} \left(1 - \hat{\alpha} - \hat{\beta} + \frac{3}{4} \hat{\gamma} \right) k^* \right] \left(h \frac{\partial \overline{u}^*}{\partial y^*} - \frac{\overline{u}^*}{R^*} \right) \\ & + (2 - \hat{\alpha} + \hat{\beta}) (\sigma_i^* - \sigma_j^*) \frac{\overline{u}^*}{R^*} - C_1 \beta^* h \omega^* \tau^* . \end{aligned} \quad (3-38.c)$$

- axial stress.

$$\overline{u}^* \frac{\partial \sigma_i^*}{\partial x^*} + h \overline{v}^* \frac{\partial \sigma_i^*}{\partial y^*} = \frac{2}{3} (2 - 2\hat{\alpha} + \hat{\beta}) h P + (2 - \hat{\alpha} - \hat{\beta}) \tau^* \frac{\overline{u}^*}{R^*} - C_1 \beta^* h \omega^* \sigma_i^* . \text{ and } (3-38.d)$$

- normal stress

$$\overline{u}^* \frac{\partial \sigma_j^*}{\partial x^*} + h \overline{v}^* \frac{\partial \sigma_j^*}{\partial y^*} = -\frac{2}{3} (1 - \hat{\alpha} + 2\hat{\beta}) h P - (2 - \hat{\alpha} - \hat{\beta}) \tau^* \frac{\overline{u}^*}{R^*} - C_1 \beta^* h \omega^* \sigma_j^* . \quad (3-38.e)$$

The turbulent energy in the lower partition is represented by e , therefore, $(k-e)$ is the upper partition energy. The non-dimensional upper partition energy, $(k-e)^*$, is defined as

$$(k-e)^* = \frac{k-e}{U_{ref}^2} ,$$

and is governed by

$$\overline{u}^* \frac{\partial (k-e)^*}{\partial x^*} + h \overline{v}^* \frac{\partial (k-e)^*}{\partial y^*} = (1 - \hat{\alpha} - \hat{\beta}) h P - \beta^* h \omega^* k^* \left(\frac{k-e}{k} \right)^{3/2} . \quad (3-38.f)$$

Similar to the $k-\omega$ model, the eddy viscosity is defined as

$$\nu_t^* = \frac{k^*}{\omega^*} .$$

and the model constants are

$$\sigma^* = 1/2, \quad \beta^* = 9/100, \quad \sigma = 1/2, \quad \beta = 3/40, \quad \alpha = 4/5, \quad \xi = 1, \\ \hat{\alpha} = 42/55, \quad \hat{\beta} = 6/55, \quad \hat{\gamma} = 1/4, \text{ and } C_0 = 4.$$

The boundary values at the wall for k and ω are the same as for the k - ω model, and for the Reynolds stresses:

$$\tau_w^* = 0, \quad \sigma_{vw}^* = 0, \quad \sigma_{yw}^* = 0, \text{ and } (k-e)_w^* = 0.$$

The boundary values of k , ω and $(k-e)$ at the freestream edge are again obtained from the reduced form of the k , ω and $(k-e)$ equations, which are

$$\bar{u}_e^* \frac{dk_e^*}{dx^*} = - h_e \beta^* \omega_e^* k_e^*, \quad (3-39.a)$$

$$\bar{u}_e^* \frac{d\omega_e^*}{dx^*} = - h_e \beta \omega_e^* \left(\omega_e^* + \xi \left| \frac{d\bar{u}_e^*}{dy^*} - \frac{\bar{u}_e^*}{hR^*} \right| \right), \quad (3-39.b)$$

$$\bar{u}_e^* \frac{d}{dx^*} (k-e)_e^* = - h_e \beta^* \frac{\omega_e^*}{\sqrt{k_e^*}} (k-e)_e^*, \quad (3-39.c)$$

and the Reynolds stresses are

$$\tau_e^* = 0, \quad \sigma_{ve}^* = 0, \text{ and } \sigma_{ve}^* = 0.$$

Although the multiscale model is technically a Reynolds stress model, Wilcox (1988b) found that there was only a 50% increase in computing time relative to the k - ω model for steady boundary-layer computation.

3.4 Flow-Computation Algorithm

A FORTRAN computer code called TSL (Thin Shear Layer) was written to calculate flow development for this study. The TSL program is listed in Appendix A and its main features are presented in this section.

3.4.1 Grid Generation

Geometrical grid point distribution was used in the normal to the wall, or y , direction, that is

$$\delta y_{i+1} = (1 - \psi) \delta y_i, \quad (3-40)$$

where δy is the spacing between adjacent grid points, and ψ is the stretching parameter, whose typical value was 0.05 - 0.1 .

The maximum number of grid points in the y direction was 250, with a typical number between 150 to 200 points. The first grid point next to the wall ($i=2$) was placed at $y_2^+ < 1$ based on an initial calculation of C_f . For most of the flows in this study, C_f decreased as the flow moved downstream. Therefore y_2^+ remained below 1 throughout the calculation range. In the cases where the initial C_f could not be calculated accurately due to imprecise starting profiles, the initial value of y_2^+ was adjusted to maintain $y_2^+ < 1$ in the main downstream region of interest. The grid point farthest from the wall ($i=n$) was

kept at $y_w/\delta > 1.75$ for boundary layers, and $y_w/y_{1/2} > 5$ for wall jets. Grid points were added as necessary to the top of the calculation domain if the shear layer grew with downstream distance.

For selected cases, the initial y_2^* and maximum streamwise step size were either halved or doubled to ensure that the results were grid independent.

3.4.2 Control Volume Discretization

The system of equations was discretized using the control volume method described in Patankar (1980) with control volumes as shown in Figure 3-2. Since the control volume sizes varied in the y direction, care had to be taken in specifying the locations of the control volume faces. For convenience of calculating the diffusion coefficients, the face edges were chosen to be midway between adjacent nodes.

The streamwise momentum equation and the turbulence model equations can be cast in a standard form as:

$$\frac{\partial}{\partial x^*}(\bar{u}^* \phi^*) + \frac{\partial}{\partial y^*}(\bar{v}^* \phi^* - G \frac{\partial \phi^*}{\partial y^*}) = S_p \phi^* + S_c \quad (3-41)$$

where G is the diffusive coefficient, and S_p and S_c represent the source terms which include the production and dissipation terms.

Figure 3-2 Control volume for discretization.

The discretized equation for the standard equation (3-41) is

$$a_P \phi_P^* = a_W \phi_W^* + a_N \phi_N^* + a_S \phi_S^* + b \quad (3-42)$$

where a_P , a_W , a_N , a_S and b are coefficients calculated from the convection, diffusion and source terms. ϕ_P^* , ϕ_W^* , ϕ_N^* and ϕ_S^* are the unknown scalar variables ($\bar{u}^*$, k^* , ε^* , ω^* , etc.) at the nodes P, W, N and S as shown in Figure 3-2. The detailed derivation of equations (3-41), (3-42) and the formulae for calculating a_P , a_W , a_N , a_S and b are provided in Appendix B.

Since equation (3-41) is parabolic, the TSL program employs an implicit space-marching algorithm. When solving for the unknowns at a constant x , the upstream values of the unknowns are already known. Therefore, ϕ_W^* is known and ϕ_P^* , ϕ_N^* and ϕ_S^* are the unknowns. The system of equations such as (3-42) at all the nodes at a constant x , forms a tri-diagonal matrix which can be solved with the well-known Thomas algorithm.

3.4.3 Modularization

Since all the discretized equations were of the same form, they could be solved with a common solver. Not only did the common solver make coding simpler, but it also enabled the use of analytical solutions to validate the calculated results. By computing laminar flows, most aspects of the program were tested, except for the turbulence models. In particular the common solver was used to solve the streamwise momentum equation and the computed results were compared with analytical solutions to verify correct

functioning of most aspects of the computer code. Validation work will be discussed later.

3.4.4 Sequential Iteration

Since the discretized equations were nonlinear, they had to be solved by iteration. Two methods of iteration are commonly used, sequential iteration and block iteration. For sequential iteration, each equation is solved separately, one at a time. In block iteration, a group of several equations are solved together at the same time. For this study, sequential iteration was chosen because it was simpler to implement and more flexible in terms of modifications of the equations. This technique is known to be slower to converge but for the parabolic space-marching algorithm, this was unimportant.

3.4.5 Additional Iteration for Self Similarity

Self similar flows are attractive as test cases because they do not require accurate starting conditions. For plane wall jets, self similarity requires constant U_e/U_m along the streamwise direction (Irwin, 1973); this is governed by the streamwise pressure gradient. For self similar curved wall jets in still air, y_w/R has to be constant along the flow (Guitton and Newman, 1977) and the wall curvature is the controlling factor. In the TSL program, an extra iteration loop is provided for self similar wall jets, to keep the

characteristic parameter (U_c/U_m or y_w/R) constant by adjusting the controlling factor (streamwise pressure gradient or the wall curvature).

3.4.6 Starting Problem

Experimental data are never complete enough to start a numerical calculation without additional assumptions, particularly for the values of ε or ω . Inaccurate starting conditions result in initial transients that can potentially affect integral parameters, such as y_w and U_m , everywhere downstream. For self-similar flows, starting is not a problem, as mentioned above. In this study five methods were used to start a calculation:

- assuming the profiles of k and l ,
- assuming the profiles of τ and l ,
- using given profiles of k and l ,
- using given (or experimental) Reynolds stress profiles and calculating ω or ε from the Boussinesq eddy viscosity assumption, and
- using results from previous runs.

The choice of method for any given case depended on the available information at the starting station and usually involved substantial trial and error to find the best technique. Further details on the starting problem are given in Appendix C.

3.5 Validation

3.5.1 Solver Validation

The common solver was validated by comparing the calculated results with analytical solutions. A number of analytical solutions for laminar flows were available in literature and some selected cases were chosen for this purpose.

3.5.1.1 Blasius Flow

The most basic flow was the laminar flow over a flat plate at zero pressure gradient, of which the analytical solution is the Blasius solution. The comparison between calculated results and the Blasius solution is shown in Figure 3-3. The calculation was done over the range of Reynolds number based on x , $\frac{U_\infty x}{\nu}$, from 1 to 1000. The calculated results compare very well with the analytical solution indicating that the common solver was implemented correctly.

Figure 3-3 Comparison of calculated results with Blasius solution.

3.5.1.2 Falkner-Skan Flow

The next test case was the laminar flow over a flat plate under an adverse pressure gradient. The analytical solutions for various pressure gradients are called the Falkner-Skan solutions. The comparison between the calculated results and the Falkner-Skan solution is shown in Figure 3-4. The calculations were carried out for the case of Reynolds number, $\frac{U_\infty L}{\nu} = 100$, and $m = -0.08257$. The parameter m indicated the change of the edge velocity according to $U_\infty \propto x^m$, which in turn, indicated the pressure gradient. The Falkner-Skan velocity distribution and values of the integral parameters were obtained from White (1991). This particular case is the most severe adverse pressure gradient just short of separation for which the analytical solutions are provided. Figure 3-4 shows that the calculated results match the analytical solution well, therefore, the effect of pressure gradient was incorporated correctly in the common solver.

3.5.1.3 Plane Laminar Wall Jet

The analytical solution for this case was presented first by Glauert (1956). From the Glauert relations, it can be shown that:

$$\frac{U_m}{U_\infty} = \left(\frac{x - x_0}{x_0} \right)^{-1/2}, \quad (3-43.a)$$

$$\frac{y_{1/2}}{h} = \left(\frac{x - x_0}{x_0} \right)^{3/4}, \quad (3-43.b)$$

Figure 3-4 Comparison of TSL calculated results with Falkner-Skan solution.

$$C_f = \frac{5.22}{Re_0} \left(\frac{x + x_0}{x_0} \right)^{-1.4} \text{ , and} \quad (3-43.c)$$

$$\frac{x}{b} = \frac{Re_0}{17.25} \text{ .} \quad (3-43.d)$$

where U_j and b are the U_m and $y_{1/2}$ at the initial position, $x=0$, and the initial Reynolds number, $Re_0 = \frac{U_j b}{\nu}$. The derivation of the above relations is provided in Appendix D. The calculation was carried out for $Re_0=100$ and the results are compared with the Glauert solution in Figure 3-5. The calculation compared well with the analytical solution, confirming that the solution technique performed satisfactorily with the wall-jet like velocity profile.

3.5.1.4 Laminar Boundary Layer over a Cylinder

Unfortunately, no exact analytical solution exists for this case. An approximate solution, based on series expansion, is provided by Schlichting (1968). The solution is approximate because the plane continuity equation is used and the normal momentum equation is neglected. The analytical solution based on the edge potential flow velocity is given by

$$U_e = 2 U_\infty \sin \phi \text{ .} \quad (3-44)$$

where ϕ is the downstream angular distance measured from the front stagnation point (see Figure 3-6). The above edge velocity relation provides the following edge pressure coefficient.

Figure 3-5 Comparison between calculated results and Glauert solution.

$$C_{pe} = 2 \cos \phi \quad . \quad (3-45)$$

For the calculation of this flow, the above pressure coefficient distribution was assumed to exist at the wall of the cylinder (i.e. the boundary layer thickness was neglected). Although this was physically unrealistic, it was actually more consistent with the approximate analytical solution.

The comparison between the calculated results and the approximate analytical solution is shown in Figure 3-6. The scaled velocity profiles (u/u_p vs y/δ^*) indicate good comparison between the calculated results and the analytical solutions at all of the three streamwise positions shown in the plot. The scale parameters, δ^* and U_{pw} , display some discrepancies. This is not unexpected since the approximate analytical solution neglected the boundary layer thickness by dropping the normal momentum equation, whereas the TSL program does incorporate the normal momentum equation. On the basis of the velocity profiles and C_f distribution, the solver was judged to have handled the curvature terms adequately.

Figure 3-6 Laminar boundary layer over a cylinder.

3.5.2 Validation of the Implementation of Turbulence Models

3.5.2.1 Baldwin-Lomax Model

The literature contains few results calculated using this model for flows which could be calculated by the TSL program. While TSL can only compute two-dimensional, thin-shear layer flows, the Baldwin-Lomax results generally available in literature are for three-dimensional flows and flows with complicated geometries. The only appropriate test case found was a boundary layer flow under increasingly adverse pressure gradient (Samuel and Joubert, 1974) as computed by Wilcox (1993b). Only the computed C_f is shown by Wilcox and the exact method of calculation was not discussed. The TSL calculation for this flow was performed by using a modified Thompson velocity profile (Galbraith and Head, 1975) as the starting profile. The comparison between the TSL computed result and Wilcox's result is shown in Figure 3-7. The discrepancy near the starting point is probably due to different starting conditions between the two calculations. Without details of the starting conditions for the Wilcox calculation, it is not possible to determine the exact cause of this discrepancy. The difference between the two results decreases further downstream, suggesting that the Baldwin-Lomax model is implemented correctly in the TSL program.

Figure 3-7 Comparison of calculated C_{fx} for Samuel-Joubert flow using the Baldwin-Lomax model.

The curvature correction for the Baldwin-Lomax model follows the suggestion of Shrewsbury (1989), who reported that the value 25 for the empirical constant α in equation (3-19) gave the best agreement between calculated results and experimental data on a circulation control airfoil. Unfortunately, the flow that Shrewsbury computed cannot be calculated with the TSL program, therefore, the implementation of the Baldwin-Lomax model has not been directly validated for curved flow cases. However, the form of the curvature correction term is quite straight forward and no difficulty is expected from its implementation.

3.5.2.2 Dash k- ϵ Model

The first validation case is a plane wall jet in still air. The test conditions are described by Wilson and Goldstein (1976) and the calculated results are published by Dash et al. (1983). The comparison of the results between Dash's calculation and TSL program is shown in Figure 3-8.

Since plane wall jets in still air are self-similar, the calculated results are independent of the starting conditions. The technique used to arrive at the self-similar solutions was to use the results at the end of a run as the starting profiles for the next run. The process was repeated until the profiles of the similarity variables no longer changed with streamwise distance. There are noticeable differences between the calculations of

Dash et al. and results from TSL program. Critical parameters affecting the calculation and results are summarized in Table 3-1.

Several other combinations of grid sizes were tried without significant changes to the results calculated by TSL program. It is not clear what causes the TSL calculated results to be different from those of Dash et al. However, as will be seen later, the difference between the two calculations is much smaller than the difference between calculations using different turbulence models.

Parameters	Dash et.al. (1983)	TSL
Location of interface point, $y_1^{(1)}$	50	50
Stretching parameter, $\psi^{(2)}$	0.2	0.1
Location of 2 nd grid point, $y_2^{(3)}$	n/a	0.00024
No. of grid points normal to the wall	50	~ 100
Streamwise step size in multiple of jet width ⁽⁴⁾	0.05	0.05
$\frac{dy_{1,2}}{dx}$	0.089	0.095
$\frac{y_m}{y_{1,2}}$	0.1	0.08

Notes ⁽¹⁾ See section 3.3.2 .

⁽²⁾ See equation (3-40).

⁽³⁾ See section 3.4.1.

⁽⁴⁾ Assuming jet width = $2y_{1,2}$

Table 3-1 Comparison of parameters affecting calculations of plane wall jets in still air for Dash et al. (1983) and present calculation using TSL program.

Figure 3-8 Comparison of calculated results between Dash's (1983) and TSL using k-ε model for plane wall jet in still air.

The second validation case is a curved wall jet over a cylinder in still air. Similar to the first case, the reference calculated results are provided by Dash et al. (1983) and the test conditions by Wilson and Goldstein (1976). The comparison between Dash's results and those from TSL program is shown in Figure 3-9.

The difference between the two calculations for this case is even larger than the previous case of a plane wall jet in still air. Dash et al. also provide calculated results for which only the curvature effects arising from coordinate transformation was considered, but the curvature correction as specified by equation (3-27) was neglected. In the TSL program, this is equivalent to setting $C_c=0$ in equation (3-27). The comparison for this case is shown in Figure 3-10. Again the two calculations are different both with ($C_c=0.16$) and without ($C_c=0$) extra curvature correction. This indicates that the difference lies in the way the coordinate transformation was carried out in each set of the system of equations. It should be pointed out here that there are some differences between Dash's equations for k and ϵ and the corresponding equations in the TSL program. Dash's equations are as follows.

$$\bar{u} \frac{\partial k}{\partial x} + h\bar{v} \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left(\frac{h\nu_c}{\sigma_k} \frac{\partial k}{\partial y} \right) + P - \epsilon \quad , \text{ and} \quad (3-46.a)$$

$$\bar{u} \frac{\partial \epsilon}{\partial x} + h\bar{v} \frac{\partial \epsilon}{\partial y} = \frac{\partial}{\partial y} \left(\frac{h\nu_c}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial y} \right) + \frac{\epsilon}{k} (C_1 P - C_2 \epsilon) \quad . \quad (3-46.b)$$

Figure 3-9 Comparison of calculated results between TSL and Dash et al. (1983), using k- ϵ model for wall jet over a cylinder.

Figure 3-10 Comparison of curvature effects on the calculated results of Dash et al. (1983) and TSL using the $k-\epsilon$ model for a wall jet over a cylinder.

Comparing these equations to equation (3-26.a) and (3-26.b), it can be seen that the factor h in the source terms on the right hand side appears to be missing from the equations (3-46.a) and (3-46.b). When the above equations were used instead of (3-26.a) and (3-26.b), the TSL calculated results come closer to Dash's results, as shown in Figure 3-10. The small differences that remained are probably due to slightly different starting conditions. Therefore, it appears that the TSL program implements the Dash k - ϵ model, and that TSL handles the curvilinear coordinate transformation more accurately than Dash et al. (1983) did. However, the difference between the two calculations only appears at high values of y_1/R , where the thinness assumption becomes questionable.

According to Dash et al., the location of the matching point between the inner mixing length layer and the outer k - ϵ layer, as specified by y_1^* , has only small effect on the predicted results, as long as the matching point stays within the log-law layer ($20 \leq y_1^* \leq 200$). The calculated results from the TSL program at various values of y_1^* are shown in Figure 3-11. An interesting observation is that the inner boundary values of k and ϵ provided by equations (3-33.a) and (3-33.b) appear to be incompatible with the limiting values of k and ϵ as y approaches the matching point from the outer layer. The effect is the large step change, in Figure 3-11, of the values of k and ϵ between the first and second grid points above the matching point. The velocity profile in terms of inner variables y^* and u^* also demonstrates a discontinuous slope. Nevertheless, the effect on the growth of the wall jet thickness is not excessive and only occurs at high y_1/R , where again the thinness assumption is questionable.

Figure 3-11 Effects of location of matching point on the results of Dash k - ϵ model for a wall jet over a cylinder.

3.5.2.3 Wilcox k- ω Model

For this turbulence model, the reference calculated results to compare with the TSL program were obtained from the program EDDYBL (Wilcox, 1993b). EDDYBL is a compressible, two-dimensional and axisymmetric boundary-layer program that embodies, among other turbulence models, the Wilcox k- ω model and the Wilcox multiscale model. The first validation case is a turbulent boundary layer over a flat plate. The unit Reynolds number is 296,000/ft and the calculations were carried out from $x=1$ to 14.2 ft. The calculated results from EDDYBL and TSL programs are compared in Figure 3-12. The two calculations produced comparable results in general. The most noticeable differences are in the values of C_f and δ^+ . It can be seen that the cause of the differences is the value of the wall smoothness parameter, k_R^- , which is used to determine the wall boundary value for ω according to equation (3-37.b). Wilcox (1993b) states that a smooth surface can be represented by $k_R^- \leq 5$. As seen in Figure 3-12, C_f and δ^+ display asymptotic behaviour when k_R^- becomes small but k_R^- of 5 seems too large to produce the asymptotic results. Furthermore, the asymptotic values of C_f and δ^+ do not match with the EDDYBL results. It is not clear what value of k_R^- was employed by EDDYBL. Nevertheless, the implementation of this model in the TSL program seems satisfactory. $k_R^- = 0.01$ were used in the rest of calculations since it was sufficiently small to produce the asymptotic solution of $k_R^- \rightarrow 0$.

Figure 3-12 Comparison of calculated results using the k - ω model from EDDYBL and TSL programs for turbulent boundary layer over a flat plate.

The second validation case is a turbulent boundary layer over a cylinder. A fictitious test case was set up with $\frac{U_\infty R}{\nu} = 148,000$ and zero pressure gradient at the freestream edge. Calculations were performed between x/R from 2 to 3.8. The calculated results from EDDYBL and TSL programs are compared in Figure 3-13. An exact match between the two calculations cannot be expected in this case because EDDYBL does not include the normal momentum equation while TSL does. The effects of k_R^- discussed previously are also present. The calculation of integral parameters, δ^+ , θ , and H , may also be different in the two programs. Following So and Mellor (1972), these calculations in the TSL program take into account the radial variation of the potential flow velocity profile in the freestream region. It is not clear whether this effect is accounted for in EDDYBL. However, judging from the similar trends between the two sets of results, there does not appear to be any discrepancy in the implementation of the curvature effects for the $k-\omega$ turbulence model in the TSL program.

3.5.2.4 Wilcox Multiscale

Similar to the validation of the $k-\omega$ model, the reference data for the validation of the implementation of the Multiscale model were obtained from the program EDDYBL (Wilcox, 1993b).

Figure 3-13 Comparison of the calculated results using $k-\omega$ model, from EDDYBL and TSL programs for the turbulent boundary layer over a circular cylinder with zero circumferential pressure gradient.

The first validation case is the turbulent boundary layer over a flat plate. Unit Reynolds number was 296,000 per ft and streamwise calculation domain was from $x=1$ to $x=14.2$ ft. Figure 3-14 shows the comparison between the calculated results from EDDYBL and TSL programs using the Multiscale model. The results indicate a similar situation to that of the $k-\omega$ model. The two programs generated similar results. The main differences were in the values of C_f and δ^+ and seem to have been caused by the choice of the value for k_R^- . The TSL results displayed asymptotic behaviour when $k_R^- \rightarrow 0$ and $k_R^- = 0.01$ was sufficiently small to produce asymptotic results.

The second validation case is a turbulent boundary layer over a cylinder under zero circumferential pressure gradient. The Reynolds number based on radius, $\frac{U \cdot R}{\nu} = 148,000$ and the streamwise calculation range was from $x/R=2$ to 3.8. The calculated results from EDDYBL and TSL programs are compared in Figure 3-15. The two sets of results displayed similar trends, indicating that the model implementation in TSL program was satisfactory. The possible sources of the differences between the results are the normal momentum equation, which was incorporated in TSL but neglected in EDDYBL; the effect of k_R^- , which in TSL was 0.01 but whose value in EDDYBL is unknown; and the calculation of δ^+ and θ , for which in TSL the potential flow characteristics of the freestream were accounted for but in EDDYBL might or might not have been.

Figure 3-14 Comparison of calculated results using multiscale model from EDDYBL and TSL programs for turbulent boundary layer over a flat plate.

Figure 3-15 Comparison of calculated results from EDDYBL and TSL programs using the multiscale model for turbulent boundary layer over a circular cylinder under zero circumferential pressure gradient.

3.6 Preliminary Tests of Unmodified Turbulence Models

3.6.1 Turbulent Boundary Layer over a Flat Plate

All four turbulence models were tested on the identical case of a turbulent boundary layer over a flat plate under zero pressure gradient. The unit Reynolds number is 296,000 per ft. and the calculations cover the range of Reynolds number based on x , $\frac{U_\infty x}{\nu}$, from 0.296×10^6 to 4.2×10^6 . The calculated results were compared with experimental data from Klebanoff (1955), and the integral parameter correlations suggested by White (1991) in Figure 3-16. The inner-variables plot (y^+ vs u^+) was also compared to the universal log-law-of-the-wall, with the constants recommended by Patel (1965). The results from all four models are seen to be very similar and reasonably comparable to the experimental data.

3.6.2 Samuel and Joubert Flow

The next test case is a turbulent boundary layer over a flat plate under increasingly adverse pressure gradient. The test conditions and the experimental results were provided by Samuel and Joubert (1974). The calculations started at $x=1.1$ m., using the experimental data at nearest streamwise positions as starting profiles. The pressure

Figure 3-16 Turbulent boundary layer over a flat plate, calculated results of unmodified models vs experimental data.

coefficient, C_p , at the wall was integrated from the experimental data for dC_p/dx . The calculated results from all four models are compared with the experimental data in Figure 3-17. All of the models produced very similar results. The BL model predicted slightly more accurate C_f than the others, and the MS model slightly better velocity profile. All models significantly underpredicted k and $\overline{u}^2$.

