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STRESS, STRAIN AND FAILURE ANALYSIS
OF
TUNGSTEN CARBIDE
THICK-WALLED CYLINDERS
FOR
HIGH-PRESSURE EQUIPMENTS

by

Claudio R. Boer, Dott. Ing., M.A.S. (Eng.)

A Thesis

Submitted to the School of Graduate Studies
in Partial Fulfillment of the Requirements
for the Degree
Doctor of Philosophy

Carleton University

May, 1978

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The undersigned recommend to the Faculty of Graduate Studies and Research acceptance of the thesis "Stress, Strain and Failure Analysis of Tungsten Carbide Thick-Walled Cylinders for High Pressure Equipments" by Claudio Roberto Boer, Dott. Ing., M.A.S.. (Eng.) in partial fulfillment of the requirement for the Degree of Doctor of Philosophy.

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TO MY WIFE

ABSTRACT

The present study is a contribution toward the design with brittle materials.

The consideration of material properties wisely, in design optimization is a fundamental engineering problem. The increasing demand for high-pressure systems stimulates research toward more accurate knowledge of material properties and the employment of alloys which are especially attractive for this use. Therefore it was decided to analyse a tungsten carbide-cobalt cylinder as a component of high pressure equipment because the cylinder is typical of the geometry commonly employed, while the material presents special problems and difficulties that could be encountered when using brittle material.

To analyse the stress distribution and to illustrate how the stresses can interact in the design with the strains, an interactive computer package of programs for finite element analysis has been developed. A description of the computer package is provided in the thesis.

Using the generalized computer package plus yield-brittle failure criteria based on experimental data systematically collated from the literature, the present thesis serves to illustrate in an organized way how loading conditions, material, properties, geometric configurations and boundary conditions can be successfully integrated to identify areas of possible crack initiation or growth as a function of geometry and loads.

The necessity to increase the pressure volume size for production purposes, leads to even more complex problems; a probability of survival of the component based on a statistical approach to the design, allows optimization of the geometric configuration with the loading conditions.

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NOMENCLATURE

Only the symbols used more than once in the text are listed below

a	semicrack length
c	radius at the elastic-plastic interface
C_{WC}	contiguity of WC particles
d	average diameter
d_i	internal diameter
d_o	external diameter
D_{WC}	mean diameter of WC particles
E	Young's modulus
F	probability of failure
$F(\sigma)$	probability of failure at stress σ
h	height
H	spherical component of the stress tensor
k	plane strain yield shear stress
K	ratio of outside to inside diameter
K_{IC}	critical stress intensity factor
L_{Co}	mean free path of Cobalt interlayer
m	Weibull's exponent
p	hydrostatic or internal pressure
p_i	internal pressure
p_o	external pressure
p_a	axial pressure
P	applied load
P_f	probability of failure
P_o	probability of failure per unit volume

r	general radius
r_i	internal radius
r_o	external radius
R	risk of rupture
S	probability of survival
S_o	probability of survival per unit volume
SRF	Strength Reduction Factor (ratio of mean failure stresses in biaxial loading to uniaxial loading conditions)
t	thickness
u	axial displacement
v	number of unit volumes
V	total volume
W	applied load
Y	tensile yield stress
B	crack angle
γ_i	effective surface energy
$\bar{\epsilon}$	equivalent strain
ϵ_c	compressive strain
ϵ_t	tensile strain
ν	Poisson's ratio
σ	applied stress
$\bar{\sigma}$	equivalent stress
σ_1, σ_2	principal stress
σ_3	
σ_c	compressive stress
σ_{cc}	yield stress in biaxial compression

σ_e effective stress (by maximum tensile strain criteria)
 σ_f effective stress (by Von Mises criteria)
 σ_H hydrostatic or mean stress
 σ_0 characteristic stress of Weibull line
 σ_r radial stress
 σ_{re} radial stress at interface c
 σ_{rp} radial stress in plastic region
 σ_t tensile stress
 σ_{tc} tangential stress at interface c
 σ_{tp} tangential stress in plastic region
 σ_{tt} yield stress in biaxial tension
 σ_u limiting tensile stress in Weibull relationship
 $\sigma_x, \sigma_y, \sigma_z$ stress in x, y, z directions
 σ_{yc} yield stress in uniaxial compression
 σ_{yt} yield stress in uniaxial tension
 σ_θ tangential stress
 $\sigma_{\theta y}, \sigma_{ry}$ stress along the principal diameters of the disc test
 $\sigma_{\theta x}, \sigma_{rx}$
 τ_y shear yield stress

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Chapter 1

INTRODUCTION

1.1 Brittle Material Design

In recent years there has been considerable interest in the structural use of materials with very high strength or capacity to resist high temperatures, or both.

Usable materials that possess such characteristics are generally brittle composites such as the well-known tungsten carbide-cobalt combination. In some cases refractory ceramics have also been considered. Characteristically, these materials show little or no plastic deformation before failure and they have little toughness to arrest crack growth. This characteristic has resulted in a general lack of confidence by designers and an unwillingness to risk the potential dangers of high pressure situations without better failure criteria than are now available.

Conventional design with ductile metallic materials involves a number of approximations which are made in the interest of analytical and experimental simplicity, and which generally involve the use of average values of applied stresses, applied loads, etc. Such methods have, indeed, been very successful because structures built with ductile materials are not sensitive to local effects and are, consequently, better able to accommodate the approximations made by the designer.

All of the considerations involved in brittle material design result from the condition that the material shows little or no yielding prior to failure. The consequences of this assumption are illustrated and summarized in Figure 1.1 [1.1], which shows a series of dependences and the design consequences that follow from each. The first row gives the selection of dependences and vertically below these are the consequently new or improved design techniques.

1) The initial assumption of no yielding leads to failure at points of maximum stress, regardless of the fact that these high stresses might be localized. As a consequence, additional requirements are imposed on the stress analysis methods since they must define these localized maximum stresses. New concepts must be introduced into the design and important changes are required in the methods for experimentally determining the mechanical properties of the materials.

2) One of the sources of very localized high stresses may be flaws within the material, and thus the apparent strength of the material becomes dependent on the size, type and frequency of occurrence of such flaws. This requires an improved understanding of material fracture mechanisms and the establishment of appropriate material fracture theories, particularly under complex stress conditions.

3) Since the material strength is affected by the presence of flaws, which are in themselves random phenomena, a variability in material among supposedly identical specimens, can be expected. Such variations are present in all materials, but the inability to

BRITTLE MATERIAL DESIGN PRINCIPLES

Figure 1.1

New or Improved Design Techniques
Required with Brittle Materials

relieve high local stresses by local yielding makes this condition in brittle materials sufficiently serious that it must be considered in the design.

In Figure 1.2 [1.2] the above-described situation is schematically represented for the case of a thick-walled cylinder (a). In (b) it is given an assumed distribution of heterogeneities which in turn will lower the maximum stress level for heterogeneous materials as in (c). If the applied stress level (a tensile hoop stress, for example) is such as to exceed the maximum allowable stress, a condition of failure is present.

It thus becomes necessary to use a statistical rather than a deterministic definition of material strength. This variability also requires attention, by the designer, to the material processing and quality control to minimize such variability, and hence maximize the strength that can be expected with a specified level of reliability. Variability also introduces the consideration of other methods of achieving high working stresses in conjunction with high reliability, and these methods will generally affect the design processes.

4) As a consequence of the variation in the mechanical properties of brittle materials, a single strength value cannot be assigned, but the usable strength level must be associated with an acceptable probability of failure.

By conducting tests on a large number of samples, strength data can be obtained from which a curve can be plotted to show an expected failure rate against stress level. The definition of

Schematic representation of the stress levels in a flow sensitive material

Figure 1.2

material strength as a matter of probability rather than certainty is perhaps one of the most important differences between brittle material design technology and the technology used with ductile materials. In addition to the analytical techniques required to calculate the probability of failure for a complex structure, this statistical concept introduces changes in design criteria philosophy, with the need to define an acceptable failure probability, and the possibility of trade-off between allowable stress level and failure probability.

The brittle material design principles presented above, lead to an expectation that new optimization concepts in the design should be introduced and evaluated. This is particularly so when a high degree of sophistication is required in the design and analysis of the structure in which the brittle materials (especially ceramics) are increasingly employed, for example in high temperature turbines [1.3] and in certain new types of automotive turbine [1.4].

Consider, for example, a turbine rotor. First, the engineering requirements of aerodynamic performance, inertia, etc., are established. The rotor material and the fabrication process are selected and the rotor sized using crude analytical models and then evaluated against predetermined loading modes representative of engine operation. The rotor geometry is roughly optimized for minimum stress within specified inertia constraints and the preliminary feasibility for the candidate material verified. Using the conventional deterministic approach, it is shown that the disk stresses are within the material's capabilities. With this information, preliminary drawings

are released for the verification of the fabrication process.

Next, an attachment concept is selected and the rotor models refined to reflect the interaction between the attachment and the rotors. At this stage, very detailed heat transfer and stress analysis are performed and the rotor's geometry further optimized to assure that the criterion of minimum stress within specified constraints is fully realized. Figure 1.3 [1.3] is a schematic representation of the method being used during this design phase of refining rotor geometry and finalizing a rotor attachment concept.

1.2 Design of Components for Very High Pressure Manufacturing Equipment

The present study will direct attention to the use of brittle materials for components of a very high pressure apparatus.

In the design and development of very high pressure equipment for laboratory and industrial purposes, the design pressure is generally maximized subject to the constraints imposed by the high pressure volume requirements and the limitations of the component material properties. Particularly, attention will be directed to cemented carbides since these materials are quite suitable for applications where severe loads are expected. In addition to wear resistance combined with good toughness, cemented carbide is characterized by such properties as high compressive strength, hardness, modulus of elasticity and resistance to corrosion; the only serious disadvantage associated with tungsten carbide is that the material exhibits an in-

Figure 1.3

creased brittleness as the cobalt percentage is decreased. It is unfortunately necessary to utilize cemented carbide having a low cobalt percentage to ensure the adequate rigidity of the system; this design requirement introduces numerous problems pertaining to the design technique and criteria used to accommodate the brittle behaviour of the tungsten carbide in a design application of this type.

The design of pressure vessels for operation at very high pressures is therefore a complex problem involving many considerations including the definition of the operating and permissible stress levels, criteria of failure and material behaviour. In the present study these three aspects will be pursued to show the problems and possible solutions of a successful design of a high pressure chamber.

The accepted mode for generating high pressures is basically the piston-cylinder arrangement (Figure 1.4), which, for specific applications, can be modified to the belt and anvil apparatus by appropriate use of compressible gaskets (wonderstone). Figure 1.5 [1.5].

The concept of massive support [1.5] can be utilized to extend the working limits of a material by increasing the cross-sectional area of the component across which the load is applied. An illustration of this approach is provided by the piston and chamber of the belt apparatus as shown in Figure 1.5.[1.10], where the high pressure acting on the small nose area of the piston is distributed across and supported by an increased area which extends to the base of the piston. The same principle has been utilized in the chamber design where the pressure in the high-pressure volume of the chamber is supported by the entire

Schematic of High Pressure piston-cylinder arrangement

Figure 1.4

Schematic section through 'Belt' apparatus, (General Electric Co., Schenectady, U.S.A.) [1.10]

Figure 1.5

height of this component. A basic understanding of the problems associated with the design of high pressure equipment hardware can however be adequately demonstrated by examining the behaviour of the more fundamental piston cylinder arrangement [1.2].

For the present study, a schematic model of the high pressure system was chosen [1.6]. The high pressure working volume is defined within the cylinder shown in Figure 1.4 having an internal diameter d_i , external diameter d_o , and height h . The cylinder may be subjected to an internal pressure p_i and radial support p_e obtained by the shrink fitting of steel support rings.

The dominant design constraints for this and other systems can be attributed to:

(i) non-availability of suitable high pressure materials for the piston and cylinder. While tungsten carbide-cobalt (WC-Co) composites are adequate for piston and cylinder operating in the $0.6 \times 10^6 \text{ lb/in}^2$ (4 GPa) region, the behaviour of these materials is less predictable at pressure in excess of 1.0×10^6 (7 GPa) (See Chapters 3 and 5).

(ii) an inadequate understanding of the flow and fracture mode of WC-Co particularly under multi-axial stress systems. This limitation is further compounded in the high pressure application where brittle fracture can occur as well as plastic yielding of the cobalt matrix.

(iii) incompatibility which arises in the overall system design as a result of attempts to compensate for the short comings

in (i) and (ii) above, i.e. modification of the cylinder to a belt type apparatus to utilize the massive support principle, sectoried dies and pyrophyllite gaskets [1.7], [1.8], [1.5], [1.6].

1.3 Objectives of the Thesis

The following are the objectives of the present thesis:

- (a) the establishment of brittle material design principles and the illustration of these principles with the design of thick-walled cylinders made of WC-Co for high pressure applications,
- (b) the comparison of mechanical properties tests for brittle materials and their analysis for the determination of failure loci,
- (c) the development of an interactive computer package called CLASS (Computer Language Analysis Stress Strain) for the stress and strain analysis of the thick-walled cylinder (and of any axisymmetric solids in general),
- (d) the assessment of the stress and strain patterns by the interactive computer package to study the interdependence among the required internal pressure, the radial and axial support, and other boundary conditions,
- (e) the development of yield-brittle failure criteria based on experimental data systematically collected from the literature,
- (f) the study of loading conditions, material properties, geometric configurations and boundary conditions, using an integrated approach between the interactive computer package and the failure

criteria developed in (e) above, to identify areas of possible crack initiation or growth as function of geometry and load,

(g) the formulation of a probabilistic failure criteria which will permit the probability of survival of a structural component subjected to a given load condition to be evaluated. This objective is accomplished within the interactive package by a modular program which allows the calculation of the probability of survival with data inputs obtained by a finite element stress analysis,

(h) the optimization of the magnitude of the test chamber as a function of load conditions, load sequence and material properties, using the reliability factor developed in (g) above.

The importance of the interactive package, CLASS, at each step of the present analysis should be stressed at this point. The complexity of the problem and the large number of solutions required for the design optimization of the component imposed the necessity of the development of such a package.

1.4 Historical Note on High Pressure Research

We will not attempt to give a historical summary on the problem of reaching high pressure because many review papers and books have been published on this subject. Of particular interest is the first chapter in the book edited by Pugh [1.7] and the general review paper on progress in high pressure research by Bridgman [1.9].

Primary experimental research on pressure started in the seventeenth century under the impetus of such men as Galileo,

Descartes and Pascal. The effect of pressure on the volume of liquids, solids and gases was studied for some time but most of the experimental work undertaken up to about 1900 was limited to the range of pressures that could be produced by water pumps.

Experimental work involving higher pressures was started after about 1850. Attempts to liquefy gases, the effect of pressure on chemical reaction and the synthesis of new substances were the most notable experiments in the second part of the nineteenth century.

Very important work on high pressure research has so far been done during this, the twentieth century. P.W. Bridgman is a leading figure in this field because of the wide range of problems he investigated and the pressure ranges attained by the concomitant technical innovations that he introduced. He was awarded the Nobel Prize in 1946. The pioneering work of Bridgman gave considerable impetus to the study of the effects of high pressure, providing a new dimension for research into many phenomena, particularly in the fields of geology, physics and chemistry, while industrial applications such as the production of polythene and of synthetic diamonds [1.10] have made a substantial impact on everyday life. The study of the effects of high pressure in the metallurgical and engineering fields is a more recent development and in the last 15 years the results achieved have shown a wide-range of industrial applications of hydrostatic pressure.

These industrial developments have created widespread interest in the technology of high pressure in recent years and much work has been initiated on the design and performance of high pressure

containers and associated equipment, as well as on the effects of pressure on the mechanical and metallurgical behaviour of materials [1.8], [1.10].

The present study is intended to be a contribution towards the solution of some of the previously mentioned technological problems.

Chapter 2

MECHANICAL PROPERTY TEST METHODS FOR BRITTLE MATERIALS

2.1 Introduction

The basic mechanical property data used for structural design is the uniaxial tensile and compressive stress-strain curve, which is normally obtained from axially loaded specimens. With brittle materials problems result in attempting to conduct tensile and compressive strength test because of the inability of the test materials to absorb local stress concentrations. To obtain the true strength of the material, therefore, very uniform stress distributions must be developed across the test sections. Specimens and apparatus must permit extremely accurate load alignment and maintenance of alignment during testing. Specimens must be fabricated to close tolerances, and specimen shapes must be developed which will accommodate the stress concentrations that normally arise at load application points. Experience shows that these requirements are difficult to meet, but the testing problem is further complicated by the high cost of fabricating many of the specimens due to extreme hardness. Thus the requirement for extremely accurate close tolerance specimens, together with the need for statistical strength data requiring a large number of such specimens, can result in extremely expensive programs.

Many types of tensile test have been tried to avoid the problems described above, generally involving different types of load application grips, with arrangements for alignment and adjustment, and soft

spacing materials to accommodate local surface imperfections. Strain gages on the test section, in combination with adjustable grips, have also been used to obtain accurate load alignment. None of these methods has proven particularly satisfactory except perhaps the use of strain gages. In some programs the bending strains on tensile test specimens resulting from misalignment have been measured and the values are surprisingly high. Values between 20 and 35% of the tensile strains are apparently common when special precautions to obtain accurate alignment are taken, and values of 50% and above are easily obtained in the absence of such precautions. Probably the magnitude of this difficulty is related to the size of the specimen which is generally small in the interests of reducing fabrication costs and conserving material. The small test section increases the difficulty since many of the tolerances in the test equipment are of a fixed magnitude and independent of specimen size.

A number of methods are currently used to evaluate the tensile strength of brittle materials:

2.2 Tensile Hoop Test

A method which has gained popularity is a tensile test which makes use of internal pressurization of a thin, short cylinder [2.1]. The tensile stress in a cylinder wall corresponding to an internal hydrostatic pressure up to the point of failure is computed from the formula

$$\sigma = \frac{p r_i^2}{r_o^2 - r_i^2} \left(1 + \frac{r_o^2}{r^2} \right), \quad (2.1)$$

where p = hydrostatic pressure in psi,

r_i = internal radius in inches,

r_o = external radius in inches,

r = radius at σ_r .

For radius-to-wall-thickness ratio equal to or greater than 10, a close approximation to the maximum stress is customarily obtained by

$$\sigma = \frac{p d}{2 t}, \quad (2.2)$$

where d = average diameter in inches,

t = wall thickness in inches

Different designs of the apparatus [2.2], [2.3] to apply the internal hydrostatic pressure were developed in recent years. (See appendix A1).

Another test which uses a thick wall cylinder ring has been widely used in Civil Engineering and is an ASTM standard [2.4]. In this test, the specimen is a ring which is compressed across a diameter. The stresses developed depend on the ratio of inside and outside diameters and on the specimen thickness.

2.3 Stress-Wave Test

To avoid the problems connected with conventional tensile testing, an interesting test has been devised which uses a stress-wave technique to determine the tensile properties of homogeneous materials only. [2.5]. In that sense it is considered only an illustration.

In the usual one-dimensional experiment, a uniform impulse is applied to one end of a long, slender, freely supported bar. In the ideal case, a dispersionless longitudinal stress wave will propagate down the bar, and measurement of the displacement of the free end of the bar can be used to determine the applied stress. Under these conditions at any section in a uniform prismatic bar

$$\sigma = \rho c \frac{\partial u}{\partial t}, \quad (2.3)$$

where: σ = applied stress,

ρ = mass per unit volume,

c = velocity of propagation of the wave,

$$c = \sqrt{E/\rho}$$

E = the modulus of elasticity,

u = axial displacement of a point on the bar,

$\frac{\partial u}{\partial t}$ = particle velocity of a point on the bar.

One of the advantages of these tests is that the stress at any section is uniform and therefore provides an excellent way to study the effects of surface finish.

2.4 Tests for the State of Multiaxial Stress

The uniaxial stress test is limited by two factors:

- (1) Load misalignment (eliminated in the ring test)
- (2) impossibility of obtaining data useful for the determination of yield criteria

The case of mixed stresses, one being tensile, one compressive (for biaxial stress state) is of great importance in this regard, particularly when, as in the case of WC-Co, the fracture behaviour is dependent on the hydrostatic or mean stress. A biaxial or triaxial stress state is very common in nature but it represents a difficult challenge for the engineer because it is difficult to devise an experiment for a specimen for which an analytical solution in the two or three principal stress state is known.

The following tests can be utilized:

- (1) Brazilian test;
- (2) diametral compression of elliptical discs;
- (3) compression and indentation of square discs;
- (4) diametral compression of a disc with a central hole;
- (5) disc under biaxial tension;
- (6) the sphere test;
- (7) tubes under internal pressure and axial loading.

2.4.1 The Brazilian Test

This test is very simple and gives accurate results if the necessary precautions are taken. The information obtained with this

test will be used to determine a point on the failure locus.

This test was first proposed, apparently, by Barcellos [2.6] and since then has often been referred to as the Brazilian Disc Test. This test has been used for many years in the study of the mechanical properties of rocks and more recently it was found suitable for examination of other brittle materials.

The specimen may be either cylindrical or disc shaped, and is compressed across a diameter, the load which produces failure being recorded. With a cylindrical specimen, plane strain conditions pertain at the centre. Plane stress conditions may exist in a disc shaped specimen providing that a suitable ratio exists between the diameter and the thickness. This may be ensured by making the thickness of the test piece no more than one quarter of its diameter [2.7].

In the Brazilian test the greatest difficulty is associated with load application. The problems are that uniform gradual loading must be produced, bending effects must be minimized on thin specimens, and spurious fractures initiating from the point of contact of the specimen and the loading platen [2.8], [2.9] must be avoided. In fact a failure due to shear stresses intersects the loaded diameter at an angle and is followed by secondary failures causing an irregular fracture pattern. If such a failure occurs, the test is not valid.

The ideal fracture sequence in a Brazilian test on hard brittle materials is illustrated in Figures 2.1 [2.9]. To reduce to a minimum the bending effects, an accurately machined jig of the type shown in Figure 2.2 [2.10] is essential. In this jig, the top anvil

IDEAL FRACTURE PROPAGATION IN A DISC SPECIMEN

Figure 2.1

Diagram of diametral compression tension test fixture.

Figure 2.2

is a very close fit to the steel body, and graphite is used to reduce the friction at the contact surfaces. The bearing surfaces of the top and bottom anvils must be very carefully ground and assembled in order to be as near parallel as possible. Bending failure may be detected if the broken specimen shows spalling or flaking from one side. The shear failures and fractures which initiate at the loading point may be avoided in two ways: by placing soft padding materials between the specimen and the platens or by loading through a flat surface machined onto the periphery of the specimen, or both.

The function of the soft padding is to smooth out the asperities between the platen and the surface of the specimen. The material for this padding is best selected by means of a series of preliminary experiments and should give the highest loads at failure, with the least spread in the results. For tungsten carbide, brass shims 0.010 in. thick gives the best results [2.7].

The flat machined onto the periphery of the specimen has an effect similar to the soft padding, namely to distribute the load over an area instead of along a line, with a reduction of compression and shear stresses at the application points. In Appendix A2, the stress distribution for the different loading conditions are given [2.11], [2.12], [2.13].

If the load is distributed over too great an area, the stresses in the control portion of the specimen will be affected. Therefore if the tensile strength of the material has to be found, the load must not be distributed over a wide area.

Conversely, a different tension-compression pattern can be obtained by a different distribution of the loads [2:14]. Suppose that the loads are W per unit length and that the acute angle between their directions is ω as in Figure 2.3 (a), then Figure 2.3 (b) shows the nature of the stresses for $\omega = 90^\circ$. Within the curve I, both principal stresses in the plane of the disc are compressive; on the curve I one of them vanishes; in the region outside curve I, one principal stress is tensile and this has its greatest value on curve II. Figure 2.3 (c) is similarly drawn for the case $\omega = 60^\circ$. In this case the tensile region just reaches the axis of the cylinder. The variation of the maximum tensile stress σ_t attained in the cylinder is given in Table 2.1, together with the radius r at which it occurs, R being the radius of the cylinder. $\omega = 0$ corresponds to the ordinary Brazilian test with loads $2W$.

If an additional confining pressure is applied to the specimen, the transition from tensile to compressive values of the least principal stress could be studied [2:15]. This situation with applied hydrostatic pressure and loading in a diametral plane is described as a 'confined Brazilian test'. If the additional hydrostatic pressure or the spherical component of the stress tensor is called H , the stresses at the center of the disc (or cylinder in plane strain) become (see Appendix A2)

$$\left. \begin{aligned} \sigma_1 = \sigma_x = \sigma_{\theta y} = \sigma_{rx} &= \frac{2P}{\pi td} + H; \\ \sigma_2 &= H; \\ \sigma_3 = \sigma_y = \sigma_{\theta x} = \sigma_{ry} &= -\frac{6P}{\pi td} + H. \end{aligned} \right\} \quad (2.4)$$

(A) CYLINDER COMPRESSED IN TWO DIAMETRAL PLANES INCLINED AT ω .
 (B) $\omega = 90^\circ$ CURVE I, LIMIT OF TENSILE FIELD. CURVE II, LOCUS OF MAXIMUM TENSILE STRESS.
 (C) $\omega = 60^\circ$ CURVE I, LIMIT OF TENSILE FIELD. CURVE II, LOCUS OF MAXIMUM TENSILE STRESS.

VALUE AND POSITION OF THE MAXIMUM TENSILE STRESS ATTAINED.

ω	0	30°	45°	60°	90°
RO_1/W	0.622	0.452	0.253	0.112	0.039
r/R	0	0	0	0.5	0.85

Figure 2.3 and Table 2.1

An important advantage of the Brazilian test is that values may be obtained for the elastic constants by means of strain gauges mounted at the center of the specimen [2.16]. A ninety-degree rosette gauge can be mounted so that one gauge is along the loaded diameter whilst the other is at right angles. Using two gauges, one mounted on each side of the specimen, an average value may be obtained and bending effects eliminated. If the stresses at the center of the specimen are indicated as σ_t (tensile at right angles to the loaded diameter) and σ_c (compressive along the loaded diameter), two equations may be written for the tensile and compressive strains:

$$\left. \begin{aligned} \epsilon_t &= \frac{1}{E} (\sigma_t - \nu \sigma_c), \\ \epsilon_c &= \frac{1}{E} (\sigma_c - \nu \sigma_t), \end{aligned} \right\} \quad (2.5)$$

and, from Equation

$$\sigma_c = -3\sigma_t \quad (\text{see Appendix A2})$$

$$\left. \begin{aligned} \epsilon_t &= \frac{\sigma_t}{E} (1 + 3\nu), \\ \epsilon_c &= \frac{-\sigma_t}{E} (3 + \nu). \end{aligned} \right\} \quad (2.6)$$

If the applied load P is known, σ_t may be calculated and Equation (2.6) enable values for the Young's modulus and Poisson ratio to be obtained.

2.4.2 Diametral Compression of Elliptical Discs

A natural extension of the Brazilian test is the compression of an elliptical disc along the major or minor axis [2.17]. An advantage of this test is that the ratio between the compressive and tensile stresses may be varied by varying the ratio of the major and minor axes of the ellipse.

Table 2.2 has been obtained using the results of [2.17]. This text has received very little attention.

2.4.3 Compression and Indentation of Square Discs

This test was originally devised following a study of the effect of localized forces on coal [2.18]. The specimen takes the form of a square which is compressed between flat parallel platens, as shown in Figure 2.4.

The stress distributions set up in this system were analyzed by the photoelastic method. Tensile and compressive stresses were developed at the center of the specimen and the ratio of these stresses depended on the ratio of the indenter width to the side length of the specimen. The results are expressed in Table 2.3 for various indenter widths. The stresses developed may be determined with reference to Table 2.4, which gives values for the tensile stress at the center of the specimen generated by an applied stress of 50 units at the boundary, for various indenter widths.

TABLE 2.2
THE STRESSES IN AN ELLIPTICAL DISC UNDER
SYMMETRICAL LOADING

a_x/a_y	$-\sigma_x/\frac{P}{2a_y}$	$\sigma_y/\frac{P}{2a_y}$	σ_x/σ_y
0.5	3.75	1.1	3.41
0.67	2.80	0.865	3.24
1.0	1.91	0.635	3.00
1.5	1.30	0.35	3.71
2.0	1.15	0.17	6.76

a_x = axis parallel to load

a_y = axis perpendicular to load

σ_x = stress parallel to load

σ_y = stress perpendicular to load

P = applied load

Table 2.3

The ratio of compressive stress to tensile stress
in an indentation or square disc test

Ratio $\frac{\text{Indenter Width}}{\text{Specimen Side Length}} = \frac{b}{a}$	Ratio $\frac{\text{Compression Stress } \sigma_c}{\text{Tensile Stress } \sigma_t}$
0.125	2.25
0.250	2.85
0.375	3.60
0.500	4.70
0.750	8.50

Table 2.4

The tensile stresses at center of square disc

Ratio $\frac{\text{Indenter Width}}{\text{Specimen Side Length}} = \frac{b}{a}$	Tensile stress at center for applied stress of 50 units (psi)
0.125	4.6
0.250	7.0
0.375	8.3
0.500	7.65
0.750	5.40

SCHEMATIC DIAGRAM OF INDENTATION TEST

Figure 2.4

SPHERE LOADED ACROSS A DIAMETER

Figure 2.5

Example:	Specimen dimensions	$a = 0.75"$
		$t = 0.1875"$
	Load applied	$P = 2000 \text{ lbs.}$
	Indenter ratio	$\frac{b}{a} = 0.5$

The applied stress is $\frac{2000}{0.1875 \times 0.5} = 21,333 \text{ psi.}$

From Table 2.4, it can be seen that for an applied stress of 50 psi, there is a tensile stress of 7.65 psi. Hence the tensile stress at center of this new specimen is $21,333 \times 7.65/50 = 3264 \text{ psi}$. From Table 2.3, the compressive stress is $3264 \times 4.70 = 15,341 \text{ psi}$.

