**ON WHICH FINITE SUPERSOLVABLE GROUPS
ARE SUBNORMALLY MONOMIAL**

by

JAMES HEGINBOTTOM, B. A.

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School of Mathematics and Statistics
Ottawa-Carleton Institute of Mathematics and Statistics

Carleton University
Ottawa, Ontario, Canada
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ABSTRACT

We examine the relationship between finite supersolvable groups and subnormally monomial groups. We start by proving the existence of supersolvable groups which are not subnormally monomial. Next, we show that if the commutator subgroup of a supersolvable group is abelian then the group is subnormally monomial. We also show when a direct product of supersolvable groups is subnormally monomial. We show that all groups of order $2p^3$, for p an odd prime, are subnormally monomial except for one. Finally, we show that all supersolvable groups whose order is cubefree except for the least prime divisor are subnormally monomial.

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1. INTRODUCTION

In this work a *representation* is a homomorphism from a group into the general linear group of a finite-dimensional complex vector space. Given a finite group G , a subgroup H of G , and a representation σ of H , one may form an induced representation $\rho = \text{Ind}_H^G \sigma$ of G (§7.2 [Ser77]). If σ is one-dimensional and ρ is irreducible, then ρ is said to be *monomial*. If every irreducible representation of a group is monomial, then the group is said to be *monomial* (these are sometimes referred to as M-groups). Every monomial group is solvable (Theorem 15.7 [Dor71]). If a monomial representation ρ of a group G is induced from a representation σ of a subnormal subgroup H , so there exist subgroups of G such that

$$H = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G,$$

then ρ is said to be *subnormally monomial*. If every irreducible representation of a group is subnormally monomial, then the group is said to be *subnormally monomial* (these are sometimes referred to as SM-groups). For a monomial group to not be subnormally monomial it must have an irreducible representation ρ such that if H is a subgroup of G with a one-dimensional representation σ and $\rho = \text{Ind}_H^G \sigma$, then H is not subnormal in G .

A group is *supersolvable* if it has a series of subgroups where each subgroup is normal in the group and the factor groups are cyclic,

$$\{1\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_k = G.$$

This series is referred to as a composition series. For finite groups, it is immediate that the factor groups may be taken to be of prime order. Obviously, every supersolvable group is solvable. In the case of finite groups, every nilpotent group is supersolvable (§8.3 [Ser77]). Finally, every group of prime order, or p -group, is nilpotent (Theorem 14 [Ser77]). This gives the inclusion sequence

$$p\text{-group} \subset \text{nilpotent} \subset \text{supersolvable} \subset \text{solvable}.$$

This thesis is concerned with connecting some of these structural properties of a group with the group's representation theoretic properties.

In Section 3, we shall show that all supersolvable groups are monomial. As stated above, every finite nilpotent group is supersolvable. Since every subgroup of a nilpotent group is subnormal (Theorem 1, §6.1 [DF91]), every finite nilpotent group is subnormally monomial. Thus we have an infinite class of finite supersolvable groups which are subnormally monomial. This immediately poses the question, “Do there exist any supersolvable groups which are not subnormally monomial?” We shall show in Section 4 that the answer to this question is: “Yes, such groups do exist”.

The above information on solvable and monomial groups may be represented by the following inclusion diagram

Figure 1

The focus of this thesis is finite groups which are supersolvable but not nilpotent. The goal is to provide simple conditions for determining when such groups are subnormally monomial and when they are just monomial.

In the beginning of our research we used the GAP system [GAP] to examine selected groups of orders less than two thousand. In looking at supersolvable groups, which are not nilpotent, we quickly determined that there do exist ones which are not subnormally monomial. The smallest has order 54 and is the only one of this order. There is also one such group with order 250 and one with order 686. Multiple such groups were found to exist with orders

108, 162, 216, 270, 324, 378, 432, 500, and 540,

although not all groups of order less than 540 were examined. Looking at the prime decompositions of these orders, we noticed that there is a unique such

group for the orders $2 \cdot 3^3$, $2 \cdot 5^3$, and $2 \cdot 7^3$. Also, every such group has order divisible by $2p^3$ for some odd prime p . This lead us to our first conjectures.

Conjecture 1.1. *For each odd prime p , there exists a unique supersolvable group having order $2p^3$, which is not subnormally monomial.*

Conjecture 1.2. *If G is a finite supersolvable group which is not subnormally monomial then its order is divisible by $2p^3$ for some odd prime p .*

These two conjectures set the direction for the work done. We shall examine the relationship between finite supersolvable groups and subnormally monomial groups by dividing these groups into classes based on their order. The first class we consider is comprised of groups whose order is $2p^3$ for some odd prime p . We prove Conjecture 1.1 showing that there is a unique group of this order which is not subnormally monomial. As all the groups of this order are supersolvable, the others are all subnormally monomial (Theorem 6.11). The second class we consider is comprised of groups whose order is not divisible by a cube except for their least prime factor. We show that these groups are all subnormally monomial. This class is a subset of groups whose order is not divisible by $2p^3$ for any odd prime p and so provides a partial result for Conjecture 1.2. We also prove some more general results about subnormally monomial supersolvable groups.

Some work has been done in this area. In [How84a] and [How84b] it is stated that the class of subnormally monomial groups is closed under direct products (Lemma 3.1 [How84b]) although no proof is provided. We prove this result in Theorem 5.2.

Define CSF to be the class of chiefly sub-Frobenius groups (Definition 1.1 [How84a]) and define X to be the class of all solvable groups G satisfying the

following condition: for all primes p and for all $A \leq G$, $O_p(A)$, the unique maximal normal p -subgroup of A (Theorem 8.2 [Gor68]), has no central p -factor. Then the class of all subnormally monomial groups is equal to the intersection $CSF \cap X$ (Theorem 3.8 [How84b]). This classification requires knowledge of the group's structural properties.

An algorithm for determining whether a group is subnormally monomial or not is given in [Hor95]. As this algorithm iterates over the irreducible representations of the group and requires finding a series of maximal normal subgroups for each, it is potentially expensive.