3.6.3 Curved Boundary Layer

This test case involves a turbulent boundary layer over a convex wall under zero pressure gradient. It is part of the Stanford II (Kline, Cantwell and Lilley, 1982) test cases under the title of Flow 0233. The test geometry, shown in Figure 3-18, consists of a straight wall entry section, followed by a 90° turn curved section, then a straight wall recovery section. The first available experimental profiles, at $x = -0.3$ m., were used as starting profiles. The experimental data for the C_p at the wall confirmed that it remained essentially constant, therefore, zero pressure gradient was assumed for the calculations. The calculated results of the four turbulence models are compared to the experimental data in Figure 3-19. The KW model seems to have an overall edge on other models, particularly in the prediction of C_f . The BL model is notably inferior for the predictions of C_f , as well as for the shear stress, τ . Otherwise, all models are generally similar.

Figure 3-17 Turbulent boundary layer over a flat plate under increasingly adverse pressure gradient. calculated results from unmodified models vs experimental data.

Figure 3-18 Test geometry of turbulent boundary layer over a convex surface.

Figure 3-19 Turbulent boundary layer over a convex surface with recovery under zero pressure gradient. calculated results from unmodified models vs experimental data.

It should be mentioned that all turbulence model equations contain singular terms, such as $\frac{\varepsilon}{k}$ or $\frac{\omega}{k}$. These terms can potentially become very large near the freestream edge where k is small. The sensitivity to this problem varies from model to model, and from flow to flow. For this flow, the problem became significant for the KW model. The effects of the singularity could be controlled by increasing ω_e , the value of ω at the freestream edge. An initial value, ω_{ex} , of ω_e can be specified as part of the initial profile of ω . The downstream values of ω_e are then calculated from equations (3-36.a) and (3-36.b). Figure 3-20 shows that the effects of ω_{ex} were limited to reducing the tendency of the k profile to "blow up" near the freestream edge. The effects on other parameters were minimal. $\omega_{ex}^* = 10$ was used to produce the calculated results in Figure 3-19.

3.6.4 Plane Wall Jet in Still Air

For this case the calculations were intended to simulate the experiment of Wilson (1970) for which detailed turbulence quantities were available. Since the plane wall jet in still air is self-similar, the starting profiles were obtained from the results of the previous run. The process was repeated until the solutions become self-similar. The calculated results are compared to Wilson's experimental data in Figure 3-21 and flow development parameters are compared to consensus of experimental data, as reviewed by Launder and Rodi (1981), in Table 3-2.

Figure 3-20 Effects of initial values of ω_e on the predictions of the $k-\omega$ model for curved boundary layer with convex curvature (flow 0233).

	$\frac{dy_{1/2}}{dx}$	$\frac{y_m}{y_{1/2}}$	$\frac{y_{x0}}{y_m}$
Consensus of experiments	.073	.15	.6
BL model	.065	.095	1
KE model	.095	.08	1
KW model	.115	.065	1
MS model	.15	.06	.9

Table 3-2 Flow development parameters of plane wall jet in still air.

The BL model produced the best overall results for this case, followed by the KE model. The KW and MS models seemed to have significant difficulty. The profiles of the velocity and turbulence quantities calculated by the KW model approached the freestream values much farther from the wall than the experimental data, or the calculated results of other models. The MS model overpredicted the wall jet growth rate, as well as the turbulence quantities, by a large margin. Several investigators reported that the $k-\omega$ model is sensitive to the value of ω in the freestream, ω_e (Wilcox, 1991; Menter, 1991). Several starting values of ω_e were tried without significant changes to the final self-similar results of either the $k-\omega$ or the multiscale models, confirming the finding from other studies that the ω_e sensitivity only becomes a problem for free-shear flows, not for wall-bounded flows.

The inner velocity profiles (u^+ vs y^+) of all four models show only a small region of agreement with the universal log law, in this case with constants according to Patel (1965). This is consistent with the finding in a review of experimental data by Launder

Figure 3-21 Plane wall jet in still air. calculated results of unmodified models vs experimental data.

and Rodi (1981). They reported conflicting opinions among experimenters on whether the universal log law applied to wall jets in still air. Even in experiments where this agreement was found, the range of y^+ to which it applied reduced with increasing downstream distance.

3.6.5 Plane Wall Jet under Adverse Pressure Gradient

The test conditions for this case duplicate the experiment of Irwin (1973), which dealt with a self-similar plane wall jet with $U_e/U_m = 0.38$. The resulting streamwise pressure gradient corresponds to the variation of freestream velocity given by $U_e \propto x^{-1/4}$. Again, since the flow is self-similar, the results are independent of the starting conditions. The actual pressure at each streamwise position was iterated to maintain constant U_e/U_m as described in Section 3.4.5. The calculated results from all four models are shown in Figure 3-22 and flow development parameters in Table 3-3.

	$\frac{dy_{1/2}}{dx}$	$\frac{y_m}{y_{1/2}}$	$\frac{y_{r0}}{y_m}$
Experiment (Irwin, 1973)	.044	.211	n/a
BL model	.045	.16	1
KE model	.048	.145	1
KW model	.043	.155	1
MS model	.053	.145	.9

Table 3-3 Flow development parameters of self-similar plane wall jet with $U_e/U_m=0.38$.

Figure 3-22 Plane self-similar wall jet with $U_e/U_m=0.38$. calculated results from unmodified models vs experimental data.

The remarkable observation for this case is that all models performed much better than in the case of a plane wall jet in still air. One possible interpretation is that wall jets in a moving stream are more similar to boundary layer flows than wall jets in still air, and all models perform better in boundary layer flows than in wall jet flows. Supporting evidence for this is the conformity of the inner velocity profile to the universal log law. Irwin (1973) reported good agreement between measured inner velocity profile data ($\bar{u}$ vs $\bar{y}$) and the universal log law with the constants according to Patel (1965). The calculated results also reproduced this agreement as shown in Figure 3-22. In contrast, the similar agreement is much more restricted in plane wall jets in still air (Figure 3-21).

3.6.6 Curved Wall Jet over a Log Spiral in Still Air

This case simulates the experiment of Guitton (1970) on a wall jet over a log spiral with $x/R=2/3$, which is also called Flow 0263 in the Stanford II proceedings (Kline, Cantwell and Lilley, 1982). The flow is self-similar, thus independent of starting conditions, and the pressure gradient is zero. In the actual experiment, the wall jet flowed over a spiral surface where the ratio of downstream distance to local radius of curvature was constant, $x/R=2/3$. The resulting self-similar wall jet had a constant $y_{1/2}/R$ value of 0.104. In the TSL program, the local radius of curvature at any x was iterated to arrive at the value that yielded $y_{1/2}/R=0.104$ (see Section 3.4.5). The calculated results are shown in Figure 3-23 and flow development parameters in Table 3-4.

	$\frac{dy_{1/2}}{dx}$	$\frac{y_m}{y_{1/2}}$	$\frac{y_{r0}}{y_m}$
Experiment (Guitton and Newman, 1977)	.155	.159	n/a
BL model	.075	.074	1
KE model	.16	.065	.85
KW model	.24	.042	.93
MS model	.35	.028	.85

Table 3-4 Flow development parameters of curved wall jet over log spiral with $x/R=2.3$.

Only the KE model produced reasonable results in comparison to the experimental data. The BL model significantly underpredicted the wall jet growth rate and the maximum velocity decay. The KW and MS models encountered the same difficulties as in the case of a plane wall jet in still air: namely, the profiles approaching freestream values too far from the wall in the case of the KW model, and too high growth rate for the MS model. Furthermore, the profiles of all parameters, including u , of the MS model displayed a spurious double-peak feature.

3.7 Tuning Models

The two models that are the main foci of this study, namely the $k-\omega$ and the multiscale models, did not perform well in the cases of wall jets in still air. The following sections describe the efforts to tune these models for wall jet flows.

Figure 3-23 Curved wall jet over a log-spiral with $x/R=2/3$, calculated results of unmodified model vs experimental data.

3.7.1 Tuning Flows

Self-similar wall jets were selected as tuning flows because they are independent of starting conditions. The flows that were used as tuning flows and the sources of experimental data are listed in Table 3-5.

Flow	Description	Experiment	Notes
FWJ	Plane wall jet in still air	Wilson (1970)	Additional data from review by Launder and Rodi (1981).
F0261	Plane wall jet in moving stream with $U_e/U_m=0.38$	Irwin (1973)	
GN4	Plane wall jet in moving stream with $U_e/U_m=0.77$	Gartshore and Newman (1969)	Only jet half width and maximum velocity decay data are available.
F0263	Curved wall jet over a log spiral with $x/R=2.3$	Guittou and Newman (1977)	
GUIT2	Curved wall jet over a log spiral with $x/R=1$	Guittou and Newman (1977)	

Table 3-5 The tuning flows and sources of experimental data.

3.7.2 Tuning Strategy and Selection of Matching Points

An effective method to evaluate the turbulence model constants is to apply the model equations at some convenient points in the flow, where the equations simplify to a manageable form, then solve the simplified equations in association with the experimental data (Wilcox, 1993b). This approach was applied using the experimental data on self-similar wall jets mentioned in the previous section, at the following matching points:

- Maximum shear stress point As shown in Figure 3-24, turbulence quantities, k , $\overline{u'^2}$, $\overline{v'^2}$ and $\overline{u'v'}$, attain their maximum values at approximately the same non-dimensional distance from the wall in any particular flow. At such a location the wall-normal derivatives, $\frac{\partial}{\partial y}$, would vanish, and the model equations simplify. Furthermore, this point is sufficiently far from the wall that fairly accurate experimental data are available for the evaluation of the model constants.
- Zero shear stress point At this point the production terms in the model equations vanish. This point is usually close to, but not necessarily coincident with the maximum velocity point. Unfortunately, this point is too close to the wall and the available experimental data are neither plentiful nor accurate.

Figure 3-24 Experimental data on self-similar wall jets.

FWJ : Plane wall jets in still air.

F0261: Self-similar plane wall jet with $U_e/U_m=0.38$.

F0263: Curved wall jet over a log-spiral with $x/R=2/3$.

- Wall layer The existence of the universal log-law of the wall for wall jets is still controversial. Many experimenters suggested that wall jets do not conform with the universal log law of the wall (Kruka and Eskinazi, 1964; Tailland and Mathieu, 1967) while others found at least some agreement with the universal log law (Irwin, 1973; Guitton and Newman, 1977). Özarapoglu (1973) performed a thorough study of the validity of the universal law of the wall in several turbulent flows, including plane wall jets, and concluded that the universal log law did apply for plane wall jets. He argued that the discrepancy reported in literature was due to "small but significant" errors caused by turbulence and streamline displacement as a result of interaction between the Pitot probe and the wall. He also noted that the tendency to fit the best straight line through too many data points in the $u^+ \text{ vs } y^+$ graph resulted in a large variation of the values of the constants in the log law relation. Launder and Rodi (1981) extensively reviewed experimental data on wall jets and came to a similar conclusion.

In view of this situation, and also the fact that without the universal log law there would not be enough information for the evaluation of the model constants, the universal log law was assumed to apply in the present work. Some retroactive justification was provided by the fact that the modified models did work reasonably well for wall jets.

3.8 Modifying the Wilcox k- ω Model

3.8.1 Model Equations at the Matching Points

For self-similar wall jets such as those of the tuning flows, the flow variables are non-dimensionalized locally as:

$$\begin{aligned} u^{**} &= \frac{u - U_e}{U_m - U_e} = \frac{u - U_e}{U_\tau} \quad , \\ v^{**} &= \frac{v}{U_\tau} \quad , \\ k^{**} &= \frac{k}{U_\tau^2} \quad , \text{ and} \\ \omega^{**} &= \frac{y_{1,2}}{U_\tau} \omega \quad . \end{aligned} \tag{3-47}$$

The non-dimensional variables u^{**} , v^{**} , k^{**} and ω^{**} are assumed to be functions of $\eta = \frac{y}{y_{1,2}}$ only. At the zero-shear-stress point, the non-dimensional k^{**} and ω^{**} are assumed to be constants. The model equations for k and ω at the zero-shear-stress point then become

$$2\left(u_1^{**} + \frac{U_e}{U_\tau}\right) \frac{y_{1,2}}{U_\tau} \frac{dU_\tau}{dx} + C_k \frac{y_{1,2}}{R} \left(u_1^{**} + \frac{U_e}{U_\tau}\right) \frac{u_1^{**}}{\omega_1^{**}} = -\beta^* h_1 \omega_1^{**} \quad , \text{ and} \tag{3-48.a}$$

$$\left(u_1^{**} + \frac{U_e}{U_\tau}\right) \left(\frac{y_{1,2}}{U_\tau} \frac{dU_\tau}{dx} - \frac{dy_{1,2}}{dx}\right) = -\beta^* h_1 \omega_1^{**} \quad . \tag{3-48.b}$$

The subscript 1 refers to values at the zero-shear-stress point and the derivation of the above equations, as well as all other equations in this section is given in Appendix E.

Assuming that the curvature term (term proportional to C_k) in equation (3-48.a) is much smaller than the convection term (the first term on left hand side), then the ratio β^* becomes

$$\frac{\beta}{\beta^*} = \frac{1}{2} \left(1 - \frac{\frac{dy_{1,2}}{dx}}{\frac{y_{1,2}}{U_\tau} \frac{dU_\tau}{dx}} \right). \quad (3-49)$$

At the maximum-shear-stress point, k^{**} is assumed to be quadratic and ω^{**} a constant, that is

$$k^{**} = K\eta^2 + K_1\eta + K_0, \text{ and} \quad (3-50.a)$$

$$\omega^{**} = W, \quad (3-50.b)$$

where K , K_1 , K_0 , and W are constants. Substituting into the k and ω equations of the KW model, (3-35.a) and (3-35.b), and simplifying, one gets

$$2 \left(u_z^{**} + \frac{U_\tau}{U_\tau} \right) \frac{y_{1,2}}{U_\tau} \frac{dU_\tau}{dx} + C_k \frac{y_{1,2}}{R} \left(u_z^{**} + \frac{U_\tau}{U_\tau} \right) \frac{u_z^{**}}{W} = 2\sigma^* h_z \frac{K}{W} + \frac{h_z}{W} \left[u_z^{**} - \frac{y_{1,2}}{h_z R} \left(u_z^{**} + \frac{U_\tau}{U_\tau} \right) \right] - \beta^* h_z W. \quad (3-51.a)$$

$$\left(u_z^{**} + \frac{U_\tau}{U_\tau} \right) \left(\frac{y_{1,2}}{U_\tau} \frac{dU_\tau}{dx} - \frac{dy_{1,2}}{dx} \right) = \alpha \frac{h_z}{W} \left[u_z^{**} - \frac{y_{1,2}}{h_z R} \left(u_z^{**} + \frac{U_\tau}{U_\tau} \right) \right]^2 - \beta h_z W. \quad (3-51.b)$$

The subscript 2 refers to values at the maximum-shear-stress point. By substituting equation (E-9) in Appendix E which is derived from the Bousinnesque definition of the eddy viscosity and simplifying, the constants β^* and α are found from

$$\beta^* = \left(\frac{\tau}{k} \right)_z \left[1 - 2 \left\{ \frac{\sigma^* \left(\frac{K}{W} \right) \left(u_z^{**} - \frac{U_c}{U_\tau} \right) \frac{y_{1,z}}{U_\tau} \frac{dU_\tau}{dx}}{P} - \frac{\frac{C_k}{2} \left(u_z^{**} - \frac{U_c}{U_\tau} \right) \frac{u_z^{**}}{W} \frac{y_{1,z}}{R}}{h_z P} \right\} \right] \text{ and (3-52)}$$

$$\alpha = \frac{\beta}{\beta^*} \left[1 - 2 \left\{ \frac{\sigma^* \left(\frac{K}{W} \right)}{P} - \frac{\frac{C_k}{2} \left(u_z^{**} - \frac{U_c}{U_\tau} \right) \frac{u_z^{**}}{W} \frac{y_{1,z}}{R}}{h_z P} \right\} \right] \text{ (3-53)}$$

$$\text{where } P = \left(\frac{\tau}{k} \right)_z \left[u_z^{**} - \frac{y_{1,z}}{hR} \left(u_z^{**} - \frac{U_c}{U_\tau} \right) \right] .$$

In the log-law region Wilcox (1993) shows that

$$\kappa^2 = \frac{\sqrt{\beta^*}}{\sigma} \left(\frac{\beta}{\beta^*} - \alpha \right) \text{ (3-54)}$$

Collecting equations (3-49), (3-52), (3-53) and (3-54), the model constants can be calculated as follows:

$$\frac{\beta}{\beta^*} = \frac{1}{2} (1 - r) \text{ , (3-55.a)}$$

$$\beta^* = \left(\frac{\tau}{k} \right)_z \left[1 - 2(t - q - s) \right] \text{ , (3-55.b)}$$

$$\alpha = \frac{\beta}{\beta^*} \left[1 + 2(t - q) \right] \text{ , and (3-55.c)}$$

$$\sigma = \frac{\sqrt{\beta^*}}{\kappa} \left(\frac{\beta}{\beta^*} - \alpha \right) \text{ , (3-55.d)}$$

$$\text{where } r = \frac{\frac{dy_{1,z}}{dx}}{\frac{y_{1,z}}{U_\tau} \frac{dU_\tau}{dx}} \text{ , (3-55.e)}$$

$$s = \frac{\left(u_z^{**} + \frac{U_e}{U_\tau}\right) \frac{y_1}{U_\tau} \frac{dU_\tau}{dx}}{\left(\frac{\tau}{k}\right)_z \left[\left(hu^{**}\right)_z - \frac{y_1}{R} \left(u_z^{**} + \frac{U_e}{U_\tau}\right) \right]} . \quad (3-55.f)$$

$$t = \frac{\sigma \left(\frac{K}{W}\right)}{\left(\frac{\tau}{k}\right)_z \left[\left(hu^{**}\right)_z - \frac{y_1}{R} \left(u_z^{**} + \frac{U_e}{U_\tau}\right) \right]} . \text{ and} \quad (3-55.g)$$

$$q = \frac{\frac{C_k}{2} \left(u_z^{**} + \frac{U_e}{U_\tau}\right) \frac{y_1}{R} \frac{u_z^{**}}{W}}{\left(\frac{\tau}{k}\right)_z \left[\left(hu^{**}\right)_z - \frac{y_1}{R} \left(u_z^{**} + \frac{U_e}{U_\tau}\right) \right]} . \quad (3-55.h)$$

The non-dimensional parameters r, s, t and q can be interpreted as follows:

- r indicates conservation of momentum because it is the ratio of the rate of growth of wall jets to the rate of decay of maximum excess velocity. If the wall friction effect is small, then r should be constant for all wall jets. This is the case as shown in Table 3-6.

- s is the ratio of the convection term to the production term. For each tuning flow, experimental data are available to calculate s as listed in Table 3-6.

- t is the ratio of the diffusion term to the production term. It contains the term K/W from equations (3-50.a) and (3-50.b), which cannot be determined from experimental data. Therefore, t has to be determined by trial and error.

- q is the ratio of the curvature term to the production term. It is proportional to C_k which is the "extra" curvature correction factor in equation (3-35.a). This term also contains W from equation (3-50.b) which cannot be determined from experimental data, and has to be determined by trial and error.

Since there is no relation involving σ^* and C_k in equations (3-55), these constants have to be chosen arbitrarily. Therefore, 4 parameters, σ^* , C_k , t and q , have to be determined by trial and error; 2 parameters, r and s , are calculated from experimental data; and the rest of the model constants are calculated from equations (3-55). σ^* , C_k , t and q were chosen to match the predictions of the wall jet growth rate, dy/dx , and the wall-normal profiles of u , k and τ for each tuning flow. Plane wall jets were considered first since the curvature parameters, q and C_k , can be omitted. Once the values of σ^* and t were determined, then curved wall jets were computed to find q and C_k .

Parameters	FWJ	F0261	GN4	F0263	GUIT2
$\frac{y_{1,2}}{R}$	0	0	0	0.104	0.277
$\frac{U_c}{U_r}$	0	0.605	3	0	0
$\frac{U_c}{U_m}$	0	0.377	0.75	0	0
u_2^{**}	0.64	0.64	n/a	0.75	0.75
$(hu^{**})_2$	-0.675	-0.675	n/a	-0.72	-0.94
$\left(\frac{\tau}{k}\right)_2$	-0.31	-0.32	n/a	-0.29	-0.35
$\frac{dy_{1,2}}{dx}$	0.073	0.044	0.0305	0.156	0.276
$\frac{y_{1,2}}{U_r} \frac{dU_r}{dx}$	-0.0365	-0.198	-0.015	-0.0835	-0.145
r	-2	-2.222	-2.033	-1.868	-1.903
s	-0.112	-0.114	n/a	-0.271	-0.271

Table 3-6 Experimental values of r and s parameters for tuning flows.

In order to simplify the calculations of r and s , approximate values of experimental data were used as shown in Table 3-7. The approximate growth rate in Table 3-7 was obtained from the correlation of experimental data by Launder and Rodi (1981) which is given by

$$\frac{dy_{1,2}}{dx} = 0.073 + 0.8 \frac{y_{1,2}}{R} - 0.3 \left(\frac{y_{1,2}}{R} \right)^2 - 0.0974 \frac{U_c}{U_m} + 0.0543 \left(\frac{U_c}{U_m} \right)^2, \quad (3-56)$$

	FWJ	F0261	GN4	F0263	GUIT2
$\frac{y_{1,2}}{R}$	0	0	0	0.104	0.277
$\frac{U_e}{U_\tau}$	0	0.605	3	0	0
$\frac{U_e}{U_m}$	0	0.377	0.75	0	0
u_2^{**}	0.7	0.7	0.7	0.7	0.7
$(hu^{**})_z$	-0.75	-0.75	-0.75	-0.75	-0.75
$\left(\frac{\tau}{k}\right)_z$	-0.32	-0.32	-0.32	-0.32	-0.32
$\frac{dy_{1,2}}{dx}$	0.073	0.044	0.0305	0.153	0.272
$\frac{y_{1,2}}{U_\tau} \frac{dU_\tau}{dx}$	-0.035	-0.021	-0.015	-0.073	-0.129
r	-2.1	-2.1	-2.1	-2.1	-2.1
s	-0.101	-0.114	-0.224	-0.194	-0.3

Table 3-7 Approximate values of r and s parameters for tuning flows.

3.8.2 Predicted Results of the Modified k- ω Model

This section presents the results calculated for the tuning flows, using the modified k- ω model, henceforth referred to as the KW1 model.

- Plane Wall Jets.

Plane Wall Jet in Still Air As mentioned in the previous section, two

parameters, σ^* and t , had to be determined by trial and error. It turned out

that an effective strategy for selecting σ^* and t was to choose σ^* so that the width of the wall-normal profiles for u , k and τ matched the corresponding experimental profiles, then choose t to match the calculated dy^+ / dx to the experimental data. For this particular flow, the effect of σ^* on dy^+ / dx was significant which made the task of choosing σ^* and t complicated and time consuming. After much trial and error, a satisfactory set of σ^* and t was found. This set is listed in Table 3-8, as well as the resulting model constants and calculated results, in comparison with the unmodified k - ω model and the experimental data. Further comparison is shown in Figure 3-25 where the assumption regarding the functional form of k and ω at the matching points could be verified. The first assumption is that at the maximum-shear-stress point

$$\frac{\partial k}{\partial y^+} = \frac{\partial \omega}{\partial y^+} = 0 .$$

This assumption is seen to be satisfied in Figure 3-25 (d) and (e). The next assumption is that ω and k are constant at the zero-shear-stress point. As shown in Figure 3-25 (g), this assumption is the least satisfactory. $\tau = 0$ occurs at $y/y_{\tau}^+ = 0.1$ where k has already begun to rise from a constant plateau near the wall, and ω has not yet fallen to the first plateau where it attains a minimum value. Nevertheless, judging from the vastly improved results relative to the original k - ω model, in comparison

with the experimental data, the modification of the k- ω model for the plane wall jet in still air case is a success.

	KW	KW1	Experiments
Model constants.			
t		-0.43	
σ^*	0.5	0.36	
σ	0.5	0.268	
β^*	0.09	0.114	
β	0.075	0.177	
α	0.556	1.417	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.115	0.073	0.073
$\frac{y_m}{y_{1,2}}$	0.065	0.1	0.15
$\frac{y_{\tau_0}}{y_m}$	1	1	0.76

Table 3-8 Comparison of the KW and KW1 models for flow FWJ.

Self-Similar Plane Wall Jet with $U_\infty/U_m=0.38$ The same strategy was used to determine σ^* and t , that is, adjusting σ^* to match profile width of u , k and τ , then adjusting t to match $dy_{1,2}/dx$. The sensitivity of $dy_{1,2}/dx$ to σ^* is much less for this case and finding a satisfactory set of σ^* and t was

Figure 3-25 Comparison of the predictions of modified vs original $k-\omega$ model for plane wall jets in still air.

considerably simpler than for the flow FWJ. The final values of the model constants, as well as comparison with the original k- ω model and experimental data are shown in Table 3-9 and Figure 3-26.

	KW	KW1	Experiments
Model constants.			
ι		-0.78	
σ^*	0.5	0.7	
σ	0.5	0.477	
β^*	0.09	0.110	
β	0.075	0.170	
α	0.556	1.308	
Calculated results			
$\frac{dy_{12}}{dx}$	0.045	0.044	0.044
$\frac{y_m}{y_{12}}$	0.155	0.17	0.211
$\frac{y_{z_o}}{y_m}$	1	1	n a

Table 3-9 Comparison of the KW and KW1 models for flow F0261.

The general conclusions regarding the validity of the assumptions of the tuning model are similar to the case of flow FWJ. An additional observation for this case is the fact that both the original and the modified

Figure 3-26 Comparison of the predictions of modified vs original $k-\omega$ model for self-similar plane wall jet with $U_e/U_m = 0.38$.

k- ω models produce fairly similar results, although some of the model constants are quite different. This suggests that there may be more than one way to modify the model constants to obtain the same results.

Self-Similar Plane Wall Jet with $U_c/U_m \approx 0.77$ The final set of σ^* and t was again determined by trial and error. The resulting model constants are listed in Table 3-10. Comparison between the calculated results of the modified k- ω model, the original k- ω model, and the experimental data are shown in Table 3-10 and Figure 3-27.

	KW	KW1	Experiments
Model constants.			
t		-0.098	
σ^*	0.5	1	
σ	0.5	0.647	
β^*	0.09	0.128	
β	0.075	0.199	
α	0.556	1.246	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.024	0.307	0.305
$\frac{y_m}{y_{1,2}}$	0.049	0.46	0.5
$\frac{y_{z_0}}{y_m}$	1	1	n/a

Table 3-10 Comparison of the KW and KW1 models for flow GN4.

Figure 3-27 Comparison of the predictions of modified vs original k - ω model for plane self-similar wall jets with $U_e/U_m = 0.77$.

The same general conclusions as with the two previous flows can be reached for this flow regarding the validity of the tuning model assumptions, and the success of producing calculated results that are comparable to experimental data

- Curved Wall Jets.

