Generalized Delaunay triangulations: Graph-theoretic properties and Algorithms

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Generalized Delaunay triangulations: Graph-theoretic properties and Algorithms

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A thesis presented to obtain the
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Supervised by

Prosenjit Bose, and
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To all women in science.

"...a scientist must also be absolutely like a child. If he sees a thing, he must say that he sees it, whether it was what he thought he was going to see or not. See first, think later, then test. But always see first. Otherwise you will see only what you were expecting."

Douglas Adams - The Hitchhiker’s Guide to the Galaxy
This thesis studies different generalizations of Delaunay triangulations, both from a combinatorial and algorithmic point of view. The Delaunay triangulation of a point set $S$, denoted $DT(S)$, has vertex set $S$. An edge $uv$ is in $DT(S)$ if it satisfies the empty circle property: there exists a circle with $u$ and $v$ on its boundary that does not enclose points of $S$. Due to different optimization criteria, many generalizations of the $DT(S)$ have been proposed. Several properties are known for $DT(S)$, yet, few are known for its generalizations. The main question we explore is: to what extent can properties of $DT(S)$ be extended for generalized Delaunay graphs?

First, we explore the connectivity of the flip graph of higher order Delaunay triangulations of a point set $S$ in the plane. The order-$k$ flip graph might be disconnected for $k \geq 3$, yet, we give upper and lower bounds on the flip distance from one order-$k$ triangulation to another in certain settings.

Later, we show that there exists a length-decreasing sequence of plane spanning trees of $S$ that converges to the minimum spanning tree of $S$ with respect to an arbitrary convex distance function. Each pair of consecutive trees in the sequence is contained in a constrained convex shape Delaunay graph. In addition, we give a linear upper bound and specific bounds when the convex shape is a square.

With focus still on convex distance functions, we study Hamiltonicity in $k$-order convex shape Delaunay graphs. Depending on the convex shape, we provide several upper bounds for the minimum $k$ for which the $k$-order convex shape Delaunay graph is always Hamiltonian. In addition, we provide lower bounds when the convex shape is in a set of certain regular polygons.

Finally, we revisit an affine invariant triangulation, which is a special type of convex shape Delaunay triangulation. We show that many properties of the standard Delaunay triangulations carry over to these triangulations. Also, motivated by this affine invariant triangulation, we study different triangulation methods for producing other affine invariant geometric objects.
Esta tesis estudia diferentes generalizaciones de la triangulación de Delaunay, tanto desde un punto de vista combinatorio como algorítmico. La triangulación de Delaunay de un conjunto de puntos $S$, denotada $DT(S)$, tiene como conjunto de vértices a $S$. Una arista $uv$ está en $DT(S)$ si satisface la propiedad del círculo vacío: existe un círculo con $u$ y $v$ en su frontera que no contiene ningún punto de $S$ en su interior. Debido a distintos criterios de optimización, se han propuesto varias generalizaciones de la $DT(S)$. Hoy en día, se conocen bastantes propiedades de la $DT(S)$, sin embargo, poco se sabe sobre sus generalizaciones. La pregunta principal que exploramos es: ¿Hasta qué punto las propiedades de la $DT(S)$ se pueden extender para generalizaciones de gráficas de Delaunay?

Primero, exploramos la conectividad de la gráfica de flips de las triangulaciones de Delaunay de orden alto de un conjunto de puntos $S$ en el plano. La gráfica de flips de triangulaciones de orden $k \geq 3$ podría ser disconexa, sin embargo, nosotros damos una cota superior e inferior para la distancia en flips de una triangulación de orden $k$ a alguna otra cuando $S$ cumple con ciertas características.

Relacionado con transformaciones entre dos triangulaciones, está el problema de transformar un árbol generador de $S$ a otro. Nosotros probamos que existe una secuencia de árboles generadores sin cruces tal que la suma total de la longitud de las aristas con respecto a una distancia convexa arbitraria es decreciente y converge al árbol generador mínimo con respecto a la distancia correspondiente. Cada par de árboles consecutivos en la secuencia se encuentran en una triangulación de Delaunay con restricciones. Adicionalmente, damos una cota superior lineal para la longitud de la secuencia y cotas específicas cuando el conjunto convexo es un cuadrado.

Aún concentrados en distancias convexas, estudiamos hamiltonicidad en las gráficas de Delaunay de distancia convexa de $k$-orden. Dependiendo en la distancia convexa, exhibimos diversas cotas superiores para el mínimo valor de $k$ que satisface que la gráfica de Delaunay de distancia convexa de orden-$k$ es hamiltoniana. También damos cotas inferiores para $k$ cuando el conjunto convexo pertenece a un conjunto de ciertos polígonos regulares.

Finalmente, re-visitamos una triangulación afín invariante, la cual es un
caso especial de triangulación de Delaunay de distancia convexa. Probamos que muchas propiedades de la triangulación de Delaunay estándar se preservan en estas triangulaciones. Además, motivados por esta triangulación afín invariante, estudiamos diferentes algoritmos que producen otros objetos geométricos afín invariantes.
First of all, I would like to thank Jit and Rodrigo. Both of you have been excellent supervisors and more than I could have wished for. I am so grateful for all of your lessons, your patience, your jokes, and your support. For teaching me that science can be hard but also exciting and fun, for believing in me and for your work. For the Argentinian and the Quebecois accents, and the funny questions about Mexican slang. For all this journey, thank you! Hopefully we can have our solomillo con foie soon.

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Computational geometry is an area of discrete mathematics and computer science dealing with the study of geometric problems. For instance, the real world problem of finding the closest hospital to home can be rephrased as: given a point $p$ and a set of points $S$ in the plane, find the closest point of $S$ to $p$. The field of computational geometry is concerned with the systematic study of algorithms and data structures associated with geometric objects, with an emphasis on exact algorithms that are asymptotically fast. An important feature of this discipline is that often the classical mathematical characterization of geometric objects does not lead to the design of efficient algorithms. As such, a focus on computational aspects is a prerequisite for guaranteeing efficient algorithms.

Combinatorial geometry and discrete geometry are branches of geometry that deal with combinations and arrangements of geometric objects and discrete properties of these objects, respectively. Combinatorial geometry includes aspects of topology, graph theory, number theory, and other disciplines. In particular, geometric graph theory lies in the union of the fields of computational geometry, combinatorial geometry and graph theory. More precisely, geometric graph theory is concerned with graphs defined by geometric means. For instance, once we find which hospital is the closest to home, the next problem is to find the fastest route from home to the hospital. This problem can be translated in the following fashion: given two vertices in a connected graph $G$ in which each edge has weight equal to the distance between its endpoints, find the path with smallest weight in $G$ from $p$ to $q$.

A geometric graph has a vertex set, a set of points in the plane and edges are pairs of points joined by straight line segments. This definition of edge is in contrast with the definition in the combinatorial setting where an edge is defined as a binary relation rather than as a geometric object. The study of
We define a \textit{planar graph} as a geometric graph where no pair of edges properly intersect, i.e., their interiors are disjoint. A \textit{triangulation} of a point set $S$ in the plane is a planar graph with vertex set $S$ where each of its faces define a triangle except maybe its outer face. Triangulations are one of the most studied objects in discrete and computational geometry due to their many applications in different fields like mesh generation, computer aided geometric design, and geographic information science \cite{4,18,44,95}, among others.

A point set in the plane can have many different triangulations, as shown in Figure 1.1. More precisely, there are point sets with $n$ points that have $\Omega(8.65^n)$ distinct triangulations \cite{52}, while the best upper bound is currently $30^n$ \cite{94}. In the literature, much effort has been dedicated to study the efficient computation of triangulations that are optimal with respect to a desired property or criteria. Examples of properties studied include: having the minimum sum of edge-length among all possible triangulations \cite{53,77,83}, or maximizing the minimum angle among all possible triangulations \cite{73,96}, among others. The \textit{Delaunay triangulation} is the most widely used and studied type of triangulation.

The study of the Delaunay triangulation comes from two different paths that are joined. The oldest path comes from the study of \textit{Voronoi diagrams}, also known as \textit{Dirichlet tessellations}, whose origins date back to the 17th century by René Descartes \cite{47}. Roughly speaking, a Voronoi diagram of a set of sites (points) in a certain space, is defined as a partition of the space into regions, where each region “belongs” to a site, and for each site $p$, its region consists of elements that are “influenced” by $p$ in some way. See Figure 1.2. For instance, the sites can represent the hospitals in Barcelona and the influence is the walking distance to each hospital. Voronoi diagrams have not only been important in mathematics but in other areas, such as physics, mineralogy, geography \cite{107,101,106}, among a host of others; for this reason we can find them in the literature with different names. When the space is the plane and the influence is measured using the Euclidean distance, we obtain the name \textit{Voronoi diagrams}.
after Georgi Feodosjewitsch Voronoi [105] who formally introduced this concept.

Voronoi [105] also defines the geometric dual of the Voronoi diagram, where two points will be connected if and only if their Voronoi regions have a boundary in common. The second path comes from Boris Delaunay [46] who defines a geometric graph of a point set $S$ in the plane by joining two points $p$ and $q$ by an edge if they satisfy the empty circle property, that is there exists a circle that contains $p$ and $q$ on its boundary and no other point of $S$ lies in its interior. This structure is called Delaunay graph or Delaunay tessellation after Boris Delaunay. This structure is a triangulation and unique, called Delaunay triangulation and denoted $DT(S)$, when the point set $S$ is in general position, that is, no three points are collinear and no four points are cocircular. The Delaunay triangulation turns out to be the dual of the Voronoi diagram.

As mentioned before, the Delaunay triangulation is one of the most studied triangulations due to its fascinating properties. One such property follows directly from one of its definitions, that it is the dual of the Voronoi diagram, which by itself is already interesting and complex. A second property, is that the Delaunay triangulation maximizes the minimum angle of all angles in the triangles among all triangulations of a given point set. This property describes “well shaped” triangles, which is one of the main reasons for its popularity. In particular, in mesh generation this well shaped property is important for modeling objects in 3D because it allows some kind of smoothness. For a survey, we refer to [25, 95].

Another interesting property of Delaunay triangulations is that it contains different important proximity graphs. A proximity graph is simply a graph where a pair of vertices are joined by an edge if they satisfy a certain geometric property. In particular, for the Delaunay triangulation, the context of proximity is associated with a distance function. It is also a 1.99-spanner [108] (i.e., for any pair of vertices $x, y$, the shortest path between $x$ and $y$ in the Delaunay triangulation has length that is at most 1.99 times the Euclidean distance from $x$ to $y$). See [14, 87] for an encyclopedic treatment of this structure and its
Introduction

many properties.

For these reasons and more, the Delaunay triangulation is one of the most studied structures in computational geometry. However, for different applications there are certain desired properties that the Delaunay triangulation almost but does not quite possess. Thus, several generalizations of the Delaunay graph have been introduced.

In this thesis we explore different combinatorial properties of some generalizations of the Delaunay graph, together with some of their different proximity subgraphs. In the remainder of this chapter, we first give some basic notions about graphs that will be fundamental throughout this thesis, followed by a brief introduction to the hierarchy of subgraphs of the Delaunay triangulation. Then, we introduce different generalizations of the Delaunay graph. Later, we give an introduction to convex distances together with a discussion about their relation with convex Gabriel graphs, and finally, we give the outline of this thesis.

1.1 Basic notions

We assume that the reader has some knowledge on the basic notions of computational geometry. If this is not the case, we refer to the books by Shamos and Preparata [89], and by de Berg et al. [43]. In this section we give some basic definitions that will be used in this thesis. Most of them are standard in graph theory but we include their definitions for the sake of completeness. See Bondy and Murty [23] for a comprehensive overview of the terminology used on graph theory.

A graph \( G = (V, E) \) is a set \( V \) of vertices and a set \( E \) defined by a subset of elements in \( V \times V \) called edges. If it is not clear from the context, the vertex set of a graph \( G \) is denoted by \( V(G) \) and edge set by \( E(G) \). All graphs considered are undirected, finite and simple, unless stated otherwise.

A graph \( H \) is a subgraph of \( G \), denoted as \( H \subseteq G \), if \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). If \( V(H) = V(G) \) we say that \( H \) is a spanning subgraph of \( G \). Let \( A \subseteq V \), the subgraph of \( G \) induced by \( A \), denoted \( G[A] \), is the graph with vertex set \( A \) where two vertices are joined by an edge if and only if they are joined by an edge in \( G \). A graph is complete if for each pair of vertices \( u \) and \( v \) there exists the edge \( uv \) in \( E(G) \).

A path in \( G \) is a finite non-empty sequence of pairwise distinct vertices \( P = v_0v_1 \ldots v_{n-1}v_n \), such that \( v_iv_{i+1} \) is an edge in \( E \) for all \( i \in \{0, \ldots, n-1\} \). A cycle is the union of a path and the edge joining the first and last vertex of the path. An acyclic graph is one that contains no cycles.

Two vertices \( u \) and \( v \) are connected in \( G \) if there exists a path between them
in $G$. A graph $G$ is connected if every pair of vertices in $G$ is connected. A tree is a connected acyclic graph. A spanning tree of $G$ is a spanning subgraph of $G$ that is a tree. Consider a subset $A$ of $V(G)$, we denote by $G \setminus A$ the induced subgraph $G[V(G) \setminus A]$. The graph $G$ is $k$-connected if there exists a subset $C$ of $V(G)$ of size $k$ such that $G \setminus C$ is disconnected. For instance, a tree is 1-connected, a cycle is 2-connected and a triangulation can be 2-, 3- and 4-connected.

A path that contains every vertex of $G$ is called a Hamiltonian path; similarly, a Hamiltonian cycle of $G$ is a cycle that contains every vertex of $G$. If a graph $G$ contains a Hamiltonian cycle then $G$ is called Hamiltonian. A graph is 1-tough if, for any $k$, removing $k$ vertices from the graph splits the graph into at most $k$ components. An important relation between the concept of 1-toughness and Hamiltonian graphs is that every Hamiltonian graph is 1-tough. However, not every 1-tough graph is Hamiltonian.

A graph $G$ is weighted if each edge $e$ in $E(G)$ is associated to a real number $w(e)$, called its weight. The weight of $G$ is the total sum of the weight of all its edges. Given a weighted graph $G = (V, E)$ and a real number $t \geq 1$, a $t$-spanner of $G$ is a spanning subgraph $G'$ such that for every edge $uv$ in $G$, there exists a path from $u$ to $v$ in $G'$ whose weight is no more than $t$ times the weight of the edge $uv$ in $G$. When $G'$ is a $t$-spanner of the complete graph and each edge is weighted with its Euclidean length, we denote $t = sr(G')$ and call $sr(G')$ the spanning ratio of $G'$. We say that $G'$ is a $t$-spanner—or simply spanner—if $sr(G') = t$ is a constant.

Throughout this thesis we will assume that all graphs are geometric graphs, i.e., the edges are line segments.

Let $S$ be a point set in the plane. Let $G = (S, E)$ be a plane graph with vertex set $V(G) = S$ and edges $E(G)$. We say that the edge $pq$ is visible in $G$ if and only if one of the following holds: (i) $pq \in E(G)$, or (ii) $pq$ does not cross edges in $E(G)$. When $pq$ is visible in $G$, we say that $p$ and $q$ see each other in $G$ or that $p$ is visible to $q$ in $G$ and vice versa. Let $e$ be an edge in $E(G)$, we say that $e$ is blocking a point $p \in S$ from a point $q \in S$, if the line segment $pq$ crosses $e$ in its interior. We define the visibility graph of $S$ in $G$ as the graph with vertices $S$ and the set of all visible edges in $G$ as the edge set. Notice that the visibility graph is always a connected graph.

1.2 A hierarchy of subgraphs

As mentioned before, one of the nice properties of the Delaunay triangulation is the containment of some important proximity graphs. In this section we introduce such graphs.
The minimum spanning tree was introduced by Borůvka [26, 27, 84] in order to design an efficient electric distribution network of Moravia. A minimum spanning tree of a weighted graph $G$ is a spanning tree of $G$ with minimum weight. The minimum spanning tree has been widely studied due to its several applications in the design of networks, in the approximation of the traveling salesman problem, cluster analysis [13, 42, 61, 70, 90], and more. Consider a point set $S$ in the plane and a graph $G$ with vertex set $S$. The Euclidean minimum spanning tree of $G$, denoted $\text{MST}(G)$, is the minimum spanning tree of $G$ where the weight of each edge $e$ in $G$ is the Euclidean distance between the endpoints of $e$. When $G$ is the complete graph, it is called the Euclidean minimum spanning tree of $S$ and it is denoted by $\text{MST}(S)$.

In order to describe geographical variation data and develop statistical methods for categorizing sets of populations sampled from different localities, Gabriel with Sokal [106] introduced the following geometric graph. Given a point set $S$ in the plane, two points $p$ and $q$ in $S$ are joined by an edge if the smallest circle that contains $p$ and $q$ on its boundary does not contain other points of $S$ in its interior. This graph is called the Gabriel graph after K. Ruben Gabriel and it is denoted $\text{GG}(S)$.

Motivated by pattern recognition, Toussaint introduced the relative neighborhood graph [102], slightly changing the notion of “relatively close” neighbours defined by Lankford [71]. A relative neighborhood graph of a point set $S$ in the plane, denoted $\text{RNG}(S)$, is a geometric graph with vertex set $S$, where a pair of points $p$ and $q$ in $S$ are connected by an edge if and only if no other point of $S$ is at lower Euclidean distance to both of them. Geometrically speaking, two
points \(p\) and \(q\) in \(S\) are connected by an edge in \(\text{RNG}(S)\) if the lune defined by the intersection of the disks centered at \(p\) and \(q\) with radius the Euclidean distance \(d(p, q)\) does not contain points of \(S\) in its interior. Toussaint showed that for any point set \(S\), the relative neighborhood graph of \(S\) is a subgraph of the Delaunay triangulation of \(S\) [102]. In fact, one of the nicest properties about all of these graphs is the following hierarchical order in containment:

\[
\text{MST}(S) \subseteq \text{RNG}(S) \subseteq \text{GG}(S) \subseteq \text{DT}(S)
\]  

(1.1)

Figure 1.3 depicts an example of this hierarchy of subgraphs.

Finally, note that the definitions of the Relative neighborhood graph, Gabriel graph and Delaunay triangulation satisfy an empty region property. That is, a pair of points of \(S\) are joined by an edge if there exists certain closed region that contains the two points but no other point of \(S\) in its interior.

1.3 Generalized Delaunay graphs

The Delaunay triangulation, as mentioned before, is an important triangulation due to its several applications in mesh generation, finite element methods, GIS, graphics, and more. For a survey we refer to [14, 37, 25, 87, 95]. However, the fact that the Delaunay triangulation of a point set \(S\) in general position is unique can become an issue in certain application domains where extra flexibility is needed.

For instance, triangulations are often used to model terrains. In this case, the points in \(S\) are samples of a 3D surface, thus they also have an elevation. Yet, the Delaunay triangulation ignores the elevation information, potentially resulting in poor terrain models where important terrain features, such as valleys or ridge lines, are ignored [44, 63]. Then it would be nice if the triangulation could preserve the additional information as well. This leads us to consider the construction of a “nice” triangulation for a set of points and line segments. This motivated Lee [74, 75] to define the “Generalized Delaunay Triangulation”, such graph was also called “obstacle triangulation” by Chew [41] due to its applications in motion planning with obstacles. Finally, Chew [40] introduced such graph as the constrained Delaunay triangulation.

Let \(S\) be a point set in the plane and \(G\) be a geometric graph with vertex set \(S\) and edges called constraints. The constrained Delaunay triangulation of \(G\), denoted \(\text{DG}(G)\), is a geometric graph with vertex set \(S\) where each pair of points \(p\) and \(q\) in \(S\) are joined by an edge if either \(pq\) is a constraint or there exists a circle that contains \(p\) and \(q\) that does not enclose points of \(S\) that are visible to both \(p\) and \(q\). Note that when \(G\) has no edges, then we obtain the Delaunay triangulation. Moreover, since this definition depends on the empty
region property, then one can similarly define the constrained Gabriel Graph and constrained Relative neighborhood graph.

Gudmundsson et al. [62] were also motivated by the terrain interpolation problem. They were trying to describe a nice triangulation that contains certain constraints, in which case the constrained Delaunay triangulation is not always the optimal. Thus, they proposed the higher order Delaunay triangulations, a generalization of the Delaunay triangulation that intends to provide well shaped triangles, while giving flexibility to choose from a larger set of triangulations. A triangulation $T$ of $S$ is an order-$k$ Delaunay triangulation—or, simply, order-$k$ triangulation—if the circumcircle of each triangle of $T$ encloses at most $k$ points of $S$ in its interior. Note that an order-0 triangulation is a standard Delaunay triangulation. Order-$k$ triangulations have been used for terrain modeling, minimum interference networks and triangulation of polygons [17, 97, 91]. Note that for $k \geq 1$, there might be more than one order-$k$ triangulation. See Figure 1.4.

A very similar generalization of the Delaunay triangulation is the $k$-order Delaunay Graph. A $k$-order Delaunay graph—or, simply, $k$-Delaunay graph—of a point set $S$ in the plane, denoted $k$-$DG(S)$, is a geometric graph with vertex set $S$ where each pair of points $p$ and $q$ of $S$ are connected by an edge if and only if there exists a circle that contains both $p$ and $q$ such that the circle encloses at most $k$ points of $S$. This graph was introduced by Tung-Hsin Su and Ruei-Chuan Chang [100], which they used to improve the running time of solving the Euclidean bottleneck biconnected edge subgraph problem and Euclidean bottleneck matching problem. Note that the $k$-$DG(S)$ is not necessarily a planar graph. Also, from the definition, one can easily show that for any $k \geq 0$, $k$-$DG(S) \subset (k + 1)$-$DG(S)$, see Figure 1.5. In particular, note that the 0-Delaunay graph is the standard Delaunay triangulation. In addition, this generalization is again defined under the empty region property, then a similar definition exists for the $k$-Gabriel Graph, denoted $k$-$GG(S)$ and the $k$-Relative neighborhood graph, denoted $k$-$RNG(S)$. In fact, the hierarchy order $k$-$RNG(S) \subseteq k$-$GG(S) \subseteq k$-$DG(S)$ is still preserved.

For some applications, the Euclidean distance does not provide the desired
Generalized Delaunay graphs

1.3

DT\((S)\) \subseteq 1-DG\((S)\) \subseteq 2-DG\((S)\)

Figure 1.5: The 0-, 1- and 2-Delaunay graphs of a point set.

measure. In this thesis we are particularly interested in convex distance functions, which we will formally define in Section 1.4. Consider a compact convex set \(C\) that contains the origin as its center. Roughly speaking, the \(C\)-distance from \(p\) to \(q\), denoted \(d_C(p, q)\), is calculated in the following way. Center \(C\) at \(p\) and scale (expand or contract) \(C\) until its boundary touches \(q\), then \(d_C(p, q)\) is the scaling factor of the resulted homothet of \(C\).

There are two ways of defining the Delaunay triangulation — the geometric dual of the Voronoi Diagram, and with the empty circle property — these definitions are equivalent for the Euclidean distance. However, this might not be true for convex distance functions, since the symmetry property of a metric might not hold, i.e., \(d_C(p, q)\) is not necessarily equal to \(d_C(q, p)\). The definition of an edge in the empty circle property looks for the distance of the center towards the endpoints of the edge. This is in contrast to the definition of a bisector in the Voronoi diagram, which looks for the distance to each of the two sites towards points at same convex distance, i.e., for each \(p, q \in S\), the bisector of \(p\) and \(q\) is defined by all points \(x\) such \(d_C(p, x) = d_C(q, x)\). Let \(C'\) be the reflected shaped of \(C\) about the origin. Then, since \(d_C(p, q) = d_{C'}(q, p)\), the dual of the Voronoi diagram in the \(C\)-distance is equivalent to the structure with edges that satisfy the empty \(C'\) shape property. This leads to a surprising property, that the dual of a \(C\)-distance Voronoi diagram is invariant under movements of the center of \(C\). See [14, 33, 79].

In this thesis we will study properties of the Convex shape Delaunay graph by the empty disk property. Thus, we define the Convex shape Delaunay graph —or, simply, \(C\)-Delaunay graph— of a point set \(S\) in the plane, denoted \(DG_C(S)\), as the graph with vertex set \(S\) and for each pair \(p\) and \(q\) in \(S\), the edge \(pq\) is in \(DG_C(S)\) provided that there exists a homothet of \(C\) that contains \(p\) and \(q\) on its boundary and no other point of \(S\) in its interior. Note that when \(C\) is a circle, then \(DG_C(S)\) is the standard \(DT(S)\). It is known that the \(C\)-Delaunay graph

\[A \text{ homothet of } C \text{ is obtained by scaling } C \text{ with respect to the origin, followed by a translation.}\]
is a geometric planar graph \cite{33}. Several classes of Convex Delaunay graphs have been studied in the literature. For instance, Chew \cite{41} showed that the \( \triangle \)-Delaunay graph (i.e., where the shape is an equilateral triangle instead of a disk), denoted \( DG_\triangle(S) \), is a 2-spanner and that the \( \square \)-Delaunay graph (i.e., where the disk is replaced by a square), denoted \( DG_\square(S) \), is a \( \sqrt{10} \)-spanner. Bose et al. \cite{33} proved that the \( \mathcal{C} \)-Delaunay graph is a \( c \)-spanner where the constant \( c \) depends only on the perimeter and width of the convex shape \( \mathcal{C} \).

Similarly, the \( \mathcal{C} \)-Gabriel graph of a point set \( S \), denoted \( GG_\mathcal{C}(S) \), is defined as the graph with vertex set \( S \) where two points \( p \) and \( q \) of \( S \) are joined by an edge if and only if there exists a smallest homothet of \( \mathcal{C} \) with \( p \) and \( q \) on its boundary such that no other point of \( S \) is in its interior. The \( \mathcal{C} \)-Relative neighborhood graph of a point set \( S \), denoted \( RNG_\mathcal{C}(S) \), is a graph with vertex set \( S \) and for each pair \( p \) and \( q \) in \( S \), the edge \( pq \) is in \( RNG_\mathcal{C}(S) \) if and only if the intersection of the homothets of \( \mathcal{C} \) centered at \( p \) and \( q \) with scaling factor \( d_\mathcal{C}(p,q) \) and \( d_\mathcal{C}(q,p) \), respectively, contains no points of \( S \) in its interior. Finally, the \( d_\mathcal{C} \)-minimum spanning tree of a point set \( S \), denoted \( MST_\mathcal{C}(S) \) in the plane, is the minimum spanning tree of the complete subgraph of \( S \) with weight for each edge \( e \) equal to the \( \mathcal{C} \)-distance between its endpoints. Aurenhammer and Paulini \cite{15} recently showed that the subgraph hierarchy also holds for any compact convex set \( \mathcal{C} \), i.e., \( MST_\mathcal{C}(S) \subseteq RNG_\mathcal{C}(S) \subseteq GG_\mathcal{C}(S) \subseteq DT_\mathcal{C}(S) \).

Note that the definition of the constrained Delaunay graph can be generalized to constrained \( \mathcal{C} \)-Delaunay graph by replacing the circle with a homothet of \( \mathcal{C} \). Consider a planar graph \( G \) with vertex set \( S \). The constrained \( \mathcal{C} \)-Delaunay graph of \( S \), denoted \( DT_\mathcal{C}(G) \), is formally defined as follows. For each pair \( u \) and \( v \) in \( S \), an edge \( uv \in DT_\mathcal{C}(G) \) if either \( uv \in E(G) \) or there exists a homothet of \( \mathcal{C} \) that contains \( u \) and \( v \) on its boundary and no point of \( S \) in its interior is visible to both \( u \) and \( v \). Bose et al. \cite{35} proved that these graphs are planar. Moreover, they proved that regardless of the shape there exists a constant \( t \) such that the constrained \( \mathcal{C} \)-Delaunay graph is a \( t \)-spanner.

Finally, we can define the most general of the generalized Delaunay graphs used in this thesis: the \( k \)-order \( \mathcal{C} \)-Delaunay graph. Similarly to the Euclidean distance (where the empty shape is a circle) a \( k \)-order \( \mathcal{C} \)-Delaunay graph of a point set \( S \) in the plane, denoted \( k-DG_\mathcal{C}(S) \), is the graph with vertex set \( S \) such that, for each pair of points \( p,q \in S \), the edge \( pq \) is in \( k-DG_\mathcal{C}(S) \) if and only if there exists a \( \mathcal{C} \)-disk that has \( p \) and \( q \) on its boundary and contains at most \( k \) points of \( S \) different from \( p \) and \( q \). When \( k = 0 \) and \( \mathcal{C} \) is a circle, \( k-DG_\mathcal{C}(S) \) is the standard Delaunay triangulation. In addition, we analogously define the \( k \)-order \( \mathcal{C} \)-Gabriel graph and the \( k \)-order Relative neighborhood of a point set \( S \), denoted \( k-GG_\mathcal{C}(S) \) and \( k-RNG_\mathcal{C}(S) \), respectively. Except that in \( k-GG_\mathcal{C}(S) \), the homothets considered are restricted to be smallest homothets of \( \mathcal{C} \) with \( p \) and \( q \) on the boundary. In fact, \( MST_\mathcal{C}(S) \subseteq k-RNG_\mathcal{C}(S) \subseteq k-GG_\mathcal{C}(S) \subseteq k-DG_\mathcal{C}(S) \).
The main topic in this thesis is the study of different combinatorial and algorithmic properties of these generalized Delaunay triangulations.

1.4 Convex distances and the C-Gabriel graph

In this section we define the convex distance functions that will be used throughout this thesis.

A norm of $\mathbb{R}^2$ is a nonnegative function $\rho: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ with the following properties, for all $\lambda \in \mathbb{R}^+$ and $u, v \in \mathbb{R}^2$: (a) $\rho(u + v) \leq \rho(u) + \rho(v)$, (b) $\rho(\lambda v) = \lambda \rho(v)$ and, (c) if $\rho(v) = 0$ then $v$ is the zero vector. A metric is a distance function $d: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ such that for all $u, v, w \in \mathbb{R}^2$ the following properties hold: (a) $d(u, v) = 0 \iff u = v$, (b) $d(u, v) = d(v, u)$ and, (c) $d(u, w) \leq d(u, v) + d(v, w)$. When the function $d(u, v) = \rho(u - v)$ is a norm, then it is called a normed metric.

Let $p$ and $q$ be two points in the plane. Let $C$ be a compact convex set that contains the origin, denoted $\bar{o}$, in its interior. We denote the boundary of $C$ by $\partial C$. The convex distance $d_C(p, q)$ is defined as follows: If $p = q$, then $d_C(p, q) = 0$. Otherwise, let $C_p$ be the convex set $C$ translated by the vector $\overrightarrow{p}$ and let $q'$ be the intersection of the ray from $p$ through $q$ and $\partial C_p$. Then, $d_C(p, q) = \frac{d(p, q)}{d(p, q')}$ (see Figure 1.6) where $d$ denotes the Euclidean distance.

![Figure 1.6: Convex distance from $p$ to $q$.](image)

The convex set $C$ is the unit $C$-disk of $d_C$ with center $\bar{o}$, i.e., every point $p$ in $C$ satisfies that $d_C(\bar{o}, p) \leq 1$. The $C$-disk with center $c$ and radius $r$ is defined as the homothet of $C$ centered at $c$ and with scaling factor $r$. The triangle inequality holds: $d_C(p, q) \leq d_C(p, z) + d_C(z, q), \forall p, q, z \in \mathbb{R}^2$. However, this distance may not define a metric when $C$ is not point-symmetric about the origin\(^2\) since there may be points $p, q$ for which $d_C(p, q) \neq d_C(q, p)$. When $C$ is point-symmetric with respect to the origin, $d_C$ is called a symmetric convex distance function and it is a metric. We will refer to such distance functions as symmetric convex.

\(^2\) A shape $C$ is point-symmetric with respect to a point $x \in C$ provided that for every point $p \in C$ there is a corresponding point $q \in C$ such that $pq \in C$ and $x$ is the midpoint of $pq$. 
Moreover, $d_C(\hat{o}, p)$ defines a norm of a metric space. In addition, if a point $p$ is on the line segment $ab$, then $d_C(a, b) = d_C(a, p) + d_C(p, b)$ (see [14, Chapter 7]).

Recall that an edge $pq$ is in the $k$-order $C$-Delaunay graph of $S$, provided that there exists a $C$-disk that has $p$ and $q$ on its boundary and contains at most $k$ points of $S$ different from $p$ and $q$.

As mentioned earlier, the definition of Gabriel graphs requires the notion of a smallest homothet containing two points on its boundary. To be able to use our techniques, it is convenient to be able to associate a distance to the size of such smallest homothets, but $d_C$ fails on defining such distance because $d_C$ might not be symmetric when the shape is not point-symmetric. To circumvent this issue, Aurenhammer and Paulini [15] showed how to define, from any convex shape $C$, another shape that results in a distance function that is always symmetric: The set $\hat{C}$ is defined as the Minkowski sum of $C$ and its shape reflected about its center. For an example, see Figure 1.7. The shape $\hat{C}$ is point-symmetric and the $d_{C\hat{}}$-distance from $p$ to $q$ is given by the scaling factor of a smallest homothet of $C$ containing $p$ and $q$ on its boundary. The diameter and width of $\hat{C}$ is twice the diameter and width of $C$, respectively. Moreover, if $C$ is point-symmetric, $d_{C\hat{}}(p, q) = \frac{d_{C}(p, q)}{2}$.

Thus, for any pair of points $p, q \in S$, edge $pq$ is in the $k$-order $C$-Gabriel graph of $S$ provided that there exists a $C$-disk with radius $d_{C\hat{}}(p, q)$ that has $p$ and $q$ on its boundary and contains at most $k$ points of $S$ different from $p$ and $q$. From the definition of $k$-$GG_C(S)$ and $k$-$DG_C(S)$ it follows that $k$-$GG_C(S) \subseteq k$-$DG_C(S)$, and it can be a proper subgraph. See Figure 1.8a for an example. Further, $\hat{C}$ always contains $\hat{C}$ in its interior. However, for some non point-symmetric convex $C$ it is not true that $GG_{C\hat{}} \subseteq GG_C$; see Figure 1.8b for an example.

For simplicity, denote by $C_r(a, b)$ a $C$-disk of radius $r$ with the points $a$ and $b$ on its boundary. For the special case of a diametral disk; i.e., when $r = d_{C\hat{}}(a, b)$,
Figure 1.8: (a) $\mathcal{C}$ is a regular hexagon. Edge $pq$ is in $2DG_{\mathcal{C}}(S)$ but it is not in $2GG_{\mathcal{C}}(S)$. (b) Edge $pq$ is in $GG_{\mathcal{C}}(S)$ but it is not in $GG_{\mathcal{C}}(S)$. (c) Many $\mathcal{C}$-disks $\mathcal{C}(a, b)$ may exist for $a$ and $b$.

we denote it by $\mathcal{C}(a, b)$. Note that $\mathcal{C}(a, b)$ may not be unique when the boundary $\partial \mathcal{C}$ is not smooth, see Figure 1.8c. In addition, we denote by $D_{\mathcal{C}}(c, r)$ the $\mathcal{C}$-disk centered at point $c$ with radius $r$.

1.5 Outline

In Chapter 2 we consider the flip graph of $k$-Delaunay triangulations of a point set $S$. The flip graph of $S$ has one vertex for each possible triangulation of $S$, and an edge connecting two vertices when the two corresponding triangulations can be transformed into each other by a flip (i.e., exchanging the diagonal of a convex quadrilateral by the other one). The flip graph is an essential structure in the study of triangulations, but until now it had been barely studied for order-$k$ Delaunay triangulations. In this work we show that, even though the order-$k$ flip graph can be disconnected for $k \geq 3$, any order-$k$ triangulation can be transformed into some other order-$k$ triangulation by at most $k - 1$ flips, such that the intermediate triangulations are of order at most $2k - 2$, in the following settings: (1) for any $k \geq 0$ when $S$ is in convex position, and (2) for any $k \leq 5$ and any point set $S$. Our results have several implications on the flip distance between order-$k$ triangulations, as well as on their efficient algorithmic enumeration.

This chapter is based on the following publications:


In Chapter 3 we look at constrained $C$-Delaunay triangulations and $d_C$-minimum spanning trees, for an arbitrary compact convex set $C$. We first extend a known result about standard Delaunay triangulations to $C$-Delaunay graphs. Let $ST(S)$ be the set of all plane spanning trees of a planar point set $S$ of size $n$. We prove that for each element $T$ of $ST(S)$, there exists a weight-decreasing sequence of trees $T_0, \ldots, T_k$ in the $d_C$-distance such that $T_0 = T$, $T_k = MST_C(S)$ and $T_i$ is the $d_C$-minimum spanning tree of $DG_C(T_{i-1})$. Hence, $T_i$ does not cross $T_{i-1}$ for all $i = 1, \ldots, k$. Later, we look at the $\Box$-distance — also known as the $L_\infty$-metric — and give an $\Omega(\log n)$ lower bound for the length of the sequence. Finally, we show an upper bound of $\log n$ on the length of the sequence for a case in which the constraints are crossing the edges of the $MST_\Box(S)$ in a specific manner.

This chapter is based on the following publication:

[31] Prosenjit Bose, Pilar Cano, and Rodrigo I. Silveira. Sequences of spanning trees for $L_\infty$-Delaunay triangulations. In 34th European Workshop on Computational Geometry: extended abstracts (EuroCG), 2018

In Chapter 4 we study Hamiltonicity in the $C$-Delaunay graphs. More specifically, we provide upper bounds on the minimum value of $k$ for which $k$-$GG_C(S)$ is Hamiltonian. Since $k$-$GG_C(S) \subseteq k$-$DG_C(S)$, all results carry over to $k$-$DG_C(S)$. In particular, we give upper bounds of 24 for every $C$ and 15 for every point-symmetric $C$. We also improve these bounds to 7 for squares, 11 for regular hexagons, 12 for regular octagons, and 11 for even-sided regular $t$-gons (for $t \geq 10$). These constitute the first general results on Hamiltonicity for convex shape Delaunay and Gabriel graphs.

In addition, we show lower bounds of $k = 3$ and $k = 6$ on the existence of a bottleneck Hamiltonian cycle in the $k$-order Gabriel graph for squares and hexagons, respectively. Finally, we construct a point set such that for an infinite family of regular polygons $P_t$, the Delaunay graph $DG_{P_t}$ does not contain a Hamiltonian cycle.