An important feature of this test is that the tensile stresses developed are all close to the center of the specimen. All of the stresses near the specimen boundary are compressive. As for the Brazilian test, the square specimen may be used to determine the elastic constants of the material. Two equations may be written to determine the tensile and compressive strains developed:

$$\left. \begin{aligned} \epsilon_t &= \frac{1}{E} (\sigma_t + \nu \sigma_c) = \frac{\sigma_t}{E} (1 + N\nu), \\ \epsilon_c &= \frac{1}{E} (\sigma_c + \nu \sigma_t) = \frac{\sigma_t}{E} (N + \nu). \end{aligned} \right\} \quad (2.7)$$

The tensile stress σ_t may be calculated as in the example above and N is the ratio between the compressive and tensile stresses developed. The value of N therefore depends on the ratio b/c and this is given in Table 2.3. Other experimental tips may be found in

[2.7] and [2.18].

2.4.4 Diametral Compression of a Disc with Central Hole

In view of the uncertainty associated with the origin of the primary fracture, several investigators have sought to promote diametral tensile failure by drilling a small hole along the axis of the disc (or the cylinder). [2.20], [2.21].

The maximum tensile stress σ_t and the maximum compressive stress σ_c are expressed, approximately by

$$\left. \begin{aligned} \sigma_t &= \frac{W}{\pi a t} (6 + 38q^2), \\ \sigma_c &= \frac{-W}{\pi a t} (10 + 10q^2), \end{aligned} \right\} \quad (2.8)$$

where W = applied diametral load,

a = external radius,

$q = \frac{b}{a}$,

b = internal radius,

t = thickness.

The load W at failure is not greatly affected by the presence of a small axial hole, but tensile strengths by this method are apparently much greater than those determined with a solid disc [2.20].

2.4.5 Disc Under Biaxial Tension

In order to obtain an evaluation of the biaxial tensile properties of brittle materials a test can be used [2.22] which uses

a simple disc specimen which is supported around its perimeter and subjected to pressure on one face. The pressure can be applied with a block of elastomer as in Figure 2.6 or by a neoprene diaphragm which transmits a uniform hydrostatic pressure across the entire surface of the disc [2.22]. The strains on the tension face of the disc can be measured by strain gauges.

Simple plate theory [D1.1] for a freely supported edge condition gives the following results for radial and tangential components of the stress at radius r ,

$$\sigma_r = \frac{3Pa^2}{8t^2} \left[(3 + \nu) \left(1 - \frac{r^2}{a^2} \right) \right], \quad (2.9)$$

$$\sigma_t = \frac{3Pa^2}{8t^2} \left[(3 + \nu) - (1 + 3\nu) \frac{r^2}{a^2} \right], \quad (2.10)$$

where σ_r and σ_t are the radial and tangential components of surface stress, P is the applied pressure, a is the outer radius and t is the disc thickness. An elasto-plastic solution for this problem can be found in References [D1.2], [D1.3].

2.4.6 The Sphere Tests

One of the simplest experiments is a transversely loaded sphere as shown in Figure 2.5. An analytical solution exists [2.23], [2.24] for the stress distribution in an elastic sphere under two equal and opposite concentrated loads, applied at the end points of a diameter.

SCHEMATIC DIAGRAM OF BIAXIAL DISC TEST

Figure 2.6

The equations are quite complicated but at the center of the sphere the principal stresses in the x, y, z directions may be obtained by:

$$\left. \begin{aligned} \sigma_z &= -\frac{P}{\pi r_0^2} \left[\frac{42 + 15\nu}{14 + 10\nu} \right], \\ \sigma_x &= \sigma_y = \frac{P}{\pi r_0^2} \left[\frac{21}{28 + 20\nu} \right] \end{aligned} \right\} \quad (2.11)$$

In the spherical test the stresses are a function of the Poisson's ratio also. The stresses may be calculated by three-dimensional photoelasticity [2.25] and the results follow very close to the analytical solution.

Other failure experiments for triaxial stresses can be found in [2.14] and Figure 2.7 presents some typical configurations.

2.5 The Determination of a Failure Criterion by Means of Mechanical Property Tests

The stress distribution in the wall of a high pressure cylinder is generally triaxial. Therefore, to determine the failure of the component it is necessary to relate the three principal stresses to a unique stress called effective stress. The development of the relationship or failure criterion used in the present study is described in Chapter 4.

The state of stress within the cylinder is assessed in terms

FOR COMBINED STRESSES

Combined stress specimens with uniform stress distribution.

Figure 2.7

NOTE: S_1, S_2, S_3 are the principal stresses $\sigma_1, \sigma_2, \sigma_3$

of the effective stress based on maximum tensile strain conditions, σ_e , for brittle failure. In Chapter 4 is discussed also the influence of the hydrostatic stress, σ_H , on the effective stress, σ_e , for uniaxial, biaxial and triaxial tests. These results are presented in Figure 2.8 where the effective stress, σ_e , has been plotted as a function of the hydrostatic pressure using the experimental results from tests on WC-Co compositions having 6 and 12 percent cobalt contents respectively [2.12], [2.26].

The uniaxial tensile stress is obtained by using a hoop-tensile test, the biaxial stress by means of the disc test, the triaxial stress by using a sphere test and the uniaxial compressive stress by a compression test on a cylindrical specimen. Other points of the failure curve can be obtained by carefully planning the experiments. The failure areas which can be investigated by the known tests are indicated in Figure 2.8. These failure areas were obtained by the calculations performed in Table 2.5. Other areas can be determined by other tests like the confined Brazilian disc test or the cylinder under axial and internal pressures.

The mechanical behaviour of WC-Co can be explained knowing the mechanical properties and microstructural parameters which determine the superior rigidity and strength of this material (see Appendix A3). In particular, with an increase in the cobalt content the influence of the hydrostatic pressure component is reduced, in accordance with yield criteria formulated for ductile materials.

WC-Co FAILURE LOCI

Figure 2.8

Table 2.5

The ratio of hydrostatic stress to effective stress
for a number of test conditions.

Elliptical Disc

a_x/a_y	$-\sigma_x / \frac{P}{2a_y}$	$\sigma_y / \frac{P}{2a_y}$	σ_x/σ_y	σ_H	σ_e	σ_H/σ_e
0.5	-3.75	1.1	3.41	-0.88	2.98	-0.30
0.67	-2.80	0.865	3.24	-0.65	2.27	-0.29
1.0	-1.91	0.635	3.00	-0.43	1.59	-0.27
1.5	-1.30	0.35	3.71	-0.32	1.48	-0.22
2.0	-1.15	0.17	6.76	-0.33	0.745	-0.44

Square Disc

b/a	$\sigma_c / \frac{P}{(b/a)}$	$\sigma_t / \frac{P}{(b/a)}$	σ_c/σ_t	σ_H	σ_e	σ_H/σ_e
0.125	-1.66	0.74	2.25	-0.31	1.57	-0.2
0.250	-1.60	0.56	2.85	-0.35	1.36	-0.26
0.375	-1.59	0.44	3.60	-0.38	1.24	-0.31
0.500	-1.44	0.31	4.70	-0.38	1.03	-0.37
0.750	-1.22	0.14	8.50	-0.36	0.75	-0.48

Full compression test $\sigma_H/\sigma_c = 0.33$ $\sigma_H/\sigma_e = -0.66$

Tensile-ring test $\sigma_H/\sigma_t = 0.33$ $\sigma_H/\sigma_e = 0.30$

Sphere test $\sigma_H/\sigma_e = -0.32$

Biaxial tension disc test $\sigma_H/\sigma_e = 1.34$

NOTATION: σ_H = hydrostatic stress

σ_e = effective stress

σ_c = compressive stress

a, b (see Figure 2.4)

σ_t = tensile stress

σ_x = stress along a_x axis

σ_y = stress along a_y axis

CHAPTER 3

STRESS AND STRAIN ANALYSIS FOR BRITTLE MATERIALS

3.1 Introduction

The assumption that there is little or no yielding prior to failure, leads to particular requirements being imposed on the stress analysis methods [1.6]. It should be possible to do a more detailed analysis of the stress distribution, particularly in areas of high stress gradient.

The design of very high pressure vessels reflects this kind of complex problem which involves many considerations including the definition of the operating and permissible stress levels [1.5]. The loads to which the individual components of the pressure chamber are subjected are dependent upon the required internal pressure, the radial support, the axial support, the position of the piston relative to the pressure chamber and the geometric configuration of the chamber itself.

Having assessed the relative magnitudes of these loads, it is necessary to examine the stress and strain patterns produced within the components by the application of these boundary conditions. This can be achieved by analytical solutions when available (see the following two paragraphs) or by force polyghs (see paragraphs 3.4) or by resorting to the finite element technique (see paragraph 3.5) and considering the material to behave as a continuum.

3.2 Analytical Solution for Elastic Stresses in a Thick-Walled Cylinder

The elastic stresses in a thick-walled cylinder containing the residual stresses are given by the well known Lamé equation [3.1].

These equations are obtained by solving the differential equation obtained by summing forces in the radial direction on a differential element of the cylinder. This derivation is found in most standard texts on advanced strength of materials [3.1] and yields the following equations for the stresses resulting from internal pressure p_i and external pressure p_o ,

$$\sigma_{\theta} = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} + \frac{(p_i - p_o) r_o^2 r_i^2}{r_o^2 - r_i^2} \frac{1}{r^2} = K_1 + \frac{K_2}{r^2} \quad (3.1)$$

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} - \frac{(p_i - p_o) r_o^2 r_i^2}{r_o^2 - r_i^2} \frac{1}{r^2} = K_1 - \frac{K_2}{r^2} \quad (3.2)$$

where σ_{θ} is the tangential stress,

σ_r is the radial stress,

r_o is the radius of the outer diameter,

r_i is the radius of the bore,

r is some intermediate radius.

The schematic of Figure 1.4 shows an open end condition (the ends of the cylinder are free to move in the longitudinal direction) and therefore for the elastic cylinder, this results in the plane stress condition (considering small thickness sections)

$$\sigma_z = 0 \quad (3.3)$$

where σ_z is the axial stress.

3.3 Analytical Solution for Elastic-Plastic Stresses in a Thick-Walled Cylinder

The pressure at which yielding will occur in the wall of a monobloc cylinder is obtained by the incorporation of equations (3.1) and (3.2) in the appropriate yield criterion. The most generally used yield criteria are the maximum shear stress of Tresca and the strain energy of distortion of Maxwell-Mises [3.2], [3.3].

The Tresca criterion (as related to a thick-walled cylinder) is defined by

$$\frac{\sigma_{\theta} - \sigma_r}{2} = \tau_y, \quad (3.4)$$

where τ_y is the shear yield stress. It is commonly assumed that $2\tau_y = Y$ where Y is the tensile yield stress; then Equation 3.4 becomes:

$$\sigma_{\theta} - \sigma_r = Y. \quad (3.5)$$

If we assume internal pressure only, the stresses in the elastic region are obtained from the Lamé Equations (3.1) and (3.2) combined with the fact that the radial stress at the elastic-plastic boundary ($r = c$) equals the yield pressure for the elastic portion of the cylinder. This yields

$$\frac{\sigma_{\theta c}}{Y} = \frac{1}{2} \frac{c^2}{r_o^2} \left(\frac{r_o^2}{r^2} + 1 \right) \quad (3.6)$$

$$\frac{\sigma_{rc}}{\gamma} = -\frac{1}{2} \frac{c^2}{r_o^2} \left(\frac{r_o^2}{r^2} - 1 \right) \quad (3.7)$$

The stresses in the plastic region can now be determined by combining the yield condition (3.5) and the equilibrium equation

$$\frac{\sigma_{\theta} - \sigma_r}{r} - \frac{d\sigma_r}{dr} = 0. \quad (3.8)$$

Solving the differential equation and applying the boundary conditions

$$\text{at } r = r_i, \sigma_r = -p_i$$

$$\text{and at } r = c, \sigma_{rp} = \sigma_{rc}$$

we obtain the stresses in the plastic region

$$\frac{\sigma_{\theta p}}{\gamma} = -\ln \frac{c}{r} + \frac{1}{2} \left(1 + \frac{c^2}{r_o^2} \right), \quad (3.9)$$

$$\text{and, } \frac{\sigma_{rp}}{\gamma} = -\ln \frac{c}{r} - \frac{1}{2} \left(1 - \frac{c^2}{r_o^2} \right). \quad (3.10)$$

The internal pressure required to produce plastic flow to a depth c is therefore given by

$$\frac{p_i}{\gamma} = \ln \frac{c}{r_i} + \frac{1}{2} \left(1 - \frac{c^2}{r_o^2} \right). \quad (3.11)$$

Therefore the pressure at the onset of yielding at the bore of the cylinder is obtained by setting $c = r_i$ in equation (3.11), which

yields:

$$p_i = \frac{Y}{2} \left(1 - \frac{r_i^2}{r_o^2} \right), \quad (3.12)$$

while the pressure to produce complete plastic flow or complete overstrain is given by equation (3.11) setting $c = r_o$, which yields:

$$p_i = Y \ln \frac{r_o}{r_i} \quad (3.13)$$

3.4 Force Plane Diagrams for Stressed Thick-Walled Cylinders

In ref. [3.4], the force polygon was introduced for representing a slip-line field in such a way that the force on any plane in the region of plane plastic strain could be immediately determined. In ref. [3.5], the approach was applied to determine the force polygon as it applied to some relatively complicated metal working operations. The last example investigated in ref. [3.5], was that of a thick-walled cylinder under an internal pressure (only) that rendered it wholly plastic. The author devoted a paper to examining the application of the force polygon notion to the elastic and the elastic-plastic loading of thick walled cylinders [3.6]. No force polygons have previously been given for non-wholly plastic cases. A brief general description and justification of the force polygon method is given in Appendix B1.

The main advantage of the method is the graphical solution and representation of the stresses. The procedure described in Appendix B1 can easily be coded in a computer program to plot the force polygons for any radius ratio and internal or external or in-

ternal-external pressure.

For the intended purposes in the present study the application of the force polygon was found to be inconvenient. In fact a force polygon diagram should be drawn for each section perpendicular to the axis to be able to represent the state of stress throughout the wall of the cylinder. This 'slicing' procedure is quite time consuming and it would still be necessary to know the distribution of at least one of the stresses along the radius of the cylinder. For this reason it was decided not to analyse the cylinder extensively by the new method, but to select the more versatile finite element technique instead.

3.5 Finite Element Method (F.E.M.) of Stress Analysis

3.5.1 General

The analytical equations and the force polygon approach given in the previous paragraphs are used to give first approximate solutions to the stresses developed in the most critical section of the high pressure chamber and they are the first step towards the design optimization of the equipment. But for a successful design with brittle materials, stress analysis methods which avoid the limitations of both the "simple" strength-of-materials theory and the classical theory of elasticity must be used; the former limitation involves extremely simple component geometry and loading conditions and the latter, while giving precise results, is limited by mathematical complexity to a few specialized loading conditions and geometric shapes. Fortunately, the

F.E.M. is an analytical tool which meets these requirements [3.7]. It involves division of the structure into a large number of elements of sufficiently simple shape that relationships between loads, stresses, and deformations of each element can be easily written; nevertheless, these elements can be of sufficiently small size that all complexities of the component geometry and loading can be accurately represented. Equations expressing equilibrium and displacement compatibility at the junctions of the elements are then written in matrix form and solved for the internal displacements and stresses. Since a large number of elements is required for accurate representation of a typical structure a computer is generally required for solution of the matrix equations. Similarly it is necessary to automate the development of stress-deformation relationships for the various types of element that are used in the analysis, so that the matrices can also be assembled by the computer.

3.5.2 The Interactive Computer Program CLASS

The study of the high pressure cylinder requires a more accurate analysis of the displacements and stress distribution within the wall of the component and, therefore, a finite element approach has been utilized. But, due to the complexity and repetitive nature of the problem, involving a large number of solutions for different cases of geometry, loading conditions and loading sequence, the assembly of the input data and the analysis of the results by the finite element method becomes extremely difficult, time consuming and prone to errors.

In this study the stress-strain analysis was performed using a specially developed interactive, graphics-oriented computer program calledCLASS (Computer Language Analysis Stress Strain). It consists of a set of modular programs in which each program implements a specific task in the overall numerical-analysis package. All of the programs are auto-prompting and, to a considerable extent, auto-instructional with all of the intricacies of data file manipulation hidden from the user.

One of the aims of CLASS is to provide a numerical analysis tool for the non-computer-oriented stress analyst which can be manipulated with ease in a conversational format.

Changes in boundary conditions, material properties, geometry, or other parameters can be quickly made in the specific program which stores them for use by other programs. The problem can then be resolved without the need of re-entering those parameters which are unchanged.

The general flow-chart of the computer program which simulates a stress-strain experiment is illustrated in Figure 3.1. The phases of the simulation include fabricating the model, describing the material properties, applying the boundary conditions, solving the problem and interpreting the results. The program architecture, execution and an example are given in Appendix B2.

Flow-chart of the interactive program CLASS

Figure 3.1

3.5.3 Model Definition

The cross-section of the component is constructed using the automatic grid-generating capabilities of CLASS. Points on the boundary of the component are used as input to generate a mesh of triangular elements. The component is divided by partition lines and between partitions it is possible to define the number of rows of elements and the number of elements per row.

The boundaries can be defined straight or curved and the elements can be dimensioned with arithmetic or parabolic progression. The program calculates the coordinates of the nodes and prints these values and plots the mesh on a CRT or a hard copy can be obtained by use of the CALCOMP routines and plotter. A detailed description of the program and of the variables involved is given in Appendix B3.

3.5.4 Boundary Conditions

An important step in a finite element stress analysis is the definition of the boundary conditions of the model.

Due to symmetry in the model, many times it is possible to consider only part of the model. This is accomplished by fixing the nodes on the symmetrical boundary in one or more directions to simulate continuity of the material. The finite element program used in the present study can handle both zero and non-zero displacements. This results in a greater flexibility in the analysis of the model and allows the simulation of many particular situations like constrained expansion (due to temperature gradient) or movement of the supports.

Other boundary conditions are the loads which are acting on the model. These loads are forces concentrated on the nodes of the elements or are distributed pressures, along the sides of the elements.

Two modules are included in CLASS to define the boundary conditions, the first (DISXEQ) establishes the node displacements and the second (FORXEQ) imposes the loads.

The module DISXEQ requires the following information:

- (1) the node number for which a displacement is specified
- (2) the displacements in the radial or in the axial direction or both.

The module FORXEQ assumes that the externally imposed forces act as distributed forces (pressures) on lines connecting any two adjacent nodes. The following information is required:

- (1) a node number at one end of the line on which the distributed force acts.
- (2) a node number at the other end of the line. This node must be adjacent to that given in (1) above.
- (3) the pressure intensity in the radial direction along the specified line.
- (4) the pressure intensity in the axial direction along the specified line.

3.5.5 Material Variables

A separate module called MATXEQ is provided in CLASS to input the data on the material of the model under study.

For an elastic finite element solution the Young's modulus and the Poisson's ratio must be included in the list. If the yield stress is also given, together with the information for the selected yield criterion, (see 3.5.7) the program calculates an effective stress based on this yield criterion and compares it with the given yield stress. Yielding occurs if the equivalent stress exceeds the initially specified yield value for the material.

After yielding occurs, plastic flow is assumed to take place in a direction perpendicular to the yield surface in stress space. This is the usual assumption in plasticity theory.

The possible yield criteria are given in Table 3.1. If a plastic finite solution is required the law of strain hardening must be provided. The strain hardening rule relates the equivalent stress and a corresponding equivalent strain of a material undergoing plastic deformation. Any rule can be used in the program which expresses this relation in terms of an analytical function of equivalent strain.

$$\bar{\sigma} = H \bar{\epsilon}^n, \quad (3.14)$$

where H and n are constants that must be provided during the execution of the module MATXEQ.

Table 3.1

YIELD CRITERIA

1. VON MISES

$$\sqrt{\frac{1}{2} [(\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2 + (\sigma_\theta - \sigma_r)^2 + 6 \tau_{rz}^2]} \leq \bar{\sigma}$$

2. HILL'S - ANISOTROPIC

$$\sqrt{\frac{3}{2} \left[\frac{F(\sigma_\theta + \sigma_z)^2 + G(\sigma_z - \sigma_r)^2 + H(\sigma_r - \sigma_\theta)^2 + 2L \tau_{rz}^2}{(F + G + H)} \right]} \leq \bar{\sigma}$$

3. DUBEY - HILLIER

$$\sqrt{\pm \alpha \left(\frac{\sigma_r + \sigma_\theta + \sigma_z}{3} \right)^2 + \frac{1}{2} [(\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2 + (\sigma_\theta - \sigma_r)^2 + 6 \tau_{rz}^2]} \leq \bar{\sigma}$$

$$\sigma_r + \sigma_\theta + \sigma_z > 0 \quad + \text{ve sign used}$$

$$\sigma_r + \sigma_\theta + \sigma_z < 0 \quad - \text{ve sign used}$$

4. MODIFIED COULOMB

$$\beta \left(\frac{\sigma_r + \sigma_\theta + \sigma_z}{3} \right) + \sqrt{\frac{1}{2} [(\sigma_r - \sigma_z)^2 + (\sigma_z - \sigma_\theta)^2 + (\sigma_\theta - \sigma_r)^2 + 6 \tau_{rz}^2]} \leq \bar{\sigma}$$

5. TENSION CUT-OFF

$$\sigma_\theta \leq \bar{\sigma}$$

$$\sigma_1 \leq \bar{\sigma}$$

$$\sigma_2 \leq \bar{\sigma} \quad \text{where } \sigma_1, \sigma_2 = \text{principal stress in r-z plane}$$

6. MAXIMUM TENSILE STRAIN

$$\bar{\sigma}_1 - \nu (\sigma_2 + \sigma_3) \leq \bar{\sigma} \quad \text{where } \nu = \text{Poisson's Ratio, } \sigma_1 > \sigma_2 > \sigma_3$$

F, G, H, L, α , β are constants

3.5.6 Selection of Options

The module CONTROL provides the different options for the execution of the finite element program. The following data are required, showing some choice in analyses and operating conditions:

- (1) Title of the job,
- (2) Elastic or plastic analysis,
- (3) Element number of an element whose convergence characteristics are desired (the equivalent stress in this element is printed for every iteration of the plastic analysis),
- (4) Desired yield criterion (see Table 3.1),
- (5) Plots of the results (stress contours, etc.),
- (6) Calculation of probability of survival,
- (7) Convergence accuracy specified for Gauss-Seidel iteration (typical value is 0.00001),
- (8) Convergence accuracy specified for plastic iteration (typical value is 0.01),
- (9) Percentage of full load to be applied initially (typical value is 80%),
- (10) Percentage of full load to be added in each load step (typical value is 10%).

3.5.7 Input Data Verification

The module VERXEQ is used to verify that the input data are correct. No additional input data are required for this module. An

output with all the data necessary for the execution of the finite element program is printed. This avoids a costly execution with incorrect data and the concomitant time loss to check the data.

3.5.8 Finite Element Program

The module FEMXEQ sets up the necessary temporary file and permanent files to solve the problem by the finite element method and to store the results of the solution. No inputs from the terminal are given: the necessary data are taken from the file previously composed by the different modules of CLASS.

The program^{*} is capable of handling axisymmetric bodies composed of triangular, constant strain, finite elements. The plastic flow rules used are those related to the given yield criterion by the requirement of normality between the yield surface and the strain vector in stress-strain space. The program can handle both zero and non-zero displacements. In addition, it can provide a straightforward elastic solution.

A detailed description of the program will not be given here because it is not vital to the purpose of the present study, however, some information on the storage of the stiffness matrix is given in Appendix B2, this being a crucial point.

* Note: the finite element code is an adaptation and modification of the program described in [A3.4].

3.5.9 Stress Field Analysis

A difficult task in a finite element method is the analysis of the results. A very long listing of stresses, deformations, reactions, etc. is printed; the more refined is the mesh (leading to a more accurate solution), the longer is the output and the more difficult is the interpretation of the results. The module CNXEQ of CLASS helps the stress analyst in his interpretation task. It consists of a main program and a series of subroutines to plot the contours of the stresses:

Information is required on the following:

- (1) the title of the plot which will be printed at the bottom of the graph page,
- (2) the plot size
- (3) the desired number of contour levels
- (4) the choice of specifying maximum and minimum values of contour levels or an automatic specification after a search by the computer
- (5) the selection of the stress to be mapped:
 - 1 = radial stress (σ_1)
 - 2 = tangential stress (σ_{hoop})
 - 3 = axial stress (σ_2)
 - 4 = effective stress by maximum tensile strain criterion (σ_e),
 - 5 = effective stress by Von Mises criterion (σ_f),
 - 6 = brittleness ratio (σ_e/σ_f),
 - 7 = mean or hydrostatic stress ($\sigma_H = \frac{\sigma_1 + \sigma_{hoop} + \sigma_2}{3}$),

- (6) option of printing the stresses,
- (7) the selection of a CRT display or hardcopy CALCOMP plotter unit.

The program searches for the minimum and maximum of the node coordinates then prints these values; it calculates the scale and scales the coordinates. It then finds the centre of gravity of each element and it uses this point as the location of the stress that is constant throughout the element.

The contour plot is inscribed within a set of boundary lines with tick marks along the boundary lines for easier location of a particular point of interest.

The actual program limits the plots to a quadrilateral shape, this limitation is imposed by the contour subroutine of the CALCOMP plot package. As soon as a new, more flexible, routine becomes available the program will be updated.

3.5.10 Strain Analysis

In many applications, it is very important to know the deformation of the model under the applied load conditions.

The module SHAPEXEQ of CLASS was written to plot the deformed shape of the component under study. This module consists of two main programs. The first (load module TRSCALEMN) reads the node coordinates of the undeformed and deformed shape and it expands the scale of the strains to exaggerate the deformed shape. The second (load module SHAPELMN) reads the scaled displacements and plots the undeformed and

the scaled-up deformed shape. To avoid confusion, the outer periphery of the component is plotted only and a small symbol is drawn at each node coordinate.

-Two parameters are needed:

- (1) the scale for all deformations (generally values between 10 and 1000 are necessary depending on the applied external loads);
- (2) the plot dimensions.

A choice of CRT display or hardcopy plot is possible and the execution of either is similar to that given in Appendix B3 for the Grid Generator Program.

3.6 Finite Element Stress Strain Analysis of the High Pressure Chamber

The computer interactive package CLASS was used to perform a series of simulated tests for different geometry of the chamber, loading conditions, boundary conditions, and loading sequence. A large number of solutions was obtained, but the ability of CLASS to interpret large quantities of such data eased the consequent reduction of results [1.6].

- The grid-generation capabilities of CLASS were used to create the element mesh illustrated in Figure 3.2 where nodes 1. to 9 are positioned on rollers as a result of the symmetry of the cylinder.

Utilizing the symmetry conditions does permit a more generalized presentation of the results, however this condition does require both

LOADING CONDITIONS ILLUSTRATED ON FINITE ELEMENT REPRESENTATION OF THE HIGH PRESSURE CYLINDER

Figure 3.2

pistons A and B, shown in Figure 1.4, to move inwards simultaneously as the pressure is generated; this is an operational mode more commonly associated with the belt apparatus. For the more conventional piston-cylinder (P/C) arrangement, the lower piston (piston B in Figure 1.4) is usually stationary relative to the cylinder with the overall increase of pressure in the working volume being directly associated with the movement of the upper piston A. This unsymmetric loading condition of the internal pressure will not be considered at present.

The type of loading considered for the internal pressure is illustrated in Figure 3.2. With the height, $h/2$, sub-divided into six equal elements, the internal pressure can be applied across any number of these elements, i.e., 2, 3 or 4 of the six elements. Arranging the model in this manner enables an increase in pressure within the working volume to be easily accommodated as the pistons move inward. The relationship between pressure increase and piston displacement being directly dependent upon the compressibility characteristics of the materials in the working volume.

The model in its present form provides considerable insight into the operational behaviour of the P/C high pressure system when subjected to different internal pressure, radial support and axial loads. The analysis has been confined to the elastic domain, which is appropriate for the range of pressures investigated when using a tungsten carbide (WC-Co) matrix material having a low cobalt content.

3.7 Analysis of the Results

A number of solutions, as detailed in Table 3.2, for different input variables pertaining to the cylinder loading and geometry have been obtained.

These solutions are presented with particular emphasis being given to the following design functions:

- (i) influence of radial support
- (ii) barrelling of the cylinder
- (iii) effect of axial load on the cylinder
- (iv) quasi static load considerations.

