Structure of Semigroup Algebras

by

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Abstract

The concept of a graph semigroup is introduced and the structure of semigroup algebras of finite semigroups is studied. Developed techniques lead directly to a description of the structure of many interesting subsemigroups of finite inverse semigroups.

The main theorem describes the structure of full subsemigroups of finite inverse semigroups in terms of graph semigroups. The connection to graph semigroups is particularly well reflected in algebras of such semigroups, as they are shown to be direct products of semigroup algebras of graph semigroups. This provides a significant generalization of the known results on finite commutative and finite inverse semigroups.

A major application of this theory is an explicit description of the structure of semigroups of partial isomorphisms that are compatible with certain partial orders. We determine the combinatorial and algebraic invariants associated with the relevant graph semigroups in terms of posets of integer partitions, products of symmetric groups and their permutation representations.

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Introduction

The present work grew out of the desire to gain a better understanding of algebras of finite semigroups. We try to strike a balance between two points of view: a semigroup theory point of view, which sees just the underlying semigroup, and utmost ring theory view, which considers a semigroup algebra just as a ring, disregarding completely its default set of generators.

Ideally, from the representation theory view, one would like to have the structural description of a semigroup algebra in terms of its primary homological invariants (simple modules and their extensions) and secondary structure, such as radical layers or relations.

However, a semigroup typically contains a plenitude of nontrivial groups, and this fact is directly reflected in the properties of the semigroup algebra. Group algebras, as a rule, do not provide us with an immediate description of the above-mentioned kind, and their properties are heavily dependent on arithmetical properties of the coefficient ring. Aside from that, even when successfully presenting a group algebra in its simplest form from that point of view, one may pay a price: the individual pieces do not necessarily have a nice set of generators.

A few remarks of a general historical nature may help to understand the motivation for the approach outlined below.

Modern representation theory of finite-dimensional algebras of the past forty years has eventually made its way into the theory of modular and integral representations of groups, offering new perspective to the field that has been based primarily on character theory. This is easily attested by recent work on blocks of group algebras.

Algebraic semigroup theory, being a distinctly less traditional field of research than group theory, has begun systematic study of linear representations of semigroups only in the second half of the twentieth century. This contrasts with group representations, which have been a subject of intense study ever since the times of Schur and Frobenius. The history of representations of commutative groups would go even further, of course. On the semigroup side, it was the work of W. Munn and I. S. Ponizovskii published in the late fifties that constituted by far the most significant contribution to the subject for a very long time. Their methods are character based and thus yield immediately definitive answers to questions concerning semisimple representations of semigroup algebras over fields of characteristic zero.

After that, investigations of semigroup algebras seemed to have put an emphasis on extending the applicability of the same methods to larger classes of semigroups. In particular, the area had received hardly any benefit from the tremendous developments that took place in the representation theory of finite-dimensional algebras.

Deeper structural results are much harder to obtain, however, using techniques of irreducible characters. One not only has to pay attention to the way simple factors of a semigroup are “glued” together, but also take into account their non-regular elements.

Our approach is to make as much use as possible of the underlying semigroup structure, resisting temptation to make simplifying assumptions on the arithmetical

properties of the coefficients or making use of the special properties that structural groups of the studied semigroup may possess. In this respect, the techniques for handling semigroup algebras that we put forth are an antithesis to the approaches based on irreducible characters. The attitude of not delving into the structure of the groups they contain is not a new one — most of the theory of semigroups aims to describe their structure “modulo groups”.

We succeeded in developing techniques that are applicable even to non-regular semigroups. The class most amenable to our approach is the class of semigroups with commuting idempotents.

The first two chapters serve to build up the necessary machinery, as many concepts needed are not available off-the-shelf in sufficient generality. The necessary details are provided. Chapter 1 ends with a construction of a functor of which we make frequent use later.

The substance of the second chapter is the definition of the graph semigroup. It is this concept which allows us to exploit results and techniques of the contemporary representation theory of algebras.

Direct decompositions of semigroup algebras are studied in the following chapter. Having the concept of graph semigroup at hand, it turns out to be possible to describe the structure of some classes of semigroup algebras. We prove that suitable subsemigroups of finite inverse semigroups are built up from graph semigroups of special form. The most prominent inverse semigroups, the symmetric inverse semigroups, and some of their subsemigroups are shown to have a particularly simple decomposition.

Chapter 4 is devoted to semigroups of partial isomorphisms of posets of a special kind. These are semigroups with far more complex structure than that of the

examples examined in Chapter 3. Here the techniques and concepts developed in all the previous chapters converge, and most important invariants of these semigroups are determined.

Various examples are collected in the last chapter. Some of these serve as an illustration of the theoretical results, but some are semigroups for which there is currently no comprehensive theory that would yield the structure of their algebras. Despite that, we provide explicit isomorphisms to algebras of graph semigroups.

Chapter 1

Semigroups and Algebras

Whenever maps are considered, their composition is written “algebraically”: thus fg means that f is applied first. In the relatively infrequent case that we do need to refer to particular points being mapped, we write them on the left, as in $(x)fg$, which stands for the image of x under the map fg . The traditional “functional” notation is retained when writing ‘operators’ such as $Z(A)$ (the centre of A).

One should find all the notation in the first and third section fairly standard, with the possible exception of the symbols used for various “composing” operations for semigroups — the terminology in the literature is not entirely universal.

The section on strong bimodules presents the standard construction of “adjoining identity”, but it is taken here to the utmost generality (within the realm of associative algebras).

The fourth section makes use of this construction in order to define a functor that enables us to get around some of the difficulties that one encounters when the coefficient algebras are not unital. This greatly simplifies the forthcoming treatment of semigroup algebras.

1.1 Basic Semigroup Notation

Semigroups present slight notational challenges that are not encountered in group or ring theory, mostly due to the possible presence or absence of zero and identity elements. We recall the elementary notation, mostly quite standard in semigroup literature.

The zero, respectively identity elements of a semigroup S are always denoted by 0 , respectively 1 ; if ambiguities might arise, we write 0_S and 1_S instead. The one-element semigroup is not considered to possess a zero.

We define S^1 to be S if $1 \in S$; otherwise, $S^1 = S \cup \{1\}$ with multiplication extended by $1s = s1 = s$ for all $s \in S \cup \{1\}$.

Similarly, let S^0 be S if $0 \in S$; otherwise, $S^0 = S \cup \{0\}$ with multiplication $0s = s0 = 0$ for all $s \in S \cup \{0\}$.

For a subset U of S , we denote by $\langle U \rangle$ the semigroup generated by U , while the ideal I of S generated by U is denoted by $\langle\langle U \rangle\rangle$. The associated Rees congruence is the relation $\rho_I = I \times I \cup \text{id}_S$, the factor-semigroup S/ρ_I will be written simply as S/I . If I is a nonempty proper ideal, we identify elements of S/ρ_I with $S \setminus I \cup \{0\}$.

The symbol $E(S)$ denotes the set of all idempotents of S . The relation $\leq$ defined on $E(S)$ by $e \leq f$ if $ef = e = fe$ (for two idempotents e, f) is a partial order.

Notation 1.1.1. Let S, T be semigroups. In case S (resp. T) does not have a zero, consider the set $\{0_S\}$ (resp. $\{0_T\}$) empty.

The *direct product* $S \times T$ is the semigroup on $S \times T$ with component-wise multiplication.

The *disjoint sum* $S \sqcup T$, often called a 0-disjoint union, is the semigroup on $S \cup T \cup \{0\}$ (elements of S, T and $\{0\}$ are considered distinct) with the product

extended by setting $S \cdot T = 0 = T \cdot S$ and with 0 acting as the zero element.

The *contracted direct product* $S \dot{\times} T$ is the semigroup $S \times T/I$, where I is the ideal $S \times \{0_T\} \cup \{0_S\} \times T$ of $S \times T$.

The *contracted disjoint sum* $S \dot{\sqcup} T$ is the semigroup $S \sqcup T/I$, where I is the ideal $\{0_S\} \cup \{0_T\} \cup \{0\}$ of $S \sqcup T$.

These constructs can be extended in obvious manner to arbitrary families of semigroups. We explicitly state the definition just for the contracted disjoint sum as this is the construct of which we make frequent use in subsequent chapters.

For a family of semigroups S_i indexed by $i \in \Lambda$, we write $\bigsqcup_{i \in \Lambda} S_i$ for the factor-semigroup $\bigsqcup_{i \in \Lambda} S_i/I$ where I is the ideal $I = \bigcup_{i \in \Lambda} \{0_{S_i}\} \cup \{0\}$. Again, $\{0_{S_i}\}$ is interpreted as empty set in case S_i does not possess a zero.

We find the contracted versions $\dot{\times}$ and $\dot{\sqcup}$ of much greater utility when discussing semigroups, and more in the spirit of the definition of S^0 and S^1 . They will be simply referred to as the direct product and the disjoint sum of semigroups, eliding the word ‘contracted’. The traditional meaning of symbols $\times$ and $\sqcup$ is retained so as not to cause misunderstanding.

The notation regarding *Green’s relations* on a semigroup is entirely standard. We mention it for the sake of completeness.

Notation 1.1.2. Let S be a semigroup, $s, t \in S$. We write

$$\begin{aligned} s \leq_{\mathcal{L}} t & \quad \text{if} \quad S^1 s \subseteq S^1 t, \\ s \leq_{\mathcal{R}} t & \quad \text{if} \quad s S^1 \subseteq t S^1, \\ s \leq_{\mathcal{J}} t & \quad \text{if} \quad S^1 s S^1 \subseteq S^1 t S^1. \end{aligned}$$

These three relations on S are quasi-orders.

Each equivalence $\Gamma \in \{\mathcal{L}, \mathcal{R}, \mathcal{J}\}$ on S is defined by

$$s\Gamma t \quad \text{if} \quad s \leq_{\Gamma} t \text{ and } t \leq_{\Gamma} s,$$

and the set S/Γ of Γ -classes is partially ordered by the induced relation $\leq_{\Gamma}$. We use Γ_s to denote the Γ -class of s , and writing $s <_{\Gamma} t$ (or $\Gamma_s <_{\Gamma} \Gamma_t$) means that $s \leq_{\Gamma} t$ and $\Gamma_s \neq \Gamma_t$.

The equivalence $\mathcal{H}$ is defined as $\mathcal{L} \cap \mathcal{R}$, while the equivalence $\mathcal{D}$ is given as the smallest equivalence containing both $\mathcal{L}$ and $\mathcal{R}$. As with $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$, we will write $\mathcal{H}_s$ and $\mathcal{D}_s$ for the equivalence classes containing s .

The *principal factor* associated to an element $s \in S$ (or to a $\mathcal{J}$ -class $\mathcal{J}_s$) is the semigroup J/I , where $J = S^1 s S^1$ is the *principal ideal* generated by s , and I is the ideal generated by all $t \in S$ such that $t <_{\mathcal{J}} s$.

An element $s \in S$ is *regular* if there is a $t \in S$ such that $sts = s$. If no such t exists, s is called *non-regular*. Elements of a given $\mathcal{D}$ -class are either all regular or all non-regular, thus the terms ‘regular’ and ‘non-regular’ apply to $\mathcal{D}$ -classes as well.

Many special classes of semigroups will be introduced later. Here we only mention notation for semigroups commonly occurring in statements and proofs — the semigroup of elementary matrix units and the symmetric group.

Notation 1.1.3. We use the symbol $\mathcal{M}_{\Lambda}$ to denote the *elementary matrix units semigroup* on the set Λ :

$$\mathcal{M}_{\Lambda} = \{m_{ij} \mid i, j \in \Lambda\} \cup \{0\},$$

with product of nonzero elements defined by $m_{ij}m_{kl} = m_{il}$ if $j = k$ and 0 if $j \neq k$. It is advantageous in situations such as this one to make use of the Kronecker symbol

δ_{jk} and write $m_{ij}m_{kl} = \delta_{jk}m_{il}$, covering both cases. Thus δ_{jk} symbolically ‘acts’ as identity if $j = k$ and otherwise it ‘acts as zero’; the symbol is meaningful as soon as zero exists in the considered structure.

If the set Λ is finite, with cardinality $|\Lambda| = n$, we write simply $\mathcal{M}_n$.

The symmetric group on n elements is written as S_n .

For other commonly used concepts of semigroup theory that are not defined here, such as regular, inverse, or completely 0-simple semigroups and Rees semigroups $\mathcal{M}^0(G; \Lambda, \Upsilon; P)$, one may consult any modern text on semigroup theory, for instance [2, 9, 12].

1.2 Strong Bimodules

Rings that are most relevant to representation theory of any kind almost always carry an additional structure. Most often, this additional structure makes the ring an algebra over another (usually “simpler”) ring.

We will adopt the policy of using the term “algebra” even though we do not strictly require an additional structure on a ring. This terminology will serve as a reminder that whenever we consider morphisms, we are really working in a category of all morphisms preserving the additional structure in case it exists. Since existence of such a structure is frequently entirely obvious from the context, this policy will save us from having to point out the category we are working with. Since any ring is a $\mathbb{Z}$ -algebra, there is really no loss of generality in working with algebras.

Algebras arising from semigroups often do not possess identities. We present a generalization of the construction that is typically employed to ‘adjoin identity’ to an

algebra. A collection of statements that will be useful in subsequent considerations follows the definition.

Definition 1.2.1 (Strong bimodule). Let A, B be associative algebras (possibly non-unital). We call A a *strong B -bimodule* if A is a B -bimodule such that for all $a, a' \in A$ and $b, b' \in B$, the following equalities hold:

$$(aa')b = a(a'b),$$

$$b(aa') = (ba)a',$$

$$(ab)a' = a(ba').$$

Notice that the first two conditions require that A be an A - B -bimodule and B - A -bimodule (with the default action of A) and are automatically satisfied in cases where B acts centrally on A (for instance, in the case of A being a B -algebra in the classical sense). The last condition is of a different nature.

If moreover B has an identity and its action on A is unital, we call A a *unital strong B -bimodule*.

Remark 1.2.2. Let C be an algebra which splits additively as $C = A \times B$ with the set A (identified with $A \times \{0\}$) being an ideal of C and the set B (that is, $\{0\} \times B$) a subalgebra. Then the action of B makes A a strong B -bimodule. We denote this situation symbolically by $C = B \triangleleft A$, and we say that B is a *retract* of C .

We will now show that all strong bimodules arise in this way by constructing suitable algebra C using a strong B -bimodule A .

Definition 1.2.3 (Algebra A^{1B}). Let A, B be algebras such that A is a strong B -bimodule. We define A^{1B} to be the cartesian product $A \times B$ with the component-wise

additive structure, endowed with the multiplication

$$(a_1, b_1)(a_2, b_2) = (a_1a_2 + a_1b_2 + b_1a_2, b_1b_2).$$

In the simplest case of zero algebra $A = \{0\}$, we obtain $A^1 \cong B$.

Lemma 1.2.4. *A^{1B} is an associative algebra. The algebra B , embedded as $\{(0, b) \mid b \in B\}$, is a subalgebra of A^{1B} . The algebra A , embedded as $\{(a, 0) \mid a \in A\}$ in A^{1B} , is an ideal of A^{1B} and its left and right B -submodule.*

Proof. It is a matter of routine calculation to verify the claim:

$$\begin{aligned} (a_1a_2 + a_1b_2 + b_1a_2, b_1b_2)(a_3, b_3) = \\ ((a_1a_2 + a_1b_2 + b_1a_2)a_3 + (a_1a_2 + a_1b_2 + b_1a_2)b_3 + (b_1b_2)a_3, b_1b_2b_3) \end{aligned}$$

At the same time,

$$\begin{aligned} (a_1, b_1)(a_2a_3 + a_2b_3 + b_2a_3, b_2b_3) = \\ (a_1(a_2a_3 + a_2b_3 + b_2a_3) + a_1(b_2b_3) + b_1(a_2a_3 + a_2b_3 + b_2a_3), b_1b_2b_3) \end{aligned}$$

and the structure of strong B -bimodule is exactly what is needed to obtain an associative product.

The remaining claims easily follow. For instance, $(a_1, b_1)(a_2, 0) = (a_1a_2 + b_1a_2, 0)$ shows that the subalgebra A is a left ideal. $\square$

Identity element and the centre

The proof of the following lemma is apparent from the definition of multiplication on A^{1B} .

Lemma 1.2.5. *Let A be a strong B -bimodule.*

- i) *If A^{1B} has an identity element then B has an identity.*
- ii) *If A is a unital strong B -bimodule, then A^{1B} has an identity element $(0, 1_B)$.*

In the second case, we denote the identity element $(0, 1_B)$ of A^{1B} simply by 1.

Remark 1.2.6. Observe that in the unital case, the definition of multiplication is equivalently described as being extended B - B -linearly to A^{1B} by requiring that $1 \cdot 1 = 1$ and $1 \cdot (a, 0) = (a, 0) \cdot 1 = (a, 0)$, where 1 stands for the element $(0, 1_B)$ of A^{1B} .

The next claim relates the centre of A^{1B} to the centres of A and B . For the sake of simplicity of notation, we identify A and B with the corresponding subsets of A^{1B} .

Lemma 1.2.7. *Let A be a strong B -bimodule.*

- i) *If $Z(B)$ acts centrally on A , then $Z(A^{1B}) \subseteq Z(A) \times Z(B)$.*
- ii) *If B acts centrally on A , then $Z(A^{1B}) = Z(A) \times Z(B)$ as a set.*

Proof. Suppose $(a_1, b_1) \in Z(A^{1B})$. Then certainly $b_1 \in Z(B)$. Consider (a_2, b_2) with $a_2 \in A$ arbitrary, and $b_2 \in Z(B)$ ($b_2 = 0$ would suffice). As the two pairs must commute in A^{1B} , we have

$$a_1 a_2 - a_2 a_1 = b_2 a_1 - a_1 b_2 + b_1 a_2 - a_2 b_1.$$

Since the right-hand side is zero by the hypothesis of the lemma, we see that $a_1 \in Z(A)$. Thus $Z(A^{1B}) \subseteq Z(A) \times Z(B)$.

The second claim is verified easily, since the expression on the left-hand side of the above equation is zero if B acts centrally. $\square$

Lemma 1.2.8. *Let A be a strong B -bimodule. If the algebra A has an identity then $A^{1_B} \cong A \times B$ (algebra isomorphism).*

Proof. Throughout the proof, the pair (a, b) always refers to elements of A^{1_B} , not to the decomposition on the right side of the isomorphism.

Let e be the identity of A . Now observe that we can define two subalgebras A_1, A_2 of A^{1_B} by

$$A_1 = \{(a + be, 0) \mid a \in A, b \in B\} \cong A,$$

$$A_2 = \{(-be, b) \mid b \in B\} \cong B.$$

and their intersection is just $\{(0, 0)\}$. They are easily verified to be ideals of A^{1_B} . Hence $A^{1_B} \cong A \times B$ as algebras. $\square$

Lemma 1.2.9. *Let A, C be strong B -bimodules for some algebra B . If A has an identity, then $(A \times C)^{1_B} \cong A \times C^{1_B}$.*

Note that the Lemma 1.2.8 is a special case of this statement.

Proof. Denoting by 1_A the identity element of A , it is just a matter of routine calculation to show that the map given by

$$((a, c), b) \mapsto (a + 1_A b, (c, b)),$$

with $a \in A, b \in B$ and $c \in C$ provides the desired isomorphism of algebras. $\square$

Idempotents

The relation between the idempotents of A and those of A^{1_B} is also straightforward if B does not have nontrivial idempotents.

Lemma 1.2.10. *Let A be a unital strong B -bimodule. Assume that B has no other idempotents than 0 and 1. Then the element (a, b) is an idempotent of A^{1B} if and only if either $b = 0$ and $a^2 = a$, or $b = 1$ and $a^2 = -a$.*

Proof. As $(a, b)^2 = (a^2 + ab + ba, b^2)$ and the equation $b = b^2$ can only have the two solutions 0, 1 in B , the claim is clear. $\square$

Remark 1.2.11. The claim of the lemma can be restated as follows: All the idempotents of A^{1B} are either those already in A (meaning elements $(e, 0)$ with $e^2 = e$) or their complements (that is, elements $1 - (e, 0)$).

Remark 1.2.12. If B has nontrivial idempotents, there could be more idempotents that are not ‘coming’ from A .

1.3 Semigroup Algebras

We begin with defining two variants of a semigroup algebra. The coefficients are allowed to have values in an arbitrary associative algebra.

Definition 1.3.1 (Semigroup algebra $A[S]$ or AS). Let S be a semigroup and A an algebra.

The *semigroup algebra* $A[S]$ of S over A is the set of all functions $f : S \rightarrow A$ having finite (or empty) support, with convolution multiplication: if $f : S \rightarrow A$ assigns $s \mapsto a_s$ and $g : S \rightarrow A$ assigns $s \mapsto b_s$, then the product fg is given by $s \mapsto \sum_{pq=s} a_p b_q$, summing over all pairs $p, q \in S$ such that $pq = s$. If no such pair exists (s is not factorizable), we consider the empty sum to stand for the zero element of A .

The *contracted semigroup algebra* AS (written as $A(S)$ if it might help to avoid ambiguous reading) of S over A has its product defined identically, but considering only functions defined on *nonzero* elements of S .

We will freely refer to both variants as ‘the semigroup algebra’, the distinction will be kept only in notation.

Remark 1.3.2. The semigroup algebra of S over A is an associative algebra. As an A -module (left or right), the semigroup algebra is free. It has a natural structure of an A - A bimodule, since functions f from S to A can be multiplied by ‘constants’ $a_1, a_2 \in A$ (either on the right, or on the left) and $(a_1 f)a_2 = a_1(fa_2)$. Note that the semigroup algebra is certainly a strong A -bimodule.

More generally, if A is a strong B -bimodule or a unital strong B -bimodule, then any semigroup algebra over A has the same property.

Non-isomorphic semigroups may have isomorphic semigroup algebras — for some pairs of semigroups, this is true regardless of the coefficient ring. We are usually interested in isomorphisms that respect the natural bimodule structure of semigroup algebras. We call an algebra homomorphism $\phi : AS \rightarrow AT$ an *A -algebra homomorphism* if it is a homomorphism of algebras that is a morphism of A -bimodules as well: $(ax)\phi = a(x\phi)$ and $(xa)\phi = (x\phi)a$ for all $a \in A, x \in AS$. It is easy to see that such homomorphisms actually preserve the structure of a strong bimodule.

This terminology is a convenient shorthand. We will be mostly using it for isomorphisms of semigroup algebras. One just has to keep in mind that speaking of an A -algebra isomorphism does not imply that A is meant to be commutative or unital.

Remark 1.3.3. Observe that there are some slight inconveniences caused by the def-

inition not requiring that the coefficient algebra possesses an identity.

- i) First of all, if A has an identity 1, this fact is often exploited to identify elements of $s \in S$ with the elements 1_s of the semigroup algebra $A[S]$. These elements form a multiplicative subsemigroup of $A[S]$ isomorphic to S , and this identification sometimes allows for simpler (or more suggestive) notation. We merely note here that the same identification can be easily accomplished using any (fixed) nonzero idempotent of A .
- ii) Another issue is whether the semigroup algebra, when looked upon as a (semigroup) graded algebra, has the property that the s -component of a factorizable s is generated by products of components of its factorizations. This is automatically true in case A is unital, but in extreme cases such as the coefficient algebra having zero multiplication, the structure of S is not reflected by the semigroup algebra at all. There are, however, assumptions much weaker than the existence of identity, such as $A \cdot A = A$, that are sufficient for the algebra AS to be “strongly graded” in the sense outlined above.

