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ANALYSIS OF INVENTORY SYSTEMS WITH FAILURES

by

SRINIVASAN RENGARAJAN

A thesis submitted to the Faculty of Graduate Studies and
Research in partial fulfilment of the requirement for the
degree of

Master of Science

Department of Mathematics and Statistics

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June 30, 1983

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ABSTRACT

A recent problem facing modern organisation is to control and maintain inventories of perishable items. Subramaniam and Kumarasamy(1981) developed a continuous review (S,s) policy inventory model for items with exponential lifetimes and exponential interarrival times between unit demands. They showed that the limiting distribution of the stock level is uniform over the set $(s+1, s+2, \dots, s+Q)$. This result is incorrect. In this thesis, we derive the correct expressions for the steady state distributions of the stock level under the same assumptions. We also generalise this model to the one with gamma and hyperexponential lifetimes. Alternative methods of determining the steady state distributions using regenerative process techniques and continuous time Markov chain techniques are also discussed.

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INTRODUCTION.

An important problem facing decision makers in modern organisation is to control and maintain inventories of perishable and deteriorating items. Analysis of inventories of items where utility does not remain constant over time has invoked a number of different concepts of deterioration. It is possible to identify problems in which all the items in the inventory become obsolete at some fixed point in time (style goods problem), and problems where the product deteriorates throughout the planning horizon. The class of products subject to ongoing deterioration can be broken down to those products with a maximum usable lifetime and those without. Thus perishability corresponds to the case where an item may be retained in stock for a fixed period of time with no loss in utility. However, if it has not been used to meet a demand before the expiration date, it must be discarded. Deterioration or decay corresponds to the situation where a fraction of stock onhand expires each period or the utility of each item decreases over time. Thus for example, spare parts for military aircraft are style goods, since they become obsolete when a replacement model is introduced. Food stuffs and photographic films are common examples of units with limited life time. In the medical sector, there are few

inventories maintained that are not subject to expiration. Most drugs and pharmaceuticals are produced with expiration dates. Volatile liquids such as alcohol and gasoline, radioactive medicines and some electronic components are products which decay and do not have maximum lifetime. The earliest attempt at modelling such problems were concerned with optimal production decision rule.

In this thesis, we study the stochastic process associated with a specified (s, S) inventory policy for perishable items whose lifetimes are stochastic. Items of this type fail after a random amount of time. The randomness in lifetime may be due to the complex environmental conditions under which the items are stored. In chapter-1, we discuss the basic (s, S) policy inventory model with arbitrary interarrival time distribution between unit demands. Chapter-2 extends this model to the case where the items stored have exponential lifetimes. Stationary distributions of the stock level are derived explicitly. Chapter-3 deals with alternative methods to derive the stationary distribution using regenerative process and continuous time Markov chain techniques. In Chapter-4, we extend the model developed in Chapter-3 to one where the lifetime of an item has a gamma distribution. Chapter-5 is devoted to items with hyperexponential lifetimes. We also discuss an

inventory model where the failures are due to the random lifetime and bulk disposal. In conclusion, we summarise the models discussed in this thesis and the possibilities of further studies of this problem.

Chapter 1

A Continuous Review Inventory System with Stochastic Demands

1.1 Orientation

Inventory systems may be broadly classified as continuous review systems or periodic review systems. In continuous review systems, the system is monitored continuously over time whereas in periodic review systems, the system is monitored only at discrete, equally spaced instants of time. Processes generated in each of these two systems correspond, respectively, to continuous parameter processes and discrete parameter processes. The simplest type of inventory system is the one which has a single installation point for stocking a single commodity. The inflow of commodity is governed by certain specified policies which can be from production or from a purchase order. The outflow of the commodity is induced by a demand associated with either customer orders or production orders. In general it is impossible or uneconomical to match exactly the inflow with the outflow, hence inventory is created at storage points.

The operation of an inventory system is usually associated with some costs. There are three types of costs which affect the inventory system depending on the mode of procurement and the occurrences of demands, and known as procurement cost, holding cost and shortage cost. The procurement cost has two components: a fixed component that is independent of the quantity ordered and a variable component that is a function of the quantity ordered. The holding cost expresses the cost of carrying inventory over time. This cost will be assumed to be proportional to the stock onhand and the length of the time over which such stock is carried. It's usually expressed in dollars per unit per unit time. The shortage or the penalty cost is the cost incurred in the system when the system is out of stock. This cost is also expressed in dollars per unit per unit time.

An inventory policy is a set of decision rules which directs the inflow of materials into the inventory system. These policies always dictate when and how much to order and they are based on the movement of the stock in the inventory system. One of the inventory policies which is often used and more efficient is the (s, S) inventory policy. Under this policy, whenever the inventory level is less than or equal to a value s , a purchase is made to bring the inventory level to S . For a continuous review inventory system, the procurement

quantity Q is always fixed and is equal to $(S-s)$. The (s, S) policy incorporates two decision variables s and S . The variable s identifies "when to order" while the variables s and S identify "how much to order". The inventory problem under this policy, is the determination of the optimal values of s and S satisfying some objective function.

The objective function in an inventory problem may take several forms. Usually, these involve minimisation of a cost function or maximisation of a profit function. The planning period or the planning horizon which is the length of the time over which the system is assumed to operate may be finite or infinite. For finite horizon period, the total cost involved over the entire planning horizon may be the criterion or the average cost per unit time may be taken. In the case of infinite planning horizon, the long run average cost per unit time is the criterion of optimisation. In stochastic models, the total expected costs are the measures used. Under the (s, S) policy, the objective function is expressed as a function of s and S and the optimisation problem consists of determining the optimal values of s and S .

(7) In an inventory system, an element of randomness may be present due to the inability to provide exact information about the parameters of the system. The

parameters are described in terms of random variables to account for the uncertainty. Of all the parameters, demand is the one which could affect the structure of the inventory system most drastically. Though in general, the fluctuation in demand may incorporate some trends and seasonalities, we will assume that the stochastic process generating the demand is time invariant.

Two distinct approaches may be taken in formulating and solving the stochastic inventory problem. In the first approach, the system is viewed as a multistage process and dynamic programming techniques are used to find the optimal policy that minimises the expected cost over the duration of the process. When the planning horizon is infinite, an ordering policy of a given type is chosen and the stationary behaviour of the inventory level process is analysed to get the characteristics of the inventory system such as expected frequency of orders and expected quantity onhand, which are then used to derive the stationary expected total cost per unit time. This cost is minimised to get the optimal values of the decision variables. The second approach is more efficient than the first one in the sense of computational complexity.

For a continuous review inventory system, where the demand process is defined by a discrete-valued process

with units withdrawn from stock one at a time and where the interarrival time between successive demands are exponentially distributed, the transient and the steady state distribution of the inventory level were derived by Hadley & Whittin(1) under the (s, S) policy. They showed that the limiting distribution of the inventory level is uniform over the set $\{s, s+1, \dots, s+Q\}$. In what follows, we will show that the limiting distribution of the inventory level is uniform over the set $\{s, s+1, \dots, s+Q\}$ and is independent of the distribution of the interarrival times. This was demonstrated by Sivazlian(2).

1.2 The Model

We consider a continuous review (s, S) policy inventory system with the following assumptions.

(i) The demand occurs as a discrete-valued process such that units are demanded from stock one at a time. Let $X_1, X_2, \dots$ be the interarrival times between the successive unit demands. The sequence of r.v.'s $\{X_j\}$ are i.i.d. r.v.'s with a common distribution function $\Phi(\cdot)$ and a density $\phi(\cdot)$

(ii) The quantity ordered to replenish the stock is a constant $(S-s)$ under the (s, S) policy. We also assume

-3-

that there is no lead time involved with the delivery of orders. Since the system is monitored continuously, there is no shortage. As soon as the inventory level reaches a value s , an order is placed and an amount $Q=S-s$ is received in stock immediately.

(iii) Let K be the fixed cost per order and C be the variable cost per unit. The holding cost is h per unit per unit time.

Under these assumptions, the objective is to determine the optimal values of s and S by balancing the procurement cost against the holding cost. We follow the second approach to study this system, i.e. by studying the stationary behaviour of the inventory level process without reference to the cost structure. The cost structure is then imposed, and the stationary total expected cost per unit time is minimised to get the values of s and S . The sample function of the inventory pattern in a continuous review inventory system with immediate delivery of orders operating under (s, S) policy is given in fig-1.

Fig. 1

1.3 Analysis of the Stochastic Process

Let $H(t)$ be the stock level at a particular time t , $t > 0$. Initially at time $t=0$, we assume that a demand has just taken place resulting in a stock level of $s+i$, $i=1, 2, \dots, Q$ (see fig-1). Then $\{H(t), t \geq 0\}$ is a discrete valued stochastic process taking values $s+1, s+2, \dots, s+Q$. Let $\bar{H}(t) = H(t) - s$ and $P\{H(t)=n\}$ be the probability that at time t , the stock level is $s+n$, $n=1, 2, \dots, Q$.

Lemma 1.3.2 : Let $\bar{P}(n, s)$ and $\bar{\phi}(s)$ be the Laplace transforms of $P\{\bar{H}(t)=n\}$ and $\phi(t)$ respectively. Then,

$$\bar{P}(n,s) = \begin{cases} \frac{1}{s} \left([\bar{\phi}(s)]^{1-n} [1-\phi(s)] + [\bar{\phi}(s)]^{Q-n+1} \frac{1-\bar{\phi}(s)}{1-[\bar{\phi}(s)]^Q} \right), & n=1,2,\dots,1 \\ \frac{1}{s} [\bar{\phi}(s)]^{Q-n+1} \left(\frac{1-\bar{\phi}(s)}{1-[\bar{\phi}(s)]^Q} \right), & n = 1+1, \dots, Q \end{cases} \quad \underline{\underline{1.3.1}}$$

Proof : Consider a sequence of events consisting of the epochs at which an order of Q is placed and received immediately. Let Y_1 be the time elapsed from the origin until the first order is placed, Y_2 be the time elapsed between the first and second order, and so on. Now, the sequence $\{Y_k\}$, $k=1, 2, \dots$, forms a modified renewal process in which the distribution function of Y is given by

$$P\{Y_1 \leq y\} = \int_0^y \phi^{*(1)}(u) du \quad 1.3.2$$

where $\phi^{*(i)}$ denotes the i -th fold convolution of ϕ with itself, and the distribution function of the r.v.'s Y_i is given by

$$P\{Y_1 \leq y\} = \int_0^y \phi^{*(Q)}(u) du, \quad i > 1. \quad 1.3.3$$

The probability that the first order will be placed between times t and $t+dt$ is

$$P\{t < Y_1 \leq t_1 + dt\} = \phi^{*(1)}(t)dt, \quad i = 1, 2, \dots, Q. \quad 1.3.4$$

The probability that the m -th order, $m=2, 3, \dots$, will be placed between time t and $t+dt$ is

$$\begin{aligned}
P\{t < Y_1 + Y_2 + \dots + Y_m \leq t + dt\} \\
&= [\phi^{*(1)}(t)] * [\phi^{*(m-1)0}(t)] dt \\
&= \phi^{*[1 + (m-1)Q]}(t) dt \quad m = 2, 3, \dots \quad 1.3.5
\end{aligned}$$

Let $P\{H(t)=n/k, t-\theta\}$ be the probability that the stock level is $s+n$, $n=1, 2, \dots, Q$, at time t , and the $(k+1)$ th order is not yet placed given that the k -th order was placed at $t-\theta$.

