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INVERSE SPECTRAL THEORY FOR GENERAL MATRIX OPERATORS

by

Kazem Ghanbari

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics
Carleton University
Ottawa, Ontario
June, 2001

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**INVERSE SPECTRAL THEORY FOR
GENERAL MATRIX OPERATORS**

Submitted by
Kazem Ghanbari, B.Sc., M.Sc.

in partial fulfilment of the requirements
for the degree of Doctor of Philosophy

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Abstract

In this thesis we study the spectral analysis corresponding to generalized eigenvalue problem $Ay = \lambda By$. We apply the concept of positive definite sequence (according to Ahiezer and Krein [2]) and the concept of m-functions to solve the generalized inverse eigenvalue problem (GIEP) $Ay = \lambda By$ in both finite and infinite dimensional cases (Chapter 1 and Chapter 2). We investigate the GIEP for symmetric matrices A and B (Chapter 3). We also study a moment problem corresponding to the infinite system $Ay = \lambda By$, where A is a tridiagonal block matrix and B is a diagonal block matrix.

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Introduction

Inverse eigenvalue problems, i.e., those that concern the reconstruction of matrices from a prescribed set of spectral data, are a very important subclass of inverse problems that arise in mathematical modeling. There has been challenging work on this area in the last 20 years. Inverse eigenvalue problems mostly are connected to the theory of orthogonal polynomials. An inverse eigenvalue problem appears to be more challenging when the objective matrix is a specifically structured matrix. Jacobi matrices are among those interesting structured matrices that are connected with three term recurrence relations. In this thesis we study some inverse spectral problems of the Jacobi operator

$$Jy_n = c_n y_{n+1} + a_n y_n + c_{n-1} y_{n-1}$$

which is a discrete analogue of a Sturm-Liouville operator, and their investigation has many similarities with the inverse spectral theory of Sturm-Liouville operators. The matrix representation of the Jacobi operator is a Jacobi matrix, thus we will be involved with the inverse problem of the type $Ay = \lambda By$, where A is a Jacobi matrix and B is a given symmetric matrix in general, or a diagonal matrix in particular.

Chapter 1 investigates a generalized inverse eigenvalue problem $Ay = \lambda By$, where A is a semi-infinite Jacobi matrix and B is a semi-infinite diagonal matrix with nonzero diagonal entries. The finite dimensional case of this problem has been studied by many

authors (e.g. [3, 5, 22, 23, 24, 25]). We use the concept of positive definite sequences to establish the machinery to deal with the infinite dimensional case.

Chapter 2 gives a new approach of dealing with the generalized inverse eigenvalue problem (GIEP) $Ay = \lambda By$, where A is a Jacobi matrix and B is a diagonal matrix. We study the GIEP by means of the most powerful tools from the spectral theory of $-\frac{d^2}{dx^2} + q(x)$, namely m -functions of Weyl, while the usual approach of dealing with the GIEP in the discrete cases are orthogonal polynomials as we do in Chapter 1. We propose a recursive way of computing the entries of A via the rational polynomial $m(z)$.

Chapter 3 considers the generalized inverse eigenvalue problem (GIEP) $Ay = \lambda By$, where A and B are real symmetric matrices. We investigate the GIEP in both singular and non-singular case of B . We also obtain an existence theorem in the case where A and B are Jacobi with B positive definite.

In Chapter 4 we study a classical moment problem and a generalized inverse eigenvalue problem for block Jacobi matrices. By defining a matrix valued inner product, we set up a class of orthogonal matrix polynomials which satisfy a three term recurrence relation, resulting in a block Jacobi matrix that has a prescribed sequence of square matrices as its moments. In the second part we study a generalized inverse eigenvalue problem for block Jacobi matrices. By defining an appropriate spectral function we obtain some conditions for solvability of the GIEP.

Chapter 0

Basic Concepts

In this chapter we discuss some basic concepts that we need in the following chapters. Most of the them are associated to semi-infinite Jacobi matrices, and they are direct extensions of the corresponding concepts for finite dimensional matrices. These can be found in textbooks on matrix theory.

0.1 Jacobi Matrices

Definition 0.1.1 Every semi-infinite tridiagonal matrix of the form

$$A = \begin{pmatrix} a_0 & c_0 & 0 & . & . & . \\ c_0 & a_1 & c_1 & 0 & . & . \\ 0 & c_1 & a_2 & c_2 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix} \quad (1)$$

is said to be a *Jacobi* matrix if $c_i > 0$, for $i = 0, 1, \dots$. For every k , the finite square $k \times k$ submatrix A_k in the upper left corner of A

is called a *leading principal* submatrix of A , i.e.,

$$A_k = \begin{pmatrix} a_0 & c_0 & 0 & . & . & . \\ c_0 & a_1 & c_1 & 0 & . & . \\ 0 & c_1 & a_2 & c_2 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & c_{k-2} \\ . & . & . & . & c_{k-2} & a_{k-1} \end{pmatrix}$$

An $n \times n$ matrix D is said to be *positive definite* if each leading principal submatrix of D has positive determinant. A Jacobi matrix A is said to be *positive definite* if all leading principal submatrices of A are positive definite.

Definition 0.1.2 Let A be a Jacobi matrix and $B = \text{diag}(b_0, b_1, \dots)$ with $b_i \neq 0$ for $i = 0, 1, \dots$. Let λ be a complex number.

- (i) If the equation $A\mathbf{x} = \lambda\mathbf{x}$ has a nontrivial solution $\mathbf{x}$, then λ is called an *eigenvalue* of A and $\mathbf{x}$ is called the corresponding *eigenvector*. The set of all eigenvalues of A is denoted by $\sigma(A)$.
- (ii) If the equation $A\mathbf{x} = \lambda B\mathbf{x}$ has a nontrivial solution $\mathbf{x}$, then λ is called a *generalized eigenvalue*. The set of all generalized eigenvalues is denoted by $\sigma(A, B)$.

0.2 Three-term Recurrence Relations and GIEP

Let $y_n(\lambda)$ be the solution of the three term recurrence relation

$$\begin{aligned} c_n y_{n+1}(\lambda) &= (b_n \lambda - a_n) y_n(\lambda) - c_{n-1} y_{n-1}(\lambda), \quad n = 0, 1, \dots \\ y_{-1}(\lambda) &= 0, \quad c_{-1} y_0(\lambda) = 1 \end{aligned} \quad (2)$$

where $\{a_n\}_{n \geq 0}$ is an arbitrary sequence of real numbers, $\{b_n\}_{n \geq 0}$, and $\{c_n\}_{n \geq 0}$ are sequences of nonzero real numbers with $c_n > 0$, for $n = 0, 1, \dots$.

We can display the recurrence relation (2) as

$$Ay = \lambda By \quad (3)$$

where

$$A = \begin{pmatrix} a_0 & c_0 & 0 & \cdot & \cdot & \dots \\ c_0 & a_1 & c_1 & 0 & \cdot & \dots \\ 0 & c_1 & a_2 & c_2 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix} \quad (4)$$

$$B = \text{diag} (b_0, b_1, \dots) \quad (5)$$

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots)^T. \quad (6)$$

If λ_r is a generalized eigenvalue and $y^{(r)} = (y_0(\lambda_r), y_1(\lambda_r), \dots)^T$ is the corresponding eigenvector, then the number

$$\rho_r = \left\{ \sum_0^{\infty} b_n |y_n(\lambda_r)|^2 \right\}$$

is called the *normalization constant*. The set of all normalization constants of the recurrence relation (2) is denoted by $\rho(A, B)$.

Let $\{\lambda_r\}_{r \geq 0}$ and $\{\rho_r\}_{r \geq 0}$ be real numbers so that $\lambda_i \rho_i > 0$, for $i = 0, 1, \dots$. The GIEP (Generalized Inverse Eigenvalue Problem) in this thesis is concerned with finding a positive definite semi-infinite Jacobi matrix A which satisfies (2) with

$$(i) \sigma(A, B) = \{\lambda_0, \lambda_1, \dots\},$$

$$(ii) \rho(A, B) = \{\rho_0, \rho_1, \dots\}, \text{ given at the outset.}$$

We call this problem an infinite dimensional GIEP in this thesis. The GIEP may have a unique solution or no solution depending on the given data, called *spectral data*. In Chapter 1 we obtain some conditions on the spectral data that will ensure the existence of a unique Jacobi matrix, A .

0.3 Quadratic Forms and Positive Definite Sequences

An infinite real *quadratic form* is an expression of the form

$$\sum_{i,k=0}^{\infty} a_{ik} \xi_i \xi_k, \quad (a_{ik} = a_{ki}), \quad \text{where the } \xi_i, \xi_k \in \mathbb{R}.$$

The infinite quadratic form is called *positive* if all its partial sums

$$\sum_{i,k=0}^n a_{ik} \xi_i \xi_k, \quad (n = 0, 1, \dots)$$

are positive for any choice of the $\xi_i \neq 0$. As a result, the Jacobi matrix A is positive definite if and only if the corresponding quadratic form is positive. If $\{s_k\}_{k \geq 0}$ is a sequence of real numbers the infinite quadratic form $\sum s_{i+k} \xi_i \xi_k$ is called a quadratic *Hankel* form.

Definition 0.3.1 An infinite sequence $\{s_k\}_0^\infty$ is said to be *positive definite* if the quadratic Hankel form $\sum s_{i+k} \xi_i \xi_k$ is positive. This is equivalent to the fact [2] that the determinants

$$D_k = \det \begin{pmatrix} s_0 & s_1 & \dots & s_k \\ s_1 & s_2 & \dots & s_{k+1} \\ . & . & \dots & . \\ . & . & \dots & . \\ s_k & s_{k+1} & \dots & s_{2k} \end{pmatrix} \quad (k = 0, 1, \dots)$$

are all positive. Let $R_n(\lambda) = \sum_{j=0}^n p_j \lambda^j$ be any nonnegative polynomial on $\mathbb{R}$. Then $R_n(\lambda)$ must be of even degree, say $n = 2m$, and it can be represented in the form

$$R_n(\lambda) = [A_m(\lambda)]^2 + [B_m(\lambda)]^2$$

where each of the polynomials

$$A_m(\lambda) = \sum_{k=0}^m x_k \lambda^k, \quad B_m(\lambda) = \sum_{k=0}^m y_k \lambda^k$$

has real coefficients [2]. Let $\{s_j\}_0^\infty$ be a given set of real numbers. Then, with the p_j , x_i , y_i defined above,

$$\sum_{j=0}^n p_j s_j = \sum_{i,k=0}^m s_{i+k} x_i x_k + \sum_{i,k=0}^m s_{i+k} y_i y_k.$$

The last equation implies that:

Lemma 0.3.1 The sequence $\{s_j\}_0^\infty$ is positive definite if $\sum_{j=0}^n p_j s_j$ is positive for every nonnegative polynomial

$$R_n(\lambda) = \sum_{j=0}^n p_j \lambda^j, \text{ for } n = 0, 2, \dots$$

Alaca [3] studied the finite version of the GIEP. She proved the following lemma that we will use in the first chapter, see [3, Lemma 1.4.1].

Lemma 0.3.2 Let A be a positive definite $m \times m$ Jacobi matrix and $B = \text{diag}(b_0, b_1, \dots, b_{m-1})$. If

$$\sigma(A, B) = \{\lambda_0, \lambda_1, \dots, \lambda_{m-1}\}, \text{ and } \rho(A, B) = \{\rho_0, \rho_1, \dots, \rho_{m-1}\},$$

then $\lambda_i \rho_i > 0$, for $i = 0, 1, \dots, m-1$. Moreover, $y_i(\lambda)$ and $y_j(\lambda)$ satisfy a *dual orthogonality* in the following sense

$$\int_{-\infty}^{\infty} y_i(\lambda) y_j(\lambda) d\tau_m(\lambda) = b_i^{-1} \delta_{ij}, \quad (i, j = 0, 1, \dots, m-1),$$

where $\tau_m(\lambda)$ is called the *spectral function*, and it is given by

$$\tau_m(\lambda) = \sum_{\lambda_r \leq \lambda} \frac{1}{\rho_r}$$

0.4 Power Moments and the Classical Moment Problem

Let $d\rho$ be a probability measure on $(-\infty, \infty)$. *Power moments* associated to the measure $d\rho$ are real numbers defined by

$$\mu_n = \int_{-\infty}^{\infty} \lambda^n d\rho(\lambda), \quad n = 0, 1, \dots$$

We will see that the power moments are closely related to a certain Jacobi matrix A , (Chapters 1 and 2).

The Hamburger Moment Problem. Given a sequence $\{\mu_n\}_{n \geq 0}$ of real numbers, when is there a measure $d\rho$ on $(-\infty, \infty)$ such that

$$\mu_n = \int_{-\infty}^{\infty} \lambda^n d\rho(\lambda), \quad n = 0, 1, \dots$$

and if such $d\rho$ exists, when is it unique?

This problem has been studied by many authors, (e.g., [3, 2, 37]). The most comprehensive research was published by Simon [37]. He achieved an existence and uniqueness theorem. Indeed, he proved that:

Theorem 0.4.1 A necessary and sufficient condition for the existence of a solution of the moment problem is that $\det(H_k) > 0$ for $k = 0, 1, 2, \dots$, where H_k is the Hankel matrix $H_k = [\mu_{i+j}]$, for $0 \leq i, j \leq k$.

For a given set $\{\mu_n\}_{n \geq 0}$ of moments there exists a Jacobi matrix A of the form (1) where a_n and c_n are explicitly computed in terms of the determinants of moments. We will discuss this in Chapter 1. Conversely, given a matrix A of the form (1), there exists moments $\{\mu_n\}_{n \geq 0}$ such that

$$\mu_n = \langle \delta_0, A^n \delta_0 \rangle = \int_{-\infty}^{\infty} \lambda^n d\rho(\lambda) \quad (7)$$

where $\delta_0 = (1, 0, 0, \dots)$ is in ℓ_2 , (see [37]).

Every matrix of the form (1), where a_n and c_n are all square matrices of the same size, is called a *block Jacobi* matrix. We will study an analog of the Hamburger moment problem in Chapter 4. In this case the moments are a sequence of square matrices of the same size, say $\{h_n\}_{n \geq 0}$, and we find a formula similar to (7), where $\delta_0 = (I_{p \times p}, 0, \dots)$ with $I_{p \times p}$ as the $p \times p$ identity matrix.

Chapter 1

An Infinite Dimensional GIEP

1.1 Introduction

In the present chapter we give an explicit solution to an infinite dimensional generalized inverse eigenvalue problem (GIEP) by establishing an appropriate spectral function with respect to a given set of spectral data. The finite dimensional case of this problem has been studied by many authors (e.g [3, 5]). We only discuss the infinite dimensional version of GIEP in this chapter.

1.2 Three-term Recurrence Relation and Its Associated Spectral Function

Let $y_n(\lambda)$ be solution of

$$\begin{aligned} c_n y_{n+1}(\lambda) &= (b_n \lambda - a_n) y_n(\lambda) - c_{n-1} y_{n-1}(\lambda), \quad n = 0, 1, \dots, \\ y_{-1}(\lambda) &= 0, \quad c_{-1} y_0(\lambda) = 1, \end{aligned} \tag{1}$$

where $\{a_n\}_{n \geq 0}$ is an arbitrary sequence of real numbers, $\{b_n\}_{n \geq 0}$ and $\{c_n\}_{n \geq 0}$ are sequences of nonzero real numbers.