Note that $AS^0 = AS$. The algebras $A[S]$ and AS are plainly the same object if S does not have a zero. In the complementary case, the relationship is as follows.

Lemma 1.3.4. *Assume S has a zero 0. Then the map $A[S] \cong A \times AS$, induced by $a_s \mapsto (a, a_s)$ for all $s \neq 0$ and $a_0 \mapsto (a, 0_{AS})$, is an isomorphism of algebras.*

Consequently, the structure of $A[S]$ is in essence completely determined by AS . It is not just the direct product decomposition that makes the lemma so useful — it is also the fact that each direct factor has a nice set of generators. In case A has a nonzero idempotent e , we see that the set $\{e_s - e_0 \mid s \in S\}$ is a multiplicative

subset of $A[S]$ isomorphic to S . In general, a *multiplicative subset* of an algebra is a subset with a slightly weaker property: the product of two elements of the subset, if nonzero, is again an element of the subset.

We now mention a number of elementary properties of the semigroup algebra construction.

Lemma 1.3.5. *Let A, B be algebras and S a semigroup. Then there are canonical isomorphisms*

$$\text{i) } (A \times B)[S] \cong A[S] \times B[S],$$

$$\text{ii) } (A \times B)(S) \cong AS \times BS.$$

For algebras A_i ($i \in \Lambda$), the symbol $\prod_{i \in \Lambda} A_i$ denotes the product of A_i 's (with coordinate-wise operations) and $\bigoplus_{i \in \Lambda} A_i$ its subalgebra consisting of elements that have all but finitely many coordinates zero.

Lemma 1.3.6. *Let A be an algebra and S, T, S_i semigroups for all $i \in \Lambda$. Then we have canonical A -algebra isomorphisms*

$$\text{i) } A[S][T] \cong A[T][S] \cong A[S \times T],$$

$$\text{ii) } AS(T) \cong AT(S) \cong A(S \dot{\times} T),$$

$$\text{iii) } AS \times AT \cong A(S \sqcup T),$$

$$\text{iv) } \bigoplus_{i \in \Lambda} AS_i \cong A(\bigsqcup_{i \in \Lambda} S_i).$$

Proofs immediately follow from the definitions.

Lemma 1.3.7. *If S is a semigroup without identity, then $AS^1 \cong (AS)^{1_A}$.*

Proof. Follows from the remarks following Definition 1.2.3 on page 10. $\square$

Remark 1.3.8. Since semigroup algebras are often unital even if S has no identity element, the assumption $S \neq S^1$ cannot be easily avoided by a simple modification of the definition of A^{1A} (so as to not adjoin anything if A is already unital).

If A has no idempotent, there is no guarantee that the algebra AS^1 has an identity.

We conclude this section with two lemmas, whose proofs can be found, for example, in [18].

The first statement shows that the existence of identity in a semigroup algebra AS poses structural limitations on both the algebra A and the semigroup S .

Lemma 1.3.9. *Let AS be a semigroup algebra with identity. Then the following is true.*

- i) *A has an identity.*
- ii) *There is a finite set U of idempotents of S such that $US = S = SU$.*

These two necessary conditions are sufficient for semigroups whose idempotents commute. Moreover, we know the general form of the identity element in this case.

Lemma 1.3.10. *Let S be a semigroup with commuting idempotents and A an algebra. If A and S satisfy the two conditions of the previous lemma, then AS possesses identity and this element is an integer linear combination of (a finite number of) some idempotents of S .*

By ‘integers’, we mean finite sums of the identity element 1 of A .

1.4 Induced Isomorphisms

The main result of this section will be very useful for dealing with semigroup algebras in cases where the coefficient algebra does not have an identity element. We will entirely avoid any use of tensor products, relying on the construction of A^{1B} instead.

If B is unital, any B -algebra homomorphism from BS (to BT , say) is determined by images of elements 1_s of BS ($s \in S$). If A is a unital strong B -bimodule, the algebra BS is a subalgebra of $A^{1B}S$ and there are two natural ways of extending the homomorphism to a morphism of A^{1B} -modules, since A^{1B} is generated by $(0, 1)$ (both as a left and a right A^{1B} -module). If B acts centrally (that is $ba = ab$ for all $a \in A, b \in B$), not only does the ambivalence disappear, but we will get an A^{1B} -bimodule homomorphism. In this case, the morphism clearly maps elements of the AS (as ideal of $A^{1B}S$) to elements of ideal AT of $A^{1B}S$, so we obtain a morphism of A -bimodules as well. The only question is whether such a morphism will be an algebra homomorphism as well.

The following theorem shows that this is indeed the case if B is in the centre of A^{1B} . Moreover, in case the original homomorphism of B -algebras is an isomorphism, the induced A^{1B} -homomorphism and A -homomorphism are also isomorphisms. Note that the assumption essentially requires A to be a B -algebra in the classical sense — except that A does not necessarily possess an identity.

Theorem 1.4.1. *Let A be a unital strong B -bimodule, with the commutative unital algebra B acting centrally on A . Assume S, T are semigroups and $\varphi : BS \rightarrow BT$ is a homomorphism of B -algebras. Then the following is true.*

- i) φ extends to an A^{1B} -algebra homomorphism $\varphi^{A^{1B}} : A^{1B}S \rightarrow A^{1B}T$.

ii) The restriction $\varphi^A : AS \rightarrow AT$ of $\varphi^{A^{1B}}$ is an A -algebra homomorphism.

iii) If φ is an isomorphism, then both $\varphi^{A^{1B}}$ and φ^A are isomorphisms.

Proof. Throughout the proof, the symbol 1 is used to denote both the identity of B and that of A^{1B} . This is hardly a misuse, as inside A^{1B} these two elements can be identified.

i) Let $s, s' \in S$. We will use a simplified notation for writing the images of elements under φ , leaving the (necessarily finite) sums non-delimited. The final symbol (say 1_t , for the sake of example) determines that the non-delimited sum is over a finite set of elements of T , its individual elements denoted by t .

$$1_s \mapsto \sum b_t 1_t \quad 1_{s'} \mapsto \sum b_{t'} 1_{t'} \quad 1_{ss'} \mapsto \sum \left(\sum_{tt'=t''} b_t b_{t'} \right) 1_{t''}$$

The 1 stands for the identity of B in the expressions above. The last assignment is, of course, a consequence of the fact that φ is an algebra homomorphism.

Using the same notation, it is easy to write down where “basic” elements of $A^{1B}S$ are mapped to under the uniquely determined extension $\varphi^{A^{1B}}$ of φ . This time, 1 denotes the identity of A^{1B} , and $c, c' \in A^{1B}$.

$$c1_s \mapsto \sum cb_t 1_t \quad c'1_{s'} \mapsto \sum c'b_{t'} 1_{t'} \quad cc'1_{ss'} \mapsto \sum cc' \left(\sum_{tt'=t''} b_t b_{t'} \right) 1_{t''}$$

To verify that the extension to A^{1B} -morphism is an algebra homomorphism, we only have to check that the last expression agrees with the product in $A^{1B}T$ of the right-hand sides of the first two lines. We have

$$\sum cb_t 1_t \sum c'b_{t'} 1_{t'} = \sum \left(\sum_{tt'=t''} cb_t c'b_{t'} \right) 1_{t''} = \sum \left(\sum_{tt'=t''} cc' b_t b_{t'} \right) 1_{t''},$$

the last transformation being possible due to the centrality of the action of B on A . The final expression is the same element as the image of $cc'1_{ss'}$, so the map $\varphi^{A^{1B}}$ is indeed a homomorphism of A^{1B} -algebras.

- ii) As observed before, the ideal AS of $A^{1B}S$ gets mapped into the ideal AT of $A^{1B}T$, therefore the restriction φ^A is an A -algebra homomorphism.
- iii) If φ is an isomorphism, there is a B -algebra homomorphism $\psi : BT \rightarrow BS$ such that $\varphi\psi = \text{id}_{BS}$ and $\psi\varphi = \text{id}_{BT}$. The composition of extensions $\varphi^{A^{1B}}\psi^{A^{1B}}$ certainly maps 1_s to 1_s (as these belong to the subalgebra BS), and it is an A^{1B} -bimodule map from $A^{1B}S$ to itself. So it must be the map $\text{id}_{A^{1B}S}$, as this is the only such map of A^{1B} -bimodules. Similarly, $\psi^{A^{1B}}\varphi^{A^{1B}}$ must be the map $\text{id}_{A^{1B}T}$.

Finally, since $AS\varphi^{A^{1B}} \subseteq AT$ and $AT\psi^{A^{1B}} \subseteq AS$, we have $AS\varphi^{A^{1B}}\psi^{A^{1B}} \subseteq AS$. But the last containment is equality, thus all the containments are equalities, and so $\varphi^A\psi^A = \text{id}_{AS}$. Analogously, $\psi^A\varphi^A = \text{id}_{AT}$.

□

Remark 1.4.2. Cursory examination of the proof reveals that we have actually proved a little more. Instead of assuming that all of B acts centrally on A , it would be enough to assume that all the elements b_t (which, in fact, define the B -homomorphism φ) act centrally on A^{1B} . However, this statement would be only cosmetically stronger, since the set of all elements of B that act centrally on A^{1B} forms a commutative unital subalgebra of B .

Evidently, A^{1B} and A are functors between the respective categories of algebras. The more appropriate categories would be, however, the categories of all B , A , or

A^{1B} -algebras, as the construction of the functor does not require the algebras to be semigroup algebras.

As any algebra is a unital strong $\mathbb{Z}$ -bimodule, we obtain the following.

Corollary 1.4.3. *An isomorphism of integral semigroup algebras $\mathbb{Z}S$ and $\mathbb{Z}T$ implies that $AS \cong AT$ for arbitrary algebra A .*

Most frequently, we will use the theorem in order to show that a semigroup algebra splits into a product of several semigroup algebras.

Corollary 1.4.4. *Let S, T, R be semigroups and A, B algebras as in the theorem above. If $BS \cong BR \times BT$ (B -algebra isomorphism), then $AS \cong AR \times AT$ (A -algebra isomorphism).*

Example 1.4.5. Consider, for a fixed $k > 1$, the cyclic nilpotent semigroup $S = \langle n \mid n^k = 0 \rangle$ and the two-element group $G = \mathbb{Z}_2$. For any algebra A , we shall examine the semigroup algebra $A(S \dot{\times} G)$. Lemma 1.3.6 on page 17 provides the isomorphisms

$$A(S \dot{\times} G) \cong AG(S) \cong AS(G).$$

The semigroup algebra is nilpotent — the second form makes this fact transparent. Therefore, there are no nonzero idempotents in the semigroup algebra. Yet, it will frequently split into a product of simpler algebras.

If B is any field such that $\text{char } B \neq 2$, we have $BG \cong B \times B$. Therefore we can split every group algebra of G as long as its coefficients are from an algebra over B . If A is such an algebra, we obtain $AG \cong A \times A$ (even if A does not have an identity) and consequently

$$A(S \dot{\times} G) \cong AS \times AS.$$

The isomorphisms can be made just as explicit as the B -isomorphism from which it is constructed. Only the lack of a canonical generator in A prevents us from using the shorthand commonly used for unital coefficient algebras, where it is enough to give the values of the isomorphism on the ‘canonical’ basis of BS .

Observe that no splitting would be possible had we tried to avoid non-unital situation by considering $(S \times G)^1$.

Chapter 2

Graph Semigroups

The general notion of graph semigroup extends the definition given in [3]. It is motivated by the extraordinary success of quiver-based methods in the representation theory of finite-dimensional algebras. Path algebras over quivers and the associated vector-space representations have been developed by Gabriel in [6] and then extended in [7], using the notion of a species, so as to be applicable to finite length abelian categories. Dlab and Ringel in [4] used this generalization in order to classify hereditary algebras of finite and tame type, bringing out to the full extent the theme of Dynkin and Euclidean (exceptional) diagrams — patterns that are now shared by all the distinct areas of mathematics known as representation theory.

Transformation representations of semigroups have been studied under various names and in varying degrees of generality. Among the terminology used for these representations of a semigroup S are the following: partial S -operands ([17]), S -polygons ([21]), pointed S -acts ([22]), S -sets ([1]), S -acts ([10]), with the last two terms being the most common. Exact assumptions on the nature of the action of S on the given set vary, but by far the most common is the case of S being a monoid

(semigroup with identity), 1 acting by identical transformation. Treatment of more general classes of semigroup or bi-representations is scarce.

Notable contributions to the study of transformation representations are two papers of Talwar [22, 23], where a theory of Morita-type equivalence for semigroups is developed, requiring much milder assumptions on semigroups than the existence of an identity element.

We have opted for the terminology of module theory used in [5]. The notion of transformation module would basically correspond to 0-centred S -operands of the classic two-volume reference work of Clifford and Preston [2].

We should point out that we do not explore in the present work the subject of transformation representations of graph semigroups. We merely use the concept in order to describe a way of building semigroups from (structurally simpler) parts. Similarly, we do not attempt to relate the relevant constructions to functorial properties of suitable module categories. Our aim is to exploit the construction to point out frequently occurring “symmetries” of semigroups.

2.1 Transformation Bimodules

Definition 2.1.1 (Semigroup action). Let S be a semigroup and X a set. We let $X^0 = X \cup \{0\}$, where 0 is a distinct element not belonging to X — an ‘external’ zero. We denote by $T_X^{\text{fix } 0}$ the subsemigroup of the full transformation semigroup on X^0 consisting of all maps fixing the distinguished element 0. Note that $T_X^{\text{fix } 0}$ has a zero: it is the constant map sending every element to 0.

We say that S *acts on the right* on X if we have a semigroup homomorphism $\Omega : S \rightarrow T_X^{\text{fix } 0}$ which is 0-preserving (that is, if S has a zero, it has to be mapped to

the zero of $T_X^{\text{fix } 0}$). The element of X^0 which is the result of applying $(s)\Omega$ to $x \in X$ is denoted simply by xs . In other words, we require that the map $X^0 \times S \rightarrow X^0$ satisfies

$$\text{i) } x(s_1 s_2) = (xs_1)s_2 \text{ for all } s_1, s_2 \in S, x \in X^0.$$

$$\text{ii) } 0s = 0 \text{ for all } s \in S.$$

$$\text{iii) } x0 = 0 \text{ for all } x \in X \text{ if } S \text{ has zero element } 0.$$

Whether the symbol 0 stands for the semigroup zero or for the distinguished element is usually apparent from the context.

Similarly we define when S acts on the left on a set X . This time, the action is given by a 0-preserving homomorphism of the opposite semigroup of S (the same set with the multiplication reversed) into $T_X^{\text{fix } 0}$, the result of the action of s on x is denoted by sx . Thus, aside from the “zero-preservation” identities, we require that $(s_1 s_2)x = s_1(s_2 x)$.

A set X equipped with a right (resp. left) action of S will be called a right (resp. left) *transformation S -module*. If S possesses an identity which acts on X by the identity transformation, we call the module X (resp. the action) a *unital S -module* (resp. action).

Remark 2.1.2. We do not object if the set X is empty. Such bimodules are just as useful as the trivial vector spaces in linear algebra are in saving us from the duty of stating extra assumptions when discussing, for instance, direct decompositions.

In contrast to some other treatments of the action of a semigroup on a set, we do not consider the 0 as part of the set X . One can see easily that the definition of the action is equivalent to providing a 0-preserving homomorphism into the semigroup

of partial transformations on X (this semigroup has the ‘empty’ map as its zero). In this equivalent formulation, there would be no need for an ‘external’ element. However, many of the foregoing constructs would be harder to formulate.

We purposely make no effort to denote the ‘external’ zero as something specific to the set X (or even the homomorphism Ω). Quite on the contrary — any time several sets are considered in the same context (that is, the same semigroup acts on them), we identify all the external elements.

Remark 2.1.3. Note that given a subsemigroup T of S and a set X with an action of S , we do not necessarily get an induced action of T on X in case that S has a zero and T has a zero distinct from that of S . There is no such problem with the subsemigroup $T \cup \{0_S\}$ of S , however.

Definition 2.1.4 (Transformation bimodule). Let S, T be semigroups. A set X (possibly empty) is a *transformation S - T -bimodule* if

- i) S acts on X on the left.
- ii) T acts on X on the right.
- iii) The left action of S and the right action of T are compatible, that is, $(sx)t = s(xt)$ for all $s \in S, t \in T, x \in X^0$.

If $S = T$, we call X simply an S -bimodule.

If S , resp. T , is the one-element semigroup and its action on X is unital, we see that an S - T -bimodule X is nothing else but a left S -module, resp. a right T -module.

Definition 2.1.5 (Unitary bimodule). Let X be a transformation S - T -bimodule. We call X *left unital* if it is unital as an S -module, and *right unital* if it is unital as a T -module. X is *unital* if it is unital on both sides.

Definition 2.1.6 (Subbimodules, factor-bimodules). A *subbimodule* of an S - T -bimodule X is a subset Y of X such that Y^0 is invariant under the left and right actions.

A *bimodule map* is a map compatible with the bimodule structure (a partial map, if thought of as a map on X only). For $f : X \rightarrow Y$, it amounts to requiring that

$$(sx)f = s(xf), (xt)f = (xf)t, \quad \text{for all } x \in X^0, s \in S, t \in T.$$

The set of all elements of X that are mapped to 0 is a (possibly empty) subbimodule of X . The image Xf is a subbimodule of Y , and is called *factor-bimodule* of X , since we can think of it as the set of ff^{-1} -equivalence classes of X induced by the map f .

As with Rees congruence on a semigroup, we can associate to every subbimodule Y of X a factor-bimodule on the set $X \setminus Y$, with bimodule structure given by the actions inherited from X , but identifying all elements of Y^0 with the 0 of the factor-bimodule. This bimodule is denoted by X/Y .

Definition 2.1.7 (Disjoint union, direct product). The *disjoint union* of two S - T -bimodules X and Y is the obvious bimodule structure on $X \cup Y$ and is denoted by $X \sqcup Y$. The elements of X and Y are treated as distinct, the ‘outside’ element is again 0.

There is no difficulty in extending this to arbitrary families of S - T -bimodules X_λ indexed by a set Λ . The disjoint union of the bimodules X_λ is the set

$$\bigsqcup_{\lambda \in \Lambda} X_\lambda = \bigcup_{\lambda \in \Lambda} (X_\lambda, \lambda)$$

with semigroups acting only on the first components of all the pairs. Here the sets were made disjoint explicitly, but we will try to avoid needless formalism if there is not a clear danger of misinterpretation.

The *direct product* of two S - T -bimodules X and Y is denoted by $X \times Y$, semigroups acting coordinate-wise on the cartesian product. For a family of bimodules X_λ , the product is denoted by $\prod_{\lambda \in \Lambda} X_\lambda$.

It is perhaps worth noting that these natural definitions would have to be written with a lot more care to make sure the zeroes got identified had we considered 0 as internal part of each of the sets X_λ (cf. Notation 1.1.1 on page 6).

The following claim is an immediate consequence of the definition.

Lemma 2.1.8. *If each X_λ is left (resp. right) unital, then both $\bigsqcup_{\lambda \in \Lambda} X_\lambda$ and $\prod_{\lambda \in \Lambda} X_\lambda$ are left (resp. right) unital.*

The semigroup S itself is equipped with a natural bimodule structure, given by left and right multiplication.

Definition 2.1.9 (Regular bimodule). Let I be a nonempty, nonzero ideal of S . We let $I^b = I \setminus \{0\}$ if S has zero and $I^b = I$ otherwise. The set I^b can be viewed as an S -bimodule, even though it is not necessarily an ideal of S anymore.

In the case $I = S$, we call S^b the *regular bimodule* of S .

2.2 S -product

There are several other ways of building a bimodule from a pair of bimodules. Two constructions that will allow us to introduce the fundamental notion of this chapter — the graph semigroup — are examined next.

First, we observe that if X is an S_1 - T_1 -bimodule and Y is an S_2 - T_2 -bimodule, then we can naturally regard both bimodules as $(S_1 \sqcup S_2)$ -($T_1 \sqcup T_2$)-bimodules, with “the other” semigroups acting trivially, and therefore we can form their disjoint sum: $X \sqcup Y$ is an $(S_1 \sqcup S_2)$ -($T_1 \sqcup T_2$)-bimodule.

The other construction is essentially the tensor product for transformation bimodules.

Definition 2.2.1 (S -product). Let R, S, T be semigroups. Assume X is an R - S -bimodule and Y an S - T -bimodule. The S -product $X \times_S Y$ of X and Y is the set of equivalence classes on the set-theoretic $X \times Y$ given by

$$X \times_S Y = X \times Y / \sim_S,$$

where $\sim_S$ is the equivalence generated by the relation identifying (xs, y) with (x, sy) for all $x \in X, y \in Y, s \in S$.

The set $X \times_S Y$ is naturally an R - T -bimodule since the relation $\sim_S$ is compatible with the action of semigroups R and T . It is enough to realize that the generating relation is a left congruence under the action of T , and a right congruence under the action of R . This immediately follows from the fact that X and Y are bimodules.

Remark 2.2.2. The map $f : X \times Y \rightarrow X \times_S Y$, assigning to a pair consisting of an element of X and an element of Y its equivalence class in $X \times_S Y$, is S -balanced:

$$f(xs, y) = f(x, sy) \text{ for all } x \in X, y \in Y, s \in S.$$

Moreover, from the definition of $\times_S$, f is universal with respect to this property: the kernel equivalence of any map $X \times Y \rightarrow Z$ which is a map of left R -modules as well as a map of right T -modules and is S -balanced will necessarily contain $\sim_S$ and thus the map will factor through $X \times_S Y$.

Observe that $X \times_S Y = X \times_{S^0} Y$.

Another trivial observation is that the operation of S -product results in empty set if all pairs are $\sim_S$ equivalent to 0 (the equivalence class containing all $(x, 0)$ and $(0, y)$).

Standard properties of ‘tensor’ product hold, and we will state them for ease of reference as a lemma. We do not include quite obvious statement on “associativity” of the disjoint union.

Lemma 2.2.3. *Let R, S, T and U be semigroups. Suppose X, X_λ are R - S -bimodules for all $\lambda \in \Lambda$ and Y, Y_v are S - T -bimodules for all $v \in \Upsilon$. Let Z be a T - U -bimodule.*

Then there is an isomorphism of R - U -bimodules

$$(X \times_S Y) \times_T Z \cong X \times_S (Y \times_T Z),$$

and isomorphisms of R - T -bimodules

$$\begin{aligned} \left(\bigsqcup_{\lambda \in \Lambda} X_\lambda \right) \times_S Y &\cong \bigsqcup_{\lambda \in \Lambda} (X_\lambda \times_S Y), \\ X \times_S \left(\bigsqcup_{v \in \Upsilon} Y_v \right) &\cong \bigsqcup_{v \in \Upsilon} (X \times_S Y_v), \\ \left(\bigsqcup_{\lambda \in \Lambda} X_\lambda \right) \times_S \left(\bigsqcup_{v \in \Upsilon} Y_v \right) &\cong \bigsqcup_{\lambda \in \Lambda, v \in \Upsilon} (X_\lambda \times_S Y_v). \end{aligned}$$

The following lemma applies, for example, in the frequent case of taking the S -product with a regular bimodule, where the semigroup has an identity. We say that an ideal I of S is *right cyclic* if there is an $a \in I$ such that $I = aI$, and we call it *left cyclic* if $I = Ib$ for some $b \in I$.