This chapter is based on the following publications:


In Chapter [5] we revisit an affine invariant triangulation defined by Gregory M. Nielson [86], that uses the inverse of the covariance matrix of \( S \) to define an affine invariant norm, denoted \( A_S \), and an affine invariant triangulation, denoted \( DT_{A_S}(S) \). The \( A_S \)-norm is a special kind of \( C \)-distance where \( C \) is replaced by a type of ellipse. We revisit the \( A_S \)-norm from a geometric perspective, and show that \( DT_{A_S}(S) \) can be seen as a standard Delaunay triangulation of a transformed point set based on \( S \). We prove that it retains all of its well-known properties. In addition, motivated by this norm and the problem of finding affine geometric methods, we provide different affine invariant sorting methods of a point set \( S \) and of the vertices of a polygon \( P \) that can be combined with well-known algorithms in order to obtain other affine invariant methods.

This chapter is based on the following publication:


Finally, in the concluding remarks of this thesis we introduce some open problems that were not studied in this thesis but are related to some of our results and generalized Delaunay triangulations.
A fundamental operation to work with triangulations is the edge flip. It consists in removing an edge, shared by two triangles that form a convex quadrilateral, and inserting the other diagonal of the quadrilateral. A flip transforms a triangulation $T$ into another triangulation $T'$ that differs by exactly one edge and two triangles. The flip operation leads naturally to the definition of the flip graph of $S$. Each triangulation of $S$ is represented by a vertex in this graph, and two vertices are adjacent if their corresponding triangulations differ by exactly one flip.

The importance of flips in triangulations comes from the fact, first proved by Lawson [72], that the flip graph is connected. Moreover, the sequence of edge flips connecting any two triangulations has length $O(n^2)$. In fact, it was later shown [73, 96] that any triangulation of $S$ can be converted into $DT(S)$ by performing $O(n^2)$ flips. Each flip in this transformation also results in an increase of the angle vector of the triangulation, i.e., the vector of all angles of each of its triangles in increasing order. It is also known that the quadratic upper bound on the diameter of the flip graph is tight in general [58, 66], although it goes down to $\Theta(n)$ if the points in $S$ are in convex position [99]. In general, computing the distance in the flip graph between two given triangulations is a difficult problem, whose complexity was open until recently, when it was shown to be APX-hard [78, 88]. This has drawn considerable attention to the study of certain subgraphs of the flip graph, which define the flip graph of certain classes of triangulations, for instance, bounded degree triangulations [7] or triangulations with perfect matchings [65]. We refer to [36] for a survey.

In this chapter we study the flip graph of higher order Delaunay triangulations. Most previous work on such triangulations focused on algorithmic questions related to finding order-$k$ triangulations that are optimal with respect
to extra criteria [104, 97, 98], or on evaluating their effectiveness in practical settings [20, 45]. One of the few theoretical aspects studied is the asymptotic number of order-k triangulations [82]. In that work, the authors showed that for points drawn uniformly at random, already for $k = 1$ one can expect an exponential number of different order-$k$ triangulations. However, almost nothing is known about the flip graph of order-$k$ triangulations, except that it is connected only for $k \leq 2$ [1].

Abellanas et al. [2] studied the flip graph of triangulations that consists of only order-$k$ edges. In their work an order-$k$ (Delaunay) edge $e$ is defined as an edge for which there exists a circle through the endpoints of $e$ that encloses at most $k$ points of $S$. All edges in an order-$k$ triangulation are order-$k$. However, the converse is not true: a triangle composed of three order-$k$ edges can have order greater than $k$. In fact, the lowest order triangulation containing an order-$k$ edge can have order up to $2k - 2$ [63]. Similarly as for order-$k$ triangulations, Abellanas et al. [2] showed that the flip graph of triangulations of point sets with edges of order $k$ is connected for $k \leq 1$, but might be disconnected for $k \geq 2$. On the other hand, they proved that for point sets in convex position the flip graph is always connected [2]. However, their proof implies an exponential bound in the diameter of the flip graph. The only previous work on the flip graph of order-$k$ triangulations is by Abe and Okamoto [1], in the context of enumeration algorithms. They observed that for $k \leq 2$, the fact that the flip graph is connected implies that the reverse enumeration framework by Avis and Fukuda [16] can be applied to enumerate all order-$k$ triangulations, spending polynomial time on each of them.

**Our contributions.** We present several structural properties of the flip graph of order-$k$ triangulations. For points in convex position, we show that for any $k > 2$ there exist a point set in convex position for which the flip graph is not connected. However, we prove that no order-$k$ triangulation is too far from all the other order-$k$ triangulations, in the sense that for any order-$k$ triangulation $T$ there exists another order-$k$ triangulation $T'$ at distance at most $k - 1$ in the flip graph of order-$(2k - 2)$ triangulations. It is noteworthy that each flip on the path from $T$ to $T'$ increases the angle vector of the triangulation. The bottom line is that while order-$k$ triangulations are not connected via the flip operation, they become connected if a slightly relaxed condition is considered. For points in generic (non-convex) position, we prove the same result for up to $k \leq 5$, although we conjecture that it holds for all $k$. Our results have several implications on the flip distance between order-$k$ triangulations, as well as on their efficient algorithmic enumeration.
2.1 Preliminaries and general observations

In this section we give some definitions and observations that will be useful for the rest of the chapter.

Let $S$ be a point set in the plane. Throughout this chapter we assume that set $S$ is in general position, i.e., that no three points of $S$ lie on a line and no four points of $S$ lie on a circle. Let $T$ be a triangulation of $S$, and let $\triangle uyv$ be a triangle in $T$ with vertices $u, y, v$. We will denote by $\bigcirc uyv$ the open disk defined by the enclosed area of the circumcircle of $\triangle uyv$ (i.e., the unique circle through $u$, $y$, and $v$). Thus, $\partial \bigcirc uyv$ denotes the circumcircle of $\triangle uyv$. Triangle $\triangle uyv$ is a triangle of order $k$, also called an order-$k$ triangle, if $\bigcirc uyv$ contains at most $k$ points of $S$. A triangulation where all triangles are order-$k$ is an order-$k$ (Delaunay) triangulation. Hence, a triangulation $T$ is not order-$k$ if $\bigcirc uvy$ contains more than $k$ points of $S$ for some $\triangle uyv$ in $T$. The set of all order-$k$ triangulations of $S$ will be denoted $T_k(S)$.

Let $e = uv$ be an edge in $T$. Edge $e$ is flippable if $e$ is incident to two triangles $\triangle uxv$ and $\triangle uyv$ of $T$ and $uxvy$ is a convex quadrilateral. The edge $e$ is called illegal if $\bigcirc uxv$ contains $y$. Note that this happens if and only if $\bigcirc uyv$ contains $x$. See Fig. 2.1b. Otherwise, the edge $uv$ is called legal. It is easy to see that an illegal edge is flippable. The angle vector $\alpha(T)$ of a triangulation $T$ is the vector whose components are the angles of each triangle in $T$ ordered in increasing order. Let $T' \neq T$ be another triangulation of $S$. We say that $\alpha(T) > \alpha(T')$ if $\alpha(T) > \alpha(T')$ in lexicographic order. It is well-known that if $T'$ is the triangulation obtained by flipping an illegal edge of $T$, then $\alpha(T') > \alpha(T)$ \[58\]. Moreover, since $DT(S)$ maximizes the minimum angle, it follows that $DT(S)$ is the only triangulation where all the edges are legal \[96\]. This property leads to a simple algorithm for computing the $DT(S)$: start from

![Diagram](image)

Figure 2.1: (a) An illegal edge $uv$, with region $\bigcirc_{uv}$ in gray. (b) The union of the dashed filled disks $\bigcirc uxv$ and $\bigcirc xvy$ is contained in the union of the gray disks $\bigcirc uxv$ and $\bigcirc uyv$. 
Higher order Delaunay triangulations and flips

an arbitrary triangulation of \( S \), and flip illegal edges until none is left. This also implies that the flip graph of all triangulations of \( S \) is connected, since any triangulation can be transformed into the \( DT(S) \) by a finite number of flips. Let \( G(T_k(S)) \) denote the flip graph of \( T_k(S) \).

Next we present several important facts. We start with a well-known observation, illustrated in Fig. 2.1.b.

**Observation 2.1.1.** Let \( \triangle uxy \) and \( \triangle uvx \) be two adjacent triangles in a triangulation of \( S \). If edge \( uv \) is illegal, then \( (\bigcirc uxy \cup \bigcirc uvx) \subset (\bigcirc uxy \cup \bigcirc uvx) \).

In the context of order-\( k \) triangulations, Obs. 2.1.1 implies the following.

**Lemma 2.1.2** (Abe and Okamoto [1]). Let \( T \) be a triangulation of \( S \), let \( uv \) be an illegal edge of \( T \), and let \( \triangle uxy \) and \( \triangle uvx \) be the triangles incident to \( uv \) in \( T \). If \( \triangle uxy \) is of order \( k \), and \( \triangle uvx \) is of order \( l \), then triangles \( \triangle uxy \) and \( \triangle uvx \) are of order \( k' \) and \( l' \), respectively, for some \( k' + l' \leq k + l - 2 \).

Now we need an extra piece of notation, which we will use extensively. For a triangle \( \triangle uyv \), we let \( \mathcal{O}_y^{uv} \) denote the open region bounded by edge \( uv \) and the arc of \( \partial \bigcirc uyv \) that does not contain \( y \). See Fig 2.1.a.

In what remains of this section we will consider a triangulation \( T \) of order \( k \geq 3 \), and an illegal edge \( uv \) adjacent to triangles \( \triangle uxy \) and \( \triangle uvx \).

Throughout this chapter we will often refer to points of \( S \) that are contained—or not—in a certain region. For brevity, we will sometimes omit “of \( S \)” and simply refer to points in a certain region, as we do in next observation.

**Observation 2.1.3.** If \( \mathcal{O}_y^{uv} \) does not contain points, then \( (\bigcirc uxy \cap \bigcirc xyv \cap \mathcal{O}_y^{uv}) \subset (\bigcirc uxy \cap \mathcal{O}_y^{uv}) \) does not contain points.

**Proof.** Consider the intersection \( \partial \bigcirc uxy \cap \partial \bigcirc xyv = \{x, y\} \). Thus, \( \bigcirc uxy \cap \bigcirc xyv \) is defined by the area between the arcs from \( x \) to \( y \) of the circles \( \partial \bigcirc uxy \) and \( \partial \bigcirc xyv \) that do not contain \( u \) and \( v \), respectively. Since \( y \) lies in \( \bigcirc uvx \) and \( x \) lies in \( \bigcirc uyv \), the intersection \( \bigcirc uxy \cap \bigcirc xyv \) is contained in \( \bigcirc uvx \cap \bigcirc uyv \), which does not contain any point in \( \mathcal{O}_y^{uv} \).

Consider the triangulation \( T' \) resulting from flipping \( uv \) in \( T \) such that \( \triangle uxy \) is not of order \( k \). Thus, \( T' \) is not of order \( k \). For the sake of simplicity, for any region \( R \) in the plane, we denote by \( |R| \) the number of points of \( S \) in the interior of \( R \).

Using that \( |\bigcirc uxy \setminus \{y\}| \leq k - 1 \), \( |\bigcirc uyv \setminus \{x\}| \leq k - 1 \) and \( |\bigcirc uxy| \geq k + 1 \), a rather simple counting argument implies the following.

**Observation 2.1.4.** Each of \( \mathcal{O}_y^{uv} \setminus \bigcirc uvx \) and \( \mathcal{O}_x^{uv} \setminus \bigcirc uyv \) contains at least two points.
Lemma 2.1.6. If \( \triangle uxy \) contains \( k - 1 \) points, then \( \mathcal{S}_y^{ux} \) contains \( k - 1 \) points. Furthermore, \( \mathcal{S}_y^{ux} \cap \mathcal{O}uxv \) does not contain any point.

Proof. From Obs. 2.1.4 there are at least two points in \( \mathcal{S}_y^{ux} \). Since there are points in \( \mathcal{S}_y^{ux} \), from Obs. 2.1.5 such points are only in \( \mathcal{S}_y^{ux} \). Let \( p_1 \) be the point in \( \mathcal{S}_y^{ux} \) such that \( \mathcal{O}up_1x \) does not contain any point of \( \mathcal{S}_y^{ux} \). Then, the arc of \( \partial \mathcal{O}up_1x \) from \( u \) to \( x \) contains \( p_1 \) lies in \( \mathcal{S}_y^{ux} \). Thus, the other arc of \( \partial \mathcal{O}up_1x \) from \( x \) to \( u \) lies outside \( \mathcal{O}uxy \). Therefore, the region \( \mathcal{O}uxy \setminus \mathcal{S}_y^{ux} \) is contained in \( \mathcal{O}up_1x \). Then, \( \mathcal{O}up_1x \) contains \( k \) points (the \( k - 1 \) points in \( \mathcal{O}uxy \setminus \mathcal{S}_y^{ux} \)). By the same arguments, for any \( p' \in \mathcal{S}_y^{ux} \setminus \{p_1\} \), \( \mathcal{O}up'x \) contain at least \( k + 1 \) points (the points in \( \mathcal{O}up_1x \) and \( p_1 \)). Thus, for any \( p' \in \mathcal{S}_y^{ux} \setminus \{p_1\} \), \( \triangle up'x \) is not in \( \mathcal{O}uxy \). It remains to show that \( \triangle up_1x \) is in \( \mathcal{O}uxy \). For the sake of a contradiction suppose that \( \triangle up_1x \notin \mathcal{O}uxy \). Thus, \( u \) and \( x \) are not adjacent to any point in \( \mathcal{S}_y^{ux} \) in \( \mathcal{O}uxy \). Then, there exists at least one \( p' \in \mathcal{S}_y^{ux} \) such that \( p' \) is adjacent to the endpoints of an edge \( ab \) crossing twice \( \partial \mathcal{O}uxy \) and is blocking \( x \) or \( u \) from \( p' \) (note that \( a \) can be \( u \) and \( b \) can be \( x \) but \( ab \neq \mathcal{O}uxy \)). Hence, at least one of \( a \) or \( b \) lies outside \( \mathcal{O}uxy \). Consider the disk \( \mathcal{O}ap'b \). Since \( ab \) crosses \( \partial \mathcal{S}_y^{ux} \), the arc of \( \partial \mathcal{O}ap'b \) from \( a \) to \( b \) that contains \( p' \) crosses twice the
Figure 2.2: For $k = 4$ we have $\epsilon = 4^\circ$: (a) Shows the intersection points $a_1, a'_1, b_1$ and $b'_1$ given in the proof of Theorem 2.2.1. (b) Shows an example of regions $R_1$ and $R'_1$ given in the proof of Theorem 2.2.1, represented by the blue raising tilling pattern and the grey filled area, respectively; in this case $R_1 \cap R'_1 = R_1$

arc of $\partial \circ uxy$ from $u$ to $x$ that defines $\mathcal{O}_{uxy}^{ux}$. Hence, $\circ uxy \setminus \mathcal{O}_{uxy}^{ux}$ is contained in $\circ ap'b$. Therefore, $\circ ap'b$ contains at least $k + 1$ points of $S$ (the $k - 1$ points in $\circ uxy \setminus \mathcal{O}_{uxy}^{ux}$, $y$ and either $u$ or $x$), which is a contradiction. \hfill \Box

### 2.2 Points in convex position

In this section we show that $k - 1$ flips are sufficient to transform any order-$k$ triangulation of a convex point set into some other order-$k$ triangulation, such that all the intermediate triangulations are of order $2k - 2$.

Before that, we show that our result is tight in how large the flip distance between two order-$k$ triangulations can be.

**Theorem 2.2.1.** For any $k > 2$ there is a set $S_k$ of $2k + 2$ points in convex position such that $G(\mathcal{T}_k(S_k))$ is not connected. Moreover, there is a triangulation $T_k$ in $\mathcal{T}_k(S_k)$ such that in order to transform $T_k$ into any other triangulation in $\mathcal{T}_k(S_k)$ one needs to perform at least $k - 1$ flips.

**Proof.** In this construction, all angles will be smaller than $180^\circ$, so by $\angle xyz$ we denote the smaller of the two angles formed by the rays $yx$ and $yz$. By angle between two lines we mean the smaller or the two such angles.

We start with a horizontal line segment $uv$ and a point $p_1$ above it, such that $\angle(vup_1) = 15^\circ$ and $\angle(p_1uv) = 35^\circ$. Let $q_1$ be the reflection of $p_1$ with respect to the line through $uv$. Then, $\angle(up_1v) = \angle(vq_1u) = 130^\circ$. Thus, $\angle(up_1q_1) = \angle(p_1q_1u) = 75^\circ$ and $\angle(vq_1p_1) = \angle(q_1p_1v) = 55^\circ$. Let $\epsilon > 0$ be such that $(205^\circ - 2(k - 1)\epsilon^\circ) > 180^\circ$. Let $\ell_1$ be $\circ up_1v$ and $\ell_1$ be the line through $p_1v$. See Fig. 2.3.
Figure 2.3: Each disk $\bigcirc_i$ contains $q_1$ and $\mathcal{O}^{uq_1}_{p_1}$. The colored rays represent the supporting line of the edge $e_i = p_ip_{i-1}$ for all $i > 1$ and respecting $\bigcirc_i$.

Let $a_1$ be the intersection point of $\ell_1$ and the line obtained by rotating the line through $q_1p_1$ about $q_1$ counterclockwise by $\epsilon^\circ$. Thus, $\angle a_1q_1v = 55^\circ + \epsilon^\circ$. Let $a_1'$ be the point defined as $q_1a_1 \cap (\partial \bigcirc_1)$. Let $b_1$ be the point defined as the intersecting of $\ell_1$ and the line obtained by rotating counterclockwise $\epsilon^\circ$ about $u$ the line through $up_1$. Hence, $\angle q_1ub_1 = 35^\circ + \epsilon^\circ$. Let $b_1'$ be the point defined as $ub_1 \cap (\partial \bigcirc_1 \setminus u)$. Notice that by construction, the points $a_1$ and $b_1$ do not lie in $\bigcirc_1$. Moreover, there is a 2-dimensional region $R_1$ outside $\bigcirc_1$ defined by the edges $a_1a_1'$, $p_1a_1$ and the arc of $\partial \bigcirc_1$ with endpoints $p_1$ and $a_1'$, that does not contain $u$. Similarly, there is a 2-dimensional region $R'_1$ outside $\bigcirc_1$ defined by the edges $b_1b_1'$, $p_1b_1$ and the arc of $\partial \bigcirc_1$ with endpoints $p_1$ and $b_1'$, that does not contain $u$. See Fig. 2.2a. Notice, that since $a_1$ and $b_1$ are outside $\bigcirc_1$, then $R_1 \cap R'_1$ defines a non-empty 2-dimensional region. Let $p_2$ be a point in the interior of $R_1 \cap R'_1$. See Fig. 2.2b. Let $\delta_1$ be the angle $\angle(p_1up_2)$ and let $\delta_1'$ be the angle $\angle(p_1q_1p_2)$. By definition of $p_2$, $\delta_1 < \epsilon$ and $\delta_1' < \epsilon$. Then,

$$\angle(up_2p_1) = \angle(up_2q_1) + \angle(q_1p_2p_1) \geq \angle(up_1p_1) + \angle(q_1a_1v) \geq \angle(up_1q_1) + \delta_1' - \delta_1 + \angle(q_1p_1v) - \delta_1' = 130 - \delta_1.$$

Thus, $\angle(up_2p_1) + \angle(p_1q_1u) > \angle(up_1v) - \delta_1 + \angle(p_1q_1u) - \delta_1' = 130 - \delta_1 + 75 - \delta_1' \geq 205 - 2\epsilon > 180$. Therefore, $\bigcirc_2 = \bigcirc up_{p_2p_1}$ contains $q_1$. Moreover, $\bigcirc_2$ contains $\mathcal{O}^{uq_1}_{p_1}$.

Recursively, for $i \in \{1, \ldots, k - 1\}$ we find a point $p_{i+1}$ with the following properties: (1) $p_{i+1}$ lies in $R_i$ and $R'_i$ between the supporting line $\ell_i$ of edge $p_ip_{i-1}$ and outside $\bigcirc_i = \bigcirc up_{p_ip_{i-1}}$, where $p_0 = v$. (2) $\angle(up_{i+1}q_1) + \angle(p_{i+1}q_1u) > 130 - \delta_i + 75 - \delta_i' - 2(i-1)\epsilon \geq 205 - 2i\epsilon > 180$. For all $i > 2$, note that by construction $p_j$ does not lie in $\bigcirc_i$ for all $j < i - 1$. Also, note that $\mathcal{O}^{uq_1}_{p_1}$ is contained in $\bigcirc_i$. 

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See Fig. 2.3. Symmetrically, we define a point set \( \{q_1, q_2, \ldots, q_k\} \), such that \( q_i \) is the reflection of \( p_i \) with respect to the horizontal line passing through \( u \) and \( v \).

Let \( S' := \{p_1, p_2, \ldots, p_k\} \) and \( S'' := \{q_1, q_2, \ldots, q_k\} \). We define \( S_k := \{u, v\} \cup S' \cup S'' \). Consider the triangulation \( T = \{\Delta up_1v, \Delta uq_1v\} \cup \{\Delta up_{i+1}p_i, \Delta uq_{i+1}q_i : \forall i \in \{1, \ldots, k-1\}\} \). See Fig. 2.4. By construction of \( S_k \), each of the triangles is of order \( k \), which is, each triangle formed by points in \( S \setminus S' \) contains exactly the \( k \) points of \( S' \) in the interior of its circumcircle and symmetrically each triangle formed by points in \( S \setminus S'' \) contains in the interior of its circumcircle exactly the \( k \) points of \( S'' \). Moreover, the triangulations \( T \setminus S' \) and \( T \setminus S'' \) are the Delaunay triangulations of the point sets \( S \setminus S' \) and \( S \setminus S'' \), respectively.

Let us show that for any order-\( k \) triangulation \( T' \) different from \( T \), the difference between \( T \) and \( T' \) consists of a set of edges with one endpoint in \( S' \) and the other endpoint in \( S'' \). In other words, we will show that any edge \( p_i p_j \) or \( q_i q_j \) with \( i - 1 > j \) is in a triangle that is not order \( k \). Note, that if this is true, then flipping any edge of \( T \setminus S' \) or \( T \setminus S'' \) will result in a triangulation that is not of order \( k \). In addition, the only edge of \( T \) that can be transformed by one flip flipped to an edge with one endpoint in \( S' \) and the other in \( S'' \) is \( uv \). By construction, the triangle \( \Delta up_1q_1 \) is of order \( 2k - 2 \). This means that flipping any edge of \( T \) results in a triangulation that is not order-\( k \). Therefore \( G(T_k(S_k)) \) is not connected.

Let us show that for \( k - 1 \geq i - 1 > j \), any triangulation of \( S_k \) containing edge \( p_i p_j \) is not order-\( k \), the other case for \( q_i q_j \) is symmetric. To show this, we consider triangle \( \Delta p_i p_j \) for any \( k \geq i > t > j \geq 0 \). Note that there exists a \( t \) with \( i > t > j \), since \( p_i p_j \) is a diagonal of the convex hull of \( S_k \) where one of the two triangles containing \( p_i p_j \) has a vertex \( p_i \) with \( i > t > j \).

By construction, points \( p_i \) and \( p_j \) are not in \( \bigcup up_i p_{t-1} \). Then the arc \( A \) of \( \partial \bigcup p_i p_{t-1} \) with endpoints \( p_i \) and \( p_j \) that contains \( p_t \) crosses \( \partial \bigcup up_i p_{t-1} \) twice, that is, once in \( p_t \) and one more time in one of the two arcs: (1) the arc with endpoints \( p_i \) and \( p_t \) that does not contain \( p_j \), or (2) the arc with endpoints \( p_i \) and \( p_j \) that does not contain \( p_i \). Arc \( A \) has a portion that lies in \( \bigcup up_i p_{t-1} \) \( \bigcup q_{p_1} \), since the points \( p_i, p_t, p_{t-1}, p_j \) lie in \( \bigcup q_{p_1} \). In addition, since \( p_i \) and \( p_j \) do not lie in \( \bigcup up_i p_{t-1}, \bigcup p_t p_j \) contains all the points in \( S'' \cup \{u\} \) whose total number is \( k + 1 \). Thus, flipping any edge of \( T \) (including \( uv \) as mentioned before) results in a triangulation that is not order-\( k \). Therefore, two things have been showed: (1) the flip graph \( G(T_k(S_k)) \) is not connected, and (2) any order-\( k \) triangulation differs from \( T \) by only edges with one endpoint in \( S' \) and the other in \( S'' \).

It remains to show that \( k - 1 \) flips are necessary to transform \( T \) into another order-\( k \) triangulation \( T' \). We have shown above, that each edge of \( T' \) that is not in \( T \) must have one endpoint in \( S' \) and one in \( S'' \). Thus, edge \( uv \) has to be flipped at some point. Note that if there is an edge \( p_t q_r \), then there is a
Let $S$ be a point set in convex position. Let $T$ be an order-$k$ triangulation of $S$. We say that $T$ is minimal if flipping any illegal edge in $T$ results in a triangulation that is not order-$k$. Let $uv$ be a diagonal in $T$ and let $\triangle uxy$ and $\triangle uyz$ be the triangles incident to it. Since $S$ is in convex position, the diagonal $uv$ partitions $T$ into two sub-triangulations that only share edge $uv$. Let $T^x_{uv}$ (respectively, $T^y_{uv}$) denote the sub-triangulation that contains triangle $\triangle upq$ with $i \geq t$ and $j \geq r$: since $u$ is not adjacent to $v$ and $u$ and it is adjacent to $p_k$ and $q_k$, the minimum $i$ and minimum $j$ such that $u$ is adjacent to $p_i$ and $q_j$ creates a triangle $\triangle upq$. For $k - 1 \geq i \geq 1$ and $k - 1 \geq j \geq 1$, let us show that $\partial \triangle upq$ contains all points $p_t$ with $t > i$ and all points $q_r$ with $r > j$, i.e., triangle $\triangle upq$ is not of order $2k - i - j - 1$. This would end the proof, since any edge $p_iq_j$ with $2k - i - j \leq k$ crosses at least $k - 1$ edges from $T$, i.e., edge $p_iq_j$ crosses the edges $uv$, $up_t$ and $uq_r$ for all $t < i$ and $r < j$. This would conclude that $k - 1$ flips are needed in order to transform $T$ into another triangulation of order $k$.

Notice that disk $\partial \triangle upq$ crosses disk $\partial \triangle upq_{i-1}$ twice in $u$ and in $p_t$. See Fig. 2.4.a. Since $q_j$ is in $\partial \triangle upq_{i-1}$, the arc of $\partial \triangle upq_{i-1}$ between $u$ and $p_t$ that contains $q_j$ lies in $\partial \triangle upq_{i-1}$. In addition, since the point $q_j$ lies in $\partial \triangle upq_{i-1}$ for all $t > i$, the arc from $u$ to $q_j$ of $\partial \triangle upq_{i-1}$ that does not contain $p_i$ lies in $\partial \triangle upq_{i-1}$ for all $t > i$. Also, since $p_i$ is not in $\partial \triangle upq_{i-1}$, it follows that the arc between $u$ and $p_i$ that contains $q_j$ crosses $\partial \triangle upq_{i-1}$ twice, and a portion of this arc lies in $\partial \triangle upq_{i-1}$. Thus, the arc of $\partial \triangle upq_{i-1}$ between $u$ and $p_t$ that does not contain $p_{t-1}$ lies in $\partial \triangle upq_{i-1}$. Hence, $p_i$ lies in $\partial \triangle upq_{i-1}$. Similarly, $q_r$ is in $\partial \triangle upq_{i-1}$ for all $r > j$. Therefore, since for any order-$k$ triangulation different from $T$ differs by only edges with one endpoint in $S'$ and the other in $S''$, in order to get some triangle $\triangle upq$ of order $k$, we have to flip $k - 1$: the edge $uv$ and $k - 2$ edges of the form $up_t$ and $uq_r$ with $2k - i - j \leq k$, $t < i$ and $r < j$.

![Figure 2.4: (a) An order-$k$ triangulation at distance at least $k - 1$ from any other order-$k$ triangulation ($k=4$). (b) The gray area corresponds to $T^x_{uv}$.](image-url)
Figure 2.5: (a) The \( \bigcirc uxy \) contains 12 points, i.e., \( \triangle \) is not order 11. But the rest of the triangles are of order 8. The gray area is \( \Theta_{ux} \) and the dashed area corresponds to \( tp_{ux} \), which in this case is contained in \( \Theta_{ux} \). (b) Triangulation \( tp_{ux} \cup \{ \triangle uxy \} \) obtained from the one in (a).

Recall that \( |uxy| \) refers to the number of points of \( S \) in the interior of \( \Theta_{ux} \). The following lemma will be important for showing the main theorem of this section.

**Lemma 2.2.2.** Let \( T \) be a triangulation with the following properties:

1. Exactly one triangle \( \triangle uxy \) in \( T \) is not of order \( k > 2 \),
2. \( \Theta_{ux} \) contains at least two points,
3. \( tp_{ux} \) consists only of legal edges, where \( \triangle up_{1}x \) is the triangle in \( T \) adjacent to \( \triangle uxy \) through edge \( ux \). See Fig 2.5a.

Then, there exists a triangulation \( T' \) in \( T_{k}(S) \) which can be reached from \( T \) by flipping at most \( |\Theta_{ux}| - 1 \) illegal edges of \( tp_{ux} \). Moreover, each intermediate triangulation is of order \( |\Theta_{ux}| + k - 1 \).

*Proof.* Consider the triangulation \( tp_{ux} \cup \{ \triangle uxy \} \), which is a sub-triangulation of \( T \). See Fig. 2.5b. Since \( \Theta_{ux} \) contains at least two points in \( tp_{ux} \), it follows that \( ux \) is an illegal edge in \( tp_{ux} \cup \{ \triangle uxy \} \). All other edges in \( tp_{ux} \cup \{ \triangle uxy \} \) are legal since \( tp_{ux} \) consists of legal edges and \( uy, xy \) are convex hull edges by construction.

Let us prove the lemma by induction on \( |\Theta_{ux}| \).

Consider the base case: \( |\Theta_{ux}| = 2 \). Since \( ux \) is an illegal edge, \( p_{1} \in \Theta_{ux} \). Consider the triangulation \( T_{1} = (T \setminus \{ux\}) \cup \{p_{1}y\} \). Since \( ux \) is an illegal edge, by Obs. 2.1.1 (\( \bigcirc p_{1}y \cup \bigcirc p_{1}xy \) \( \subset \) (\( \bigcirc uxy \) \( \cup \) \( \bigcirc up_{1}x \)). Since \( tp_{ux} \) consists of legal edges, \( tp_{ux} \) is a Delaunay triangulation. Thus, \( \bigcirc up_{1}x \) does not contain any
Both new triangles $\triangle up_1y$ and $\triangle p_1xy$ are not of order 8. The sub-triangulation in the gray area contains exactly one triangle, $\triangle up_1y$, that is not of order 8.

Point from $T_{ux}^{p_1}$. Hence, $\bigcirc up_1y$ contains at most one point in $\mathcal{O}_{ux}^{up_1} \subset \mathcal{O}_{ux}$ and at most $k - 1$ points in $\bigcirc up_1y \setminus \mathcal{O}_{ux}^{up_1}$ (the points from $\bigcirc up_1x \setminus \{y\}$). Similarly, for $\bigcirc p_1xy$. Thus, $\triangle up_1y$ and $\triangle p_1xy$ are of order $k$.

Inductive hypothesis: Assume that lemma holds when $|uxy| \leq m - 1$ for $m \geq 3$.

Let $|\mathcal{O}_{ux}^{ux}| = m$. Since $ux$ is an illegal edge, $p_1 \in \mathcal{O}_{ux}^{ux}$. Consider the triangulation $T_1 = (T \setminus \{ux\}) \cup \{p_1y\}$. Then, $(\bigcirc up_1y \cup \bigcirc p_1xy) \subset (\bigcirc uxy \cup \bigcirc up_1x)$. In addition, since $T_{ux}^{p_1}$ consists of legal edges, $\bigcirc up_1x$ does not contain any point from $T_{ux}^{p_1}$. Hence, $\bigcirc up_1y$ contains at most $m - 1$ points in $\mathcal{O}_{ux}^{up_1} \subset \mathcal{O}_{ux}$ (the $m$ points in $\mathcal{O}_{ux}^{ux}$ minus $p_1$) and at most $k - 1$ points in $\bigcirc up_1y \setminus \mathcal{O}_{ux}^{up_1}$ (the points from $\bigcirc up_1x \setminus \{y\}$). A similar argument holds for $\bigcirc p_1xy$. On the other hand, by Obs. 2.1.3 the disks $\bigcirc up_1y$ and $\bigcirc p_1xy$ do not share points in $\mathcal{O}_{ux}^{ux}$. Hence, $|\mathcal{O}_{ux}^{ux}| \leq |\mathcal{O}_{ux}^{ux} - \mathcal{O}_{ux}^{p_1x}| - 1 \leq m - 1$ and $|\mathcal{O}_{ux}^{ux}| \leq |\mathcal{O}_{ux}^{ux} - \mathcal{O}_{ux}^{up_1}| - 1 \leq m - 1$.

Thus, the order of triangles $\triangle up_1y$ and $\triangle p_1xy$ is $|\mathcal{O}_{ux}^{ux}| + k - 2$. If the new triangles $\triangle up_1y$ and $\triangle p_1xy$ are of order $k$, then the lemma follows. If exactly one of $\triangle up_1y$ and $\triangle p_1xy$ is not of order $k$, then by the inductive hypothesis the lemma follows. Assume that both $\triangle up_1y$ and $\triangle p_1xy$ are not of order $k$. See Fig. 2.6.

Since $\bigcirc up_1y \setminus \mathcal{O}_{ux}^{up_1}$ contains at most $k - 1$ points, $|\mathcal{O}_{ux}^{up_1}| \geq 2$. Similarly, $|\mathcal{O}_{ux}^{p_1x}| \geq 2$. Consider $p_2$ and $p_3$ in $S$ such that $\triangle up_2p_1$ is adjacent to $\triangle up_1y$, and $\triangle p_1p_3x$ is adjacent to $\triangle p_1xy$. Then, the points in $\mathcal{O}_{ux}^{up_1}$ are vertices of $T_{up_1}^{p_2}$.
and the points in \( \mathcal{O}_{y}^{p_{1}x} \) are vertices of \( T_{p_{1}x}^{p_{3}} \).

Consider the triangulation \( T_{2} = (T_{1} \setminus T_{p_{1}x}^{p_{3}}) \cup \{p_{1}x\} \), which consists of the triangulation \( T_{1} \) of \( S \) minus the points in \( T_{p_{1}x}^{p_{3}} \setminus \{p_{1}, x\} \). See the triangulation of the gray area in Fig. 2.6. Note that \( \triangle up_{1}y \) is the only triangle that is not of order \( k \) in \( T_{2} \). Indeed, since the only points in \( \mathcal{O}_{p_{1}xy} \setminus \mathcal{O}_{y}^{p_{1}x} \), the triangle \( \triangle p_{1}xy \) is of order \( k - 1 \) in \( T_{2} \). Thus, \( \triangle up_{1}x \) is the only triangle that is not of order \( k \) in \( T_{2} \). Since \( T_{up_{1}}^{p_{2}} \subset T_{ux}^{p_{1}}, T_{up_{1}}^{p_{2}} \) consists of legal edges. Thus, the inductive hypothesis holds for \( T_{2} \). That is, \( T_{2} \) is at distance at most \( |\mathcal{O}_{y}^{up_{1}}| - 1 \) from some triangulation \( T_{2}' \) of order \( k \) such that the only edges flipped are in \( T_{up_{1}}^{p_{2}} \). Moreover, the triangulations between \( T_{2} \) and \( T_{2}' \) are of order \( |\mathcal{O}_{y}^{up_{1}}| + k - 1 \). Consider the triangulation \( T_{3} = (T_{1} \setminus T_{2}) \cup T_{2}' \).

Since \( T_{3} \) consists of edges in \( T_{1} \) and at most \( |\mathcal{O}_{y}^{up_{1}}| - 1 \) flipped edges from \( T_{up_{1}}^{p_{2}} \), the triangulation \( T_{3} \) is at distance at most \( |\mathcal{O}_{y}^{up_{1}}| - 1 \) from \( T_{1} \) by flipping only illegal edges in \( T_{up_{1}}^{p_{2}} \). In addition, the triangles between \( T_{1} \) and \( T_{3} \) are of order \( \max\{\mathcal{O}_{y}^{up_{1}}, \mathcal{O}_{y}^{p_{1}x}\} \) + \( k - 1 \). Note that \( \triangle p_{1}xy \in T_{3} \). By construction, \( \triangle p_{1}xy \) is the only triangle that is not of order \( k \) in \( T_{3} \) since \( T_{2}' \) is of order \( k \) and the only that are not order-\( k \) in \( T_{1} \) were \( \triangle up_{1}y \) and \( \triangle p_{1}xy \). Thus, \( T_{3} \) is of order \( |\mathcal{O}_{y}^{p_{1}x}| + k - 1 \). Finally, since \( T_{p_{3}}^{p_{1}x} \subset T_{ux}^{p_{1}}, T_{p_{1}x}^{p_{3}} \) consists of legal edges. Thus, by inductive hypothesis \( T_{3} \) is at distance at most \( |\mathcal{O}_{y}^{p_{1}x}| - 1 \) from a triangulation \( T_{3}' \) of order \( k \) by flipping only illegal edges in \( T_{p_{1}x}^{p_{3}} \). Also, the triangulations between \( T_{3} \) and \( T_{3}' \) are of order \( |\mathcal{O}_{y}^{p_{1}x}| + k - 1 \). Thus, \( T_{1} \) is at distance at most \( |\mathcal{O}_{y}^{up_{1}}| + |\mathcal{O}_{y}^{p_{1}x}| - 2 \leq |\mathcal{O}_{y}^{ux}| - 2 = m - 2 \) to a \( k \)-\( DT(S) \) by flipping only illegal edges in \( (T_{up_{1}}^{p_{2}} \cup T_{p_{3}}^{p_{1}x}) \subset T_{ux}^{p_{1}} \). Therefore, there exists a triangulation \( T' \) of order \( k \) which resulted after flipping at most \( |\mathcal{O}_{y}^{ux}| - 1 \) illegal edges from \( T_{p_{1}x}^{p_{3}} \) in \( T \). Moreover, the triangulations between \( T \) and \( T' \) are of order \( |\mathcal{O}_{y}^{ux}| + k - 1 \).