For $k=1, 2, \dots$ we have,

$$\begin{aligned}
P\{\bar{H}(t) = Q/k, t-\theta\} &= 1 - P\{x_1 \leq \theta\} \\
P\{\bar{H}(t) = Q - 1/k, t - \theta\} &= P\{X_1 \leq \theta\} - P\{X_1 + X_2 \leq \theta\} \\
P\{\bar{H}(t) = 1/k, t-\theta\} &= P\{X_1 + X_2 + \dots + X_{Q-1} \leq \theta\} \\
&\quad - P\{X_1 + X_2 + \dots + X_Q \leq \theta\}.
\end{aligned}$$

Thus in general for $n=1, 2, \dots, Q-1$,

$$P\{\bar{H}(t) = n/k, t-\theta\} = \int_0^\theta \phi^{*(Q-n)}(u) du - \int_0^\theta \phi^{*(Q-n+1)}(u) du \quad 1.3.6$$

and for $n=Q$,

$$P\{\bar{H}(t) = Q/k, t-\theta\} = 1 - \int_0^\theta \phi(u) du \quad 1.3.7$$

Now, let $P\{\bar{H}(t)=n/0\}$ be the probability that the stock level is $s+n$ at time t and the first order has not been

placed; then for $n=i+1, i+2, \dots, Q$,

$$P\{\bar{H}(t) = n/0\} = 0, \quad 1.3.8$$

For $n=i$,

$$P\{\bar{H}(t) = i/0\} = 1 - \int_0^t \phi(u) du, \quad 1.3.9$$

For $n=1, 2, \dots, i-1$,

$$P\{\bar{H}(t) = n/0\} = \int_0^t \phi^{*(i-n)}(u) du - \int_0^t \phi^{*(i-n+1)}(u) du, \quad 1.3.10$$

We are interested in finding out an expression for $P\{H(t)=n\}$ and this can be obtained from the following relation

$$P\{\bar{H}(t) = n\} = P\{\bar{H}(t) = n/0\} + \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^{\theta=t} P\{\bar{H}(t) = n/k, t-\theta\} P\{t-\theta \leq Y_1 + Y_2 + \dots + Y_k \leq t - \theta + d\theta\} \right\} \quad 1.3.11$$

We compute this expression for various values of n . Now for $n=1, 2, \dots, i-1$, using expressions 1.3.6 and 1.3.10 in 1.3.11 we have,

$$P\{\bar{H}(t) = n\} = \int_0^t \phi^{*(i-n)}(u) du - \int_0^t \phi^{*(i-n+1)}(u) du + \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^{\theta=t} \left[\int_0^{\theta} \phi^{*(Q-n)}(u) du - \int_0^{\theta} \phi^{*(Q-n+1)}(u) du \right] \phi^{*[1+(k-1)Q]}(t-\theta) d\theta \right\} \quad 1.3.12$$

The corresponding expression for the Laplace transform

$\bar{P}(n, s)$ is,

$$\begin{aligned} \bar{P}(n, s) &= \frac{1}{s} [\bar{\phi}(s)]^{1-n} - \frac{1}{s} [\bar{\phi}(s)]^{1-n+1} + \sum_{k=1}^{\infty} \left\{ \frac{1}{s} [\bar{\phi}(s)]^{Q-n} - \frac{1}{s} [\bar{\phi}(s)]^{Q-n+1} \right. \\ &\quad \left. \times [\bar{\phi}(s)]^{i+(k-1)Q} \right\} \quad 1.3.13 \\ &= \frac{1}{s} [\bar{\phi}(s)]^{1-n} [1 - \bar{\phi}(s)] + \frac{1}{s} [\bar{\phi}(s)]^{Q-n+1} \frac{1 - \bar{\phi}(s)}{1 - [\bar{\phi}(s)]^Q} \end{aligned}$$

For $n=i$ using 1.3.8 and 1.3.9 in 1.3.11 we have,

$$\begin{aligned} P\{\bar{H}(t) = n\} &= 1 - \int_0^t \phi(u) du \\ &+ \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^t \left\{ \int_0^{\theta} \phi^{*(Q-1)}(u) du - \int_0^{\theta} \phi^{*(Q-1+1)}(u) du \right\} \right. \\ &\quad \left. \times \phi^{*[i+(k-1)Q]}(t-\theta) d\theta \right\} \quad 1.3.14 \end{aligned}$$

Hence for $n=i$,

$$\bar{P}(n, s) = \frac{1}{s} [1 - \bar{\phi}(s)] + \frac{1}{s} [\bar{\phi}(s)]^Q \frac{1 - \bar{\phi}(s)}{1 - [\bar{\phi}(s)]^Q} \quad 1.3.15$$

For $n=i+1, i+2, \dots, Q-1$ using 1.3.6 and 1.3.8 in 1.3.11 we have,

$$\begin{aligned} P\{\bar{H}(t) = n\} &= \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^t \left\{ \int_0^{\theta} \phi^{*(Q-n)}(u) du - \int_0^{\theta} \phi^{*(Q-n+1)}(u) du \right\} \right. \\ &\quad \left. \times \phi^{*[i+(k-1)Q]}(t-\theta) d\theta \right\} \quad 1.3.16 \end{aligned}$$

and consequently

$$\bar{P}(n,s) = \frac{1}{s} [\bar{\phi}(s)]^{Q-n+1} \frac{1-\bar{\phi}(s)}{1-[\bar{\phi}(s)]^Q} \quad 1.3.17$$

Finally for $n=0$ using 1.3.7 and 1.3.8 in 1.3.11 we have,

$$P\{\bar{H}(t) = n\} = \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^t [1 - \int_0^{\theta} \phi(u) du]^Q * [1+(k-1)Q] (t-\theta) d\theta \right\} \quad 1.3.18$$

and

$$\bar{P}(n,s) = \frac{1}{s} [\bar{\phi}(s)]^1 \frac{1-\bar{\phi}(s)}{1-[\bar{\phi}(s)]^Q} \quad 1.3.19$$

Combining 1.3.13, 1.3.15, 1.3.17 and 1.3.19 we may write

in general

$$\bar{P}(n,s) = \begin{cases} \frac{1}{s} [\bar{\phi}(s)]^{1-n} [1-\bar{\phi}(s)] + \frac{1}{s} [\bar{\phi}(s)]^{Q-n+1} \frac{1-\bar{\phi}(s)}{1-[\bar{\phi}(s)]^Q}, & n=1, 2 \dots 1 \\ \frac{1}{s} [\bar{\phi}(s)]^{Q-n+1} \frac{1-\bar{\phi}(s)}{1-[\bar{\phi}(s)]^Q}, & n=1, +1, \dots Q \end{cases} \quad 1.3.20$$

Thus, we have derived the time dependent distribution of the number of units in stock. Let p_n be the probability that exactly $s+n$ units are in stock in the steady state. Then, we have the following theorem.

Theorem 1.3.1 : In the steady state, the distribution of number of units in stock is uniform over the set $\{s+1, s+2, \dots, s+Q\}$ and is independent of the interarrival time distribution between unit demands.

Proof: Using the final value theorem, P_n can be determined from the Laplace transform of $P(\bar{H}(t)=n)$ as follows.

$$P_n = \lim_{t \rightarrow \infty} P\{\bar{H}(t) = n\} = \lim_{s \rightarrow 0} s \bar{P}(n, s) \quad 1.3.21$$

From 1.3.20 it is easy to verify for all $n=1, 2, \dots$

$$P_n = \lim_{s \rightarrow 0} \frac{1 - \bar{\phi}(s)}{1 - [\bar{\phi}(s)]^Q} \quad 1.3.22$$

By applying l'Hopital's rule once to the above expression we get

$$P_n = \lim_{s \rightarrow 0} \frac{-\bar{\phi}'(s)}{-Q\bar{\phi}'(s)[\bar{\phi}(s)]^{Q-1}} = 1/Q, \quad n = 1, 2, \dots, Q \quad 1.3.23$$

Thus in steady state, the number of units in stock are uniformly distributed over the interval $\{s+1, s+2, \dots, s+Q\}$ and are independent of the interarrival time distribution.

1.4 Optimal Decision Rule

Having found the steady state probabilities of the stock level in the system, our interest now is to incorporate these results to determine the optimal values of the decision variables s and S . As we said earlier, since there is no shortage, the objective is to balance the procurement cost against the holding cost. We select the steady state total expected cost per unit time as the objective function. This function has to be minimized to determine the optimal values of s and S .

If D denotes the expected demand per unit time, then $D = 1/E(X)$, where $E(X)$ is the expected interarrival time between two successive demands. Then, the expected time elapsed between two successive orders using renewal theory arguments is

$$E(Y) = Q E(X) = Q/\bar{D} \quad , \quad 1.4.1$$

Thus the expected number of orders placed per unit time is

$$1/E(Y) = \bar{D}/Q \quad , \quad 1.4.2$$

The expected stock level at any given time is

$$E(H) = s + \sum_{n=1}^Q n P_n \quad , \quad 1.4.3$$

Using 1.3.25 in 1.4.3 we have,

$$E(H) = s + 1/Q \sum_{n=1}^Q n$$

$$= s + \frac{Q+1}{2} \quad ; \quad 1.4.4$$

The total expected cost per unit time expressed as a function of s and Q is

$$F(s, Q) = \frac{K + C Q}{E(Y)} + h E(H) \quad , \quad 1.4.5$$

Using 1.4.2 and 1.4.4 we obtain for $s > 0$, $Q > 1$,

$$F(s, Q) = \frac{K\bar{D}}{Q} + C\bar{D} + h \left[s + \frac{Q+1}{2} \right] \quad , \quad 1.4.6$$

This expression has to be minimised with respect to s and Q . It is clear that $F(s, Q)$ is a separable function of s and Q and that the optimal value of s is $s=0$. The optimal value of Q is obtained by minimising the function

$$\bar{F}(Q) = \frac{K\bar{D}}{Q} + h \frac{Q}{2} \quad , \quad 1.4.7$$

over the set of positive integers. It may be verified that the optimal value of Q , Q^* , exists and that, for such values, the following conditions prevail

$$\bar{F}(Q^*) - \bar{F}(Q^*+1) \leq 0$$

$$\bar{F}(Q^*) - \bar{F}(Q^*-1) \leq 0 \quad , \quad 1.4.8$$

From 1.4.7, these conditions give

$$Q^*(Q^*-1) \leq \frac{2K\bar{D}}{h} \leq Q^*(Q^*+1) \quad , \quad 1.4.9$$

Once we know the values of K , D and h , the optimal value of Q can be calculated easily from 1.4.9

The cost expression $F(s, Q)$ is only affected by the first moment of the interarrival times and is independent of any other statistical characteristics. When the demand is deterministic, i.e. when the time interval between demands is a constant, for large value of Q , the optimal value of Q , Q^* , is approximately given by

$$Q^* = \sqrt{\frac{2KD}{h}}$$

1.4.10

which is the basic Wilson's economic lot size formula.