Definition 1.2.1 A *spectral function* $\tau(\lambda)$ associated with the recurrence formula (1) is to be non-decreasing and right continuous, satisfying the boundedness requirement

$$\int_{-\infty}^{\infty} \lambda^{2n} d\tau(\lambda) < \infty, \quad n = 0, 1, \dots \quad (2)$$

for all n , and the orthogonality relation

$$\int_{-\infty}^{\infty} y_p(\lambda) y_q(\lambda) d\tau(\lambda) = b_p^{-1} \delta_{pq}, \quad p, q = 0, 1, \dots \quad (3)$$

Theorem 1.2.1 *If the Jacobi matrix A corresponding to the recurrence relation (1) is positive definite then there is at least one spectral function for the recurrence relation (1).*

Proof. Let A_m be any principal submatrix of A , for $m = 1, 2, \dots$. Then, A_m is positive definite for all m . We take a sequence of finite dimensional spectral functions $\tau_m(\lambda)$, $m = 1, 2, \dots$ given by Lemma 0.3.2. Hence with $i = j = 0$,

$$\int_{-\infty}^{\infty} y_0(\lambda)^2 d\tau_m(\lambda) = b_0^{-1} \quad \text{implies} \quad c_{-1}^{-2} \int_{-\infty}^{\infty} d\tau_m(\lambda) = b_0^{-1}$$

for all m . Thus $\tau_m(\lambda)$ is uniformly bounded; therefore we can select a subsequence $m_1, m_2, \dots$ such that $\tau_m(\lambda)$ converges to a limit, say $\tau(\lambda)$. Since $\tau_m(\lambda)$ is a non-decreasing sequence, it follows that $\tau(\lambda)$ is non-decreasing and bounded. Therefore we can make here the limiting transition in integrals with polynomial integrands ([5], Theorem 1.6.1), namely

$$\int_{-\infty}^{\infty} p(\lambda) d\tau_{m_n}(\lambda) \rightarrow \int_{-\infty}^{\infty} p(\lambda) d\tau(\lambda)$$

where $p(\lambda)$ is any polynomial. In particular,

$$\int_{-\infty}^{\infty} y_p(\lambda) y_q(\lambda) d\tau_{m_n}(\lambda) \rightarrow \int_{-\infty}^{\infty} y_p(\lambda) y_q(\lambda) d\tau(\lambda).$$

This completes the proof because we have dual orthogonality in finite dimensional case (see Lemma 0.3.2). ■

Theorem 1.2.2 *For every λ , the recurrence relation (1) has at least one nontrivial solution of summable square, in the sense that*

$$\sum_0^{\infty} b_n |y_n(\lambda)|^2 < \infty$$

Proof. See ([5], theorems 5.4.2, 5.6.1). ■

Definition 1.2.2 Let $\{y_n(\lambda)\}_{n \geq 0}$ be an arbitrary solution of the recurrence relation (1). If

$$\sum_0^{\infty} b_n |y_n(\lambda)|^2 < \infty$$

for some $\lambda \in \mathbb{C}$, then we say that the Limit-Circle case holds for the recurrence relation (1).

Theorem 1.2.3 *Suppose that the limit circle case holds for one value $\lambda = \lambda_0 \in \mathbb{C}$. Then this is true for all $\lambda \in \mathbb{C}$.*

Proof. See ([5], Theorem 5.6.1). ■

1.3 Eigenvalues in the Limit-Circle Case

Assuming that the limit-circle case holds, we take it that all solutions of (1) are summable square, and uniformly so in any finite λ -region. We take it that

$$\sum_0^{\infty} b_n |y_n(\lambda)|^2 < \eta(\lambda), \quad (4)$$

where $\eta(\lambda)$ is some function which is bounded for bounded λ . Adopting some fixed real μ as an eigenvalue of (1), we define the eigenval-

ues as the zeros of

$$(\lambda - \mu) \sum_0^{\infty} b_n y_n(\lambda) y_n(\mu). \quad (5)$$

By (4), the function (5) is an entire function of λ . It does not vanish identically, since its derivative is not zero when $\lambda = \mu$. Hence its zeros will have no finite limit. Moreover, these zeros will be real, being the limits of the zeros of

$$(\lambda - \mu) \sum_{n=0}^m b_n y_n(\lambda) y_n(\mu) = c_{m-1} \{y_m(\lambda) y_{m-1}(\mu) - y_m(\mu) y_{m-1}(\lambda)\}. \quad (6)$$

The zeros of the latter are real [3, theorem 4.3.1], and the zeros of (5) are the limits of the zeros of (6), by Rouché's theorem.

Having defined the eigenvalues λ_r as the roots of

$$\lim_{m \rightarrow \infty} c_{m-1} \{y_m(\lambda) y_{m-1}(\mu) - y_m(\mu) y_{m-1}(\lambda)\} = 0, \quad (7)$$

we may define a spectral function as a step function whose jumps are at the λ_r and of the amount

$$\frac{1}{\rho_r} = \left\{ \sum_{n=0}^{\infty} b_n |y_n(\lambda_r)|^2 \right\}^{-1}. \quad (8)$$

More formally we can express this spectral function as follows:

$$\tau(\lambda) = \sum_{\lambda_r \leq \lambda} \frac{1}{\rho_r}. \quad (9)$$

Remark 1.3.1 In general, we may have continuous spectrum for the recurrence relation (1). If we consider the discrete Schrodinger operator

$$Hy_n = y_{n+1} + y_{n-1} + a_n y_n$$

where, $\lim_{n \rightarrow \infty} |a_n| = 0$, then H has a continuous spectrum (see, for example, [26]).

Remark 1.3.2 Note that we can display the recurrence relation (1) as

$$Ay = \lambda By \quad (10)$$

where

$$A = \begin{pmatrix} a_0 & c_0 & 0 & . & . & \dots \\ c_0 & a_1 & c_1 & 0 & . & \dots \\ 0 & c_1 & a_2 & c_2 & 0 & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix} \quad (11)$$

$$B = \text{diag} (b_0, b_1, \dots) \quad (12)$$

$$y(\lambda) = (y_0(\lambda), y_1(\lambda), \dots). \quad (13)$$

Lemmas 1.3.1 *Let A and B be the matrices given by (11) and (12), and $y(\lambda)$ be the vector given by (13). Let $\{\lambda_r\}_{r \geq 0}$ be the sequence of eigenvalues and $\{\rho_r\}_{r \geq 0}$ be the sequence defined by (8).*

(i) *If $i \neq j$ then $y(\lambda_i)$ is orthogonal to $y(\lambda_j)$ in the sense that*

$$\sum_{p=0}^{\infty} b_p y_p(\lambda_i) y_p(\lambda_j) = \rho_i \delta_{ij} \quad (14)$$

(ii) *If $p, q \geq 0$ then*

$$\sum_{r=0}^{\infty} y_p(\lambda_r) y_q(\lambda_r) \rho_r^{-1} = b_p^{-1} \delta_{pq} \quad (15)$$

Property (ii) is called the *dual orthogonality*. We denote the sequences $\{\lambda_r\}_{r \geq 0}$ and $\{\rho_r\}_{r \geq 0}$ by $\sigma(A, B)$ and $\rho(A, B)$, respectively.

Proof. See ([5], p. 133). ■

1.4 Infinite Dimensional GIEP

Let $\{b_n\}_{n \geq 0}$ be a sequence of nonzero real numbers and let $B = \text{diag}(b_0, b_1, \dots)$. The objective is to find a positive definite semi-infinite Jacobi matrix A of the form (11) which satisfies (10) with

$$\begin{aligned} (i) \quad & \sigma(A, B) = \{\lambda_0, \lambda_1, \dots\}, \\ (ii) \quad & \rho(A, B) = \{\rho_0, \rho_1, \dots\}. \end{aligned}$$

We call this problem an infinite dimensional GIEP.

Remark 1.4.1 If there is a solution for the GIEP with $\sigma(A, B) = \{\lambda_0, \lambda_1, \dots\}$ and $\rho(A, B) = \{\rho_0, \rho_1, \dots\}$, then $\lambda_i \rho_i > 0$. For, if λ_i is in $\sigma(A, B)$, and $y^{(i)}$ is the corresponding eigenvector, then $Ay^{(i)} = \lambda_i B y^{(i)}$ hence

$$0 < y^{(i)T} A y^{(i)} = \lambda_i y^{(i)T} B y^{(i)} = \lambda_i \rho_i. \blacksquare$$

Lemma 1.4.1 The equation (3) with $p \neq q$ is equivalent to

$$\int_{-\infty}^{\infty} y_p(\lambda) \lambda^q d\tau(\lambda) = 0, \quad 0 \leq q \leq p-1. \quad (16)$$

Proof. See ([5], Theorem 4.6.1). $\blacksquare$

Definition 1.4.1 Let $\tau(\lambda)$ be the spectral function defined by (9). The scalars

$$\mu_j = \int_{-\infty}^{\infty} \lambda^j d\tau(\lambda), \quad j = 0, 1, \dots \quad (17)$$

are called the *moments* of $\tau(\lambda)$.

Note that formula (17) is equivalent to

$$\mu_j = \sum_{r=0}^{\infty} \frac{\lambda_r^j}{\rho_r}, \quad j = 0, 1, \dots \quad (18)$$

and since $\lambda_r \rho_r > 0$, the odd moments μ_{2n+1} are all positive.

Definition 1.4.2 Let $\tau(\lambda)$ be the spectral function defined by (9) and let μ_j be the moments of $\tau(\lambda)$ given by (18). We define M_n and Δ_n by

$$M_n = \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{pmatrix} \quad (19)$$

$$\Delta_n = \det(M_n), \quad \Delta_0 = 1. \quad (20)$$

Lemma 1.4.2 *Using the notation in the previous definition, we have*

$$\begin{aligned} \Delta_{n+1} = & \mu_n \sum_{k=1}^n (-1)^k \mu_{n-1+k} A_{1k} + \mu_{n+1} \sum_{k=1}^n (-1)^{k+1} \mu_{n-1+k} A_{2k} \\ & + \dots + \mu_{n+r} \sum_{k=1}^n (-1)^{r+k} \mu_{n-1+k} A_{r+1,k} + \dots \\ & + \mu_{2n-1} \sum_{k=1}^n (-1)^{n-1+k} \mu_{n-1+k} A_{nk} + \mu_{2n} \Delta_n \end{aligned} \quad (21)$$

where A_{ij} is the determinant of the matrix obtained from M_n by deleting row i and column j . If $n = 1$ we set $A_{11} = 1$.

Proof. By induction on n . ■

1.5 Positive Definite Sequences

In this section we use the concept of positive definite sequences given by Ahiezer and Krein [2] to prove that the sequence of moments of the spectral function $\tau(\lambda)$ is a positive definite sequence.

Definition 5.1 Let $J = (a, b)$ ($-\infty \leq a < b \leq \infty$) be an interval in $\mathbb{R}$. An infinite sequence $\{s_k\}_{k \geq 0}$ is said to be positive definite if

and only if for every $n = 0, 1, \dots$

$$\sum_{j=0}^n p_j s_j > 0 \quad (22)$$

for every nonnegative polynomial $R_n(\lambda) = \sum_{j=0}^n p_j \lambda^j$ defined in $J = (a, b)$ (see Lemma 0.3.1).

Theorem 1.5.1 *Let $\{r_n\}_{n \geq 1}$ be a sequence of positive real numbers and let $\xi_1 < \xi_2 < \dots$ be a sequence of real numbers such that $\sum r_i \xi_i^k < \infty$, for all $k \geq 0$. We put*

$$s_k = \sum_{i=1}^{\infty} r_i \xi_i^k, \quad k = 0, 1, \dots \quad (23)$$

Then, the sequence $\{s_k\}_{k \geq 0}$ is positive definite in every interval (a, b) satisfying $-\infty \leq a < \xi_1 < \xi_2 < \dots < b \leq \infty$.

Proof. Let $\varphi(\lambda)$ be any real nonnegative polynomial on the interval (a, b) , say, $\varphi(\lambda) = \sum_{k=0}^n p_k \lambda^k$. We have

$$\sum_{k=0}^n p_k s_k = \sum_{k=0}^n \sum_{i=1}^{\infty} p_k r_i \xi_i^k = \sum_{i=1}^{\infty} r_i \left(\sum_{k=0}^n p_k \xi_i^k \right) = \sum_{i=1}^{\infty} r_i \varphi(\xi_i) > 0.$$

■

Corollary 1.5.1 *Let $\{\lambda_i\}_{i \geq 0}$ and $\{\rho_i\}_{i \geq 0}$ be the eigenvalues and the normalization constants of the GIEP. Then the sequence $\{\mu_i\}_{i \geq 1}$ is a positive definite sequence in $(-\infty, \infty)$.*

Proof. We define the real numbers

$$\begin{aligned} r_k &= \frac{\lambda_{k-1}}{\rho_{k-1}}, \quad k \geq 1, \\ \xi_k &= \lambda_{k-1}, \quad k \geq 1. \end{aligned} \quad (24)$$

If we set

$$s_k = \sum_{i=1}^{\infty} r_i \xi_i^k, \quad k = 0, 1, \dots \quad (25)$$

then $r_k > 0$ for $k \geq 1$. Thus, by Theorem 1.5.1 the sequence $\{s_k\}_{k \geq 0}$ is positive definite on $(-\infty, \infty)$. By (18), $s_k = \mu_{k+1}$, ($k = 0, 1, \dots$) and so $\{\mu_i\}_{i \geq 1}$ is a positive definite sequence on $(-\infty, \infty)$. ■

Definition 1.5.2 We say that an infinite real quadratic form

$$\sum_{i,k=0}^{\infty} a_{ik} \xi_i \xi_k, \quad (a_{ik} = a_{ki}) \quad (26)$$

is *positive* if all its partial sums

$$\sum_{i,k=0}^n a_{ik} \xi_i \xi_k, \quad (n = 0, 1, \dots) \quad (27)$$

are positive.

Theorem 1.5.2 *The sequence $\{s_n\}_{n \geq 0}$ is positive definite in the interval $(-\infty, \infty)$ if the infinite quadratic form*

$$\sum_{i,k=0}^{\infty} s_{i+k} \xi_i \xi_k$$

is positive.

Proof. See [Lemma 0.3.1]. ■

Theorem 1.5.3 *Let $\{\lambda_i\}_{i \geq 0}$ and $\{\rho_i\}_{i \geq 0}$ be the eigenvalues and the normalization constants of the GIEP. Let $\lambda_i \rho_i > 0$, and let $\{\mu_i\}_{i \geq 0}$ be a sequence of moments given by (18). Then the quadratic form*

$$\sum_{i,k=0}^{\infty} \mu_{i+k+1} \xi_i \xi_k$$

is positive.

Proof. Since $\lambda_i \rho_i > 0$, by Corollary 1.5.1 the sequence $\{\mu_i\}_{i \geq 1}$ is positive definite. By Theorem 1.5.2, the infinite quadratic form

$$\sum_{i,k=0}^{\infty} \mu_{i+k+1} \xi_i \xi_k$$

is positive. ■

Theorem 1.5.4 *Let $\{\lambda_i\}_{i \geq 0}$ and $\{\rho_i\}_{i \geq 0}$ be the eigenvalues and normalization constants of the GIEP. Let $\lambda_i \rho_i > 0$, and let $\{\mu_i\}_{i \geq 0}$ be a sequence of moments given by (18). Then*

$$\det \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_2 & \mu_3 & \dots & \mu_{n+1} \\ . & . & \dots & . \\ . & . & \dots & . \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n-1} \end{pmatrix} > 0$$

for $n = 1, 2, \dots$.

Proof. By Theorem 1.5.3, the quadratic form

$$\sum_{i,k=0}^{\infty} \mu_{i+k+1} \xi_i \xi_k$$

is positive. Then, the determinants of the principal submatrices are positive, i.e,

$$\det \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_2 & \mu_3 & \dots & \mu_{n+1} \\ . & . & \dots & . \\ . & . & \dots & . \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n-1} \end{pmatrix} > 0$$

for $n=1,2, \dots$. ■

1.6 Construction of a Solution for GIEP

In this section we conclude that, if the *limit circle* case holds for the recurrence relation (1), then we find an explicit solution for the GIEP.