Lemma 2.2.4. *Let I be a nonzero ideal of S and X be an S - T -bimodule. If the ideal I is left cyclic and X is finite, then $I^b \times_S X = IX$ as S - T -bimodules.*

Dual statement holds for right cyclic ideals of T .

Proof. The set $IX = \{ix \mid i \in I, x \in X\}$ is an S - T -bimodule, and the map from (set-product) $I \times X \rightarrow IX$ given by $(i, x) \mapsto ix$ satisfies the fundamental property of S -product (described in Remark 2.2.2 on page 30) and is onto. Thus there is a surjective map of S - T -bimodules $I \times_S X \rightarrow IX$. The cyclicity assumption implies that $(a, ix) \sim_S (ai, x)$, where a is a generator of I . Thus there cannot be more than $|X|$ of $\sim_S$ -equivalence classes forming $I \times_S X$, and the induced surjective map must be an isomorphism. $\square$

If we start with an S -bimodule X , we may inductively form S -bimodules $X \times_S X$, $X \times_S X \times_S X$, $\dots$. These modules will be written as $X^{\times_S^n}$ (for each positive integer n ; when $n = 1$, the bimodule is the X itself). For formal reasons, we will find it convenient to put $X^{\times_S^0} = S^b$, the regular S -bimodule.

The following lemma generalizes the fundamental property of the S -product to multiple S -products in the important special case of S -powers of a single bimodule.

Cartesian powers X^n of a bimodule X are considered as left S -modules under the left action of S on the first component. Similarly, the right action of S on the last component makes X^n a right S -module.

Lemma 2.2.5. *Let X, Y be S -bimodules, $n \geq 1$ a positive integer. Suppose*

$$f : X^{\times_n} = X \times \dots \times X \rightarrow Y$$

is a map of left S -modules as well as a map of right S -modules and is S -balanced:

$$f(x_1, \dots, x_i s, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_i, s x_{i+1}, \dots, x_n)$$

for all $s \in S, x_j \in X (j = 1, \dots, n)$ and all $i = 1, \dots, n - 1$.

Then f factors through $X^{\times_S^n}$, inducing an S -bimodule map $f' : X^{\times_S^n} \rightarrow Y$.

2.3 Stable Bimodules

The definition of a bimodule is quite general, allowing an arbitrary semigroup to act on sets in complex ways even if the semigroup is far from being a group (for instance, having many divisors of zero). The disadvantage in considering arbitrary bimodules is that the action of semigroups may be arbitrarily degenerate. Consequently, structural properties of acting semigroups will be reflected extremely poorly by some bimodules.

The notion of a stable bimodule is introduced in order to curtail such possibility. Stable bimodules satisfy a mild regularity condition which prevents the semigroup action from being unduly degenerate. Yet the class of stable bimodules is far more inclusive than the class of unital bimodules.

Definition 2.3.1 (Stable bimodule). We say that an S - T -bimodule X is *stable* if for every $x \in X$ there is some $s \in S$ and $t \in T$ such that $sx = x$ and $x = xt$.

Any unital bimodule is obviously stable, but the converse is certainly not true — definition of stable bimodules does not even require existence of identities in acting semigroups.

It follows that subbimodules and factor-bimodules of arbitrary stable bimodule are stable as well. Also the two main constructs of the preceding sections are well-behaved.

Lemma 2.3.2. *Let Λ be a set indexing a family of S - T -bimodules X_λ .*

- i) *If each S - T -bimodule X_λ is stable, then $\bigsqcup_{\lambda \in \Lambda} X_\lambda$ is a stable S - T -bimodule.*
- ii) *If X is a stable R - S -bimodule and Y a stable S - T -bimodule, then $X \times_S Y$ is a stable R - T -bimodule.*

Unlike unital modules, stable modules are not closed under direct products. However, there is an important property that stable modules share with unital modules.

As the next lemma shows, stable modules canonically decompose if one can decompose the semigroup acting on them.

Lemma 2.3.3. *Let W be a stable S - T -bimodule. If $S = \bigsqcup_{i \in \Lambda} S_i$, $T = \bigsqcup_{j \in \Upsilon} T_j$, then W uniquely decomposes as $\bigsqcup_{i \in \Lambda, j \in \Upsilon} W_{ij}$, where W_{ij} is an S_i - T_j -bimodule such that*

$$(S \setminus S_i)W_{ij} = 0 = W_{ij}(T \setminus T_j).$$

Proof. Let $x \in W$ and let $s \in S$, $t \in T$ be such that $sx = x = xt$. As s, t are clearly nonzero, there are $i \in \Lambda$, $j \in \Upsilon$ such that $s \in S_i$, $t \in T_j$. Now take arbitrary $s' \in S \setminus S_i$. We have

$$s'x = s'(sx) = (s's)x = 0x = 0,$$

and similarly, any $t' \in T \setminus T_j$ gives $xt' = 0$. The indices i, j are thus unique, and the map $x \mapsto (i, j)$ established in this way partitions W into a disjoint union of (possibly empty) sets W_{ij} , which are easily seen to be S_i - T_j -bimodules. The claim follows. $\square$

Frequently, bimodules are built from subbimodules or factors of a regular bimodule. It is therefore useful to state a few conditions guaranteeing that such a bimodule be stable.

Remark 2.3.4. Many semigroups have the following property: there is a subset U of idempotents of S such that $US = S = SU$. The regular bimodule of such a semigroup is certainly stable.

We recall that S is a *regular semigroup* if every element $s \in S$ is regular. S is called an *eventually regular semigroup* if for every $s \in S$, there is an $n \geq 1$ such that s^n is a regular element of S . All finite and all regular semigroups, as well as periodic semigroups (semigroups not containing a free semigroup on one generator) are eventually regular.

In the class of eventually regular semigroups, the converse to Remark 2.3.4 on the page before holds.

Lemma 2.3.5. *Let S be an eventually regular semigroup. Then S^b is stable if and only if there is a subset U of idempotents of S such that $US = S = SU$.*

Proof. We need to prove the reverse implication only. Let $x \in S^b$ and $s \in S$ be such that $sx = x$. Then $s^n x = x$ for all $n \geq 1$, and there is at least one of them for which s^n is regular. But for a regular element, there certainly is an idempotent e such that $es^n = s^n$. Thus $ex = x$. Dually, one can show that $xf = x$ for some idempotent f of S . $\square$

Remark 2.3.6. It is easy to see that if S^b is stable then S cannot have any nontrivial factors without identity that would be one of the following:

- i) A nilpotent semigroup;
- ii) A left-cancellative semigroup;
- iii) A right-cancellative semigroup.

2.4 Free Graph Semigroups

Let Q be an oriented graph (often called *quiver*). There may be multiple edges between the same pair of vertices, as well as *loops* — edges starting and ending at the same vertex.

The set of all oriented paths in Q , including those of length zero (which are in one-to-one correspondence with vertices of Q), together with an extra element 0 forms a semigroup. The product of two paths is their concatenation if the first path ends in the same vertex where the second path begins. In all other cases, the product is defined to be 0.

The resulting semigroup will be called the *path semigroup* of Q .

Definition 2.4.1 (Modulated graph). Let Q be a finite oriented graph. A *semigroup modulation* on the graph Q is an assignment of a semigroup S_i to each vertex i and a transformation S_i - S_j -bimodule W_{ij} to each edge $i \rightarrow j$ of Q .

The orientation of edges determines which semigroup acts on the left (the one corresponding to the starting vertex of the arrow) and which one acts on the right (the semigroup assigned to the ending vertex). This is a helpful convention, as we are going to write the composition of paths from the left (first path) to the right (second path).

Definition 2.4.2 (Free graph semigroup). Let Q be an oriented graph with a modulation given by semigroups S_i and S_i - S_j -bimodules W_{ij} . Let S be the contracted disjoint sum of the semigroups S_i , and W the disjoint union of all bimodules W_{ij} . This W is an S -bimodule and we put

$$FQ = FQ(S, W) = \bigsqcup_{n \geq 0} W^{\times_S^n} \cup \{0\}.$$

The adjoined zero element is identified with the ‘external’ zero of each bimodule. We call FQ the *free graph semigroup* over Q given by the modulation (S, W) .

“Internalizing” zero this way makes FQ a semigroup, since now we can multiply any two elements of $W^{\times_S^{n_1}}$ and $W^{\times_S^{n_2}}$, if these sets are nonempty, using the maps of pairs of elements induced by the S -product. Such product will either belong to $W^{\times_S^{(n_1+n_2)}}$ or it will be 0. The product of 0 with any element of FQ (in any order) is defined to be the 0 itself.

Definition 2.4.3 (Trivial modulation). Assume that Q is an oriented graph. Let S_i be the one-element semigroup for each vertex i of Q , and W_{ij} be a one-element set on which both S_i and S_j act as identity (for each edge $i \rightarrow j$ of Q). We call this modulation the *trivial modulation* on Q .

Remark 2.4.4. With the trivially modulated Q , the free graph semigroup FQ is isomorphic to the path semigroup of Q .

This is not the only connection of path semigroups and free graph semigroups that one can immediately observe.

While the notation makes it clear that FQ is ‘graded’ by nonnegative integers, much more is true. Since W is a disjoint union, its every nonempty S -power $W^{\times_S^n}$ splits, by repeated application of Lemma 2.2.3 on page 31, into a disjoint union of S -products of modules W_{ij} , each product having n terms. There is exactly one path of the path semigroup of Q corresponding to each of these terms, and this gives a ‘grading’ on FQ by the path semigroup of Q .

A much weaker grading on FQ will be very useful to consider. The simple fact that the product of two paths of a free path semigroup is zero if and only if the paths do not concatenate has an interesting consequence. If we declare two nonzero paths

$\sim$ -equivalent whenever they both start at the same vertex and also their end vertices coincide — this is clearly an equivalence on nonzero paths — and consider the zero path to be an equivalence class by itself, then this equivalence is a congruence due to the aforementioned property.

Definition 2.4.5 (Path-class semigroup). The factor-semigroup $P^{\text{cl}}Q$ of the free path semigroup by the congruence $\sim$ will be called the *path-class semigroup* of Q .

It is immediate that the path-class semigroup is a subsemigroup of the semigroup $\mathcal{M}_\Lambda$ of elementary matrix units on the set Λ of vertices of Q (cf. Notation 1.1.3 on page 8). Namely,

$$P^{\text{cl}}Q \cong \{m_{ij} \in \mathcal{M}_\Lambda \mid \text{there is an oriented path from } i \text{ to } j \text{ in } Q\} \cup \{0\}.$$

Remark 2.4.6. Every element of the free graph semigroup FQ can be clearly assigned a unique element of the path-class semigroup $P^{\text{cl}}Q$. In this way, the set FQ decomposes as a disjoint union of sets FQ_{ij} (or FQ_0) which can be thought of as being indexed by elements of $P^{\text{cl}}Q$. The nonempty sets of the decomposition will be called *path-class components* of FQ .

Each path-class component FQ_{ij} has a structure of an S_i - S_j -bimodule and this structure completely determines its S -bimodule structure (since ‘the other’ semigroups S_k act trivially on FQ_{ij}). In the next section, path-class components are defined for a wider class of semigroups.

Example 2.4.7. Consider the following graph with a modulation given by semigroups S_1, S_2, S_3 and appropriate bimodules $W_{13}, W'_{13}, W_{12}, W_{23}$.

In this case, $P^{\text{cl}}Q$ has seven elements:

$$P^{\text{cl}}Q = \{m_{11}, m_{22}, m_{33}, m_{12}, m_{13}, m_{23}, 0\}.$$

For instance, the path-class component FQ_{13} is isomorphic to

$$W_{13} \sqcup W'_{13} \sqcup (W_{12} \times_{S_2} W_{23})$$

as an S_1 - S_3 -bimodule.

We end this section with summarizing basic properties of FQ that do not involve its path-class components.

Lemma 2.4.8. *Let FQ be the free graph semigroup of a graph Q with a modulation given by S and W .*

- i) S is a subsemigroup of FQ .
- ii) $NQ = FQ \setminus S \cup \{0\}$ is an ideal of FQ .
- iii) The ideal NQ is generated by $W \cup \{0\}$.
- iv) The factor-semigroup FQ/NQ is isomorphic to S^0 .

The ideal NQ of FQ will be called the *radical* of FQ . It is easy to observe that the radical is a nilpotent semigroup whenever Q has only a finite number of vertices and has no oriented cycles.

We recall here that since S^0 is a subsemigroup of FQ sharing the same zero element, we can regard every FQ -bimodule as S^0 -bimodule as well.

Lemma 2.4.9. *Let FQ be the free graph semigroup of a graph Q with a modulation given by S and W .*

i) We have the chain of FQ -bimodules

$$(NQ \cdot NQ)^b \subset (NQ)^b \subset FQ^b.$$

ii) There are the following isomorphisms for the corresponding factor-bimodules (both as S^0 -bimodules and FQ -bimodules):

$$FQ^b/(NQ)^b \cong (S^0)^b, \quad (NQ)^b/(NQ \cdot NQ)^b \cong W.$$

2.5 Graph Semigroups

Definition 2.5.1 (Radical-square supported congruence). Let FQ be a free graph semigroup, NQ its radical, and ρ a congruence on FQ . We say that ρ is a *radical-square supported congruence* if all nontrivial ρ -equivalence classes lie in the ideal $NQ \cdot NQ$.

The main point of this definition is that if we consider the factor-semigroup FQ/ρ and call the ideal NQ/ρ its radical, then the two factor-semigroups obtained from FQ and FQ/ρ by factoring out their radical-squares are going to be isomorphic. Moreover, the isomorphism, being induced by ρ , identifies images of radicals of the two factor-semigroups.

It is easily seen that the semigroup FQ/ρ will satisfy the two lemmas of the previous section (reading FQ/ρ instead of FQ and NQ/ρ instead of NQ).

One would ideally like to have also some of the nice properties of the free graph semigroup, such as suitable grading, preserved in the square of the radical of the semigroup FQ/ρ . This would provide a lot more information on the structure of the factor than just knowing that it is identical to the free graph semigroup in the

upper two ‘layers’. None of the gradings, however, will be preserved in general by all congruences on the free graph semigroup.

Definition 2.5.2 (Path-class and homogeneous congruences). Let ρ be a radical-square supported congruence on a free graph semigroup FQ with a modulation given by S and W .

One says that ρ is a *path-class congruence* if each nonzero ρ -equivalence class lies in some path-class component FQ_{ij} .

We call ρ a *homogeneous congruence* if it is a path-class congruence for which each nonzero ρ -equivalence class lies in $W^{\times n}$ for some $n > 1$.

The following definition applies to any semigroup that is a factor of a free graph semigroup by a path-class congruence, and generalizes what has been introduced in Remark 2.4.6 on page 38 for free graph semigroups.

Definition 2.5.3 (Path-class components). Let ρ be a path-class congruence on $FQ = FQ(S, W)$. Then $T = FQ/\rho$ is a disjoint union of *path-class components* of T : these are the sets

$$T_{ij} = \{\text{nonzero } \rho\text{-classes of } FQ_{ij}\}$$

and $T_0 = \{0_T\}$, indexed by elements of the path-class semigroup (some of the sets may be empty, including T_0).

The sets $T_{ii} \cup \{0_T\}$ are clearly subsemigroups of T , whenever nonempty. The sets T_{ij} are bimodules under the action of these semigroups. For radical-square supported congruences, $S_i \subseteq T_{ii}$, and if there is an edge $i \rightarrow j$ in Q , then $W_{ij} \subset T_{ij}$.

It is immediate that factors of free graph semigroups by path-class congruences inherit a “grading” given by path-class semigroup in the following sense.

Multiplying elements t, t' from two components results either in zero or in an element from the component indexed by the product (in the path-class semigroup) of the two path-class component indices of t and t' .

A more refined grading is available for factors given by homogeneous congruences — these factors can be “graded” by the path semigroup of the underlying graph.

Now the question is whether one could give a characterization of semigroups which arise as factors of free graph semigroups by well-behaved congruences without appealing to congruences on FQ . This can be accomplished, indeed, if we assume that the bimodule W is stable (see Definition 2.3.1 on page 33).

Remark 2.5.4. Lemma 2.3.3 on page 34 has a notable consequence. One can assign essentially unique (up to isomorphism of modulated graphs) modulated graph to a stable S -bimodule W as soon as a decomposition $S = \bigsqcup_{i \in \Lambda} S_i$ is fixed. Vertices will correspond to S_i 's, edges to nonempty W_{ij} 's. For a particular decomposition of S , we may therefore speak about ‘the graph’ of W .

Now it is easy to show that with a stable bimodule, we cannot have other congruences than path-class congruences.

Lemma 2.5.5. *Let W be a stable S -bimodule, FQ a free graph semigroup modulated by S and W . Then every radical-square supported congruence on FQ is a path-class congruence.*

Proof. Let ρ be a radical-square supported congruence on FQ . Let $a, b \in FQ$ be nonzero elements from $NQ \cdot NQ$ such that $a \rho b$.

There are some $x, y \in W$ and $a', b' \in NQ$ for which $a = xa'$, $b = yb'$. As W is stable, we have $x \in W_{ij}$ and $y \in W_{kl}$ for uniquely determined vertices i, j, k, l . Let

$s \in S$ be such that $sx = x$. Since $a\rho b$, we have also $sa = sxa' = xa' = a\rho sb = syb'$. Thus if $i \neq k$, we obtain from $sy = 0$ that $a\rho 0$, and thus $b\rho 0$.

In the same vein, with an action on the right, one gets that whenever a and b do not have the same ending vertex, then both a and b belong to the ρ -class of 0.

Consequently, if $a\rho b$, they are in the same class-path component. $\square$

Let us now introduce notation that will help to formulate the notion of graph semigroup.

Notation 2.5.6. We will write $T = S \triangleleft N$ if S is a subsemigroup and N an ideal of T such that $T = S \cup N$ and $S \cap N \subseteq \{0\}$. As with algebras (cf. Remark 1.2.2 on page 10), we call S a *retract* of T .

If N is nonempty, we have the obvious identification $S^0 \cong T/N$, and N^b is an S^0 -bimodule, with $(N \cdot N)^b$ as its subbimodule. One would like to reconstruct the structure of T using just the subsemigroup S^0 and the S^0 -bimodule $W = N^b / (N \cdot N)^b$. Let us call the elements of W *irreducible*, as these are, by definition, precisely all nonzero elements of N which are not a product of two such elements.

We shall now add a requirement that there are ‘enough’ elements in W . Namely, the set W ought to ‘generate’ the ideal N . Note that W is canonically a subset of N , and by saying W generates N^b we mean that the smallest subsemigroup containing W includes all of N with the possible exception of 0. This is a slight misuse of the notation, since the bimodule structures of W and N^b play no role in this assumption.

Definition 2.5.7 (Graph semigroup). Let $T = S \triangleleft N$, with S a subsemigroup, N an ideal of T . Denote by W the bimodule of irreducibles, that is, the S^0 -bimodule $N^b / (N \cdot N)^b$. We call T a *graph semigroup* if W is a stable S^0 -bimodule generating N^b .

The two lemmas at the end of the previous section imply that any factor of a free graph semigroup by a path-class congruence is a graph semigroup provided the S -bimodule W giving the modulation is stable. The main theorem of this chapter gives the converse of this claim — all graph semigroups can be obtained in this manner.

Let us recall the observation stated in Remark 2.5.4 on page 42: a stable bimodule determines a modulated graph for every decomposition of the acting semigroup as a contracted disjoint sum.

Theorem 2.5.8 (Characterization of graph semigroups). *Let $T = S \triangleleft N$ be a graph semigroup, W its bimodule of irreducible elements. Then T is a factor-semigroup of a free graph semigroup (on a graph of W) by a path-class congruence.*

Proof. Let Q be a graph of W with respect to a decomposition of S as a disjoint sum, and let $FQ = FQ(S, W)$ be its free graph semigroup. Since FQ is generated by the set $X = \{0\} \cup S \cup W = \{0\} \cup W^{\times_S^0} \cup W^{\times_S^1}$, a map from X to the semigroup T will extend in at most one way to a semigroup homomorphism $FQ \rightarrow T$.

Let $g : X \rightarrow T$ be the map identifying elements of X with their counterparts in T — it is an S -bimodule map. Now define for each $n \geq 1$ a map f_n from the n -th cartesian power of W into T as follows:

$$\begin{aligned} f_n : W^{\times^n} = W \times \cdots \times W &\rightarrow T \\ (w_1, \dots, w_n) &\mapsto w_1 \cdots w_n. \end{aligned}$$

That is, f_n maps each n -tuple of w_i 's to the product of their counterparts in T . This map certainly satisfies the assumptions of Lemma 2.2.5 on page 32, and the induced bimodule maps f'_n will be exactly the maps which g induces on $W^{\times^n}$. If we

denote by $f_0 = f'_0$ the restriction of g to S^0 and by $f_1 = f'_1$ the restriction to W , it is immediate that the bimodule maps f'_n inherit the property of f_n 's:

$$(w_1, \dots, w_{n+m})f_{n+m} = (w_1, \dots, w_n)f_n \cdot (w_{n+1}, \dots, w_{n+m})f_m$$

for all $n, m \geq 1$ and therefore g does indeed give rise to a semigroup homomorphism from FQ to T — we will denote the homomorphism simply by g .

The image of $W^{\times n}$ under g is all of $W^n = W \cdot W \cdots W \cdot W$ (n times, here W as a subset of T) for every $n \geq 2$. Moreover, $W^n \subseteq N \cdot N$. This fact has two consequences.

First, since g is one-to-one on X , the kernel congruence of g must be radical-square supported. By Lemma 2.5.5 on page 42, it must be a path-class congruence.

Second, because $N \subseteq \{0\} \cup W \cup W \cdot W \cup \dots$, the homomorphism g is surjective. $\square$

Remark 2.5.9. One may observe that the assumption that W be stable was not really needed to establish that T is a factor of a graph semigroup by a radical-square supported congruence. All we needed the assumption for was to make sure the congruence is path-class congruence and, in case S has a nontrivial decomposition, to obtain “grading” on T by a nontrivial path-class semigroup. It is exactly these fine points that will be crucial for the discussion in the next section.

2.6 Diagonal Bimodules and M-Variants

The grading of a graph semigroup by its path-class semigroup is very useful if we want to describe one type of commonly occurring structural symmetries.

We have seen that stable modules decompose according to a decomposition of the acting semigroup as a disjoint sum. This kind of decomposition, however, can

be carried out only so far — many semigroups will not decompose any further in the same manner.

Let us now consider the situation where some of the components S_i are a (contracted) direct product of two semigroups. For instance, semigroups of the form $\mathcal{M}_\Lambda \dot{\times} G$ frequently occur (where Λ is a set and G a group), as these are fundamental building blocks of inverse semigroups. In order to take advantage of such splitting to infer a further structure of a graph semigroup, much more will be needed. The main reason is that the mere existence of a splitting of two semigroups as a direct product does not automatically entail any compatible splitting of a bimodule over these semigroups.