Chapter 2

A Continuous Review Inventory System with Exponential Demands and Exponential Failures

2.1 Orientation

This chapter is devoted to the extension of the inventory model discussed in the last chapter where the items stored in the inventory have random life times. The enormous literature available on inventory theory have assumed that the lifetimes of the items stored are infinite. This assumption is not always true. This may be reasonable when the planning horizon is short. For reasonably long planning horizons, the effect of lifetime of the items must be considered while deriving the optimal decision rule. The assumption of infinite lifetime is very crucial for certain items in developing an inventory model. Items like medicines, tin foods, vegetables, blood, biological products, human organs, radioactive materials, electronic components and volatile liquids do not have infinite lifetimes. Some of the items listed above have finite lifetimes (perishable items), some decay continuously and they do not have any limit on their lifetimes (deteriorating items) and some have random

lifetimes. Analysis of optimal policies for a fixed life product was undertaken by Van-Zyl (3) and later by Nahmias and Pierskalla (4) for the case in which the lifetime was exactly two periods. These models were extended by Fries(5), Nahmias and Pierskalla (6) and Nahmias (7) to allow for a useful lifetime of m periods. These models involved dynamic programming with a state variable of dimension $m-1$. For $m>2$, the computation and the implementation of the policies were too difficult and Nahmias (8) developed a heuristic approximation which reduced the dimensionality of the problem considerably. These studies were mostly concerned with optimal decision rules. Emmons (9) first developed an EOQ model for decaying inventories. He extended the basic EOQ model for exponentially decaying inventories. Covert and Philip(10) and Philip(11) further extended these models for varying rate of deterioration. Some probabilistic models with stationary probability distribution of demand were developed by Shah and Jaiswal(12) and Jani, Jaiswal and Shah(13). These models were the simple extension of the Naddor's(14) models for non-deteriorating items. Later, Nahmias and Wang(15) demonstrated that the decaying inventory models serve as an approximation to perishable inventory models. Kumarasamy and Subramaniam(16) studied a continuous review (s,S) inventory system for items with random lifetimes. They

assumed exponential lifetime for the items stored and the treatment was similar to the one discussed in the last chapter for non-failing items. They showed that the steady state probabilities of the inventory level is still uniform under the assumption of exponential lifetimes and exponential interarrival time between unit demands. This result is incorrect. This result is correct only if we assume that the probability of failure of a unit in a small time interval dt is λdt instead of exponential lifetimes for the items stored in the inventory. The results under the modified assumptions are given in the following theorems.

Theorem 2.1.1 : Let $V_1, V_2, V_3, \dots$ be the first, second, third, ..., cycle times respectively. Then, the density function $f_{V_k}(\cdot)$ is given by

$$f_{V_1}(t) = \frac{e^{-(\lambda+\mu)t} (\lambda+\mu)^1 t^{1-1}}{(1-1)!} \quad 2.1.1$$

$$f_{V_k}(t) = \frac{e^{-(\lambda+\mu)t} (\lambda+\mu)^Q t^{Q-1}}{(Q-1)!}, \quad k = 2, 3, \dots \quad 2.1.2$$

under the assumption that the interarrival time between unit demands is exponentially distributed with parameter μ and the probability of a failure of a unit in a small time interval dt is λdt .

Theorem 2.1.2 : If $Z_m = V_1 + V_2 + \dots + V_m$, then the probability density function of Z_m is given by

$$f_{Z_m}(t) = \frac{e^{-(\lambda+\mu)t} (\lambda+\mu)^{i+(m-1)Q} t^{i+(m-1)Q-1}}{[i+(m-1)Q-1]!} \quad 2.1.3$$

Theorem 2.1.3 : If $\phi_i(n,t)$ denotes the probability that the stock level $H(t)=s+n$ at time t and no replenishment has taken place in $(0,t)$, given that $H(0)=s+i$ then,

$$\begin{aligned} \phi_i(n,t) &= 0, \text{ for } n = i+1, i+2, \dots, i+Q \\ &= \frac{e^{-(\lambda+\mu)t} [(\lambda+\mu)t]^{i-n}}{(i-n)!} \quad 1 \leq n \leq i \end{aligned} \quad 2.1.4$$

Theorem 2.1.4 : If $\phi(n,t)$ represents the probability that the stock level at time t is $s+n$ and no other replenishment has taken place in $(0,t)$, given that a replenishment has taken place at time $t=0$ then,

$$\phi(n,t) = \frac{e^{-(\lambda+\mu)t} [(\lambda+\mu)t]^{s-n}}{(s-n)!}, \quad n=1, 2, \dots, Q \quad 2.1.5$$

Theorem 2.1.5 :

$$P\{H(t) = s+n\} = \phi_i(n,t) + \int_0^t h_1^V(u) \phi(n,t-u) du \quad 2.1.6$$

where $h_1^V(u)$ is the renewal density of replenishment epochs. i.e.

$$h_1^V(u) = \sum_{m=1}^{\infty} \frac{e^{-(\lambda+\mu)u} (\lambda+\mu)^{i+(m-1)Q} u^{i+(m-1)Q-1}}{[i+(m-1)Q-1]!} \quad 2.1.7$$

Theorem 2.1.6 : The steady state distribution P_n which is the limit of the $P\{H(t)=s+n\}$ as $t \rightarrow \infty$ is given by

$$\bar{P}(n,s) = \begin{cases} \frac{A^{1-n}}{(s+A)^{1-n+1}} + \frac{A^{S-n+1-Q}}{(s+A)^{S-n+1-Q}} \cdot \frac{A^Q}{(s+A)^{Q-A^Q}} & \text{for } 1 \leq n \leq 1 \\ \frac{A^{S-n+1-Q}}{(s+A)^{S-n+1-Q}} \cdot \frac{A^Q}{(s+A)^{Q-A^Q}} & , \text{ for } n > 1 \end{cases} \quad 2.1.8$$

where $A = \lambda + \mu$. Then $P_n = 1/Q$. Thus the inventory level in the steady state is uniform over the set $\{s+1, s+2, \dots, s+Q\}$.

In the following sections, we derive explicit results for the transient and steady state distributions for the stock level under the assumption of exponential lifetimes for the items stored in the inventory.

2.2 The Model

All the assumptions stated in the last chapter hold good for this model also. Here, the depletion of stock onhand is due to demand as well as failures. Thus the decrease in stock level is due to either a demand or a failure, which are assumed to occur one at a time. The maximum inventory level is fixed at S . As soon as the stock level drops to s an order is placed for a constant quantity $Q=S-s$ and received immediately. Instead of

arbitrary interarrival time between unit demands, we assume exponential interarrival times with parameter μ

The items stored in inventory have exponential lifetimes with parameter λ . Initially at time $t=0$, we assume a demand or a failure has just taken place resulting in a stock level of $s+i$, $i=1, 2, \dots, Q$. Then the process $\{H(t), t \geq 0\}$ is again a discrete valued process taking values $s+1, s+2, \dots, s+Q$. Let $\bar{H}(t) = H(t) - s$ and $P(\bar{H}(t) = n)$ be the probability that at time t the stock level is $s+n$, $n=1, 2, \dots, Q$. To follow the same kind of argument as that of chapter-1, we have to derive the distribution of interarrival time between any two consumptions. We also assume that the demand process and failure process are independent of each other. In the next section, we derive the distribution of the interconsumption time.

2.3 Distribution of the Interconsumption times

Let X_j be the time taken for j -th drop of one unit from the stock level. When there are i items onhand, each item has an equal probability of failure in a small interval of time. Let $T_1, T_2, \dots$ be the lifetime of the first, second, ...etc. units respectively each distributed exponentially with a common density $\lambda e^{-\lambda t}$ and let Z be the interarrival time between unit demands with a density $\mu e^{-\mu t}$. To determine the time taken for the

stock level to drop from i to $i-1$, we proceed as follows. Let x_1 be the time taken for the stock level to drop from i to $i-1$. This may happen due to a failure or a demand. The distribution of x_1 is given by

$$\begin{aligned} P\{X_1 \leq t\} &= P\{\min(T_1, T_2, \dots, T_i, Z) \leq t\} \\ &= 1 - P\{\min(T_1, T_2, \dots, T_i, Z) > t\} \\ &= 1 - P\{T_1 > t, T_2 > t, \dots, T_i > t, Z > t\}. \end{aligned} \quad 2.3.1$$

Since the r.v.'s T_i 's are i.i.d r.v.'s and independent of Z , 2.3.1 simplifies to

$$P\{X_1 \leq t\} = 1 - [P\{T_1 > t\}]^i P\{Z > t\} \quad 2.3.2$$

The r.v.'s T_i and Z are exponential with parameters λ and μ and therefore 2.3.2 becomes

$$P\{X_1 \leq t\} = 1 - e^{-(i\lambda + \mu)t} \quad 2.3.3$$

Hence the time taken for the first drop from stock level i to $i-1$ is exponentially distributed with parameter $(i\lambda + \mu)$. Similarly, we can show that the time taken for the stock level to drop from $i-1$ to $i-2$ is exponential with parameter $(i-1)\lambda + \mu$. Thus, the r.v.'s $\dots$ are distributed exponentially with parameters $(i\lambda + \mu)$, $(i-1)\lambda + \mu$, $\dots$, $(\lambda + \mu)$ respectively.