Curved Wall Jet over a Log-Spiral with $x/R=2.3$ For curved wall jets, two more parameters, q and C_k , must be found by trial and error in addition to σ^* and t . With four parameters to optimize at the same time, there was no straight forward strategy to match the calculated results with the experimental data. It was decided to simplify the task by holding σ^* and t the same as the for the FWJ flow. One possible justification was that both F0263 and FWJ flows were wall jets in still air. Even with only two parameters, q and C_k , the choice of each one was still admittedly arbitrary. After much trial and error, it was found that q could simply be set to 0 and C_k adjusted to match dy/dx . The final set of parameter values, as well as the resulting model constants are listed in Table 3-11. The calculated results of the modified $k-\omega$ model are shown in Figure 3-28, along with results from the original $k-\omega$ model and the experimental data.

Figure 3-28 Comparison of the predictions of modified (KW1) vs original (KW) $k-\omega$ model for curved wall jet over a log-spiral with $x/R=2/3$.

The most controversial assumption is again that k and ω are constant at $\tau = 0$. As shown in Figure 3-28(g), this assumption is not quite true, but the calculated results are still reasonable.

	KW	KW1	Experiments
Model constants.			
q		0	
C_k	4.5	1.05	
r		-0.043	
σ^*	0.5	0.36	
σ	0.5	0.289	
β^*	0.09	0.133	
β	0.075	0.207	
α	0.556	1.417	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.24	0.155	0.155
$\frac{y_m}{y_{1,2}}$	0.042	0.075	0.159
$\frac{y_{\tau_0}}{y_m}$	0.93	0.77	n/a

Table 3-11 Comparison of the KW and KW1 models for flow F0263.

Curved Wall Jet over a Log-Spiral with $\kappa R=1$ Fortunately, the same set

of q , C_k , σ^* and t for flow F0263, worked well for this flow too.

Table 3-12 lists the resulting model constants, as well as the flow

development parameters in comparison with the original $k-\epsilon$ model and

the experimental data. Wall-normal profiles are shown in Figure 3-29.

	KW	KW1	Experiments
Model constants.			
q		0	
C_k	4.5	1.05	
t		-0.043	
σ^*	0.5	0.36	
σ	0.5	0.312	
β^*	0.09	0.155	
β	0.075	0.240	
α	0.556	1.417	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.378	0.269	0.271
$\frac{y_m}{y_{1,2}}$	0.037	0.12	0.21
$\frac{y_{c_0}}{y_m}$	0.8	0.4	n/a

Table 3-12 Comparison of the KW and KW1 models for flow GUIT2.

Figure 3-29 Comparison of the predictions of modified (KW1) vs original (KW) $k-\omega$ model for curved wall jet over a log-spiral with $x/R=1$.

Again, the assumption of constant k and ω at $t=0$, did not hold.

Furthermore, the predicted k and τ are well above the experimental data, but still well below those of the original k - ω model.

3.8.3 Discussion

Modification of the k - ω model to match experimental data on the self-similar wall jets was generally a success. With appropriate model constants, the k - ω model can reproduce the correct wall jet growth rate, as well as reasonable wall-normal profiles. However, appropriate values of the model constants for each tuning flow are different, as shown in Table 3-13. Some of these tuning flows, particularly the plane wall jet in still air, are very sensitive to changes in the values of the model constants. Therefore, a single set of constants that work reasonably well for all flows is unlikely.

It is possible to correlate the findings of the previous section so that the parameters, r , s , t , q , C_k and σ^* can be evaluated for any self-similar wall jets. This can be done by use of the following equations, which are best fit curves to the results.

	KW	KW1				
		FWJ	F0261	GN4	F0263	GUIT2
$\frac{U_c}{U_m}$		0	0.377	0.75	0	0
$\frac{y_{1,2}}{R}$		0	0	0	0.104	0.277
r		-2.1	-2.1	-2.1	-2.1	-2.1
s		-0.101	-0.114	-0.224	-0.194	-0.299
t		-0.043	-0.078	-0.098	-0.043	-0.043
q		0	0	0	0	0
C_k	4.5	1.05	1.05	1.05	1.05	1.05
σ^*	0.5	0.36	0.7	1	0.36	0.36
σ	0.5	0.268	0.477	0.647	0.289	0.312
β^*	0.09	0.114	0.11	0.128	0.133	0.155
β	0.075	0.177	0.17	0.199	0.207	0.24
α	0.556	1.417	1.308	1.246	1.417	1.417

Table 3-13 Model constants of the original k - ω (KW) model and the modified k - ω (KW1) model for each tuning flow.

$$r = -2.1 \quad , \quad (3-57.a)$$

$$s = \frac{\left(u_z^{**} - \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx}}{\left(\frac{\tau}{k}\right)_z \left[\left(hu^{**}\right)_z - \frac{y_{1,z}}{R} \left(u_z^{**} - \frac{U_c}{U_r}\right) \right]} \quad , \quad (3-57.b)$$

$$t = -0.043 - 0.085 \frac{U_c}{U_m} \quad , \quad (3-57.c)$$

$$q = 0 \quad , \quad (3-57.d)$$

$$C_k = 1.05 \quad , \text{ and} \quad (3-57.e)$$

$$\sigma^* = 0.36 - 0.853 \frac{U_c}{U_m} \quad , \quad (3-57.f)$$

$$\text{where } \frac{U_m}{U_c} = 1 - \frac{U_r}{U_c} \quad , \quad (3-57.g)$$

$$\frac{dy_{1,z}}{dx} = 0.073 - 0.8 \frac{y_{1,z}}{R} - 0.3 \left(\frac{y_{1,z}}{R} \right)^2 - 0.0974 \frac{U_c}{U_m} + 0.0543 \left(\frac{U_c}{U_m} \right)^2 \quad , \quad (3-57.h)$$

$$u_z^{**} = 0.7 \quad , \quad (3-57.i)$$

$$\left(hu^{**}\right)_z = -0.75 \quad , \text{ and} \quad (3-57.j)$$

$$\left(\frac{\tau}{k}\right)_z = -0.32 \quad , \quad (3-57.k)$$

Since a self-similar plane wall jet is characterized by a constant U_c/U_m , and a self-similar curved wall jet by a constant $y_{1,z}/R$, the above equations can be used to calculate r , s , t , q , C_k and σ^* for any self-similar wall jet, then the model constants can be calculated by equations (3-55). Using the above correlations, a number of self-similar plane wall jets were calculated, and the results were plotted against the correlation of experimental data given by Launder and Rodi (1981) in Figure 3-30. Similarly, self-similar curved wall jet

results are shown in Figure 3-31 in comparison with a correlation of experimental data by Guitton and Newman (1977). These results confirm that the modification works well for all self-similar wall jets.

A preliminary attempt was made to use the above correlations to calculate non-self similar wall jets. The first problem with non-self similar wall jets is how to obtain the starting profiles. Experimental data are not detailed enough to obtain accurate values for ω . Approximation using the Boussinesq assumption created too large initial transients that were felt over most of the downstream region of interest. The approach taken here was to obtain self-similar wall jet profiles for the experimental value of U_e/U_m for plane wall jets, or y_w/R for curved wall jets, at the starting position using correlations described by equations (3-57). These profiles were then used as starting profiles for non-self similar wall jet calculations. The turbulence model constants for non-self similar wall jets were calculated by assuming local equilibrium at all points along the flow and adjusting the model constants continuously according to the local value of U_e/U_m , or y_w/R , as the flow moved downstream. The attempt was not successful, as shown by a sample results in Figure 3-32 for a plane wall jet in uniform moving stream. It is not possible to identify whether the problem was caused by initial transients due to incorrect starting profiles, or by the deficiency of the model itself. No valid modification for non-self similar wall jets is yet available.

Figure 3-30 Calculated growth rate of self-similar plane wall jets by the modified $k-\omega$ (KW1) model.

Figure 3-31 Calculated growth rate of self-similar curved wall jets by the modified $k-\omega$ (KW1) model.

Figure 3-32 Growth rate of a plane wall jet in uniform moving stream. comparison between experimental data and calculation using the modified $k-\omega$ (KW1) model.

3.9 Modifying the Multiscale Model

A similar approach to that taken for the $k-\omega$ model, was used to re-evaluate the model constants of the multiscale model. As with the $k-\omega$ model, the modifications were based on experimental data on self-similar wall jets.

3.9.1 Model Equations at the Matching Points

The self-similar non-dimensional variables for the multiscale model are:

$$\begin{aligned}
 u^{**} &= \frac{u - U_c}{U_m - U_c} = \frac{u - U_c}{U_\tau} \quad , \\
 v^{**} &= \frac{v}{U_\tau} \quad , \\
 k^{**} &= \frac{k}{U_\tau^2} \quad , \\
 \omega^{**} &= \frac{U_m}{U_\tau} \omega \quad , \\
 \tau^{**} &= \frac{\tau}{U_\tau^2} = \frac{-\overline{u'v'}}{U_\tau^2} \quad , \\
 \sigma_u^{**} &= \frac{\sigma_u}{U_\tau^2} = \frac{\overline{u'^2} - \frac{2}{3}k}{U_\tau^2} \quad , \\
 \sigma_v^{**} &= \frac{\sigma_v}{U_\tau^2} = \frac{\overline{v'^2} - \frac{2}{3}k}{U_\tau^2} \quad , \text{ and} \\
 (k - e)^{**} &= \frac{k - e}{U_\tau^2} \quad .
 \end{aligned} \tag{3-58}$$

These self-similar variables, u^{**} , v^{**} , k^{**} , etc., are assumed to be functions of $\eta = \frac{y}{y_{1,z}}$

only. When the self-similar variables are substituted into the multiscale model equations

(3-38.a) through (3-38.f), the model equations become:

- the turbulent kinetic energy.

$$\begin{aligned} \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right) \left(2 \frac{y_{1,z}}{U_\tau} \frac{dU_\tau}{dx} k^{**} - \frac{dy_{1,z}}{dx} \eta k^{**}\right) - h v^{**} k^{**} \\ = \sigma \frac{\hat{c}}{\hat{c}\eta} \left(h \frac{k^{**}}{\omega^{**}} k^{**}\right) + \tau^{**} \left(h u^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right)\right) - \beta^* h \omega^{**} k^{**}. \end{aligned} \quad (3-59.a)$$

- the specific dissipation rate.

$$\begin{aligned} \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right) \left[\left(\frac{y_{1,z}}{U_\tau} \frac{dU_\tau}{dx} - \frac{dy_{1,z}}{dx}\right) \omega^{**} - \frac{dy_{1,z}}{dx} \eta \omega^{**}\right] + h v^{**} \omega^{**} = \sigma \frac{\hat{c}}{\hat{c}\eta} \left(h \frac{k^{**}}{\omega^{**}} \omega^{**}\right) \\ - \alpha \frac{\omega^{**}}{k^{**}} \tau^{**} \left(h u^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right)\right) - \beta \omega^{**} \left(h \omega^{**} - \left[h u^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right)\right]\right). \end{aligned} \quad (3-59.b)$$

- the Reynolds shear stress.

$$\begin{aligned} \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right) \left(2 \frac{y_{1,z}}{U_\tau} \frac{dU_\tau}{dx} \tau^{**} - \frac{dy_{1,z}}{dx} \eta \tau^{**}\right) - h v^{**} \tau^{**} \\ = \left(C_{xx} \sigma_\tau^{**} + C_{xy} \sigma_y^{**} + C_{xk}\right) \left(h u^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right)\right) \\ + C_\epsilon \frac{y_{1,z}}{R} (\sigma_y^{**} - \sigma_\tau^{**}) \left(u^{**} + \frac{U_\epsilon}{U_\tau}\right) - C_1 \beta^* h \omega^{**} \tau^{**}. \end{aligned} \quad (3-59.c)$$

- the streamwise stress deviator,

$$\begin{aligned} \left(u^{**} - \frac{U_c}{U_r}\right) \left(2 \frac{V_{1,z}}{U_r} \frac{dU_r}{dx} \sigma_r^{**} - \frac{dV_{1,z}}{dx} \eta \sigma_r^{**}\right) - h v^{**} \sigma_r^{**} = C_r \tau^{**} \left(h u^{**} - \frac{V_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r}\right)\right) \\ - 2 C_r \frac{V_{1,z}}{R} \tau^{**} \left(u^{**} - \frac{U_c}{U_r}\right) - C_1 \beta^* h \omega^{**} \sigma_r^{**} . \end{aligned} \quad (3-59.d)$$

- the normal stress deviator,

$$\begin{aligned} \left(u^{**} - \frac{U_c}{U_r}\right) \left(2 \frac{V_{1,z}}{U_r} \frac{dU_r}{dx} \sigma_r^{**} - \frac{dV_{1,z}}{dx} \eta \sigma_r^{**}\right) - h v^{**} \sigma_r^{**} = C_r \tau^{**} \left(h u^{**} - \frac{V_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r}\right)\right) \\ - 2 C_r \frac{V_{1,z}}{R} \tau^{**} \left(u^{**} - \frac{U_c}{U_r}\right) - C_1 \beta^* h \omega^{**} \sigma_r^{**} , \text{ and} \end{aligned} \quad (3-59.e)$$

- the upper partition energy,

$$\begin{aligned} \left(u^{**} - \frac{U_c}{U_r}\right) \left(2 \frac{V_{1,z}}{U_r} \frac{dU_r}{dx} (k - e)^{**} - \frac{dV_{1,z}}{dx} \eta (k - e)^{**}\right) - h v^{**} (k - e)^{**} \\ = C_e \tau^{**} \left(h u^{**} - \frac{V_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r}\right)\right) - \beta^* h \omega^{**} k^{**} \left(\frac{k - e}{k}\right)^{\frac{1}{2}} . \end{aligned} \quad (3-59.f)$$

$$\text{where} \quad C_1 = 1 - C_0 \left(\frac{k - e}{k}\right)^{\frac{1}{2}} , \quad (3-59.g)$$

$$C_{rr} = -\hat{\beta} , \quad (3-59.h)$$

$$C_{11} = 1 - \hat{\alpha} , \quad (3-59.i)$$

$$C_{33} = \frac{2}{3} \left(1 - \hat{\alpha} - \hat{\beta} + \frac{3}{4} \hat{\gamma}\right) , \quad (3-59.j)$$

$$C_r = \frac{2}{3} (2 - 2\hat{\alpha} + \hat{\beta}) , \quad (3-59.k)$$

$$C_v = -\frac{2}{3} (1 - \hat{\alpha} + 2\hat{\beta}) , \quad (3-59.l)$$

$$C_e = 2 - \hat{\alpha} + \hat{\beta} , \text{ and} \quad (3-59.m)$$

$$C_\epsilon = 1 - \hat{\alpha} - \hat{\beta} . \quad (3-59.n)$$

Similar to the approach outlined in Section 3.8.1, the following assumptions are made:

- In the near wall region, convection terms are small, $k u_\tau^2$ is constant, diffusion of $(k-e)$ is small and the u profile conforms to the universal log law, or

$$\frac{\partial u}{\partial y} = \frac{u_\tau}{k y} .$$

- At the $\tau = 0$ position, k^{**} and ω^{**} are constant and u^{**} is small.
- At the maximum-shear-stress position, ω^{**} is constant and all wall-normal derivatives of turbulence variables are zero, that is

$$\frac{\partial k}{\partial y} = \frac{\partial \omega}{\partial y} = \frac{\partial (k-e)}{\partial y} = \frac{\partial \tau}{\partial y} = \frac{\partial \sigma_\epsilon}{\partial y} = \frac{\partial \sigma_\omega}{\partial y} = 0 .$$

Under these assumptions, the model constants can be evaluated as follows.

$$\hat{\alpha} = \frac{\left\{ - \left[\frac{y_{1z}}{U_r} \frac{dU_r}{dx} \left(u_z^{**} + \frac{U_c}{U_r} \right) + \frac{C_1}{2} (\beta^* h \omega^{**})_z \right] \left| \begin{array}{l} \frac{2 \left(\frac{\sigma_r}{k} \right)_z + \left(\frac{\sigma_i}{k} \right)_z}{\left(\frac{\tau}{k} \right)_z} (hu^{**})_z \\ \frac{\left(\frac{\sigma_r}{k} \right)_z - 2 \left(\frac{\sigma_i}{k} \right)_z}{\left(\frac{\tau}{k} \right)_z} - \frac{y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \end{array} \right| \right.}{\left. - \left[(hu^{**})_z + \frac{2y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right] \left| \left[(hu^{**})_z - \frac{y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right] \right|} \right\}}{\left((hu^{**})_z - \left[\frac{y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right] \right)^2} \quad (3-60.a)$$

$$\hat{\beta} = \frac{\left\{ - \left[\frac{y_{1z}}{U_r} \frac{dU_r}{dx} \left(u_z^{**} - \frac{U_c}{U_r} \right) + \frac{C_1}{2} (\beta^* h \omega^{**})_z \right] \left| \begin{array}{l} \frac{\left(\frac{\sigma_r}{k} \right)_z - 2 \left(\frac{\sigma_i}{k} \right)_z}{\left(\frac{\tau}{k} \right)_z} (hu^{**})_z \\ - \frac{2 \left(\frac{\sigma_r}{k} \right)_z - \left(\frac{\sigma_i}{k} \right)_z}{\left(\frac{\tau}{k} \right)_z} - \frac{y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \end{array} \right| \right.}{\left. - \frac{y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \left| \left[(hu^{**})_z - \frac{y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right] \right|} \right\}}{\left((hu^{**})_z - \left[\frac{y_{1z}}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right] \right)^2} \quad (3-60.b)$$

$$\hat{\tau} = 2 \left| \frac{\hat{\beta} \left\{ \left(\frac{\sigma_i}{k} \right)_z - \frac{2}{3} \right\} - (1 - \hat{\alpha}) \left\{ \left(\frac{\sigma_i}{k} \right)_z - \frac{2}{3} \right\}}{\left\{ 2 \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \left(u_z^{**} + \frac{U_e}{U_r} \right) + C_1 (\beta^* h \omega^{**})_z \right\} \left\{ \left(\frac{\tau}{k} \right)_z - (2 - \hat{\alpha} + \hat{\beta}) \frac{y_{1,z}}{R} \left(u_z^{**} + \frac{U_e}{U_r} \right) \right\} \left\{ \left(\frac{\sigma_i}{k} \right)_z - \left(\frac{\sigma_r}{k} \right)_z \right\}} - \frac{(hu^{**})_z - \frac{y_{1,z}}{R} \left(u_z^{**} + \frac{U_e}{U_r} \right)}{\left(hu^{**} \right)_z - \frac{y_{1,z}}{R} \left(u_z^{**} + \frac{U_e}{U_r} \right)} \right| . \quad (3-60.c)$$

$$\beta^* = \frac{C_{xx}C_r - C_{xy}C_x - C_{yk}(1 - C_xC_e)}{(1 - C_0C_e)^2} . \quad (3-60.d)$$

$$\beta = \frac{1 - \frac{\frac{dy_{1,z}}{dx}}{\frac{y_{1,z}}{U_r} \frac{dU_r}{dx}}}{\frac{\frac{2}{\beta^*} - \frac{\frac{y_{1,z}}{R}}{\frac{y_{1,z}}{U_r} \frac{dU_r}{dx}}}} . \quad (3-60.e)$$

$$\alpha = \frac{\left(u_z^{**} + \frac{U_e}{U_r} \right) \left(\frac{y_{1,z}}{U_r} \frac{dU_r}{dx} - \frac{dy_{1,z}}{dx} \right) - \left(\frac{\beta}{\beta^*} \right) (\beta^* h \omega^{**})_z - \beta \xi \left| (hu^{**})_z + \left(u_z^{**} + \frac{U_e}{U_r} \right) \frac{y_{1,z}}{R} \right|}{\left(\frac{\tau}{k} \right)_z \left| (hu^{**})_z - \left(u_z^{**} + \frac{U_e}{U_r} \right) \frac{y_{1,z}}{R} \right|}} . \quad (3-60.f)$$

$$\sigma = \frac{\sqrt{\beta^*}}{\kappa^2} \left(\frac{\beta}{\beta^*} - \alpha + \frac{\xi \beta}{\sqrt{\beta^*}} \right) . \quad (3-60.g)$$

$$\text{where} \quad C_1 = 1 + C_0 \left(\frac{k-e}{k} \right)_z^{\frac{3}{2}} . \quad (3-60.h)$$

$$(\beta^* h \omega^{**})_z = -2 \frac{y_{1,z}}{U_x} \frac{dU_x}{dx} \left(u_z^{**} + \frac{U_e}{U_x} \right) \frac{\left(\frac{\sigma_z^{**} \sigma_\psi}{k} \right)_z - \frac{2}{3} \left(\frac{k-e}{k} \right)_z}{C_1 \left(\frac{\sigma_z^{**} \sigma_\psi}{k} \right)_z - \frac{2}{3} \left(\frac{k-e}{k} \right)_z^{\frac{3}{2}}} . \quad (3-60.i)$$

The subscript 2 refers to the maximum τ point and the derivation of these equations is given in Appendix F.

A number of experimental data from the tuning flows that are needed in equations (3-60), are listed in Table 3-14. To simplify the calculations of these quantities for an arbitrary self-similar wall jet, approximate correlations that fit the experimental data are given in equation (3-61). The results of these approximate relations for the tuning flows are listed in Table 3-15. The remaining parameters needed to calculate the model constants, σ^* , C_0 and $((k-e)/k)_2$, have to be obtained by trial and error while matching calculated results with experimental data.

$$\left(\frac{\sigma_e}{k}\right)_2 = 0.2 - 0.45 \left(1 - \frac{1}{1 + \frac{y_{1,2}}{R}}\right), \quad (3-61.a)$$

$$\frac{dy_{1,2}}{dx} = 0.073 + 0.8 \frac{y_{1,2}}{R} - 0.3 \left(\frac{y_{1,2}}{R}\right)^2 - 0.0974 \frac{U_e}{U_m} + 0.0543 \left(\frac{U_e}{U_m}\right)^2, \quad (3-61.b)$$

$$\frac{y_{1,2}}{U_e} \frac{dU_e}{dx} = \frac{\frac{dy_{1,2}}{dx}}{-2.1}. \quad (3-61.c)$$

Parameters	Tuning Flows				
	FWJ	F0261	GN4	F0263	GUIT2
$\frac{y_{1,2}}{R}$	0	0	0	0.104	0.277
$\frac{\bar{U}_e}{\bar{U}_x}$	0	0.605	3	0	0
$\frac{\bar{U}_e}{\bar{U}_m}$	0	0.377	0.75	0	0
u_2^{**}	0.64	0.64	n/a	0.75	0.75
$(h u^{**})_2$	-0.675	-0.675	n/a	-0.72	-0.94
$\left(\frac{\tau}{k}\right)_2$	-0.31	-0.32	n/a	-0.29	-0.35
$\left(\frac{\sigma_r}{k}\right)_2$	0.25	0.25	n/a	0.04	0.1
$\left(\frac{\sigma_r}{k} - \frac{\sigma_1}{k}\right)_2$	0.05	0.02	n/a	-0.08	0.1
$\frac{dy_{1,2}}{dx}$	0.073	0.044	0.0305	0.156	0.276
$\frac{y_{1,2}}{\bar{U}_x} \frac{d\bar{U}_x}{dx}$	-0.0365	-0.0198	-0.015	-0.0835	-0.145

Table 3-14 Experimental data for tuning flows.

Parameters	Tuning Flows				
	FWJ	F0261	GN4	F0263	GUIT2
$\frac{y_{1,2}}{R}$	0	0	0	0.104	0.277
$\frac{U_c}{U_r}$	0	0.605	3	0	0
$\frac{U_c}{U_m}$	0	0.377	0.75	0	0
u_z^{**}	0.7	0.7	0.7	0.7	0.7
$(h u^{**})_z$	-0.75	-0.75	-0.75	-0.75	-0.75
$\left(\frac{\tau}{k}\right)$	-0.32	-0.32	-0.32	-0.32	-0.32
$\left(\frac{\sigma_r}{k}\right)_z$	0.2	0.2	0.2	0.158	0.102
$\left(\frac{\sigma_r}{k} - \frac{\sigma_i}{k}\right)_z$	0.05	0.05	0.05	0.05	0.05
$\frac{dy_{1,2}}{dx}$	0.073	0.044	0.030	0.153	0.272
$\frac{y_{1,2}}{U_x} \frac{dU_x}{dx}$	-0.035	-0.021	-0.015	-0.073	-0.129

Table 3-15 Approximate data for tuning flows.

3.9.2 Calculated Results Using the Modified Multiscale Model

- Plane Wall Jets

Plane Wall Jet in Still Air With 3 parameters. σ^* , C_0 and

$((k-e)/k)_2$, needing to be chosen, the selection process became very

subjective. It was decided to rely on the results of the modification of the

$k-\omega$ model for the value of σ^* , thus reducing the number of parameters

needing to be chosen to two, C_0 and $((k-e)/k)_2$. For plane wall jets, there

are many combinations of C_0 and $((k-e)/k)_2$ that produce roughly similar

results. The difference between different combinations is more

pronounced in the case of curved wall jets which will be discussed later.

The constant value of $C_0=6$ was chosen as the best overall choice for all of

the tuning flows. The final choice of the parameters and the resulting

model constants for the original multiscale (MS) model and the modified

multiscale (MS1) model, as well as some key results, are listed for the

plane wall jet in still air in Table 3-16. More detailed results are shown in

Figure 3-32. The modified model performs much better than the original

model and produces results comparable to experimental data. The main

problem appears to be the behavior of k and ω near the $\tau=0$ point. Figure

3-32(f) shows that the assumption of constant k and ω near this point is

not completely verified. A similar problem occurs in the case of the

modified k- ω model but does not create large effects on the model prediction.

	MS	MS1	Experiments
Model constants.			
$\left(\frac{k-e}{k}\right)_2$		0.39	
C_0	4	6	
σ^*	0.5	0.36	
σ	0.5	0.319	
β^*	0.09	0.114	
β	0.075	0.177	
α	0.8	1.915	
$\hat{\alpha}$	0.764	0.692	
$\hat{\beta}$	0.109	0.123	
$\hat{\gamma}$	0.25	0.399	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.15	0.073	0.073
$\frac{y_m}{y_{1,2}}$	0.06	0.11	0.15
$\frac{y_{c_0}}{y_m}$	0.9	0.9	0.6

Table 3-16 The multiscale model constants and key results for flow FWJ.

Figure 3-33 Comparison of the predictions of the modified (MS1) and the original (MS) multiscale models with experimental data for plane wall jets in still air.

Self-Similar Plane Wall Jet with $U_c/U_m = 0.38$ Similar to the previous case, the value of σ^* was obtained from the modified k- ω model for the same flow, and C_{ij} was set to 6. $((k-e)/k)_2$ was adjusted to obtain the experimental growth rate. The resulting model constants of the modified multiscale (MS1) model are compared to the original multiscale (MS) model in Table 3-17. The key calculated results of both models are compared with the experimental data in Table 3-17 and detailed results are shown in Figure 3-33. Again the modified model works quite well.