We shall define what is a natural bimodule having such compatible structure.

Definition 2.6.1 (Diagonal bimodule). Let X be an S - T -bimodule, and Y an S' - T' -bimodule. Then the *diagonal bimodule* $X \times Y$ is the cartesian product $X \times Y$ endowed with the action

$$(s, s')(x, y)(t, t') = (sxt, s'yt')$$

for all $(s, s') \in S \times S'$, $(t, t') \in T \times T'$, $(x, y) \in X \times Y$. This set $X \times Y$ can be regarded both as an $(S \times S')$ - $(T \times T')$ -bimodule or as an $(S \dot{\times} S')$ - $(T \dot{\times} T')$ -bimodule.

Diagonal bimodules are well behaved under S -products and disjoint unions.

Lemma 2.6.2. *Let X, Y be as above, so that $X \times Y$ is the diagonal $(S \dot{\times} S')$ - $(T \dot{\times} T')$ -bimodule. Similarly, assume $V \times W$ is the diagonal $(R \dot{\times} R')$ - $(S \dot{\times} S')$ -bimodule built from appropriate bimodules. Then*

$$(V \times W) \times_{S \dot{\times} S'} (X \times Y) \cong (V \times_S X) \times (W \times_{S'} Y)$$

as $(R \dot{\times} R')$ - $(T \dot{\times} T')$ -bimodules.

The left-hand side is an $S \times S'$ -product of diagonal bimodules, the right-hand side is a diagonal bimodule build from an S -product and an S' -product.

This property could be used to construct a *diagonal product* of two graph semigroups having compatible graphs. Such construction would yield a semigroup whose path-class components are the diagonal bimodules built from the corresponding path-class components of the two semigroups.

We shall consider below only a very specific kind of the diagonal product — one which allows us to build “Morita-equivalent” variants of a semigroup.

But first, we shall look at the semigroups of matrix units. Recall from Notation 1.1.3 on page 8 that $\mathcal{M}_\Lambda$ denotes the semigroup of elementary matrix units on a set Λ and δ_{ij} is the symbolic Kronecker delta.

Remark 2.6.3. Let us for a moment denote by $\{1\}$ the one-element semigroup. If X is a left S -module and Y a right T -module, we can regard X as S - $\{1\}$ -bimodule and Y as a $\{1\}$ - T -bimodule (the $\{1\}$ acting as identity). Thus we may built an S - T -bimodule $X \times_{\{1\}} Y$, whose underlying set is the cartesian product $X \times Y$.

A simple, but frequently occurring $\mathcal{M}_\Upsilon$ - $\mathcal{M}_\Lambda$ -bimodule is the one built from a nonzero $\mathcal{L}$ -class of $\mathcal{M}_\Upsilon$ and a nonzero $\mathcal{R}$ -class of $\mathcal{M}_\Lambda$. Because of its importance, we define the bimodule explicitly.

Definition 2.6.4 (Bimodule $M_{\Upsilon\Lambda}$). Let Υ, Λ be nonempty sets. We will denote by $M_{\Upsilon\Lambda}$ the set $\Upsilon \times \Lambda$, written equivalently as $\{x_{mn} \mid m \in \Upsilon, n \in \Lambda\}$, with the $\mathcal{M}_\Upsilon$ - $\mathcal{M}_\Lambda$ -bimodule structure given by

$$m_{ij}x_{mn} = \delta_{jm}x_{in}, \quad x_{mn}m_{kl} = \delta_{nk}x_{ml}$$

for all $i, j, m \in \Upsilon$ and $k, l, n \in \Lambda$.

One can see that the bimodules $M_{\Upsilon\Lambda}$ are stable: the action of idempotents m_{mm} of $\mathcal{M}_\Upsilon$ on the left and m_{nn} of $\mathcal{M}_\Lambda$ on the right fixes the element x_{mn} of $M_{\Upsilon\Lambda}$. Also the following elementary observation is not surprising.

Lemma 2.6.5. *Let $\Upsilon, \Lambda, \Omega$ be nonempty sets. Then there are the following isomorphisms of bimodules:*

- i) $M_{\Lambda\Lambda} \cong \mathcal{M}_\Lambda^b$,
- ii) $M_{\Upsilon\Lambda} \times_{\mathcal{M}_\Lambda} M_{\Lambda\Omega} \cong M_{\Upsilon\Omega}$.

Proof. i) This part follows directly from the definitions.

- ii) Let us demonstrate the second claim by determining the $\sim_{\mathcal{M}_\Lambda}$ -classes of $M_{\Upsilon\Lambda} \times M_{\Lambda\Omega}$.

First, if $(x_{ij}, x_{kl}) \sim_{\mathcal{M}_\Lambda} (x_{i'j'}, x_{k'l'})$, then certainly $i = i'$ and $l = l'$, since these outer indices are invariant under the action defining $\sim_{\mathcal{M}_\Lambda}$.

Second, if (x_{ij}, x_{kl}) has $j \neq k$, then it is $\sim_{\mathcal{M}_\Lambda}$ -equivalent to 0: $x_{ij}m_{jj} = x_{ij}$, but $m_{jj}x_{kl} = 0$.

Third, for fixed i and l , all elements (x_{ij}, x_{jl}) are equivalent:

$$(x_{ij}, x_{jl}) = (x_{ij}, m_{jk}x_{kl}) \sim_{\mathcal{M}_\Lambda} (x_{ij}m_{jk}, x_{kl}) = (x_{ik}, x_{kl})$$

for arbitrary $k \in \Lambda$. Moreover, these are easily seen to be all elementary $\sim_{\mathcal{M}_\Lambda}$ -relations applicable to elements (x_{ij}, x_{jl}) , so these classes are nonzero and correspond bijectively with the elements of $M_{\Upsilon\Omega}$.

The bimodule structure of the $\sim_{\mathcal{M}_\Lambda}$ -classes clearly coincides with that of $M_{\Upsilon\Omega}$.

□

Remark 2.6.6. One may notice that we did not actually need all elements of $\mathcal{M}_\Lambda$ in order to obtain the $\mathcal{M}_\Lambda$ -equivalence classes described in the proof. All that suffices in place of $\mathcal{M}_\Lambda$ is any subsemigroup of $\mathcal{M}_\Lambda$ that contains all the idempotents of $\mathcal{M}_\Lambda$ and every pair m_{ii}, m_{jj} of idempotents is ‘connected’: there is a sequence $e_0 = m_{ii}, e_1, \dots, e_{t-1}, e_t = m_{jj}$ of idempotents of $\mathcal{M}_\Lambda$ such that for every $1 \leq p \leq t$, there is an m in the subsemigroup for which $e_{p-1}m = m = me_p$ or $me_{p-1} = m = e_pm$.

We will encounter one such semigroup in the next chapter. We define just the finite version here.

Notation 2.6.7. The *triangular matrix units semigroup* is the subsemigroup $\mathcal{T}_n$ of $\mathcal{M}_n$ consisting of 0 and all m_{ij} with $i \leq j$.

Let Q be an oriented graph. Assign to each vertex i a nonempty set Λ_i . We can now use the modulation M of Q given by the semigroups $\mathcal{M}_{\Lambda_i}$ and the bimodules $M_{\Lambda_i \Lambda_j}$ (for each edge $i \rightarrow j$ of Q) to specify the diagonal product sketched above.

Recall that S_{ij} denote the path-class components of a graph semigroup S (see Definition 2.5.3 on page 41).

Definition 2.6.8 (M-variant). Let M be as above, and let S be a graph semigroup over Q . We define the *M-variant* of S (determined by Λ_i ’s) to be the semigroup $\tilde{S}$ which is the union of $\{0\}$ and diagonal bimodules $\tilde{S}_{ij}$ for nonzero $ij \in P^{\text{cl}}Q$, where

$$\tilde{S}_{ij} \cong M_{\Lambda_i \Lambda_j} \times S_{ij}.$$

The product in $\tilde{S}$ is given by $\tilde{S}_{ii}^0$ -products, where

$$\tilde{S}_{ii}^0 \cong \mathcal{M}_{\Lambda_i} \dot{\times} S_{ii}^0$$

for every vertex i of Q . The element $\{0\}$ of $\tilde{S}$ ‘internalizes’ the 0 of all the bimodules, as usual.

Note that the regular bimodule of the semigroup $\tilde{S}_{ii}^0$ is exactly the bimodule $\tilde{S}_{ii}$. The multiplication in $\tilde{S}$ is easily understood in view of Lemma 2.6.2 on page 46.

Remark 2.6.9. Suppose the graph semigroup S has a modulation

$$\{S_i, W_{ij}\}$$

where i indexes vertices of Q as before and ij now indexes all edges $i \rightarrow j$ in the graph Q (not elements of $P^{\text{cl}}Q$). It is easy to observe that the semigroup $\tilde{S}$ is a graph semigroup on the same graph Q , its modulation is given by

$$\{\mathcal{M}_{\Lambda_i} \times S_i, M_{\Lambda_i \Lambda_j} \times W_{ij}\},$$

with i denoting vertices of Q and $i \rightarrow j$ edges of Q .

Chapter 3

Decompositions of Semigroup Algebras

Very simple procedures are used to split algebras into direct products. All are eventually based on central idempotents. There are, however, details not to be overlooked, as only some idempotents will give a splitting into semigroup algebras (that would be easily recognised as such). Knowing the form of the used central idempotent, the resulting isomorphism of semigroup algebras can be described explicitly.

An interesting class of semigroups where existence of suitable central idempotents is guaranteed are (finite) inverse semigroups. We demonstrate the decompositions afforded by their semigroup algebras. Due to the simple form of the relevant central idempotents, such decompositions hold for arbitrary coefficient algebra. Similar results for some notable subsemigroups of symmetric inverse subsemigroups are obtained as well.

We specialize the concept of a graph semigroup. The resulting notion of R-

graph semigroup axiomatizes a semigroup where regular and non-regular elements are suitably separated. This is an important step in our approach, since it allows us to deal effectively with non-regular classes of a semigroup. This contrasts with most techniques that as a first step reduce to principal factors or use irreducible characters, de facto discarding nonregular classes. R-graph semigroups turn out to be very useful for the description of algebras of suitable subsemigroups of finite inverse semigroup. The structure of these semigroup algebras is generally much more complicated than that of algebras of inverse semigroups.

3.1 Factoring Algebras

In this section, we shall frequently speak of elements central on a subset of an algebra or semigroup, meaning that they commute with all elements of that subset. Similarly, saying that an element is an identity on a subset (typically an ideal) means nothing else but the condition that multiplying by the element on either side results in identity operation when restricted to the subset.

In the class of unital algebras, all decompositions into a direct product arise from central idempotents. This is not true for non-unital algebras. Nevertheless, we pay a great deal of attention to the existence and construction of central idempotents — they will be fundamental to our considerations in this chapter.

First of all, we introduce convenient notation.

Definition 3.1.1 (Ideals A^e and A_e). Let A be an algebra, $e \in A$ a central idempotent. Denote by A^e the ideal $\langle\langle e \rangle\rangle$ generated by e . The symbol A_e will stand for the annihilator ideal $\text{Ann}(e) = \{a \in A \mid eae = 0\} = \{a \in A \mid ea = 0\} = \{a \in A \mid ae = 0\}$.

Note that since e is central, we have always $ea = ae$ for every $a \in A$. We prefer writing this expression as ea since this form is symmetric. The algebra A splits as $A^e \times A_e$. The direct factor A^e is a unital algebra (the identity being e), while A_e is unital if and only if A is unital.

Lemma 3.1.2. *Let A be an algebra, I an ideal of A and $e \in I$ an idempotent. If e is central in I , then it is central in A .*

Proof. Let $a \in A$. Then

$$ea = e(ea) = (ea)e = e(ae) = aee = ae,$$

the second and fourth transformations being possible due to the fact that $ea, ae \in I$ and e is assumed to commute with elements of I . $\square$

Remark 3.1.3. One can see that a weaker assumption that e commutes with elements of eA and Ae would be sufficient for the validity of the statement.

We now continue to investigate the problem of manufacturing central idempotents using just idempotents. The following proposition will make it possible to lift a central idempotent from a factor-algebra if additional assumptions hold.

Let $a \in A$ be an arbitrary element, and $I < A$ an ideal. The condition that the class of a in the factor-algebra A/I be central is easily expressed using the usual commutator notation — it is equivalent to the condition $[a, A] \subseteq I$.

Proposition 3.1.4. *Let A be an algebra, I an ideal of A and $e \in I$ an identity on I . Let f be an idempotent in A such that $[f, A] \subseteq I$. Then the element*

$$f' = f - efe$$

is a central idempotent in A . Furthermore, the central idempotents f' and e are orthogonal.

Remark 3.1.5. While the existence of such an idempotent f' in A is hardly surprising given that the ideal I can be split off as a direct factor of A , we are leaving the statement in the present form. It is the form of the idempotent which will be a recurring theme of this chapter. In essence, it characterizes the isomorphism $A \cong A/I \times I$.

Proof. As e is an identity on I , it is a central idempotent in A by the previous lemma. Hence $fe = ef = efe$, and we easily obtain

$$f'^2 = f^2 - fefe - efef + efe^2fe = f^2 - efe,$$

proving that f' is indeed an idempotent. It is immediate that $f'e = 0 = ef'$, that is, the idempotents f' and e are orthogonal.

It remains to show that f' is central. Take an arbitrary $a \in A$. Then

$$[f', a] = [f, a] - [efe, a] = [f, a] - e[f, a].$$

The last equality is thanks to the fact that e is central in A . But we are actually assuming that e acts as identity on I , and $[f, a]$ is in I by our second assumption. Thus $[f', a] = 0$. □

Remark 3.1.6. We note again that the assumptions could have been weakened. We need e to be a central idempotent acting as identity on $[f, A]$ only. In other words, $[f, A] \subseteq eIe$ (with e a central idempotent in I) is a sufficient assumption.

If the ideal I is the complementary ideal of a retract $A = B \triangleleft I$, the commutator assumption is trivially satisfied for central idempotents of B .

Corollary 3.1.7. *Let $A = B \triangleleft I$. If f is an central idempotent of B and e an identity on I , then $f' = f - efe$ is a central idempotent of A , orthogonal to e .*

Remark 3.1.8. Observe that even if f is an identity on B , it may fail to be an identity of A in case where the action of B on I is not unital. If I is a unital strong B -bimodule, then B has an identity if and only if A has an identity (and if the identities exist, they are the same element inside A , of course).

3.2 Factoring Semigroup Algebras

We now turn our attention to semigroup algebras. Unlike group algebras, semigroup algebras are naturally closed under direct products, since the operation of ‘disjoint union’ is available in the category of semigroups — see Lemma 1.3.6 on page 17.

It is of importance to know whether meaningful classes of algebras are closed under taking direct factors. For instance, if a semigroup algebra splits as a product of two algebras, are the two factors semigroup algebras as well? Unfortunately, this is not the case, as the following example shows.

Example 3.2.1. Let $A = \mathbb{R}Q_8$, the group algebra of quaternion group over the field of real numbers. We denote the elements of Q_8 using the familiar basis elements of the quaternion division algebra:

$$Q_8 = \{\widehat{1}, \widehat{-1}, \widehat{i}, \widehat{-i}, \widehat{j}, \widehat{-j}, \widehat{k}, \widehat{-k}\}.$$

The idempotent $e = \frac{1}{2}(\widehat{1} - \widehat{-1})$ is central in $\mathbb{R}Q_8$. The algebra A_e is a semigroup algebra, since $A_e \cong \mathbb{R}^4$.

The algebra A^e , however, is not a semigroup algebra over $\mathbb{R}$, since $A^e \cong \mathbb{H}$ — the division algebra of quaternions — does not have a multiplicative basis.

The same example, with idempotent $1 - e$ instead, shows that A_e may fail to be a semigroup algebra. We may notice, however, that the quaternion algebra is an example of a twisted semigroup algebra. There are many other examples of blocks of group algebras which have the structure of a twisted semigroup algebra. A natural question arises:

Is the class of twisted semigroup algebras closed with respect to direct factors?

Having seen that not all central idempotents will lead to factorizations that stay within the class of semigroup algebras, we are aiming to find some central idempotents that will have the desired property.

For the sake of simplicity of definitions, we confine our coefficient algebras to be unital for a short while. We will see later that in many interesting cases, we shall be able to deduce the corresponding general result on the structure of semigroup algebra using the Theorem 1.4.1 on page 19.

Definition 3.2.2 (Sets S^e and S_e). Let B be a unital algebra. For $A = BS$ (or $B[S]$), let $e \in A$ be a central idempotent. Define

$$S^e = \{s \in S \mid ese = s \text{ or } s = 0\},$$

$$S_e = \{s - ese \mid s \in S \setminus S^e\}.$$

The set S^e can be regarded either as a subset of S or a subset of A (identifying zero with 0 in A if necessary), whichever is convenient. The set S_e is a subset of A .

For the remainder of this section, S is a semigroup and B a unital algebra.

Lemma 3.2.3. *Let $A = BS$ or $B[S]$ be a semigroup algebra, $e \in A$ a central idempotent. Then*

- i) S^e is an ideal of S .
- ii) S_e is a multiplicative subset of A .

Proof. As before, we are freely identifying elements of S with the corresponding elements of A .

- i) Suppose $ese = s$. Multiplying on the right with arbitrary $t \in S$, we have $eset = este = st$, so $st \in S^e$. Analogously, S^e is invariant under left multiplication by elements of S .
- ii) If $s_1, s_2 \in S \setminus S^e$, then

$$(s_1 - es_1e)(s_2 - es_2e) = s_1s_2 - es_1s_2e,$$

because the terms $-s_1es_2e$ and $es_1e^2s_2e$ cancel each other, and $es_1es_2 = es_1s_2e$. Hence the product is either 0 in A (if $s_1s_2 \in S^e$) or else the product is in S_e as claimed.

□

Recall that a *proper ideal* is an ideal not coinciding with the whole semigroup.

Proposition 3.2.4. *Let $A = B[S]$ be a semigroup algebra, $e \in A$ a central idempotent. If the ideal S^e is proper and generates A^e as a B -module, then both A^e and A_e are semigroup algebras and*

- i) $A^e \cong B[S^e]$, its multiplicative basis is S^e ;
- ii) $A_e \cong B(S/S^e)$, its multiplicative basis is S_e .

Thus $BS \cong B[S^e] \times B(S/S^e)$.

Saying that a set is a basis (for a B -module) means here simply that the module is the free B -module generated by that set.

Proof. With the multiplicativity of bases established, the proof is simple.

- i) Since we have $S^e = S \cap A^e$, the set S^e is a basis of the free B -module A^e . The multiplicative property from the previous lemma implies that A^e is the semigroup algebra of S^e .
- ii) Consider the map $p_2 : A \rightarrow A_e$ given by $s \mapsto s - ese$ and the map $p : A \rightarrow B(S/S^e)$ given by $s \mapsto s$ if $s \notin S_e$ and $s \mapsto 0$ otherwise. Both maps are algebra homomorphisms, and our assumption implies that the kernel of p is A^e , that is, the same as the kernel of p_2 . Thus the (partial) map $S/S^e \rightarrow S_e$ determined by $s \mapsto s - ese$ for s nonzero in S/S^e gives rise to an isomorphism of semigroup algebras.

□

For contracted semigroup algebras, the proposition takes the following form. One has to guard against the case $S^e = \{0\}$ here, as the (otherwise useful) convention that one-element semigroup does not possess zero makes $BS^e \cong B$, even if $A^e = \{0\}$.

Proposition 3.2.5. *Let $A = BS$ be a semigroup algebra, $e \in A$ a central idempotent. If S^e is proper, nonzero, and generates A^e as a B -module, then both A^e and A_e are semigroup algebras and*

- i) $A^e \cong BS^e$, nonzero elements of S^e form its multiplicative basis;
- ii) $A_e \cong B(S/S^e)$, its multiplicative basis is S_e .

Thus $BS \cong BS^e \times B(S/S^e)$.

Of course, thanks to Theorem 1.4.1 on page 19, such a splitting works also for a semigroup algebra whose coefficient algebra is a strong B -bimodule.

Remark 3.2.6. The main hypothesis of the above proposition cannot be satisfied in many interesting cases. For example, if S is a left (or right) cancellative semigroup then $ese = se = s$ can only be true if e is an idempotent of S , since for all other idempotents of BS , the support of se will have at least two elements. An arbitrary group algebra, for instance, will never satisfy the assumptions, since for any central idempotent e other than 0 or 1, the set S^e is empty.

Nevertheless, there are important classes of semigroups to which Proposition 3.2.5 on the page before is easy to apply. The easiest of all is the case when the idempotent e is actually an identity of the semigroup algebra of an ideal in S .

3.3 Identities on Ideals

Let us look now at the situation where there is no doubt that S^e generates A^e .

Lemma 3.3.1. *Let I be a proper nonzero ideal in S , and $e \in BI$ be an identity on the ideal BI of BS . Then*

$$BS \cong BI \times B(S/I),$$

the isomorphism being the same as in Proposition 3.2.5 on the preceding page.

Finding an identity on a general ideal of a semigroup is not an easy task, as no characterization of semigroups whose algebras are unital is known, aside from certain special cases.

We may exploit these known cases. One of them is the case of semigroups whose idempotents commute, and we will see that one can ascertain quite a lot in this case. The other class of semigroups where one can, in a way, characterize which semigroups produce unital algebras is the class of completely 0-simple semigroups. This is obviously a very relevant case if we intend to take for the ideal I a minimal nonzero ideal of a finite semigroup.

We recall the essence of Rees–Sushkevich Theorem (see [2], for instance). A semigroup is completely 0-simple if and only if it is a *regular Rees semigroup*. That is, a semigroup of the form $\mathcal{M}^0(G; \Lambda, \Upsilon; P)$, where G is a group, and P a *sandwich matrix* (with entries in G^0 , its rows and columns indexed by Λ, Υ) such that no row or column of P consists entirely of zeroes.

The theorem below was proved by Munn (and extended by Ponizovskii in [18] to the general coefficient setting). For any algebra A , the symbol $\text{Mat}_{n \times n}(A)$ denotes the $n \times n$ *matrix algebra* with entries in A .

Theorem 3.3.2 (Munn’s Criterion). *Let $S = \mathcal{M}^0(G; \Lambda, \Upsilon; P)$. Then AS has identity if and only if $|\Lambda| = |\Upsilon| = n < \infty$ and P is invertible in the ring $\text{Mat}_{n \times n}(AG)$.*

The condition on invertibility of the matrix is not easy to verify, even in particular cases. Note that it entails that A is unital.

In case the conditions are satisfied, the identity of AS is just the inverse of P , interpreted as the corresponding A -linear combination of elements of S .

We will not state separately that a minimal ideal satisfying the condition of the theorem splits off, but we note the following consequence for finite semigroups (or finite ideals of semigroups).

Lemma 3.3.3. *Let S be a finite semigroup. If there is an algebra B such that BS is unital, then every maximal $\mathcal{J}$ -class of S is regular, square and the matrix obtained from the P -matrix of the class by replacing nonzero entries with 1 is nonsingular as integer matrix.*

Proof. As factors of unital algebras are unital, consideration of factors corresponding to the maximal $\mathcal{J}$ -classes immediately yields that these must, for a given B , satisfy Munn's criterion. Passing to the quotient algebra of $\text{Mat}_{n \times n}(BG)$ by the algebra $\text{Mat}_{n \times n}(JG)$, where JG is the augmentation ideal of the group algebra BG , we obtain that the image of the matrix P must be a nonsingular matrix (over the prime subring of B). Thus it will be nonsingular also as a matrix over $\mathbb{Z}$. $\square$

This lemma has an obvious application: we know which ideals *not to use* when we look for possible ways of splitting a semigroup algebra in the manner of Lemma 3.3.1 on page 59.