2.4 Analysis of the Stochastic Process

Consider the sequence of events consisting of the times at which an order in the amount of Q is placed and received immediately. The lead time is assumed to be zero. The sequence of r.v.'s $\{Y_k\}$ forms a modified renewal process where Y_1 is the time elapsed between the origin and the first order and $Y_2, Y_3, \dots$ are the time elapsed between successive order points respectively. The distribution of Y_1 is given by

$$P\{Y_1 \leq y\} = \int_0^y f_1 * f_2 * \dots * f_1(u) du \quad 2.4.1$$

where

$$\begin{aligned} f_1(t) &= (1 - \lambda + \mu) e^{-(1 - \lambda + \mu)t} \\ f_2(t) &= ((1 - 1)\lambda + \mu) e^{-((1 - 1)\lambda + \mu)t} \\ &\dots \dots \dots \\ f_1(t) &= (\lambda + \mu) e^{-(\lambda + \mu)t} \end{aligned} \quad 2.4.2$$

and $*$ stands for the convolution.

The common distribution of $Y_2, Y_3, \dots$ is given by

$$P\{Y \leq y\} = \int_0^y f_1 * f_2 * \dots * f_Q(u) du \quad 2.4.3$$

The probability that the first order will be placed

between t and $t+dt$ is

$$P(t < Y_1 \leq t + dt) = f_1 * f_2 * \dots * f_1(t) dt, \quad i=1, 2, \dots, Q. \quad 2.4.4$$

The probability that the m -th order will be placed in the time interval t and $t+dt$ is

$$P(t < Y_1 + Y_2 + \dots + Y_m \leq t + dt) \\ = \{(f_1 * f_2 * \dots * f_1) * (f_1 * f_2 * \dots * f_Q)^{(m-1)}\}(t) dt, \quad m=2, 3, \dots \quad 2.4.5$$

We are interested in finding an expression for $P(\bar{H}(t)=n)$. As usual let $\bar{P}(n,s)$ be its Laplace transform. Then we have the following lemma 2.4.1.

Lemma 2.4.1 :

$$\bar{P}(n,s) = \frac{1}{s} \prod_{j=0}^{i-n-1} \frac{(i-j)\lambda + \mu}{s + (i-j)\lambda + \mu} \left\{ 1 - \frac{n\lambda + \mu}{s + n\lambda + \mu} \right\} \\ + \sum_{k=1}^{\infty} \left\{ \frac{1}{s} \left(\prod_{j=0}^{Q-n-1} \frac{(Q-j)\lambda + \mu}{s + (Q-j)\lambda + \mu} \right) \left(1 - \frac{n\lambda + \mu}{s + n\lambda + \mu} \right) \right. \\ \left. \times \left(\prod_{j=1}^i \frac{j\lambda + \mu}{s + j\lambda + \mu} \right)^* \prod_{j=1}^Q \left(\frac{j\lambda + \mu}{s + j\lambda + \mu} \right)^{k-1} \right\}, \quad n = 1, 2, \dots, Q \quad 2.4.6$$

Proof 2.4.1 : $P(\bar{H}(t)=n)$ is calculated using the following

relation

$$P\{\bar{H}(t) = n\} = P\{\bar{H}(t) = n/0\} + \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^t P\{\bar{H}(t) = n/k, t-\theta\} \cdot P\{t - \theta < Y_1 + Y_2 + \dots + Y_k \leq t - \theta + d\theta\} \right\} \quad 2.4.7$$

where $P\{\bar{H}(t)=n/0\}$ is the probability that the inventory position is $s+n$ at time t and the first order has not been placed and $P\{\bar{H}(t)=n/k, t-\theta\}$ is the probability that the inventory position is $s+n$ ($n=1, 2, \dots, Q$) at time t and the $(k+1)$ -th order is not yet placed, given that the k -th order was placed at time $t-\theta$ (see fig-1).

Now, we compute the values of the unknown expression in 2.4.7 using the same probabilistic arguments as that of chapter-1.

For $k=1, 2, \dots$ we have,

$$P\{\bar{H}(t) = Q/k, t-\theta\} = 1 - P\{X_1 \leq \theta\}$$

$$P\{\bar{H}(t) = Q-1/k, t-\theta\} = P\{X_1 \leq \theta\} - P\{X_1 + X_2 \leq \theta\},$$

.....

$$P\{\bar{H}(t) = 1/k, t-\theta\} = P\{X_1 + X_2 + \dots + X_{Q-1} \leq \theta\}$$

$$- P\{X_1 + X_2 + \dots + X_Q \leq \theta\}.$$

In general, for $n=1, 2, \dots, Q-1$

$$\begin{aligned} P\{\bar{H}(t) = n/k, t - \theta\} &= \int_0^\theta f_1 * f_2 * \dots * f_{Q-n}(t) dt \\ &\quad - \int_0^\theta f_1 * f_2 * \dots * f_{Q-n+1}(t) dt \end{aligned} \quad 2.4.8$$

and for $n=Q$

$$P\{\bar{H}(t) = Q/k, t - \theta\} = 1 - \int_0^\theta f_1(t) dt \quad 2.4.9$$

Now for $n=i+1, i+2, \dots, i+Q$

$$P\{\bar{H}(t) = n/0\} = 0 \quad 2.4.10$$

For $n=i$

$$P\{\bar{H}(t) = i/0\} = 1 - \int_0^t f_1(u) du \quad 2.4.11$$

and for $n=1, 2, \dots, i-1$

$$\begin{aligned} P\{\bar{H}(t) = n/0\} &= \int_0^t f_1 * f_2 * \dots * f_{i-n}(u) du \\ &\quad - \int_0^t f_1 * f_2 * \dots * f_{i-n+1}(u) du \end{aligned} \quad 2.4.12$$

For $n=1, 2, \dots, i-1$ using 2.4.8, 2.4.12 and 2.4.7

$$\begin{aligned} P\{\bar{H}(t) = n\} &= \int_0^t f_1 * f_2 * \dots * f_{i-n}(u) du - \int_0^t f_1 * f_2 * \dots * f_{i-n+1}(u) du \\ &\quad + \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^t \left\{ \int_0^\theta f_1 * f_2 * \dots * f_{n-Q}(t) dt - \int_0^\theta f_1 * f_2 * \dots * f_{n-Q+1}(t) dt \right\} \right. \\ &\quad \left. \times \{f_1 * f_2 * \dots * f_i\} * \{f_1 * f_2 * \dots * f_Q\}^{*(k-1)}(t-\theta) d\theta \right\} \end{aligned} \quad 2.4.13$$

Let the Laplace transform of $P(\bar{H}(t)=n)$ be defined as

$$\bar{P}(n,s) = \int_0^{\infty} e^{-st} P(\bar{H}(t) = n) dt. \quad 2.4.14$$

We know that

$$LT(f_1(t)) = \frac{1\lambda + \mu}{s + 1\lambda + \mu}$$

$$LT(f_2(t)) = \frac{(1-1)\lambda + \mu}{s + (1-1)\lambda + \mu} \quad 2.4.15$$

.....

and so on .

Using 2.4.15 in 2.4.14 through 2.4.13 we get,

$$\begin{aligned} \bar{P}(n,s) = & \frac{1}{s} \prod_{j=0}^{i-n-1} \left\{ \frac{(i-j)\lambda + \mu}{s + (i-j)\lambda + \mu} \right\} \left\{ 1 - \frac{n\lambda}{s + n\lambda + \mu} \right\} \\ & + \sum_{k=1}^{\infty} \left\{ \frac{1}{s} \prod_{j=0}^{Q-n-1} \left\{ \frac{(Q-j)\lambda + \mu}{s + (Q-j)\lambda + \mu} \right\} \left\{ 1 - \frac{n\lambda + \mu}{s + n\lambda + \mu} \right\} \left\{ \prod_{j=1}^i \frac{j\lambda + \mu}{s + j\lambda + \mu} \right\} \right. \\ & \left. \times \left\{ \prod_{j=1}^Q \frac{j\lambda + \mu}{s + j\lambda + \mu} \right\}^{k-1} \right\}. \end{aligned} \quad 2.4.16$$

For $n=i$, using 2.4.6, 2.4.11 and 2.4.7 we get

$$\begin{aligned} P(\bar{H}(t) = n) = & 1 - \int_0^{\theta} f_1(t) dt \\ & + \sum_{k=1}^{\infty} \left\{ \int_{\theta=0}^t \left\{ \int_0^{\theta} f_1 * f_2 * \dots * f_{Q-1}(t) dt - \int_0^{\theta} f_1 * f_2 * \dots * f_{Q-1+1}(t) dt \right\} \right. \\ & \left. \times P\{t - \theta < Y_1 + Y_2 + \dots + Y_k \leq t - \theta + d\theta\} \right\}. \end{aligned} \quad 2.4.17$$

and the Laplace transform is

$$\begin{aligned} \bar{P}(n,s) = & \frac{1}{s} \left(1 - \frac{1\lambda+\mu}{s+1\lambda+\mu} \right) \\ & + \sum_{k=1}^{\infty} \left\{ \frac{1}{s} \prod_{j=0}^{Q-1-k} \left\{ \frac{(Q-j)\lambda+\mu}{s+(Q-j)\lambda+\mu} \right\} \left\{ 1 - \frac{1\lambda+\mu}{s+1\lambda+\mu} \right\} \left\{ \prod_{j=1}^1 \frac{j\lambda+\mu}{s+j\lambda+\mu} \right\} \right. \\ & \left. \times \left\{ \prod_{j=1}^Q \frac{j\lambda+\mu}{s+j\lambda+\mu} \right\}^{k-1} \right\} \end{aligned} \quad 2.4.18$$

Similarly, for $n=i+1, i+2, \dots, Q-1$, using 2.4.8, 2.4.10 and 2.4.7

$$\begin{aligned} \bar{P}(n,s) = & \sum_{k=1}^{\infty} \left\{ \frac{1}{s} \prod_{j=0}^{Q-n-1} \left\{ \frac{(Q-j)\lambda+\mu}{s+(Q-j)\lambda+\mu} \right\} \left\{ 1 - \frac{n\lambda+\mu}{s+n\lambda+\mu} \right\} \right. \\ & \left. \times \left\{ \prod_{j=1}^1 \frac{j\lambda+\mu}{s+j\lambda+\mu} \right\} \left\{ \prod_{j=1}^Q \frac{j\lambda+\mu}{s+j\lambda+\mu} \right\}^{k-1} \right\} \end{aligned} \quad 2.4.19$$

Finally for $n=0$ using 2.4.9, 2.4.10 and 2.4.7

$$\bar{P}(n,s) = \sum_{k=1}^{\infty} \left\{ \frac{1}{s} \left\{ 1 - \frac{Q\lambda+\mu}{s+Q\lambda+\mu} \right\} \left\{ \prod_{j=1}^1 \frac{j\lambda+\mu}{s+j\lambda+\mu} \right\} \left\{ \prod_{j=1}^Q \frac{j\lambda+\mu}{s+j\lambda+\mu} \right\}^{k-1} \right\} \quad 2.4.20$$

We have derived the time dependent distribution and its Laplace transform for the stock level. In the next

section, we derive the steady state distribution of the stock level.

