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A New Approach to Infinite Dimensional Holomorphy

by

Abolfazl Monadi, B.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

Department of Mathematics and Statistics
Ottawa-Carleton Institute of Mathematics and Statistics

Carleton University

Ottawa, Ontario

August, 1994

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A New Approach to Infinite Dimensional Holomorphy

submitted by

Abolfazl Monadi, B.Sc., M.Sc.

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

Professor V. Dlab, Chair

Department of Mathematics and Statistics

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Carleton University

1994

Abstract

We introduce a new approach to infinite dimensional holomorphy. Cast in the setting of closed-embedded linear convergence spaces and based on a categorical definition of derivative, our theory applies beyond the traditional open domains. It reaches certain domains with empty interior (that arise naturally in Fréchet spaces) and gives a fully fledged differential calculus. On open domains our approach provides a new characterization of holomorphic maps. Thus earlier theories become expanded as well as strengthened.

We also extend the new formulations of ‘Inverse Function Theorem’, ‘Implicit Function Theorem’ and ‘Newton’s Method’ initiated by L.D. Nel in 1991 and 1994. The new formulations are significantly different from previous versions in the non-normable setting in that neighborhoods become replaced by tangential ‘quasineighborhoods’ which need not be open and may even have empty interior.

Acknowledgements

I would like to thank first and foremost my thesis supervisor Professor L.D. Nel for his insightful and critical comments and discussions at each twist and turn in the long and arduous trek to the completion of this work. His patience is definitely legendary.

I would like to thank the Ferdowsi University of Mashhad, Islamic Republic of Iran for a scholarship to pursue my study in Canada.

I also gratefully acknowledge the receipt of the Ratna Devi-Dhan Chand Marwah award in 1993.

Finally and most importantly, I am especially indebted to my wife 'Simintaj' and my lovely children 'Maryam, Ali, Amir and Mona' that their constant moral support and patience made it possible for me to pursue my goals.

Introduction

The main purpose of this dissertation is to demonstrate a new approach to infinite dimensional holomorphy.

There have been numerous definitions of holomorphic maps in the infinite dimensional setting. There is one based on functions defined by 'power series', where holomorphic maps between locally convex spaces are locally expandable in Taylor-like series in terms of homogeneous polynomials (see [2]); there is one based on 'differentials', where holomorphic maps are functions with Fréchet derivative in the context of maps between Banach spaces (see [4]); and there is one based on the notion of ' G -holomorphy', where the maps are weakly holomorphic along affine lines in the context of $\mathcal{C}_c$ -maps (see [1]). Holomorphy has also been studied recently in [10] and [13]. For further historical information about this concept we refer to [23].

These previous approaches all have one common weakness: the formulation works only for open domains. Yet there are naturally arising domains that are not open e.g. consider the usual complex Fréchet space $\mathcal{H}_c(\mathbb{C}, \mathbb{C})$ of entire functions and let $U \subset \mathcal{H}_c(\mathbb{C}, \mathbb{C})$ be the subspace formed by all never vanishing maps. The 'elementary' function $f : U \rightarrow U$, $f(z) = 1/z$, is excluded by the above older theories of holomorphy (see also [10]) since its domain U is not open, in fact U has empty interior (see [13]).

Our holomorphy will enable us to extend the new formulation of 'Inverse Function theorem', 'Implicit Function Theorem' and 'Newton's Method' initiated in [18] and [14]. The new formulations are significantly different from previous versions in the nonnormable setting (cf. [6], [7]) in that neighborhoods become replaced by tangential 'quasi neighborhoods' which need not be open and may even have empty

interior (see Chapter 3).

The present approach will be based on a new real differentiation theory [21]. In this theory the real differential calculus departs from a 'foundational isomorphism', more precisely, a natural isometric linear isomorphism that represents the familiar Banach space of continuous curves on a compact interval. This foundational isomorphism contains the germ of differentiation and its inverse the germ of integration. Several calculus properties are just quick corollaries of this isomorphism.

One could get a complex version of this real theory in more than one way: (1) establish a complex analogue of the foundational isomorphism (see Chapter 1) and (2) depart directly from the real differential calculus (see Chapter 2).

Let us elaborate on the two mentioned ways of getting a complex differentiation theory. The foundational isomorphism, as formulated for real scalars does not exist in the complex case. By replacing the space of continuous curves with a certain space of analytic functions, namely those that have a local Taylor expansion, we prove in Chapter 1 a version of the foundation theorem. This theorem enables us to define 'analytically differentiable functions' on domains which may be neither open nor convex.

As in the classical complex analysis, one cannot expect to get all properties of analytic functions via power series (see [4]): one needs at some point to bring in the powerful tool of integration along curves. In our theory this takes place in Chapter 2. The real foundational isomorphism quite naturally brings integration to the very center of attention. The theory of holomorphic maps that we build here departs from the idea of 'holomorphy = real continuous differentiability + complex linearity of the derivative'. Here, real continuous differentiability is the specialization of the categorical approach to calculus developed recently in [16] and [17] that applies

to the toponome $\mathcal{C}_c$ (convergence spaces). In the elementary context the above requirement of holomorphy essentially reformulates the Cauchy-Riemann equations.

The concept of holomorphy in our context is formally stronger than that of [1], and is equivalent to G -holomorphy on τ -open domains. In Chapter 2 we also show that being weakly holomorphic is equivalent to being holomorphic along affine lines.

Holomorphy on domains with empty interior is also established in [13], which is the specialization of [17] that applies to the complex toponome $\mathcal{C}_h$ of holological spaces (see [10]). The $\mathcal{C}_h$ -based holomorphy has intimate connections with bornology and so with Banach spaces, while the $\mathcal{C}_c$ -based theory has a close relation to locally convex spaces and to continuous convergence. Thus the two theories supplement each other.

the elementary filter $\mathcal{F}$, generated by all $S_n = \{s_\nu : \nu_i \geq n_i, i \in [1..k]\}, n \in \mathbb{N}^k$, is Cauchy, where $s_\nu = \sum_{j=0}^\nu b_j q^j$. Let $\varepsilon > 0$ be given. Then there is $N \in \mathbb{N}^k$ such that

$$(*) \quad S_N - S_N \subseteq \varepsilon \mathbb{D}.$$

We claim that $(SS_N^\lambda - SS_N^\lambda)(L) \subseteq \varepsilon \mathbb{D}$, for all λ in the given polydisk, where SS_N^λ is one of the generators of the filter $\mathcal{F}_\lambda$ (see definition 1.1.9).

Let $t, p \in \mathbb{N}^k$ with $t_i, p_i \geq N_i, i \in [1..k]$. Put $d_i = \min\{t_i, p_i\}$ and $e_i = \max\{t_i, p_i\}$ and $d = (d_1, d_2, \dots, d_k), e = (e_1, e_2, \dots, e_k)$.

Suppose ss_t^λ and ss_p^λ are the partial sums of order t and p of the series $\sum a_n \lambda^n$. Then $|(ss_t^\lambda - ss_p^\lambda)(u)| \leq s_e - s_d$, for all $u \in L$ and λ in the given polydisk. Since $e_i \geq d_i \geq N_i$, by $(*)$ $s_e - s_d \leq \varepsilon$. Therefore, $|(ss_t^\lambda - ss_p^\lambda)(u)| \leq \varepsilon$, for all $u \in L$ and λ in the polydisk. Hence $(SS_N^\lambda - SS_N^\lambda)(u) \in \varepsilon \mathbb{D}$, for all λ in the polydisk. The proof of the theorem is complete.

Lemma 1.1.11 *Let $\Omega \subseteq \mathbb{C}$ be open and $f \in \mathcal{A}_c(\Omega, E)$. Let f have the local representation $f(\xi) = \sum_{n=0}^\infty a_n(\xi - \xi_0)^n$ valid in some open ball $\Lambda = B(\xi_0, r)$, where $\xi_0 \in \Omega, a_n \in E$ ($n \in \mathbb{N}$). Define $A : \Lambda \times \Lambda \rightarrow E$ by, $A(\alpha, \beta) = \sum_{n=0}^\infty (n+1)^{-1} a_n \sum_{k=0}^n (\alpha - \xi_0)^k (\beta - \xi_0)^{n-k}$. Then $A \in \mathcal{A}_c(\Lambda \times \Lambda, E)$.*

Proof. We reduce the lemma to the case that $0 \in \Omega$ and we consider the local representation of f at 0. It is enough to prove this case of the lemma for the case that E is regarded as a closed linear subspace of E^{**} .

Fix $\alpha, \beta \in \Lambda$. By theorem 1.1.10, it is enough to show that the set $M = \{a_{n+k}(n+k+1)^{-1} \alpha^n \beta^k : n+k > N, n, k \in \mathbb{N}\}$ is bounded in E^{**} , for some $N \in \mathbb{N}$. For this purpose, let $\gamma \in \mathbb{C}$ be such that $|\gamma| = \max\{|\alpha|, |\beta|\}$. By assumption $f(\gamma) = \sum_{n=0}^\infty a_n \gamma^n$ converges. Let $\mathcal{F}$ be the filter which is generated by $SS_n, n \in \mathbb{N}$, where $SS_n = \{ss_k = \sum_{i=0}^k a_i \gamma^i : k \geq n\}$. Then $\mathcal{F}$ is Cauchy in E^{**} . Therefore for every $\mathcal{L} \rightsquigarrow \ell$, in E^* , $(\mathcal{F} - \mathcal{F})(\mathcal{L}) \rightsquigarrow 0$ in $\mathbb{C}$. Let $\varepsilon > 0$ be given. Then there is $N \in \mathbb{N}$ and $L \in \mathcal{L}$ such that $(SS_N - SS_N)(L) \subseteq \varepsilon \mathbb{D}$. Hence $|(ss_k - ss_{k+1})(u)| =$

Chapter 0

Preliminaries

The outline to follow explains the setting of our theory: the category $\mathcal{C}_c$ for continuous maps and $c\mathcal{LC}_c$ for linear continuous maps. We also explain the recent theory of real differential calculus on which our holomorphy becomes based.

$\mathcal{C}_c$ -spaces are sets X equipped with convergent filters subject to a single axiom: the principal ultrafilter at x converges to x . $\mathcal{C}_c$ -maps are continuous functions $f : X \rightarrow Y$ between $\mathcal{C}_c$ -spaces: if $\mathcal{G} \rightsquigarrow x$ in X ($\mathcal{G}$ converges to x) then $f(\mathcal{G}) \rightsquigarrow f(x)$ in Y . We refer to [19] and [20] for an elementary introduction to categorical concepts and to $\mathcal{C}_c$ -based functional analysis. Here we recall just some basic facts for convenient reference. The scalar field $\mathbb{K}$ (real or complex) carries the $\mathcal{C}_c$ -structure derived from the usual metric topology. A linear $\mathcal{C}_c$ -space is a $\mathcal{C}_c$ -space E on which addition $E \times E \rightarrow E$ and scalar multiplication $\mathbb{K} \times E \rightarrow E$ has been defined so as to be $\mathcal{C}_c$ -maps, subject to the usual linear space axioms. Such spaces, together with $\mathbb{K}$ -linear continuous maps between them, build the category $\mathbb{K}\mathcal{LC}_c$, abbreviated $\mathcal{LC}_c$ when $\mathbb{K}$ is understood. Let X be a $\mathcal{C}_c$ -space, let E and F be $\mathcal{LC}_c$ -spaces. Then $\mathcal{C}_c(X, F)$ denotes the $\mathcal{LC}_c$ -space of all continuous maps $X \rightarrow F$, equipped with the continuous

convergence structure and pointwise linear operations. The $\mathcal{LC}_c$ -subspace $[E, F]$ of $\mathcal{C}_c(E, F)$ consists of all $\mathcal{LC}_c$ -maps $E \rightarrow F$. The special case $[E, \mathbb{K}]$, the dual space, is abbreviated E^* .

For effective analysis one needs spaces to be complete and separated. Moreover, the categorical differentiation theory requires a category such that, along with E , the mapping spaces $[E, F]$ and $\mathcal{C}_c(X, F)$ remain in the category; more precisely, the spaces should build an analyte [20]. This tall order cannot be filled by the classical settings. However, the full subcategory $\mathbf{c}\mathcal{LC}_c$ of closed-embedded $\mathcal{LC}_c$ -spaces comes to the rescue. It is determined by all E for which the canonical map $\alpha_E : E \rightarrow E^{**}$, $\alpha(x)(u) = u(x)$ is a closed embedding. Every complete locally convex space is a reflexive $\mathbf{c}\mathcal{LC}_c$ -space (see [20] for a proof applicable to both real and complex scalars).

Every $\mathcal{C}_c$ -subspace of a $\mathbf{c}\mathcal{LC}_c$ -space is a $\mathcal{C}_{c*}$ -space i.e. it satisfies the remaining Choquet axioms: if $\mathcal{F} \rightsquigarrow x$ then so does every refinement of $\mathcal{F}$; $\mathcal{G} \rightsquigarrow x$ holds whenever every ultrafilter refinement of $\mathcal{G}$ converges to x .

In a $\mathcal{C}_c$ -space X a subset U is *open* when for every $x \in U$ and every $\mathcal{F} \rightsquigarrow x$ in X we have $U \in \mathcal{F}$.

Recall that the following are components of natural isomorphisms in $\mathcal{C}_c$.

Cartesian Exponent Law

$$\begin{aligned} \dagger_{XYZ} : \mathcal{C}_c(X \times Y, Z) &\longrightarrow \mathcal{C}_c(X, \mathcal{C}_c(Y, Z)), \quad \dagger_{XYZ}(f)(x)(y) \stackrel{\text{def}}{=} f^1(x)(y) \stackrel{\text{def}}{=} f(x, y), \\ \ddagger_{XYZ} : \mathcal{C}_c(X, \mathcal{C}_c(Y, Z)) &\longrightarrow \mathcal{C}_c(X \times Y, Z), \quad \ddagger_{XYZ}(g)(x, y) \stackrel{\text{def}}{=} g^1(x, y) \stackrel{\text{def}}{=} g(x)(y), \end{aligned}$$

Exponential Commutation Law

$$\S_{XYZ} : \mathcal{C}_c(X, \mathcal{C}_c(Y, Z)) \longrightarrow \mathcal{C}_c(Y, \mathcal{C}_c(X, Z)), \quad \S_{XYZ}(h)(y)(x) \stackrel{\text{def}}{=} h^1(y)(x) \stackrel{\text{def}}{=} h(x)(y).$$

The following are related natural transformations in $\mathcal{C}_c$.

$$\eta_X^Y : X \longrightarrow \mathcal{C}_c(\mathcal{C}_c(X, Y), Y), \quad \eta_X^Y(x)(k) = k(x)$$

$$\text{comp}_{YZ}^X : \mathcal{C}_c(Y, X) \times \mathcal{C}_c(X, Z) \longrightarrow \mathcal{C}_c(Y, Z), \quad \text{comp}_{YZ}^X(f, g) = g \circ f$$

so on.

The categorical definition of derivative in [17] is not in terms of a limit of difference quotients. So, not surprisingly, our concept of holomorphic map ($\mathcal{H}_c$ -map) is formally quite different from that of earlier theories. However, on open domains it is usually equivalent to some known concept, thus providing a new characterization — new even in the long studied special case of a complex function of a single complex variable (see section 2).

Holomorphy has already been studied in the context of $\mathcal{C}_c$ -maps in [1]. There, on τ -open sets, they depart from the notion of G-holomorphic map (weakly holomorphic + holomorphic along affine lines). Our concept ($\mathcal{H}_c$ -map) is formally much stronger and is defined in a larger class of domains (see 2.1.1), but in section 3 we prove it equivalent to G-holomorphy on τ -open domains. An interesting spin-off fact emerges: weakly holomorphic = holomorphic along affine lines, in this context too.

In section 4 we present an interesting result, namely a Taylor-like expansion of holomorphic maps in infinite dimensional spaces, proved in [1] but with a different method.

Holomorphy on domains with empty interior is also accommodated in [13], the specialization of [17] that applies to the complex toponome $\mathcal{C}_h$ of holological spaces [10]. Whereas the $\mathcal{C}_h$ -based holomorphy has intimate connection with bornology, therefore with Banach duals, the $\mathcal{C}_c$ -based theory relates closely to complete locally convex spaces and continuous convergence. Thus the two theories supplement each other.

The use of domains with empty interior becomes crucial in this context, as the above special function $z \mapsto 1/z$ already suggests. For this approach the $\mathcal{C}_c$ -based theory seems preferable to the $\mathcal{C}_h$ -version.

linear maps.

We build our infinite dimensional theory on the $\mathbb{C}$ -valued analytic maps. All of these can be found in [4] with one exception: the theorem of Vitali (13.11.1 in [8]) that affirms completeness of the space of bounded analytic functions into a Banach space. Note that this completeness proof uses only elementary properties of power series. Relative to these prerequisites, the theory to follow is largely selfcontained.

Chapter 1

Complex Calculus via Foundation Theorems

In this chapter we develop, for $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -spaces, a complex differentiation theory analogous to the new approach in [21]. In this approach, real differential calculus arises from a foundation theorem, more precisely a natural isometric linear isomorphism, the ‘Foundational Isomorphism’.

The real foundation theorem tells us, for every nondegenerate compact interval I there is a natural isomorphism $\mathbf{ed}^I$ in $\mathbf{Ban}$ with isometric components $\mathbf{ed}^I : \mathbf{ad}\mathcal{C}_c(I \times I, E) \rightarrow \mathcal{C}_c(I, E)$, $\mathbf{ed}(A)(\lambda) = A(\lambda, \lambda)$, where $\mathbf{ad}\mathcal{C}_c(I \times I, E)$ is the Banach subspace of all elements of $\mathcal{C}_c(I \times I, E)$ that satisfy the Additivity Law (see definition 1.2.1).

It is immediately clear that the Foundational Isomorphism, as formulated for real scalars, does not exist in the complex case. However, when spaces of analytic functions are cast in the role of continuous functions then one may hope for a complex version. So we start with the set of analytic functions, namely those that have local Taylor expansion.

In the first section we consider power series in $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -spaces and some of the basic properties needed for our theory. Two versions of the foundation theorem, namely for Banach-spaces and $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -spaces, are presented in section 2. Finally, in the last

section we develop our main goal, a complex differential calculus: integral, derivative and some of their properties.

1.1 Power Series in $c\mathcal{L}_c$ -spaces

In this section we study power series in $c\mathcal{L}_c$ -spaces. We generalize some of the known facts about power series on Banach spaces which are mostly presented in [4].

Until further notice, possibly with subscripts, E, F and G denote $c\mathcal{L}_c$ -spaces, Ω an open subset of $\mathbb{C}^k$ and $\mathcal{O}$ the neighbourhood filter at zero in $\mathbb{C}$.

Definition 1.1.1 Let $x_n \in E$ ($n \in \mathbb{N}^k$). Then the series $\sum_{n \in \mathbb{N}^k} x_n$ is said to be *convergent* in E iff the elementary filter generated by its partial sums converges to an element $x \in E$. In this case, we write $\sum_{n \in \mathbb{N}^k} x_n = x$.

Lemma 1.1.2 Let $\sum_{n \in \mathbb{N}^k} x_n$ be convergent in E and $\ell \in [E, F]$. Then $\sum_{n \in \mathbb{N}^k} \ell \cdot x_n$ is convergent in F and $\sum_{n \in \mathbb{N}^k} \ell \cdot x_n = \ell \cdot \sum_{n \in \mathbb{N}^k} x_n$.

Proof. Let $s_n = \sum_{i=0}^n x_i, t_n = \sum_{i=0}^n \ell \cdot x_i = \ell \cdot s_n, S_n = \{s_k : k \geq n\}, T_n = \{t_k : k \geq n\}$ and $\mathcal{F}, \mathcal{G}$ be the filters that are generated by S_n 's and T_n 's respectively. Then by assumption $\mathcal{F} \rightsquigarrow \sum_{n \in \mathbb{N}^k} x_n \stackrel{\text{def}}{=} x$.

Therefore $\ell \cdot \mathcal{F} \rightsquigarrow \ell \cdot x$. Clearly $\mathcal{G} = \ell \cdot \mathcal{F} \rightsquigarrow \ell \cdot x$. Hence $\sum_{n \in \mathbb{N}^k} \ell \cdot x_n = \ell \cdot x = \ell \cdot \sum_{n \in \mathbb{N}^k} x_n$, as required.

Notation 1.1.3 For $n = (n_1, n_2, \dots, n_k) \in \mathbb{N}^k$ and $\xi_i \in \mathbb{C}$ ($i \in [1..k]$), we write $\xi_1^{n_1} \xi_2^{n_2} \dots \xi_k^{n_k} = \xi^n$, and we write $\lambda = (\lambda_i)$ for $(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k$.

Definition 1.1.4

- (a) A subset M of E is called *bounded* iff $M\mathcal{O} \rightsquigarrow 0$ in E .
- (b) A $\mathcal{C}_c$ -map $f : \Omega \longrightarrow E$ is said to be *weakly bounded* iff $\ell \circ f : \Omega \rightarrow \mathbb{C}$ is bounded for all $\ell \in E^*$.

(c) A map $f : \Omega \longrightarrow E$ is *analytic* means f has local power series representation i.e. for every $\lambda = (\lambda_i) \in \Omega$ there exist $r \in \mathbb{E}^k$ and a multisequence $a : \mathbb{N}^k \rightarrow E$ such that $f(\xi) = \sum_{n \in \mathbb{N}^k} a_n (\xi - \lambda)^n$ for all ξ in the polydisk $|\xi_i - \lambda_i| < r_i$ ($i \in [1..k]$).

(d) $\mathcal{A}_c(\Omega, E)$ is the $\mathcal{C}_c$ -subspace of $\mathcal{C}_c(\Omega, E)$ formed by all analytic maps $f : \Omega \longrightarrow E$ and $\text{wb}\mathcal{A}_c(\Omega, E)$ is the $\mathcal{C}_c$ -subspace of $\mathcal{A}_c(\Omega, E)$ formed by all weakly bounded analytic maps.

Definition 1.1.5 For a Banach space E , let $\text{b}\mathcal{A}_c(\Omega, E)$ denote the space of bounded analytic functions on Ω . Equip $\text{b}\mathcal{A}_c(\Omega, E)$ with the supnorm. Then by Vitali's theorem 13.14.1 in [8] and Banach valued version of Cauchy's convergence criterion 3.2 in [22] this space is complete. In fact it is a Banach algebra with identity.

Lemma 1.1.6 *Let $f \in \mathcal{A}_c(\Omega, E)$. Then $\ell \circ f \in \mathcal{A}_c(\Omega, \mathbb{C})$, for all $\ell \in E^*$. Moreover, if $f \in \text{wb}\mathcal{A}_c(\Omega, E)$ then $\ell \circ f \in \text{b}\mathcal{A}_c(\Omega, \mathbb{C})$, for all $\ell \in E^*$.*

Proof. Trivial in view of 1.1.2.

In the following paragraphs we prove some of the properties of analytic functions long known for $\mathbb{C}$ -valued maps. Similar generalizations for the Banach spaces valued case are well known (see e.g. [4]). In the present version we have to replace norms in the formulations and proofs by corresponding filter concepts. Let us start with the principle of analytic continuation.

Proposition 1.1.7 (*The Principle of Analytic Continuation*)

Let Λ, Ω be open connected subsets of $\mathbb{C}^k$ and let f, g belong to $\mathcal{A}_c(\Lambda, E)$ and $\mathcal{A}_c(\Omega, E)$ respectively. If $f = g$ on a nonempty open set $U \subseteq \Lambda \cap \Omega$, then $f = g$ on $\Lambda \cap \Omega$.

Proof. By assumption $\ell \circ f = \ell \circ g$ on U , for all $\ell \in E^*$. On the other hand by lemma 1.1.6, $\ell \circ f, \ell \circ g$ are analytic in classical sense, for all $\ell \in E^*$. Since $\ell \circ f = \ell \circ g$ on $U \subseteq \Lambda \cap \Omega$, by the known principle of analytic continuation for scalar

valued maps, $\ell \circ f = \ell \circ g$ on $\Lambda \cap \Omega$, for all $\ell \in E^*$. After canceling the monofamily formed by these ℓ , we get $f = g$ on $\Lambda \cap \Omega$.

Lemma 1.1.8 *Let Ω, Λ be bounded open subsets of $\mathbb{C}$ such that $\Lambda \subseteq \Omega$ and $f \in \mathcal{A}_c(\Omega, E)$. Then $f|_\Lambda \in \text{wb}\mathcal{A}_c(\Lambda, E)$.*

Proof. By compactness of $\bar{\Lambda}$, $(\ell \circ f)(\Lambda)$ is bounded in $\mathbb{C}$, for all $\ell \in E^*$. Hence $f|_\Lambda \in \text{wb}\mathcal{A}_c(\Lambda, E)$.

Definition 1.1.9 Let $a_n \in E$ ($n \in \mathbb{N}^k$), $\xi_i \in \mathbb{C}$, $r_i > 0$ ($i \in [1..k]$) and the series $\sum_{n \in \mathbb{N}^k} a_n(\lambda - \xi)^n$ be given. Let $SS_n^\lambda = \{ss_m^\lambda : ss_m^\lambda = \sum_{j=0}^m a_j(\lambda - \xi)^j, m_i \geq n_i, i \in [1..k]\}$ and $\mathcal{F}_\lambda$ be the filter that is generated by these SS_n^λ , $n \in \mathbb{N}^k$. The series $\sum_n a_n(\lambda - \xi)^n$ is *uniformly Cauchy* in the polydisk $|\lambda_i - \xi_i| \leq r_i$ ($i \in [1..k]$) iff for every filter $\mathcal{L} \rightsquigarrow \ell$ in E^* and every $\varepsilon > 0$, there are $M, N \in \mathbb{N}^k$ and $L \in \mathcal{L}$ such that $L(SS_N^\lambda - SS_M^\lambda) \subseteq \varepsilon \mathbb{D}$, for all λ in the given polydisk.

It is clear from the above definition that if $\sum_n a_n(\lambda - \xi)^n$ is uniformly Cauchy in a polydisk, then it converges at every point of that polydisk.

Theorem 1.1.10 *Let $a_n \in E$ ($n \in \mathbb{N}^k$), $0 \neq \xi_i \in \mathbb{C}$ ($i \in [1..k]$) and $M = \{a_n \xi^n : n \in \mathbb{N}^k\}$ be bounded in E . Let $0 < r_i < |\xi_i|$ ($i \in [1..k]$). Then $\sum_n a_n \lambda^n$ is uniformly Cauchy in the closed polydisk $|\lambda_i| \leq r_i$ ($i \in [1..k]$).*

Proof. It is enough to prove our assertion for the case that E is regarded as a closed linear subspace of E^{**} . Let $\mathcal{L} \rightsquigarrow \ell$ in E^* be arbitrary. Fix $\alpha > 0$. As M is bounded in E^{**} , $M(\mathcal{L})\mathcal{O} \rightsquigarrow 0$ in $\mathbb{C}$. Then there are $L \in \mathcal{L}$ and $P \in \mathcal{O}$ such that $M(L)P \subseteq \alpha \mathbb{D}$. Let $0 \neq \eta \in P$. Then $|a_n(u)\xi^n \eta| \leq \alpha$ or equivalently $|a_n(u)\xi^n| \leq \alpha/|\eta|$, for all $u \in L$ and $n \in \mathbb{N}^k$. Put $b = \alpha/|\eta|$. Let $\lambda_i \in \mathbb{C}$ be such that $|\lambda_i| \leq r_i$, $i \in [1..k]$. Then $|a_n(u)\lambda^n| \leq |a_n(u)r^n| \leq |a_n(u)\xi^n|q^n \leq bq^n$, for all $n \in \mathbb{N}^k$ and $u \in L$, where $q_i = r_i/|\xi_i| < 1$, $i \in [1..k]$. Since $\sum_n bq^n$ is convergent,

the elementary filter $\mathcal{F}$, generated by all $S_n = \{s_\nu : \nu_i \geq n_i, i \in [1..k]\}, n \in \mathbb{N}^k$, is Cauchy, where $s_\nu = \sum_{j=0}^\nu b q^j$. Let $\varepsilon > 0$ be given. Then there is $N \in \mathbb{N}^k$ such that

$$(*) \quad S_N - S_N \subseteq \varepsilon \mathbb{D}.$$

We claim that $(SS_N^\lambda - SS_N^\lambda)(L) \subseteq \varepsilon \mathbb{D}$, for all λ in the given polydisk, where SS_N^λ is one of the generators of the filter $\mathcal{F}_\lambda$ (see definition 1.1.9).

Let $t, p \in \mathbb{N}^k$ with $t_i, p_i \geq N_i, i \in [1..k]$. Put $d_i = \min\{t_i, p_i\}$ and $e_i = \max\{t_i, p_i\}$ and $d = (d_1, d_2, \dots, d_k), e = (e_1, e_2, \dots, e_k)$.

Suppose ss_t^λ and ss_p^λ are the partial sums of order t and p of the series $\sum a_n \lambda^n$. Then $|(ss_t^\lambda - ss_p^\lambda)(u)| \leq s_e - s_d$, for all $u \in L$ and λ in the given polydisk. Since $e_i \geq d_i \geq N_i$, by $(*)$ $s_e - s_d \leq \varepsilon$. Therefore, $|(ss_t^\lambda - ss_p^\lambda)(u)| \leq \varepsilon$, for all $u \in L$ and λ in the polydisk. Hence $(SS_N^\lambda - SS_N^\lambda)(u) \in \varepsilon \mathbb{D}$, for all λ in the polydisk. The proof of the theorem is complete.

Lemma 1.1.11 *Let $\Omega \subseteq \mathbb{C}$ be open and $f \in \mathcal{A}_c(\Omega, E)$. Let f have the local representation $f(\xi) = \sum_{n=0}^\infty a_n(\xi - \xi_0)^n$ valid in some open ball $\Lambda = B(\xi_0, r)$, where $\xi_0 \in \Omega, a_n \in E$ ($n \in \mathbb{N}$). Define $A : \Lambda \times \Lambda \rightarrow E$ by, $A(\alpha, \beta) = \sum_{n=0}^\infty (n+1)^{-1} a_n \sum_{k=0}^n (\alpha - \xi_0)^k (\beta - \xi_0)^{n-k}$. Then $A \in \mathcal{A}_c(\Lambda \times \Lambda, E)$.*

Proof. We reduce the lemma to the case that $0 \in \Omega$ and we consider the local representation of f at 0. It is enough to prove this case of the lemma for the case that E is regarded as a closed linear subspace of E^{**} .

Fix $\alpha, \beta \in \Lambda$. By theorem 1.1.10, it is enough to show that the set $M = \{a_{n+k}(n+k+1)^{-1} \alpha^n \beta^k : n+k > N, n, k \in \mathbb{N}\}$ is bounded in E^{**} , for some $N \in \mathbb{N}$. For this purpose, let $\gamma \in \mathbb{C}$ be such that $|\gamma| = \max\{|\alpha|, |\beta|\}$. By assumption $f(\gamma) = \sum_{n=0}^\infty a_n \gamma^n$ converges. Let $\mathcal{F}$ be the filter which is generated by $SS_n, n \in \mathbb{N}$, where $SS_n = \{ss_k = \sum_{i=0}^k a_i \gamma^i : k \geq n\}$. Then $\mathcal{F}$ is Cauchy in E^{**} . Therefore for every $\mathcal{L} \rightsquigarrow \ell$, in E^* , $(\mathcal{F} - \mathcal{F})(\mathcal{L}) \rightsquigarrow 0$ in $\mathbb{C}$. Let $\varepsilon > 0$ be given. Then there is $N \in \mathbb{N}$ and $L \in \mathcal{L}$ such that $(SS_N - SS_N)(L) \subseteq \varepsilon \mathbb{D}$. Hence $|(ss_k - ss_{k+1})(u)| =$

$|a_{k+1}(u)\gamma^{k+1}| \leq \varepsilon$, for all $k \geq N$ and $u \in L$. We claim that $M(L)\mathbb{D} \subseteq \mathbb{C}\mathbb{D}$ as required for the boundedness of M . To see that, consider the following inequalities:

$$|(n+k+1)^{-1}a_{n+k}(u)\alpha^n\beta^k\eta| \leq |a_{n+k}(u)\gamma^{n+k}| \leq \varepsilon,$$

for all $u \in L$, $\eta \in \mathbb{D}$ and $n+k \geq N$. By theorem 1.1.10 $\sum_{n+k \geq N} (n+k+1)^{-1}a_{n+k}\alpha^n\beta^k$ converges. So does $\sum_{n,k} (n+k+1)^{-1}a_{n+k}\alpha^n\beta^k = \sum_n (n+1)^{-1}a_n \sum_{k=0}^n \alpha^k\beta^{n-k}$.