Using Lemma 2.2.2 we now proceed to prove the upper bound.

**Theorem 2.2.3.** For a point set \( S \) in convex position and \( k \geq 2 \), let \( T \neq DT(S) \) be a triangulation in \( T_{k}(S) \). Then, there exists \( T' \) in \( T_{k}(S) \) such that there is a path between \( T \) and \( T' \) in \( G(T_{2k-2}(S)) \) of length at most \( k - 1 \), where each edge of the path corresponds to flipping an illegal edge.

**Proof.** Note that for \( k = 2 \), follow trivially since \( G(T_{2}(S)) \) is connected. Assume \( k \geq 3 \). If \( T \) is not minimal, then \( T \) contains an illegal edge \( e \) such that flipping \( e \) results in an order-\( k \) triangulation. Thus, we assume that triangulation \( T \) is minimal. Observe first that there must be an illegal edge \( uv \) in \( T \) incident to triangle \( \triangle uxy \) such that all edges of \( T_{ux}^{xy} \) are legal. Indeed, since \( T \) is not an order-0 triangulation, \( T \) contains an illegal edge. Any triangulation of \( S \) has at least two ears, i.e., triangles with two edges in the convex hull of \( S \), and for any ear in \( T \) all three of its edges are legal, otherwise \( T \) is not minimal.
Let $uv$ be an illegal edge incident to triangle $\triangle uxy$ such that all edges in $T_{uv}$ are legal. Let $\triangle uyv$ be the other triangle in $T$ incident to $uv$. Consider triangulation $T_1 = (T \setminus \{uw\}) \cup \{xy\}$. Since $T$ is minimal, $T_1$ is not order-$k$. The only triangles in $T_1$ that could be not order-$k$ are the new triangles $\triangle uxy$ and $\triangle yuv$. Without loss of generality assume that $\triangle uxy$ is not order-$k$. By Lemma 2.2.2 it follows that $\triangle uxy$ is the only one that is not order-$k$. In addition, $\triangle uxy$ is order-$2(k-2)$. By Obs. 2.1.4 it follows that $2 \leq |\mathcal{O}^{\triangle uxy}_y| \leq k-1$.

By Lemma 2.2.2 it follows that $T_1$ can be transformed into an order-$k$ triangulation $T'$ by flipping at most $k-2$ illegal edges. Moreover, in the sequence of triangulations from $T_1$ to $T'$ every triangulation is of order $2k-2$. The claim follows.

Observe that due to the known property of illegal edges, each flip in the path from $T$ to $T'$ increases the angle vector of the triangulation. In addition, if $T$ is an order-$k$ triangulation, then the edges of $T$ ar of order $k$. Independently Abellanas et al. [2] and Gudmundsson et al. [62] showed that there are $O(kn)$ order-$k$ edges. It follows from Theorem 2.2.3 that $T$ can be transformed into $DT(S)$ by a sequence of at most $O((2k-2)n) = O(kn)$ flips, since all the flipped edges are illegal and of order at most $2k-2$, which implies that no order-$(2k-2)$ edge is flipped twice.

**Corollary 2.2.4.** Let $S$ be a point set in convex position. Any triangulation in $T_k(S)$ can be transformed into $DT(S)$ by a sequence of at most $O(kn)$ triangulations of order at most $2k-2$.

**Proof.** Let $T \in \mathcal{T}_k$. Let us show that there is a sequence of triangulations $T_0 = T, T_1, \ldots, T_m = DT(S)$. By Sibson [96], $T$ can be transformed into $DT(S)$ by a finite number of illegal flips. For $i \geq 0$, if $T_i$ is not minimal then there exists an illegal edge $u_iv_i$ in two triangles $\triangle u_ix_iv_i$ and $\triangle u_iy_iv_i$ such that $T_{i+1} = T_i \setminus \{u_iv_i\} \cup \{x_iy_i\}$ is an order-$k$ triangulation. If $T_i$ is minimal, then flipping any illegal edge in $T_i$ will result into a triangulation that is not order-$k$. It follows from Theorem 2.3.1 that there exists an order-$k$ triangulation $T_j$ such that $T_i$ can be transformed into $T_j$ by flipping at most $k-1$ only illegal edges and the intermediate triangulations are order-$(2k-2)$. So, add the sequence of triangulations between $T_i$ and $T_j$: $T_i, T_i + 1, \ldots, T_j$ where $j < i + k$. Since all of the flipped edges (including the ones of order-$(2k-2)$) are illegal, the sequence of order-$(2k-2)$ triangulations is strictly increasing with respect to the angle vector and $DT(S)$ is the only maximum element, at most all of the order-$(2k-2)$ edges are flipped in order to reach $DT(S)$. By Abellanas et al. [2] and Gudmundsson et al. [62], there are $O((2k-2)n) = O(kn)$ flipped edges.

Theorem 2.2.3 also implies an enumeration algorithm for all order-$k$ triangulations. However, using the preprocessing method given by Silveira and van
Kreveld [97] where all order-$k$ triangles and order-$k$ edges in all order-$k$ triangulation of a point set in general position can be computed in $O(k^2 n \log k + kn \log n)$ expected time, implies a faster algorithm.

**Lemma 2.2.5.** For any point set $S$ in convex position, there exists an algorithm that enumerates all order-$k$ triangulations of $S$ in polynomial time per triangulation.

The lemma follows since we can store all the order-$k$ possible triangles in order-$k$ triangulations given by Silveira and van Kreveld [97] in a hash table. Then, for each order-$k$ triangle we subdivide the problem into three subsets of $S$ and count all possible triangulations of each subset without repeating triangles already visited.

### 2.3 General point sets

In this section we consider a general point set $S$. We show that a triangulation of order $k = 3, 4$ or $5$ of $S$ can be transformed into some other order-$k$ triangulation of $S$ by flipping at most $k - 1$ illegal edges, and that the intermediate triangulations are of order $2k - 2$. Moreover, since we flip only illegal edges, after each flip that transforms a triangulation $T$ to $T'$, we have that $\alpha(T') > \alpha(T)$. Thus, if we keep applying this procedure, we will eventually reach $DT(S)$.

**Theorem 2.3.1.** Let $S$ be a point set in general position and let $T$ be a triangulation in $T_k(S)$ for $2 \leq k \leq 5$. There exists $T'$ in $T_k(S)$ such that there is a path from $T$ to $T'$ in $G(T_{2k-2}(S))$ of length at most $k - 1$, where each edge of the path corresponds to flipping an illegal edge.

In order to prove Theorem 2.3.1 we consider whether $T$ is minimal. If not, the statement follows trivially. If $T$ is minimal, then $k > 2$. Also, for any illegal edge $uv$ in $T$, flipping $uv$ produces a new and unique triangle $\triangle uxy$ that is not order-$k$. Next, we show that the statement holds for such two cases and with

![Figure 2.7](image-url)

Figure 2.7: In both cases, $k = 5$. (a) There are four points in the gray region $\bigcirc_{uxy} \setminus \bigcirc_{ux}$. (b) There are exactly three points in $\bigcirc_{uy}^x$ and exactly three points in $\bigcirc_{ux}^y$. 
that we conclude with the theorem.

**Lemma 2.3.2.** For \( k = 3, 4, 5 \), the following are the only two possible cases:

(a) There are exactly \( k - 1 \) points in \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \) or exactly \( k - 1 \) points in \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \). See Fig 2.7a. (b) There are exactly \( k - 2 \) points in \( \Phi_{y}^{ux} \) and exactly \( k - 2 \) points in \( \Phi_{x}^{uy} \).

**Proof.** By Obs. 2.1.1 \( \bigcirc_{uy} \) has at most \( 2k - 2 \) points of \( S \). First, we consider \( k = 3, 4 \) and show that in order for the \( \bigcirc_{uy} \) to have more than \( k \) points, then only case (a) can happen. Recall that since \( \bigcirc_{uy} \) contains more than \( k \) points, by Obs. 2.1.4 both regions \( \Phi_{y}^{ux} \) and \( \Phi_{x}^{uy} \) have at least \( 2 \) points. In addition, each of \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \subset \bigcirc_{uyv} \) and \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \subset \bigcirc_{uyv} \) can contain at most \( k - 1 \) points.

Consider \( k = 3 \). Since by Obs. 2.1.1 \( \bigcirc_{uy} \) has at most \( 2k - 2 = 2(3) - 2 = 4 \) points, it follows from Obs. 2.1.4 that there are exactly \( 2 \) points in \( \Phi_{y}^{ux} \) and exactly \( 2 \) in \( \Phi_{x}^{uy} \). Thus, both regions \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) and \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \) have exactly \( k - 1 = 3 - 1 = 2 \) points.

Now, consider \( k = 4 \). Since there are at least \( 2 \) points in each of \( \Phi_{y}^{ux} \) and \( \Phi_{x}^{uy} \), and \( \bigcirc_{uy} \) contains at least \( k + 1 = 5 \) points, it follows that at least one of the regions \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) and \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \) contains \( 3 \) (thus \( k - 1 \)) points.

Therefore, we can conclude that for \( k = 3, 4 \) in order for \( \bigcirc_{uy} \) to contain more than \( k \) points, there must be exactly \( k - 1 \) points in \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) or exactly \( k - 1 \) points in \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \).

When \( k = 5 \), there are two cases to consider. The first one is when there are exactly \( k - 1 \) points in \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) or in \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \). This case can happen, since \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) is contained in \( \bigcirc_{uyv} \), which can have up to \( k - 1 \) points. Similarly, \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \) is contained in \( \bigcirc_{uyv} \), which can have up to \( k - 1 \) points different from \( x \).

The only other case for \( k = 5 \) is when there are \( k - 2 = 3 \) points in \( \Phi_{y}^{ux} \) and \( k - 2 = 3 \) points in \( \Phi_{x}^{uy} \). To observe this, note that if there are fewer than \( k - 1 \) points in \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) and \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \), then there are at most \( k - 2 = 3 \) points in both regions \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) or \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \). On the other hand, since \( k = 5 \) and \( \bigcirc_{uy} \) contains at least \( k + 1 = 6 \) points, \( | \bigcirc_{uy} \setminus \Phi_{y}^{ux} | + | \bigcirc_{uy} \setminus \Phi_{x}^{uy} | \geq 6 \). Thus, both \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) and \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \) contain at least \( k - 2 = 3 \) points, otherwise one of \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) and \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \) contains at least \( 4 = k - 1 \) points, falling into the previous case. It remains to show that the 3 points in \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) and \( \bigcirc_{uy} \setminus \Phi_{x}^{uy} \) are actually in \( \Phi_{y}^{ux} \) and \( \Phi_{x}^{uy} \), respectively. Note that if \( \Phi_{y}^{ux} \) does not contain points, then it follows that there are 3 points in both \( \Phi_{y}^{ux} \) and \( \Phi_{x}^{uy} \). For the sake of a contradiction suppose that there are points in \( \Phi_{x}^{uy} \). Since \( \bigcirc_{uy} \setminus \Phi_{y}^{ux} \) contains exactly 3 points and \( \Phi_{x}^{uy} \) contains at
least 2 points, it follows that $\Theta_{u}^{xy}$ contains exactly one point and $\Theta_{x}^{uy}$ contains exactly 2 points. Similarly, since $\bigcirc uxy \setminus \Theta_{x}^{xy}$ contains exactly 3 points and $\Theta_{y}^{xy}$ contains at least 2 points, it follows that $\Theta_{u}^{xy}$ contains exactly 1 point and $\Theta_{x}^{ux}$ contains exactly 2 points. Thus, $\bigcirc uxy$ has $|\Theta_{y}^{ux}| + |\Theta_{x}^{uy}| + |\Theta_{u}^{xy}| = 2 + 1 + 2 = 5$ points, which contradicts the assumption that $\bigcirc uxy$ contains more than $k = 5$ points. 

Therefore, in order to show that with at most $k - 1$ flips the triangulation $T$ can be transformed into some other triangulation $T'$ of order $k$ such that only illegal edges are flipped, for $k = 3, 4, 5$, we consider two cases:

A) $[k = 3, 4, 5]$ There are exactly $k - 1$ points in $\bigcirc uxy \setminus \Theta_{x}^{xy}$ or exactly $k - 1$ points in $\bigcirc uxy \setminus \Theta_{y}^{xy}$. See Fig 2.7a.

B) $[k = 5]$ There are exactly $k - 2 = 3$ points in $\Theta_{u}^{ux}$ and exactly $k - 2 = 3$ points in $\Theta_{y}^{xy}$. See Fig 2.7b.

Case A

For $k = 3, 4, 5$; there are exactly $k - 1$ points in $\bigcirc uxy \setminus \Theta_{x}^{xy}$ or exactly $k - 1$ points in $\bigcirc uxy \setminus \Theta_{y}^{xy}$. Since the case when there are exactly $k - 1$ points in $\bigcirc uxy \setminus \Theta_{y}^{xy}$ is symmetric to the one in $\bigcirc uxy \setminus \Theta_{x}^{xy}$, without loss of generality we assume that there are exactly $k - 1$ points in $\bigcirc uxy \setminus \Theta_{x}^{xy}$.

We show that $T$ can be transformed into a different order-$k$ triangulation $T'$ by a sequence of at most $k - 1$ flips of only illegal edges. Moreover, the intermediate triangulations are of order $2k - 2$.

Lemma 2.3.3. Let $S$ be a point set in general position and let $T$ be a minimal triangulation in $T_{k}(S)$, for $3 \leq k \leq 5$. Let $uv$ be an illegal edge of $T$ with adjacent triangles $\triangle uxv$ and $\triangle uyx$. Consider the triangulation $T_{1} = (T \setminus \{uv\}) \cup \{ux\}$ such that:

1. $\triangle uxy$ is not of order $k$,
2. there are $k - 1$ points in $\bigcirc uxy \setminus \Theta_{y}^{yx}$.

Then, there exists a triangulation $T'$ in $T_{k}(S)$ which can be reach from $T$ by flipping at most $k - 1$ only illegal edges. Moreover, the intermediate triangulations are of order $2k - 2$.

Proof. Recall by Lemma 2.1.2 that $\bigcirc uxy$ contains at most $2k - 2$ points. Thus, $T_{1} \in T_{2k-2}(S)$.

From Lemma 2.1.6 there exists a point $p_{1} \in \Theta_{y}^{ux} \setminus \bigcirc uxy$ such that $\triangle up_{1}x$ is in $T$ and $\bigcirc up_{1}x$ does not contain a point from $\Theta_{y}^{ux}$. Then, the edge $ux$ is an illegal edge. Moreover, since the arc from $u$ to $x$ in $\partial \bigcirc up_{1}x$ that contains $p_{1}$ is contained in $\Theta_{u}^{ux} \subset \bigcirc uxy$, $\bigcirc up_{1}x$ contains all of the points in $\bigcirc uxy \setminus \Theta_{y}^{yx}$.
including \( y \). Thus, \( \bigcirc up_1 x \) contains exactly \( k \) points, which are the points in \( \bigcirc uxy \ \setminus \Theta_y^{ux} \) including \( y \).

Consider the triangulation \( T_2 \) obtained from \( T_1 \) after flipping \( ux \): \( T_2 = (T_1 \setminus \{ux\}) \cup \{p_1 y\} = (T \setminus \{uv,ux\}) \cup \{xy,p_1 y\} \). Since \( ux \) is an illegal edge, by Obs. 2.1.1 \((\Theta_{up_1 y} \cup \Theta_{p_1 xy}) \subset (\Theta_{uxy} \cup \Theta_{up_1 x})\). Hence, \( \Theta_{up_1 y} \) and \( \Theta_{p_1 xy} \) contain at most \( 2k - 3 \) points (the at most \( 2k - 2 \) points in \( \Theta_{uxy} \cup \Theta_{up_1 x} \) minus \( p_1 \)). Thus, \( T_2 \in \mathcal{T}_{2k-3}(S) \subset \mathcal{T}_{2k-2}(S) \). Note that for \( k = 3 \) the lemma already follows, since \( 2k - 3 = 6 - 3 = 3 \). For the remaining of this proof, we assume that \( k = 4, 5 \). If \( T_2 \in \mathcal{T}_k(S) \), then the lemma follows. Assume that \( T_2 \) is not of order \( k \).

Thus, there is one triangle \( \triangle \) (either \( \triangle up_1 y \) or \( \triangle p_1 xy \)) that is not order-\( k \). Since \( p_1 \in \Theta_y^{ux} \), the regions \( \Theta_y^{up_1} \) and \( \Theta_y^{p_1 x} \) are in \( \Theta_y^{ux} \). Therefore, by Obs. 2.1.4 the interior of circumcircle \( \bigcirc \triangle \) of \( \triangle \) contains at least 2 points from \( \Theta_y^{ux} \setminus \{p_1\} \). From Obs. 2.1.3 it follows that \( \Theta_{up_1 y} \) and \( \Theta_{p_1 xy} \) do not share a point from \( \Theta_y^{ux} \setminus \{p_1\} \). Since \( k = 4, 5 \), there are at most \( 3 = 5 - 2 \) points in \( \Theta_y^{ux} \setminus \{p_1\} \). Then, exactly one triangle \( \triangle \) of \( \triangle up_1 y \) and \( \triangle p_1 xy \) is not order-\( k \).

Now, we consider the following two cases depending how many points are contained in \( \bigcirc \triangle \ \setminus \Theta_y^{ux} \): \( k = 1 \) or fewer.

**Case 1** The disk \( \bigcirc \triangle \) contains \( k - 1 \) points in \( \bigcirc \triangle \ \setminus \Theta_y^{ux} \). Recall that the points contained in \( \bigcirc up_1 x \) are the \( k - 1 \) points in \( \bigcirc uxy \ \setminus \Theta_y^{ux} \) and \( y \). Hence, the \( k - 1 \) points in \( \bigcirc \triangle \ \setminus \Theta_y^{ux} \) are the \( k - 1 \) points in \( \bigcirc uxy \ \setminus \Theta_y^{ux} \). Thus, for either \( \triangle = \triangle up_1 y \) or \( \triangle = \triangle p_1 xy \), this is an instance of case \( A \) again, but now with at most \( k - 2 \) points in either of \( \Theta_y^{up_1} \) or \( \Theta_y^{p_1 x} \), respectively.

Assume \( \triangle = \triangle up_1 y \) (the case \( \triangle = \triangle p_2 p_1 y \) is analogous). Using the same arguments as in the beginning of this proof, there exists a point \( p_2 \) in \( \Theta_y^{up_1} \subset \Theta_y^{ux} \) such that \( \triangle up_2 p_1 \in \mathcal{T} \) and \( \bigcirc up_2 p_1 y \) does not contain points in \( \Theta_y^{up_1} \). Then, the edge \( up_1 \) is an illegal edge in \( T_2 \). Consider \( T_3 = (T_2 \setminus \{up_1\}) \cup \{p_2 y\} \). Analogously, as before we show that \( T_3 \) has order \( 2k - 3 - 1 = 2k - 4 \), since \((\bigcirc up_2 y \cup \bigcirc p_2 p_1 y) \subset (\bigcirc up_1 y \cup \bigcirc up_2 p_1) \). Therefore, \( T_3 \in \mathcal{T}_{2k-4}(S) \subset \mathcal{T}_{2k-2}(S) \). Note that for \( k = 4, 5 \), \( \mathcal{T}_{2k-4}(S) = \mathcal{T}_4 \). Thus, for \( k = 4 \) the lemma follows.

Consider \( k = 5 \). If \( T_3 \in \mathcal{T}_5 \), then the lemma follows. Otherwise, consider \( T_3 \notin \mathcal{T}_5 \). Hence, \( T_3 \in \mathcal{T}_6(S) \) since \( 2k - 4 = 6 \). Thus, one triangle \( \hat{\triangle} \) of the new triangles \( \triangle up_2 y \) or \( \triangle p_2 p_1 y \) is of order 6. Since \( p_2 \in \Theta_y^{up_1} \subset \Theta_y^{ux} \setminus \{p_1\} \), we have that \( \Theta_y^{up_2} \subset \Theta_y^{ux} \setminus \{p_1\} \) and \( \Theta_y^{p_2 p_1} \subset \Theta_y^{ux} \setminus \{p_1\} \). Then, by Obs. 2.1.4 \( \bigcirc \hat{\triangle} \) contains the 2 points in \( \Theta_y^{ux} \setminus \{p_1,p_2\} \), namely, in either the regions \( \Theta_y^{up_2} \subset \Theta_y^{ux} \setminus \{p_1,p_2\} \) or \( \Theta_y^{p_2 p_1} \subset \Theta_y^{ux} \setminus \{p_1,p_2\} \). Hence, exactly one of \( \triangle up_2 y \) and \( \triangle p_2 p_1 y \) can have order 6. Also, since \( \bigcirc \hat{\triangle} \) has exactly 2 points in \( \Theta_y^{ux} \), \( \bigcirc \hat{\triangle} \) contains the \( k - 1 = 4 \) points in \( \bigcirc up_1 y \setminus \Theta_y^{up_1} \subset \bigcirc up_2 p_1 \). Therefore, \( \bigcirc \hat{\triangle} \ \setminus \Theta_y^{ux} \) contains \( k - 1 \) points,
Figure 2.8: In both cases: $k = 5$, the gray filled disk is $\triangle uxy$ and both $\triangle uyv$ contain 4 points. (a) Disk $\triangle p1xy$ does not contain the 4 points in $\triangle uxy \setminus \triangle uyv$. (b) The blue disk $\triangle up1y$ contains the 4 points in $\triangle uxy \setminus \triangle uyv$.

which is again an instance of Case A. Analogously, there is an illegal edge $e$ (either $up2$ or $p2p1$) such that the triangulation $T_4 = T_3 \setminus \{e\} \cup \{e'\}$ resulting after flipping $e$ by $e'$ in $T_3$ is of order at most $2k - 4 - 1 = 5$. Therefore, the lemma follows for this case.

Case 2) The disk $\triangle uyv$ contains fewer than $k - 1$ points in $\triangle uyv$. In addition, since $\triangle uyv \subseteq \triangle uxy$ contains at most $k - 1$ points, the region of $\triangle uyv$ contained in $\triangle uyv$ contains at most $k - 2$ points (the points that are in $\triangle uyv$ minus $p1$). Hence, note that this is a special instance of Case B. Thus, this case can only happen for $k = 5$. Moreover, $\triangle uyv$ contains exactly 3 points in $\triangle uyv$ (either the 3 points in $\triangle uyv$ or the 3 points in $\triangle p1y$) and exactly 3 points in $\triangle uyv$ (either the 3 in $\triangle uyv$ or the 3 in $\triangle p1y$). In addition, $\triangle uyv \subseteq \triangle uxy$ must contain $k - 1 = 4$ points in order for either $\triangle uyv$ or $\triangle p1y$ to have exactly 3 points. Since $\triangle uxy \setminus \triangle uyv$ contains $k - 1$ points, these must lie in $\triangle uyv \subseteq \triangle uxy$, since $\triangle uyv \subseteq \triangle uxy \cap \triangle uyv$. But $\triangle uyv \setminus \triangle uyv$ already contains $k$ points: $k - 1$ in $\triangle uyv$ and $x$. Notice that $\triangle up1y$ contains $\triangle uyv$, since the arc from $u$ to $y$ of $\partial \triangle up1y$ that contains $p1$ is in $\triangle uxy$. Thus, $\triangle uyv$ contains exactly $k - 1 = 4$ points. Therefore we have $\triangle up1y \setminus \triangle uxy$. See Fig. 2.8.

Let us show that the edge $p1x$ is an illegal edge. Suppose for the sake of a contradiction that $p1x$ is not illegal. Then, none of the points in $\triangle p1y$ is in a triangle with both $x$ and $p1$. Hence, there exists a point $p'$ in $\triangle p1y$ that is in a triangle $\triangle ap'b \subseteq T$ such that the edge $ab$ is blocking either $p1$ or $x$ from all the points in $\triangle p1y$. Thus, either both endpoints are different from $x$ and $p1$ or either $p1 = a$ or $x = b$ but $p1x \neq ab$. Since $\triangle uyv$ contains 4 points and $\triangle p1y$ contains 3 points, at least one of the endpoints of $ab$ is not in $\triangle uxy \cup \triangle uyv$. If $ab$ is blocking $p1$ from $p'$ then the arc from $a$ to $b$ of $\partial \triangle ap'b$ that contains $p'$ crosses twice $\triangle uyv$ and has a portion in $\triangle p1y \subseteq \triangle uyv$. Thus, $\triangle ap'b$ contains $y, p1, u$ and the 4 points in $\triangle uyv$, contradicting that $\triangle ap'b$ is in $T$. If $ab$ is blocking $x$
from $p'$, then the arc from $a$ to $b$ of $\partial \triangle ap'b$ that contains $p'$ has a portion in $\triangle p_1x \subset \triangle uv$. Since at least one of the endpoints of $ab$ is not in $\triangle uyv$, $\triangle ap'b$ contains $y, x, v$ and the 3 points in $\triangle p_1y$, contradicting that $\triangle ap'b$ is in $T$.

Therefore, there exists a point $p_2$ in $\triangle p_1x$ such that $\triangle p_1p_2x$ is in $T$, and $p_1x$ is illegal. Consider the triangulation $T_3 = T_2 \setminus \{p_1x\} \cup \{p_2y\}$.

Case 2a) Disk $\triangle p_1p_2x$ contains at most 3 points from $\triangle uy$. Since the disks $\triangle p_2xy$ and $\triangle p_1p_2y$ are in the union of $\triangle p_1xy \cup \triangle p_1p_2x$, which contains at most 7 points including $p_2$ and $y$, the order of the triangles $\triangle p_2xy$ and $\triangle p_1p_2y$ is 5. Thus, the lemma follows.

Case 2b) Disk $\triangle p_1p_2x$ contains exactly 4 points from $\triangle uy$. Hence, $T_3$ has order $|\triangle uy| + |\triangle p_1x \setminus \{p_2\}| = 4 + 2 = 6$. Therefore, $T_3 \in \mathcal{T}_6(S) \subset \mathcal{T}_{2k-2}(S)$. If $T_3 \in \mathcal{T}_5(S)$, then the lemma follows. Otherwise, consider the case when $T_3$ is of order 6. There is at most one triangle $\triangle$ from $\triangle p_1p_2y$ and $\triangle p_2yx$ of order 6. Thus the circumcircle of $\triangle$ contains the 4 points in $\triangle uy$, implying that there are 4 points in $\triangle \setminus \triangle p_1x$, which is an instance of Case A again. Using similar arguments as in case 1, the lemma follows.
Case B

We consider the resulting triangulation \( T_1 \) after flipping \( uv \) such that Case B occurs: (1) \( \triangle uxy \) is not order-5, (2) there are exactly three points in both \( \Theta_y^{ux} \) and \( \Theta_x^{uy} \).

We show that there exists an order-5 triangulation \( T' \) that can be reached from \( T \) in the flip graph of \( \mathcal{T}_{2k-2}(S) \) by flipping at most 4 illegal edges.

**Lemma 2.3.4.** Let \( S \) be a point set in general position and let \( T \) be a minimal triangulation in \( \mathcal{T}_5(S) \). Let \( uv \) be an illegal edge of \( T \) with adjacent triangles \( \triangle uxy \) and \( \triangle uyx \). Consider the triangulation \( T_1 = (T \setminus \{uv\}) \cup \{ux\} \) such that:

1. \( \triangle uxy \) is not order-5,
2. there are exactly three points in both \( \Theta_y^{ux} \) and \( \Theta_x^{uy} \).

Then, there exists a triangulation \( T' \) in \( \mathcal{T}_5(S) \) at flip distance at most 3 from \( T \) in \( G(\mathcal{T}_5(S)) \) such that all of the flipped edges are illegal.

**Proof.** Since edge \( uv \) is an illegal edge in \( T \), Since \( \triangle uxy \) is of order 6, it follows from Lemma 2.1.2 that \( \triangle xuv \) is of order 2, since \( 2k - 2 = 8 \). Thus, \( T_1 \in \mathcal{T}_{2k-2}(S) \).

Next we consider two cases, depending on whether one of \( ux \) or \( uy \) is illegal or not.

**Case 1)** One of \( ux \) or \( uy \) is an illegal edge. Assume \( ux \) is an illegal edge (the case when \( uy \) is illegal is symmetric). There exists a point \( p_1 \) in \( \Theta_y^{ux} \) such that \( \triangle up_1x \) is in \( T \). Consider the triangulation \( T_2 = T_1 \setminus \{ux\} \cup \{p_1y\} \). If \( T_2 \in \mathcal{T}_5(S) \) then the lemma follows. Assume that \( T_2 \) is not of order 5.

Note that the disk \( \cap up_1y \) is contained in \( \cap uxy \cup \cap uyv \) since the arc from \( u \) to \( y \) in \( \partial \cap up_1y \) that contains \( p_1 \) is in \( \cap uyv \) and the other arc from \( y \) to \( u \) is contained in \( \cap uxy \). Similarly, \( \cap p_1xy \) is in \( \cap uxy \cup \cap uyv \) since the arc from \( p_1 \) to \( y \) of \( \partial \cap p_1xy \) that contains \( x \) is contained in \( \cap uyv \) and the other arc from \( y \) to \( p_1 \) is contained in \( \cap uyv \) with one portion in \( \cap uxy \). Moreover, since \( p_1 \in \Theta_y^{ux} \), \( \cap up_1x \) can contain an extra point in \( \cap uxy \setminus (\cap uyv \cup \cap uxy) \). Thus, \( T_2 \) is of order 6, since each of \( \cap up_1y \) and \( \cap p_1xy \) can contain at most 6 points (the 5 points in \( \cap uxy \setminus \{p_1\} \) plus the extra point in \( \cap up_1x \)). Since \( |\cap up_1x| + |\cap uxy| = 5 + 6 - 2 = 9 \), by Lemma 2.1.2 it follows that at most one of the triangles \( up_1y \) and \( p_1xy \) is of order 6, say \( \triangle up_1y \). Hence, there are 4 points in \( \cap up_1y \setminus \Theta_y^{up_1} \) which is an instance of Case A. So, analogously as in the proof of Lemma 2.3.3 the lemma follows. Analogously, we prove it when \( \triangle p_1xy \) is of order 6.

**Case 2)** None of the edges \( ux, uy \) is illegal. Since \( ux \) is not illegal, the triangle \( \triangle utx \) of \( T \) incident to \( ux \) must have its third vertex \( t \) outside \( \cap uxy \), so \( t \) cannot be any of the three points in \( \Theta_y^{ux} \). Hence, there are edges blocking...
Let \( a_1 b_1 \) be the blocking edge such that \( \triangle a_1 p_1 b_1 \in T \). See Fig. 2.9. For point \( a_1 \) two cases are possible: \( a_1 = u \), or \( a_1 \) is outside \( \partial \circ uxy \). Similarly, either \( b_1 = x \), or \( b_1 \) is outside \( \partial \circ uxy \). Observe that \( a_1 b_1 \) intersects \( \partial \circ uxy \) twice, and both intersection points are on the closed arc from \( u \) to \( x \) that defines the boundary of \( \circ uxy \). Now, consider the cases whether \( a_1 \) and \( b_1 \) are in \( \circ uxy \).

If both \( a_1 \) and \( b_1 \) were outside \( \partial \circ uxy \), then \( \partial \circ a_1 p_1 b_1 \) would contain at least six points: \( u, x, y \) and the three points that are in \( \circ uxy \), see Fig. 2.9a. This would make \( \partial \circ a_1 p_1 b_1 \) a triangle of \( T \) of order 6, a contradiction. Thus one of \( a_1, b_1 \) must be respectively \( u \) or \( x \), and the other one must be outside \( \partial \circ uxy \). Symmetrically, define triangle \( \triangle a_2 q_1 b_2 \) such that \( q_1 \in \circ uxy \) and \( a_2 b_2 \) is blocking \( u \) or \( y \) from all the points in \( \circ uxy \).

Consider \( a_1 = a_2 = u \). See Fig. 2.9b. Since both \( \partial \circ a_1 p_1 b_1 \) and \( \partial \circ a_2 q_1 b_2 \) contain \( x \) and \( y \), at least one of the following holds: \( b_1 \in \circ a_2 q_1 b_2 \) or \( b_2 \in \circ a_1 p_1 b_1 \). This again contradicts our assumption that \( T \) is of order 5.

Consider \( b_1 = x \) and \( b_2 = y \). See Fig. 2.9b. Since both \( \partial \circ a_1 p_1 b_1 \) and \( \partial \circ a_2 q_1 b_2 \) contain \( u, x \) and \( y \), at least one of the following holds: \( a_1 \in \circ a_2 q_1 b_2 \) or \( a_2 \in \circ a_1 p_1 b_1 \). This also contradicts our assumption that \( T \) is of order 5.

Finally, consider \( a_1 = u \) and \( b_2 = y \) (the remaining case is symmetric, \( a_2 = u \) and \( b_1 = x \)). See Fig. 2.9b. Notice that \( b_1 \) has to lie in the interior of \( \circ uyv \). Otherwise, \( \circ up_1 b_1 \) contains at least 6 points (the three points in \( \circ uy \), and \( \{x, y, v\} \)), which is a contradiction. On the other hand, notice that triangulation \( T \) must contain the triangle \( \triangle uxb_1 \). Otherwise, \( b_1 \) form a triangle with an edge \( ub_3 \), such that \( b_3 \) is outside \( \circ uyv \), since \( \circ uyv \) contains already 5 points (the three points in \( \circ uy \), and \( \{x, b_1\} \)). Hence, \( \triangle uxb_1 b_3 \) is of order 6, since part of the arc of \( \partial \circ ub_1 b_3 \) from \( u \) to \( b_3 \) that contains \( b_1 \) is in \( \circ uxy \) and \( \circ uyv \), the disk \( \circ ub_1 b_3 \) must contain the three points in \( \circ uy \), and \( \{x, y, v\} \), which is a contradiction. Therefore, \( \triangle uxb_1 \) is in \( T \). Moreover, \( \circ uxb_1 \) contains the three points in \( \circ uy \) since the arc from \( u \) to \( x \) of \( \partial \circ uxb_1 \) that does not contain \( b_1 \) is contained in \( \circ uxy \setminus \circ uyv \). Hence, the edge \( ub_1 \) is an illegal edge in \( T_1 \).

Now, consider the triangulation \( T_2 = T_1 \setminus \{ub_1\} \cup \{p_1 x\} \).

Let us show that \( \triangle p_1 b_1 x \) is of order 4. First, notice that if a point is in the intersection \( \circ uxy \cap \circ uyv \), then such point has to be in \( \circ uyx \); otherwise \( \circ uyv \) contains more than 5 points (the three points in \( \circ uy \), \{x, b_1\} plus the extra point in \((\circ uyv \cap \circ uxy) \setminus \circ uyx\)). Notice that the arc of \( \partial \circ p_1 b_1 x \) from \( b_1 \) to \( p_1 \) that contains \( x \) is in \( \circ uyv \). Hence, if \( \partial(\circ p_1 b_1 x) \) intersects twice \( \partial(\circ uxy) \) (otherwise \( \circ p_1 b_1 x \) and \( \circ uxy \) are tangent), \( (\circ p_1 b_1 x \cap \circ uxy) \subset u \) or \( x \) from all of the vertices in \( \circ uyx \). Thus, there exists a point \( p_1 \) in \( \circ uyx \) that is incident to one of the edges that is blocking \( u \) or \( x \) from all the points in \( \circ uxy \).
Higher order Delaunay triangulations and flips

2

\((\circ uxv \cap \circ uyv)\). Thus, \(\circ p_1 b_1 x\) can only contain points in \(\mathcal{O}_{x}^{uv} \setminus \{p_1, x, b_1\}\) and the points in \(\circ ubx_1 \setminus (\circ uxv \cup \{p_1\})\). Since \(\circ ubx_1\) contains the points in \(\mathcal{O}_{x}^{uv} \setminus \{x, b_1\}\), \(\circ ubx_1\) contains at most 4 points (the at most 5 points in \(\circ ubx_1\) minus \(p_1\)).

Now, let us show that triangle \(\triangle up_1 x\) is of order 6. Since \(T\) is minimal, then \(\triangle up_1 x\) is not of order 5. Notice that \(p_1\) lies in \(\mathcal{O}_{y}^{uv} \setminus \circ uxv\), otherwise \(\circ up_1 b_1\) contains more than 5 points: Since the arc from \(u\) to \(b_1\) of \(\partial \circ up_1 b_1\) that contains \(p_1\) crosses twice \(\partial \circ uxv\) and a portion of such arc is in \(\mathcal{O}_{x}^{uv}\), \(\circ up_1 b_1\) contains the 3 points in \(\mathcal{O}_{x}^{uv}\), and \(\{x, y, v\}\), which is a contradiction. Thus, disk \(\circ up_1 x\) must contain the points in \((\circ up_1 b_1 \cap \circ uxv) \setminus \{x\}\) which are the 3 points in \(\mathcal{O}_{x}^{uv}\) and \(y\). In addition, since the arc of \(\partial \circ up_1 x\) from \(u\) to \(x\) containing \(p_1\) is in \(\mathcal{O}_{y}^{uv}\), the disk \(\circ up_1 x\) can also contain the 2 points in \(\mathcal{O}_{x}^{uv} \setminus \{p_1\}\). Hence, \(\triangle up_1 x\) is of order 6. Thus, \(T_2 \in \mathcal{T}_6(S) \subset \mathcal{T}_8(S)\). Since \(p_1\) is in \(\mathcal{O}_{y}^{uv}\), the edge \(ux\) is an illegal edge in \(T_2\).

Consider the triangulation \(T_3 = T_2 \setminus \{ux\} \cup \{p_1 y\}\). Notice that the two new triangles \(\triangle up_1 y\) and \(\triangle p_1 xy\) are of order 5 since the union of \(\circ up_1 x \cup \circ uyx\) only contains the points in \(\circ uxy\). Therefore, \(T_3\) is of order 5. \(\square\)

Finally, Theorem 2.3.1 follows.