2.5 Steady State Distribution

Let P_n be the probability that the stock level is exactly $s+n$ units ($n=1, 2, \dots, Q$) in the steady state. Then, we have the following theorem 2.5.1.

Theorem 2.5.1 : In the steady state, the distribution of the stock level depends on the onhand inventory and it is given by

$$P_n = \frac{(1/n\lambda + \mu)}{\sum_{i=1}^Q (1/i\lambda + \mu)}, \quad n = 1, 2 \dots Q \quad 2.5.1$$

Proof : From the expressions 2.4.16, 2.4.18, 2.4.19 and 2.4.20 of the lemma 2.4.1, it can be easily verified that

$$\begin{aligned} P_n &= \lim_{t \rightarrow \infty} p\{\bar{H}(t) = n\} = \lim_{s \rightarrow 0} \bar{s}p(n, s) \\ &= \lim_{s \rightarrow 0} \frac{1 - \{(n\lambda + \mu)/(s + n\lambda + \mu)\}}{1 - \prod_{j=1}^Q (j\lambda + \mu/s + j\lambda + \mu)} \\ &= \frac{(1/n\lambda + \mu)}{\sum_{i=1}^Q (1/i\lambda + \mu)} \end{aligned} \quad 2.5.2$$

Thus the steady state probabilities clearly indicate that

they depend on the onhand inventory and they are not uniform over the set $(s+1, s+2, \dots, s+Q)$ as claimed by Kumarasamy and Subramaniam(16).

2.6 Optimal Decision Rule

The criterion for optimisation is the steady state expected total cost per unit time. Since the delivery of orders is instantaneous, no shortage is allowed and hence the objective is to balance the procurement cost against the inventory holding cost and the cost due to failure. Let h be the holding cost per unit per unit time, K be the fixed ordering cost and C be the cost of a single unit. At any particular instant of time, the stock level $N(t)$ corresponds to the onhand inventory $H(t)$. Since the sequence of r.v.'s $\{Y_n\}$ forms a modified renewal process, the expected number of orders per unit time is given by the renewal theorem(17)

$$E(Z) = 1/E(Y)$$

2.6.1

where Y is the interoccurrence time between consecutive orders. The expected value of Y can be found from the Laplace transform of the density function of Y

$$LT[f(y)] = \prod_{i=1}^Q (1\lambda + \mu)/(s + 1\lambda + \mu)$$

2.6.2

Now,

$$E(Y) = -d/ds \text{ LT } [f(y)]|_{s=0}$$

$$= \sum_{i=1}^Q \frac{1}{i\lambda + \mu}$$

2.6.3

In the steady state, the expected amount consumed per unit time is $Q/E(Y)$ and the expected number of failures per unit time is $u/E(Y) - \mu$. Now, the expected stock level $E(N)$ is given by

$$\begin{aligned} E(N) &= s + \sum_{n=1}^Q n P_n \\ &= s + \frac{\sum_{n=1}^Q (n/n\lambda + \mu)}{\sum_{i=1}^Q (1/i\lambda + \mu)} \end{aligned}$$

2.6.4

The steady state total cost per unit time $C(s, Q)$ is,

$$\begin{aligned} C(s, Q) &= \frac{K+CQ}{E(Y)} + h E(N) \\ &= \frac{K+CQ}{E(Y)} + h \left\{ s + \sum_{n=1}^Q n P_n \right\} \end{aligned}$$

2.6.5

From this expression, we observe that $C(s, Q)$ is a separable function of s and Q and the minimum of s, s^* is zero. The resulting objective function which is to be

minimised with respect to Q is

$$C(Q) = \frac{1}{\sum_{i=1}^Q \frac{1}{i\lambda + \mu}} \left\{ K + C Q + h \sum_{n=1}^Q \frac{n}{n\lambda + \mu} \right\} \quad 2.6.6$$

Since Q is an integer, the optimal value of Q , Q^* , must satisfy the following conditions:

$$\begin{aligned} C(Q^*+1) - C(Q^*) &\geq 0 \\ \text{and } C(Q^*-1) - C(Q^*) &\geq 0 \end{aligned} \quad 2.6.7$$

If we know the values of λ , μ , K , C , h , we can tabulate the values of $C(Q)$ for different Q 's and from this the optimal Q satisfying 2.6.7 can be found. When there is no failure, i.e. when $\lambda = 0$, the cost function 2.6.6 agrees with that of 1.4.6.

Chapter 3

Alternative Methods for the Steady State Distribution of the Stock Level

3.1 Regenerative Processes and their Applications

In the last chapter, we discussed in detail the derivation of the steady state distribution through rigorous probabilistic arguments. This procedure can be avoided by observing that the process $\{H(t), t \geq 0\}$ possesses a regenerative structure. A stochastic process $\{H(t), t \geq 0\}$ is a regenerative process if there exists a regenerative epoch S_1 , such that the continuation of the process beyond S_1 is a probabilistic replica of the process beginning at time 0 (17). The existence of the epoch S_1 implies the existence of the epochs $S_2, S_3, \dots$ with the same regenerative property. Let S_i be the i -th regenerative epoch. By defining $S_i = 0, Y_i = S_i - S_{i-1}, i=1, 2, \dots$, we have, as usual, $S_i = Y_1 + Y_2 + \dots + Y_i + S_0$. The definition of the regenerative epochs implies that the Y_i 's are i.i.d. r.v.'s, so S_i can be interpreted as the i -th renewal epoch of the renewal process generated by $\{Y_i\}$.

Many models in Operations Research have this

regenerative property. The (s, S) inventory models discussed in the last two chapters regenerate when an order is placed, so the stochastic process representing the amount in inventory is a regenerative process. The very important property of most of the regenerative processes is the existence of the limits of the processes. A key result which is often used in future discussions is stated in the following theorem.

Theorem 3.1.1 : Let $\{H(t), t \geq 0\}$ be a regenerative process with the state space $S \subset I$. Let $P_k(t) = P\{H(t) = k\}$, and assume that $v = E(Y_1) < \infty$. If either

i) the sample paths $H(., \omega)$ are in the set $D[0, \infty)$ with probability one or

ii) $P\{Y_1 \leq t\}$ has a density on some interval

then,

$$\lim_{t \rightarrow \infty} P_k(t) = P_k, \quad k \in S$$

exists with $P_k > 0$ and $\sum_{k \in S} P_k = 1$. Moreover, let T_k be the amount of time $H(., \omega) = k$ during $[S_0, S_1]$, then,

$$P_k = E(T_k) / v.$$

As we had seen, the renewal process which emerged from the (s, S) inventory models is a modified or a delayed renewal process. Here, the first regenerative

epoch need not have the same distribution as that of the remaining regenerative epochs. If $s_1 > 0$, then $(H(t), t \geq 0)$ is a delayed regenerative process and theorem 3.1.1 holds good for this process if $P\{s_0 < \infty\} = 1$.

3.1.1. (s,S) Inventory System with no Failures

This section is aimed at consolidating the ideas developed in the last section as an alternative technique to derive the steady state distribution of the stock level when there are no failures in the system. In other words, the lifetime of the items stored do not have any effect on the inventory system or the effect is negligible. We consider the basic (s,S) inventory system as that of chapter-1 to demonstrate the use of the regenerative approach. Two basic (s,S) inventory models are analysed here. The first one assumes exponential interarrival time between unit demands and the second one assumes arbitrary interarrival time distribution. In both the cases, we show that the steady state distribution of the stock level is uniform over the set $\{s+1, s+2, \dots, s+Q\}$ and once again prove that the steady state distribution of the stock level is independent of the interarrival distribution between unit demands. We will be using theorem 3.1.1 to derive the steady state distribution of the stock level. One of the main conditions to be

verified in applying theorem 3.1.1 is to check the finiteness of the mean value of the cycle time. Under the (s, S) policy, the maximum number of units ordered at any replenishment epoch is Q and the time taken to deplete this amount is gamma distributed under the assumption of exponential interarrival time between unit demands, and the expected value of the cycle time is finite. In the case of arbitrary interarrival time distribution also it is reasonable to expect the finiteness of the expected cycle length due to the finiteness of the maximum order quantity.

As usual, let $H(t)$ denote the position inventory level at time t and let X_n be the interarrival time between unit demands. Since the units are depleted one by one, the process $\{H(t), t \geq 0\}$ is a discrete-valued continuous-parameter stochastic process. This process is also a delayed regenerative process with order points as regenerative epochs. We assume instantaneous delivery of orders. Let $P(H(t)=n)=P_n(t)$, where $\bar{H}(t)=H(t)-s$. The limiting distribution of P_n of the process $\{H(t), t \geq 0\}$ exists according to the theorem 3.1.1 and is given by

$$P_n = \lim_{t \rightarrow \infty} P_n(t) = \frac{E(X_n)}{v}$$

3.1.1

where v is the expected value of the cycle length and X_n is the time to occurrence for the stock to drop from n to

$n-1$. This r.v. is exponentially distributed with parameter μ . Hence

$$E(X_n) = 1/\mu$$

3.1.2

Let Y be the cycle time. Each cycle except the first one carries Q units and the time to demand for each unit is exponential. So the distribution of Y is nothing but the sum of Q independent exponential r.v.'s each with mean $1/\mu$. The expected value of Y is therefore Q/μ . Using these results in 3.1.1, we get the steady state distribution of stock level P_n as

$$P_n = 1/Q$$

3.1.3

Hence the distribution of the stock level in the steady state is uniform over the set $\{s+1, s+2, \dots, s+Q\}$. Other characteristics like expected number of orders per unit time, expected stock level can be derived as in chapter-1.

When the interarrival time is arbitrarily distributed, to derive the limiting time distribution on the same line using theorem 3.1.1, we have to verify the finiteness of the expected cycle time. Once again, the distribution of the cycle time is the sum of Q independent and identically distributed r.v.'s. So the expected value of the cycle time is $QE(X)$, where $E(X)$ is the expected value of the interarrival times. It is reasonable to assume the expected value to be finite and

hence the expected value of the cycle time is finite. Using these facts in theorem 3.1.1, the steady state probability of the stock level is

$$P_n = \frac{E(X)}{QE(X)} = 1/Q$$

3.1.4

This result agrees with that of chapter-1.