3.7.1 Influence of Radial Support

Figures 3.3 (a) - (d) illustrate the distributions of the circumferential, axial and radial stresses for the applied conditions detailed in Table 3.2. The wall thickness of the cylinder in these four solutions is considerably less than that which would be used in a practical situation, however this geometry has been selected to demonstrate the following two significant points of interest in the design of high pressure systems:

- (i) the first follows logically from the geometry of the loaded system in that any increase in the radial support is accompanied by a reduction of the excessive tensile circumferential stresses;
- (ii) the second stipulates that an increase in the radial support pressure can suppress the tensile circumferential

Table 3.2

Solution Code	A200*	B200	A2X0	B2X0	B200	B2X0	B2X0*	B2X0**	B2XA	B2XA*	A4X0*	B4X0	C4X0
See Appendix B4													
Internal pressure, P_1 $\text{lb/in}^2 \times 10^3$	294	588	294	588	588	588	588	588	588	588	294	588	882
(CPa)	2.0	4.0	2.0	4.0	4.0	4.0	4.0	4.0	4.0	4.0	2.0	4.0	6.0
Number of elements for application of P_1	2 $\longrightarrow$												
Radial support pressure, P_0 $\text{lb/in}^2 \times 10^3$	0	0	-100	-100	0	-100	-100	-100	-100	-100	-100	-100	-100
(CPa)			-0.69	-0.69		-0.69	-0.69	-0.69	-0.69	-0.69	-0.69	-0.69	-0.69
Ratio of external to internal diameter of cylinder	2/1 $\longrightarrow$ 3/1 $\longrightarrow$												
Axial load, P_a (CPa)	0	0	$\longrightarrow$										
Radial displacement of node 55 (37)* $\text{in} \times 10^{-3}$	.032	.064	-1.08	-1.05	.041	-0.51	-0.0	-0.26	-0.46	0.0	0.0	0.25	0.84
(μm)	0.82	1.64	-27.5	-26.7	10.4	-13.0		-6.5	-11.7			6.3	21.3
Axial stress in element 16, $\text{lb/in}^2 \times 10^3$	101.6	203.3	102.2	203.9	43.3	44.5	34.6	46.7	4.05	-3.03	19.7	48.1	71.5
(CPa)	0.71	1.40	0.70	1.40	0.29	0.30	0.24	0.32	0.03	-0.02	0.13	0.33	0.49

INPUT VARIABLES FOR CODED SOLUTIONS

MATERIAL: TUNGSTEN CARBIDE (WC-CO), WITH POISSON'S RATIO $\nu = 0.236$ AND YOUNG'S MODULUS, $E = 90 \times 10^6 \text{ lb/in}^2$ (6200 GPa)

** NODES OF ELEMENTS CONSTRAINED IN RADIAL DIRECTION

** TAPERED CYLINDER

(a) Solution A200 ($d_0 = 2''$) (b) Solution B200 ($d_0 = 2''$)

Stress distributions through the cylinder wall at the centre plane

Figure 3.3

Stress distributions through the cylinder wall at the centre plane

Figure 3.3

stresses at the outer periphery of the cylinder can actually exceed the magnitude of the tensile circumferential stress.

This axial tensile stress component can be of considerable importance at the tungsten carbide-steel interface, where the quality of the surface finish of the respective mating surfaces will dictate the level of stress concentration and enhance the probability of crack initiation, propagation and fracture.

Figures 3.4 (a) and (b) illustrate the deformation profile of a cross-section for the cylinder with and without radial support obtained by the DEFEXQ module of CLASS. While the effect of radial support is advantageous to the circumferential stress, the cylinder is now subjected to an axial 'bending' condition which can give rise to an excessive degree of barrelling at the bore of the cylinder.

3.7.2 Barrelling of the Cylinder

The effect of barrelling, as a result of excessive radial support, is to impede the 'free' movement of the piston during the loading sequence of pressure generation in the working volume. This incompatibility between the piston and cylinder during loading can result in an undesirable localized clamping action of the piston at those points on the bore of the cylinder where the inward radial displacement is greatest. This action can result in the frequent breakage of the piston at this point of clamping; a condition which is further aggravated by the barrelling of the piston as an axial load is supplied to this component. To accommodate this interference condition, the nodes on the bore which displace radially inward can

(a) Deformation of cylinder with no radial support. Solution B200

(b) Deformation of cylinder with radial support illustrating the excessive barreling condition. Solution B2X0

be constrained to move only in the axial direction as illustrated by the solution for the deformation profile shown in Figure 3.5 (a). This type of boundary condition is undoubtedly an over-simplification of the problem since the interaction of these nodes with the piston would require the simultaneous solution of both high pressure components.

To eliminate, or at least reduce, this practical difficulty of barrelling requires the P/C arrangement to be modified to a belt and anvil type configuration as shown by the insert in Figure 3.2. The introduction of the taper at the upper section of the bore enables the radial interference to be reduced, however, numerous other modifications then became necessary. The profile of the piston must be altered to the flat-nosed conical anvil type to interface with the tapered face of the cylinder. While this change permits considerable strengthening of the anvil as a result of the massive support concept, limitations are unfortunately also introduced, i.e. the need for compressible gaskets between the mating surfaces of the high pressure components is introduced with the added limitation of reducing the axial stroke length of the piston.

The profile deformation of a tapered cylinder is illustrated in Figure 3.5 (b). This result is directly comparable to Figures 3.4 (b) and 3.5 (a); an inward radial displacement of one node has occurred as shown in Figure 3.5 (b) since the internal loading has only been applied across two elements (see Table 3.2). For an internal pressure distributed along the entire boundary of the bore no barrelling can take place.

(b) Deformation of tapered cylinder.
Solution B2X0**

(a) Deformation of cylinder with
nodes 37, 46 and 55 constrained
in the radial direction. Solution B2X0*

3.7.3 Effect of Axial Load

Ignoring the overall system design difficulties which would be associated with the introduction of an axial load acting on the cylinder, it is useful to ascertain what beneficial effects might stem from the imposition of such a loading condition as illustrated in Figure 3.2. Figure 3.6 illustrates the deformation profiles for the conditions listed in Table 3.2. Barrelling of the cylinder still occurs, and a useful comparison of the maximum radial displacements of these and other results is given in the lower half of Table 3.2, together with the axial stress values in element 16 on the outer periphery of the cylinder.

While the radial displacements recorded in Table 3.2 might be considered to be small, they do serve to illustrate the effect of the different variables discussed. Any further increase in internal pressure or radial support would increase the magnitude of both the radial displacements and the particular axial stresses, however the analysis would then have to be extended to the elasto-plastic condition to accommodate the plastic yielding which would occur in and around the bore of the cylinder where the radial pressure is applied.

3.7.4 Quasi-Static Condition of Loading

With the downward movement of the piston during the generation of pressure the external loading conditions on the cylinder with regard to the magnitude and location of the internal pressure are continuously changing. The deformation profiles of Figures 3.4 - 3.7 which relate

2

(a) Deformation of the cylinder with axial load. Solution 02XA

(b) Deformation of cylinder with nodes 37, 46 and 55 constrained in the radial direction and axial load. Solution 02XA

(a) Deformation of the cylinder corresponding to Solution A4X0* with node 55 fixed in the radial directions.

(b) Deformation of the cylinder corresponding to Solution B4X0

to different loading and boundary conditions provide a useful insight to ascertain the extent of dimensional changes (displacements) at the bore during loading sequences.

Chapter 4

FAILURE CRITERIA FOR QUASI-BRITTLE MATERIALS

IN HIGH PRESSURE APPLICATIONS

4.1 Introduction

The design of components for high pressure equipment made of a tungsten carbide-cobalt composite presents numerous difficulties as no adequate failure criterion has been developed to accommodate the flow and fracture behaviour exhibited by WC-Co composites (see Appendix A3) when subjected to the triaxial stress conditions illustrated in the previous chapter. The quasi-brittle nature of WC-Co composites requires that consideration be given to both the brittle and ductile modes of failure [4.1].

In high pressure cylinders, plastic deformation of the cobalt binder phase can accompany the development of radial cracks in the central part of the cylinder bore. Cracks, normal to the axis of the bore, can also develop adjacent to the ends of the pistons in the bore and at other locations within the cylinder. The location, initiation and propagation of these cracks will depend to a varying degree on the availability of micro cracks present in the material as voids or defects, on the state of stress, on the geometric configuration of the cylinder and on the microstructural properties of the material [4.2], [4.3], [4.4], [4.5].

The effects of different loading conditions were previously analyzed in Chapter 3 in which an elastic finite element approach was

used with the high pressure cylinder being effectively modelled as a right circular cylinder [1.6]. In the present chapter the state of stress within the cylinder is assessed in terms of the effective stress based on maximum tensile strain conditions and the effective stress evaluated from maximum energy distortion considerations. The application of these effective stress values has enabled a brittleness index to be defined; contour plots of this index provide a useful indication for assessing the possible fracture mode of the high pressure cylinders.

The interactive package CLASS was again extensively used to generate and analyze different solutions, to calculate the effective stresses and the brittleness index and to plot the contours of the results.

4.2 Ductile and Brittle Fracture

Fracture in materials can be classified as either brittle or ductile in nature. As a tensile load acting on an element of material is increased, one of two events must eventually occur: either the shear stress acting on the slip planes will cause slip, i.e. plastic deformation, with no volume change, resulting in a shear type fracture, or the strongest cohesive bonds between the separate atoms will break down, permitting a sudden cleavage type fracture, i.e. brittle elastic fracture. Numerous fractures are appropriately described as being partially brittle and partially ductile, because certain portions of the fractured surface are orientated approximately parallel to the planes of maximum shear stress, while other portions of the fractured

surface are oriented towards the planes of maximum tensile stress.

As the primary concern of this investigation is to develop an appropriate fracture criterion for the quasi-brittle WC-Co composite material, it is useful to examine those factors which influence brittle fracture:

- (i) The effects of temperature and dynamic loading exert considerable influence on the type of fracture anticipated; however these effects are of secondary importance in the development of the fracture criterion at this time and will not be further considered.
- (ii) Any increase in volume enhances the brittle nature of the component. This 'size effect' can be explained in terms of the improved probability of the existence of critical flaws in the larger volume, and the inability of brittle materials to flow plastically thereby relieving the stress at a point of severe stress concentration [2.27], [4.6].
- (iii) Brittle materials which tend to fail with a cleavage type fracture when subjected to triaxial tensile stresses, can plastically deform when subjected to triaxial compressive stresses [4.7].
- (iv) As a further consequence of (iii) above, it has been demonstrated that the maximum tensile strength and tensile strain cannot be considered as material properties or constants with respect to brittle material behaviour, but rather are required to be interpreted in the context of the state of stress

at the critical point of fracture [2.27], [4.8].

(v) Since the WC-Co composites exhibit a considerable stress differential between the uniaxial tensile and compressive yield stresses (see Appendix A3), it is logical to assume that these materials will also be substantially stronger in biaxial compression than in uniaxial tension. When this is the case, we have the implication that failure models are no longer valid if they depend on deviatoric stresses only. A comparison of a cylindrical failure surface with a pyramidal surface is illustrated in Figure 4.1. The cylindrical surface is a function of the level of the deviatoric stresses only, whereas the pyramidal surface is also partially dependent upon the hydrostatic pressure component of the defined state of stress [4.9].

4.3 Failure Criteria

In the present work a failure theory based on existing theories which relate, independently, to both brittle fracture and plastic yield considerations has been developed with particular reference to the design of high pressure equipment. The proposed model is partially constructed on the basis of and assessed in terms of available experimental material testing techniques and relevant data pertaining to the properties of tungsten carbide and other brittle or ductile materials, or composites of the two.

Comparison of a cylindrical failure surface to a pyramidal surface

4.3.1 Failure Criteria Independent of Hydrostatic Stress

From experimental results it has been confirmed that the maximum normal strain failure theory can be used to describe the fracture behaviour of brittle materials. This criterion, which has been used in the design of gun barrels and thick-walled cylinder, is given by Equation (4.1) where it is assumed that the principal stresses are as $\sigma_1 > \sigma_2 > \sigma_3$ and the effective stress simply depends on:

$$\sigma_e = \sigma_1 - \nu (\sigma_2 + \sigma_3) \quad (4.1)$$

However, in the case of WC-Co composites it has been demonstrated that under particular stress conditions yielding can occur and, in such cases, a failure theory based on plastic yield of the cobalt binder phase must be considered. The maximum distortion energy criterion, or Mises criterion, can be used to define the onset of plastic yielding and is given by Equation (4.2):

$$\sigma_f = \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}. \quad (4.2)$$

To illustrate the differences between the two theories, a simple right circular cylinder under plane stress conditions has been analyzed to evaluate the internal pressure necessary to initiate fracture (yield) at the inner bore. The results are presented in Figure 4.2 for a range of cylinder radii using both the theories given by Equations (4.1) and (4.2), together with the more conservative maximum shear stress theory (Tresca) and the thin-walled cylinder analysis for comparative purposes. As the internal pressure results have been

Comparison of failure criteria for a right circular cylinder.

Figure 4.2

normalized with respect to the yield stress in uniaxial tension, similar results would be obtained when using the fracture stress in uniaxial tension to normalize the pressure. Of particular interest in these results is the influence of the choice of Poisson's ratio such that the maximum normal strain theory, when using the value of $\nu = 0.5$, shifts closer to the Mises criterion, illustrating that the $\epsilon_{\max}$ theory can provide a considerably more sensitive assessment of the brittle nature of a material as compared to other criteria.

4.3.2 Influence of Hydrostatic Pressure

The implications of macroscopic failure criteria which are independent of hydrostatic stress have been presented in Reference [4.9]. For the WC-Co composites, which are considerably stronger in compression as compared to tension, any failure criterion based on the deviatoric stress components would therefore provide conservative estimates when subjected to states of stress comprising large compressive hydrostatic levels of stress. For these reasons it would appear that the failure surface for WC-Co composites would correspond to the pyramidal type surface shown in Figure 4.1.

The influence of hydrostatic stress on the effective stress, σ_e , derived from the maximum normal strain criterion has been investigated by Shaw et al [2.12] for uniaxial, biaxial and triaxial tests. These results were presented in Figure 2.8, and for the reader's convenience, in Figure 4.3. The effective stress, σ_e , has been plotted as a function of the hydrostatic pressure using the experimental

WC-Co FAILURE CURVES

Figure 4.3

results from tests on WC-Co composites having 6 and 12 percent cobalt contents respectively [2.12], [2.26].

It is evident from Figure 4.3 that the tensile strength is not a material property of brittle materials but that failure is dependent upon the state of stress at the point of fracture. Furthermore, with an increase in the cobalt content the influence of the hydrostatic pressure component is reduced, in accordance with yield criteria formulated for ductile materials.

4.4 Analysis of High Pressure Cylinders

Utilizing this representation of σ_e as a function of σ_H , the stress states within a high pressure component can be conveniently assessed. The states of stress for a series of high pressure cylinders subjected to the different loading conditions given in Appendix B4 have been superimposed on the experimental results of Figure 4.3. These results have been obtained by use of CLASS, so that the influence of increased internal pressure, axial load and radial support pressure can be evaluated. The mesh is illustrated in Figure 3.2; the distribution of the elements in Figures 4.4 - 4.7 refers to the number sequence given on this mesh, i.e. only a single row of elements is represented in Figures 4.4 - 4.7 for the coded solutions provided in Appendix B4.

From these results, presented in Figures 4.4 - 4.7, it is possible to assess the relative points of weakness in the component for different loading conditions and wall thickness.

The results may be summarized as follows:

Solutions B200 and B2X0 (Figure 4.4) A direct comparison of these solutions indicates the extent of the shift of the curve attributable to increased radial support. While the effect of this support is to suppress fracture, the thickness of the cylinder wall should be increased to ensure that all points drop below the bounding fracture limit. One might anticipate that the fracture site would probably occur at the bore, but for the conditions listed in Appendix B4, this has moved to the outer periphery of the cylinder as evidenced by the implications of curve B2X0. The stress condition surrounding element 16 is of considerable significance as this area, which is rather sensitive to changes in radial pressure and axial load, is aggravated by the fact that it interfaces with the steel support rings. The application of an axial load to the cylinder can therefore be effectively used, as shown in Table 3.2, to suppress the axial stresses at this location thereby reducing the hydrostatic component and shifting the curve below the fracture limit.

Solutions B2X0 (Figure 4.5) An increase in the wall thickness from $d_o = 2$ inches to $d_o = 3$ inches results in a considerable improvement by moving the entire stress condition in the cylinder towards a more compressive hydrostatic condition.

Solutions B4X0 and C4X0 (Figure 4.6) With the internal pressure applied across an increased number of elements at the bore, the total

WC-Co FAILURE CURVES

Figure 4.4

WC-Co FAILURE CURVES

Figure 4.5

WC-Co FAILURE CURVES

Figure 4.6

WC-Co FAILURE CURVES

Figure 4.7

load on the cylinder is increased. Under these conditions the effective stress component deteriorates substantially at the bore, primarily as the result of the increased tensile circumferential stress shown by the curves for solutions B2X0 and B4X0. A further comparison of the effect of increased load at the bore is provided by solution C4X0 with an internal pressure of 0.9×10^6 lb/in (6GPa). These results indicate the importance of having the internal pressure applied across a small percentage of the overall cylinder height, particularly for ultra high pressures.

Of particular significance is the considerable shift which can be obtained by the application of radial support pressure (solution B200 to solution B2X0) and by an increase in the wall thickness of the cylinder (solution B2X0 for $d_o = 2.0$ and $d_o = 3.0$ in). While these states of stress for elements 1 through to 16 can be assessed in terms of the effective stress, σ_e , derived from the maximum normal strain considerations, it is useful to examine the variation in the effective stress, σ_f , derived in accordance with the maximum strain energy criterion.

For the state of pure tensile stress $\sigma_e = \sigma_f$ and the two failure criteria from Equations (4.1) and (4.2) coincide for both the 6 and 12% cobalt compositions. Considerably different results are obtained however for pure compression where $\sigma_1 = \sigma_2 = 0$. For the latter case $\sigma_e = -\nu\sigma_3$ and $\sigma_f = \sigma_3$ so that the value of σ_f will be double the value of σ_e for $\nu = 0.5$. From the theoretical considerations therefore, we have $0.5 < \sigma_e / \sigma_f < 1$, where the maximum brittleness

The effect of hydrostatic stress on the failure criteria

Figure 4.8

is obtained from the pure tension conditions.

These results are illustrated in Figure 4.8. The values for the ratio of σ_e/σ_f , which is a measure of the degree of brittle behaviour anticipated [2.12], has also been plotted as a function of the hydrostatic pressure. The larger this ratio the greater the relative tendency for brittle fracture to initiate, hence with an increasing compressive hydrostatic component the material would be expected to undergo plastic flow prior to exhibiting a brittle tendency.

The specific results for σ_e, σ_f and the ratio σ_e/σ_f as a function of the hydrostatic component have been presented in Figure 4.9 for two particular solutions namely B2X0 with an outside diameter $d_o = 2$ inches and B4X0 with an outside diameter $d_o = 3$ inches. These results are coded in accordance with Appendix B4 and represent the stress condition in the elements 1 through 16 as shown in Figure 3.2. For the smaller diameter cylinder it is clearly possible to have both brittle fracture and yielding occurring at the outer periphery of the cylinder, i.e. element 16. This condition is substantially improved for the B4X0 solution primarily as a result of the increased wall thickness even though the total loading, which arises from the application of the internal pressure, has been applied across a height double that of the B2X0 solution. Furthermore, at the inner bore, both solutions are contained within the 6% cobalt σ_f value, but have clearly exceeded the σ_e value. Since cracks at this location will almost certainly be normal to the circumferential stress,

Figure 4.9

these cracks can effectively reduce the monoblock tungsten carbide to a sectoried die such that the tensile circumferential stresses are diminished in magnitude. Whilst this condition might be acceptable at the inner bore in the region of elements 1 and 2, the degree of brittle behaviour defined by σ_e/σ_f must be investigated throughout the component particularly in those regions where a negative hydrostatic component does not exist.

4.5 Location of Fracture

To provide an assessment over the entire body of the high pressure cylinder, contour maps of σ_e/σ_f and σ_H have been plotted to coincide with the finite element mesh illustrated in Figure 3.2. The results are presented in Figure 4.10 and 4.11 for a series of loading conditions which provide considerable insight into the possible failure modes of these components. The results are summarized as follows:

B2X0 and B4X0. For both solutions, having an external diameter $d_o = 3$ inches, the effect of the internal pressure being applied across four elements for solution B4X0, as opposed to two elements for solution B2X0, can be investigated. For solution B4X0 the effect of barrelling has been reduced [1.2] as compared with B2X0, so that the brittleness index is substantially reduced in the vicinity of the free surface of the cylinder. This could be interpreted as reducing the probability of fracture across a plane normal to the axial direction at this location, however the hydrostatic component has now changed to a tensile value, for B4X0, at the bore near the free surface.

- Maximum
- Minimum

Contour maps of hydrostatic stress σ_H
Figure 4.11

B4X0 and B4Y0. A comparison of these solutions in which B4Y0 has an increased radial support pressure improves the condition at the inner bore. However, the increased radial pressure again focusses attention at the outer periphery of the cylinder, and specific attention to the quality of the tungsten carbide-steel interface could enhance the possibility of fracture initiating at this location in accordance with observations from the literature [4.2 - 4.5].

B2X0, B4Y0 and C2Y0. These solutions illustrate the usefulness of the approach presented in this study such that changes in the radial support pressure, internal pressure and the extent of internal pressure application can be investigated with a view to maximization of the internal pressure or working volume conditions.

B2X0 and B2XA. A comparison of the σ_e/σ_f contours for these two solutions provides an indication of the influence of axial load, a physical condition which is not usually employed in high pressure systems. The effect is to reduce the brittleness ratio over a considerable volume of the cylinder thereby improving the probability of survival of the cylinder. Utilization of this concept involving axial loading is difficult to implement in a working system.

PB2Y0. This particular solution has been included for comparative purposes only. Utilizing an elasto-plastic finite element model a small amount of plastic deformation was permitted at the inner bore of the cylinder where the yield stress levels were exceeded. The result again indicates the negative influence of excessive radial support particularly when the internal pressure loading is relatively low.

4.6 Conclusions

The present study illustrates a possible fracture criterion procedure for analyzing the behaviour of quasi-brittle materials more specifically with respect to high pressure equipment. An assessment of the σ_e and σ_f values as a function of the hydrostatic pressure coupled with the contour maps of the brittleness ratio can provide a useful indication of the possible location and mode of fracture. Considerably more information of an experimental nature is required to substantiate how and to what extent the external variables of radial and axial support loading as well as the internal pressure loading can influence or shift the fracture initiation site thereby giving greater credibility to this model. However, it is encouraging that on the evidence of the results presented, while being relatively qualitative in nature, reasonable correspondence with observations from the literature is observed.

Chapter 5

PROBABILISTIC EVALUATION OF THE MAGNITUDE OF THE TEST CHAMBER AND OF THE LOADING SEQUENCE OF HIGH PRESSURE CYLINDERS

5.1 Introduction

In the process of the overall optimization design of high pressure circular chambers, the probability of survival of a component is particularly important in the study of the interaction effects of different design considerations.

The successful attempt to reach very high pressure is generally achieved in pressure chambers of relatively small volume. For production purposes it would be very interesting to reach high pressures in larger volumes. Experiments in such a direction have always encountered a number of difficulties. If the ratio of the outside diameter to the inside diameter of the cylinder ($K = d_o/d_i$) is kept constant, the stress distribution does not vary throughout the wall thickness if the loads do not change. It is experimentally verified instead that large cylinders of relatively brittle material like WC-Co break after only a few loading and unloading cycles.

The sensitivity of the mechanical properties to flaws, as a consequence of the lack of ductility, is much greater in brittle materials. These flaws include pores, microcracks, inclusions of foreign materials, etc. and will generally be random in size and distribution. They will produce local stress concentrations which will depend on the size and configuration of the flaws and, since the

apparent material strength will be controlled by the peak stress, it follows that a randomness will exist in the strength of the material.

X Experience shows that the total variability in the mechanical properties of brittle materials is great enough that a single strength value cannot be assigned, but rather that the usable strength level must be associated with an acceptable probability of failure.

In addition to the analytical techniques required to calculate the probability of failure for a complex structure, this statistical concept introduces changes in design criteria philosophy, with the need to define an acceptable failure probability and the possibility of a trade-off between allowable stress level and failure probability.

The dependence of the strength level on the volume of the material must also be introduced, since the greater the volume the greater the probability of a critical flaw. Obviously, this statement is tempered by the stress distribution, and the prediction of strength becomes dependent not on the maximum stress at some particular location but at the summation of the failure probabilities of each element of the component. For instance, a low stress distributed over most of the volume of the component may contribute significantly to the failure probability despite very localized high stresses, since the low stress is more likely to be associated with a larger flaw.

Attempts to specify the strength of brittle materials as a function of failure probability were made by Weibull in 1939 [5.1] and,

until recently, little additional work had been conducted.

Weibull established a series type model to describe a material containing flaws of random size and distribution, and he selected the simplest mathematical relationship which would fit the model.

The expression contains parameters which must be evaluated experimentally from samples of the material of interest. Despite the apparent inadequacies of the assumed material model and the simplicity of the mathematical expression, the Weibull description of the failure probability of the material has been reasonably well confirmed in practice although experimental work is currently limited.

5.2 Weibull Statistical Approach

Statistical treatments of the experiments are essential when brittle materials are tested since there is always a large and unavoidable variation in the results. However, only if the results obtained from an experimental programme are analysed using Weibull statistics [4.6] and the data plotted on Weibull probability paper, (See Appendix C1) it is possible to predict the load at which a given percentage of the components will fail.

The Weibull expression for statistical failure may be expressed in two different ways - [5.2], [5.3]. The following is a derivation of Weibull's equations for uniformly distributed flaws.

The risk of rupture R , and the probability of survival S , are related by the equation

$$R = - \ln S \quad (5.1)$$

If P_0 is the probability of a failure at a given stress level per unit volume, the probability of survival per unit volume is

$$S_0 = 1 - P_0 \quad (5.2)$$

If we suppose that v of such unit volumes are arranged in series and the stress is applied, the probability of survival for all the volumes is the product of the probabilities that the individual volumes will not break and is represented by,

$$S = (1 - P_0)^v \quad (5.3)$$

or by $\ln S = v \ln(1 - P_0) \quad (5.4)$

From equation (5.4) and (5.1), it is found that

$$R = -v \ln(1 - P_0),$$

and at a differential scale we obtain

$$dR = -\ln(1 - P_0) dv \quad (5.5)$$

Weibull experimentally found that

$$-\ln(1 - P_0) = \left(\frac{\sigma - \sigma_u}{\sigma_0} \right)^m \quad (5.6)$$

where σ is the tensile stress on the elemental volume,

σ_u is the smallest tensile strength which any element can have,

and where σ_0 , m are material constants.

Clearly, in most cases $\sigma_u = 0$ (particularly with brittle materials).

and (5.6) can be reduced to

$$-\ln(1 - P_0) = \left(\frac{\sigma}{\sigma_0} \right)^m \quad (5.7)$$

Then from (5.7) and (5.5) we obtain

$$dR = \left(\frac{\sigma}{\sigma_0} \right)^m dv$$

and

$$R = \int_V \left(\frac{\sigma}{\sigma_0} \right)^m dv \quad (5.8)$$

Equation (5.8) shows the risk of rupture over the entire volume V (by integration). The stress σ can be constant within the entire volume V (a homogeneous stress) or can be variable (nonhomogeneous stress).

The probability of survival S is from (5.1):

$$S = e^{-R} = \exp \left[- \int_V \left(\frac{\sigma}{\sigma_0} \right)^m dv \right]$$

and the probability of rupture is:

$$P = 1 - S = 1 - \exp \left[- \int_V \left(\frac{\sigma}{\sigma_0} \right)^m dv \right] \quad (5.9)$$

From equation (5.7) it can be seen that since $0 \leq P \leq 1$, it follows that $0 \leq \left(\frac{\sigma}{\sigma_0} \right)^m \leq \infty$.

The integral of equation (5.9) or (5.8) can be solved and written as:

$$R = \int_V \left(\frac{\sigma}{\sigma_0} \right)^m dv = KV \left(\frac{\sigma}{\sigma_0} \right)^m \quad (5.10)$$

where V = the volume subject to tensile stress,

K = a factor which depends on the nonhomogeneity of the stress (or stress gradient)

If the volume is subjected to uniform tensile stress (or homogeneous stress) $K = 1$.

If the stress distribution over the component is known, it is possible to calculate the value of K and therefore the influence of the stress gradient.

Important considerations can be deduced from equation (5.10) if we consider two volumes V_1 and V_2 having the same material constants σ_0 and m , but subject to differing stresses σ_1 and σ_2 and differing stress gradient factors K_1 and K_2 . We can write:

$$R_1 = K_1 V_1 \left(\frac{\sigma_1}{\sigma_0} \right)^m,$$

$$\text{and } R_2 = K_2 V_2 \left(\frac{\sigma_2}{\sigma_0} \right)^m;$$

and for the same risk of rupture $R_1 = R_2$:

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{K_2}{K_1} \right)^{1/m} \left(\frac{V_2}{V_1} \right)^{1/m} \quad (5.11)$$

Therefore:

a) if $K_1 = K_2$ (i.e. the stress configurations are the same):

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{V_2}{V_1} \right)^{1/m} \quad (5.12)$$

The breaking strength varies with specimen size as $V^{-1/m}$. This has also been found empirically by others [5.4].

b) If $V_1 = V_2$ (i.e. the same volume in each case is subjected to tensile stress)

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{K_2}{K_1} \right)^{1/m} \quad (5.13)$$

The breaking strength varies with different stress gradients as $K^{-1/m}$.

All different test configurations can be compared with a pure tensile specimen where $K_T = 1$, therefore

$$\frac{\sigma_1}{\sigma_T} = \left(\frac{K_T}{K_1} \right)^{1/m}$$

$$\text{and } \frac{\sigma_1}{\sigma_T} = K_1^{-1/m} \quad (5.14)$$

Some values of K_1 are shown in Appendix C2. The values for the ratios σ_1/σ_T for some values of m (Weibull modulus) are given in Table 5.1. The values in the table were obtained considering the total volume of the specimen (not just the volume subject to tensile stress).

5.3 The Significance of Weibull's modulus m for WC-Co

The theory presented in the previous paragraph and in Appendix C3 correlated the Weibull's modulus with the crack distribution in the material under consideration.