Both [How84b] and [Hor95] provide methods for finding finite subnormally monomial groups. [How84b] classifies these groups using complicated structural properties. [Hor95] provides a potentially expensive algorithm for finding such groups. Our goal is to classify subnormally monomial groups based on the order of the group given the sole structural property of the group being supersolvable.

As well as being an interesting problem unto itself, the material covered in this thesis is motivated by work being done in number theory. The Langlands Program (see [Ber03]) attempts to associate representations of Galois groups to automorphic representations. As all supersolvable groups are Galois [Sha54], these groups are a piece of this puzzle. One proposition which has been put forward is

Proposition. (*Proposition 7.2 [AC89] and [Dad88]*) *Let E/F be a solvable Galois extension of number fields, with Galois group G . Assume that ρ is an irreducible representation of G which is subnormally monomial. Then ρ is automorphic, associated to a cuspidal representation.*

If G is supersolvable and subnormally monomial then the hypotheses of this proposition are met by all the irreducible representations.

2. BACKGROUND MATERIAL ON REPRESENTATION THEORY

Given a finite group G , a subgroup H of G , and a representation $\sigma : H \rightarrow GL(W)$, one may describe an induced representation $\rho = \text{Ind}_H^G \sigma$ of G in the following way. Let R be a set of representatives for the left cosets of H in G . Then every element $g \in G$ may be uniquely expressed as $g = hr$ for some $h \in H$ and some $r \in R$. For each right coset Hr , we can define a space W_r which is isomorphic to W and define the space $V = \oplus W_r$ for all $r \in R$. Since for all $h \in H$, $\sigma(h)W \subseteq W$, we can thus define $\rho(g)W_{r_i} = \rho(hr)W_{r_i} = \sigma(h)\rho(r)W_{r_i} = \sigma(h)W_{r_j}$ for every $g \in G$, where $r_i, r_j \in R$, $r_j = rr_i$ and $\sigma(h)W_{r_j} \cong \sigma(h)W$. This then defines the induced representation $\rho : G \rightarrow GL(V)$.

The *character* χ of a representation ρ of a finite group G is defined element-wise $\chi(g) = \text{tr}(\rho(g))$ for all $g \in G$. Two representations are equivalent if and only if they have identical characters (Corollary 2 of Theorem 4 [Ser77]). It immediately follows that the dimension of a representation is the value of its character at the identity. A character is said to be irreducible if its representation is irreducible. For two characters ϕ and ψ of the group G there is an inner product of characters (§2.3 [Ser77])

$$(2.1) \quad (\phi \mid \psi) = \frac{1}{|G|} \sum_{t \in G} \phi(t) \overline{\psi(t)}.$$

The value of this inner product (which gives nonnegative integers) is the multiplicity of ψ in ϕ (Theorem 4 [Ser77]). We therefore have that if χ is a character

and

$$(\chi | \chi) = 1,$$

then χ is an irreducible character. We can also determine information about induced representations through the characters involved. If G is a finite group, H is a subgroup of G , and σ is a representation of H with character ψ , then the formula

$$(2.2) \quad \chi(s) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}st \in H}} \psi(t^{-1}st),$$

for $s \in G$, gives the character of the representation $\text{Ind}_H^G \sigma$ (§7.2 [Ser77]). From this formula we can see that if the representation with character ψ has dimension n , then the dimension of the induced representation with character χ is $n \cdot [G : H]$. Both equations 2.1 and 2.2 are used extensively throughout this thesis.

3. SUPERSOLVABLE GROUPS ARE MONOMIAL

The main results of this thesis rely on the fact that every supersolvable group is monomial. In this section we present the key theorems and lemmas which prove this result following the work [Ser77]. We present expanded versions of the proofs in [Ser77] and some extra material which is used in later sections.

We first prove that every irreducible representation of a finite group G is induced from a representation of a subgroup H . We then refine the definition of this subgroup. In many of the proofs which follow we use this subgroup to prove

the desired results. Lemma 3.3 allows us to assume that irreducible representations are faithful under certain conditions and is also used later in this thesis. Finally, Theorem 3.5 proves that all supersolvable groups are monomial, a result upon which this thesis relies.

Throughout this section G is a group and all groups are assumed to be finite. Also, following [Ser77], we will identify representations with their underlying vector spaces.

Proposition 3.1. (*Proposition 24 [Ser77]*) *Let A be a normal subgroup of G , and let $\rho : G \rightarrow GL(V)$ be an irreducible representation of G . Then either*

- (a) there exists a subgroup H of G , unequal to G and containing A , and an irreducible representation σ of H such that ρ is induced from σ ; or*
- (b) the restriction of ρ to A is isotypic (i.e., is the direct sum of equivalent irreducible representations).*

Proof. Let $V = \oplus V_i$ be the canonical decomposition of ρ restricted to A into a direct sum of isotypic representations. Thus each $V_i = n_i W_i$ where the W_i are irreducible subrepresentations, $n_i \in \mathbb{Z}^+$ and $W_j = W_i$ if and only if $V_j = V_i$. Let $s \in G$. Then we have that

$$\rho(s) V_i = \rho(s) n_i W_i = n_i \rho(s) W_i.$$

As A is normal $s^{-1}as \in A$ for any $a \in A$ and

$$\rho(s)^{-1} \rho(a) \rho(s) W_i = \rho(s^{-1}as) W_i \subseteq W_i.$$

This implies that $\rho(a)\rho(s)W_i \subseteq \rho(s)W_i$. Letting $\rho(s)W_i = W_{i'}$, a subrepresentation of A , we now have that

$$n_i \rho(s)W_i = n_i W_{i'} = V_{i'}.$$

Similarly, $\rho(s^{-1})V_{i'} = V_i$ implying $n_i = n_{i'}$. Thus $\rho(s)$ permutes the V_i and, since V is irreducible, G permutes them transitively (i.e., for any V_i and V_j there exists $s \in G$ such that $\rho(s)V_i = V_j$). Fix an isotypic component V_j . If V_j is equal to V then the restriction of ρ to A is isotypic and case (b) holds. Otherwise, let

$$H = \{s \in G \mid \rho(s)V_j = V_j\}.$$

Obviously the identity is in H . Furthermore, if $s, t \in H$ we have that $\rho(st)V_j = \rho(s)\rho(t)V_j = \rho(s)V_j = V_j$, so H is closed. Finally

$$\rho(s)V_j = V_j$$

implies that

$$V_j = \rho(s^{-1})\rho(s)V_j = \rho(s^{-1})V_j,$$

proving that H is a subgroup of G . For $a \in A$ we have that $\rho(a)V_j = V_j$ by the definition of V_j and so A is a subgroup of H . Next, we have that $H \neq G$ since G acts transitively and ρ restricted to A is not isotypic. Let $\sigma : H \rightarrow GL(V_j)$ be the representation of H defined by restricting ρ to H and its action to V_j . Theorem 14.1 (Clifford's Theorem) [Dor71] proves that $\rho = \text{Ind}_H^G \sigma$ and thus, since ρ is irreducible, σ is irreducible and case (a) holds. $\square$