1.2 Foundation Theorems

Having established the basics of analytic maps into $c\mathcal{LC}_c$ -spaces, we now embark on the derivation of a complex Foundation Theorem. In fact two versions of foundation theorem are presented in this section. These theorems enable us to develop the complex calculus in the next section. Let us start with the following definitions and categorical results.

Definition 1.2.1 Let E be a $c\mathcal{LC}_c$ -space. Define $\text{ad}\mathcal{A}_c(\Omega \times \Omega, E)$ to be the $\mathcal{C}_c$ -subspace of all A in $\mathcal{A}_c(\Omega \times \Omega, E)$ that satisfy the *Additivity Law*:

$$(\beta - \alpha)A(\alpha, \beta) + (\gamma - \beta)A(\beta, \gamma) + (\alpha - \gamma)A(\gamma, \alpha) = 0.$$

Clearly A is symmetric i.e. $A(\alpha, \beta) = A(\beta, \alpha)$. We define $\text{adl}\mathcal{A}_c(\Omega \times \Omega, E)$ and $\text{adwb}\mathcal{A}_c(\Omega \times \Omega, E)$ in a similar manner.

Theorem 1.2.2 Let $\Omega = \bigcup_{\alpha \in \Delta} \Omega_\alpha$ where every Ω_α is an open disk in the complex plane and $\overline{\Omega_\alpha} \subset \Omega$, $\alpha \in J$. Suppose the following triangles of subspace embeddings

represent an inductive colimit in $\mathcal{C}_c$. Then the functor $\text{wb}\mathcal{A}_c(-, E)$ converts this inductive colimit to a projective limit in $\mathcal{LC}_c$ with projective components

$$p_\alpha : \mathcal{A}_c(\Omega, E) \longrightarrow \text{wb}\mathcal{A}_c(\Omega_\alpha, E), \quad p_\alpha(f) = f|_{\Omega_\alpha}.$$

Proof. The only non-trivial part is the universal property. To see that, let $q_\alpha : G \longrightarrow \text{wb}\mathcal{A}_c(\Omega_\alpha, E)$ be the components of a rival q of p .

Define $\ell : G \longrightarrow \mathcal{A}_c(\Omega, E)$ by $\ell(g)(\lambda) = q_\alpha(g)(\lambda)$ if $\lambda \in \Omega_\alpha$. Clearly ℓ is linear, unique and $p_\alpha \circ \ell = q_\alpha$ for all α . Moreover, ℓ takes values in $\mathcal{A}_c(\Omega, E)$ by its definition. We need to show that ℓ is a $\mathcal{C}_c$ -map. Let $\mathcal{F} \rightsquigarrow g$ in G and $\mathcal{A} \rightsquigarrow \lambda$ in Ω . We must show that $\ell(\mathcal{F})(\mathcal{A}) \rightsquigarrow \ell(g)(\lambda)$. As $\lambda \in \Omega_\alpha$ for some α , $\mathcal{A} \cap \Omega_\alpha \rightsquigarrow \lambda$ in Ω_α . Hence $q_\alpha(\mathcal{F})(\mathcal{A} \cap \Omega_\alpha) = \ell(\mathcal{F})(\mathcal{A} \cap \Omega_\alpha) \rightsquigarrow q_\alpha(g)(\lambda) = \ell(g)(\lambda)$. Since $\ell(\mathcal{F})(\mathcal{A}) \supseteq \ell(\mathcal{F})(\mathcal{A} \cap \Omega_\alpha)$, so $\ell(\mathcal{F})(\mathcal{A}) \rightsquigarrow \ell(g)(\lambda)$. This completes the proof.

Theorem 1.2.3 *By the given assumptions in the previous theorem, the maps $p_\alpha : \text{ad}\mathcal{A}_c(\Omega \times \Omega, E) \longrightarrow \text{adwb}\mathcal{A}_c(\Omega_\alpha \times \Omega_\alpha, E), p_\alpha(f) = f|_{\Omega_\alpha \times \Omega_\alpha}$ are also projective limit components in $\mathcal{LC}_c$.*

Proof. Let $q_\alpha : G \longrightarrow \text{adwb}\mathcal{A}_c(\Omega_\alpha \times \Omega_\alpha, E)$ be components of a rival of p . Define $\ell : G \longrightarrow \text{ad}\mathcal{A}_c(\Omega \times \Omega, E)$ as follows: let $g \in G$ and $(\lambda, \mu) \in \Omega \times \Omega$. If $(\lambda, \mu) \in \Omega_\alpha \times \Omega_\alpha$ for some α , define $\ell(g)(\lambda, \mu) = q_\alpha(g)(\lambda, \mu)$. Otherwise $\lambda \in \Omega_\alpha, \mu \in \Omega_\beta$ for some α, β . As we have an inductive colimit diagram, there is Ω_γ such that $\lambda, \mu \in \Omega_\gamma$. In this case, define $\ell(g)(\lambda, \mu) = q_\gamma(g)(\lambda, \mu)$. Again, it is obvious that $\ell(g) \in \text{ad}\mathcal{A}_c(\Omega \times \Omega, E)$, ℓ is unique and linear, and $p_\alpha \circ \ell = q_\alpha$ for all α . We can show that ℓ is a $\mathcal{C}_c$ -map with reasoning similar to that used in the proof of theorem 1.2.2.

Definition 1.2.4 We define an n -cluster in $\mathbb{C}$ recursively as follows: every open disk Ω is a 1-cluster in $\mathbb{C}$. For $n > 1$, call Ω an n -cluster if there exists a list $\Omega_1, \dots, \Omega_n$ of open disks, call them *local disks of Ω* , such that $\Omega = \bigcup_{k=1}^n \Omega_k$ and the following conditions hold:

(C1) Every Ω_k has a non empty intersection with some other Ω_j , $j \neq k$.

(C2) There exists Ω_j such that $\bigcup_{k \neq j} \Omega_k$ is an $(n-1)$ -cluster in $\mathbb{C}$ with local disks $\Omega_1, \dots, \Omega_{j-1}, \Omega_{j+1}, \dots, \Omega_n$.

Theorem 1.2.5 (*Banach-valued Version of Foundation Theorem*)

Let Θ be a path connected and simply connected open subset of $\mathbb{C}$. Then for every n -cluster Ω in Θ there is a natural isomorphism $\mathbf{ed}_E^\Omega$ in the category **Ban** with components $\mathbf{ed}_E^\Omega : \text{adb}\mathcal{A}_c(\Omega \times \Omega, E) \longrightarrow \text{b}\mathcal{A}_c(\Omega, E)$, $\mathbf{ed}_E^\Omega(A)(\lambda) = A(\lambda, \lambda)$. Its inverse is called $\mathbf{av}_E^\Omega$. Moreover, every component is isometric.

Proof. The idea of the proof is similar to the corresponding theorem in [21]. However, the proof that $\mathbf{ed}_E^\Omega$ is surjective is different and more complicated than in the real case. The map $\mathbf{ed}_E^\Omega$ is clearly linear and continuous. The naturality requirement is also obviously satisfied. To prove $\|\mathbf{ed}_E^\Omega(A)\| = \|A\|$, it is enough show that $\|A(\alpha, \beta)\| \leq \|\mathbf{ed}_E^\Omega(A)\| = \sup_{\lambda \in \Omega} \|A(\lambda, \lambda)\|$ for all $\alpha, \beta \in \Omega$.

Let $\alpha \neq \beta$. Define $\lambda_k = \alpha + (k/n)(\beta - \alpha)$ ($k = 0, 1, \dots, n$). From the Additivity Law we get that $(\beta - \alpha)A(\alpha, \beta) = \sum_{k=1}^n (\lambda_k - \lambda_{k-1})A(\lambda_{k-1}, \lambda_k)$. Hence

$$(1) \quad \|A(\alpha, \beta)\| \leq |\beta - \alpha|^{-1} \sum_{k=1}^n |\lambda_k - \lambda_{k-1}| \|A(\lambda_{k-1}, \lambda_k)\|.$$

Fix $\varepsilon > 0$. By compactness of the line segment $[\alpha, \beta]$, A is uniformly continuous on $[\alpha, \beta] \times [\alpha, \beta]$. So there exists $\delta > 0$ such that $\|A(\lambda, \gamma) - A(\mu, \nu)\| \leq \varepsilon$ whenever $|\lambda - \mu| \leq \delta$ and $|\gamma - \nu| \leq \delta$. Take n so that $|\lambda_k - \lambda_{k-1}| = |\beta - \alpha|n^{-1} < \delta$. Then

$$(2) \quad \|A(\lambda_{k-1}, \lambda_k) - A(\lambda_k, \lambda_k)\| \leq \varepsilon,$$

for all k . It follows that $\|A(\lambda_{k-1}, \lambda_k)\| \leq \|A(\lambda_k, \lambda_k)\| + \varepsilon \leq \|\mathbf{ed}_E^\Omega(A)\| + \varepsilon$.

Using (1), we see $\|A(\alpha, \beta)\| \leq \|\mathbf{ed}_E^\Omega(A)\| + \varepsilon$, hence $\|A(\alpha, \beta)\| \leq \|\mathbf{ed}_E^\Omega(A)\|$. Thus $\mathbf{ed}_E^\Omega$ is a linear isometry, hence injective. We prove the surjectiveness of $\mathbf{ed}_E^\Omega$ in two steps as follows:

Step 1. Ω is an open disk.

We can clearly reduce to the special case $\Omega = B(0, r)$. Suppose f is a bounded

analytic function on Ω . Then $f(\xi) = \sum_{n=0}^{\infty} a_n \xi^n$, $|\xi| < r$ where $a_n \in E$. Define $\mathbf{av}_E^{\Omega}(f)(\alpha, \beta) = \sum_{n=0}^{\infty} a_n (n+1)^{-1} \sum_{k=0}^n \beta^{n-k} \alpha^k$. Then $(\mathbf{ed}_E^{\Omega} \circ \mathbf{av}_E^{\Omega})(f)(\alpha) = \mathbf{av}_E^{\Omega}(f)(\alpha, \alpha) = f(\alpha)$ and by lemma 1.1.11, $\mathbf{av}_E^{\Omega}(f) \in \mathcal{A}_c(\Omega \times \Omega, E)$. We also need to show that $\mathbf{av}_E^{\Omega}(f)$ is bounded. For this purpose, let M be a bound for f , i.e. $|f(\xi)| \leq M$, for all $\xi \in \Omega$. Let $\alpha, \beta \in \Omega$ and $\alpha \neq \beta$. As $\mathbf{av}_E^{\Omega}(f)$ is analytic on $\Omega \times \Omega$, it is continuous. In particular, it is uniformly continuous on the compact set $[\alpha, \beta] \times [\alpha, \beta]$. By repeating the given argument in the first paragraph of the proof for $A = \mathbf{av}_E^{\Omega}(f)$, we get that (see (2)):

$$\|A(\lambda_{k-1}, \lambda_k)\| \leq \|A(\lambda_k, \lambda_k)\| + \varepsilon, k \in [1..n].$$

By inequality (1) and equation $A(\lambda_k, \lambda_k) = f(\lambda_k)$ we obtain:

$$\begin{aligned} \|A(\alpha, \beta)\| &\leq |\beta - \alpha|^{-1} \sum_{k=1}^n |\lambda_k - \lambda_{k-1}| (\|f(\lambda_k)\| + \varepsilon) \\ &\leq |\beta - \alpha|^{-1} \sum_{k=1}^n |\beta - \alpha| n^{-1} (M + \varepsilon) = M + \varepsilon. \end{aligned}$$

Therefore $\|A(\alpha, \beta)\| \leq M$ for all $\alpha, \beta \in \Omega$.

Step 2. Ω is an n -cluster with local disks $\Omega_1, \dots, \Omega_n$.

In this case the map $\mathbf{av}_E^{\Omega}$ that we are looking for, must also satisfy the identity

$$(3) \quad \mathbf{av}_E^{\Omega}(f)|_{\Omega_j \times \Omega_j} = \mathbf{av}_E^{\Omega_j}(f|_{\Omega_j}),$$

for all $j = 1, 2, \dots, n$.

We prove this step by induction. If $n = 2$, then by step 1 $\mathbf{av}_E^{\Omega_1}$ and $\mathbf{av}_E^{\Omega_2}$ exist. Let $f \in b\mathcal{A}_c(\Omega, E)$ and $f_k = f|_{\Omega_k}$. Then $f_k \in b\mathcal{A}_c(\Omega_k, E)$. Put $B_{kk} \stackrel{\text{def}}{=} \mathbf{av}_E^{\Omega_k}(f_k)$. Then $B_{11}|_{(\Omega_1 \cap \Omega_2) \times (\Omega_1 \cap \Omega_2)} = B_{22}|_{(\Omega_1 \cap \Omega_2) \times (\Omega_1 \cap \Omega_2)}$ (see 9.4.3 in [4]).

Pick $\xi_1 \in (\Omega_1 \cap \Omega_2)$. Define the map $B_{12} : \Omega_1 \times \Omega_2 \rightarrow E$ as follows: for $\lambda \in \Omega_1, \mu \in \Omega_2, \mu \neq \lambda$, $B_{12}(\lambda, \mu) = (\xi_1 - \lambda)(\mu - \lambda)^{-1} B_{11}(\lambda, \xi_1) + (\mu - \xi_1)(\mu - \lambda)^{-1} B_{22}(\xi_1, \mu)$ and for $\xi \in (\Omega_1 \cap \Omega_2)$ put $B_{12}(\xi, \xi) \stackrel{\text{def}}{=} B_{11}(\xi, \xi) = B_{22}(\xi, \xi)$. Define $B_{21} : \Omega_2 \times \Omega_1 \rightarrow E$ by $B_{21}(\mu, \lambda) = B_{12}(\lambda, \mu)$. So the maps B_{kj} agree where their domains overlap. Define $B : \Omega \times \Omega \rightarrow E$ by putting $B|_{\Omega_k \times \Omega_j} = B_{kj}$ ($k, j = 1, 2$). By definition of B , it is clear that B is analytic and by the principle of analytic

continuation it is well defined.

Claim 1. B satisfies the Additivity Law.

Proof 1. Let $\alpha, \beta, \gamma \in \Omega$. It is enough to deal with the case $\alpha, \beta \in \Omega_1, \gamma \in \Omega_2$ and $\alpha \neq \gamma, \beta \neq \gamma$, since the other cases are similar or trivial. Consider the following algebraic manipulation, where ξ_1 is the complex number that appeared in the definition of B :

$$\begin{aligned} & (\beta - \alpha)B(\alpha, \beta) + (\gamma - \beta)B(\beta, \gamma) + (\alpha - \gamma)B(\gamma, \alpha) = \\ & [-(\xi_1 - \beta)B_{11}(\beta, \xi_1) - (\alpha - \xi_1)B_{11}(\xi_1, \alpha)] + \\ & (\gamma - \beta)[(\xi_1 - \beta)(\gamma - \beta)^{-1}B_{11}(\beta, \xi_1) + (\gamma - \xi_1)(\gamma - \beta)^{-1}B_{22}(\xi_1, \gamma)] + \\ & (\alpha - \gamma)[(\xi_1 - \alpha)(\gamma - \alpha)^{-1}B_{11}(\alpha, \xi_1) + (\gamma - \xi_1)(\gamma - \alpha)^{-1}B_{22}(\xi_1, \gamma)] = 0. \end{aligned}$$

Define $\mathbf{av}_E^\Omega(f) = B$.

Claim 2. $\mathbf{ed}_E^\Omega \circ \mathbf{av}_E^\Omega = \text{id}_{\mathbf{b}\mathcal{A}_c(\Omega, E)}$, $\mathbf{av}_E^\Omega \circ \mathbf{ed}_E^\Omega = \text{id}_{\mathbf{ad}\mathbf{b}\mathcal{A}_c(\Omega \times \Omega, E)}$ and $\mathbf{av}_E^\Omega$ is an $\mathcal{LC}_c$ -map.

Proof 2. Since $\text{sub}_1 : \Omega_1 \longrightarrow \Omega$ and $\text{sub}_2 : \Omega_2 \longrightarrow \Omega$ are inductive colimit components, the claim follows via theorem 1.2.2.

Now fix k and suppose for every n -cluster Ω with local disks $\Omega_1, \dots, \Omega_n$, $n \leq k$, $\mathbf{av}_E^\Omega$ exists in terms of chosen ξ_j ($0 < j < n$), and for every $f \in \mathbf{b}\mathcal{A}_c(\Omega, E)$, $\mathbf{av}_E^\Omega(f)$ satisfies equation (3). We must prove the theorem for a $(k+1)$ -cluster in $\mathcal{C}$. Let Ω be an $(k+1)$ -cluster with local disks $\Omega_1, \dots, \Omega_{k+1}$ in $\mathcal{C}$. Let $\Omega_1, \dots, \Omega_{k+1}$ be relabelled such that $\Omega_j \cap \Omega_{j+1} \neq \emptyset$, $1 \leq j < k+1$. Put $\Lambda = \bigcup_{j=1}^k \Omega_j$. By the induction hypothesis $\mathbf{av}_E^\Lambda$ exists and it is defined via complex numbers $\xi_m \in (\Omega_m \cap \Omega_{m+1})$, $m = 1, 2, \dots, k-1$. Now similar to the argument given in the starting point, we define $\mathbf{av}_E^\Omega$ by using fixed $\xi_k \in (\Omega_k \cap \Omega_{k+1})$. If Ω_{k+1} intersects Ω_p , for some $p \neq k$, then $\mathbf{av}_E^\Omega$ can be also defined via a fixed point $\xi_{k+1} \in \Omega_{k+1} \cap \Omega_p$. Now our task is to show these two definitions for $\mathbf{av}_E^\Omega$ coincide.

Let $\mathbf{av}_k^\Omega$ and $\mathbf{av}_p^\Omega$ be the two average functions which are defined via ξ_k and ξ_{k+1} respectively. To prove $\mathbf{av}_k^\Omega = \mathbf{av}_p^\Omega$, let $f \in \mathcal{A}(\Omega, E)$ and $x, y \in \Omega$. It is enough to show:

$$\mathbf{av}_k^\Omega(f)(x, y) = \mathbf{av}_p^\Omega(f)(x, y),$$

for the case that $x \in \Omega_{k+1}$ and $y \in \Omega_n$, for some $n \neq k$, since the other cases are similar. By definition of $\mathbf{av}_k^\Omega, \mathbf{av}_p^\Omega$, the above equation is equivalent to:

$$(4) \quad (\xi_k - x)(y - x)^{-1} \mathbf{av}_E^\Lambda(f)(x, \xi_k) + (y - \xi_k)(y - x)^{-1} \mathbf{av}_E^{\Omega_{k+1}}(f)(\xi_k, y) = \\ (\xi_{k+1} - x)(y - x)^{-1} \mathbf{av}_E^\Lambda(f)(x, \xi_{k+1}) + (y - \xi_{k+1})(y - x)^{-1} \mathbf{av}_E^{\Omega_{k+1}}(f)(\xi_{k+1}, y).$$

In view of the definitions, this can be shown to be equivalent to each of the following equations:

$$(\xi_k - x) \mathbf{av}_E^\Lambda(f)(x, \xi_k) + (x - \xi_{k+1}) \mathbf{av}_E^\Lambda(f)(\xi_{k+1}, x) + \\ (y - \xi_k) \mathbf{av}_E^{\Omega_{k+1}}(f)(\xi_k, y) + (\xi_{k+1} - y) \mathbf{av}_E^{\Omega_{k+1}}(f)(y, \xi_{k+1}) = 0, \\ (\xi_k - \xi_{k+1}) \mathbf{av}_E^\Lambda(f)(\xi_{k+1}, \xi_k) + (\xi_{k+1} - \xi_k) \mathbf{av}_E^{\Omega_{k+1}}(f)(\xi_k, \xi_{k+1}) = 0, \\ (\xi_p - \xi_{k+1}) A_p(\xi_{k+1}, \xi_p) + (\xi_{p+1} - \xi_p) A_{p+1}(\xi_p, \xi_{p+1}) + \cdots + \\ (\xi_{k+1} - \xi_k) A_{k+1}(\xi_k, \xi_{k+1}) = 0,$$

where $A_j = \mathbf{av}_E^{\Omega_j}(f|_{\Omega_j})$, $p \leq j \leq k+1$. For simplicity we represent the above equation by:

$$(5) \quad A[\xi_p, \xi_{p+1}, \dots, \xi_{k+1}, \xi_p] = 0$$

Now we prove our assertion by the simple connectedness of Θ and the fact that for every $\alpha_1, \alpha_2, \dots, \alpha_j \in B = B(z, r)$, $\mathbf{av}_E^B(f)[\alpha_1, \alpha_2, \dots, \alpha_j, \alpha_1] = 0$.

Let γ_1 be the curve that traces out the segments $[\xi_p, \xi_{p+1}], \dots, [\xi_k, \xi_{k+1}]$ in succession and γ_2 be the curve which has the line segment $[\xi_p, \xi_{k+1}]$ as its trace; then γ_1 and γ_2 are homotopic. So there is a continuous function $\Gamma : I^2 \rightarrow \Omega$ satisfying:

$$\Gamma(s, 0) = \gamma_1(s) \text{ and } \Gamma(s, 1) = \gamma_2(s)$$

$$\Gamma(0, t) = \Gamma(1, t), \quad 0 \leq t \leq 1.$$

Since I^2 is compact the distance $r = d(\Gamma(I^2), \mathbb{C} - \Omega)$ is positive. Since Γ is uniformly continuous there is an integer m such that if $\|(s, t) - (s', t')\| < 2/m$ then $|\Gamma(s, t) - \Gamma(s', t')| < r$. Choose s_i so that $\Gamma(s_i, 0) = \gamma_1(s_i) = \xi_i$, $i = p, p+1, \dots, k+1$ and put $\mathbb{F} = \{0, 1/m, 2/m, \dots, 1\} \cup \{s_p, s_{p+1}, \dots, s_{k+1}\}$. Then we can relabel so that $\mathbb{F} = \{w_0 = 0, w_1, \dots, w_q = 1\}$ ($q < m + k - p + 2$). Put $\eta_{v,j} = \Gamma(w_i, w_j)$ and $R_{v,j} = [w_v, w_{v+1}] \times [w_j, w_{j+1}]$, for $0 \leq v, j < q-1$. Since the diameter of the rectangle $R_{v,j}$ is less or equal $\sqrt{2}/m$, it follows that $\Gamma(R_{v,j}) \subseteq B(\eta_{v,j}, r)$. So if we let $P_{v,j}$ be the closed polygon $[\eta_{v,j}, \eta_{v+1,j}, \eta_{v+1,j+1}, \eta_{v,j+1}, \eta_{v,j}]$, then because disks are convex $P_{v,j} \subset B(\eta_{v,j}, r)$. Put $A_{v,j} = \mathbf{av}_E^{R(\eta_{v,j}, r)}(f)$, then we claim that:

$$(6) \quad A[\xi_1, \xi_2, \dots, \xi_{k+1}, \xi_1] = \sum_{v=0}^{q-1} \sum_{j=0}^{q-1} A_{v,j} [\eta_{v,j}, \eta_{v+1,j}, \eta_{v+1,j+1}, \eta_{v,j+1}, \eta_{v,j}].$$

To prove (6) consider the following algebraic manipulation:

$$\begin{aligned} A[\xi_p, \xi_{p+1}, \dots, \xi_{k+1}, \xi_p] \\ &= \sum_{v=0}^{q-1} (\eta_{v+1,0} - \eta_{v,0}) A_{v,0}(\eta_{v,0}, \eta_{v+1,0}) + \sum_{v=0}^{q-1} (\eta_{v,q} - \eta_{v+1,q}) A_{v,q}(\eta_{v+1,q}, \eta_{v,q}) \\ &= \sum_{v=0}^{q-1} G_v, \end{aligned}$$

where

$$\begin{aligned} G_0 &= [(\eta_{1,0} - \eta_{0,0}) A_{0,0}(\eta_{0,0}, \eta_{1,0}) + (\eta_{1,q} - \eta_{1,0}) B_1(\eta_{1,0}, \eta_{1,q}) \\ &\quad + (\eta_{0,q} - \eta_{1,q}) A_{0,q}(\eta_{1,q}, \eta_{0,q})], \\ G_v &= [(\eta_{v+1,0} - \eta_{v,0}) A_{v,0}(\eta_{v,0}, \eta_{v+1,0}) + (\eta_{v+1,q} - \eta_{v+1,0}) B_{v+1}(\eta_{v+1,0}, \eta_{v+1,q}) \\ &\quad + (\eta_{v,q} - \eta_{v+1,q}) A_{v,q}(\eta_{v+1,q}, \eta_{v,q}) + (\eta_{v,0} - \eta_{v,q}) B_v(\eta_{v,0}, \eta_{v,q})], \\ G_{q-1} &= [(\eta_{q,0} - \eta_{q-1,0}) A_{q-1,0}(\eta_{q-1,0}, \eta_{q,0}) + (\eta_{q-1,q} - \eta_{q,q}) A_{q-1,q}(\eta_{q,q}, \eta_{q-1,q}) \\ &\quad + (\eta_{q-1,0} - \eta_{q-1,q}) B_{q-1}(\eta_{q-1,q}, \eta_{q-1,0})] \end{aligned}$$

and

$$(\eta_{v,q} - \eta_{v,0}) B_v(\eta_{v,0}, \eta_{v,q}) = \sum_{j=0}^{q-1} (\eta_{v,j+1} - \eta_{v,j}) A_{v-1,j}(\eta_{v,j}, \eta_{v,j+1}), \quad v \in [1, (q-1)].$$

On the other hand:

$$G_v = \sum_{j=0}^{q-1} A_{v,j} [\eta_{v,j}, \eta_{v+1,j}, \eta_{v+1,j+1}, \eta_{v,j+1}, \eta_{v,j}].$$

Hence the validity of (6) follows. Since the right hand side of (6) clearly vanishes, the proof of step 2 is complete.

The proof of the theorem is complete.

Theorem 1.2.6 (*c $\mathcal{LC}_c$ -valued Version of the Foundation Theorem*)

For every path connected and simply connected open subset Ω of $\mathbb{C}$, there is a natural isomorphism $\mathbf{ed}_E^\Omega$ in the category $\mathcal{LC}_c$ with components $\mathbf{ed}_E^\Omega : \mathbf{ad}\mathcal{A}_c(\Omega \times \Omega, E) \longrightarrow \mathcal{A}_c(\Omega, E)$, $\mathbf{ed}_E^\Omega(A)(\lambda) = A(\lambda, \lambda)$. Its inverse is called $\mathbf{av}_E^\Omega$.

Proof. Similar to the corresponding theorem 1.2.5, $\mathbf{ed}_E^\Omega$ is clearly an $\mathcal{LC}_c$ -map. The naturality is also obvious. To prove it is injective, let $A, B \in \mathcal{A}_c(\Omega \times \Omega, E)$ such that $\mathbf{ed}_E^\Omega(A) = \mathbf{ed}_E^\Omega(B)$. Then $(\ell \circ A)(\lambda, \lambda) = (\ell \circ B)(\lambda, \lambda)$, for all $\ell \in E^\star$ and $\lambda \in \Omega$. By 9.1.3 in [4], $\ell \circ A = \ell \circ B$, for all $\ell \in E^\star$. After cancelling the monofamily of such ℓ , we arrive at $A = B$ as required. We prove the surjectiveness of $\mathbf{ed}_E^\Omega$ in three steps as follows:

Step 1. Ω is an open disk and $\mathbf{ed}_E^\Omega : \mathbf{adwb}\mathcal{A}_c(\Omega \times \Omega, E) \longrightarrow \mathbf{wb}\mathcal{A}_c(\Omega, E)$.

For simplicity, suppose $\Omega = B(0, r)$. Let $f \in \mathbf{wb}\mathcal{A}_c(\Omega, E)$. Then it has the representation $f(\xi) = \sum_n a_n \xi^n$, $|\xi| < r$ and $a_n \in E$, for all n .

We define $\mathbf{av}_E^\Omega(f)$ as follows (see proof of 1.2.5 for a similar definition):

$$\mathbf{av}_E^\Omega(f)(\alpha, \beta) = \sum_{n=0}^{\infty} (n+1)^{-1} a_n \sum_{k=0}^n \alpha^k \beta^{n-k}, \quad |\alpha|, |\beta| < r.$$

We need to show that $\mathbf{av}_E^\Omega$ takes values in $\mathbf{adwb}\mathcal{A}_c(\Omega \times \Omega, E)$. Clearly $\mathbf{av}_E^\Omega$ satisfies the Additivity Law. By lemma 1.1.11, $\mathbf{av}_E^\Omega(f) \in \mathcal{A}_c(\Omega \times \Omega, E)$. Hence $\mathbf{av}_E^\Omega(f) \in \mathbf{ad}\mathcal{A}_c(\Omega \times \Omega, E)$. We must also show that $\mathbf{av}_E^\Omega(f)$ is weakly bounded. As $f \in \mathbf{wb}\mathcal{A}_c(\Omega, E)$, $\ell \circ f : \Omega \longrightarrow \mathbb{C}$ is bounded, for all $\ell \in E^\star$. By lemma 1.1.2, $(\ell \circ f)(\xi) = \sum_{n=0}^{\infty} \ell(a_n) \xi^n$. Similarly, $(\ell \circ \mathbf{av}_E^\Omega(f))(\alpha, \beta) = \sum_{n=0}^{\infty} (n+1)^{-1} \ell(a_n) \sum_{k=0}^n \alpha^k \beta^{n-k}$. By an argument similar to the given one in the step 1 of the theorem 1.2.5, $(\ell \circ \mathbf{av}_E^\Omega(f))(\Omega \times \Omega)$ is bounded in $\mathbb{C}$, for all $\ell \in E^\star$.

It remains to prove the continuity of $\mathbf{av}_E^\Omega$. As before, we may regard E as a closed linear subspace of E^{**} . Let $\mathcal{F} \rightsquigarrow f$ in $\mathbf{wb}\mathcal{A}_c(\Omega, E)$, $\mathcal{L} \rightsquigarrow \ell$ in E^* be arbitrary but fixed and $(\alpha, \beta) \in \Omega \times \Omega$. Let $\mathcal{F}_\mathcal{L}$ be the filter which is generated by $\{F_L : F \in \mathcal{F}, L \in \mathcal{L}\}$, where $F_L = \{g_u : \Omega \longrightarrow \mathbb{C} | g_u(\lambda) = g(\lambda)(u), g \in F, u \in L\}$. In fact $\mathcal{F}_\mathcal{L}$ is a filter in $\mathbf{b}\mathcal{A}_c(\Omega, \mathbb{C})$. Since $\mathcal{F} \rightsquigarrow f$, $\mathcal{F}(\alpha + \mathcal{O})(\mathcal{L}) \stackrel{\text{def}}{=} \mathcal{F}_\mathcal{L}(\alpha + \mathcal{O}) \rightsquigarrow f(\alpha)(\ell) \stackrel{\text{def}}{=} f_\ell(\alpha)$. Hence $\mathcal{F}_\mathcal{L} \rightsquigarrow f_\ell$ in $\mathbf{b}\mathcal{A}_c(\Omega, \mathbb{C})$. For continuity of $\mathbf{av}_E^\Omega$ we must show that

$$(1) \quad \mathbf{av}_E^\Omega(\mathcal{F})(\mathcal{H})(\mathcal{L}) \rightsquigarrow \mathbf{av}_E^\Omega(f)(\alpha, \beta)(\ell) \text{ in } \mathbb{C},$$

for all filters $\mathcal{H} \rightsquigarrow (\alpha, \beta)$ in $\Omega \times \Omega$. By continuity of $\mathbf{av}_\mathbb{C}^\Omega$ we have:

$$(2) \quad \mathbf{av}_\mathbb{C}^\Omega(\mathcal{F}_\mathcal{L})(\mathcal{H}) \rightsquigarrow \mathbf{av}_\mathbb{C}^\Omega(f_\ell)(\alpha, \beta) \text{ in } \mathbb{C},$$

for all filters $\mathcal{H} \rightsquigarrow (\alpha, \beta)$ in $\Omega \times \Omega$.