Table 5.1

The ratio of the mean extreme stress at failure in various cases to the mean stress at failure in pure tension: $\frac{\sigma_l}{\sigma_T}$

m	2	4	8
4-point bending	2.45	1.78	1.44
3-point bending	4.24	2.66	1.89
2:1 hollow cylinder	1.69	1.58	1.41

For WC-Co the modulus m is readily obtained and its value is quite reproducible. This can be seen in Figure 5.1(a), taken from [5.5] for a WC-Co specimen prepared by sintering having 22 vol% tested in three-point bending, Figure 5.1(b) [2.12] for WC-Co 16% employing the hydrostatic tensile ring test and in Figure 5.1(c) [2.12] for WC-Co 16% using the Brazilian disc test.

The value of m varies with the number of tests (for WC-Co 6%, $m = 22$ for 8 tests and $m = 10$ for 80 tests).

Generally m increases with the volume fraction of Co and for a given volume fraction, it is largest for the material with the smallest grain size. This has to be expected because WC with a lower Co % is more brittle and therefore there is more scatter in the fracture test results. Alternatively, a higher content of Co makes the material more ductile, relieves the local stress concentrations, and the test results are more consistent.

The value of m is therefore an indication of the dispersion of the defects leading to fracture and one can look for correlations between m and the microstructural parameters which control initiation or at least propagation of fracture. The increase of m with the contiguity of the carbide phase (Figure 5.2 [5.7]) shows on the one hand the brittleness of the grain boundaries and on the other hand the ease with which a microcrack, once initiated, can propagate along the grains to achieve a critical size.

Another parameter which has an influence on the strength of WC-Co is the porosity. There is no doubt that the presence of pores decreases the mechanical strength of cemented carbide. It is to be

(a) Weibull plot for WC-Co22 vol.% ($\bar{D}_{wc} = 1.1 \mu\text{m}$) alloy.

(b) Weibull plot for hydrostatic ring test data. (c) Weibull plot for disk test data.

Figure 5.1

expected that the reduction in strength could be affected both by the size and number of pores and also by the presence of slag-like impurities.

Therefore, it is possible to treat the homogeneous material in a piece of carbide according to Weibull and the stress field introduced in the Weibull statistics will vary according to the presence of pores in the structure. This has led to the conclusion that the increase in the probability of failure due to the presence of pores compared with the probability of failure in a homogeneous structure free from pores is dominantly a function only of the volume fraction of pores and is substantially independent of the extent, size, or number of the individual pores [5,8]. Calculating the stresses around a pore, the risk of rupture for one pore, and the interaction between pores, it is possible to determine a relationship between the average mechanical strength $\bar{\sigma}_{r,B}$ of a sample with pores, (the risk of rupture being B) and the average mechanical strength of a sample without pores $\bar{\sigma}_{r,1}$ (the risk of rupture being 1). Then we obtain

$$\frac{\bar{\sigma}_{r,B}}{\bar{\sigma}_{r,1}} = n^{-p} f(p, m), \quad (5.15)$$

where p = volume fraction of pores,

n^{-p} = factor that takes into account the long range interaction between pores,

m = Weibull's modulus,

$f(p, m)$ = a function of the risk of rupture and pore size and p .

Variation of Weibull coefficient, m , with contiguity of carbide phase, C_{wc} , for various WC-Co series.

Figure 5.2

Theoretical percentage of full mechanical strength for samples with different amounts and sizes of pores.

Figure 5.3

In Figure 5.3 the ratio given by equation (5.15) is shown as a function of p for different sizes of pores, for $m = 9$.

5.4 Statistical Fracture Approach to Polyaxial State of Stress

The geometry and loading conditions of the problem under investigation, a right circular tungsten carbide cylinder which approximates to the pressure chamber used in very high pressure equipment, give rise to a complex state of stress throughout the wall of the cylinder.

The Weibull statistical approach illustrated in the previous sections does not generally specify the nature of the defects causing failure, nor does it provide a unique and rigorous method of deducing the statistics of fracture under a general stress state unless it is simple tension.

If a statistical theory of failure of the polyaxial state of stress is developed it will be for a constant stress state. The finite-element method of structural analysis is an ideal mechanism for implementing such a statistical theory since each element can be made small enough so that the stress state is nearly constant throughout. Thus the probability of failure can be found for each element and applied toward the calculation of a composite probability of failure which, for a given material, depends only on stress state and volume, both of which are available at the element level.

The first and simplest approach to the statistical failure under polyaxial stress can be expressed as [5.9] noting that in this case σ_u of equation 5.6 is non-vanishing.

$$R = \exp \left[- \int_V \left[\frac{\bar{\sigma} - \sigma_u}{\sigma_0} \right]^m dv \right], \quad (5.16)$$

where R is defined as the reliability of the component of volume V subjected to a given stress condition $\bar{\sigma} = \bar{\sigma}(x, y, z)$.

The implementation of Equation (5.16) becomes difficult due to the interpretation of the stress, $\bar{\sigma}$, which should adequately represent an equivalent stress condition compatible with the requirements necessary to initiate some fracture mechanism.

Generally the $\bar{\sigma}$ can be interpreted in three principal directions:

- i) Stress considerations should be of primary concern when the material under study has pores which are roughly spherical in form. In fact, in this case the flaws do not have a particular sensitivity to any direction as is the case for a sharp crack.
- ii) On the other hand, sharp cracks have a preferential direction of propagation; therefore a parameter which takes into consideration the directional sensitivity should be considered in a theory of fracture.
- iii) Clearly the equivalent $\bar{\sigma}$, which considers both stress and a geometrical crack parameter, will give a better approximation, particularly in very strong polyaxial states of stress.

If the appropriate fracture criterion has been selected and converted into the equivalent stress $\bar{\sigma}$, the probability of survival for each volume element of the finite element sub-structure can be

calculated. This approach can be extended to all elements and the total (or composite) reliability of the entire component can be evaluated by [1.3]

$$R = \exp \left[- \sum_{i=1}^N \left(\frac{\bar{\sigma} - \sigma_u}{\sigma_o} \right)^m V_i \right], \quad (5.17)$$

where V_i refers to the volume of the i th element, $i = 1, N$.

This approach therefore enables the reliability of structural components of virtually unlimited geometric complexity to be evaluated for any imposed boundary conditions.

As previously discussed, the equivalent $\bar{\sigma}$ as discussed in more detail in Appendix C4 can be interpreted in various ways. In [1.3] the design of a turbine engine for automotive application is discussed. For this particular case the rotor material is ceramic (Si_3N_4 or SiC) and therefore brittle. The designer assessed the strength of this material not in terms of a conventional margin of safety, but rather he associated the usable strength level of the ceramic structure with an acceptable probability of failure. A treatment similar to that outlined previously was followed and the $\bar{\sigma}$ was interpreted as the stress in those directions in which tensile stress occurs. By this analysis it was possible to compute the risk of rupture intensity contour maps for the disk of the turbine rotor.

5.5 Probabilistic Approach to the Optimization Design of High Pressure Cylinders

The computer interactive package CLASS was used to evaluate

the probability of survival of a right circular cylinder for high pressure equipment. One of the modules, PROBXEQ, allows the calculation of the reliability of the component with data inputs obtained by a finite element stress analysis performed by the other modular programs of CLASS.

The probability of survival is calculated according to equation (5.17). In most cases the circumferential stress (σ_θ) is of sufficient tensile magnitude to obscure the effect of any smaller tensile stress in the r-z plane. Therefore the representative stress $\bar{\sigma}$ is given by:

$$\begin{aligned} 1) \quad \bar{\sigma} &= \sigma_\theta && \text{if } \sigma_\theta \geq 0 \text{ (tensile) ,} \\ \bar{\sigma} &= 0 && \text{if } \sigma_\theta \leq 0 \text{ (compressive) .} \end{aligned}$$

Under particular loading conditions and if a certain amount of plastic deformation is considered, the tensile circumferential stress is lowered and other stresses (σ axial, generally) become increasingly tensile. In these cases the effective stress is expressed by:

$$\begin{aligned} 2) \quad \bar{\sigma} &= \max(\sigma_1, \sigma_\theta, \sigma_2) && \text{if } \sigma_1 \text{ or } \sigma_\theta \text{ or } \sigma_2 > 0, \\ \bar{\sigma} &= 0 && \text{if } \sigma_1 < \sigma_\theta < \sigma_2 \leq 0, \end{aligned}$$

where $\sigma_1, \sigma_\theta, \sigma_2$ are the three principal stresses.

The program calculates the $\bar{\sigma}$, verifies that $\bar{\sigma} > \sigma_u$ and then calculates the probability of survival for the element. If this process is extended to cover all elements, we can determine a reliability for the whole component. The structure of the program permits and encourages experiments by varying the values of the three statistical parameters and the effective stress.

Pressure chambers of three different inside diameters ($d_i = 1", 2", 4"$), of the same height of 1.5" and of constant diameter ratio $K = d_o/d_i = 3$, have been analysed for different conditions of loading.

Elastic and elastic-plastic finite element solutions were obtained using the finite element grid network shown in Figure 3.2 for the boundary conditions and material parameters presented in Table 5.2.

Different grid geometries could lead to significantly different numerical results attributable to the high stress gradients within the walls of the cylinders, but the trend does not change [5.9] therefore the selected grid was deemed to be sufficiently accurate to illustrate the technique.

The results of these analyses are shown in Table 5.2 and Figures 5.4 - 5.7 with reference to change in radial support, axial support, material parameters, loading sequence, bulk support and size effect.

5.5.1 The effect of radial support

In Figure 5.4 is illustrated the effect of radial support with an increase in cylinder size. The beneficial effect should be noticed of increasing radial support for every cylinder size [1.6]. From Table 5.2, the reliability for cylinders with no external support is zero (solutions B400); if the radial support is high (solutions B4Y0) the reliability is near 1.

Table 5.2

Reliability of thick-walled cylinders under different stress conditions

Loading Conditions (See Appendix B4)	$d_i = 1"$		$d_i = 2"$		$d_i = 4"$	
	$m = 5$	$m = 10$	$m = 5$	$m = 10$	$m = 5$	$m = 10$
B4Q0	0.0	0.0	0.0	0.0	0.0	0.0
B4X0	0.8054	0.8566	0.5854	.9063	.2288	.8192
B4XA	.8173	.8751	.5964	.9123	.3067	.8771
B4XB	.8032	.8556	.6189	.9233	.3777	.9131
B4Y0	.9917	.9996	.9886	.9999	.8487	.9969
B4YA	.9904	.9995	.9889	.9999	.8835	.9982
B4YB	.9889	.9993	.9893	.9999	.9276	.9993
C2Y0	.9999	1.0000	.9999	1.0000	.9999	1.000
C2YA	1.0000	1.0000	1.0000	1.0000	.9999	1.000
PC2Y0	.9999 (.9989)	.9999	Plastic Elements 1, 2, 17		Yield stress = 680 ksi	
PC2Y0	.9998 (.9987)	.9999	Plastic Elements 1, 2, 3, 17		Yield stress = 580 ksi	
PC2Y0	.9997 (.9986)	.9999	Plastic Elements 1, 2, 3, 17, 19		Yield stress = 550 ksi	

Note: Strain hardening exponent $n = 0.20$

Proportional limit $H = 250$ ksi

$\sigma_u = 0$

$\sigma_0 = 222$ ksi

$d_i =$ internal diameter

$m =$ Weibull's modulus

The effect of radial support, material parameters, bulk support and component size.

Figure 5.4

Figure 5.5

Figure 5.6

Figure 5.7

5.5.2 The effect of bulk support and the size effect

In Figure 5.4 except for the curve with $m = 5$ and $p_0 = 100$ ksi (solution B4X0), all curves show a tendency to have a maximum for the cylinder with $d_i = 2"$; this can be attributed to the massive or bulk support concept combined with the effect of cylinder size. The "size effect" refers to the decrease in the reliability as the component size increases due to the fact that the number of flaws is directly proportional to the volume of the component under investigation.

The application of a load, P_i , across a fraction of the height of the cylinder, enables a considerable percentage of the total cylinder to be in a relatively unstressed state as compared with the material directly in support of the applied loads. This large volume of relatively unstressed material provides useful bulk support in that the applied loads can be distributed across and supported by this material which extends to the outer radius of the cylinder when support is provided over the entire height of the cylinder.

The cylinder with internal diameter $d_i = 2"$ represents the balance between bulk support and size effect for the current load application.

5.5.3 The effect of increased variability in the material parameters

In Figure 5.4 the reliability of the three cylinders is presented for both values of the material property $m = 5$ and $m = 10$.

The greater the value of m , the smaller the variation in the critical stress level associated with the flaws. A comparison of the curves in Figure 5.4 as well as in Figure 5.5 - 5.7 illustrates the decreased reliability of the cylindrical component as the value of m is reduced.

This effect is more pronounced for the large cylinders and can be attributed to the size effect being more important than the bulk support effect. These results indicate the degree to which any variation in the material properties can influence the reliability of a component and therefore this emphasizes the necessity for a statistical rather than a deterministic interpretation of the material strength to be employed. [5.1], [5.7].

5.5.4 The effect of axial support

Figures 5.5, 5.6, 5.7 show the effect of increasing axial support on the reliability of the component. For the larger components the effect of axial load is towards an increased reliability, particularly for the diameter $d_1 = 4"$. For the small diameter $d_1 = 1"$ the axial load has little or no effect on the reliability. This can be attributed again to the bulk effect because, for larger components, the axial support is distributed over a larger surface and therefore contributes more to support the internal load.

5.5.5 Loading sequence

In Table 5.2 are reported the results for the loading conditions C2Y0 and C2YA. For all the cylinder sizes the reliability is practically

equal to 1. Even if the internal pressure is increased, the reduced height onto which the pressure is applied creates an increased volume of unstressed material and therefore the bulk support becomes dominant over all the other effects.

This type of result is particularly important in the loading sequence of a high pressure cylinder in which the internal pressure is continuously increased as the cylinder contents are compressed.

From the results it seems clear that the design of a larger size pressure chamber should not be based only on the maximum pressure to be attained, but also on the intermediate loads considering the effects of bulk support and size.

5.5.6 The effect of plastic deformation

The Weibull approach should strictly be applied only on an elastically stressed brittle body. However, three elastic-plastic finite element solutions were obtained for the condition of higher stress (C2Y0) to determine the influence of a plastically deformed volume on the component reliability. The results are given in Table 5.2 for the cylinder with internal diameter $d_i = 1"$.

As the yield strength was lowered and therefore as the number of finite elements becoming plastic was increased, the reliability of the component decreased slightly. The reliability appears lower (values in brackets) if it is evaluated not on the hoop stress alone, but on the maximum tensile stress as well. This result is of particular importance when very high pressures have to be reached and therefore large portions of the cylinder wall become plastic under the influence

of greater hydrostatic compression (see Chapter 4).

6.1 Conclusions

A résumé of the important developments contained in this thesis is provided here under, no importance being attached to their order:

- (i) Outlines for the design technology of brittle materials have been given, showing the importance of coordination among tests for mechanical properties, stress analysis techniques, failure criteria and a statistical approach to design.
- (ii) Due to the complexity of the problem involving the coordination of loading conditions, material properties geometric configurations and boundary conditions, the design of tungsten carbide circular cylinders for high pressure equipment was used to illustrate the principles of brittle material design.
- (iii) A review of tests for brittle materials has been provided; experimental data from the literature were systematically collected and employed to form a failure criterion locus for tungsten carbide cobalt material.
- (iv) A new computer package for stress and strain analysis of structural components was written. The structure of the programs is such as to allow an interactive operation by the user, thus allowing certain checks and options to be exercised during a lengthy computation. The task of creating a mesh of

triangular finite elements is therefore much simplified.

The boundary conditions (displacements and forces), the material variables and other controlling parameters can be easily selected to obtain a finite element solution.

The results can be easily examined and displayed with the graphical capabilities of the interactive package.

- (v) The interactive program was used to analyse the stress distribution and the deformations of the circular cylinders under different loading conditions, loading sequences and geometric configurations.

From an inspection of the solutions it can be concluded that:

- (a) the tensile hoop stresses may be decreased by the application of radial support (in the form of shrink-fitted steel rings).
- (b) the axial stresses developed as a consequence of excessive barrelling of the cylinder may be reduced by axial external support.
- (c) it was demonstrated that it is also important to consider the effect of the strains to avoid excessive barrelling of the cylinders with consequent clamping on the piston and breakage of this component.
- (d) the hydrostatic stress was found to play an important role in the failure history of highly stressed WC-Co components.

- (vi) A new failure criterion for WC-CO was developed. The criterion takes into consideration the brittle behaviour of this material together with the plastic deformations exhibited at high hydrostatic stress. Using the interactive package, in conjunction with the failure criterion, to study the high pressure chambers under different loading conditions it was possible to understand more clearly the failure history of these components.
- (vii) A brittleness ratio was used as an index of the brittle state of stress. Using the interactive package, maps of the brittleness ratio and hydrostatic stress can be plotted. These maps give the location of starting failure which is not always at the inside wall of the cylinder but which depending on the loading conditions, can be at the outside wall.
- (viii) An overall optimization design of the pressure chamber was performed using the probability of survival as the criterion to predict the failure behaviour of cylinders of different sizes and under varying loading conditions. This probability of survival is based on the finite element solution of stresses and strains. It was determined that a balance must be reached between the bulk support effect, size effect, radial and axial support in order to avoid an early failure of the cylinder with the larger inner radius.

6.2 Recommendations for future work

- (1) The present study has shown the validity of the failure criterion which is a function of the hydrostatic stress. Using data

available from published literature, a failure locus was developed for tungsten carbide-cobalt material; other experiments would improve the validity of the failure criterion in the areas not covered by the present tests.

- (2) The use of the computer program CLASS demonstrated the ease and value of an interactive approach to stress analysis by finite element methods. The same program could be expanded to include other types of elements and to permit a finite element analysis of component configurations other than axisymmetric ones.
- (3) Under definite loading conditions, the present study has shown that the tungsten carbide-steel interface should receive a particularly attentive study. The finite element program can be extended to include solutions for non-symmetric loading in order to study this problem.
- (4) The brittleness ratio incorporated in CLASS has proved to be an important indicator of the area of fracture initiation, thus aiding the localization of attention to fracture growth. The author is currently investigating the possibility of developing a model to incorporate the brittleness ratio with basic fracture mechanics concepts and Weibull probability considerations to predict the life of high pressure components.
- (5) The force polygon was not used in the present study as a design criterion but it did prove to be an important tool for the study of stresses. A further convenience offered by this technique is the easy automation of force-polygon plotting, using a simple computer programme that would accommodate different cylinder sizes and loading configurations.

Appendix A1

TENSILE RING TEST EQUIPMENT.

The testing technique developed in [2.1] is basically a modified hoop-stress loading which takes advantage of two characteristic properties of hydrostatic pressure: the fact that the direction of pressure is always normal to the confining surface and the fact that the pressure is perfectly uniform over the entire area of contact. The specimen configuration employed in the test is a short, thin-walled cylinder. Application of hydrostatic pressure through an elastic envelope to the inside surface of the cylinder, causes it to expand and creates a tangential tensile stress in the wall (see Figure A1.1).

The same principle was also used in [2.2]. Major modifications of the design include:

- (i) the use of a rubber O-ring in place of the rubber sleeve to permit a wider range of diameters, much higher pressures and large elongations,
- (ii) the use of a rigid, adjustable frame to eliminate the need for spacers and to eliminate the requirement for a fixed specimen height, and
- (iii) the incorporation of an extensometer to measure changes in circumference.

Figure A1.2 shows the test fixture. A short ring-shaped specimen, together with a concentric O-ring on the inside, is gradually pressurized with hydraulic fluid while being held between two rigidly

(a)
Tensile testing specimen holder; (a) exploded view,
(b) assembled unit.

Figure A1.1

Ring tensile tester without extensometer.

Figure A1.2

fixed, parallel surfaces. The pressure chamber is sealed by the O-ring which also transmits the hydraulic pressure uniformly over the inner surface of the specimen. Before the pressure is applied, two adjusting screws are hand tightened to obtain zero initial clearance between the parallel surfaces and the ring.

These screws force two wedges together, raise the lower surface plate, and clamp the specimen and O-ring against the upper surface.

Pressures up to 25,000 psi can be reached with a precise and rigid design of the steel frame to prevent extrusion of the O-ring. Both the previous methods make use of short tubular specimens (ring specimens) which are subjected to hoop stress loading. The apparatus described hereafter is a modification of the two previous ones and is intended for use where a pressure vessel capable of high pressures, such as isostatic pressing equipment, is available. High pressure equipment is used for a variety of purposes in laboratories e.g. power pressing, extrusion, etc. In [2.3] a jig was made for use in an isostatic press with a capacity of 207 MNm^{-2} (30000 lb in^{-2}); the design is shown in Figure A1.3. The whole is machined from mild steel, the bottom face of the cover plate being ground and face XX being polished on the lathe. Grooves A and B are expansion chambers. The position of the ring specimen is shown in Figure A1.4. Standard elastomer (nitrate) O rings are placed inside the ring specimen to provide sealing between the ring and the steel holder. An alteration to the apparatus so that rings of different heights can be tested is illustrated in Figure (A1.5). This requires construction from rather thicker plates.

Plan view of ring tester base
(recessed areas shown cross-hatched)

Sectional view of ring tester
components.

Figure A1.3

Detail of centre section of assembled ring tester.

Figure A1.4

Proposed modification to ring tester to allow adjustment for samples of differing height.

• Figure A1.5

The testing technique is as follows. The specimen and the O rings are put into position, the cover plat is bolted on and the whole is placed in the isostatic press. The pressure is slowly increased until the ring fractures. At this point one or more O rings also rupture and the pressurizing fluid flows into the expansion chambers A and B, the volume of which is sufficient to cause an easily discernible drop in pressure. Thus the pressure at which fracture of the ring occurs can be determined.

Appendix A2

BRAZILIAN DISC TEST STRESS DISTRIBUTIONS

Figure A2.1 illustrates the different mode of loading:

A is called point of line loading, B is a flat loading and C is a distributed loading.

For point loading an analytical solution is available [2.11].

The equations in rectangular coordinates are (see Figure A2.2):

$$\begin{aligned}\sigma_x &= -\frac{2P}{\pi t} \left[\frac{(R-y)x^2}{r_1^4} + \frac{(R+y)x^2}{r_2^4} - \frac{1}{d} \right], \quad (a) \\ \sigma_y &= -\frac{2P}{\pi t} \left[\frac{(R-y)^3}{r_1^4} + \frac{(R+y)^3}{r_2^4} - \frac{1}{d} \right], \quad (b) \\ \tau_{xy} &= \frac{2P}{\pi t} \left[\frac{(R-y)^2 x}{r_1^4} - \frac{(R+y)^2 x}{r_2^4} \right], \quad (c)\end{aligned}\tag{A2.1}$$

in which

$$\begin{aligned}r_1^2 &= x^2 + (R-y)^2, \quad (a) \\ r_2^2 &= x^2 + (R+y)^2. \quad (b)\end{aligned}\tag{A2.2}$$

On the X axis (the diameter perpendicular to the loads) the equations simplify as:

$$\begin{aligned}\sigma_x &= -\frac{2P}{\pi t d} \left[\frac{16d^2 x^2}{(d^2 + 4x^2)^2} - 1 \right] = \\ &= \frac{2P}{\pi t d} \left[\frac{d^2 - 4x^2}{d^2 + 4x^2} \right] \quad (a) \\ \sigma_y &= -\frac{2P}{\pi t d} \left[\frac{4d^4}{(d^2 + 4x^2)^2} - 1 \right] \quad (b) \\ \tau_{xy} &= 0. \quad (c)\end{aligned}\tag{A2.3}$$

- A Idealized diametral compression tests under point loads.
 B Specimen with small flats used for tests.
 C Geometry for which an analytical solution is available

Figure A2.1

On the Y axis (the diameter on which the loads act) the equations become:

$$\sigma_x = \frac{2P}{\pi t d} \quad (a)$$

$$\sigma_y = -\frac{2P}{\pi t} \left[\frac{2}{d-2y} + \frac{2}{d+2y} - \frac{1}{d} \right] \quad (b) \quad (A2.4)$$

$$\tau_{xy} = 0 \quad (c)$$

Figure A2.2, A2.3 shows the stress distribution. The stresses on the two diameters are therefore principal stresses also, because $\tau_{xy} = 0$.

The maximum stresses at the center of the disc are given by:

$$(\sigma_x)_{x=0, y=0} = \frac{2P}{\pi t d} \quad (a) \quad (A2.5)$$

$$(\sigma_y)_{x=0, y=0} = -\frac{6P}{\pi t d} \quad (b)$$

If the loading configuration of Figure A2.1(b) is used, then the stress distribution on such flats is unknown. A Hertz circular distribution was assumed in [2.12] and the stresses distributions are shown in Figure A2.3. The mean stress $\bar{p}$ is $\frac{4}{\pi}$ times the maximum stress for such a distribution.

The mean stress on a flat will therefore be

$$\bar{p} = -\frac{p}{0.2dt} \quad (A2.6)$$

and hence the maximum stress at $x = 0$ will be

$$p_0 = -\frac{4}{\pi} \frac{p}{0.2dt} = -\frac{20}{\pi} \frac{p}{dt} = 10\sigma_x \quad (A2.7)$$

Curves showing the stress distribution along the horizontal section of symmetry for a disc under diametral compression.

Curves showing the stress distribution along the vertical section of symmetry.

Figure A2.2

Stress distribution for transversely loaded disc.

Figure A2.3

This stress, which is the limiting compressive stress σ_y at the surface when a flat is used, is seen to be exactly 10 times the tensile stress (σ_x) given by the equation:

$$\sigma_x = \frac{2P}{\pi dt} \quad (A2.8)$$

Any real loading fixture will distribute the loads over an area. A general analytical solution is available [2.13] which, in polar coordinates, reads (see notation in Figure A2.1(c))

$$\begin{aligned} \sigma_r &= -\frac{2P}{\pi} \left\{ \alpha + \sum_{n=1}^{\infty} \left[1 - \left(1 - \frac{1}{n} \right) \left(\frac{r}{R} \right)^2 \right] \left(\frac{r}{R} \right)^{2n-2} \sin 2n\alpha \cos 2n\theta \right\} \\ \sigma_\theta &= -\frac{2P}{\pi} \left\{ \alpha - \sum_{n=1}^{\infty} \left[1 - \left(1 + \frac{1}{n} \right) \left(\frac{r}{R} \right)^2 \right] \left(\frac{r}{R} \right)^{2n-2} \sin 2n\alpha \cos 2n\theta \right\} \\ \tau_{r\theta} &= \frac{2P}{\pi} \left\{ \sum_{n=1}^{\infty} \left[1 - \left(\frac{r}{R} \right)^2 \right] \left(\frac{r}{R} \right)^{2n-2} \sin 2n\alpha \sin 2n\theta \right\} \end{aligned} \quad (A2.9)$$

The stresses along the principal diameters are given by the following equations:

(a) Stresses along the vertical diameter:

$$\begin{aligned} \sigma_{\theta y} &= \frac{2P}{\pi} \left[\frac{(1-r^2/R^2) \sin 2\alpha}{(1-2r^2/R^2 \cos 2\alpha + r^4/R^4)} - \tan^{-1} \frac{(1+r^2/R^2)}{(1-r^2/R^2)} \tan \alpha \right] \\ \sigma_{ry} &= -\frac{2P}{\pi} \left[\frac{(1-r^2/R^2) \sin 2\alpha}{(1-2r^2/R^2 \cos 2\alpha + r^4/R^4)} + \tan^{-1} \frac{(1+r^2/R^2)}{(1-r^2/R^2)} \tan \alpha \right] \\ \tau_{r\theta} &= 0. \end{aligned} \quad (A2.10)$$

(b) Stresses along the horizontal diameter:

$$\begin{aligned}\sigma_{\theta x} &= -\frac{2P}{\pi} \left[\frac{(1-r^2/R^2)\sin 2\alpha}{(1+2r^2/R^2\cos 2\alpha + r^4/R^4)} + \tan^{-1} \frac{(1-r^2/R^2)}{(1+r^2/R^2)} \tan \alpha \right] \\ \sigma_{rx} &= +\frac{2P}{\pi} \left[\frac{(1-r^2/R^2)\sin 2\alpha}{(1+2r^2/R^2\cos 2\alpha + r^4/R^4)} - \tan^{-1} \frac{(1-r^2/R^2)}{(1+r^2/R^2)} \tan \alpha \right] \\ \tau_{r\theta} &= 0.\end{aligned}\quad (A2.11)$$

The stresses along these principal planes corresponding to the diameters through the OX and OY axes are plotted in Figure A2.4.

The curves converge to relatively high compressive stress values at the rim, equal to the applied pressure which is in fact a boundary condition. Between r/R values of zero (at centre 0) and 0.33 the curve $\sigma_{\theta y}$ is essentially constant, while $\sigma_{ry}, \sigma_{\theta x}, \sigma_{rx}$ all suffer little change in the region of the centre of the specimen.

The stresses at the centre, (0) are: ($r/R = 0$, $P = pat$, $\alpha = a/d$, $p\alpha = P/dt$)

$$\begin{aligned}\sigma_{\theta y} = \sigma_{rx} &= \frac{2p\alpha}{\pi} = \frac{2P}{\pi dt} \\ \sigma_{\theta y} = \sigma_{ry} &= -\frac{6p\alpha}{\pi} = -\frac{6P}{\pi dt}\end{aligned}\quad (A2.12)$$

An interesting point to note, is that these expressions show that point loads or short distributed loads applied to circular elements develop identical stresses at the centre of the specimen.

Theoretical stress distribution along the vertical (OY) and horizontal (OX) diameters. $.2p/\pi = 1$

Figure A2.4

Appendix A3

MECHANICAL BEHAVIOUR OF CEMENTED CARBIDES

A3.1 Introduction

To implement an elastic-plastic analysis requires a clear and detailed understanding of the behaviour of tungsten carbide.