3.1.2 (s,S) Inventory System with Exponential Failures

The model discussed in this section is the same as that of chapter-2. Again, the comparatively simple approach, the regenerative approach, is used to derive the limiting probabilities of the stock level. The discrete-valued, continuous-parameter process $\{H(t), t \geq 0\}$ is a delayed regenerative process with order points as regenerative epochs. From section 2.3, we know that the successive interconsumption times are exponentially distributed with parameters $(Q\lambda + \mu)$, $((Q-1)\lambda + \mu)$, ..., etc. respectively when the cycle starts with Q units. To apply theorem 3.1.1, we have to verify the finiteness of the expected cycle time. The distribution of the cycle time is the convolution of Q exponential r.v.'s. The

expected value of Y from 2.6.3 is

$$E(Y) = \sum_{i=1}^Q \frac{1}{i\lambda + \mu} \quad 3.1.5$$

which is finite. The steady state distribution of the stock level being equal to n from the theorem 3.1.1 is

$$P_n = \frac{E(X_n)}{E(Y)} \quad 3.1.6$$

where $E(X_n)$ is the average time the process $\{H(t), t \geq 0\}$ spent in the state n . Since the interconsumption times are exponentially distributed,

$$P_n = (1/n\lambda + \mu) \left/ \sum_{i=1}^Q \frac{1}{i\lambda + \mu} \right. \quad 3.1.7$$

The steady state probabilities agree with that of chapter-2.

Thus, we have demonstrated the use of regenerative processes in deriving the limiting distribution of the stock level. This method is easy and one need not follow the tedious way of calculating the time dependent distribution of the stock level through Laplace transform techniques. In the next chapter, we demonstrate the use of another powerful technique to obtain the limiting distribution of the stock level.

3.2. Continuous time Markov chain techniques

The technique developed in this section is again aimed at calculating the steady state distribution of the stock level in an easier way. The same (s, S) policy inventory system is considered. The process $\{H(t), t \geq 0\}$ associated with this inventory system turned out to be a discrete space continuous time stochastic process taking values $s+1, s+2, \dots, s+Q$. When the r.v.'s involved in deriving the process have the memoryless property, the process $\{H(t), t \geq 0\}$ turns out to be a continuous time Markov chain (CTMC). Determining the steady state distribution of the CTMC is relatively simple and the whole exercise boils down to solving a system of linear equations. The stochastic process $\{H(t), t \geq 0\}$ is a CTMC with state space $S \subset I$, if each $H(t)$ assumes values only in S and

$$\begin{aligned} P\{H(t_{n+1}) = j / H(t_0) = i_0, \dots, H(t_n) = i_n\} \\ = P\{H(t_{n+1}) = j / H(t_n) = i_n\} \end{aligned} \quad 3.2.1$$

holds for $n \in I$, $0 < t_0 < t_1 < \dots < t_{n+1}$ and all $i_0, i_1, \dots, i_n, j \in S$ (18). A CTMC is called finite when S is finite. Alternatively, the stochastic process $\{H(t), t \geq 0\}$ is a CTMC if it satisfies

$$\begin{aligned} P\{H(t+s) = j / H(u) = i, u < s, H(s) = i\} \\ = P\{H(t+s) = j / H(s) = i\} \end{aligned} \quad 3.2.2$$

for any $t > 0$ and $s > 0$. If the function on the RHS of 3.2.2 is independent of s , i.e., the CTMC is homogeneous, then it is usually denoted by $P_{ij}(t)$ and called the transition probability function from state i to j . Usually, a CTMC is specified in terms of its transition rates. For any transition matrix the limits

$$q_{ii} = \lim_{t \rightarrow 0} \frac{1 - P_{ii}(t)}{t} \quad \text{and} \quad q_{ij} = \lim_{t \rightarrow 0} \frac{P_{ij}(t)}{t}, \quad i \neq j \quad 3.2.3$$

exist with $q_{ii} \geq 0$ and $0 \leq q_{ij} < \infty$. A Q-matrix is defined as $Q = (q_{ij})$. A digraph of a CTMC is formed by associating a node with each state. If $q_{ij} > 0$, a directed arc from node i to j is drawn. A vector $\pi = (\pi_j, j \in S)$ is a stationary probability vector of the CTMC if

$$\pi Q = 0$$

$$\text{with } \sum_{j \in S} \pi_j = 1, \pi_j \geq 0, j \in S, \quad 3.2.4$$

In the next two sections we demonstrate how this technique is used to derive the steady state probabilities of the stock level.

3.2.1 (s,S) Inventory System with no Failures

This model assumes exponential interarrival time

between unit demands and that the life time of the items do not have any effect on the inventory system. Other assumptions of the model in chapter-1 hold good for this model also. Under the assumptions of instantaneous delivery of orders, the process $\{H(t), t \geq 0\}$ is a CTMC. The digraph of the CTMC is given in the following figure with the transition rates.

Fig. 2

The steady state probabilities π_i $i=1, 2, \dots, Q$ of this CTMC are obtained by solving the system of equations,

$$\sum_{i=1}^Q \pi_i q_{ij} = 0, \quad j = 1, 2, \dots, Q$$

$$\text{with } \sum_{i=1}^Q \pi_i = 1, \quad \pi_i \geq 0$$

3.2.5

where q_{ij} is defined as follows.

$$q_{i+1,i} = \mu, \quad i = 1, 2, \dots, Q-1$$

$$q_{i1} = -\mu, \quad i = 1, 2, \dots, Q$$

$$q_{1Q} = \mu$$

$$q_{ij} = 0 \quad \text{for other } i \text{ and } j$$

3.2.6

Using 3.2.6, 3.2.5 reduces to

$$-\mu \pi_1 + \mu \pi_2 = 0$$

$$-\mu \pi_2 + \mu \pi_3 = 0$$

3.2.7

$$-\mu \pi_Q + \mu \pi_1 = 0$$

$$\pi_1 + \pi_2 + \dots + \pi_Q = 1.$$

Solving the system of equations for $\pi_1, \pi_2, \dots, \pi_Q$ we get,

$$\pi_1 = \pi_2 = \dots = \pi_Q = 1/Q$$

3.2.8

which is the same as that of section 3.1.1.

3.2.2 (s,S) Inventory System with Exponential Failures

The same model as that of chapter-2 is discussed here. The assumptions are the same. The process $(H(t), t \geq 0)$ is CTMC. The digraph of the CTMC and its transition rates are given in the figure 3.

Fig. 3

The steady state distributions are obtained by solving

the following system of equations.

$$\sum_{i=1}^Q \pi_i q_{ij} = 0, j = 1, \dots, Q$$

$$\text{with } \sum_{i=1}^Q \pi_i = 1, \pi_i \geq 0$$

where q_{ij} is defined as follows.

3.2.9

$$q_{i+1,i} = (i+1)\lambda + \mu, i = 1, 2, \dots, Q-1$$

$$q_{1,i} = -(i\lambda + \mu), i = 1, 2, \dots, Q$$

$$q_{1Q} = (\lambda + \mu)$$

$$q_{ij} = 0, \text{ for other } i \text{ and } j$$

3.2.10

Using 3.2.10, 3.2.9 reduces to

$$-(\lambda + \mu)\pi_1 + (2\lambda + \mu)\pi_2 = 0$$

$$-(2\lambda + \mu)\pi_2 + (3\lambda + \mu)\pi_3 = 0$$

$$\dots$$

$$-(Q\lambda + \mu)\pi_Q + (\lambda + \mu)\pi_1 = 0$$

3.2.11

$$\pi_1 + \pi_2 + \dots + \pi_Q = 1.$$

Solving the system of equations recursively in terms of

π_1 we get,

$$\pi_2 = \frac{\lambda + \mu}{2\lambda + \mu} \pi_1$$

$$\pi_3 = \frac{\lambda + \mu}{3\lambda + \mu} \pi_1$$

$$\dots$$

$$\pi_Q = \frac{\lambda + \mu}{Q\lambda + \mu} \pi_1.$$

3.2.12

Substituting 3.2.12 in the equation

$$\pi_1 + \pi_2 + \dots + \pi_Q = 1$$

we get

$$\pi_1 = \frac{1/\lambda + \mu}{\sum_{i=1}^Q 1/i\lambda + \mu}$$

$$\pi_i = \frac{1/2\lambda + \mu}{\sum_{i=1}^Q 1/i\lambda + \mu}$$

$$\pi_Q = \frac{1/Q\lambda + \mu}{\sum_{i=1}^Q 1/i\lambda + \mu}$$

The steady state probabilities obtained above agree with those of chapter-2.

This technique is only possible when the r.v.'s involved with the system have the memoryless property. This is the reason we did not consider arbitrary interarrival time distributions when there were no failures. Actually, the alternative methods are handy when we move from the assumption of exponentiality to a more general one. The next two chapters discuss this aspect in detail.

Chapter 4

A Continuous Review (U,S) Inventory System with Poisson Demands and Gamma Failures

4.1 Orientation

The inventory systems discussed so far have assumed exponential lifetimes for the items stored in the inventory. It is natural to pose a question whether the limiting distribution of the stock level can be derived explicitly under a similar set-up discussed in the previous chapters for arbitrary lifetime and interarrival time distributions. There are two major difficulties in dealing with the arbitrary distributions. First, we have lost the memoryless property of the r.v.'s involved in the system and hence the CTMC techniques can not be used and secondly, the age of the items carried over to the next cycle in the name of safety stock are different from the age of the items received during the cycle. In the case of exponential r.v.'s, the rate of failure was a constant and we did not have to differentiate between the old and new items. To avoid the difficulty of keeping track of the ages of the items carried over to the successive cycles, we set the value of the safety stock

$s=0$, and the inventory policy acts as follows: whenever the inventory level drops down to zero, an order for a quantity $S=Q$ is placed and received immediately.

We assume exponential distribution for the interarrival time between unit demands and gamma-2 distribution or two stage Erlangian distribution for the lifetimes of the units stored. The failure of the units occur in two stages, each stage being distributed exponentially with parameter λ . We further assume that the customers do not differentiate between the items in different stages, i.e. a demand can be met with any one of the items from the two stages. The items stored in the inventory will be either in the stage one or stage two. Items in the stage one are not failed at all, whereas the items in the stage two have completed one stage of failure process. The units are failed completely only when they go through both the stages of failure. The concept of units failing in stages can be extended to any arbitrary number of stages. In the next two sections, we derive the steady state distribution of the stock level using the regenerative process technique and CTMC technique. The regenerative process technique becomes quite cumbersome when the order size is very large.