**proof of Theorem 2.3.1.** Note that for \(k = 2\), the theorem follows since \(G(\mathcal{T}_2(S))\) is connected. If \(T\) is not minimal, then there exists an illegal edge \(e\) such that the resulting triangulation \(T'\) after flipping \(e\) is of order \(k\). Assume that \(T\) is a minimal triangulation. Hence, by Lemmas 2.3.3 and 2.3.4 there exists an order-\(k\) triangulation \(T'\) that is flip distance at most \(k - 1\) in \(G(\mathcal{T}_{2k-2}(S))\) such that all the flipped edges are illegal. \(\square\)

Finally, since there are \(O(kn)\) order-\(k\) edges (see [2, 62]), it follows from Theorem 2.3.1 that for \(k \leq 5\), any order-\(k\) triangulation can be transformed into \(DT(S)\) by a sequence of at most \(O(kn)\) triangulations of order at most \(2k - 2\).

**Corollary 2.3.5.** Let \(S\) be a point set in general position. For \(k \leq 5\), any triangulation in \(\mathcal{T}_k(S)\) can be transformed into \(DT(S)\) by a sequence of at most \(O(kn)\) triangulations of order \(2k - 2\).

**Proof.** Let \(T \in \mathcal{T}_k\). Let us show that there is a sequence of triangulations \(T_0 = T, T_1, \ldots, T_m = DT(S)\). By Sibson [96], \(T\) can be transformed into \(DT(S)\) by a finite number of illegal flips. For \(i \geq 0\), if \(T_i\) is not minimal then there exists an illegal edge \(u_i v_i\) in two triangles \(\triangle u_i x_i v_i\) and \(\triangle u_i y_i v_i\) such that \(T_{i+1} = T_i \setminus \{u_i v_i\} \cup \{x_i y_i\}\) is an order-\(k\) triangulation. If \(T_i\) is minimal, then flipping any illegal edge in \(T_i\) will result into a triangulation that is not order-\(k\). It follows from Theorem 2.3.1 that there exists an order-\(k\) triangulation \(T_j\) at flip distance
at most \( k - 1 \) in \( G(\mathcal{T}_{2k-2}(S)) \) from \( T_i \) such that all the flipped edges are illegal. So, add the sequence of triangulations between \( T_i \) and \( T_j \): \( T_i, T_i+1, \ldots, T_j \) where \( j < i + k \). Since all of the flipped edges (including the ones of order-(\( 2k - 2 \)) are illegal, the sequence of order-(\( 2k - 2 \)) triangulations is strictly increasing with respect to the angle vector and \( DT(S) \) is the only maximum element, at most all of the order-(\( 2k - 2 \)) edges are flipped in order to reach \( DT(S) \). By Abellanas et al. [2] and Gudmundsson et al. [62], there are \( O((2k - 2)n) = O(kn) \) flipped edges.

Using Theorem 2.3.1 the reverse search framework of Avis and Fukuda [16] and pre-processing the order-\( k \) triangles of a points set \( S \) in general position, we can enumerate all of the triangulations of order-\( k \) in expected polynomial time per triangulation.

**Corollary 2.3.6.** For \( k \leq 5 \), there exists an algorithm that enumerates all \( k \)-order Delaunay triangulations of any point set in general position in expected \( O(k^4n^2) \) time per triangulation.

**Proof.** Let \( S \) be a point set in general position. First, we compute the Delaunay triangulation of \( S \) which takes \( O(n \log n) \) time. Also, we use the pre-processing algorithm that computes all the triangles and edges that are in triangulations of order \( k \) of \( S \) defined by Silveira and van Kreveld [97], which takes \( O(k^2n \log k + kn \log n) \) expected time. We place each triangle and corresponding edges in a hash table. Now, in order to enumerate all of the \( k \)-order Delaunay triangulations of \( S \) for \( k \leq 5 \), we do a reverse search method as the one define by Avis and Fukuda [16] where \( DT(S) \) defines the root element of order-\( k \) triangulations of \( S \). For a triangulation \( T \) in \( \mathcal{T}_k(S) \) in order to find its child \( T' \) in \( \mathcal{T}_k(S) \) and the parent of \( T' \) of a rooted spanning tree of \( \mathcal{T}_k(S) \), we do the following:

For each vertex \( v \) in \( T \), we look at any pair of incident edges \( e_1 = (v, a) \) and \( e_2 = (v, b) \) to \( v \) such that \( ab \notin T \) and there are at most \( k - 1 \) edges in \( T \) crossing \( ab \). Then, in expected \( O(1) \) time, we can check in the hash table if \( \triangle avb \) is of order \( k \). Since we do this for each vertex, in at most \( \sum_{v \in S} \deg_T(v)(\deg_T(v) - 1) = O(n^2) \) we can check which are the possible order-\( k \) triangles that \( T \) can reach after \( O(k - 1) \) edges being flipped, in order to reach another triangulation of order \( k \). By [62] there are \( O(kn) \) edges of order \( k \) that belong to a triangulation of order \( k \). Hence, there are \( O(kn) \) possible triangles of order \( k \) that are not in \( T \) where for each triangle \( \triangle avb \) exactly one of its edges \( ab \) is not in \( T \) and is crossed by at most \( k - 1 \) edges of \( T \). Thus, we have to flip at most \( k - 1 \) edges from \( T \) in order to get a triangulation that contains \( \triangle avb \). In order to know which triangulations of order \( k \) can be obtained after flipping at most \( k - 1 \) edges of \( T \), we do the following. Per each possible triangle \( \triangle avb \) of order \( k \) with
exactly one edge \( ab \) not in \( T \), we find the vertex \( p_1 \) of edge \( vp_1 \) that crosses \( ab \) in \( T \) with the following properties: 1) \( \triangle ap_1 b \) is of order \( k \), and 2) \( \circap_1 b \) does not contain point \( p \) such that \( vp \in T \) and crosses \( ab \). This takes \( O(k) \) expected time.

We do the same for each point \( p \) adjacent to \( v \) and edge \( vp \) crosses either \( ap_1 \) or \( p_1 b \). Since there are at most \( k - 1 \) edges crossing \( ab \), this process takes \( O(k^2) \) expected time. Thus, in at most \( O(k^3n) \) time we can find possible children of \( T \).

Per each possible child \( T' \) of \( T \), we can check in \( O(kn) \) time the triangulation \( T'' \) of order \( k \) with maximum angle vector with \( \alpha(T'') > \alpha(T) \). Therefore, there exists an algorithm that enumerates the elements of \( T_k(S) \) in \( O(k^4n^2) \) expected time per triangulation.

\[ \square \]

2.4 Conclusions

In this chapter we presented the first general results on the flip graph of order-\( k \) Delaunay triangulations. We showed that already for points in convex position, the flip graph may not be connected. This is in contrast to the flip graph of triangulations that consist of order-\( k \) edges, for which the flip graph is always connected [2]. Our main result is that \( k - 1 \) flips are sometimes necessary and always sufficient to transform an order-\( k \) triangulation into some other order-\( k \) triangulation, for any \( k \geq 2 \) if the points are in convex position. Moreover, we proved that these \( k - 1 \) flips go through triangulations of order \( 2k - 2 \).

This is a noteworthy result, and one of the first results on order-\( k \) Delaunay triangulations proven for any value of \( k \). In the setting of general point sets, we also showed that for \( k = 3, 4, 5 \), the order-\( k \) triangulations are at flip distance at most \( k - 1 \) from some other order-\( k \) triangulation within the flip graph of order-(\( 2k - 2 \)) triangulations. This result also implies that the flip distance between any two order-\( k \) triangulations is \( O(kn) \), which is consistent with the fact that the diameter of the flip graph of all triangulations is \( \Theta(n^2) \).

Our results imply an enumeration algorithm using the Avis and Fukuda framework [16], generalizing the results that Abe and Okamoto obtained for \( k \leq 2 \) [1]. For the case of convex position this is not of practical importance, as in that case one can obtain a more efficient method by first pre-computing all order-\( k \) triangles, and then recursively enumerating all order-\( k \) triangulations. However, our results imply the first non-trivial enumeration results for points in generic position for \( 3 \leq k \leq 5 \). It should be mentioned that small values of \( k \) are the most important ones in practice, since for small orders the triangle shape is still close to Delaunay, but at the same time they are enough to obtain significantly better triangulations [91].

Clearly, the main question left open is what happens in general when \( k \geq 6 \). For larger orders our techniques present issues due to a large increase in the
number of cases that need to be considered. However, we conjecture that the same results obtained for convex position hold in general, and in particular, that any order-\(k\) triangulation is at flip distance at most \(k - 1\) from another order-\(k\) triangulation.
Flips in triangulations are an instance of a more general question: Given a class of spanning graphs \( G \) of a point set \( S \), which local operations can transform one element of \( G \) into another? Several operators have been introduced for different classes of spanning graphs that are not triangulations, such as perfect matchings, plane Hamiltonian cycles and plane spanning trees \([6, 9, 36]\). One class of graphs that has received much attention in these type of operators, is the set of plane spanning trees of a point set.

Consider the set of all plane spanning trees \( T \) of \( S \), denoted \( ST(S) \). Avis and Fukuda \([16]\) considered a graph with vertex set \( ST(S) \) where two vertices are connected if the corresponding trees can be transformed into each other by an edge move, i.e., the corresponding two trees differ by exactly one edge. They showed that this graph is connected with diameter \( O(n) \). In other words, Avis and Fukuda showed that any two distinct plane spanning trees can be transformed into each other by performing a linear number of edge moves. Observe that the edge move is a very general operation, since once we remove one edge of a tree, there might be a quadratic number of possible replacements.

A more local operation for transforming two different plane trees is the edge slide, which consists of replacing the edge \( uv \) from \( T \in ST(S) \) by an edge \( vw \) such that \( w \) is adjacent to \( u \) in \( T \). See Figure 3.1. Consider the graph \( G(ST(S)) \) with vertex set \( ST(S) \) and a pair of vertices are joined if the corresponding trees can be transformed into each other by a slide edge operation. Aichholzer et al. \([5]\) showed that \( G(ST(S)) \) is connected. Later, it was shown that the diameter of \( G(ST(S)) \) is \( O(n^2) \) \([8]\).

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1A planar graph \( G \) is called plane when \( V(G) \) is mapped to a set of points in the plane and edges are mapped to segments whose interiors do not intersect. A plane graph refers to the planar embedding of a planar graph.
In order to prove that \( G(\mathcal{ST}(S)) \) is connected, Aichholzer et al. [5] used another operation that transforms one plane spanning tree into a complementary plane spanning tree. This result is interesting on its own, since they showed that any plane spanning tree of \( S \) can be transformed into the Euclidean minimum spanning tree of \( S \) by repeatedly computing Euclidean minimum spanning trees of the constrained Delaunay triangulations. More precisely, the following iterative procedure converges to the Euclidean minimum spanning tree of \( S \). Start with an arbitrary spanning tree \( T_0 \) of \( S \). Compute the Delaunay triangulation of \( S \) taking the edges of \( T_0 \) as constraints. Next, compute the Euclidean minimum spanning tree \( T_1 \) of this constrained triangulation, and repeat. Aichholzer et al. [5] also showed a tight bound of \( \Theta(\log n) \) on the length of this sequence.

A natural question with respect to \( \mathcal{C} \)-distances is whether there exists a sequence that transforms any plane spanning tree of \( S \) into the \( d_{\mathcal{C}} \)-minimum spanning tree of \( S \) such that two consecutive plane trees are contained in the constrained \( \mathcal{C} \)-Delaunay graph.

**Our contributions.** In this chapter we first show that the result by Aichholzer et al. [5] can be generalized to arbitrary \( \mathcal{C} \)-distance functions, i.e., there exists a sequence of trees \( T_0, T_1, \ldots T_k \) such that \( T_i = MST_{\mathcal{C}}(DT_{\mathcal{C}}(T_{i-1})) \) and this procedure converges in \( O(n) \) steps to the \( d_{\mathcal{C}} \)-minimum spanning tree of \( S \), where \( |S| = n \). The main ingredient of our result is, as in [5], a fixed tree theorem (Theorem 3.2.1) that states that once one iteration of the above procedure does not produce a change, then it must have reached the \( d_{\mathcal{C}} \)-minimum spanning tree of \( S \). We note, however, that our proofs are different from those used for the Euclidean metric, since several key lemmas in [5] rely on properties of circles. Later, we show a lower bound of \( \Omega(\log n) \) on the sequence of trees that converges to the \( d_{\square} \)-minimum spanning tree. Finally, for \( \square \)-distance we prove that for a special case of plane spanning trees, the length of this sequence is \( O(\log n) \).
3.1 General observations

In this section we give different results that will be essential for the proof of the fixed tree theorem.

Let $S$ be a point set in the plane and let $C$ be a compact convex set that contains the origin in it interior. Throughout this chapter we will assume that all pairwise $C$-distances between points in $S$ are different. Let $G$ be a planar graph with vertex set $S$ and edges $E(G)$. Let $p$ and $q$ be two points in $S$. We say that a path is a pq-path in $G$ if it is contained in $G$ and it starts from $p$ and ends in $q$.

The following property is a known characterization of the edges in the minimum spanning tree.

**Property 3.1.1.** For any graph $G$, an edge $e \in G$ is not present in $\text{MST}_{C}(G)$ if and only if there is a path in $G$ between the endpoints of $e$ that solely consists of edges shorter than $e$.

Next we show that for each pair of points $p$ and $q$ that are visible to each other there exists a path that connects them in $\text{DG}_{C}(G)$ fully contained in $C_{r}(p, q)$ for all $r \geq d_{C}(p, q)$.

**Lemma 3.1.2.** Let $p, q$ be two visible vertices such that $pq \in G$. Then, for all $r \geq d_{C}(p, q)$ the $C$-disk $C_{r}(p, q)$ contains a pq-path in $\text{DG}_{C}(G)$.

**Proof.** Let $r \geq d_{C}(p, q)$ and let $C$ be a $C_{r}(p, q)$. The proof is by induction on the number of points of $S$ contained in the interior of the disk $C$. If $C$ does not contain points in its interior, then by definition, there exists the edge $pq \in \text{DG}_{C}(S)$.

Now, suppose that the result holds for $k \geq 0$ points contained in $C_{r}(p, q)$. Assume that $C$ contains $k + 1$ vertices of $S$ in its interior.

Note that if none of the points in the interior of $C$ is visible to both of $p$ and $q$, then by definition the edge $pq \in \text{DG}_{C}(G)$. Consider the case when at least one point $v \in S$ contained in $C$ is visible to both points. Since $v$ is in the interior of $C$, there exist $C' = C_{r'}(p, v)$ and $C'' = C_{r''}(v, q)$ that are fully contained in $C$. Hence, $C'$ and $C''$ contain at most $k$ points in their interior. Then, by inductive hypothesis, there exist a $pv$-path $P'$ and a $vq$-path $P''$ that are contained in $C' \subset C$ and $C'' \subset C$, respectively. Therefore, there is a $pq$-path in $P' \cup P''$ that is contained in $C$ (note that $P' \cup P''$ might repeat edges, i.e., it is not necessarily a $pq$-path but it contains one).

Observe that for any edge $uv \in \text{MST}_{C}(S)$ it holds that any $C(u, v)$ does not contain any point of $S$ in its interior, otherwise, from Lemma 3.1.2 there exists an $uv$-path with edges solely shorter than $uv$, contradicting Property 3.1.1.
We say that \( p \) and \( q \) are separated in a \( C \)-disk \( C \) if \( p \) and \( q \) are in \( C \) and there exists a constraint \( c \in E(G) \) crossing \( C \) such that \( c \) divides \( C \) into two different sets such that \( p \) and \( q \) are in different sets, we refer to Figure 3.2. Using Lemma 3.1.2 we can show that if two points are contained in a homothet \( C \) of \( C \) where no edge is separating them (note that, despite this, the points might not be visible to each other), then there exists a path between the two points in \( DG_C(G) \) that is contained in \( C \).

**Lemma 3.1.3.** Let \( C \) be a \( C_r(p,q) \) for some \( r \geq d_C(p,q) \) such that there is no constraint separating \( p \) and \( q \) in \( C \), then there exists a \( pq \)-path of \( DG_C(G) \) contained in \( C \).

**Proof.** Since there is no constraint separating \( p \) and \( q \), there exists a path \( P = p_0, p_1, \ldots, p_m \) in the visibility graph in \( C \), where \( p_0 = p \) and \( p_m = q \). The vertices \( p_i \) and \( p_{i-1} \) can see each other, since the edge \( p_{i-1}p_i \) is in the visibility graph for all \( i \in \{1, \ldots, n\} \). Also, since \( p_{i-1} \) and \( p_i \) are in \( C \), there is an \( r_i \) such that \( C_i = C_{r_i}(p_{i-1}, p_i) \) is contained in \( C \) for each \( i \). Thus by Lemma 3.1.2 there exists a \( p_{i-1}p_i \)-path \( P_i \) in \( C_i \) for all \( i \). Therefore, there is a \( pq \)-path in \( P_1 \cup P_2 \cup \ldots \cup P_m \) that is contained in \( C \) (note that \( P_1 \cup P_2 \cup \ldots \cup P_m \) might repeat edges, i.e., it is not necessarily a \( pq \)-path but it contains one).

A characteristic of circles is that two different circles intersect either in exactly one point or exactly two points where one arc of a circle is enclosed by the other and vice versa. Something similar is true for convex curves.

**Proposition 3.1.4** (Ge Xia [108]). Let \( \partial C \) be the boundary of \( C \), which is a closed convex curve in the plane. The intersection of two distinct homothets of \( \partial C \) is the union of two sets, each of which is either a segment, a single point, or empty.

We know that if two points \( z \) and \( w \) lie in the interior of a \( C(p,q) \), then \( d_C(z,w) \leq d_C(p,q) \) and that there exists a \( C_r(z,w) \) contained in \( C(p,q) \). How-
ever, it is possible that $C(z, w)$ is not fully contained in $C(p, q)$. See Figure 3.3a.

We observe that if the points $z$ and $w$ lie in the line segment $pq$ then there exists a $C(z, w)$ in the interior of $C(p, q)$.

![Figure 3.3](image)

Figure 3.3: (a) The unique $C(z, w)$ is not fully contained in $C(p, q)$. (b) The line segment $zw$ lies in the line segment $pq$. The origin corresponds to the midpoint of $pq$. The gray ellipse is the homothet $C'$ of $C(p, q)$ scaled by $\frac{d(z, w)}{d(p, q)} = \frac{1}{\lambda}$, and the ellipse $C(z, w)$ is the homothet of $C'$ translated by a vector $\vec{b}$.

**Observation 3.1.5.** Let $z$ and $w$ be two points that lie in the line segment $pq$, then for each $C = C(p, q)$ there exists a $C(z, w)$ contained in $C$, and vice versa, for each $C' = C(z, w)$ there exists a $C(p, q)$ that contains $C'$.

**Proof.** Since scaling of $C$ does not depend on where the center of $C$ is, we consider without loss of generality a convex set $C'$, homothet of $C$, such that the origin $\bar{o}$ is in the midpoint of $pq$. Since $d(z, w) = \lambda d(p, q)$ we can scale $C$ by $1/\lambda$ with a linear transformation $\alpha$ and translate the new scale $C'$ by a vector $\vec{b}$, denoted $C' + \vec{b}$, that maps $\bar{o}$ to the midpoint $b$ of $zw$, then $\alpha(p)\alpha(q) + \vec{b} = zw$. See Figure 3.3b. By construction, $C' + \vec{b}$ is a $C_r(z, w)$. In addition, $C' + \vec{b}$ is a $C(z, w)$, otherwise there exists a smaller radius (scaling factor) $r$ such that $C_r(z, w)$ is smaller than $C'$, then scaling it by $\lambda$ we obtained a smaller $C(p, q)$ than $C$, which is a contradiction. Note that if we show that $d_{C'}(\bar{o}, u) \leq d_{C'}(\bar{o}, p)$ for all $u \in (C' + \vec{b})$, then $C' + \vec{b}$, that is a $C(z, w)$, is contained in $C$. Let $u \in (C' + \vec{b})$, then $d_{C'}(\bar{o}, u) \leq d_{C'}(\bar{o}, b) + d_{C'}(b, u) \leq d_{C'}(\bar{o}, b) + d_{C'}(b, z) \leq d_{C}(\bar{o}, b) + d_{C}(b, p) = d_{C'}(\bar{o}, p)$. The second part of the observation is proved analogously.

\[\square\]
Figure 3.4: Edges e and e' are edges crossing uv ∈ MST_C: (a) The edge e is closer to v than e', and e' is closer to u than e. (b) None of C(p, q) and C(p', q') contain u, but both contain v.

Let t = uv be an edge of the MST_C(S) crossed by a set of constraints. Let C_t be a C(u, v). Let e = pq and e' = pq be two edges in DG_C(G) that cross t. We say that e is closer than e' to v if d(e ∩ t, v) ≤ d(e' ∩ t, v). See Figure 3.4a.

**Lemma 3.1.6.** Let e = pq and e' = p'q' be edges of DG_C(T) crossing uv such that u is not contained in any C(p', q') and, e is closer to v than e'. Then, there exists a C(p, q) that contains v. See Figure 3.4b.

**Proof.** Let C_t be a C(u, v). Since uv ∈ MST_C(S), p and q lie outside C_t. Consider the intersection points z', w' of e' ∩ ∂C_t. Let C_{e'} be a C(p', q'). By Observation 3.1.5, there exists a C_{z', w'} = C(z', w') contained in C_{e'}. Since z' and w' lie on ∂C_t, z'w' partitions C_t into two convex subsets of C_t. Thus, by Proposition 3.1.4, one of the two convex subsets defined by z'w' in C_t is totally contained in C_{z', w'}: Since C_{e'} does not contain u, C_{z', w'} does not contain u neither. Hence, C_{z', w'} contains the subset of C_t defined by z'w' that contains v.

Since e crosses t closer to v than e', the intersection points z and w of e ∩ ∂C_t are contained in C_{z', w'}. Hence, d_C(z, w) ≤ d_C(z', w'). Note that if there exists a C(z, w) that contains v in its interior, then, by Observation 3.1.5, there exists a C(p, q) that contains v in its interior. For the sake of a contradiction suppose that none of the C(z, w) contains v. Let C_{zw} be a C(z, w). Since z and w are on ∂C(u, v) and v ∉ C_{zw}, by Proposition 3.1.4, it follows that C_{zw} contains the subset of C_t that contains u. Since e' is closer to u than e, C_{zw} contains z' and w'. Then, d_C(z, w) = d_C(z', w'). Thus, C_{zw} is a C(z', w') that contains u. Therefore, by Observation 3.1.5, it follows that there exists a C(p', q') that
3.2 Fixed tree theorem

Let \( ST(S) \) be the set of all crossing-free spanning trees of \( S \). For each element \( T \in ST(S) \), we define the sequence \( T_0, T_1, T_2, \ldots \) where \( T_0 = T, \ T_i = MST_C(DG_C(T_{i-1})) \) for all \( i > 0 \). In this section we will prove the convergence of this sequence to the canonical element \( MST_C(S) \) of \( ST(S) \). In other words, we show that any spanning tree of \( S \) can be transformed into the \( MST_C(S) \) by a finite sequence of crossing-free spanning trees.

The following theorem shows that the fixed tree in the sequence of \( T_0, T_1 \ldots \) is unique and it is the \( MST_C(S) \).

**Theorem 3.2.1 (Fixed tree theorem).** Let \( T \in ST_S \). \( T = MST_C(DG_C(T)) \) if and only if \( T = MST_C(S) \).

**Proof.** The “if” part is trivial by definition of \( MST_C(S) \). Let us prove the “only if” part. Let \( T = MST_C(DT_C(T)) \), and assume for the sake of a contradiction that \( T \neq MST_C(S) \). Then, there exists an edge \( t = uv \in MST_C(S) \) that does not belong to \( T \). Note that there is at least one edge in \( T \) that crosses \( t \): otherwise, \( t \in DG_C(T) \). Thus, \( t \) must be in \( MST_C(DG_C(T)) \), a contradiction. Hence, there is at least one edge in \( T \) that crosses \( t \).

Let \( C_t \) be a \( C(u, v) \). Each edge in \( DG_C(T) \) crossing \( t \) crosses \( C_t \) and has its endpoints outside \( C_t \). Let \( c = ab \in T \) be the constraint closest to \( v \). We consider whether there exists a \( C(a, b) \) that contains \( v \).

If there exists a \( C_c = C(a, b) \) that contains \( v \), then none of the edges of \( T \) separates \( v \) from \( a \) and \( b \) in \( C_c \). Otherwise, such edge crosses \( t \) and is closer to \( v \) than \( c \), a contradiction. Hence, by Lemma 3.1.3 there is an \( av \)-path, \( P_a \), and a \( vb \)-path, \( P_b \), contained in \( C_c \). Therefore there is an \( ab \)-path in \( P_a \cup P_b \) that solely consists of edges shorter than \( c \), contradicting our hypothesis by Property 3.1.1.

If there does not exist a \( C(a, b) \) that contains \( v \), then by Proposition 3.1.4 any \( C(a, b) \) must contain \( u \). Consider the closest constraint \( c' = a'b' \) to \( u \). Thus, by Lemma 3.1.6 there exists a \( C_{c'} = C(a', b') \) that contains \( u \). Again, there is no edge in \( C_{c'} \) that separates \( u \) from \( a' \) and \( b' \). Hence, by Lemma 3.1.3 there is an \( a'u \)-path, \( P_{a'} \), and an \( ub' \)-path, \( P_{b'} \), contained in \( C_{c'} \). Therefore there is an \( a'b' \)-path in \( P_{a'} \cup P_{b'} \) that solely consists of edges shorter than \( c \), contradicting our hypothesis by Property 3.1.1. Therefore \( T = MST_C(S) \).

Consider again an arbitrary tree \( T \in ST(S) \) and a sequence \( T_0, T_1, \ldots \), such that \( T_0 = T, T_i = MST_C(DG_C(T_{i-1})) \) for all \( i \geq 1 \). Notice that this is a \( d_C \)-length-decreasing sequence, since \( T_i \) has smaller \( d_C \)-weight than \( T_{i-1} \), unless both are identical trees. Also, by definition of \( T_i \), the trees \( T_i \) and \( T_{i-1} \) do not
cross, since they belong to the same plane graph, namely $DG_C(T_{i-1})$, for all $i > 0$. Since the size of $S$ is finite, there are only a finite number of crossing-free spanning trees of $S$. Thus, as a consequence of Theorem 3.2.1 we obtain a $d_C$-length-decreasing sequence of trees in $ST(S)$ which reaches a fixed point $T_k = MST_C(S)$ in a finite number of steps. We will call this sequence the canonical sequence of $T$.

**Theorem 3.2.2.** For any $T \in ST(S)$ there exists a finite sequence $T_0, T_1, \ldots, T_k$ in $ST(S)$, such that $T_0 = T$ and $T_k = MST_C(S)$.

Figure 3.6 shows a canonical sequence that converges in 4 steps where $C$ is an axis-aligned square.

### 3.2.1 Upper bound

The next question to answer is what the length of the canonical sequence is. In this section we give a linear upper bound for the length of such sequence.

First, we show that for any $i \geq 0$, if an edge is in the constrained Delaunay graph of $T_i$ and does not appear in $T_{i+1}$, then such edge cannot appear in any of the following trees of the canonical sequence.

**Lemma 3.2.3.** Let $e = pq$ be an edge in $DG_C(T_i)$ that is not in tree $T_{i+1}$, with $i \geq 0$. Then $d \notin T_{i+j}$ for all $j > 1$.

**Proof.** Recall that $T_{i+1} = MST_C(DG_C(T_i))$. Since $e \notin T_{i+1}$, then by Property 3.1.1 there exist a $pq$-path in $DG_C(T_i)$ that solely consists of edges of...
$d_C$-length smaller than $d_C(p, q)$. In particular, $T_{i+1}$ contains a $pq$-path with this property, such a path is in $DG_C(T_{i+1})$, hence $e \notin MST_C(DG_C(T_{i+1})) = T_{i+2}$. By induction, $e$ can never reappear in the following trees.

Next, we show that the canonical sequence converges in $O(n)$ steps, where $n$ is the size of $S$. The proof is similar to the one in Aichholzer et al. [5] for circles, yet we make use of special properties of arbitrary convex shapes.

**Theorem 3.2.4.** The canonical sequence has length $O(n)$, where $|S| = n$.

**Proof.** Let $k \geq 2$ and $j = k - 2$. Then, there exists an edge $t = uv \in MST_C(S)$ that is not in $T_j$. Thus, by Proposition 3.1.4 there exists a constraint $c = ab$ in $T_j$ such that $C(a, b)$ contains either $u$ or $v$, without loss of generality say $v$. Hence, by Lemma 3.1.6 the edge $c_j = a_jb_j$ closest to $v$ has the property that $C_j = C(a_j, b_j)$ contains $v$. Then, $v$ is not separated by an edge in $T$ from $a_j$ and $b_j$ in $C_j$. Thus, by Lemma 3.1.3 there exists an $a_jb_j$-path that goes through $v$ contained in $C_j$. Hence, there exists a point $p_j$ in $C_j$ that is visible to both $a_j$ and $b_j$ such that $\Delta_j = \Delta a_jp_jb_j \in DG_C(T_j)$. Since $a_jp_jb_j$ is an $a_jb_j$-path that consists of edges solely shorter than $c_j$, the point $p_j$ had to be separated by an edge $c' = a'b'$ of $T_{j-1}$ for every $C(a_j, b_j)$. But $c'$ cannot cross $c_j$, since $c_j \in DG_C(T_{j-1})$. See Figure 3.5. Note that there exists a $C(a', b')$ that contains $p_j$ in its interior: For the sake of a contradiction suppose that $p_j$ is not contained in any $C(a', b')$. We note that $c'$ partitions $C_j$ into two parts, one of which, by Proposition 3.1.4, is contained in any $C(a', b')$, that is the one that does not contain $p_j$. Since, $c'$ separates $p_j$ from either $a_j$ or $b_j$, $c_j$ is contained in any $C(a', b')$. Now, consider the intersecting points $z$ and $w$ in $c' \cap \partial C_j$ (note that at most one of the following can happen: $a_j = z = a'$ or $b_j = w = b'$). Thus, $d_C(z, w) \leq d_C(a_j, b_j)$. Also, by Observation 3.1.5 for each $C = C(a', b')$ there is a $C_{zw} = C(z, w)$ contained in $C$. Then, by Proposition 3.1.4 $C_{zw}$ contains $c_j$. Therefore, $d_C(a_j, b_j) = d_C(z, w)$. Thus, $C_j$ is a $C(z, w)$. Hence, $C(z, w)$ contains $p_j$, a contradiction.

Consider the closest edge $c_{j-1} = a_{j-1}b_{j-1}$ to $p_j$ that separates $p_j$ from $a_j$ or $b_j$ in $C_{j-1}$. Then, there exists a $C(a_{j-1}, b_{j-1})$ that contains $p_j$. Moreover, there is no edge in $T_{j-1}$ that can separate $p_j$ from $a_{j-1}$ and $b_{j-1}$ in $C_{j-1}$, otherwise, $c_j$ wouldn’t be the closest edge to $p_j$ crossing $p ja_j$ and $C_j$ in $T_{j-1}$. Therefore, there exists a $p_{j-1}$ in $C_{j-1}$, such that $\Delta_j = \Delta a_{j-1}p_{j-1}b_{j-1}$ is in $DT_C(T_{j-1})$. Consequently, we obtained a sequence $\Delta_j, \Delta_{j-1}, \ldots, \Delta_0$ of empty triangles.

Observe that for each point $p_i$ the edge $c_j$ blocks the visibility to each $c_{i+1}, \ldots, c_j$. Thus, we can map each index $i$ in $\{0, \ldots, j - 1\}$ to an endpoint of edge $c_{i+1}$ that is different from the one in $c_i$ (they can share at most one endpoint). Since $p_0$ and $c_0$ do not participate, then $j \leq n - 3$. Therefore, $k \leq n - 1$. \qed
3.3 Bounds for square spanning trees

In this section we study the canonical sequence when $C$ is an axis-aligned square.

We denote the $x$- and $y$-coordinate of a point $p$ by $p_x$ and $p_y$, respectively. An important property of the □-distance is that it is the same as the $L_\infty$-metric, i.e., for any pair $p, q \in S$, $d(\square(p, q)) = \max\{|p_x - q_x|, |p_y - q_y|\}$. In particular, any $C(p, q)$ contains $p$ and $q$ on opposite sides of $\partial C(p, q)$.

![Figure 3.6](image-url)

Figure 3.6: Example of a sequence with a spanning tree of a 10-point set that converges to $MST\square(S)$ in 4 steps. The dashed edges represent the appearing edges at stage $i + 1$.

3.3.1 A lower bound

In this section we give a lower bound on the length of the canonical sequence when $C$ is a square.

A natural question, once we know that this sequence converges, is how fast it reaches the $MST\square(S)$. As a first step for answering this question we give a lower bound based on a construction shown in Figure 3.6, similar to the one given in [5], which has length $\Theta(\log n)$.

Consider points $u$ and $b$ such that the edge $t = uv$ is a diagonal of the unique square $Q_t = C(u, v)$. Let $n = 2^m + 2$ for some $m \in \mathbb{N}$. Let $b$ be a point below $Q_t$. Consider points $a_1, a_2, \ldots, a_{n-3}$ above $Q_t$ such that each $a_j$ has the following properties: (1) the line segment $c_j = a_jb$ crosses the top and bottom sides of $Q_t$, (b) $c_j$ is closer to $u$ than $c_s$ for all $j < s$, and (c) if $j$ is odd, then $c_j$ has length $\ell$, otherwise, $c_j$ has length $\frac{\ell}{3^{i-1}}$ where $j \equiv 2i$ modulo $2^{i+1}$.
some $i \in \{2, \ldots, m\}$. Refer to Figure 3.6. Let $S := \{u, v, b, a_1 \ldots, a_{n-3}\}$. By construction, $t \in \text{MST}_\square(S)$.

Now, consider the spanning tree $T$ of $S$ with edges $c_1, \ldots, c_{n-3}, ub$ and $vb$. Then, $t$ is crossed by all the constraints $c_1, \ldots, c_{n-3}$. Notice that at step $i - 1$, the $d\square$-longest constraints of $T_{i-1}$ are the constraints with length $\ell_{3i-1}$ and these constraints are the only ones of $T_{i-1}$ not present at $T_i$ for $1 \leq i < m$. Indeed, let $c_k = ab$ be a constraint crossing $t$ in $T_{i-1}$ where $k \not\equiv 2i \text{ modulo } 2^{i+1}$, then $d\square(a_i, b) \leq \frac{\ell_{3i-1}}{3^{i+1}}$, where $c_i = a_ib$ for all $1 \leq i < k$. Thus, any $a_ib$-path different from edge $a_ib$ would contain an edge with length $\frac{2\ell_{3i-1}}{3^{i+1}}$, refer to Figure 3.7. Also, the constraints $c_j$ where $j \equiv 2i \text{ modulo } 2^{i+1}$ disappear at stage $i - 1$, since there exists a path between $c_j$’s endpoints with edges shorter than $c_j$, refer to Figure 3.7. Hence the tree undergoes $m + 1 = \log_2(n - 2) + 1$ iterations.

Figure 3.7: The dashed edges have $d\square$-length $\frac{2\ell_{3i-1}}{3^{i+1}}$, while each of the $a_4b$-paths different from $a_4b$ passes through one of the endpoints of $c_2$ or $c_6$ that are different from $b$.

Theorem 3.3.1. There exists a point set $S$ and $T \in \mathcal{ST}(S)$, such that its canonical sequence has length $\Theta(\log n)$.

3.3.2 An upper bound

In this section we show a special case when the upper bound of $O(\log n)$ on the length of the canonical sequence when $\mathcal{C}$ is a square.

Note that if we show that an arbitrary edge of $\text{MST}_\square(S)$ appears in $O(\log n)$ steps, then the sequence of $d\square$-spanning trees has length $O(\log n)$. In this section we will show that an arbitrary edge $uv$ of the $\text{MST}_\square(S)$ that is only crossed by constraints that cross opposite sides of a square $\partial\mathcal{C}(u, v)$, then $uv$ appears in a tree of the canonical sequence in at most $O(\log n)$ steps.

Let $t = uv$ be an edge of the $\text{MST}_\square(S)$ crossed by a set of constraints. Let $Q_t$ be a $\mathcal{C}(u, v)$. Without loss of generality we assume that $d\square(u, v) = |u_x - v_x|$ and that $u$ and $v$ lie on the right and left side of $Q_t$, respectively. Let $e$ be an edge crossing $t$. We say that $e$ is a diagonal when $e$ crosses $t$ in consecutive sides.
of \( \partial Q_t \), or call it \textit{vertical} and \textit{horizontal} if \( e \) crosses opposite sides that are top- and bottom-sides or left- and right-sides of \( \partial Q_t \), respectively. In this section, we show that if an edge \( t \in MST_{\square}(S) \) is only crossed by vertical constraints, then \( t \) appears in \( O(\log n) \) steps. The case when the constraints are horizontal is symmetric.

![Figure 3.8: Left: quadrants at \( p \). Right: both \( q \) and \( q' \) are in the same quadrant at \( p \). Point \( q' \) is in \( \Box(p,q) \).](image)

Recall that the quadrants at a point \( p \) of \( S \) are given by the four quadrants defined by the lines \( x = p_x \) and \( y = q_y \). See Figure 3.8(left). Consider two edges \( pq \) and \( pq' \) such that \( d_{\square}(p,q') < d_{\square}(p,q) \). Observe that if \( q \) and \( q' \) are in the same quadrant at \( p \), then \( q' \) lies in a \( \square(p,q) \): since the \( Q = \Box(p,q) \) with vertex \( p \) lies in the same quadrant at \( p \) as \( q \) and \( q' \). Thus, since \( d_{\square}(p,q') < d_{\square}(p,q) \), it follows that \( q' \) lies in \( Q \). Therefore, \( d_{\square}(q,q') < d_{\square}(p,q) \). See Fig. 3.8(right).

**Observation 3.3.2.** Consider edges \( pq \) and \( pq' \) incident to \( p \). If \( q \) and \( q' \) are in the same quadrant at \( p \), then \( d_{\square}(q,q') < \max\{d_{\square}(p,q),d_{\square}(p,p')\} \).

Consider a vertical edge \( e = pq \) and the intersection points \( z, w \) in \( e \cap \partial Q_t \). We have that \( d_{\square}(z,w) = |z_y - w_y| = d_{\square}(u,v) \). Thus, the rectangle \( R_{zw} \) with \( z \) and \( w \) as opposite vertices has height greater or equal than its width. The rectangle \( R_{zw} \) is a homothet of the rectangle with opposite vertices \( p \) and \( q \). Therefore, \( d_{\square}(p,q) = |p_y - q_y| \).