In most respects, cemented carbide has a behaviour which is qualitatively similar to that of metals: its plastic behaviour conforms to the laws of classical plasticity; its ductility decreases with increasing hardness and increasing hydrostatic tension component.

Two reviews detailing the parameters which influence the mechanical properties of tungsten carbide - cobalt alloys were published during the period 1968-1970, by Kreimer [A3.1] and Exner and Gurland [A3.2]. In addition, Exner and Gurland [A3.3] published a guide to the literature on tungsten carbide spanning the years 1930 - 1970. More recently, a report [A3.4] was published with a recent literature survey, with particular attention devoted to structural application of tungsten carbide. In the last two years, few papers have been published on the influence of parameters on the strength of WC-Co.

A3.2 Manufacture of WC-Co

It is important to examine in general terms a typical process sequence of manufacture as given in Figure A3.1. The process appears to be reasonably straight forward but numerous material and processing variables, each having a distribution about some mean value, are interlocked to form a "homogeneous and reproducible" material. The particular

distribution associated with each variable leads to the production of sintered carbide component having an individual set of properties.

This confirms [A3.4].

- i) the extent to which material variables influence the mechanical properties of the carbide, and
- ii) the need to consider statistical methods in the mechanical property evaluation.

But (ii) was taken into consideration in Chapter 5 and the following paragraphs will attempt to explain point (i) above, with the cobalt content, grain size, porosity, cobalt layers, and plastic deformation being the most relevant material variables.

A3.3 Cobalt Content

The cobalt content is the most important variable in producing cemented tungsten carbide of different mechanical properties. Cemented carbides for technical applications contain from 5 to 25 wt% (8.5 - 32 vol. %) cobalt. The present state of knowledge can be summarized as in Figure A3.2(a) and A3.2(b). The hardness decreases in the previously defined range with increasing cobalt content, and the compressive strength reaches a maximum at 5% and then drops sharply when cobalt content is raised [A3.2]. The transverse rupture strength in contrast improves with increasing cobalt content up to a maximum at 20 ± 5 wt % Co. The maximum strength composition depends strongly on other variables, such as the carbide grain size. Similar results are reported in a very recent publication [5.6] and the data are given in Table A3.1.

The relatively high tensile or transverse stresses found in alloy with increasing cobalt content can be explained [A3.5]

MANUFACTURING VARIABLES OF CEMENTED CARBIDES

Figure A3.1

INFLUENCE OF COBALT CONTENT AND GRAIN SIZE ON

- 1) HARDNESS
- 2) TRANSVERSE RUPTURE STRENGTH
- 3) MODULUS OF ELASTICITY

Figure A3.2(q)

INFLUENCE OF COBALT CONTENT AND GRAIN SIZE ON
1) COMPRESSIVE STRENGTH, AND
2) LIMIT OF PROPORTIONALITY IN COMPRESSION

Figure A3.2(b)

Table A3.1
Characteristics of WC-Co Materials

$\bar{D}_{WC}$ μm	$V_V(Co)$ %	$\sigma_{rf}^{bend.}$ N/mm^2	$\sigma_{rc}^{comp.}$ N/mm^2	$\epsilon_{rf}^{bend.}$ %	$\epsilon_{rc}^{comp.}$ %
2.2	5	1200	4250	0.19	0.91
	10	2100	4200	0.42	1.18
	15	2700	3900	0.60	1.28
	22	2800	3600	0.89	1.81
	30	2650	3450	1.05	
	37	2500	3200	1.07	
1.1	5	1060		0.16	
	10	1690	4400	0.27	0.84
	15	2280	4200	0.42	0.99
	22	2530	4000	0.60	1.34
	30	2750	3900	0.86	1.67
	37	2670	3600	1.03	
1.0	5	1450			
	10	1880			
	15	2310			
	22	2700			
	30	2930			
	37	2900			
0.7	5	1940		0.30	
	10	2250		0.36	
	15	2500		0.44	
	22	2850	4600	0.59	1.56
	30	2910	4450	0.83	1.89
	37	3090	4200	1.25	2.23

$\bar{D}_{WC}$ = mean diameter of tungsten carbide crystals

$V_V(Co)$ = cobalt volume fraction

$\sigma_{rf}^{bend.}$ = rupture stress in bending

$\sigma_{rc}^{comp.}$ = rupture stress in compression

$\epsilon_{rf}^{bend.}$ = strain to fracture in bending

$\epsilon_{rc}^{comp.}$ = strain to fracture in compression

because the conditions obtained in it approach those of isostatic extension with a very limited capacity for plastic deformation, or in other words the WC particle acts as constraints to the flow of the cobalt phase giving therefore raise to high tensile stresses in the cobalt phase.

The character of rupture in uniaxial compression is radically different, since rupture takes place only after appreciable, sometimes even very substantial, deformation, which markedly alters the properties of the alloy. The deformation of a hard alloy in compression (which is much greater than its tensile deformation) causes many carbide grain contacts to rupture, lowering the strenght of the skeleton and, consequently, of the cobalt phase. With rise in cobalt content, the two alloy skeletons, cobalt and carbide, lose strength more rapidly in compression than in tension or bending. This is responsible for the shift of the strength maximum in the compression strength curves toward very low cobalt contents. At the same time, the process of damage build up in the carbide phase become interlinked in a complex manner with the strain hardening of both phases. This, combined with the high compressive strength characterstic of tungsten carbide, imparts to the alloy a high resistance to compressive loads.

A3.4 Porosity

The properties of cemented carbides vary with density as do those of other sintered materials. In hard metal technology, high densities (up to 99.5% and above) [A3.2] are achieved by optimal processing conditions, but a uniformly distributed residual porosity is usually present and is commonly accepted as not being harmful but WC-Co alloys have been produced with minimal porosity and it has been

stated that much higher and more uniform transverse rupture strength values are obtained.

Quantitative data were used to develop statistical concepts [A3.6] which were presented in Chapter 5 . The increase in the probability of failure due to the presence of pores compared with the probability of failure in a homogeneous structure free from pores, is dominantly a function only of the volume fraction of pores and is substantially independent of the extent, size, or number of individual pores.

A3.5 Carbide Grain Size

Studies of the influence of carbide particle size on the mechanical properties of tungsten carbide-cobalt alloys have resulted in the following observations. (See Figures A3.2). Hardness and compressive strength decreases with increasing particle size. The transverse rupture strength has a pronounced maximum which lies at a particle size of 3 μm for a 12 wt % Co alloy [A3.2]. These results were also obtained by [A3.7] and by [5.6] as shown in Table A3.1. The influence of the mean diameter of the tungsten carbide crystals is much more marked in the values of rupture stress determined in compression than in those obtained in bending.

With increase in carbide size (or binder content), the number of contacts between carbide particle decreases [A3.8] and consequently, the average stress transferred to respective carbide particles through the skeleton structure increases. Under such circumstances, transgranular fracture may occur in some stress concentrated regions in the carbide phase before the initiation of shear rupture along some

intergranular regions. One can see that this may result in lowering of the compressive strength of the alloy.

A3.6 Intergranular Cobalt Layers or Cobalt Mean Free Paths

A plot of rupture stress vs. volume fraction of cobalt does not show specifically the part played by carbide grain size. Thus several authors have already used the mean free path in the Co phase, L_{Co} [A3.1, A3.2, 5.6, A3.9]. In particular, Gurland and Bardzil [A3.9] were the first to establish the technically very significant fact that, for a given cobalt content, the toughness exhibits a pronounced maximum for a certain grain size or mean-free-path value (Figure A3.3). In Figure A3.4 [5.6] we have reported a similar graph obtained by different authors that confirm the validity of the theory. As can be seen the graph shows very clearly the maxima, which lie at a value of $1 \mu m$. The actual values of the stress maxima depend on the mean diameters $\bar{D}_{WC}$ of the carbide crystals.

A3.7 Plastic Deformation and Fracture

The behaviour and especially the permanent deformation of WC-Co composites under pressure are important in the study of the problems of the fracture of high-pressure apparatus in tungsten carbide cobalt [A3.10].

Fractographic patterns of WC-Co 6 wt % fractured pistons reveal essentially intergranular fracture for the smaller carbide crystals. For the cobalt phase, in spite of its low volume ratio, small ductile shear rupture dimples are observed. The metallographic study shows therefore on the one hand that fracture is initiated by the fracture

TRAVERSE RUPTURE STRENGTH OF WC-Co COMPOSITIONS
AS A FUNCTION OF MEAN FREE PATH.

Figure A3.3

RUPTURE STRESS IN BENDING σ_{rf} , AS FUNCTION OF MEAN
FREE PATH IN Co PHASE OF DIFFERENT SERIES OF WC-Co.

Figure A3.4

of the largest crystals and on the other hand that the propagation proceeds preferentially in the Co phase.

In the model for fracture of WC-Co proposed by Gurland [A3.11] and Nabarro and Luyckx [A3.12] it was suggested that fracture is initiated by plastic flow of Co. This model can be used employing the microstructural parameters which can be measured directly as shown in [5.6]. The Co deforms plastically causing dislocation to pile-up against the carbide crystals.

A pile-up of dislocations in terms of an energy per unit length, U can be accounted for by a function of $\bar{L}_{Co}^2$, i.e.

$$U = f(\bar{L}_{Co}^2),$$

where $\bar{L}_{Co}$ is the mean free path in cobalt binder phase. This pile-up leads in general to the fracture of the carbide crystals when its energy is equal to the energy required to produce two new surfaces in the carbide crystals, i.e. $2 \times \bar{D}_{WC}$ where γ is the surface energy per unit area, and $\bar{D}_{WC}$ is the mean grain size of the tungsten carbide crystals. This leads to the microstructural parameter $\bar{L}_{Co}^2 / \bar{D}_{WC}$. In [5.6] it was sought to represent rupture stress as a function of the parameter $\bar{L}_{Co}^2 / \bar{D}_{WC}$. This quantity is rather significant because it takes into account at the same time the size of the regions of Co matrix and the mean diameter of the tungsten carbide crystals. In Figure A3.5 it can be seen that the experimental points are grouped closely about a mean curve, irrespective of the mean diameter of the WC crystals.

In the case of compression, high strains and stresses are reached owing to the appearance of slip lines in the carbide crystals. However, the good agreement between the ratio $\sigma_{rc} / \sigma_{rf}$ and $\bar{L}_{Co}^2 / \bar{D}_{WC}$

Rupture stress in bending, σ_{rf} vs. parameter $\sqrt{L_{Co}/D_{wc}}$

Variation of ratio of rupture stress in compression, σ_{rc} and in bending σ_{rf} as function of parameter $\sqrt{L_{Co}/D_{wc}}$

Figure A3.5

Figure A3.6

(Figure A3.6) shows that even here fracture is initiated by plastic flow of the Co. A similar behaviour was predicted by the application of FEM computer program for elastic-plastic analysis of two-dimensional problems in plane strain to plate models, obtained from electron micrographs, which have been divided into a pattern of triangular elements [A3.13]. These results show the importance of compatibility (continuity in deformation) for the analysis of heterogeneous materials. It is shown that plastic deformation is possible in the binder phase despite the presence of a carbide network, due to the elastic deformability of the network. In a previous paper [A3.14] it was shown that the influence of grain boundary cracks (between carbide grains) on the stiffness of the WC-Co material is small.

A3.8 Temperature and Fatigue Effects

Within this study the temperature as a parameter was not taken into consideration but the effects on the behaviour of WC-Co composite are briefly discussed here. The data in the literature are very rare and scattered.

The fracture strength is little affected by temperature up to 450°C for composition ranging from 10 to 31% vol. Co [A3.15, A3.16].

A study of thermal fatigue cracks [A3.17] reached the following conclusions:

1. The number of thermal fatigue cracks increase with increasing cobalt content and increasing WC grain size.

Initiation of these fatigue cracks starts at the surface after a certain incubation time, which has a tendency to increase with increasing cobalt content.

Curve 1 Ultimate compressive strength.

Curve 2 Fatigue strength at 2×10^6 cycles

Figure A3.7

2. The propagation rate of the fatigue cracks decreases with increasing cobalt content and increasing WC grain size. For alloys with a low cobalt content the grain size influence is great, but this influence decreases as the cobalt content is increased.
3. The fatigue cracks propagate preferentially in the WC-WC grain boundaries and the WC-Co phase boundaries, where the propagation rate is high. The rate-controlling factor seems to be the interparticle distance, i.e. the mean free path of the binder phase. In these interlayers the cracks propagate in a ductile manner.

Again, therefore, the metallurgical parameter L_{Co} seems to be the important factor as it is shown in Figure A3.7 [A3.7] where the compressive strength is plotted versus the mean free path. All points fall rather well on one single curve both for the ultimate compressive strength and for the fatigue (2×10^6 cycles) data. The values in Figure A3.7 indicate also that even if the strength values in fatigue are lower than in static compression, no drastic reduction had taken place. The fatigue strength is 75 - 80% of the static value.

Appendix B1

FORCE POLYGON DIAGRAMS

FOR

STRESSED THICK-WALLED CYLINDERS

B1.1 The Upper-Bound Method, the Slip-Line Field and the Force Polygon

The theory of slip-line fields for the plane plastic straining of rigid-perfectly plastic material has long been well established [B1.1] and applied to the study of incipient and steady-state deformation processes, e.g. indentation and extrusion respectively. The development of the sliding block method by Johnson [B1.2] for providing upper bounds to loads for plane strain deformations processes has greatly simplified the treatment of problems in metal working plasticity. This method is described at length in [B1.3]. The analyst has to propose a kinematically admissible velocity field which is a pattern of "tangential velocity discontinuity" (t.v.d.) for the metal forming process in which he is interested: this is generally called the physical plane diagram. Then he has to draw the valid associated hodograph which is a velocity diagram. From these two figures, the rate at which work is done by a punch, ram or set of dies, can easily be arrived at and an upper bound (or overestimate) for externally applied force or torque can be obtained [B1.4].

This method of sliding blocks is excellent for "explaining" flow, but it lacks the power to show force distribution in the zone of plastic deformation. The method presented in [3.4] and in [3.5] derives such information from the physical plane diagram using the idea of the closed force polygon. The approach can always be applied to a plane strain upper bound solution or to a slip-line field. If an upper bound

solution is used, the pressures or forces deduced will be approximated; however, for well approximated s.l.f. the method applies and can give reasonable estimates of local pressure without the use of Hencky's equations.

B1.2 Justifying the Closed Force-Polygon Approach

This paragraph demonstrates that considering closed force polygons - which represent force equilibrium for every block in a pattern of t.v.d.s. for a process - leads to the same result as does the well-known work dissipation rate method which utilizes both t.v.d. pattern and its associated hodograph [3.4].

Consider Figure B1.1(a) and the portion of stream tube of rigid-perfectly plastic material of unit depth and height h , moving horizontally to the left with speed V due to force P . At crossing line $\delta\delta$, the direction of motion is altered with an angle α and the material progress frictionlessly along plane DD ; line $\delta\delta$ is the line of t.v.d. Let the net reaction of DD on the re-directed material be denoted by $(R_1 - R_2)$, where R_1 and R_2 are the total normal forces on either side of the stream tube; no frictional forces are involved. If k is the plane strain yield stress and s the length of $\delta\delta$, the shear force along $\delta\delta$ is ks ; N denotes the normal force on this plane. If there is a steady uniform motion of the strip moving parallel to DD , i.e. it is in force equilibrium, then the closed force polygon can be drawn as in Figure B1.1(b) and hence

$$(R_1 - R_2) \sin(\theta - \alpha) = ks, \quad (B1.1)$$

and thence

$$(R_1 - R_2) \sin \alpha = p = (T_2 - T_1) \tan \alpha, \quad (B1.2)$$

where forces T_1 and T_2 are normal to the stream tube.

Elementary stream-tube to justify the force polygon diagram.

Figure B1.1

Eliminating ($R_1 - R_2$), the punch force P is

$$P = k s_1 \frac{\sin \alpha}{\sin (\theta - \alpha)} \quad (B1.3)$$

If a hodograph is drawn for Figure B1.1(a), as in Figure B1.1(c) then

$$\frac{V^*}{u} = \frac{\sin \alpha}{\sin (\theta - \alpha)} \quad (B1.4)$$

Thus, from eq. (B1.3) and (B1.4)

$$P = k s_1 \frac{V^*}{u} \quad \text{or } P \cdot u = k s_1 V^* \quad (B1.5)$$

This last equation is usually written down directly [B1.3] without any reference to force equilibrium considerations. Above, it has been derived from the condition for force equilibrium for the stream tube of Figure B1.1(a). Therefore, the same result is arrived at whatever a rate of plastic work dissipation is used (i.e. using a hodograph) or a force equilibrium consideration.

An extrusion process is considered in Figure B1.2(a), the corresponding closed force polygon in Figure B1.2(b), and the hodograph in Figure B1.2(c). In this case, these are two lines of t.v.d. but still we obtained the same results. Resolving the four forces for $k s_1$, $k s_2$, N_1 and N_2 , vertically and horizontally gives,

$$\frac{k s_2}{\sin \phi} + k s_1 \sin \theta + N \cos \alpha = R \cos \alpha \quad (B1.6)$$

and

$$N \sin \theta + k s_1 \cos \theta = P \quad (B1.7)$$

Eliminating N , noting that $R = P \cot \alpha$, and simplifying, we have,

$$\frac{k s_1}{\sin \theta} + \frac{k s_2}{\sin \phi} = P (\cot \alpha - \cot \theta). \quad (B1.8)$$

Force polygon for an upper-bound solution in extrusion

Figure BI.2

Using the hodograph,

$$\frac{V_1^*}{u} = \frac{\sin \alpha}{\sin(\theta - \alpha)} \quad \text{and} \quad \frac{V_2^*}{u} = \frac{V_2^*}{V_1^*} \cdot \frac{V_1^*}{u} = \frac{\sin \theta \cdot \sin \alpha}{\sin \phi \cdot \sin(\theta - \alpha)} \quad (\text{B1.9})$$

Therefore,

$$\begin{aligned} P &= ks_1 \frac{\sin \alpha}{\sin(\theta - \alpha)} - ks_2 \frac{\sin \theta \sin \alpha}{\sin(\theta - \alpha) \sin \phi} = \\ &= ks_1 \frac{V_1^*}{u} + ks_2 \frac{V_2^*}{u} \end{aligned} \quad (\text{B1.10})$$

and hence

$$P \cdot u = ks_1 V_1^* + ks_2 V_2^* \quad (\text{B1.11})$$

B1.3 Construction of the Force Plane Diagram for a Thick-Walled Cylinder

The force polygon presented in Ref. [3.5] applied to a thick-walled cylinder of radius ratio $R_0/R_i = 2.72$, rendered fully plastic by an internal pressure of $2k$, where k is the plane strain yield shear stress.

Figure B1.3(a) shows the slip line field for a cylinder which is fully plastic where the cylinder has a radius ratio of 4.81; the internal pressure at plastic collapse is

$$p = 2k \ln R_0/P_i = 1.57 (2k) \quad (\text{B1.12})$$

The polar equation of the logarithmic spirals - lines of maximum shear stress - in the wall of a cylinder are given by

$$r = R_i e^{\theta} \quad (\text{B1.13a})$$

and in particular

$$R_0 = R_i e^{\theta_0} \quad (\text{B1.14b})$$

Figure B1.3

The radius ratio chosen for investigation in this paper is

$$R_0/R_i = 4.81, \text{ and}$$

(B1.14)

$$\theta_0 = \ln (R_0/R_i) = \pi/2.$$

Figure B1.3(b) shows how the non-dimensional hoop stress σ_θ/Y varies with radius for the four values of the non-dimensional internal pressure p/Y corresponding to fully elastic, elastic-plastic and fully plastic cylinders[†]. For the partially plastic cylinder, the stresses in the elastic region are obtained from the Lamé equations[†]; the radial stress at the elastic-plastic boundary just brings the elastic outer zone up to yield at the boundary. This gives

$$\frac{\sigma_\theta}{Y} = \frac{1}{2} \frac{c^2}{R_0^2} \left(\frac{R_0^2}{r^2} + 1 \right); \quad (B1.15)$$

c denotes the radius of the elastic-plastic interface.

In the plastic region, we have to satisfy the equilibrium equation and the Tresca criterion [B1.4], [B1.5]:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \text{ and } \sigma_\theta - \sigma_r = Y. \quad (B1.16)$$

Solving, yields:

$$\frac{\sigma_\theta}{Y} = -\ln \frac{c}{r} + \frac{1}{2} \left(1 + \frac{c^2}{r_0^2} \right) \quad (B1.17)$$

[†] It is assumed that the radius ratio does not change significantly when the cylinder is under internal pressure.

If $c = R_i$ the cylinder wall is fully elastic and if $c = R_o$ the wall is fully plastic.

The internal pressure p is given by:

$$\frac{p}{Y} = \ln \frac{c}{R_i} + \frac{1}{2} \left(1 - \frac{c^2}{R_o^2} \right) \quad (B1.18)$$

The force on each radial element 0-1, 1-2, ..., 5-6 is obtained by integration of the area under the curve of σ_θ/Y . Figures B1.4(a), B1.4(b), B1.4(c), B1.4(d) are the force polygons for the four given cases and are all drawn to the same scale.

All points on radius 00 (0° radius) or 0, 1, 2, 3, 4, 5, 6, see Figure B1.3(a), transform to points or 06 on Figures B1.4(a), B1.4(b), B1.4(c), B1.4(d), points on 00' (90° radius) transform to 0'6' and the points at which the internal pressure over the inside surface acts i.e. 6, α'' , ..., ϵ'' , 6', to points on the circular arc 6, α' , ..., ϵ' , 6' (Figures B1.4(a), B1.4(b), B1.4(c), B1.4(d)). The force on 6 α'' 6' is the pressure p time chord length 66' and is perpendicular to 66': this appears as vector 66' in Figures B1.4(a), (b), (c), (d). The force on any segment in the wall of the tube e.g. 8-16 is given by the line joining 16 (on 0'6') and 8 (on the radius between 0' α' and 0'6').

The two heavy marked lines drawn in each force polygon are the locus of points corresponding to the two lines of maximum shear stress α and β of Figure B1.3(a). Notice the change of curvature under different stress conditions.

An element (1, 2, 14, 13) of the cylinder wall is drawn in Figures B1.5(a), B1.5(b), B1.5(c), B1.5(d), showing the magnitude of

3

4

Figure Bl.4

Force Polygons

Figure Bl.4

Radial and tangential forces on a cylinder wall element

Figure B1.5

the forces acting on it as given by the corresponding force polygons.

B1.4 Interpretation of Curves

If we take an oblique plane, such as 31' for example in Figure B1.3(a), the force acting upon it can be calculated from the σ_r curves or directly from the force polygons as vector 31'.

For the chosen radius ratio $R_0/R_i = 4.81$, the hoop stress undergoes a change of sign at the inner radius of the cylinder when high internal pressure is applied [3.3].

Figure B1.4(c) reflects this situation; the forces 45, 56 and the corresponding vectors 4'5' and 5'6' acting on the most inner annuli of the cylinder, have opposite directions to the other forces. The total forces on the vertical sections of the wall, see Figure B1.4(c), are given by:

$$\begin{aligned} T_1 &= 01 + 12 + 23 + 34 + 45 + 56 \\ &= 0.971 + 0.522 + 0.228 + 0.047 - 0.067 - 0.130 = \\ &= 1.571 \text{ (2k)} \end{aligned} \quad (\text{B1.19})$$

The resultant force on the quarter cylinder (per unit length) is by (B1.19)

$$F = 1.571 \cdot \sqrt{2} \text{ (2k)} = 2.222 \text{ (2k)} \quad (\text{B1.20})$$

The total force created by the internal pressure p and acting at 45° angle on the inside portion of the wall under consideration can be obtained by integrating, (see Figure B1.6)

$$F = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} p R_i \cos \theta \, d\theta = 2 p R_i \sin \frac{\pi}{4} \quad (\text{B1.21})$$

Figure Bl.6

Schematic for calculation of total force F'

The internal pressure p is obtained from equation (B1.18) with $c = 4.81$ and gives $p = 1.571 (2k)$. Substituting in equation (B1.21) we obtain

$$F' = 2 \times 1.571 (2k) \times 1 \times \sin \frac{\pi}{4} = 2.222 (2k) \quad (B1.22)$$

Therefore it is verified that $F = F'$

In Figure B1.7(a) is drawn the slip-line field (or lines at maximum shear stress) for a cylinder wall with radius ratio $R_0/R_i = 2.193$. The full s.l.f. for this case is contained in an angle of $\pi/4$. This drawing as well as the following force polygons are drawn to the same scale as that of the previous one with a radius ratio $R_0/R_i = 4.81$. This gives an immediate impression of the forces involved.

In Figure B1.7(b) the various σ_y stress distributions are given. Figures B1.8(a), (b), (c), (d) are the force polygons for the four different internal pressures p/y given in Figure B1.7 (c).

It will be noticed that when an internal pressure is applied which renders the wall of the cylinder fully plastic, there is no change in sign of the hoop stress at the inner bore of the cylinder.

The force in cylinder wall section 15, 4, (see Figure B1.7(a)) is given, to same scale, by line length 15, 4 in the force polygons of Figure B1.8.

Figure B1.7

Appendix B2

INTERACTIVE COMPUTER PROGRAM CLASS

B2.1 Program architecture of CLASS

The interactive computer package CLASS is divided in three major sections. (see Figure B2.1):

- 1) model definition,
- 2) finite element analysis,
- 3) analysis of results

Each section is composed of a certain number of executive modules.

There is a certain degree of hierarchy in the execution of each module; some modules must be executed before and some after.

When a brand new problem is started the first module to be executed will be NEWMESH. The other modules of the Section 1) may be executed at any time and in any order but always after the execution of NEWMESH. The module FEMXEQ should be executed as the last step of Section 2). The other modul FEMXEQ should be executed at any time and in any order but always after at least one execution of NEWMESH.

The modules of Section 3) can be executed at any time and in any order but after at least one execution of FEMXEQ. Each XEQ module is composed of a series of control commands for the organization of the input and/or output files and of one or more load modules (see Table B2.1). The load modules are the executable form of the source programs containing the code for all the algorithms of the grid generation, finite element analysis and displacement and stress display.

CLASS Structure Organization

Figure B2.1

The code is written in FORTRAN language and only standard code was used to allow the installation of the CLASS package in most computer centre facilities.

A brief description of the XEQ modules has been given in the previous pages. The list of load modules with a brief description of their function and the input/output files that each load module uses is given in the Table B2.2 and B2.4 according to the particular XEQ modules.

B2.2 The storage of the stiffness matrix

Some information on the storage of the stiffness matrix is given being a crucial point.

The stiffness matrix is stored in an array having length equal to twice the number of nodes in the structure. The width of the stiffness matrix will be $2(n+1)$ if the number of boundary lines conveying on a node is n . Currently the maximum allowed is $n = 8$.

An additional width of $2(n+1)$ columns is required for coded integers which specify the proper position of this band (the stiffness matrix has been compressed to a width of $2(n+1)$) when the elements are expanded to form the true (square) stiffness matrix.

If sufficient core storage is available in the computer on which the program is to be run, the stiffness matrix may be stored in the central memory of the computer and the variables related to the overall stiffness matrix are organized in a common block STFNS which takes the form (for 100 nodes):

```
/STFNS/AMFS(200,18),IPAT(200,18),JKY2(200) ,
```

Table B2.1

XEQ modules (in alphabetical order)

C CHARGE
 !SET F:6
 !SET F:5/DATAGRID;OUT;SAVE
 load module———!GRIDIN.
 !SET F:5/DATAGRID;IN
 !SET F:6/OUTPUT;OUT;SAVE
 !SET F:7/GRIDOUT;OUT;SAVE
 load module———!GRIDBLN.
 C GRIDOUT OVER GRIDOUT1
 !SET F:6/OUTPUT;IN
 !SET F:7/GRIDOUT;INOUT;SAVE
 load module———!CREATEBLN.
 G OUTPUT 1
 !SET F:7/GRIDOUT;IN
 !SET F:3/PLASDAT;INOUT;SAVE
 load module———!CHARGEEND.

C CNAE
 !SET F:5
 !SET F:6
 !SET F:8/DEF;IN
 !SET F:9/PRNC;IN
 !SET F:11/WEIBB;IN
 load module———!PHE

C CONTROL
 !SET F:3/PLASDAT;INOUT;SAVE
 !SET F:5
 load module———!ENVARLEN.

C DEVAL
 !SET F:5/DATAGRID;IN
 !SET F:6/OUTPUT;OUT;SAVE
 !SET F:7/SPICNCL;OUT;SAVE
 load module———!GRIDBLN.

C DISA
 !SET F:3/PLASDAT;INOUT;SAVE
 !SET F:5
 load module———!DISLEN.

C FLASGX
 !SET F:1/STFNS;OUTIN;REL
 !SET F:5/PLASDAT;IN
 !SET F:6/PLASOUT;OUT;SAVE
 !SET F:8/DEF;OUT;SAVE
 !SET F:9/STFNS;OUT;SAVE
 !SET F:10/PRNC;OUT;SAVE
 !SET F:11/WEIBB;OUT;SAVE
 PRINTAT ENG
 C FLASGX TO BL

Table B2.1 (cont.)