Self-Similar Plane Wall Jet with $U_c/U_m = 0.77$ From the k- ω modification work, $\sigma^* = 1$. With $C_{ij} = 6$, the correct growth rate was obtained with $((k-e)/k)_2 = .66$. The model constants are listed in Table 3-18, as well as some key calculated results. The detailed results are shown in Figure 3-34.

	MS	MS1	Experiments
Model constants.			
$\left(\frac{k-e}{k}\right)_-$		0.418	
C_{ij}	4	6	
σ^*	0.5	0.7	
σ	0.5	0.46	
β^*	0.09	0.115	
β	0.075	0.179	
α	0.8	1.848	
$\hat{\alpha}$	0.764	0.681	
$\hat{\beta}$	0.109	0.128	
$\hat{\gamma}$	0.25	0.414	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.053	0.045	0.044
$\frac{v_m}{v_{1,2}}$	0.145	0.19	0.211
$\frac{v_{\infty}}{v_m}$	0.9	0.9	n/a

Table 3-17 The multiscale model constants and key results for flow F0261.

Figure 3-34 Comparison of the predictions of the modified (MS1) and the original (MS) multiscale models with experimental data for self-similar plane wall jet with $U_e/U_m = 0.38$.

	MS	MS1	Experiments
Model constants.			
$\left(\frac{k-e}{k}\right)_2$		0.66	
C_η	4	6	
σ^*	0.5	1	
σ	0.5	0.698	
β^*	0.09	0.132	
β	0.075	0.204	
α	0.8	1.79	
$\hat{\alpha}$	0.764	0.429	
$\hat{\beta}$	0.109	0.229	
$\hat{\gamma}$	0.25	0.742	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.025	0.0304	0.0305
$\frac{y_m}{y_{1,2}}$	0.53	0.5	0.5
$\frac{y_{z,0}}{y_m}$	0.9	0.9	n/a

Table 3-18 The multiscale model constants and key results for flow GN4.

Figure 3-35 Comparison of the predictions of the modified (MS1) and the original (MS) multiscale models with experimental data for self-similar plane wall jet with $U_c/U_m = 0.77$.

- Curved Wall Jets.

Curved Wall Jet over a Log Spiral with $x/R=2/3$ The model constants that result in the correct growth rate as well as some key results are listed in Table 3-19. More detailed results are shown in Figure 3-35. The velocity profile in Figure 3-35(b) shows some discrepancy in the outer part of the wall jet. The calculated velocity profile falls to zero at a noticeably lower value of y/y_w than the experimental data and the velocity profile shape is somewhat unrealistic. Profiles of the Reynolds stresses show a slight amount of extra stresses above the point of maximum τ , where the results of the original model shows the second peaks. However, the overall results are still adequate.

Curved Wall Jet over a Log Spiral with $x/R=1$ The model constants and key results for this case are listed in Table 3-20. Detailed results are shown in Figure 3-36. The deficiency of the predicted velocity profile and the double-peak feature of the τ profile become more pronounced.

Furthermore, to produce the correct growth rate, $((k-e)/k)_2$ has to be as high as .85. Since $((k-e)/k)_2$ is the ratio of upper partition energy to the total turbulence energy, its maximum value is 1 and 0.85 is quite high. With a lower value of C_0 , the corresponding $((k-e)/k)_2$ would be lower, but the problem of the velocity and shear stress profiles would then worsen. It can be concluded that modifying the multiscale model by changing the

model constants alone, is not entirely satisfactory for curved wall jets. The predicted velocity profile shape, in particular, is poor, although it is much better than that predicted by the original multiscale model.

	MS	MS1	Experiments
Model constants.			
$\left(\frac{k-e}{k}\right)_2$		0.575	
C_{11}	4	6	
σ^*	0.5	0.36	
σ	0.5	0.838	
β^*	0.09	0.127	
β	0.075	0.181	
α	0.8	1.533	
$\hat{\alpha}$	0.764	0.661	
$\hat{\beta}$	0.109	0.049	
$\hat{\gamma}$	0.25	0.4	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.35	0.154	0.155
$\frac{y_m}{y_{1/2}}$	0.028	0.08	0.159
$\frac{y_{c0}}{y_m}$	0.85	0.75	n/a

Table 3-19 The multiscale model constants and key results for flow F0263.

Figure 3-36 Comparison of the predictions of the modified (MS1) and the original (MS) multiscale models with experimental data for curved wall jet over a log-spiral with $x/R=2/3$.

	MS	MS1	Experiments
Model constants.			
$\left(\frac{k-e}{k}\right)_2$		0.85	
C_v	4	6	
σ^*	0.5	0.36	
σ	0.5	1.62	
β^*	0.09	0.17	
β	0.075	0.223	
α	0.8	1.192	
$\bar{\alpha}$	0.764	0.595	
$\hat{\beta}$	0.109	-0.106	
$\hat{\gamma}$	0.25	0.704	
Calculated results			
$\frac{dy_{1,2}}{dx}$	0.71	0.273	0.271
$\frac{y_m}{y_{1,2}}$	0.018	0.075	0.21
$\frac{y_{\tau_0}}{y_m}$	0.8	0.55	n/a

Table 3-20 The multiscale model constants and key results for flow GUIT2.

Figure 3-37 Comparison of the predictions of the modified (MS1) and the original (MS) multiscale models with experimental data for curved wall jet over a log-spiral with $x/R=1$.

3.9.3 Alternative Models for (k-e)/k

In order to determine the cause of the difficulty that the modified multiscale (MS1) model experiences in curved wall jets, the profiles of individual terms of a typical multiscale model equation are shown in Figure 3-56. From equation (3-59.d), the production and dissipation terms are given by

$$Production = C_\tau \tau^{**} \left(hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_\tau} \right) \right) + 2 C_\tau \frac{y_{1,z}}{R} \tau^{**} \left(u^{**} + \frac{U_c}{U_\tau} \right) \quad (3-62.a)$$

$$Dissipation = \left[1 + C_0 \left(\frac{k-e}{k} \right)^{3/2} \right] \beta^* h \omega^{**} \sigma_\tau^{**} \quad (3-62.b)$$

For self-similar wall jets, the production and dissipation are expected to be in balance in most regions of the flow. Figure 3-37 shows noticeable imbalance between these terms in the outer part of the flow, around $1 < y/y_{1,z} < 1.6$. Also shown in Figure 3-37 is the upper partition energy ratio (k-e)/k, which drops to zero near the freestream edge due to the boundary condition that (k-e) = 0 there. Since dissipation is modelled as being proportional to (k-e)/k, this may be the reason that causes the imbalance between production and dissipation. On physical grounds, (k-e)/k may approach any finite value between 0 and 1 at the freestream edge. Therefore, three alternative models for the calculation of (k-e)/k are proposed:

Figure 3-38 Term by term comparison of the σ_k equation of the MS1 model for flow GUIT2.

- constant $(k-e)/k$.
- algebraic $(k-e)/k$. and
- differential $(k-e)/k$.

For the constant $(k-e)/k$ model, $(k-e)/k$ is assumed to be constant across the whole wall jet. The value of $(k-e)/k$ is found by trial and error while matching the calculated growth rate with experimental data. The derivation of the governing equation for $(k-e)/k$ is shown in Appendix G. If an assumption is made that, at the maximum-shear-stress point the production of $(k-e)/k$ equals the dissipation, then it can be shown, in a similar method as that described in Appendix F, that the model constants for this model can be evaluated by using equations (3-60), but substituting the following for equations (3-60.d) and (3-60.i). This model is henceforth referred to as the MS2 model.

$$\beta^* = \frac{C_{\tau_1} C_{\tau_1} - C_{\tau_1} C_{\tau_1} - C_{\tau_1} C_{\tau_1}}{C_{\tau_1}^2} \quad \text{, and} \quad (3-63.a)$$

$$(\beta^* h W)_z = \frac{\frac{2}{3} \left(\frac{k-e}{k} \right)_z \left(\frac{\tau}{k} \right)_z \left[\left(h u^{**} \right)_z - \frac{y_{1,z}}{R} \left(u_z^{**} + \frac{U_c}{U_\tau} \right) \right] - 2 \frac{y_{1,z}}{U_\tau} \frac{dU_\tau}{dx} \left(u_z^{**} + \frac{U_c}{U_\tau} \right) \left(\frac{\sigma_\tau - \sigma_1}{k} \right)_z}{C_1 \left(\frac{\sigma_\tau + \sigma_1}{k} \right)_z - \frac{2}{3} \left(\frac{k-e}{k} \right)_z \left(\sqrt{\left(\frac{k-e}{k} \right)_z - 1} \right)} \quad (3-63.b)$$

The derivation of the algebraic $(k-e)/k$ model, henceforth called the MS3 model, follows the approach outlined by Rodi (1976) to obtain algebraic Reynolds stress equations. It is assumed that

$$\frac{D}{Dt}(k-e) - D_{(k-e)} = \frac{k-e}{k} \left(\frac{Dk}{Dt} - D_k \right) . \quad (3-64)$$

where $\frac{D}{Dt}(k-e)$ is the convection of (k-e).

$D_{(k-e)}$ is the diffusion of (k-e).

$\frac{Dk}{Dt}$ is the convection of k, and

D_k is the diffusion of k.

Equation (3-64) leads to the following algebraic relation for (k-e)/k.

$$\left(\frac{k-e}{k} - C_e \right) \frac{\tau}{k} \left[hu'' - \frac{y_{12}}{R} \left(u'' + \frac{U_e}{U_e} \right) \right] - \left(\frac{k-e}{k} \right) \left(1 - \sqrt{\frac{k-e}{k}} \right) \beta^* h \omega'' = 0 . \quad (3-65)$$

which replaces equation (3-38.f). A detailed derivation is given in Appendix H.

The model constants for the MS3 model are calculated from equations (3-60)

except that equation (3-60.i) is replaced with equation (3-63.b).

Appendix G shows that the exact equation for $(k-e)/k$ is

$$u \frac{\partial}{\partial x} \left(\frac{k-e}{k} \right) - h v \frac{\partial}{\partial y} \left(\frac{k-e}{k} \right) = \frac{k-e}{k^2} \sigma \cdot \frac{\partial}{\partial x} \left(\frac{k}{\omega} \frac{\partial k}{\partial x} \right) + \left(C_e - \frac{k-e}{k} \right) \frac{\tau}{k} \left(h \frac{\partial u}{\partial y} - \frac{u}{R} \right) - \beta^* h \omega \left(\frac{k-e}{k} \right) \left(\sqrt{\frac{k-e}{k}} - 1 \right) . \quad (3-66)$$

In order to use the common solver to solve equation (3-66), the diffusion term has to be approximated differently as

$$u \frac{\partial}{\partial x} \left(\frac{k-e}{k} \right) + h \nu \frac{\partial}{\partial y} \left(\frac{k-e}{k} \right) = \hat{\sigma} \frac{\partial}{\partial y} \left(\frac{k}{\omega} \frac{\partial}{\partial y} \left(\frac{k-e}{k} \right) \right) - \left(C_e - \frac{k-e}{k} \right) \frac{\tau}{k} \left(h \frac{\partial u}{\partial y} - \frac{u}{R} \right) - \beta^* h \omega \left(\frac{k-e}{k} \right) \left(\sqrt{\frac{k-e}{k}} - 1 \right). \quad (3-67)$$

where $\hat{\sigma}$ is an additional model constant. The differential (k-e)/k model, referred to as the MS4 model, substitutes equation (3-67) for equation (3-38.f.) The model constants are evaluated using equations (3-60) with equation (3-63.b) replacing (3-60.i).

Figure 3-38 displays the calculated results of the four modified multiscale models for flow GUIT2. All 4 models were adjusted to produce the correct growth rate.

Figure 3-38(d) shows that (k-e)/k for the constant (k-e)/k model (MS2) is virtually identical to that of the differential (k-e)/k model (MS4), thus the results from both models are very similar.

None of the models are clearly superior to others. MS2 and MS4 produce more correct velocity profiles but MS1 predicts more accurate magnitude of Reynolds stresses. Perhaps the MS2 model is the most attractive simply because it is the simplest, and its prediction is no worse than others. Nevertheless, none of the modified (k-e)/k equations provides great improvement over the equation of the MS1 model.

Figure 3-39 Comparison of calculated results from four modified multiscale models and experimental data for curved wall jet over a log-spiral with $x/R=1$.

3.9.4 Discussion of the Multiscale Model

The predictions of the multiscale model for self-similar wall jets are no better than those of the $k-\omega$ model, and for curved wall jets they are even poorer. Different model constants are still needed for each flow, as shown in Table 3-21, therefore it is unlikely that the model can be used to predict general, non-self similar wall jets.

Model constants	MS	MS1				
		FWJ	F0261	GN4	F0263	GUIT2
$\left(\frac{k-e}{k}\right)$		0.39	0.418	0.66	0.575	0.85
C	4	6	6	6	6	6
σ^*	0.5	0.36	0.7	1	0.36	0.36
σ	0.5	0.319	0.46	0.698	0.838	1.62
β^*	0.09	0.114	0.115	0.132	0.127	0.17
β	0.075	0.177	0.179	0.204	0.181	0.223
α	0.8	1.915	1.848	1.79	1.533	1.192
$\hat{\alpha}$	0.764	0.692	0.681	0.429	0.661	0.595
$\hat{\beta}$	0.109	0.123	0.128	0.229	0.049	-0.106
$\hat{\gamma}$	0.25	0.399	0.414	0.742	0.4	0.704

Table 3-21 Model constants of the original multiscale (MS) model and the modified multiscale (MS1) model for each tuning flow.

An interesting insight can be gained by adding the σ_x and σ_y equations at the maximum τ point (see Appendix F) to obtain

$$\begin{aligned} \frac{2}{3} C_\epsilon \left(\frac{\tau}{k} \right)_2 \left[\left(h u^{**} \right)_2 - \frac{y_{12}}{R} \left(u^{**} + \frac{U_\epsilon}{U_\tau} \right) \right] \\ = \left[2 \frac{y_{12}}{U_\tau} \frac{dU_\tau}{dx} \left(u^{**} + \frac{U_\epsilon}{U_\tau} \right) + C_1 (\beta^* h W)_2 \right] \left(\frac{\sigma_\tau + \sigma_\epsilon}{k} \right)_2. \end{aligned} \quad (3-68)$$

For typical values of all parameters, equation (3-68) requires that $\left(\frac{\sigma_\tau - \sigma_\epsilon}{k} \right)_2 > 0$. The experimental data on $\left(\frac{\sigma_\tau + \sigma_\epsilon}{k} \right)_2$ are shown in Table 3-14 and it is seen to be quite small, even negative in the case of flow F0263. The effect of $\left(\frac{\sigma_\tau + \sigma_\epsilon}{k} \right)_2$ on the values of model constants calculated by equation (3-60) is quite large which makes it hard to decide what value of $\left(\frac{\sigma_\tau - \sigma_\epsilon}{k} \right)_2$ to use. This may indicate some difficulty in the physics of the model and complicates the task of tuning the model to wall jets.

3.10 Discussions of the Turbulence Modelling Work

3.10.1 The Need for More Experimental Data

The evaluation of the model constants requires extensive experimental data. In equations (3-55) for the k - ω model and (3-60) for the multiscale model, experimental data on u'' , u''' , τ/k , σ_v/k and σ_y/k at the maximum τ point are needed. For plane wall jets in still air, Launder and Rodi (1981) show in their review of available experimental data that the scatter of the Reynolds stresses is large. More experiments are needed to determine the scatter of the Reynolds stress data for other self-similar wall jets.

At the zero-shear-stress point, the assumption of constant k'' and ω'' is not consistent with the calculated results. The experimental data in this region are not reliable enough to use as a guide, and more detailed data are needed to help in formulating a better approximation.

For plane wall jets, the results of this study support the idea that the universal log law is applicable to wall jets. All of the calculations in this study are based on the existence of the universal log law, and the results are generally comparable to experimental data for plane wall jets. For curved wall jets, the results are not as successful, particularly for the multiscale model. Meroney and Bradshaw (1975)

investigated curved boundary layer and reported a modified law of the wall. More experiments focusing on the law of the wall in curved wall jets can help to determine whether a better assumption exists.

Boundary layers can be thought of as wall jets with $U_e/U_m=1$. Since both the $k-\omega$ and the multiscale models with the original model constants, work well for boundary layers, one would expect that the model constants of the modified models would approach the original constants as $U_e/U_m \rightarrow 1$. This is not the case as shown in Table 3-13 and Table 3-21, which implies that wall jets may be fundamentally different from boundary layers. More experiments on self-similar wall jets with U_e/U_m near 1 are needed to assess this point.

3.10.2 Diffusion Model for k

The usual gradient diffusion model for k is inconsistent in the log-law layer for flows under pressure gradient. With non-zero pressure gradient, the shear stress is not constant near the wall, as can be shown by applying the momentum equation at the wall (Sjolander, 1991).

$$\left. \frac{\partial \tau}{\partial y} \right|_{y=0} = \frac{1}{\rho} \frac{dP_e}{dx} \quad (3-69)$$

Wilcox (1993) shows that the solution of the $k-\omega$ model equations in the log-law layer produces

Clearly, τ/k cannot be constant if k is constant and τ is not. This indicates that

$$k = \frac{u_z^2}{\sqrt{\beta^*}} \text{ , and } \frac{\tau}{k} = \sqrt{\beta^*} \quad (3-70)$$

there may be a problem in using the gradient diffusion term in the k equation. It is not clear how this inconsistency affects the calculation but a more consistent model is desirable.

3.10.3 Discrepancy near Maximum Velocity Point

The Boussinesq eddy viscosity assumption requires that zero shear stress and maximum velocity are coincident. For a plane wall jet in still air, experiments show that shear stress actually falls to zero at only 60% of the distance to y_m (Launder and Rodi, 1981). As a result, the $k-\omega$ model, even after modification, consistently overpredicts C_f and underpredicts y_m . The multiscale model does not use the Boussinesq eddy viscosity assumption, but the tuning model is based on an assumption that the velocity gradient is small at the zero-shear stress point. Therefore, the modified multiscale model does not predict neither C_f nor y_m more accurately than the $k-\omega$ model.

3.10.4 Potential Use of the Tuning Model

Equations such as (3-55) and (3-60) show the relationships between model constants and measurable quantities. With more detailed and reliable experimental data, it may be possible to calculate the model constants without having to use the trial and error method. This technique can be used to test new, or modified turbulence models to verify whether common constants satisfy the tuning model equations for all self-similar wall jets.

3.10.5 Difficulties with Curved Wall Jets

All turbulence models tested in this study experienced more difficulties with curved wall jets than with plane wall jets. Even when the models were modified to produce the correct growth rate, profiles of other parameters still differed significantly from experimental data. Guitton (1970) detected spanwise variation of total pressure indicative of some three-dimensional flow structures in his experiment on curved wall jets over log spirals. If there are three-dimensional structures in the flow, they can significantly affect the flow characteristics and may account for the difficulties the turbulence model has in predicting the flow. The following chapter describes an experiment that was carried out on a curved wall jet over a cylinder to determine whether or not such three-dimensional structures exist.

Chapter 4

**EXPERIMENT ON A NOMINALLY TWO-DIMENSIONAL
CURVED WALL JET**

An experiment was carried out on a nominally two-dimensional curved wall jet over a convex surface to determine whether there were any three-dimensional structures that could affect flow development.

4.1 Background

4.1.1 Görtler Vortices

It is well known that an inviscid flow over a curved surface is unstable when angular momentum decreases with increasing radius of curvature. This condition is called the Rayleigh Circulation Criterion (Rayleigh, 1916) and is illustrated in Figure 4-1. In viscous flows, the Rayleigh Circulation Criterion is found to be a necessary condition for instability. For boundary layers this condition of centrifugal instability exists in the flow over a concave surface, Figure 4-2(a). For wall jets, the unstable condition exists in the inner part (between the wall and velocity maximum) of a wall jet over a concave surface, Figure 4-2(c), and in the outer part (between the velocity maximum and the edge) of a wall jet over a convex surface, Figure 4-2(d).

Figure 4-1 Illustrations of velocity distributions that lead to centrifugal instabilities (Reproduced from Saric, 1994).

Figure 4-2 Shear layers and their stability according to the inviscid stability criterion (Reproduced from Saric, 1994).

Case A, C, and D are potentially unstable.

Streamwise vortices are found in many flows where the condition of instability exists and they are given different names depending on the type of the basic flow (Saric, 1994). For example, Taylor vortices refer to those in closed parallel flows, such as between two concentric rotating cylinders (Taylor, 1923). Dean vortices are found in open parallel flows, such as in a curved channel (Dean, 1928). In open and slightly non-parallel flows, such as boundary layers and wall jets, the vortices are called Görtler vortices (Görtler, 1941) and are characterized as pairs of counter-rotating streamwise vortices (Figure 4-3).

4.1.2 Görtler Vortices in Laminar Flows

Because laminar flow is mathematically tractable, Görtler vortices in laminar flow have been studied extensively as a stability problem. This work is summarized in Appendix I. The results are often presented in a stability diagram such as Figure 4-4 for the Blasius boundary layer over a concave surface.

The diagram shows the relationships between Görtler number (G), wavelength parameter (Λ), and spatial amplification (σ), where

$$G = \left(\frac{U_\infty \delta_r}{\nu} \right) \left(\frac{\delta_r}{R} \right)^{1/2}, \quad (4-1)$$

$$\Lambda = \left(\frac{U_\infty \lambda_r}{\nu} \right) \left(\frac{\lambda_r}{R} \right)^{1/2}, \quad (4-2)$$

U_∞ = freestream velocity,

Figure 4-3 Görtler vortices within a boundary-layer flow over a concave wall
(Reproduced from Saric, 1994).

δ_r = boundary layer thickness.

ν = molecular kinematic viscosity.

R = radius of curvature. and

λ = period of the paired vortices (see definition in Appendix I).

The neutral curve, where an existing disturbance does not grow with downstream distance, is labelled $\sigma=0$ in Figure 4-4. For a concave surface with a constant radius, the Görtler number increases with downstream distance, therefore, the diagram shows that the flow becomes less stable as it progresses downstream. At some point it crosses the neutral curve and becomes unstable. The Görtler number at this point is called the critical Görtler number. Also shown in Figure 4-4 are experimental data points where Görtler vortices have been observed. The experiments generally agree with the analytical prediction and confirm the mathematical assumption that the wave number remains constant with downstream distance. The wave number indicates the number of vortex pairs per unit span and cannot be predicted by stability analysis. In experiments it is found that the wave number is determined by the initial upstream disturbance.

4.1.3 Görtler Vortices in Turbulent Flows

Mathematical analysis of Görtler instability in turbulent flows is made difficult by the lack of a suitable turbulence model. A number of assumptions have to be made regarding the eddy viscosity to enable analyses such as those of Sandmayr (1972) on the

Figure 4-4 Stability diagram for Blasius boundary layer over a concave surface (reproduced from Saric, 1994).

boundary layer over a concave surface and of Fujisawa, Kobayashi, and Shirai (1985) on the wall jet over a concave wall. A recent review by Floryan (1991) compares these analyses with experimental results and states that they are qualitatively correct.

Experiments on Görtler vortices cover the cases of a turbulent boundary layer over a concave surface (Tani, 1962) and a turbulent wall jet over a concave wall (Kobayashi and Fujisawa, 1983). The data show that the vortices tend to wander about preferred lateral positions which are determined by steady nonuniformities in the oncoming flow (Floryan, 1991). This makes definite demonstration of the vortex existence difficult. In laminar flows, the usual indication of the vortices is the sinusoidal spanwise variation of the streamwise velocity, u . In turbulent flows, the variation of u is neither regular nor steady. In fact for the case of a turbulent wall jet over a concave wall, the spanwise variation of u was not detected at all. Note that in this case only the flow in the relatively thin inner layer ($y^+ < y_m^+$) is expected to be unstable. The existence of Görtler vortices was inferred by the sudden increase of turbulence intensity (Figure 4-5). A regular vortex pattern can be generated by placing small vortex generators in the settling chambers but the amplitudes of these steady disturbances must be large compared to the unsteady disturbances. The critical Görtler number for the appearance of the vortices in turbulent flows is still uncertain, but generally seems to correspond to the location where $\delta^+ R \geq 0.01$ for boundary layers and $y_m^+ R \geq 0.17$ for wall jets.

Figure 4-5 Variation of maximum streamwise turbulence intensity for a wall jet over concave surface

- Tailand and Mathieu (1967)
- △ Wilson and Goldstein (1976)
- Spettel et. al. (1972)
- Kobayashi and Fujisawa (1983)
- 1 : $R = 0.250$ m
- 2 : $R = 0.078$ m

Reproduced from Kobayashi and Fujisawa (1983).

4.1.4 Scope of the Present Experiment

This experiment concerns a turbulent wall jet over a convex surface. For this type of flow, Görtler vortices have not yet been experimentally demonstrated. Being the first experiment of its kind, it was preliminary in nature. The main objective was to establish whether Görtler-type vortices existed in this category of flow and to obtain some preliminary information on their behaviour. This chapter reports on the observed behaviour of the vortices which were in fact observed, the sources of the vortices, and on the attempts made to manipulate the vortices.

4.2 Apparatus

4.2.1 Test Model

The model was a circular cylinder with a diameter of 127 mm, and a span of 495 mm, as shown in Figure 4-6. A high velocity jet exited from a full-span slot which had an average opening height of 0.94 mm. The air flow was provided by two 1-HP blowers which were connected to the air inlets at opposite ends of the hollow cylinder. Separate variacs controlled the speed of each blower. There were static pressure taps every 10° at mid span of the cylinder starting from the slot. This model was already in existence and further details about it are available in a thesis by Gooden (1977).

Figure 4-6 Photograph of the test model.

As shown in Figure 4-6, the cylinder was mounted in the wind tunnel test section used by Gooden. However, the present experiment was done in still air and the wind tunnel was not used. The floor and ceiling of the test section formed the end walls and helped to maintain two dimensional flow. The cylinder was fairly close to the edge of these end walls, with the edges only two radii away from the centre of the cylinder (Figure 4-7). Therefore, removable wall extensions were made to increase the distance between the centre of the cylinder and the edges of the end walls to 5 radii, as shown in Figure 4-7.

4.2.2 Instrumentation

4.2.2.1 Seven-Hole Probe Instrumentation

The majority of the flow data were collected using a seven-hole probe instrumentation system. The seven-hole probe, shown in Figure 4-8, senses different pressures from each one of the holes and enables conversion of the information into flow directions and local dynamic and total pressures by comparing with the probe calibration data. Figure 4-9 shows the setup of the seven-hole probe instrumentation system. The probe was mounted on a two-axis traversing gear, which had a positioning resolution of 0.003 mm, and maximum travels of 270 mm, spanwise and 95 mm, radial. The probe pressure signals were connected to a 12-port scanivalve which was in turn connected to the "P" pressure transducer. The "Q" pressure transducer measured the gage pressure

Figure 4-7 Sketch of the removable wall extension.

Figure 4-8 Seven-hole pressure probe geometry (reproduced from DeCecco, 1995).