Theorem 1.6.1 *Let $\tau(\lambda)$ be the spectral function defined by (9), and let $B = \text{diag}(b_0, b_1, \dots)$, where $\{b_n\}_{n \geq 0}$ is a sequence of nonzero real numbers. If the relation*

$$b_n \Delta_n \Delta_{n+1} > 0$$

for all $n \geq 0$, then there exists a countable set of orthogonal polynomials $\{y_n(\lambda)\}_{n \geq 0}$ (in the sense of the dual orthogonality property (15)), where the polynomials are determined up to a change of sign. Moreover, the polynomials $\{y_n(\lambda)\}_{n \geq 0}$ are dense in L_τ^2 , where L_τ^2 is the set of all real functions $f(t)$ such that

$$\int_{\mathbb{R}} [f(t)]^2 d\tau(\lambda) < \infty.$$

Proof. For the first part, we seek polynomials of the form

$$y_n(\lambda) = \beta_n(\lambda^n + \sum_{k=0}^{n-1} \alpha_{nk} \lambda^k) \quad k = 0, 1, \dots \quad (28)$$

where $y_0(\lambda) = \beta_0$ and $\beta_n \neq 0$. Using (15) and (17) we get

$$\beta_0^2 = \frac{1}{\mu_0 b_0} = \frac{\Delta_0}{b_0 \Delta_1}$$

which is positive by assumption. It follows from (28) that

$$(y_n(\lambda))^2 = \beta_n^2 [\lambda^{2n} + \sum_{k=0}^{n-1} \alpha_{nk} \lambda^{n+k}] + \beta_n y_n(\lambda) \sum_{k=0}^{n-1} \alpha_{nk} \lambda^k \quad (29)$$

$$n = 1, 2, \dots$$

Combining (15), (16), and (29) follows that

$$b_n^{-1} = \beta_n^2 (\mu_{2n} + \sum_{k=0}^{n-1} \alpha_{nk} \mu_{n+k}). \quad (30)$$

This gives β_n in terms of the α_{nk} and the moments. To determine α_{nk} , we substitute $y_n(\lambda)$ given by (28) in (16) and we obtain

$$\mu_{n+k} + \sum_{k=0}^{n-1} \alpha_{nk} \mu_{n+k} = 0 \quad (31)$$

where $0 \leq k \leq n-1$ and $n = 1, 2, \dots$. Using matrix notation this means

$$M_n(\alpha_{n0}, \alpha_{n1}, \dots, \alpha_{n,n-1})^T = (-\mu_n, -\mu_{n+1}, \dots, -\mu_{2n-1})^T. \quad (32)$$

Therefore

$$\alpha_{nr} = \left(\sum_{k=1}^n (-1)^{k+r} \mu_{n-1+k} A_{r+1,k} \right) / \Delta_n, \quad (33)$$

where $0 \leq r \leq n-1$ and $n = 1, 2, \dots$. Substituting equation (21) in (32), we get

$$\Delta_{n+1} = \Delta_n (\mu_{2n} + \sum_{k=0}^{n-1} \alpha_{nk} \mu_{n+k}). \quad (34)$$

Using (30) in (34) we obtain

$$\beta_n^2 = \frac{\Delta_n}{\Delta_{n+1} b_n} \quad (35)$$

which is positive by assumption, and this completes the proof. For the second part see ([5], p.141). ■

Theorem 1.6.2 *Let the assumptions of Theorem 1.6.1 hold. Then the Jacobi matrix A of the form*

$$A = \begin{pmatrix} a_0 & c_0 & 0 & . & . & \dots \\ c_0 & a_1 & c_1 & 0 & . & \dots \\ 0 & c_1 & a_2 & c_2 & 0 & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix}$$

where the entries are of the form

$$a_0 = -b_0 \alpha_{1,0}$$

$$a_n = b_n (\alpha_{n,n-1} - \alpha_{n+1,n}), \quad n = 1, 2, \dots$$

$$\alpha_{nr} = \left(\sum_{k=1}^n (-1)^{k+r} \mu_{n-1+k} A_{r+1,k} \right) / \Delta_n, \quad 0 \leq r \leq n-1, \quad n = 1, 2, \dots$$

$$\begin{aligned} c_n &= b_n \beta_n / \beta_{n+1}, \quad n = 0, 1, \dots \\ \beta_n^2 &= \Delta_n / \Delta_{n+1} b_n, \quad n = 0, 1, \dots \end{aligned}$$

is a solution for the GIEP, which assumes $\tau(\lambda)$ as its spectral function, where

$$\begin{aligned} y(\lambda) &= (y_0(\lambda), y_1(\lambda), \dots)^T \\ y_n(\lambda) &= \beta_n(\lambda^n + \sum_{r=0}^{n-1} \alpha_{nr} \lambda^r), \quad n = 0, 1, \dots \end{aligned}$$

Moreover, the matrix A is positive definite in the sense of Definition 1.5.2.

Proof. Substituting $y_n(\lambda)$ given by (28) in the recurrence relation (1) and comparing the corresponding coefficients of the powers of λ , we obtain

$$\begin{aligned} a_0 &= -b_0 \alpha_{1,0} \\ c_n \beta_{n+1} &= b_n \beta_n \\ c_n \beta_{n+1} \alpha_{n+1,n} + a_n \beta_n &= b_n \beta_n \alpha_{n,n-1}, \quad n \geq 1. \end{aligned}$$

By substituting the second equation into the third equation we obtain

$$a_n = b_n(\alpha_{n,n-1} - \alpha_{n+1,n})$$

which completes the first part of the proof. For the second part, let D_n be the determinant of the leading principal submatrices of A . By taking induction on n it can be checked that

$$D_n = b_0 b_1 \dots b_{n-1} (-1)^n \alpha_{n,0}, \quad \text{for } n = 1, 2, \dots \quad (36)$$

This is equivalent to

$$D_n = \frac{b_0 \Delta_0}{\Delta_1} \frac{b_1 \Delta_1}{\Delta_2} \dots \frac{b_{n-1} \Delta_{n-1}}{\Delta_n} (-1)^n \alpha_{n,0} \Delta_n. \quad (37)$$

By using (33) we obtain

$$(-1)^n \alpha_{n,0} \Delta_n = \det \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_2 & \mu_3 & \dots & \mu_{n+1} \\ . & . & \dots & . \\ . & . & \dots & . \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n-1} \end{pmatrix}.$$

Therefore, combining (37) with the assumptions of Theorem 1.6.1 we see that $D_n > 0$ if and only if

$$\det \begin{pmatrix} \mu_1 & \mu_2 & \dots & \mu_n \\ \mu_2 & \mu_3 & \dots & \mu_{n+1} \\ . & . & \dots & . \\ . & . & \dots & . \\ \mu_n & \mu_{n+1} & \dots & \mu_{2n-1} \end{pmatrix} > 0$$

which is true by Theorem 1.5.4, for $n = 1, 2, \dots$. This completes the proof. ■

Chapter 2

Generalized Inverse Eigenvalue Problem and m -Functions

2.1 Introduction

In this chapter we study the generalized inverse eigenvalue problem (GIEP), $Ax = \lambda Bx$, in which A is a Jacobi matrix of the form

$$A = \begin{pmatrix} a_1 & c_1 & 0 & . & . & . \\ c_1 & a_2 & c_2 & 0 & . & . \\ 0 & c_2 & a_3 & c_3 & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & c_{N-1} \\ . & . & . & . & c_{N-1} & a_N \end{pmatrix} \quad (1)$$

$B = \text{diag}(b_1, b_2, \dots, b_N)$, where $b_i \neq 0$ and $c_i > 0$, for $i = 1, 2, \dots, N$. In the literature the usual approach in solving GIEP is related to orthogonal polynomials as we used in Chapter 1. In the present chapter we use another approach, namely m -functions. We obtain a recursive formula to compute the entries of A via the rational

polynomial $m(z)$, called m -function in this thesis, defined by

$$m(z) = -\frac{1}{b_1} \prod_{i=1}^{N-1} (z - \mu_i) \prod_{j=1}^N (z - \lambda_j)^{-1}.$$

The main point is that a_1 and c_1 can be found from the asymptotic form of $m(z)$ (Theorem 2.3.1), as a result of which the m -function for the pencil $(A_{[2,N]}, B_{[2,N]})$ can be found. The zeros and poles of the resulting m -function serve as the spectral information for the reduced pencil $(A_{[2,N]}, B_{[2,N]})$ in exactly the same way as $m(z)$ for (A, B) . Therefore, the asymptotic formula can be applied to recover a_2 and c_2 , and the process repeats until all entries of A are found. We prove that the same procedure works for a semi-infinite bounded Jacobi matrix A .

2.2 m -functions

Let A be a Jacobi matrix of the type described above. We begin by defining some polynomials of a complex variable z which we denote by $\{p_n(z)\}_1^{N+1}$ and $\{\psi_n(z)\}_0^N$. The $p_n(z)$ is a polynomial of degree $n - 1$ defined by

$$\begin{aligned} c_n p_{n+1}(z) + a_n p_n(z) + c_{n-1} p_{n-1}(z) &= z b_n p_n(z) \\ 1 \leq n \leq N, p_0(z) &= 0, p_1(z) = 1. \end{aligned} \quad (2)$$

Inductively, it is clear that

$$p_{j+1}(z) = \frac{b_1 b_2 \dots b_j}{c_1 c_2 \dots c_j} z^j + \alpha_1 z^{j-1} + \dots \quad (3)$$

Lemma 2.2.1 *Let $\{p_j(z)\}$ be defined by (2). Then,*

$$p_{j+1}(z) = \frac{1}{c_1 c_2 \dots c_j} \det(z B_{[1,j]} - A_{[1,j]}), \quad j \geq 1 \quad (4)$$

where, $B_{[1,j]}$ and $A_{[1,j]}$ are the $j \times j$ principal sub-matrices in the upper left corners of B and A , respectively.

Proof. By (3),

$$\frac{c_1 c_2 \dots c_j}{b_1 b_2 \dots b_j} p_{j+1}(z), \quad \frac{1}{b_1 b_2 \dots b_j} \det(zB_{[1,j]} - A_{[1,j]}),$$

are monic polynomials of degree j . Thus, it suffices to prove that they have the same roots and multiplicities. But $p_{j+1}(z) = 0$ if and only if there is a vector $u = (u_1, u_2, \dots, u_j)$ with $u_1 = 1$ so that

$$(A_{[1,j]} - zB_{[1,j]})u^T = 0.$$

It is obvious that every eigenvector of the pair $(A_{[1,j]}, B_{[1,j]})$ has $u_1 \neq 0$. Thus, the zeros of $p_{j+1}(z)$ are precisely the eigenvalues of the pair $(A_{[1,j]}, B_{[1,j]})$. Since the eigenvalues are simple, the multiplicities are one, and this completes the proof. ■

We now define $\psi_n(z)$ by

$$c_n \psi_{n+1} + a_n \psi_n(z) + c_{n-1} \psi_{n-1}(z) = z b_n \psi_n(z) \quad (5)$$

$$1 \leq n \leq N, \quad \psi_N(z) = 1, \psi_{N+1}(z) = 0.$$

For convenience we define $a_0 = 1$ to enable us to define $\psi_0(z)$. The polynomial $\psi(z)$ is just like $p(z)$ but it is defined at the right end. Using an argument similar to the one in Lemma 2.2.1 we have

$$\psi_{N-j}(z) = \frac{1}{c_{N-1} c_{N-2} \dots c_{N-j}} \det(zB_{[N-j+1,N]} - A_{[N-j+1,N]}), \quad (6)$$

where, $A_{[j+1,N]}$ is a sub-matrix obtained from A by deleting $A_{[1,j]}$.

Definition 2.2.1 Given any two sequences u_n and v_n , the Wronskian $W_n(u, v)$ is defined by

$$W_n(u, v) = c_n [u_n v_{n+1} - u_{n+1} v_n]. \quad (7)$$

It is well known that for every two solutions of (2), W_n is constant (see, for example, [5]).

Definition 2.2.2 (*Green's function*) Let $\{e_n\}$ be the standard basis for $\mathbb{R}^N$. The Green's function related to the pair (A, B) is

given by

$$G(z, m, n) = \langle e_m, (A - zB)^{-1} e_n \rangle \quad (8)$$

where, $1 \leq m, n \leq N$, and $\Im(z) \neq 0$. We have the following formula:

Lemma 2.2.2 *The Green's function $G(z, m, n)$ is obtained by*

$$G(z, m, n) = [W_n(p, \psi)]^{-1} p_{\min(m, n)}(z) \psi_{\max(m, n)}(z). \quad (9)$$

Proof. It is easy to check that if $G(z, m, n)$ is defined by (9), then

$$\sum_k (A_{m, k} - zB_{m, k}) G(z, k, n) = \delta_{m, n}.$$

This implies that $G(z, m, n)$ is indeed the matrix of the resolvent (A, B) , i.e.,

$$G(z, m, n) = (A - zB)_{m, n}^{-1}. \blacksquare$$

Definition 2.2.3 (m -function) *The most basic m -function related to the pair (A, B) is defined by*

$$m(z) = \langle e_1, (A - zB)^{-1} e_1 \rangle. \quad (10)$$

Definition 2.2.4 *The spectral measure $d\tau$ for the pair (H, e_1) is defined by*

$$\langle e_1, H^\ell e_1 \rangle = \int_{\mathbb{R}} \lambda^\ell d\tau(\lambda), \quad \ell = 0, 1, 2, \dots$$

Remark 2.2.1 In terms of the spectral function τ ,

$$m(z) = \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - b_1 z}, \quad (11)$$

where $d\tau$ is a probability measure.

Lemma 2.2.3 *If ψ is the solution of (5), then*

$$m(z) = -\frac{\psi_1(z)}{c_0 \psi_0(z)}. \quad (12)$$

Proof. By the definition of the Green's function, (8), and the assumptions $p_0(z) = 0$, $p_1(z) = 1$, we get

$$m(z) = \langle e_1, (A - zB)^{-1} e_1 \rangle = G(z, 1, 1) = -\frac{\psi_1(z)}{c_0 \psi_0(z)}. \blacksquare$$

Theorem 2.2.1 *If $\sigma(A, B) = \{\lambda_1, \dots, \lambda_N\}$, and $\sigma(A_{[2,N]}, B_{[2,N]}) = \{\mu_1, \dots, \mu_{N-1}\}$, then*

$$m(z) = -\frac{1}{b_1} \prod_{i=1}^{N-1} (z - \mu_i) \prod_{j=1}^N (z - \lambda_j)^{-1} \quad (13)$$

where, $\lambda_1 < \lambda_2 < \dots < \lambda_N$, and $\mu_1 < \mu_2 < \dots < \mu_{N-1}$.

Proof. If we set $j = N$, and $j = N - 1$ in (6), respectively, we get

$$\psi_1(z) = \frac{1}{c_{N-1} c_{N-2} \dots c_1} \det(zB_{[2,N]} - A_{[2,N]}), \quad (14)$$

$$\psi_0(z) = \frac{1}{c_{N-1} c_{N-2} \dots c_1 c_0} \det(zB_{[1,N]} - A_{[1,N]}). \quad (15)$$

Combining (14), (15), and (12) we get

$$\begin{aligned} m(z) &= -\frac{\psi_1(z)}{c_0 \psi_0(z)} = -\frac{\det(zB_{[2,N]} - A_{[2,N]})}{\det(zB_{[1,N]} - A_{[1,N]})} \\ &= -\frac{|B_{[2,N]}|}{|B|} \prod_{i=1}^{N-1} (z - \mu_i) \prod_{j=1}^N (z - \lambda_j)^{-1} \\ &= -\frac{b_2 \dots b_N}{b_1 b_2 \dots b_N} \prod_{i=1}^{N-1} (z - \mu_i) \prod_{j=1}^N (z - \lambda_j)^{-1}. \blacksquare \end{aligned}$$

Corollary 2.2.1 *If $\{\lambda_i\}_{i=1}^N$ and $\{\mu_j\}_{j=1}^{N-1}$ are given sets of real numbers satisfying the interlacing property*

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \dots < \mu_{N-1} < \lambda_N, \quad (16)$$

then there exists a unique Jacobi matrix A , with

$$\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_N\}.$$

Proof. By Theorem 2.2.1, $\{\lambda_i\}_{i=1}^N$ and $\{\mu_j\}_{j=1}^{N-1}$ determine $m(z)$, and by (11), they determine $\tau(\lambda)$. Given the spectral function $\tau(\lambda)$, we can apply the procedure of Chapter 1 to determine the required Jacobi matrix A . ■

Definition 2.2.5 *Using the notations introduced above, we define*

$$m(z, n) = \langle e_{n+1}, (A_{[n+1, N]} - zB_{[n+1, N]})^{-1} e_{n+1} \rangle \quad (17)$$

$$\Im(z) \neq 0, \quad n = 0, 1, \dots, N-1.$$

Thus, $m(z) = m(z, 0)$ and as in Lemma 2.2.3 we have

$$m(z, n) = -\frac{\psi_{n+1}(z)}{c_n \psi_n(z)}. \quad (18)$$

Combining (18) with (5) we obtain

$$c_n^2 m(z, n) + \frac{1}{m(z, n-1)} = a_n - zb_n. \quad (19)$$

2.3 Solving GIEP via m -functions

In this section, we will use m -functions to show how to recover a Jacobi matrix from the spectral function $\tau(\lambda)$ and the diagonal matrix B .