	C PLASXEN !SET F:1/STFHS;OUTIN;IEL !SET F:5/PLASDAT;I !SET F:6/PLASOUT;OUT;SAVE !SET F:8/DEF;OUT;SAVE !SET F:9/STHS;OUT;SAVE !SET F:10/PPNC;OUT;SAVE !SET F:11/WEIDS;OUT;SAVE PRINTAT LRG C PERCON TO ME !PLAS7LEN. C PLASOUT TO LP PRIN C PERCON12 TO ME
load module	!C FORXEN !SET F:5/PLASDAT;INOUT;SAVE !SET F:5
load module	!FORLEN.
	!C MATXEN !SET F:3/PLASDAT;INOUT;SAVE !SET F:5 ME
load module	!MATLEN.
	!C GRID3EN !SET F:6 !SET F:5/DATAGRID;INOUT;SAVE
load module	!GRID3LEN. !SET F:5/DATAGRID;I !SET F:6/OUTPUT;OUT;SAVE !SET F:7/GRIDOUT;OUT;SAVE
load module	!GRID3LEN. C GRIDOUT OVER GRIDOUT1 !SET F:6/OUTPUT;I !SET F:7/GRIDOUT;INOUT;SAVE
load module	!CRELATELEN. C OUTPUT TO ME !SET F:7/GRIDOUT;I !SET F:3/PLASDAT;INOUT;SAVE
load module	!CHREL3LEN.
	!C REL3LEN !SET F:6 !SET F:5/DATAGRID;OUT;SAVE
load module	!GRIDLEN. !SET F:5/DATAGRID;I !SET F:6/OUTPUT;OUT;SAVE !SET F:7/GRIDOUT;OUT;SAVE
load module	!GRID3LEN. C GRIDOUT OVER GRIDOUT1 !SET F:6/OUTPUT;I !SET F:7/GRIDOUT;INOUT;SAVE
load module	!CRELATELEN. C OUTPUT TO ME !SET F:7/GRIDOUT;I !SET F:3/PLASDAT;OUT;SAVE
load module	!INITLEN.

Table B2.1 (cont.)

	<u>IC ODDXCH</u>
	!SET F:7/G/IDCCT;1.
	!SET F:3/PLABD;1;INCCT;3
load module	!CINSHL;1.
	<u>IC PROBABIT</u>
	!SET F:5
	!SET F:6
	!SET F:2/DE;CCT;SAVE
	!SET F:7/G/IDCCT;1.
	!SET F:6/DE;1.
	!SET F:9/PRAC;1.
	!SET F:11/DE;1.
load module	!DEISL;1.
	!SET F:2/DE;1.
load module	!DEFOUL;1.
	<u>IC PROBA</u>
	!SET F:5
	!SET F:6
	!SET F:2/DE;CCT;SAVE
	!SET F:7/G/IDCCT;1.
	!SET F:6/DE;1.
	!SET F:9/PRAC;1.
	!SET F:11/DE;1.
load module	!DEIPL;1.
	<u>IC SHAPE</u>
	!SET F:5/G/IDCCT;1.
	!SET F:6/OCIP;1.
	!SET F:6/DE;1.
	!SET F:7/DE;CCT;SAVE
	!SET F:9
load module	!TRSCALE;1.
	!SET F:7/DE;CCT;1.
load module	!SHAPE;1.
	<u>IC VERX</u>
	!SET F:3/PLACIA;1.
	!SET F:6
load module	!VERX;1.

Table B2.2

Listing of Load Modules and Data Files for Section 1 of CLASS

<u>NEWMESH</u>		
<u>Load Module</u>	<u>Function</u>	<u>Files</u>
GRIDIN	Interactive construction of a new mesh	DATAGRID (output) Terminal (input) (output)
GRID3LMN	Grid Generator Program (Calcomp Plot Subroutines)	DATAGRID (input) OUTPUT (output) GRIDOUT (output)
CREATELMN	Create the input file for F.E.M.	OUTPUT (input) GRIDOUT (input/output)
INITLMN	Initialize the input file for F.E.M.	GRIDOUT (input) PLASDAT (output)
<u>MODMESH</u>		
<u>Load Module</u>	<u>Function</u>	<u>Files</u>
MODGR1MN	Interactive modification of the mesh	DATAGRID (input/output) Terminal (input/output)
GRID3LMN	Grid Generator Program (Calcomp Plot Subroutines)	DATAGRID (input) OUTPUT (output) GRIDOUT (output)
CREATELMN	Create the input file for F.E.M.	OUTPUT (input) GRIDOUT (input/output)
INITLMN	Initialize the input file for F.E.M.	GRIDOUT (input) PLASDAT (output)
<u>DEVXEQ</u>		
<u>Load Module</u>	<u>Function</u>	<u>Files</u>
GRID3LMN	Grid Generator Program (Calcomp Plot Subroutines)	DATAGRID (input) OUTPUT (output) GRIDOUT (output)

Table B2.3

Listing of Load Modules and Data Files for Section 2 of CLASS

CONTROL

<u>Load Module</u>	<u>Function</u>	<u>File</u>
FEMVARLMN	Selection of the options for the finite element analysis	PLASDAT (input/output) Terminal (input/output)

MATXEQ

<u>Load Module</u>	<u>Function</u>	<u>File</u>
MATERLMN	Definition of the material variables for the finite element analysis	PLASDAT (input/output) Terminal (input/output)

DISXEQ

<u>Load Module</u>	<u>Function</u>	<u>File</u>
DISLMN	Definition of the imposed displacements for the finite element analysis	PLASDAT (input/output) Terminal (input/output)

FORXEQ

<u>Load Module</u>	<u>Function</u>	<u>File</u>
FORLMN	Definition of the external forces for the finite element analysis	PLASDAT (input/output) Terminal (input/output)

VERXEQ

<u>Load Module</u>	<u>Function</u>	<u>File</u>
VERLMN	Verification and printing of the input data for the finite element analysis	PLASDAT (input) Terminal (output)

FEMXEQ

<u>Load Module</u>	<u>Function</u>	<u>File</u>
PLAS7LMN	Finite Element Analysis: Stress and Strain calculations	STFNS (output/input) temporary PLASDAT (input) PLASOUT (output) DEF (output) STRS (output) PRNG (output) WEIBB (output)

Table B2.4

Listing of Load Modules and Data Files for Section 3 of CLASS

<u>Load Module</u>	<u>Function</u>	<u>Files</u>
PHLMN	Plots the stress contours	Terminal (input/output) DEF (input) PRNC (input) WEIB (input)

<u>Load Module</u>	<u>Function</u>	<u>Files</u>
TRSCALELMN	Scales the deformations	GRIDOUT1 (input) OUTPUT (input) DEF (input) DEFSC (output) Terminal (input/output)
SHAPELMN	Plots the deformed shape	DEFSC (input)

<u>Load Module</u>	<u>Function</u>	<u>Files</u>
WEIBLMN	Calculates the probability of survival of the component	Terminal (input/output) GRIDOUT (input) DEF (input) PRNC (input) WEIBB (input)

where AMFS is the stiffness matrix and JKV2 and IPAT are the pointer matrices.

If the program has to be run on-line on the Carleton University XEROX-SIGMA9 computer, the core storage is insufficient for these arrays (and for the other arrays necessary for execution); therefore the stiffness matrix must be stored on a slower mass-storage device (tape or disk). This was accomplished by storing the stiffness matrix and large pointer matrix on a random access disk file. When needed, an entire row of the stiffness and pointer matrices is transferred into the computer central memory. The file is manipulated by the subroutines DIRECT READ and DIRECT WRITE. Informations on these subroutines is provided on the following pages. When in central memory, a row of the stiffness matrix is contained in AMFS (18) and a row of the pointer matrix in IPAT (18). An index of length $(2 \times (\text{number of nodes}) + 1)$ must be provided for the random access file. For this storage mode, common block STFNS has the appearance (for 100 nodes):

/STFNS/AMFS(18), IPAT(18), INDEX(207), JKV2 (200)

Listing of the programs and subroutines which must be changed in order to increase the arrays size is given in the technical Report CRB-CLASS-1.

B2.3 File manipulation subroutines

The following two subroutines were used for mass storage filing of the stiffness matrix.

The subroutines DIRECT READ and DIRECT WRITE perform random access I/O operations; whole records are randomly accessed. The records are keyed records, which have no sequence; they only have keys (labels)

The keyed I/O subroutines perform the following functions:

CALL DIRECT READ (1, AMFS, 54, 1) reads from the file on tape 1, 54 computer words at index location 1. These will be stored in 54 words following the first element of AMFS (due to storage arrangement this implies 36 words will fill AMFS (18) [$18 \times 2 = 36$ because double precision] providing the 1st row of the stiffness matrix, and 18 words will fill IPAT (18) providing the corresponding pointer).

CALL DIRECT WRITE (1, AMFS, 54, 1) where the 4 parameters have the same meaning as in DIRECT-READ. This subroutine causes writing of 54 words into the file.

B2.4 Modules execution and input of the data

The operation of CLASS does not require the learning of any particular computer language or command sequence. On the on-line system connected with the SIGMA-9 computer of the Computer Center at Carleton University, the execution of the module is done by the TEL (Terminal Executive Language) command !XEQ:

!XEQ module name, account number,

where module name is the name of one of the modules in Table B2.1.

account number is the account under which all the modules are stored (it can be obtained from the Computing Center Consulting Office).

Each module is a series of commands to create input and/or output files, and to execute the programs in load module form. All of the intricacies of data file manipulation are hidden from the user and each program is auto prompting.

After each question, the computer prompts the user with a question mark and waits for the users answer.

The data are of three types of format:

- 1) Alphanumeric
- 2) Integer
- 3) Floating point,

The alphanumeric data are reserved for titles, file names, devices etc. Any character can be entered with blanks or spaces, as desired, within the field format.

The integer data are numbers, without decimal points. They can be entered separated by commas as

?100,5,21,

or separated by blanks within the specified field. For example if the format is 3I4, then the data can be entered as

?00001 0022 1111

The floating point numbers are numbers with decimal points. The numbers may be entered as decimal numbers separated by commas or in scientific notation (i.e. E format), separated by commas or within the specified field.

For example if the format required is 2E5.2,E10.4,F10.5 the numbers can be entered, as

?10.5,22.,1000.5,.222,

or ?10.50,22.0,1.0005E+03,0.222,

or ?10.5022.001.0005E+03 0000.22200.

Obviously, the easiest way to enter the data is the free-format type with numbers divided by commas.

B2.5 Example

An example is presented in order to illustrate the use and capabilities of CLASS. The stress and strain analysis of an inlet type of a tube subject to an axial pressure at the outer edge was performed. A schematic drawing is given in Figure B2.2. There is one line of symmetry (ϕ) which is taken as the Z axis and the R axis is perpendicular and passes through the two centres of curvature O' and O'' . The mesh is generated for the right hand section of the inlet. This section is divided into two partitions (see Figure B2.3); the first is included between the two straight lines AB and EF and the two curves ACE and BDF; the second partition is the rectangle EFGH. The base line is the segment AB, the first partition line is EF and the second partition line is GH. The coordinates of each point are given in the table in Figure B2.3. The mesh can now be generated with the module NEWMESH:

```
!XEQ NEWMESH
```

The input/output example is shown at pages 196 to 211. The lines typed by the user are underlined, all the other comments and questions are printed by the computer. At page 190 it was chosen to plot the mesh on the CRT. (DEVICE ...TVNORM), in the Mackenzie Building (LOCATION...M). Under the options, it was decided not to print the results and therefore only a brief summary is typed at the terminal by the computer. The list includes the following, in order:

- 1) title, [TEST FOR **CLASS**]
- 2) date, [15:46 MAR 31, '78]
- 3) number of nodes = 21, and number of elements = 24,
- 4) The plot dimensions [XMAX = 10:0000, YMAX = 10:000]
- 5) The scale factor [true dimensions/plot dimensions] = .5

Technical drawing for the example

Figure B2.2

Points coordinates

	R	Z
A	1.0	0.
B	2.0	0.
C	1.25	1.0
D	2.25	1.0
E	3.0	2.0
F	3.0	1.5
G	4.0	2.0
H	4.0	1.5

Identification of boundary lines

Figure B2.3

The plot has the same scale factor in the R-axis and the Z-axis to avoid deformations of the true shape.

On the CRT the mesh was without errors but it was decided to reduce the size to fit on a standard page (8.5" x 11"). The module MODMESH performs the modifications desired (see page 200 to 201). The mesh was again plotted on the screen and thereafter it was decided to obtain a hardcopy on the CALCOMP plotter. The module DEVXEQ (see page 201) was used for this purpose.

The hardcopy of the mesh is in Figure B2.4. This concludes the creation of the mesh and the calculation of the node coordinates and the elements node numbers. A file is created with this data which is the input file for the finite element program. To run the F.E.M. program it is necessary to implement the file with more data. The first module of the F.E.M. set of programs to be run is CONTROL (see page 202). This module controls the options of execution and output. Next, it was chosen to run the module MATXEQ (see page 203) to select the material variables.

The selection of the boundary conditions is another important and sometimes tedious (if there are many of these conditions) step in the preparation of the F.E.M. input data. The forces and node displacements for the present example are illustrated in Figure B2.5. The module DISXEQ updates the file with the imposed displacements and the module FORXEQ with the external forces.

The input data file is now ready for execution. The data can be checked with the module VERXEQ (see page 207).

TEST FOR **CLASS**
 SCALE FACTOR .500 16:14 APR 04. '78

Mesh output from automatic grid generator program

Figure B2.4

Displacements

node	R	Z
1	free	0.0
2	free	0.0
3	free	0.0

Forces

nodes	R	Z
13 - 16	—	-15 ksi
16 - 19	—	-15 ksi

Identification of Forces and Displacements
Figure B2.5

Finally the finite element calculations can be performed with the module FEMXEQ (see page 209). The printout, which includes node displacements and element stresses, is obtained at the line-printer due to the large format and lengthy listing.

The displacements can be plotted with the module SHAPEXEQ (see page 210). This module consists of two separate programs; the first calculates the new node coordinates scaling the displacements by a factor selected by the user; the second plots the undeformed shape and the deformed shape. Only the outer boundary is plotted with nodes being indicated by square symbols in the undeformed shape and by circles in the deformed shape. The module SHAPEXEQ was run again to obtain a hardcopy (see page 211). The result is given in Figure B2.6.

The step-by-step procedure for a successful execution of CLASS is summarized in Table B2.5.

Deformations plot for the example

Figure B2.6

Table B2.5

Step-by-step execution of CLASS

- Step 1 - Start from the technical drawing.
- Step 2 - Find the symmetry lines and the R and Z axes.
- Step 3 - Locate the base line and divide the component into partitions.
- Step 4 - Find the coordinates of the base and partition line, and the points for the curvature.
- Step 5 - Generate the mesh and, if necessary, check the mesh and input the modifications.
- Step 6 - Locate the nodes on which displacements must be imposed and the lines on which external pressures are applied.
- Step 7 - Input the control variables and material variables.
- Step 8 - Verify the input data.
- Step 9 - Run the finite element program.
- Step 10- Check the output.
- Step 11- Plot the displacements and the stresses.

Example for CLASS! ED MECH

```

***** PROGRAM TO INPUT THE DATA FOR THE *****
***** MECH GENERATOR PROGRAM *****

```

```

♦♦DIMENSIONS OF THE PLOT (LMAX,IMAX) 2F5.1
10.,10.

```

```

♦♦TITLE OF THE PLOT (NAME) 40 CHARACTER:
TEST FOR ♦♦CLASS♦♦

```

```

♦♦PRINT RESULTS (IM1)
♦♦PLOT GRID (IM2)
♦♦PLOT BOUNDARY NODES (NUMBER) (IM3)
♦♦PLOT ELEMENT NUMBERS (IM4)
♦♦PUNCH CARDS (IM5)
1=YES,0=NO (THE FORMAT IS 5I1)
01111

```

!! SPECIFY THE BASE LINE !!

```

♦♦NUMBER OF ELEMENTS ON THE BASE LINE (N) 12
2
♦♦INDICATOR OF SPACING OF ELEMENTS ON THE BASE LINE
=1 FOR ARITHMETIC SPACING
=2 FOR PARABOLIC SPACING (FORMAT 12)

```

```

1
♦♦INDICATOR OF CURVATURE OF THE BASE LINE
=0 FOR NO CURVATURE
=1 FOR CURVATURE (FORMAT 12)

```

```

♦♦RATIO OF FIRST TO LAST (OR MID) ELEMENT IN
ARITHMETIC (OR PARABOLIC) SPACING (F5.3)

```

```

1.
♦♦COORDINATES OF THE -LEFT- HAND CORNER NODE
OF THE BASE LINE (P1,Q1) 2F8.4

```

```

1.,0.
♦♦COORDINATES OF THE -RIGHT- HAND CORNER NODE
OF THE BASE LINE (P2,Q2) 2F8.4

```

```

2.,0.
♦♦COORDINATES OF A POINT ON THE BASE LINE
TO DEFINE CURVATURE (IF CURVED) (PCC,QCC) 2F8.4

```

!! SUBSEQUENT DATA SET !!

♦♦NUMBER OF LAYERS OF ELEMENTS BETWEEN PRESENT AND
SUCCEEDING MAJOR PARTITION LINE (2) :

1 4

♦♦NUMBER OF TRIANGLES GAINED OF LAST PER LAYER ON
LEFT (2, 1, 0, -1) AND RIGHT (1, 0, -1, -2)
RESPECTIVELY (212) :

1 1

♦♦INDICATOR OF SPACING OF ELEMENTS ON CENTRE,
LEFT AND RIGHT BOUNDARIES RESPECTIVELY.
=1 MEANS ARITHMETIC SPACING.
=2 MEANS PARABOLIC SPACING (312) :

1 1 1

♦♦INDICATOR OF CURVATURE OF CENTRE, LEFT AND RIGHT
BOUNDARIES RESPECTIVELY.
=0 MEANS NO CURVATURE
=1 MEANS CURVATURE (312) :

0 1 1

♦♦RATIO OF FIRST TO LAST (OR MID) ELEMENTS IN
ARITHMETIC OR PARABOLIC SPACING (3F5, 3) :

1 1.1.1.1.

♦♦COORDINATES OF LEFT HAND CORNER NODE OF
NEXT MAJOR PARTITION LINE (2F8, 4) :

1 3.1.2.1.

♦♦COORDINATES OF RIGHT HAND CORNER NODE OF
NEXT MAJOR PARTITION LINE (2F8, 4) :

1 3.1.1.5.

♦♦COORDINATES OF A POINT ON THE NEXT MAJOR PARTITION
LINE TO DEFINE CURVATURE (IF CURVED) (2F8, 4) :

1 1

♦♦COORDINATES OF A POINT ON THE LEFT BOUNDARY
TO DEFINE CURVATURE (IF CURVED) (2F8, 4) :

1 1.25.1.1.

♦♦COORDINATES OF A POINT ON THE RIGHT BOUNDARY
TO DEFINE CURVATURE (IF CURVED) (2F8, 4) :

1 2.25.1.1.

+++++ ANOTHER PARTITION LINE? 1=YES, 2=NO

1

!! SUBSEQUENT DATA SET !!

♦♦NUMBER OF LAYERS OF ELEMENTS BETWEEN PRESENT AND
SUCCEEDING MAJOR PARTITION LINE (I2)

2

♦♦NUMBER OF TRIANGLES GAINED OR LOST PER LAYER ON
LEFT (2, 1, 0, -1) AND RIGHT (1, 0, -1, -2)
RESPECTIVELY (2I2)

♦♦INDICATOR OF SPACING OF ELEMENTS ON CENTRE,
LEFT AND RIGHT BOUNDARIES RESPECTIVELY.
=1 MEANS ARITHMETIC SPACING
=2 MEANS PARABOLIC SPACING (3I2)

1 1 1

♦♦INDICATOR OF CURVATURE OF CENTRE, LEFT AND RIGHT
BOUNDARIES RESPECTIVELY.
=0 MEANS NO CURVATURE
=1 MEANS CURVATURE (3I2)

♦♦RATIO OF FIRST TO LAST (OR MID) ELEMENTS IN
ARITHMETIC (OR PARABOLIC) SPACING (2F5, 3)

1.1.1.

♦♦COORDINATES OF LEFT HAND CORNER NODE OF
NEXT MAJOR PARTITION LINE (2F8, 4)

4.2.

♦♦COORDINATES OF RIGHT HAND CORNER NODE OF
NEXT MAJOR PARTITION LINE (2F8, 4)

4.1.5.

♦♦COORDINATES OF A POINT ON THE NEXT MAJOR PARTITION
LINE TO DEFINE CURVATURE (IF CURVED) (2F8, 4)

♦♦COORDINATES OF A POINT ON THE LEFT BOUNDARY
TO DEFINE CURVATURE (IF CURVED) (2F8, 4)

♦♦COORDINATES OF A POINT ON THE RIGHT BOUNDARY
TO DEFINE CURVATURE (IF CURVED) (2F8, 4)

***** ANOTHER PARTITION LINE? 1=YES 2=NO

2

===== END =====

♦STOP♦ 0

DEVICE... FTWORM
 LOCATION... FM
 ♦STOP♦
 ..COPYING

*** START CREATE ***

*** END OF CREATE ***

♦STOP♦ 0

1

TEST FOR ♦♦CLASS♦♦

DATE.....15:46 MAR 31, 78

NUMBER OF NODES = 31

NUMBER OF ELEMENTS = 24

AMAX = 10.0000

VMAX = 10.0000

SCALE FACTOR = .20000

*** START OF INIT FOR NEW MECH ONLY ***

*** END OF INIT ***

♦STOP♦ 0

♦♦♦MECH TERMINATED♦♦♦

MECH MODMECH

*** START THE MECH MODIFICATION ***

◆◆GIVE CODE TO CHANGE FORMAT=11....
 OPTION:-----1
 BASE LINE SPECIFICATION:-----2
 BASE LINE COORDINATE:-----3
 PARTITION SPECIFICATION:-----4
 PARTITION COORDINATE:-----5

1

◆◆DO YOU WANT TO CHANGE 11....
 PLOT DIMENSION: 1-----1
 TITLE ?-----2
 OUTPUT OPTION: 1-----3

1

◆◆DIMENSION OF THE PLOT (MINIMUM) 2F5.1
 NOW ARE 10. 10.

8. 5. 1

◆◆DO YOU WANT ANY OTHER MODIFICATION? 1=YES, 0=NO

1

*** END OF MODIFICATION ***

◆STOP◆ 0
 DEVICE... STVHORM
 LOCATION... 711
 ◆STOP◆ 0
 ..COPYING

*** START CREATE ***

*** END OF CREATE ***

◆STOP◆ 0

TEST FOR ♦♦CLACC♦♦

DATE.....16:11 APR 04, 78

NUMBER OF NODES = 31

NUMBER OF ELEMENTS = 34

UMAX = 8.00000

YMAX = 5.00000

SCALE FACTOR = .50000

*** START OF INIT FOR CHANGE OF MECH ONLY ***

*** END OF INIT ***

♦♦TOP♦♦ 0

♦♦♦NEO TERMINATED♦♦♦

!NEO DEVED

DEVICE... PLOT

FILE NAME... CLACC

♦♦TOP♦♦ 0

♦♦♦NEO TERMINATED♦♦♦

!BATCH PLOTIT.APLGRAE NUM=85605170 NAME=BOEP FID=:CLACC

ID=0025 SUBMITTED 16:15 APR 04, 78

WAITING: 170TD RUN

NEO CONTROL

*** PROGRAM TO INPUT CONTROL VARIABLES ***

++TITLE OF THE JOB: 1084+
 TEST FOR ◆◆CLAC◆◆

++SELECT THE MODE OF SOLUTION FOR FEM PROGRAM
 ELASTIC = 0+1+
 PLASTIC = 1+1+

++SELECT THE ELEMENT NUMBER WHOSE CONVERGENCE
 CHARACTERISTICS ARE DECIDED
 THE EQUIVALENT STRESS IN THIS ELEMENT IS PRINTED
 EVERY PLASTIC ITERATION (FORMAT: I3+)

1+.

++SELECT THE DECIDED YIELD CRITERIA
 1 FOR VONMISE
 2 FOR HILL ANISOTROPIC
 3 FOR DUREY HILLIER
 4 FOR MODIFIED COULOMB
 5 FOR TENSION CUT-OFF
 6 FOR MAX. TENSILE STRAIN

1+.

++IF THE PLOT OF THE RESULTS IS DECIDED, SET 1

1+.

++IF THE PROBABILITY OF SURVIVAL HAS TO BE
 CALCULATED, SET 1

1+.

++CONVERGENCE CRITERION FOR GAUSS-SEIDEL ITERATION
 TYPICAL VALUE IS 0.00001 (F10.7)
 0.00001+.

++CONVERGENCE CRITERION FOR PLASTIC ITERATION
 TYPICAL VALUE IS 0.01 (F10.5)
 0.01+.

++PERCENTAGE OF FULL LOAD TO BE APPLIED INITIALLY
 TYPICAL VALUE IS 80% (F10.0)

80+.

++PERCENTAGE OF FULL LOAD TO BE ADDED IN EACH LOAD STEP
 TYPICAL VALUE IS 10% (F10.0)

10+.

*** END OF FEMVAR ***

◆STOP◆ 0
 ◆◆NEO TERMINATED◆◆

! MED MATER

*** PROGRAM TO INPUT MATERIAL VARIABLES ***

++ELASTIC YOUNG MODULUS AND POISSON RATIO (E10.4)
1.3E+08, .3

++YIELD STRESS INITIAL YIELD VALUE OF EQUIVALENT
 STRESS FOR GIVEN FAILURE CRITERION (E10.4)
110000.

++MOHR COEFFICIENT RATIO OF TENSILE TO COMPRESSIVE YIELD
 STRESS USED TO CALCULATE MOHR NUMBER (E10.4)
1.

++H AND N IN STRAIN HARDENING RELATION
 $STRESS = H * STRAIN ** N$ (E10.4)
1., 1.

++GIVE MATERIAL CONSTANTS TO SPECIFY PARAMETERS FOR
 THE SPECIFIED YIELD CRITERIA (SPECIFIED IN FEMVAR)

++VON MISES CRITERIA (NONE ARE NEEDED)

*** END OF MATER ***

♦STOP♦ 0
 ♦♦♦MED TERMINATED♦♦♦

IMED DIPLD

*** PROGRAM TO INPUT EXTERNALLY ***

*** IMEDIED DISPLACEMENTS..... ***

I = 1

++NODE NUMBER FOR WHICH DISPLACEMENT IS SPECIFIED: I4

I1

++TYPE 1 IF DISPLAC. IN RADIAL DIREC. IS SPEC. (11)

TYPE 0 IF " " " " IS NOT SPEC. (11)

I1

++TYPE 1 IF DISPLAC. IN AXIAL DIREC. IS SPEC. (11)

TYPE 0 " " " " IS NOT SPEC. (11)

I1

++DISPLACEMENT IN AXIAL DIRECTION: F10.0

F10.0

==TYPE 1 IF YOU WANT TO SPECIFY ANOTHER DISPL. (11)

I1

I = 2

++NODE NUMBER FOR WHICH DISPLACEMENT IS SPECIFIED: I4

I2

++TYPE 1 IF DISPLAC. IN RADIAL DIREC. IS SPEC. (11)

TYPE 0 IF " " " " IS NOT SPEC. (11)

I1

++TYPE 1 IF DISPLAC. IN AXIAL DIREC. IS SPEC. (11)

TYPE 0 " " " " IS NOT SPEC. (11)

I1

++DISPLACEMENT IN AXIAL DIRECTION: F10.0

F10.0

==TYPE 1 IF YOU WANT TO SPECIFY ANOTHER DISPL.==11

1
I = 3

==NODE NUMBER FOR WHICH DISPLACEMENT IS SPECIFIED==14

3 Note: mistake in typing
3. Correction

==TYPE 1 IF DISPLAC. IN RADIAL DIREC. IS SPEC.==11
TYPE 0 IF " " " " IS NOT SPEC.==11

==TYPE 1 IF DISPLAC. IN AXIAL DIREC. IS SPEC.==11
TYPE 0 " " " " IS NOT SPEC.==11

1

==DISPLACEMENT IN AXIAL DIRECTION==10.0

0.

==TYPE 1 IF YOU WANT TO SPECIFY ANOTHER DISPL.==11

*** END OF EQU-D1 ***

♦♦♦TOP♦ 0
♦♦♦END TERMINATED♦♦♦

BOUND FOR

```

*** PROGRAM TO INPUT EXTERNALLY ***
*** INPUT FORCE1, THE FORCE1 ***
*** ACT AD PRESSURE ON LINE1 CON. ***
*** AND 2 ADJACENT NODE1. ***

```

-----SPECIFIED PRESSURE NUMBER

I = 1

++NODE NUMBER AT THE ONE END OF LINE ON WHICH PRESSURE
ACT1:14

113.

++NODE NUMBER AT OTHER END OF LINE ADJACENT:14

116.

++PRESSURE INTENSITY IN RADIAL DIRECTION ALONG THE
SPECIFIED LINE:113.8

++PRESSURE INTENSITY IN AXIAL DIRECTION ALONG THE
SPECIFIED LINE:113.8

1-15000..

==TYPE 1 IF YOU WANT TO SPECIFY ANOTHER FORCE:11

1

-----SPECIFIED PRESSURE NUMBER

I = 2

++NODE NUMBER AT THE ONE END OF LINE ON WHICH PRESSURE
ACT1:14

116.

++NODE NUMBER AT OTHER END OF LINE ADJACENT:14

119.

++PRESSURE INTENSITY IN RADIAL DIRECTION ALONG THE
SPECIFIED LINE:113.8

++PRESSURE INTENSITY IN AXIAL DIRECTION ALONG THE
SPECIFIED LINE:113.8

1-15000..