The above Proposition only requires that the subgroup A be normal in G . If A is abelian, then case (b) is equivalent to saying that $\rho(a)$ is a homothety for each $a \in A$ (Remark following Proposition 24 [Ser77]). In other words, there exists a constant $c_a \in \mathbb{C}$ for each $a \in A$ such that $\rho(a) = c_a I_n$, where I_n is the identity matrix. In this work we are only concerned with situations where A is abelian.

When the subgroup A of Proposition 3.1 is normal and abelian we can say more about the subgroup H which has the representation σ which induces to ρ . This additional information is used in the proofs of two of the results of this thesis. Given a representation ρ_j of A we define $\rho_j^s(a) = \rho_j(s^{-1}as)$ for $a \in A$ and $s \in G$.

Lemma 3.2. *Let A be a normal abelian subgroup of a group G . Let ρ be an irreducible representation of G such that the restriction of ρ to A is not isotypic. Then there exists a subgroup H of G , unequal to G and containing A , and an irreducible representation σ of H , such that ρ is induced from σ . Furthermore, we have that $H = \{s \in G \mid \rho_i^s \cong \rho_i\}$, where $\rho_i : A \rightarrow GL(V_i)$ is defined by restricting ρ to A and to its action on V_i , one of the isotypic components of the canonical decomposition of ρ .*

Proof. The existence of H and σ is shown in Proposition 3.1. Let $h \in H$. We have that $H = \{s \in G \mid \rho(s)V_i = V_i\}$. Since $A \subseteq H$, this implies that for all

$a \in A$

$$\begin{aligned}
 \rho_i(h^{-1}ah)V_i &= \rho_i(h^{-1})\rho_i(a)\rho_i(h)V_i \\
 &= \rho_i(h^{-1})\rho_i(a)V_i \\
 &= \rho_i(h^{-1})V_i \\
 &= V_i \\
 &= \rho_i(a)V_i,
 \end{aligned}$$

and so $h \in \{s \in G \mid \rho_i^s \cong \rho_i\}$ and $H \subseteq \{s \in G \mid \rho_i^s \cong \rho_i\}$.

For the reverse inclusion let $s \in G$ such that $\rho_i^s \cong \rho_i$. Then for all $a \in A$, there exists $T : V_i \rightarrow V_i$ such that

$$\rho_i(s^{-1}as) = T \cdot \rho_i(a) \cdot T^{-1}$$

for some $a' \in A$ as A is normal in G . As A is abelian we have $\rho_i(a) = c_a I_n$ for some constant c_a and the identity matrix I_n . Thus $T \cdot \rho_i(a) \cdot T^{-1} = T \cdot c_a I_n \cdot T^{-1} = c_a T T^{-1} = c_a I_n$. Since $\rho_i(a)V_i = V_i$ for all $a \in A$ we have that $\rho_i(s^{-1}as)V_i = V_i$ and so $s \in H$. $\square$

Lemma 3.3. *Let G be a group and let ρ be an irreducible representation of G which is not faithful (i.e. $\ker \rho \neq \{e\}$). Let $N = \ker \rho$ and define $\overline{G} = G/N$ and $\overline{\rho}(xN) = \rho(x)$ (this can be checked to be well-defined). Assume there exists a subgroup $\overline{H}$ of $\overline{G}$ and an irreducible representation $\overline{\sigma}$ of $\overline{H}$ such that $\text{Ind}_{\overline{H}}^{\overline{G}} \overline{\sigma} = \overline{\rho}$. Then there exists a subgroup H of G and an irreducible representation σ , with the same dimension as $\overline{\sigma}$, of H such that $\rho = \text{Ind}_H^G \sigma$.*

Proof. By definition, $\rho(m)$ is the identity for all $m \in N$. By the Lattice Isomorphism Theorem [DF91], there exists a subgroup H of G containing N such that $H/N = \overline{H}$. We define $\sigma(x) = \overline{\sigma}(xN)$ for all $x \in H$. We first show that σ is a representation. Let $x, y \in H$. Then

$$\sigma(xy) = \overline{\sigma}(xyN) = \overline{\sigma}(xNyN) = \overline{\sigma}(xN)\overline{\sigma}(yN) = \sigma(x)\sigma(y),$$

as required. We now show that σ is irreducible. Let χ_σ be the character of σ and let $\chi_{\overline{\sigma}}$ be the character of $\overline{\sigma}$. So $\chi_\sigma(x) = \text{Tr}(\sigma(x)) = \text{Tr}(\overline{\sigma}(xN)) = \chi_{\overline{\sigma}}(xN)$. Letting R be a system of representatives for H/N , we have

$$\begin{aligned} (\chi_\sigma | \chi_\sigma) &= \frac{1}{|H|} \sum_{t \in H} \chi(t) \chi(t^{-1}) \\ &= \frac{1}{|H|} \sum_{t \in H} \chi_{\overline{\sigma}}(tN) \chi_{\overline{\sigma}}(t^{-1}N) \\ &= \frac{1}{|H|} \sum_{r \in R} \sum_{n \in N} \chi_{\overline{\sigma}}(rnN) \chi_{\overline{\sigma}}(n^{-1}r^{-1}N) \\ &= \frac{1}{|H|} \sum_{n \in N} \sum_{r \in R} \chi_{\overline{\sigma}}(rN) \chi_{\overline{\sigma}}(r^{-1}N) \\ &= \frac{1}{|H/N|} \sum_{r \in R} \chi_{\overline{\sigma}}(rN) \chi_{\overline{\sigma}}(r^{-1}N) \\ &= (\chi_{\overline{\sigma}} | \chi_{\overline{\sigma}}) \\ &= 1, \end{aligned}$$