Example 3.3.4. Consider a finite band S (that is, a semigroup consisting entirely of idempotents). Every $\mathcal{J}$ -class of such S is regular, and its P -matrix is simply a rectangular array of 1's. Thus the class cannot satisfy the criterion unless P is of size 1×1 . In this case, the criterion will be satisfied if and only if the coefficient algebra is unital.

3.4 Semigroup S^∞

Every semigroup has a series of ideals

$$S \supseteq S^2 \supseteq S^3 \supseteq \dots \supseteq S^n \supseteq \dots$$

with set-differences of consecutive terms having a clear meaning in terms of factorizability of elements: $s \in S^n \setminus S^{n+1}$ iff s can be written as a product of n elements of S , but not as a product of $n + 1$ elements. There are, generally, elements that can be written as a product of arbitrarily many elements of S .

Definition 3.4.1 (Ideal S^∞). Denote by S^∞ the set $S^\infty = \bigcap_{n \geq 1} S^n$.

The set S^∞ is a subsemigroup of S , and has a number of obvious properties. Most importantly, S^∞ is an ideal of S , and it contains all the idempotents: $E(S) \subseteq S^\infty$.

We have encountered, in defining graph semigroup, the condition that an ideal be generated by its ‘irreducible’ elements. If we borrow the symbol W to denote the set $S \setminus S^2$, the question is whether S is generated by W . It turns out that all one has to worry about is the set S^∞ .

Lemma 3.4.2. *Let S be a semigroup and $W = S \setminus S^2$. Then*

- i) $S = \bigcup_{n \geq 1} (S^n \setminus S^{n+1}) \cup S^\infty$ is a disjoint union.
- ii) $W^n \supseteq S^n \setminus S^{n+1}$ for every $n \geq 1$.

Proof. The first part follows from the preceding discussion. In order to show the second claim, take $s = s_1 \dots s_n \in S^n$. If $s \notin W^n$, there must be some $1 \leq i \leq n$ such that $s_i \notin W$. That means $s_i \in S^2$, and so $s \in S^{n+1}$. $\square$

One may observe that we have actually proved the claim below.

Lemma 3.4.3. *The following are equivalent:*

- i) W generates S .
- ii) W generates S^∞ .

iii) $S \setminus S^\infty$ generates S^∞ .

In the last two conditions, ‘generates’ has to be interpreted liberally, meaning that every element of S^∞ is a product of elements from the respective generating set (and not implying that there are no other elements generated by the set).

Recall from Lemma 1.3.9 on page 18 that if a semigroup algebra of S has an identity, then there is a finite set $U \subseteq E(S)$ of idempotents such that $US = S = SU$. In particular, $S^2 = S$. Thus $S = S^\infty$ is a necessary condition for a semigroup in order for its semigroup algebra to have an identity. If $S \neq S^\infty$, no subsemigroup of S properly containing S^∞ can have a unital semigroup algebra.

Lemma 3.4.4. *Let S be a semigroup with commuting idempotents such that $S \neq S^\infty$, and A be an arbitrary algebra. If there is a finite set $U \subseteq E(S)$ such that $US^\infty = S^\infty = S^\infty U$, then*

$$AS \cong AS^\infty \times A(S/S^\infty).$$

If S itself is finite, $S \neq S^\infty$, one can see that S^∞ is the largest ideal in S such that the associated factor-semigroup is nilpotent.

Proof. By Lemma 1.3.10 on page 18, we obtain that there is an e which is an identity on the integral semigroup algebra $\mathbb{Z}S^\infty$. Since S^∞ is an ideal of S , Lemma 3.3.1 on page 59 gives us

$$\mathbb{Z}S \cong \mathbb{Z}S^\infty \times \mathbb{Z}(S/S^\infty).$$

Applying Corollary 1.4.3 on page 22, we obtain the required isomorphisms of semigroup algebras over general A . □

3.5 Subsemigroups of Inverse Semigroups

The decomposition of the semigroup algebra of a finite inverse semigroup over an arbitrary field has been found already by Rukolaïne in [20]. This fact proved to be easy to miss, perhaps due to the fact that finite inverse semigroups have been frequently discussed in earlier literature, but most of the work has been concerned only with characterization of semisimplicity. Such a characterization follows already from the general reduction to 0-simple factors by Munn in [13]. We learned about the work of Rukolaïne thanks to the excellent survey of Ponizovskii [18]. See [15] for results concerning the structure of Munn algebras (a generalization of semigroup algebras of Rees semigroups).

The only completely 0-simple semigroups with commuting idempotents are those with P -matrix equal (up to equivalence) to a square identity matrix. That is, such a semigroup is isomorphic to the Rees semigroup $\mathcal{M}^0(G; \Lambda, \Lambda, \text{Id}_{\Lambda \times \Lambda})$. A semigroup of this form is often called *Brandt semigroup*. Using Notation 1.1.1 on page 6, such a semigroup is easily written as $\mathcal{M}_\Lambda \dot{\times} G$.

If its matrix P is finite, the Brandt semigroup satisfies Munn's criterion for any unital algebra B . In this case, identifying the identity on the semigroup algebra is particularly easy. The following lemma accumulates most pertinent consequences of the previously proved statements.

Lemma 3.5.1. *Let S be a semigroup which has a minimal nonzero ideal I isomorphic to a finite Brandt semigroup, and A denotes the semigroup algebra BS for a unital algebra B . Let $e_1, \dots, e_p$ be all nonzero idempotents of I and let G be the structure group of I . Then*

- i) $e = e_1 + \cdots + e_p$ is an identity on BI , thus a central idempotent of BS .
- ii) $S^e \cong I$ generates A^e as a B -module.
- iii) $S^e \cong \mathcal{M}_p \dot{\times} G$, and $A^e \cong \text{Mat}_{p \times p}(BG) \cong B(\mathcal{M}_p \dot{\times} G)$.

Because the central idempotent e that was used to split the semigroup algebra is an integer combination of elements of the semigroup, this lemma applies to the integer semigroup algebra $\mathbb{Z}S$ in particular. By Corollary 1.4.3 on page 22 and Corollary 1.4.4 on page 22, we have the following consequence.

Corollary 3.5.2. *Let S, I, p, G be as above and let A be an arbitrary algebra. Then*

$$AS \cong A(\mathcal{M}_p \dot{\times} G) \times A(S/I).$$

Inverse semigroups are regular semigroups whose idempotents commute. In case of a finite inverse semigroup S , we can simply continue splitting off minimal nonzero ideals one by one, since the arising factors of S are inverse semigroups as well. In case $0 \notin S$, the minimal ideal of S will be isomorphic to a finite group G , and the corresponding ideal of semigroup algebra splits off, having the obvious identity element. As $AG \cong AG^0 \cong A(\mathcal{M}_1 \dot{\times} G)$, there is no need to treat this case separately.

Consequently, simple inductive application of the preceding lemma yields immediately the following theorem.

Theorem 3.5.3. *Let S be a finite inverse semigroup, A an arbitrary algebra. For each nonzero $\mathcal{J}$ -class D_i of S , with $1 \leq i \leq t$, let p_i be its number of idempotents, G_i its structure group. Then the semigroup algebra AS splits into a product of matrix rings over group algebras as follows:*

$$AS \cong \text{Mat}_{p_1 \times p_1}(AG_1) \times \cdots \times \text{Mat}_{p_t \times p_t}(AG_t),$$

or, equivalently,

$$AS \cong A(\mathcal{M}_{p_1} \dot{\times} G_1) \times \cdots \times A(\mathcal{M}_{p_t} \dot{\times} G_t).$$

Remark 3.5.4. The semigroup algebra of a finite inverse semigroup S is therefore Morita equivalent to the product of group algebras of the structure groups of S . We leave it aside to what extent does this claim make sense when the algebra of coefficients is nonunital.

Polygraph Semigroups

The definition of graph semigroup allows us to construct some semigroups from their retracts and the bimodule of irreducibles, but the class of graph semigroups is completely unrestricted as we are allowing arbitrary retracts.

We will now define a restriction of the general concept that takes into account regularity of elements.

Definition 3.5.5 (R-graph semigroup). Let $S = R \triangleleft N$ be a graph semigroup, with R providing the modulation of vertices. We say that S is an *R-graph semigroup* if $R = \text{Reg}(S)$ (the set of all regular elements of S).

In the case of R-graph semigroup, we have that R , as it is a retract of S , is a union of $\mathcal{J}$ -classes of S . Thus $\mathcal{D}$ -classes contained in a particular $\mathcal{J}$ -class of S are either all regular, or all non-regular. R by itself is a regular semigroup. Of course, all regular semigroups are R-graph semigroups.

Definition 3.5.6 (Polygraph semigroup). We say that S is a *polygraph semigroup* if it can be build as a finite extension of R-graph semigroups. That is, there

is a finite ideal chain

$$S = I_t \supset I_{t-1} \supset \cdots \supset I_1 \supset I_0$$

such that I_i/I_{i-1} is an R-graph semigroup for each $1 \leq i \leq t$ and $I_0 = \{0\}$ or $\emptyset$ depending on whether $0 \in S$.

A subsemigroup is called *full* if it contains all the idempotents of the containing semigroup.

We are now ready to prove the following much more general result.

Theorem 3.5.7 (Unital Decomposition). *Let S be a finite semigroup which is either commutative or a full subsemigroup of a finite inverse semigroup. Denote by n the length of the longest chain of regular nonzero $\mathcal{J}$ -classes of S (with respect to the order $<_{\mathcal{J}}$). Then*

- i) S^∞ is a polygraph semigroup, and its semigroup algebra splits into a direct product of n algebras of R-graph semigroups S_i :

$$AS^\infty = AS_1 \times \cdots \times AS_n.$$

- ii) If $S \neq S^\infty$, then S is an extension of S^∞ by a nilpotent semigroup S/S^∞ , and the semigroup algebra AS splits into a product of AS^∞ and a nilpotent semigroup algebra:

$$AS = AS^\infty \times A(S/S^\infty).$$

Proof. Observe that the subsemigroup S^∞ trivially satisfies all the assumptions too.

Since a product of two idempotents is an idempotent due to the fact that idempotents commute, the product of two regular elements is regular as well. Consequently,

the ideal I generated by all *minimal regular nonzero* $\mathcal{J}$ -classes of S^∞ is an R -retract $R \triangleleft N$. N is stable as an R -bimodule — this is clear in case of commutative S , since I is generated by idempotents. In case S is a full subsemigroup of an inverse semigroup T , every $s \in N$ belongs to some minimal nonzero ideal J of T . Such J is a Brandt semigroup, thus there are $e_1, e_2 \in J$ such that $e_1 s = s = s e_2$. But it is easy to see that e_1, e_2 must be both in R due to the assumption that S is full in T . Note that S/I is again a full subsemigroup of a factor of T by the disjoint union of all minimal nonzero ideals of T . This allows us to proceed inductively.

The retract ideal N is certainly generated by its irreducibles (see Lemma 3.4.3 on page 62), since it is nilpotent. It follows by induction that S^∞ is a polygraph semigroup.

Since R is regular and its idempotents commute, the sum of its idempotents is an identity on I (for the integral semigroup algebra). Therefore we can apply Lemma 3.3.1 on page 59 to split the semigroup algebra as long as I is a proper ideal. Proceeding inductively, we obtain the desired decomposition, with all semigroup algebras in the decomposition of S^∞ unital.

Combining this result on S^∞ with Lemma 3.4.4 on page 63, we obtain the second part of the theorem. $\square$

Remark 3.5.8. Of course, we have also some information about the modulation of the graphs of the non-nilpotent factors: for each R -graph semigroup $S_i = R_i \triangleleft N_i$, we can split R_i into a disjoint union of semigroups of the form $\mathcal{M}_p \dot{\times} G$.

In case of a commutative S_i , the semigroup R_i splits into a disjoint union of commutative groups. Moreover, there cannot be any edges between two distinct vertices in the graph of commutative S_i , since that would imply existence of noncommuting

elements in S_i . Thus the only edges that can occur are loops.

Obviously, the theorem could be somewhat strengthened — as long as we can show S^∞ is a polygraph semigroup (with commuting idempotents), we obtain the above decomposition regardless of whether S/S^∞ is finite or infinite (or nilpotent, for that matter).

Remark 3.5.9. The results of the theorem were known for the case of a commutative semigroup. Nearly equivalent assertion on the structure of commutative semigroups (satisfying suitable finiteness conditions) has been shown in [16]. The related result on the structure of the semigroup algebra of a finite commutative semigroup was obtained in [19].

3.6 Symmetric Inverse Semigroup

Definition 3.6.1 (Symmetric inverse semigroup). The *symmetric inverse semigroup* $\mathcal{I}_X$ on a finite set X is the semigroup of all *partial isomorphisms* of X . Its elements are all one-to-one maps $\varphi : U \rightarrow V$ with $U, V \subseteq X$. Whenever we write $\varphi : U \rightarrow V$, it is understood that U is the domain of the map φ and V its range. It is convenient to allow the case $U = V = \emptyset$, the corresponding ‘empty map’ acting as the zero element of $\mathcal{I}_X$. The product of $\varphi : U \rightarrow V$ and $\psi : W \rightarrow Z$ is the composition

$$\varphi' \circ \psi' : \varphi^{-1}(V \cap W) \rightarrow \psi(V \cap W)$$

of the maps induced by restricting the range of φ , respectively the domain of ψ :

$$\varphi' : \varphi^{-1}(V \cap W) \rightarrow V \cap W,$$

$$\psi' : V \cap W \rightarrow \psi(V \cap W).$$

The composition is the ‘empty map’ if $V \cap W = \emptyset$.

Remark 3.6.2. We observe that if $n = |X|$ then the semigroup $\mathcal{J}_X$ has

$$|\mathcal{J}_X| = \sum_{t=0}^n \binom{n}{t}^2 t!$$

elements.

Definition 3.6.3 (Ordered partial isomorphisms). Let $<$ be a linear order on X .

The semigroup $\mathcal{O}_X$ of *ordered partial isomorphisms* is the subsemigroup of $\mathcal{J}_X$ of all the partial isomorphisms which preserve the order $<$. That is,

$$\mathcal{O}_X = \{\varphi \in \mathcal{J}_X \mid x\varphi < y\varphi \text{ whenever } x < y\}.$$

Once again, we consider the zero element of $\mathcal{J}_X$ to be order-preserving.

Remark 3.6.4. Note that if U, V are any two subsets of X of the same cardinality, there is exactly one element of $\mathcal{J}_X$ which belongs to $\mathcal{O}_X$. This observation immediately yields the following formula for the size of $\mathcal{O}_X$:

$$|\mathcal{O}_X| = \sum_{t=0}^n \binom{n}{t}^2 = \binom{2n}{n},$$

where n is the cardinality of X .

Definition 3.6.5 (Lexicographic order). Let X be a finite set, linearly ordered by relation $<$. The *lexicographic order* on subsets of X will be denoted by $\prec$. Given

two subsets U and V of X ,

$$U = \{i_1 < i_2 < \dots < i_k\},$$

$$V = \{j_1 < j_2 < \dots < j_l\},$$

we write $U \prec V$ if there is $0 \leq m \leq \min\{k, l\}$ such that

$$i_t = j_t \text{ for all } 1 \leq t \leq m \text{ and } \begin{cases} k < l & \text{if } m = \min\{k, l\}, \\ i_{m+1} < j_{m+1} & \text{if } m < \min\{k, l\}. \end{cases}$$

As usual, we write $U \preceq V$ whenever $U \prec V$ or $U = V$.

The subsets of X are linearly ordered by $\preceq$, the empty set being the smallest and the set X the largest element. Having defined the lexicographic order, we proceed to introduce another subsemigroup of $\mathcal{I}_X$.

Definition 3.6.6 (Ascending partial isomorphisms). The semigroup $\mathcal{A}_X$ of *ascending ordered partial isomorphisms* is the subsemigroup of $\mathcal{O}_X$ consisting of order-preserving maps whose domain precedes or equals the range in the lexicographic order:

$$\mathcal{A}_X = \{\varphi : U \rightarrow V \mid \varphi \in \mathcal{O}_X \text{ and } U \preceq V\}.$$

Remark 3.6.7. As for the previously defined semigroups, it is easy to count the size of $\mathcal{A}_X$, given that $n \geq 1$ is the size of the set X :

$$|\mathcal{A}_X| = \frac{1}{2} \sum_{t=0}^n \left(\binom{n}{t}^2 + \binom{n}{t} \right) = \frac{1}{2} \binom{2n}{n} + 2^{n-1}.$$

As is apparent from the definitions, we have canonical inclusions

$$\mathcal{A}_X \subseteq \mathcal{O}_X \subseteq \mathcal{I}_X$$

and these inclusions extend straightforwardly to embeddings of the corresponding semigroup algebras. The structure of these semigroup algebras will be examined next.

3.7 Structure of $\mathcal{I}_X$, $\mathcal{O}_X$, $\mathcal{A}_X$

In the present section, the symbol n always stands for the cardinality of the set X , and A is an algebra. Note that all the statements are formulated for contracted semigroup algebras, therefore the “ranks” of the semigroup algebras as free A -modules (whenever the notion of rank is meaningful for the algebra A) are one less than the cardinalities of underlying semigroups.

$\mathcal{I}_X$ and $\mathcal{O}_X$ are both inverse, each has $n + 1$ principal ideals which form a chain, and t -th principal factor is the matrix semigroup of order $\binom{n}{t}$ with entries in the structure group of the corresponding $\mathcal{J}$ -class. Thus we obtain the following two theorems as a special case of Theorem 3.5.3 on page 65.

Theorem 3.7.1. *The algebra $A\mathcal{I}_X$ decomposes as follows:*

$$A\mathcal{I}_X \cong \prod_{t=1}^n \text{Mat}_{\binom{n}{t} \times \binom{n}{t}}(A\mathcal{S}_t) \cong \prod_{t=1}^n A(\mathcal{M}_{\binom{n}{t}} \times \mathcal{S}_t).$$

Hence the algebra is Morita equivalent to the product of group algebras of the symmetric groups $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_n$.

Theorem 3.7.2. *The algebra $A\mathcal{O}_X$ has the following structure:*

$$A\mathcal{O}_X \cong \prod_{t=1}^n \text{Mat}_{\binom{n}{t} \times \binom{n}{t}}(A) \cong \prod_{t=1}^n A\mathcal{M}_{\binom{n}{t}}.$$

Consequently, if A is a division algebra, the semigroup algebra $A\mathcal{O}_X$ is semisimple.

Notation 3.7.3. The $p \times p$ (upper) *triangular matrix algebra* whose entries are from an algebra A is denoted by the symbol $\text{Triang}_p(A)$. Using Notation 2.6.7 on page 49, we can write $\text{Triang}_p(A) = A\mathcal{T}_p$.

The semigroup $\mathcal{A}_X$ is not inverse, but it does satisfy the assumptions of Theorem 3.5.7 on page 67. One can see easily that the semigroup R of the ideal I constructed in the proof of the theorem is formed exactly by the elements from $\mathcal{O}_X \cap I$. Ordering the idempotents of I in the lexicographic order, we can see that the ideal I in the t -th step is isomorphic to the semigroup $\mathcal{T}_{(n)}$.

Theorem 3.7.4. *The algebra $A\mathcal{A}_X$ has the following structure:*

$$A\mathcal{A}_X \cong \prod_{t=1}^n \text{Triang}_{(n)}(A) \cong \prod_{t=1}^n A\mathcal{T}_{(n)}.$$

Remark 3.7.5. Thus the individual direct factors are semigroup algebras of free graph semigroups on Dynkin graphs of type A (with “uniform” orientation of edges) with trivial modulation.

If A is a division algebra, $A\mathcal{A}_X$ is a hereditary algebra, its quiver is a disjoint union of n Dynkin diagrams $\mathbb{A}_{(n)}, \mathbb{A}_{(n)}, \dots, \mathbb{A}_{(n)}$.

Chapter 4

Partial Isomorphisms of Chain Collections

We have observed that composition of order-preserving maps is again order preserving and used this fact to define the subsemigroup $\mathcal{O}_X$ of $\mathcal{I}_X$. Composability of order-preserving maps does not require that the order on X be linear — indeed, all that is needed is just the transitivity of the relation that is used to delimit the subset of maps.

Instead of total order on the underlying set, we consider an order that is a collection of chains. The decomposition theorems of the previous chapter apply to these semigroups, thus we are lead to describing the relevant R-graph semigroups.

First, using the notion of M-variant, the problem of finding the objects defining appropriate graph semigroups is reduced. After that, several problems of combinatorial nature need to be overcome. The combinatorial objects encountered are quite interesting in their own sake, however. These are sets of partitions of a fixed integer, constrained in a specific way.

We determine the underlying graph and its modulation — the semigroups assigned to vertices and the structure of the bimodules of irreducible maps.

4.1 Semigroups of Order-Preserving Maps

We follow the common practice of using the word ‘chain’ as a shorthand for or a linearly ordered set. We will consider partial orders on X that are “collections” of chains. That is, the set X is partitioned into a disjoint union of nonempty subsets, each of which is linearly ordered, and elements of distinct subsets are mutually incomparable. Isomorphism classes of such collections of chains on a set of size n obviously correspond to integer partitions of n .

Definition 4.1.1 (Semigroups $\mathcal{J}_X^\pi$). Suppose $\leq$ is a partial order on X which makes X a disjoint union of chains. Let $\pi = [\pi_1, \dots, \pi_p]$ be the partition of $n = |X|$, denoted by $\pi \vdash n$, that is given by sizes of (maximal) chains of X , $\pi_1 \geq \pi_2 \geq \dots \geq \pi_p$. We define $\mathcal{J}_X^\pi$ as the subset of $\mathcal{J}_X$ consisting of all the maps preserving the order $\leq$:

$$\mathcal{J}_X^\pi = \{\varphi \in \mathcal{J}_X \mid x\varphi < y\varphi \text{ whenever } x < y\}.$$

As we noted in the beginning, $\mathcal{J}_X^\pi$ is a subsemigroup of $\mathcal{J}_X$. It contains all the idempotents of $\mathcal{J}_X$ — we will make great use of this fact in order to study the structure of semigroup algebra of $\mathcal{J}_X^\pi$.

Remark 4.1.2. Two of the previously examined semigroups $\mathcal{J}_X$ and $\mathcal{O}_X$ are just special cases of the semigroup $\mathcal{J}_X^\pi$: $\mathcal{J}_X = \mathcal{J}_X^\pi$ for $\pi = [1, \dots, 1]$ (no pair of distinct elements of X is comparable) and $\mathcal{O}_X = \mathcal{J}_X^\pi$ for $\pi = [n]$ (linear order on X).