4.2 Regenerative Process Approach

The inventory position $H(t)$ at any time t under the assumption of instantaneous delivery of orders is a discrete-valued, continuous-parameter stochastic process and is also a regenerative process with regenerative epochs as order points. We use the key theorem 3.1.1 to derive the steady state distributions. To make the analysis simple, we restrict the order size to two units. Whenever the inventory level reaches zero, an order of size two is placed and received immediately. Let X_1, X_2 be the interconsumption times for the first and second consumption respectively and T_1, T_2 be the lifetimes of the items with a common density $\lambda^2 t e^{-\lambda t}$. Let Z be the interarrival time between unit demands with a density $\mu e^{-\mu t}$. We also assume that the failure process is independent of the demand process. The distribution of X_1 is given by

$$\begin{aligned} P\{X_1 \leq x\} &= P\{\min(T_1, T_2, Z) \leq x\} \\ &= 1 - P\{\min(T_1, T_2, Z) > x\}, \end{aligned} \tag{4.2.1}$$

T_i 's are i.i.d r.v.'s and are independent of Z . Hence 4.2.1 simplifies to

$$\begin{aligned} P\{X_1 \leq x\} &= 1 - P\{T_1 > x, T_2 > x, Z > x\} \\ &= 1 - P\{T_1 > x\} P\{T_2 > x\} P\{Z > x\} \end{aligned}$$

$$= 1 - e^{-(2\lambda+\mu)x}(1+\lambda x)$$

4.2.2

The Laplace transform of the density function $\phi(\cdot)$ of X_1 is given by

$$\bar{\phi}_1(s) = \frac{2\lambda+\mu}{(s+\mu+2\lambda)^2} + \frac{2(\lambda^2\mu+2\lambda^3)}{(s+\mu+2\lambda)^3} + \frac{\mu}{(s+\mu+2\lambda)}$$

4.2.3

The expected value of X_1 is determined using the property of the Laplace transform

$$E(X_1) = -[d/ds \bar{\phi}_1(s)]_{s=0}$$

$$= \frac{10\lambda^2 + 6\lambda\mu + \mu^2}{(\mu+2\lambda)^3}$$

4.2.4

Since the failure distribution does not have the memoryless* property, the time taken for the second consumption X_2 depends on X_1 . Now, the conditional distribution of X_2 given X_1 is

$$P\{X_2 \leq x/X_1\} = P\{\min(T-X_1, Z/T) > X_1\} \leq x\}$$

$$= 1 - P\{T-X_1 > x/\Gamma > X_1\} P\{Z > x/T > X_1\}$$

where T is the lifetime of an item.

$$= 1 - \left[\frac{e^{-(\lambda+\mu)x} (1+\lambda(x+X_1))}{1+\lambda X_1} \right] \quad 4.2.5$$

The conditional density of X_2 given X_1 is

$$\phi_2(x/X_1) = \frac{e^{-(\lambda+\mu)x}}{1+\lambda X_1} \left\{ x(\lambda^2 + \lambda\mu) + X_1(\lambda^2 + \lambda\mu) + \mu \right\} \quad 4.2.6$$

The marginal distribution of X_2 is found by multiplying 4.2.6 by the marginal density of X_1 and integrating w.r.t X_1 .

$$\phi_2(x) = \int_0^\infty \phi_2(x/Y) \phi_1(Y) dY \quad 4.2.7$$

$$= e^{-(\lambda+\mu)x} \left\{ \frac{x\lambda(\lambda+\mu)^2}{2\lambda+\mu} + \frac{\lambda(\lambda+\mu)}{2\lambda+\mu} + \frac{\mu(\lambda+\mu)}{2\lambda+\mu} \right\} \quad 4.2.8$$

The Laplace transform of the density of X_2 is

$$\bar{\phi}_2(s) = \frac{\lambda(\lambda+\mu)^2}{(2\lambda+\mu)(s+\lambda+\mu)^2} + \frac{\lambda(\lambda+\mu)}{(2\lambda+\mu)(s+\lambda+\mu)} + \frac{\mu(\lambda+\mu)}{(2\lambda+\mu)(s+\lambda+\mu)} \quad 4.2.9$$

The expected value of x_2 is

$$E(X_2) = \frac{(3\lambda + \mu)}{(2\lambda + \mu)(\lambda + \mu)} \quad . \quad 4.2.10$$

The expected cycle time is

$$E(Y) = E(X_1) + E(X_2) \quad , \quad 4.2.11$$

According to the theorem 3.1.1, the steady state probabilities of the states one and two are

$$P_1 = \frac{E(X_2)}{E(Y)} \text{ and } P_2 = \frac{E(X_1)}{E(Y)} \quad . \quad 4.2.12$$

Using 4.2.4 and 4.2.10 in 4.2.11 and hence in 4.2.12 we get,

$$P_1 = \frac{(3\lambda + \mu)(2\lambda + \mu)^2}{(3\lambda + \mu)(2\lambda + \mu)^2 + (\lambda + \mu)(10\lambda^2 + 6\lambda\mu + \mu^2)}$$

$$P_2 = \frac{(10\lambda^2 + 6\lambda\mu + \mu^2)(\lambda + \mu)}{(3\lambda + \mu)(2\lambda + \mu)^2 + (\lambda + \mu)(10\lambda^2 + 6\lambda\mu + \mu^2)} \quad . \quad 4.2.13$$

When $Q > 2$, this method can be extended by conditioning each time to find the distribution of the interconsumption time and hence to steady state probabilities of the stock level. In principle this

procedure can be done, but this becomes more complex when Q is very large. We will not recommend this procedure to get closed expressions for the steady state probabilities in general. In the next section, we use CTMC technique to derive the steady state distribution.

4.3 Continuous time Markov chain technique

We consider the same model as that of the last section. The process $\{H(t), t \geq 0\}$ is not a CTMC under the assumption of two stage Erlangian distribution for the lifetime of the items stored in the inventory and exponential distribution for the interarrival times between unit demands. But the possible states of the inventory system form a CTMC. We assume that only one of the processes can happen at a time, i.e., either a demand or a failure can happen. The digraph of the CTMC with

Each state in the digraph is denoted by a two letter symbol, (a,b). The first symbol a indicates the number of items in the first stage, i.e. units which are not failed. The second letter b gives the number of units in the second stage, i.e. the units which have gone through one stage of failure. An item can eventually fail only when it is in the second stage. Actually, the CTMC gives the various possible states of the system and not the stock level. The steady state distribution of the CTMC are obtained by solving the following system of equations.

$$-(2\lambda + \mu)\pi_{20} + (\mu + \lambda)\pi_{01} + \mu\pi_{10} = 0$$

$$-(2\lambda + \mu)\pi_{11} + 2\lambda\pi_{20} = 0$$

$$-(\mu + 2\lambda)\pi_{02} + \lambda\pi_{11} = 0$$

4.3.1

$$(\mu + 2\lambda)\pi_{02} - (\mu + \lambda)\pi_{01} + \frac{\mu}{2}\pi_{11} + \lambda\pi_{10} = 0$$

$$\mu\pi_{20} + \left(\frac{\mu}{2} + \lambda\right)\pi_{11} - (\lambda + \mu)\pi_{10} = 0$$

$$\pi_{20} + \pi_{11} + \pi_{02} + \pi_{01} + \pi_{10} = 1$$

Solving this system of equations we get,

$$\pi_{20} = \frac{(2\lambda + \mu)^2 (\lambda + \mu)}{2(2\lambda + \mu)^3 + 2\lambda(\lambda + \mu)(3\lambda + \mu)} \quad 4.3.2$$

$$\pi_{11} = \frac{2\lambda(2\lambda + \mu)(\lambda + \mu)}{2(2\lambda + \mu)^3 + 2\lambda(\lambda + \mu)(3\lambda + \mu)} \quad 4.3.3$$

$$\pi_{12} = \frac{2\lambda^2(\lambda+\mu)}{2(2\lambda+\mu)^3 + 2\lambda(\lambda+\mu)(3\lambda+\mu)} \quad 4.3.4$$

$$\pi_{01} = \frac{2\lambda(2\lambda+\mu)^2}{2(2\lambda+\mu)^3 + 2\lambda(\lambda+\mu)(3\lambda+\mu)} \quad 4.3.5$$

$$\pi_{10} = \frac{(2\lambda+\mu)^2(\lambda+\mu)}{2(2\lambda+\mu)^3 + 2\lambda(\lambda+\mu)(3\lambda+\mu)} \quad 4.3.6$$

The actual steady states of the system are obtained by adding 4.3.2, 4.3.3 and 4.3.4, 4.3.5 and 4.3.6:

$$\pi_1 = \pi_{01} + \pi_{10} \quad 4.3.7$$

$$\pi_2 = \pi_{20} + \pi_{11} + \pi_{02}$$

$$\pi_1 = \frac{(2\lambda+\mu)^2(3\lambda+\mu)}{(3\lambda+\mu)(2\lambda+\mu)^2 + (\lambda+\mu)(10\lambda^2 + 6\lambda\mu + \mu^2)} \quad 4.3.8$$

$$\pi_2 = \frac{(10\lambda^2 + 6\lambda\mu + \mu^2)(\lambda+\mu)}{(3\lambda+\mu)(2\lambda+\mu)^2 + (\lambda+\mu)(10\lambda^2 + 6\lambda\mu + \mu^2)}$$

Expression 4.3.8 is the same as that of 4.2.13. When $\lambda \neq 0$, i.e. when there are no failures, the steady state

probabilities $\pi_1 = \pi_2 = 1/2$.

This method can be easily extended for any Q . The system of equations for the steady state distribution is

$$\begin{aligned} \frac{(i+1)}{i+j+1} \mu \pi_{i+1,j} + (i+1)\lambda \pi_{i+1,j-1} + ((j+1)\lambda + \frac{j+1}{i+j+1} \mu) \pi_{i,j+1} \\ - ((i+j)\lambda + \mu) \pi_{i,j} = 0 \quad i, j \neq 0; i+j \leq Q \end{aligned} \quad 4.3.9$$

$$((j+1)\lambda + \mu) \pi_{0,j+1} + \frac{1}{j+1} \mu \pi_{1,j} - (j\lambda + \mu) \pi_{0,j} = 0, \quad j \neq 0; j \leq Q; \pi_{0,Q+1} = 0$$

$$\mu \pi_{i+1,0} + (\lambda + \frac{1}{i+1} \mu) \pi_{i,1} - (\mu + i\lambda) \pi_{i,0} = 0, \quad i \neq 0, i \neq Q; i < Q$$

$$(\lambda + \mu) \pi_{0,1} + \mu \pi_{1,0} - (\mu + Q\lambda) \pi_{Q,0} = 0.$$

with the additional equation that all the steady state probabilities sum to one. With any arbitrary Q , there are $(Q(Q+3)/2)+1$ states in the system and we have $Q(Q+3)/2$ unknowns. In principle, the system of equations can be solved. But the procedure becomes complex and it is very tedious to get a closed expression for the steady state probabilities. We only discussed the two stage Erlangian here. The same ideas can be extended to any k -stage Erlangian distribution. It is almost impossible to draw the digraph for any general Q and one has to rely on

computers to solve the system numerically. The next chapter deals with the hyperexponential lifetimes. That distribution combined with the Erlangian distribution can approximate any arbitrary distribution for the lifetime.