Theorem 2.3.1 *The function $m(z)$ has the following asymptotic formula as $|z|$ goes to infinity ,*

$$m(z) = -\frac{1}{b_1 z} - \frac{a_1}{(b_1 z)^2} - \frac{(\frac{b_1}{b_2})c_1^2 + a_1^2}{(b_1 z)^3} - O(z^{-4}) \quad (20)$$

Proof. By (18) we have

$$m(z, 1) = -\frac{\psi_2(z)}{c_1 \psi_1(z)} = -\frac{\det(zB_{[3, N]} - A_{[3, N]})}{\det(zB_{[2, N]} - A_{[2, N]})}.$$

Therefore, $m(z, 1) = -\frac{1}{b_2 z} + O(z^{-2})$. By (19) we obtain

$$m(z) = \frac{1}{a_1 - b_1 z - c_1^2 m(z, 1)}, \quad (21)$$

and this implies that

$$\begin{aligned}
 m(z) &= -\frac{1}{b_1 z} \left[1 - \frac{a_1}{b_1 z} - \frac{c_1^2}{b_1 b_2 z^2} + O(z^{-3}) \right]^{-1} \\
 &= -\frac{1}{b_1 z} \left[1 + \frac{a_1}{b_1 z} + \frac{c_1^2}{b_1 b_2 z^2} + \left(\frac{a_1}{b_1 z} \right)^2 + O(z^{-3}) \right] \\
 &= -\frac{1}{b_1 z} - \frac{a_1}{(b_1 z)^2} - \frac{\left(\frac{b_1}{b_2} \right) c_1^2 + a_1^2}{(b_1 z)^3} - O(z^{-4}). \blacksquare
 \end{aligned}$$

Corollary 2.3.1 *In terms of the spectral measure $d\tau$, comparing the expansion (11) and (20), we obtain*

$$a_1 = \int_{\mathbb{R}} \lambda d\tau, \quad \left(\frac{b_1}{b_2} \right) c_1^2 = \int_{\mathbb{R}} \lambda^2 d\tau - a_1^2. \quad (22)$$

Remark 2.3.1 If $d\tau(\lambda)$ is not a probability measure, then the formulae given by (22) become,

$$a_1 = \frac{\int_{\mathbb{R}} \lambda d\tau}{\int_{\mathbb{R}} d\tau}, \quad \left(\frac{b_1}{b_2} \right) c_1^2 = \frac{\int_{\mathbb{R}} \lambda^2 d\tau}{\int_{\mathbb{R}} d\tau} - a_1^2.$$

Theorem 2.3.2 (Trace formula) *Let $\sigma(A, B) = \{\lambda_1, \dots, \lambda_N\}$, and $\sigma(A_{[2,N]}, B_{[2,N]}) = \{\mu_1, \dots, \mu_{N-1}\}$. Then*

$$a_1 = b_1 \left(\sum_{j=1}^N \lambda_j - \sum_{i=1}^{N-1} \mu_i \right) \quad (23)$$

$$2 \left(\frac{b_1}{b_2} \right) c_1^2 + a_1^2 = b_1^2 \left(\sum_{j=1}^N \lambda_j^2 - \sum_{i=1}^{N-1} \mu_i^2 \right) \quad (24)$$

Proof. By Theorem 2.2.1 we have

$$\begin{aligned}
 m(z) &= -\frac{1}{b_1} \prod_{i=1}^{N-1} (z - \mu_i) \prod_{j=1}^N (z - \lambda_j)^{-1} \\
 &= -\frac{1}{b_1 z} \prod_{i=1}^{N-1} \left(1 - \frac{\mu_i}{z} \right) \prod_{j=1}^N \left(1 - \frac{\lambda_j}{z} \right)^{-1} \\
 &= -\frac{1}{b_1 z} - \frac{\alpha}{(b_1 z)^2} - \frac{\beta}{(b_1 z)^3} + O(z^{-4}),
 \end{aligned}$$

where,

$$\alpha = b_1 \left(\sum_{j=1}^N \lambda_j - \sum_{i=1}^{N-1} \mu_i \right) \quad (25)$$

$$\beta = b_1^2 \left(\sum_{j=1}^N \lambda_j^2 + \sum_{j < k}^N \lambda_j \lambda_k + \sum_{i < m}^{N-1} \mu_i \mu_m - \sum_{j=1}^N \lambda_j \sum_{i=1}^{N-1} \mu_i \right) \quad (26)$$

Comparing with (20), we get $a_1 = \alpha$, and using (24) we have

$$\beta = b_1^2 \left(\frac{1}{2} \sum_{j=1}^N \lambda_j^2 - \frac{1}{2} \sum_{i=1}^{N-1} \mu_i^2 + \frac{\alpha^2}{2b_1^2} \right).$$

Therefore,

$$b_1^2 \left(\sum_{j=1}^N \lambda_j^2 - \sum_{i=1}^{N-1} \mu_i^2 \right) = 2\beta - \alpha^2 = 2 \left(\frac{b_1}{b_2} \right)^2 c_1^2 + a_1^2. \blacksquare$$

Remark 2.3.2 If we consider the case where B is the identity matrix, then (23) and (24) become

$$\text{Tr}(A) - \text{Tr}(A_{[2,N]}) = a_1$$

$$\text{Tr}(A^2) - \text{Tr}(A_{[2,N]}^2) = 2c_1^2 + a_1^2.$$

Corollary 2.3.2 We can now explain the procedure for recovering the matrix A from the given spectral function $\tau(\lambda)$, or equivalently $m(z) = \int_{\mathbb{R}} \frac{d\tau(\lambda)}{(\lambda - b_1 z)}$, and the diagonal matrix B as follows:

- (i) Use the formulae (22) to recover a_1 and c_1^2 .
- (ii) Using (21) we obtain,

$$m(z, 1) = c_1^{-2} \left(a_1 - b_1 z - \frac{1}{m(z)} \right),$$

which is the m -function for the pair $(A_{[2,N]}, B_{[2,N]})$.

- (iii) Go to step (i) to find a_2 and c_2^2 , and then (21) to find $m(z, 2)$, etc.

In order to illustrate Corollary 2.3.2, we give an example for the case $N = 3$.

Example 2.3.1 Let $N = 3$ and let $B = \text{diag}(b_1 = 1, b_2 = -1, b_3 = 1)$. With the notations of Chapter 1, let

$$\begin{aligned}\sigma(A, B) &= \{\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 3\}, \\ \rho(A, B) &= \{\rho_1 = -1, \rho_2 = 1, \rho_3 = 1\}.\end{aligned}$$

As discussed in Chapter 1, we define the spectral function $\tau(\lambda)$ as follows

$$\tau(\lambda) = \sum_{\lambda_i \leq \lambda} \frac{1}{\rho_i}, \quad \text{for } i = 1, 2, 3.$$

Obviously $\tau(\lambda)$ is a non-decreasing right continuous step function with jumps $\frac{1}{\rho_i}$ at λ_i , for $i = 1, 2, 3$. Using (22) gives $a_1 = 6$ and $c_1 = 2\sqrt{6}$. By (ii) of Corollary 2.3.2 we get

$$m(z, 1) = \frac{1}{24} \left(6 - z - \frac{1}{m(z)} \right)$$

where $m(z)$ can be obtained by using (11) as follows:

$$m(z) = \int \frac{d\tau(\lambda)}{\lambda - b_1 z} = \frac{1}{z+1} + \frac{1}{2-z} + \frac{1}{3-z}$$

Substituting this formula in the previous one we get

$$m(z, 1) = \frac{1}{24} \left(6 - z - \frac{(z+1)(z-2)(z-3)}{z^2 + 2z - 11} \right)$$

which can be simplified as $|z| \rightarrow \infty$, i.e.,

$$m(z, 1) = \frac{z - 5/2}{z^2 + 2z - 11} = \frac{1}{z} - \frac{9/2}{z^2} - \frac{20}{z^3} + O(z^{-4}).$$

On the other hand since $m(z, 1)$ is an m -function for the pair $(A_{[2,N]}, B_{[2,N]})$, we have an asymptotic expression for $m(z, 1)$ similar to (20):

$$m(z, 1) = -\frac{1}{(b_2 z)} - \frac{a_2}{(b_2 z)^2} - \frac{(b_2/b_3)c_2^2 + a_2^2}{(b_2 z)^3} + O(z^{-4})$$

where $b_2 = -1$. Therefore comparing the two last formulae gives a_2 and c_2 , i.e. $a_2 = \frac{9}{2}, c_2 = \frac{1}{2}$. For a_3 it suffices to put $\lambda_1 = -1$ in $\det(A - \lambda B) = 0$ which gives $a_3 = \frac{5}{2}$. Therefore the desired Jacobi matrix A is

$$A = \begin{pmatrix} 6 & 2\sqrt{6} & 0 \\ 2\sqrt{6} & 9/2 & 1/2 \\ 0 & 1/2 & 5/2 \end{pmatrix}.$$

Example 2.3.2 In the previous example the matrix B is indefinite. Now we consider an example where B is positive definite. Let $N = 2$, and $B = \text{diag}(b_1 = 1, b_2 = 2)$. Let

$$\begin{aligned} \sigma(A, B) &= \{\lambda_1 = -2, \lambda_2 = 3\}, \\ \rho(A, B) &= \{\rho_1 = 1, \rho_2 = 1/2\}. \end{aligned}$$

If we define $\tau(\lambda)$ as in the previous example, then $d\tau$ is not a probability measure, since we have $\int_{\mathbb{R}} d\tau = 3$. By using Remark 2.3.1 we obtain $a_1 = 4/3$, and $c_1 = \pm 10/3$. Consequently, $m(z)$ has the following asymptotic representation, near $z = \infty$,

$$m(z) = -\frac{1}{z} - \frac{4/3}{z^2} - \frac{22/3}{z^3} + O(z^{-4}).$$

Therefore we obtain the following solutions,

$$A_{\pm} = \begin{pmatrix} 4/3 & \pm 10/3 \\ \pm 10/3 & -2/3 \end{pmatrix}.$$

Remark 2.3.2 We can combine the method of m -functions summarized by Corollary 2.3.2 and Theorem 3.2.3 to recover the symmetric matrix A for the GIEP $Ax = \lambda Bx$, where B is a symmetric real matrix.

Remark 2.3.3 The Generalized Inverse Eigenvalue Problem is challenging if $c_i = 1$, for $i = 1, 2, \dots, N-1$. For example, if we consider

the easy case $N = 2$ with B identity the matrix, then a necessary and sufficient condition for solvability of the inverse problem $Ax = \lambda x$ is that $|\lambda_1 - \lambda_2| \geq 2$, where

$$A = \begin{pmatrix} a_1 & 1 \\ 1 & a_2 \end{pmatrix}.$$

Moreover, if $|\lambda_1 - \lambda_2| \geq 2$, then there exists two solutions. For, if A is given, we have

$$\lambda_1 = \frac{a_1 + a_2}{2} - \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + 1}, \quad \lambda_2 = \frac{a_1 + a_2}{2} + \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + 1}.$$

Clearly we get $|\lambda_1 - \lambda_2| = 2\sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + 1} \geq 2$. Conversely, if $|\lambda_1 - \lambda_2| \geq 2$, then we have

$$\det(\lambda I - A) = \lambda^2 - (a_1 + a_2)\lambda + a_1 a_2 - 1.$$

We want to find a_1, a_2 such that $\lambda_1 + \lambda_2 = a_1 + a_2$, $\lambda_1 \lambda_2 = a_1 a_2 - 1$. Therefore we obtain an equation in terms of a_1 as follows

$$a_1^2 - (\lambda_1 + \lambda_2)a_1 + \lambda_1 \lambda_2 - 1 = 0$$

which has the following discriminant

$$\Delta = (\lambda_1 + \lambda_2)^2 - 4.$$

Therefore $\Delta > 0$ if and only if $|\lambda_1 - \lambda_2| \geq 2$. Hence there are two solutions for a_1 , and two solutions for a_2 . For $N \geq 3$ the problem gets more complicated.

If A and B are symmetric matrices of order N with B positive definite, then we can find an explicit representation for $G(\lambda, i, j)$. It is well known (see e.g., [6]) that if $\{\lambda_i\}_1^N$ and $\{x^{(i)}\}_1^N$ are the eigenpairs of $Ax = \lambda Bx$, then A, B can be simultaneously diagonalized in the form

$$X^T A X = D, \quad X^T B X = I$$

where $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$, $X = [x^{(1)}, x^{(2)}, \dots, x^{(N)}]$, and therefore

$$A - \lambda B = (X^{-1})^T (D - \lambda I) X^{-1}.$$

Provided that $\lambda \neq \lambda_i$, $i = 1, 2, \dots, N$ we may invert this to give

$$(A - \lambda B)^{-1} = X(D - \lambda I)^{-1} X^T. \quad (27)$$

This gives the following lemma.

Lemma 2.3.1 *If the eigenvalue λ_k is simple. Then*

$$x_i^{(k)} x_j^{(k)} = \lim_{\lambda \rightarrow \lambda_k} (\lambda_k - \lambda) e_i^T (A - \lambda B)^{-1} e_j.$$

Proof. Let $\alpha_{ij}(\lambda)$ be the (i, j) th entry of $(A - \lambda B)^{-1}$. Then

$$\begin{aligned} \alpha_{ij}(\lambda) &= e_i^T (A - \lambda B)^{-1} e_j \\ &= \langle e_i, (A - \lambda B)^{-1} e_j \rangle \\ &= G(\lambda, i, j). \end{aligned}$$

Using (27) this gives

$$G(\lambda, i, j) = \sum_{k=1}^N \frac{x_i^{(k)} x_j^{(k)}}{\lambda_k - \lambda}.$$

Thus, provided that λ_k is simple, we get

$$x_i^{(k)} x_j^{(k)} = \lim_{\lambda \rightarrow \lambda_k} (\lambda_k - \lambda) G(\lambda, i, j). \blacksquare$$

We now apply this lemma to obtain the representation of $m(\lambda)$ in terms of eigenvalues, in general.

Theorem 2.3.3 *If A and B are symmetric matrices of order N with B positive definite and $\sigma(A, B) = \{\lambda_1, \dots, \lambda_N\}$, $\sigma(A_{[2,N]}, B_{[2,N]}) = \{\mu_1, \dots, \mu_{N-1}\}$ with the interlacing property (16), then*

$$m(\lambda) = \sum_{k=1}^N \frac{[x_1^{(k)}]^2}{\lambda_k - \lambda},$$

where

$$[x_1^{(k)}]^2 = \frac{|B_{[2,N]}|}{|B_{[1,N]}|} \prod_{j=1}^{N-1} (\mu_j - \lambda_k) \prod_{j \neq k}^N (\lambda_j - \lambda_k)^{-1}.$$

Proof. Consider the following equation

$$(A - \lambda B)y = e_1. \quad (28)$$

If $\lambda \neq \lambda_i$ ($i = 1, 2, \dots, N$), then

$$y_1 = e_1^T (A - \lambda B)^{-1} e_1 = \alpha_{11}(\lambda) = m(\lambda).$$

By applying Cramer's rule to (28) we get

$$y_1 = \frac{\det(A_{[2,N]} - \lambda B_{[2,N]})}{\det(A - \lambda B)} = \frac{P_{N-1}(\lambda)}{P_N(\lambda)}.$$

Since $\{\mu_i\}_1^{N-1}$ and $\{\lambda_i\}_1^N$ are zeros of $P_{N-1}(\lambda)$ and $P_N(\lambda)$, respectively, we obtain

$$m(\lambda) = \alpha_{11}(\lambda) = y_1 = \frac{|B_{[2,N]}|}{|B_{[1,N]}|} \prod_{j=1}^{N-1} (\mu_j - \lambda) \prod_{j=1}^N (\lambda_j - \lambda)^{-1},$$

and the previous lemma with $i = j = 1$, gives

$$[x_1^{(k)}]^2 = \frac{|B_{[2,N]}|}{|B_{[1,N]}|} \prod_{j=1}^{N-1} (\mu_j - \lambda_k) \prod_{j \neq k}^N (\lambda_j - \lambda_k)^{-1}. \blacksquare$$

2.4 Existence of Solution for GIEP

The algorithm given above shows that a given pair $(d\tau, B)$ comes from at most one A . We want to prove the existence of a solution for GIEP by the method above, i.e., given any spectral function $\tau(\lambda)$, this method yields a Jacobi matrix A whose spectral function is precisely $\tau(\lambda)$.