As $\mathcal{J}_X^\pi$ is certainly a full subsemigroup of $\mathcal{J}_X$, Theorem 3.5.7 on page 67 applies. We obtain immediately that $\mathcal{J}_X^\pi$ is a polygraph semigroup since $\mathcal{J}_X^\pi = \mathcal{J}_X^\pi \cdot \mathcal{J}_X^\pi$ — no nilpotent factor is present. Individual graph semigroups — the subfactors of $\mathcal{J}_X^\pi$ constructed in the proof of the theorem — are precisely the semigroups of all order-preserving maps of a fixed rank k ($1 \leq k \leq n = |X|$) with multiplication modulo maps of lower rank. We shall denote these graph semigroups by $\tilde{P}_k^\pi$.

To be able to analyze the structure of semigroups $\tilde{P}_k^\pi$, we shall need more notation. First, we define what is meant by the term subpartition.

Definition 4.1.3 (Subpartition). Let $\pi = [\pi_1, \dots, \pi_p] \vdash n$ and $\lambda = [\lambda_1, \dots, \lambda_l] \vdash k$ be partitions. We call λ a *subpartition* of π if $l \leq p$ and $\lambda_i \leq \pi_i$ for all $i = 1, \dots, l$.

In the following discussion, we shall often identify each partition π with a set X^π that is partially ordered by the collection of chains of sizes corresponding to the partition. For two partitions λ, μ , we may consider the set of all injective maps $X^\lambda \rightarrow X^\mu$ which are order-preserving — that is, maps of ordered sets. We will simply refer to this set as ‘ordered maps from λ to μ ’, bypassing the need to speak about the corresponding ordered sets. Note that with this convention, the set $\mathcal{J}_X^\pi$ would be declared simply as the set of ordered maps from π to π .

Definition 4.1.4 (Sets $P_{\lambda\mu}$, G_λ , symbols $\ll$, p_λ^π). Let $\lambda, \mu \vdash k$ be partitions. We denote by $P_{\lambda\mu}$ the set of all *ordered injective maps* from λ to μ . The symbol G_λ will denote the set $P_{\lambda\lambda}$. We write $\lambda \ll \mu$ if $P_{\lambda\mu} \neq \emptyset$.

The integer p_λ^π denotes the number of subsets of X^π of size k such that the order induced from X^π on the considered subset is a collection of chains of type λ .

Remark 4.1.5. Equivalently, the number p_λ^π could be described as the number of inequivalent embeddings of the poset X^λ into the poset X^π , where two embeddings

are equivalent if one of them can be written as a composition of an isomorphism of X^λ followed by the other embedding.

Note that we consider fully defined maps only when speaking of maps of partitions. As maps of sets, elements of $P_{\lambda\mu}$ are bijective, but their set-theoretical inverses may not be ordered maps.

We shall now consider the structure of $\tilde{P}_k^\pi$ in more detail. Suppose $k > 0$ and $\varphi \in \tilde{P}_k^\pi$, mapping the subset U of X^π of type λ onto the subset V of X^π of type μ , where $\lambda, \mu \vdash k$.

Let us fix, for each $\kappa \vdash k$ that is a subpartition of π , a bijection between all subsets of X^π of size k and type κ and the set $\{1, 2, \dots, p_\kappa^\pi\}$. Also, for each of these subsets, we fix an order-isomorphism of the subset to the set X^κ (such an isomorphism is not unique in case there are at least two chains of the same size). Having done that, we see that φ uniquely determines not only λ and μ , but also an element $\bar{\varphi}$ of $P_{\lambda\mu}$ and two integers i_φ, j_φ corresponding to U and V : $1 \leq i_\varphi \leq p_\lambda^\pi$ and $1 \leq j_\varphi \leq p_\mu^\pi$.

Thus, each map $\varphi \in \tilde{P}_k^\pi$ uniquely determines the data

$$\lambda \ll \mu \vdash k \text{ (subpartitions of } \pi), \bar{\varphi} \in P_{\lambda\mu}, 1 \leq i_\varphi \leq p_\lambda^\pi, 1 \leq j_\varphi \leq p_\mu^\pi.$$

Conversely, any such data determines uniquely U, V and an order-preserving map $\varphi : U \rightarrow V$.

Let now $\psi \in \tilde{P}_k^\pi$ be another map, $\psi : W \rightarrow Z$. We have $\varphi\psi = 0$ in $\tilde{P}_k^\pi$ unless $V = W$. Thus, the product is nonzero only if ψ is determined by

$$\mu \ll \nu \vdash k \text{ (subpartitions of } \pi), \bar{\psi} \in P_{\mu\nu}, 1 \leq i_\psi \leq p_\mu^\pi, 1 \leq j_\psi \leq p_\nu^\pi$$

and $j_\varphi = i_\psi$. In this case, the product corresponds to

$$\lambda \ll \nu \vdash k \text{ (subpartitions of } \pi), \bar{\varphi}\bar{\psi} \in P_{\lambda\nu}, 1 \leq i_\varphi \leq p_\lambda^\pi, 1 \leq j_\psi \leq p_\nu^\pi.$$

In this way, we have established that the set $\tilde{P}_k^\pi$ can be partitioned as follows:

$$\tilde{P}_k^\pi = \{0\} \cup \bigcup_{\substack{\lambda \ll \mu \vdash k \\ \text{subpartitions of } \pi}} M_{p_\lambda^\pi p_\mu^\pi} \times P_{\lambda\mu}.$$

The purpose of the upcoming sections is to show that such a decomposition holds in much stronger sense.

Example 4.1.6. Let $\pi = [3, 1] \vdash n = 4$. Thus the set X^π has two chains, we shall write it as $X = \{1, 2, 3\} \cup \{4\}$, and assume that the subsets of integers are ordered naturally ($1 < 2 < 3$ in this case). There are only two partitions of $k = 2$, both are subpartitions of π : $\lambda = [1, 1]$, $\mu = [2]$. There are three subsets of X^π of size k whose type is $[2]$, and another three whose type is $[1, 1]$:

$$\{1, 2\}, \{1, 3\}, \{2, 3\} \text{ of type } [2],$$

$$\{1, 4\}, \{2, 4\}, \{3, 4\} \text{ of type } [1, 1].$$

Thus $p_\lambda^\pi = 3 = p_\mu^\pi$.

Let $\varphi : \{1, 4\} \rightarrow \{2, 3\}$ be the map $1 \mapsto 3, 4 \mapsto 2$. Clearly $\lambda = [1, 1]$, $\mu = [2]$, and using the order of enumeration shown above, we have $i_\varphi = 1, j_\varphi = 3$. Supposing that the fixed bijection of $\{1, 4\}$ to $[1, 1]$ maps 1 to the first and 4 to the second chain, the map $\bar{\varphi} : [1, 1] \rightarrow [2]$ sends the first chain to the larger element of the 2-chain of $[2]$.

This example will be revisited again in the next section.

4.2 Elementary Properties

The following proposition summarizes the rudimentary properties of the sets $P_{\lambda\mu}$.

Proposition 4.2.1. *Let $\lambda, \mu, \nu \vdash k$ be partitions of k .*

- i) *If $a \in P_{\lambda\mu}$ and $b \in P_{\mu\nu}$, then $ab \in P_{\lambda\nu}$.*
- ii) *G_λ (that is, $P_{\lambda\lambda}$) is a finite group.*
- iii) *The relation $\ll$ is a partial order (on partitions of k).*
- iv) *$P_{\lambda\mu}$ is a transformation G_λ - G_μ -bimodule, the action given by composition of maps.*
- v) *Point stabilizers are trivial for the left action of G_λ on $P_{\lambda\mu}$, as well as for the right action of G_μ on $P_{\lambda\mu}$.*

Proof. Suppose $\lambda, \mu, \nu \vdash k$.

- i) Immediately follows from transitivity of partition orders.
- ii) Let $a \in P_{\lambda\lambda}$. The bijection a must send every ordered pair from λ (read: pair of elements from X^λ) to an ordered pair in λ . But the set (of pairs) is finite, thus every ordered pair in λ is an image under a of an ordered pair in λ . Therefore the set-theoretical inverse a^{-1} is a map of ordered sets. This fact together with i) implies that G_λ is a group.
- iii) Transitivity follows from i). The sets G_λ are nonempty, as they each contain the identity map. If $\lambda \ll \mu, \mu \ll \lambda$, there are $a \in P_{\lambda\mu}, b \in P_{\mu\lambda}$, and the map ab is invertible by ii). Thus a and b are isomorphisms of ordered sets, proving $\lambda = \mu$.

- iv) Having $g \in G_\lambda, h \in G_\mu, a \in P_{\lambda\mu}$, it follows from associativity of map composition that $(ga)h = g(ah)$.
- v) Suppose $g \in G_\lambda, a \in P_{\lambda\mu}$. Then $ga = a = 1a$ implies that g is the identity map, as a is a bijection in the category of sets. Dually on the right side.

□

Remark 4.2.2. Observe that v) implies that the cardinality of $P_{\lambda\mu}$ is always a multiple of $|G_\lambda|$, as well as a multiple of $|G_\mu|$.

We now return to the setting of Example 4.1.6 on page 78 and the discussion that precedes it. In this case, the structure of the semigroup $\tilde{P}_k^\pi$ can be easily determined.

Example 4.2.3. Let $\pi = [3, 1] \vdash n = 4$. Partitions $\lambda = [1, 1], \mu = [2]$ are the only two partitions of $k = 2$, both are subpartitions of π . It is clear that $G_\lambda \cong \mathcal{S}_2, G_\mu \cong \mathcal{S}_1$, and $P_{\lambda\mu} \cong \mathcal{S}_2$, where the left action of $\mathcal{S}_2$ on $P_{\lambda\mu}$ is given by multiplication. The action of $\mathcal{S}_1$ on the right is unital.

Recalling that $p_\lambda^\pi = 3 = p_\mu^\pi$, we can describe the semigroup $\tilde{P}_k^\pi$ as the free graph semigroup on the modulated graph

$$\begin{array}{c} \mathcal{M}_3 \dot{\times} G_\mu \\ \uparrow M_{33} \times P_{\lambda\mu} \\ \mathcal{M}_3 \dot{\times} G_\lambda \end{array}$$

(with the diagonal bimodule $M_{33} \times P_{\lambda\mu}$). Note that the graph comes from the order $\ll$ (arrows go in the direction where maps $P_{\lambda\mu}$ exist)

$$\begin{array}{c} \mu = [2] \\ \uparrow \\ \lambda = [1, 1] \end{array}$$

and that $\tilde{P}_k^\pi$ is an M-variant of the semigroup

$$\begin{array}{c} G_\mu \\ \uparrow P_{\lambda\mu} \\ G_\lambda \end{array}$$

which we shall briefly denote by P_k^π .

One rather special property of this semigroup $\tilde{P}_k^\pi$ was that we could express it as $\mathcal{M}_3 \dot{\times} P_k^\pi$. In general, this is not true, although $\tilde{P}_k^\pi$ is always an M-variant of some P_k^π .

Example 4.2.4. Let $\pi = [2, 1, 1] \vdash n = 4$. Again, $\lambda = [1, 1]$, $\mu = [2]$ are the only two partitions of $k = 2$ and they are subpartitions of π , thus the underlying graph of $\tilde{P}_k^\pi$ is the same as before, including the groups G_λ , G_μ and the bimodule $P_{\lambda\mu}$ (that is, the entire modulation of P_k^π). However, we have $p_\mu^\pi = 1$ in this case (only one subset of type μ), while $p_\lambda^\pi = 5$. The semigroup $\tilde{P}_k^\pi$ is another M-variant of the very same P_k^π , namely the free graph semigroup on the modulated graph

$$\begin{array}{c} G_\mu \\ \uparrow M_{51} \times P_{\lambda\mu} \\ \mathcal{M}_5 \dot{\times} G_\lambda \end{array}$$

where the bimodule $M_{51} \times P_{\lambda\mu}$ is diagonal (and $G_\mu \cong \mathcal{M}_1 \dot{\times} G_\mu$, of course).

Remark 4.2.5. Yet another M-variant of the semigroup P_k^π is the semigroup $\tilde{P}_k^\pi$ for $\pi = [2, 2]$, $k = 2$. In this case, $p_\lambda^\pi = 4$, $p_\mu^\pi = 2$.

In these examples, the bimodule involved had a particularly simple structure. This is not always so.

Example 4.2.6. The modules $P_{\lambda\mu}$ are not necessarily transitive (even when considered as bimodules). For instance, let us look at the partitions $\lambda = [2, 1]$, $\mu = [3]$ of 3. Now $P_{\lambda\mu}$ has three elements, but both G_λ and G_μ are trivial.

As a permutation bimodule, the set $P_{\lambda\mu}$ may not be faithful. Consider $\lambda = [3, 2, 1, 1]$, $\mu = [4, 1, 1]$. Let $g \in G_\lambda, h \in G_\mu$ be the order isomorphisms that permute the two 1-chains (in their respective groups). Now clearly $ga = ah$ for arbitrary $a \in P_{\lambda\mu}$, so the stabilizer of every element is a nontrivial subgroup of $G_\lambda \times G_\mu$.

The structure of groups G_λ is easily determined for a general partition. A little bit of notation is introduced before stating the elementary structural result.

Definition 4.2.7 (Group $G_{\lambda:j}$). Let $\lambda = [\lambda_1, \dots, \lambda_l] \vdash k$ be a partition. For each $1 \leq j \leq k$, denote by the symbol $\lambda : j$ all the chains of λ whose length is exactly j . Let $|\lambda : j|$ stand for the number of these chains, and $G_{\lambda:j}$ be the subgroup of G_λ of all maps permuting the chains of $\lambda : j$ and fixing all the chains of other sizes.

Each subgroup $G_{\lambda:j}$ is clearly isomorphic to the symmetric group $\mathfrak{S}_{|\lambda:j|}$.

Lemma 4.2.8. *Let $\lambda \vdash k$ be a partition. Then G_λ is the direct product of its symmetric subgroups $G_{\lambda:j}$:*

$$G_\lambda \cong G_{\lambda:1} \times \cdots \times G_{\lambda:k}.$$

Thus the group G_λ is trivial if and only if lengths of all the chains in λ are distinct.

Proof. Maps of G_λ are invertible, thus a chain can only be mapped into another one of the same size, and the order of elements in the chain is exactly preserved. One can only permute entire chains. $\square$

Elementary properties of the order $\ll$ are given next.

Lemma 4.2.9. *Suppose $\lambda, \mu, \nu \vdash k$, and l, m, n are the respective numbers of chains in each partition.*

- i) $\lambda \ll \mu$ if and only if there is an equivalence relation on the list $[\lambda_1, \dots, \lambda_l]$ such that $[\mu_1, \dots, \mu_m]$ is the list of sums of equivalence classes.
- ii) If $\lambda \ll \mu \ll \nu$, then $n \leq m \leq l$ and $(l - n) = (l - m) + (m - n)$.

One would normally prefer to say ‘partition of’ instead of an ‘equivalence relation on’, but it might be better here to avoid overloading the term ‘partition’.

Proof. A straightforward argument establishes both claims.

- i) Since maps must preserve the order, an entire chain must always be mapped into another chain. Thus a map from $P_{\lambda\mu}$ induces a surjective map of chains. The kernel relation of this induced map is the sought equivalence. Counting the sizes is simply summing, as all the maps are bijections of sets.
- ii) The inequality $m \leq l$ is an easy consequence of the above. The trivial equality shows that the bracketed ‘degrees’ are additive.

□

4.3 Irreducible Maps

Having dealt with groups G_λ , we proceed to examine structure of the sets $P_{\lambda\mu}$ further.

First of all, let us introduce the relation $\stackrel{1}{\ll}$ on partitions of k that will be of great use in the sequel.

Definition 4.3.1 (Relation $\overset{1}{\ll}$, irreducible maps). Let $\lambda, \mu \vdash k$ and let l, m denote their respective number of chains. We write $\lambda \overset{1}{\ll} \mu$ if $\lambda \ll \mu$ and $l - m = 1$.

The elements of $P_{\lambda\mu}$ with $\lambda \overset{1}{\ll} \mu$ are called *irreducible maps*.

Thus if $\lambda \overset{1}{\ll} \mu$, then there are exactly two chains in λ that are mapped onto a chain in μ . These three chains will be called *exceptional chains* — two are in λ , one in μ .

Remark 4.3.2. Observe that if $\lambda \overset{1}{\ll} \nu$ and $\lambda \ll \mu \ll \nu$, then necessarily either $\lambda = \mu$ or $\mu = \nu$. In other words, ν is a cover of λ in the order $\ll$. It follows from the second part of Lemma 4.2.9 on the page before that all covers are given by $\overset{1}{\ll}$.

The next lemma is intuitively obvious, but formal proof requires some careful counting.

Lemma 4.3.3. *Let $\lambda \overset{1}{\ll} \mu$. Then the sizes of exceptional chains are uniquely determined.*

Proof. First we will argue that it is enough to show uniqueness of the size of exceptional chains in μ . Indeed, if we know that size, we simply select one such chain of μ as exceptional, and pair off all the remaining chains of μ with chains of the same size in λ in arbitrary way. There will be always two chains left unpaired in λ , and their sizes will be independent of the pairing chosen.

Uniqueness of the size of exceptional chains in μ will be proved by contradiction. Suppose there are two maps, a and b , with the size of their exceptional chain in μ denoted by m'_a and m'_b , respectively, and $m'_a \neq m'_b$. Each of these two maps induces a bijection of some $m - 1$ chains of λ onto $m - 1$ non-exceptional chains of μ . These bijections preserve sizes of chains. As $m'_a \neq m'_b$, the total sizes of non-exceptional

chains of μ differ for a and b . So the domains of the two induced bijections cannot be the same set — the union of the two domains must cover at least m chains of λ . But now we have shown that the assumptions for the existence of a complete matching from chains of μ to chains of λ of the same size are satisfied. Thus, by the well known theorem of P. Hall, there is a size-preserving bijection between some m chains of λ and the m chains of μ . This is a contradiction, since the sole unmatched chain in λ proves that the partitions λ and μ cannot be both partitions of the same integer. $\square$

Notation 4.3.4. When $\lambda \stackrel{1}{\ll} \mu$, we will denote the sizes of exceptional chains in λ by l' , l'' and the size of an exceptional chain in μ by m' .

The next claim follows easily by inspection of the proof of the previous lemma. The action of G_λ and G_μ on $P_{\lambda\mu}$ does not change the way an exceptional μ -chain is ‘partitioned’ into two subsets (given by images of exceptional λ -chains). Only in the case $l' = l''$, the two subsets can be interchanged by the action of G_λ , reducing the number of orbits by the factor of 2.

Lemma 4.3.5. *Let $\lambda \stackrel{1}{\ll} \mu$ and consider the G_λ - G_μ -bimodule $P_{\lambda\mu}$.*

- i) *Point stabilizers of $P_{\lambda\mu}$ are determined uniquely up to conjugation in $G_\lambda \times G_\mu$.*

We denote by $G_{\lambda\mu}^{\text{stab}}$ one of them.

- ii) *If $l' \neq l''$, the bimodule $P_{\lambda\mu}$ is a union of $\binom{m'}{l'}$ identical copies of the transitive permutation bimodule $G_\lambda \times G_\mu / G_{\lambda\mu}^{\text{stab}}$.*

- iii) *If $l' = l''$, the bimodule $P_{\lambda\mu}$ is a union of $\frac{1}{2} \binom{m'}{l'}$ identical copies of the transitive permutation bimodule $G_\lambda \times G_\mu / G_{\lambda\mu}^{\text{stab}}$.*

Lemma 4.3.6. *Consider $\lambda \ll \nu$ and take an arbitrary $c \in P_{\lambda\nu}$. Suppose $\lambda \stackrel{1}{\ll} \mu$. Then c factors through $P_{\lambda\mu}$ (that is, $c = ab$ for some $a \in P_{\lambda\mu}, b \in P_{\mu\nu}$) if and only if there are two λ -chains of exceptional sizes l', l'' that are merged by c (that is, mapped into a single chain).*

Proof. It is clear that the stated condition is necessary for such a factorization to exist, since the map a merges two such chains and every b will map them into a single chain of ν .

We will prove that the condition is sufficient. Assume we do have two exceptional chains in λ — denote them by C', C'' — and assume that $(C' \cup C'')_c \subseteq D$ for some chain D of ν . We select an arbitrary chain in μ of the exceptional size m' as the μ -exceptional chain. Call it C . Define the map a as follows:

- i) Pick an arbitrary size-preserving bijection between non-exceptional chains of λ and μ , and then define a as identity on these chains (that is, we are using the bijection to identify the chains).
- ii) For the two exceptional chains C', C'' : map them into the exceptional chain of C of μ so that their order in that chain coincides with their relative order in D — the linear order in D restricted to $(C' \cup C'')_c$.

Having defined a , it is easy to construct suitable map b :

- i) On non-exceptional chains of μ , define b to coincide with c .
- ii) On C , let it be the only order-preserving map that fills the unoccupied positions in D . That is, map C onto the subset $(C' \cup C'')_c$ of D .

Now certainly $b \in P_{\mu\nu}$ and $c = ab$. □

Proposition 4.3.7. *Let $\lambda \ll \nu$ and $c \in P_{\lambda\nu}$. Then either $\lambda = \nu$ or there is a $t \geq 1$ and $\mu_0, \dots, \mu_t$ such that*

$$\lambda = \mu_0 \stackrel{1}{\ll} \mu_1 \stackrel{1}{\ll} \dots \stackrel{1}{\ll} \mu_t = \nu \text{ and } c \in P_{\mu_0\mu_1} \cdot P_{\mu_1\mu_2} \cdots P_{\mu_{t-1}\mu_t}.$$

Proof. The claim easily follows from Lemma 4.3.6 on the preceding page by induction on the degree $t = l - n$. If $t > 0$, we take any two chains that are merged by c and Lemma 4.3.6 on the page before to find a factorization $c = ab$, and by Lemma 4.2.9 on page 83, the degree of b is $t - 1$, hence inductive hypothesis applies. $\square$

Proposition 4.3.8. *Let $\lambda \ll \nu$, $\lambda \neq \nu$ and $c \in P_{\lambda\nu}$. If both λ and ν are subpartitions of some π , then there exists a subpartition μ of π such that $\lambda \stackrel{1}{\ll} \mu \ll \nu$ and c factors through $P_{\lambda\mu}$.*

Proof. The problem is to find two chains of λ which get merged by c in such a way that μ determined by this choice (that is, the μ with $\lambda \stackrel{1}{\ll} \mu$ and the two chains exceptional) will be a subpartition of π . Such μ certainly satisfies $\mu \ll \nu$ and the Lemma 4.3.6 on the page before assures us that the required factorization exists.

Let $C' = \lambda_k$ be the largest chain among the chains of λ which do get merged by c with another chain, with the index k as small as possible among such chains of maximal size.

Let C'' be any chain that is merged with the selected C' by the map c (its index among λ_i 's is thus $> k$). We claim that the μ having C', C'' exceptional is a subpartition of π .

Indeed, it follows from the choice of k that $[\lambda_1, \lambda_2, \dots, \lambda_{k-1}]$ is a subpartition of ν , and there is another chain in ν of size at least $|C'| + |C''|$. Since ν is a subpartition of π , we have that the partition μ_1 containing chains $\lambda_1, \dots, \lambda_{k-1}$ and the chain of size $|C'| + |C''|$ is a subpartition of $[\pi_1, \dots, \pi_k]$.

On the other hand, the partition μ_2 containing all the chains $\lambda_{k+1}, \dots, \lambda_l$ (l is the number of chains in λ) except for the chain C'' is a subpartition of $[\pi_{k+1}, \dots, \pi_l]$.