**Observation 3.3.3.** If \( pq \) is a vertical edge in \( DG_{\square}(T) \), then \( d_{\square}(p,q) = |p_y - q_y| \).

Moreover, there exists a \( C(z,w) \) that contains both \( u \) and \( v \). Hence, by Observation 3.1.5 there exists a \( C(p,q) \) that contains both \( u \) and \( v \). Thus, the following observation holds.

**Observation 3.3.4.** If \( e = pq \) is a vertical edge crossing \( t = uv \in MST_{\square}(S) \), then \( d_{\square}(p,v) < d_{\square}(p,q) \) and \( d_{\square}(v,q) < d_{\square}(p,q) \).

Next, we observe that if two edges \( e \) and \( e' \) are vertical edges crossing \( t \), then
Figure 3.9: (a) Point $z$ lies in the supporting line of $pq$ and $z_y = p'_y$. (b) Rectangle $R$ has vertices $z$ and $p'$ on its top-side and $q$ lies on its boundary.

the $d_{\square}$ distance of the two endpoints below $Q_t$ is shorter than the $d_{\square}$-length of the largest of $e$ and $e'$.

**Observation 3.3.5.** Let $e = pq$ and $e' = p'q'$ be two vertical edges crossing $Q_t$ with the next properties: (a) $p_y > q_y$ and $p'_y > q'_y$, and, (b) $d_{\square}(p, q) < d_{\square}(p', q')$. Then, the following hold:

1. If $p \neq p'$, then $d_{\square}(p, p') < d_{\square}(p', q')$.
2. If $q \neq q'$, then $d_{\square}(q, q') < d_{\square}(p', q')$.

**Proof.** Since $d_{\square}(p, q) < d_{\square}(p', q')$, at least one of $p \neq p'$ and $q \neq q'$ happens. Since both cases are symmetric, without loss of generality we assume that $p \neq p'$. Note that the edges $p'q$ and $pq'$ are also vertical edges, since they are either edges (if $q = q'$) or diagonals of the polygon $pqq'p'$. Also, note that one of $p'q$ or $pq'$ has smaller $d_{\square}$-length than $p'q'$, since either $q_y > q'_y$ or $p_y < p'_y$, otherwise $p'q'$ has smaller $d_{\square}$-length than $pq$, which is a contradiction. Consider the edge $e''$ from \{p'q, pq'\} that has smaller $d_{\square}$-length than $p'q'$. Without loss of generality assume that $e'' = p'q$ and $p_y < p'_y$. Observe that if $p$ and $p'$ belong to the same quadrant at $q$, then by Observation 3.3.2 the statement holds. Assume that $p$ and $p'$ are in different quadrants at $q$. Consider the point $z$ on the supporting line of $e$ with $z_y = z'_y$. Let $R$ rectangle be the rectangle that contains $q$ on its boundary and the points $z$ and $p'$ are vertices of $R$. See Figure 3.9. Rectangle $R$ has height $|p'_y - q_y| = |z'_y - q_y|$ and width $|p'_x - z_x| \geq |p'_x - p_x|$. Consider the intersection points $w$ of the top side of $Q_t$ and $pq$, and the intersection point $w'$ of top side of $Q_t$ and $p'q$. Consider the rectangle $R'$ that has $w$ and $w'$ as vertices and contains
$q$ on its bottom side. See Fig. 3.9b. Then, $R'$ has width $|w_x - w'_x| \leq d_{\Box}(u, v)$ and height greater than $d_{\Box}(u, v)$. Thus, $|q_y - w'_x| < |w_x - w'_x|$. The rectangle $R'$ is a homothet of $R$. Hence, the height of $R$ is greater than its width. Therefore, $d_{\Box}(p, p') < |p'_y - q_y| = d_{\Box}(p', q) \leq d_{\Box}(p', q')$.\hfill\qed

Let $C_i := \{c^i_1, c^i_2, \ldots, c^i_m\}$ be the set of vertical constraints in $T_i$ crossing $t$ sorted in the following fashion: $d(c^i_j \cap t, u) < d(c^i_{j+1} \cap t, u)$ for all $1 \leq j < m$.

Observe that $c^i_1$ and $c^i_m$ are the closest edges to $u$ and $v$, respectively. In addition, $m \leq n - 3$, since the edges in $T_i$ with endpoints $u$ and $v$ do not cross $uv$ and $T_i$ has $n - 1$ edges. Let $c^i_j \in C_i$, we denote $c^i_j = a_jb_j$ where $a_{jy} > b_{jy}$ for all $j$, i.e., $a_j$ is above $b_j$. Let $s_t, s_b, s_r, s_l$ denote the top-side, bottom-side, right-side and left-side of $\partial Q_t$, respectively.

Consider two consecutive constraints $c^i_j$ and $c^i_{j+1}$. Let $z$ be the intersection point $c^i_j \cap s_t$ and $z'$ be the intersection point $c^i_{j+1} \cap s_t$. Let $P$ be the polygon $a_jzz'a_{j+1}$. Since the line segments $a_jz, zz'$ and $z'a_{j+1}$ are contained in $c^i_j, s_t$ and $c^i_{j+1}$, respectively, the constraints crossing $P$ are only crossing the segment $a_ja_{j+1}$, we refer to Figure 3.10b. Consider the convex hull $CH_{a_j}$ of all the points of $S$ contained in $P$. Notice that if $a_j \neq a_{j+1}$, then $a_ja_{j+1}$ is an edge of $CH_{a_j}$. If $CH_{a_j}$ is different than a point (i.e., $CH_{a_j} \neq a_j$ when $a_j = a_{j+1}$), then we denote by $ch_{a_j}$ the longest $a_ja_{j+1}$-path in $CH_{a_j}$ (note that $CH_{a_j}$ can be an edge $a_ja_{j+1}$). Similarly, we define a chain between $b_j$ and $b_{j+1}$ denoted $ch_{b_j}$. See Figure 3.10b.

Consider the intersection points $c^i_j \cap t$ and $c^i_{j+1} \cap t$, we say that an edge $e$ is between $c^i_j$ and $c^i_{j+1}$ if $e$ crosses $t$ in the line segment $(c^i_j \cap t)(c^i_{j+1} \cap t)$. See Figure 3.10b.

The following lemma shows that for every two consecutive constraints $c^i_j$ and $c^i_{j+1}$ constraints in $C_i$, then at most one vertical constraint appears in $T_{i+1}$ between $c^i_j$ and $c^i_{j+1}$ (possibly one of $c^i_j$ and $c^i_{j+1}$).

**Lemma 3.3.6.** Consider two consecutive vertical constraints $c^i_j$ and $c^i_{j+1}$ crossing $t$ and consider the chains $ch_{a_j}$ and $ch_{b_j}$. Let $a$ be the bottom-most point of $ch_{a_j}$ and $b$ the top-most point of $ch_{b_j}$. If there is a constraint $c \in C_{i+1}$ between $c^i_j$ and $c^i_{j+1}$, then such a constraint is the only constraint of $C_{i+1}$ that appears between $c^i_j$ and $c^i_{j+1}$. Moreover, $c = ab$.

**Proof.** By Observation 3.3.3 the length of any vertical constraint is given by the $y$-coordinates of its endpoints, then $ab$ is the shortest possible edge between $c^i_j$ and $c^i_{j+1}$. Let us show that the edge $c = ab$ crossing $t$ is the only possible constraint of $T_{i+1}$ between $c^i_j$ and $c^i_{j+1}$. For the sake of a contradiction suppose that there exists a constraint $c' = a'b' \in T_{i+1}$ with $d_{\Box}(a', b') > d_{\Box}(a, b)$ that is between $c^i_j$ and $c^i_{j+1}$. Consider squares $Q_a = C(a, a')$ and $Q_b = C(b, b')$. Note
that the bottom side of \( \partial Q_a \) lies at most at the height of \( a_y \) and the top side of \( \partial Q_b \) lies at least at the height of \( b_y \). In addition, by Observation 3.3.5 the square \( Q_a \) has side-length less than \( C(a', b') \). Similarly, \( Q_b \) has side-length less than \( C(a', b') \). Since \( a \) defines the bottom-most point of the chain \( ch_{a_j} \), there is no edge in \( T_i \) that separates \( a \) from \( a' \) in \( Q_a \). Similarly, since \( b \) is the top-most point in the chain \( ch_{b_j} \), there does not exist a constraint in \( T_i \) that separates \( b \) from \( b' \) in \( Q_b \). Thus, by Lemma 3.3.3 there exists an \( a'a \)-path \( P_a \) and a \( bb' \)-path \( P_b \) in \( DG_{\Diamond}(T_i) \) contained in \( Q_a \) and \( Q_b \), respectively. Therefore, there is an \( a'b' \)-path in \( P_a \cup \{ab\} \cup P_b \) that solely consists of edges shorter than \( a'b' \), contradicting Property 3.1.1. 

In the following lemma we show that if there appears a constraint \( c \) of \( C_{i-1} \) between two consecutive constraints \( c_i^j \) and \( c_i^{j+1} \) in \( C_i \) such that \( c \) is neither \( c_i^j \) or \( c_i^{j+1} \), or adjacent to both \( c_i^j \) and \( c_i^{j+1} \), then there must have been two constraints in \( C_{i-1} \) between \( c_i^j \) and \( c_i^{j+1} \). See Figure 3.11.

**Lemma 3.3.7.** Let \( i \geq 1 \). If a constraint \( c = ab \) in \( C_{i+1} \) is between \( c_i^j = a_jb_j \) and \( c_{i+1}^j = a_{j+1}b_{j+1} \), and at least one of \( a \) and \( b \) is not in \( \{a_j, a_{j+1}, b_j, b_{j+1}\} \), then there are at least two constraints in \( C_{i-1} \) that are between \( c_i^j \) and \( c_{i+1}^j \).

**Proof.** Assume without loss of generality that \( a \) is not in \( \{a_j, a_{j+1}, b_j, b_{j+1}\} \).
Figure 3.11: There are two blue vertical constraints in $T_{i-1}$ that are between two consecutive red constraints $c^i_j = a_jb_j$ and $c^i_{j+1} = a_{j+1}b_{j+1}$ in $T_i$. The constraint $c = ab$ is in $C_{i+1}$ and $a \notin \{a_j, a_{j+1}\}$.

Then, there must be an edge $c' = a'b'$ blocking $a$ from $b$, otherwise $ab$ is visible in $T_{i-1}$. Thus by Property 3.1.1 there exists an $ab$-path $P$ that solely consists of edges shorter than $ab$ in $T_i$. Hence, $P$ is in $DG\square(T_i)$, by Property 3.1.1 we get a contradiction.

In addition, $c'$ has to be in $C_{i-1}$, otherwise $c'$ has an endpoint with either $y$-coordinate less than $a_y$ or greater than $b_y$, contradicting Lemma 3.3.6. See Figure 3.12a.

Observe that there exists a constraint $c''$ in $T_{i-1}$ that blocks the visibility from $a$ to either $a_j$ and $b_j$, and a different constraint $c'''$ in $T_{i-1}$ that blocks the visibility from $a$ to either $a_{j+1}$ and $b_{j+1}$: suppose for the sake of a contradiction that $a$ is visible to either $b_j$ or $a_j$. Consider the consecutive constraints $c^j_i$ and $c^{i-1}_{j'}$ in $C_{i-1}$ such that $c^i_j$ appears in between $c^{i-1}_j$ and $c^{i-1}_{j'+1}$. We consider whether $a_j$ is visible to $a$. If $a_j$ is visible to $a$, then $a_j = a_{j'+1}$, otherwise there exists a point $p$ in $ch_{a_{j'}}$ with $p_y < a_{j'}y$, which contradicts Lemma 3.3.6. Then, $a$ must be blocked from $b_j$ by a constraint in $C_{i-1}$, otherwise there is a square $C(a, b_j)$ with side-length less than $d\square(a_j, b_j)$ such that $a$ and $b_j$ are not separated by a constraint in $T_{i-1}$. Hence, by Lemma 3.1.3 there exists an $a_ja$-path $P_{a_j}$.
and an \(ab_j\)-path \(P_{b_j}\) in \(DG□(T_{i-1})\) such that \(P_{a_j} \cup P_{b_j}\) solely consists of shorter edges than \(c^i_j\), contradicting Property 3.1.1. Analogously we prove that there exists another constraint blocking \(a_l\) from \(a_{j+1}\) or \(b_{j+1}\). Note that one of the constraints \(c''\) and \(c'''\) can be the constraint \(c'\) that is blocking the visibility from \(a_l\) to \(b_l\).

![Forbidden Configurations](image)

Figure 3.12: Forbidden configurations: (a) Edge \(c'\) is in \(T_{i-1}\) and \(c \in C_{i+1}\). Edge \(c'\) blocks the visibility from \(a\) to \(b = b_{j'}\) and one of the endpoints of \(c'\) has lower \(y\)-coordinate than \(a\). (b) The consecutive constraints \(c^i_j\) and \(c^i_{j+1}\) are not incident but both edges are incident to \(c^{i-1}_{j'}\). Constraint \(c^{i-1}_{j'}\) is the only constraint of \(C_{i-1}\) between \(c^i_j\) and \(c^i_{j+1}\).

We show that if there is exactly one constraint \(c^{i-1}_{j'}\) in \(C_{i-1}\) between two consecutive constraints \(c^i_j\) and \(c^i_{j+1}\) in \(C_i\) and \(c^{i-1}_{j'}\) is incident to both \(c^i_j\) and \(c^i_{j+1}\), then \(c^i_j\) and \(c^i_{j+1}\) are incident as well.

**Proposition 3.3.8.** Let \(c^i_j\) and \(c^i_{j+1}\) consecutive constraints in \(C_i\) such that exactly one edge \(c^{i-1}_{j'}\) of \(C_{i-1}\) is in between \(c^i_j\) and \(c^i_{j+1}\). If \(c^i_j\) and \(c^i_{j+1}\) are both incident to \(c^{i-1}_{j'}\), then \(c^i_j\) and \(c^i_{j+1}\) share a vertex.

**Proof.** Without loss of generality assume that \(d□(a_j, b_j) < d□(a_{j+1}, b_{j+1})\). Assume for the sake of a contradiction that \(a_{j'} = a_j\) and \(b_{j'} = b_j\). See Figure 3.12b. Since there is exactly one edge of \(C_{i-1}\) between \(c^i_j\) and \(c^i_{j+1}\), the edge \(c^i_j\) appears between \(c^{i-1}_{j'}\) and \(c^{i-1}_{j'1}\), and the edge \(c^i_{j+1}\) appears between \(c^{i-1}_{j'}\) and \(c^{i-1}_{j'1}\). On the other hand, by Observation 3.3.5 \(d□(a_j, a_{j+1}) < d□(a_{j+1}, b_{j+1})\). Consider \(Q_a = C(a_j, a_{j+1})\) and \(Q_b = C(b_j, b_{j+1})\). By Lemma 3.3.7 \(a_{j+1}\) has the lowest \(y\)-coordinate in \(ch_{a_{j'}}\) and \(b_j\) has the largest \(y\)-coordinate in \(ch_{b_{j'-1}}\). Hence, there is no edge separating \(a_j\) from \(a_{j+1}\) in \(Q_a\) and there is no edge separating
Let $b_j$ from $b_{j+1}$ in $Q_b$. Then, by Lemma 3.1.3, there exists an $a_ja_{j+1}$-path $P_a$ of $DG_{\Box}(T_{i-1})$ in $Q_a$ and a $b_jb_{j+1}$-path $P_b$ of $DG_{\Box}(T_{i-1})$ in $Q_b$. Therefore, there is an $a_j+1b_{j+1}$-path in $P_a \cup a_jb_j \cup P_b$ that solely consists of edges shorter than $c_{j+1}$, which contradicts Property 3.1.1. □

Note that if for each pair of consecutive constraints $c_{j}^{i+1}$ and $c_{j+1}^{i+1}$ in $C_{i+1}$ there are at least two constraints in $C_i$ between $c_{j}^{i+1}$ and $c_{j+1}^{i+1}$, then we obtain at most $\frac{|C_i|}{2}$ constraints in $C_{i+1}$ that appear between consecutive pairs of constraints in $C_i$. Hence, $|C_{i+1}| \leq \frac{|C_i|}{2}$.

**Observation 3.3.9.** If for each pair of consecutive constraints in $C_{i+1}$ there are at least two constraints in $C_i$ between them, then $|C_{i+1}| \leq \frac{|C_i|}{2}$.

Also note that there cannot appear any constraint $c = ab$ in $T_{i+1}$ crossing $t$ such that $c$ crosses $t$ in either the line segment $u(c_1^{i} \cap t)$ or $(c_1^{i} \cap t)v$ where $|C_i| = m$. Otherwise, without loss of generality we assume $c$ crosses the line segment $u(c_1^{i} \cap t)$. Then, by similar arguments to those in Theorem 3.2.1, there exist an $au$-path $P_a$ and an $ub$-path $P_b$ such that $P_a \cup P_b$ contains a $ab$-path that solely consists of shorter edges than $c$, which contradicts Property 3.1.1. Therefore, we only need to focus on the edges that appear between pairs of consecutive constraints in $C_i$.

**Proposition 3.3.10.** If for each pair of consecutive constraints in $C_i$ there appears exactly one edge in $C_{i+1}$ between them such that $c_1^{i+1} \cup c_2^{i+1} \cup \ldots \cup c_m^{i+1}$ defines a path $P$ in $T_{i+1}$ with $|P| \geq 2$. Then, $|C_{i+2}| \leq \frac{|C_i|}{2}$.

**Proof.** Since $C_{i+1}$ consists of edges in a path $P$ and exactly one edge appears per pair of consecutive constraints in $C_i$, it follows that $|C_{i+1}| = |C_i| - 1$. Since for each consecutive pair of constraints $c_{j}^{i+1}$ and $c_{j+1}^{i+1}$ there was exactly one constraint in $C_i$ between $c_{j}^{i+1}$ and $c_{j+1}^{i+1}$, then if there appears one edge of $C_{i+2}$ in between them, from Lemma 3.3.7 it follows that such edge has to have both endpoints in $\{a_j, b_j, a_{j+1}, b_{j+1}\}$. Since $c_{j}^{i+1}$ and $c_{j+1}^{i+1}$ are incident, then the new edge has to be exactly one of $c_{j}^{i+1}$ and $c_{j+1}^{i+1}$. Thus, $|C_{i+2}| \leq \frac{|C_i|}{2}$. □

We say that a set of consecutive pairs of constraints in $C_i$ is of *type 1* if it satisfies the hypothesis from Observation 3.3.9. We say that a set of consecutive pairs of constraints in $C_i$ is of *type 2* if it satisfies the hypothesis from Proposition 3.3.10.

**Theorem 3.3.11.** $|C_{i+3}| \leq \frac{2}{3}|C_i|$.

**Proof.** Let $k_1$ be the number of consecutive pairs of constraints in $C_i$ in a set of type 1. Let $k_2$ be the number of consecutive pairs of constraints in $C_i$ in a
set of type 2. Let $k_3$ the number of remaining consecutive pairs of constraints in $C_i$. Notice that $|C_i| = k_3 + k_1 + k_2 + 1$.

Thus, by Obs 3.3.9 and Proposition 3.3.10 $|C_{i+2}| \leq k_3 + \frac{k_1 + k_2}{2}$.

Now, let us count the remaining constraints. So, we get $k_3$ different constraints of $C_{i+1}$ in each of the remaining consecutive pairs of constraints in $C_i$ that are not in sets of type 1 and 2. Let $c_{j+1}^{i+1}$ and $c_{j+1}^{i+1}$ be a pair of these types of pairs. Since there was exactly one edge from $C_i$ in between $c_{j+1}^{i+1}$ and $c_{j+1}^{i+1}$, by Lemma 3.3.7 if there is one constraint $c$ from $C_{i+2}$ in between $c_{j+1}^{i+1}$ and $c_{j+1}^{i+1}$, then either one of these happens: (1) $c$ is one of $c_{j+1}^{i+1}$ and $c_{j+1}^{i+1}$ or (2) $c$ shares a vertex with both $c_{j+1}^{i+1}$ and $c_{j+1}^{i+1}$. If a pair of consecutive constraints in $C_{i+2}$ are disjoint such that both are in case (1), we will call these pairs type 3-1, and if one of the pair of consecutive constraints is of the case (1) and the other is of case (2), then we will call these pairs type 3-2. See Figure 3.13 for a complete example.

Hence, by Proposition 3.3.8 we get a set of paths and disjoint constraints in $C_{i+2}$. So, we have a new set of type 1 and 2 consecutive pairs of constraints in $C_{i+2}$ and a set of consecutive pairs of constraints that are disjoint of type 3-1 and 3-2 in $C_{i+2}$. Let $k_3''$ denote the number of consecutive constraints of type 1 and 2 in $C_{i+1}$ and let $k_3'_{-1}$ and $k_3'_{-2}$ the number of pairs of consecutive constraints in $C_{i+2}$ of type 3 – 1 and 3 – 2, respectively. Note that for each pair of type 3 – 1 there is a partition of a subset of pairs in $C_{i+1}$ where 4 consecutive pairs of constraints contain exactly one pair of consecutive constraints from $C_{i+2}$ (the pair of type 3-1). Refer to Figure 3.13. Also, for each pair of type 3-2 there is a partition of the pairs of consecutive constraints of $C_{i+1}$ where 3 consecutive pairs contain exactly one pair of consecutive constraints from $C_{i+2}$ (the pair of type 3-2). See Figure 3.13. Thus, we obtained that $k_3 = 2k_3'_{-1} + \frac{3}{2}k_3'_{-2} + k_3''$.

Therefore, by Proposition 3.3.10 and Observation 3.3.9 this forms at most $k_3'_{-1} + k_3'_{-2} + \frac{1}{2}k_3'' \leq \frac{2}{3}k_3$ new constraints in $C_{i+3}$ from the $k_3$ remaining pairs to count. This gives us,

$$|C_{i+3}| \leq \frac{2}{3}(k_3) + \frac{1}{2}(k_1 + k_2) \leq \frac{2|C_i|}{3} - 1$$

As a consequence of Theorem 3.3.11 we obtain that there are a constant number of edges crossing $t$ after $i = O(\log n)$ steps. From Lemma 3.3.6 we have that $|C_{i-1}| \leq |C_i| - 1$. Thus, we obtain the following corollary.

**Corollary 3.3.12.** Let $|S| = n$. Let $T \in ST(S)$ such that $T$ crosses the edges from $MST(S)$ by only vertical constraints. Then, the length of the canonical sequence of $T$ is $O(\log n)$. 

Figure 3.13: The gray edges represent the constraints in $C_{i+1}$ that do not appear in $C_{i+2}$ and the black edges are constraints in $C_{i+1} \cap C_{i+2}$. The red edges represent the constraints in $C_{i+1}$ that share vertices with two different consecutive constraints from $C_{i+1}$. Here, we make a partition of pairs of two consecutive constraints in $C_{i+1}$ by pair of consecutive constraints of type 1, 2, 3-1 and 3-2. Note that the number of edges of type 2 drops by half in the next iteration, $C_{i+3}$. 
3.4 Conclusions

For an arbitrary convex $C$, we have extended the convergence of the canonical sequence of crossing-free spanning trees for the Euclidean metric [5] to the $C$-distance. We have also given a linear upper bound for the length of the canonical sequence. For the particular case when the convex is a square, we give a $\Omega(\log n)$ and an upper bound of $O(\log n)$ where for each edge of the $d_{\square}$-minimum spanning tree, the constraints in the initial tree only cross opposite sides of at least one of the smallest squares containing such edge.

In order to show this for any tree in the $d_{\square}$-distance we considered both cases, when the constraints cross opposite sides of a smallest square containing an edge from $\text{MST}_{\square}(S)$ or the constraints cross consecutive sides of such square. When dealing with constraints crossing consecutive sides, most of the ingredients for the proof of Theorem 3.3.11 can be generalized with the exception of Lemma 3.3.7. So, the main open question is how to close the gap between $O(n)$ and $\Omega(\log n)$ for $d_{\square}$-distances. We believe that the right answer should be $O(\log n)$. However, it seems that a different approach is needed. Perhaps something using the constrained $\text{RNG}_{\square}$, similar to the upper bound of $O(\log n)$ for Euclidean distance given in [5], yet it is not clear how to use their approach since they use a lot of properties from circles that do not hold for squares.

An even more general open problem is whether there exists a convex shape where the length of the canonical sequence is actually linear.
Shamos [93] conjectured that the Delaunay triangulation contains a Hamiltonian cycle. This conjecture sparked a flurry of research activity. Although Dillencourt [48] disproved this conjecture, he showed that Delaunay triangulations are almost Hamiltonian [49], that is, they are 1-tough. Focus then shifted on determining how much to loosen the definition of the Delaunay triangulation to achieve Hamiltonicity. Chang et al. [39] showed that 19-RNG is Hamiltonian. Abellanas et al. [2] proved that 15-GG is Hamiltonian. Currently, the lowest known upper bound is by Kaiser et al. [67] who showed that 10-GG is Hamiltonian. All of these results are obtained by studying properties of bottleneck Hamiltonian cycles. Given a planar point set, a bottleneck Hamiltonian cycle is a Hamiltonian cycle whose maximum edge length is minimum among all Hamiltonian cycles of the point set. Biniaz et al. [22] showed that there exist point sets such that its 7-GG does not contain a bottleneck Hamiltonian cycle, implying that this approach cannot yield an upper bound lower than 8. Despite this, it is conjectured that 1-DG is Hamiltonian [2].

As for Hamiltonicity in convex shape Delaunay graphs, not much is known. Bonichon et al. [21] proved that every plane triangulation is Delaunay-realizable where homothets of a triangle act as the empty convex shape. This implies that there exist triangle-DG graphs that do not contain Hamiltonian paths or cycles. Biniaz et al. [21] showed that 7-DGΔ contains a bottleneck Hamiltonian cycle and that there exist point sets where 5-DGΔ does not contain a bottleneck Hamiltonian cycle. Ábrego et al. [3] showed that the DG□ admits a Hamiltonian path, while Saumell [92] showed that the DG□ is not necessarily 1-tough, and

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1 Recall that a graph $G$ is 1-tough if removing any $k$ vertices from $G$ results in $\leq k$ connected components.

2 According to the definition of $k$-RNG in [39], they showed Hamiltonicity for 20-RNG.
Table 4.1: Bounds on the minimum k for which \( k\text{-DG}_C(S) \) is Hamiltonian and for which \( k\text{-GG}_C(S) \) contains a \( d_C \)–bottleneck Hamiltonian cycle.

Therefore does not necessarily contain a Hamiltonian cycle.

Our contributions. We generalize the above results by replacing the disk with an arbitrary convex shape \( C \). We show that the \( k\)-Gabriel graph, and hence also the \( k\)-Delaunay graph, is Hamiltonian for any convex shape \( C \) when \( k \geq 24 \). Furthermore, we give improved bounds for point-symmetric shapes, as well as for even-sided regular polygons. Table 4.1 summarizes the bounds obtained. Finally, we provide some lower bounds on the existence of a Hamiltonian cycle for an infinite family of regular polygons, and bottleneck Hamiltonian cycles for the particular cases of hexagons and squares. Together with the results of Bose et al. [33], our results are the first results on graph-theoretic properties of generalized Delaunay graphs that apply to arbitrary convex shapes.

Our results rely on the use of normed metrics and packing lemmas. In fact, in contrast to previous work on Hamiltonicity for generalized Delaunay graphs, our results are the first to use properties of normed metrics to obtain simple proofs for various convex shape Delaunay graphs.

4.1 Hamiltonicity for general convex shapes

In this section we show that the 24-order \( C\)-Gabriel graph is Hamiltonian for any point set \( S \) in general position.

In this chapter we let \( S \) be a set of points in the plane satisfying the following general position assumption: For each pair \( p, q \in S \), any minimum homothet of \( C \) having \( p \) and \( q \) on its boundary does not contain any other point of \( S \) on its boundary.

Let \( \mathcal{H} \) be the set of all Hamiltonian cycles of the point set \( S \). Define the \( d_C \)-length sequence of \( h \in \mathcal{H} \), denoted \( d_C(h) \), as a sequence of edges of \( h \)
sorted in decreasing order with respect to the $d_\hat{C}$-metric. Sort the elements of $\mathcal{H}$ in lexicographic order with respect to their $d_\hat{C}$-length sequence, breaking ties arbitrarily. This order is strict. For $h_1, h_2 \in \mathcal{H}$, if $h_1$ is smaller than $h_2$ in this order, we write $h_1 \prec h_2$.

Let $h$ be the minimum element in $\mathcal{H}$, often called bottleneck Hamiltonian cycle. The approach we follow to prove our bounds, which is similar to the approach in [2, 39, 67], is to show that $h$ is contained in $k$-GG$_C(S)$ for a small value of $k$. The strategy for proving that $h$ is contained in 24-GG$_C(S)$ is to show that for every edge $ab \in h$ there are at most 24 points in the interior of any $C(a, b)$. In order to do this, we associate each point in the interior of an arbitrary fixed $C(a, b)$ to another point. Later, we show that the $d_\hat{C}$-distances between such associated points and $a$ is at least $d_\hat{C}(a, b)$. Finally, we use a packing argument to show that there are at most 24 associated points, which leads to a maximum of 24 points contained in $C(a, b)$.

Let $ab \in h$; we assume without loss of generality that $d_\hat{C}(a, b) = 1$. Let $U = \{u_1, u_2, \ldots, u_k\}$ be the set of points in $S$ different from $a$ and $b$ that are in the interior of an arbitrary fixed $C(a, b)$\footnote{Since $S$ is in general position, only $a$ and $b$ can lie on the boundary of $C(a, b)$.} When traversing $h$ from $b$ to $a$, we visit the points of $U$ in the order $u_1, \ldots, u_k$. For each point $u_i$, define $s_i$ to be the point preceding $u_i$ in $h$. See Figure 4.1a.
Figure 4.2: $C(a, b)$ has radius 1, $D_C(\bar{o}, 1)$ is the $\hat{C}$-disk with radius 1 centered at $\bar{o}$ and $D_C(\bar{o}, 2)$ is the $\hat{C}$-disk with radius 2 centered at $\bar{o}$. The points $s'_i$ and $s'_j$ are projections of $s_i$ and $s_j$ on $\partial D_C(\bar{o}, 2)$, respectively. The dashed $\hat{C}$-disk is centered at $s'_j$ and has radius 1.

Proof. If $s_1 = b$, then $d_C(a, s_1) = 1$ and $d_C(s_1, u_1) < 1$. Otherwise, define $h' = (h \setminus \{ab, s_iu_i\}) \cup \{as_i, u_ib\}$. For sake of a contradiction suppose that $d_C(a, s_i) < \max\{d_C(s_i, u_i), 1\}$. It holds that $d_C(a, s_i) < \max\{d_C(s_i, u_i), d_C(a, b)\}$ since $d_C(a, b) = 1$. Also, $d_C(u_i, b) < 1$ since $u_i \in C(a, b)$. Thus, $\max\{d_C(a, s_i), d_C(u_i, b)\} < \max\{d_C(s_i, u_i), d_C(a, b)\}$. Therefore $h' < h$, which contradicts the definition of $h$. \hfill $\square$

Claim [4.1.1] implies that, for each $i \in \{1, \ldots, k\}$, $s_i$ is not in the interior of $C(a, b)$.

Claim 4.1.2. Let $1 \leq i < j \leq k$. Then $d_C(s_i, s_j) \geq \max\{d_C(s_i, u_i), d_C(s_j, u_j), 1\}$.

Proof. For sake of a contradiction suppose that $d_C(s_i, s_j) < \max\{d_C(s_i, u_i), d_C(s_j, u_j), 1\}$. Consider the Hamiltonian cycle $h' = h \setminus \{(a, b), (s_i, u_i), (s_j, u_j)\} \cup \{(s_i, s_j), (u_i, a), (u_j, b)\}$. As in Claim 4.1.1 we have that $d_C(u_i, a) < 1$ and $d_C(u_j, b) < 1$. So, $\max\{d_C(s_i, s_j), d_C(u_i, a), d_C(u_j, b)\} < \max\{d_C(s_i, u_i), d_C(s_j, u_j), d_C(a, b)\}$. Therefore, $h' < h$ which contradicts the minimality of $h$. \hfill $\square$

The $d_C$-distance from a point $v$ to a region $C$ is given by the minimum $d_C$-distance from $v$ to any point $u$ in $C$. Notice that if $v \notin C$, then the $d_C$-distance from $v$ to $C$ is defined by a point on $\partial C$. This can be seen by taking an $\varepsilon \in \mathbb{R}^+$.
small enough such that $D_C(v, \varepsilon)$ does not intersect $C$, and making $\varepsilon$ grow until $D_C(v, \varepsilon)$ hits $C$.

**Observation 4.1.3.** Let $u \notin D_C(\bar{o}, r)$ for some $r \in \mathbb{R}^+$ and let $p$ be the intersection point of $\partial D_C(\bar{o}, r)$ and line segment $\bar{o}u$. Then, the $d_C$-distance from $u$ to $D_C(\bar{o}, r)$ is $d_C(u, p)$.

**Proof.** Since $p$ is in $\partial D_C(\bar{o}, r)$ and $u \notin D_C(\bar{o}, r)$, the vector $u = \lambda p$ for some $\lambda > 1 \in \mathbb{R}$. In addition, the $d_C$-distance from $u$ to $D_C(\bar{o}, r)$ is at least $d_C(u, p)$. For the sake of a contradiction suppose that the $d_C$-distance from $u$ to $D_C(\bar{o}, r)$ is less than $d_C(u, p)$. Thus, there exists a point $v \in \partial D_C(\bar{o}, r)$ such that $d_C(u, v) < d_C(u, p)$, and $r \lambda = d_C(\bar{o}, u) \leq d_C(\bar{o}, v) + d_C(v, u) < d_C(\bar{o}, v) + d_C(p, u) = r + r \lambda - r = r \lambda$, which is a contradiction. \qed

Without loss of generality assume that $a$ is the origin $\bar{o}$. Since for any point $u$ in $C(a, b)$, $d_C(\bar{o}, u) = d_C(a, u) \leq 1$, we have that $D_C(\bar{o}, 1)$ contains $C(a, b)$. Also, from Claim 4.1.1, we have that $s_i$ is not in the interior of $D_C(\bar{o}, 1)$ for all $i \in \{1, \ldots, k\}$.

Let $D_C(\bar{o}, 2)$ be the $\hat{C}$-disk centered at $\bar{o} = a$ with radius 2. For each $s_i \notin D_C(\bar{o}, 2)$, define $s'_i$ as the intersection of $\partial D_C(\bar{o}, 2)$ with the ray $\bar{o}s_i$. Let $s'_i = s_i$ when $s_i$ is inside $D_C(\bar{o}, 2)$. See Figure 4.2.

**Observation 4.1.4.** If $s_j \notin D_C(\bar{o}, 2)$ (with $1 \leq j \leq k$), the $d_C$-distance from $s'_j$ to $D_C(\bar{o}, 1)$ is 1.

**Proof.** Since $s_j \notin D_C(\bar{o}, 2)$, $s'_j$ is on the boundary of $D_C(\bar{o}, 2)$ and $d_C(\bar{o}, s'_j) = 2$. Let $p$ be the intersection point of $\partial D_C(\bar{o}, 1)$ and $\bar{o}s_j$. Then $d_C(\bar{o}, p) = 1$. By Observation 4.1.3 the $d_C$-distance from $s'_j$ to $D_C(\bar{o}, 1)$ is $d_C(s'_j, p) = d_C(p, s'_j) = d_C(\bar{o}, s'_j) - d_C(\bar{o}, p) = 2 - 1 = 1$. \qed

The following claim is needed to prove our key lemma. Intuitively, this claim shows that if there is a point-symmetric $\hat{C}$-disk $\hat{C}$ of radius $r$ centered at a point $u$ such that $r \leq d_C(u, \bar{o})$, then $C$ is contained in any $\hat{C}$-disk with $\partial C \cap \bar{o}u$ on its boundary such that its center $u'$ lies on the ray $\bar{o}u$ and is farther to $\bar{o}$ than $u$. For an example, see Figure 4.1.b.

**Claim 4.1.5.** Let $C$ be a point-symmetric convex shape. Let $u$ be a point in the plane different from the origin $\bar{o}$. Let $r < d_C(u, \bar{o})$. Let $p$ be the intersection point of $\partial D_C(u, r)$ and line segment $\bar{o}u$. Let $u' = \lambda u$, with $\lambda > 1 \in \mathbb{R}$, be a point defined by vector $u$ scaled by a factor of $\lambda$. Then $D_C(u, r) \subset D_C(u', d_C(u', p))$. (See Figure 4.1.b.)

**Proof.** Let $q \in D_C(u, r)$; then $d_C(u, q) \leq d_C(u, p)$. Since $u$ is on the line segment $u'p$, we have that $d_C(u', p) = d_C(u', u) + d_C(u, p)$. Hence $d_C(u', q) \leq d_C(u', u) + d_C(u, p)$. Therefore, $d_C(u, r) \subset D_C(u', d_C(u', p))$. \qed
\[ d_C(u, q) \leq d_C(u', u) + d_C(u, p) = d_C(u', p). \] Therefore, \( D_C(u, r) \) is contained in \( D_C(u', d_C(u', p)) \).

Using the previous claims we can prove a key lemma stating that for every pair of points \( s_i' \) and \( s_j' \), we have that \( d_C(s_i', s_j') \geq 1 \). From this lemma we can conclude that any pair of \( \hat{C} \)-disks with radius \( \frac{1}{2} \) centered at \( s_i' \) and \( s_j' \) are internally disjoint, which allows us to bound \( |U| \) via a packing argument.