Claim 1. $\mathbf{av}_\mathbb{C}^\Omega(f_\ell)(\alpha, \beta) = (\mathbf{av}_E^\Omega(f)(\alpha, \beta))(\ell)$.

Proof 1. By definition of $\mathbf{av}_E^\Omega$ and lemma 1.1.2, we have:

$$\begin{aligned} (3) \quad f_\ell(\xi) &= f(\xi)(\ell) = (\sum_n a_n \xi^n)(\ell) \\ &= \mathbf{eval}_\ell(\sum_n a_n \xi^n) = \sum_n \mathbf{eval}_\ell(a_n \xi^n) \\ &= \sum_n a_n(\ell) \xi^n \end{aligned}$$

and

$$\begin{aligned} (4) \quad \mathbf{av}_\mathbb{C}^\Omega(f_\ell)(\alpha, \beta) &= \sum_{n=0}^{\infty} (n+1)^{-1} a_n(\ell) \sum_{k=0}^n \alpha^k \beta^{n-k} \\ &= (\sum_{n=0}^{\infty} (n+1)^{-1} a_n \sum_{k=0}^n \alpha^k \beta^{n-k})(\ell) \\ &= (\mathbf{av}_E^\Omega(f)(\alpha, \beta))(\ell). \end{aligned}$$

Let $\mathcal{H} \rightsquigarrow (\alpha, \beta)$ be arbitrary.

Claim 2. $\mathbf{av}_\mathbb{C}^\Omega(\mathcal{F}_\mathcal{L})(\mathcal{H}) = \mathbf{av}_E^\Omega(\mathcal{F})(\mathcal{H})(\mathcal{L})$.

Proof 2. It is enough to show that:

$$(5) \quad \mathbf{av}_\mathbb{C}^\Omega(F_L)(H) = \mathbf{av}_E^\Omega(F)(H)(L).$$

for all $F \in \mathcal{F}, L \in \mathcal{L}$ and $H \in \mathcal{H}$. For this purpose, choose F, L, H as above and let $g \in F, u \in L$ and $(\eta, \lambda) \in H$. Let $g(\xi) = \sum_n b_n \xi^n$, $b_n \in E$ ($n \in \mathbb{N}$), be the representation of $g, |\xi| < r$. Similar to (3) and (4) we have $g_u(\xi) = \sum_n b_n(u) \xi^n$ and $\mathbf{av}_E^u(g_u)(\eta, \lambda) = (\mathbf{av}_E^u(g)(\eta, \lambda))(u)$. Therefore, the validity of (5) follows and the proof of claim 2 is complete.

Now (1) follows via claim 1, claim 2 and (2).

Step 2. Ω is an n -cluster in $\mathbb{C}$ and $\mathbf{ed}_E^\Omega : \mathbf{adwb}\mathcal{A}_c(\Omega \times \Omega, E) \longrightarrow \mathbf{wb}\mathcal{A}_c(\Omega, E)$.

By repeating the given argument in the theorem 1.2.5, and proposition 1.1.7, the proof will follow.

Step 3. Ω is an arbitrary path connected and simply connected subset of $\mathbb{C}$.

Fix arbitrary $\alpha \in \Omega$. Then for ever $\beta \in \Omega$ there is a path γ_β from α to β . The image of this path can be covered by finitely many open disks $\Omega_1^\beta, \Omega_2^\beta, \dots, \Omega_{n_\beta}^\beta$ such that $\overline{\Omega_{n_\beta}^\beta} \subseteq \Omega$ and $\Omega_j^\beta \cap (\cup_{k \neq j} \Omega_k^\beta) \neq \emptyset$, for $j = 1, 2, \dots, n_\beta$. Put $\Omega^\beta = \cup_{j \in [1..n_\beta]} \Omega_j^\beta$ then $\Omega = \cup_{\beta \in \Omega} \Omega^\beta$. On the other hand we can write $\Omega = \cup_\theta \Omega_\theta$, where Ω_θ is a finite union of Ω^β 's. Clearly every Ω_θ is a k -cluster, for some $k \in \mathbb{N}$. Since the following triangles of subspace embeddings

$$\begin{array}{ccc}
 \Omega_\theta & \xrightarrow{\text{sub}_\theta} & \Omega \\
 \searrow \text{sub}_{\theta\lambda} & & \nearrow \text{sub}_\lambda \\
 & \Omega_\lambda &
 \end{array}$$

represent an inductive colimit in $\mathcal{C}_c$, it will by theorem 1.2.2 be converted to a projective limit with the following components:

$$\mathcal{A}_c(\text{sub}_\theta, E) : \mathcal{A}_c(\Omega, E) \longrightarrow \mathbf{wb}\mathcal{A}_c(\Omega_\theta, E)$$

Similarly, the subspace embeddings $\Omega_\theta \times \Omega_\theta \longrightarrow \Omega_\lambda \times \Omega_\lambda \longrightarrow \Omega \times \Omega$ build an inductive colimit. By theorem 1.2.3, that inductive colimit diagram is converted to a projective limit with the following components:

$$\text{ad}\mathcal{A}_c(\text{sub}_\theta \times \text{sub}_\theta, E) : \text{ad}\mathcal{A}_c(\Omega \times \Omega, E) \longrightarrow \text{adwb}\mathcal{A}_c(\Omega_\theta \times \Omega_\theta, E).$$

We can also prove routinely that the following diagram commutes:

$$\begin{array}{ccc} \text{wb}\mathcal{A}_c(\Omega_\theta, E) & \xrightarrow{\text{av}_E^{\Omega_\theta}} & \text{adwb}\mathcal{A}_c(\Omega_\theta \times \Omega_\theta, E) \\ \downarrow \text{wb}\mathcal{A}_c(\text{sub}_{\theta\lambda}, E) & & \downarrow \text{adwb}\mathcal{A}_c(\text{sub}_{\theta\lambda} \times \text{sub}_{\theta\lambda}, E) \\ \text{wb}\mathcal{A}_c(\Omega_\lambda, E) & \xrightarrow{\text{av}_E^{\Omega_\lambda}} & \text{adwb}\mathcal{A}_c(\Omega_\lambda \times \Omega_\lambda, E) \end{array}$$

Hence, by the universal property of projective limits, there are $\mathcal{LC}_c$ -maps av_E^Ω and ed_E^Ω such that the following rectangle commutes:

$$\begin{array}{ccc} \mathcal{A}_c(\Omega, E) & \xrightarrow{\text{av}_E^\Omega} & \text{ad}\mathcal{A}_c(\Omega \times \Omega, E) \\ \downarrow \mathcal{A}_c(\text{sub}_\theta, E) & & \downarrow \text{ad}\mathcal{A}_c(\text{sub}_\theta \times \text{sub}_\theta, E) \\ \text{wb}\mathcal{A}_c(\Omega_\theta, E) & \xrightarrow{\text{av}_E^{\Omega_\theta}} & \text{adwb}\mathcal{A}_c(\Omega_\theta \times \Omega_\theta, E) \end{array}$$

Since every Ω_θ is a k -cluster for some k , by step 2, we have an isomorphism $\text{ed}_E^{\Omega_\theta}$. By uniqueness of limit-induced maps, it follows that ed_E^Ω and av_E^Ω are mutual inverses. The proof of step 3 is complete.

The proof of the theorem is complete.

1.3 $\mathcal{A}_c^r$ -maps on Analytically Tangential Domains

In this section we introduce the new domains, namely analytically tangential domains for which analyticity will be developed. We will also show how well these domains serve as domains for calculus. In this section proofs and definitions are mostly similar to the corresponding ones in [17]. This is not surprising in view of the formulations in [21] and [17] of purely categorical nature.

Henceforth let Ω be a path connected and simply connected domain in $\mathbb{C}$ and E, F and G be $\mathcal{CLC}_c$ -spaces.

Definition 1.3.1 A map $g \in \mathcal{C}_c(\Omega, E)$ is called an *analytic path* iff there exists $A_g \in \mathcal{A}_c(\Omega \times \Omega, E)$ such that

$$(1) \quad g(z) - g(\xi) = (z - \xi)A_g(\xi, z) \quad (z, \xi \in \Omega).$$

Clearly $A_g \in \text{ad}\mathcal{A}_c(\Omega \times \Omega, E)$ and is uniquely determined by g . Define

$$(2) \quad g'(\xi) \stackrel{\text{def}}{=} \text{ed}_E^\Omega(A_g)(\xi) = A_g(\xi, \xi). \text{ Then}$$

$$(3) \quad g'(\xi) = \lim_{(\lambda, \mu) \rightsquigarrow (\xi, \xi)} \frac{g(\mu) - g(\lambda)}{\mu - \lambda}.$$

Note that this is a formally stronger statement than to require the existence of $\lim_{\lambda \rightsquigarrow \xi} \frac{g(\lambda) - g(\xi)}{\lambda - \xi}$ at each ξ .

Definition 1.3.2 For $f \in \mathcal{A}_c(\Omega, E)$ we define

$$(4) \quad \oint_\alpha^\beta f \stackrel{\text{def}}{=} \oint_\alpha^\beta f(\xi) d\xi \stackrel{\text{def}}{=} (\beta - \alpha) \text{av}_E^\Omega(f)(\alpha, \beta) \quad (\alpha, \beta \in \Omega).$$

From the above definition we clearly obtain a $\mathcal{C}_c$ -map

$$(5) \quad \oint : \Omega \times \Omega \times \mathcal{A}_c(\Omega, E) \longrightarrow E, \oint(\alpha, \beta, f) = \oint_\alpha^\beta f, \text{ linear in the variable } f.$$

Naturality of av_E^Ω , just restated, gives:

$$(6) \quad \oint_\alpha^\beta u \cdot f(\xi) d\xi = u \cdot \oint_\alpha^\beta f(\xi) d\xi \quad (u \in [E, F]). \text{ In particular, if } f \in \mathcal{A}_c(\Omega, \mathcal{C}_c(X, E)) \text{ and } u = \text{eval}_x, \text{ then}$$

$$\oint_{\alpha}^{\beta} f(\xi)(x)d\xi = (\oint_{\alpha}^{\beta} f(\xi)d\xi)(x).$$

The Additivity Law just restated, gives the finite additivity:

$$(7) \quad \oint_{\alpha}^{\beta} f = \oint_{\alpha}^{\gamma} f + \oint_{\gamma}^{\beta} f.$$

By definition of an analytic path and of the integral, see (1) and (4), we have at once a version of the Fundamental Theorem of Calculus:

$$(8) \quad g(\beta) - g(\alpha) = \oint_{\alpha}^{\beta} f(\xi)d\xi, \text{ where } g' = f$$

and also it is easy to see that

$$(9) \quad \text{If } f \in \mathcal{A}_c(\Omega, E) \text{ then } g' = f \text{ holds for some analytic path } g, \text{ unique up to a constant function; in particular for } g(\lambda) = \oint_{\alpha}^{\lambda} f(\xi)d\xi.$$

Proposition 1.3.3 *If f_1 and f_2 in $\mathcal{A}_c(\Omega, E)$ are such that $\oint_{\alpha}^{\beta} f_1(\xi)d\xi = \oint_{\alpha}^{\beta} f_2(\xi)d\xi$ for all $\alpha, \beta \in \Omega$, then $f_1 = f_2$.*

Proof. Fix $\alpha \in \Omega$. Then for all $\lambda \neq \alpha$ and $\ell \in E^*$ we have $\ell \cdot \mathbf{av}_E^{\Omega}(f_1)(\alpha, \lambda) = \ell \cdot \mathbf{av}_E^{\Omega}(f_2)(\alpha, \lambda)$. If $\lambda \rightsquigarrow \alpha$, then we have $\ell \cdot \mathbf{av}_E^{\Omega}(f_1)(\alpha, \alpha) = \ell \cdot \mathbf{av}_E^{\Omega}(f_2)(\alpha, \alpha)$, hence $\ell \cdot f_1(\alpha) = \ell \cdot f_2(\alpha)$. After cancelling the monofamily formed by such ℓ , we obtain the result.

Definition 1.3.4 Let $U \subseteq E \in \mathbf{cLCC}_c$ and $p : \Omega \longrightarrow U$ be an analytic path. A vector of the form $p'(\alpha)$ will be called as usual a *tangent vector* of U at $x = p(\alpha)$; accordingly the associated linear map $Dp(\alpha) : \mathbb{C} \longrightarrow E$, $Dp(\alpha) \cdot \lambda = \lambda p'(\alpha)$, will be called an *analytic tangent linear map* of U at x .

An *analytically tangential domain* in E means a $\mathcal{C}_c$ -subspace $U \subseteq E$ such that, at every $x \in U$, the analytic tangent linear maps form an epifamily $\mathbb{C} \longrightarrow E$ in $\mathbf{cLCC}_c$; this family called an *analytic tangent epifamily* to U at x . Equivalently, U is analytically tangential if at every $x \in U$ the tangent linear vectors of U form a functionally dense subset of E' , i.e. if for every $\ell \in E^*$, $\ell(p'(\alpha)) = 0$ for all paths p with $x = p(\alpha)$, then $\ell = 0$ (see 3b in [17]).

A domain $U \subseteq E$ is said to be *analytically path-open* if $p^{-1}(U)$ is open, for all analytic paths $p : \Omega \rightarrow E$.

Examples:

- (1) Every open subspace of E is an analytically tangential domain.
- (2) Every analytically path-open domain is analytically tangential.
- (3) In the Fréchet space $\mathcal{H}_c(\mathbb{C}, \mathbb{C})$, the set of never vanishing maps is analytically tangential (see [13] for more details).

From now on, suppose that $U \subset E$ denotes an analytically tangential domain.

Lemma 1.3.5 *Suppose $S, T \in \mathcal{C}_c(U, [E, F])$ are such that for every analytic path $p : \Omega \rightarrow U$ we have $S(p(-)) \cdot p'(-), T(p(-)) \cdot p'(-) \in \mathcal{A}_c(\Omega, F)$ and $\oint_\alpha^\beta S(p(\xi)) \cdot p'(\xi) d\xi = \oint_\alpha^\beta T(p(\xi)) \cdot p'(\xi) d\xi$ for all $\alpha, \beta \in \Omega$. Then $S = T$.*

Proof. By proposition 1.3.3, $S(p(\xi)) \cdot p'(\xi) = T(p(\xi)) \cdot p'(\xi)$. Now at every $x \in U$ we have an analytic tangent epifamily $Dp(\xi) : \mathbb{C} \rightarrow E$ ($p(\xi) = x$) and we have just seen that $S(x) \circ Dp(\xi) = T(x) \circ Dp(\xi)$. By canceling the epifamily we obtain $S(x) = T(x)$ as required.

Definition 1.3.6 A $\mathcal{C}_c$ -map $f : U \rightarrow F$ is an *analytically differentiable* map or an $\mathcal{A}_c$ -map if there exists a $\mathcal{C}_c$ -map $\hat{D}f : U \rightarrow [E, F]$ satisfying

$$f(p(\beta)) - f(p(\alpha)) = \oint_\alpha^\beta \hat{D}f(p(\xi)) \cdot p'(\xi) d\xi,$$

for all analytic paths $p : \Omega \rightarrow U$ and all $\alpha, \beta \in \Omega$. For a given f , the preceding lemma guarantees uniqueness of such $\hat{D}f$ (the derivative of f).

Let $\mathcal{A}_c(U, E)$ be the subset of $\mathcal{C}_c(U, E)$ formed by all $\mathcal{A}_c$ -maps $f : U \rightarrow E$. We define inductively $\mathcal{A}_c^1(U, E) = \mathcal{A}_c(U, E)$ and, for $r \geq 1$, $\mathcal{A}_c^{r+1}(U, E)$ formed by all f such that $\hat{D}f \in \mathcal{A}_c^r(U, [E, F])$. Also $\mathcal{A}_c^\infty(U, F) = \bigcap_{r \geq 1} \mathcal{A}_c^r(U, F)$.

Corollary 1.3.7 *A $\mathcal{C}_c$ -map $p : \Omega \rightarrow E$ is an $\mathcal{A}_c$ -map iff it is an analytic path.*

Theorem 1.3.8 (*Localization Theorem*)

Suppose the $\mathcal{C}_c$ -subspace $X \subseteq E$ is a union $X = \bigcup_{i \in I} V_i$ of analytically tangential domains such that X carries the final $\mathcal{C}_c$ -structure induced by the family of insertion maps $\text{ins}_i : V_i \rightarrow X$. Then X is analytically tangential. Suppose furthermore every V_i is analytically path-open in X . Then a function $f : X \rightarrow G$ is an $\mathcal{A}_c^r$ -map whenever every restriction $f_i = f \circ \text{ins}_i : V_i \rightarrow G$ is an $\mathcal{A}_c^r$ -map ($r \in [0, \infty]$).

Proof. The proof is essentially the same as the proof of the corresponding theorem in real calculus (4e in [17]).

It follows from the definition that X is analytically tangential. To prove the second part we proceed by induction on r . For $r = 0$ the assumption immediately gives the result. Assuming the validity of the theorem for all $\mathcal{A}_c^r$ -maps, take $f : X \rightarrow G$ such that every restriction f_i is $\mathcal{A}_c^{r+1}$. Then every $\hat{D}f_i : V_i \rightarrow [E, G]$ is $\mathcal{A}_c^r$. If $V_i \cap V_j$ is non-empty, it is again analytically tangential. By uniqueness of derivatives, the maps $\hat{D}f_i$ and $\hat{D}f_j$ agree on the $V_i \cap V_j$, so these maps can be glued together to form a $\mathcal{C}_c$ -map $\hat{D}f : X \rightarrow [E, G]$. Since the restrictions $\hat{D}f_i$ are $\mathcal{A}_c^r$ -maps, the inductive assumption (applied to $\hat{D}f$) tells us that $\hat{D}f$ is likewise a $\mathcal{A}_c^r$ -map. It remains to be shown only that $\hat{D}f$ is indeed the derivative of f . So take $p : \Omega \rightarrow X$ and $\alpha, \beta \in \Omega$. Let $q : [a, b] \subset \mathbb{R} \rightarrow \Omega$ be a path from α to β . Then $(p \circ q)^{-1}(V_i)$ ($i \in I$) form an open covering of $[a, b]$. By compactness of $[a, b]$ there is a finite subcovering, hence by connectedness of $[a, b]$ we can express this interval as a finite union of the form $[a, b] = \bigcup_{k=1}^m [\gamma_{k-1}, \gamma_k]$ such that $\gamma_0 = a, \gamma_m = b$ and $p \circ q$ carries the segment $[\gamma_{k-1}, \gamma_k]$ into one single $V_{i(k)}$ ($k \in [1..m]$). Hence $\oint_a^b \hat{D}f(p(q(\xi))) \cdot p'(\xi) d\xi = \sum_k \oint_{q(\gamma_{k-1})}^{q(\gamma_k)} \hat{D}f_{i(k)}(p(q(\xi))) \cdot p'(\xi) d\xi = \sum_k [f(p(q(\gamma_k))) - f(p(q(\gamma_{k-1})))] = (f \circ p \circ q)(\gamma_m) - (f \circ p \circ q)(\gamma_0) = f(p(\beta)) - f(p(\alpha))$.

Proposition 1.3.9 *If $f \in \mathcal{A}_c(\Lambda, E)$, then $f' \in \mathcal{A}_c(\Lambda, E)$, where Λ is an open subset of $\mathbb{C}$.*

Proof. For each α , there exist a_n and $r_n > 0$ such that $f(\xi) = \sum_n a_n (\xi - \alpha)^n$

converges for all $\xi \in B_\alpha = B(\alpha, r_\alpha)$. Since $\Lambda = \bigcup_{\alpha \in \Lambda} B_\alpha$, the localization theorem 1.3.8, implies that $f \in \mathcal{A}_c$ whenever all restrictions $f_\alpha = f \circ \text{sub}_\alpha : B_\alpha \rightarrow E$ are. For simplicity let $\alpha = 0$ and we use f, B and r in the role of f_α, B_α and r_α respectively. We reduce the proposition to the case that E is regarded as a closed $\mathcal{C}_c$ -subspace of E^{**} . We will show that $\sum_{n \geq 0} n a_n \xi^{n-1}$ converges in E , for all $\xi \in B$ and its sum equals $f'(\xi)$. By theorem 1.1.9 it is enough to show that there is $N \in \mathbb{N}$ such that $M = \{n a_n \xi^{n-1} : n \geq N\}$ is bounded. To see that, let $\xi \in B$ and $s, t \in \mathbb{R}$ such that $0 < |\xi| < t < s < r$. As $\sum_n a_n s^n$ converges, for a given $\varepsilon > 0$ and $\mathcal{L} \rightsquigarrow \ell$ in E^* , there are $N_1 \in \mathbb{N}$ and $L \in \mathcal{L}$ such that $|a_n(u)s^n| \leq t\sqrt{\varepsilon}$, for all $u \in L$ and $n \geq N_1$. On the other hand

$$(*) \quad |n a_n(u) \xi^{n-1}| \leq n |a_n(u)| t^{n-1} = (t^{-1} |a_n(u) s^n|)(n q^n),$$

where $q = (t/s) < 1$. Put $\delta = q^{-1} - 1 > 0$. Then $q^{-n} = (1 + \delta)^n \geq \binom{n}{2} \delta^2$. Hence $0 < n q^n \leq \frac{2}{(n-1)\delta^2}$. Therefore there is $N_2 \in \mathbb{N}$ such that $|n q^n| \leq \sqrt{\varepsilon}$, for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$, then by $(*)$ $|n a_n(u) \xi^{n-1}| \leq \varepsilon$, for all $n \geq N$ and $u \in L$. This is nothing but the boundedness of M .

It remains to show that $f'(\xi) = \sum_{n \geq 0} n a_n \xi^{n-1}$. To see that, fix $\xi \in B$ and let h be such that $|h| + |\xi| < r$. Then $f(\xi + h) = f(\xi) + h f_1(\xi) + \cdots + h^k f_k(\xi) + \cdots$ is a convergent series in h , where by the above argument each f_k is a convergent series and $f_k(\xi) = (1/k!) \sum_{n \geq k} n(n-1) \cdots (n-k) a_n \xi^{n-k}$. Therefore $(f(\xi + h) - f(\xi))/h = f_1(\xi) + h(f_2(\xi) + \cdots)$ converges to $f_1(\xi)$ whenever $h \rightsquigarrow 0$, as required.

Theorem 1.3.10 *Let $\Lambda \subseteq \mathbb{C}$ be open. Then $\mathcal{A}_c(\Lambda, E) = \mathcal{A}_c^\infty(\Lambda, E)$.*

Proof. Let $f \in \mathcal{A}_c(\Lambda, E)$. By localization theorem 1.3.8, we reduce the theorem to the case that $\Lambda = B(\alpha, r)$ and $f(\xi) = \sum_n a_n (\xi - \alpha)^n$. For simplicity let $\alpha = 0$. By proposition 1.3.9 and induction, $f^{(k)}(\xi) = \sum_{n \geq k} n(n-1) \cdots (n-k) a_n \xi^{n-k}$ converges for all $\xi \in \Lambda$. Let us show that the map $g : \Lambda \rightarrow [\mathbb{C}, F]$, $g(\xi) \cdot \lambda = \lambda f'(\xi)$ serves in the role of $\hat{D}f$, i.e. for analytic paths $p : \Omega \rightarrow \Lambda$ we have

$$(**) \quad f(p(\mu)) - f(p(\lambda)) = \oint_{\lambda}^{\mu} g(p(\xi)) \cdot p'(\xi) d\xi.$$

Indeed, by local representation of f we have

$$f(p(\mu)) - f(p(\lambda)) = \sum_n a_n [(p(\mu))^n - (p(\lambda))^n].$$

The map $q_n : \Omega \rightarrow \mathbb{C}$, given by $q_n(\xi) = (p(\xi))^n$, is an analytic path with $q_n'(\xi) = n(p(\xi))^{n-1} \cdot p'(\xi)$. Since $q_n(\mu) - q_n(\lambda) = \oint_{\lambda}^{\mu} q_n'(\xi) d\xi$, we obtain $f(p(\mu)) - f(p(\lambda)) = \sum_{n \geq 1} n a_n \oint_{\lambda}^{\mu} (p(\xi))^{n-1} \cdot p'(\xi) d\xi$. By continuity of the integral, this is nothing but (**). Moreover the definition of $\hat{D}f = g$ shows that $\hat{D}f$ is also an $\mathcal{A}_c$ -map. Similarly, if n -th derivative of f is $\mathcal{A}_c$, then so is the $(n+1)$ -th derivative. Hence, by induction $\mathcal{A}_c(\Lambda, E) = \mathcal{A}_c^{\infty}(\Lambda, E)$.

Theorem 1.3.10 can be generalized for analytic maps on open domain U in $\mathbb{C}^k$ as follows:

Theorem 1.3.11 *For an open set U in $\mathbb{C}^k$, we have $\mathcal{A}_c(U, E) = \mathcal{A}_c^{\infty}(U, E)$.*

Proof. By localization theorem 1.3.8, we can reduce to the special case $U = \Lambda_1 \times \cdots \times \Lambda_k$, where $\Lambda_i = B(\alpha_i, r_i)$ ($i \in [1..k]$). For simplicity let $\alpha_i = 0$, for all i .

If $f \in \mathcal{A}_c(U, E)$, then it has the representation $f(\xi) = \sum_n a_n \xi^n$. Define $g : U \rightarrow [\mathbb{C}^k, E]$ by

$$g(\xi_1, \dots, \xi_k) \cdot (\lambda_1, \dots, \lambda_k) = \sum_{i=1}^k \lambda_i \sum_{n \in \mathbb{N}^k} n_i a_n \xi_1^{n_1} \cdots \xi_{i-1}^{n_{i-1}} \xi_i^{n_i-1} \xi_{i+1}^{n_{i+1}} \cdots \xi_k^{n_k}.$$

We claim that g serves in the role of $\hat{D}f$, i.e. for analytic path $p : \Omega \rightarrow U$ we have

$$(*) \quad f(p(\mu)) - f(p(\lambda)) = \oint_{\lambda}^{\mu} g(p(\xi)) \cdot p'(\xi) d\xi.$$

Indeed, by local representation of f we have

$$f(p(\mu)) - f(p(\lambda)) = \sum_{i=1}^k \sum_n n_i a_n [(p_i(\mu))^{n_i} - (p_i(\lambda))^{n_i}] \prod_{j=1}^{i-1} (p_j(\lambda))^{n_j} \prod_{r=i+1}^k (p_r(\mu))^{n_r},$$

where $p_i = \text{pro}_i \circ p$. Define $q_i : \Omega \rightarrow \Lambda_i$ by $q_i(\xi) = (p_i(\xi))^{n_i}$ ($i \in [1..k]$). Since $q_i(\mu) - q_i(\lambda) = \oint_{\lambda}^{\mu} q_i'(\xi) d\xi$, we obtain

$$f(p(\mu)) - f(p(\lambda)) = \sum_{i=1}^k \sum_n n_i a_n (\oint_{\lambda}^{\mu} (p_i(\xi))^{n_i-1} \cdot p_i'(\xi) d\xi) \prod_{j=1}^{i-1} (p_j(\lambda))^{n_j} \prod_{r=i+1}^k (p_r(\mu))^{n_r}.$$

By continuity of integral, this is nothing but (*).

Now we will show that $\hat{D}f \in \mathcal{A}_c(U, [\mathbb{C}^k, E])$. Define $g_1 : U \rightarrow [\mathbb{C}^k, [\mathbb{C}^k, E]]$ by

$g_1(\xi) \cdot (\lambda) \cdot (\gamma) = \sum_{(i,j)} \lambda_i \lambda_j \sum_n n_i n_j a_n (\prod_{s=1}^{i-1} \xi_s^{n_s}) \xi_i^{n_i-1} (\prod_{r=i+1}^{j-1} \xi_r^{n_r}) \xi_j^{n_j-1} (\prod_{t=j+1}^k \xi_t^{n_t})$,
 where $\xi = (\xi_1, \dots, \xi_k)$, $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\gamma = (\gamma_1, \dots, \gamma_k)$. By an argument similar to the above and equation (5) followed by definition 1.3.2, one can show that $\dot{D}^2 f = g_1$. Hence inductively the required result will follow.

Definition 1.3.12

(a) A $\mathcal{C}_c$ -subspace U of E is *convexwise* open if it is the codomain of a final epifamily in $\mathcal{C}_c$ of subspace embeddings, $\text{sub}_j : U_j \rightarrow U$ ($j \in J$), with every U_j a convex open domain (the significance of this concept will become clear in chapter 2, see 2.3.3)

(b) A $\mathcal{C}_c$ -map $f : U \rightarrow F$ is *holomorphic along affine lines* or *analytic along affine lines* if, for every $(x, h) \in U \times E$, the map $g : \Upsilon \rightarrow F$, $g(\lambda) = f(x + \lambda h)$ is holomorphic, where Υ is the open subset of all $\lambda \in \mathbb{C}$ for which $x + \lambda h \in U$.

Open questions 1.3.13

(1) For which kind of domains U in infinite dimensional space E ,

$${}^{\sim}\mathcal{A}_c(U, F) = \mathcal{A}_c^{\sim}(U, F) {}^{\sim}?$$

(2) Let U be a convexwise open domain. Is the following statement true?

$${}^{\sim}f \in \mathcal{A}_c(U, F) \text{ iff } f \text{ is analytic along affine lines} {}^{\sim}.$$

Chapter 2

Holomorphy in Convergence Spaces

In the preceding chapter, for nonnormable spaces, we developed the complex differential theory based on Banach-spaces and $\mathbf{c}\mathcal{L}\mathcal{C}_r$ -spaces theory by using the basic facts about the set of analytic functions and power series in $\mathbf{c}\mathcal{L}\mathcal{C}_r$ -spaces. Similar to the classical case one can not expect to get all of the results via power series. In the present chapter we give another approach to differential calculus and holomorphy concept. This theory is based on the idea that ‘holomorphy = real continuous differentiability + complex linearity of the derivative’. In elementary context the latter requirement essentially reformulates the Cauchy-Riemann equations (see 2.1.3). Real continuous differentiability is in the sense of chapter 0. A special motivation (corresponding to an aim of [17]) is to make the theory strong enough to admit certain maps on domains with empty interior. An example of such a map is $f : U \rightarrow U$, $f(z)(\zeta) = 1/z(\zeta)$, where U is the subspace of never-vanishing maps in the classical Fréchet space $\mathcal{H}_c(\mathbb{C}, \mathbb{C})$ of entire holomorphic maps. Since U has empty interior (see [13]), this ‘elementary function’ is excluded from older theories that prescribe open domains for their calculus to work.

We show in section 1 that the usual basic theorem of differential calculus do indeed result from the indicated point of departure: the chain rule, the holomorphy of continuous linear and multilinear maps, the properties of partial derivatives and

so on.

The categorical definition of derivative in [17] is not in terms of a limit of difference quotients. So, not surprisingly, our concept of holomorphic map ($\mathcal{H}_c$ -map) is formally quite different from that of earlier theories. However, on open domains it is usually equivalent to some known concept, thus providing a new characterization — new even in the long studied special case of a complex function of a single complex variable (see section 2).

Holomorphy has already been studied in the context of $\mathcal{C}_c$ -maps in [1]. There, on τ -open sets, they depart from the notion of G-holomorphic map (weakly holomorphic + holomorphic along affine lines). Our concept ($\mathcal{H}_c$ -map) is formally much stronger and is defined in a larger class of domains (see 2.1.1), but in section 3 we prove it equivalent to G-holomorphy on τ -open domains. An interesting spin-off fact emerges: weakly holomorphic = holomorphic along affine lines, in this context too.