Figure 4-9 Seven-hole probe instrumentation system setup.

inside the cylinder plenum chamber, which indicated the total pressure of the jet at the slot exit. The signals from both the "P" and "Q" transducers were connected to a data acquisition system. A computer controlled the positioning of the probe and the switching of the scanivalve, as well as the collection of data from the data acquisition system. More technical details of the seven-hole probe instrumentation system are given in Appendix J.

4.2.2.2 Air Supply Instrumentation

The air supply system was instrumented to measure the flow rate, air pressure, and temperature, as shown in Figure 4-10. The air flow rate was measured with two orifice-plate flowmeters, one for each blower. A water-filled u-tube manometer measured the pressure drop across each orifice plate. A dial thermometer, located downstream of the orifice, measured the temperature of the air. The flow rate was calculated using the calibration curves of British standard 1042 and was accurate to within $\pm 3\%$ (Gooden, 1977). The pressure of the air was measured in the plenum chamber inside the cylinder. The plenum pressure tap was connected to the "Q" pressure transducer, as part of the seven-hole probe instrumentation described in the previous section, and also to a single-tube, water-filled manometer to provide a continuous visual readout.

Figure 4-10 Air supply instrumentation system.

4.3 Experimental Procedure

4.3.1 Test Conditions

Test conditions were determined by the power settings of the blowers. The maximum power settings were limited by either the calibrated range of the "P" transducer or by the maximum capacity of the blowers. The resulting slot Reynolds number for all runs were in the range of 3800 to 4100 based on slot opening height.

4.3.2 Data Collection

Data were collected in three radial planes located at 80°, 120° and 160° downstream from the slot. These planes, as well as the xyz coordinate system, are shown in Figure 4-11. Because a scanivalve was used, the data acquisition system had to wait a certain amount of time before starting to sample any pressure port in order to eliminate the transient effects due to port switching. Furthermore, since the flow was turbulent, the data acquisition system, which sampled at the rate of 5 samples second, had to sample for a sufficiently long period of time to get a steady average. The "wait" and "sampling" periods varied depending on the regions of the flow. The suitable values were determined by a trial and error process and are listed in Table 4-1.

Figure 4-11 Locations of survey planes and the positioning coordinate system.

The data collection program, called the RTFA (Real Time Flow Analysis) program, was written to implement the above consideration. The RTFA program is listed in Appendix K.

Plane	y (mm.)	Wait (s.)	Sampling (s.)
80°	0-9.5	20	20
	9.5-22	40	40
120°	0-31.5	30	30
160°	0-12.6	30	20
	12.6-25.2	30	30
	25.2-31.5	30	40
	31.5-37.9	30	50
	37.9-44.2	30	60

Table 4-1 Wait and Sampling times.

4.3.3 Seven-Hole Probe Data Reduction

The data reduction technique follows closely that of Benner (1994). The output of the procedure consists of 4 non-dimensional parameters, α , β , C_{pt} , and C_{pq} . α and β are the probe angle of attack and sideslip shown in Figure 4-12. C_{pt} and C_{pq} are the probe non-dimensional total and dynamic pressure coefficients defined as:

$$C_{pt} = \frac{p_t - P_{ref}}{Q_{ref}}, \text{ and}$$

$$C_{pq} = \frac{q}{Q_{ref}}.$$

Figure 4-12 Tangential coordinate system used for low angle regime
(reproduced from DeCecco, 1995)

where p_t is the local total pressure, and q is the local dynamic pressure. The P_{ref} and Q_{ref} are reference static and dynamic pressures. For this experiment ambient pressure was chosen as P_{ref} because the jet issued into still air. The jet plenum chamber gage pressure was selected as Q_{ref} because it represented the dynamic pressure of the jet at the slot exit.

The wall jet in this experiment would be expected to be nominally two dimensional. The streamwise vortices would be expected to appear as three dimensional structures superimposed on the two-dimensional basic flow. To accentuate such structures, a space averaging scheme was implemented as part of the data reduction. All values of C_{p0} , C_{pq} and velocity components, u , v and w calculated from α and β , measured along a spanwise line at constant height, y , from the cylinder surface were averaged. The resulting space-averaged value was subtracted from each individual data value at that height. At any point the difference from the space-average was the net effect of three dimensional structures. This technique also eliminated most of the bias errors, such as those due to radial pressure gradient, radial velocity gradient in the basic two-dimensional flow, wall proximity, and probe alignment, because only differences between measurements remained important. A program called POST, listed in Appendix L, was written to implement the space-averaging scheme.

The following notation is used to refer to the results of the space averaging scheme:

C_{p0} , C_{pq} , u , v and w refer to the actual local probe total pressure coefficient, probe dynamic pressure coefficient, and streamwise, radial, and spanwise velocity component respectively.

$\hat{C}_{p0}$, $\hat{C}_{pq}$, $\hat{u}$, $\hat{v}$ and $\hat{w}$ refer to the space-averaged values of the respective parameters at a constant height from the cylinder.

C''_{p0} , C''_{pq} , u'' , v'' and w'' refer to the difference between actual and the space averaged values of the respective parameters.

4.3.4 Probe Alignment

The probe alignment procedure took into account the fact that only the relative differences between data points were important. The cylinder was levelled vertically using a bubble level. The probe was then levelled horizontally, so that it was perpendicular to the cylinder. The probe was then brought as close to the cylinder as possible and traversed through the entire span of the survey area, with adjustments made if necessary to ensure that it remained at essentially the same height from the cylinder surface throughout the spanwise range. On every test run, the zero-height reference did not vary by more than 0.3 mm, across the span of the survey area. Circumferential

alignment was achieved by setting one side of a rectangular setting plate against the cylinder, and the adjacent side against the traversing gear base plate. Care was taken to ensure that the setting plate touched the cylinder exactly at the location where the measurements were to be taken, and that the probe was perpendicular to the base plate.

4.3.5 Calibration

Both axes of the traversing gears, the pressure transducers, and the seven-hole probe were all calibrated.

The traversing gear was built specifically for this experiment and had never been calibrated. The calibration was done by mounting a pen on the traversing gear and, under computer control, drawing a two dimensional grid 30,000 steps in the radial direction and 80,000 steps in the spanwise direction, with a 1000 step grid spacing in both directions. The actual lengths at selected grid intersections were measured by a machinists ruler and are plotted in Figure 4-13. The calibration factors were found to be 315 steps mm, spanwise and 312.5 steps mm, radial. The difference between the straight lines in Figure 4-13 and actual measurement at each point was less than the ± 0.5 mm, accuracy limit of the machinists ruler. The positioning repeatability was also assessed by redrawing some grid lines several times. No noticeable double line was observed, therefore, the repeatability was better than the line width, which was 0.5 mm.

Figure 4-13 Traversing gear calibration curve.

The "P" pressure transducer was calibrated against a micromanometer to obtain a typical calibration curve as shown in Figure 4-14(a). Figure 4-14(b) shows the difference between the actual measured pressure and calculated pressure based on the best fit straight line through the data. This difference, or errors, was taken as an indication of transducer calibration accuracy. The calibration accuracy of the "P" transducer was ± 1.5 Pa, or $\pm 0.04\%$ of the minimum Q_{ref} . The "Q" pressure transducer was calibrated against a single-tube water-filled manometer and a typical calibration curve is shown in Figure 4-15(a). Figure 4-15(b) shows that the calibration accuracy of the "Q" transducer was ± 0.03 kPa, or $\pm 0.8\%$ of the minimum Q_{ref} .

This particular seven-hole probe was the same one used by DeCecco (1995) who quoted an accuracy of $\pm 1^\circ$ for α and β , and $\pm 3\%$ for C_{p0} and C_{pq} . The probe calibration, as reported by DeCecco, was done at a reference velocity of 30 m/s. Since the velocities measured in this experiment was often considerably different from 30 m/s, the probe calibration was spot checked in the calibration rig at two representative velocities of 18.5 and 7.9 m/s. The calibration rig, as described by Yaras (1987), provided a steady flow at known angle and a precision probe positioning mechanism. At 18.5 m/s the probe calibration was checked at 10° intervals over a range of $\pm 30^\circ$ for both α and β . For 7.9 m/s, the interval was again 10° and the range was $\pm 35^\circ$ for α and β . The results, reduced using DeCecco's calibration database, are presented in Figure 4-16 and Figure 4-17.

Note ΔP_{linear} is the difference between the actual measurements and the values obtained from the best fit straight line.

Figure 4-14 A typical calibration curve for the "P" pressure transducer.

Note ΔP_{linear} is the difference between the actual measurements and the values obtained from the best fit straight line.

Figure 4-15 A typical calibration curve for the "Q" pressure transducer.

At 18.5 m/s, Figure 4-16 shows that the accuracy of $\pm 1^\circ$ for α and β and $\pm 3\%$ for C_{pu} was well satisfied within $\pm 20^\circ$ for α and β with very few exceptions. The accuracy of C_{pq} was $\pm 5\%$ with some exceptions over the same α and β ranges. Although for some α and β ranges, notably when both α and β are more negative than -20° , the C_{pq} error was as large as -10% , all the errors were of the same sense, i.e. negative errors. In the present experiment relative values were more important than absolute values, therefore, this error was not considered excessive.

The errors increased for the case of 7.9 m/s reference velocity. Within the range of $\pm 30^\circ$ for α and β , the errors were $\pm 2^\circ$ for α and β , and $\pm 6\%$ for C_{pu} with few exceptions. The errors on C_{pq} were again larger, generally about $\pm 10\%$, with a maximum of -20% at some points.

Figure 4-16 Seven-hole probe calibration errors at $U_{rel} = 18.5$ m/s. using DeCecco's (1995) calibration database.

Figure 4-17 Seven-hole probe calibration errors at $U_{ref} = 7.9$ m/s, using DeCecco's (1995) calibration database.

4.4 Results and Discussion

4.4.1 Slot Total Pressure

A total pressure survey across the span at the slot is shown in Figure 4-18. The total pressure was measured with a circular pitot tube with 1.6 mm. outside tip diameter and 0.8 mm. inside tip diameter. This Pitot tube was connected to a water-filled manometer. Figure 4-18 shows that the ratio of the local total pressure to the average total pressure across the span varied by less than $\pm 2\%$, with most of the measurements falling within a $\pm 1\%$ band. Therefore, total pressure across the span at the slot exit was essentially uniform.

4.4.2 Wall Static Pressure

The streamwise distribution of cylinder wall static pressures is shown in Figure 4-19, along with similar data from Fekete (1963) and data from an experiment in 1972 on the present test model (Kind, 1972). The shape of the wall static pressure distribution conformed fairly well to other experiments, indicating that there were no gross flow disruptions, such as early separation.

Figure 4-18 Spanwise total pressure distribution at the slot exit.

Figure 4-19 Comparison of cylinder wall static pressure distribution between present experiment, previous experiment on the same test model (Kind and Gooden, 1972), and Fekete (1963) experiment.

4.4.3 Effects of Wall Extensions

Figure 4-20 shows the comparison, with and without wall extensions, of the velocity vectors in the radial plane at the 120° downstream position. There was no significant difference in the flow patterns as a result of the wall extensions. Therefore the remainder of the measurements were done without the extensions.

4.4.4 Growth Rate and Velocity Decay

The measurements of $y^+ R$ and U_m/U_i are plotted versus circumferential position on the cylinder in Figure 4-21. The y^+ of the present measurements were calculated from the velocity profile obtained from the space-averaged values at each of the survey planes. In order to obtain a comparison with a truly two-dimensional flow, a numerical simulation was carried out using the Thin Shear Layer program (described in Chapter 3). The numerical simulation used the Dash et al. (1983) k - ϵ turbulence model which was adjusted to match the Fekete's (1963) experimental results, as described in Appendix M. The adjusted model was then used to simulate the present experiment and produced the results comparable to measurements, as shown in Figure 4-21.

Figure 4-20 Comparison of flow velocity patterns in the radial plane at $\theta = 120^\circ$, with and without wall extensions.

Figure 4-21 Growth rate and maximum velocity decay of a wall jet over a cylinder in still air.

4.4.5 Observed Cross Flow Patterns and Behaviour

Figure 4-22 shows the velocity vectors in radial planes at 3 downstream positions. Classic Görtler vortices, i.e. counter rotating vortex pairs, clearly appeared in the outer layer of the wall jet in all three survey planes. This finding is consistent with the supposition that the flow in the outer layer of convex wall jets is prone to centrifugal instability and is the first definitive evidence of the existence of such vortices in wall jets over a convex surface. Flow parameters at the 3 positions are listed in Table 4-2.

Parameters	$\theta = 80^\circ$	$\theta = 120^\circ$	$\theta = 160^\circ$
$\frac{x}{h}$	100	150	200
$\frac{y_{1,2}}{h}$	13	23	39
$\frac{U_m}{U_\infty}$	0.29	0.22	0.16
$\frac{x}{R}$	1.4	2.1	2.8
$\frac{y_{1,2}}{R}$	0.182	0.322	0.546
Eddy viscosity ¹ , $\frac{\nu}{U_m h}$	0.130	0.204	0.302
Reynolds no., $Re_{1,2} = \frac{U_m y_{1,2}}{\nu}$	15,080	20,240	24,960
Görtler no., $G_{1,2} = \left(\frac{U_m y_{1,2}}{\nu} \right) \left(\frac{y_{1,2}}{R} \right)^{1/2}$	12.3	14.1	15.3

¹ Eddy viscosity was obtained from numerical simulation at $y = y_{1,2}$

Table 4-2 Flow parameters at the survey location: $Re_h = \frac{U_\infty h}{\nu} = 4,000$, $\frac{h}{R} = 0.014$.

Figure 4-22 Velocity vectors in radial planes.
 $Re_h = 4000$. $b R = 0.014$

Since the vortices were present in the first downstream plane, the critical Görtler number, which corresponds to the location where the vortices first appeared, was less than the smallest measured values tabulated in Table 4-2. Unfortunately, the size of the probe relative to the thickness of the wall jet did not permit measurements further upstream which might enable determination of the critical Görtler number.

In terms of the magnitude of the velocity in the radial plane, some vortices were more dominant than others. The sizes of these dominant vortices were comparable to the thickness of the jet and their spanwise positions remained approximately constant with downstream distance. As a result of the growth of the stationary dominant vortices, some of the weaker vortices disappeared and the number of vortex pairs per unit span, or the wave number, reduced as the flow moved downstream. The actual mechanism of the elimination of the vortices is unknown. They might have merged with the nearby vortices or may simply have dissipated.

The effect of the streamwise vortices on momentum exchange is shown in Figure 4-23, which juxtaposes the patterns of the flow directions in the radial plane with the spanwise distribution of the total pressure at constant height from the surface of the cylinder. The locations of upward flow are seen to correspond with the locations of high u and the downward flow to locations of low u . This indicates that upward flow brought high momentum fluid from near the wall up into the region of lower momentum further away from the wall, and that the downward flow brought low momentum fluid from the outer region of the wall jet into the region nearer the wall. A similar spanwise variation of

Figure 4-23 Correlation between spanwise distribution of total pressure and the flow patterns in the radial plane at $\theta = 120^\circ$

total pressure distribution had been observed in earlier studies, such as Fekete (1963), Kind (1967) and Guitton (1970). Therefore streamwise vortices might have been present in those studies as well.

No apparent correlation between vertical flow and the total pressure variations existed close to the wall, as illustrated by the contrast between the C''_{pu} distribution at $y=15.8$ mm, and that at $y=0$ mm. Below the maximum velocity point, the radial gradient of angular momentum is positive and the centrifugal instability mechanism seems to have disappeared, with little penetration of the effects of the vortices into the near-wall region. The spanwise distribution of other parameters related to streamwise momentum, namely C''_{pu} and $u'' U_{\tau}$, were similar to the spanwise distributions of C''_{pu} , as shown in Figure 4-24.

The effects of the streamwise vortices on streamwise momentum exchange can be compared to those due to turbulence by examining the product $u''v''$. This term represents the vertical flux of streamwise momentum as a result of the vortices, and is similar to the Reynolds shear stress (RS), $\overline{u'v'}$, which is a momentum flux due to turbulence. Therefore $u''v''$ may be thought of as an equivalent Reynolds shear stress (ERS). Figure 4-25 shows the correlation of $u''v''$ with the flow velocity vectors. Each line in Figure 4-25(b) represents the ERS at constant height from the wall. The ERS was positive almost everywhere which meant that it had the same sign as the Reynolds shear stress in the

Figure 4-24 Spanwise distribution of momentum related parameters at $\theta = 120^\circ$ and $y = 15.8 \text{ mm.}$

Figure 4-25 Correlation of the equivalent Reynolds shear stress with flow patterns at $\theta = 120^\circ$

outer part of the wall jet. Thus, the momentum exchange as a result of the vortices would add to the momentum exchange due to turbulence.

Using the Reynolds stress from the results of the aforementioned numerical simulation, the ratio of the equivalent Reynolds shear stress, $u''v''$, to the calculated Reynolds shear stress, $\overline{u'v'}$, is shown in Figure 4-26. Although $u''v''$ is seen to be large comparing to $\overline{u'v'}$ at several locations, generally $u''v''$ was less than 10% of $\overline{u'v'}$ in most regions of the flow. Therefore, the effects of the vortices are not expected to be dominant over the effects of turbulence. However, the magnitude of $u''v''$ may have been underestimated. The process of determining $u''v''$ involved very long time-averaging, in the order of several hours. This process would not detect short-period oscillation, such as the effects of vortices which jitter in space. These vortices would have gone undetected, or their effects would have been underestimated. Furthermore, the large-scale motion of the vortices would increase the mixing rate of turbulent eddies from different heights from the surface, resulting in higher $\overline{u'v'}$ at and around the locations of the vortices. This effect could be even larger than the direct effect of $u''v''$ and can be confirmed only if $\overline{u'v'}$ is measured experimentally.

Figure 4-26 Contour of the ratio of measured equivalent Reynolds shear stress to calculated Reynolds shear stress in radial planes.

4.4.6 Sources of the Vortices

The next series of tests were aimed at identifying the sources of the streamwise vortices and determining if they could be eliminated. In order to quickly visualize the vortices, tufts were installed. Tufts on the surface of the cylinder were found to be ineffective, presumably because the vortices did not penetrate down to the surface, as discussed above. Tufts were, therefore, attached to a spanwise-oriented rod that was placed some distance away from the cylinder, as shown in Figure 4-27. The low static pressure created by entrainment, pulled the tufts into the outer part of the wall jet. The length of the tufts that were used put their ends at approximately the $\theta=80^\circ$ position. If the tufts were pulled into a region where streamwise vortices were present, their movement would become much more erratic than when they were in a region without vortices. Thus, the locations of the vortices could be detected by sliding the rod in the spanwise direction and observing the movement of the tufts. In cases where some measure of the magnitude of the vortices was needed, a spanwise total pressure survey was conducted at some distance above the cylinder wall. The distance from the wall was selected based on the downstream position so that the spanwise distribution of C''_{p0} would correlate well with the strength of the vortices, such as in Figure 4-23. Because of its availability, the seven-hole probe was used for the total pressure survey by reading only the pressure from the centre hole.

Figure 4-27 Photograph of the tuft installation.

Possible sources of the vortices that were investigated included:

- possible effects associated with wall jet separation from the cylinder wall.
- effects of the end wall flow.
- irregularity of the cylinder surface.
- irregularity of entry flow into the slot.
- irregularities of the slot surfaces.

The wall jet separated from the cylinder surface probably somewhere beyond $\theta=200^\circ$, as could be deduced from the wall static pressure distribution in Figure 4-19. The separation point could be controlled by putting a blocker at some downstream position, as shown in Figure 4-28. The blocker used was a 85x40x40 mm. block of wood. By changing the downstream, as well as the spanwise, position of the blocker, the separation pattern could be altered. It was found that the tufts did not indicate any difference in the locations of the vortices unless the blocker was put quite close to the tufts.

The effect of the end walls was modified by using a fence, which was a piece of cardboard with a cutout matching the curvature of the cylinder, and a width that was much larger than the thickness of the wall jet. By putting the fence at different spanwise positions, the distance between the known vortices and the end wall flow could be varied. Again, the locations of the vortices did not change unless the fence was put very close to them.

a) without the blocker

b) with blocker

Figure 4-28 Effects of a blocker on the separation point.

The test model had a smooth finished surface. Magnetic disks with diameter of about 5 mm. and thickness of 1 mm. were used to simulate wall roughness. It was found that unless the disks were closer than 20 mm. to the slot exit. they did not generate any flow disturbances detectable by the tufts. This was probably because the wall irregularity had to extend into the outer part of the wall jet. where centrifugal instability existed. to affect the flow. From this observation it was concluded that the existing vortices were unlikely to have been caused by wall irregularity.

The cross section of the test model. showing the flow passage between the plenum chamber and the slot exit is shown in Figure 4-29. As shown in the figure. a layer of 20 mm. thick permeable foam was placed across the entry of the slot to reduce any non-uniformity that might have been present in the slot entry flow. Tufts did not indicate any difference in the existing pattern of vortices.

An exploded view of the test model is shown in Figure 4-30. When the upper lip was removed. some dirt deposits. presumably remaining from flow visualization work carried out by Gooden (1977) and invisible from outside the slot. were found on the inside surface near the slot exit. Figure 4-31 shows the spanwise locations of these deposits and their approximate thicknesses in comparison with the spanwise distribution of the total pressure at $\theta=80^\circ$. before and after the dirt deposits were removed. The deposits seemed to have caused some of the vortices. particularly the pair at the $z=-55$ mm. position which disappeared after the deposits were removed. Other vortices. such as

Figure 4-29 Cross section of the test model and the installation of permeable foam cloth.

Figure 4-30 Exploded view of the test model.

Figure 4-31 Locations of deposits found on the inside surface of the upper slot lip, and the total pressure distribution at $\theta = 80^\circ$ and $y = 9.6 \text{ mm.}$, before and after the deposits were removed.

those at $z=55$ mm. and $z=120$ mm. still remained. The fact that at least some of the deposits were able to cause large flow disturbances indicated that the upper lip, or at least the part of it quite near the slot exit, was a very critical surface.

Further tests on the effect of the slot lip surface irregularity were done by putting small pieces of 0.06 mm. thick adhesive tape on the upper and lower lips at the exit. The result in Figure 4-32 shows large disturbances coincided with the location of tape on the upper lip. The tape on the lower lip produced much smaller disturbances. This is consistent with other findings that the disturbances which affected the outer part of the wall jet were more critical than those that affected the inner part.

The next test concentrated on the effect of the upper lip. A plastic clip, 0.22 mm. thick and 186 mm. wide, was put over the upper lip. After a spanwise survey of the total pressure was completed, the clip was moved by half the period of the largest observed total pressure fluctuation, and the survey was repeated. Figure 4-33 shows that the pattern of total pressure variation translated in spanwise direction by the same distance the clip was moved. This clearly shows that the variation of the total pressure was caused by the irregularity pattern of the clip. Therefore, the irregularity of the upper slot lip was the likely source of the streamwise vortices.

Figure 4-32 Effects of 0.06 mm thick adhesive tape at the slot lips, on the spanwise total pressure distribution at $\theta = 80^\circ$ and $y = 9.5$ mm.

Figure 4-33 Effects of 0.22 mm thick plastic clip over the upper lip, on the spanwise total pressure distribution at $\theta = 80^\circ$ and $y = 9.5$ mm.

The next test was aimed at effectively replacing the upper slot lip with a new edge. A 1.6 mm. thick aluminium plate was placed across the whole span at an angle against the edge of the upper lip. Two angles were chosen, 90° (Figure 4-34(a)) and 45° (Figure 4-34(b)). At each angle, a different edge of the plate acted as the upper lip. Figure 4-35 shows the comparison of the spanwise total pressure distributions with the different edges as the upper slot lip. There were significant differences in the distributions with the different edges, supporting the hypothesis that irregularity of lip was the source of the streamwise vortices.

The original slot lip was a machined surface, and the edges of the aluminium plate were fresh cut by a power sheet-metal shear having continuous straight ground shear blades. None of the edges had any obvious nicks or marks. Yet all three edges produced the large peaks and valleys in the total pressure distribution which are indicative of the existence of streamwise vortices. Moreover the patterns of the variation were distinctly different for each of the three edges. This suggested that, at least for this condition of b/R , these vortices might be unavoidable in practical application of wall jets having convex curvature. Similar observations were reported by Fekete (1963) and Guitton (1970). They had to take extreme care in the manufacturing of their upper slot lips to avoid large variations in the spanwise total pressure distribution.

Figure 4-34 Installation of an aluminium plate to form a new edge of the upper slot lip.

Figure 4-35 Comparison of spanwise total pressure distribution at $\theta = 80^\circ$ and $y = 9.5$ mm. for 3 different upper slot lip edges.

4.4.7 Manipulation of the Vortices

In some applications, such as boundary layer control and circulation control, a high mixing rate is desirable. Since the streamwise vortices were found to increase the mixing rate, a technique to generate them may sometimes be useful. As earlier results suggest that disturbing the outer part of the curved wall jet produces stronger vortices than disturbing the flow next to the wall, this part of the testing concentrated on disrupting the shear layer between the outer edge of the wall jet and the entrainment flow at the slot exit.

The flow disrupters that were used were equilateral triangular-shaped magnets, 1 mm. thick and 6 mm. long on each side. They were attached to the top of the upper slot lip and extended out over the slot exit as shown in Figure 4-36. Two configurations of disrupter were tested:

- configuration A (Figure 4-37(a)) with 2 mm. overhang, and
- configuration B (Figure 4-37(b)) with approximately twice the overhang area of configuration A.

For both configurations, the disrupters were placed at four spanwise locations, -95 mm., -31.5 mm., 31.5 mm., and 95 mm. These locations were chosen to avoid the existing streamwise vortices.

Figure 4-36 Installation of a triangular flow disruptor.

Figure 4-37 Configuration of disruptor installation.

The comparison of spanwise total pressure distribution between no disrupters (original), disrupter configuration A (mod A), and configuration B (mod B) is shown in Figure 4-38. The total pressure was measured at the 80° downstream position and 9.5 mm. above the cylinder wall. Configuration A produced almost no difference from the no-disruptor distribution, except near the disruptor at the 95 mm. spanwise location. This showed that configuration A was near the minimum overhang necessary to produce flow disturbances. Each disruptor was probably placed at slightly different overhang distances, also the shear layer might have been at slightly different locations at each of the disrupters, resulting in some of the disruptors touching the shear layer, and others not, as illustrated in Figure 4-39. Configuration B produced distinct peaks coinciding with each of the disrupters, indicating that streamwise vortex pairs were present at those locations. Therefore disrupting the free shear layer at the slot exit appeared to be a viable approach to triggering longitudinal vortices. Whether triggering these vortices produces an on off effect, or whether a continuous variation of vortex strength can be produced, is unclear.

Figure 4-38 Effects of flow disruptors on the spanwise total pressure distribution at $\theta = 80^\circ$ and $y = 9.5$ mm.

a) Disruptor touching shear layer.

b) Disruptor not touching shear layer.