**Lemma 4.1.6.** For any pair \( s_i \) and \( s_j \) with \( i \neq j \), we have that \( d_C(s_i', s_j') \geq 1 \).

**Proof.** If both \( s_i \) and \( s_j \) are in \( D_{\hat{C}}(\bar{o}, 2) \), then from Claim 4.1.2 we have that \( d_C(s_i', s_j') = d_C(s_i, s_j) \geq 1 \). Otherwise, we assume, without loss of generality, that \( d_C(\bar{o}, s_i) \geq d_C(\bar{o}, s_j) \). Then, \( s_j \notin D_{\hat{C}}(\bar{o}, 2) \). Since \( s_j' \) is on the line segment \( \bar{o}s_j \), we have \( s_j = \lambda s_j' \) for some \( \lambda > 1 \). Let \( p \) be the intersection point of \( \partial D_{\hat{C}}(\bar{o}, 1) \) and \( \bar{o}s_j \). Since \( \hat{C} \) defines a norm, we have \( d_C(\lambda s_j', \bar{o}) = \lambda d_C(s_j', \bar{o}) \). By Observation 4.1.4 we have that \( d_C(s_j, p) = d_C(s_j, \bar{o}) - d_C(p, \bar{o}) = \lambda d_C(s_j', \bar{o}) - 1 = 2\lambda - 1 \). From Observation 4.1.3 it follows that the \( d_C \)-distance from \( s_j \) to \( D_{\hat{C}}(\bar{o}, 1) \) is equal to \( d_C(s_j, p) \). Further, \( d_C(s_j, s_j') = d_C(s_j, \bar{o}) - d_C(s_j', \bar{o}) = 2\lambda - 2 \). Let us prove that \( d_C(s_j, \bar{o}) - d_C(s_j', \bar{o}) \leq 1 \). For sake of a contradiction suppose that \( d_C(s_j, \bar{o}) - d_C(s_j', \bar{o}) < 1 \). Let \( D_{s_j'} = D_{\hat{C}}(s_j, 1) \). By Observation 4.1.4 \( d_C(s_j', p) = 1 \). Therefore, \( p \) is on \( \partial D_{s_j'} \). Now, we consider two cases:

**Case 1)** \( s_i \in D_{\hat{C}}(\bar{o}, 2) \). Then \( d_C(\bar{o}, s_i) \leq 2 \). Since \( d_C(s_j', s_j) < 1 \), we have \( s_i \in D_{s_j'} \). From Claim 4.1.5 it follows that \( D_{s_j'} \) is contained in \( D_{\hat{C}}(s_j, d_C(s_j, p)) \). Thus, \( s_i \in D_{\hat{C}}(s_j, d_C(s_j, p)) \) and \( d_C(s_j, s_j') = d_C(s_j, s_i) \leq d_C(s_j, p) \). Since \( S \) is in general position, \( u_j \) is in the interior of \( D_{\hat{C}}(\bar{o}, 1) \). Hence, \( d_C(s_j, s_i) \leq d_C(s_j, p) < d_C(s_j, u_j) \), which contradicts Claim 4.1.2.

**Case 2)** \( s_i \notin D_{\hat{C}}(\bar{o}, 2) \). Then \( d_C(\bar{o}, s_i) > 2 \). Thus, \( s_i = \delta s_i' \) for some \( \delta > 1 \in \mathbb{R} \). Moreover, since \( d_C(\bar{o}, s_j) \geq d_C(\bar{o}, s_i) \) and \( s_i' \), \( s_j' \) are on \( \partial D_{\hat{C}}(\bar{o}, 2) \), \( \delta \leq \lambda \). Hence, \( s_i \) is on the line segment \( s_i'(\lambda s_i') \). Let \( D_{s_j} = D_{\hat{C}}(s_j, 2\lambda - 1) \). Note that \( \lambda < 2\lambda - 1 \) because \( \lambda > 1 \). Since \( \hat{C} \) defines a norm, \( d_C(s_j, \lambda s_j') = d_C(\lambda s_j', s_j') = \lambda d_C(s_j', s_j') < \lambda < 2\lambda - 1 \). Hence, \( \lambda s_i' \in D_{s_j} \). In addition, since \( d_C(s_j, p) = 2\lambda - 1 \), from Claim 4.1.5 it follows that \( D_{s_j'} \subseteq D_{s_j} \). Therefore, \( s_i' \in D_{s_j} \). Thus, the line segment \( s_i'(\lambda s_i') \) is contained in \( D_{s_j} \). Hence, \( s_i \in D_{s_j} \). Then, \( d_C(s_j, s_i) \leq 2\lambda - 1 = d_{\hat{C}}(s_j, p) < d_{\hat{C}}(s_j, u_j) \) which contradicts Claim 4.1.2.

**Theorem 4.1.7.** For any set \( S \) of points in general position and convex shape \( \mathcal{C} \), the graph \( 24-GG_{\mathcal{C}}(S) \) is Hamiltonian.

**Proof.** For each \( s_i \) we define the \( \hat{C} \)-disk \( D_i = D_{\hat{C}}(s_i', \frac{1}{2}) \). We also set \( D_0 := D_{\hat{C}}(\bar{o}, \frac{1}{2}) \) (recall that we can assume without loss of generality that \( a = \bar{o} \)). By Lemma 4.1.6, each pair of \( \hat{C} \)-disks \( D_i \) and \( D_j \) \((0 < i < j \leq k)\) are internally disjoint. Note that, if \( s_i' \) is on \( \partial D_{\hat{C}}(\bar{o}, 2) \), then \( D_0 \) and \( D_i \) are internally disjoint.
Figure 4.3: The $\hat{C}$-disks $D_0, D_i, D_j$ and $D_t$ of radius $\frac{1}{2}$ are centered at $a, s'_i, s'_j$ and $s'_t$, respectively. Such $\hat{C}$-disks are contained in the $\hat{C}$-disk $D_{\hat{C}}(\bar{o}, \frac{5}{2})$.

On the other hand, if $s'_i$ is in the interior of $D_{\hat{C}}(\bar{o}, 2)$, then by definition $s'_i = s_i$. Thus, by Claim 4.1.1 $D_0$ is internally disjoint from $D_i$. See Figure 4.3. Since $s'_i \in D_{\hat{C}}(\bar{o}, 2)$ for all $i$, each disk $D_i$ is inside $D_{\hat{C}}(\bar{o}, \frac{5}{2})$. In $D_{\hat{C}}(\bar{o}, \frac{5}{2})$, there can be at most $\frac{\text{Area}(D_{\hat{C}}(\bar{o}, \frac{5}{2}))}{\text{Area}(D_0)} = (\frac{\frac{5}{2}}{\frac{1}{2}})^2 \frac{\text{Area}(\hat{C})}{\text{Area}(\hat{C})} = 25$ internally disjoint disks of type $D_i$. Thus, since $D_0$ is centered at $a$, there are at most 24 points $s'_i$ in $D_{\hat{C}}(\bar{o}, 1)$. As a consequence, there are at most 24 points of $S$ in the interior of $C(a, b)$, and the bottleneck Hamiltonian cycle of $S$ is contained in $24-GG_{C}(S)$.

4.2 Hamiltonicity for point-symmetric convex shapes

In this section we improve Theorem 4.1.7 for the case where $C$ is convex and point-symmetric. We use similar arguments to those in Section 4.1.

Consider $h$ defined as before, i.e., $h$ is the minimum Hamiltonian cycle in $\mathcal{H}$. Let $ab$ be an edge in $h$ and consider an arbitrary fixed $C(a, b)$. In this section it will be more convenient to assume without loss of generality that $d_C(a, b) = 2$ and that $C(a, b)$ is centered at the origin $\bar{o}$. Thus, $C(a, b) = D_C(\bar{o}, 1)$, see Figure 4.4. Consider again the set $U = \{u_1, \ldots, u_k\}$ defined as in Section 4.1 and let $s_i$ be the predecessor of $u_i$ in $h$.

Using that $d_C(a, b) = 2d_{\hat{C}}(a, b)$ when $C$ is point-symmetric, we can prove the
following claims.

**Claim 4.2.1.** \( d_C(s_i, a) \geq \max\{d_C(s_i, u_i), 2\} \).

**Proof.** By Claim 4.1.1 we have that \( d_C(s_i, a) = 2d_C(s_i, a) \geq 2\max\{d_C(s_i, u_i), 1\} = \max\{2d_C(s_i, u_i), 2\} = \max\{d_C(s_i, u_i), 2\} \). \( \square \)

**Claim 4.2.2.** Let \( 1 \leq i < j \leq k \), then \( d_C(s_i, s_j) \geq \max\{d_C(s_i, u_i), d_C(s_j, u_j), 2\} \).

**Proof.** By Claim 4.1.2 we have that \( d_C(s_i, s_j) = 2d_C(s_i, s_j) \geq 2\max\{d_C(s_i, u_i), d_C(s_j, u_j), 1\} = \max\{2d_C(s_i, u_i), 2d_C(s_j, u_j), 2\} = \max\{d_C(s_i, u_i), d_C(s_j, u_j), 2\} \). \( \square \)

![Figure 4.4: \( C(a, b) \) has radius 1 and it is centered at \( \bar{o} \). The point \( s_j \) is not in \( D_C(\bar{o}, 3) \), so \( s'_j \) is the intersection point of \( \bar{o}s_j \cap \partial D_C(\bar{o}, 3) \). The dotted \( C \)-disk is centered at \( s'_j \) and has radius 2.](image)

From Claim 4.2.1 we have that \( s_i \) is not in the interior of \( D_C(\bar{o}, 1) = C(a, b) \) for all \( i \in \{1, \ldots, k\} \). Let \( D_C(\bar{o}, 3) \) be the \( C \)-disk centered at \( \bar{o} \) with radius 3. For each \( s_i \notin D_C(\bar{o}, 3) \), define \( s'_i \) as the intersection of \( \partial D_C(\bar{o}, 3) \) with the ray \( \overrightarrow{\bar{o}s_i} \). We let \( s'_i = s_i \) when \( s_i \) is inside \( D_C(\bar{o}, 3) \). See Figure 4.4.

The following lemma is similar to Lemma 4.1.6. We show that every pair \( s'_i \) and \( s'_j \) are at \( d_C \)-distance at least 2. This lemma allows us again to reduce our problem to a packing problem.

**Lemma 4.2.3.** For any pair \( s_i \) and \( s_j \) with \( i \neq j \), we have that \( d_C(s'_i, s'_j) \geq 2 \). Moreover, if at least one of \( s_i \) and \( s_j \) is not in \( D_C(\bar{o}, 3) \), then \( d_C(s'_i, s'_j) > 2 \).

**Proof.** If both \( s_i \) and \( s_j \) are in \( D_C(\bar{o}, 3) \), then from Claim 4.2.2 we have that \( d_C(s'_i, s'_j) = d_C(s_i, s_j) \geq 2 \). Otherwise, assume without loss of generality that \( d_C(\bar{o}, s_j) \geq d_C(\bar{o}, s_i) \). Then \( s_j \notin D_C(\bar{o}, 3) \) and \( s'_j \) is on the line segment \( \bar{o}s_j \).
Thus, \( s_j = \lambda s_j' \) for some \( \lambda > 1 \). Let \( p \) be the intersection point of \( \partial C(a,b) \) and \( \partial s_j \). By Observation 4.1.3, the \( d_C \)-distance from \( s_j' \) to \( C(a,b) \) is \( d_C(s_j', p) = d_C(s_j', \partial) - d_C(p, \partial) = 2 \). Since \( d_C \) defines a norm, \( d_C(s_j, p) = d_C(s_j, \partial) - d_C(p, \partial) = \lambda d_C(s_j', \partial) - 1 = 3\lambda - 1 \), and this corresponds to the \( d_C \)-distance from \( s_j \) to \( C(a,b) \). Further, \( d_C(s_j, s_j') = d_C(s_j, \partial) - d_C(s_j', \partial) = 3\lambda - 3 \). For sake of contradiction we suppose that \( d_C(s_j', s_j) \leq 2 \). Thus, \( s_j' \) is in \( D(s_j, 2) \). We consider the following two cases.

**Case 1**  \( s_i \in D_C(\bar{\partial}, 3) \). Then \( d_C(\bar{\partial}, s_i) \leq 3 \). Since \( d_C(s_i', s_j') \leq 2 \), \( s_i = s_i' \in D_C(s_j', 2) \). From Claim 4.1.5 follows that \( D_C(s_j', 2) \subset D_C(s_j, 3\lambda - 1) \). Thus, \( s_i \in D_C(s_j, 3\lambda - 1) \). Hence, \( d_C(s_j, s_i') = d_C(s_j, s_i) \leq d_C(s_j, p) \). Since \( S \) is in general position, \( u_j \) is in the interior of \( C(a,b) \). Therefore, \( d_C(s_j, s_i) \leq d_C(s_j, p) < d_C(s_j, u_j) \), which contradicts Claim 4.2.2.

**Case 2**  \( s_i \notin D_C(\bar{\partial}, 3) \). Then \( s_i' \in \partial D_C(\bar{\partial}, 3) \) and \( s_i = \delta s_i' \) for some \( \delta > 1 \). Moreover, since \( d_C(\bar{\partial}, s_j) \geq d_C(\bar{\partial}, s_i) \) and \( s_i', s_j' \) are on the boundary of \( D_C(\bar{\partial}, 3) \), \( \delta \leq \lambda \). Hence, \( s_i \) is on the line segment \( s_i'(\lambda s_i') \). Note that \( 2\lambda < 3\lambda - 1 \) because \( \lambda > 1 \). Since \( d_C \) defines a norm, \( d_C(s_j, \lambda s_i') = \lambda d_C(s_j', s_i') \leq 2\lambda < 3\lambda - 1 \). Hence, \( \lambda s_i' \in D_C(s_j, 3\lambda - 1) \). In addition, from Claim 4.1.5 it follows that \( D_C(s_j', 2) \subset D_C(s_j, 3\lambda - 1) \). Thus, \( s_i' \in D_C(s_j, 3\lambda - 1) \) and the line segment \( s_i'(\lambda s_i') \) is contained in \( D_C(s_j, 3\lambda - 1) \). Then, \( s_i \in D_C(s_j, 3\lambda - 1) \) and \( d_C(s_j, s_i) \leq 3\lambda - 1 = d_C(s_j, p) < d_C(s_j, u_j) \), which contradicts Claim 4.2.2.

**Theorem 4.2.4.** For any set \( S \) of points in general position and point-symmetric convex shape \( C \), the graph 15-GG\(_C\)(\( S \)) is Hamiltonian.

**Proof.** For each \( s_i \in S \) we define the \( C \)-disk \( D_i = D_C(s_i', 1) \). We also set \( D_0 := D_C(a, 1) \). From Lemma 4.2.3, each pair of \( C \)-disks \( D_i \) and \( D_j \) are internally disjoint, for \( 0 < i < j \leq k \). Note that, if \( s_i' \) is on \( \partial D_C(\bar{\partial}, 3) \), then \( D_0 \) and \( D_i \) are internally disjoint. On the other hand, if \( s_i' \) is in the interior of \( D_C(\bar{\partial}, 3) \), then by definition \( s_i' = s_i \). Thus, by Claim 4.2.1 \( D_0 \) is internally disjoint from \( D_i \). Consider \( D_C(\bar{\partial}, 4) \). Since, \( s_i' \in D_C(\bar{\partial}, 3) \) for all \( i \in \{1, \ldots k\} \), then each disk \( D_i \) is inside \( D_C(\bar{\partial}, 4) \). Hence, in \( D_C(\bar{\partial}, 4) \) there can be at most \( \frac{\text{Area}(D_C(\bar{\partial}, 4))}{\text{Area}(C)} = \frac{4^2 \text{Area}(C)}{\text{Area}(C)} = 16 \) internally disjoint disks of type \( D_i \). Since \( D_0 \) is centered at \( a \), there are at most 15 points \( s_i' \) in \( D_C(\bar{\partial}, 3) \). Therefore, there are at most 15 points of \( S \) in \( C(a,b) \), and the bottleneck Hamiltonian cycle of \( S \) is contained in 15-GG\(_C\)(\( S \)).

**4.3 Hamiltonicity for regular polygons**

An important family of point-symmetric convex shapes is that of regular even-sided polygons. When \( C \) is a regular polygon \( P_t \) with \( t \) sides, for \( t \) even, we
can improve the previous bound by analyzing the properties of the shape for different values of $t$.

### 4.3.1 Hamiltonicity for squares

![Diagram of squares](image)

Figure 4.5: Lines $x = -1$, $x = 1$, $y = -1$, and $y = 1$ split $D□(3, \bar{o})$ into nine unit squares: $C(a, b), D_0, \ldots, D_7$.

First, we consider the case when the polygon is a square. In this case, we divide $D□(\bar{o}, 3)$ into 9 disjoint squares of radius 1 and show that there can be at most one point of $\{a, s'_1, \ldots, s'_k\}$ in each such square. We use lines $x = -1, x = 1, y = -1, y = 1$ to split $D□(\bar{o}, 3)$ into 9 squares of radius 1. Refer to Figure 4.5. Let $D_0, D_1, \ldots, D_7$ be the squares of radius 1 in $D□(\bar{o}, 3)$ different from $C(a, b)$, ordered clockwise, and where $D_0$ is the top-right corner square. In the following lemma we prove that there is at most one point of $\{a, s'_1, \ldots, s'_k\}$ in each $D_i$. Let indices be taken modulo 8. Note that each $D_i$ shares a side with $D_{i-1}$, and for each odd $i$, $D_i$ shares a side with $C(a, b)$. Moreover, there exists a $D_i$ that contains $a$ on its boundary. We will associate any point in $D□(\bar{o}, 3)$ (not in the interior of $C(a, b)$) to a unique square $D_i$ in the following way: Let $p$ be a point in $D_i$. If $p$ does not lie on the shared boundary of $D_i$ and some other $D_j$, then $p$ is associated to $D_i$. If $i$ is odd and $p$ is the intersection point $D_i \cap D_{i-1} \cap D_{i-2}$, then $p$ is associated to $D_{i-2}$ ($p$ can be $a$ or $b$). Otherwise, if $p$ is on the edge $D_i \cap D_{i-1}$, then $p$ is associated to $D_{i-1}$.

**Observation 4.3.1.** Any two points at $d□$-distance 2 in a unit square must be on opposite sides of the square.

**Lemma 4.3.2.** There is at most one $s'_i$ associated to each $D_i$. Moreover, the $D_i$ containing the point $a$ on its boundary has no $s'_j$ associated to it.

**Proof.** Suppose that there are two points $s'_j$ and $s'_m$ associated to $D_i$. From Lemma 4.2.3 we have that $d□(s'_j, s'_m) \geq 2$. Also, since $D_i$ is a unit square,
Figure 4.6: Cases (a) and (b) contradict Claim 4.2.2. Case (c) contradicts our general position assumption. In case (d) only one point, $s_j$, is associated to $D_7$.

$d_{\square}(s'_j, s'_m) \leq 2$. Therefore, $d_{\square}(s'_j, s'_m) = 2$. Then Lemma 4.2.3 implies that $s_j$ and $s_m$ must be inside $D_C(a, b)$. In addition, by Observation 4.3.1 the points $s_j$ and $s_m$ are on opposite sides of the boundary of $D_i$. For simplicity we will assume that $d_{\square}(\bar{a}, s_j) \geq d_{\square}(\bar{a}, s_m)$. If $i$ is even, then the $d_{\square}$-distance of $s_j$ to $C(a, b)$ is exactly 2. We refer to Figure 4.6a and 4.6b. Recall that, by our general position assumption, $u_j$ is in the interior of $C(a, b)$. Thus, the $d_{\square}$-distance from $s_j$ to $C(a, b)$ is less than $d_{\square}(s_j, u_j)$, i.e., $d_{\square}(s_j, u_j) > 2$. Hence, $d_{\square}(s_j, s_m) = 2 < d_{\square}(s_j, u_j)$ which contradicts Claim 4.2.2. Therefore, if $i$ is even, there is at most one point in $D_i$, which is associated to it.

If $i$ is odd, then $s_j$ is either on $D_i \cap D_{i-1}$ or $D_i \cap D_{i+1}$, or on $D_i \cap \partial D_C(\bar{a}, 3)$ (see Figure 4.6c and 4.6d). If $s_j$ is on $D_i \cap D_{i-1}$ or $D_i \cap D_{i+1}$, then only one of $s_j$ and $s_m$ is associated to $D_i$. If $s_j$ is on $D_i \cap \partial D_C(\bar{a}, 3)$, then by Observation 4.3.1 $s_m$ is on $C(a, b)$, which by our general position assumption implies that $s_m = b$, since $s_m \neq a$. Thus, $d_{\square}(s_j, u_j) > 2 = d_{\square}(s_j, s_m)$, which contradicts Claim 4.2.2. Therefore, there is only one point associated to $D_i$.

Finally, if $D_i$ contains $a$, then there is no point $s'_j$ in $D_i$. Indeed, assume for sake of a contradiction that $s'_j \in D_i$. Then, $s_j$ is not in $D_i$, otherwise, $d_{\square}(a, s_j) < d_{\square}(s_j, u_j)$, contradicting Claim 4.2.1. Thus, $s'_j$ is on $D_i \cap \partial D_C(\bar{a}, 3)$ and $s_j = \lambda s'_j$ for some $\lambda > 1$. Hence, $d_{\square}(s'_j, a) = 2$, which means that $a \in D_{\square}(s'_j, 2)$. Let $p$ be the point $\bar{a}s'_j \cap \partial C(a, b)$. By Claim 4.1.5 $D_{\square}(s'_j, 2) \subset D_{\square}(s_j, d_{\square}(s_j, p))$. So, $a \in D_{\square}(s_j, d_{\square}(s_j, p))$ and $d_{\square}(s_j, a) < d_{\square}(s_j, u_j)$, contradicting Claim 4.2.1.

**Theorem 4.3.3.** For any set $S$ of points in general position, the graph $7-GG_{\square}(S)$ is Hamiltonian.

**Proof.** From Lemma 4.3.2 we have that for each $0 \leq i \leq 7$ there is at most one point of $\{a, s'_1, \ldots, s'_k\}$ associated to $D_i$, and any square containing $a$ has no $s'_i$ associated to it. Since there is at least one $D_i$ containing $a$, there are at most 7 points $s'_j$ in $D_{\square}(\bar{a}, 3)$. Therefore, there are at most 7 points of $S$ in the interior of $C(a, b)$, and the bottleneck Hamiltonian cycle of $S$ is contained in
4.3.2 Hamiltonicity for regular hexagons

Figure 4.7: The bold hexagon is the boundary of \( D_{P_6}(\bar{o}, 3) \). Such hexagon is divided into 13 interior-disjoint regions: 6 quadrangles—a third of a unit \( P_6 \)-disk—and 7 unit \( P_6 \)-disks.

The analysis for the case of hexagons is similar to the previous one. First we divide the hexagon \( D_{P_6}(\bar{o}, 3) \) into 13 different regions \( C(a, b), D_0, \ldots, D_5, Q_0, \ldots, Q_5 \), shown in Figure 4.7. Let indices be taken modulo 6. We will associate a point in \( D_{P_6}(\bar{o}, 3) \) (not in the interior of \( C(a, b) \)) to a region \( D_i \) or \( Q_i \) in the following fashion. If a point is in the interior of \( D_i \) or \( Q_i \) we say that such point is associated to \( D_i \) or \( Q_i \), respectively. If a point is on the edge \( D_i \cap D_{i-1} \) or edge \( D_i \cap Q_{i-1} \), then such point is associated to \( D_{i-1} \) and \( Q_{i-1} \), respectively. In the case when a point is the vertex \( D_i \cap D_{i-1} \cap Q_{i-1} \), we say that such point is associated to \( D_{i-1} \). When a point is on the edge \( D_i \cap Q_i \) then we associate it with \( D_i \). See Figure 4.8.

Figure 4.8: The dashed boundary of \( D_5 \) is associated to \( D_4 \) and the dotted one is associated to \( Q_4 \). The rest of \( D_5 \) is associated to \( D_5 \).
Observation 4.3.4. Any two points at $d_{\mathcal{P}_6}$-distance 2 in a unit hexagon $D$ must be on opposite sides of $D$.

In the following lemma we show that the hexagon $D_{\mathcal{P}_6}(\bar{o}, 3)$ contains at most 11 points $s'_1, \ldots, s'_k$.

Lemma 4.3.5. There is at most one point $s'_j$ associated to each region of type $D_i$ or $Q_i$. Moreover, there is no point $s'_j$ in the hexagon $D_i$ that contains $a$.

Proof. If a point is in the interior of $D_i$ or $Q_i$ then by Observation 4.3.4 there is no other point in the same region.

Note that if $Q_i$ contains two points at $d_{\mathcal{P}_6}$-distance 2, then by Observation 4.3.4 such points are exactly $D_i \cap Q_i \cap \partial D_{\mathcal{P}_6}(\bar{o}, 3)$ and $D_{i+1} \cap Q_i \cap \partial D_{\mathcal{P}_6}(\bar{o}, 3)$. Since the points on $D_i \cap Q_i$ are associated to $D_i$, the intersection point $D_i \cap Q_i \cap \partial D_{\mathcal{P}_6}(\bar{o}, 3)$ is not associated to $Q_i$. Thus, there is at most one point associated to $Q_i$. If $D_i$ contains a point $s'_j$ that is on $D_i \cap \partial D_{\mathcal{P}_6}(\bar{o}, 3)$, then there cannot be another $s'_m \in D_i$: Otherwise, by Observation 4.3.4 $s'_m$ would be on the boundary of $\mathcal{P}_6(a, b)$, in which case $s'_m = b$ due to our general position assumption. Since, $d_{\mathcal{P}_6}(s'_j, s'_m) = 2$, it follows from Lemma 4.2.3 that $s_j$ would be in $D_{\mathcal{P}}(\bar{o}, 3)$. Thus, $d_{\mathcal{P}_6}(s_j, s_m) = 2 < d_{\mathcal{P}_6}(s_j, u_j)$ which contradicts Claim 4.2.2. Consequently, if $D_i$ contains two points $s'_j$ and $s'_m$ then by Observation 4.3.4 either: 1) one is on the edge $D_i \cap Q_i$ and the other is on the edge $D_i \cap D_{i-1}$ (see Figure 4.9a); or 2) one is on the edge $D_i \cap D_{i+1}$ and the other is on the edge $D_i \cap Q_{i-1}$ (see Figure 4.9b). In either case, just one point is associated to $D_i$.

Figure 4.9: In both, (a) and (b), $D_5$ contains exactly two points at $d_{\mathcal{P}_6}$-distance 2.

Finally, if $D_i$ contains $a$, then there is no point $s'_j$ in $D_i$. Indeed, suppose for sake of contradiction that $s'_j \in D_i$. Then, $s_j$ is not in $D_i$ because, by Claim 4.2.1 $d_{\mathcal{P}_6}(a, s_j) \geq d_{\mathcal{P}_6}(s_j, u_j)$. Thus, $s'_j$ is on $D_i \cap \partial D_{\mathcal{P}_6}(\bar{o}, 3)$ and $s_j = \lambda s'_j$ for some $\lambda > 1$. Hence, $d_{\mathcal{P}_6}(s'_j, a) = 2$ and $a \in D_{\mathcal{P}_6}(s'_j, 2)$. Let $p$ be the intersection point $\bar{o}s_j \cap \partial \mathcal{P}_6(a, b)$. By Claim 4.1.5 $D_{\mathcal{P}_6}(s'_j, 2) \subset D_{\mathcal{P}_6}(s_j, d_{\mathcal{P}_6}(s_j, p))$ and,
thus, $a \in D_{P6}(s_j, d_{P6}(s_j, p))$. We obtain that $d_{P6}(s_j, a) < d_{P6}(s_j, u_j)$, which again contradicts Claim 4.2.1.\hfill \Box

The following theorem holds.

**Theorem 4.3.6.** For any set $S$ of points in general position, the graph $11-GG_{P6}(S)$ is Hamiltonian.

### 4.3.3 Hamiltonicity for regular even-sided $t$-gons where $t \geq 8$

For the remaining regular polygons with an even number of sides, we use the circumcircle\footnote{The **circumcircle** of a polygon $P$ is the smallest circle that contains $P$.} of $D_{P_l}(\bar{o}, 3)$ in order to give an upper bound on the number of points in $D_{P_l}(\bar{o}, 3)$ at pairwise Euclidean distance at least 2. Without loss of generality we assume that the incircle\footnote{The **incircle** of a polygon $P$ is the largest circle in the interior of $P$ that is tangent to each side of $P$.} of the unit $P_l$-disk has Euclidean radius 1. See Figure 4.10.

In this section we will first treat the case $t \geq 10$, and afterwards the case $t = 8$.

**Theorem 4.3.7.** For any set $S$ of points in general position and regular polygon $P_l$ with even $t \geq 10$, the graph $11-GG_{P_l}(S)$ is Hamiltonian.

*Proof.* Let $P_l$ be a polygon with $t \geq 10$ sides and $t$ even. Then $D_{P_l}(\bar{o}, 3)$ is inscribed in a circle of radius $r = \frac{3}{\cos(\frac{\pi}{l})}$. Since the function $\cos(\frac{\pi}{l})$ is an increasing function for $t \geq 2$, we have that $r \leq \frac{3}{\cos(\frac{\pi}{10})}$. Therefore, $D_{P_l}(\bar{o}, 3)$ is...
inside the circumcircle of a decagon with incircle of radius 3. In addition, from Lemma 4.2.3 we know that for any pair of points $s_i', s_j'$ in $D_{\mathcal{P}_t}(\bar{o}, 3)$, $d_{\mathcal{P}_t}(s_i', s_j') \geq 2$. Since the incircle of the 2-unit $\mathcal{P}_t$-disk has Euclidean radius 2, we have that $d(s_i', s_j') \geq 2$. Hence, it suffices to show that there are at most 12 points in $D_{\mathcal{P}_t}(\bar{o}, 3)$ at pairwise Euclidean distance at least 2. Fodor [56] proved that the minimum radius $R$ of a circle having 13 points at pairwise Euclidean distance at least 2 is $R \approx 3.236$, which is greater than $\frac{3}{\cos(\frac{\pi}{10})} \approx 3.154$. Thus, $D_{\mathcal{P}_t}(\bar{o}, 3)$ contains at most 12 points at pairwise distance at least 2 from all $s_i'$'s, there are at most 11 points of $S$ inside $\mathcal{P}_t(a, b)$.

For the case of octagons, the radius of the circumcircle of $D_{\mathcal{P}_8}(\bar{o}, 3)$ is greater than 3.236, so we cannot use the result in [56]. However, we can use a similar result from Fodor [57] to prove an analogous theorem:

**Theorem 4.3.8.** For any set $S$ of points in general position, the graph $12-GG_{\mathcal{P}_8}(S)$ is Hamiltonian.

**Proof.** From Lemma 4.2.3 we know that, for any pair of points $s_i', s_j'$ in $D_{\mathcal{P}_8}(\bar{o}, 3)$, $d_{\mathcal{P}_8}(s_i', s_j') \geq 2$. Since the incircle of the 2-unit $\mathcal{P}_8$-disk has Euclidean radius 2, we have that $d(s_i', s_j') \geq 2$. Hence, it suffices to show that there are at most 13 points in $D_{\mathcal{P}_8}(\bar{o}, 3)$ at pairwise Euclidean distance at least 2. The regular octagon $D_{\mathcal{P}_8}(\bar{o}, 3)$ is inscribed in a circle of radius $r = \frac{3}{\cos(\frac{\pi}{8})} \approx 3.247$. By a result of Fodor [57], the smallest radius $R$ of a circle containing 14 points at pairwise Euclidean distance at least 2 is $R \approx 3.328$. Hence, $D_{\mathcal{P}_8}(\bar{o}, 3)$ contains at most 13 points at pairwise Euclidean distance at least 2. Since $a$ is also at $d_{\mathcal{P}_8}$-distance at least 2 from all $s_i'$'s, there are at most 12 points of $S$ inside $\mathcal{P}_8(a, b)$. 

4.4 Bottleneck Hamiltonian cycles in $k-GG_{\square}$ and $k-GG_{\mathcal{P}_6}$

In this section we give lower bounds on the minimum values of $k$ for which the graphs $k-GG_{\square}$ and $k-GG_{\mathcal{P}_6}$ contain a bottleneck Hamiltonian cycle. This is useful to understand to what extent we can use the bottleneck Hamiltonian cycle for showing Hamiltonicity in a $k-GG_C$ in order to improve the known upper bounds on $k$. The proofs are very similar to those in [22, 21, 67].

**Lemma 4.4.1.** There exists a point set $S$ with $n \geq 17$ points such that $2-GG_{\square}(S)$ does not contain any $d_{\square}$-bottleneck Hamiltonian cycle of $S$. 


Figure 4.11: The diagonal-pattern square is a $C(a, b)$ with $a$ as a vertex, and the gray-filled square is a $C(a, b)$ with $b$ as vertex. The union of both squares contains all the possible $C(a, b)$. The bold edges belong to $h$ in the proof of Lemma 4.4.1.

Proof. Consider the point set $S$ in Figure 4.11. The length of edge $ab$ is $d_\square(a, b) = 1$, and the two dashed squares have radius 1 and are centered at $a$ and $b$. Notice that any $C(a, b)$ contains at least 3 points from $U = \{u_1, u_2, u_3, u_4\}$, so $ab \notin 2-GG_\square(S)$.

Let $R = \{r_1, r_2, r_3, r_4, t_1, \ldots t_7\}$. For each point in $R$ there is a red square centered at such point with radius $1 + \varepsilon$, where $\varepsilon$ is a small positive value. Thus, $d_\square(r_i, u_i) = 1 + \varepsilon$, $d_\square(r_i, a) > 1 + \varepsilon$, $d_\square(r_i, b) > 1 + \varepsilon$ and $d_\square(r_i, r_j) > 1 + \varepsilon$, for $i \neq j$. The cycle $h=(a, b, u_1, r_1, u_2, r_2, t_1, t_2, t_3, t_4, t_5, t_6, t_7, r_3, u_3, r_4, u_4, a)$ is Hamiltonian and the maximum length of its edges in the $d_\square$-distance is $1 + \varepsilon$. Hence, any $d_\square$-bottleneck Hamiltonian cycle of $S$ has at most $1 + \varepsilon$ maximum edge $d_\square$-length.
We will show that the edge $ab$ is in every $d_{\Box}$-bottleneck Hamiltonian cycle of $S$. Let $h'$ be a $d_{\Box}$-bottleneck Hamiltonian cycle. Since the $d_{\Box}$-distance from $a$ and $b$ to any point in $R$ is greater than $1+\varepsilon$, in $h'$, $a$ and $b$ can only be connected between each other or to the points in $U$. Note that $u_2$ has to be connected to $r_1$ and $r_2$ in $h'$, since otherwise $r_1$ or $r_2$ would be adjacent to an edge whose $d_{\Box}$-length is greater than $1+\varepsilon$. Similarly, $u_3$ has to be connected to $r_3$ and $r_4$ in $h'$, since otherwise $r_3$ or $r_4$ would be adjacent to an edge of $d_{\Box}$-length greater than $1+\varepsilon$. Finally, $a$ and $b$ have to be connected to each other, since otherwise both would be adjacent to $u_1$ and $u_4$, which does not produce a Hamiltonian cycle.

In summary, $ab$ is included in any $d_{\Box}$-bottleneck Hamiltonian cycle, and since $ab \notin 2-GG_{\Box}(S)$, the lemma holds.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.12.png}
\caption{The gray hexagon is the unique $C(a, b)$, and it contains 6 points of $S$. The bold edges belong to $h$ in the proof of Lemma 4.4.2.}
\end{figure}

**Lemma 4.4.2.** There exists a point set $S$ with $n \geq 22$ points such that $5-GG_{P_6}(S)$ does not contain any $d_{P_6}$-bottleneck Hamiltonian cycle of $S$.

**Proof.** We proceed in the same fashion as in the previous proof. Consider the
point set $S$ in Figure 4.12. The length of edge $ab$ is $d_{P_6}(a, b) = 1$, and the dashed hexagons have radius 1 and are centered at $a$ and $b$. Notice that there is exactly one $C(a, b)$, and it contains all points from $U = \{u_1, \ldots, u_6\}$. Therefore, $ab \notin 5$-$GGP_6(S)$. Let $R = \{r_1, \ldots, r_6, t_1, \ldots, t_8\}$. For each point in $R$ there is a red regular hexagon centered at such point with radius $1 + \varepsilon$, where $\varepsilon$ is a small positive value. Thus, $d_{P_6}(r_i, u_i) = 1 + \varepsilon$, $d_{P_6}(r_i, a) > 1 + \varepsilon$, $d_{P_6}(r_i, b) > 1 + \varepsilon$ and $d_{P_6}(r_i, r_j) > 1 + \varepsilon$, with $i \neq j$. The cycle $h = (a, b, u_1, r_1, u_2, r_2, u_3, r_3, t_1, t_2, \ldots, t_8, r_4, u_4, r_5, u_5, r_6, u_6, a)$ is Hamiltonian, and the maximum length of its edges in the $d_{P_6}$-distance is $1 + \varepsilon$.

Let $h'$ be a $d_{P_6}$-bottleneck Hamiltonian cycle. Let us show that $ab \in h'$. Since the $d_{P_6}$-distance from $a$ and $b$ to any point in $R$ is greater than $1 + \varepsilon$, in $h'$, $a$ and $b$ can only be connected between each other or to the points in $U$. Note that $u_1$ has to be adjacent to $r_2$ and $r_3$; otherwise, $r_2$ or $r_3$ would be adjacent to an edge of $d_{P_6}$-length greater than $1 + \varepsilon$. Similarly, $u_2$, $u_4$, $u_5$ have to be adjacent to $r_2$ and $r_3$, to $r_4$ and $r_5$, and to $r_5$ and $r_6$, respectively. Finally, $a$ and $b$ have to be connected to each other, otherwise both would be adjacent to $u_1$ and $u_6$ which does not produce a Hamiltonian cycle. Therefore, $ab$ is included in any $d_{\square}$-bottleneck Hamiltonian cycle, and since $ab \notin 5$-$GGP_6(S)$, the lemma holds.

\[ \square \]

4.5 Non-Hamiltonicity for regular polygons

Until now we have discussed upper and lower bounds for $k$, so that $k$-$GGC$ contains a bottleneck Hamiltonian cycle. As mentioned in Section 1.4, $k$-$GGC \subseteq k$-$DG_{\square}$, thus all upper bounds given in the previous sections hold for $k$-order $C$-Delaunay graphs as well, but not the lower bounds. In this section we present point sets for which $DG_{P_t}$ is not Hamiltonian. For $t = 4$, Saumell [92] showed that for any $n \geq 9$ there exists a point set $S$ such that $DG_{\square}(S)$ is non-Hamiltonian, so we focus on $t \geq 5$.

First, we present particular cases of $t \geq 5$ for which $DG_{P_t}$ is non-Hamiltonian. Later on, we present a generalization of these point sets and show the non-Hamiltonicity for an infinite family of $DG_{P_t}$.