==TYPE 1 IF YOU WANT TO SPECIFY ANOTHER FORCE:11

*** END OF BOUND FOR ***

♦STOP♦♦♦

♦♦♦END TERMINATED♦♦♦

*** START VERIFY THE INPUT FILE TO THE F.E.M. ***

♦ TITLE OF THE JOB TEST FOR ♦♦CLAD♦♦

♦ NUMBER OF NODES 21

♦ NUMBER OF ELEMENTS 24

♦ NODE COORDINATES

1.00000	.00000	1.50000	.00000	2.00000	.00000	1.14079	.00000
1.57891	.69161	2.01702	.49125	2.57367	1.42873	1.59166	1.185
2.21004	.94330	2.22803	1.85921	2.29052	1.67739	2.55109	1.295
3.00000	2.00000	3.00000	1.75000	3.00000	1.50000	3.50000	2.000
3.50000	1.75000	3.50000	1.50000	4.00000	2.00000	4.00000	1.750
4.00000	1.50000						

♦ NODE NUMBER AT EACH ELEMENT FROM 1 TO
NOE = 24

4	2	1	4	5	2	5	3	2	5	3	7	5	4	7	9
8	6	5	8	9	6	10	3	7	10	11	9	11	4	3	11
13	11	10	13	14	11	14	12	11	14	15	12	16	14	13	12
17	15	14	17	18	15	19	17	16	19	20	17	20	18	17	20
															21

♦ NUMBER OF KNOWN FORCES 2

♦ NODE #	♦ NODE #	♦ RADIAL F.	♦ AXIAL F.
13	16	.000000E 00	-.150000E 05
16	19	.000000E 00	-.150000E 05

♦ NUMBER OF KNOWN DISPLACEMENTS

NODE #	1	AXIAL DISPLACEMENT	.000000
NODE #	2	AXIAL DISPLACEMENT	.000000
NODE #	3	AXIAL DISPLACEMENT	.000000

♦ELASTIC ANALYSIS SELECTED

♦EQUIVALENT STRESS IS PRINTED FOR ELEM. ** 1

♦VONMISES YIELD CRITERIA

♦GAUSS-SEIDELL CONVERGENCE .0000100

♦PLASTIC CONVERGENCE .01000

♦PERCENTAGE OF FULL LOAD 80.

♦PERCENTAGE AT EACH STEP 10.

♦YOUNG MODULUS 1.3000E 08

♦POISSON RATIO .3000E 00

♦YIELD STRESS .1100E 08

♦TENSILE TO COMPRESSIVE STRESS RATIO 1.00000

♦ STRESS = .1000E 01 STRAIN ♦♦ .1000E 01

♦AA,AB,AC,AD,AE = .0000E 00 .0000E 00 .0000E 00 .0000E 00 .0000E 00

♦END OF JOB ♦♦♦♦

*** END OF VERIFY ***

♦STOP♦

♦♦♦JOB TERMINATED♦♦♦

NEO CHAPENED

*** START OF SCALING ***

++DEFORMATION SCALE FACTOR=0.25000

10.0

*** END OF SCALING ***

♦STOP♦ 0

*** START OF SHAPE ***

++PLOT DIMENSION XMAX YMAX 255.1==

11.0

TEST FOR ♦♦ELAS♦♦

DEVICE... 7INORM

LOCATION... 1M

XMAX = 11.0000

YMAX = 8.00000

SCALE FACTOR = .25000

*** END OF SHAPE ***

♦STOP♦ 0

♦♦NEO TERMINATED♦♦

!XED CHAPENED

*** START OF SCALING ***

++DEFORMATION SCALE FACTOR (MM/M)

110.

*** END OF SCALING ***

♦STOP♦ 0

*** START OF CHAPE ***

++PLOT DIMENSIONS: XMAX=XMAX/255.1==

11.08.

TEST FOR ♦♦CLAD♦♦

DEVICE... PLOT

FILE NAME... !CHAPE

XMAX = 11.0000

YMAX = 8.00000

SCALE FACTOR = .25000

♦♦PLOT FILE GENERATED, SEE PLOTIT.APLGRAF!!!

*** END OF CHAPE ***

♦STOP♦ 0

♦♦XED TERMINATED♦♦

!BATCH PLOTIT.APLGRAF NY UM=AS605170, NAME=BOEP, FID=: CHAPE

ID=000A SUBMITTED 16:01 APR 04, 78
WAITING: 15310 RUN

Appendix B3

FINITE ELEMENT GRID GENERATOR PROGRAM

B3.1 General Description

The program generates a triangular finite element mesh on a boundary contour which is given as input.

Nodal coordinates, element node numbers and boundary node numbers are calculated and may be printed out, and/or recorded onto a magnetic disc then a plot of the mesh can be produced.

The structure for which a grid is to be generated must have a boundary which can be described by a lower boundary line (also called base-line), two side lines and an upper boundary line. (See Figure B3.1). The upper and lower boundaries may be straight or circular and the sides may be composed of a series of straight or circular sections with the same number of sections on each side.

Initially the structure is divided by a series of major partition lines extending between the two sides at the dividing points between the sections of the sides (see Figure B3.1).

Each major partition is then divided into a series of quadrilaterals by two sets of grid lines. One set extends between the lower and upper major partition lines. The curvature and slope of these grid lines are determined from a weighted average of the corresponding values for the two side sections (see Figure B3.2(a)). The other set of grid lines extend between the two sides and their curvature and slope are determined in a similar manner from the values for the upper and lower major partition lines (see Figure B3.2(b)).

Schematic of boundary lines

Figure B3.1

Sequence of mesh generation

(a)

(b)

(c)

The resulting quadrilaterals are diagonally divided to produce the triangular elements (see Figure B3.2(c)).

Curvature of any boundary line is specified by giving the coordinates of the two end points and any other point on the line. A circle is then fitted to these three points.

The spacing of nodal points on these lines can be of two types. The first method (arithmetic spacing) produces spaces between consecutive nodes increasing and decreasing in an arithmetic series

$$l_1, l_2, l_3, \dots, l_i, \dots, l_n$$

$$Q, Q+D, Q+2d, Q+(i-1)d, Q+(n-1)d$$

$$1, 2, 3, \dots, i, \dots, n$$

where $l_1, l_2, \dots, l_i$ is the length of the i th element;

Q is the length of the first element,

n is the number of elements,

d is a constant.

For this spacing the ratio between the lengths of the first (l_1) and the last element (l_n) on the line must be specified.

The alternative method (parabolic spacing) produces spaces which increase (or decrease) to the middle of the line, then decrease (or increase) both in arithmetic series. For this spacing, the ratio between the length of the middle element (or elements) and either the first or last element (both identical) must be given.

B3.2 The NEWMESH Module

The module NEWMESH is used to create the data file for the grid generator program. The user answers interactively to the questions appearing at the terminal.

The first set of questions specifies the dimensions of the page in which the plot will be fitted (if plotting is desired), the title of the job and the output options.

After this, the next set of questions are used to build-up the base-line (first data set) which are the number of elements along the base-line, the spacing of the nodes, the curvature of the base-line and the coordinates of the two nodes at the ends of the line.

The second data set completes the structure up to the first major partition line, the third data set up to the second major partition line, etc., until the structure is complete (see Figure B3.3).

Each subsequent data set specify the grid up to the next major partition line giving information on the curvature and spacing of nodes on the two sides up to the next major partition line and of the curvature and spacing on the next partition line itself. The spacing used on the partition line bounding this section on the lower side is the same as that used for the previous section (or the base line for the first section). In addition, the number of triangles to be gained or lost on the right and left sides (between each layer of triangles) must be specified.

It is possible to identify the output device where the plot of the mesh has to be directed by answering the question DEVICE ...? The different options are given by depressing the RETURN key but the most common are TVDISC, TVNORM and HDPLOT. If the plot is to be shown on a CRT the location of the device must be given (A = Administration Building, M = Mackenzie building). If a hardcopy is desired (HDPLOT), the name of the file used to identify the tape has to be given and the following command must be entered:

```
! BATCH PLOTIT.APLGRAPH NUM=acc. number, NAME=acc. name, FID=file name
```

Partitions and corresponding data sets

Figure B3.3

B3.2.1 Printout.

The program prints out the following information (see example):

- Title of the job
- Date of execution of the job
- Total number of nodes in the mesh
- ~~Total~~ number of elements in the mesh
- (optional) Coordinates of the nodes (node number, x-coordinate, y-coordinate)
- (optional) Node numbers of the elements (element number, node number at the corners)
- Dimensions of the plot (XMAX, YMAX)
- Scale factor of the plot (= actual size/plot size)

B3.3 The DEVXEQ Module

If the mesh was checked on the CRT and no modifications are necessary, it is possible to obtain a hardcopy without answering all the questions again. The module DEVXEQ is used for this purpose to change the DEVICE and input the tape file name with the data for the Calcomp plotter.

B3.4 The MODMESH Module

If modifications are required after execution of the NEWMESH module, the user will find useful the MODMESH module. The program follows a hierarchical approach, with a one digit codification (Table B3.1).

Table B3.1

<u>First Level Modifications</u>	<u>First Level Code</u>
Options	1
Base line specifications	2
Base line coordinates	3
Partition specifications	4
Partition coordinates	5

After the first level modification has been selected with the appropriate code, the program enquires for a second level modification according to the selected code (Table B3.2).

The program updates the input/output data file for the finite element program. Then the user specifies if another modification is required.

When the mesh has reached the desired specifications, the MODMESH module proceeds in the same manner as the NEWMESH module.

B3.5 The CHANGEMESH Module

If a finite element analysis was already run with a previous mesh and therefore the option, material variables, specified displacements and external forces have already been defined and a new mesh has to be input, the CHANGEMESH may be used. This module will create a new mesh and it will build a file with the previously defined finite element data.

Table B3.2

Code for Second Level Modification of the Mesh

<u>First level Code</u>	<u>Second level code</u>	<u>Second level modifications</u>
1. (Options)	1	Plot dimensions
	2	Title
	3	Output options
2. (Base line Specifications)	1	Number of elements
	2	Type of spacing
	3	Curvature
	4	Ratio of spacing
3. (Base line coordinates)	1	Left hand corner
	2	Right hand corner
	3	Curvature point
4. (Partition specifications)	NP	Partition Number (Max. = 50)
	1	Number of layers
	2	Number triangles gain or lost
	3	Spacing at centre, left or right
	4	Curvature at centre, left or right
	5	Ratio of spacing centre, left or right
5. (Partition coordinates)	NP	Partition number (Max. = 50)
	1	Coordinates of left hand corner
	2	Coordinates of right hand corner
	3	Coordinates of curvature point at center
	4	Coordinates of curvature point at left
	5	Coordinates of curvature point at right

* Appendix B4

Internal Pressure		Number of Elements		Radial Support		Axial Load	
P_i		for application P_i (see figure 4)		pressure P_e		P_a	
ksi (kbar)	Code		Code	ksi	Code	ksi	Code
294 (20)	A			0	0	0	0
588 (40)	B	2/6	2	-100	X	-50	A
882 (60)	C	4/6	4	-150	Y		

All cylinders have an internal diameter $d_i = 1$ inch

Code for loading conditions:

Example: B2X0

$P_i = 588$ ksi, Code B

2 elements for application of P_i , Code 2

$P_e = 100$ ksi, code X

$P_a = 0$ psi, Code 0

Appendix C1

WEIBULL PROBABILITY PAPER

Weibull paper is a special kind of probability paper that is very convenient for analyzing a wide variety of industrial data, particularly that involving fracture.

Weibull found empirically that the percentage (S) of a group of specimens that would survive a tensile stress σ was given by:

$$S = \exp \left[- \left(\frac{\sigma}{\sigma_0} \right)^m \right] \quad (C1.1)$$

where σ_0 is a constant for the test series called characteristic stress,

m is a constant for the test series called the Weibull slope.

Working with the fraction failed (F) we have:

$$F = 1 - S = 1 - \exp \left[- \left(\frac{\sigma}{\sigma_0} \right)^m \right] \quad (C1.2)$$

$$\text{or} \quad \exp \left(\frac{\sigma}{\sigma_0} \right)^m = \frac{1}{1-F} \quad (C1.3)$$

and therefore:

$$\ln \ln \left(\frac{1}{1-F} \right) = m \ln \sigma - m \ln \sigma_0 \quad (C1.4)$$

Weibull paper is ruled with $\ln \ln \frac{1}{1-F}$ as the ordinate and $\ln \sigma$ as the abscissa (Figure C1.1).

From equation C1.4 it is evident that when $\sigma = \sigma_0$, $F = 63.2$.

In using Weibull paper the data are first ordered in increasing fracture load and the values of P determined for each rank order.

If a constant frequency distribution is assumed, each fraction

FORM OF THE WEIBULL PLOT

Figure CII

F will assume the value $\frac{i}{n}$, where i is test specimen number, n is total number of specimens. In reality a group of test results are generally normally distributed and values of F based on this type of distribution are termed median ranks. Tables of median ranks for different sample sizes (number of tests) are available in the literature and some are reported in this Appendix as Tables C1.1, C1.2, C1.3.

It may be shown that the j th rank (F_j) for a case where there were n observations will be

$$F_j = \frac{j-0.3}{n+0.4} \quad (C1.5)$$

For example, with a 5 test sample ($n = 5$), then the first median rank ($j=1$) would be:

$$F_1 = \frac{1-0.3}{5+0.4} = \frac{0.7}{5.4} = 12.95 \quad (C1.6)$$

and the second median rank would be:

$$F_2 = \frac{2-0.3}{5+0.4} = \frac{1.7}{5.4} = 31.38 \quad (C1.7)$$

The Weibull procedure is an effective way of estimating the distribution of results that would be obtained in a very large test series from a relatively small sample.

F and P may be plotted on Weibull paper (Figure C1.2). Now, P_{10} is the load P at which only 10% of a large test series will have broken. For the example $P_{10} = 53$ lb. Using a small sample size, an uncertainty in the observations has to be expected. A convenient way of visually displaying the uncertainty is to provide confidence limits on the Weibull plot.

TABLE C1.1Median Ranks (F_{50})

Rank Order	Sample Size		
	<u>5</u>	<u>10</u>	<u>20</u>
1	12.95	6.70	3.41
2	31.38	16.23	8.25
3	50.00	25.86	13.15
4	68.62	35.51	18.06
5	87.06	45.17	22.97
6		54.83	27.88
7		64.49	32.80
8		74.14	37.71
9		83.77	42.63
10		93.30	47.54
11			52.46
12			57.37
13			62.29
14			67.21
15			72.12
16			77.03
17			81.95
18			86.85
19			91.75
20			96.59

TABLE C1.2

5% Ranks F_5

Rank Order	Sample Size		
	<u>5</u>	<u>10</u>	<u>20</u>
1	1.02	0.51	0.26
2	7.64	3.68	1.81
3	18.93	8.73	4.22
4	34.26	15.00	7.14
5	54.93	22.24	10.41
6		30.35	13.96
7		39.34	17.73
8		49.31	21.71
9		60.58	25.87
10		74.11	30.22
11			34.69
12			39.36
13			44.20
14			49.22
15			54.44
16			59.90
17			65.63
18			71.74
19			78.40
20			86.09

TABLE C1.3

95% Ranks F_{95}

Rank Order	Sample Size		
	<u>5</u>	<u>10</u>	<u>20</u>
1	45.07	25.89	13.91
2	65.74	39.42	21.61
3	81.08	50.69	28.26
4	92.36	60.66	34.37
5	98.98	69.65	40.10
6		77.76	45.56
7		85.00	50.78
8		91.27	55.80
9		96.32	60.64
10		99.49	65.31
11			69.81
12			74.14
13			78.29
14			82.27
15			86.05
16			89.59
17			92.87
18			95.78
19			98.19
20			99.74

A WEIBULL PLOT FOR THE ILLUSTRATIVE EXAMPLE HAVING A SAMPLE OF 5. THE BROKEN LINES ARE CONFIDENCE LIMITS.

Assigning the median rank value to each load, means that the true value will be above this value 50% of the time and below this value 50% of the time. The 5% rank is a lower value assigned to each load, so determined that the probability of the true value lying below this value is 5%. Similarly, the 95% rank is a higher value so determined that the probability of the true value lying below this value is 95%. Values of F_5 and F_{95} are given in the table C1.2, C1.3, for sample sizes of 5, 10 and 20.

When the observed values of the data are plotted against F_5 and F_{95} instead of F_{50} the dotted curves shown in Figure C1.2 are obtained.

Increasing the number of specimens, causes the band to decrease.

Appendix C2

DETERMINATION OF K STRESS GRADIENT FACTOR

By stress analysis, the stress configuration for some geometric and loading conditions can be determined. (+)

C2.1. Three Point Bend Specimen

In this case equation (5.10) is evaluated only in the tensile region shown shaded in Figure C2.1. The tensile stress at any point is (Figure C2.1).

$$\sigma = \frac{My}{I} = \frac{\left(\frac{Px}{2}\right) \cdot y}{\frac{1}{12}bh^3} = \frac{6Pxy}{bh^3} \quad (C2.1)$$

Substituting into equation (3.10):

$$\begin{aligned} R_{B3} &= 2 \int_0^{\ell/2} \int_0^{h/2} \left(\frac{6Pxy}{bh^3\sigma_o} \right)^m b dx dy = \quad (C2.2) \\ &= \frac{1}{(m+1)^2} \cdot \frac{2b\ell h}{4} \left(\frac{6P\ell h}{4bh^3\sigma_o} \right)^m = \\ &= \frac{1}{(m+1)^2} \cdot V \left(\frac{\sigma_{\max}}{\sigma_o} \right)^m \end{aligned}$$

$$\text{where } V = \frac{b\ell h}{2}$$

$$\sigma_{\max} = \frac{3P\ell}{2bh^2}$$

(+) A detailed description of Weibull approach and determination of K factors for different bending specimens may be found in [C2.1].

Three point loading transverse rupture test and resultant bending moment (M).

Figure C2.1

Four point loading transverse rupture test and resultant bending moment (M).

Therefore $K = \frac{1}{(m+1)^2}$ (C2.3)

or $K = \frac{1}{2(m+1)^2}$ if the volume V is the total volume of the specimen.

C2.2 Four-Points Bend Specimen

In this case the tensile stress is constant between the inner supports and the bending moment is (Figure C2.2):

$$\sigma = \frac{My}{I} = \frac{Pay}{\frac{1}{12}bh^3} = \frac{12Pay}{bh^3} \quad (C2.4)$$

Substituting into equation (5.10):

$$\begin{aligned} R_{B4} &= 2 \int_a^{a+l/2} \int_0^{h/2} \left(\frac{12Pay}{bh^3 \sigma_o} \right)^m b dx dy = \\ &= \frac{1}{m+1} \frac{2b\ell h}{4} \left(\frac{12 P a h}{2bh^3 \sigma_o} \right)^m = \\ &= \frac{1}{m+1} V \left(\frac{\sigma_{\max}}{\sigma_o} \right)^m \end{aligned}$$

where: $V = \frac{b\ell h}{2}$ $\sigma_{\max} = \frac{6P\ell}{bh^2}$

Therefore $K = \frac{1}{m+1}$

or $K = \frac{1}{2(m+1)}$

if the volume V is the total volume in between the supports.

C2.3 Hollow Cylinder with Internal Pressure

The tensile stress is circumferential:

$$\sigma_{\theta} = \frac{p r^2}{1 - \rho^2} \left(1 + \frac{R_2^2}{r^2} \right) = \frac{\sigma_{\max}}{1 + \frac{R_2^2}{R_1^2}} \left(1 + \frac{R_2^2}{r^2} \right) \quad (C2.6)$$

where $\rho = \frac{R_1}{R_2}$ (R_1 = internal radius
 R_2 = external radius)

l = cylinder length

$\sigma_{\max}$ = maximum of σ_{θ} at $r = R_1$

Equation (5.10) becomes:

$$R_c = \frac{2\pi l \left(\frac{\sigma_{\max}}{\sigma_0} \right)^m \rho^{2m}}{(1 + \rho^2)^m} \int_{R_1}^{R_2} \left(1 + \frac{R_2^2}{r^2} \right)^m r dr \quad (C2.7)$$

For a given m , the integrand can be expanded binomially¹ and integrated term by term.

For example for $m = 2$

$$R_c = \frac{\rho^4}{(1 + \rho^2)^2} \left[1 - \frac{4}{(1 - \rho^2)} \ln \rho + \frac{1}{\rho^2} \right] v \left(\frac{\sigma_{\max}}{\sigma} \right)^2 \quad (C2.8)$$

therefore:

$$K = \frac{\rho^4}{(1 + \rho^2)^2} \left[1 - \frac{4}{1 - \rho^2} \ln \rho + \frac{1}{\rho^2} \right] \quad (C2.9)$$

¹ The integral can be written as:

$$\int_{R_1}^{R_2} \left(1 + \frac{R_2^2}{r^2} \right)^m r dr = \int_{\rho^2}^1 \left(1 + \frac{1}{q} \right)^m dq \quad \text{with } q = \frac{r^2}{R_2^2}$$

and

$$\left(1 + \frac{1}{q} \right)^m = 1 + m q^{-1} + \frac{m(m-1)}{2} q^{-2} \dots$$

Appendix C3

GENERAL STATISTICAL APPROACH TO BRITTLE FRACTURE

Weibull was the first to establish a mathematical basis for the failure of brittle materials¹. In his theory he used an empirical formula which, with remarkable engineering intuition, fits most of the very common material. It is important to note that the parameter m , (the Weibull modulus, see paragraph 5.2 and 5.3) which is a material characteristic, has no real relationship to the micro- or macrostructure of the material.

In very recent years, other statistical approaches have been developed. Some have a sound basis in statistics while others have more physical claims.

An important development of the Weibull approach, is the consideration of a polyaxial state of stress. In the Weibull formula (5.9) the applied stress σ has to be interpreted as uniaxial tension or an effective stress from a biaxial or triaxial state of stress. This will be considered in a separate paragraph because each principal stress has a different effect on the failure probability.

The Weibull approach of 1939 has been repeated following the theory of extreme value statistics which states that "in a brittle material with no capacity to prevent high local stress concentrations by plastic flow, the worst flaw generating the highest stress concentration governs the fracture stress of the part". The total behaviour is thus governed by the single worst flaw, hence the name "extreme value statistics". [5.4]. It was found that the exponent m , which was considered as an empirical constant, was related to the properties of the flaw size

¹ Leonardo da Vinci, was the first who noted in a crude experiment that long iron rods are, on the average, weaker than short ones, without giving an explanation.

distribution of a material [C3.1]. A theory which predicts the stress necessary to propagate an inclined crack, as shown in Figure C3.1, has also been developed [C3.2, C3.3]. The initial crack growth occurs when the strain energy density of a body attains a critical value. Using this concept, the strength, σ , of a brittle material is given by:

$$\sigma^2 a = L(\beta, \nu), \quad (C3.1)$$

where a = semi-crack length,
 β = crack angle, and
 ν = Poisson's ratio.

$L(\beta, \nu)$ is a function of β and ν . For $\nu = 0.25$ (brittle materials) equation (C3.1) may be rewritten as

$$\sigma^2 a = 1/2 K_{IC}^2 \beta^{-1} \quad (C3.2)$$

where K_{IC} is the critical stress intensity factor of the material.

If $f(a)$ is the probability density of the semi crack length and, assuming that any crack angle is equally likely, the probability density of β is $\frac{2}{\pi}$ for $0 < \beta < \frac{\pi}{2}$.

Therefore, the probability of failure $F(\sigma)$, at a stress, σ , for one crack is given by:

$$F(\sigma) = \iint_{\frac{L(\beta, \nu)}{a} < \sigma^2} f(a) da d\beta, \quad (C3.3)$$

in which σ , β
 and a are
 related by

$$0 < \beta < \frac{\pi}{2}$$

From equations (C3.1) and (C3.2) we have

$$L(\beta, \nu) = 1/2 K_{IC}^2 \beta^{-1} \quad (C3.4)$$

Experimental results [C3.4] suggest that $f(a)$ can be closely fitted by the following expression:

$$f(a) = \frac{c^{n-1}}{(n-2)!} a^{-n} e^{-c/a} \quad (C3.5)$$

where c is a scaling parameter,

and n is the rate at which density tends to zero.

Expression (C3.5) was obtained from some results on glass. The same approach can be taken for other materials of brittle nature

in which case equation (C3.3) will take the form:

$$F(\sigma) = \iint_{\pi} \frac{c^{n-1} a^{-n} e^{-c/a}}{(n-2)!} da d\beta \quad (C3.6)$$

$$\frac{K_{IC}^2}{2\beta a} \leq \sigma^2$$

$$0 \leq \beta \leq \frac{\pi}{2}$$

The limits of integration are:

$$\frac{K_{IC}^2}{\pi \sigma} \leq a \leq \infty$$

$$\frac{K_{IC}^2}{2a\sigma^2} \leq \beta \leq \frac{\pi}{2}$$

and after one integration the expression becomes

$$F(\sigma) = \int_{\frac{K_{IC}}{\pi\sigma^2}}^{\infty} \left[1 - \frac{K_{IC}^2}{\pi a \sigma^2} \right] \frac{c^{n-1} a^{-n} e^{-c/a}}{(n-2)!} da. \quad (C3.7)$$

For N cracks, the probability of failure, P_f , is given by

$P_f = 1 - \text{probability of survival of all } N \text{ cracks and therefore:}$

$$P_f = 1 - [1 - F(\sigma)]^N. \quad (C3.8)$$

P_f can be calculated for different values of σ and the mean stress $\bar{\sigma}$.

is given by:

$$\bar{\sigma} = \int_0^1 \sigma dP_f = \int_0^1 (1 - P_f) d\sigma \quad (C3.9)$$

Assuming N large and σ small it is found [C3.1] that:

$$P_f \cong 1 - \exp \left[-N \frac{c^{n-1}}{n!} \left(\frac{\pi \sigma^2}{K_{IC}^2} \right)^{n-1} \right] \quad (C3.10)$$

which can be written as:

$$P_f = 1 - \exp \left[-k_1 N (\sigma)^{2n-2} \right] \quad (C3.11)$$

where k_1 is a constant for a given material.

In the Weibull analysis, the expression (5.9) can be written as:

$$P_f = 1 - \exp \left[-V \left(\frac{\sigma}{\sigma_0} \right)^m \right] \quad (C3.12)$$

Since the number of cracks is proportional to the volume of material, the above expression can be rewritten as:

$$P_f = 1 - \exp \left[-k_2 N (\sigma)^m \right] \quad (C3.13)$$

Therefore, m and n are related by the equation:

$$m = 2n - 2$$

(k_2 in equation (C3.13) is a material constant)

The value of n can be obtained by fitting experimental data to the curve given in equation (C3.5).

Therefore, the Weibull modulus, m , can now be related to the properties of flaw size distribution in a material. For any particular material, this distribution may be obtained by non-destructive or other methods.

For a large number of cracks, N , the Weibull analysis is a good approximation to the general theory described above. The lowest limit for N (for a given value of n or m) for which the Weibull analysis is a good approximation is given in Table C3.1).

Table C3.1

Lowest values of N for which Weibull analysis is
a good approximation

n	$m = 2n - 2$	N
2	2	3
3	4	28
4	6	60
6	10	110

An inclined crack under a uniform tensile stress

Figure C3.1

Appendix C4

STATISTICAL THEORIES FOR POLYAXIAL STATE OF STRESS

If an element is subjected to a biaxial state of tension, the analytical treatment given in the Appendix C3 for uniaxial tension can be extended to include this situation [C4.1]. The theory is based on the crack growth of a body containing an inclined crack using strain energy density concepts. The initial crack growth is determined when the strain energy density of a body attains a minimum value.

If σ_1 and σ_2 are the biaxial stresses (see Figure C4.1), β is the crack angle and ν is the Poisson's ratio, then the strength σ_2 can be expressed as:

$$\sigma_2^2 a = L(\beta, R, \nu) \quad (C4.1)$$

where $R = \sigma_1/\sigma_2$ and a is the semicrack length.

For a particular value of ν (i.e. 0.25 for brittle materials) equation (C4.1) may be re-written in the terms of the critical stress intensity factor, K_{IC} , as

$$\sigma_2^2 a = \frac{2}{\pi} K_{IC}^2 U(\beta, R) \quad (C4.2)$$

Equation (C4.2) can be expressed as a family of curves as in Figure (C4.2) when σ_1 and σ_2 are both tensile.

It is well known experimentally that in the case of uniaxial tensile loading, the subsequent direction of crack growth, for an

An inclined crack in a biaxial tensile stress system.

Figure C4.1

Figure C4.2

inclined crack, always turns away from the initial direction of propagation and runs normal to the applied load, thus producing a catastrophic failure.

In compression, the initial crack does not propagate catastrophically and grows only slowly to a certain length, as the load is gradually increased [C4.3]. This can be explained since the subsequent crack path gradually aligns itself with the axis of the compression loadings thus requiring a very high stress to propagate it further. It follows that in compression, failure of one flaw does not lead to total failure. The statistical approach to tensile biaxial system follows step by step the approach given in the previous paragraph on the uniaxial case and can be found in Reference [C4.1]. The same results are obtained i.e.

$$m = 2n - 2 \quad (C4.3)$$

where m is the Weibull modulus and n is the parameter of crack size distribution.

The affect of a biaxial stress on the fracture strength of a brittle body, containing randomly orientated flaws, can be explained with respect to the uniaxial stress effects by defining a suitable term, called the "strength reduction factor" (SRF) as equal to the ratio of the mean failure stresses (higher stress in the case of biaxial loading) in biaxial loading to uniaxial loading conditions. It was shown that, for a given material, the SRF remains fairly constant over the range of cracks between 40 and 1000. Therefore the SRF is independent of the volume, and thus, the average values of SRF for different biaxial loading systems can be tabulated (see Table C4.1).

Table C4.1

Average values of SRF for different biaxial
loading systems.

n	m	SRF (%)		
		R = 1.0	R = 0.6	R = 0.4
2	2	71.7	86.9	93.6
4	6	82.1	95.5	98.6
6	10	85.7	97.3	99.3

SRF = Strength Reduction Factor = $\frac{\sigma_{\max} \text{ (in biaxial tension)}}{\sigma \text{ (in uniaxial tension)}}$

$$R = \sigma_1 / \sigma_2$$

m = Weibull modulus

$$m = 2n - 2$$

These values contained in this table are very useful for a design engineer; the failure stresses in uniaxial conditions can be readily obtained from handbooks or experimentally and converted to biaxial conditions by the use of Table C4.1.