In section 4 we present an interesting result, namely a Taylor-like expansion of holomorphic maps in infinite dimensional spaces, proved in [1] but with a different method.

Holomorphy on domains with empty interior is also accommodated in [13], the specialization of [17] that applies to the complex toponome $\mathcal{C}_h$ of holological spaces [10]. Whereas the $\mathcal{C}_h$ -based holomorphy has intimate connection with bornology, therefore with Banach duals, the $\mathcal{C}_c$ -based theory relates closely to complete locally convex spaces and continuous convergence. Thus the two theories supplement each other.

The use of domains with empty interior becomes crucial in this context, as the above special function $z \mapsto 1/z$ already suggests. For this approach the $\mathcal{C}_c$ -based theory seems preferable to the $\mathcal{C}_h$ -version.

2.1 Holomorphy on Tangential Domains

Until further notice, possibly with subscripts, E , F and G denote $\mathbf{cLC}_c$ spaces, U and V tangential domains in $E_{\mathbb{R}}$ and $F_{\mathbb{R}}$ respectively, Λ a nondegenerate real interval.

As we mentioned in the introduction, holomorphy has been already studied in [1] on *finitely open domains*, namely subsets U of E such that $U \cap L$ is open in the euclidean topology on L for every finite dimensional subspace L of E . The following lemma shows that the class of tangential domains properly contains these domains.

Lemma 2.1.1 *Every finitely open domain is tangential. But the converse is false.*

Proof. Let U be a finitely open domain in E . Fix $x \in U$ and $h \in E$. Then by the finite openness of U there is $\varepsilon > 0$ such that $\{x + \lambda h : |\lambda| \leq \varepsilon\} \subseteq U$. Define $p_h : [-\varepsilon, \varepsilon] \rightarrow U$ by $p_h(\lambda) = x + \lambda h$. Then $p_h'(\lambda) = h$. Hence U is a tangential domain.

To show the converse is not true, let $E = \mathcal{H}_c(\mathbb{C}, \mathbb{C})$ and U be the subspace formed by all never vanishing maps $\mathbb{C} \rightarrow \mathbb{C}$. Define f, g by $f(\xi) = e^{2\xi}$ and $g(\xi) = -e^{\xi}$, then $f, g \in U$. Since $\{\lambda : f(\xi) + \lambda g(\xi) \neq 0, \text{ for all } \xi \in \mathbb{C}\} = \{0\}$ is not open, U is not finitely open. As U is tangential (see 3d in [13]), we arrive at the desired conclusion.

Definition 2.1.2 $\mathcal{H}_c(U, F)$ is the subset of $\mathcal{C}_c^1(U, F)$ formed by all f such that for each $a \in U$ the real derivative $Df(a) : E_{\mathbb{R}} \rightarrow F_{\mathbb{R}}$ implies an $\mathcal{LC}_c$ -map $Df(a) : E \rightarrow F$ (i.e. a $\mathbb{C}$ -linear continuous map). We define inductively $\mathcal{H}_c^1(U, F) = \mathcal{H}_c(U, F)$ and, for $r \geq 1$, $\mathcal{H}_c^{r+1}(U, F)$ is the subset of $\mathcal{C}_c^{r+1}(U, F)$ formed by all f such that $\hat{D}f \in \mathcal{H}_c^r(U, [E, F])$. Also, $\mathcal{H}_c^\infty(U, F) = \bigcap_{r \geq 1} \mathcal{H}_c^r(U, F)$.

The following proposition shows the idea of the above definition i.e. holomorphy = real continuous differentiability + complex linearity of derivative in the one-dimensional context.

Proposition 2.1.3 *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \rightarrow \mathbb{C}$ be a map. Then f is holomorphic in classical complex analysis iff it is real continuously differentiable and its derivative is $\mathbb{C}$ -linear.*

Proof. Let $f : \Omega_{\mathbb{R}} \rightarrow \mathbb{R}^2$ (i.e. $\mathbb{C}_{\mathbb{R}}$) be real continuously differentiable with $\mathbb{C}$ -linear derivative. Suppose $f(\xi, \eta) = \begin{bmatrix} u(\xi, \eta) \\ v(\xi, \eta) \end{bmatrix}$. Then $f'(\xi, \eta)$ has the following representation as a linear map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ by:

$$\begin{bmatrix} \partial u / \partial \xi & \partial u / \partial \eta \\ \partial v / \partial \xi & \partial v / \partial \eta \end{bmatrix} \stackrel{\text{def}}{=} A(\xi, \eta).$$

Every $\lambda \in \mathbb{C}$ can be represented by $\lambda = \lambda_1 + i\lambda_2 = \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix}$. Then $\mathbb{C}$ -multiplication is equivalent to matrix-multiplication. The matrix $A(\xi, \eta)$ is $\mathbb{C}$ -linear iff for every $\lambda \in \mathbb{C}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in $\mathbb{R}^2$, with $c \neq 0$ we have $A(\lambda c + b) = \lambda A \cdot c + A \cdot b$. The latter requirement becomes

$$\begin{bmatrix} (c_1 \lambda_1 - c_2 \lambda_2 + b_1) \frac{\partial u}{\partial \xi} + (c_1 \lambda_2 + c_2 \lambda_1 + b_2) \frac{\partial u}{\partial \eta} \\ (c_1 \lambda_1 - c_2 \lambda_2 + b_1) \frac{\partial v}{\partial \xi} + (c_1 \lambda_2 + c_2 \lambda_1 + b_2) \frac{\partial v}{\partial \eta} \end{bmatrix} = \\ \begin{bmatrix} b_1 \frac{\partial u}{\partial \xi} + b_2 \frac{\partial u}{\partial \eta} + c_1 \lambda_1 \frac{\partial u}{\partial \xi} - c_1 \lambda_2 \frac{\partial v}{\partial \xi} + c_2 \lambda_1 \frac{\partial u}{\partial \eta} - c_2 \lambda_2 \frac{\partial v}{\partial \eta} \\ b_1 \frac{\partial v}{\partial \xi} + b_2 \frac{\partial v}{\partial \eta} + c_1 \lambda_2 \frac{\partial u}{\partial \xi} + c_1 \lambda_1 \frac{\partial v}{\partial \xi} + c_2 \lambda_2 \frac{\partial u}{\partial \eta} + c_2 \lambda_1 \frac{\partial v}{\partial \eta} \end{bmatrix},$$

which reduces to

$$c_2 \lambda_2 (\partial v / \partial \eta - \partial u / \partial \xi) + c_1 \lambda_2 (\partial u / \partial \eta + \partial v / \partial \xi) = 0$$

$$c_2\lambda_2(\partial v/\partial\xi + \partial u/\partial\eta) + c_1\lambda_2(\partial u/\partial\xi - \partial v/\partial\eta) = 0.$$

Since for every $\lambda_2 \in \mathbb{R}$ the above equations are satisfied and $c_1^2 + c_2^2 \neq 0$, one arrives at $\partial v/\partial\eta - \partial u/\partial\xi = 0$ and $\partial u/\partial\eta + \partial v/\partial\xi = 0$, which are the Cauchy-Riemann equations. Therefore f is holomorphic. The converse is obvious.

We obviously need to study how this holomorphy relates to previous concepts. We do this in stages, starting with the following theorem.

Theorem 2.1.4 *Suppose the tangential domain U is convex and $f \in \mathcal{C}_c(U, F)$. Then the following statements are equivalent:*

- (A) $f \in \mathcal{H}_c(U, F)$.
- (B) *There exists a $\mathcal{C}_c$ -map $\hat{D}f : U \rightarrow [E, F]$ such that $f(y) - f(x) = \int_0^1 \hat{D}f(x + \theta(y-x)) \cdot (y-x) d\theta$ ($x, y \in U$).*
- (C) *f has a difference factorizer i.e. a $\mathcal{C}_c$ -map $\hat{\Phi} : U \times U \rightarrow [E, F]$ such that $f(y) - f(x) = \hat{\Phi}(x, y) \cdot (y-x)$ ($x, y \in U$).*

Proof. The reasoning largely repeats that of proof 4c in [17]. It is just a matter of changing Df to $\hat{D}f$ and replacing the real difference factorizer by a complex one. In proving (B) $\Rightarrow$ (C), the $\mathbb{C}$ -linearity of $\hat{\Phi}$ follows from that of $\hat{D}f$ and theorem 2d(0) in [17].

We call a $\mathcal{C}_c$ -map $\ell : U \rightarrow F$ linear (even when its domain is not a whole linear space) when it is the restriction of some linear map $\ell : E \rightarrow F$; similarly for multilinear maps.

Theorem 2.1.5 *For integers $1 \leq r \leq \infty$ the following hold.*

- (1^r) *If $f : U \rightarrow F$ is constant, then $f \in \mathcal{H}_c^r(U, F)$ and $\hat{D}f(x) = 0$.*
- (2^r) *If $\ell \in \mathcal{C}_c(U, F)$ is $\mathbb{C}$ -linear, then $\ell \in \mathcal{H}_c^r(U, F)$ and $\hat{D}\ell(x) = \ell$.*
- (3^r) *If $m \in \mathcal{C}_c(U_1 \times U_2 \times \dots \times U_n, F)$ is complex n -linear, then*

$m \in \mathcal{H}_c^r(U_1 \times U_2 \times \dots \times U_n, F)$ and

$$\hat{D}m(x) \cdot (h_1, \dots, h_n) = \sum_{i=1, \dots, n} m(x_1, \dots, x_{i-1}, h_i, x_{i+1}, \dots, x_n).$$

(4') If $0 \notin U \subset \mathbb{C}$ then the map $g : U \rightarrow \mathbb{C}$, $g(\zeta) = 1/\zeta$ is $\mathcal{H}_c^r$ and $\hat{D}g(\zeta) \cdot \eta = -\eta/\zeta^2$.

(5') If $f : U \rightarrow V$ and $g : V \rightarrow G$ are $\mathcal{H}_c^r$ -maps, then so is $g \circ f : U \rightarrow G$ and $\hat{D}(g \circ f)(x) = \hat{D}g(f(x)) \circ \hat{D}f(x)$.

(6') $\mathcal{H}_c^r(U, F)$ is a $\mathbb{C}$ -linear subspace of $\mathcal{C}_c(U, F)$ and it is a subalgebra when F is an algebra.

Proof. This is a straightforward consequence of the corresponding result in the real $\mathcal{C}_c$ -based differentiation theory (see theorem 4d in [17]).

The local nature of holomorphy is not clear from the new definition. The following result addresses this important issue.

Theorem 2.1.6 (*Localization Theorem*)

Let X be a $\mathcal{C}_c$ -subspace of E . Suppose $X = \bigcup_{i \in I} U_i$, a union of tangential domains in $E_{\mathbb{P}}$, and the epifamily of subspace insertions $\text{sub}_i : U_i \rightarrow X$ is final in $\mathcal{C}_c$. Then X is tangential in $E_{\mathbb{P}}$.

Suppose furthermore every U_i is path-open in X . Then a function $f : X \rightarrow F$ is $\mathcal{H}_c^r$ whenever all the restrictions $f \circ \text{sub}_i : U_i \rightarrow F$ are $\mathcal{H}_c^r$ -maps ($1 \leq r \leq \infty$).

Proof. The argument proceeds as in the categorical differentiation theory; see 1c of [17].

When $f \in \mathcal{C}_c(U \times V, G)$, the single variable transforms $f^{\dagger 1} : U \rightarrow \mathcal{C}_c(V, G)$ and $f^{\dagger 2} : V \rightarrow \mathcal{C}_c(U, G)$ are defined respectively as follows: $f^{\dagger 1}(x)(y) = f(x, y) = f^{\dagger 2}(y)(x)$. In this chapter we use $D_1 f : U \times V \rightarrow [E, G]$, $D_2 f : U \times V \rightarrow [F, G]$ for the partial derivatives (see [17]) and denote them $\hat{D}_1 f$ and $\hat{D}_2 f$ when their values are $\mathbb{C}$ -linear. In the special case when U and V are domains in $\mathbb{C}$, we put $\partial_1 f(\lambda, \mu) = \hat{D}_1 f(\lambda, \mu) \cdot 1$; thus $\partial_1 f$ takes values in G rather than $[\mathbb{C}, G]$ and $\partial_1 f(\lambda, \mu) = (f^{\dagger 1})'(\lambda)(\mu)$; similarly for $\partial_2 f$.

The following is a Hartogs-like characterization which, in the presence of continuity assumptions, is much easier to prove than the deep classical Hartogs theorem. Nevertheless, in many situations it is all we need.

Proposition 2.1.7 *For a $\mathcal{C}_c$ -map $f : U \times V \rightarrow G$ the following statements are equivalent:*

- (A) $f \in \mathcal{H}_c(U \times V, G)$.
- (B) $f^{\dagger 1} \in \mathcal{H}_c(U, \mathcal{C}_c(V, G))$ and $f^{\dagger 2} \in \mathcal{H}_c(V, \mathcal{C}_c(U, G))$.

Moreover, these statements relate the derivatives as follows:

$$\dot{D}f(x, y) \cdot (h, k) = \dot{D}_1 f(x, y) \cdot h + \dot{D}_2 f(x, y) \cdot k,$$

$$(\dot{D}f^{\dagger 1}(x) \cdot h)(y) = \dot{D}_1 f(x, y) \cdot h,$$

$$(\dot{D}f^{\dagger 2}(y) \cdot k)(x) = \dot{D}_2 f(x, y) \cdot k.$$

Proof. See 4g in [17]. The only thing needed in addition to the categorical result for real scalars is the $\mathbb{C}$ -linearity of the derivatives and that follows in a straightforward manner.

2.2 Finite Dimensional Open Domains

Henceforth, Ω denotes a nonempty open subspace of $\mathbb{C}$.

Holomorphic maps on $\mathcal{D}$ allow numerous characterizations. These are long known for maps into Banach spaces. The equivalence of (H3) through (H9) in the following theorem was proved in [2] for maps into a sequentially complete locally convex space. That proof proceeded largely via estimates for seminorms — a technique not available when the codomain is a $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -space. So we were forced to find a different proof, which turned out to be simpler than that of [2]. This theorem, and indeed most of this chapter, goes through without complication if $\mathbf{c}\mathcal{L}\mathcal{C}_c$ is replaced by the

larger analytic of sequentially closed-embedded spaces (see 29.i in [20] and the last paragraph of 1f in [17] for more details).

For paths $c : [0,1] \rightarrow \Omega$ such that $c(0) = c(1)$ and $f \in \mathcal{C}_c(\Omega, F)$, we put

$$\int_c f(\gamma) d\gamma = \int_0^1 f(c(\tau)) \cdot c'(\tau) d\tau.$$

A *circuit in Ω around λ* means a path c as above which maps into some convex neighborhood of λ in Ω and satisfies $\int_c (\gamma - \lambda)^{-1} d\gamma = 2\pi i$; thus c winds once around λ in the positive sense. Typical examples of such c are of course the maps $c(\tau) = \lambda + \rho e^{2\pi i \tau}$, with $|\rho|$ small enough.

Theorem 2.2.1 *For $f \in \mathcal{C}_c(\Omega, F)$ the following statements are equivalent. In these statements, λ, μ, ζ vary in Ω and c is a circuit in Ω around all of these points.*

$$(H1) \quad f \in \mathcal{H}_c(\Omega, F).$$

$$(H2) \quad \text{For some } A \in \mathcal{C}_c(\Omega \times \Omega, F), \quad f(\mu) - f(\lambda) = (\mu - \lambda)A(\lambda, \mu).$$

$$(H3) \quad \text{For some } f' \in \mathcal{C}_c(\Omega, F), \quad f'(\zeta) = \lim_{(\lambda, \mu) \rightsquigarrow (\zeta, \zeta)} [f(\mu) - f(\lambda)]/(\mu - \lambda).$$

$$(H4) \quad \int_c f(\gamma) d\gamma = 0.$$

$$(H5) \quad f(\lambda) = (2\pi i)^{-1} \int_c (\gamma - \lambda)^{-1} f(\gamma) d\gamma.$$

$$(H6) \quad f \text{ has complex derivatives of all orders and they are given by the Cauchy formulas } f^{(m)}(\lambda) = m!(2\pi i)^{-1} \int_c (\gamma - \lambda)^{-m-1} f(\gamma) d\gamma.$$

$$(H7) \quad f \text{ has local Taylor expansions } f(\zeta) = \sum_{r \geq 0} (f^{(r)}(\lambda)/r!)(\zeta - \lambda)^r.$$

$$(H8) \quad f \text{ has local representations } f(\zeta) = \sum_{r \geq 0} a_r(\zeta - \lambda)^r, \text{ where } a_r \in F.$$

$$(H9) \quad \text{All composites } u \circ f : \Omega \rightarrow \mathbb{C} \text{ are holomorphic in the usual elementary sense, where } u \text{ varies in some monofamily of } \mathcal{LC}_c\text{-maps } F \rightarrow \mathbb{C} \text{ (e.g. } u \in F^* \text{)}.$$

Proof. (H1) $\Rightarrow$ (H2). Put $\Delta = \{(\zeta, \zeta) \mid \zeta \in \Omega\}$, $V = (\Omega \times \Omega) \setminus \Delta$. Then $\Omega \times \Omega = V \cup \bigcup_{\zeta \in \Omega} (B_\zeta \times B_\zeta)$, a union of open subsets, where each B_ζ is an open ball contained in Ω and centered on ζ . We construct $A : \Omega \times \Omega \rightarrow F$ by defining its restriction to each member of the above open covering, as follows. For $(\lambda, \mu) \in V$ we put $A_V(\lambda, \mu) = [f(\mu) - f(\lambda)]/(\mu - \lambda)$. For $(\lambda, \mu) \in B_\zeta \times B_\zeta$ we put $A_\zeta(\lambda, \mu) =$

$\int_0^1 \dot{D}f(\lambda + \theta(\mu - \lambda)) \cdot 1 d\theta$, where $\dot{D}f : \Omega \rightarrow [\mathbb{C}, F]$ is the $\mathcal{C}_c$ -map provided by (H1). Since B_ζ is convex, theorem 2.1.4 guarantees that $f(\mu) - f(\lambda) = (\mu - \lambda)A_\zeta(\lambda, \mu)$ and $A_\zeta(\zeta, \zeta) = \dot{D}f(\zeta) \cdot 1$. Thus the maps $A_V : V \rightarrow F$ and $A_\zeta : B_\zeta \times B_\zeta \rightarrow F$ are $\mathcal{C}_c$ -maps and they clearly agree where they overlap. Therefore they induce a unique $\mathcal{C}_c$ -map $A : \Omega \times \Omega \rightarrow F$ such that $A|_V = A_V$ and $A|_{(B_\zeta \times B_\zeta)} = A_\zeta$. Moreover, A satisfies the identity of (H2) by construction.

(H2) $\Rightarrow$ (H1). Given A as in (H2), we obtain the map $\dot{D}f \in \mathcal{C}_c(\Omega, [\mathbb{C}, F])$ by putting $\dot{D}f(\lambda) \cdot 1 = A(\lambda, \lambda)$. To see that $\dot{D}f$ works as derivative, consider a path $p : \Lambda \rightarrow \Omega$. Then (see 2d(3) in [17]) there exists $S \in \mathcal{C}_c(\Lambda \times \Lambda, \mathbb{C})$ such that $p(\beta) - p(\alpha) = (\beta - \alpha)S(\alpha, \beta)$ for all $\alpha, \beta \in \Lambda$ and $p'(\tau) = S(\tau, \tau)$. In view of (H2) we find that $f(p(\beta)) - f(p(\alpha)) = (p(\beta) - p(\alpha))A(p(\alpha), p(\beta)) = (\beta - \alpha)S(\alpha, \beta)A(p(\alpha), p(\beta))$. By 2d(3) of [17], applied in the opposite direction, $f \circ p$ is a path, with $(f \circ p)'(\tau) = S(\tau, \tau)A(p(\tau), p(\tau))$. Hence $f(p(\beta)) - f(p(\alpha)) = \int_\alpha^\beta (f \circ p)'(\tau) d\tau = \int_\alpha^\beta \dot{D}f(p(\tau)) \cdot p'(\tau) d\tau$, as required for (H1).

(H2) $\Rightarrow$ (H3). Given A as in (H2) we obtain the needed f' by putting $f'(\zeta) = A(\zeta, \zeta)$.

(H3) $\Rightarrow$ (H2). Given f' as in (H3), express $\Omega \times \Omega$ as the disjoint union $V \cup \Delta$ (nondiagonal points and diagonal points as above). Put $A(\lambda, \mu) = [f(\mu) - f(\lambda)]/(\mu - \lambda)$ when $(\lambda, \mu) \in V$ and $A(\zeta, \zeta) = f'(\zeta)$. Let us show $A \in \mathcal{C}_c(\Omega \times \Omega, F)$. Continuity at points in V is clear. Consider a point $(\zeta, \zeta) \in \Delta$ and let $\mathcal{M}$ be an ultrafilter such that $\mathcal{M} \rightsquigarrow (\zeta, \zeta)$ in $\Omega \times \Omega$. Let N_ζ be a neighborhood of (ζ, ζ) in $\Omega \times \Omega$. Then $N_\zeta \in \mathcal{M}$. Moreover, as an ultrafilter, $\mathcal{M}$ has either Δ or V as member. In case $\Delta \in \mathcal{M}$, we have also $\Delta \cap N_\zeta \in \mathcal{M}$ and $A(\Delta \cap N_\zeta) = f'(\Delta \cap N_\zeta)$, so $A(\mathcal{M}) \rightsquigarrow A(\zeta, \zeta) = f'(\zeta)$, by the continuity of f' . In case $V \in \mathcal{M}$, it follows at once from (H3) that $A(\mathcal{M}) \rightsquigarrow A(\zeta, \zeta) = f'(\zeta)$. Thus A preserves convergence of ultrafilters. Since we are dealing here with $\mathcal{C}_{c*}$ -spaces, A preserves convergence of all filters.

(H3) $\Rightarrow$ (H4) $\Rightarrow$ (H5). As in [2], by composing with $u \in F^\star$, the problem is reduced to the case of a map $\Omega \rightarrow \mathbb{C}$, where the implications hold by the classical theory. Since these $u : F \rightarrow \mathbb{C}$ form a monofamily, they can be canceled and the implications for F -valued maps follow.

(H5) $\Rightarrow$ (H6). We use induction on $r \in \mathbb{N}$. The case $r = 0$ is just the assumed statement (H5). Now take $r \geq 0$ and suppose the r -th derivative exists and is given by the formula in (H6). Then we have

$$f^{(r)}(\mu) - f^{(r)}(\lambda) = r!(2\pi i)^{-1} \int_c [(\gamma - \mu)^{-(r+1)} - (\gamma - \lambda)^{-(r+1)}] f(\gamma) d\gamma.$$

Using the identity $y^{-(r+1)} - x^{-(r+1)} = (xy)^{-(r+1)}(y - x) \sum_{0 \leq j \leq r} x^{r-j} y^j$ and the continuity of the integral, we see $[f^{(r)}(\mu) - f^{(r)}(\lambda)]/(\mu - \lambda)$ converges to $(r+1)!(2\pi i)^{-1} \int_c (\gamma - \lambda)^{-(r+2)} f(\gamma) d\gamma$ as $\mu \rightsquigarrow \lambda$. So (H6) follows by induction.

(H6) $\Rightarrow$ (H3). This follows at once from the continuity of the integral with respect to the parameters in the integrand.

(H6) $\Rightarrow$ (H7). Choose $0 < \rho < \text{dist}(\lambda, \partial\Omega)$, $B = B(\lambda, \rho)$ and for the moment we specialize $c(\tau) = \lambda + \rho e^{2\pi i \tau}$. Then for $\zeta \in B$ and $\gamma = c(\tau)$ we have $|\zeta - \lambda|/|\gamma - \lambda| < 1$. So the series $\sum_{r \geq 0} (\zeta - \lambda)^r / (\gamma - \lambda)^{r+1}$ converges in $\mathbb{C}$. It follows, by (H6) and continuity of the integral, that

$$\sum_{r=0}^{\infty} (f^{(r)}(\lambda)/r!)(\zeta - \lambda)^r = (2\pi i)^{-1} \int_c f(\gamma) \sum_{r=0}^{\infty} (\zeta - \lambda)^r (\gamma - \lambda)^{-(r+1)} d\gamma.$$

We conclude that the Taylor series of f converges near λ . To see that its sum is $f(\zeta)$, we look at $(u \circ f)(\zeta)$, where $u \in F^\star$, we use $(u \circ f)^{(r)}(\lambda) = (u \circ f^{(r)})(\lambda)$ and the known Taylor expansion of this $\mathbb{C}$ -valued holomorphic map. After canceling the monofamily formed by these u , we arrive at (H7).

(H7) $\Rightarrow$ (H8) $\Rightarrow$ (H9) are obvious implications.

(H9) $\Rightarrow$ (H5). For every $u \in F^\star$ we have, by $\mathbb{C}$ -valued holomorphy,

$$u \cdot f(\lambda) = (u \circ f)(\lambda) = (2\pi i)^{-1} \int_c (\gamma - \lambda)^{-1} u \cdot f(\gamma) d\gamma.$$

We can take the linear map u out of the integral. Canceling the monofamily formed

by these u , we get (H5). This completes the proof.

For all its comprehensive appearance, the preceding theorem does not yet tell us whether $\mathcal{H}_c(\Omega, F) = \mathcal{H}_c^\infty(\Omega, F)$. Let us now address this question.

Scholium 2.2.2 *If $f \in \mathcal{H}_c(\Omega, F)$, then the map A appearing in 2.2.1(H2) is unique, it has the local representation*

$$A(\lambda, \mu) = (2\pi i)^{-1} \int_c (\gamma - \lambda)^{-1} (\gamma - \mu)^{-1} f(\gamma) d\gamma$$

and $A \in \mathcal{H}_c(\Omega \times \Omega, F)$. Hence $\mathcal{H}_c(\Omega, F) = \mathcal{H}_c^\infty(\Omega, F)$.

Proof. Uniqueness follows at once from the fact that $A(\lambda, \mu) = [f(\mu) - f(\lambda)]/(\mu - \lambda)$ holds on a dense subset of $\Omega \times \Omega$. By using this expression in conjunction with the Cauchy formula 2.2.1(H5) for $f(\mu)$ and $f(\lambda)$, we arrive at the mentioned local representation after just a little algebraic manipulation. Applying this representation, we have, for $\zeta \neq \lambda$,

$$[A(\zeta, \mu) - A(\lambda, \mu)]/(\zeta - \lambda) = (2\pi i)^{-1} \int_c f(\gamma)/[(\gamma - \zeta)(\gamma - \lambda)(\gamma - \mu)] d\gamma.$$

Hence $(A^{\dagger 1})'(\lambda)(\mu) = \partial_1 A(\lambda, \mu) = (2\pi i)^{-1} \int_c (\gamma - \lambda)^{-2} (\gamma - \mu)^{-1} f(\gamma) d\gamma$. So $A^{\dagger 1} : \Omega \rightarrow \mathcal{C}_c(\Omega, F)$ is a $\mathcal{C}_c$ -map whose derivative is likewise a $\mathcal{C}_c$ -map. It follows by 2.2.1 via the localization theorem 2.1.6 that it is an $\mathcal{H}_c$ -map. Similarly, $A^{\dagger 2} \in \mathcal{H}_c(\Omega, \mathcal{C}_c(\Omega, F))$. Applying 2.1.7, we get $A \in \mathcal{H}_c(\Omega \times \Omega, F)$. Since $f'(\lambda) = A(\lambda, \lambda)$, it follows that $f' \in \mathcal{H}_c(\Omega, F)$. Similarly, if the r -th derivative $f^{(r)}$ is $\mathcal{H}_c$, then so is the $(r + 1)$ -th derivative. Hence, by induction, $\mathcal{H}_c(\Omega, F) = \mathcal{H}_c^\infty(\Omega, F)$.

Let us now turn to the special case $E = \mathbb{C}^n$ for some $n > 1$. We begin by noting the following fact, which follows from 2.1.7 by induction.

Corollary 2.2.3 *For $f \in \mathcal{C}_c(\Omega_1 \times \dots \times \Omega_n, F)$ to be in $\mathcal{H}_c(\Omega_1 \times \dots \times \Omega_n, F)$ it is necessary and sufficient that for every $i = 1, \dots, n$ we have $f^{\dagger i} \in \mathcal{H}_c(\Omega_i, \mathcal{C}_c(\prod_{j \neq i} \Omega_j, F))$.*

In conjunction with the localization theorem 2.1.6, this gives a sense in which a $\mathcal{C}_c$ -map on an open domain U in $\mathbb{C}^n$ is holomorphic iff it is holomorphic in each

variable separately. It is a weaker sense than the classical one, but good enough for our purposes. Indeed, in conjunction with theorem 2.2.1 it gives the following important fact.

Theorem 2.2.4 *For an open set U in $\mathbb{C}^n$ and $f \in \mathcal{C}_c(U, F)$ the following statements are equivalent:*

$$(A) \ f \in \mathcal{H}_c(U, F).$$

$$(B) \ f \in \mathcal{H}_c^\infty(U, F).$$

(C) *all composites $u \circ f : U \rightarrow \mathbb{C}$ are $\mathcal{H}_c$ -maps, where u varies in some monofamily of $\mathcal{LC}_c$ -maps $F \rightarrow \mathbb{C}$.*

Proof. By the localization theorem 2.1.6, we can reduce to the special case $U = \Omega_1 \times \dots \times \Omega_n$.

(A) $\Rightarrow$ (B). If $f \in \mathcal{H}_c(U, F)$, we deduce via 2.2.1, 2.2.2 and 2.2.3 that every partial derivative is again an $\mathcal{H}_c$ -map. This can be repeated indefinitely to give $f \in \mathcal{H}_c^\infty(U, F)$.

(B) $\Rightarrow$ (C). Trivial.

(C) $\Rightarrow$ (A). In view of 2.2.3, we can get (A) by showing each $f^{\dagger i} : \Omega_i \rightarrow \mathcal{C}_c(\Omega_{i*}, F)$ is $\mathcal{H}_c$, where $\Omega_{i*} = \prod_{j \neq i} \Omega_j$. From $u \circ f \in \mathcal{H}_c(U, \mathbb{C})$, we derive first that $(u \circ f)^{\dagger i} : \Omega_i \rightarrow \mathcal{C}_c(\Omega_{i*}, \mathbb{C})$ is $\mathcal{H}_c$, hence so is $\text{eval}_x \circ (u \circ f)^{\dagger i} : \Omega_i \rightarrow \mathbb{C}$ ($x \in \Omega_{i*}$), where $\text{eval}_x : \mathcal{C}_c(\Omega_{i*}, \mathbb{C}) \rightarrow \mathbb{C}$ is the evaluation at x . If the $u : F \rightarrow \mathbb{C}$ form a monofamily of linear maps, then so do $\mathcal{C}_c(\Omega_{i*}, u) : \mathcal{C}_c(\Omega_{i*}, F) \rightarrow \mathcal{C}_c(\Omega_{i*}, \mathbb{C})$ and so do the maps $\text{eval}_x \circ \mathcal{C}_c(\Omega_{i*}, u)$. Since $\text{eval}_x \circ \mathcal{C}_c(\Omega_{i*}, u) \circ f^{\dagger i} = \text{eval}_x \circ (u \circ f)^{\dagger i}$, we can apply 2.2.1(H9) to arrive at the desired conclusion that $f^{\dagger i}$ is $\mathcal{H}_c$.

Remark 2.2.5 In view of 2.2.3 and 2.2.4, several further characterizations of the maps in $\mathcal{H}_c(U, F)$ follow immediately from the known similar characterizations in the $\mathbb{C}$ -valued case. For example, those in terms of iterated Cauchy formulas or local Taylor expansions. We omit the details and refer to texts on several complex variables or to [2] for explicit formulations.

2.3 Infinite Dimensional Open Domains

Let us now turn to the case of open domains U in a general $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -space E , necessarily more complicated than the foregoing. Recall that in a $\mathcal{C}_c$ -space X a subset U is open when for every $x \in U$ and every $\mathcal{F} \rightsquigarrow x$ in X we have $U \in \mathcal{F}$.