4.5.1 Non-Hamiltonicity for regular polygons with small number of sides

In this section we prove that $DG_{P_t}$ fails to be Hamiltonian for every point set when $t = 5, 6, \ldots, 11$ (see Figure 4.13).

**Lemma 4.5.1.** For any $n \geq 7$ and any $t \in \{5, 6, \ldots, 11\}$, there exists an $n$-point set $S$ such that $DG_{P_t}(S)$ is non-Hamiltonian.
Figure 4.13: For each $t \in \{5, 6, 7, 8, 9, 10, 11\}$ the graph $DG_{\mathcal{P}_t}(S)$ is non-Hamiltonian.

**Proof.** Let $t \in \{5, 6, \ldots, 11\}$. Consider the graph $DG_{\mathcal{P}_t}(S)$ in Figure 4.13 for such $t$. Note that such graph is indeed a $\mathcal{P}_t$-Delaunay graph, since for each edge there exists a $\mathcal{P}_t$-disk that contains its vertices on its boundary and is empty of other points of $S$. Also, note that some edges from the convex hull of $S$ do not appear in such graphs. Finally, notice that there exists an area $r$ that is not contained in any of the $\mathcal{P}_t$-disks associated to the edges of the outer face or the triangle $\triangle p_1p_2p_3$. Such area can have an arbitrary number of points in its interior, say $n - 6$. Now, let $G' = DG_{\mathcal{P}_t}(S) \setminus \{p_1, p_2, p_3\}$. The graph $G'$ consists of 4 connected components, so $DG_{\mathcal{P}_t}(S)$ is not 1-tough. Since every Hamiltonian graph is 1-tough, $DG_{\mathcal{P}_t}(S)$ is non-Hamiltonian. 

### 4.5.2 An infinite family of regular polygons such that $DG_{\mathcal{P}_t}$ is non-Hamiltonian

Based on the point sets given in the previous section, we construct an $n$-point set $S$, with $n \geq 7$, such that the following theorem holds.

**Theorem 4.5.2.** Let $\mathcal{P}_t$ be a regular $t$-gon, where $t > 3$ and $t$ is an odd number and multiple of three. For any $n \geq 7$, there exists an $n$-point set $S$ such that $DG_{\mathcal{P}_t}(S)$ is non-Hamiltonian.

Our construction is a generalization of the ones in the previous section. However, in order to be able to prove that $DG_{\mathcal{P}_t}(S)$ has the desired structure for arbitrary large values of $t$, we have to define it in a very precise way.
Before we proceed to prove Theorem 4.5.2, we need some new definitions and a few auxiliary claims.

Let $P_t$ be a regular $t$-gon, where $t = 3(2m + 1)$ for some positive integer $m$. Without loss of generality, we assume that $P_t$ is oriented so that its bottom side is horizontal. We also assume that its vertices are given in counterclockwise order.

![Figure 4.14: An equilateral triangle pointing downwards.](image)

Consider three points $p_1, p_2$ and $p_3$ in the plane that define an equilateral triangle $T$ as in Figure 4.14. Let $c$ be the circumcenter of the triangle $T$. Let $C_1, C_2$ and $C_3$ be three circles circumscribing the triangles $\triangle p_1p_2c$, $\triangle p_2p_3c$ and $\triangle p_3p_1c$, respectively. These three circles are Johnson circles\(^6\) they have the same radius $r$, and they intersect at $c$. Let $c_1, c_2$ and $c_3$ be the centers of $C_1, C_2$ and $C_3$, respectively. Notice that the line segments $p_2c_2$ and $c_3p_1$ are vertical, and that $\angle cc_ip_{i+1} = \frac{\pi}{3}$ and $\angle p_ic_3c = \frac{\pi}{3}$, for all $i$ modulo 3. See Figure 4.15.

![Figure 4.15: The circles $C_1, C_2, C_3$ contain triangles $\triangle p_1p_2c$, $\triangle p_2p_3c$ and $\triangle p_3p_1c$, respectively. The big triangle $T' = \triangle p_4uv$ is the anticomplementary triangle of $T$.](image)

Consider the anticomplementary triangle\(^7\) $T' = \triangle p_4uv$ of $T$ defined as in

\[^6\]A set of Johnson circles is a set of three circles of the same size that mutually intersect each other in a single point. For a survey about the properties of Johnson circles, we refer the reader to the Johnson Theorem [80].

\[^7\]Let $C$ be the circle with center $C_1 \cap C_2 \cap C_3$ and radius $2r$. The anticomplementary triangle
Figure 4.15 Let $\mathcal{P}_t^1, \mathcal{P}_t^2$ and $\mathcal{P}_t^3$ be the three $t$-gons inscribed in $C_1, C_2$ and $C_3$, respectively. See Figure 4.16.

Figure 4.16: Inscribed $t$-gons for $t = 9$. The angles $\alpha$ and $\alpha'$ are less than $\frac{\pi}{9}$.

Claim 4.5.3. The points $p_1, p_2$ and $p_3$ are on $\partial \mathcal{P}_t^3 \cap \partial \mathcal{P}_t^1, \partial \mathcal{P}_t^1 \cap \partial \mathcal{P}_t^2$ and $\partial \mathcal{P}_t^2 \cap \partial \mathcal{P}_t^3$, respectively.

Proof. Recall that $t = 3(2m + 1)$ with $m > 0$. Let $a_1b_1$ be the bottom side of $\partial \mathcal{P}_t^1$. Since the line segment $c_1c$ is vertical, $c_1c$ bisects $a_1b_1$. Thus, the angle formed by $c_1c$ and the $i$-th vertex of $\partial \mathcal{P}_t^1$ is given by $\frac{\pi}{t} + \frac{2(i-1)}{t}$. In particular, for $i = m + 1$ we obtain $\frac{2m\pi}{t} + \frac{\pi}{t} = \frac{(2m+1)\pi}{3(2m+1)} = \frac{\pi}{3}$, which is precisely $\angle cc_1p_2$. Hence, $p_2 \in \partial \mathcal{P}_t^1$. The proof for $p_1 \in \partial \mathcal{P}_t^1$ is symmetric.

Since the bottom sides of $\partial \mathcal{P}_t^2$ and $\partial \mathcal{P}_t^3$ are horizontal, the top-most vertices of $\partial \mathcal{P}_t^2$ and $\partial \mathcal{P}_t^3$ are $p_2$ and $p_1$, respectively. Therefore $p_1 \in \partial \mathcal{P}_t^1 \cap \partial \mathcal{P}_t^3$ and $p_2 \in \partial \mathcal{P}_t^1 \cap \partial \mathcal{P}_t^2$.

On the other hand, since the top-most point of $\partial \mathcal{P}_t^2$ is $p_2$, the angle formed by $p_2c_2$ and the $i$-th vertex of $\partial \mathcal{P}_t^2$ is given by $\frac{2\pi}{t}$. In particular, for $i = 2m + 1$ we obtain $\frac{2(2m+1)\pi}{3(2m+1)} = \frac{2\pi}{3}$, which is precisely $\angle p_2c_2p_3$. Thus, $p_3 \in \partial \mathcal{P}_t^2$. Similarly, we can show $p_3 \in \partial \mathcal{P}_t^3$.

Given points $a$ and $b$, we next show how to define a polygon which we call the $\mathcal{P}_t$-of-influence of $a$ and $b$. Recall that the vertices $v_1, \ldots, v_t$ of $\mathcal{P}_t$ are oriented counterclockwise, where $v_1$ is the top-most one. The $i$-th oriented edge of $\mathcal{P}_t$ is defined by $e_i = v_iv_{i+1}$. We define the oriented line $\ell_i$ as the supporting line of the edge $e_i$ with the same orientation as $e_i$. For each $\ell_i$, we consider two oriented lines parallel to $\ell_i$, one passing through $a$ and another through $b$. Among all these lines, we only take those having $a$ and $b$ on its left or on the line. Now, $\triangle$ of $T$ has as vertices the three tangent points of $C$ with the Johnson circles.
consider the left half-planes defined by such oriented lines; the intersection of these half-planes defines the $P_t$-of-influence of $a$ and $b$. See Figure 4.17a. Since a point $p$ is in a $P_t$-disk if $p$ is on the left of each supporting line $\ell_i$ or on $\ell_i$, any $P_t$-disk containing $a$ and $b$ contains their $P_t$-of-influence.

Let $p_5$ and $p_6$ be two points on the boundary of $P_t^2$ and $P_t^3$, respectively, such that $\alpha = \angle p_5p_3u < \frac{\pi}{t}$ and $\alpha' = \angle v p_3 p_6 < \frac{\pi}{t}$. See Figure 4.16

**Claim 4.5.4.** Any $P_t$-disk containing $p_5$ and $p_6$ on its boundary contains $p_3$ in its interior.

**Proof.** Note that if $p_3$ is in the interior of the $P_t$-of-influence of $p_6$ and $p_5$ then the claim follows. Let us show first that $p_3$ is in the $P_t$-of-influence of $p_6$ and $p_5$ (but not necessarily in its interior). We denote by $\ell_i^p$ the parallel line to $\ell_i$ passing through point $p$. Without loss of generality assume that $p_5$ is above the horizontal line passing through $p_6$. Note that $\alpha'$ is equal to the inner angle at $p_6$ formed by the horizontal line passing through $p_6$ and edge $p_6p_3$, and this angle is less than $\frac{\pi}{t}$. Also, note that for $h = \ell_t^{\frac{1}{2}} + 1$, $\ell_h$ is horizontal. Finally, observe that the angle formed by consecutive $\ell_i^p$ and $\ell_{i+1}^p$ is $\frac{2\pi}{t}$. Then, $p_3$ is contained in the wedge defined by $\ell_h^{p_6}$ and $\ell_{h-1}^{p_6}$ with inner angle $\frac{2\pi}{t}$ that lies above $\ell_h^{p_6}$. Refer to Figure 4.17b. Since $\alpha' < \frac{\pi}{t}$, this wedge contains the wedge defined by edge $p_6p_3$ and $\ell_h^{p_6}$, which contains $p_5$. Thus, $p_5$ is on the left of $\ell_h^{p_6}$ and on the right of $\ell_{h-1}^{p_6}$. The lines of the form $\ell_i^{p_6}$ that have $p_5$ on its left are the ones encountered when rotating $\ell_h^{p_6}$ along $p_6$ counterclockwise until it hits $p_5$; the total angle of rotation is $\pi$ minus the inner angle formed by $p_5p_6$ and
Therefore, these lines are $\ell_{h}^{P6}$ and $\ell_{i}^{P6}$ containing $p_{5}$ has angle $\frac{\pi}{7}$, $p_{3}$ also lies on such wedge and $p_{3}$ is on the left of $\ell_{i}^{P6}$. Moreover, the angle of the cone containing $p_{5}$ formed by $\ell_{h}^{P6}$ and $\ell_{i}^{P6}$, for any $i \in \{h + 1, \ldots, t\}$, is at least $\frac{\pi}{7}$. Hence, $p_{3}$ lies on the left of $\ell_{i}^{P6}$ for all $i \in \{h, \ldots, t\}$. Similarly, we show that $\ell_{i}^{P6}$ has $p_{6}$ on its left if and only if $i = \{1, \ldots, \frac{t-1}{2}\}$, and these lines also have $p_{3}$ on its left. Thus, $p_{3}$ is in the $P_{t}$-of-influence of $p_{5}$ and $p_{6}$. Moreover, since $p_{3}$ is strictly on the left of all the mentioned relevant lines, $p_{3}$ is in the interior of the $P_{t}$-of-influence of $p_{5}$ and $p_{6}$. Therefore, $p_{3}$ is in the interior of any $P_{t}$-disk containing $p_{5}$ and $p_{6}$. \hfill \square

Now, we proceed to prove Theorem 4.5.2

Proof of Theorem 4.5.2 Since the bottom side $e_{b}$ of $P_{t}^{1}$ is horizontal and the intersection of the three circles $C_{1}, C_{2}, C_{3}$ is only the point $c$, there is an empty space in $C_{1}$ bounded by $e_{b}$ and the circular arcs of $C_{2}$ and $C_{3}$ with endpoints $c$ and the intersection points $e_{b} \cap C_{2}$ and $e_{b} \cap C_{3}$. Let us call such area $A_{c}$. See Figure 4.18. Let $S'$ be a set of $n - 6$ points in general position contained in $A_{c}$. Let $S = \{p_{1}, p_{2}, p_{3}, p_{4}, p_{5}, p_{6}\} \cup S'$.

Since for $i = 1, 2, 3$ the $P_{t}$-disk $P_{i}^{t}$ contains no point of $S$ in its interior, from Claim 4.5.3 it follows that the edges $p_{1}p_{2}, p_{2}p_{3}, p_{3}p_{1}$ are in $DG_{P_{t}}(S)$\footnote{Notice that each of the $P_{t}^{i}$ satisfies the following property: For any two of the three points of $S$ on its boundary, the $P_{t}$-disk can be slightly perturbed so that the two chosen points remain on its boundary and the third point lies in the exterior of the $P_{t}$-disk.} Also, since for each of the edges $p_{5}p_{2}, p_{2}p_{4}, p_{4}p_{1}, p_{1}p_{6}, p_{6}p_{3}$ and $p_{3}p_{5}$, its endpoints lie on $\partial P_{i}^{t}$ for some fixed $i \in \{1, 2, 3\}$, such edges are in $DG_{P_{t}}(S)$. By Claim 4.5.4 $p_{5}p_{6} \notin DG_{P_{t}}(S)$. Hence, the outerface of $DG_{P_{t}}(S)$ is given by the edges $p_{5}p_{2}, p_{2}p_{4}, p_{4}p_{1}, p_{1}p_{6}, p_{6}p_{3}$ and $p_{3}p_{5}$.
The graph $DG_{P_4}(S)$ is not 1-tough because $DG_{P_4}(S) \setminus \{p_1, p_2, p_3\}$ consists of four connected components, namely, $\{p_4\}$, $\{p_5\}$, $\{p_6\}$ and $DG_{P_4}(S')$. Therefore, $DG_{P_4}(S)$ is not Hamiltonian.

4.6 Conclusions

In this chapter we have presented the first general results on Hamiltonicity for higher-order convex-shape Delaunay and Gabriel graphs. By combining properties of metrics and packings, we have achieved general bounds for any convex shape, and improved bounds for point-symmetric shapes, as well as for even-sided regular polygons. For future research, we point out that our results are based on bottleneck Hamiltonian cycles, in the same way as all previously obtained bounds [2, 39, 67]. However, in several cases, as we show in Section 4.4, this technique is reaching its limit. Therefore a major challenge to effectively close the existing gaps will be to devise a different approach to prove Hamiltonicity of Delaunay graphs.
Affine invariance of various geometric structures is an important property in areas like graphics and computer aided geometric design. In the context of triangulations, consider a triangulation algorithm $T$, which given a point set $S$ computes a triangulation $T(S)$. We say that $T$ is affine invariant if and only if for any invertible affine transformation $\alpha$ (see Section 5.1 for a formal definition), the triangulations $\alpha(T(S))$ and $T(\alpha(S))$ are the same; i.e., triangle $\triangle pqr$ is in $T(S)$ if and only if triangle $\triangle \alpha(p)\alpha(q)\alpha(r)$ is in $T(\alpha(S))$. Note that $\alpha$ is not known to the triangulation algorithm.

Besides the Delaunay triangulation, another famous triangulation is the minimum weight triangulation, denoted $\text{MW} T$, which minimizes the sum of the Euclidean-length of its edges. The Delaunay triangulation may fail to be a minimum weight triangulation by a factor of $\Theta(n)$ where $n$ is the size of $S$ [69]. It is easy to see that neither the Delaunay triangulation nor the minimum weight triangulation is affine invariant in general (see Figure 5.1). This is because non-uniform stretching can make a point previously outside of a circumcircle become inside, or edge lengths can increase non-uniformly.

Affine invariance is also important in the analysis and visualization of data, to guarantee for instance that different units of measurement do not influence the geometric structure, such as a triangulation that is computed. For this reason, Nielsion [85] defined an affine invariant normed metric $A_S$ of a point set $S$, denoted $A_S$-norm, where for each point $v \in S$ and any affine transformation $\alpha$, $A_S(x) = A_{\alpha(S)}(\alpha(x))$. The $A_S$-norm produces ellipses (see Figure 5.2) as the boundary of the $A_S$-norm disk and using this notion Nielsion [86] defined an $A_S$-Delaunay triangulation that is affine invariant. Nielsion’s approach does not distinguish if the point set is rotated or reflected. While this is not an issue to obtain an affine invariant Delaunay triangulation, it makes the method
unsuitable to construct other triangulations or geometric objects, like the ones discussed in Section 5.3. Surprisingly the whole topic of affine invariant geometric constructions has gone virtually unnoticed in the computational geometry literature. Note that the use of the title “Affine invariant triangulations” have also been used by Haesevoets et al. [64], but with different meaning, to study affine invariant triangulation methods of the union of given triangles at different times (two triangles can overlap in different regions).

**Our contributions.** We revisit the $A_S$-norm and explain the geometry behind it in order to understand how the $A_S$-Delaunay triangulation behaves. We show that such triangulations have a spanning ratio related to the spanning ratio of the standard Delaunay triangulation, and that the hierarchy of subgraphs of the Delaunay triangulation, such as the minimum spanning tree [102] or the relative neighborhood graph [102] is also affine invariant. In addition, we show how to use the $A_S$-norm to compute different affine invariant orderings of a point set (radial order, sweep line ordering, and a polygon traversal ordering). Using these affine invariant orderings as subroutines, we can adapt standard geometric algorithms for computing a triangulation of a point set or a polygon to become affine invariant.

### 5.1 Preliminaries

An **affine transformation** $\alpha : X \rightarrow Y$ is a function of the form $\alpha(v) = Mv + b$ where $X$ and $Y$ are affine space mapped, $M$ is a linear transformation on each vector in $X$, and $b$ is a vector in $Y$. In this chapter we will work in $\mathbb{R}^2$, i.e., $X = Y = \mathbb{R}^2$, $M$ is a matrix in $\mathbb{R}^2 \times \mathbb{R}^2$ and $b$ is a vector in $\mathbb{R}^2$. For the rest of this chapter we will not distinguish a point from a vector unless notation is confusing, and we will assume that $\alpha$ is invertible, i.e., it is a non-degenerate function and $det(M) \neq 0$. Let $S$ be a point set, we denote $\alpha(S)$ the
Figure 5.2: Point set $S'$ is an affine transformation of $S$. Each color of the ellipses represents the corresponding boundary of the $A_S$- and $A_{S'}$-disk centered at the red point (mean) and containing the corresponding point of each transformation.

set \{\alpha(v) : v \in S\}. The following proposition states some well-known properties of affine transformations.

**Proposition 5.1.1.** Let $\alpha(v) = Mv + b$ be an affine invertible transformation on the affine space $X$ and let $S$ be a point set in $X$. Then function $\alpha$:

1. maps lines (resp., line segments) to lines (resp., line segments),
2. preserves parallelism between lines and line segments,
3. maps a simple $n$-gon to a simple $n$-gon,
4. preserves the ratio of lengths of two parallel segments,
5. preserves the ratio of areas of two object, and
6. maps the mean of $S$ to the mean of $\alpha(S)$.

Let $S = \{p_1, p_2, \ldots, p_n\}$ be a point set in the plane of size $n$. The order type of $S$ is a mapping that assigns to each ordered triple $i, j, k \in \{1, \ldots, n\}$ the orientation of $p_i, p_j$ and $p_k$ (either clockwise or counterclockwise). It can be shown that order types are preserved up to a change of sign, by checking for each triple $p_i, p_j, p_k \in S$ the signed area of the triangles $\triangle p_ip_jp_k$ and $\triangle \alpha(p_i)\alpha(p_j)\alpha(p_k)$ given by the following cross product $\alpha(p_i - p_j) \times \alpha(p_k - p_j) = \det(M)((p_i - p_j) \times (p_k - p_j))$.

We denote the coordinates of each point $p_i$ in $S$ by $(p_{ix}, p_{iy})$. Nielson [85] defines an affine invariant normed metric, that we call $A_S$-norm, in the following way.
Let

\[
\begin{align*}
\mu_x &= \frac{1}{n} \sum_{i=1}^{n} p_{ix}, \\
\mu_y &= \frac{1}{n} \sum_{i=1}^{n} p_{iy}, \\
\sigma_x^2 &= \frac{1}{n} \sum_{i=1}^{n} (p_{ix} - \mu_x)^2, \\
\sigma_y^2 &= \frac{1}{n} \sum_{i=1}^{n} (p_{iy} - \mu_y)^2, \\
\sigma_{xy} &= \frac{1}{n} \sum_{i=1}^{n} (p_{ix} - \mu_x)(p_{iy} - \mu_y).
\end{align*}
\]

Note that the mean \(\mu = (\mu_x, \mu_y)\) is the barycenter of \(S\). The covariance matrix of a point set \(S\) is defined as \(\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}\) and its inverse as

\[
\Sigma^{-1} = \frac{1}{\sigma_x^2 \sigma_y^2 - \sigma_{xy}^2} \begin{pmatrix} \sigma_y^2 & -\sigma_{xy} \\ -\sigma_{xy} & \sigma_x^2 \end{pmatrix}.
\]

Then, the \(A_S\)-norm metric is defined as \(A_S(p_x, p_y) = (p_x, p_y) \Sigma^{-1} \begin{pmatrix} p_x \\ p_y \end{pmatrix}\).

The matrix \(\Sigma^{-1}\) is also known as the concentration matrix [51], which defines a norm with respect to the normal (Gaussian) distribution defined by \(S\). The eigenvectors of \(\Sigma\) and \(\Sigma^{-1}\) are the same and these vectors define the principal orthogonal directions of how spread the point set is with respect to its mean (barycenter) \(\mu\). In other words, if we compute the Gaussian manifold defined by the bivariate normal distribution given by the point set \(S\) and then cut the Gaussian manifold with a plane parallel to the plane \(z = 0\), then we obtain an ellipse. See Figure 5.2. Such an ellipse has principal orthogonal axes defined by the eigenvectors of \(\Sigma^{-1}\). Thus, the boundary of an \(A_S\)-disk will be defined by a homothet of the resulting ellipse. In addition, the boundary of the unit \(A_S\)-disk will be represented by the ellipse with principal axes being parallel to the eigenvectors of \(\Sigma^{-1}\), and the magnitude of each principal axis will be given by the square root of the eigenvalue of the corresponding unit eigenvector.

Throughout this chapter, we say that a point set \(S\) is in general position if no three points are collinear and all points in \(S\) are at different \(A_S\)-norm distance from the mean \(\mu\). Since the boundary of the \(A_S\)-disk is an ellipse, Nielson computes the \(A_S\)-Delaunay triangulation, denoted \(DT_{A_S}(S)\), where an edge \(uv\) is in \(DT_{A_S}(S)\) provided that it satisfies the \(A_S\)-disk empty property, and shows the following.

**Theorem 5.1.2** (Nielson [86]). The triangulation represented by \(DT_{A_S}(S)\) is affine invariant.
5.2 The $A_S$-Delaunay triangulation revisited

In this section we discuss the connection between the standard Delaunay triangulation and the $A_S$-Delaunay triangulation.

Let $S$ be a point set in $\mathbb{R}^2$ in general position. Consider the $2 \times n$ matrix $N$ such that for each point $p$ in $S$ there is one column in $N$ represented by the vector $\bar{p} - \bar{\mu}$. Then, one can check that $\Sigma = \frac{1}{n}NN^T$. If a point set $S' = \alpha(S)$ and $\alpha(\bar{p}) = M\bar{p} + b$ with $p \in S$, is an affine transformation of the point set $S$, then its mean is given by $\alpha(\mu)$ and the covariance matrix $\Sigma'$ of $S'$ is given by $\Sigma' = M\Sigma M^T$, recall that $M$ is a $2 \times 2$ matrix in $\mathbb{R}^2$ with $\det(M) \neq 0$.

Since $S$ is in general position, $\det(\Sigma) \neq 0$. Thus, $\Sigma$ is invertible. Moreover, since $\Sigma$ is a square symmetric matrix, $\Sigma = Q\Lambda Q^T$ where $Q$ is the matrix of eigenvectors of $\Sigma$, $\Lambda$ is the diagonal matrix of eigenvalues and $Q^{-1} = Q^T$. Recall that if a matrix $M$ is diagonal, then $M = M^T$ and that $M^t$ with $t \in \mathbb{R}^+$ has entries $M^t_{i,j}$. Therefore, we can also rewrite the covariance matrix as $\Sigma = (QA_\frac{1}{2})(QA_\frac{1}{2})^T$. Furthermore, $Q$ represents a rotation matrix and $A_\frac{1}{2}$ represents the scaling factor of a point set where the covariance matrix is the identity matrix $I$. Looking carefully at this representation of $\Sigma$ and $\Sigma'$ above, we obtain that $(QA_\frac{1}{2})^{-1}S$ is an affine transformation of $S$ with $I$ as its covariance matrix. We refer to the point set $(QA_\frac{1}{2})^{-1}S$ as the point set $S$ normalized. Note that the unit $A_{(QA_\frac{1}{2})^{-1}S}$-disk is the Euclidean unit disk. This implies that the $A_{(QA_\frac{1}{2})^{-1}(S)}$-Delaunay triangulation of $(QA_\frac{1}{2})^{-1}S$ is given by the Delaunay triangulation of $(QA_\frac{1}{2})^{-1}S$, which together with Theorem 5.1.2 proves the following proposition.

**Proposition 5.2.1.** Let $S$ be a point set in general position in $\mathbb{R}^2$ and let $\Sigma = QAQ^T$ be its covariance matrix where $Q$ is the matrix of eigenvectors of $\Sigma$ and $\Lambda$ is the diagonal matrix of eigenvalues. Then, $DT((QA_\frac{1}{2})^{-1}S) = DT_{A_S}(S)$.

Let $S' = \alpha(S)$ be an affine transformation of the point set $S$ with covariance matrix $\Sigma' = Q'\Lambda'Q'^T$. Notice that even though the point sets $(QA_\frac{1}{2})^{-1}S$ and $(Q'\Lambda'\frac{1}{2})^{-1}S'$ have the Euclidean metric as the $A_S$-norm, both point sets can be different, since the unit eigenvectors are the ones mapped to the unit base $\{(1,0),(0,1)\}$. In other words, $(QA_\frac{1}{2})^{-1}S$ and $(Q'\Lambda'\frac{1}{2})^{-1}S'$ are the same point set up to rotations and reflections.

For instance, in Figure 5.3 we have on the left a point set that defines the Euclidean metric as its $A_S$-norm, in the middle we have the point set resulted from a rotation and scaling of the point set on the left. Finally, the point set on the right is a normalization of the point set in the middle. Note that the two point sets on the right and left are different by a rotation.
Another example is depicted in Figure 5.4 where we have two point sets in the middle that can be transformed into each other by an affine transformation – one is a reflection of the other by the $x$-axis. When we normalize each of the two middle point sets, we obtain the point set on the extreme left and right, respectively. Note that such these two point sets are different. Moreover, the point sets on the left and right have different order type.

On the other hand, a nice implication of Proposition 5.2.1 is that the $A_S$-Delaunay triangulation behaves in many ways like a standard Delaunay triangulation. For instance, from the results given by Dillencourt [49], it follows that the $DT_{A_S}(S)$ contains a perfect matching when $|S|$ is even, and that it is 1-tough.

It is known that the standard Delaunay triangulation is a spanner, see [50, 68, 108]. The following theorem shows that $DT_{A_S}(S)$ is also a spanner whose spanning ratio is a constant factor times the spanning ratio of the standard Delaunay triangulation. The constant factor depends on the eigenvalues of the co-variance matrix.
Theorem 5.2.2. Let $S$ be a point set in general position and let $\Sigma = Q \Lambda Q^T$ be the covariance matrix of $S$. Let $\lambda_{\text{max}}$ and $\lambda_{\text{min}}$ be the maximum and minimum eigenvalues of $\Sigma$, respectively. Then,

$$sr(DT_{AS}(S)) \leq \left(\frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}\right)^{\frac{1}{2}} \cdot sr(DT((Q \Lambda^{\frac{1}{2}})^{-1}S)).$$

Proof. Let $S' = (Q \Lambda^{\frac{1}{2}})^{-1}S$. The triangulation $DT(S')$ is a standard Delaunay triangulation. For every pair of points $u, v \in S'$ let $\delta_{uv}$ be a shortest path from $u$ to $v$ contained in $DT(S')$. By definition the spanning ratio $\frac{\sum_{(p_i, p_j) \in \delta_{uv}} d(p_i, p_j)}{d(u, v)} \leq sr(DT(S')).$

Note that the only thing that changes the spanning ratio is when the graph $DT(S')$ is stretched with different scaling factors in the $x$- and $y$-coordinates. As discussed previously, such scaling is defined by the square root of the eigenvalues of $\Sigma$ given in the diagonal matrix $\Lambda$. Hence, for any $u, v \in S'$ we have $d(\Lambda^{\frac{1}{2}} u, \Lambda^{\frac{1}{2}} v) \leq \lambda_{\text{max}}^{\frac{1}{2}} \cdot d(u, v)$ and $d(\Lambda^{\frac{1}{2}} u, \Lambda^{\frac{1}{2}} v) \geq \lambda_{\text{min}}^{\frac{1}{2}} \cdot d(u, v)$. Therefore, $sr(DT_{AS}(S)) \leq \left(\frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}\right)^{\frac{1}{2}} \cdot sr(DT(S')).$  

5.3 Primitives for other affine invariant geometric constructions

A natural line of study to follow is to consider other geometric objects with affine invariant construction algorithms, such as algorithms for triangulating a point set besides the $A_S$-Delaunay triangulation, triangulating a simple polygon, or computing a $k$-angulation of a point set, among others. In this section we identify some ingredients for defining such methods.

At first sight, when such constructions rely on a metric, then the $A_S$-norm can be used, for instance to compute an affine invariant Delaunay triangulations or a minimum weight triangulation. In general, these types of algorithms can be split into two categories: (1) based on an empty disk property, and (2) based on the rank of the length of the $\binom{n}{2}$ edges of a point set. Later we present different examples of algorithms that result to be affine invariant using the $A_S$-norm.

However, this does not always work. Many algorithmic techniques rely on the given order of the points, such as a radial order or the order obtained by sweeping the point set in a given direction. If we can use the $A_S$-norm in order to normalize $S$ by mapping all of its points to a point set $S'$ where the $A_{S'}$-norm is the Euclidean distance, then we can apply the standard algorithms in $S'$ for obtaining an affine invariant radial order or sweep-line order. Yet, as noted in Section 5.2 this is not enough. This only solves the scaling factors on the $x$- and $y$-coordinates, but it does not solve rotations or reflections. For instance,
if \( S \) is a point set such that \( A_S \)-norm is the Euclidean distance and \( S' \) is a rotation of \( S \), then the boundary of the \( A_{S'} \)-disk is also defined by a circle, and there is no transformation needed. However, the resulting order of a sweep-line with respect to the \( x \)-coordinates for \( S \) and \( S' \) will not necessarily be the same. Another simple example is when \( P \) is a simple polygon with vertex set \( S \), such that \( S \) has as \( A_S \)-disk the Euclidean disk. Consider point set \( S' \) obtained by reflecting \( P \) with respect to the \( x \)-axis, and the resulting polygon \( P' \). Again, \( S' \) will have as \( A_{S'} \)-disk the Euclidean disk. However, the clockwise order of \( S \) and \( S' \) might be different.

Thus, the next question is, what do we need in order to create affine invariant sorting methods? For any sorting method, we always need to define which point is the initial point, so, we have to be able to choose a point \( \varphi \) in the point set \( S \), such that for any affine transformation \( \alpha \), we always choose \( \alpha(\varphi) \) as the initial point. Second, for the cases of radial order and sweep-line we need a ray \( \overrightarrow{\varphi \mu} \) and a line \( \ell_\varphi \), respectively, such that for any affine transformation \( \alpha \), we always choose \( \alpha(\varphi)\alpha(\mu) \) and line \( \alpha(\ell_\varphi) \), respectively. Finally, since there might be reflections, which change clockwise for counterclockwise and left for right of a ray, we need to be able to choose the right orientation (clockwise or counterclockwise, right or left) whether there is a reflection or not. We can solve this problem if we are able to choose a point \( \delta \), such that for any affine transformation \( \alpha \), we always choose \( \alpha(\delta) \). Then the direction is given depending on which side of the plane \( \alpha(\delta) \) lies with respect to \( \alpha(\varphi)\alpha(\mu) \).

We say that a function \( f \) of \( S \) is affine invariant if and only if \( \alpha(f(S)) = f(\alpha(S)) \) for any affine transformation \( \alpha \). For instance, the median \( f(S) = \mu \) is an affine invariant function. Hence, note that if we manage to define affine invariant functions that compute a point, a ray, and an oriented line, there is no need for mapping \( S \) to a point set \( S' \) with \( A_{S'} \)-norm as the Euclidean metric.

Let \( S \) be a point set in general position. Let \( u, v \) be two points in \( S \). We say that point \( u \) is on the right of \( \overrightarrow{wz} \) if the signed area of \( \triangle zwu \) is positive. Otherwise, \( u \) is on the left of \( \overrightarrow{wz} \). Let \( \alpha(x) = Mx + b \) be an affine transformation of \( \mathbb{R}^2 \). Using that \( \alpha(w - z) \times \alpha(w - u) = \det(M)((w - z) \times (w - u)) \) one can check that if \( \det(M) < 0 \), then the orientation of \( \alpha(u) \) with respect to \( \alpha(w)\alpha(z) \) changes. Otherwise, the orientation remains the same. We say that \( u \) and \( v \) lie on the same side of a directed line segment \( \overrightarrow{wz} \) if \( \text{sign}((w - z) \times (w - u)) = \text{sign}((w - z) \times (w - v)) \).

**Observation 5.3.1.** Let \( \overrightarrow{wz} \) be a directed line segment in \( \mathbb{R}^2 \) and let \( v, u \) be two points in \( S \). Let \( \alpha \) be an affine transformation. The points \( u \) and \( v \) are on the same side of \( \overrightarrow{wz} \) if and only if \( \alpha(u) \) and \( \alpha(v) \) are on the same side of \( \alpha(w)\alpha(z) \).
Figure 5.5: The bold polygons define the convex hull of two point sets that can be transform into each other by an affine transformation. Equal color indicate pairs of points and their affine transformations. The triangle defined by the points $\mu$, black and purple is the triangle with largest area. In addition, the triangle defined by the points $\mu$, yellow and green is the triangle with second largest area.

**Proof.** Since $u$ and $v$ are on the same side of $\overrightarrow{wz}$, $\text{sign}((w - z) \times (w - u)) = \text{sign}((w - z) \times (w - v))$. Hence, 
\[
\text{sign}(\alpha(w) - \alpha(z)) \times (\alpha(w) - \alpha(u))) = \text{sign}(\text{det}(M))\text{sign}((w - z) \times (w - u)) = \text{sign}(\text{det}(M))\text{sign}((w - z) \times (w - v)) = \text{sign}((\alpha(w) - \alpha(z)) \times (\alpha(w) - \alpha(v))).
\]

Note that there may be several ways of implementing affine invariant functions that compute the desired primitives for sorting: a point $\wp \in S$, a ray $\overrightarrow{\wp\mu}$ and a point $\delta$ that defines an orientation with respect to $\overrightarrow{\wp\mu}$. In the rest of this section we provide two different procedures for defining the desired primitives.

In the following observation, we use the convex hull and the barycenter of $S$ in order to compute the primitives.

**Observation 5.3.2.** Let $S$ be a point set in general position in the plane. Let $\mu$ be the mean of $S$ and let $\text{CH}(S)$ be the convex hull of $S$. For each edge $p_ip_{i+1} \in \text{CH}(S)$, consider the triangle $p_ip_{i+1}\mu$. Assume that the areas of these triangles are pairwise different. Let $\vartheta$ and $\delta$ be the barycenters of the triangles with largest and second largest area, respectively, denoted $\triangle_{B_1}$ and $\triangle_{B_2}$, respectively. Consider the ray $\overrightarrow{\mu\vartheta}$. If $\delta$ is on the left of $\overrightarrow{\mu\vartheta}$, then let $\wp$ be the vertex of $\triangle_{B_1}$ that is on the left of $\overrightarrow{\mu\vartheta}$. Otherwise, let $\wp$ be the vertex of $\triangle_{B_1}$ that is on the right of $\overrightarrow{\mu\vartheta}$. The functions $f_1(S) = \vartheta, f_2(S) = \delta$ and $f_3(S) = \wp$ are affine invariant. See Figure 5.5

**Proof.** Let $S' = \alpha(S)$ be an affine transformation of $S$. Consider the convex hull $\text{CH}(S')$ of $S'$ and for each edge $p_ip_{i+1}' \in \text{CH}(S')$ we consider the triangles $\triangle_{B_1}'$ and $\triangle_{B_2}'$ of $\text{CH}(S')$ we consider the triangles $\triangle_{B_1}'$ and $\triangle_{B_2}'$ of $\text{CH}(S')$ we consider the triangles $\triangle_{B_1}'$ and $\triangle_{B_2}'$ that have largest and second largest area, respectively. Let $\triangle_{B_1}'$ and $\triangle_{B_2}'$ be the triangles with largest and second largest area.
area of the triangles defined by the edges of $CH(S')$ and $\alpha(\mu)$. Let $f_1(S') = \vartheta'$ and $f_2(S') = \delta'$ be the barycenters of $\triangle'_{B_1}$ and $\triangle'_{B_2}$, respectively. If $\delta'$ is on the left of $\overrightarrow{\alpha(\mu)\vartheta}$, then let $f_3(S') = \varphi'$ be the vertex in $\triangle'_{B_1}$ that is on the left of $\overrightarrow{\alpha(\mu)\vartheta}$. Otherwise, let $f_3(S') = \varphi'$ be the vertex $\triangle'_{B_1}$ that is on the right of $\overrightarrow{\alpha(\mu)\vartheta}$.