Every Borel measure μ on $\mathbb{R}$ corresponds to a spectral function

induced by μ as follows,

$$\tau(\lambda) = \begin{cases} -\mu((\lambda, 0]), & \text{if } \lambda < 0 \\ 0 & \text{if } \lambda = 0 \\ \mu((0, \lambda]), & \text{if } \lambda > 0 \end{cases}$$

which is a monotone increasing right continuous function on $\mathbb{R}$. Associated to τ is the Hilbert space $L^2(\mathbb{R}, d\tau)$ with inner product

$$\langle f, g \rangle = \int_{\mathbb{R}} f(\lambda)g(\lambda)d\tau(\lambda).$$

If $\int_{\mathbb{R}} d\tau(\lambda) < \infty$ we can define the *Borel transformation* of $d\tau(\lambda)$ by

$$F(z) = \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_{\pm},$$

where $\mathbb{C}_{\pm} = \{z \in \mathbb{C} \mid \pm \Im(z) > 0\}$.

Definition 2.4.1 A holomorphic function $F : \mathbb{C}_+ \rightarrow \mathbb{C}_+$ is called a *Herglotz function*.

Theorem 2.4.1 F is a Herglotz function if and only if

$$F(z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right) d\tau(\lambda), \quad z \in \mathbb{C}_{\pm},$$

where $a, b \in \mathbb{R}$, with $b \geq 0$, and $d\tau(\lambda)$ is a nonzero measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} \frac{d\tau(\lambda)}{1 + \lambda^2} < \infty.$$

Proof. See [41], Theorem B.2. ■

Remark 2.4.1 Every Borel transformation is a Herglotz function because it maps the upper half plane $\mathbb{C}_+$ into itself as can be seen from

$$\Im(F(z)) = \Im(z) \int_{\mathbb{R}} \frac{d\tau(\lambda)}{|z - \lambda|^2}.$$

Moreover, if α is a positive real number, then $F(\alpha z)$ is a Herglotz function.

Definition 2.4.2 Let Ω be a topological space, and let μ be an outer measure. We define its *support* as the closed set

$$\text{spt}(\mu) = \Omega - \bigcup \{G : \mu(G) = 0, \quad G \text{ open}\}.$$

Remark 2.4.2 If μ is a probability measure on $\mathbb{R}$ with N support points, then the spectral function induced by μ is a non-decreasing step function having jumps at the support points.

Lemma 2.4.1 Suppose that

$$m(z) = \int_{\mathbb{R}} \frac{d\tau(\lambda)}{\lambda - b_1 z},$$

where $d\tau(\lambda)$ is a probability measure on $[-c, c]$ whose support contains more than one point, and $b_1, b_2, \dots, b_N$ are given nonzero real numbers. Define

$$a_1 = \int_{\mathbb{R}} \lambda d\tau(\lambda), \quad \left(\frac{b_1}{b_2}\right) c_1^2 = \int_{\mathbb{R}} \lambda^2 d\tau(\lambda) - a_1^2. \quad (29)$$

Define $m_1(z)$ by

$$m_1(z) = c_1^{-2} \left[a_1 - b_1 z - \frac{1}{m(z)} \right].$$

Then

$$m_1(z) = \int_{\mathbb{R}} \frac{d\tau_1(\lambda)}{\lambda - b_2 z}, \quad (30)$$

where $d\tau_1(\lambda)$ is a probability measure also supported on $[-c, c]$. Moreover, $d\tau$ is supported on exactly N points if and only if $d\tau_1$ is supported on exactly $N - 1$ points.

Proof. By (29) and an expansion of a geometric series, (20) holds. Thus by combining (ii) of Corollary 2.3.3 and asymptotic

formula of $m(z, 1) = -\frac{1}{b_2 z} + O(z^{-2})$ we obtain

$$\tilde{m}(z) = (-m(z))^{-1} = b_1 z - a_1 - \frac{c_1^2}{b_2 z} + O(z^{-2}). \quad (31)$$

Since $m(z)$ is a Herglotz function, so is $\tilde{m}(z)$. Thus, by the Herglotz Representation Theorem, Theorem 2.4.1,

$$\tilde{m}(z) = \hat{c} + \hat{d}z + \int_{\mathbb{R}} \frac{d\tilde{\tau}(\lambda)}{\lambda - b_2 z} \quad (32)$$

for a measure $d\tilde{\tau}$. By (31) and (32), $\hat{c} = -a_1$, $\hat{d} = b_1$, and $\int_{\mathbb{R}} d\tilde{\tau} = c_1^2$. Thus,

$$m_1(z) = \int_{\mathbb{R}} \frac{d\tau_1(\lambda)}{\lambda - b_2 z},$$

and $d\tau_1 = c_1^{-2} d\tilde{\tau}$ is also a probability measure. Furthermore $d\tau$ is supported on N points if and only if $m(z) = \frac{P_{N-1}(z)}{Q_N(z)}$, where $P_{N-1}(z)$ and $Q_N(z)$ are polynomials of degree $N-1$ and N , respectively, which proves the last part of the theorem. ■

Theorem 2.4.2 Every N -Point supported probability measure is the spectral measure of a unique Jacobi matrix.

Proof. By the algorithm given in the previous lemma, we can find suitable c_j^2, a_j inductively. Since $d\tau$ has N -point support, the process terminates after $N-1$ steps, where $d\tau_N$ has a single support point, and we define c_N to be that point. Let $d\tilde{\tau}$ be the spectral measure for the Jacobi matrix A that has just been constructed. In order to complete the proof, we show that $d\tau = d\tilde{\tau}$. Let $\tilde{m}(z) = \int_{\mathbb{R}} \frac{d\tilde{\tau}}{\lambda - b_1 z}$. Then by the construction algorithm above,

$$\tilde{m}(z) = \frac{-1}{b_1 z - a_1 + c_1^2 \left[\frac{-1}{b_2 z - a_2 + c_2^2 \dots} \right]}$$

That is, $m, \tilde{m}$ have the same partial fraction expansion, although the remainders could be different. This means that the Taylor series for $\tilde{m}(z)$ near $z = \infty$ agrees with that for $m(z)$ near $z = \infty$ so $m(z) = \tilde{m}(z)$, thus $d\tau = d\tilde{\tau}$. ■

2.5 GIEP for Semi-infinite Jacobi Matrices

In Chapter 1, we studied the GIEP $Ax = \lambda Bx$, where A is a semi-infinite Jacobi matrix and B is a nonsingular diagonal matrix. We used the concept of positive definite sequences to construct an appropriate spectral function, consequently to construct the semi-infinite Jacobi matrix A . In this section we recover the semi-infinite Jacobi matrix A by using the approach of the present chapter. In order to take advantage of discrete operator theory, we restrict A and B^{-1} to be bounded. In fact, we restate Theorem 2.3.1 for the case $N = \infty$. First, we introduce some notations. Let $\mathbb{H}$ be a Banach space with norm $|\cdot|$, and let $\ell(\mathbb{N}, \mathbb{H})$ be the set of $\mathbb{H}$ -valued sequences $(u_n)_{n \in \mathbb{N}}$. We define

$$\begin{aligned} \ell^p(\mathbb{N}, \mathbb{H}) &= \{u \in \ell(\mathbb{N}, \mathbb{H}) \mid \sum_{n \in \mathbb{N}} |u_n|^p < \infty\}, \quad 1 \leq p < \infty \\ \ell^\infty(\mathbb{N}, \mathbb{H}) &= \{u \in \ell(\mathbb{N}, \mathbb{H}) \mid \sup |u_n| < \infty\} \end{aligned} \quad (33)$$

It is well-known that $\ell^p(\mathbb{N}, \mathbb{H})$, $1 \leq p \leq \infty$ is itself a Banach space with the following norms

$$\|u\|_p = \left(\sum_{n \in \mathbb{N}} |u_n|^p \right)^{1/p}, \quad 1 \leq p < \infty, \quad \|u\|_\infty = \sup |u_n| \quad (34)$$

In addition, if $\mathbb{H}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$, then $\ell^2(\mathbb{N}, \mathbb{H})$ is a Hilbert space with inner product

$$\langle u, v \rangle = \sum_{n \in \mathbb{N}} \langle u_n, v_n \rangle_{\mathbb{H}}, \quad u, v \in \ell^2(\mathbb{N}, \mathbb{H})$$

and corresponding norm

$$\|u\| = \|u\|_2 = \sqrt{\langle u, u \rangle}. \quad (35)$$

In what follows, we consider the Hilbert space $\ell^2(\mathbb{N}, \mathbb{R})$ in which the inner product and relevant norm are given by

$$\langle u, v \rangle = \sum u_n v_n, \quad \|u\| = \sqrt{\langle u, u \rangle}, \quad u, v \in \ell^2(\mathbb{N}).$$

Define a Jacobi operator A on $\ell^2(\mathbb{N})$ by

$$\begin{aligned} (Au)(n) &= c_n u_{n+1} + a_n u_n + c_{n-1} u_{n-1}, \quad n \geq 2, \\ &= c_1 u_2 + a_1 u_1, \quad n = 1, \end{aligned} \quad (36)$$

where, $a_n, c_n \in \mathbb{R}$ and $c_n > 0$, for $n = 1, 2, \dots$. Combining the definitions of A and the Wronskian $W_n(u, v)$ given by (7) gives the following formula, known as *Green's formula*,

$$\sum_{j=m}^n (u(Av) - (Au)v)(j) = W_n(u, v) - W_{m-1}(u, v). \quad (37)$$

Theorem 2.5.1 *Suppose*

$$a, c \in \ell^\infty(\mathbb{N}), \quad c_n > 0, \quad n = 1, 2, \dots \quad (38)$$

where $a = (a_n)_{n \in \mathbb{N}}$, and $c = (c_n)_{n \in \mathbb{N}}$. Then the Jacobi operator $A : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})$ given by (36) is a bounded self-adjoint operator. Moreover, a, c bounded is equivalent to A bounded since we have

$$\|a\|_\infty, \|c\|_\infty \leq \|A\|, \text{ and } \|A\| \leq 2\|c\|_\infty + \|a\|_\infty$$

where $\|A\|$ denote the operator norm of A .

Proof. Combining the assumption $c \in \ell^\infty(\mathbb{N})$ with $W_n(u, v) = c_n (u_n v_{n+1} - u_{n+1} v_n)$ implies that $\lim_{n \rightarrow \infty} W_n(u, v) = 0$, because the sequence $c = (c_n)_{n \in \mathbb{N}}$ is bounded and $u, v \in \ell^2(\mathbb{N})$ hence $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = 0$. This, and use of Green's formula implies that A is self-adjoint, i.e., for $u, v \in \ell^2(\mathbb{N})$ we have

$$\begin{aligned} \langle u, Av \rangle - \langle Au, v \rangle &= \sum_{j=1}^{\infty} (u(Av) - (Au)v)(j) \\ &= [u_1 A(v)(1) - A(u)(1)v_1] + \sum_{j=2}^{\infty} (u(Av) - (Au)v)(j) \\ &= c_1 [u_1 v_2 - v_1 u_2] + \lim_{n \rightarrow \infty} [W_n(u, v) - W_1(u, v)] \\ &= \lim_{n \rightarrow \infty} W_n(u, v) = 0. \end{aligned}$$

For the second part, consider $c_n^2 + c_{n-1}^2 + a_n^2 = \|A\delta_n\|^2 \leq \|A\|^2$, where δ_n is the vector in $\ell^2(\mathbb{N})$ with n^{th} entry 1 and the others 0. On the other hand, by definition of A , we have

$$\langle u, Au \rangle \leq (2\|c\|_\infty + \|a\|_\infty) \|u\|^2$$

which gives the last part of the theorem, i.e.

$$\|A\| \leq 2\|c\|_\infty + \|a\|_\infty. \blacksquare$$

This theorem leads one to prove Theorem 2.3.1 for the case $N = \infty$.

Theorem 2.5.2 *Let $b^{-1} = (b_n^{-1})_{n \in \mathbb{N}}$ be in $\ell^\infty(\mathbb{N})$. Then $m(z)$ has the following asymptotic formula as $|z|$ goes to infinity,*

$$m(z) = -\frac{1}{b_1 z} - \frac{a_1}{(b_1 z)^2} - \frac{(\frac{b_1}{b_2})c_1^2 + a_1^2}{(b_1 z)^3} - O(z^{-4})$$

Proof. By Theorem 2.5.1, A is a self-adjoint bounded operator, thus the norm convergent expression implies

$$\begin{aligned} (A - zB)^{-1} &= -z^{-1}B^{-1} (1 - z^{-1}(B^{-1}A))^{-1} \\ &= -z^{-1}B^{-1} - z^{-2}B^{-2}A - z^{-3}B^{-1}(B^{-1}A)^2 + O(z^{-4}). \end{aligned}$$

By definition of $m(z)$, we obtain

$$m(z) = \langle e_1, (A - zB)^{-1} e_1 \rangle = -\frac{1}{b_1 z} - \frac{a_1}{(b_1 z)^2} - \frac{(\frac{b_1}{b_2})c_1^2 + a_1^2}{(b_1 z)^3} - O(z^{-4}),$$

which completes the proof. $\blacksquare$

Therefore, we can use Corollary 2.3.2 for the case $N = \infty$ to recover the semi-infinite Jacobi matrix A by a given spectral measure $d\tau(\lambda)$.

Chapter 3

GIEP for symmetric Matrices

3.1 Introduction

In this chapter we study the general case of the inverse generalized eigenvalue problem $Ay = \lambda By$, where B is a symmetric matrix. We investigate the GIEP where A and B are real symmetric matrices. If A and B are Jacobi matrices with B positive definite, then the existence of a solution is obtained.

3.2 GIEP for Symmetric Matrices

In this chapter GIEP stands for the *generalized inverse eigenvalue problem* $Ax = \lambda Bx$, i.e., we are given a set of real numbers $\{\lambda_i\}_{i=1}^m$, as eigenvalues, a symmetric real matrix B , and the objective is to find a symmetric real matrix A such that $\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$. We study the GIEP in two cases.

Problem I Let B be a non-singular symmetric matrix, and let $\{\lambda_i\}_{i=1}^m$ be a given set of real numbers. We want to find a symmetric matrix A such that $\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$.

Problem II Solve Problem I if B is singular.

First we consider Problem I. Since B is non-singular, real, and symmetric, all eigenvalues of B are nonzero real numbers, say $\{\mu_i\}_{i=1}^m$.

Definition 3.2.1 We say that a real square matrix C is orthogonal if $C^T C = I$, where C^T is the transpose of C .

Theorem 3.2.1 *Let B be a real symmetric matrix. Then there is an orthogonal matrix T such that*

$$T^T B T = \begin{pmatrix} \mu_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_m \end{pmatrix} \quad (1)$$

where, $\mu_1, \mu_2, \dots, \mu_m$ are the eigenvalues of B . Equivalently,

$$(x, Bx) = \sum_{i=1}^m \mu_i y_i^2$$

where $y = T^T x$.

Proof. See ([6], p.54, Theorem 2). ■

Lemma 3.2.1 *Let $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ be a given set of nonzero real numbers. Let*

$$D = \begin{pmatrix} 1/\lambda_1 & 0 & 0 & \dots & 0 \\ 0 & 1/\lambda_2 & 0 & \dots & 0 \\ 0 & 0 & 1/\lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1/\lambda_m \end{pmatrix} \quad (2)$$

Then $\{\lambda_i\}_{i=1}^m$ are eigenvalues of $x = \lambda D x$, where $x = (x_1, x_2, \dots, x_m)^T$ is a vector.

Proof. This is clear. ■

Theorem 3.2.2 *Let B be a nonsingular symmetric real $m \times m$ matrix with eigenvalues $\{\mu_i\}_{i=1}^m$. Let $\{\lambda_i\}_{i=1}^m$ be a given set of real numbers such that $\lambda_i \mu_i > 0$. Then there exists a symmetric real $m \times m$ matrix A such that $\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$.*

Proof. By Theorem 3.2.1, there exists an orthogonal matrix P such that

$$P^T B P = \begin{pmatrix} \mu_1 & 0 & 0 & \dots & 0 \\ 0 & \mu_2 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \mu_m \end{pmatrix} \quad (3)$$

We define the matrix Q as follows

$$Q = \begin{pmatrix} 1/\sqrt{\lambda_1 \mu_1} & 0 & 0 & \dots & 0 \\ 0 & 1/\sqrt{\lambda_2 \mu_2} & 0 & \dots & 0 \\ 0 & 0 & 1/\sqrt{\lambda_3 \mu_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & 1/\sqrt{\lambda_m \mu_m} \end{pmatrix}. \quad (4)$$

Then we obtain

$$Q^T P^T B P Q = \text{diag}(1/\lambda_1, 1/\lambda_2, \dots, 1/\lambda_m) \quad (5)$$

If we put $T = PQ$, then we get $T^T B T = D$, where D is the matrix given by (2). Therefore we may write the equation

$$x = \lambda D x \quad (6)$$

as follows,

$$x = \lambda T^T B T x. \quad (7)$$

If we define $T^T A T = I$, then obviously A is symmetric since $(T^{-1})^T = (T^T)^{-1}$. Therefore, we may rewrite (7) in the form

$$T^T A T x = \lambda T^T B T x \quad (8)$$

which is equivalent to

$$Ay = \lambda By \quad (9)$$

where $y = Tx$. Therefore $\{\lambda_i\}_{i=1}^m$ are the eigenvalues of the inverse generalized eigenvalue problem $Ay = \lambda By$, i.e, $\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$. ■

Corollary 3.2.1 (Existence of a Solution) *Let B be a nonsingular real $m \times m$ matrix with eigenvalues $\{\mu_i\}_{i=1}^m$. If $\{\lambda_i\}_{i=1}^m$ is a given set of nonzero real numbers, then there exists a symmetric real $m \times m$ matrix A such that $\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$.*

Proof. If $\lambda_i \mu_i < 0$, for some i , then it suffices to replace $1/\sqrt{\lambda_i \mu_i}$ by $1/\sqrt{-\lambda_i \mu_i}$ in (4), and repeat the proof of Theorem 3.2.2. ■

Example 3.2.1 Assume $\{\lambda_1 = -1, \lambda_2 = 2\}$ is given. Let

$B = \begin{pmatrix} -2 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}$ be a given matrix.