We call a $\mathcal{C}_c$ -map $f : U \rightarrow F$ *holomorphic along affine lines* if, for every (x, h) in $U \times E$, the map $g : \Upsilon \rightarrow F$, $g(\lambda) = f(x + \lambda h)$ is holomorphic, where Υ is the open subset of all $\lambda \in \mathbb{C}$ for which $x + \lambda h \in U$. Henceforth, $\mathbb{D}$ denotes the open unit disk $|\lambda| < 1$ in $\mathbb{C}$. Thus $x + \varepsilon \mathbb{D}h$ is the set of all $x + \lambda h$ with $|\lambda| < \varepsilon$.

Theorem 2.3.1 *Suppose U is a convex open set in E , $f \in \mathcal{C}_c(U, F)$ and f is holomorphic along affine lines. Then $f \in \mathcal{H}_c(U, F)$.*

Proof. By 2.1.4(B) it is enough to exhibit a $\mathcal{C}_c$ -map $\hat{D}f : U \rightarrow [E, F]$ that gives the identity $f(y) - f(x) = \int_0^1 \hat{D}f(x + \theta(y - x)) \cdot (y - x) d\theta$. We go through several intermediate steps until reaching such a $\hat{D}f$ in Claim 6 below. Let x and y vary in U , h and k in E . Put

$$W_\delta = \{(x, h) \in U \times E \mid x + \delta \mathbb{D}h \subset U\}.$$

Then $\delta \geq \varepsilon$ implies $W_\delta \subset W_\varepsilon$. We interpret the ordered set $\mathbb{P}_+$ of positive reals in the usual way as an inductive preorder category in which the morphism $\delta\varepsilon : \delta \rightarrow \varepsilon$ exists precisely when $\delta \geq \varepsilon$. Then there is an obvious functor $W : \mathbb{P}_+ \rightarrow \mathcal{C}_c$ that carries $\delta\varepsilon : \delta \rightarrow \varepsilon$ to the subspace embedding $W_{\delta\varepsilon} : W_\delta \rightarrow W_\varepsilon$. We also have the subspace embedding maps $\text{sub}_\varepsilon : W_\varepsilon \rightarrow U \times E$ such that $\text{sub}_\varepsilon \circ W_{\delta\varepsilon} = \text{sub}_\delta$.

Claim 1. The maps $\text{sub}_\varepsilon : W_\varepsilon \rightarrow U \times E$ provide a colimit for the functor W .

Proof 1. To get this inductive colimit it is clearly enough to show that the maps $\text{sub}_\delta : W_\delta \rightarrow U \times E$ ($\delta > 0$) form a final epifamily in $\mathcal{C}_c$. So let $\mathcal{F} \rightsquigarrow (x, h)$ in $U \times E$. Our task is to find $\varepsilon > 0$ and a filter $\mathcal{F}_\varepsilon$ on W_ε such that $\mathcal{F}_\varepsilon \rightsquigarrow (x, h)$ in W_ε and

the filter base $\text{sub}_\varepsilon(\mathcal{F}_\varepsilon)$ generates $\mathcal{F}$ (see [19] for background about this). The map $s : U \times E \times \mathbb{C} \rightarrow E$ given by $s(x, h, \lambda) = x + \lambda h$ is continuous and $s(x, h, 0) = x \in U$. Hence $s(\mathcal{F} \times \mathcal{O}) \leadsto x$ in U , where $\mathcal{O}$ is the neighborhood filter of 0 in $\mathbb{C}$. Since U is open, we have $U \in s(\mathcal{F} \times \mathcal{O})$. So there exists $M \in \mathcal{F}$ and $\varepsilon > 0$ such that, simultaneously, $s(M \times \varepsilon\mathbb{D}) \subset U$ and $x + \varepsilon\mathbb{D}h \subset U$. It follows that $(x, h) \in \text{sub}_\varepsilon(W_\varepsilon)$, so we have an epifamily. Let $\mathcal{F}_\varepsilon$ be the trace of $\mathcal{F}$ on W_ε . Then $M \in \mathcal{F}_\varepsilon$ and $\mathcal{F}_\varepsilon$ satisfies the requirements mentioned above. So Claim 1 holds.

For every $\delta > 0$ we define the $\mathcal{C}_c$ -map $\dot{d}_\delta f : W_\delta \rightarrow F$ by putting

$$\dot{d}_\delta f(x, h) = (2\pi i)^{-1} \int_c \gamma^{-2} f(x + \gamma h) d\gamma,$$

where c is a circuit in $\delta\mathbb{D}$ around 0. In view of 2.2.1(H9) and the elementary theory of complex variables, this definition does not depend on the choice of c .

Claim 2. If $\delta > \varepsilon$, then $\dot{d}_\varepsilon f \circ W_{\delta\varepsilon} = \dot{d}_\delta f$.

Proof 2. This follows at once from the fact that the circuit c used in the definition of $\dot{d}_\varepsilon f$ can serve just as well in the definition of $\dot{d}_\delta f$.

In view of Claim 2 there is a colimit-induced $\mathcal{C}_c$ -map

$$\dot{d}f : U \times E \rightarrow F,$$

uniquely determined by the equations $\dot{d}f \circ \text{sub}_\delta = \dot{d}_\delta f$.

Claim 3. $\dot{d}f(x, \lambda h) = \lambda \dot{d}f(x, h)$ ($\lambda \in \mathbb{C}$).

Proof 3. Take δ so that $(x, h) \in W_\delta$ and ε so that, simultaneously, $(x, \lambda h) \in W_\varepsilon$ and $|\lambda\varepsilon| < \delta$. Let c be a circuit in $\varepsilon\mathbb{D}$ around 0 and put $b = \lambda c$. Then b is a circuit in $\delta\mathbb{D}$ around 0 and we obtain, after substitution in the integrals, that $\lambda \dot{d}f(x, h) = \lambda \dot{d}_\delta f(x, h) = \lambda (2\pi i)^{-1} \int_b \gamma^{-2} f(x + \gamma h) d\gamma = (2\pi i)^{-1} \int_c \gamma^{-2} f(x + \gamma \lambda h) d\gamma = \dot{d}_\varepsilon f(x, \lambda h) = \dot{d}f(x, \lambda h)$.

Claim 4. $\dot{d}f(x, k) = \lim_{\lambda \searrow 0} [f(x + \lambda h + \lambda k) - f(x + \lambda h)]/\lambda$ and

$$df(x, h) = \lim_{\lambda \searrow 0} [f(x + \lambda h) - f(x)]/\lambda.$$

Proof 4. Take $\delta > 0$ so that, simultaneously, $x + \delta\mathbb{D}h \subset U$, $x + \delta\mathbb{D}k \subset U$,

$x + \delta\mathbb{E}(h + k) \subset U$. Then, for nonzero λ and μ in $\delta\mathbb{D}$ and c a circuit in $\delta\mathbb{D}$ around λ , μ and 0, we have

$$f(x + \lambda h + \mu k) = (2\pi i)^{-1} \int_c (\gamma - \mu)^{-1} f(x + \lambda h + \gamma k) d\gamma.$$

Subtracting the special case $\mu = 0$ from the case $\mu = \lambda$, we obtain after some algebraic manipulation

$$f(x + \lambda h + \lambda k) - f(x + \lambda h) = (\lambda/2\pi i) \int_c (\gamma - \lambda)^{-1} \gamma^{-1} f(x + \lambda h + \gamma k) d\gamma.$$

By continuity of the integral, the first limit is $(2\pi i)^{-1} \int_c \gamma^{-2} f(x + \gamma k) d\gamma$, which is nothing but $\hat{d}f(x, k)$. The second limit is obtained by a similar (simpler) argument.

Claim 5. $\hat{d}f(x, h + k) = \hat{d}f(x, h) + \hat{d}f(x, k)$.

Proof 5. We have the identity

$$f(x + \lambda h + \lambda k) - f(x) = f(x + \lambda h + \lambda k) - f(x + \lambda h) + f(x + \lambda h) - f(x).$$

Thus $\hat{d}f(x, h + k)$ can be expressed as a sum of two limits, namely those appearing in Claim 4. So Claim 5 follows from Claim 4.

Define $\hat{D}f : U \rightarrow [E, F]$ by $\hat{D}f(x) \cdot h = \hat{d}f(x, h)$. Continuity of $\hat{D}f$ is automatic. Complex linearity of $\hat{D}f(x)$ follows from Claims 3 and 5.

Claim 6. $f(y) - f(x) = \int_0^1 \hat{D}f(x + \theta(y - x)) \cdot (y - x) d\theta$.

Proof 6. With x and y fixed, put $h = y - x$. Since U is convex, $x + \theta h \in U$ for all $\theta \in [0, 1]$. The set Υ of $\lambda \in \mathbb{C}$ such that $x + \lambda h \in U$ is open and it contains I , the closed line segment in $\mathbb{C}$ from 0 to 1. Take $\varepsilon > 0$ so that $I + \varepsilon\mathbb{D} \subset \Upsilon$. Then, by the assumption on f , we have the following holomorphic maps:

$$g : \Upsilon \rightarrow F, g(\lambda) = f(x + \lambda h),$$

$$g_\theta(\lambda) : \varepsilon\mathbb{D} \rightarrow F, g_\theta(\lambda) = g(\theta + \lambda), (\theta \in I).$$

Clearly, $g_\theta'(\lambda) = g'(\theta + \lambda)$; in particular, $g_\theta'(0) = g'(\theta)$. But we also have $g_\theta'(0) = (2\pi i)^{-1} \int_c \gamma^{-2} f(x + \theta h + \gamma h) d\gamma$, where c is a circuit in $\varepsilon\mathbb{D}$ around 0. It follows that

$g'(0) = \dot{d}f(x+\theta h, h)$, hence $f(y) - f(x) = g(1) - g(0) = \int_0^1 \dot{D}f(x+\theta(y-x)) \cdot (y-x) d\theta$, as required for Claim 6. This completes the proof of theorem 2.3.1.

Remark 2.3.2 The preceding proof illustrates a situation where $\mathcal{C}_c$ is preferable to the smaller toponome $\mathcal{C}_{c*}$ (governed by all three Choquet axioms) as a setting for analysis. In $\mathcal{C}_{c*}$, the proof of Claim 1 would have been more complicated: every additional $\mathcal{C}_{c*}$ -axiom complicates the final structures (see 22.f in [19]). The extra work would not have made the map $\dot{d}f$ any more continuous. Meanwhile, our spaces, being $\mathcal{C}_c$ -subspaces of $c\mathcal{L}\mathcal{C}_c$ -spaces, automatically have all $\mathcal{C}_{c*}$ -properties, and more.

We call a $\mathcal{C}_c$ -subspace T of E *convexwise open* if it is the codomain of a final epifamily in $\mathcal{C}_c$ of subspace embeddings, $\text{sub}_j : U_j \rightarrow T$ ($j \in J$), with every U_j a convex open domain. Note that such T is open. Indeed, if $\mathcal{F} \rightsquigarrow x$ in T , then in the presence of the given final epifamily, $\mathcal{F}$ must be generated by some $\text{sub}_j(\mathcal{F}_j)$, where $\mathcal{F}_j \rightsquigarrow x$ in U_j . It follows that $U_j \in \mathcal{F}_j$, hence $T \in \mathcal{F}$, so T is open.

Theorem 2.3.3 *Let U be convexwise open. For a map $f \in \mathcal{C}_c(U, F)$ the following statements are equivalent:*

- (A) $f \in \mathcal{H}_c(U, F)$.
- (B) $u \circ f \in \mathcal{H}_c(U, \mathbb{C})$, where u varies in some monofamily of $\mathcal{L}\mathcal{C}_c$ -maps $F \rightarrow \mathbb{C}$.
- (C) f is holomorphic along affine lines.
- (D) $f \in \mathcal{H}_c^\infty(U, F)$.

Proof. We are given a final epifamily of embeddings $\text{sub}_j : U_j \rightarrow U$ ($j \in J$) with convex open domains.

(A) $\Rightarrow$ (B) follows immediately.

(B) $\Rightarrow$ (C). Every $u \circ f \circ \text{sub}_j : U_j \rightarrow \mathbb{C}$ is an $\mathcal{H}_c$ -map, hence holomorphic along affine lines. By 2.2.1, every $f \circ \text{sub}_j$ is holomorphic along affine lines. It follows via 2.1.6 that f is then also holomorphic along affine lines.

(C) $\Rightarrow$ (D). In view of 2.1.6 and the convexwise openness, we can reduce to the case where U is a convex open set. From (C) it follows by theorem 2.3.1 that $f \in \mathcal{H}_c(U, F)$, so we have $\dot{D}f \in \mathcal{C}_c(U, F)$. It is now enough to show that $\dot{D}f \in \mathcal{H}_c(U, [E, F])$ (i.e. that f is an $\mathcal{H}_c^2$ -map), because then the rest follows by straightforward induction. Since $\dot{d}f : U \times E \rightarrow F$ is linear in its second variable, we can express $\dot{D}f(x) = (\dot{d}f)^{\dagger 1}$ and apply 2.1.7 as soon as we know $\dot{d}f \in \mathcal{H}_c(U \times E, F)$. By theorem 2.3.1, our task therefore reduces to showing that $\dot{d}f$ is holomorphic along affine lines. So let us fix the points (x, h) in $U \times E$, (k, l) in $E \times E$, and consider the map $g : \Upsilon \rightarrow F$, $g(\lambda) = \dot{d}f((x, h) + \lambda(k, l))$, where Υ is the maximal open subset of $\mathbb{C}$ on which $g(\lambda)$ makes sense. We need to show $g \in \mathcal{H}_c(\Upsilon, F)$ and we do this via 2.2.1(H3). Fix $\zeta \in \Upsilon$, then choose $\varepsilon > 0$ and $\delta > 0$ so that for all $\lambda \in \zeta + \varepsilon\mathbb{B}$ we have $(x + \lambda k, h + \lambda l) \in W_\delta$, where W_δ is the set introduced in proof 2.3.1, but now relative to $U \times E$ in the role of U . Then, as in that proof, we have the local representation

$$g(\lambda) = (2\pi i)^{-1} \int_{\mathbb{C}} \gamma^{-2} f(x + \lambda k + \gamma h + \gamma \lambda l) d\gamma.$$

The integrand is continuously differentiable with respect to λ . So the continuous differentiability of g (criterion 2.2.1(H3)) follows at once from the continuity of the integral with respect to parameters.

(D) $\Rightarrow$ (A) follows trivially.

Remarks 2.3.4

(1) In [1] a domain U in E is called τ -open if for every $x \in U$ there exists a convex open V such that $x + V \subset U$. It is readily seen that U is τ -open if and only if it is convexwise open (cf. proof of Claim 1 in 2.3.1).

(2) Theorem 2.3.3 sheds light on the notion of G-holomorphic map employed in [1]. As used there, G-holomorphy of $f \in \mathcal{C}_c(U, F)$, where U is τ -open, means f is weakly holomorphic and holomorphic along affine lines, where *weakly holomorphic* means all composites $\ell \circ f : U \rightarrow \mathbb{C}$ ($\ell \in F^*$) are holomorphic. Theorem 2.3.3 reveals

that the two conditions that jointly define G-holomorphy are in fact equivalent. This equivalence is known in other contexts, more precise reference [2].

(3) Note that 2.3.3(B) is strictly weaker than ‘weakly holomorphic’. Here (and also in theorem 2.2.1) the difference is significant. For example, before a special $\mathcal{C}_c$ -map $f : U \rightarrow \mathcal{H}_c(V, \mathbb{C})$ can be tested for holomorphy via weak holomorphy, one needs to know what all functionals $\ell \in \mathcal{H}_c(V, \mathbb{C})^*$ look like. This information is unavailable except for the simplest V . On the other hand, to test f via 2.3.3(B), one merely needs to know enough functionals to form a monofamily. This presents no problem: just take the evaluations $g \mapsto g(x)$, ($x \in V$).

(4) The equivalence of (A) and (C) is another Hartogs-like theorem under continuity assumptions (cf. 3.6 in [2]).

2.4 Taylor-like Expansion of Holomorphic Maps

In view of 2.3.3 several interesting results proved in [1] for G-holomorphic maps apply to our $\mathcal{H}_c$ -maps, notably the results about Taylor-like expansions as a series of homogeneous polynomials. We present the Taylor-like expansion of holomorphic maps, with a different proof, in this section. Let us start with the following facts about the $\mathcal{C}_c$ -map f :

Proposition 2.4.1 *Let Λ, Γ be nondegenerate fundamental domains (i.e. non-degenerated real interval) and $f \in \mathcal{C}_c(\Lambda \times \Gamma, E)$. Then we have*

$$(1) \quad \int_{\Lambda} \left(\int_{\Gamma} f(\tau, \theta) d\theta \right) d\tau = \int_{\Gamma} \left(\int_{\Lambda} f(\tau, \theta) d\tau \right) d\theta.$$

Proof. Consider the following diagrams:

$$\begin{array}{ccc}
\mathcal{C}_c(\Lambda, \mathcal{C}_c(\Gamma, E)) & \xrightarrow{\mathbf{av}_{\mathcal{C}_c(\Gamma, E)}^\Lambda} & \text{ad}\mathcal{C}_c(\Lambda \times \Lambda, \mathcal{C}_c(\Gamma, E)) \\
\downarrow \xi_{\Lambda \Gamma E} & & \downarrow \xi_{(\Lambda \times \Lambda) \Gamma E} \\
\mathcal{C}_c(\Gamma, \mathcal{C}_c(\Lambda, E)) & \xrightarrow{\mathcal{C}_c(\Gamma, \mathbf{av}_E^\Lambda)} & \mathcal{C}_c(\Gamma, \text{ad}\mathcal{C}_c(\Lambda \times \Lambda, E))
\end{array}$$

and

$$\begin{array}{ccc}
\text{ad}\mathcal{C}_c(\Lambda \times \Lambda, \mathcal{C}_c(\Gamma, E)) & \xrightarrow{\text{ad}\mathcal{C}_c(\Lambda \times \Lambda, \mathbf{av}_E^\Gamma)} & \text{ad}\mathcal{C}_c(\Lambda \times \Lambda, \text{ad}\mathcal{C}_c(\Gamma \times \Gamma, E)) \\
\downarrow \xi_{(\Lambda \times \Lambda) \Gamma E} & & \downarrow \xi_{(\Lambda \times \Lambda) (\Gamma \times \Gamma) E} \\
\mathcal{C}_c(\Gamma, \text{ad}\mathcal{C}_c(\Lambda \times \Lambda, E)) & \xrightarrow{\mathbf{av}_{\mathcal{C}_c(\Lambda \times \Lambda, E)}^\Gamma} & \text{ad}\mathcal{C}_c(\Gamma \times \Gamma, \text{ad}\mathcal{C}_c(\Lambda \times \Lambda, E))
\end{array}$$

By an argument similar to 3c2 [16], the first rectangle is commutative. We can also show that $\xi_{(\Lambda \times \Lambda) (\Gamma \times \Gamma) E} \circ \text{ad}\mathcal{C}_c(\Lambda \times \Lambda, \mathbf{av}_E^\Gamma) \circ \xi_{\Gamma (\Lambda \times \Lambda) E}$ satisfies the identities given in 3c1 [16] (i.e. its evaluation at every f in the domain satisfies the Additivity Law), so it is equal to $\mathbf{av}_{\mathcal{C}_c(\Lambda \times \Lambda, E)}^\Gamma$. Hence

$$(2) \quad \xi_{(\Lambda \times \Lambda) (\Gamma \times \Gamma) E} \circ \text{ad}\mathcal{C}_c(\Lambda \times \Lambda, \mathbf{av}_E^\Gamma) \circ \mathbf{av}_{\mathcal{C}_c(\Gamma, E)}^\Lambda = \mathbf{av}_{\mathcal{C}_c(\Lambda \times \Lambda, E)}^\Gamma \circ \mathcal{C}_c(\Gamma, \mathbf{av}_E^\Lambda) \circ \xi_{\Lambda \Gamma E}.$$

Define $g : \Lambda \rightarrow \mathcal{C}_c(\Gamma, E)$ by $g(\lambda)(\gamma) = f(\lambda, \gamma)$. Clearly g is a $\mathcal{C}_c$ -map. Now by evaluating both sides of equation (2) at g and applying 2d(0) in [17] we arrive at (1).

Corollary 2.4.2 *Let $f \in \mathcal{C}_c(\Lambda \times \Gamma, E)$ be such that for every $\theta \in \Lambda$, the $\mathcal{C}_c$ -map $f(\theta, -) : \Gamma \rightarrow E$ is a path. If $g(\gamma) = \int_a^\beta f(\theta, \gamma) d\theta$ then $g'(\gamma) = \int_a^\beta \partial_2 f(\theta, \gamma) d\theta$, where $\partial_2 f(\theta, -) = (f(\theta, -))'$.*

Proof. As $f(\theta, -)$ is a path, $f(\theta, \gamma) = \int_\lambda^\gamma \partial_2 f(\theta, \tau) d\tau$. Hence

$$g(\gamma) = \int_a^\beta \int_\lambda^\gamma \partial_2 f(\theta, \tau) d\tau d\theta.$$

By applying proposition 2.4.1 we have $g(\gamma) = \int_{\lambda}^{\gamma} (\int_{\alpha}^{\beta} \partial_2 f(\theta, \tau) d\theta) d\tau$. Now by 2d(2) in [17] we arrive at the required identity.

Remarks 2.4.3

(a) Proposition 2.4.1 can be also proved by classical real analysis. In fact for every $\ell \in E^*$,

$$\int_{\alpha}^{\beta} \int_{\lambda}^{\gamma} \ell \cdot f(\tau, \theta) d\theta d\tau = \int_{\lambda}^{\gamma} \int_{\alpha}^{\beta} \ell \cdot f(\tau, \theta) d\tau d\theta.$$

Since such ℓ form a monofamily, we get the required conclusion.

(b) For convexwise open $\mathcal{C}_c$ -subspaces U of E and $f \in \mathcal{H}_c(U, F)$, one can show that $\dot{D}^m f$ has the following local representation:

$$\dot{D}^m f(x) \cdot h_1 \cdot h_2 \cdots h_m = (2\pi i)^{-m} \int_{c_1} \int_{c_2} \cdots \int_{c_m} \frac{f(x + \sum_{i=1}^m \gamma_i h_i)}{\gamma_1^2 \gamma_2^2 \cdots \gamma_m^2} d\gamma_1 d\gamma_2 \cdots d\gamma_m,$$

where $x \in U$, $h_i \in E$, $x + \sum_{i=1}^m \varepsilon_i \mathbb{D} h_i \subseteq U$ for some $\varepsilon_i > 0$ and c_i is a circuit around zero in $\varepsilon_i \mathbb{D}$ for $1 \leq i \leq m$. The proof is similar to 2.3.1. One applies 2.4.1 and induction.

Definition 2.4.4

(a) A subset U of E is said to be *closed* in E if it contains all $x \in E$ such that some filter $\mathcal{F} \rightsquigarrow x$ with $U \in \mathcal{F}$.

(b) $\mathcal{H}_c^r(U, F)$ is defined to carry the *initial $\mathcal{C}_c$ -structure* induced by the linear functions $\text{ins}^r : \mathcal{H}_c^r(U, F) \rightarrow \mathcal{C}_c^r(U, F)$ ($r = 1, 2, \dots$).

Theorem 2.4.5 *For a convexwise open subset U of E , $\mathcal{H}_c(U, F)$ is a closed $\mathcal{C}_c$ -subspace of $\mathcal{C}_c(U, F)$.*

Proof. First we prove $\mathcal{H}_c(U, F)$ is closed in $\mathcal{C}_c(U, F)$. To see that, let $\mathcal{F} \rightsquigarrow f$ in $\mathcal{C}_c(U, F)$ and $\mathcal{H}_c(U, F) \in \mathcal{F}$. We must show that $f \in \mathcal{H}_c(U, F)$. In view of 2.2.1 and 2.3.3, it is enough to show $\ell \circ f$ is holomorphic along affine lines for all $\ell \in E^*$. To prove the latter, fix $x \in U$ and $h \in E$. Then there exists $\varepsilon > 0$

such that $x + \varepsilon \mathbb{D}h \subseteq U$. Fix $\ell \in E^*$. As $\mathcal{C}_c(U, \ell) : \mathcal{C}_c(U, F) \rightarrow \mathcal{C}_c(U, \mathbb{C})$ is a $\mathcal{C}_c$ -map, $\mathcal{C}_c(U, \ell)(\mathcal{F}) \rightsquigarrow \mathcal{C}_c(U, \ell)(f)$ in $\mathcal{C}_c(U, \mathbb{C})$ i.e. $\ell \circ \mathcal{F} \rightsquigarrow \ell \circ f$ in $\mathcal{C}_c(U, \mathbb{C})$. Define $a_{xh} : \varepsilon \mathbb{D} \rightarrow U, a_{xh}(\lambda) = x + \lambda h$. Then we have the $\mathcal{C}_c$ -map $\mathcal{C}_c(a_{xh}, \mathbb{C}) : \mathcal{C}_c(U, \mathbb{C}) \rightarrow \mathcal{C}_c(\varepsilon \mathbb{D}, \mathbb{C}), \mathcal{C}_c(a_{xh}, \mathbb{C})(g) = g \circ a_{xh}$. Hence $\mathcal{C}_c(a_{xh}, \mathbb{C})(\ell \circ \mathcal{F}) \rightsquigarrow \mathcal{C}_c(a_{xh}, \mathbb{C})(\ell \circ f)$ in $\mathcal{C}_c(\varepsilon \mathbb{D}, \mathbb{C})$ or equivalently $(\ell \circ \mathcal{F}) \circ a_{xh} \rightsquigarrow (\ell \circ f) \circ a_{xh}$ in $\mathcal{C}_c(\varepsilon \mathbb{D}, \mathbb{C})$. On the other hand $\ell \circ \mathcal{H}_c(U, F) \circ a_{xh} \in (\ell \circ \mathcal{F} \circ a_{xh}) \cap \mathcal{H}_c(\varepsilon \mathbb{D}, \mathbb{C})$. Since $\varepsilon \mathbb{D}$ is locally compact, $\mathcal{C}_c(\varepsilon \mathbb{D}, \mathbb{C})$ carries the compact open topology. So by classical analysis, $\mathcal{H}_c(\varepsilon \mathbb{D}, \mathbb{C})$ is a closed subspace of $\mathcal{C}_c(\varepsilon \mathbb{D}, \mathbb{C})$. We conclude that $\ell \circ f \circ a_{xh} \in \mathcal{H}_c(\varepsilon \mathbb{D}, \mathbb{C})$ for all $\ell \in E^*$. Therefore f is holomorphic via theorem 2.3.3.

It remains to prove $\mathcal{H}_c(U, F)$ is a $\mathcal{C}_c$ -subspace of $\mathcal{C}_c(U, F)$. To show that, we start with a filter $\mathcal{F}$ in $\mathcal{H}_c(U, F)$ and a holomorphic map $f : U \rightarrow F$ such that $J \circ I(\mathcal{F}) \rightsquigarrow f$ in $\mathcal{C}_c(U, F)$, where $J : \mathcal{C}_c^1(U, F) \rightarrow \mathcal{C}_c(U, F)$ and $I : \mathcal{H}_c(U, F) \rightarrow \mathcal{C}_c^1(U, F)$ are inserion maps. Consider the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{H}_c(U, F) & \xrightarrow{I} & \mathcal{C}_c^1(U, F) \\
 \hat{D} \downarrow & & \downarrow D \\
 \mathcal{C}_c(U, [E, F]) & \xrightarrow{\mathcal{C}_c(U, \text{ins})} & \mathcal{C}_c(U, [E_{\mathbb{P}}, F_{\mathbb{P}}])
 \end{array}$$

where $\hat{D}, D$ are $\mathcal{C}_c$ -maps with $\hat{D}(f) = \hat{D}f$ and $D(f) = Df$ and $\text{ins} : [E, F] \rightarrow [E_{\mathbb{P}}, F_{\mathbb{P}}]$ is the inserion map. We must show that

$$(1) \quad \mathcal{F} \rightsquigarrow f \text{ in } \mathcal{H}_c(U, F).$$

But $\mathcal{H}_c(U, F)$ is a $\mathcal{C}_c$ -subspace of $\mathcal{C}_c^1(U, F)$, so it is enough to show $I(\mathcal{F}) \rightsquigarrow f$ in $\mathcal{C}_c^1(U, F)$. By definition of $\mathcal{C}_c$ -structure of $\mathcal{C}_c^1(U, F)$, $\{J, D\}$ is an initial family in $\mathcal{LC}_c$. Hence (1) is equivalent to $J \circ I(\mathcal{F}) \rightsquigarrow f$ in $\mathcal{C}_c(U, F)$ and $D \circ I(\mathcal{F}) \rightsquigarrow Df$ in $\mathcal{C}_c(U, [E_{\mathbb{P}}, F_{\mathbb{P}}])$. As the above rectangle is commutative $\mathcal{C}_c(U, \text{ins}) \circ \hat{D} = D \circ I$, and $\mathcal{C}_c(U, \text{ins})$ is a $\mathcal{C}_c$ -map, we only need to show that $\hat{D}\mathcal{F} \rightsquigarrow \hat{D}f$ in $\mathcal{C}_c(U, [E, F])$ which

is equivalent to: if $\mathcal{A}$ and $\mathcal{H}$ are filters in U and E respectively such that $\mathcal{A} \rightsquigarrow x \in U$ and $\mathcal{H} \rightsquigarrow h \in E$, then $\dot{D}\mathcal{F}(\mathcal{A}) \cdot \mathcal{H} \rightsquigarrow \dot{D}f(x) \cdot h$. To see that, let $\varepsilon > 0$ be such that $x + \varepsilon \mathbb{D}h \subseteq U$ and c be a circuit around zero in $\varepsilon \mathbb{D}$. By the continuity of the integral and theorem 2.2.1 $J \circ I(\mathcal{F})(\mathcal{A} + \gamma \mathcal{H}) \rightsquigarrow f(x + \gamma h)$. Therefore, for all γ , $\mathcal{F}(\mathcal{A} + \gamma \mathcal{H}) \rightsquigarrow f(x + \gamma h)$, as refinement. So we arrive at:

$$\dot{D}\mathcal{F}(\mathcal{A}) \cdot \mathcal{H} = (2\pi i)^{-1} \int_c \mathcal{F}(\mathcal{A} + \gamma \mathcal{H}) \gamma^{-2} d\gamma \rightsquigarrow (2\pi i)^{-1} \int_c f(x + \gamma h) \gamma^{-2} d\gamma = \dot{D}f(x) \cdot h,$$
 as required.

Remark 2.4.6 Proposition 2.4.5 shows that, for convexwise open $\mathcal{C}_c$ -subspaces of E , the structure given to $\mathcal{H}_c(U, F)$ in [1] is equivalent to the structure defined in 2.4.4(b).

In the following theorem the equivalence of (A) and (B) is already proved in [1]. We give a different proof, simpler than that of [1]. Let us first define the homogeneous polynomials.

Definition 2.4.7 A mapping $f : E \rightarrow F$ is called a *homogeneous polynomial of degree k* or *k -homogeneous* iff there exists a multi $\mathbb{C}$ -linear $\mathcal{C}_c$ -map $m : E^k \rightarrow F$ such that $f(h) = m(h, h, \dots, h), h \in E$.

Theorem 2.4.8 Let U be a convexwise open $\mathcal{C}_c$ -subspace of E . For $x \in U$, the following conditions are equivalent:

(A) $f \in \mathcal{H}_c(U, F)$.

(B) f has local Taylor-like expansion. Precisely, there is a unique sequence of homogeneous polynomials $P_{m,x} : E \rightarrow F$ such that $f(x + h) = \sum_{m=0}^{\infty} P_{m,x}(h)$, where h varies on an absolutely convex open subset V of E containing zero (i.e. $\mathbb{D}V \subseteq V$).

(C) f has a local power series expansion. Precisely, $f(x + h) = \sum_{m=0}^{\infty} \frac{1}{m!} A_{m,x} \cdot h^m$, where the $A_{m,x}$'s are m -linear $\mathcal{C}_c$ -maps and h varies on an absolutely convex open subset of E containing zero.