Hence μ , being the concatenation of partitions μ_1 and μ_2 , is a subpartition of π . $\square$

Combining the last two propositions, we obtain the following.

Corollary 4.3.9. *Let $\lambda \ll \nu$, with λ, ν subpartitions of π . Suppose $c \in P_{\lambda\nu}$. Then either $\lambda = \nu$ or there is an integer $t \geq 1$ and some $\mu_0, \dots, \mu_t$, all subpartitions of π , such that*

$$\lambda = \mu_0 \stackrel{1}{\ll} \mu_1 \stackrel{1}{\ll} \dots \stackrel{1}{\ll} \mu_t = \nu \text{ and } c \in P_{\mu_0\mu_1} \cdot P_{\mu_1\mu_2} \cdots P_{\mu_{t-1}\mu_t}.$$

Consequently, there are no other irreducible maps than the ones from sets $P_{\lambda\mu}$ with $\lambda \stackrel{1}{\ll} \mu$, even if we are restricted to subpartitions of some π .

Example 4.3.10. Take $\pi = [7, 2, 1, 1] \vdash 11$ and $\lambda = [4, 2, 1, 1], \nu = [7, 1] \vdash 8$. All equivalences on λ induced by maps from $P_{\lambda\nu}$ have only one nontrivial class $\{4, 2, 1\}$. Maps from $P_{\lambda\nu}$ factor through $[4, 3, 1]$, but this is not a subpartition of π . However, the maps do factor through $[5, 2, 1]$ or $[6, 1, 1]$, and these are subpartitions of π .

4.4 Semigroups P_k^π

We shall now introduce more notation that will be crucial for our description of the structure of the semigroup algebra of $\mathcal{J}_X^\pi$.

Definition 4.4.1 (Partition set $\mathbb{P}_k^\pi$). Let $\pi \vdash n$ be a fixed partition of n . We denote by $\mathbb{P}_k^\pi$ the set of all partitions of k that are subpartitions of π :

$$\mathbb{P}_k^\pi = \{k \vdash \lambda \mid \lambda \text{ is a subpartition of } \pi\}.$$

The set $\mathbb{P}_k^\pi$ is partially ordered by $\ll$.

It is easy to characterize the rare case of $\mathbb{P}_k^\pi$ being just a single point. A partition π of n is called *rectangular* if it is of the form $n = m + m + \cdots + m$.

Lemma 4.4.2. $\mathbb{P}_k^\pi$ has exactly one point if and only if one of the following conditions is true:

- i) $k \leq 1$.
- ii) The partition π is rectangular and $n - k \leq 1$.

The next example shows a typical poset $\mathbb{P}_k^\pi$.

Example 4.4.3. Let $\pi = [5, 3, 2, 1, 1] \vdash 12$ and $k = 8$. Then $\mathbb{P}_k^\pi$ is the poset

The Hasse diagram is given by drawing $\stackrel{1}{\ll}$. Partitions that appear ‘lowest’ in the diagram are ‘smallest’ in the order $\ll$.

Now we are ready to define three sets G_k^π , N_k^π , P_k^π .

Definition 4.4.4 (Sets G_k^π , N_k^π , P_k^π). Let $\pi \vdash n$ and $1 \leq k \leq n$. We put

$$P_k^\pi = \bigcup_{\lambda, \mu \in \mathbb{P}_k^\pi} P_{\lambda\mu} \cup \{0\},$$

$$G_k^\pi = \bigcup_{\lambda \in \mathbb{P}_k^\pi} G_\lambda \cup \{0\},$$

$$N_k^\pi = P_k^\pi \setminus G_k^\pi \cup \{0\}.$$

It immediately follows from Proposition 4.2.1 on page 79 that the sets G_k^π , N_k^π and P_k^π are semigroups under composition of order-preserving maps, zero being the product of two maps whose composition is not defined. Obviously, G_k^π is build-up from other semigroups and we could have written it as a contracted disjoint sum $G_k^\pi = \bigsqcup_{\lambda \in \mathbb{P}_k^\pi} G_\lambda$. The action of G_k^π on N_k^π is nontrivial for nonempty N_k^π , so P_k^π is not a contracted disjoint sum of G_k^π and N_k^π .

Note that the poset $\mathbb{P}_k^\pi$ has a finite depth. Simple application of Proposition 4.2.1 on page 79 and Corollary 4.3.9 on page 88 yields the statement below.

Theorem 4.4.5. *Let $\pi \vdash n$ be a fixed partition of n .*

i) N_k^π is a nilpotent ideal of P_k^π , its nilpotency index equal to the depth of the poset $\mathbb{P}_k^\pi$.

ii) $N_k^\pi N_k^\pi$ consists of all non-invertible, non-irreducible maps:

$$N_k^\pi N_k^\pi = \bigcup_{\substack{\lambda, \mu \in \mathbb{P}_k^\pi \\ \lambda \neq \mu, \lambda \not\leq \mu}} P_{\lambda\mu} \cup \{0\}.$$

iii) N_k^π is generated by irreducible maps:

$$N_k^\pi = \langle \bigcup_{\substack{\lambda, \mu \in \mathbb{P}_k^\pi \\ \lambda \leq \mu}} P_{\lambda\mu} \rangle.$$

The obvious consequence for the semigroup algebra of P_k^π over an algebra A is the following.

Corollary 4.4.6. *AP_k^π is semisimple if and only if $\mathbb{P}_k^\pi$ has only one point λ and AG_λ is semisimple.*

The ideal N_k^π is, without a doubt, a stable G_k^π -bimodule. Moreover, G_k^π consists of all regular elements of P_k^π , while N_k^π consists of all nilpotent elements of P_k^π .

Corollary 4.4.7. *The semigroups P_k^π are R -graph semigroups.*

We will rephrase the result once more. The poset $\mathbb{P}_k^\pi$ is understood to determine an oriented graph by means of its Hasse diagram — direction of arrows is from ‘smaller’ to ‘larger’ elements.

Theorem 4.4.8 (Structure of P_k^π). *The semigroup P_k^π is isomorphic to a graph semigroup over the poset $\mathbb{P}_k^\pi$ with the modulation*

$$\left\{ (G_\lambda, P_{\lambda\mu}) \mid \lambda, \mu \in \mathbb{P}_k^\pi, \lambda \stackrel{1}{\ll} \mu \right\}.$$

Recall that the structure of G_λ is given in Lemma 4.2.8 on page 82 and that of $P_{\lambda\mu}$ in Lemma 4.3.5 on page 85.

Referring to discussion preceding Example 4.1.6 on page 78, we can express precisely the structure of $\tilde{P}_k^\pi$ — this semigroup is an M -variant of P_k^π .

Theorem 4.4.9 (Structure of $\tilde{P}_k^\pi$). *The semigroup $\tilde{P}_k^\pi$ is isomorphic to a graph semigroup over the poset $\mathbb{P}_k^\pi$ with the modulation*

$$\left\{ (\mathcal{M}_{p_\lambda^\pi} \dot{\times} G_\lambda, M_{p_\lambda^\pi p_\mu^\pi} \times P_{\lambda\mu}) \mid \lambda, \mu \in \mathbb{P}_k^\pi, \lambda \stackrel{1}{\ll} \mu \right\},$$

where $M_{p_\lambda^\pi p_\mu^\pi} \times P_{\lambda\mu}$ are diagonal bimodules.

Recall that the integers $p_\lambda^\pi \geq 1$ were introduced in Definition 4.1.4 on page 76.

According to Theorem 3.5.7 on page 67, we obtain the following description of the semigroup algebra of $\mathcal{J}_X^\pi$.

Theorem 4.4.10 (Structure of $A\mathcal{J}_X^\pi$). *Let π be a partition of n , and A be an algebra. Then*

$$A\mathcal{J}_X^\pi \cong \prod_{k=1}^n A\tilde{P}_k^\pi,$$

where each semigroup $\tilde{P}_k^\pi$ is an M -variant of P_k^π .

Example 4.4.11. Let $\pi = [7, 2, 1, 1] \vdash 11$ and $k = 8$. Then $\mathbb{P}_k^\pi$ is the poset below.

The semigroup P_k^π is a graph semigroup over the modulated graph

where the bimodules $T, V, \bar{V}, W, \bar{W}$ have 1, 2, 2, 6, 6 elements, respectively. T, V, W are unital right $\mathcal{S}_1$ -modules, and are regular left $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ -modules, in respective order. The bimodule $\bar{V}$ is isomorphic to the regular bimodule of $\mathcal{S}_2$. The bimodule $\bar{W}$ is, as a left module, isomorphic to the regular left $\mathcal{S}_3$ -module, and the right action of $\mathcal{S}_2$ comes from the identification of $\mathcal{S}_2$ with a subgroup of $\mathcal{S}_3$.

Denoted powers signify that the bimodule assigned to an edge is a disjoint union of the indicated number of copies of the given bimodule.

Remark 4.4.12. One may notice that for a fixed k , there are only finitely many distinct posets $\mathbb{P}_k^\pi$ that may arise, since they are all subposets of the poset of all partitions of k with the order given by $\ll$. For instance, with $k = 8$, $\mathbb{P}_k^\pi$ will be always a subposet of

(cf. Example 4.4.3 on page 89 and Example 4.4.11 on the page before).

Chapter 5

Examples

The main objective of this chapter is to provide numerous examples illustrating the concepts developed in earlier chapters.

The representation theory of finite semigroups is to some extent controlled by representations of free graph semigroups. For this reason, we include several simple examples.

Linear representations of semigroups are briefly introduced. Transformation representations fit into this more general framework, and have a great advantage of allowing more efficient description than a general linear representation.

Unlike in previous chapters, we describe in some detail the most relevant representation-theoretic aspects of particular finite semigroups. For the sake of concreteness, representations of semigroups are often considered over division algebras.

5.1 Trivial Modulation

We shall give some of the simplest examples of graph semigroups. As we mentioned in Remark 2.4.4 on page 37, every path semigroup over a quiver Q is a special case of a free graph semigroup.

More generally, consider a graph semigroup over a graph Q which is trivially modulated (cf. Definition 2.4.3 on page 37). As the semigroups labelling vertices have only one element, and the bimodules labelling edges have one element too, it is advantageous to identify all these with the corresponding elements of the graph semigroup and think of them simply as generators of the semigroup.

Note that vertices correspond to mutually orthogonal idempotents in such a graph semigroup. We will henceforth adopt the convention of labelling vertices (modulated by trivial semigroups) with symbols $e_1, e_2, \dots$, denoting by E the set formed by all e_i 's if this notation is useful. The edges connecting two such vertices, and modulated by the trivial unital bimodule — these are all the remaining generators of a semigroup in case the whole graph is trivially modulated — will be most often labelled by symbols g_i . Their collection is sometimes denoted by W , if this is the set of all irreducibles.

The first example shows that free semigroups with identity are particular examples of path semigroups.

Example 5.1.1. The graph Q in this example has one vertex, labelled e_1 , and three loops, labelled g_1, g_2, g_3 .

The free graph semigroup of this modulated graph, that is, the path semigroup of

the “bouquet” of three loops, is the free semigroup with identity (namely e_1) on three generators g_1, g_2, g_3 .

Remark 5.1.2. Free graph semigroups $S = FQ(E, W)$ over a trivially modulated graph Q have the following property: for every nonzero $a \in S \setminus E$, there is a *unique* presentation

$$a = g_{j_1} g_{j_2} \cdots g_{j_p} \quad \text{with } g_{j_q} \in W \text{ for } 1 \leq q \leq p.$$

Observe that in such an S , the sets E and W are uniquely determined: E as the set of all nonzero idempotents, W as the set of irreducible nonzero elements of $S \setminus W$.

The next example presents two graph semigroups over the same modulated graph.

Example 5.1.3. Semigroups $S = \{e_1, e_2, g_1, g_2, 0\}$ and $S' = \{e_1, e_2, g_1, g_2, a, 0\}$ are given by the multiplication tables below.

S	e_1	e_2	g_1	g_2
e_1	e_1		g_1	
e_2		e_2		g_2
g_1		g_1		
g_2	g_2			

and

S'	e_1	e_2	g_1	g_2	a
e_1	e_1		g_1		a
e_2		e_2		g_2	
g_1		g_1		a	
g_2	g_2				
a		a			

For clarity, zero products are shown as a blank space in multiplication tables. Rows and columns corresponding to 0 (and 1, if applicable) are not shown.

Let $E = \{e_1, e_2\}$, $W = \{g_1, g_2\}$. We can see that both S and S' are graph semigroups according to Definition 2.5.7 on page 43. Firstly, we can write both S and S' as $E^0 \triangleleft \langle\langle W \rangle\rangle$. Secondly, W is the bimodule of irreducibles, and is clearly stable as E^0 -bimodule in both cases.

Thus each semigroup S , S' is a graph semigroup over the trivially modulated graph Q :

$$\begin{array}{ccc} & g_1 & \\ e_1 & \xrightarrow{\quad} & e_2 \\ & g_2 & \end{array}$$

By Theorem 2.5.8 on page 44, these semigroups are factors of the path semigroup of Q . It is easy to see that the appropriate path-class congruences are generated by the relations $g_1g_2 = 0$, $g_2g_1 = 0$ (for S) and $g_2g_1 = 0$ (for S').

As the next example shows, even graph semigroups over trivially modulated graphs Q may possess nonzero idempotents distinct from idempotents corresponding to vertices of Q .

Example 5.1.4. Let $i, p \geq 1$ be integers. The *cyclic semigroup* with index i and period p is the semigroup S with presentation

$$S = \langle s \mid s^i = s^{i+p} \rangle.$$

We now consider S^1 .

- i) If $i = 1$, then $S^1 = S = G$ is a cyclic group with p elements, the element $e = s^p$ is its identity element.

ii) If $i > 1$, then S^1 is the graph semigroup over the graph

with trivial modulation ($E = \{e_1\}, W = \{s\}$), given by the relation $s^i = s^{i+p}$. Of course, the element e_1 is the adjoined identity of S^1 . As the set $G = \{s^k \mid i \leq k < i+p\}$ is a (cyclic) subgroup of S , there is an idempotent $e \in S$ — the identity of the subgroup G .

If the period $p > 1$, this e cannot be a zero of S , hence it is a nonzero idempotent of S^1 , distinct from the identity e_1 .

Consider now a unital algebra B . Note that in all the cases, e is certainly a central idempotent of the semigroup algebra BS . The subgroup G is an ideal of S . We denote by N the semigroup S/G and by $\bar{s}$ the generator of N .

If $i = 1$, N is one-element semigroup. If $i > 1$, however, N is a nilpotent semigroup, since $\bar{s}^i = 0$ in N .

It is now most convenient to split the discussion into three cases.

- i) If $i = 1$, we have $BG \cong BS$.
- ii) If $i > 1$, $p = 1$, then $BN^1 \cong BS$.
- iii) If $i > 1$, $p > 1$, then e satisfies all the assumptions of Proposition 3.2.5 on page 58, with $S^e \cong G$. Thus

$$BG \times BN^1 \cong BS.$$

Note the exact form of the isomorphism. It is the identity map on G , but the

generators $\bar{s}, 1$ of N^1 are mapped as follows:

$$\begin{aligned} 1 &\mapsto e_1, \\ \bar{s} &\mapsto s - es. \end{aligned}$$

As follows from Theorem 1.4.1 on page 19, the assumption that B be unital is unnecessary — it just allows us to express the isomorphism with the simplest notation possible.

Similarly, BS is either a group algebra or a nilpotent algebra or a product of the two kinds of algebras.

Remark 5.1.5. Demonstrated properties of S and S^1 would also follow by an (elementary) application of the powerful Theorem 3.5.7 on page 67.

5.2 Linear Representations

So far, we have used transformation modules of a semigroup purely as a tool for construction of more complex semigroups from simpler building blocks. Transformation modules, however, play the same role as in the group theory — they reveal useful information about the properties of the acting semigroup.

Example 5.2.1. To illustrate a case where the concept of a transformation S -module is useful, consider the *Kronecker semigroup* $S = \{e_1, e_2, g_1, g_2, 0\}$ given

by the multiplication table

	e_1	e_2	g_1	g_2
e_1	e_1		g_1	g_2
e_2		e_2		
g_1			g_1	
g_2				g_2

Equivalently, S is the free graph semigroup over the graph

with trivial modulation.

Here, all “discrete” indecomposable representations are given by transformation S -modules. For each $n \geq 1$, the set

$$X_n = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, y_{n+1}\}$$

with the S -action given by

$$\begin{array}{llll}
 x_i \cdot e_1 = x_i, & x_i \cdot e_2 = 0, & x_i \cdot g_1 = y_i, & x_i \cdot g_2 = y_{i+1}, \\
 y_j \cdot e_1 = 0, & y_j \cdot e_2 = y_j, & y_j \cdot g_1 = 0, & y_j \cdot g_2 = 0
 \end{array}$$

is an indecomposable S -module. Together with the “transposed” S -modules

$$X'_n = \{x_1, x_2, \dots, x_n, x_{n+1}, y_1, y_2, \dots, y_n\}$$

where g_2 now acts by $x_{i+1} \cdot g_2 = y_i$, they exhaust all discrete indecomposable representations of S , as showed by Kronecker in [11].

Stable transformation modules of a finite graph semigroup with trivial modulation are easy to represent graphically. The diagram

encodes much of the information on the action of S on X_n . The action of the irreducible element g_1 is represented by a solid arrow, the action of g_2 by a dashed arrow.

Non-existence of an arrow (starting in a particular element of the underlying set of the module) in this diagram implies that the action of the corresponding element g_i is zero. Or, if we view the action g_i as a partial action, such situation naturally indicates that given point is not in the domain of definition.

The only information missing is the action of e_i 's. This action is better diagrammed not using arrows, but rather denoting which idempotent acts as identity, as there is exactly one such idempotent for each element of our module.

As the names of elements of X_n carry no vital information, the transformation S -module X_n is most efficiently encoded by the diagram

and the transposed module X'_n could be represented as

Although the concept of a transformation S -module may be in certain situations useful, it is natural, as in the case of groups, to deal with a more appropriate concept of representations of semigroups.

Linear representations (as opposed to transformation representations) of a semigroup over an algebra B encode a B -linear action of the semigroup on a free B -module. This time, there is no need for an ‘external zero’ element — the zero of the free module serves that function.

A *linear representation* of a semigroup S on a free B -module V is a semigroup homomorphism $S \rightarrow \text{End}_B(V)$ of S to the multiplicative semigroup of B -endomorphisms of V such that 0 of S (if it exists) is mapped into the zero endomorphism of V .

As B is unital, such a semigroup homomorphism obviously extends to an algebra homomorphism $BS \rightarrow \text{End}_B(V)$, and conversely, an algebra homomorphism restricts to a semigroup homomorphism.

A linear representation V of S will be called simply an S -module.

A transformation S -module X gives rise to an S -module V over B in an obvious fashion: we let V be the free B -module on the set X , and extend the action of S on X to B -linear maps from V to itself.

S -modules arising in this way are sometimes called *monomial*. Not every S -module can be obtained in this way, however.

Example 5.2.2. Let us now consider the free graph semigroup S over the trivially modulated graph

$$\begin{array}{ccccc} & & e_2 & & \\ & & \downarrow g_2 & & \\ e_1 & \xrightarrow{g_1} & e_4 & \xleftarrow{g_3} & e_3 \end{array}$$

Thus, $S = \{e_1, e_2, e_3, e_4, g_1, g_2, g_3, 0\}$, with the multiplication in S determined by requiring that e_1, e_2, e_3, e_4 be orthogonal idempotents and

$$e_1 g_1 e_4 = g_1, \quad e_2 g_2 e_4 = g_2, \quad e_3 g_3 e_4 = g_3.$$

Consider the transformation representation of S on the set $X_1 = \{x_1, x_2, x_3, x_4\}$ given by the diagram below.

There is no need to distinguish different arrows graphically in this case. For instance, the arrow from 3 to 4 must represent the action of g_3 , due to the fact that $e_3 g_1 = 0 = e_3 g_2$ in the semigroup S .

The corresponding linear representation could be given as follows. Let V be the free B -module on the set X . The following action of S on the ordered basis $\mathcal{X} = (x_1, x_2, x_3, x_4)$ of V uniquely determines the B -module V :

$$\begin{aligned}
 \mathcal{X} \cdot e_1 &= (x_1, 0, 0, 0), & \mathcal{X} \cdot g_1 &= (x_4, 0, 0, 0), \\
 \mathcal{X} \cdot e_2 &= (0, x_2, 0, 0), & \mathcal{X} \cdot g_2 &= (0, x_4, 0, 0), \\
 \mathcal{X} \cdot e_3 &= (0, 0, x_3, 0), & \mathcal{X} \cdot g_3 &= (0, 0, x_4, 0), \\
 \mathcal{X} \cdot e_4 &= (0, 0, 0, x_4).
 \end{aligned}$$

Now let $X' = \{x_1, x_2, x_3, x_4, x_5\}$, and $\mathcal{X}' = (x_1, x_2, x_3, x_4, x_5)$ be the corresponding ordered basis of a free B -module V' on X' . The action of S on V' , given on

generators $\mathcal{X}'$ by

$$\begin{aligned}\mathcal{X}' \cdot e_1 &= (x_1, 0, 0, 0, 0), & \mathcal{X}' \cdot g_1 &= (x_4, 0, 0, 0, 0), \\ \mathcal{X}' \cdot e_2 &= (0, x_2, 0, 0, 0), & \mathcal{X}' \cdot g_2 &= (0, x_5, 0, 0, 0), \\ \mathcal{X}' \cdot e_3 &= (0, 0, x_3, 0, 0), & \mathcal{X}' \cdot g_3 &= (0, 0, x_4 + x_5, 0, 0), \\ \mathcal{X}' \cdot e_4 &= (0, 0, 0, x_4, x_5),\end{aligned}$$

gives V' the structure of an S -module. V' is an indecomposable module, just as V is. This time, however, the representation is not monomial.

Graphical depiction of the representation V' presents a slight problem, since it is not a representation induced by a transformation module. Nevertheless, it can be given by the diagram

Note that to completely specify the representation, one has to know to which linear combination of the targets of the split arrow is the source mapped to by the relevant action. In this case, supplying that information does not excessively clutter the diagram.

5.3 Illustrative Examples

In this final section, we are going to provide a few more examples of semigroups of modest size, illustrating both the strengths and limitations of the results proved in previous chapters.

In describing the representations of given semigroups, we shall exploit some results of the theory of representations of finite-dimensional associative algebras of [4].

Unless there are explicit further assumptions, B stands for an arbitrary unital algebra. When mentioning classification of indecomposable representations, however, we are dealing exclusively with categories of finitely generated linear representations of semigroups over a division algebra.

For a nonzero element s of a semigroup S , we denote by $\bar{s}$ the associated nonzero element of the relevant principal factor of S .

Example 5.3.1. Let us identify the symmetric inverse semigroup $\mathcal{I}_X$ of all partial one-to-one maps of a two-element set X with $S = \{0, 1, e, f, a, a^{-1}, b\}$. Its multiplication table is

	e	f	a	a^{-1}	b
e	e		a		a
f		f		a^{-1}	a^{-1}
a		a		e	e
a^{-1}	a^{-1}		f		f
b	a^{-1}	a	f	e	1

and there are two nonzero $\mathcal{J}$ -classes: $D_1 = \{e, f, a, a^{-1}\} <_{\mathcal{J}} D_2 = \{1, b\}$; the corresponding principal factors are the matrix units semigroup $\mathcal{M}_2$ and the symmetric group $\mathcal{S}_2$.