Therefore $\sigma(B) = \{\mu_1 = -3, \mu_2 = 1\}$. All the assumptions of Theorem 3.2.2 are satisfied. Based on the algorithm given in Theorem 3.2.2, we get

$$P = \frac{1}{2} \begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}, \quad Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore, $T = PQ = \begin{pmatrix} \frac{-1}{2} & \frac{1}{4}\sqrt{2} \\ \frac{1}{6}\sqrt{3} & \frac{1}{4}\sqrt{6} \end{pmatrix}$, and so $TT^T = \begin{pmatrix} \frac{3}{8} & \frac{1}{24}\sqrt{3} \\ \frac{1}{24}\sqrt{3} & \frac{11}{24} \end{pmatrix}$.

Finally,

$$A = (TT^T)^{-1} = \begin{pmatrix} \frac{11}{4} & \frac{-1}{4}\sqrt{3} \\ \frac{-1}{4}\sqrt{3} & \frac{9}{4} \end{pmatrix}$$

where A is a symmetric real matrix with

$$\sigma(A, B) = \{\lambda_1 = -1, \lambda_2 = 2\}$$

as required.

Remark 3.2.1 Let B be a symmetric $m \times m$ real matrix. According to Theorem 3.2.1, there exists an orthogonal matrix T such that $T^T B T = \text{diag}(\mu_1, \mu_2, \dots, \mu_m)$, where $\mu_1, \mu_2, \dots, \mu_m$ are the eigenvalues of B . If we put $D = \text{diag}(\mu_1, \mu_2, \dots, \mu_m)$, then B is positive definite (or indefinite) if and only if D is positive definite (indefinite). As we mentioned in the previous chapter it has been proved ([3], [18], and [5]) that under some restrictions on D , we can find a Jacobi matrix J with $\sigma(J, D) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}$, where $\{\lambda_1, \lambda_2, \dots, \lambda_m\}$ is a given set of real numbers.

Theorem 3.2.3 *We use the notation in Remark 3.2.1. If J is a Jacobi solution of the inverse generalized eigenvalue problem with*

$$\sigma(J, D) = \{\lambda_1, \lambda_2, \dots, \lambda_m\} \quad (10)$$

then $A = T^T J T$ is a solution of Problem I with

$$\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_m\}.$$

Moreover, A is Jacobi if T is diagonal. Therefore, the determination of Jacobi solutions of the inverse generalized eigenvalue problem $Ax = \lambda Bx$ in the diagonal case of B leads one to determine the solutions of Problem I.

Proof. Since J is a solution, then for every λ in $\sigma(J, D)$ there is a nonzero vector x such that $Jx = \lambda Dx$. If we put $A = T^T J T$, then $J = T^T A T$, since T is orthogonal. Using Remark 3.2.1 we have $T^T B T = D$, hence $Jx = \lambda Dx$ is equivalent to

$$T^T A T x = \lambda T^T B T x \quad (11)$$

which is equivalent to $Ay = \lambda By$, where $y = Tx$. Therefore,

$$\sigma(A, B) = \{\lambda_1, \lambda_2, \dots, \lambda_m\} \quad (12)$$

which completes the proof. It is obvious that the solution A is positive definite if and only if J is positive definite. ■

Now, we are ready to treat *Problem II*. We will treat the problem by considering the multiplicity of the zero eigenvalue of B .

Problem II

We first consider the diagonal case of B , and then extend the idea to the general case. Note that every $m \times m$ diagonal matrix with main diagonal consist of a combination of zeros and non-zero entries $\mu_1, \mu_2, \dots, \mu_k$ is similar to the matrix $D = \text{diag}(\mu_1, \mu_2, \dots, \mu_k, 0, 0, \dots, 0)$, where $1 \leq k \leq m - 1$. It is clear by elementary row operations.

Theorem 3.2.4 *Let $D = \text{diag}(\mu_1, \mu_2, \dots, \mu_k, 0, 0, \dots, 0)$ be a real $m \times m$ diagonal matrix such that $\mu_i \neq 0$, for $1 \leq i \leq k$, and $\mu_i = 0$, for $k + 1 \leq i \leq m$. Let $\{\lambda_i\}_{i=1}^k$ be a given set of real numbers, and let $D_0 = \text{diag}(\mu_1, \mu_2, \dots, \mu_k)$. Let $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be a given set of real numbers. If there exists a $k \times k$ Jacobi matrix J_1 such that $\sigma(J_1, D_0) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$, then there is an infinite number of $m \times m$ matrices of the form*

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix} \quad (13)$$

such that

$$\sigma(J, D) = \{\lambda_1, \lambda_2, \dots, \lambda_k\} \quad (14)$$

where, $J_0 = \text{diag}(\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_m)$, in which $\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_m$ are arbitrary positive real numbers.

Proof. According to Remark 3.2.1 we can find a positive definite $k \times k$ Jacobi matrix J_1 such that

$$\sigma(J_1, D_0) = \{\lambda_1, \lambda_2, \dots, \lambda_k\} \quad (15)$$

Having J_1 , we assume $z(\lambda) = (z_1(\lambda), z_2(\lambda), \dots, z_k(\lambda))$ to be the corresponding eigenvector. Then the matrix (13) with eigenvector

$y(\lambda) = (z_1(\lambda), z_2(\lambda), \dots, z_k(\lambda), 0, \dots, 0)^T$ of size $m \times 1$ is a solution of Problem II, i.e., $Jy = \lambda Dy$ with

$$\sigma(J, D) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}. \blacksquare \quad (16)$$

Obviously J is positive definite because J_1 and J_0 are positive definite.

Theorem 3.2.5 *Let B be a singular $m \times m$ symmetric real matrix with*

$$\sigma(B) = \{\mu_1, \mu_2, \dots, \mu_m\} \quad (17)$$

where, $\mu_i \neq 0$, for $1 \leq i \leq k$, and $\mu_i = 0$, for $k+1 \leq i \leq m$. Let $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ be a given set of real numbers. If the assumption and notations of Theorems 3.2.3 and 3.2.4 hold, then the matrix $A = TJT^T$ is a solution of Problem II. Moreover, A is positive definite if and only if J is positive definite.

Proof. By Theorem 3.2.3 we have

$$\sigma(J, D) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}, \quad (18)$$

and $Jx = \lambda Dx$. By Theorem 3.2.1, $T^T BT = D$ for an orthogonal T . Hence, we get $JT^T Tx = \lambda T^T BTx$, which is equivalent to

$$(TJT^T)y = \lambda By \quad (19)$$

where, $y = Tx$. Therefore, TJT^T is a solution and

$$\sigma(TJT^T) = \{\lambda_1, \lambda_2, \dots, \lambda_k\} \quad (20)$$

which completes the proof. $\blacksquare$

Remark 3.2.2 If we replace J_0 in Theorem 3.2.4 by the Jacobi matrix J'_0 of the form

$$J'_0 = \begin{pmatrix} \alpha_{k+1} + \alpha_{k+2} & -\alpha_{k+2} & 0 & . & . & . \\ -\alpha_{k+2} & \alpha_{k+2} + \alpha_{k+3} & -\alpha_{k+3} & 0 & . & . \\ 0 & -\alpha_{k+3} & . & . & 0 & . \\ . & 0 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & -\alpha_{m-1} & \alpha_{m-1} + \alpha_m & -\alpha_m \\ . & . & . & 0 & -\alpha_m & \alpha_m \end{pmatrix},$$

and define

$$J' = \begin{pmatrix} J_1 & 0 \\ 0 & J'_0 \end{pmatrix}$$

then J' is a Jacobi matrix and

$$\sigma(J', D) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}.$$

Therefore, Theorem 3.2.4 holds for the Jacobi matrix J' instead of J . Moreover, the corresponding eigenvector is

$$y(\lambda) = (z_1(\lambda), z_2(\lambda), \dots, z_{k-1}(\lambda), z_k(\lambda), z_k(\lambda), \dots, z_k(\lambda))^T.$$

Since, J'_0 can be obtained from J by means of elementary row and column operations, this does not affect the value of the determinant.

In this section we prove an existence theorem for the GIEP $Ax = \lambda Bx$, where A and B are Jacobi matrices with B positive definite. We use the procedure given by H. Hald [22] to prove this theorem. Let $\{\mu_j\}_1^{m-1}$ be a set of real numbers with the interlacing property

$$\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \dots < \mu_{m-1} < \lambda_m \quad (21)$$

According to H. Hald [22], there is a unique Jacobi matrix J of the

form

$$J = \begin{pmatrix} a_1 & c_1 & 0 & . & . & . \\ c_1 & a_2 & c_2 & 0 & . & . \\ 0 & . & . & . & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & c_{m-2} & a_{m-1} & c_{m-1} \\ . & . & . & 0 & c_{m-1} & a_m \end{pmatrix} \quad (22)$$

such that

$$\begin{aligned} \sigma(J) &= \{\lambda_1, \lambda_2, \dots, \lambda_m\} \\ \sigma(J_{m-1}) &= \{\mu_1, \mu_2, \dots, \mu_{m-1}\} \end{aligned}$$

where, J_{m-1} is a submatrix of J obtained by deleting the last row and the last column of J . Moreover, a_i, c_i are obtained by the following relations:

$$\begin{aligned} p_0(\lambda) &= 1 \\ p_1(\lambda) &= a_1 - \lambda \\ &\vdots \\ p_i(\lambda) &= (a_i - \lambda)p_{i-1}(\lambda) - c_{i-1}^2 p_{i-2}(\lambda) \\ p_m(\lambda) &= \prod_j (\lambda_j - \lambda), \quad p_{m-1}(\lambda) = \prod_j (\mu_j - \lambda) \end{aligned} \quad (23)$$

where, $p_i(\lambda)$ is the leading principal minor of order i of $J - \lambda I$.

Now we prove the following theorem by using (23), which proves the existence of a Jacobi matrix J for *Problem I* whenever B is a positive definite Jacobi matrix.

Theorem 3.2.6 *Let B be a given tridiagonal positive definite Jacobi matrix of the form*

$$B = \begin{pmatrix} b_1 & d_1 & 0 & . & . & . \\ d_1 & b_2 & c_2 & 0 & . & . \\ 0 & . & . & . & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & d_{m-2} & b_{m-1} & d_{m-1} \\ . & . & . & 0 & d_{m-1} & b_m \end{pmatrix} \quad (24)$$

and let $\{\lambda_i\}_{i=1}^m$ and $\{\mu_i\}_{i=1}^{m-1}$ be the set of real numbers satisfying the interlacing condition (21). Then, there exists a Jacobi matrix J of the form (22) such that

$$\begin{aligned} \sigma(J, B) &= \{\lambda_1, \lambda_2, \dots, \lambda_m\} \\ \sigma(J_{m-1}, B_{m-1}) &= \{\mu_1, \mu_2, \dots, \mu_{m-1}\}. \end{aligned}$$

Proof. The equations $Jx = \lambda Bx$ and $[(J - \lambda B) + \lambda I]x = \lambda x$ are equivalent, this gives the following interchanges in (23),

$$a_i \rightarrow a_i - \lambda b_i + \lambda, \quad c_i \rightarrow c_i - \lambda d_i, \quad p_i(\lambda) \rightarrow t_i p_i(\lambda),$$

where $t_i = \det B_i$, since $\det(J_i - \lambda B_i) = \det B_i \det(B_i^{-1} J_i - \lambda I)$. Therefore, the equation corresponding to (23) is

$$\begin{aligned} t_i p_i(\lambda) &= (a_i - \lambda b_i) t_{i-1} p_{i-1}(\lambda) - (c_{i-1} - \lambda d_{i-1})^2 t_{i-2} p_{i-2}(\lambda) \\ i &= 2, 3, \dots, m. \end{aligned} \quad (25)$$

We use induction on i . For $i = 2$, we have to find real numbers a_1, c_1, a_2 satisfying

$$a_1 - \lambda b_1 = b_1(\mu_1 - \lambda) \quad (26)$$

$$(a_2 - \lambda b_2)(a_1 - \lambda b_1) - (c_1 - \lambda d_1)^2 = (b_1 b_2 - d_1^2)(\lambda_1 - \lambda)(\lambda_2 - \lambda). \quad (27)$$

Comparing the corresponding coefficients of powers of λ in both sides of (27) gives

$$a_2b_1 + b_2a_1 - 2d_1c_1 = (b_1b_2 - d_1^2)(\lambda_1 + \lambda_2) \quad (28)$$

$$a_1a_2 - c_1^2 = (b_1b_2 - d_1^2)\lambda_1\lambda_2 \quad (29)$$

By (26), we get $a_1 = b_1\mu_1$. Substituting a_1 in (28) and (29) and eliminating a_2 gives

$$c_1^2 - 2d_1\mu_1c_1 + b_1b_2\mu_1^2 + (b_1b_2 - d_1^2)[\lambda_1\lambda_2 - \mu_1(\lambda_1 + \lambda_2)] = 0.$$

This is a quadratic equation for c_1 with discriminant

$$\begin{aligned} \Delta &= 4d_1^2\mu_1^2 - 4b_1b_2\mu_1^2 - 4(b_1b_2 - d_1^2)[\lambda_1\lambda_2 - \mu_1(\lambda_1 + \lambda_2)] \\ &= 4(b_1b_2 - d_1^2)[- \mu_1^2 + (\lambda_1 + \lambda_2)\mu_1 - \lambda_1\lambda_2] \\ &= 4(b_1b_2 - d_1^2)(\mu_1 - \lambda_1)(\lambda_2 - \mu_1), \end{aligned}$$

which is positive because of the positive definiteness of B and the interlacing condition (21). Thus, the quadratic equation has two distinct roots. Substituting one of the roots in (29) and considering $a_1 = b_1\mu_1$ gives a_2 , and hence we can construct J . Suppose that the theorem is true for $i \leq m-1$. By (23) we have

$$p_m(\lambda) = \prod_{i=1}^m (\lambda_i - \lambda), \quad p_{m-1}(\lambda) = \prod_{i=1}^{m-1} (\mu_i - \lambda)$$

and there exist real numbers a_m and c_{m-1} such that

$$t_m p_m(\lambda) = (a_m - \lambda b_m) t_{m-1} p_{m-1}(\lambda) - (c_{m-1} - \lambda d_{m-1})^2 t_{m-2} p_{m-2}(\lambda).$$

The induction assumption implies that there exists a Jacobi matrix

J_{m-1} of the form

$$J_{m-1} = \begin{pmatrix} a_1 & c_1 & 0 & . & . & . \\ c_1 & a_2 & c_2 & 0 & . & . \\ 0 & . & . & . & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & c_{m-3} & a_{m-2} & c_{m-2} \\ . & . & . & 0 & c_{m-2} & a_{m-1} \end{pmatrix}$$

such that

$$\det(J_{m-1} - \lambda B_{m-1}) = t_{m-1} p_{m-1}(\lambda), \quad \det(J_{m-2} - \lambda B_{m-2}) = t_{m-2} p_{m-2}(\lambda).$$