Proof.

(A) $\Rightarrow$ (B). By the continuity of linear operations in E there is an absolutely convex subset V of E containing zero such that $x + 2V \subseteq U$. Pick $h \in V$, then $x + 2\mathbb{D}h \subseteq U$. Define $g : 2\mathbb{D} \rightarrow F$ by $g(\lambda) = f(x + \lambda h)$. Since g is holomorphic, by 2.2.1(H7) $f(x + \lambda h) = \sum_{m=0}^{\infty} ((2\pi i)^{-1} \int_{2\mathbb{D}} f(x + \gamma h) \gamma^{-m-1} d\gamma) \lambda^m$. Let $\lambda = 1$, then

$$f(x + h) = \sum_{m=0}^{\infty} P_{m,x}(h),$$

where $P_{m,x}(h) = g^{(m)}(0)/m!$. It is enough to show, for every m , $P_{m,x}$ is a homogeneous polynomial of degree m . Clearly

$$\begin{aligned} P_{m,x}(h) &= (m!)^{-1} g^{(m)}(0) = (m!)^{-1} \frac{\partial^m}{\partial \xi_1 \partial \xi_2 \cdots \partial \xi_m} f(x + \sum_{i=1}^m \xi_i h) \Big|_{\xi_1 = \cdots = \xi_m = 0} \\ &= (m!)^{-1} (2\pi i)^{-m} \int_{c_1} \int_{c_2} \cdots \int_{c_m} f(x + \sum_{i=1}^m \xi_i h) (\prod_{i=1}^m \gamma_i^{-2}) d\gamma_1 d\gamma_2 \cdots d\gamma_m, \end{aligned}$$

where c_i is a circuit around zero, for all $i, 1 \leq i \leq m$. Hence by remark 2.1.3(b) $P_{m,x}(h) = (m!)^{-1} \hat{D}^m f(x) \cdot h \cdot h \cdots h \stackrel{\text{def}}{=} (m!)^{-1} \hat{D}^m f(x) \cdot h^m$, which is certainly m -homogeneous.

(B) $\Rightarrow$ (C). Obvious.

(C) $\Rightarrow$ (A). Follows by theorem 2.4.5 and the fact that every homogeneous polynomial is holomorphic. This completes the proof.

From the proof of theorem 2.4.8 we get the following important result:

Scholium 2.4.9 *Let U be a convexwise open $\mathcal{C}_c$ -subspace of E and $f \in \mathcal{H}_c(U, F)$. Then f has a local Taylor-like expansion*

$$f(x + h) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{D}^m f(x) \cdot h^m,$$

where h varies on an absolutely convex open subset of E containing zero.

2.5 The Exponential Laws for Holomorphic Maps

We suppose from now on $\Lambda \subseteq \mathbb{R}$ is a fundamental domain. E, F and G are $\mathcal{CLC}$ -spaces and $U \subseteq E, V \subseteq F$ are tangential domains.

In this section we establish the exponential laws for holomorphic maps along the lines of the corresponding laws for smooth maps. Recall that $\mathcal{H}_c^r(U, F)$ carry the initial $\mathcal{C}_c$ -structure induced by the linear functions $\text{ins}^r : \mathcal{H}_c^r(U, F) \rightarrow \mathcal{C}_c^r(U, F), r \in [1, \infty)$ (see 2.4.4).

We obviously need to show that the hom-set $\mathcal{H}_c^r(U, F)$ lies in the category $\mathbf{c}\mathcal{LC}_c$. We do this in the following proposition.

Proposition 2.5.1 *The $\mathcal{LC}_c$ -subspace $\mathcal{H}_c^r(U, F)$ of $\mathcal{C}_c^r(U, F)$ is a $\mathbf{c}\mathcal{LC}_c$ -space ($r \in [0, \infty]$).*

Proof. Similar to the arguments given in 5b[17], let $J(\Lambda) = \Lambda \times \Lambda \times \mathcal{C}_c^1(\Lambda, U)$. Consider the following equalizer in $\mathcal{LC}_c$:

$$E \xrightarrow{\text{ins}} \mathcal{C}_c(U, F) \times \mathcal{C}_c(U, [E, F]) \xrightleftharpoons[m]{\ell} P \stackrel{\text{def}}{=} \prod_{\Lambda \in \mathbf{Fun}} \prod_{(\alpha, \beta, p) \in J(\Lambda)} F$$

where $\mathbf{Fun}$ denotes the set of all fundamental domain $\Lambda \subseteq \mathbb{R}$ and

$$\begin{aligned} \ell((f, T)_{(\Lambda, \alpha, \beta, p)}) &= (f \circ p)(\beta) - (f \circ p)(\alpha) \\ m((f, T)_{(\Lambda, \alpha, \beta, p)}) &= \int_{\alpha}^{\beta} T(p(\tau)) \cdot p'(\tau) d\tau \end{aligned}$$

Then E is just the subspace on which the maps ℓ and m agree and in view of the uniqueness of derivatives and $\mathbb{C}$ -linearity of the elements of $[E, F]$, it consists of precisely the pairs $(f, \dot{D}f)$ with $f \in \mathcal{H}_c(U, F)$. Since E is isomorphic in $\mathcal{LC}_c$ to the space $\mathcal{H}_c(U, F)$ via the isomorphism $f \mapsto (f, \dot{D}f)$ and $\mathbf{c}\mathcal{LC}_c$ is a reflective subcategory of $\mathcal{LC}_c$ which is closed under limits, we conclude $\mathcal{H}_c(U, F) \in \mathbf{c}\mathcal{LC}_c$. The rest follows by induction.

Lemma 2.5.2 *$[E, F]$ is a $\mathcal{C}_c$ -subspace of $[E_{\mathbb{R}}, F_{\mathbb{R}}]$.*

Proof. As $[E_{\mathbb{R}}, F_{\mathbb{R}}]$ and $[E, F]$ are $\mathcal{C}_c$ -subspaces of $\mathcal{C}_c(E, F)$, the insertion maps $\alpha : [E_{\mathbb{R}}, F_{\mathbb{R}}] \rightarrow \mathcal{C}_c(E, F)$ and $\beta : [E, F] \rightarrow \mathcal{C}_c(E, F)$ are initial $\mathcal{C}_c$ -maps. Let $\gamma :$

$[E, F] \rightarrow [E_{\mathbb{P}}, F_{\mathbb{P}}]$ be the insertion function. Then $\beta = \alpha \circ \gamma$. Hence γ is initial, as inner factor.

Notation 2.5.3 We denote the insertion $\gamma : [E, F] \rightarrow [E_{\mathbb{P}}, F_{\mathbb{P}}]$ in the above lemma by $\text{ins}_{\mathbb{P}}$.

Remarks 2.5.4

(a) By the following commutative diagram the insertion map

$$\text{inlc}^{r,r-1} : \mathcal{H}_c^r(U, F) \rightarrow \mathcal{H}_c^{r-1}(U, F)$$

is a $\mathcal{C}_c$ -map:

$$\begin{array}{ccc}
 \mathcal{H}_c^r(U, F) & \xrightarrow{\text{inlc}^{r,r-1}} & \mathcal{H}_c^{r-1}(U, F) \\
 \downarrow (\mathcal{C}_c\text{-map}) \text{ins}^r & \searrow \mathcal{C}_c\text{-map} & \downarrow \text{ins}^{r-1} \text{ (initial)} \\
 \mathcal{C}_c^r(U, F) & \xrightarrow{\text{inc}^{r-1}} & \mathcal{C}_c^{r-1}(U, F)
 \end{array}$$

(b) The map $\hat{D} : \mathcal{H}_c^r(U, F) \rightarrow \mathcal{H}_c^{r-1}(U, [E, F]), f \mapsto \hat{D}f$ is an $\mathcal{LC}_c$ -map.

Consider the following diagram:

$$\begin{array}{ccc}
 \mathcal{H}_c^r(U, F) & \xrightarrow{\hat{D}} & \mathcal{H}_c^{r-1}(U, [E, F]) \\
 \downarrow \text{ins}^r & \searrow \mathcal{C}_c\text{-map} & \downarrow \mathcal{C}_c^{r-1}(U, \text{ins}_{\mathbb{P}}) \circ \text{ins}^{r-1} \text{ (initial)} \\
 \mathcal{C}_c^r(U, F) & \xrightarrow{D} & \mathcal{C}_c^{r-1}(U, [E_{\mathbb{P}}, F_{\mathbb{P}}])
 \end{array}$$

Let $f \in \mathcal{H}_c^r(U, F)$. Then $(\mathcal{C}_c^{r-1}(U, \text{ins}_{\mathbb{P}}) \circ \text{ins}^{r-1} \circ \hat{D})(f) = (\text{ins}_{\mathbb{P}} \circ \text{ins}^{r-1})(\hat{D}f) = \hat{D}f$ and $(D \circ \text{ins}^r)(f) = D(f) = Df$. Since $f \in \mathcal{H}_c^r(U, F)$, so $Df(x)$ is $\mathbb{C}$ -linear for all $x \in U$. Therefore Df underlies $\hat{D}f$ i.e. the above square is commutative. Hence $\hat{D}$ is a $\mathcal{C}_c$ -map.

(c) The maps

$$\hat{D} : \mathcal{H}_c^r(U, F) \rightarrow \mathcal{H}_c^{r-1}(U, [E, F]),$$

$$\text{inhc}^{r,r-1} : \mathcal{H}_c^r(U, F) \rightarrow \mathcal{H}_c^{r-1}(U, F)$$

form an initial $\mathcal{C}_c$ -family. This follows in view of the following commutative rectangle and the known initiality of the family in the bottom row (see 5a in [17]):

$$\begin{array}{ccccc}
 & & G & & \\
 & \swarrow \hat{D} \circ h & \downarrow h & \searrow \text{inhc}^{r,r-1} \circ h & \\
 \mathcal{H}_c^{r-1}(U, [E, F]) & \xleftarrow{\hat{D}} & \mathcal{H}_c^r(U, F) & \xrightarrow{\text{inhc}^{r,r-1}} & \mathcal{H}_c^{r-1}(U, F) \\
 \downarrow \ell \circ \text{ins}^{r-1} & & \downarrow \text{ins}^r \text{ (initial)} & & \downarrow \text{ins}^{r-1} \\
 \mathcal{C}_c^{r-1}(U, [E_{\mathbb{P}}, F_{\mathbb{P}}]) & \xleftarrow{D} & \mathcal{C}_c^r(U, F) & \xrightarrow{\text{ins}^{r,r-1}} & \mathcal{C}_c^{r-1}(U, F)
 \end{array}$$

where $\ell = \mathcal{C}_c^{r-1}(U, \text{ins}_{\mathbb{P}})$. In fact for a map h , if $\hat{D} \circ h$ and $\text{inhc}^{r,r-1} \circ h$ are $\mathcal{C}_c$ -map, then $\text{ins}^{r-1} \circ \text{inhc}^{r,r-1} \circ h$ and $\ell \circ \text{ins}^{r-1} \circ \hat{D} \circ h$ are $\mathcal{C}_c$ too. Hence by initiality of the family $\{D, \text{ins}^{r,r-1}\}$, $\text{ins}^r \circ h$ is also $\mathcal{C}_c$. Therefore h is a $\mathcal{C}_c$ -map via initiality of $\{\text{ins}^r\}$. This means that the family $\{\hat{D}, \text{inhc}^{r,r-1}\}$ is initial.

Proposition 2.5.5 *Let $G : \mathbf{cLC}_c \rightarrow \mathbf{cLC}_c$ be a functor which preserves $\mathbf{cLC}_c$ -initial families. Suppose $F : \mathbf{cLC}_c \rightarrow \mathbf{cLC}_c$ is another functor and $\mathbf{n}_E : F(E) \rightarrow G(E)$ are embeddings ($E \in \mathbf{cLC}_c$) and components of a natural transformation. Then F also preserves initial families.*

Proof. Let $f_i : E \rightarrow E_i$ ($i \in I$) be an initial family in $\mathbf{cLC}_c$. Then $Gf_i : GE \rightarrow GE_i$ ($i \in I$) is also an initial family and $\mathbf{n}_E \circ Ff_i = Gf_i \circ \mathbf{n}_E$ ($i \in I$). Since $\mathbf{n}_E$ is an embedding, $Gf_i \circ \mathbf{n}_E$ is an initial family. Therefore $\mathbf{n}_E \circ Ff_i$ is an initial family too. Hence Ff_i is an initial family, as inner factor.

Theorem 2.5.6 *(The Exponential Laws For $\mathcal{H}_c^r$ -maps)*

For integers $r \geq 1$ the following hold:

(a^r) *The functor $\mathcal{C}_c(U, -) : \mathbf{cLC}_c \rightarrow \mathbf{cLC}_c$ lifts to a functor $\mathcal{H}_c^r(U, -) : \mathbf{cLC}_c \rightarrow \mathbf{cLC}_c$ and this functor preserves $\mathcal{C}_c$ -initial families of $\mathbb{C}$ -maps, hence it preserves cartesian products.*

(b^r) *There is a natural $\mathbf{cLC}_c$ -isomorphism $h\mathfrak{F}_{FUG}^r : [F, \mathcal{H}_c^r(U, G)] \rightarrow \mathcal{H}_c^r(U, [F, G])$ which lifts $\mathfrak{F}_{FUG} : [F, \mathcal{C}_c(U, G)] \rightarrow \mathcal{C}_c(U, [F, G])$.*

(c^r) *The following statements about $f \in \mathcal{C}_c(U \times V, G)$ are equivalent:*

(α^r) $f \in \mathcal{H}_c^r(U \times V, G)$.

(β^r) $f^{\dagger 1} \in \mathcal{H}_c^s(U, \mathcal{H}_c^{r-s}(V, G))$ and $f^{\dagger 2} \in \mathcal{H}_c^s(V, \mathcal{H}_c^{r-s}(U, G))$ for $s = 0, 1, 2, \dots, r$.

(d^r) *There exists a natural transformation $h\ddagger 1_{U \times V, G}^{r,s} : \mathcal{H}_c^r(U \times V, G) \rightarrow \mathcal{H}_c^s(U, \mathcal{H}_c^{r-s}(V, G))$ ($s = 0, 1, \dots, r$) which lifts $\ddagger 1$ and a similar statement holds for $h\ddagger 2^{r,s}$.*

(e^r) *There exists a natural transformation $h\ddagger 1_{U \times V, G}^r : \mathcal{H}_c^r(U, \mathcal{H}_c^r(V, G)) \rightarrow \mathcal{H}_c^r(U \times V, G)$ which lifts the corresponding $\ddagger$ in $\mathcal{C}_c$.*

(f^r) *The function $h\mathbf{comp}_{U \times V, G}^r : \mathcal{H}_c^r(U, V) \times \mathcal{H}_c^r(V, G) \rightarrow \mathcal{H}_c^r(U, G)$, $h\mathbf{comp}^r(f, g) = f \circ g$ is a $\mathcal{C}_c$ -map.*

(β') The following statements about $f \in \mathcal{C}_c(U, F)$ with U tangentially convex are equivalent:

$$(\alpha^r) f \in \mathcal{H}_c^{r+1}(U, F).$$

(β^r) There exists $\dot{D}f \in \mathcal{H}_c^r(U, [E, F])$ such that

$$f(y) - f(x) = \int_0^1 \dot{D}f(x + \theta(y - x)) \cdot (y - x) d\theta \quad (x, y \in U).$$

(γ^r) There exists $\dot{\Phi} \in \mathcal{H}_c^r(U \times V, [E, F])$ such that

$$f(y) - f(x) = \dot{\Phi}(x, y) \cdot (y - x) \quad (x, y \in U)$$

Proof. The proof largely repeats the reasoning of 5c in [17] or 4b1 in [16]. We only show the continuity of the indicated maps.

As regards (α^r), by 2.1.5(2 r) for every $\mathbb{C}$ -linear $\mathcal{C}_c$ -map $w : F \rightarrow G$ the linear function $\mathcal{C}_c(U, w), f \mapsto w \circ f$ carries $\mathcal{H}_c^r(U, F)$ into $\mathcal{H}_c^r(U, G)$. Thus the linear lifted function $\mathcal{H}_c^r(U, w) : \mathcal{H}_c^r(U, F) \rightarrow \mathcal{H}_c^r(U, G)$ exists. As $\mathcal{C}_c^r(U, w)$ is a $\mathcal{C}_c$ -map by 4b1(α^r) in [16], from the following commutative diagram we get that the function $\mathcal{H}_c^r(U, w)$ is an $\mathcal{LC}_c$ -map. Hence $\mathcal{H}_c^r(U, -)$ is a functor:

$$\begin{array}{ccc}
 \mathcal{H}_c^r(U, F) & \xrightarrow{\mathcal{H}_c^r(U, w)} & \mathcal{H}_c^r(U, G) \\
 \downarrow \text{ins}^r & \searrow \mathcal{C}_c\text{-map} & \downarrow \text{ins}^r \text{ (initial)} \\
 \mathcal{C}_c^r(U, F) & \xrightarrow{\mathcal{C}_c^r(U, w)} & \mathcal{C}_c^r(U, G)
 \end{array}$$

By putting $G = \mathcal{C}_c^r(U, -)$ and $F = \mathcal{H}_c^r(U, -)$ in proposition 2.5.5 preservation of $\mathcal{C}_c$ -initial families follows.

As regards (b^r), one can show (similarly to 5c(b^r) in [17]) that $h\mathfrak{H}_{F \cup G}^r$ as a function exists. By the following commutative square and 5c(b^r) in [17] we get that $h\mathfrak{H}_{F \cup G}^r$ is a $\mathcal{C}_c$ -map:

$$\begin{array}{ccc}
 [F, \mathcal{H}_c^r(U, G)] & \xrightarrow{h\mathfrak{H}^r} & \mathcal{H}_c^r(U, [F, G]) \\
 \downarrow \text{ins}_{\mathbb{P}} \circ [F, \text{ins}^r] & \searrow \mathcal{C}_c\text{-map} & \downarrow \mathcal{C}_c^r(U, \text{ins}_{\mathbb{P}}) \circ \text{ins}^r \text{ (initial)} \\
 [F_{\mathbb{P}}, \mathcal{C}_c^r(U, G)_{\mathbb{P}}] & \xrightarrow{\mathfrak{H}^r} & \mathcal{C}_c^r(U, [F_{\mathbb{P}}, G_{\mathbb{P}}])
 \end{array}$$

As regards (c^r), (d^r), the statements are similar to 5c(c^r) in [17]. To show $h \nmid 1^{r,s}$ is a $\mathcal{C}_c$ -map consider the following diagram. Similarly for $h \nmid 2^{r,s}$.

$$\begin{array}{ccc}
 \mathcal{H}_c^r(U \times V, G) & \xrightarrow{h \nmid 1^{r,s}} & \mathcal{H}_c^s(U, \mathcal{H}_c^{r-s}(V, G)) \\
 \downarrow \text{ins}^r & \searrow \mathcal{C}_c\text{-map} & \downarrow \text{ins}^s \circ \mathcal{H}_c^s(U, \text{ins}^{r-s}) \text{ (initial)} \\
 \mathcal{C}_c^r(U \times V, G) & \xrightarrow{\nmid 1^{r,s}} & \mathcal{C}_c^s(U, \mathcal{C}_c^{r-s}(V, G))
 \end{array}$$

As regards (e^r), (f^r), proofs are similar to the corresponding ones in 5c in [17]. The continuity of the indicated maps follow respectively from the diagrams below.

$$\begin{array}{ccc}
 \mathcal{H}_c^r(U, \mathcal{H}_c^r(V, G)) & \xrightarrow{h\sharp^r} & \mathcal{H}_c^r(U \times V, G) \\
 \downarrow \text{ins}^r \circ \mathcal{H}_c^r(U, \text{ins}^r) & \searrow \mathcal{C}_c\text{-map} & \downarrow \text{ins}^r \text{ (initial)} \\
 \mathcal{C}_c^r(U, \mathcal{C}_c^r(V, G)) & \xrightarrow{\sharp^r} & \mathcal{C}_c^r(U \times V, G)
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{H}_c^r(U, V) \times \mathcal{H}_c^r(V, G) & \xrightarrow{h\text{comp}_{UVG}^r} & \mathcal{H}_c^r(U, G) \\
 \downarrow \text{ins}^r \times \text{ins}^r & \searrow \mathcal{C}_c\text{-map} & \downarrow \text{ins}^r \text{ (initial)} \\
 \mathcal{C}_c^r(U, V) \times \mathcal{C}_c^r(V, G) & \xrightarrow{\text{comp}_{UVG}^r} & \mathcal{C}_c^r(U, G)
 \end{array}$$

As regards (g^r) , the reasoning is again similar to 5c in [17].

2.6 More Tangential Domains With Empty Interior

In the following we present fascinating tangential domains: nonopen with empty interior, yet arising quite naturally. We also present a large class of $\mathcal{H}_c^\infty$ -maps defined on these domains.

Lemma 2.6.1 $\mathcal{H}_c(\mathbb{D}, \mathbb{D})$ is a tangential domain in the $\mathcal{CLC}$ -space $\mathcal{H}_c(\mathbb{D}, \mathbb{C})$ which is not only nonopen but its interior is empty.

Proof. Let us show $\mathcal{H}_c(\mathbb{D}, \mathbb{D})$ is tangential. Fix $a \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})$. For every $h \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})$ we can form the path $c : [0, 1] \rightarrow \mathcal{H}_c(\mathbb{D}, \mathbb{D})$, $c(\tau) = a + \tau(h - a)$ with $c(0) = a$, $c'(0) = h - a$. Thus the tangent vectors at a contains the set $\mathcal{H}_c(\mathbb{D}, \mathbb{D}) - a$. By Runge's classical theorem (see e.g. [3]), the set $\mathcal{P}$ of polynomials on $\mathbb{D}$ form a dense subset of $\mathcal{H}_c(\mathbb{D}, \mathbb{C})$. Therefore $\mathcal{P} - a$ is likewise dense. For every nonzero polynomial $p(\zeta) = \sum_{i=0}^n \alpha_i \zeta^i$ we can take $\beta > \max_i |\alpha_i|$ and write:

$$p(\zeta) - a(\zeta) = \beta \sum_{i=0}^n [(\alpha_i/\beta)\zeta^i - a(\zeta)] + (1 - n\beta)(0 \cdot \zeta - a(\zeta)).$$

This equation shows that the linear span of $\mathcal{H}_c(\mathbb{D}, \mathbb{D}) - a$ contains the dense set $\mathcal{P} - a$. We conclude $\mathcal{H}_c(\mathbb{D}, \mathbb{D})$ is tangential.

Let us now show that 0 is not an interior point of $\mathcal{H}_c(\mathbb{D}, \mathbb{C})$. It is enough to show that every basic neighborhood $V(r\bar{\mathbb{D}}, \varepsilon\mathbb{D})$ ($0 < r < 1$, $\varepsilon > 0$) of 0 has a member which is not in $\mathcal{H}_c(\mathbb{D}, \mathbb{D})$. Since $r^n \rightarrow 0$, we can choose an integer n such that $r^n < \varepsilon/2 < 1$ and form the function $f(\zeta) = 2\zeta^n$. Then $f \in V(r\bar{\mathbb{D}}, \varepsilon\mathbb{D})$. On the other hand if $1 > |\zeta| > 2^{-1/n}$, then $|\zeta|^n > 1/2$ or $|f(\zeta)| > 1$. Therefore $f \notin \mathcal{H}_c(\mathbb{D}, \mathbb{D})$.

Finally, let us show that $\mathcal{H}_c(\mathbb{D}, \mathbb{D})$ has empty interior. Let $g \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})$. Then $g + V(r\bar{\mathbb{D}}, \varepsilon\mathbb{D})$ is a basic neighborhood at g . Let f be the function defined in the above paragraph. Then $g + f \in g + V(r\bar{\mathbb{D}}, \varepsilon\mathbb{D})$. If $1 > |\zeta| > 2^{-1/n}(1 + |g(\zeta)|)^{1/n}$, then $|2\zeta^n| = |g(\zeta)| > 1$. Hence $|f(\zeta) + g(\zeta)| \geq |f(\zeta)| - |g(\zeta)| > 1$. Therefore $g + f \notin \mathcal{H}_c(\mathbb{D}, \mathbb{D})$, i.e. g is not an interior point of $\mathcal{H}_c(\mathbb{D}, \mathbb{D})$ for all $g \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})$.

Proposition 2.6.2 For $f \in \mathcal{H}_c(\mathbb{D}, \mathbb{C})$, $\mathcal{H}_c(\mathbb{D}, f) \in \mathcal{H}_c^{**}(\mathcal{H}_c(\mathbb{D}, \mathbb{D}), \mathcal{H}_c(\mathbb{D}, \mathbb{C}))$, with derivative given by $(\hat{D}\mathcal{H}_c(\mathbb{D}, f))(x) \cdot h(\alpha) = \hat{D}f(x(\alpha)) \cdot h(\alpha) = h(\alpha)f'(x(\alpha))$.

Proof. Since f is holomorphic with convex domain, by theorem 2.1.4 there is a $\mathcal{C}$ -map $\hat{\Psi}_1 : \mathbb{D} \times \mathbb{D} \rightarrow [\mathbb{C}, \mathbb{C}]$ such that $f(\alpha) - f(\beta) = \hat{\Psi}_1(\alpha, \beta) \cdot (\beta - \alpha)$ ($\alpha, \beta \in \mathbb{D}$) and $\hat{\Psi}_1(\alpha, \alpha) \cdot \lambda = \hat{D}f(\alpha) \cdot \lambda = \lambda f'(\alpha)$. Define $\hat{\Phi}_1 : \mathcal{H}_c(\mathbb{D}, \dots) \times \mathcal{H}_c(\mathbb{D}, \mathbb{D}) \rightarrow$

$[\mathcal{H}(\mathbb{D}, \mathbb{C}), \mathcal{H}(\mathbb{D}, \mathbb{C})]$ by, $(\hat{\Phi}_1(x, y) \cdot h)(\alpha) = \hat{\Psi}_1(x(\alpha), y(\alpha)) \cdot h(\alpha)$. Then $\hat{\Phi}_1$ is a $\mathcal{C}_c$ -map, via exponential laws in the category $\mathcal{C}_c$. So by theorem 2.1.4, $\mathcal{H}_c(\mathbb{D}, f)$ is a holomorphic map. Moreover,

$$\begin{aligned} (\hat{D}\mathcal{H}_c(\mathbb{D}, f)(x) \cdot h)(\alpha) &= (\hat{\Phi}_1(x, x) \cdot h)(\alpha) \\ &= \hat{\Psi}_1(x(\alpha), x(\alpha)) \cdot h(\alpha) = h(\alpha)f'(x(\alpha)). \end{aligned}$$

As $\mathbb{D}$ is open, by theorem 2.3.3, $f \in \mathcal{H}_c^\infty(\mathbb{D}, \mathbb{C})$. We will show that $\mathcal{H}_c(\mathbb{D}, f)$ is an $\mathcal{H}_c^\infty$ -map by induction. To do this, we will first show that $\mathcal{H}_c(\mathbb{D}, f)$ is an $\mathcal{H}_c^2$ -map. The inductive step is similar to showing $\mathcal{H}_c(\mathbb{D}, f)$ is $\mathcal{H}_c^2$ and will be omitted, despite the fact that it is more complicated. So we only need prove that $\hat{D}\mathcal{H}_c(\mathbb{D}, f)$ is an $\mathcal{H}_c$ -map. Since $\hat{D}f : \mathbb{D} \rightarrow [\mathbb{C}, \mathbb{C}]$ is an $\mathcal{H}_c$ -map, there is a $\mathcal{C}_c$ -map $\hat{\Psi}_2 : \mathbb{D} \times \mathbb{D} \rightarrow [\mathbb{C}, [\mathbb{C}, \mathbb{C}]]$ such that $\hat{D}f(x) - \hat{D}f(y) = \hat{\Psi}_2(x, y) \cdot (y - x)$. Define $\hat{\Phi}_2 : \mathcal{H}_c(\mathbb{D}, \mathbb{D}) \times \mathcal{H}_c(\mathbb{D}, \mathbb{D}) \rightarrow [\mathcal{H}_c(\mathbb{D}, \mathbb{C}), [\mathcal{H}_c(\mathbb{D}, \mathbb{C}), \mathcal{H}_c(\mathbb{D}, \mathbb{C})]]$ by, $(\hat{\Phi}_2(g, k) \cdot h_1 \cdot h_2)(\alpha) = \hat{\Psi}_2(g(\alpha), k(\alpha)) \cdot h_1(\alpha) \cdot h_2(\alpha)$. Similar to $\hat{\Phi}_1$, $\hat{\Phi}_2$ is also a $\mathcal{C}_c$ -map. On the other hand

$$\begin{aligned} ([\hat{D}\mathcal{H}_c(\mathbb{D}, f)(g) - \hat{D}\mathcal{H}_c(\mathbb{D}, f)(k)] \cdot h_2)(\alpha) \\ &= \hat{D}f(g(\alpha)) \cdot h_2(\alpha) - \hat{D}f(k(\alpha)) \cdot h_2(\alpha) \\ &= \hat{\Psi}_2(g(\alpha), k(\alpha)) \cdot (k(\alpha) - g(\alpha)) \cdot h_2(\alpha) \\ &= (\hat{\Phi}_2(g, k) \cdot (k - g) \cdot h_2)(\alpha) \end{aligned}$$

for all $\alpha \in \mathbb{D}$ and $f, g \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})$. Hence by theorem 2.1.4, $\mathcal{H}_c(\mathbb{D}, f)$ is an $\mathcal{H}_c^2$ -map.

Theorem 2.6.3 *Let Λ, Ω be simply connected open proper subsets of $\mathbb{C}$. Then $\mathcal{H}_c(\Lambda, \Omega)$ is a nonopen tangential domain with empty interior in $\mathcal{H}_c(\Lambda, \mathbb{C})$.*

Proof. By the Riemann Mapping Theorem VII.4.2 in [3], there are biholomorphic maps $f : \mathbb{D} \rightarrow \Omega$ and $g : \mathbb{D} \rightarrow \Lambda$. We carry out our task by the following two steps:

Step 1. $\mathcal{H}_c(\mathbb{D}, \Omega)$ is a nonopen tangential domain in $\mathcal{H}_c(\mathbb{D}, \mathbb{C})$.

Proof 1. Define $F : \mathcal{H}_c(\mathbb{D}, \mathbb{D}) \rightarrow \mathcal{H}_c(\mathbb{D}, \mathbb{C})$, by $F(h) = f \circ h$. Then by the above proposition F is an $\mathcal{H}_c^\infty$ -map with $(\hat{D}F(x) \cdot h)(\alpha) = \hat{D}f(x(\alpha)) \cdot h(\alpha) = h(\alpha)f'(x(\alpha))$.

As f is biholomorphic, so $f'(x(\alpha)) \neq 0$ and $f' \circ x$ is holomorphic for all $x \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})$. Since $F(\mathcal{H}_c(\mathbb{D}, \mathbb{D})) = \mathcal{H}_c(\mathbb{D}, \Omega)$, via theorem 4i in [17], it is enough to show $\dot{D}F(x)$ is an epimorphism in the category $\mathbf{c}\mathcal{L}\mathcal{C}_c$ for all $x \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})$. To see that, fix $\mathcal{L}\mathcal{C}_c$ -maps $G, H : \mathcal{H}_c(\mathbb{D}, \mathbb{C}) \rightarrow \mathbb{C}$ such that $H \circ \dot{D}F(x) = G \circ \dot{D}F(x)$. Then $H(\dot{D}F(x) \cdot h) = G(\dot{D}F(x) \cdot h)$ for all $h \in \mathcal{H}_c(\mathbb{D}, \mathbb{C})$. Fix $k \in \mathcal{H}_c(\mathbb{D}, \mathbb{C})$ and put $h = k/(f' \circ x)$, then $H(k) = G(k)$. Hence $H = G$, so we have an epimorphism.

Step 2. $\mathcal{H}_c(\Lambda, \Omega)$ is a nonopen tangential domain in $\mathcal{H}_c(\Lambda, \mathbb{C})$.