and so σ is irreducible. We need to show that $\rho = \text{Ind}_H^G \sigma$. Since $\rho(x) = \overline{\rho}(xN)$, this is equivalent to showing that

$$\text{Ind}_H^G \sigma(x) = \text{Ind}_{\overline{H}}^{\overline{G}} \overline{\sigma}(xN),$$

for all $x \in G$. Let R be a system of representatives for G/N . Then, examining the characters of these two representations, we have that

$$\begin{aligned}
\chi_{\text{Ind}_H^G \sigma}(x) &= \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}xt \in H}} \chi_{\sigma}(t^{-1}xt) \\
&= \frac{1}{|H|} \sum_{r \in R} \sum_{\substack{n \in N \\ (rn)^{-1}x(rn) \in H}} \chi_{\sigma}((rn)^{-1}x(rn)) \\
&= \frac{1}{|H|} \sum_{r \in R} \sum_{\substack{n \in N \\ (rn)^{-1}x(rn) \in H}} \chi_{\bar{\sigma}}((rn)^{-1}x(rn)N),
\end{aligned}$$

(§7.2 [Ser77]). The summands satisfy

$$\chi_{\bar{\sigma}}((rn)^{-1}x(rn)N) = \chi_{\bar{\sigma}}(r^{-1}NxNrN) = \chi_{\bar{\sigma}}(r^{-1}xrN).$$

If $(rn)^{-1}x(rn) \in H$ then clearly $r^{-1}xrN \in H/N$. Also, $r^{-1}xrN \in H/N$ implies that

$$r^{-1}xrn = hn',$$

for some $n' \in N$. Since $N \subseteq H$, $hn' \in H$ we have that $n^{-1}r^{-1}xrn = n^{-1}hn' \in H$ and so $(rn)^{-1}x(rn) \in H$. Thus $(rn)^{-1}x(rn) \in H$ if and only if $r^{-1}xrN \in H/N$.

Therefore we can rewrite the earlier summation as

$$\begin{aligned}
\chi_{\text{Ind}_H^G \sigma}(x) &= \frac{1}{|H|} \sum_{r \in R} \sum_{n \in N} \chi_{\bar{\sigma}}(r^{-1}xrN) \\
&\quad r^{-1}xrN \in H/N \\
&= \frac{1}{|H/N|} \sum_{tN \in G/N} \chi_{\bar{\sigma}}(t^{-1}xtN) \\
&\quad r^{-1}xrN \in H/N \\
&= \chi_{\text{Ind}_{H/N}^{G/N} \bar{\sigma}}(x).
\end{aligned}$$

which completes the proof. $\square$

Lemma 3.4. (*Lemma 4, §8.5 [Ser77]*) *Let G be a nonabelian supersolvable group. Then there exists a normal abelian subgroup of G which is not contained in the center of G .*

Proof. The quotient $H = G/Z(G)$ is supersolvable. Thus it has a composition series in which the first nontrivial term H_1 is a cyclic normal subgroup of H . The Lattice Isomorphism Theorem [DF91] assures us that there exists a subgroup G_1 of G such that $G_1/Z(G) = H_1$ and G_1 is normal in G . Furthermore, we have that $G_1 = \langle x, Z(G) \rangle$, for some $x \in G$, and, since every element in $Z(G)$ commutes with every element in G , G_1 is abelian. $\square$

Theorem 3.5. (*Theorem 16 [Ser77]*) *Let G be a supersolvable group. Then each irreducible representation of G is induced from a representation of dimension 1 of a subgroup of G (i.e. G is monomial).*

Proof. We prove the theorem by induction on the order of G . Suppose ρ is an irreducible representation of G which is not faithful, then the elements of $\ker \rho$ form a nontrivial normal subgroup N of G and we can form the quotient group G/N . This quotient group is supersolvable (Theorem 7.2.4 [Sco64]) and smaller than G , so by the induction hypothesis, is monomial. Applying Lemma 3.3, we have a subgroup H of G and a one-dimensional representation σ of H such that σ induces to ρ .

Suppose ρ is faithful. If G is abelian, such a ρ is one-dimensional and there is nothing to prove. Suppose G is not abelian. According to Lemma 3.4, G must have a normal abelian subgroup A which is not contained in the center of G . Since ρ is faithful, $G \cong \rho(G)$. This implies that $\rho(A)$ is not contained in the center of $\rho(G)$. Hence there exists $a \in A$ such that $\rho(a)$ is not a scalar matrix (as all scalar matrices are necessarily in the center of $\rho(G)$). The restriction of ρ to A is thus not isotypic. By Proposition 3.1, this implies that ρ is induced from an irreducible representation of a proper subgroup H of G . The theorem now follows by applying the induction hypothesis to H , as induction of representations is transitive (Remark 3 after Proposition 18 [Ser77]). $\square$

4. A SUPERSOLVABLE GROUP WHICH IS NOT SUBNORMALLY MONOMIAL

From Theorem 3.5 we know that all supersolvable groups are monomial. We now prove that there exist supersolvable groups which are not subnormally monomial. Specifically, for each odd prime p , we prove the existence of a supersolvable group of order $2p^3$ which is not subnormally monomial. We begin by defining the group.

Throughout this section, let p be an odd prime. From §5.5 [DF91] we have that the group

$$P = \langle b, c, d \mid b^p = c^p = d^p = 1, b^{-1}cb = cd, db = bd, dc = cd \rangle$$

has order p^3 . Let $B = \langle b \rangle$, $C = \langle c \rangle$, and $D = \langle d \rangle$, so $\langle c, d \rangle = C \times D$. P can be regarded as a semidirect product

$$P \cong \mathbb{Z}_p \rtimes_{\phi} (\mathbb{Z}_p \times \mathbb{Z}_p)$$

where $\phi : B \rightarrow \text{Aut}(C \times D)$ is defined by $\phi(b)(c) = cd$ and $\phi(b)(d) = d$.