Let us show that $\alpha(f_1(S)) = \alpha(\vartheta) = \vartheta' = f_1(S')$, $\alpha(f_2(S)) = \alpha(\delta) = \delta' = f_2(S')$ and $\alpha(f_3(S)) = \alpha(\varphi) = \varphi' = f_3(S')$. From Proposition 5.1.1[6] the mean of $S'$ is $\alpha(\mu)$. From Proposition 5.1.1[3] the convex hull of $S$ is mapped to the convex hull $S'$. Also, from Proposition 5.1.1[3], for each edge $p_ip_{i+1} \in CH(S)$, the triangle $\triangle p_ip_{i+1}\mu$ is mapped to $\triangle(\alpha(p_i)\alpha(p_{i+1})\alpha(\mu))$. Let $\triangle$ be a triangle defined by the endpoints of an edge in $CH(S)$ and $\mu$, that is different from $\triangle_{B_1}$. Since $\text{Area}(\triangle_{B_1}) > \text{Area}(\triangle)$, $\frac{\text{Area}(\triangle_{B_1})}{\text{Area}(\triangle)} > 1$. From Proposition 5.1.1[5] $\frac{\text{Area}(\alpha(\triangle_{B_1}))}{\text{Area}(\alpha(\triangle)))} > 1$. Thus, $\text{Area}(\alpha(\triangle_{B_1})) > \text{Area}(\alpha(\triangle))$. Therefore, $\alpha(\triangle_{B_1}) = \triangle_{B_1}'$. By the same arguments, $\alpha(\triangle_{B_2}) = \triangle_{B_2}'$. Hence, $\vartheta'$ and $\delta'$ are the barycenters of $\alpha(\triangle_{B_1})$ and $\alpha(\triangle_{B_2})$, respectively. From Proposition 5.1.1[6] it follows that $\alpha(\vartheta) = \vartheta'$ and $\alpha(\delta) = \delta'$. From Proposition 5.1.1[1] we have that $\overrightarrow{\mu\vartheta}$ is mapped to $\overrightarrow{\alpha(\mu)\vartheta}$. From Observation 5.3.1 follows that $\alpha(\varphi) = \varphi'$.

Under the assumption that the area of the triangles defined by each edge of the $CH(S)$ and $\mu$ are pairwise different, we can define affine invariant functions that compute a point $\varphi$ of $S$, a ray $\overrightarrow{\varphi\mu}$, a line defined by $\mu$ and any of $\varphi$ and $\vartheta$; and a direction defined by a ray $\overrightarrow{\varphi\mu}$ and point $\delta$. On the other hand, notice that using the area of a triangle formed by the mean $\mu$ of $S$ and any pair of points, we obtain an affine invariant order of the edges of the complete graph of $S$ such that each edge has weight equal to the area of the triangle formed by its endpoints and $\mu$. Under the assumption that pairwise triangles have different area, we can also obtain affine invariant algorithms that are based on the rank of the edges of the complete graph.

We will say that a point $u$ is the $A_S$-closest point to $\mu$ if $u$ minimizes the $A_S$-distance to $\mu$. Using the fact that $A_S$-norm is affine invariant, we observe that there is another method for distinguishing the desired primitives for affine invariant radial and sweep-line ordered.

**Observation 5.3.3.** Let $S$ be a point set in general position and let $\mu$ be the mean of $S$ and let $\alpha(S)$ be an affine transformation of $S$ with mean $\mu'$. The $k$-th $A_S$-closest point to $\mu$ is mapped to the $k$-th $A_{\alpha(S)}$-closest point to $\mu'$.

**Proof.** Since $S$ is in general position, all of the points in $S$ are at different $A_S$-distance to $\mu$. Hence, we can order the points in $S$ in increasing order with respect to the $A_S$-distance from each point to $\mu$. Since $A_S$ is an affine
Figure 5.6: An affine invariant ordering of the point sets $S$ (left) and its affine transformation $S' = \alpha(S)$ (right).

Invariant norm, the points in $\alpha(S)$ are also at different $A_{\alpha(S)}$-distance to the mean $\alpha(\mu) = \mu'$. This again defines an order of $\alpha(S)$. Since the $A_S$-distance is invariant under affine transformations, it follows that the increasing order with respect to the $A_S$-distance from each point to $\mu$ is affine invariant. Thus, the $k$-th $A_S$-closest point to $\mu$ is mapped to the $A_{\alpha(S)}$-closest point to $\mu'$.

From Observation 5.3.3 it follows that we can also use the $A_S$-norm in order to define affine invariant functions $f_1(S)$ and $f_2(S)$, such that each function computes a point in the following fashion. Let $\mu$ be the mean of $S$, $f_1(S) = \wp$ be the $A_S$-closest point to $\mu$. Let $f_2(S) = \delta$ be the second $A_S$-closest point to $\mu$ if it is not on the line defined by $\mu$ and $\wp$. Otherwise, let $f_2(S) = \delta$ be the third $A_S$-closest point to $\mu$.

5.4 Affine invariant sorting algorithms of a point set

In this section we present two different sorting algorithms that together with the affine invariant primitives we defined before result into two different affine invariant sorting methods.

5.4.1 Affine invariant radial ordering

Let $\mu$, $\delta$ and $\wp$ be three non-collinear points, such that $\wp$ is in $S$. Consider the following radial sorting procedure of $S$.

$\text{RADIALLSORT}(S, \mu, \delta, \wp)$: Sort points radially around $\wp$ in the direction of $\delta$, starting from $\mu$. See Figure 5.6.

Using Observations 5.3.1 and 5.3.3 we can prove that the radial order produced by $\text{RADIALLSORT}$ is affine invariant.
Figure 5.7: (a) $p'_i$ is not in $T = \triangle p'_{i-1} \phi' \alpha(p_i)$ and $u$ is the intersecting point of $\phi'p'_i$ and $\alpha(p_i)p'_{i-1}$. (b) $p'_i$ is in the interior of $T = \triangle v \phi' \alpha(p_i)$ and the Euclidean distance from $\phi'$ to $v$ is the same from $\phi'$ to $p'_{i-1}$.

**Theorem 5.4.1.** Let $S$ be a point set in general position and let $f_1(S) = \mu$, $f_2(S) = \delta$ and $f_3(S) = \phi$ be three affine invariant functions such that $\mu, \delta, \phi$ are three non-collinear points and $\phi \in S$. Then, `RadiallySort`(S, $\mu, \delta, \phi$) computes an affine invariant radial order.

**Proof.** Let $\alpha$ be an affine transformation. Let $p_0 = \phi, p_2, \ldots, p_{n-1}$ be the order of $S$ given by `RadiallySort`(S, $\mu, \delta, \phi$) and let $p'_0 = \phi', p'_2, \ldots, p'_{n-1}$ be the order of $\alpha(S)$ given by `RadiallySort`($\alpha(S)$, $f_1(\alpha(S)) = \mu'$, $f_2(\alpha(S)) = \delta'$, $f_3(\alpha(S)) = \phi'$), respectively. Then, $\alpha(p_0) = \alpha(\phi) = \phi' = p_0, \alpha(\mu) = \mu'$ and $\alpha(\delta) = \delta'$. It remains to show that $p'_i = \alpha(p_i)$ for all $1 \leq i \leq n - 1$. If $p_1$ is on the ray $\overrightarrow{\phi' \mu}$, then $\alpha(p_1)$ is on the ray $\overrightarrow{\phi' \mu'}$. Thus, $p'_1 = \alpha(p_1)$. Assume that $p_1$ is not on $\overrightarrow{\phi' \mu}$.

Assume by induction that the order is affine invariant for the first $i - 1$ points and let us show $\alpha(p_i) = p'_i$. Assume for the sake of a contradiction that $\alpha(p_i) \neq p'_i$. First, let us show that $p'_i$ is on the same side as $\alpha(p_i)$ with respect to $\overrightarrow{\phi'p'_{i-1}}$ (we consider the ray $\overrightarrow{\phi' \mu'}$ when $i = 1$). Without loss of generality assume that $\delta'$ is to the right of $\overrightarrow{\phi' \mu'}$, since the other case is symmetric. If $\alpha(p_i)$ is to the right of $\overrightarrow{\phi'p'_{i-1}}$, then $p'_i$ has to be to the right of $\overrightarrow{\phi'p'_{i-1}}$ since $p'_i$ is hit first while rotating $\overrightarrow{\phi'p'_{i-1}}$ counterclockwise. If $\alpha(p_i)$ is to the left of $\overrightarrow{\phi'p'_{i-1}}$, then by Observation 5.3.1 $p_i$ is to the left (resp., to the right) of $\overrightarrow{\phi p_{i-1}}$ if $\delta$ is to the right (resp., to the left) of $\overrightarrow{\phi \mu}$. Thus, there is no point of $S$ to the right (resp., to the left) of $\overrightarrow{\phi \mu}$ if $\delta$ is to the left (to the right) of $\overrightarrow{\phi \mu}$. Hence, by Observation 5.3.1 there is no point to the left of $\overrightarrow{\phi'p'_{i-1}}$, in particular, $p'_i$ cannot be on that side.
Therefore, \( \alpha(p_i) \) and \( p'_i \) lie on the same side with respect to \( \overrightarrow{p'_{i-1}} \).

Let \( \ell \) be the line containing \( \varphi' \) and \( p'_{i-1} \) and let \( v \) be the point on \( \ell \) on opposite direction of \( \overrightarrow{p'_{i-1}} \) with respect to \( \varphi' \), such that the length from \( \varphi' \) to \( v \) is the same as the length from \( \varphi' \) to \( p'_{i-1} \). See Figure 5.7.

We define a triangle \( T \) in the following fashion.

- If \( \alpha(p_i) \) and \( p'_i \) are to the right of \( \overrightarrow{p'_{i-1}} \), then \( T = \triangle p'_{i-1} \varphi' \alpha(p_i) \).
- If \( \alpha(p_i) \) and \( p'_i \) are to the left of \( \overrightarrow{p'_{i-1}} \), then \( T = \triangle v \varphi' \alpha(p_i) \).

Then, either \( p'_i \) is in \( T \) or not. See Figure 5.7. Again, if \( p'_i \) is contained in \( T \) then from Proposition 5.1.1(3), \( \alpha^{-1}(p'_i) \) is contained in \( T \), which contradicts that \( p_i \) appears immediately after \( p_{i-1} \) when sorting \( S \) around \( \varphi \). On the other hand, if \( p'_i \) is not in \( T \), then line segment \( \overrightarrow{p'_{i-1}} \) intersects \( T \) on the edge opposite to \( \varphi' \). Let \( u \) be the intersection point of \( \overrightarrow{p'_{i-1}} \) and the edge of \( T \) opposite to \( \varphi' \). The point \( u \) is in triangle \( T \) and in line segment \( \overrightarrow{\varphi'p'_i} \). Thus, by Properties 1 and 3 of Proposition 5.1.1 \( \alpha^{-1}(u) \) is in \( \alpha(T) \) and on line segment \( \overrightarrow{\varphi\alpha^{-1}(p'_i)} \), which again contradicts the fact that \( p_i \) appears first while ordering \( S \) around \( \varphi \). Therefore, \( p'_i = \alpha(p_i) \).

**5.4.2 Affine invariant sweep-line ordering**

Let \( \mu, \delta \) and \( \varphi \) be three non-collinear points, such that \( \varphi \) is in \( S \). Consider the following sweep-line order method.

SweepLine\((S, \mu, \delta, \varphi)\): Let \( \ell^\perp \) be the line containing \( \mu \) and \( \varphi \). Let \( t \) and \( b \) be a farthest point from the line \( \ell^\perp \) that lies on the same and opposite side of \( \delta \) with respect to \( \ell^\perp \), respectively. Then, sort the vertices in \( S \) by sweeping with lines parallel to \( \ell^\perp \) from \( t \) towards \( b \). If two vertices \( u, v \) lie on the same line parallel to \( \ell^\perp \), we say that \( v \) appears before \( u \) if and only if \( \delta \) is on the left of \( \overrightarrow{\varphi\mu} \) and \( v \) is on the left of \( \overrightarrow{u\mu} \).

Note that a consequence of Observation 5.3.1 is that if two vertices of \( P \) lie on the same side with respect to \( \ell^\perp \), then the corresponding vertices in \( \alpha(P) \) lie again on the same side with respect to \( \alpha(\ell^\perp) \).

**Observation 5.4.2.** Let \( \ell \) be a line in \( \mathbb{R}^2 \) and let \( v, u \) be two points in \( S \). Let \( \alpha \) be an affine transformation. If \( u \) and \( v \) are on the same side of \( \ell \), then \( \alpha(u) \) and \( \alpha(v) \) are on the same side of \( \alpha(\ell) \).

The following theorem shows that such ordering is affine invariant when the primitives are affine invariant. See Figure 5.8.
Figure 5.8: Each colored line is a line parallel to the black line containing the corresponding \( f_3(S) = \varphi \) and \( f_1(S) = \mu \). Each color corresponds to the same line in the transformation \( \alpha \). SweepLine\((S, \mu, \delta, \varphi)\) is affine invariant.

**Theorem 5.4.3.** Let \( S \) be a point set in general position and let \( f_1(S) = \mu, f_2(S) = \delta \) and \( f_3(S) = \varphi \) be three affine invariant functions such that \( \mu, \delta, \varphi \) are three non-collinear points and \( \varphi \in S \). Then, \( \text{SweepLine}(S, \mu, \delta, \varphi) \) computes an affine invariant sweep-line of \( S \).

**Proof.** Let \( \alpha(x) = Mx + b \) be an affine transformation. Let \( f_1(\alpha(S)) = \mu', f_2(\alpha(S)) = \delta', f_3(\alpha(S)) = \varphi' \). Then, \( \alpha(\mu) = \mu', \alpha(\varphi) = \varphi', \alpha(\delta) = \delta' \). Let \( p_0, p_1, \ldots, p_{n-1} \) be the resulting order of \( \text{SweepLine}(S, \mu, \delta, \varphi) \) and let \( p'_0, p'_1, \ldots, p'_{n-1} \) be the resulting order of \( \text{SweepLine}(\alpha(S), \mu', \delta', \varphi') \). Let us show that \( \alpha(p_i) = p'_i \) for all \( i \in \{0, 1, \ldots, n - 1\} \).

Let \( \ell' \perp \) be the line containing \( \mu' \) and \( \varphi' \). From Proposition 5.1.1(1), \( \alpha(\ell') = \ell' \). We denote by \( \ell'_v \perp \) the line parallel to \( \ell' \perp \) containing the point \( v \in S \). Let \( t' \in \alpha(S) \) be a farthest point from \( \ell' \perp \) that lies on the same side of \( \delta' \) with respect to \( \ell' \perp \) and let \( b' \) be a farthest point from \( \ell' \perp \) that lies on the opposite side of \( \delta' \) with respect to \( \ell' \perp \). Thus, all points of \( \alpha(S) \) that are not on \( \ell'_v \perp \) lie on the same side of \( \ell'_v \perp \). Also, all points of \( \alpha(S) \) that are not on \( \ell'_b \perp \) lie on the same side of \( \ell'_b \perp \).

On the other hand, \( t \) and \( \delta \) are on the same side of \( \ell' \perp \). Thus, by Observation 5.4.2 \( \alpha(t) \) and \( \delta' \) are on the same side of \( \ell'_t \perp \). Since all points of \( S \) that are not on \( \ell'_t \perp \) lie on the same side of \( \ell'_t \perp \), by Observation 5.4.2 it follows that all points of \( \alpha(S) \) that are not on \( \ell'_t \perp_{\alpha(t)} \) lie on the same side of \( \ell'_t \perp_{\alpha(t)} \). Since both \( \alpha(t) \) and \( t' \) lie on the same side of \( \delta' \) with respect to \( \ell' \perp \), it follows that both lines \( \ell'_t \perp_{\alpha(t)} \) and \( \ell'_t \perp_{\alpha(t)} \) lie on the same side with respect to \( \ell' \perp \). Finally, since there
is exactly one line parallel to \( \ell^\perp \) on the same side of \( \delta' \) with respect to \( \ell'^\perp \) that passes through a point of \( S \) such that all points of \( S \) that are not in such line are on the same side, it follows that \( \ell'^\perp_{\alpha(t)} = \ell'^\perp_t \).

Similarly, since all points of \( S \) that are not on \( \ell'^\perp_b \) lie on the same side of \( \ell'^\perp_b \), by Observation 5.4.2 it follows that all points of \( \alpha(S) \) that are not on \( \ell'^\perp_{\alpha(b)} \) lie on the same side of \( \ell'^\perp_{\alpha(b)} \). In addition, since \( b \) is not on \( \ell'^\perp_t \), \( \alpha(b) \) is not on \( \ell'^\perp_t \). Therefore, \( \ell'^\perp_{\alpha(b)} = \ell'^\perp_b \). Thus, the initial and the final lines while sweeping \( S \) with lines parallel to \( \ell'^\perp \) are preserved under affine transformations. Assume that the order of the lines parallel to \( \ell'^\perp \) up to the \((i-1)\)-th line while sweeping \( S \) is preserved under affine transformations. Let \( \ell'^\perp_i \) be the \( i \)-th line parallel to \( \ell'^\perp \) while sweeping \( S \). From Observation 5.4.2 it follows that for each point \( v \) of \( S \) that lies on the same side of \( \ell'^\perp_i \) as \( t \), \( \alpha(v) \) lies on the same side of \( \alpha(\ell'^\perp_i) \) as \( t' \). Since all points of \( S \) that are on the same side of \( \ell'^\perp_i \) as \( t \) have been swept by the first \( i-1 \) lines parallel to \( \ell'^\perp \), it follows that \( \alpha(\ell'^\perp_i) \) is the \( i \)-th line parallel to \( \ell'^\perp \) while sweeping \( \alpha(S) \).

It remains to prove that if two points \( v \) and \( u \) of \( S \) lie on the same line parallel to \( \ell'^\perp \), then the order given is affine invariant. Let \( v \) and \( u \) be two points in \( S \) that lie on the same line parallel to \( \ell'^\perp \) such that \( v \) appears before \( u \) while ordering \( S \). From Proposition 5.1.1(1), \( \alpha(u) \) and \( \alpha(v) \) lie on the same line parallel to \( \ell'^\perp \). Assume \( \delta \) is on the left of \( \overrightarrow{q \mu} \), then \( v \) is on the left of \( \overrightarrow{u \mu} \). Consider the following cases.

\text{Case 1)} If \( \det(M) > 0 \), then \( \delta' \) is on the left of \( \overrightarrow{q' \mu'} \) and \( \alpha(v) \) is on the left of \( \alpha(u)\mu' \). Therefore, \( \alpha(v) \) appears before \( \alpha(u) \).

\text{Case 2)} If \( \det(M) < 0 \), then \( \delta' \) is on the right of \( \overrightarrow{q' \mu'} \) and \( \alpha(v) \) is on the right of \( \alpha(u)\mu' \). Therefore, \( \alpha(v) \) appears before \( \alpha(u) \).

The case when \( \delta \) is on the right of \( \overrightarrow{q \mu} \) is symmetric. \( \square \)

### 5.5 Applications to affine invariant geometric objects

In this section we give different application of the \( A_S \)-norm and methods given in Section 5.4 that result in affine invariant algorithms.

#### 5.5.1 Affine invariant algorithms based on \( A_S \)-norm

Consider a norm \( \rho : \mathbb{R}^2 \to \mathbb{R}^+ \). In this section we mention different geometric objects of a point set \( S \) constructed using the \( \rho \)-length of all possible \( \binom{n}{2} \) edges in \( S \). Using the fact that the \( A_S \)-norm is affine invariant, we have that the order of the edges between each pair of points in \( S \) given by the \( A_S \)-norm is also
affine invariant. Therefore, using such ordering gives us a set of affine invariant algorithms for computing geometric objects.

The first object is the MWT of $S$ that we already defined, which minimizes the sum of the $\rho$-length of its edges. The Greedy triangulation of $S$ is a triangulation that adds at each step an edge in strict increasing order of the $N$-length of the edges provided that the new edge cannot intersect in the interior of a previously inserted edge. The $\rho$-minimum spanning tree of $S$, denoted $\text{MST}_\rho(S)$, is a spanning tree of $G$ with minimum total edge length. The $\rho$-closest pair of $S$ computes the edge with lowest $\rho$-length. The $k$-$\rho$-nearest neighbor graph of $S$, denoted $\text{k-NNG}_\rho(S)$, is the graph with vertex set $S$ and an edge $uv \in \text{k-NNG}_\rho(S)$ provided that the $\rho$-distance between $u$ and $v$ is the $k$-th smallest $\rho$-distance from $u$ to any other point in $S$.

Consider an affine transformation $\alpha$, since $\alpha(S(u-v)) = \alpha(S)(\alpha(u)-\alpha(v))$, we obtain that if $A_S(u-v) \leq A_S(u'-v')$, then $A_{\alpha(S)}(\alpha(u)-\alpha(v)) \leq A_{\alpha(S)}(\alpha(u')-\alpha(v'))$. Hence, the increasing and decreasing order of the $A_S$-length of the edges in $S$ and $\alpha(S)$ is the same.

**Observation 5.5.1.** There exists an affine invariant order of the $\binom{n}{2}$ edges in any point set in general position.

Therefore, any greedy algorithm defined by the $A_S$-length of the edges of $S$ is affine invariant, i.e., geometric objects that can be obtained by the rank of its edges with respect to the $A_S$-norm. In particular, the $\text{MST}_{A_S}(S)$ is one of them.

Finally, since the $A_S$-disks are affine invariant, we obtain that the empty $A_S$-disk property is affine invariant.

**Corollary 5.5.2.** Let $S$ be a point set in general position. Then, the following objects are affine invariant: $\text{GG}_{A_S}(S)$, $\text{RNG}_{A_S}(S)$, $\text{k-DG}_{A_S}(S)$, $\text{k-GG}_{A_S}(S)$, $\text{k-RNG}_{A_S}(S)$.

In general, any object constructed based on the empty region property defined by the $A_S$-disk, will result affine invariant.

Note that, from Aurenhammer and Paulini [15] it follows that $\text{MST}_{A_S}(S) \subset \text{GG}_{A_S}(S) \subset \text{GG}_{A_S}(S) \subset \text{DT}_{A_S}(S)$.

### 5.5.2 An affine invariant Graham triangulation

One of the most popular algorithms for computing the convex hull of a point set $S$ is Graham’s scan [60]. A nice property of this algorithm is that a modification of the algorithm can also produce a triangulation, sometimes called the Graham triangulation [55]. In this section we present an affine invariant version of Graham’s scan using the $A_S$-norm metric.

Our method is based on the algorithm for Graham triangulation [103]. The
method consists in first choosing a point \( p \) and then radially sorting the remaining points around \( p \), say \( p_1, \ldots, p_{n-1} \). Once the points are sorted, the algorithm adds edge \( pp_i \) for all \( i \in \{1, \ldots, n-1\} \). Finally, the algorithm visits each remaining point \( p_i \) in order, and adds edge \( p_ip_j \) if and only if \( p_j \) is visible from \( p_i \), for all \( j > i \).

The Graham triangulation can be computed in linear time when \( S \) is given radially ordered. Moreover, since the edges of the triangulation are added according to the radial sort of \( S \), it follows that if the radial order is affine invariant, then the triangulation is affine invariant.

The following result is an implication of Theorem 5.4.1.

**Corollary 5.5.3.** There exists an affine invariant Graham triangulation for any point set \( S \) in general position.

### 5.5.3 An affine invariant Hamiltonian triangulation

When the point set \( S \) has at least one point in the interior of its convex hull, then \( S \) can be triangulated by the insertion method, which consists of computing the convex hull of \( S \) and then inserting points from the interior in arbitrary order. Every time a point \( v \) is inserted, the edges connecting \( v \) with the points defining the face that contains \( v \) are added.

Note that since the convex hull of a point set is preserved under affine transformations, it follows that if the points that are in the interior of the convex hull are inserted in an affine invariant order, then the insertion method computes an affine invariant triangulation of \( S \). Hence, applying the insertion method to \( S \) such that the interior points in the convex hull of \( S \) are inserted in the order given by \textsc{RadiallySort} computes an affine invariant triangulation.

**Corollary 5.5.4.** There exists an insertion method that computes an affine invariant triangulation for any point set in general position.

These triangulations are Hamiltonian, i.e., their duals\(^1\) contain a Hamiltonian path, and are of interest for fast rendering in computer graphics \( [10, 55] \).

**Corollary 5.5.5.** There exist methods that compute affine invariant triangulations that are Hamiltonian for any point set in general position.

\(^1\)The dual of a graph \( G \) has as vertices the faces of \( G \) and an edge between vertices is added if and only if the two faces share an edge in \( G \).
5.5.4 An affine invariant triangulation of a polygon by ear clipping

In this section we define a traversal of a polygon that is affine invariant.

An ear of a polygon is a triangle formed by three consecutive vertices \( p_1, p_2 \) and \( p_3 \) such that the line segment \( p_1p_3 \) is a diagonal\(^2\) of the polygon. It is a well-known fact that every simple polygon contains two ears (see Meisters [81]). By recursively locating and chopping an ear, one can triangulate any simple polygon. This method is known as ear clipping.

It follows from Properties 1 and 3 of Proposition 5.1.1 that the diagonals of a simple polygon are preserved under affine transformations. Thus, the ears of a simple polygon are also preserved. Hence, if the ear clipping procedure locates an ear by traversing the polygon in an affine invariant order, then such procedure computes an affine invariant triangulation. The traversal of the polygon in an affine invariant order depends only on finding an affine invariant starting point, and on deciding correctly whether to traverse it clockwise or counterclockwise. The latter depends only on whether the affine transformation contains an odd number of reflections.

Let \( P \) be a simple polygon with vertex sequence \( S := \{v_1, \ldots, v_n\} \), such that \( S \) is in general position. Let \( \mu, \delta \) and \( \wp \) be three non-collinear points such that \( \wp \) is in \( S \). Consider the following traversal of \( P \).

\( \text{TRAVERSAL}(P, \mu, \delta, \wp) \): If \( \delta \) is on the left of \( \overrightarrow{\mu \wp} \), then order \( S \) by traversing \( P \) from \( \wp \) in counterclockwise order. Otherwise, order \( S \) by traversing \( P \) from \( \wp \) in clockwise order. See Figure 5.9.

Using arguments similar to the ones used to prove Theorem 5.4.1, we show that \( \text{TRAVERSAL} \) is affine invariant.

**Theorem 5.5.6.** Let \( P \) be a simple polygon with vertex sequence \( S := \{v_1, \ldots, v_n\} \), such that \( S \) is in general position. Let \( f_1(S) = \mu, f_2(S) = \delta \) and \( f_3(S) = \wp \) be three affine invariant functions such that \( \mu, \delta, \wp \) are three non-collinear points and \( \wp \in S \). Then, \( \text{TRAVERSAL}(P, \mu, \delta, \wp) \) traverses \( P \) in an affine invariant order.

**Proof.** Let \( \alpha(P) \) be an affine transformation of \( P \). The vertices of \( \alpha(P) \) are \( \alpha(S) \). Consider \( \text{TRAVERSAL}(\alpha(P), f_1(\alpha(S)) = \mu', f_2(\alpha(S)) = \delta', f_3(\alpha(S)) = \wp') \). By definition, we have that \( \alpha(\mu) = \mu', \alpha(\wp) = \wp', \alpha(\delta) = \delta' \).

Let \( p_0 = \wp, p_1, \ldots, p_{n-1} \) be the resulting order of \( \text{TRAVERSAL}(P, \mu, \delta, \wp) \) and let \( p'_0 = \wp', p'_1, \ldots, p'_{n-1} \) be the resulting order of \( \text{TRAVERSAL}(\alpha(P), \mu', \delta', \wp') \). Let us show that \( \alpha(p_i) = p'_i \) for all \( i \in \{0, 1, \ldots, n - 1\} \).

\(^2\)A diagonal of a polygon is a line segment between two non-consecutive vertices that is totally contained inside the polygon.
Figure 5.9: An affine invariant traversal of two simple polygons with point set $S$ and $\alpha(S)$ from Figure 5.2.

Since the ordering is defined while traversing the simple polygon, it suffices to show that $\alpha(p_0) = p'_0$ and that the vertices are traversed in the same order. By definition, $p'_0 = \varphi' = \alpha(\varphi) = \alpha(p_0)$. In addition, the orientation of the points of $P$ changes in $\alpha(P)$ when there is an odd number of reflections in $\alpha$. Hence, if the orientation of the points of $P$ changes, then $\delta'$ is on the opposite direction of $\overrightarrow{\varphi'\mu'}$ as $\delta$ from $\overrightarrow{\varphi\mu}$. Therefore, $\text{TRAVERSAL}(P, \mu, \delta, \varphi)$ traverses $P$ in the opposite direction of $\text{TRAVERSAL}(\alpha(P), \mu', \delta', \varphi')$ traverses $P'$ if the orientation of the points in $P$ is changed in $\alpha(P)$. Thus, the vertices are traversed in the same order if the orientation of the points in $P$ is opposite to the one in $\alpha(P)$. Otherwise, if the orientation does not change, then $\text{TRAVERSAL}(\alpha(P), \mu', \delta', \varphi')$ traverses the points clockwise (resp., counterclockwise) if and only if $\text{TRAVERSAL}(P, \mu, \delta, \varphi)$ traverses the points clockwise (resp., counterclockwise). \hfill\qed

The following result is an implication of Theorem 5.5.6.

**Corollary 5.5.7.** There exists an affine invariant ear clipping triangulation for any simple $P$ with vertex set in general position.

### 5.5.5 An affine invariant triangulation of a polygon by sweep-line

In this section we give another affine invariant triangulation algorithm for any simple polygon.

A simple polygon $P$ is *monotone with respect to a line* $\ell$ if for any line $\ell^\perp$ perpendicular to $\ell$, the intersection of $P$ with $\ell^\perp$ is connected. A line $\ell$ divides the plane into two half planes, and we say that two points lie on the same side of $\ell$ if they lie on the same half plane. Let $v$ be a vertex of $P$ and $\ell_v$ be the...
Figure 5.10: Each colored line is a line parallel to the black line. The red diagonals partition the simple polygon into monotone polygons. Each diagonal contains a cusp with respect to the black line. Each color corresponds to the same line in the transformation.

line containing \( v \) that is parallel to \( \ell \). We say that \( v \) is an \( \ell \)-cusp if \( v \) is a reflex vertex of \( P \) and its neighbors in \( P \) lie on the same side of \( \ell_v \). A characterization of a monotone polygon is stated in the following property.

**Property 5.5.8** (Edelsbrunner [54]). A polygon is \( \ell \)-monotone if and only if it does not contain an \( \ell_\perp \)-cusp.

An \( \ell \)-monotone polygon can be triangulated by sweeping its vertices with line \( \ell_\perp \) (see, e.g., Garey et. al. [59]).

If \( P \) is not \( \ell \)-monotone, then one can split \( P \) into \( \ell \)-monotone subpolygons by adding diagonals in order to break all \( \ell_\perp \)-cusps. Then each resulting \( \ell \)-monotone polygon can be triangulated independently. The way to add these diagonals is due to Lee et. al. [76] and also uses a sweep line of \( \ell_\perp \). For each vertex \( v \) in \( P \) take the line \( \ell_v ^\perp \) and partition \( P \) into trapezoids defined by the intersection of the \( \ell_v ^\perp \) lines and the edges of \( P \). If a vertex \( v \) is an \( \ell_\perp \)-cusp, add the diagonal of \( P \) to the vertex \( u \) that is not in the same side as the neighbors of \( v \) with respect to \( \ell_v ^\perp \) and \( u \) is the opposite vertex in a trapezoid containing \( v \). See Figure 5.10.

Using Observation 5.4.2 we show that \( \ell_\perp \)-cusps are preserved under affine transformations.

**Lemma 5.5.9.** Let \( \alpha(P) \) be an affine transformation of \( P \). Each \( \ell_\perp \)-cusp of \( P \) is mapped to an \( \alpha(\ell_\perp) \)-cusp in \( \alpha(P) \).

**Proof.** Let \( v \) be an \( \ell_\perp \)-cusp of \( P \). Let us show that \( \alpha(v) \) is an \( \alpha(\ell_\perp) \)-cusp in
\( \alpha(P) \). Let \( v^- \) and \( v^+ \) be the neighbors of \( v \) in \( P \). If \( v^- \) and \( v^+ \) are on the same side of \( \ell^\perp_v \) then by Observation 5.4.2, \( \alpha(v^+) \) and \( \alpha(v^-) \) lie on the same side of \( \ell^\perp_v \). Moreover, the triangle \( \triangle v^- v^+ \) is not in \( P \). From Proposition 5.1.1(3) follows that the triangle \( \triangle \alpha(v^-) \alpha(v) \alpha(v^+) \) is not in \( \alpha(P) \). Hence, the interior angle of \( \alpha(v) \) in \( \alpha(P) \) is greater than \( \pi \). So, \( \alpha(v) \) is an \( \alpha(\ell^\perp) \)-cusp in \( \alpha(P) \).

Therefore, if \( P \) is \( \ell \)-monotone and \( \ell^\perp \) is perpendicular to \( \ell \), then \( \alpha(P) \) is \( \ell^* \)-monotone where \( \ell^* \) is perpendicular to \( \alpha(\ell^\perp) \).

Thus in order to compute an affine invariant triangulation of \( P \) it remains to give an affine invariant sweep-line order of its vertices. Therefore, from Theorem 5.4.3 we obtain the following corollary.

**Corollary 5.5.10.** There exists an affine invariant algorithm for triangulating a polygon \( P \) with vertex set in general position using a SWEEPLINE preprocessing of its vertices.

### 5.6 Conclusions

In this chapter we initiated the study of affine invariant geometric algorithms, a topic absent in the computational geometry literature. We revisited Nielsen’s affine invariant norm, whose unit disk represents how spread the point set is with respect to its mean. We also proposed affine invariant point sorting methods, which are necessary for other affine invariant geometric constructions. Our methods heavily rely on being able to distinguish three points. To this end, we gave two methods for distinguishing such points in Section 5.3. However, for this we had to require that the points are in certain general position. Otherwise, the point set becomes highly symmetric, which introduces a problem of indistinguishability. In particular, an interesting open question is to what extent such restriction can be removed, while still being able to distinguish rotations and reflections. Finally, we believe that finding affine invariant methods to construct other geometric objects is a promising direction for future research.
In this thesis we presented different contributions in the study of generalized Delaunay graphs. Each chapter presents a different problem in a specific generalization of the Delaunay graph. At the end of each chapter we presented different future lines of study to follow in each particular problem. We devote this last chapter to introduce a few other different open problems related to generalized Delaunay graphs.

In Chapter 4 we have shown upper bounds on the minimum $k$ for which the $k$-order convex shape Delaunay graph is Hamiltonian. Restricting the triangulation in its connectivity, a long-standing open problem is whether 3-connected Delaunay triangulations are Hamiltonian [48]. By loosening slightly the cycle property, the next question is whether the Delaunay triangulation always contains a Hamiltonian path. This question was answered in the affirmative when the convex distance is the square distance ($L_\infty$-metric) [3]. An interesting problem in this setting, is to determine for which convex shapes this is also true (in particular, circles).

A closely related problem is the one of perfect matchings. Using the fact that Delaunay triangulations are 1-tough, Dillencourt [49] showed that the Delaunay triangulation always contains a perfect matching. When focus shifted to convex shape Delaunay graphs, Ábrego et al. [3] proved that the □-Delaunay graph always contains a perfect matching. However, not all convex shape Delaunay graphs have a perfect matching. At least that is the case for triangles, since any triangulation is triangle-Delaunay-realizable [24]. Biniaz et al. [21] proved that the 2-order △-Delaunay graph (i.e., when the triangle is equilateral) always contains a perfect matching. Our results on Hamiltonicity in Chapter 4 imply bounds for the minimum $k$ for which the $k$-order convex shape Delaunay graph contains a perfect matching. However, we think that it must be possible to generalize the 1-toughness result of Dillencourt for point-symmetric convex shape Delaunay graphs and thus the containment of a perfect matching. In particular, the short proof given recently by Biniaz [19] gives some hope to believe this is true.

If we consider the $k$-order Delaunay graphs, any of the graph-theoretic properties have been studied by Abellanas et al. [2] and Bose et. al [34]. In particular,
Worst-case possible values for the studied properties. $SR, D, \kappa, \chi, \text{ and } \theta_c$ denote the spanning ratio, diameter, connectivity, chromatic number, and constrained geometric thickness, respectively. The results hold for specific ranges of $k$; see the complete statements throughout the paper.

<table>
<thead>
<tr>
<th>Gabriel graphs</th>
<th>Delaunay graphs</th>
</tr>
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<tbody>
<tr>
<td>$\theta\left(\sqrt{\frac{n}{k}}\right) \leq \max_{</td>
<td>S</td>
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<tr>
<td>$\left\lceil \frac{n}{k+1}\right\rceil \leq \max_{</td>
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<td>$\min_{</td>
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<tr>
<td>$3(k+1) \leq \max_{</td>
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<td>$3 \leq \max_{</td>
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<tr>
<td>$\frac{3k}{2} \leq \max_{</td>
<td>S</td>
</tr>
</tbody>
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Figure 5.11: This table appears as Table 1 in Bose et al. [34].

Bose et al. [34] gave several bounds for the spanning ratio, diameter, chromatic number and constrained geometric thickness, in both: $k$-Delaunay graphs and $k$-Gabriel graphs. However, most of these bounds have gaps that need to be closed. See Figure 5.11 Thus, the question is which are the correct bounds? or can we get better bounds for each: lower and upper bounds? A different but related line of study concerns the bounds in [34]. Which of these bounds are preserved for $k$-order $C$-Delaunay graphs? For instance, we believe that the bounds given for the diameter of the $k$-$DG(S)$ must hold for $k$-order $C$-Delaunay graphs, since their proof does not use a particular property of a circle but properties of homothets of a circle.

Other properties of study are the ones given by Binia et al. [21], in which they studied the $k$-order $\triangle$-Delaunay graphs. More precisely, they showed bounds on the minimum $k$ for which the $k$-$DG_{\triangle}(S)$ contains a $d_{\triangle}$-bottleneck of the following settings: biconnected spanning graph, Hamiltonian cycle and perfect matching. In Chapter 4 we gave bounds for the minimum $k$ in which the $k$-order $C$-Delaunay graph contains a $d_C$-bottleneck Hamiltonian cycle. Yet, besides trying to improve such bounds, there is the line of what happens with $d_C$-bottleneck biconnected spanning graphs and perfect matchings. Particularly, we believe that the bounds given by Binia et al. [21] on $d_{\triangle}$-bottleneck biconnected spanning graphs should also easily be generalized to any $C$-distance.

In summary, generalized Delaunay graphs are of great interest, with a vast sea of open questions. Thus, we invite everyone interested in Delaunay graphs to dive into this sea in order to find some solutions and, most likely, come up with more new questions waiting to be answered.


"So long, and thanks for all the fish."