Now, consider the matrix J of the form

$$J = \begin{pmatrix} a_1 & c_1 & 0 & . & . & . \\ c_1 & a_2 & c_2 & 0 & . & . \\ 0 & . & . & . & 0 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & c_{m-2} & a_{m-1} & c_{m-1} \\ . & . & . & 0 & c_{m-1} & a_m \end{pmatrix} \quad (30)$$

We have

$$\begin{aligned} \det(J - \lambda B) &= (a_m - \lambda b_m) t_{m-1} p_{m-1}(\lambda) - (c_{m-1} - \lambda d_{m-1})^2 t_{m-2} p_{m-2}(\lambda) \\ &= t_m p_m(\lambda), \end{aligned}$$

and $\det(J_{m-1} - \lambda B_{m-1}) = t_{m-1} p_{m-1}(\lambda)$. Therefore, J is the desired Jacobi matrix. ■

Chapter 4

The Classical Moment Problem and GIEP for Block Jacobi Matrices

4.1 Introduction

As we discussed in Chapter 1, the theory of orthogonal polynomials is intimately connected with the Theory of Moments. Simon [37] investigated this connection by using difference operator theory. In this comprehensive paper, he provided some effective conditions for the existence and uniqueness of solutions for the classical moment problem. For a given sequence of real numbers $\{\mu_n\}_{n \geq 0}$, and for polynomials

$$p_n(x) = \sum_{i=0}^n a_i x^i, \quad \text{and} \quad p_m(x) = \sum_{j=0}^m b_j x^j$$

he defines the inner product

$$\langle p_n(x), p_m(x) \rangle = \sum_{i=0}^n \sum_{j=0}^m b_j \mu_{i+j} a_i$$

which plays an important role throughout the paper. In the present chapter we discuss a similar moment problem in which $\{\mu_n\}_{n \geq 0}$ is a

sequence of $p \times p$ matrices. In fact, we investigate a typical *moment problem* in which we are given a sequence of $p \times p$ invertible matrices $B_0, B_1, \dots$, and a sequence of $p \times p$ matrices $h_0, h_1, \dots$ such that the block Hankel matrix

$$H_k = [h_{i+j}], \quad i, j = 0, 1, \dots, k. \quad (1)$$

is invertible, for $k = 0, 1, \dots$. The objective is to find a tridiagonal block matrix of the form

$$L = \begin{pmatrix} A_0 & C_0 & 0 & \cdot & \cdot & \dots \\ D_0 & A_1 & C_1 & 0 & \cdot & \dots \\ 0 & D_1 & A_2 & C_2 & 0 & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{pmatrix} \quad (2)$$

such that $Lx = \lambda Bx$ and the sequence $h_0, h_1, \dots$ is the sequence of moments of the pair (L, B) in the following sense (Theorem 0.4.1),

$$h_0(M^k)_{00} = h_k, \quad k = 0, 1, \dots \quad (3)$$

where M is a tridiagonal block matrix such that $L = BM$, $B = \text{diag}(B_0, B_1, \dots)$, and $(M^k)_{00}$ is the entry located in the first row and the first column on M^k .

4.2 Strongly Regular Sequences

Definition 4.2.1 *A doubly indexed sequence*

$$q_{ij}, \quad i, j = 0, 1, \dots$$

of elements of $\mathbb{R}^{p \times p}$ is said to be strongly regular if the block matrix

$$Q_k = \begin{pmatrix} q_{00} & q_{01} & \dots & q_{0k} \\ q_{10} & q_{11} & \dots & q_{1k} \\ \vdots & \vdots & \dots & \vdots \\ q_{k0} & q_{k1} & \dots & q_{kk} \end{pmatrix} \quad (4)$$

is invertible.

Example 4.2.1 In the case of $p = 1$ we introduced a particular case of the notion of a strongly regular sequence in Chapter 1, by considering a positive definite sequence of Hankel matrix

$$H_k = [\mu_{i+j+1}], \quad i, j = 0, 1, \dots, k.$$

We denote Q_k^{-1} by Γ_k and the ij block entry of Γ_k by $\gamma_{ij}^{(k)}$. Thus, Γ_k is of the form

$$\Gamma_k = \begin{pmatrix} \gamma_{00}^{(k)} & \gamma_{01}^{(k)} & \dots & \gamma_{0k}^{(k)} \\ \gamma_{10}^{(k)} & \gamma_{11}^{(k)} & \dots & \gamma_{1k}^{(k)} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ \gamma_{k0}^{(k)} & \gamma_{k1}^{(k)} & \dots & \gamma_{kk}^{(k)} \end{pmatrix}$$

If Q_k is invertible for $k = 0, 1, \dots, n$, the matrix Q_n will be called *nondegenerate*.

Lemma 4.2.1 *Let Q_n be a nondegenerate block matrix. Then $\gamma_{kk}^{(k)}$ is an invertible $p \times p$ matrix for $k = 0, 1, \dots, n$.*

Proof. See Lemma 3.1 of [14]. ■

To find the solution of the *moment problem* we define a matrix valued inner product to produce a sequence of orthogonal matrix polynomials. Let $\mathbb{P}$ denote the set of polynomials in λ with $p \times p$ matrix coefficients. With the previously introduced notation we define a sequence of polynomials as follows:

$$R_k(\lambda) = \sum_{j=0}^k \gamma_{kj}^{(k)} \lambda^j. \quad (5)$$

In fact $R_k(\lambda)$ is the matrix polynomial corresponding to the last row of Γ_k .

Now we define a matrix valued *inner product* $\langle \cdot, \cdot \rangle : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{R}^{p \times p}$ given by the following formula

$$\left\langle \sum_{i=0}^n \alpha_i \lambda^i, \sum_{j=0}^m \beta_j \lambda^j \right\rangle = \sum_{i=0}^n \sum_{j=0}^m \beta_j^T q_{ji} B_i \alpha_i \quad (6)$$

where $n, m = 0, 1, \dots$; $\alpha_i, \beta_j \in \mathbb{R}^{p \times p}$, $i = 0, 1, \dots, n$; $j = 0, 1, \dots, m$.

The following properties of the inner product are immediate results of the definition.

$$\langle \lambda^n I_p, \lambda^m I_p \rangle = q_{mn} B_n, \quad n, m = 0, 1, \dots$$

$$\left\langle \sum_{i=1}^2 X_i(\lambda) T_i, \sum_{j=1}^2 Y_j(\lambda) S_j \right\rangle = \sum_{i=1}^2 \sum_{j=1}^2 S_j^T \langle X_i(\lambda), Y_j(\lambda) \rangle T_i \quad (7)$$

where $T_i, S_i \in \mathbb{R}^{p \times p}$ and $X_i(\lambda), Y_j(\lambda)$ are in $\mathbb{P}$.

Lemma 4.2.2 *Let $R_k(\lambda)$ be the polynomials given by (5). Then*

$$\langle \lambda^k I_p, R_n^T(\lambda) \rangle = B_k \delta_{kn} I_p, \quad n = 0, 1, \dots; \quad k = 0, 1, \dots, n \quad (8)$$

where

$$R_n^T(\lambda) = \sum_{j=0}^n \gamma_{nj}^{(n)T} \lambda^j = \sum_{j=0}^n \gamma_{jn}^{(n)} \lambda^j.$$

Proof. The proof is an immediate consequence of $\Gamma_n Q_n = I$, since we have

$$\langle \lambda^k I_p, R_n^T(\lambda) \rangle = \sum_{j=0}^n \gamma_{nj}^{(n)} q_{jk} B_k = (\Gamma_n Q_n)_{nk} B_k. \blacksquare$$

4.3 Block Hankel Matrices and the Moment Problem

In order to extend the moment problem from real matrices to block matrices, we restrict attention to the particular case where all the

matrices Q_n are block Hankel, i.e.,

$$q_{ij} = h_m$$

for any pair of indices $i, j \geq 0$ such that $i + j = m$, $m = 0, 1, \dots$. As a result, a sequence of $p \times p$ matrices $h_0, h_1, \dots$ for which all the block Hankel matrices

$$H_k = [h_{i+j}], \quad i, j = 0, 1, \dots, k \quad (9)$$

are invertible will be referred to as a *strongly regular sequence*.

Lemma 4.3.1 *Let H_k be given by (9). Then*

$$\langle \lambda X(\lambda), Y(\lambda) \rangle = \langle X(\lambda), \lambda Y(\lambda) \rangle, \quad X, Y \in \mathbb{P} \quad (10)$$

Proof. Let $X(\lambda) = \sum_{i=0}^k \alpha_i \lambda^i$ and $Y(\lambda) = \sum_{j=0}^k \beta_j \lambda^j$. Then, assuming that $\alpha_{-1} = 0$ and $\beta_{-1} = 0$, we have

$$\langle \lambda X(\lambda), Y(\lambda) \rangle = \left\langle \sum_{i=0}^k \alpha_i \lambda^{i+1}, \sum_{j=0}^k \beta_j \lambda^j \right\rangle = \sum_{i=0}^{k+1} \sum_{j=0}^k \beta_j^T h_{i+j-1} B_{i-1} \alpha_{i-1}.$$

The same calculation shows that

$$\langle X(\lambda), \lambda Y(\lambda) \rangle = \sum_{i=0}^{k+1} \sum_{j=0}^k \beta_j^T h_{i+j-1} B_{i-1} \alpha_{i-1}$$

which completes the proof. ■

Theorem 4.3.1 *Let $h_0, h_1, \dots$ be a strongly regular sequence of $p \times p$ matrices, and let $R_k(\lambda)$ be the polynomials given by (5). Let $B_0, B_1, \dots$ be a sequence of invertible $p \times p$ matrices. Then, there is a tridiagonal block matrix L of the form (2) such that*

$$C_k R_{k+1}(\lambda) + A_k(\lambda) R_k(\lambda) + D_{k-1} R_{k-1}(\lambda) = \lambda B_k R_k(\lambda) \quad (11)$$

where $R_{-1} = 0$ and A_k, C_k , and D_k are specified by the following formulae

$$C_k = B_k \tau_k \tau_{k+1}^{-1},$$

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$$A_k = B_k \tau_k (\tau_k^{-1} \delta_k - \tau_{k+1}^{-1} \delta_{k+1}) \tau_k^{-1}, \quad k \geq 0 \quad (12)$$

$$D_{k-1} = B_k^2 B_{k-1}^{-1}, \quad k \geq 1 \quad (13)$$

in which

$$\tau_k = \gamma_{kk}^{(k)}, \quad \delta_k = \gamma_{kk-1}^{(k)}, \quad k \geq 0$$

subject to the condition $\delta_0 = 0$.

Proof. According to the definition of the matrix polynomial $R_j(\lambda)$, the coefficient of λ^j is invertible for every $j = 0, 1, \dots$. Thus, $\lambda B_k R_k(\lambda)$ admits a unique representation as

$$\lambda B_k R_k(\lambda) = \sum_{j=0}^{k+1} F_j^{(k)} R_j(\lambda); \quad (14)$$

where $F_j^{(k)}$ are constant $p \times p$ matrices. For $k = 0$ we get

$$\lambda B_0 \gamma_{00}^{(0)} = F_0^{(0)} \cdot \gamma_{00}^{(0)} + F_1^{(0)} (\gamma_{10}^{(1)} + \gamma_{11}^{(1)} \cdot \lambda),$$

thus, comparing the coefficients of λ , we obtain

$$C_0 = F_1^{(0)} = B_0 \gamma_{00}^{(0)} \{\gamma_{11}^{(1)}\}^{-1} = B_0 \tau_0 \tau_1^{-1}.$$

For $k \geq 1$ the three term recurrence relation (11) is easily justified by the usual orthogonality arguments based on formulae (8) and (10). Formula (12) comes from comparing the coefficients of λ^k and λ^{k+1} in the identity

$$\lambda B_k R_k(\lambda) = F_{k-1}^{(k)} R_{k-1}(\lambda) + F_k^{(k)} R_k(\lambda) + F_{k+1}^{(k)} R_{k+1}(\lambda),$$

while (13) comes from the following equalities:

$$\begin{aligned}
 F_{k-1}^{(k)} B_{k-1} &= F_{k-1}^{(k)} \langle \lambda^{k-1} I_p, R_{k-1}^T(\lambda) \rangle \\
 &= \langle \lambda^{k-1} I_p, (F_{k-1}^{(k)} R_{k-1})^T(\lambda) \rangle, \quad \text{from (7)} \\
 &= \langle \lambda^{k-1} I_p, (\lambda B_k R_k^T(\lambda)) \rangle, \quad \text{by the orthogonality property} \\
 &= B_k \langle \lambda^k, R_k^T(\lambda) \rangle, \quad \text{using (7),(10)} \\
 &= B_k^2, \quad \text{from (8).}
 \end{aligned}$$

Therefore, $D_{k-1} = F_{k-1}^{(k)} = B_k^2 B_{k-1}^{-1}$, as required. ■

Theorem 4.3.1 shows that any strongly regular sequence $h_0, h_1, \dots$ generates a block matrix of the form (2) by the following procedure. We proceed by stating a supplementary theorem to investigate the *moment problem* stated in this chapter.

Corollary 4.3.1

1. From the initial data $h_0, h_1, \dots$ we construct invertible block Hankel matrices

$$H_k = [h_{i+j}], \quad i, j = 0, 1, \dots, k, \quad k = 0, 1, \dots$$

2. Find all the last block row vectors

$$[\gamma_{k0}^{(k)} \quad \gamma_{k1}^{(k)} \quad \dots \quad \gamma_{kk}^{(k)}]$$

of matrices $\Gamma_k = H_k^{-1}$, $k = 0, 1, \dots$

3. Take invertible corners $\tau_k = \gamma_{kk}^{(k)}$ and next-to-the-corner block entries

$$\delta_k = \gamma_{k \ k-1}^{(k)} \text{ of the matrices } \Gamma_k, \quad k = 0, 1, \dots, \quad \delta_0 = 0.$$

4. Compute A_k , C_k and D_k via the formulae

$$\begin{aligned}
 C_k &= B_k \tau_k \tau_{k+1}^{-1}, \\
 A_k &= B_k \tau_k (\tau_k^{-1} \delta_k - \tau_{k+1}^{-1} \delta_{k+1}) \tau_k^{-1}, \\
 D_{k-1} &= B_k^2 B_{k-1}^{-1}.
 \end{aligned}$$

Theorem 4.3.2 *Let $B_0, B_1, \dots$ be a sequence of $p \times p$ invertible matrices, and let $h_0, h_1, \dots$ be a strongly regular sequence of $p \times p$ matrices. Let L be a block matrix of the form (2), where the entries A_j, C_j , and D_j are specified by Corollary 4.3.1. Then all the matrices C_k and D_k are invertible, and the initial sequence may be constructed by the formulae*

$$h_0.(M^k)_{00} = h_k, \quad (15)$$

where M is a semi-infinite tridiagonal block matrix such that $L = BM$ with the understanding that $(L^0)_{00} = I_p$.

Remark 4.3.1 We recall the fact from operator theory that the spectral measure $\rho(\lambda)$ of the pair (δ_1, L) is defined by

$$\langle \delta_1, L^k \delta_1 \rangle = \int_{\mathbb{R}} \lambda^k d\rho(\lambda) = \mu_k$$

where, $\delta_1 = (I_{p \times p}, 0, 0, \dots)$. Therefore it is obvious that

$$(L^k)_{00} = \langle \delta_1, L^k \delta_1 \rangle.$$

Proof. The invertibility of C_k and D_k are obvious from (12) and Lemma 4.2.1. Let

$$R(\lambda) = [R_0(\lambda) \ R_1(\lambda), \dots, R_k(\lambda), \dots]^T \quad (16)$$

denote the infinite block column vector, where the matrix polynomials $R_k(\lambda)$ are given by (5). Then it follows from (2) and (11) that

$$LR(\lambda) = \lambda BR(\lambda) \quad (17)$$

in the natural sense of block matrix multiplication, where we identify objects block by block. Clearly (17) implies that

$$M^k R(\lambda) = \lambda^k R(\lambda), \quad k = 1, 2, \dots \quad (18)$$

Therefore, by comparing the upper $p \times p$ blocks we get

$$(M^k)_{00}R_0(\lambda) + (M^k)_{01}R_1(\lambda) + \cdots + (M^k)_{0k}R_k(\lambda) = \lambda^k R_0(\lambda). \quad (19)$$

The basic properties of the inner product associated with the initial strongly regular sequence $h_0, h_1, \dots$ along with relation (8) gives

$$\begin{aligned} \left\langle I_p, \left(\sum_{j=0}^k (M^k)_{0j} R_j(\lambda) \right)^T \right\rangle &= \sum_{j=0}^k (M^k)_{0j} \langle I_p, R_j^T(\lambda) \rangle \\ &= \sum_{j=0}^k (M^k)_{0j} B_j \delta_{0j} \cdot I_p \\ &= (M^k)_{00} B_0 \end{aligned} \quad (20)$$

and

$$\begin{aligned} \langle I_p, (\lambda^k R_0(\lambda))^T \rangle &= \langle I_p, \lambda^k I_p \gamma_{00}^{(0)} \rangle \\ &= \gamma_{00} \langle I_p, \lambda^k I_p \rangle \\ &= \gamma_{00} h_k B_0 \\ &= h_0^{-1} h_k B_0. \end{aligned} \quad (21)$$

Comparing (20) and (21) we obtain

$$h_0 \cdot (M^k)_{00} = h_k. \blacksquare$$

Corollary 4.3.2 (Construction of a scalar symmetric Jacobi matrix via a given set of spectral data) *Let $h_0, h_1, \dots$ be a positive definite sequence of real numbers. Then there is a Jacobi matrix J of the form*

$$J = \begin{pmatrix} a_0 & c_0 & 0 & . & . & \dots \\ c_0 & a_1 & c_1 & 0 & . & \dots \\ 0 & c_1 & a_2 & c_2 & 0 & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix} \quad (22)$$

such that

$$h_k = h_0 (J^k)_{00}$$

where, $a_i \in R, c_k > 0, k = 0, 1, \dots$.