Let $\alpha : P \rightarrow P$ be defined by $\alpha(b) = b^{-1}$, $\alpha(c) = c^{-1}$, and $\alpha(d) = d$. It is easily checked that $\alpha \in \text{Aut}(P)$ and that it is an involution. Let $A = \langle a \mid a^2 = 1 \rangle$ and let $\psi : A \rightarrow \text{Aut}(P)$ be defined by $\psi(a) = \alpha$. This gives us our desired group.

Definition 4.1. Let p be an odd prime. Define the group J_p to be

$$J_p = \mathbb{Z}_2 \rtimes_{\psi} (\mathbb{Z}_p \rtimes_{\phi} (\mathbb{Z}_p \times \mathbb{Z}_p)),$$

where ψ and ϕ are defined as above. We fix a presentation of this group with

$$(4.1) \quad J_p = \langle a, b, c, d \mid a^2 = b^p = c^p = d^p = 1, a^{-1}ba = b^{-1}, a^{-1}ca = c^{-1}, \\ a^{-1}da = d, b^{-1}cb = cd, b^{-1}db = d, c^{-1}dc = d \rangle.$$

We will use this presentation throughout this thesis. We will also refer to the various cyclic subgroups formed by each of the generators as A , B , C , and D , as above.

We now show that J_p is supersolvable. By Theorem 3.5, this implies that it is monomial. Although it is possible to prove that J_p is supersolvable based solely on its order, we provide a constructive proof here.

Theorem 4.2. *The group J_p is supersolvable.*

Proof. For a finite group G to be supersolvable it must have a sequence

$$\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$$

of subgroups of G , with (i) G_i normal in G , and (ii) G_i/G_{i-1} cyclic for $1 \leq i \leq n$ (§8.3 [Ser77]). The chain of subgroups

$$\{1\} \subset \langle d \rangle \subset \langle c, d \rangle \subset \langle b, c, d \rangle \subset J_p$$

has all these properties.

(i) Each subgroup in the chain is normal in J_p . It is obvious that $\{1\}$, $\langle b, c, d \rangle$ and J_p are each normal in J_p . Since $\langle d \rangle$ is the center of the group it is a normal subgroup. Thus all that is needed is to show that $\langle c, d \rangle \triangleleft J_p$. Examining the actions of a and b on the generators of this subgroup we see that

$$a^{-1}ca = c^{-1} \in \langle c, d \rangle$$

$$a^{-1}da = d \in \langle c, d \rangle$$

$$b^{-1}cb = cd \in \langle c, d \rangle$$

$$b^{-1}db = d \in \langle c, d \rangle.$$

This demonstrates that $\langle c, d \rangle$ is closed under conjugation and thus normal in J_p .

(ii) Each of the successive factor group is cyclic. So $\langle d \rangle / \{1\} \cong \mathbb{Z}_p$, $\langle c, d \rangle / \langle d \rangle \cong \mathbb{Z}_p$, $\langle b, c, d \rangle / \langle c, d \rangle \cong \mathbb{Z}_p$, and $J_p / \langle b, c, d \rangle \cong \mathbb{Z}_2$. This is clear since the order of each factor group is a prime number.

Therefore J_p is supersolvable. $\square$

We now demonstrate that J_p is not subnormally monomial. To do this, we show that J_p has an irreducible representation which is not induced from a one-dimensional representation of any subnormal subgroup. We show first that J_p has an irreducible representation of dimension p . This then must be induced from a one-dimensional representation of a subgroup of index p (see Section 2), and so of order $2p^2$, which is therefore a maximal subgroup of J_p . If any such subgroup is subnormal, then it is normal. We will show that J_p does not have a normal subgroup of index p and so prove that J_p is not subnormally monomial.

In order to show that J_p has an irreducible representation of dimension p , we prove that it has an irreducible character χ with $\chi(1) = p$ (see Section 2). To obtain this character, we use theorems from [JL01] and [Ser77] that allow us to compute the irreducible characters of semidirect products and induced characters. We fix specific irreducible characters of various subgroups of J_p and use them to create an irreducible character of the desired dimension.

Let $\zeta = e^{2\pi i/p}$ be the first p -th root of unity.

Lemma 4.3. (*§5.1 [Ser77]*) *The character table for the cyclic group $\mathbb{Z}_p$, where p is prime and r generates the group, is given by*

$$\chi_h(r^k) = e^{2\pi i h k / p} = \zeta^{hk},$$

where $h \in 0, \dots, p-1$.

Using Lemma 4.3, we fix two specific irreducible characters. We define an irreducible character ψ_D of the subgroup $\langle d \rangle$ by $\psi_D(d^i) = \zeta^i$. We define the irreducible character ψ_C of the subgroup $\langle c \rangle$ to be the trivial character, $\psi_C(c^j) = 1$.

Lemma 4.4. *An irreducible character for the subgroup $\langle c, d \rangle$ is given by*

$$\psi_{CD}(c^j d^i) = \zeta^i,$$

for all $0 \leq i, j \leq p-1$.

Proof. This follows from Theorem 19.18 [JL01], which states that the irreducible characters of a direct product are the products of the irreducible characters of the component subgroups. Thus we have that $\psi_{cd} = \psi_c \times \psi_d$. $\square$

Lemma 4.5. *An irreducible character for the subgroup $\langle a, c, d \rangle$ is given by*

$$\psi(a^k c^j d^i) = \zeta^i.$$

Proof. We have that $\langle a, c, d \rangle = \langle a \rangle \rtimes \langle c, d \rangle$, where $\langle c, d \rangle$ is abelian. Thus we meet the criteria needed to use Proposition 25 [Ser77], which allows us to compute the irreducible characters of a semidirect product when the normal subgroup is abelian.