Proof 2. Define $L : \mathcal{H}_c(\mathbb{D}, \Omega) \rightarrow \mathcal{H}_c(\Lambda, \mathbb{C})$, by $L(h) = h \circ g^{-1}$. As L is $\mathbb{C}$ -linear, by theorem 2.1.5(2'), L is an $\mathcal{H}_c$ -map with $\dot{D}L(x) = L$ for all $x \in \mathcal{H}_c(\mathbb{D}, \Omega)$. Similar to step 1, it is enough to show $\dot{D}L(x)$ is an epimorphism in $\mathbf{c}\mathcal{L}\mathcal{C}_c$. But this is immediate since $L : \mathcal{H}_c(\mathbb{D}, \mathbb{C}) \rightarrow \mathcal{H}_c(\Lambda, \mathbb{C})$ is an $\mathcal{L}\mathcal{C}_c$ -isomorphism.

The proof of the theorem is complete.

2.7 Comparison of $\mathcal{A}_c^r$ -maps and $\mathcal{H}_c^r$ -maps

We have developed two theories of complex calculus in the preceding chapter and this chapter. One naturally asks how these relate to each other. As they are defined in different domains, namely analytically tangential domains and tangential domains, we obviously ask how these domains relate.

The following result answers this question:

Proposition 2.7.1 *Every analytically tangential domain is a tangential domain.*

Proof. Let E be a $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -space and $U \subset E$ be an analytically tangential domain in E . Fix $x \in U$. Then the analytic tangent vectors of U at x form a functionally dense subset of E . Let $p : \Omega \rightarrow U$ be an analytic path with $p(\alpha) = x$ for some $\alpha \in \Omega$. There exists $r > 0$ such that $B(\alpha, r) \subseteq \Omega$. Define $q_p, r_p : [0, 1] \rightarrow \Omega$ by $q_p(\tau) = \alpha + r\tau/2$ and $r_p(\tau) = \alpha + ir\tau/2$. Then $(p \circ q_p)(0) = (p \circ r_p)(0) = x$ and

$(p \circ q_p)'(0) = (r/2)p'(\alpha)$, $(p \circ r_p)'(0) = (ir/2)p'(\alpha)$. Hence for every analytic path in U we get two paths in U . We claim that the set of all tangent vectors that arise from the above analytic paths are functionally dense in E . To see that, let $\ell : E \rightarrow \mathbb{R}$ be $\mathbb{R}$ -linear. Define $\ell_{\mathbb{C}} : E \rightarrow \mathbb{C}$ such that $\ell_{\mathbb{C}}(y) = \ell(y) - i\ell(iy)$. Then $\ell_{\mathbb{C}}$ is $\mathbb{C}$ -linear. Let $\ell((p \circ q_p)'(0)) = \ell((p \circ r_p)'(0)) = 0$, for all analytic paths p with $p(\alpha) = x$. Then $\ell(p'(\alpha)) = \ell(ip'(\alpha)) = 0$. Therefore $\ell_{\mathbb{C}}(p'(\alpha)) = 0$, for all p . As the set $\{p'(\alpha) : p \text{ an analytic path in } U \text{ with } p(\alpha) = x\}$ is functionally dense in E , we obtain $\ell_{\mathbb{C}} = 0$. Hence $\ell = 0$, as required.

By theorem 2.2.1, scholium 2.2.2 and theorem 1.3.10 we have the following important fact:

Corollary 2.7.2 *A $\mathcal{C}_c$ -map $g : \Omega \rightarrow E$ is an analytic path iff it is holomorphic. Hence $\mathcal{A}_c^\infty(\Omega, E) = \mathcal{H}_c^\infty(\Omega, E)$.*

Lemma 2.7.3 *Let $f : \Omega \rightarrow E$ be an analytic path and α, β belong to a convex subset of Ω . Then*

$$\oint_{\alpha}^{\beta} f'(\xi) d\xi = \int_0^1 f'(\alpha + \theta(\beta - \alpha)) \cdot (\beta - \alpha) d\theta.$$

Proof. By equation (3) followed by definition 1.3.1, f is holomorphic. Hence by theorem 2.1.4 we obtain

$$\int_0^1 f'(\alpha + \theta(\beta - \alpha)) \cdot (\beta - \alpha) d\theta = f(\beta) - f(\alpha) = \oint_{\alpha}^{\beta} f'(\xi) d\xi.$$

Proposition 2.7.4 *Let U be a convexwise open $\mathcal{C}_c$ -subspace of E . If $f \in \mathcal{A}_c(U, F)$ then $f \in \mathcal{H}_c(U, F)$.*

Proof. As U is convexwise open, we have a final epifamily $\text{sub}_j : U_j \rightarrow U$, where domains are open. By localization theorem 2.1.6, it is enough to show that $f \circ \text{sub}_j = f_j \in \mathcal{H}_c(U_j, F)$. Since $f \in \mathcal{A}_c(U, F)$, then $f_j \in \mathcal{A}_c(U_j, F)$.

Let $x \in U_j, h \in E$. Then there is Υ open in $\mathbb{C}$ such that $x + \Upsilon h \subseteq U_j$. Define $p : \Upsilon \rightarrow U_j$ by $p(\lambda) = x + \lambda h$. Then p is an analytic path. Hence $f_j \circ p$ is also an analytic path or $p \circ f_j \in \mathcal{A}_c(\Upsilon, F)$. On the other hand $\mathcal{A}_c(\Upsilon, F) = \mathcal{H}_c(\Upsilon, F)$. Therefore $f_j \circ p \in \mathcal{H}_c(\Upsilon, F)$. That means f_j is holomorphic along affine lines. Hence the required result follows via theorem 2.3.1.

Open question 2.7.5

Let U be a convexwise open domain. Does $\mathcal{A}_c(U, F)$ contain $\mathcal{H}_c(U, F)$?

Note that this statement is equivalent to question 1.3.13(2).

Chapter 3

Existence Theorems in Nonnormable Setting

The fact that our holomorphy theory allows nonopen domains with empty interior will be seen in this chapter to be quite significant. It enables a new type of inverse function theorem and a new Newton's Method similar to that of the real differential theory in [18] and [14]. The formulations of these theorems, which reduce in Banach spaces to the classical versions, are significantly different from other known generalizations (cf. [6],[7],[9]) in that open neighborhoods become replaced by tangential 'quasineighborhoods' which need not be open and may even have empty interior.

In the present chapter we establish the inverse function theorem and Newton's method based on the holomorphy theory developed in chapter 2. Most of the materials in this chapter are similar to the corresponding ones in [18] and [14].

New categories, namely the auxiliary categories will be studied in section 1 along with some examples.

In section 2, we prepare for proving inverse function theorem then we prove it along the lines of the real version.

Finally in section 3, we state the Newton's Method and illustrate its application

to perturbation of functional equations on nonopen domains.

3.1 The Auxiliary Categories

We recall here the construction of auxiliary categories $J\mathcal{LC}_c$ and $\mathcal{JC}_c$ [18]. In what follows J will denote a preorder category i.e. a small category such that for every pair of objects i and j , there is at most one morphism $i \rightarrow j$, denoted ij when it exists. Thus a preordered category always arises from a preordered set. For example $(\mathbb{N}, \geq)$ determines a preorder category whose nonidentity morphisms can be visualized as follows:

$$\dots \rightarrow j \xrightarrow{ji} i \dots \rightarrow 1 \rightarrow 0.$$

We will use J and $\mathcal{C}_c$ to construct the new auxiliary categories $J\mathcal{LC}_c$ and $\mathcal{JC}_c$. We will also construct a few $J\mathcal{LC}_c$ -spaces. We also use $\mathbf{Lin}$ to denote the category of linear spaces and linear maps.

Definition 3.1.1 The symbol $[E]$, where E is an $\mathcal{LC}_c$ -space, will denote the constant functor $J \rightarrow \mathcal{LC}_c$ that carries all objects to E and all morphisms to id_E . A $J\mathcal{LC}_c$ -space is a triple $(E, E_{(-)}, \pi^E)$ consisting of

- (1) a $\mathcal{LC}_c$ -space E ,
- (2) a functor $E_{(-)} : J \rightarrow \mathcal{LC}_c$, the *approximation functor*, which transforms every $j \in J$ into a Banach space E_j , the *approximand space*,
- (3) a natural transformation π^E from the constant functor $[E]$ to $E_{(-)}$ which forms a limit diagram in $\mathcal{LC}_c$. Its component at $j \in J$ will be written $\pi_j = \pi_j^E : E \rightarrow E_j$.

Note that since $c\mathcal{LC}_c$ is a reflective subcategory containing all spaces E_j , their limit is likewise in $c\mathcal{LC}_c$. A $J\mathcal{LC}_c$ -map $u : (E, E_{(-)}, \pi^E) \rightarrow (F, F_{(-)}, \pi^F)$ between $J\mathcal{LC}_c$ -spaces is formed by an $\mathcal{LC}_c$ -map $u : E \rightarrow F$ which is induced by a natural

transformation $u : E_{(-)} \rightarrow F_{(-)}$ via the universal property of the natural transformation π^F . Accordingly, we have for each J -morphism $ij : i \rightarrow j$ the following commutative diagrams:

$$\begin{array}{ccccc}
 E & \xrightarrow{\pi_j} & E_j & \xrightarrow{u} & F & \xrightarrow{\pi_j} & F_j \\
 \searrow \pi_i & & \nearrow E_{ij} & & \searrow \pi_i & & \nearrow F_{ij} \\
 & E_i & & E_j & \xrightarrow{u_j} & F_j & \\
 & & \downarrow \pi_j^E & & \downarrow \pi_j^F & & \\
 & & E_j & & F_j & &
 \end{array}$$

The $J\mathcal{LC}_c$ -spaces and maps so obtained build a category of structured sets, denoted $J\mathcal{LC}_c$.

Example 3.1.2 It is well known that every complete separated locally convex space E can be represented as a projective limit of Banach spaces in the category $\mathbf{Lcx}$ of locally convex spaces (see e.g. §19.9(1) in [11]). Since $\mathbf{Lcx}$ is a full reflective subcategory of $\mathcal{LC}_c$, this projective limit can just as well be regarded as one in $\mathcal{LC}_c$. Therefore every complete separated $E \in \mathbf{Lcx}$ can be structured to become a $J\mathcal{LC}_c$ -space.

Example 3.1.3 Let F be a Banach-space and U be an open disk in $\mathbb{C}^n$. Then there are compact convex disks K_j such that $K_j \subseteq \text{int} K_{j+1}$ and $U = \bigcup_{j \in \mathbb{N}} K_j = \bigcup_{j \in \mathbb{N}} \text{int} K_j$. Let J be the preorder category $(\mathbb{N}, \geq)$, let $E = \mathcal{H}_c^r(U, F)$ and $E_{(-)} : J \rightarrow \mathcal{cLC}_c$ be the functor obtained by putting $E_j = \mathcal{H}_c^r(K_j, F)$ (see definition 2.1.2). Let $E_{ij} : E_j \rightarrow E_i$ denote the restriction map $f \mapsto f|_{K_i}$. It is known that $\mathcal{C}_c^r(K_j, F)$ is a Banach space with norm:

$$\|f\| = \max_{0 \leq k \leq r} \|D^k f\|.$$

Lemma 3.1.4 Under the above assumption $E_j = \mathcal{H}_c^r(K_j, F)$ is a Banach space. In fact it is a closed subspace of $\mathcal{C}_c^r(K_j, F)$ with the above norm.

Proof. Suppose $\{f_n; n \in \mathbb{N}\} \subseteq E_j, f \in \mathcal{C}_c^r(K_j, F)$ and $f_n \rightsquigarrow f$. For given $\varepsilon > 0$, we have $\|D^k f_n - D^k f\| < \varepsilon$, for large n and $0 \leq k \leq r$. If $A(f) = D^r f(x) \cdot (a_1, a_2, \dots, a_{j-1}, -, a_{j+1}, \dots, a_r)$, then we must show that

$$(1) \quad A(f) \cdot (g + \lambda h) = A(f) \cdot (g) + \lambda A(f) \cdot (h),$$

where $x \in K_j$, $\lambda \in \mathbb{C}$ and $g, h, a_i \in \mathbb{C}^n$ ($1 \leq i \leq r$). Consider the following inequalities:

$$\begin{aligned} & \|A(f) \cdot (g + \lambda h) - A(f) \cdot (g) - \lambda A(f) \cdot (h)\| \\ & \leq \|A(f) \cdot (g + \lambda h) - A(f_n) \cdot (g + \lambda h)\| \\ & \quad + \|A(f_n) \cdot (g) - A(f) \cdot (g)\| \\ & \quad + |\lambda| \|A(f_n) \cdot (h) - A(f) \cdot (h)\| \\ & \leq \varepsilon(|g + \lambda h| + |g| + |\lambda||h|). \end{aligned}$$

Hence we arrive at (1) which means $f \in \mathcal{H}_c^r(K_j, F)$. Thus $\mathcal{H}_c^r(K_j, F)$ is a closed subspace of $\mathcal{C}_c^r(K_j, F)$ as required.

Now, let $\pi_j^E : E \rightarrow E_j$ be also the restriction map. To show $(E, E_{(-)}, \pi^E)$ is a $\mathcal{JLC}_c$ -space, we must prove that $\pi^E : [E] \rightarrow E_{(-)}$ forms a limit diagram in $\mathcal{LC}_c$. For this purpose it is enough to show that the family $\{\pi_i^E\}$ ($i \in \mathbb{I}$) is an initial family in $\mathcal{LC}_c$. We prove this via the following lemma and proposition:

Lemma 3.1.5 *Let $H \in \mathbf{cLC}_c$. Under the above assumption the family $\text{id} \times \text{ins}_i : H \times K_i \rightarrow H \times U$ ($i \in \mathbb{I}$) is final with respect to the underlying functor $\mathcal{H}_c^r \dashrightarrow \mathbf{Set}$.*

Proof. Let $g : H \times U \rightarrow G$ be a function such that $g \circ (\text{id} \times \text{ins}_i)$ is a $\mathcal{H}_c^r$ map for all $i \in \mathbb{I}$.

$$\begin{array}{ccc}
 H \times K_i & \xrightarrow{\text{id} \times \text{ins}_i} & H \times U \\
 & \searrow g \circ (\text{id} \times \text{ins}_i) & \downarrow g \\
 & & G
 \end{array}$$

As $H \times U = \bigcup_{i \in \mathbb{I}} (H \times \text{int} K_i) = \bigcup_{i \in \mathbb{I}} (H \times K_i)$, every $H \times \text{int} K_i$ is path-open in $H \times U$, $\text{id} \times \text{ins}_i : H \times \text{int} K_i \rightarrow H \times U$ ($i \in \mathbb{I}$) is $\mathcal{C}_c$ -final and $g \circ (\text{id} \times \text{ins}_i)|_{H \times \text{int} K_{i-1}} : H \times \text{int} K_{i-1} \rightarrow G$ ($i \in \mathbb{I}$) are $\mathcal{H}_c^r$ -maps. By localization theorem 2.1.6, g is a $\mathcal{H}_c^r$ -map, as required.

Proposition 3.1.6 *Under the above assumption, the family $\{\pi_i^E\}$ ($i \in \mathbb{N}$) is an initial family with respect to the underlying functor $\mathcal{LC}_c \rightarrow \mathbf{Lin}$. Hence $(E, E_{(-)}, \pi^E)$ is a $\mathcal{JLC}_c$ -space.*

Proof. Let $k : H \rightarrow \mathcal{H}_c^r(U, F)$ be a function such that $\pi_i^E \circ k$ is an $\mathcal{LC}_c$ -map. Then by theorem 2.1.5(2^r), $\pi_i^E \circ k$ is a $\mathcal{H}_c^r$ -map for all $i \in \mathbb{I}$.

$$\begin{array}{ccc}
 \mathcal{H}_c^r(U, F) & \xrightarrow{\pi_i^E} & \mathcal{H}_c^r(K_i, F) \\
 \uparrow k & \nearrow \pi_i^E \circ k & \\
 H & &
 \end{array}$$

Consider the following commutative triangle in which $(\pi_i^E \circ k)^{\sharp^r}$ is a $\mathcal{H}_c^r$ -map (see theorem 2.5.6(c^r)):

$$\begin{array}{ccc}
 H \times K_i & \xrightarrow{\text{id} \times \text{ins}_i} & H \times U \\
 & \searrow (\pi_i^E \circ k)^{\sharp r} & \downarrow k^{\sharp} \\
 & & E
 \end{array}$$

Since the family $\{\text{id} \times \text{ins}_i\}$ ($i \in \mathbb{N}$) is $\mathcal{H}_c^r$ -final (see lemma 3.1.5), $k^{\sharp}$ is a $\mathcal{H}_c^r$ -map. Hence by theorem 2.5.6(c^r), $(k^{\sharp})^{\dagger} = k : U \rightarrow E$ is a $\mathcal{C}_c$ -map, as required.

Proposition 3.1.7 *The category $\mathcal{JLC}_c$ has finite products and the obvious forgetful functor $\mathcal{JLC}_c \rightarrow \mathcal{LC}_c$ preserves them.*

Proof. Straightforward.

Definition 3.1.8 Proceeding analogously to 3.1.1, we define a $\mathcal{JC}_c$ -space to be a triple $(X, X_{(-)}, \pi^X)$ consisting of:

- (1) a $\mathcal{C}_c$ -space X ,
- (2) a functor $X_{(-)} : \mathbf{J} \rightarrow \mathcal{C}_c$, the *approximation functor*, which transforms every $j \in \mathbf{J}$ into the *approximand space*, which is a $\mathcal{C}_c$ -subspace of a Banach space,
- (3) a natural transformation π^X from the constant functor $[X]$ to $X_{(-)}$ which furnishes a limit diagram in $\mathcal{C}_c$. Its component at $j \in \mathbf{J}$ will be written $\pi_j = \pi_j^X : X \rightarrow X_j$. A $\mathcal{JC}_c$ -map $f : (X, X_{(-)}, \pi^X) \rightarrow (Y, Y_{(-)}, \pi^Y)$ between $\mathcal{JC}_c$ -spaces is a $\mathcal{C}_c$ -map $f : X \rightarrow Y$ which is induced by a natural transformation $\mathbf{f} : X_{(-)} \rightarrow Y_{(-)}$ via the universal property of the natural transformation π^Y .

These spaces and maps build in the obvious way a category of structured sets, denoted $\mathcal{JC}_c$. If the $\mathbf{f}$ above is a natural $\mathcal{C}_c$ -embedding; and if such embedding takes the form $\mathbf{f} = \text{ins}$, a natural insertion with components $\text{ins}_j(x) = x$, then we have a $\mathcal{JC}_c$ -subspace. Thus subspaces $X_j \subset E_j \in \mathcal{JLC}_c$ give rise to $\mathcal{JC}_c$ -subspaces when $E_{j'}(X_j) \subset X_{j'}$.

Since the underlying $\mathcal{C}_c$ -space functor $\mathcal{U} : \mathcal{L}\mathcal{C}_c \rightarrow \mathcal{C}_c$ preserves limits, it induces in the obvious way a similar underlying functor $\mathcal{J}\mathcal{L}\mathcal{C}_c \rightarrow \mathcal{J}\mathcal{C}_c$ which preserves embeddings, subspaces and finite products.

Proposition 3.1.9 *$\mathcal{J}\mathcal{C}_c$ has finite products and the obvious forgetful functor $\mathcal{J}\mathcal{C}_c \rightarrow \mathcal{C}_c$ preserves them.*

Proof. Straightforward.

Definition 3.1.10 Suppose $E = (E, E_{(-)}, \pi^E)$ is a given $\mathcal{J}\mathcal{L}\mathcal{C}_c$ -space, that $X = (X, X_{(-)}, \pi^X)$ is a $\mathcal{J}\mathcal{C}_c$ -subspace and let $a \in X$. Put $a_j = \pi_j(a)$. If each approximand X_j is an open ball $B(a_j, \varepsilon) \stackrel{\text{def}}{=} \{w \in E_j : \|w - a_j\| < \varepsilon\}$ ($j \in \mathbb{J}$), then the limit space X will be denoted $B(a, \varepsilon)$ and called a *quasiball* in E with center a and radius ε . Let us emphasize that the same ε is used for all j .

Proposition 3.1.11 *Suppose $(E, E_{(-)}, \pi^E)$ is the $\mathcal{J}\mathcal{L}\mathcal{C}_c$ -space which is constructed in example 3.1.3, $g \in E$ and $\varepsilon > 0$. Then the quasiball $B(g, \varepsilon)$ coincides with $\{f \in E : \max_{k \leq r} \|f^k(\xi) - g^k(\xi)\| < \varepsilon \text{ for all } \xi \in U\}$.*

Proof. By definition of $\pi_j^E, g_j = \pi_j^E(g) = g|_{K_j}$ and the ball $B(g|_{K_j}, \varepsilon)$ in Banach-space E_j consist of all $f \in E_j$ such that $\max_{k \leq r} \|f^k(\xi) - g_j^k(\xi)\| < \varepsilon$ for all $\xi \in K_j$. Since $\pi_j^E(B(g, \varepsilon)) \subseteq B(g_j, \varepsilon)$ for all $j \in \mathbb{N}$ and $U = \bigcup_{j \in \mathbb{N}} K_j$, we get the required result.

Remarks 3.1.12

(a) In example 3.1.3 if we put $U=F=\mathbb{C}$ and $r=1$, then by proposition 3.1.11 and Liouville's theorem $B(0, \varepsilon)$ consists of all constant functions with absolute value less than ε .

(b) Let $\mathbb{J} = (\mathbb{N}, \geq)$ and define $\mathcal{J}\mathcal{L}\mathcal{C}_c$ -space $(E, E_{(-)}, \pi^E)$ by putting $E = E_j = \mathbb{C}^n$ equipped with the norm $\|\lambda\| = j|\lambda|$ and $\pi_j(\lambda) = \lambda$. Then every quasiball $B(0, \delta)$

consists of all $\lambda \in \mathbb{C}^n$ such that $\|\lambda\| = j|\lambda| < \delta$ for all $j \in \mathbb{N}$. Hence $\lambda = 0$ i.e. $B(0, \delta) = \{0\}$.

(c) In (b) if $\|\lambda\|_j = |\lambda|$ with the same underlying $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -space, quasiballs are the usual balls in $\mathbb{C}^n$.

These examples show that by choosing a suitable $\mathbf{J}\mathcal{L}\mathcal{C}_c$ -structure one can get nice quasiballs to serve as domains for $\mathcal{H}_c$ -maps. In the following example by changing the $\mathbf{J}\mathcal{L}\mathcal{C}_c$ -structure in (a) we will have nice quasiballs, in fact tangential ones.

Example 3.1.13 Let $E = \mathcal{H}_c(\mathbb{C}, \mathbb{C})$ and $E_n = \mathcal{H}_c(Q_n, \mathbb{C})$ ($n \in \mathbb{N}$), where $Q_n = n\overline{\mathbb{E}}$. By lemma 3.1.5, $\mathcal{H}_c(Q_n, \mathbb{C})$ is a Banach-space with norm:

$$\|f\|_{n*} = \max\{\sup_{\xi \in Q_n} |f(\xi)|, \sup_{\xi \in Q_n} \|\hat{D}f(\xi)\|\}.$$

Consider $E_n = \mathcal{H}_c(Q_n, \mathbb{C})$ with norm $\|f\|_n = \|f\|_{n*}/c^n$. Again $\mathcal{H}_c(Q_n, \mathbb{C})$ is a Banach-space with this norm. Under similar definition to $\pi_n^E : E \rightarrow E_n$, by proposition 3.1.6 the family $\{\pi_i^E\}$ ($i \in \mathbb{N}$) is initial with respect to the underlying functor $\mathcal{L}\mathcal{C}_c \longrightarrow \mathbf{Lin}$. Hence $(E, E_{(-)}, \pi^E)$ is a $\mathbf{J}\mathcal{L}\mathcal{C}_c$ -space.

Remark 3.1.14 It is well known that E_n , in the above example, can be considered as a $\mathcal{C}_c$ -subspace of $\mathcal{C}_c(Q_n, \mathbb{C})$ with the usual supnorm. As we saw in remark 3.1.12, from different metrics different projective limits will arise, so we prefer to use the given norm in example 3.1.13.

Example 3.1.15 Consider the following variation of example 3.1.13. Let $\xi \in Q_1$. Take E to be the $\mathbf{c}\mathcal{L}\mathcal{C}_c$ -space $\mathcal{H}_c(\mathbb{C}, \mathbb{C})_\xi$ of $\mathcal{H}_c(\mathbb{C}, \mathbb{C})$ formed by all functions f such that $f(\xi) = 0$ and similarly $E_j = \mathcal{H}_c(Q_j, \mathbb{C})_\xi$ consists of all functions in $\mathcal{H}_c(Q_j, \mathbb{C})$ that vanish at ξ . It is clear that E_j is a closed subspace of $\mathcal{H}_c(Q_j, \mathbb{C})$, so is a Banach-space. By considering the following commutative square, $\{\pi_j^E\}$ is an initial family as an inner factor of the initial family $\{\text{ins}_j^\xi \circ \pi_j^E\}$:

$$\begin{array}{ccc}
 \mathcal{H}_c(\mathbb{C}, \mathbb{C})_\xi & \xrightarrow{\pi_j^E} & \mathcal{H}_c(Q_j, \mathbb{C})_\xi \\
 \downarrow \text{(initial) ins}_j^\xi & \searrow \text{ins}_j^\xi \circ \pi_j^E & \downarrow \text{ins}_j^\xi \text{ (initial)} \\
 \mathcal{H}_c(\mathbb{C}, \mathbb{C}) & \xrightarrow[\text{(initial)}]{\pi_j} & \mathcal{H}_c(Q_j, \mathbb{C})
 \end{array}$$

Hence $(E, E_{(-)}, \pi^E)$ becomes a $\mathbf{JLC}_c$ -space.

Proposition 3.1.16 *Let $E = \mathcal{H}_c(\mathbb{C}, \mathbb{C})$ be the $\mathbf{JLC}_c$ -space described in example 3.1.13. Then every quasiball in E is tangential. The same holds for the space $E = \mathcal{H}_c(\mathbb{C}, \mathbb{C})_0$ (see 3.1.15).*

Proof. Fix quasiball $B(a, \varepsilon)$ in E . We carry out our task by the following steps:

Step 1. Every h with $h(\xi) = \frac{\varepsilon}{k!} \xi^k + a(\xi)$, for some $k \geq 0$, belongs to $B(a, \varepsilon)$.

Proof 1. Consider the following inequalities:

$$\sup_{\xi \in Q_n} |h(\xi) - a(\xi)| = \frac{\varepsilon n^k}{k!} < \varepsilon^n,$$

$$\sup_{\xi \in Q_n} \|\dot{D}h(\xi) - \dot{D}a(\xi)\| = \sup_{\xi \in Q_n} \sup_{|\lambda|=1} \left| \frac{\varepsilon}{k!} \lambda k \xi^{k-1} \right| = \frac{\varepsilon n^{k-1}}{(k-1)!} < \varepsilon^n.$$

Therefore $\|h - a\|_n = \max\{\sup_{\xi \in Q_n} |h(\xi) - a(\xi)|, \sup_{\xi \in Q_n} \|\dot{D}h(\xi) - \dot{D}a(\xi)\|\} / \varepsilon^n < \varepsilon$, for all $n \in \mathbb{N}$. Hence $h \in B(a, \varepsilon)$.

Fix $b \in B(a, \varepsilon)$. Define $p : [0, 1] \rightarrow E$, by $p(\lambda) = (1 - \lambda)b + \lambda h$, where h is one of the functions that is defined in step 1. Then $p(0) = b$ and $p'(0) = h - b$.

Step 2. p takes values in $B(a, \varepsilon)$.

Proof 2. Immediate via step 1.

Step 3. Let Q be the linear span $\{h - b : h(\xi) = a(\xi) \text{ or } a(\xi) + \frac{\varepsilon}{k!}\xi^k, \text{ for some } k \geq 0\}$. Then $\overline{Q} = E$. Hence $B(a, \varepsilon)$ is tangential.

Proof 3. As $\overline{\bigcup_{\lambda \in \mathbb{C}} \lambda(\mathcal{P} + a - b)} = E$, we need only to show that $Q \supseteq \bigcup_{\lambda \in \mathbb{C}} \lambda(\mathcal{P} + a - b)$, where $\mathcal{P} = \mathcal{P}(\mathbb{C}, \mathbb{C})$ is the set of all polynomials in $\mathbb{C}$. Let $g \in \bigcup_{\lambda \in \mathbb{C}} \lambda(\mathcal{P} + a - b)$, then $g(\xi) = \lambda_0(\sum_{k=0}^m a_k \xi^k + a(\xi) - b(\xi))$, for some $\lambda_0, a_k \in \mathbb{C}$ ($0 \leq k \leq m$). But g can be written as follows:

$$g = \sum_{k=0}^m \frac{k!}{\varepsilon} \lambda_0 a_k (h_k - b) + (\lambda_0 - \sum_{k=0}^m \frac{k!}{\varepsilon} \lambda_0 a_k)(a - b)$$

which certainly belongs to Q , where $h_k(\xi) = \frac{\varepsilon}{k!}\xi^k + a(\xi)$.

This completes the proof. A similar argument works for example 3.1.15.

3.2 The Inverse Function Theorem and Implicit Function Theorem

In this section we prove the inverse function theorem along the lines of the real version in [18]. First we establish preliminaries needed for proving this theorem. Let us start with the parametrized fixed point theorem which is recently formulated and proved in [18].

Theorem 3.2.1 (Parametrized Fixed Point Theorem)

Suppose given the quasiballs $U = B(a, \alpha) \subset E$, $V = B(b, \beta) \subset F$, a scalar $0 \leq \rho < 1$ and a $\mathcal{JC}_c$ -map $c : U \times V \rightarrow F$ such that

$$\|c_j(w, y) - c_j(w, z)\| \leq \rho \|y - z\| \text{ and } \|c_j(w, b_j) - b_j\| \leq \beta(1 - \rho)$$

hold for all $w \in U_j$, $y, z \in V_j$, ($j \in \mathbb{J}$). Then there exists a $\mathcal{K}_c$ -map $f : U \rightarrow V$ such that $f(w) = c(w, f(w))$ and $f(w)$ is the only fixed point of $c(w, \cdot)$ ($w \in U$).

Proof. see [18].

Proposition 3.2.2 *Let U be tangential in E . Let $Q : U \rightarrow [E, F]$ and $R : U \rightarrow [F, E]$ be $\mathcal{C}_c$ -maps such that for every $x \in U$ the maps $R(x) : F \rightarrow E$ and $Q(x) : E \rightarrow F$ are inverses to each other and $\mathbb{C}$ -linear. Then Q is an $\mathcal{H}_c^r$ -map if and only if R is an $\mathcal{H}_c^r$ -map ($r \in [1, \infty]$). In fact, when Q is an $\mathcal{H}_c^r$ -map, the derivative of R is given by*

$$(1) \quad \dot{D}R(x) \cdot h = -R(x) \circ (\dot{D}Q(x) \cdot h) \circ R(x).$$

Proof. By proposition 5a in [18] we obtain

$$D\dot{R}(x) \cdot h = -R(x) \circ (\dot{D}Q(x) \cdot h) \circ R(x).$$

Since $\dot{D}Q(x)$ and $R(x)$ are $\mathbb{C}$ -linear, $D\dot{R}(x)$ underlies the $\mathbb{C}$ -linear map $\dot{D}R(x)$. Hence R is $\mathcal{H}_c^r$ -map and we arrive at (1).

Remark 3.2.3 For $f \in \mathcal{H}_c^r(U, F)$ put $\dot{d}f = (\dot{D}f)^\dagger$. Then $\dot{d}f : U \times E \rightarrow F$ defined by $\dot{d}f(x, h) = \dot{D}f(x) \cdot h$ is $\mathbb{C}$ -linear in the second variable and $\dot{D}f, \dot{d}f$ determine each other.