Proof. Consider $1 = b_0 = b_1 = \dots$, so that $M = L$ and $1 = d_0 = d_1 = \dots$ in Theorem 4.3.2. Since $h_0, h_1, \dots$ is a positive definite sequence we have $\tau_k > 0$, because H_k is positive by Theorem 1.5.2, and

$$\tau_k = \frac{\det H_{k-1}}{\det H_k}.$$

Hence (12) implies that $c_k > 0, k = 0, 1, \dots$. By Theorem 4.3.2, there is a tridiagonal matrix L of the form

$$L = \begin{pmatrix} a_0 & c_0^2 & 0 & . & . & \dots \\ 1 & a_1 & c_1^2 & 0 & . & \dots \\ 0 & 1 & a_2 & c_2^2 & 0 & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & \dots \end{pmatrix} \quad (23)$$

such that $h_k = h_0(L^k)_{00}$. Then the moments of J and L are the same, because we have

$$L = \Lambda J \Lambda^{-1} \quad (24)$$

where $\Lambda = \text{diag}(1, t_0, t_1, \dots)$, $t_k = c_0 c_1 \cdots c_k$. Therefore, $L^k = \Lambda J \Lambda^{-1}$, and hence

$$h_k = h_0(L^k)_{00} = h_0[1, 0, 0, \dots] L^k [1, 0, 0, \dots]^T = h_0(J^k)_{00} \quad (25)$$

which completes the proof. ■

Example 4.3.1 Suppose the conditions of Corollary 1.5.1 hold. Then the sequence $\{\mu_i\}_{i \geq 1}$ is positive definite. By Corollary 4.3.2 there is a Jacobi matrix of the form (22) such that

$$\mu_1(J^n)_{00} = \mu_n, \quad n = 1, 2, \dots$$

4.4 GIEP for Block Jacobi Matrices

In this section we study Jacobi matrix of the form

$$A = \begin{pmatrix} A_1 & C_1 & 0 & . & . & \dots \\ C_1 & A_2 & C_2 & 0 & . & \dots \\ 0 & C_2 & A_3 & C_3 & 0 & \dots \\ . & . & . & . & . & \dots \\ . & . & . & . & . & C_{n-1} \\ . & . & . & . & C_{n-1} & A_n \end{pmatrix} \quad (26)$$

where A_j and C_j are $r \times r$ symmetric matrices on $\mathbb{R}$, and all C_j are invertible.

By establishing the spectral function for block Jacobi matrix of the form (26), theorems of existence and uniqueness for the GIEP $Ax = \lambda Bx$ and algorithms for its solution are obtained, where B is a diagonal block matrix of the form

$$B = \begin{pmatrix} B_1 & 0 & 0 & \dots & 0 \\ 0 & B_2 & 0 & \dots & 0 \\ 0 & 0 & B_3 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & B_n \end{pmatrix}, \quad (27)$$

and all B_j are $r \times r$ positive definite symmetric matrices, for $j = 1, 2, \dots, n$.

An Inner Product

We denote the ring of all $r \times r$ matrices on $\mathbb{C}$ by $\mathbb{F}$. In addition to the well known operations such as multiplication, scalar multiplication and addition, we recall the conjugate operation $'^*$ in $\mathbb{F}$:

$$(i) \quad P^* = \bar{P}^T, \quad (P^*)^* = P, \quad \text{for all } P \in \mathbb{F}$$

$$(ii) (PQ)^* = Q^*P^*$$

(iii) PP^* are semi-positive definite and self-adjoint, for all $P \in \mathbb{F}$

$$(iv) P = 0 \Leftrightarrow PP^* = P^*P = 0.$$

Let $\mathbb{H}$ denote a linear space of n -dimensional column vectors on $\mathbb{F}$, i.e.,

$$\mathbb{H} = \{f = (f_1, f_2, \dots, f_n)^T, f_i \in \mathbb{F}\} \quad (28)$$

and denote by H^* its conjugate space, that is

$$\mathbb{H}^* = \{f^* = (f_1^*, f_2^*, \dots, f_n^*)^T, f_i \in \mathbb{F}\}. \quad (29)$$

Definition 4.4.1 We denote the inner product of $f, g \in \mathbb{H}$ with respect to B by $\langle f, g \rangle$, and define it as follows:

$$\langle f, g \rangle = \sum_{i=1}^n f_i^* B_i g_i. \quad (30)$$

It is easy to see that the inner product given by (30) is a specific form of the inner product given by (6), where $f_i = \sum_{j=0}^m q_{ij} \beta_j$.

Lemma 4.4.1 For every $f, g \in H$, $\langle f, g \rangle \in \mathbb{H}$ and

$$(i) \langle f, g \rangle = \langle g, f \rangle^*$$

(ii) $\langle f, f \rangle$ is semi-positive definite for every $f \in \mathbb{H}$

$$(iii) f = 0 \Leftrightarrow \langle f, f \rangle = 0.$$

Proof. All the properties of $\langle f, g \rangle$ are obvious by the definition of $\langle f, g \rangle$. ■

The Spectral Function For a Block Jacobi Matrix

First, we introduce an eigenfunction $\phi(\lambda)$, $\lambda \in \mathbb{R}$, as follows:

$$\phi(\lambda) = (\varphi_1(\lambda), \varphi_2(\lambda), \dots, \varphi_n(\lambda)), \quad \varphi_1(\lambda) = I$$

$$A\phi(\lambda) = \lambda B\phi(\lambda) + R(\lambda), \quad R(\lambda) = (0, 0, \dots, \Phi_n(\lambda))^T \quad (31)$$

where,

$$\Phi_n(\lambda) = C_{n-1}\varphi_{n-1}(\lambda) + (A_n - \lambda B_n)\varphi_n(\lambda). \quad (32)$$

In general $\Phi_n(\lambda) \neq 0$, unless λ is an eigenvalue such that $\det \Phi_n(\lambda) = 0$, and the system $\Phi_n(\lambda)\mathbf{u} = 0$ has a nontrivial solution $\mathbf{u} = u(\lambda)$.

Let λ_k be an eigenvalue with multiplicity s_k . Therefore the homogeneous system has s_k linearly independent solutions $u^\ell(\lambda_k)$, $\ell = 1, 2, \dots, s_k$. If we take the multiplicity into account, then we may omit subscripts of u and simply write $u(\lambda_k)$.

Definition 4.4.2 *The spectral function $\rho(\lambda)$ of the block Jacobi matrix (26) is defined by*

$$\rho(\lambda) = \sum_{\lambda_k \leq \lambda} u(\lambda_k)u^*(\lambda_k), \quad \lambda \in (-\infty, \infty). \quad (33)$$

By definition $\rho(\lambda)$ is nondecreasing semi-positive definite. Denote by $\widehat{F}_\rho$ the set of all functions of λ in $\mathbb{F}$ with measure $d\rho(\lambda)$. Then we define:

$$(i) \quad P(\lambda) \stackrel{\rho}{=} Q(\lambda) \Leftrightarrow \int_{\mathbb{R}} P(\lambda) d\rho(\lambda) C(\lambda) = \int_{\mathbb{R}} Q(\lambda) d\rho(\lambda) C(\lambda), \\ \forall C(\lambda) \in \widehat{\mathbb{F}}_\rho$$

$$(ii) \quad P(\lambda) \stackrel{\rho}{=} Q(\lambda) \Leftrightarrow \int_{\mathbb{R}} C(\lambda) d\rho(\lambda) P(\lambda) = \int_{\mathbb{R}} C(\lambda) d\rho(\lambda) Q(\lambda), \\ \forall C(\lambda) \in \widehat{\mathbb{F}}_\rho$$

$$(iii) \quad P(\lambda) \stackrel{\rho}{=} Q(\lambda) \Leftrightarrow P^*(\lambda) \stackrel{\rho}{=} Q^*(\lambda)$$

Now let $\widehat{\mathbb{H}}_\rho$ denotes the space of vector functions of λ . Then $\phi(\lambda) \in \widehat{\mathbb{H}}_\rho$ and we have

$$A\phi(\lambda) \stackrel{\rho}{=} \lambda B\phi(\lambda). \quad (34)$$

For every $f \in \mathbb{H}$, we define its Fourier transform with respect to $\phi(\lambda)$ and the weight matrix B given by (27), by

$$\hat{f}(\lambda) = \phi^*(\lambda)Bf,$$

and its inverse Fourier transform by

$$f = \int_{-\infty}^{\infty} \phi(\lambda)d\rho(\lambda)\hat{f}(\lambda).$$

Thus, we have

$$f = \int_{-\infty}^{\infty} \phi(\lambda)d\rho(\lambda)\hat{f}(\lambda) = \int_{-\infty}^{\infty} \phi(\lambda)d\rho(\lambda)\phi^*(\lambda)Bf,$$

and

$$B^{-1} = \int_{-\infty}^{\infty} \phi(\lambda)d\rho(\lambda)\phi^*(\lambda). \quad (35)$$

As a result, we obtain a *dual orthogonality relation* as follows:

$$\int_{-\infty}^{\infty} \phi_j(\lambda)d\rho(\lambda)\phi_k^*(\lambda) = \delta_{jk}B_j^{-1}. \quad (36)$$

The Generalized Inverse Eigenvalue Problem

In this section we reconstruct the Jacobi block matrix A of the form (26) under the following assumptions:

(I) All C_k , $k = 1, 2, \dots, n-1$, are invertible, and are fully determined by $A_1, A_2, \dots, A_{k-1}$, and $C_1, C_2, \dots, C_{k-1}$.

In practice C_k may be given, or only depend on A_k . For completeness of the spectral function $\rho(\lambda)$ we also assume that,

(II) $\text{rank}\{\rho(\lambda_k^+) - \rho(\lambda_k^-)\} = r_k \leq r$, r_k — multiplicity of λ_k ; $\sum_k r_k = r.n$.

Theorem 4.4.1 *Under conditions (I) and (II), the matrix A of the form (26) is uniquely determined by the spectral function $\rho(\lambda)$.*

Remark 4.4.1 For a given set of real numbers, say $\{\lambda_i\}$, we can find a spectral function $\rho(\lambda)$ which satisfies condition (II).

Proof. First from (31) we have

$$A\phi(\lambda) = \lambda B\phi(\lambda) + R(\lambda).$$

This implies that,

$$\begin{aligned} A_1\varphi_1(\lambda) + C_1\varphi_2(\lambda) &= \lambda B_1\varphi_1(\lambda) \\ C_1\varphi_1(\lambda) + A_2\varphi_2(\lambda) + C_2\varphi_3(\lambda) &= \lambda B_2\varphi_2(\lambda) \\ &\vdots \\ C_{n-1}\varphi_{n-1}(\lambda) + A_n\varphi_n(\lambda) &= \lambda B_n\varphi_n(\lambda) + \Phi_n(\lambda) \stackrel{p.}{=} \lambda B_n\varphi_n(\lambda) \end{aligned} \quad (37)$$

Multiplying the first equation by $\varphi_1(\lambda)$ and using the orthogonal property of the inner product, we get

$$B_1^{-1}A_1B_1^{-1} = \langle \lambda\varphi_1, \varphi_1 \rangle = \int_{\mathbb{R}} \lambda\varphi_1(\lambda)d\rho\varphi_1^*(\lambda).$$

Now, by Property (I), C_1 can be found and

$\varphi_2(\lambda) = C_1^{-1}(\lambda B_1 - A_1)\varphi_1(\lambda)$, Thus, we get from the first equation again

$$C_1B_2^{-1} = \langle \lambda B_1\varphi_1, \varphi_2 \rangle = B_1 \int_{\mathbb{R}} \lambda\varphi_1(\lambda)d\rho\varphi_2^*(\lambda). \quad (38)$$

Therefore, if we multiply both sides of (38) by C_1 we get

$$\begin{aligned} C_1B_2^{-1}C_1 &= B_1 \int_{\mathbb{R}} \lambda\varphi_1(\lambda)d\rho(C_1\varphi_2(\lambda))^* \\ &= B_1 \int_{\mathbb{R}} \lambda\varphi_1(\lambda)d\rho(\lambda B_1 - A_1)^* \\ &= B_1 \int_{\mathbb{R}} \lambda\varphi_1(\lambda)d\rho\Phi_1^*(\lambda). \end{aligned}$$

If $A_1, A_2, \dots, A_{k-1}, C_1, C_2, \dots, C_{k-1}$, and $\varphi_1, \varphi_2, \dots, \varphi_k$ are known, then, by using the k -th equation in (37), we get

$$\begin{aligned} A_k &= B_k \langle \lambda \varphi_k, \varphi_k \rangle B_k = B_k \int_{\mathbb{R}} \lambda \varphi_k(\lambda) d\rho \varphi_k^*(\lambda) B_k, \\ \text{for } k &= 1, 2, \dots, n, \text{ and} \\ C_k B_{k+1}^{-1} C_k &= B_k \int_{\mathbb{R}} \lambda \varphi_k(\lambda) d\rho \Phi_k^*(\lambda), \\ \text{for } k &= 1, 2, \dots, n-1. \end{aligned}$$

The proof is completed by using induction. ■

Let $P_1(\lambda)$ be the identity matrix, i.e., $P_1(\lambda) = I$. For $k \geq 2$, define

$$P_k(\lambda) = C_1 C_2 \dots C_{k-1} \varphi_k.$$

Hence $\langle P_1, P_1 \rangle = B_1^{-1}$, and

$$\langle P_k, P_k \rangle = C_1 C_2 \dots C_{k-1} B_k^{-1} (C_1 C_2 \dots C_{k-1})^*, \text{ for } k \geq 2. \quad (39)$$

Theorem 4.4.2 (Solvability of the GIEP)

Condition (II) ensures all $\langle P_k, P_k \rangle$ and consequently, all C_k invertible.

Proof. If $\langle P_1, P_1 \rangle, \dots, \langle P_{k-1}, P_{k-1} \rangle$, are positive, but $\langle P_k, P_k \rangle$ is singular, i.e., $\det | \langle P_k, P_k \rangle | = 0$, then there exist a nonzero solution $Q = (q_1, q_2, \dots, q_k)^T$ for the homogeneous system $\Lambda Q = 0$, where Λ is the $k \times k$ matrix of the form

$$\Lambda = \begin{pmatrix} \langle 1, 1 \rangle & \langle 1, \lambda \rangle & \dots & \langle 1, \lambda^{k-1} \rangle \\ \langle \lambda, 1 \rangle & \langle \lambda, \lambda \rangle & \dots & \langle \lambda, \lambda^{k-1} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \lambda^{k-1}, 1 \rangle & \langle \lambda^{k-1}, \lambda \rangle & \dots & \langle \lambda^{k-1}, \lambda^{k-1} \rangle \end{pmatrix}$$

which implies

$$\langle Q(\lambda), Q(\lambda) \rangle = 0, \quad Q(\lambda) = q_k + q_{k-1} \lambda + \dots + q_1 \lambda^{k-1}$$

i.e., $\det | Q(\lambda_j) | = 0, \quad j = 1, 2, \dots, nr$. This means that $\det Q(\lambda)$ must be a polynomial of degree nr , which is impossible for $k \leq n$. ■

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