Fixing χ as the character ψ_{CD} from Lemma 4.4, we have that

$$\begin{aligned} \chi(a^{-1} c^j d^k a) &= \chi(a^{-1} c^j a a^{-1} d^k a) \\ &= \chi(c^{-j} d^k). \end{aligned}$$

Since $\chi(c^{-j}d^k) = \zeta^k = \chi(c^j d^k)$ we have that χ is an H -orbit under conjugation. Setting ρ to be the trivial representation of $\langle a \rangle$ we obtain the irreducible representation $\tilde{\rho}$ of G by composing ρ with the canonical projection $G \rightarrow H$. Thus $\tilde{\rho}$ is the trivial representation of G . We extend χ to G by setting $\chi(a^i c^j d^k) = \chi(c^j d^k)$ for $a^i \in \langle a \rangle$, $c^j d^k \in \langle c, d \rangle$, and $i, j, k \in \mathbb{Z}$. Following the proposition, if we take the tensor product of χ and $\tilde{\rho}$ and induce to G , we get an irreducible representation ψ of G . Since χ and $\tilde{\rho}$ are representations of G , ψ is also and so the induction is trivial. Thus $\psi = \chi \otimes \tilde{\rho}$ and, since $\tilde{\rho}$ is the trivial representation, $\psi = \chi$. By Proposition 25 [Ser77], ψ is an irreducible character for the subgroup $\langle a, c, d \rangle$. $\square$

We could have simply shown that ψ is a homomorphism in Lemma 4.5 but we wanted to explicitly show how this character was constructed.

Theorem 4.6. *The group J_p has an irreducible representation of dimension p .*

Proof. Let $\chi = \text{Ind}_H^G \psi$ where $G = J_p$, $H = \langle a, c, d \rangle$ and ψ is the character of Lemma 4.5. We claim that χ is the character of an irreducible representation of dimension p of J_p . We will show that it is the character of an irreducible representation by proving that

$$(\chi | \chi) = \sum_{s \in G} \chi(s) \overline{\chi(s)} = 1,$$

and that the representation has dimension p by proving that $\chi(1) = p$ (see Section 2). To do this we will partition the group into subsets and compute the values of

$\chi(s)\overline{\chi(s)}$ using the induced character formula

$$(4.2) \quad \chi(s) = \frac{1}{2p^2} \sum_{\substack{t \in G \\ t^{-1}st \in H}} \psi(t^{-1}st),$$

where $2p^2$ is the order of H (see Section 2). In doing this, we will be using some basic facts about the p -th roots of unity. The following identities hold for ζ , the first primitive p -th root of unity,

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{p-1} = 0,$$

and, if k is a fixed integer, then $\{\zeta^0, \zeta^1, \zeta^2, \dots, \zeta^p\}$ and $\{\zeta^0, \zeta^k, \zeta^{2k}, \dots, \zeta^{pk}\}$ are equal (this follows since p is a prime). This last property also holds true for $\{\zeta^1, \zeta^2, \dots, \zeta^p\}$.

Subset (i): Let $s = 1$. Obviously, $t^{-1}1t = 1 \in H$ for all $t \in G$ and so $\chi(1) = \frac{1}{2p^2} \cdot 2p^3 \cdot \psi(1) = p$. Thus $\chi(1)\overline{\chi(1)} = p \cdot \bar{p} = p^2$. This proves that the representation with this character has dimension p .

Subset (ii): Let $s = d^i$ where $i \not\equiv 0 \pmod{p}$. Since d is in the center of G we have that $t^{-1}d^i t = d^i \in H$ for all $t \in G$ and so $\chi(d^i) = \frac{1}{2p^2} \cdot 2p^3 \cdot \psi(d^i) = p\zeta^i$. Thus $\chi(d^i)\overline{\chi(d^i)} = p\zeta^i \cdot \overline{p\zeta^i} = p^2$. There are $p-1$ elements in this subset.

Subset (iii): Let $s = c^i d^j$ where $i \not\equiv 0 \pmod{p}$ and $0 \leq j \leq p-1$. Let $t = a^w b^x c^y d^z \in G$ for $w, x, y, z \in \mathbb{Z}$. Then we have

$$\begin{aligned} t^{-1}st &= d^{-z} c^{-y} b^{-x} a^{-w} c^i d^j a^w b^x c^y d^z \\ &= c^{\pm i} d^{j \pm ix} \in H, \end{aligned}$$

where the signs are positive if $w \equiv 0 \pmod{2}$ and negative otherwise. If we fix w, y and z , we have that as x runs over the values $0, \dots, p-1$, each of the values $\zeta^0, \zeta^1, \dots, \zeta^{p-1}$ appears in the Sum 4.2. Since these sum to 0, the whole sum is 0 and so $\chi(c^i d^j) = 0$. Thus $\chi(c^i d^j) \overline{\chi(c^i d^j)} = 0$. There are $p^2 - p$ elements in this subset.

Subset (iv): Let $s = b^i c^j d^k$ where $i \not\equiv 0 \pmod{p}$ and $0 \leq j, k \leq p-1$. Let $t = a^w b^x c^y d^z \in G$ for $w, x, y, z \in \mathbb{Z}$. Then we have

$$\begin{aligned} t^{-1}st &= d^{-z} c^{-y} b^{-x} a^{-w} b^i c^j d^k a^w b^x c^y d^z \\ &= b^{\pm i} c^{\pm j} d^{k \mp iy \pm jx}, \end{aligned}$$

where the signs depend on the parity of w . Since $i \not\equiv 0 \pmod{p}$, we have that $t^{-1}st \notin H$ and so $\chi(b^i c^j d^k) = 0$. Thus $\chi(b^i c^j d^k) \overline{\chi(b^i c^j d^k)} = 0$. There are $p^3 - p^2$ elements in this subset.

Subset (v): Let $s = ab^i c^j d^k$ where $0 \leq i, j, k \leq p-1$. For clarity, we divide this into two sub-cases.

First, let $t = b^x c^y d^z \in G$ for $x, y, z \in \mathbb{Z}$. Then we have

$$\begin{aligned} t^{-1}st &= d^{-z} c^{-y} b^{-x} ab^i c^j d^k b^x c^y d^z \\ &= ab^{2x+i} c^{2y+j} d^{k+jx+y(2x+i)}, \end{aligned}$$

which is in H if and only if $2x+i \equiv 0 \pmod{p}$. If i is even, set $x \equiv \frac{i}{2} \pmod{p}$ and if i is odd, set $x \equiv \frac{p+i}{2} \pmod{p}$. Thus we only have one possible choice for x for each i . This reduces the number of possible choices for t to p^2 . Therefore $\psi(t^{-1}st) = \zeta^{k+\frac{2i}{2}}$ if i is even, and $\psi(t^{-1}st) = \zeta^{k+\frac{2p+i}{2}}$ if i is odd.