Figure 4-39 Sketch of relative position between the shear layer and a disruptor.

5.1 Turbulence Modelling

A FORTRAN computer code called TSL (Thin Shear Layer) was developed to compute two-dimensional, thin-shear layer flow. Four turbulence models were implemented:

- BL: a mixing length model proposed by Baldwin and Lomax (1978);
- KE: a two-equation k - ϵ model as proposed by Dash et al. (1986);
- KW: a two-equation k - ω model as proposed by Wilcox (1988a); and
- MS: a simplified Reynolds stress model as proposed by Wilcox (1988b).

The KW and the MS models were the main foci of this study. The KW model was chosen because it was reported to perform better than the KE model for flows under adverse pressure gradient. The MS model received the emphasis because it did not rely on the Boussinesq eddy viscosity assumption which was known to be inconsistent with experimental data for wall jets.

The TSL program was extensively tested. Laminar flows where analytical solutions exist were used to validate the common solver and other general routines. The implementation of the turbulence models was validated by comparing the calculated

results with published results in the literature for the same flow, as well as with other commercially available code. Although some differences were observed between the results from the TSL program and the reference data, these are believed to have been caused by more physically realistic implementation of the equations of motion in the TSL program. Furthermore, the differences between the TSL results and the reference data are much smaller than the differences between results of different turbulence models for the same flow.

The turbulence models in their original form were used to calculate a range of flows consisting of:

- boundary layers:
 - turbulent flat plate.
 - Samuel and Joubert flow (plane boundary layer under increasing adverse pressure gradient).
 - curved boundary layer.
- wall jets:
 - plane wall jet in still air.
 - self-similar plane wall jet in a moving stream, and
 - self-similar curved wall jet over a log spiral.

All four models generally worked well for the boundary layer flows that were tested. Only the KE model worked satisfactorily for the tested wall jets.

A tuning model was developed to systematically adjust turbulence model constants for a specific wall jet flow. The tuning model is based on the following observed behaviours of wall jets.

- At the maximum shear stress point, the wall-normal gradients of turbulence intensities are also zero.
- There is a point in the flow, usually near the maximum velocity point, where the shear stress is zero.
- Although not universally accepted, there is strong evidence that the universal log-law of the wall is applicable to wall jets.

These observations lead to a number of simplifying assumptions and produce relations between model constants and measurable quantities that can be determined from experimental data. However, it turned out that there are not enough reliable experimental data to create all relations necessary to calculate every model constant directly. Some model constants had to be determined by trial and error based on matching calculated results with experimental data.

Five self-similar wall jets were selected as tuning flows for the modification of the KW and MS models. These tuning flows are:

- plane wall jet in still air, designated as flow FWJ.
- plane wall jet in a moving stream with $U_w/U_m=0.38$, designated as flow F0261.

- plane wall jet in a moving stream with $U_c/U_m=0.77$, designated as flow GN4.
- curved wall jet over a log spiral with $x/R=2/3$, designated as flow F0263, and
- curved wall jet over a log spiral with $x/R=1$, designated as flow GUIT2.

A modified $k-\omega$ model (designated as model KW1) was developed to reproduce the experimental growth rate of each tuning flow. The KW1 model generally works well for all of the tuning flows, except for the flow GUIT2 which is the most highly curved wall jet. The problem for this flow is that the predicted shear stress and turbulent kinetic energy are much higher than experimental data. The model constants of the KW1 model are different for each tuning flow. Since the calculated results are very sensitive to changes in the model constants, it is unlikely that a single set of model constants that work well for all wall jet flows can be found. Preliminary attempts to predict non-self similar wall jets using the KW1 model was unsuccessful.

A similar approach was used to develop a modified multiscale model (designated as model MS1) that reproduces the experimental growth rate of the tuning flows. The MS1 model works well for plane wall jets but experiences difficulties with curved wall jets. The calculated profiles of streamwise velocity and Reynolds stresses differ significantly from experimental data for curved wall jet, and the magnitude of the difference increases with increasing curvature.

Three additional alternative multiscale models (designated as model MS2, MS3 and MS4) were developed based on different assumptions regarding the distribution of the upper partition turbulent energy, $(k-e)/k$.

- The MS2 model assumes constant $(k-e)/k$ across the wall jet.
- The MS3 model assumes the distribution of $(k-e)/k$ is governed by an algebraic equation.
- The MS4 model assumes the distribution of $(k-e)/k$ is governed by a differential equation.

These three models were tested on flow GUIT2 and were found to produce only modest improvement over the MS1 model. Nonetheless, the MS2 model is still preferable because it is the simplest model.

The deficiency of the Boussinesq eddy viscosity assumption leads to the BL, KE and KW models, both in modified and original forms, predicting too high wall friction and too low a location of maximum velocity. The MS model does not rely on the Boussinesq assumption but it still does not resolve the problem near the maximum velocity point. This is thought to be due to the assumptions used in the tuning model.

5.2 Experiment on Curved Wall Jet over a Cylinder

An experiment was carried out on a nominally two-dimensional curved wall jet over a cylinder to determine whether there were any 3-dimensional structures that could affect flow development. A seven-hole probe was used to measure flow velocity and direction at three radial planes perpendicular to the cylinder wall. Streamwise counter-rotating vortex pairs were observed in the outer part of the wall jet at all three measuring planes.

Spanwise positions of dominant vortex pairs remained constant with downstream distance. The size of the vortices grew with downstream distance, resulting in some of the less dominant vortices being crowded out. Therefore, the wave number, or the number of vortex pairs per unit span, reduced with downstream distance.

The effect of the streamwise vortices on streamwise momentum exchange was significant, though not dominant. This effect was examined through the comparison of the Reynolds shear stress, obtained from numerical simulation, with the equivalent Reynolds shear stress which is the measured vertical flux of streamwise momentum.

The main cause of the streamwise vortices was found to be small irregularities of the upper slot lip. The sensitivity of the flow to the irregularity of the upper lip is such that a vortex-free curved wall jet over a convex surface does not seem practical.

Imitating the irregularity of the upper lip, flow disruptors mounted above the slot lip can be used to manipulate the streamwise vortices.

5.3 Main Conclusions

All of the assessed turbulence models in their original forms exhibited serious short-comings in predicting the behavior of curved wall jet flows. Somewhat unexpectedly, the k - ϵ model gave the best predictions of flow development and velocity profile shape.

Even after extensive tuning, the k - ω and multiscale methods still gave poor results for velocity and turbulence-parameter profiles for the curved wall jet flows.

Despite the fact that it does not rely on the Boussinesq eddy-viscosity assumption, the performance of the multiscale model was such that it cannot be recommended for prediction of wall jet flows. The k - ϵ method, with modifications for curvature effects, is recommended for use until improved methods become available.

The existence of three-dimensional flow structures in nominally two-dimensional convex curved wall jets was demonstrated experimentally. Elimination of such structures is unlikely to be feasible in practical applications, nor would it appear to be particularly

desirable. These structures were found to contribute significantly to cross-stream mixing. but turbulence was still the dominant cross-stream diffusion mechanism.

6.1 Further Testing and Modification of the Dash k- ϵ model

Among the four turbulence models, the Dash k- ϵ (KE) model shows the most promise for predicting wall jet flows. It should be tested further on more wall jet cases and modified as necessary using the tuning model outlined in this thesis.

6.2 Assessment of Turbulence Models for Multi-Element-Airfoil Shear Flows

The flow downstream of slots in multi-element airfoils is similar to wall jets having $U_e/U_m \approx 1$, but there is usually a velocity minimum between the U_m and U_e points on the velocity profiles. Turbulence models for such flows should be assessed using an approach similar to the present study.

6.3 Additional Experiments on Self Similar Wall Jets

More detailed and reliable experimental data on self similar wall jets are needed to improve the capability of the tuning model. In particular, information is needed on the scatter of the Reynolds stresses at the maximum shear stress point, the behaviour of turbulence kinetic energy near the zero shear stress point, and the existence of the universal law-of-the-wall for curved wall jets. Furthermore, experiments on weak wall jets, with U_e/U_m close to 1, are needed to determine if the model constants of the modified $k-\omega$ and modified multiscale models would approach those of the original models.

6.4 Continuing Experiments on Wall Jet over a Cylinder

This study shows that the longitudinal vortices in curved wall jet over a cylinder can be manipulated by disrupting the shear layer near the upper slot lip. Further experiments should be conducted to explore the possibility of using this method to control the separation point of the wall jet from the cylinder wall.

References

Alcaraz, E., 1977, "Contribution à l'étude d'un jet plan turbulent évoluant le long d'une paroi convexe à faible courbure." Thèse d'Etat, Université Claude Bernard, Lyon.

Bardina, J., Ferziger, J. H. and Reynolds, W. C., 1983, "Improved Turbulence Models Based on Large Eddy Simulation of Homogeneous." *Incompressible Turbulent Flows*, Department of Mechanical Engineering, Stanford University, Stanford, CA. Rept. TF-19.

Baldwin, B. and Lomax, H., 1978, "Thin-Layer Approximation and Algebraic Model for Separated Turbulent Flows." *AIAA Paper 78-257*.

Benner, M. W., 1994, "Losses in Axial-Flow Turbines at Off-Design Incidence." Ph.D. Thesis in progress, Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, Canada.

Bradshaw, P., 1973, "Effects of Streamline Curvature on Turbulent Flow." North Atlantic Treaty Organization, Technical Editing and Reproduction Ltd., London. *AGARDograph* No. 169.

Bradshaw, P. and Gee, M. T., 1960, "Turbulent Wall Jets with and without an External Stream." *Aero. Res. Council.*, R&M 3252.

Dash, S. M., Beddini, R. A., Wolf, D. E. and Sinha, N., 1983, "Viscous/inviscid Analysis of Curved Sub- or Supersonic Wall Jets." *AIAA Paper-83-1679*. Presented at the AIAA 16th Fluid and Plasma Dynamics Conference, Danvers, MA.

Dash, S. M., York, B. J., Sinha, M., and Dvorak, F. A., 1986, "Wall Jet Analysis for Circulation Control Aerodynamics. Part I: Fundamental CFD and Turbulence Modeling Concepts." *NASA CP2432*. Presented at Circulation Control Workshop, NASA Ames Research Center, Feb. 18-20.

Dean, W. R., 1928, "Fluid Motion in a Curved Channel." *Proceeding Royal Society London Series A15*, pp. 623-31.

DeCecco, S. F., 1995, "The Aerodynamics of Turbine Blades with Tip Damage." M.Eng. Thesis, Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, Canada.

Dvorak, F. A., 1973, "Calculation of Turbulent Boundary Layers and Wall Jets over Curved Surfaces." *AIAA Journal*, Vol. 11, pp. 517-524.

Dvorak, F. A., 1974, and Woodward, F. A., "A Viscous/Potential Flow Interaction Analysis Method for Multielement Infinite Swept Wing," *NASA CR-2476*.

Dvorak, F. A. and Choi, D. H., 1983, "Analysis of Circulation-Controlled Airfoils in Transonic Flow," *Journal of Aircraft*, Vol. 20, pp. 331-337.

Dvorak, F. A. and Kind, R. J., 1979, "Analysis Method for Viscous Flow over Circulation Controlled Airfoils," *Journal of Aircraft*, Vol. 16, pp. 23-28.

Fekete, G. I., 1963, "Coanda Flow in a Two-Dimensional Wall Jet on the Outside of a Circular Cylinder," Mech. Eng. Dept. Rep. No. 63-11, McGill University, Quebec, Canada.

Floryan, J. M., 1991, "On the Görtler Instability of Boundary Layers," *Progress of Aerospace Science*, Vol. 28, pp. 235-271.

Förthmann, E., 1934, "Über turbulente Strahlausbreitung," *Ing. Arch.*, Vol. 5, pp. 42.

Fujisawa, N., Kobayashi, R., and Shirai, H., 1985, "Centrifugal Instability of Turbulent Wall Jets along Strongly Concave Surfaces," *Journal of Applied Mechanics Transactions ASME* 52, 492-493.

Galbraith, R. A. M. and Head, M. R., 1975, "Eddy Viscosity and Mixing Length from Measured Boundary Layer Developments," *Aeronautical Quarterly*, Vol. 26, pp. 133-154.

Garthshore, I. and Newman, B. G., 1969, "The Turbulent Wall Jet in Arbitrary Pressure Gradient," *Aeronautical Quarterly*, Vol. 20, pp. 25.

Giles, J. A., Hays, A. P., and Sawyer, R. A., 1966, "Turbulent Wall Jets on Logarithmic Spiral Surfaces," *Aeronautical Quarterly*, Vol. 17, pp. 201.

Glauert, M. B., 1956, "The Wall Jet," *Journal of Fluid Mechanics*, Vol. 1, pp. 625-643.

Gooden, R. K., 1977, "Experimental Investigation of Boundary-Layer Control by Tangential Injection," M.Eng. Thesis, Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, Canada.

Görtler, H., 1941, "Instabilität laminarer Grenzschichten an Konkaven Wänden gegenüber gewissen dreidimensionalen Störungen," *ZAMM* 21: 250-52; also NACA Rep. 1375.

Guitton, D. E., 1970, "Some Contributions to the Study of Equilibrium and Non-Equilibrium Wall Jets over Curved Surfaces." Ph.D. thesis. Mech. Eng., McGill University, Quebec, Canada.

Guitton, D. E. and Newman, B. G., 1977, "Self-Preserving Turbulent Wall Jets over Convex Surfaces." *Journal of Fluid Mechanics*, Vol.81, Part 1, pp. 155-185.

Hah, C. and Lakshminarayana, B., 1980, "Prediction of Two- and Three-Dimensional Asymmetrical Turbulent Wakes, Including Curvature and Rotation Effects." *AIAA Journal*, pp. 1196-1204.

Harlow, F. H. and Nagayama, P., 1968, "Transport of Turbulence Energy Decay Rate", *Los Alamos Science Lab., Report LA-3854*.

Irwin, H. P. A. H., 1973, "Measurements in a Self-Preserving Plane Wall Jet in a Positive Pressure Gradient." *Journal of Fluid Mechanics*, Vol. 61, Part 1, pp. 33-63.

Kacker, S. C. and Whitelaw, J. H., 1971, "The Turbulence Characteristics of Two-Dimensional Wall-Jet and Wall-Wake Flows." *Journal of Applied Mechanics*, Trans. ASME, Vol. 38.

Kind, R. J., 1967, "A Proposed Method of Circulation Control." Ph.D. Thesis. University of Cambridge, Cambridge, England.

Kind, R. J., 1971, "A calculation Method for Boundary Layer Control by Tangential Blowing." *CASI Transactions*, Vol. 4, pp. 88-98.

Kind, R. J., 1972, Personal Communication.

Kind, R. J., Gooden, K. and Dvorak, F. A., 1979, "Measurements of Flows with Tangential Injection and Comparison with Prediction Methods." *AIAA Journal*, Vol. 17, pp. 730-735.

Klebanoff, P. S., 1955, "Characteristics of Turbulence in a Boundary Layer with Zero Pressure Gradient." *NACA Report 1247*.

Kline, S. J., Mokovin, M. V., Sovran, G. and Cockrell, D. J., 1968, "Computation of Turbulent Boundary Layers-1968 AFOSR-IFP Stanford Conference." Proceeding 1968 Conference, Stanford University, Stanford, California, USA.

Kline, S. J., 1969, "Turbulent Boundary Layer Prediction and Structure-The State of the Art." Rept. MD-23, Mechanical Engineering Department, Stanford University, Stanford, California, USA.

Kline S. J. Cantwell. B. J. and Lilley. G. M., (Eds.), 1982. *1980-81 AFOSR-HTTM-Stanford Conference on Complex Turbulent Flows*. Mechanical Eng. Dept., Stanford University.

Kobayashi. R. and Fujisawa. N., 1983. "Turbulence Measurements in Wall Jets Along Strongly Concave Surfaces." *Acta Mechanica*, Vol. 47, pp. 39-52.

Kruka. V. and Eskinazi. S., 1964. "The Wall-Jet in a Moving Stream." *Journal of Fluid Mechanics*, Vol. 20, pp. 555-579.

Launder. B. E., Priddin. C. H. and Sharma. B. I., 1977. "The Calculation of Turbulent Boundary Layers on Spinning and Curved Surfaces." *ASME Journal of Fluids Engineering*, pp. 231-239.

Launder. B. E. and Rodi. W., 1981. "The Turbulent Wall Jet." *Progress of Aerospace Science*, Vol. 19, pp. 81-128.

Launder. B. E. and Rodi. W., 1983. "The Turbulent Wall Jet-Measurements and Modeling." *Annual Review of Fluid Mechanics*, Vol. 15, pp. 429-459.

Meroney. R. N. and Bradshaw. P., 1975. "Turbulent Boundary-Layer Growth over a Longitudinally Curved Surface." *AIAA Journal*, Vol. 13, pp. 1448-1453.

Menter. F. R., 1991. "Influence of Freestream Values on $k-\omega$ Turbulence Model Predictions." *AIAA Journal*, Vol. 30, pp. 1657-1659.

Narayan. K. Y. and Narasimha. R., 1973. "Parametric Analysis of Turbulent Wall Jets." *Aeronautical Quarterly*, Aug. 1973, pp. 207-218.

Newman. B. G., 1961. "The Deflexion of Plane Jets by Adjacent Boundaries - Coanda Effect." from *Boundary Layer and Flow Control*, Pergamon Press, London.

Nikuradse. J., 1933. "Stromungsgesetze in rauhen Rohren." *Forsch. Arb. Ing.-Wes.*, no. 361.

Özarapoğlu. V., 1973. "Measurements in Incompressible Turbulent Flows." Ph.D. Thesis. Faculte des sciences, Universite Laval, Quebec, Canada.

Patankar. S. V., 1980. *Numerical Heat Transfer and Fluid Flow*. Hemisphere Publishing Corp.

Patel. R. P., 1962. "Self-Preserving, Two-Dimensional Turbulent Jets and Wall Jets in a Moving Stream." M.Eng. Thesis, Det. of Mech. Eng., McGill University, Montreal, Canada.

Patel, V. C., 1965, "Calibration of the Preston Tube and Limitations on Its Use in Pressure Gradients," *Journal of Fluid Mechanics*, Vol. 23, pp. 185-208.

Rayleigh, Lord., 1916, "On the Dynamics of Revolving Fluids," *Scientific Papers*, Vol. 6, pp. 447-53. Cambridge: Cambridge University Press.

Roberts, L., 1987, "A Theory for Turbulent Curved Wall Jets," *ALAA Paper 87-0004*.

Rodi, W., 1976, "A New Algebraic Relation for Calculating the Reynolds Stresses," *Mechanics of Fluid*, pp. T219-T221.

Samuel, A. E. and Joubert, P. N., 1974, "A Boundary Layer Developing in an Increasingly Adverse Pressure Gradient," *Journal of Fluid Mechanics*, Vol. 66, pp. 481-505.

Sandmyre, G., 1972 "On the Occurance of Longitudinal Vortices in Turbulent Boundary Layers at Concave Walls," *NACA TT F-14*, translation of DLR Report FB-66-41 June 1966.

Saric, W. S., 1994, "Görtler Vortices," *Annual Review of Fluid Mechanics*, Vol. 26, pp. 379-409.

Schlichting, H., 1968, *Boundary Layer Theory*, 6th Ed. McGraw-Hill, New York.

Seifert, A., Darabi, A. and Wagnanski, I., 1996, "Delay of Airfoil Stall by Periodic Excitation," *Journal of Aircraft*, Vol. 33, pp. 691-698.

Sharma, R. N., 1972, "Momentum and Mass Transfer in Turbulent Conical Wall Jets," Ph.D. Thesis, Indian Institute of Technology, Chemical Engineering Department, Kanpur.

Shrewsbury, G. D., 1989, "Numerical Study of a Research Circulation Control Airfoil Using Navier-Stokes Methods," *Journal of Aircraft*, Vol. 26, pp. 29-34.

Sjolander, S. A., 1991, "Engineering 88.501 Supplementary Course Notes," Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, Canada.

So, R. M. C. and Mellor, G. L., 1972, "An Experimental Investigation of Turbulent Boundary Layers along Curved Surfaces," *NASA CR-1940*.

So, R. M., and Mellor, G. L., 1975, "Experiments on boundary layers over a concave wall," *Aeronautical Quarterly*, Vol. 26, 25-40.

Spalding, D. B., 1964, "A Unified Theory of Friction, Heat Transfer and Mass Transfer in the Turbulent Boundary Layer and Wall Jet." *ARC* 25, pp. 925.

Spalding, D. B. and Patankar, S. V., 1967, *Heat and Mass Transfer*, Morgan-Grampian, London, England.

Spettel, F., Mathieu, J., Brison, J. F., 1972, "Tensions de Reynolds at production d'énergie cinétique turbulente dans les jets pariétaux sur parois plane et concaves." *Journal of Méchanics*, Vol. 11, pp. 403-425.

Tailland, A. and Mathieu, J., 1967, "Jet pariétal", *Journal de Méchanique*, Vol. 6, pp. 103-131.

Tani, I., 1962, "Production of Longitudinal Vortices in a Boundary Layer along a Curved Wall." *Journal of Geographic Research*, Vol. 67, pp. 3075-3080.

Taylor, G. I., 1923, "Stability of a Viscous Liquid Contained between Two Rotating Cylinders." *Phil. Trans. R. Soc. London Ser. A* 223: 289-343.

White, F. M., 1991, *Viscous Fluid Flow*, 2nd Ed. McGraw-Hill, New York.

Wilcox, D. C., 1977, "Streamline Curvature Effects on Turbulent Boundary Layers." *AIAA Journal*, Vol. 15, pp. 574-580.

Wilcox, D. C., 1988a, "Reassessment of the Scale-Determining Equation for Advanced Turbulence Models." *AIAA Journal*, Vol. 26, No. 11, pp. 1299-1310.

Wilcox, D. C., 1988b, "Multiscale Model for Turbulent Flows." *AIAA Journal*, Vol. 26, No. 11, pp. 1311-1320.

Wilcox, D. C., 1988c, "More Advanced Applications of the Multiscale Model for Turbulent Flows." *AIAA-88-0220*, Presented at the AIAA 26th Aerospace Science Meeting, Reno, Nevada, Jan. 11-14.

Wilcox, D. C., 1991, "A Half Century Historical Review of the $k-\omega$ Model." *AIAA Paper 91-0615*.

Wilcox, D. C., 1993a, "Comparison of Two-Equation Turbulence Models for Boundary Layers with Pressure Gradient." *AIAA Journal*, Vol. 31, No. 8.

Wilcox, D. C., 1993b, *Turbulence Modeling for CFD*, DCW Industries Inc.

Wille, R. and Fernholz, H., 1965, "Report on the First European Mechanics Colloquium on the Coanda Effect." *Journal of Fluid Mechanics*, Vol. 23, pp. 801.

Wilson, D. J.. 1970. "An Experimental Investigation of the Mean Velocity, Temperature and Turbulence Fields in Plane and Curved Two-Dimensional Wall Jets: Coanda Effect." Ph.D. Thesis, Mech. Eng. Dept., University of Minnesota.

Wilson, D. J. and Goldstein, R. J., 1976, "Turbulent Wall Jets with Cylindrical Streamwise Surface Curvature." *Journal of Fluid Engineering*, Trans. ASME Ser. 1, Vol. 98, pp. 550-557.

Yaras, M. I., 1987, "Flow in the Tip Gap of a Rectangular Turbine Cascade." M.Eng. Thesis, Department of Mechanical and Aerospace Engineering, Carleton University, Ottawa, Canada.

Zhou, M. D. and Wagnanski, I., 1993, "Parameters Governing the Turbulent Wall Jet in an External Stream." *AIAA Journal*, Vol. 31, pp. 848-853.

Appendix A

TSL program

Program listing available on request

Appendix B

Discretized Equations of the Equations of Motion

The streamwise momentum equation (3-9 or 3-12), the k and ϵ equations for the Dash k - ϵ model (equations 3-26.a and 3-26.b), the k and ω equations for the Wilcox k - ω model (equations 3-35.a and 3-26.b), and the k , ω , τ , σ_k , σ_ω and $(k$ - $\epsilon)$ equations for the Wilcox multiscale model (equations 3-38.a through 3-38.f) can be cast in following standard form:

$$\frac{\partial}{\partial x^*}(\bar{u}^* \phi^*) - \frac{\partial}{\partial y^*} \left(h \bar{v}^* \phi^* - G \frac{\partial \phi^*}{\partial y^*} \right) = S_p \phi^* - S_\phi \quad (\text{B-1})$$

The k^* equation (3-26.a) will be used to demonstrate the steps involved in putting it in a standard form.

First put the left hand side in the conservative form by adding k^* times the continuity equation (3-8).

$$\begin{aligned} \bar{u}^* \frac{\partial k^*}{\partial x^*} - h \bar{v}^* \frac{\partial k^*}{\partial y^*} + k^* \left[\frac{\partial \bar{u}^*}{\partial x^*} - \frac{\partial}{\partial y^*} (h \bar{v}^*) \right] &= \frac{\partial}{\partial y^*} \left(\frac{h \bar{v}^*}{\sigma_k} \frac{\partial k^*}{\partial y^*} \right) - P - \epsilon^* \\ \frac{\partial}{\partial x^*} (\bar{u}^* k^*) - \frac{\partial}{\partial y^*} (h \bar{v}^* k^*) &= \frac{\partial}{\partial y^*} \left(\frac{h \bar{v}^*}{\sigma_k} \frac{\partial k^*}{\partial y^*} \right) - P - \epsilon^* \\ \frac{\partial}{\partial x^*} (\bar{u}^* k^*) - \frac{\partial}{\partial y^*} \left(h \bar{v}^* k^* - \frac{h \bar{v}^*}{\sigma_k} \frac{\partial k^*}{\partial y^*} \right) &= P - \epsilon^* \end{aligned}$$

Substituting for P

$$\frac{\partial}{\partial x^*} (\bar{u}^* k^*) - \frac{\partial}{\partial y^*} \left(h \bar{v}^* k^* - \frac{h \bar{v}^*}{\sigma_k} \frac{\partial k^*}{\partial y^*} \right) = v_i^* \left(\frac{\partial \bar{u}^*}{\partial y^*} - \frac{\bar{u}^*}{h R^*} \right)^2 - \epsilon^* \quad (\text{B-2})$$

Comparing equation (B-2) with the standard form, equation (B-1), one obtains:

$$\begin{aligned}\phi^* &= k^* \\ G &= \frac{h\bar{v}_x^*}{\sigma_k^*} \\ S_p k^* - S_c &= v_i \left(\frac{\partial \bar{u}^*}{\partial y^*} - \frac{\bar{u}^*}{hR^*} \right) - \varepsilon^* \quad (\text{B-3})\end{aligned}$$

The choice for S_p and S_c has a significant effect on the convergence or divergence of the iterative process, and must be made with care. Patankar (1980) suggests that, for a positive definite variable, such as k , ε , ω and $(k-\varepsilon)$, unrealistic values during iteration can be prevented if $S_p \leq 0$ and $S_c \geq 0$. For equation (B-3) this condition can be met by choosing

$$\begin{aligned}S_p &= -\frac{\varepsilon^*}{k_o^*} \\ S_c &= v_i \left(\frac{\partial \bar{u}^*}{\partial y^*} - \frac{\bar{u}^*}{hR^*} \right)\end{aligned}$$

where k_o^* is the value from the last iteration.