Chapter 5

Extensions of the Continuous Review Inventory Systems

5.1 A Continuous Review Inventory System with Poisson Demands and Hyperexponential Failures

The inventory system in this section is the same as that of the last chapter except that the lifetimes of the items stored in the inventory are hyperexponentially distributed. Demand process is generated by a Poisson process. At a given time, either a demand or a failure is assumed to occur. Under the hyperexponential distribution, a certain fraction of the items fail according to an exponential distribution with parameter λ_1 and the remaining units fail according to another exponential distribution with parameter λ_2 . The failure distribution is

$$q\lambda_1 e^{-\lambda_1 x} + (1-q)\lambda_2 e^{-\lambda_2 x}$$

5.1.1

To fix the ideas clearly, we restrict the order size to two units. Under the assumption of instantaneous delivery of orders, the process $(H(t), t \geq 0)$ representing the possible states is a Continuous Time Markov chain and the

digraph representing the transition flows with the transition rates is given in the following figure.

Fig. 5

The states of the system are represented with two symbols, (a, b) . The first symbol a represents the number of units which fail according to the exponential distribution with parameter λ_1 , whereas the second symbol b indicates the number of units which fail according to the exponential distribution with parameter λ_2 . When the stock level drops to zero, an order for an amount of Q is placed and received immediately. Any of the following combinations may result when the order is received: $(2, 0)$, $(0, 2)$ and $(1, 1)$. The steady state distribution of

the CTMC is found by solving the system of equations

$$-(\mu + 2\lambda_1) \pi_{20} + (\mu + \lambda_1) q^2 \pi_{10} + (\mu + \lambda_2) q^2 \pi_{01} = 0$$

$$(\mu + 2\lambda_1) \pi_{20} - (\mu + \lambda_1) \pi_{10} + (\frac{\mu}{2} + \lambda_2) \pi_{11} = 0$$

$$2(\mu + \lambda_1) q (1-q) \pi_{10} - (\mu + \lambda_1 + \lambda_2) \pi_{11} + 2(\mu + \lambda_2) q (1-q) \pi_{01} = 0 \quad 5.1.2$$

$$(\frac{\mu}{2} + \lambda_1) \pi_{11} - (\mu + \lambda_2) \pi_{01} + (\mu + 2\lambda_2) \pi_{02} = 0$$

$$(\mu + \lambda_1) (1-q)^2 \pi_{10} + (\mu + \lambda_2) (1-q)^2 \pi_{01} - (\mu + 2\lambda_2) \pi_{02} = 0$$

Solving this recursively interms of π_{10}

$$\pi_{01} = -g \pi_{10}$$

$$\pi_{02} = a \pi_{10}$$

$$\pi_{11} = b \pi_{10}$$

$$\pi_{02} = c \pi_{10}$$

$$\& \pi_{10} = 1/a + b + 1 - g - c$$

where

$$a = \frac{(\mu + \lambda_1) q^2 - (\mu + \lambda_2) q^2 g}{\mu + 2\lambda_1} \quad 5.1.4$$

$$g = \frac{-\{(\frac{\mu}{2} + \lambda_1)(\mu + \lambda_1)(q^2 - 1) - (\mu + \lambda_1)(\frac{\mu}{2} + \lambda_2)(1 - q)^2\}}{(\mu + \lambda_2)(\frac{\mu}{2} + \lambda_2)q(2 - q) + (\frac{\mu}{2} + \lambda_1)(\mu + \lambda_2)q^2} \quad 5.1.5$$

$$b = \frac{\mu + \lambda_1}{\frac{\mu}{2} + \lambda_2} - \frac{(\mu + \lambda_1)q^2}{(\frac{\mu}{2} + \lambda_2)} + \frac{(\mu + \lambda_2)q^2g}{(\frac{\mu}{2} + \lambda_2)} \quad 5.1.6$$

$$c = \frac{\mu + \lambda_2}{\mu + 2\lambda_2} g + \frac{(\frac{\mu}{2} + \lambda_1)(\mu + \lambda_1)}{(\mu + 2\lambda_2)(\frac{\mu}{2} + \lambda_2)} - \frac{(\frac{\mu}{2} + \lambda_1)(\mu + \lambda_1)q^2}{(\mu + 2\lambda_2)(\frac{\mu}{2} + \lambda_2)} + \frac{(\frac{\mu}{2} + \lambda_1)(\mu + \lambda_2)q^2g}{(\mu + 2\lambda_2)(\frac{\mu}{2} + \lambda_2)} \quad 5.1.7$$

When $q=1$, $\lambda_1 = \lambda_2$, we have $g=0$, $a = (\mu + \lambda_1)/(\mu + 2\lambda_1)$, $b=0$ and $c=0$. The actual steady state probabilities p_1 and p_2 are

$$P_2 = \pi_{20} + \pi_{11} + \pi_{02} \quad 5.1.8$$

$$P_1 = \pi_{10} + \pi_{01}$$

Under the present assumptions

$$P_1 = \frac{\mu + 2\lambda_1}{2\mu + 3\lambda_1} \quad 5.1.9$$

$$P_2 = \frac{\mu + \lambda_1}{2\mu + 3\lambda_1} \quad 5.1.10$$

This particular case corresponds to the one with exponential failures with parameter λ and exponential interarrival time with parameter μ under the assumption of (s,S) policy where S is restricted to only two units. The same type of argument can be extended to any arbitrary order size Q. Once again, the analysis becomes more complex. We assumed only a two stage hyperexponential which is usually denoted by H_2 and schematically explained as

Fig. 6

An item entering the failure process can take branch λ_1 with probability q and branch λ_2 with probability $(1-q)$. We can generalise this technique to any R -stage hyperexponential distribution H_R and its density is given by

$$b(x) = \sum_{i=1}^R q_i \lambda_i e^{-\lambda_i x}$$

$$\text{with } \sum_{i=1}^R q_i = 1$$

5.1.12

and schematically explained in the following figure.

Fig. 7

Here, a unit entering a failure process can take any one of the branches, say i with probability q_i . Following the discussion of Erlang, the Erlangian distribution which was discussed in the last chapter can be combined with the hyperexponential distribution to approximate any arbitrary distribution of life time of the units stored in the inventory. A unit entering the failure process can select any of the R branches with probability $q_i, i=1,2,\dots,R$ and each branch has r_i number of stages of failure each distributed exponentially. The density function of the approximating distribution is

$$b(x) = \sum_{i=1}^R \frac{q_i r_i \lambda_i (r_i \lambda_i x)^{r_i-1} e^{-r_i \lambda_i x}}{(r_i-1)!} \quad 5.1.12$$

$$\text{with } \sum_{i=1}^R q_i = 1$$

The failure process can be schematically explained as

Fig. 8

At a given time, only one of the states is allowed to occupy. For this new density drawing the digraph and writing the steady state equations of the CTMC are quite complex and this large set of equations will have many boundary conditions. However, these equations will be linear in the unknowns and so the solution method is straightforward. Though there is no automated technique to evaluate these steady state probabilities, at least this procedure provides a method of approach.

5.2 A Continuous Review (Q,S) Inventory System with Exponential Failures, Poisson Demands and Bulk Disposals.

Items stored in the inventory have exponential lifetimes and the units are demanded one at a time according to a Poisson process. Moreover, due to some storing conditions, at Poisson epochs all the onhand inventory fail. The model developed here follows all the assumptions mentioned in the previous chapters. The stock level at time t , $H(t)$ is a CTMC and the digraph is given in figure 9.

Fig. 9

The steady state probabilities $\pi_1, \pi_2, \dots, \pi_Q$ satisfy the following system of equations.

$$(Q\lambda+\mu)\pi_Q = \gamma \sum_{i=1}^{Q-1} \pi_i + (\lambda+\mu)\pi_1$$

$$((Q-1)\lambda+\mu+\gamma)\pi_{Q-1} = (Q\lambda+\mu)\pi_Q$$

.....

$$(\lambda+\mu+\gamma)\pi_1 = (2\lambda+\mu)\pi_2$$

$$\pi_1 + \pi_2 + \dots + \pi_Q = 1.$$

5.2.1

Solving this system recursively in terms of π_1 ,

$$\pi_2 = \frac{(\lambda + \mu + \gamma)}{(2\lambda + \mu)} \pi_1$$

$$\pi_3 = \frac{(2\lambda + \mu + \gamma)(\lambda + \mu + \gamma)}{(2\lambda + \mu)(3\lambda + \mu)} \pi_1$$

5.2.2

$$\begin{aligned} & \dots \dots \dots \\ & \pi_{Q-1} = \frac{(\lambda + \mu + \gamma)}{(Q-1\lambda + \mu)} \pi_{Q-2} \\ & \pi_Q = \frac{(\lambda + \mu + \gamma)}{(Q\lambda + \mu)} \pi_{Q-1} \end{aligned}$$

When the failure rate λ and the bulk disposal rate γ are equal, the steady state probabilities are uniform over the set $(1, 2, \dots, Q)$, i.e.

$$\pi_1 = \pi_2 = \dots = \pi_Q = 1/Q$$

5.2.3

It is very interesting to see that the rate of bulk disposal and the failure rate cancel each other and the resulting steady state distribution is independent of the stock level onhand.

6. Conclusion

We have developed four models of the (s, S) inventory system and its variants when the items stored in inventory have random lifetime. In the first model, we assumed exponential distribution for the lifetimes and under the assumption of Poisson demands, we derived the steady state probabilities of the stock level. We also demonstrated the use of regenerative process approaches and continuous time Markov chain techniques in deriving the steady state distributions for the same model. The second model relaxed the assumption of exponential lifetime to two-stage Erlangian lifetimes and later we considered hyperexponential lifetimes. We also suggested approximating an arbitrary distribution by a convex combination of Erlangian and hyperexponential distributions. Finally, we discussed a model in which the items are subjected to bulk disposal due to drastic changes in environmental conditions. In all these models, we assumed instantaneous delivery of orders, which is not too realistic. When lead time is present in the system, the analysis becomes more complex since the decay or perishability can be applied only to the inventory onhand. Decay models can be used to approximate perishable inventory models. It would be interesting to study the system with bulk demand instead of unit

demands. Numerical algorithms could be developed for determining the optimal values of the decision variables. The immense practicality of the problem discussed so far justifies further research in this area.

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