Secondly, let $t = ab^x c^y d^z \in G$ for $x, y, z \in \mathbb{Z}$. Then we have

$$\begin{aligned} t^{-1}st &= d^{-z}c^{-y}b^{-x}aab^i c^j d^k ab^x c^y d^z \\ &= ab^{2x-i} c^{2y-j} d^{k-jx+y(2x-i)}, \end{aligned}$$

which is in H if and only if $2x - i \equiv 0 \pmod{p}$. If i is even, set $x \equiv \frac{-i}{2} \pmod{p}$ and if i is odd, set $x \equiv \frac{-p-i}{2} \pmod{p}$. Thus we only have one possible choice for x for each i . This reduces the number of possible choices for t to p^2 . Therefore $\psi(t^{-1}st) = \zeta^{k+\frac{ji}{2}}$ if i is even, and $\psi(t^{-1}st) = \zeta^{k+\frac{jp+ji}{2}}$ if i is odd. Thus for any given s in this case, all the values of $\psi(t^{-1}st)$ are a fixed power of ζ for every $t \in G$. Thus $\chi(ab^i c^j d^k) \overline{\chi(ab^i c^j d^k)} = 1$. There are p^3 elements in this subset.

Our five subsets have covered, in order, 1, $p-1$, p^2-p , p^3-p^2 and p^3 elements which sum to $2p^3$ as expected. And so we compute

$$\begin{aligned} (\chi | \chi) &= \frac{1}{2p^3} [(1)(p^2) + (p-1)(p^2) + (p^2-p)(0) + (p^3-p^2)(0) + (p^3)(1)] \\ &= \frac{1}{2p^3} [p^2 + p^3 - p^2 + p^3] \\ &= \frac{1}{2p^3} [2p^3] \\ &= 1. \end{aligned}$$

Therefore the representation of J_p with character χ is irreducible and, as was shown in set (i), has dimension p . $\square$

We now show that although J_p has a subgroup of index p (which it must have as it is monomial and has an irreducible representation of dimension p), it does not have a normal subgroup of index p .

Lemma 4.7. *The group J_p has a subgroup of index p .*

Proof. Using the presentation given in Equation 4.1, we claim that the subgroup $\langle a, c, d \rangle$ of J_p has order $2p^2$ and thus index p . Clearly this has order at least $2p^2$. Thus it is sufficient to prove that $\langle a, c, d \rangle \neq J_p$ which is equivalent to finding $g \in J_p$ such that $g \notin \langle a, c, d \rangle$. We claim that b is such an element. The facts that $dc = cd$, $da = ad$, and $ca = ac^{-1}$ demonstrate that there is no combination of powers of the generators of $\langle a, c, d \rangle$ which gives an element of the form xy for any $x, y \in J_p$. $\square$

Lemma 4.8. *Let $a \in J_p$ be an element of order 2. Then the intersection of $C_{J_p}(a)$, the centralizer of a in J_p , with the subgroup $\langle b, c, d \rangle$ is equal to $\langle d \rangle$.*

Proof. Since $d \in Z(J_p)$ (Equation 4.1), we have that $\langle d \rangle \subseteq C_{J_p}(a) \cap \langle b, c, d \rangle$ (where $C_{J_p}(a)$ is the centralizer of a in J_p). Let $b^i c^j d^k \in \langle b, c, d \rangle$ with $0 \leq i, j, k \leq p-1$ and not both $i = 0$ and $j = 0$. Then we have that

$$b^i c^j d^k a = ab^{-i} c^{-j} d^k$$

which is only equal to $ab^i c^j d^k$ if both $i = 0$ and $j = 0$ (since p is an odd prime). Since we assumed this not to be the case, we have that $b^i c^j d^k \notin C_{J_p}(a)$, which completes the proof. $\square$

Theorem 4.9. *The group J_p does not have a normal subgroup of index p .*

Proof. By Sylow's Theorem [DF91], the number of p -Sylow subgroups of J_p must be equivalent to 1 modulo p and must divide 2. Thus J_p has a unique p -Sylow subgroup. As we know that $P = \langle b, c, d \rangle$ is a p -Sylow subgroup of J_p , it must be the unique one.

Let H be a subgroup of J_p with index p and so order $2p^2$. Let K be the Sylow p -subgroup of H . K has order p^2 and, since it is also a p -subgroup of J_p , must be

a subgroup of P . Also, Sylow's Theorem tell us that K is only Sylow p -subgroup of H and so is normal in H . Therefore H must be the semidirect product of K by some subgroup of H with order 2. Let this subgroup by $A = \langle a \rangle$ and so $H = A \cdot K$.

Assume H is normal in J_p . Choose any $x \in P$ with $x \notin K$. This is possible as K is a proper subgroup of P . We have that

$$xHx^{-1} = xA \cdot Kx^{-1} = xAx^{-1} \cdot xKx^{-1}.$$

Since K is a maximal subgroup of the p -group P , it is normal in P and so $xKx^{-1} = K$. This gives us that

$$xHx^{-1} = xAx^{-1} \cdot K.$$

Since we assumed that H to be normal in J_p we have that $H = xAx^{-1} \cdot K$, implying that $xAx^{-1} \subseteq H$ and $xAx^{-1} \in H$. As xAx^{-1} is a 2-Sylow subgroup it must be conjugate to A in H . Thus there exists $y \in H$ such that $yxax^{-1}y^{-1} = a$. In fact, we can assume that $y \in K$. From Lemma 4.8 we have that $yx \in \langle d \rangle$. Assuming that $d \notin K$ we have that $bd^i, cd^j \in K$ for some $0 \leq i, j \leq p-1$. Thus $(cd^i)^{-1}(bd^i)^{-1}(cd^i)(bd^i) \in K$, and so $c^{-1}b^{-1}cb = d \in K$, contradicting our assumption. Therefore $d \in K$, and so we have that $yx \in K$. As $y \in K$ we have that $x \in K$ which contradicts our original choice of x . Thus our assumption that H is normal in J_p is false. $\square$