Returning to the standard equation, (B-1), it is discretized by integrating over the control volume of Figure B-1.

$$\oint \left[\frac{\partial}{\partial x^*} (\bar{u}^*) + \frac{\partial}{\partial y^*} \left(h\bar{v}^* \phi^* - G \frac{\partial \phi^*}{\partial y^*} \right) \right] dA = \oint (S_p \phi^* - S_c) dV$$

which for the very small two dimensional control volume of Figure B-1 can be approximated by

$$\bar{u}^* \phi^* \Delta A \Big|_u - \left(h\bar{v}^* \phi^* - G \frac{\partial \phi^*}{\partial y^*} \right) \Delta A \Big|_v = (S_p \phi^* - S_c) \Delta V$$

C.S. stands for control surfaces: C.V. stands for control volume: ΔA is the area of a surface of the control surface: ΔV is the volume of the control volume: and e, w, s and n are the four control surfaces as labeled in Figure 3-2

Patankar (1980) shows that the coefficients in the discretized equation of the standard version depend on the mass flows through the control surfaces, F , the diffusive conductance across the control surfaces, D , and the volume of the control volume, ΔV , which are determined as follows:

$$F_e = \bar{u}_e \Delta y,$$

$$F_w = \bar{u}_w \Delta y,$$

$$F_s = \bar{v}_s h_s \Delta x,$$

$$F_n = \bar{v}_n h_n \Delta x,$$

$$D_e = 0,$$

$$D_w = 0,$$

$$D_s = G_s \frac{\Delta A_s}{\delta y_s} = G_s \frac{h_s \Delta x}{\delta y_s},$$

$$D_n = G_n \frac{\Delta A_n}{\delta y_n} = G_n \frac{h_n \Delta x}{\delta y_n},$$

$$\Delta V = \frac{h_s + h_n}{2} \Delta x \Delta y.$$

The locations of the control surfaces are placed at the midpoint between the nodes simply for the convenience of calculating the coefficients. The results are such that:

$$\begin{aligned} \bar{u}_i &= \frac{\bar{u}_N + \bar{u}_E}{2}, & \bar{u}_E &= \frac{\bar{u}_i + \bar{u}_P}{2}, \\ \bar{v}_n &= \frac{\bar{v}_N + \bar{v}_P}{2}, & \bar{v}_i &= \frac{\bar{v}_N + \bar{v}_P}{2}, \\ h_n &= 1 - \frac{\bar{Y}_N - \bar{Y}_P}{2R}, & h_i &= 1 - \frac{\bar{Y}_i - \bar{Y}_P}{2R} \end{aligned}$$

where E, W, N and S refer to the nodes as labelled in Figure B-1.

Evaluation of the diffusive coefficient, G , at the control surfaces requires special attention. Patankar points out that the real objective is to find a formula that produces the correct diffusive flux across the surface. For the case where the control surface lies midway between the nodes, this formula produces

$$G_n = \frac{2G_N G_P}{G_N + G_P} \text{ and } G_i = \frac{2G_N G_P}{G_N + G_P}$$

The final discretized equation is

$$a_P \Phi_P^* = a_i \Phi_i^* + a_n \Phi_n^* + a_N \Phi_N^* + a_S \Phi_S^* - h,$$

where $a_i = D_{ij} A(P_{ij}) - \|F_{ij}, 0\| = 0$,

$$a_n = D_{nP} A(P_{nP}) - \|F_{nP}, 0\| = F_{nP},$$

$$a_N = D_{NP} A(P_{NP}) - \|F_{NP}, 0\|,$$

$$a_S = D_{SP} A(P_{SP}) - \|F_{SP}, 0\|,$$

$$h = S_P \Delta V, \text{ and}$$

$$a_P = a_i + a_n + a_N + a_S - S_P \Delta V.$$

where the symbol $[\cdot]$ represents the maximum function, and the Peclet numbers, P 's, are defined as

$$P_u = \frac{F_u}{D_u} \quad \text{and} \quad P_v = \frac{F_v}{D_v}$$

Patankar discusses several possible forms of the function $A(P)$ and recommends the Power Law scheme, given by

$$A(|P|) = \left\| 0.5 \left(1 + 0.1 |P| \right)^5 \right\|$$

as the most accurate scheme. However, in this study it was found that for some flows, the Power Law scheme produces discontinuous solutions near the freestream edge. For such cases, the Upwind scheme, which is given by

$$A(|P|) = 1$$

was used instead.

Appendix C

STARTING PROBLEM

Five starting schemes are implemented in the TSL program. Each one is described in this appendix.

Scheme 1 Assumed k and l .

In this scheme only the velocity profile is given. The turbulent kinetic energy, k , and length scale, l , are assumed as follows.

$$k = \begin{cases} k_w, & y \leq \delta \\ k_s, & y > \delta, \end{cases}$$

and $l = \min (0.41 y, 0.09 \delta)$. Then ω and ε are calculated from

$$\omega = \frac{\sqrt{k}}{l}, \text{ and}$$

$$\varepsilon = 0.09 \frac{k^{3/2}}{l}.$$

Scheme 2 Assumed τ and l .

The velocity profile is given. The Reynolds shear stress, τ , and length scale, l , are assumed. τ is assumed according to Thomas and Hasani (1989) as

$$\frac{\tau}{\tau_w} = 1 - \xi \eta - (3 - 2\xi)\eta^2 + (2 - \xi)\eta^3,$$

where $\eta = \frac{y}{\delta}$, and

$$\varepsilon = \frac{\delta}{\tau_w} \frac{d\overline{p_w}}{dx} .$$

l is assumed according to Galbraith, Sjolander and Head (1977) as

$$l_{max} = 0.09 \delta , \text{ and}$$

$$l_{inner} = \kappa_1 \left(\frac{\tau}{\tau_w} \right)^{1/2} \left| 1 - \exp \left(- \frac{y^+}{A} \right) \right| .$$

$$\text{where } A = \frac{26}{(1 - hp^*)^{1/2}} ,$$

$$p^* = \frac{y}{\rho u_*} \frac{dP}{dx} , \text{ and}$$

$$h = \begin{cases} 12.6 , & p^* > 0 \\ 14.76 , & p^* < 0 \end{cases} .$$

Eddy viscosity is calculated from the Bousinnesq assumption,

$$\nu = \frac{\tau}{\frac{\partial u}{\partial y} - \frac{u}{hR}} ,$$

and k from

$$k = \left(\frac{\nu}{l} \right)^2 .$$

Then ω and ε are calculated from the definitions of eddy viscosity,

$$\omega = \frac{k}{\nu} , \text{ and}$$

$$\varepsilon = 0.09 \frac{k^2}{\nu} .$$

Scheme 3 k and l are given.

Profiles of velocity, u , turbulent kinetic energy, k , and length scale, l , are given.

This information may come from other commercial codes, such as EDDYBL. Then ω and ε are calculated from

$$\omega = \frac{\sqrt{k}}{l}, \text{ and}$$

$$\varepsilon = 0.09 \frac{k^{3/2}}{l}.$$

Scheme 4 and 5 Reynolds stresses are given.

The Reynolds stresses may come from experimental data, or results of previous runs. The eddy viscosity is calculated from the Boussinesq assumption.

$$\nu_t = \frac{\tau}{\frac{\partial u}{\partial y} - \frac{u}{hR}}.$$

Then ω and ε are calculated from the definitions of eddy viscosity.

$$\omega = \frac{k}{\nu_t}, \text{ and}$$

$$\varepsilon = 0.09 \frac{k^{3/2}}{\nu_t}.$$

Appendix D

Relations for Laminar Plane Wall Jets

Glauert (1956) showed that, for laminar plane wall jets in still air,

$$u = \left(\frac{5F}{2\nu x} \right)^{1/2} f(\eta) \quad \text{and} \quad (D-1)$$

$$\eta = \left(\frac{5F}{32\nu^2 x^3} \right)^{1/4} y \quad (D-2)$$

where $F = \text{constant}$,

$\eta = \text{self-similar distance from the wall, and}$

$f(\eta) = \text{self-similar mass flux function.}$

Let's define x^* , b and U^* such that

$$x = x^* + x_0 \quad \text{and} \quad (D-3)$$

$$\text{at } x^* = 0: \quad \frac{U_m}{U^*} = 1 \quad \text{and} \quad \frac{y_{1/2}}{b} = 1.$$

At the maximum velocity point, from (D-1)

$$U_m = \left(\frac{5F}{2\nu x} \right)^{1/2} f_m \quad (D-4)$$

where $f = \text{maximum value of } f$.

Substitute with (D-3),

$$U_m = \left[\frac{5F}{2\nu(x^* + x_0)} \right]^{1/2} f_m \quad (D-5)$$

at $x^* = 0$, $U_m = U^*$, therefore

$$U^* = \left(\frac{5F}{2\nu x} \right)^{1/2} f_m \quad (D-6)$$

Rearrange to get

$$F = \frac{2}{5} \nu x_0 \left(\frac{U_m}{f_m} \right)^2. \quad (D-7)$$

At the wall jet half-width point $y = y_{1/2}$ and from (D-2)

$$\eta_{1/2} = \left(\frac{5F}{32 \nu^2 x^2} \right)^{1/4} y_{1/2}. \quad (D-8)$$

where $\eta_{1/2}$ = value of η at $\frac{u}{U_m} = 0.5$.

Substitute x with (D-3)

$$\eta_{1/2} = \left[\frac{5F}{32 \nu^2 (x^* - x_0)^2} \right]^{1/4} y_{1/2}. \quad (D-9)$$

at $x^* = 0$, $y_{1/2} = h$, therefore

$$\eta_{1/2} = \left(\frac{5F}{32 \nu^2 x} \right)^{1/4} h. \quad (D-10)$$

Substitute F from (D-7) and rearrange.

$$\frac{x}{h} = \frac{\text{Re}_c}{4 f_m \eta_{1/2}^2}. \quad (D-11)$$

where $\text{Re}_c = \frac{U_m h}{\nu}$.

The maximum velocity decay is given by dividing (D-5) with (D-6).

$$\frac{U_m}{U_c} = \left(\frac{x^* - x_0}{x_0} \right)^{1/2}. \quad (D-12)$$

and wall jet growth by dividing (D-9) with (D-10).

$$\frac{y_{1/2}}{h} = \left(\frac{x^* - x_0}{x_0} \right)^{3/4}. \quad (D-13)$$

The wall friction is found from the velocity gradient at the wall which is given by

$$\left(\frac{\hat{c}u}{\hat{c}y} \right)_w = \frac{U_m}{y_{12}} \frac{\hat{c}\left(\frac{u}{U_m}\right)}{\hat{c}\left(\frac{y}{y_{12}}\right)} \bigg|_w \quad (\text{D-14})$$

Divide (D-1) with (D-4),

$$\frac{u}{U_m} = \frac{f}{f_m} \quad ,$$

and (D-2) with (D-8),

$$\frac{y}{y_{12}} = \frac{\eta}{\eta_{12}} \quad .$$

therefore,

$$\frac{\hat{c}\left(\frac{u}{U_m}\right)}{\hat{c}\left(\frac{y}{y_{12}}\right)} = \frac{\hat{c}\left(\frac{u}{U_m}\right)}{\hat{c}\eta} \frac{\hat{c}\eta}{\hat{c}\left(\frac{y}{y_{12}}\right)} = \frac{\eta_{12}}{f_m} f \quad . \quad (\text{D-15})$$

The friction coefficient becomes

$$C_f = \frac{2 \hat{c}\left(\frac{\tau_w}{\rho}\right)}{U_m^2} = \frac{2 \nu \left(\frac{\hat{c}u}{\hat{c}y}\right)_w}{U_m^2} \quad .$$

Substitute with (D-14) and (D-15) and simplify,

$$C_f = \frac{2 \nu}{U_m y_{12}} \frac{\eta_{12}}{f_m} f_w = 2 \left(\frac{\nu}{U_m h} \right) \left(\frac{U_m}{U_m} \right) \left(\frac{h}{y_{12}} \right) \frac{\eta_{12}}{f_m} f_w \quad .$$

Substitute definition of Re_w (D-12) and (D-13),

$$C_f = \frac{2}{Re_w} \frac{\eta_{12}}{f_m} f_w = \left(\frac{x_w^* - x_w}{x_w} \right)^{1/4} \quad . \quad (\text{D-16})$$

From Glauert (1956), $\eta_{12}=2.03$, $f_m=0.315$ and $f_w = \frac{2}{9}$, therefore (D-11) and (D-16)

become

$$\frac{x_w}{h} = \frac{Re_w}{17.25} \quad , \text{ and}$$

$$C_f = \frac{5.22}{\text{Re}_0} \left(\frac{x^* + x_0}{x_0} \right)^{1.4}.$$

where $\text{Re}_0 = \frac{U_\infty h}{\nu}$.

Appendix E

Tuning Equations for the k - ω Model

Substituting the self-similar variables in (3-47) into the $k - \omega$ model equations (3-35) to obtain, for k at high Reynolds number,

$$\begin{aligned} & \left(u^{**} - \frac{U_\tau}{U_\tau^*} \right) \left(2 \frac{y_{1,z}}{U_\tau^*} \frac{dU_\tau}{dx} k^{**} - \frac{dy_{1,z}}{dx} \eta k^{**} \right) - h v^{**} k^{**} - C_k \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_\tau}{U_\tau^*} \right) u^{**} \frac{k^{**}}{\omega^{**}} \\ &= \sigma \frac{\tilde{c}}{\tilde{c}\eta} \left(h \frac{k^{**}}{\omega^{**}} \right) - h \frac{k^{**}}{\omega^{**}} \left| u^{**} - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_\tau}{U_\tau^*} \right) \right| - \beta^* h \omega^{**} k^{**} . \end{aligned} \quad (E-1)$$

and for ω ,

$$\begin{aligned} & \left(u^{**} - \frac{U_\tau}{U_\tau^*} \right) \left| \left(\frac{y_{1,z}}{U_\tau^*} \frac{dU_\tau}{dx} - \frac{dU_\tau}{dx} \right) \omega^{**} - \frac{dy_{1,z}}{dx} \eta \omega^{**} \right| - h v^{**} \omega^{**} \\ &= \sigma \frac{\tilde{c}}{\tilde{c}\eta} \left(h \frac{k^{**}}{\omega^{**}} \omega^{**} \right) - \alpha h \frac{k^{**}}{\omega^{**}} \left| u^{**} - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_\tau}{U_\tau^*} \right) \right| - \beta h \omega^{**} . \end{aligned} \quad (E-2)$$

The Boussinesq eddy viscosity assumption

$$\tau = \frac{k}{\omega} \left(\frac{\tilde{c}}{\tilde{c}_1} \frac{u}{h} - \frac{u}{hR} \right) ,$$

when written in self-similar variables, becomes

$$\tau^{**} = \frac{\tau}{U_\tau^*} = \frac{k^{**}}{\omega^{**}} \left| u^{**} - \frac{y_{1,z}}{hR} \left(u^{**} - \frac{U_\tau}{U_\tau^*} \right) \right| . \quad (E-3)$$

Therefore, at the zero-shear stress point,

$$u^{**} - \frac{y_{1,z}}{hR} \left(u^{**} - \frac{U_\tau}{U_\tau^*} \right) = 0 . \quad (E-4)$$

k^{**} and ω^{**} are assumed to be constant at the zero-shear stress point. Substitute $k^{**} = 0$

and (E-4) into (E-1).

$$2\left(u^{**} - \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} - C_k \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r}\right) u^{**} \frac{k^{**}}{\omega^{**}} = -\beta^* h \omega^{**} k^{**} \quad .$$

Divide by k^{**} and assume that the velocity gradient is small, so

$$2\left(u^{**} - \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} = -\beta^* h k^{**} \quad . \quad (E-5)$$

Substitute $\omega^{**} = 0$ and (E-4) into (E-2).

$$\left(u^{**} - \frac{U_c}{U_r}\right) \left(\frac{y_{1,z}}{U_r} \frac{dU_r}{dx} - \frac{dy_{1,z}}{dx}\right) \omega^{**} = -\beta h \omega^{**} \quad .$$

Divide by ω^{**} .

$$\left(u^{**} - \frac{U_c}{U_r}\right) \left(\frac{y_{1,z}}{U_r} \frac{dU_r}{dx} - \frac{dy_{1,z}}{dx}\right) = -\beta h \omega^{**} \quad (E-6)$$

Divide (E-6) by (E-5) and simplify.

$$\frac{\beta}{\beta^*} = \frac{1}{2} \left(1 - \frac{\frac{dy_{1,z}}{dx}}{\frac{y_{1,z}}{U_r} \frac{dU_r}{dx}} \right) \quad . \quad (E-7)$$

At the maximum-shear stress point, it is assumed that

$$k^{**} = K_1 \eta^{*2} - K_2 \eta^* - K_3 \quad .$$

$$\omega^{**} = \text{constant} = W, \text{ and} \quad (E-8)$$

$$k^{**} = \omega^{**} = 0 \quad .$$

Substitute (E-8) into (E-3) and rearrange.

$$\frac{\tau}{k} = \frac{u^{**} - \frac{y_{1,2}}{hR} \left(u^{**} - \frac{U_c}{U_1} \right)}{W} \quad (E-9)$$

Substitute (E-8) into (E-1).

$$\begin{aligned} & 2 \left(u^{**} - \frac{U_c}{U_1} \right) \frac{y_{1,2}}{U_1} \frac{dU_1}{dx} k^{**} - C_k \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_1} \right) u^{**} \frac{k^{**}}{W} \\ &= 2 \sigma^* h \frac{K}{W} k^{**} - h \frac{k^{**}}{\omega^{**}} \left(h \frac{k^{**}}{\omega^{**}} k^{**} \right) + h \frac{k^{**}}{W} \left[u^{**} - \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_1} \right) \right]^2 - \beta^* h W k^{**}. \end{aligned}$$

Devide by hWk^{**}

$$\begin{aligned} & \frac{2 \left(u^{**} - \frac{U_c}{U_1} \right) \frac{y_{1,2}}{U_1} \frac{dU_1}{dx}}{hW} - \frac{C_k \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_1} \right) u^{**}}{hW^2} \\ &= 2 \sigma^* \frac{K}{W^2} - \frac{\left[u^{**} - \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_1} \right) \right]^2}{W} - \beta^*. \end{aligned}$$

Substitute (E-9), rearrange and simplify.

$$\beta^* = \left(\frac{\tau}{k} \right)^2 \left[1 - \frac{2}{P} \left\{ \sigma^* \left(\frac{K}{W} \right) h - \left(u^{**} - \frac{U_c}{U_1} \right) \frac{y_{1,2}}{U_1} \frac{dU_1}{dx} - \frac{C_k}{2} \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_1} \right) \frac{u^{**}}{W} \right\} \right] \quad (E-10)$$

$$\text{where} \quad P = \left(\frac{\tau}{k} \right)^2 \left[h u^{**} - \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_1} \right) \right]$$

and subscript 2 refers to the values at the maximum-shear stress point.

Substitute (E-8) into (E-2).

$$\left(u^{**} - \frac{U_c}{U_1} \right) \left(\frac{y_{1,2}}{U_1} \frac{dU_1}{dx} - \frac{dy_{1,2}}{dx} \right) W = \alpha h \left[u^{**} - \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_1} \right) \right]^2 - \beta h W^2.$$

Devide by hW^2 and rearrange.

$$\frac{\left(u^{**} - \frac{U_c}{U_v}\right) \left(\frac{y_{1,z}}{U_v} \frac{dU_v}{dx} - \frac{dy_{1,z}}{dx}\right)}{hW} = \alpha \frac{\left|u^{**} - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_v}\right)\right|^2}{W} - \beta$$

Substitute with (E-9).

$$\frac{\left(u^{**} - \frac{U_c}{U_v}\right) \left(\frac{y_{1,z}}{U_v} \frac{dU_v}{dx} - \frac{dy_{1,z}}{dx}\right) \frac{\tau}{k}}{hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_v}\right)} = \alpha \left(\frac{\tau}{k}\right)^2 - \beta$$

Devide by $\left(\frac{\tau}{k}\right)^2$ and rearrange.

$$\frac{\left(u^{**} - \frac{U_c}{U_v}\right) \left(\frac{y_{1,z}}{U_v} \frac{dU_v}{dx} - \frac{dy_{1,z}}{dx}\right)}{P} = \alpha - \frac{\beta}{\beta^*} \frac{\beta^*}{\left(\frac{\tau}{k}\right)^2}$$

$$\text{where } P = \left(\frac{\tau}{k}\right) \left| hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_v}\right) \right|$$

Substitute (E-10) and rearrange.

$$\alpha = \frac{\beta}{\beta^*} \left| 1 - \frac{2}{P} \left\{ \sigma^* \left(\frac{K}{W} \right) h - \left(u^{**} - \frac{U_c}{U_v}\right) \frac{y_{1,z}}{U_v} \frac{dU_v}{dx} - \frac{C_k}{2} \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_v}\right) \frac{u^{**}}{W} \right\} \right|$$

$$- \frac{1}{P} \left(u^{**} - \frac{U_c}{U_v}\right) \left(\frac{y_{1,z}}{U_v} \frac{dU_v}{dx} - \frac{dy_{1,z}}{dx}\right)$$

Rearrange to get

$$\alpha = \frac{\beta}{\beta^*} \left| 1 - \frac{2}{P} \left\{ \sigma^* \left(\frac{K}{W^*} \right) h - \frac{C_s}{2} \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_c}{U_r^*} \right) \frac{u^{**}}{W^*} \right\} \right|$$

$$- \frac{2}{P} \left(u^{**} + \frac{U_c}{U_r^*} \right) \frac{y_{1,2}}{U_r^*} \frac{dU_r}{dx} \left| \frac{1}{2} \left(1 - \frac{\frac{dy_{1,2}}{dx}}{\frac{y_{1,2}}{U_r^*} \frac{dU_r}{dx}} \right) - \frac{\beta}{\beta^*} \right| .$$

Substitute with (E-7).

$$\alpha = \frac{\beta}{\beta^*} \left| 1 - \frac{2}{P} \left\{ \sigma^* \left(\frac{K}{W^*} \right) h_2 - \frac{C_s}{2} \frac{y_{1,2}}{R} \left(u_2^{**} + \frac{U_c}{U_r^*} \right) \frac{u_2^{**}}{W^*} \right\} \right| . \quad (E-11)$$

again where subscript 2 refers to value at the maximum-shear stress point.

Appendix F

Tuning Equations for the Multiscale Model

In the log-law layer the convection terms, or the left hand sides of equations (3-59), and the curvature terms are small. Therefore, from equation (3-59.a),

$$\sigma^* \frac{\partial}{\partial \eta} \left(\frac{k^{**}}{\omega^{**}} k^{**} \right) + \tau^{**} u^{**} - \tau^* \omega^{**} k^{**} = 0 \quad . \quad (F-1)$$

Following Wilcox (1993b), the solution of the multiscale model equations in the log-law layer produces

$$k = \frac{u_*^2}{\sqrt{\beta^*}} \quad , \quad \omega = \frac{u_*}{\sqrt{\beta^*} \kappa y} \quad , \quad \text{and} \quad \frac{\tau}{k} = \sqrt{\beta^*} \quad . \quad (F-2)$$

Therefore, $k^{**} = 0$ and equation (F-1) reduces to

$$\left(\frac{\tau}{k} \right) u^{**} = \beta^* \omega^{**} \quad . \quad (F-3)$$

Applying the log-law layer assumptions to equation (3-59.b) yields

$$\sigma \frac{\partial}{\partial \eta} \left(\frac{k^{**}}{\omega^{**}} \omega^{**} \right) + \alpha \frac{\omega^{**}}{k^*} \tau^{**} u^{**} - \beta \omega^{**} \left(\omega^{**} + \xi \left| u^{**} \right| \right) = 0 \quad .$$

which, by substituting (F-2), becomes

$$\sigma \sqrt{\beta^*} \kappa^2 \omega^{**} + \alpha \frac{\tau}{k} \omega^{**} u^{**} - \beta \omega^{**} \left(\omega^{**} + \xi \left| u^{**} \right| \right) = 0 \quad .$$

After simplifying the following relation results

$$\sigma = \frac{\sqrt{\beta^*}}{\kappa^2} \left(\frac{\beta}{\beta^*} - \alpha + \frac{\xi \beta}{\sqrt{\beta^*}} \right) \quad . \quad (F-4)$$

With the log-law layer assumptions, the equation (3-59.f) becomes

$$C_e \tau'' u'' - \beta'' \omega'' k'' \left(\frac{k-e}{k} \right)^{3/2} = 0 \quad .$$

which, by substituting equation (F-3), simplifies to

$$\left(\frac{k-e}{k} \right)^{3/2} = C_e \quad .$$

Substitute into the definition of C_1

$$C_1 = 1 + C_0 \left(\frac{k-e}{k} \right)^{3/2} = 1 + C_0 C_e \quad . \quad (F-5)$$

Simplified form of equation (3-59.d) in the log-law layer is

$$C_1 \tau'' u'' - C_1 \beta'' \omega'' \sigma'' = 0 \quad .$$

which, by substituting equation (F-3), becomes

$$\frac{\sigma_1}{k} = \frac{C_1}{C_1} \quad . \quad (F-6)$$

Similarly, equation (3-59.e) applied in the log-law layer becomes

$$C_1 \tau'' u'' - C_1 \beta'' \omega'' \sigma'' = 0 \quad .$$

which, when substituting with (F-3), yields

$$\frac{\sigma_1}{k} = \frac{C_1}{C_1} \quad . \quad (F-7)$$

Finally, simplify equation (3-59.c) with the log-law layer assumptions to get

$$\left(C_{\tau_1} \sigma'' - C_{\tau_1} \sigma'' + C_{\tau_k} k'' \right) u'' - C_1 \beta'' \omega'' \tau'' = 0 \quad .$$

and substituting (F-3) to obtain

$$\left(C_{\tau_1} \sigma'' - C_{\tau_1} \sigma'' + C_{\tau_k} k'' \right) u'' - C_1 \left(\frac{\tau}{k} \right) u'' \tau'' = 0 \quad .$$

Dividing by k'' , substituting (F-6) and (F-7) and simplifying yields

$$C_1 \left(\frac{\tau}{k} \right) = C_{\tau_1} \frac{C_1}{C_1} + C_{\tau_1} \frac{C_1}{C_1} + C_{\tau_k} \quad .$$