Theorem 3.5.3 on page 65 gives us the structure of BS :

$$B\mathcal{M}_2 \times B\mathcal{S}_2 \cong BS.$$

The splitting is given by the central idempotent $z_1 = e + f$, and we let $z_2 = 1 - z_1$. The isomorphism given by Proposition 3.2.5 on page 58 is easily described — we just need to identify $\mathcal{M}_2$ and $\mathcal{S}_2$ with the subsets S^{z_1} and S_{z_1} of $BS = B\mathcal{I}_X$. The isomorphism is determined by

$$\begin{array}{ll} \mathcal{M}_2 \rightarrow BS & \mathcal{S}_2 \rightarrow BS \\ \bar{s} \mapsto sz_1 = s & \bar{s} \mapsto sz_2 = s - sz_1 \\ \text{for } s \in D_1 & \text{for } s \in D_2 \end{array}$$

(and the zero of S goes to 0 of BS , of course).

One may write the above explicitly, in visually suggestive way, as

$$\mathcal{M}_2 \equiv \begin{bmatrix} e & a \\ a^{-1} & f \end{bmatrix}$$

(each nonzero element of $\mathcal{M}_2$ mapped to the element of BS occupying the corresponding entry of the matrix on the right) and

$$\mathcal{S}_2 = \{\bar{1}, \bar{b}\} \equiv \{1 - e - f, b - a - a^{-1}\}.$$

Further elaboration of the structure of S is often possible (and desirable, if we are interested in representations), but cannot be done universally, without some assumptions on the coefficient algebra B .

i) Suppose $\text{char } B = 2$ (that is, $1 + 1 = 0$ in B). Then $(\bar{1} - \bar{b})^2 = 0$ in $B\mathcal{S}_2$.

Letting T to denote the graph semigroup over the graph

$$\mathcal{M}_2 \quad \begin{array}{c} g_1 \\ \curvearrowright \\ e_1 \end{array}$$

(as before, e_1 represents the one-element semigroup, g_1 the one-element unital e_1 - e_1 -bimodule) determined by the relation $g_1 g_1 = 0$. We obtain an isomorphism $BT \cong BS$, induced by the identification of $\mathcal{M}_2$ given above and by the assignments

$$\begin{aligned} e_1 &\mapsto 1 - e - f, \\ g_1 &\mapsto 1 - e - f - b + a + a^{-1}. \end{aligned}$$

- ii) Assume that $\frac{1}{2} \in B$ (that is, 2^{-1} exists in B). Then $B\mathcal{S}_2$ splits, as we have new central idempotents $\frac{1}{2}(\bar{1} - \bar{b}), \frac{1}{2}(\bar{1} + \bar{b})$. Thus

$$B\mathcal{M}_2 \times Be_1 \times Be_2 \cong BS$$

the one-element semigroups e_1 and e_2 being mapped as follows:

$$\begin{aligned} e_1 &\mapsto \frac{1}{2}(1 - e - f - b + a + a^{-1}), \\ e_2 &\mapsto \frac{1}{2}(1 - e - f + b - a - a^{-1}). \end{aligned}$$

- iii) In other cases ($\text{char } B \neq 2$ and $2^{-1} \notin B$), one cannot give a B -isomorphism of $B\mathcal{S}_2$ to another semigroup algebra. It is sometimes possible to use the above two cases, however, if we do not insist on B -algebra isomorphisms: for instance, with $B = \mathbb{Z}/6\mathbb{Z}$, one gets $BS \cong \mathbb{F}_2 S \times \mathbb{F}_3 S$, and the $\mathbb{F}_2$ and $\mathbb{F}_3$ components may be dealt with separately.

Example 5.3.2. The symmetric inverse semigroup $S = \mathcal{I}_X$ of all partial one-to-one transformations of a three-element set X has 33 nonzero elements. There are three nonzero $\mathcal{J}$ -classes $D_1 <_{\mathcal{J}} D_2 <_{\mathcal{J}} D_3$, each D_i formed by maps of rank i . The associated principal factors are semigroups $\mathcal{M}_3$, $\mathcal{M}_3 \times \mathcal{S}_2$ and $\mathcal{S}_3$.

Let u_i be the sum of idempotents of D_i (for $1 \leq i \leq 3$), and put

$$z_1 = u_1$$

$$z_2 = (1 - z_1)u_2$$

$$z_3 = (1 - z_1 - z_2)u_3 = (1 - z_1 - z_2).$$

These elements are central idempotents of BS , obtained by twofold application of Proposition 3.2.5 on page 58 — first with z_1 , then with $z_1 + z_2$. The isomorphism

$$B\mathcal{M}_3 \times B(\mathcal{M}_3 \dot{\times} \mathcal{S}_2) \times B\mathcal{S}_3 \cong BS$$

is determined by the maps

$$\mathcal{M}_3 \rightarrow BS, \mathcal{M}_3 \dot{\times} \mathcal{S}_2 \rightarrow BS, \mathcal{S}_3 \rightarrow BS$$

given by

$$\bar{s} \mapsto z_i s$$

for $s \in D_i$, with $i = 1, 2, 3$, respectively.

Since now the structure groups $\mathcal{S}_2, \mathcal{S}_3$ have orders divisible by both 2 and 3, further decompositions of BS will depend on properties of both 2 and 3 in B , thus there are many possibilities. We examine three most common cases.

i) If both 2^{-1} and 3^{-1} are in B , we have $B \times B \cong B\mathcal{S}_2$ and $B \times B\mathcal{M}_2 \times B \cong B\mathcal{S}_3$.

Thus BS decomposes as follows:

$$B\mathcal{M}_3 \times B\mathcal{M}_3 \times B\mathcal{M}_3 \times B \times B\mathcal{M}_2 \times B \cong BS.$$

If B is a division algebra, then BS is semisimple, having (up to isomorphism) six indecomposable representations.

- ii) If we have $\text{char } B = 2$, however, we get that the graph semigroup T over the modulated graph

$$\begin{array}{ccccc} & & M_{33} \times g_1 & & g_2 \\ & & \curvearrowright & & \curvearrowright \\ \mathcal{M}_3 & & \mathcal{M}_3 \times e_1 & & \mathcal{M}_2 \\ & & \downarrow & & \downarrow \\ & & e_1 & & e_2 \end{array}$$

($M_{33} \times g_1$ is a diagonal bimodule), given by the relations $g_1 g_1 = 0$ and $g_2 g_2 = 0$, has semigroup algebra BT isomorphic to BS . Equivalently, T is an M-variant of the semigroup

$$\mathcal{S}_1 \sqcup N^1 \sqcup \mathcal{S}_1 \sqcup N^1,$$

where $\mathcal{S}_1$ is the semigroup with one element, and N is the cyclic nilpotent semigroup on two elements.

Consequently, if B is a division algebra, there are again six indecomposable representations of BS , but two of them are not simple.

- iii) When $\text{char } B = 3$, the algebra BS_3 is better described by yet another semigroup. Define a semigroup T over the graph

$$\begin{array}{ccccccc} & & & & g_1 & & \\ & & & & \curvearrowright & & \\ \mathcal{M}_3 & & \mathcal{M}_3 & & \mathcal{M}_3 & & e_1 \xrightarrow{g_1} e_2 \\ & & & & & & \curvearrowleft \\ & & & & & & g_2 \end{array}$$

by the relations $g_1 g_2 g_1 = 0$, $g_2 g_1 g_2 = 0$. Then $BT \cong BS$.

In case B is a division algebra, the algebra BS is uniserial, and has precisely nine indecomposable representations.

Example 5.3.3. Let us consider now the 15-element inverse semigroup

$$S = \{e, f, g, a, b, c, d, p, q, a^{-1}, b^{-1}, c^{-1}, d^{-1}, 0, 1\}$$

given by the following multiplication table.

	e	f	g	a	b	c	d	p	q	a^{-1}	b^{-1}	c^{-1}	d^{-1}
e	e	e	e	d			d			b^{-1}	b^{-1}		
f	e	f	e	a	b	c	d	p		b^{-1}	b^{-1}		
g	e	e	g	d			d		q	a^{-1}	b^{-1}	c^{-1}	d^{-1}
a	b	b	a	c			c		d	f	p	b^{-1}	e
b	b	b	b	c			c			p	p		
c			c						c	b		p	b
d			d						d	e		b^{-1}	e
p		p		b	b	c		p					
q			q						q	d^{-1}		c^{-1}	d^{-1}
a^{-1}	d^{-1}	a^{-1}	d^{-1}	g	e	d	q	b^{-1}		c^{-1}	c^{-1}		
b^{-1}		b^{-1}		e	e	d		b^{-1}					
c^{-1}		c^{-1}		d^{-1}	d^{-1}	q		c^{-1}					
d^{-1}	d^{-1}	d^{-1}	d^{-1}	q			q			c^{-1}	c^{-1}		

There are three nonzero $\mathcal{J}$ -classes, linearly ordered by $<_{\mathcal{J}}$:

$$D_1 = \{p, e, q, b, c, d, b^{-1}, c^{-1}, d^{-1}\},$$

$$D_2 = \{f, g, a, a^{-1}\},$$

$$D_3 = \{1\},$$

the associated principal factors are the matrix unit semigroups $\mathcal{M}_3$, $\mathcal{M}_2$, $\mathcal{M}_1$.

The elements u_i and z_i ($i = 1, 2, 3$) of BS are defined as in the previous example. In this case, D_2 has only two idempotents (f and g), however. In terms of the original semigroup generators of BS , we have

$$z_1 = p + e + q, \quad z_2 = f + g - p - 2e - q, \quad z_3 = 1 - f - g + e.$$

Again, the isomorphism

$$B\mathcal{M}_3 \times B\mathcal{M}_2 \times B\mathcal{M}_1 \cong BS$$

($B\mathcal{M}_1 \cong B\mathcal{S}_1 \cong B$, the choice is just a matter of taste) is determined by the maps

$$\mathcal{M}_3 \rightarrow BS, \mathcal{M}_2 \rightarrow BS, \mathcal{M}_1 \rightarrow BS$$

given by $\bar{s} \mapsto z_i s$ for $s \in D_i$, with $i = 1, 2, 3$, respectively.

Explicitly, these maps can be expressed by the identifications

$$\begin{aligned} \mathcal{M}_3 &\equiv \begin{bmatrix} p & b & c \\ b^{-1} & e & d \\ c^{-1} & d^{-1} & q \end{bmatrix}, \\ \mathcal{M}_2 &\equiv \begin{bmatrix} f - p - e & a - b - d \\ a^{-1} - b^{-1} - d^{-1} & g - e - q \end{bmatrix}, \\ \mathcal{M}_1 &\equiv [1 - f - g + e]. \end{aligned}$$

Thus, the semigroup algebra BS is Morita equivalent to the product of three copies of the coefficient algebra B .

Example 5.3.4. The subsemigroup $S' = \{e, f, g, a, b, c, d, 0, 1\}$ of S of the previous example is of more interest.

This semigroup is no longer inverse. Moreover, it is not a full subsemigroup of S (the idempotents p, q are missing), so we cannot simply apply Theorem 3.5.7 on page 67. Note that the $\mathcal{J}$ -classes have changed dramatically — in S' , each element forms an individual $\mathcal{J}$ -class. Nevertheless, we can still find a graph semigroup which has its semigroup algebra isomorphic to BS' .

Let T be the free graph semigroup over the trivially modulated graph

It is easily verified that the map $T \rightarrow BS'$ given by

$$\begin{aligned}
 e_1 &\mapsto f - e, \\
 e_2 &\mapsto e, \\
 e_3 &\mapsto g - e, \\
 g_1 &\mapsto b, \\
 g_2 &\mapsto a - b - d, \\
 g_3 &\mapsto d, \\
 e_4 &\mapsto 1 - f - g + e.
 \end{aligned}$$

induces an isomorphism $BT \cong BS'$.

In case B is a division algebra, BS' is a tame hereditary algebra, a direct sum of a copy of B and a path algebra of the Euclidean (or, equivalently, extended Dynkin) diagram of type $\tilde{\mathbb{A}}_2$. There is an infinite number of indecomposable representations. In fact, there are precisely three (non-isomorphic) indecomposable representations of S of dimension $3h + 1$ and $3h + 2$ for every integer $h \geq 0$ (see [4]).

If B is a finite field, the number of indecomposable representations of S' of given dimension is easy to enumerate. For example, over the Galois field $\mathbb{F}_2$ of two elements, the number of indecomposable representations of dimension $3h$ is given by the number $\alpha(h)$ of irreducible polynomials of degree h over $\mathbb{F}_2$. Referring to Gauss [8], $\alpha(h) = \frac{1}{h} \sum_{q|h} \mu(\frac{h}{q}) 2^q$, where μ is the Möbius function, and thus the

number of indecomposable representations of S' of dimension

$$3, 6, 9, 12, 15, \dots, 30, \dots, 60, \dots, 120, \dots, 150, \dots$$

is given, respectively, by

$$2, 1, 2, 3, 6, \dots, 99, \dots, 52\,377, \dots, 27\,487\,764\,474, \dots, 22\,517\,997\,465\,744, \dots$$

For example, the unique indecomposable representation of S' of dimension 6 is given by

$$\begin{aligned} e &\mapsto E_{33} + E_{44}, \quad f \mapsto E_{11} + E_{22} + E_{33} + E_{44}, \quad g \mapsto E_{33} + E_{44} + E_{55} + E_{66}, \\ a &\mapsto E_{13} + E_{15} + E_{24} + E_{25} + E_{26} + E_{35} + E_{46}, \quad b \mapsto E_{13} + E_{24}, \end{aligned}$$

where E_{ij} denote the elementary matrices ($1 \leq i, j \leq 6$).

Much clearer picture of the same representation is provided by its diagram, as a linear representation of T :

Example 5.3.5. Let $S = \{a, b, c, d, e, f, g, h, s, 0, 1\}$ be the semigroup with the fol-

lowing multiplication table.

	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>s</i>
<i>e</i>	<i>e</i>	<i>f</i>	<i>e</i>	<i>a</i>	<i>a</i>	<i>c</i>	<i>c</i>	<i>f</i>	<i>a</i>
<i>f</i>	<i>e</i>	<i>f</i>	<i>c</i>	<i>f</i>	<i>a</i>	<i>e</i>	<i>c</i>	<i>a</i>	<i>f</i>
<i>g</i>	<i>g</i>	<i>d</i>	<i>g</i>	<i>h</i>	<i>h</i>	<i>b</i>	<i>b</i>	<i>d</i>	<i>h</i>
<i>h</i>	<i>b</i>	<i>h</i>	<i>g</i>	<i>h</i>	<i>d</i>	<i>b</i>	<i>g</i>	<i>d</i>	<i>h</i>
<i>a</i>	<i>c</i>	<i>a</i>	<i>e</i>	<i>a</i>	<i>f</i>	<i>c</i>	<i>e</i>	<i>f</i>	<i>a</i>
<i>b</i>	<i>b</i>	<i>h</i>	<i>b</i>	<i>d</i>	<i>d</i>	<i>g</i>	<i>g</i>	<i>h</i>	<i>d</i>
<i>c</i>	<i>c</i>	<i>a</i>	<i>c</i>	<i>f</i>	<i>f</i>	<i>e</i>	<i>e</i>	<i>a</i>	<i>f</i>
<i>d</i>	<i>g</i>	<i>d</i>	<i>b</i>	<i>d</i>	<i>h</i>	<i>g</i>	<i>b</i>	<i>h</i>	<i>d</i>
<i>s</i>	<i>b</i>	<i>h</i>	<i>g</i>	<i>h</i>	<i>d</i>	<i>b</i>	<i>g</i>	<i>d</i>	<i>s</i>

The semigroup S is generated by two elements, e and s , satisfying $esese = e$.

There are only three nonzero $\mathcal{J}$ -classes:

$$D_1 = \{a, b, c, d, e, f, g, h\} <_{\mathcal{J}} D_2 = \{s\} <_{\mathcal{J}} D_3 = \{1\}.$$

The principal factors associated to D_2 and D_3 are both isomorphic to $\mathcal{M}_1$. The remaining principal factor is far more interesting. Denoting by I the (minimal nonzero) ideal $D_1 \cup \{0\}$, we can discern from the multiplication table that it is isomorphic to the Rees semigroup $\mathcal{M}^0(\mathcal{S}_2; 2, 2; P)$ with the sandwich matrix

$$P = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix},$$

where $\mathcal{S}_2 = \{1, t\}$. It follows from Lemma 3.3.3 on page 60 that BI does not have an identity, so we cannot split the ideal BI off in the same manner as we did in the first three examples of this section.

Observe that the element $z = s - h$ annihilates I on both sides, $zs = z = sz$ and $z^2 = z$, thus it is a central idempotent of BS . However, as $(BS)^z = Bz$ and $S^z = \{0\}$, we have $(BS)^z \neq BS^z$, thus the crucial hypothesis of Proposition 3.2.5 on page 58 does not hold. Letting $z' = 1 - z$, we certainly have that z' is an idempotent and z' acts as identity on I . Thus $I \cup \{z'\}$ is a subsemigroup of BS , isomorphic to I^1 , and $(BS)_z = B(I \cup \{z'\}) \cong BI^1$. Summarizing, we can write BS as the product of semigroup algebras

$$BI^1 \times B\mathcal{M}_1 \cong BS,$$

with $I \cup \{1\} \equiv I \cup \{1 - s + h\}$ and $\mathcal{M}_1 \equiv [s - h]$.

The structure group of the principal factor I is $\mathfrak{S}_2$, so one can expect the structure of BI (and BI^1) to depend on multiplicative properties of the element $2 (= 1 + 1)$ of B . We look at the two most important possibilities.

- i) Let $\text{char } B = 2$. Then the algebra BI^1 is isomorphic to the semigroup algebra of the graph semigroup over the quiver $e_1 \begin{matrix} \xrightarrow{g_1} \\ \xleftarrow{g_2} \end{matrix} e_2$ given by the relation $g_2g_1g_2g_1 = 0$.

Denoting by T the graph semigroup over

$$e_1 \begin{matrix} \xrightarrow{g_1} \\ \xleftarrow{g_2} \end{matrix} e_2 \quad e_3$$

with multiplication determined by the relation $g_2g_1g_2g_1 = 0$, we have an isomorphism of algebras $BT \cong BS$; explicitly, the generators of T can be mapped

as follows:

$$e_1 \mapsto 1 - s + h - e,$$

$$e_2 \mapsto e,$$

$$g_1 \mapsto g - e,$$

$$g_2 \mapsto f - e,$$

$$e_3 \mapsto s - h.$$

For instance, if B is a division algebra, then S has exactly ten indecomposable representations. The semigroup algebra BS is uniserial, but not quasi-hereditary in this case.

- ii) Let now B be such that $\frac{1}{2} \in B$. Then there is a “new” central idempotent $\frac{1}{4}(e + f + g + h - a - b - c - d)$ and the algebra BI^1 splits into the product of the semigroup algebra of the graph semigroup over $e_1 \xrightleftharpoons[g_2]{g_1} e_2$, defined by the relation $g_2g_1 = 0$, and the semigroup algebra $B\mathcal{M}_2$.

Thus the graph semigroup T over

$$e_1 \xrightleftharpoons[g_2]{g_1} e_2 \quad \mathcal{M}_2 \quad e_3$$

defined by the relation $g_2g_1 = 0$ has its semigroup algebra isomorphic to BS .

We provide an explicit isomorphism:

$$e_1 \mapsto 1 - s + h - \frac{3}{8}(e + f + g + h) + \frac{1}{8}(a + b + c + d),$$

$$e_2 \mapsto \frac{1}{8}(e + f + g + h + a + b + c + d),$$

$$g_1 \mapsto e + f - g - h + a - b + c - d,$$

$$g_2 \mapsto e - f + g - h + a + b - c - d,$$

$$\mathcal{M}_2 \equiv \begin{bmatrix} \frac{1}{4}(e + f - a - c) & e - f + a - c \\ g - h - b + d & \frac{1}{4}(g + h - b - d) \end{bmatrix},$$

$$e_3 \mapsto s - h.$$

For instance, if B is a division algebra of characteristic different from 2, the semigroup S has only seven indecomposable representations. Note that BS is quasi-hereditary in this case.

Example 5.3.6. Let us conclude this exposition by presenting three free graph semigroups S_f , S_t and S_w . Let $E = \{e_1, e_2, \dots, e_6\}$ be orthogonal idempotents and let $W = \{g_1, g_2, \dots, g_5\}$. One can define the semigroups S_f , S_t and S_w by requiring that they are semigroups generated by the set E of orthogonal idempotents and the set W , each semigroup subject to an individual set of relations.

i) $S_f = \langle E, W \rangle$ is given by

$$e_1 g_1 e_2 = g_1, \quad e_2 g_2 e_6 = g_2,$$

$$e_3 g_3 e_4 = g_3, \quad e_4 g_4 e_6 = g_4, \quad e_5 g_5 e_6 = g_5,$$

$$g_1 g_2 = a, \quad g_3 g_4 = b, \quad e_1 a e_6 = a, \quad e_3 b e_6 = b.$$

ii) $S_t = \langle E, W \rangle$ is determined by

$$\begin{aligned} e_1 g_1 e_6 &= g_1, & e_2 g_2 e_6 &= g_2, \\ e_3 g_3 e_5 &= g_3, & e_4 g_4 e_5 &= g_4, & e_5 g_5 e_6 &= g_5, \\ g_3 g_5 &= a, & g_4 g_5 &= b, & e_3 a e_6 &= a, & e_4 b e_6 &= b. \end{aligned}$$

iii) $S_w = \langle E, W \rangle$ is defined by

$$e_i g_i e_6 = g_i \quad \text{for } 1 \leq i \leq 5.$$

These semigroups are very similar as far as their basic semigroup properties are concerned: $E \cup W$ is a minimal set of generators, every element is a $\mathcal{J}$ -class by itself, exactly six of the nonzero elements are regular (namely the idempotents $e_1, \dots, e_6$), and each semigroup $S \in \{S_f, S_t, S_w\}$ satisfies $ES = S = SE$. Semigroups S_f and S_t even have the same number of elements.

Yet, from the representation-theory point of view, these semigroups are quite different. We shall demonstrate this while assuming that B is an infinite division algebra.

i) Since S_f is the path semigroup of the Dynkin diagram of type $\mathbb{E}_6$

$$\begin{array}{ccccccc} & & & e_5 & & & \\ & & & \downarrow g_5 & & & \\ e_1 & \xrightarrow{g_1} & e_2 & \xrightarrow{g_2} & e_6 & \xleftarrow{g_4} & e_4 \xleftarrow{g_3} e_3 \end{array}$$

the semigroup algebra BS_f has 36 indecomposable representations, the largest one of dimension 11.

ii) S_t , on the other hand, is given by the Euclidean diagram of type $\tilde{\mathbb{D}}_5$

and therefore BS_t of tame (infinite) representation type.

iii) Finally, the semigroup algebra of BS_w is a hereditary algebra over the graph

that is neither Dynkin nor Euclidean, and thus is of wild representation type.

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