Proposition 3.2.4 *Suppose E and F are $\mathcal{JLC}_c$ -spaces and $U \subseteq E$ a tangential domain in E which is also a $\mathcal{JC}_c$ -subspace of E . Let $f : U \rightarrow F$ be a $\mathcal{JC}_c$ -map such that $f_j \in \mathcal{H}_c(U_j, F_j)$ for all $j \in \mathbf{J}$. Then $f \in \mathcal{H}_c(U, F)$ and the complex differential $\dot{d}f : U \times E \rightarrow F$ is a $\mathcal{JC}_c$ -map with components*

$$\dot{d}f_i : U_j \times E_j \rightarrow F_j. \quad .$$

Proof. One repeats the arguments in 5b in [18] by changing the real differential df with complex differential $\dot{d}f$ and df_j with $\dot{d}f_j$.

Definition 3.2.5

(a) Let E, F be Banach spaces and V be a tangential domain in E . If $f \in \mathcal{H}_c(V, F)$, its derivative $\dot{D}f$ is a $\mathcal{C}_c$ -map on V into $[E, F]$. $\dot{D}f$ is *norm-continuous* if it is continuous as a function from V into the usual Banach space $\mathbf{Ban}(E, F)$.

Suppose the $\mathbf{c}\mathcal{LC}_c$ -spaces E and F have been structured to become $\mathbf{J}\mathcal{LC}_c$ -spaces.

(b) A *quasineighborhood* of $a \in E$ means a $\mathbf{J}\mathcal{C}_c$ -subspace $W \subseteq E$ such that for every $j \in \mathbf{J}$ the approximand space W_j is a neighborhood of a_j in E_j in the usual sense.

(c) A $\mathbf{J}\mathcal{LC}_c$ -structure on E is *differentially compatible* if every quasiball is a tangential domain (see example 3.1.13 and proposition 3.1.16).

(d) For a tangential domain U in E , a $\mathcal{C}_c$ -map $f : U \rightarrow F$ is *$\mathbb{C}$ -calm near a* if there exists a quasiball $B(a, \delta) \subseteq U$ such that f is a $\mathbf{J}\mathcal{C}_c$ -map $B(a, \delta) \rightarrow F$ with components $f_j : B(a_j, \delta) \rightarrow F_j$ such that the following hold:

- (1) $\dot{D}f_j$ is norm-continuous,
- (2) there exists $\mu > 0$ such that $\|\dot{D}f_j(a_j)\| < \mu$ ($j \in \mathbf{J}$),
- (3) for every $\varepsilon > 0$ there exists $\eta > 0$ such that for all $x \in B(a_j, \eta)$ and all $j \in \mathbf{J}$ we have:

$$\|\dot{D}f_j(x) - \dot{D}f_j(a_j)\| < \varepsilon.$$

(e) $\dot{D}f(a)$ in (d) is called *equiinvertible* if $A \stackrel{\text{def}}{=} \dot{D}f(a) : E \rightarrow F$ is a $\mathbf{J}\mathcal{LC}_c$ -isomorphism such that $\|A_j^{-1}\| \leq \xi < \infty$ holds for all $j \in \mathbf{J}$. Note that the constants δ, μ and ξ appearing above are required to work for all $j \in \mathbf{J}$ at the same time and the norm used is that of $\mathbf{Ban}(E_j, F_j)$.

Proposition 3.2.6 *Suppose $f : B(a, \delta) \rightarrow F$ to be a $\mathbf{J}\mathcal{C}_c$ -map whose components f_j are $\mathcal{H}_c$ -maps with $\sup_j \|\dot{D}f_j(a_j)\| < \infty$. Then the following implications hold:*

(a) *all f_j are $\mathcal{H}_c^2$ -maps and $v \stackrel{\text{def}}{=} \sup_j \sup_x \|\dot{D}^2 f_j(x)\| < \infty$.*

$\Downarrow$

(b) *$\dot{D}f_j$ is Lipschitz into $\mathbf{Ban}(E_j, F_j)$ with one Lipschitz constant for all j .*

$\Downarrow$

(c) *f is $\mathbb{C}$ -calm near a .*

Proof .

(a) $\Rightarrow$ (b). We have:

$$\|\dot{D}f_j(x) - \dot{D}f_j(w)\| = \left\| \int_0^1 \dot{D}^2 f_j(w + \theta(x-w)) \cdot (x-w) d\theta \right\| \leq v \|x-w\| \text{ for all } j \in J.$$

(b) $\Rightarrow$ (c). Trivial.

Example 3.2.7 Let $E=F=\mathcal{H}_c(\mathbb{C}, \mathbb{C})$, define $f : E \rightarrow E$ by $f(x) = x^2$ and choose $a \in E$ to be the constant function 1. Then $(\dot{D}f(x) \cdot h)(\xi) = 2x(\xi)h(\xi)$ and $(\dot{D}^2 f(s) \cdot h \cdot k)(\xi) = 2h(\xi)k(\xi)$. Consider E with $\mathcal{JLC}_c$ -structure in example 3.1.13, then $\sup_j \|\dot{D}f_j(a_j)\| = \sup_j \|\dot{D}f(a)\|_j = \sup_j \|2a\|_j = \sup_j (2/c^j) < 2$ and clearly f_j 's are $\mathcal{H}_c^2$ maps and $\sup_j \sup_x \|\dot{D}^2 f_j(x)\| < \infty$. Hence by proposition 3.2.6 f is $\mathbb{C}$ -calm near a . Moreover, $\dot{D}f(a)$ is equiinvertible.

Theorem 3.2.8 (Inverse Function Theorem)

Suppose $f \in \mathcal{H}_c(U, F)$ is such that there are $\mathcal{JLC}_c$ -structures for E and F , with that of E differentially compatible, such that f is $\mathbb{C}$ -calm near $a \in U$ and $\dot{D}f(a)$ is equiinvertible. Then there exists a tangential quasineighborhood V of a such that $W = f(V)$ is a tangential quasineighborhood of $f(a)$ and $f : V \rightarrow W$ has an $\mathcal{H}_c$ -inverse $g : W \rightarrow V$ with every $\dot{D}g_j$ norm-continuous. Moreover, if f is an $\mathcal{H}_c^r$ -map, then so is g ($r \in [1, \infty]$).

Proof. The proof is essentially the same as the corresponding proof in [18], but we repeat the details for convenience of the reader.

Choose $0 < \rho < 1$ once and for all. By $\mathbb{C}$ -calmness of f there exists $\delta > 0$ such that $f : B(a, \delta) \rightarrow F$ is a $\mathcal{JC}_c$ -map and there exists $\mu > 0$ such that $\|\dot{D}f_j(a_j)\| < \mu$ ($j \in J$). By equiinvertibility of $\dot{D}f(a)$ there exists k such that $\|A_j^{-1}\| \leq k$ where $A_j \stackrel{\text{def}}{=} \dot{D}f_j(a_j)$. Choose $\alpha_1 > 0$ such that $\|\dot{D}f_j(z) - \dot{D}f_j(a_j)\| < \rho/k$ for all $z \in B(a_j, \alpha_1)$. This choice is possible by the $\mathbb{C}$ -calmness of f near a .

Claim 1. For all $j \in J$ and all $w, x \in B(a_j, \alpha_1)$ we have $\|x - w - A_j^{-1}(f_j(x) - f_j(w))\| \leq \rho \|x - w\|$.

Proof 1. Put $z(\theta) = w + \theta(x - w)$. Then the left side can be expressed $\|x - w - A_j^{-1} \int_0^1 \dot{D}f_j(z(\theta))d\theta \cdot (x - w)\| = \|x - w - A_j^{-1} \int_0^1 [\dot{D}f_j(z(\theta)) - \dot{D}f_j(a_j)]d\theta \cdot (x - w) - (x - w)\| \leq \|A_j^{-1}\| \cdot \|\dot{D}f_j(z(\theta)) - \dot{D}f_j(a_j)\| \cdot \|x - w\| \leq k \cdot (\rho/k) \cdot \|x - w\|$. So claim 1 holds.

Choose $\beta > 0$ so that $0 < k\beta < \alpha_1(1 - \rho)$. Put $b = f(a)$ and $c(y, x) = x + A^{-1} \cdot (y - f(x))$.

Claim 2. There is a unique $\mathcal{JC}_c$ -map $g : B(b, \beta) \rightarrow B(a, \alpha_1)$ such that $g(y)$ is the unique fixed point of $c(y, -) : B(a, \alpha_1) \rightarrow E$. Moreover, $f(g(y)) = y$.

Proof 2. It is clear from its definition that $c : B(b, \beta) \times B(a, \alpha_1) \rightarrow E$ is a $\mathcal{JC}_c$ -map. For all $j \in J$, all $y \in B(b_j, \beta)$ and all $w, x \in B(a_j, \alpha_1)$ we have $\|c_j(y, x) - c_j(y, w)\| = \|x - w - A_j^{-1}(f_j(x) - f_j(w))\| \leq \rho\|x - w\|$, by claim 1. Moreover, $\|c_j(y, a_j) - a_j\| = \|A_j^{-1}(y - f(a_j))\| = \|A_j^{-1}(y - b_j)\| \leq k\beta < \alpha_1(1 - \rho)$. So the fixed point theorem 3.2.1 applies to c and provides a $\mathcal{JC}_c$ -map $g : B(b, \beta) \rightarrow B(a, \alpha_1)$, unique among maps such that $c(y, x) = x$ implies $x = g(y)$. But $g(y) = c(y, g(y)) = g(y) + A^{-1}(y - f(g(y)))$ so $y = f(g(y))$. Thus claim 2 holds.

Now choose $\alpha > 0$ so that $\alpha \leq \alpha_1$ and $(\mu + \rho/k)\alpha < \beta$. Put $V = B(a, \alpha)$ and $W_j = f_j(V_j)$ ($j \in J$). We have $W_j \subseteq B(b_j, \beta)$ because $\|f_j(x) - b_j\| = \|\int_0^1 \dot{D}f_j(a_j + \theta(x - a_j)) \cdot (x - a_j)d\theta\| \leq \|[\dot{D}f_j(z) - \dot{D}f_j(a_j) + \dot{D}f_j(a_j)]\| \cdot \|x - a_j\| \leq (\mu + \rho/k)\alpha < \beta$. Let W denote the $\mathcal{JC}_c$ -subspace of $B(b, \beta)$ determined by the spaces W_j ($j \in J$).

Claim 3. The restricted map $f : V \rightarrow W$ is a $\mathcal{JC}_c$ -isomorphism with inverse $g : W \rightarrow V$. Moreover, W_j is open in E_j for each j .

Proof 3. Since $f_j(V_j) = W_j$ it is clear that $f(V) \subseteq W$ and $c(f(x), x) = x + A^{-1} \cdot (f(x) - f(x)) = x$ for all $x \in V$. It follows by claim 2 that $g(f(x)) = x$ and $f(g(y)) = y$ for all $y \in W$. Note that $W_j = g_j^{-1}(V_j)$, so it must be open in E_j by continuity of g_j . So claim 3 holds.

Define $\tilde{\Phi} : V \times V \times E \rightarrow E$ by putting $\tilde{\Phi}(w, x, h) = \int_0^1 \dot{D}f(w + \theta(x - w))d\theta \cdot h$. Then

the transposed map $\tilde{\Phi}_j^\dagger : V \times V \rightarrow [E, F]$ is the difference factorizer of $f : V \rightarrow F$ and $\tilde{\Phi}$ is $\mathbb{C}$ -linear in the third variable (see theorem 2.1.4).

Claim 4. $\tilde{\Phi} : V \times V \times E \rightarrow F$ is a $\mathcal{JC}_c$ -map with components $\tilde{\Phi}_j$ such that $\tilde{\Phi}_j^\dagger : V_j \times V_j \rightarrow [E_j, F_j]$ lifts to a $\mathcal{C}_c$ -map into $\mathbf{Ban}(E_j, F_j)$ whose values $\tilde{\Phi}_j^\dagger(w, x)$ are $\mathcal{JLC}_c$ -isomorphisms $E_j \rightarrow F_j$. Hence for each $(w, x) \in V \times V$ the map $\tilde{\Phi}(w, x, -) : E \rightarrow F$ is a $\mathcal{JLC}_c$ -isomorphism.

Proof 4. That Φ is again a $\mathcal{JC}_c$ -map follows readily from the naturality of integral: $F_{ji} \int_0^1 F_j = \int_0^1 F_{ji}$ and so on. Take $(w, x) \in V_j \times V_j$ and put $z(\theta) = w + \theta(x - w)$. By $\mathbb{C}$ -calmness we have $\dot{D}f_j : V_j \rightarrow \mathbf{Ban}(E_j, F_j)$ continuous, so the same holds for $\tilde{\Phi}_j^\dagger : V_j \times V_j \rightarrow \mathbf{Ban}(E_j, F_j)$ (see theorem 2.1.4). Moreover, by the choice of α , $\|\tilde{\Phi}_j^\dagger(w, x) - A_j\| = \|\int_0^1 [\dot{D}f_j(z(\theta)) - \dot{D}f_j(a_j)] d\theta\| \leq \rho/k \leq \rho\|A^{-1}\|^{-1} \leq \|A_j^{-1}\|^{-1}$. It follows, by a well known criterion of Banach space theory, that $\tilde{\Phi}(w, x, -)$ is close enough to the invertible operator A_j to be invertible itself. Since the maps $\tilde{\Phi}_j(w, x, -)$ are natural in j , so are their inverses. Hence $\tilde{\Phi}^\dagger(w, x)$ is a $\mathcal{JLC}_c$ -isomorphism as claimed.

Claim 5. W is a tangential domain in F .

Proof 5. In view of claim 4, the derivative $\dot{D}f(x) = \tilde{\Phi}(x, x, -)$ is invertible for every $x \in V$. Therefore W , the image $f(V)$, is tangential by proposition 4i in [17].

Claim 6. $g : W \rightarrow V$ is an $\mathcal{H}_c$ -map, with $\dot{D}g(y) = \dot{D}f(g(y))^{-1}$.

Proof 6. By proposition 3.2.4 it is enough to show that every component $g_j : W_j \rightarrow V_j$ is an $\mathcal{H}_c$ -map with the derivative indicated. Since W_j is open in F_j , every $y \in W_j$ has an open ball neighborhood $B_y = B(y, \delta_y) \subseteq W_j$. We will use the localization theorem (see 2.1.6) which requires that we show $g_j : B_y \rightarrow E_j$ is a $\mathcal{H}_c$ -map with derivative $\dot{D}g_j(y) = \dot{D}f_j(g_j(y))^{-1}$ for each y . Since the balls B_y are convex we can apply the characterization of $\mathcal{H}_c$ -maps in terms of the difference factorizer (theorem 2.1.4). Accordingly, our task is to exhibit a $\mathcal{C}_c$ -map $\tilde{\Psi} : B_y \times B_y \rightarrow [F_j, E_j]$ such that $g(y_2) - g(y_1) = \tilde{\Psi}(y_1, y_2) \cdot (y_2 - y_1)$ and $\tilde{\Psi}(y_1, y_2)$ is $\mathbb{C}$ -linear for all $y_1, y_2 \in B_y$. Using claim 4, we construct $\tilde{\Psi}$ as the following com-

posite.

$$\begin{array}{ccccc}
 V_j \times V_j & \xrightarrow{\dagger(\hat{\Phi})} & \text{Inv}(E_j, F_j) & \xrightarrow{\text{inv}} & \text{Ban}(F_j, E_j) \\
 \uparrow g \times g & & & & \downarrow \text{ins} \\
 B_y \times B_y & \xrightarrow{\hat{\Psi}} & & & [F_j, E_j]
 \end{array}$$

In this construction, $\text{Inv}(E_j, F_j)$ denotes the open subset of $\text{Ban}(E_j, F_j)$ formed by all invertibles, $\text{inv}(u) = u^{-1}$, and $\text{ins}(v) = v$. These maps are all well known to be $\mathcal{C}_c$ -maps. Put $x_1 = g(y_1), x_2 = g(y_2)$. Since $\hat{\Phi}_j^\dagger$ is a difference factorizer for f_j , we have $\hat{\Phi}_j^\dagger(x_1, x_2) \cdot (x_2 - x_1) = f_j(x_2) - f_j(x_1) = y_2 - y_1$. Hence $g(y_2) - g(y_1) = \hat{\Phi}_j^\dagger(x_1, x_2)^{-1} \cdot (y_2 - y_1) = (\text{inv} \circ \hat{\Phi}_j^\dagger \circ (g \times g)(y_1, y_2)) \cdot (y_2 - y_1) = \hat{\Psi}(y_1, y_2) \cdot (y_2 - y_1)$, as required. Moreover, since $\hat{\Psi}$ factored through $\text{Ban}(F_j, E_j)$, the derivative $\hat{D}g_j$ is norm-continuous.

Claim 7. If f is an $\mathcal{H}_c^r$ -map, then so is g .

Proof 7. We proceed by induction on $r = 1, 2, \dots$. The case $r = 1$ is dealt with by claim 6. Choose $q \geq 1$ and suppose claim 7 is valid for $r \leq q$. Define $Q : V \rightarrow [E, F]$ and $R : V \rightarrow [F, E]$ by $Q(x) = \hat{D}f(x)$, $R(x) = \hat{D}f(x)^{-1}$. If f is an $\mathcal{H}_c^{q+1}$ -map, then Q and g are $\mathcal{H}_c^q$ -maps. So by proposition 3.2.2, R is an $\mathcal{H}_c^q$ -map. Hence, in view of claim 6, $\hat{D}g = R \circ g$ is an $\mathcal{H}_c^q$ -map, which means g is an $\mathcal{H}_c^{q+1}$ -map, as required. This ends the proof of the inverse function theorem.

Example 3.2.9 Let $E = F = \mathcal{H}_c(\mathbb{D}, \mathbb{C})$, define $f : E \rightarrow F$ by $f(x) = e^x$ and choose $a \in E$ to be the constant function 1. Then $(\hat{D}f(x) \cdot h)(\xi) = h(\xi)e^{x(\xi)}$ and $(\hat{D}^2 f(x) \cdot h \cdot k)(\xi) = h(\xi)k(\xi)e^{x(\xi)}$. Equip E with JLC_c -structure described in example 3.1.3, with $K_j = \frac{1}{j+1}\overline{\mathbb{D}}$. Then $\sup_j \|\hat{D}f_j(a_j)\| = \sup_j e = e$. Consider $f : B(a, 1) \rightarrow F$, then the f_j 's are $\mathcal{H}_c^2$ -maps and $\sup_j \sup_x \|\hat{D}^2 f_j(x)\| = e^2 < \infty$. Hence by 3.2.7

f is $\mathbb{C}$ -calm near a . Moreover $\dot{D}f(a)$ is equinvertible and similar to the proof of 3.1.16 one can show that the structure of E is differentially compatible. Hence by the inverse function theorem 3.2.8, there exists a tangential quasineighborhood V of a such that $W = f(V)$ is a tangential quasineighborhood of $f(a) = e^a$ and $f : V \rightarrow W$ has holomorphic inverse $g : W \rightarrow V$. In this illustrative example, g is known explicitly, namely $g(y)(\xi) = \ln y(\xi)$.

Remark 3.2.10 Equip E in the above example with the compact open topology. Then the subset $N(\mathbb{D})$ of non-vanishing holomorphic functions has empty interior. To see that let $h \in N(\mathbb{D})$ and $N(Q, V)$ be an arbitrary basic neighborhood of zero, we claim that $h + N(Q, V) \not\subseteq N(\mathbb{D})$. For this purpose, let $0 < r, \varepsilon < 1$ be such that $Q \subset B(0, r)$ and $B(0, \varepsilon) \subseteq V$. Put $\gamma = -h(\frac{r+1}{2}) \neq 0$ and $k = 3 + |\gamma|$. Then there is $n \in \mathbb{N}$ such that $\ln(\varepsilon/k) > \frac{r-1}{2}n$.

Define $g : \mathbb{D} \rightarrow \mathbb{C}$, by $g(\xi) = (\xi - \frac{r+1}{2} + \gamma)e^{n(\xi - \frac{r+1}{2})}$. Then $g(\frac{r+1}{2}) = \gamma$. Moreover, for every $\xi \in B(0, r)$ we have

$$\begin{aligned} |g(\xi)| &\leq (|\xi| + \frac{r+1}{2} + |\gamma|)e^{n(\operatorname{Re}\xi - \frac{r+1}{2})} \\ &\leq (k-1)e^{n(r - \frac{r+1}{2})} \leq (k-1)e^{\ln(\varepsilon/k)} \\ &= (k-1)(\varepsilon/k) < \varepsilon. \end{aligned}$$

This yields $g(Q) \subseteq V$, hence $g \in N(Q, V)$. On the other hand $(h+g)(\frac{r+1}{2}) = 0$ so, $h + N(Q, V) \not\subseteq N(\mathbb{D})$. Thus $N(\mathbb{D})$ has empty interior in E .

Remark 3.2.11 There is no hope for an inverse function theorem in its classical formulation. The map f in the above example 3.2.9 carries E to an image $f(E)$, the set $N(\mathbb{D})$ of non-vanishing holomorphic functions on $\mathbb{D}$ which has empty interior in E , by the above remark. Therefore for every open neighborhood V of a , $f(V)$ can not be open. Accordingly, the inverse function theorem for E , in its classical formulation, is not applicable.

Theorem 3.2.12 (Implicit Function Theorem)

Let $f \in \mathcal{H}_c(U \times V, G)$ and suppose that E, F and G be given $\mathcal{JLC}_c$ -structures such that those of E and F are differentially compatible with f $\mathbb{C}$ -calm near a point $(a, b) \in U \times V$ and $\hat{D}_2 f(a, b) : F \rightarrow G$ equinvertible. Then there exists a quasiball T centered on a , and a unique $\mathcal{H}_c$ -map $g : T \rightarrow V$ such that $f(x, g(x)) = f(a, b)$ holds for all $x \in T$. Moreover, if f is $\mathcal{H}_c^r$ -map, then so is g .

Proof. It follows from the above Inverse Function Theorem along the proof of the Banach version of the Implicit Function Theorem (see [12]).

Remark 3.2.13 Not every result in real calculus necessarily has a holomorphic analogue. For example consider proposition 6e in [18]. Since bounded entire functions are constant, in that proposition, we must replace $\mathcal{C}_c^1(\Omega, \mathbb{R})_0$ by a tangential subspace like $\mathcal{H}_c(\mathbb{D}, \mathbb{D})_0$. In this case we need a $\mathcal{JLC}_c$ -structure on $\mathcal{H}_c(\mathbb{D}, \mathbb{C})$ such that $\mathcal{H}_c(\mathbb{D}, \mathbb{D})_0$ becomes quasiopen. So for a fixed $f \in \mathcal{H}_c(\mathbb{D}, \mathbb{D})_0$, we need a quasiball $B(f, \delta)$ that lies in $\mathcal{H}_c(\mathbb{D}, \mathbb{D})_0$. For this purpose, one should define $E_f = \mathcal{H}_c(\frac{1}{j+1}\mathbb{D}, \mathbb{C})$ with norm $\|g\| = \sup_{|\epsilon| \leq \frac{1}{j+1}} \frac{|g(\epsilon)|}{1-|f(\epsilon)|}$. But this structure is not differentially compatible. Thus $\mathcal{H}_c(\mathbb{D}, \mathbb{D})_0$ can not be quasiopen.

3.3 Newton's Method

Recently a new version of Newton's Method in nonnormable spaces has been discovered in [14]. The new version which is a generalization of the Banach version of this method, is based on the auxiliary categories (see section 3.1 for definitions).

A nonnormable version of the Newton's Method which is formulated for $\mathcal{C}_c^1$ -maps, can be adapted for $\mathcal{H}_c$ -maps as well. We illustrate this adaption by giving an application to perturbation of functional equations on nonopen domains with empty interior.

Theorem 3.3.1 (Newton's Method)

Suppose E and F are $\mathbf{JLC}_c$ -spaces, U a tangential quasiopen subspace of E , $y \in U$, $f : U \rightarrow F$ an $\mathcal{H}_c$ -map underlying a $\mathbf{JC}_c$ -map having components $f_j \in \mathcal{H}_c(U_j, F_j)$ such that $\dot{D}f_j : U_j \rightarrow \mathbf{Ban}(E_j, F_j)$ is norm-continuous with $\dot{D}f_j(y_j) : E_j \rightarrow F_j$ invertible. Let μ , γ and κ be positive constants giving for every j in $\mathbf{J}$ the following estimates:

$$\|\dot{D}f_j(w) - \dot{D}f_j(x)\| \leq \kappa\|w - x\| \quad (w, x \in U_j).$$

$$\|\dot{D}f_j(y_j)^{-1}\| \leq \mu, \quad \|\dot{D}f_j(y_j)^{-1} \cdot f_j(y_j)\| \leq \gamma, \quad 2\mu\gamma\kappa \leq 1.$$

$$B[y_j, \rho] \subset U_j, \text{ where } \rho = (\mu\kappa)^{-1}[1 - (1 - 2\mu\gamma\kappa)^{1/2}].$$

Then the Newton sequence for f starting at y , namely

$$z_0 = y, \quad z_{k+1} \stackrel{\text{def}}{=} z_k - \dot{D}f(z_k)^{-1} \cdot f(z_k),$$

exists and converges to a limit y_0 in $B[y, \rho]$ such that $f(y_0) = 0$.

Proof. By repeating the given argument in the corresponding theorem 2.2 of [14] and proposition 3.2.4, the theorem follows.

Now we present an application of the Newton's Method to the perturbation of functional equations on nonopen domains with empty interior, infact where the classical versions of this method is not applicable.

Example 3.3.2 Let $E = F = \mathcal{H}_c(\Lambda, \mathbb{C})$ and $U = \mathcal{H}_c(\Lambda, \Omega)$, where Λ and Ω are simply connected domains in $\mathbb{C}$. By theorem 2.6.3, U is a tangential domain with empty interior. Equip E, F with the differentially compatible $\mathbf{JLC}_c$ -structure given in example 3.2.9. Define the map $f : U \rightarrow E$ by:

$$f(x) = x^2 + \ln(1 + \alpha_0 x)^{-2} - b,$$

where $b(\lambda) = \lambda_0 + \lambda$ and α_0, λ_0 are constants in $\mathbb{C}$. Note that $f(x)$ is a perturbation of the simpler equation $x^2 - b = 0$, which has the solution y given by $y(\lambda) = [b(\lambda)]^{\frac{1}{2}}$.

We can choose domains Λ, Ω and constants α_0, λ_0 such that the conditions of theorem 3.3.1 are satisfied. Then we have the Newton sequence for f starting at y converges to a limit in a certain quasiball that lies in U .

For example, put $\alpha_0 = 0.01$, $\lambda_0 = 69/81$, $\Lambda = \{\lambda : 0 < |\lambda| < 1/81, \operatorname{Re} \lambda > 0 \text{ and } \operatorname{Im} \lambda > 0\}$ and $\Omega = \{\omega : 8/9 < |\omega| < 1 \text{ and } \operatorname{Re} \omega > 0\}$. To see how these assumptions work for Newton's Method, consider the following arguments:

Let $\phi(\lambda) = \lambda^2 + \ln(1 + \alpha_0 \lambda)^{-2}$, then f can be represented by $f(x) = \phi \circ x - b$. As $8/9 < \sqrt{68}/9 < |y(\lambda)| < \sqrt{70}/9 < 1$ and $\operatorname{Re} y(\lambda) > 0$, for all $\lambda \in \Lambda$, we get $y \in U$. On the other hand one can show that $B[y, 0.018]$ lies in U .

By proposition 2.6.2, $\hat{D}f(x) \cdot h = (\phi' \circ x) \cdot h$, where $\phi'(\lambda) = 2\lambda - 2(100 + \lambda)^{-1}$.

For $\lambda \neq \alpha$ we have:

$|(\phi'(\lambda) - \phi'(\alpha))(\lambda - \alpha)^{-1}| = 2|1 + [(100 + \lambda)(100 + \alpha)]^{-1}| < 2(9802/9801)$, that gives:

$$|(\hat{D}f(x) - \hat{D}f(y)) \cdot h| \leq \kappa |x - y| |h|, \text{ where } \kappa = 2(9802/9801).$$

On the other hand:

$$|1/\phi'(\lambda)| = |100 + \lambda|(2|\lambda(100 + \lambda) - 1|)^{-1} < 101/171, \text{ so}$$

$$|\hat{D}f(y)^{-1} \cdot k| = |(1/(\phi' \circ y)) \cdot k| < \mu |k|, \text{ where } \mu = 101/171.$$

Since $f(y) = \ln(1 + 0.01y)^{-2}$, we have

$$|f(y)(\lambda)| = 2|\ln(1 + 0.01y(\lambda))| \leq (2)(3/2)(0.01) = 0.03 \text{ (see p.165 of [3])}.$$

Hence we obtain the estimate

$$|\hat{D}f(y)^{-1} \cdot f(y)| \leq \gamma, \text{ where } \gamma = 0.03\mu.$$

It follows that $\rho = (\mu\kappa)^{-1}[1 - (1 - 2\mu\gamma\kappa)^{1/2}] < 0.018$. In view of these estimates, the conditions of theorem 3.3.1, for $f|_{B[y, 0.018]}$, are satisfied. Therefore the Newton sequence for f starting at y converges to a limit y_0 in $B[y, 0.018] \subseteq U$.

Bibliography

- [1] S. Bjön and M. Lindström, A general approach to infinite dimensional holomorphy, *Monatsh. Math.* 101 (1986) 11-26.
- [2] J. Bochnak and J. Siciak, Analytic functions in topological vector spaces, *Studia Math.* 39 (1971) 77-112.
- [3] John B. Conway, *Functions of One Complex Variable*, Second Edition, (Springer-Verlag) (1978).
- [4] J. Dieudonné, *Foundations of Modern Analysis* (Academic Press) (1969).
- [5] S. Dineen, *Complex Analysis in Locally Convex Spaces*, (North-Holland) (1981).
- [6] R.A. Graff, Elements of nonlinear functional analysis, *Mem. Amer. Math. Soc.* 16 (1978) Number 206.
- [7] R.S. Hamilton, The inverse function theorem of Nash and Moser, *Bull Amer. Math. Soc.* (1982) 65-222.
- [8] E. Hille and R. S. Phillips, *Functional Analysis and Semigroups*, American Mathematical Society (1957).
- [9] L.V. Kantorovich and G.P. Akilov, *Functional Analysis* (Pergamon Press) (1982).
- [10] A. Kriegl and L.D. Nel, A convenient setting for holomorphy, *Cahiers Topologie Géom. Différentielle Catégoriques* 26 (1985) 273-309.

- [11] G. Köthe, *Topological Vector Spaces I* (Springer) (1969).
- [12] S. Lang, *Real Analysis* (Addison-Wesley) (1983).
- [13] K.C. Min and L.D. Nel, Infinite dimensional holomorphy via categorical calculus, *Monatsh. Math.* 111 (1991) 55-68.
- [14] K.C. Min and L.D. Nel, Newton's method in nonnormable spaces, Preprint (1994).
- [15] A. Monadi and L.D. Nel, Holomorphy in convergence spaces, *Applied Categorical Structures* 1 (1993) 233-245.
- [16] L.D. Nel, Categorical differential calculus for infinite dimensional spaces, *Cahiers Topologie Géom. Différentielle Catégoriques* 29 (1988) 257-286.
- [17] L.D. Nel, Infinite dimensional calculus allowing nonconvex domains with empty interior, *Monatsh. Math.* 110 (1990) 145-166.
- [18] L.D. Nel, Nonlinear existence theorems in nonnormable analysis, *Category Theory at Work* (Heldermann)(1991) 343-365.
- [19] L.D. Nel, *Introduction to Categorical Methods (Parts One and Two)*, Carleton-Ottawa Mathematical Lecture Note Series 11 (Carleton Univ.) (1991).
- [20] L.D. Nel, *Introduction to Categorical Methods (Part Three)*, Carleton-Ottawa Mathematical Lecture Note Series 12 (Carleton Univ.) (1992).
- [21] L.D. Nel, Differential calculus founded on an isomorphism, *Applied Categorical Structures* 1 (1993) 51-57.
- [22] R. Remmert, *Theory of Complex Functions*, Springer-Verlag, New York (1991).
- [23] Angus E. Taylor, *Historical notes on analyticity as a concept in functional analysis*, *Problems in Analysis* (Princeton University Press) (1970) 325-343.

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