Substituting $\left(\frac{\tau}{k}\right)$ with (F-2) and C_1 with (F-5) and simplifying to get

$$\beta^* = \frac{C_{\tau}C_r + C_{\tau}C_1 + C_{\tau}(1 + C_0C_r)}{(1 - C_0C_r)^2} \quad (F-8)$$

At the zero-shear stress point, k^{**} is assumed to be constant, therefore $k^{**} = 0$ and equation (3-59.a) becomes

$$\left(u^{**} - \frac{U_r}{U_r}\right) \left(2 \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} k^{**}\right) = -\beta^* h \omega^{**} k^{**}.$$

Divide by k^{**} and rearrange to get

$$h \omega^{**} = -\frac{2}{\beta^*} \left(u^{**} - \frac{U_r}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \quad (F-9)$$

ω^{**} is also assumed to be constant and equation (3-59.b) reduces to

$$\left(u^{**} - \frac{U_r}{U_r}\right) \left(\frac{y_{1,z}}{U_r} \frac{dU_r}{dx} - \frac{dy_{1,z}}{dx}\right) \omega^{**} = -\beta \omega^{**} \left[h \omega^{**} - \frac{2}{\beta} \left(h u^{**} \right)_z - \left(u^{**} - \frac{U_r}{U_r}\right) \frac{y_{1,z}}{R} \right].$$

u^{**} is assumed to be small. Substituting (F-9) and rearranging to get

$$\beta = \frac{1 - \frac{\frac{dy_{1,z}}{dx}}{\frac{y_{1,z}}{U_r} \frac{dU_r}{dx}}}{\frac{2}{\beta^*} - \frac{\frac{y_{1,z}}{R}}{\frac{y_{1,z}}{U_r} \frac{dU_r}{dx}}} \quad (F-10)$$

At the maximum-shear stress point, σ_v^{**} is assumed to be zero, and the equation (3-59.d) becomes

$$\begin{aligned} \left(u^{**} + \frac{U_c}{U_r}\right) \left(2 \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \sigma_r^{**}\right) &= C_r \tau^{**} \left[hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right] \\ &+ 2C_c \frac{y_{1,z}}{R} \tau^{**} \left(u^{**} + \frac{U_c}{U_r}\right) - C_i \beta^* h \omega^{**} \sigma_r^{**} . \end{aligned}$$

Devide by k^{**} to obtain

$$\begin{aligned} 2 \left(u^{**} + \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \frac{\sigma_r}{k} &= C_r \frac{\tau}{k} \left[hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right] \\ &+ 2C_c \frac{y_{1,z}}{R} \frac{\tau}{k} \left(u^{**} + \frac{U_c}{U_r}\right) - C_i \beta^* h \omega^{**} \frac{\sigma_r}{k} . \end{aligned} \quad (F-11)$$

Similarly, one can obtain from equation (3-59.e)

$$\begin{aligned} 2 \left(u^{**} + \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \frac{\sigma_i}{k} &= C_i \frac{\tau}{k} \left[hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right] \\ &- 2C_c \frac{y_{1,z}}{R} \frac{\tau}{k} \left(u^{**} + \frac{U_c}{U_r}\right) - C_i \beta^* h \omega^{**} \frac{\sigma_i}{k} . \end{aligned} \quad (F-12)$$

from equation (3-59.f)

$$\begin{aligned} 2 \left(u^{**} + \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \frac{k-e}{k} &= C_c \frac{\tau}{k} \left[hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right] \\ &- \beta^* h \omega^{**} \left(\frac{k-e}{k}\right)^{1/2} . \end{aligned} \quad (F-13)$$

and from equation (3-59.c)

$$\begin{aligned} 2 \left(u^{**} + \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \frac{\tau}{k} &= \left(C_{r,i} \frac{\sigma_r}{k} + C_{r,i} \frac{\sigma_i}{k} + C_{r,i}\right) \left[hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right] \\ &+ C_c \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \left(\frac{\sigma_i}{k} - \frac{\sigma_r}{k}\right) - C_i \beta^* h \omega^{**} \frac{\tau}{k} . \end{aligned} \quad (F-14)$$

Add (F-11) to (F-12).

$$2\left(u^{**} + \frac{U_e}{U_r}\right) \frac{y_{1,2}}{U_r} \frac{dU_r}{dx} \left(\frac{\sigma_r}{k} + \frac{\sigma_v}{k}\right) = (C_x + C_1) \frac{\tau}{k} \left[hu^{**} - \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_e}{U_r}\right) \right] - C_1 \beta^* h \omega^{**} \left(\frac{\sigma_r}{k} + \frac{\sigma_v}{k}\right) .$$

Substitute the definition of C_r and C_1 from equations (3-59.k) and (3-59.l) and simplify.

$$\left(u^{**} + \frac{U_e}{U_r}\right) \frac{y_{1,2}}{U_r} \frac{dU_r}{dx} \left(\frac{\sigma_r}{k} + \frac{\sigma_v}{k}\right) = \frac{2}{3} (1 - \hat{\alpha} - \hat{\beta}) \frac{\tau}{k} \left[hu^{**} - \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_e}{U_r}\right) \right] - C_1 \beta^* h \omega^{**} \left(\frac{\sigma_r}{k} + \frac{\sigma_v}{k}\right) . \quad (F-15)$$

Substitute the definition of C_r from equation (3-59.n) into equation (F-13).

$$2\left(u^{**} + \frac{U_e}{U_r}\right) \frac{y_{1,2}}{U_r} \frac{dU_r}{dx} \left(\frac{k-e}{k}\right) = (1 - \hat{\alpha} - \hat{\beta}) \frac{\tau}{k} \left[hu^{**} - \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_e}{U_r}\right) \right] - \beta^* h \omega^{**} \left(\frac{k-e}{k}\right)^{3/2} . \quad (F-16)$$

Multiply (F-16) by 2/3 and subtract from (F-15).

$$\left(u^{**} + \frac{U_e}{U_r}\right) \frac{y_{1,2}}{U_r} \frac{dU_r}{dx} \left[\frac{\sigma_r}{k} - \frac{\sigma_v}{k} - \frac{2}{3} \left(\frac{k-e}{k}\right)\right] = -\beta^* h \omega^{**} \left[C_1 \left(\frac{\sigma_r}{k} - \frac{\sigma_v}{k}\right) - \frac{2}{3} \left(\frac{k-e}{k}\right)^{3/2}\right] .$$

Rearrange and use subscript 2 to indicate values at the maximum-shear stress point.

$$(\beta^* h \omega^{**})_2 = -2 \left(u^{**}_2 + \frac{U_e}{U_r}\right) \frac{y_{1,2}}{U_r} \frac{dU_r}{dx} \frac{\left(\frac{\sigma_r}{k} - \frac{\sigma_v}{k}\right)_2 - \frac{2}{3} \left(\frac{k-e}{k}\right)_2}{C_1 \left(\frac{\sigma_r}{k} + \frac{\sigma_v}{k}\right)_2 - \frac{2}{3} \left(\frac{k-e}{k}\right)^{3/2}_2} . \quad (F-17)$$

$$\text{where } C_1 = 1 + C_0 \left(\frac{k-e}{k}\right)^{3/2}_2 .$$

Substitute the definition of C_r and C_1 from equation (3-59.k) and (3-59.m) into equation (F-11).

$$2\left(u^{**} + \frac{U_c}{U_r}\right) \frac{y_{1,2}}{U_r} \frac{dU_r}{dx} \frac{\sigma_r}{k} = \frac{2}{3} (2-2\hat{\alpha}+\hat{\beta}) \frac{\tau}{k} \left[hu^{**} - \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right] \\ + (2-2\hat{\alpha}+\hat{\beta}) \frac{y_{1,2}}{U_r} \frac{\tau}{k} \left(u^{**} + \frac{U_c}{U_r}\right) - C_1 \beta^* h \omega^{**} \frac{\sigma_r}{k}$$

Rearrange.

$$= \frac{2}{3} \frac{\tau}{k} \left[2 \left\{ hu^{**} + 2 \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right\} - \hat{\alpha} \left\{ 2 hu^{**} + \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right\} \right. \\ \left. + \hat{\beta} \left\{ hu^{**} + 2 \frac{y_{1,2}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) \right\} \right] \quad (F-18)$$

Similarly, substitute the definition of C_1 and C_2 from equations (3-59.l) and (3-59.m) into equation (F-12), the rearrange.

$$\left[2 \left(u^{**} - \frac{U_c}{U_r}\right) \frac{y_{1,2}}{U_r} \frac{dU_r}{dx} - C_2 \beta^* h \omega^{**} \right] \frac{\sigma_r}{k} \\ = - \frac{2}{3} \frac{\tau}{k} \left[\left\{ hu^{**} - 2 \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_r}\right) \right\} - \hat{\alpha} \left\{ hu^{**} - 2 \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_r}\right) \right\} \right. \\ \left. - \hat{\beta} \left\{ 2 hu^{**} - \frac{y_{1,2}}{R} \left(u^{**} - \frac{U_c}{U_r}\right) \right\} \right] \quad (F-19)$$

Solve for $\hat{\alpha}$ and $\hat{\beta}$ from (F-18) and (F-19) by Gaussian elimination.

$$\hat{\alpha} = \frac{\left\{ - \left| \frac{y_1}{U_r} \frac{dU_r}{dx} \left(u_z^{**} + \frac{U_c}{U_r} \right) + \frac{C_1}{2} (\beta^* h \omega^{**})_z \right| \left| \frac{2 \left(\frac{\sigma_r}{\tau} \right)_z + \left(\frac{\sigma_i}{\tau} \right)_z (hu^{**})_z}{\left(\frac{\sigma_r}{\tau} \right)_z + 2 \left(\frac{\sigma_i}{\tau} \right)_z} - \frac{y_1}{R} \left(u_z^{**} + \frac{U_c}{U_r} \right) \right| \right.}{\left. + \left| (hu^{**})_z + \frac{2y_1}{R} \left(u_z^{**} + \frac{U_c}{U_r} \right) \right| \left| (hu^{**})_z - \frac{y_1}{R} \left(u_z^{**} + \frac{U_c}{U_r} \right) \right| \right\}}{\left((hu^{**})_z^2 - \left| \frac{y_1}{R} \left(u_z^{**} + \frac{U_c}{U_r} \right) \right|^2 \right)^2} \quad (F-20)$$

and

$$\hat{\beta} = \frac{\left\{ - \left| \frac{y_1}{U_r} \frac{dU_r}{dx} \left(u_z^{**} - \frac{U_c}{U_r} \right) + \frac{C_1}{2} (\beta^* h \omega^{**})_z \right| \left| \frac{\left(\frac{\sigma_r}{\tau} \right)_z - 2 \left(\frac{\sigma_i}{\tau} \right)_z (hu^{**})_z}{2 \left(\frac{\sigma_r}{\tau} \right)_z - \left(\frac{\sigma_i}{\tau} \right)_z} - \frac{y_1}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right| \right.}{\left. - \frac{y_1}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \left| (hu^{**})_z - \frac{y_1}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right| \right\}}{\left((hu^{**})_z^2 - \left| \frac{y_1}{R} \left(u_z^{**} - \frac{U_c}{U_r} \right) \right|^2 \right)^2} \quad (F-21)$$

Substitute the definition of C_{xx} , C_{xy} , C_{xt} and C_t from equations (3-59.h).

(3-59.i), (3-59.j) and (3-59.n) into equation (F-14).

$$\begin{aligned}
 & 2\left(u^{**} - \frac{U_c}{U_r}\right) \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \frac{\tau}{k} \\
 &= \left| -\hat{\beta} \frac{\sigma_i}{k} + (1-\hat{\alpha}) \frac{\sigma_i}{k} - \frac{2}{3} \left(1-\hat{\alpha}-\hat{\beta} + \frac{3}{4}\hat{\gamma}\right) \right| \left| \left(hu^{**}\right)_z - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r}\right) \right| \\
 &- (2-\hat{\alpha}-\hat{\beta}) \left(\frac{\sigma_i}{k} - \frac{\sigma_i}{k} \right) \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r}\right) - C_1 \beta^* h \omega^{**} \frac{\tau}{k} .
 \end{aligned}$$

then rearrange.

$$\hat{\gamma} = 2 \left| \frac{\hat{\beta} \left\{ \left(\frac{\sigma_i}{k} \right)_z + \frac{2}{3} \right\} - (1-\hat{\alpha}) \left\{ \left(\frac{\sigma_i}{k} \right)_z + \frac{2}{3} \right\}}{\left\{ 2 \frac{y_{1,z}}{U_r} \frac{dU_r}{dx} \left(u^{**} - \frac{U_c}{U_r} \right) + C_1 (\beta^* h \omega^{**})_z \right\} \left(\frac{\tau}{k} \right)_z - (2-\hat{\alpha}-\hat{\beta}) \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r} \right) \left\{ \left(\frac{\sigma_i}{k} \right)_z - \left(\frac{\sigma_i}{k} \right)_z \right\}} \right| \left| \frac{\left(hu^{**} \right)_z - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r} \right)}{\left(hu^{**} \right)_z - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r} \right)} \right|$$

(F-22)

Equation (3-59.b), assuming $\omega^{**} = 0$, becomes

$$\begin{aligned}
 & \left(u^{**} - \frac{U_c}{U_r} \right) \left(\frac{y_{1,z}}{U_r} \frac{dU_r}{dx} - \frac{dy_{1,z}}{dx} \right) \omega^{**} \\
 &= \alpha \frac{\omega^{**}}{k} \tau^{**} \left| hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} + \frac{U_c}{U_r} \right) \right| - \beta^* \omega^{**} \left| hu^{**} - \frac{y_{1,z}}{R} \left(u^{**} - \frac{U_c}{U_r} \right) \right|.
 \end{aligned}$$

Divide by ω^{**} and rearrange.

$$\alpha = \left\{ \begin{aligned} & \left(u^{**} + \frac{U_e}{U_r} \right) \left(\frac{y_{1,z}}{U_r} \frac{dU_r}{dx} - \frac{dy_{1,z}}{dx} \right) - \left(\frac{\beta}{\beta^*} \right) (\beta^* h \omega^{**})_z \\ & - \beta \xi \left| (h u^{**})_z + \frac{y_{1,z}}{R} \left(u_z^{**} + \frac{U_e}{U_r} \right) \right| \end{aligned} \right\} . \quad (\text{F-23})$$

Appendix G

Derivation of Differential (k-e)/k Equation

Let's consider the total derivative of $(k-e)$.

$$\frac{D}{Dt}(k - e) = u \frac{\partial}{\partial x} (k - e) - hv \frac{\partial}{\partial y} (k - e) .$$

From equation (3.38.f),

$$\frac{D}{Dt}(k - e) = C_\tau \tau \left(h \frac{\partial u}{\partial x} - \frac{u}{R} \right) - \beta^* h \omega k \left(\frac{k - e}{k} \right)^{1/2} . \quad (G-1)$$

where $C_\tau = 1 - \hat{\alpha} - \hat{\beta}$.

Similarly,

$$\frac{Dk}{Dt} = u \frac{\partial k}{\partial x} - hv \frac{\partial k}{\partial y} .$$

From equation (3-38.a), assuming high Reynolds number,

$$\frac{Dk}{Dt} = \frac{\partial}{\partial y} \left(h \sigma^* \frac{k}{\omega} \frac{\partial k}{\partial y} \right) - \tau \left(h \frac{\partial u}{\partial x} - \frac{u}{R} \right) - \beta^* h \omega k . \quad (G-2)$$

We can write

$$\frac{D}{Dt}(k - e) = \frac{D}{Dt} \left(\frac{k - e}{k} \right) k = \left(\frac{k - e}{k} \right) \frac{Dk}{Dt} - k \frac{D}{Dt} \left(\frac{k - e}{k} \right) .$$

then rearrange,

$$k \frac{D}{Dt} \left(\frac{k - e}{k} \right) = \frac{D}{Dt} (k - e) - \left(\frac{k - e}{k} \right) \frac{Dk}{Dt} .$$

Substitute (G-1) and (G-2)

$$k \frac{D}{Dt} \left(\frac{k-e}{k} \right) = C_c \tau \left(h \frac{\partial u}{\partial x} - \frac{u}{R} \right) - \beta^* h \omega k \left(\frac{k-e}{k} \right)^{3/2} \\ - \left(\frac{k-e}{k} \right) \left[\sigma^* \frac{\partial}{\partial y} \left(h \frac{k}{\omega} \frac{\partial k}{\partial y} \right) - \tau \left(h \frac{\partial u}{\partial y} - \frac{u}{R} \right) - \beta^* h \omega k \right] .$$

Divide by k , rearrange and expand the total derivative.

$$k \frac{\partial}{\partial x} \left(\frac{k-e}{k} \right) - h y \frac{\partial}{\partial y} \left(\frac{k-e}{k} \right) = - \left(\frac{k-e}{k^2} \right) \sigma^* \frac{\partial}{\partial y} \left(h \frac{k}{\omega} \frac{\partial k}{\partial y} \right) - \left[C_c - \left(\frac{k-e}{k} \right) \right] \left[\frac{\tau}{k} \left(h \frac{\partial u}{\partial x} - \frac{u}{R} \right) \right. \\ \left. - \beta^* h \omega \left(\frac{k-e}{k} \right) \left(\sqrt{\frac{k-e}{k}} - 1 \right) \right] . \quad (G-3)$$

Appendix H

Derivation of Algebraic $(k-e)/k$ Equation

The main assumption leading to the algebraic $(k-e)/k$ equation is that the convection and diffusion of $(k-e)$ are related to the convection and diffusion of k by (Rodi, 1976)

$$\frac{D}{Dt}(k-e) - D_{k-e} = \left(\frac{k-e}{k}\right) \left(\frac{Dk}{Dt} - D_k\right) \quad (H-1)$$

where $\frac{D}{Dt}(k-e)$ = the convection of $(k-e)$.

D_{k-e} = the diffusion of $(k-e)$.

$\frac{Dk}{Dt}$ = the convection of k . and

Dk = the diffusion of k .

From equation (3-59.f),

$$\frac{D}{Dt}(k-e) - D_{k-e} = C_\epsilon \tau'' \left[hu'' - \frac{y_{1,z}}{R} \left(u'' - \frac{U_z}{U_\tau} \right) \right] - \beta^* h \omega'' k'' \left(\frac{k-e}{k} \right)^{3/2} \quad (H-2)$$

and from equation (3-59.a),

$$\frac{Dk}{Dt} - D_k = \tau'' \left[hu'' - \frac{y_{1,z}}{R} \left(u'' - \frac{U_z}{U_\tau} \right) \right] - \beta^* h \omega'' k'' \quad (H-3)$$

Substitute (H-2) and (H-3) into (H-1).

$$C_\epsilon \tau'' \left[hu'' - \frac{y_{1,z}}{R} \left(u'' - \frac{U_z}{U_\tau} \right) \right] - \beta^* h \omega'' k'' \left(\frac{k-e}{k} \right)^{3/2} = \left[\tau'' \left\{ hu'' - \frac{y_{1,z}}{R} \left(u'' - \frac{U_z}{U_\tau} \right) \right\} \right] - \beta^* h \omega'' k''$$

Divide by k'' , rearrange and simplify.

$$\left(\frac{k-e}{k} - C_e\right) \frac{\tau}{k} \left[hu'' - \frac{y_{1z}}{R} \left(u'' + \frac{U_e}{U_\tau} \right) \right] - \left(\frac{k-e}{k} \right) \left(1 - \sqrt{\frac{k-e}{k}} \right) \beta^* h \omega'' = 0 . \quad (\text{H-4})$$

Appendix I

Mathematical Analysis of the Görtler Vortices

In an analysis of the Görtler instability, the non-dimensional basic state velocity field is given by,

$$U = U(\xi^*, y^*), \text{ and}$$

$$V = V(\xi^*, y^*),$$

where $\xi^* = \varepsilon \frac{x}{L}$ = non-dimensional streamwise distance,

$y^* = \frac{y}{L}$ = non-dimensional wall-normal distance, and

$$\varepsilon = \left(\frac{V}{U_\tau L} \right)^{1/2} = \text{viscous parameter.}$$

Superimposed on the basic state are small, steady, spanwise periodic disturbances so that

$$\frac{u}{U_\tau} = U(\xi^*, y^*) + u(\xi^*, y^*) \cos \beta z^*,$$

$$\frac{v}{U_\tau} = \varepsilon V(\xi^*, y^*) + \varepsilon v(\xi^*, y^*) \cos \beta z^*, \quad (\text{I-1})$$

$$\frac{w}{U_\tau} = \varepsilon w(\xi^*, y^*) \sin \beta z^*, \text{ and}$$

$$\frac{p}{\rho U_\tau^2} = P + \varepsilon^2 p(\xi^*, y^*) \cos \beta z^*,$$

where $z^* = \frac{z}{L}$ = non-dimensional spanwise distance,

$$\beta = 2\pi \varepsilon \frac{L}{\lambda} = \text{non-dimensional spanwise wave number,} \quad \text{and}$$

λ = spanwise period between vortex pair (Figure I-1).

Figure 1-1 Görtler vortices within a boundary-layer flow over a concave wall
(Reproduced from Saric, 1994).

When the terms in equation (I-1) are substituted in the steady-state incompressible Navier-Stokes equations in curvilinear coordinates, the result is a system of disturbance equations (Saric, 1994) :

$$\begin{aligned} \frac{\partial u}{\partial \xi^*} - \frac{\partial v}{\partial y^*} - \beta w &= 0, \\ U \frac{\partial u}{\partial \xi^*} - u \frac{\partial U}{\partial y^*} - V \frac{\partial u}{\partial y^*} - v \frac{\partial U}{\partial y^*} - \frac{\partial^2 u}{\partial y^{*2}} - \beta^2 u &= 0, \\ U \frac{\partial v}{\partial \xi^*} - u \frac{\partial V}{\partial y^*} - V \frac{\partial v}{\partial y^*} - v \frac{\partial V}{\partial y^*} - 2G \frac{\partial u}{\partial y^*} - \frac{\partial p}{\partial y^*} - \frac{\partial^2 v}{\partial y^{*2}} - \beta^2 v &= 0, \\ U \frac{\partial w}{\partial \xi^*} - V \frac{\partial w}{\partial y^*} - \beta p - \frac{\partial^2 w}{\partial y^{*2}} - \beta^2 w &= 0, \end{aligned} \quad (I-2)$$

where $G = \left(\varepsilon \frac{L}{R} \right)^{1/2}$ = Görtler number, and

R = radius of curvature (Figure I-1).

The normal-mode solution, or local solution, of equation (I-2) is obtained by assuming the disturbance quantities of the form

$$\phi(\xi^*, y^*) = \phi(y^*) e^{\sigma \xi^*},$$

where ϕ represents u , v , w and p , and σ is the spatial amplification. The resulting equations form an eigenvalue problem with the parameters β , σ and G . The solution of the eigenvalue problem in the form $G = f(\beta, \sigma)$ is presented as stability diagram, such as Figure 4-2.

Appendix J

List of the Seven-Hole Probe Instrumentation Equipments

Equipment	Make	Model	Serial
Power supply	Hewlett-Packard	6291A	1144A02085
	Canadian Research Institute	MRP-25-1A	(130-1)-051
Solenoid Controller	Seanivalve Corp.	CTLR2P/S2-S6	2437
CPU Module	Sciometric Instruments		010439
Data Acq.	Sciometric Instruments	81	010438
Motor Controller	CU ETC Electronics	Drawing no. STEPV2C.SCH	
Motors - span	Superior Electronic	MA61FS-62019	
- radial	Eastern Air Devices	LA23ECK-81	
Pressure Transducers - "P"	MKS Instruments	220CD-00100.A2B	92225204A
- "Q"	Omega Technologies	PX170-28DV	9524378

Appendix K

RTFA program

Program listing available on request

Appendix L

POST program

Program listing available on request

Appendix M

Starting Profiles for the Numerical Simulation of a Wall Jet over a Circular Cylinder in Still Air

The numerical simulation of a wall jet over a circular cylinder in still air was done using the Dash et al. (1983) $k-\epsilon$ model described in Chapter 3. This model has a constant, C_ϵ of equation (3-27), that can be adjusted to account for the effects of curvature. For this calculation, C_ϵ was adjusted to match Fekete's (1963) experimental wall jet growth data for slot height to cylinder radius ratio, b/R , of 0.0074. Since Fekete did not provide any Reynolds stresses, the starting profiles of the locally scaled parameters u/U_m , k/U_m^2 and τ/U_m^2 came from Wilson's (1970) experiment on wall jet over a cylinder in still air at the 45° downstream position, as shown in Figure M-1. The comparison of slot Reynolds numbers and b/R for the Fekete, Wilson, and present experiments is shown in Table M-1.

Experiment	Re_b	$\frac{b}{R}$
Present	4,000	0.014
Fekete	3,000	0.0074
Wilson	13,200	0.0605

Table M-1 Comparison of slot Reynolds number and b/R .

As discussed in section 2.2, y_δ/R is the parameter that characterizes self-similar curved wall jet. Although the wall jet over a circular cylinder is not self-similar, it was decided that y_δ/R would be used to determine the position from which the starting profiles applied. In Wilson's experiment, $y_\delta/R=0.122$ at the 45° position where the

Figure M-1 Starting profiles of locally scaled variables from Wilson's (1970) experiment at the 45° downstream position.

starting profiles were obtained. The best fit line through Fekete's data, shown in Figure M-2, showed this value of y_1/R to correspond to $y_1/x=0.116$, thus,

$$x/R = 0.122/0.116 = 1.05,$$

or 60° downstream from the slot. Therefore, this position was where the starting profiles were applied. The y_1 and U_m from this location were used to obtain u , k and τ from the scaled profiles of Figure M-1 to start the calculation. The results for various values of C_ϵ are shown in Figure M-3. Comparison between calculation and experimental results was concentrated on the region $0 < \theta < 145^\circ$ because, according to Launder & Rodi (1981), two-dimensionality of the Fekete's experiment deteriorated beyond this point. Figure M-3 indicates that the appropriate value of C_ϵ is in the range $0.07 \leq C_\epsilon \leq 0.08$.

The starting profiles for the numerical simulation of the present experiment were obtained in a similar manner which is illustrated in Figure M-4. By extrapolating from the measured data, the point where $y_1/R=0.122$ was determined to be $x/R=1.03$ or 59° . The extrapolation of U_m was not very accurate because the correlation was non-linear. Therefore, the starting value of U_m was found by trial and error, so that the predictions of downstream values of U_m were reasonable. The final result by trial and error was $C_\epsilon=0.075$ and $U_m/U_1=0.34$ at the starting point. The results of numerical simulation are compared to the measured data of the present experiment in Figure 4-21.

Figure M-2 Determination of the calculation starting point for numerical simulation of Fekete's experiment.

Figure M-3 Comparison of calculations by the $k-\epsilon$ turbulence model with Fekete's (1970) experimental results.

Figure M-4 Determination of the starting point for numerical simulation of present experiment.

IMAGE EVALUATION TEST TARGET (QA-3)

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1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989

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