Weibull, to extend the uniaxial stress state treatment [5.1], used the two parameter distribution ($\sigma_u=0$) and wrote the cumulative distribution of failure probability as:

$$F(\sigma) = 1 - \exp[-B] = 1 - \exp\left[-\int_V \left(K \int_{\text{unit sphere}} \sigma_n^m dA\right) dV\right] \quad (C4.4)$$

where, by comparison with the uniaxial case, $K = (2m+1)/2\pi\sigma_0^m$

and the normal stress σ_n at each point (see Figure C4.3) is, for biaxial stress state,

$$\sigma_n = \cos^2\phi (\sigma_1 \cos^2\psi + \sigma_2 \sin^2\psi), \quad (C4.6)$$

and for triaxial stress state,

$$\sigma_n = \cos^2\phi (\sigma_1 \cos^2\psi + \sigma_2 \sin^2\psi) + \sigma_3 \sin^2\phi \quad (C4.7)$$

and $dA = \cos\phi \, d\psi \, d\phi$ and integration is carried out over the range of angles for which σ_n is tensile.

The risk of rupture B has been evaluated for a biaxial strength test [C4.2] consisting of a disc supported on the edge onto which a uniform hydrostatic pressure is acting across the entire surface (see paragraph 2.4.5). It can be written as:

$$B = 2\pi a^2 c(m) D(m) \left(\frac{\sigma_c}{\sigma_0}\right)^m, \quad (C4.8)$$

where a = test area radius,

σ_c = central stress,

σ_0 = scaling factor.

$$C(m) = (2n+1) \int_0^{\pi/2} \cos^{2m+1} \phi d\phi$$

$$D(m) = \frac{1}{\pi} \int_0^1 \left\{ \int_{-\pi/2}^{\pi/2} \left[1 - 1/2 x^2 (\alpha + \beta \cos 2\psi) \right]^m d\psi \right\} x dx$$

being

$$x = \frac{r}{a}$$

$$\alpha = 1 + \frac{1+3\nu}{3+\nu}$$

$$\beta = 1 - \frac{1-3\nu}{3+\nu}$$

Both $C(m)$ (when m is not an integral value) and $D(m)$ has to be evaluated numerically. Substituting equation (C4.8) into equation (C4.4) the failure probability curve is obtained from which the parameters $\sigma_{0,m}$ (and σ_u if it has been taken into account in equation (C4.8)) may be determined [C4.2].

The same approach and the same kind of test was used to determine the fracture initiation location in a disc under biaxial tension [C4.4].

For a specimen stressed principally at its surface, Weibull predicts (see paragraph 5.2):

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{A_2}{A_1} \right)^{\frac{1}{m}} \quad (C4.9)$$

where σ_1 is the fracture strength of a uniformly stressed body of area A_1 , σ_2 is the fracture strength of a uniformly stressed body of area A_2 , and for the purely elastic case:

$$\frac{\epsilon_1}{\epsilon_2} = \left(\frac{A_2}{A_1} \right)^{\frac{1}{m}} \quad (C4.10)$$

Geometric variables used to describe location on a unit sphere.

Figure C4.3

Equation (C4.10) can be written as:

$$m \log \epsilon_1 + \log A_1 = m \log \epsilon_2 + \log A_2 \quad (C4.11)$$

where the left hand side refers to specimen 1 and the right hand side to specimen 2 both having same probability of fracture.

The parameter:

$$P_f = m \log \epsilon + \log A \quad (C4.12)$$

is therefore a measure of the fracture probability of a body which is stressed principally at its surface and which fails due to the existence of surface flaws. Knowing the strain value ϵ along the radius, the parameter P_f can be calculated and plotted for various value of m as in Figure C4.5. (P_f normalized at $r = 0$ and assuming the strain distribution near fracture given in Figure C4.4). It can be seen that the curve all show a maximum at a position away from the centre of the disc ($r = 0$) and this is a function of the value of m . This trend was confirmed experimentally [C4.4.].

We have extended the previous approach to different specimen configuration and to include the volume effect instead. This is expressed as:

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{V_2}{V_1} \right)^{\frac{1}{m}} \quad (C4.13)$$

where again 1 and 2 stay for specimen 1 and specimen 2. Therefore the relation (C4.13) can be rewritten as:

$$P_f = m \log \sigma_E + \log V. \quad (C4.14)$$

where σ_E is the equivalent stress to be consider case by case.

Variation of strain with radius in biaxial disc test for two levels of strain.

Figure C4.4

Variation of fracture parameter P_f with radius.

Figure C4.5

If we consider a right circular cylinder of internal radius a_0 , external b_0 and height h , and we subdivide the cylinder into smaller cylinders of internal radius r and thickness dr , then

$$V = \pi h dr (2r + dr) \quad (C4.15)$$

If an internal pressure p is assumed, the stresses which will contribute to the expansion of a crack are the tangential tensile stresses σ_θ , therefore:

$$\sigma_E = \sigma_\theta \frac{p \left[\frac{b_0^2}{r^2} + 1 \right]}{\left[\frac{b_0^2}{a_0^2} - 1 \right]} \quad (C4.16)$$

and again it can be calculated as a function of the radius r for different values of m .

If the volume is reduced to a small finite element with equivalent stress σ_E , the parameter P_f can be calculated at a desired number of discrete points of the specimen under consideration and the fracture probability at a specified location can be calculated. This approach was used in the study of a high pressure chamber (see Chapters 4 and 5).

Since rather tedious calculations are required for each polyaxial stress state, in practical structures that involve continually varying stress states, there is a natural tendency to use approximations. One simple approximation [5.9] is to assume that:

$$P_s(\sigma_1, \sigma_2, \sigma_3) = P_s(\sigma_1) P_s(\sigma_2) P_s(\sigma_3) \quad (C4.17)$$

Where P_s is the probability of survival, and $\sigma_1, \sigma_2, \sigma_3$ are the principal

stresses. The equation (C4.17) would be strictly correct if all the cracks were oriented with their planes normal to any one of the principal (tensile) stresses. As it was observed in [C4.5], generally, there will be some cracks inclined to the principal axes that will be fractured by combined stresses even though capable of surviving any of the principal stresses applied individually. There are also cases in which a given crack would be fractured by more than one of the principal stresses even acting alone.

If we consider the case of equibiaxial tension, equation (C4.17) reduces to:

$$P_S(\sigma, \sigma) = [P_S(\sigma, 0)]^2 \quad (C4.18)$$

or

$$\frac{\ln P_S(\sigma, \sigma)}{\ln P_S(\sigma, 0)} = 2 \quad (C4.19)$$

This implies that the probability of failure in equibiaxial tension is twice that for uniaxial tension [C4.5]. It can be shown that if the weakest link theory is used and the polyaxial stress statistics are derived in a straightforward manner from the basic assumptions, the probability for equibiaxial tension becomes. (for integer values of m).

$$P_f(\sigma, \sigma) = 1 - \exp \left[\frac{-k(m)! \sigma^m}{(m-0.5)(m-1.5) \dots 0.5} \right] \quad (C4.20)$$

and equation (C4.19) becomes:

$$\frac{\ln P_S(\sigma, \sigma)}{\ln P_S(\sigma, 0)} = \frac{m!}{(m-0.5)(m-1.5) \dots 0.5} \quad (C4.21)$$

In Figure (C4.6) equations (C4.19) and (C4.21) are compared and it can be seen that equation (C4.19) underestimates small equibiaxial failure probability for values of $m > 1$.

In the general case of triaxial tension characterized by principal stresses σ_1 , σ_2 and σ_3 , where $\sigma_1 > \sigma_2 > \sigma_3 > 0$, the probability of fracture, given by Weibull theory or Batdorf and Crose [C4.6], takes the form

$$P_f = 1 - \exp \left[- \frac{k}{4\pi} \int_0^{2\pi} d\phi \int_0^\pi \left(\sigma_1 \cos^2 \phi \sin^2 \theta + \sigma_2 \sin^2 \phi \sin^2 \theta + \sigma_3 \cos^2 \theta \right)^m \sin \theta d\theta \right] \quad (C4.22)$$

where θ and ϕ are the polar and azimuthal angles, respectively.

As noted by Shur [C4.7], the integral in equation (C4.22) yields:

$$\int_0^{2\pi} d\phi \int_0^\pi \left(\right)^m \sin \theta d\theta = 4\pi \sigma_1^m \frac{(m!)^2}{(2m)!} \sum_{i,j,k} \frac{(2i)!(2j)!(2k)!}{(i!j!k!)^2} s^i t^k,$$

where i, j, k are any positive integers such that they satisfy the inequalities $0 \leq i, j, k \leq m$ and $i + j + k = m$, and where

$$s = \sigma_2/\sigma_1, \quad (C4.23(a))$$

$$t = \sigma_3/\sigma_1. \quad (C4.23(b))$$

Therefore, when a computing machine is used in conjunction with finite elements, an accurate Weibull-type treatment of structural problem that involves biaxial or triaxial stresses is not difficult, provided that closed-form solutions to the integral in equation (C4.22) is used. This solution is given in equation (C4.23).

If a closed-form analytical stress distribution is known, a direct solution without the use of the finite elements can be

Ratio of $\ln$ of probability of survival in a equibiaxial tension to that in uniaxial tension (equal to $P(\sigma, \sigma) / P(\sigma, 0)$ when $P_f \ll 1$).

Figure C4.6

made. Vardar and Finnie [C4.8] have used the known stress distribution in the disk test [C3.3] to evaluate the parameters m and σ_0 and compared the experimental results between three-point bending tests and Brazilian disk test. The percentage error between the predicted mean stress and the observed mean stress at fracture was of the order of 4-5%.

Appendix C5

STRENGTH-PROBABILITY-TIME (STP) RELATIONSHIP

Long term operation may not appear to be a problem in a material like ceramic which is extremely corrosion and erosion resistant but in fact the material does age in another and perhaps more catastrophic way.

Ceramic is prone to a phenomenon known as a static fatigue or delayed fracture. This is a fatigue process associated with a stationary load rather than a cyclic one. Inside the material very small sub-critical cracks are growing steadily with time under load until they become critical. This means that the component can fail at a stress well below its fracture value if it is loaded long enough. The change in the probability of survival with time is quite important in the design of the new turbines [1.3], [1.4], [C5.1] as well as in the structural application of ceramic material.

In recent years a statistical treatment of this problem has been developed and to a first approximation it is possible to incorporate this time dependence in the Weibull analysis [C5.2].

The strength (σ_f) of ceramics is controlled generally by the stress needed to propagate small micro-structural flaws, which is according to a modified Griffith equation,

$$\sigma_f = \frac{1}{Y} \left(\frac{2E\gamma_i}{a} \right)^{1/2} \quad (C5.1)$$

where Y is a geometrical constant, E Young's modulus, γ_i an effective surface energy, and a , a flaw size.

Rearranging equation, (C5.1) and defining a critical stress intensity factor K_{IC} as $(2 E \gamma_f)^{1/2}$, equation (C5.1) becomes

$$K_{IC} = \gamma \sigma_f a_c^{1/2} \quad (C5.2)$$

where the subscript c refers to the critical condition. Under constant σ , of interest in a delayed fracture test, the time to failure, τ , is given by

$$\tau = \int_{a_i}^{a_c} \frac{da}{V} \quad (C5.3)$$

where a_i is the initial crack or flaw size, a_c is the critical flaw size and V is the crack velocity which is related to the stress intensity factor by

$$V = c K_I^n, \quad (C5.4)$$

where c , and n are constants. The relationship (C5.4) can be experimentally determined. The integral in equation (C5.3) can be solved substituting equation (C5.2) and (C5.4) and, after some manipulations and simplifications we obtain

$$\tau = \frac{2 K_I^{2-n}}{\sigma^2 \gamma^2 C(n-2)} \quad (C5.5)$$

Combining equations (C5.5) and C5.2) for a particular specimen with a flaw size a_j (unknown until tested) the time to failure is

$$\tau = \frac{2 \gamma^{2-n} a_j^{(2-n)/2}}{\sigma^2 \gamma^2 C(n-2)} = \frac{c_1 a_j^{(2-n)/2}}{\sigma^n} \quad (C5.6)$$

Thus for specimens with the same initial flaw size, and the same probability of instant failure,

$$\tau \sigma^n = c_2 \quad (C5.7)$$

A family of lines can thus be constructed on a Weibull strength/probability-of-survival graph corresponding to increasing failure times as in Figure (C5.1). Note that these are equi-spread for equal logarithmic increases in failure time. An individual specimen stressed at σ_j and failure in time τ_j is, therefore, equivalent to a specimen failing at stress σ_0 in a reference time τ_0 , as indicated in Figure C5.1 according to

$$\left(\frac{\sigma_0}{\sigma_j}\right)^n = \frac{\tau_j}{\tau_0}, \quad (C5.7)$$

Method of relating delayed fracture data to "instantaneous" strength values with SPT diagram. Specimen failing at stress σ_j in time τ_j would have failed at stress σ_0 in time τ_0 .

Figure C5.1

REFERENCES

- 1.1 W.H. Dukes, Handbook of brittle material design technology, AGARDograph n. 152-71.
- 1.2 R. Venter, S. Critchley, M.C. deMalherbe, The evaluation of Tungsten Carbide Properties as applied to High Pressure Equipment, Materials Science and Engineering, 19, 201, 1975.
- 1.3 A. Paluszny, Design with Brittle Materials, Material Science and Engineering, 15, 39-50, 1974.
- 1.4 S.O. Kronogård, Advances in Automotive Gas Turbine, Mechanical Engineering, October 1977.
- 1.5 P.W. Bridgman, Collected Experimental Papers, Harvard University Press, 1964. In particular: Proc. Royal Soc. (London) A, 201, (1950).
- 1.6 C.R. Boer, R.D. Venter, M.C. deMalherbe, Analysis of Tungsten Carbide Components for Very High Pressure Manufacturing Equipment, 6th N.A.M.R.C., April 1978.
- 1.7 D. Pugh Ed., Mechanical Behaviour of Materials under Pressure, Elsevier Pub., 1970.
- 1.8 R.H. Wendorf, Modern Very High Pressure Techniques, Butterworths London, 1962.
- 1.9 P.W. Bridgman, Recent work in the field of high pressure, Rev. Modern Physics, 8, 93-291, 1946.
- 1.10 H.T. Hall, Ultra-High Pressure, High Temperature Apparatus: The Belt, The Review of Scientific Instruments, 31, February, 1960.
- 2.1 R. Sedláček, F.A. Holden, Method for tensile testing of Brittle Materials, Rev. Sci. Instrum., 1062, 33, 3, 1962.
- 2.2 W.R. Holman, E.B. Stiles, E. Fung, Tensile Test Apparatus for small ring specimens, Rev. Sci. Instrum., 44, 1967.
- 2.3 J.N. Robinson, Tensile Test Apparatus for Brittle Materials, Journal of Physics E: Scientific Instruments, 5, 1972.
- 2.4 Diametral Compression testing of Cemented Carbides, ASTM Standard, Part 9, 292-293, 1976.
- 2.5 B.W. Abbott, R.M. Cornish, A Stress-wave technique for determining the Tensile Strength of Brittle Materials, Experimental Mechanics, May, 1965.

- 2.6 F.L.L.B. Carniero and A. Barcellos, Union of Testing and Research Laboratories for Materials and Structures, No. 13 (1953).
- 2.7 P.M. Braiden, Mechanical property tests for tungsten carbide, S.M.E. Technical Paper MR73-916, 1973.
- 2.8 A. Rudnick, A.R. Hunter, F.C. Holden, An Analysis of the Diametral-Compression Test, Materials Research & Standards, 3, 4, 283-289, 1963.
- 2.9 P.S.B. Colback, An Analysis of Brittle fracture initiation and propagation in the Brazilian test, Proceedings of the First Congress of the International Society of Rock Mechanics, 1966.
- 2.10 R.M. Spriggs, L.A. Brissette, T. Vasilos, Tensile strengths of Dense Polycrystalline Ceramics by the Diametral-Compression Test, Materials Research & Standards, 4, 5, 218-219, May 1964.
- 2.11 M.M. Frocht, Photoelasticity, Vol. II, Wiley, N.Y. Pag 121-129, 1948.
- 2.12 M.C. Shaw, P.M. Braiden, G.J. DeSalvo, The disk test for Brittle Materials, Journal of Engineering for Industry, 77-87, February, 1975.
- 2.13 G. Hondros, The evaluation of Poisson's Ratio and the Modulus of Materials of a Low Tensile Resistance by the Brazilian (Indirect Tensile) Test with particular reference to concrete, Australian Journal of Applied Science, 10, 243-268, 1959.
- 2.14 J.C. Jaeger, Failure of Rocks under Tensile Conditions, Int. J. Rock Mech. Min. Sci., 4, pp. 219-227, 1967.
- 2.15 J.C. Jaeger, E.R. Hoskins, Rock Failure under the Confined Brazilian Test, J. of Geophysical Research, 71, 10, 2651-2659, 1966.
- 2.16 P.J.F. Wright, Comments on an indirect tensile test on concrete cylinders, Magazine of concrete research, 87-96, July 1955.
- 2.17 J.J. Brisbane, Axial loading of an Elliptical Disk, Journal of Applied Mechanics, 306-309, June, 1963.
- 2.18 R. Berenbaum, J. Brodie, Measurement of the tensile strength of Brittle materials, Brit. J. of Applied Physics, Vol. 10, p. 281, 1959.
- 2.19 P.M. Braiden, Measurement of the strain to failure and elastic constants in ceramics, British Society of Strain Measurement Conference, Brunel University, 1971.
- 2.20 D.W. Hobbs, An assessment of a technique for determining the tensile strength of rocks, Brit. J. Appl. Phys., Vol. 16, 25, 1965.

- 2.21 J.C. Jaeger, N.G.W. Cook, Fundamentals of Rock Mechanics, John Wiley & Sons, N.Y., 2nd ed., 1976.
- 2.22 R.E. Cooper and M.A. Waters, Statistical effects in fracture initiation location in a disc under biaxial tension, I. Journal of Fracture, Vol. 13, No. 1, 1977.
- 2.23 E. Sternberg, F. Rosenthal, The Elastic Sphere under Concentrated Loads, J. Applied Mechanics, 413-421, December 1952.
- 2.24 A.I. Lure, Three Dimensional Problems in the Theory of Elasticity, Interscience Pub. Co., pp 361-367, 1964.
- 2.25 A.J. Durelli, H.M. Hasseem, V.J. Parks, New Experimental Method in Three-Dimensional Elastostatics, Acta Mechanica 24, 219-238, 1976.
- 2.26 R.P. Felgar, J.D. Lubahn, Mechanical Behaviour of Cemented Carbides Proc. Am. Soc. Test. Mat., 57, 770-790, 1957.
- 3.1 R.C. Juvinall, Engineering considerations of Stress, Strain and Strength, McGraw-Hill C., 1967.
- 3.2 H.L.D. Pugh, (Ed), Mechanical Behaviour of Materials under Pressure Elsevier, Amsterdam, 1970.
- 3.3 R.W. Nichols (Ed), Pressure Vessel Engineering Technology, Applied Science Publishers, London, 1971.
- 3.4 W. Johnson, A.G. Mamalis, Force Polygons to Determine Upper Bounds and Force Distribution in Plane Strain Metal Forming Processes Proceedings of 18th, Int. M.T.D.R. Conference, September 1977.
- 3.5 W. Johnson, A.G. Mamalis, Some force plane diagrams for plane strain slip-line fields, Int. J. Mechanical Science, 20, 47, 1978.
- 3.6 C.R. Boer, W. Johnson, M.C. deMalherbe, Force Plane Diagrams for Elastic and Elastic-Plastically stressed thick-walled cylinder to be published.
- 3.7 A.M. Elsaie, Advanced computer technology for better engineering productivity: Computerized design analysis, Engineering Digest 24, 4, 30-32, 1978.
- 4.1 C.R. Boer, R.D. Venter, M.C. deMalherbe, Failure criteria for quas brittle materials in High Pressure Applications, (to be published).
- 4.2 C.C. Bradley, High Pressure Methods in Solid State Research, Butterworths, London, 1967.
- 4.3 F.R. Boyd, J.L. England, Apparatus for Phase-Equilibrium Measurements at Pressures up to 50 kilobars and temperature up to 1750°C, J. Geophys. Res., 65, 741, 1960.

- 4.4 R.S. Bradley, D.C. Munro, M. Whitfield, Modified stepped-piston device for ultra-high pressures, J. Sci. Instrum. 42, 714, 1965.
- 4.5 A.A. Giardini, J.E. Tydings, S.B. Levin, A very high pressure-high temperature research apparatus and the synthesis of diamonds, Am. Miner, 45, 217, 1960.
- 4.6 W. Weibull, A Statistical Distribution Function of Wide Applicability, J. Applied Mech., 293-297, 1951.
- 4.7 P.W. Bridgman, Studies in Large Plastic Flow and Fracture, McGraw-Hill, New York, 1952.
- 4.8 M. Brandes, Effect of Hydrostatic Pressure on the Coasive Strength of Brittle Materials, Int. J. of Fracture Mechanics, 1, 1, 56, 1965.
- 4.9 C.T. Candland, Implications of microscopic failure criteria which are independent of hydrostatic stress, Int. Journal of Fracture, 11, 540, 1975.
- 5.1 W. Weibull, A Statistical Theory of Strength of Materials, Ingenious Vetenshops Akadanien, Handlinger, 151, 1939.
- 5.2 J.C. Jaeger, N.C.W. Cook, Fundamentals of Rock Mechanics, 199-201, J. Wiley & Sons 1976.
- 5.3 A.S. Argon, Statistical Aspects of Fracture in Mechanical Behaviour of Materials, Addison-Wesley, Reading Mass.
- 5.4 A.J. Durelli, V. Parks, Relationship of size and stress gradient to brittle fracture stress, Proc. 4th U.S. National Congr. of Appl. Mech., 931-938, 1962.
- 5.5 J.L. Chermant, A. Deschavres, F. Osterstock, Factors influencing the rupture stress of hardmetals, Powder Metallurgy, 2, 63-69, 1977.
- 5.6 S. Amberg, H. Doxner, Porosity in Cemented Carbides, Powder Metallurgy, 1, 1-10, 1977.
- 5.7 R. Venter, M. Mullins, M.C. deMalherbe, A Probabilistic Evaluation of the Relative Behaviour of Tungsten Carbide Components in Structure Applications, Annuals of CIRP, 1975.
- 5.8 A.M. Freudenthal, Statistical Approach to Brittle Fracture, Vol. 2, H. Liebowitz, ed., Academic Press, N.Y., 1968.

- A2.1 N.A. Weil and I.M. Daniel, Analysis of Fracture Probabilities in Nonuniformly Stressed Brittle Materials, J. of Am. Ceramic Society, Vol. 47, No. 6, p. 268, 1964.
- A3.1 G.S. Kreimer, Strength of Hard Alloys, Consultants Bureau, N.Y., 1968.
- A3.2 H.E. Exner, J. Gurland, A Review of Parameters influencing some Mechanical Properties of Tungsten Carbide - Cobalt Alloys, Powder Metallurgy, V. 13, N. 25, 1970.
- A3.3 H.E. Exner, J. Gurland, A Guide to the literature on Tungsten Carbide-Cobalt Alloys, Parts 1-3, Powder Metallurgy International, V. 2, No. 2-4, 1970.
- A3.4 R. Venter, S. Critchley, M.C. de Malherbe, The evaluation of Tungsten Carbide Properties as Applied to High Pressure Equipment, Report I: DIAPROS-CARED, McMaster University, 1974.
- A3.5 V.A. Ivensen, Effect of Cobalt content on the tensile strength of WC-Co hard alloys, Translation from Poroshkovaya Metallurgiya, N. 11 (119), p. 85092, by Consultants Bureau, New York, Publ. UDC 669.018.25, 1972.
- A3.6 S. Amberg, H. Doxner, Porosity in cemented carbide, Powder Metallurgy, No. 1, 1-10, 1977.
- A3.7 I. Johansson, G. Persson and R. Hittsher, Determination of static and fatigue compressive strength of hard-metals, Powder Metallurgy, Vol. 13, N. 26, 449-464, 1970.
- A3.8 H. Doi, Y. Fujiwara, K. Migake, On the strengthening mechanism of WC-Co Alloy, Trans. J.I.M., Vol. 9, 616-622, 1968.
- A3.9 J. Gurland, R. Bardzil, Relation of Strength, Composition and grain size of sintered WC-Co Alloys, Trans. AIME of Metals, p. 311-315, February, 1955.
- A3.10 J.L. Chermant, M. Coster, G. Houtier, P. Schaufelberger, Statistical Analysis of the behaviour of cemented carbides under high pressure, Powder Metallurgy, V. 17, N. 33, 1974.
- A3.11 J. Gurland, The Fracture Strength of Sintered Tungsten Carbide Cobalt Alloys in relation to composition and particle spacing, Trans. Met. Soc. AIME, V. 227, 1, 1146-1150, 1963.
- A3.12 F.R.N. Nabarro, S.B. Luycks, The Theory of Strength of Tungsten Carbide Cobalt Compacts, Trans. SIM, V. 9, Suppl., p. 610-614, 1968.

4 OF 4

- A3.13 B.O. Sundstrom, Elastic-plastic behaviour of WC-Co Analyzed by Continuum Mechanics, Materials Science and Engineering 12, 265-276, 1973.
- A3.14 B.O. Jaensson, B.O. Sundstrom, Determination of Young's Modulus and Poisson's Ratio for WC-Co Alloys by the Finite Element Method, Materials Science and Engineering, 9, 217-222, 1972.
- A3.15 A.A. Betser, J. Gurland, Some Effects of Temperature and Heat Treatment on the strength of sintered WC-Co Alloys ASME Paper No. 66-MD-17.
- A3.16 J.T. Smith, J.D. Wood, Elevated Temperature Compressive Creep Behaviour of Tungsten Carbide-Cobalt Alloys, Acta Metallurgica, V. 16, 1219-1226, 1968.
- A3.17 M. Lagerquist, A Study of the Thermal Fatigue Crack Propagation in WC-Co cemented carbide, Powder Metallurgy, V. 18, 71-88, 1975.
- B1.1 R. Hill, The Mathematical Theory of Plasticity, Clarendon Press, pp. 354, Oxford, (1950).
- B1.2 W. Johnson, Estimation of Upper Bound Loads for Extrusion and Coining Operations, Proc. Inst. Mech. Engrs. 173, 61, 1959.
- B1.3 W. Johnson, P.B. Mellor, Engineering Plasticity, Van Nostrand Reinhold, pp. 646, London, 1973.
- B1.4 H. Ford (with J.H. Alexander, Advanced Mechanics of Materials Longmans, London, 1963.
- B1.5 W. Johnson, R. Sowerby, An Analysis of a rigid-plastic thick-walled cylinder, using slip-line field theory, Bull. Mech. Eng. Educ., 6, pp. 201-218, 1967.
- C3.1 A. Des. Jayatilaka, K. Trustrum, Statistical approach to brittle fracture, Journal of Materials Science, 12 1426-1430, 1977.
- C3.2 G.C. Sih, Strain-energy factor applied to mixed mode crack problems, Int. J. Fracture, 10, 305-321, 1974.
- C3.3 A. Des. Jayatilaka, I.J. Jenkins, S.V. Prasad, Determination of crack growth in a mixed mode loading system, 4th International Conference on Fracture, Canada (June 1977). (ICF4 Publ. Waterloo 1977), Volume 3.

- C3.4 J.D. Poloniecki, T.R. Wilshaw, Determination of surface crack size densities in glass, Nature, 229, 226-227, 1971.
- C4.1 A. des. Jayatilaka, K. Trustrum, Application of a statistical method to brittle fracture in a biaxial loading systems, Journal of Materials Science, 12, 2043-2048, 1977.
- C4.2 D.G. Rickerby, Weibull statistics for a biaxial strength test, 4th Int. Conf. of Fracture, 1133, 1977.
- C4.3 W.F. Brace and E.G. Bom bolakis, A Note of Brittle Crack Growth in Compression, J. Geophys. Res. 68, 3709, 1963.
- C4.4 R.E. Cooper, M.A. Waters, Statistical effects in fracture initiation location in a disc under biaxial tension, Int. J. of Fracture, Vol. 13, No. 1, 1977.
- C4.5 S.B. Batdorf, Some approximate treatments of fracture statistics for polyaxial tension, Int. J. of Fracture, Vol. 13, No. 1, 5-11, 1977.
- C4.6 S.B. Batdorf, and J.G. Crose, A Statistical Theory for the fracture of brittle structure subjected to non uniform polyaxial stresses, Journal of applied mechanics, 41 459, 1974.
- C4.7 D.M. Shur, Statistical criterion of the danger of Material Fracture in a complex stress state, Mashinovedenie, 1, 51, 1971.
- C4.8 O. Vardar and I. Finnie, An analysis of the Brazilian disk fracture test using the Weibull probabilistic treatment of brittle strength, Int. Journal of Fracture, Vol. 11, N. 3, 495, 1975.
- C5.1 R. Jones, Ceramics in turbine design, Engineering 1976, Sept.
- C5.2 R.W. Davidge, J.R. McLaren, G. Tapin, Strength-probability-time relationships in ceramics, Journal of Materials Science, 8, 1699, 1973.
- D1.1 Timoshenko, Strength of Materials, D. Van Nostrand, 1968
- D1.2 R. Hill, The Mathematical Theory of Plasticity, Clarendon Press, 1964.
- D1.3 L.M. Kachanov, Foundations of the Theory of Plasticity, North-Holland, 1971.

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