Theorem 4.10. *The group J_p is a supersolvable group which is not subnormally monomial.*

Proof. By Theorem 4.2, J_p is supersolvable and monomial. By Theorem 4.6, J_p has an irreducible representation of dimension p . Since J_p is monomial, this representation must be induced from a one-dimensional representation of a subgroup of J_p of index p . By Theorem 4.9, J_p has no normal subgroups of this index. This implies, since any such subgroups are maximal, that J_p has no subnormal subgroups of this index. Therefore J_p is not subnormally monomial. $\square$

5. GENERAL PROPERTIES OF SUBNORMALLY MONOMIAL GROUPS

In this section we prove two general results about supersolvable groups and when they are subnormally monomial. The first result tells us that a supersolvable group is subnormally monomial if its commutator subgroup is abelian. This simple criterion is used in proving later results. The second result tells us that if a supersolvable group is a direct product of smaller groups then it is subnormally monomial if and only if each of its component groups are subnormally monomial. This allows us to then create a large class of groups which are not subnormally monomial, namely all direct products of any supersolvable group with any of the groups J_p identified in the previous section.

Throughout this section all groups are assumed to be finite.

Theorem 5.1. *Let G be a supersolvable group. If the commutator subgroup G' is abelian then G is subnormally monomial.*

Proof. We prove this by induction on the order of G . Suppose that G' is a normal abelian subgroup of G . We have two possibilities: either G' is contained in the center of G or it is not. Assume it is contained in the center. Then by the proof of Lemma 3.4, G contains a normal abelian subgroup A which contains the center and so contains G' . If G' is not contained in the center of G , then set $A = G'$. In

either case, A is a normal abelian subgroup of G which is not contained in the center of G . Furthermore, we have that $G' \subseteq A$.

Let ρ be an irreducible representation of G . Using the proof of Theorem 3.5, there exists a subgroup H of G and an irreducible representation σ of H such that σ induces to ρ and H contains A . Hence H contains G' . By Proposition 7(4), §5.4 [DF91], every subgroup of G which contains G' is normal in G . Therefore H is normal in G . Since $H' \subseteq G'$, H' is abelian, and the induction hypothesis holds for H . Therefore σ is subnormally monomial. It follows that $\rho = \text{Ind}_H^G \sigma$ is subnormally monomial. $\square$

The converse of Theorem 5.1 is not true. The smallest groups which provide a counterexample for the converse have order 384 as can be seen using GAP.

Theorem 5.2. *Let A and B be monomial groups. Then $G = A \times B$ is subnormally monomial if and only if both A and B are subnormally monomial.*

Proof. PART 1. Assume A and B are subnormally monomial. Let ρ be an irreducible representation of G with character χ_G . By Theorem 19.18 [JL01], there must exist irreducible characters χ_A of A and χ_B of B such that $\chi_G = \chi_A \times \chi_B$. As A is subnormally monomial there must exist a subnormal subgroup C of A with monomial character χ_C such that $\chi_A = \text{Ind}_C^A \chi_C$. Similarly as B is subnormally monomial there must exist a subnormal subgroup D of B with monomial character χ_D such that $\chi_B = \text{Ind}_D^B \chi_D$.

Claim 1: $C \times D$ is a subnormal subgroup of $A \times B$. Since C is a subnormal subgroup of A and D is a subnormal subgroup of B we have subgroups $C_0, C_1, \dots, C_k$

of A and $D_0, D_1, \dots, D_l$ of D such that

$$C = C_0 \triangleleft C_1 \triangleleft \dots \triangleleft C_k = A,$$

$$D = D_0 \triangleleft D_1 \triangleleft \dots \triangleleft D_l = B.$$

Letting C_i and D_j be any two of these with $0 \leq i \leq k-1$ and $0 \leq j \leq l-1$ we have that $C_i \triangleleft C_{i+1}$ and $D_j \triangleleft D_{j+1}$. Let $(w, x) \in C_i \times D_j$ and $(y, z) \in C_{i+1} \times D_j$. Then $(y, z)(w, x)(y^{-1}, z^{-1}) = (ywy^{-1}, zxx^{-1}) \in C_i \times D_j$ and so $C_i \times D_j \triangleleft C_{i+1} \times D_j$. Similarly, $C_i \times D_j \triangleleft C_i \times D_{j+1}$. Hence $C \times D$ is subnormal in $A \times B$ with the series

$$(C_0 \times D_0) \triangleleft \dots \triangleleft (C_k \times D_0) \triangleleft (C_k \times D_1) \triangleleft \dots \triangleleft (C_k \times D_l).$$

Claim 2: $\chi_G = \chi_A \times \chi_B = \text{Ind}_{C \times D}^{A \times B} (\chi_C \times \chi_D)$. Let $H = C \times D$ and let $\chi_H = \chi_C \times \chi_D$. Let $(x, y) \in A \times B = G$. Then

$$\begin{aligned}
\chi_{\text{Ind}_H^G \chi_H}(x, y) &= \frac{1}{|H|} \sum_{\substack{(w, z) \in G \\ (w, z)^{-1}(x, y)(w, z) \in H}} \chi_H((w, z)^{-1}(x, y)(w, z)) \\
&= \frac{1}{|H|} \sum_{\substack{(w, z) \in G \\ (w^{-1}xw, z^{-1}yz) \in H}} \chi_H(w^{-1}xw, z^{-1}yz) \\
&= \frac{1}{|C \times D|} \sum_{\substack{(w, z) \in A \times B \\ (w^{-1}xw, z^{-1}yz) \in C \times D}} \chi_{C \times D}(w^{-1}xw, z^{-1}yz) \\
&= \frac{1}{|C| \cdot |D|} \sum_{\substack{w \in A \\ w^{-1}xw \in C}} \sum_{\substack{z \in B \\ z^{-1}yz \in D}} \chi_C(w^{-1}xw) \cdot \chi_D(z^{-1}yz) \\
&= \frac{1}{|C|} \sum_{\substack{w \in A \\ w^{-1}xw \in C}} \chi_C(w^{-1}xw) \cdot \frac{1}{|D|} \sum_{\substack{z \in B \\ z^{-1}yz \in D}} \chi_D(z^{-1}yz) \\
&= \text{Ind}_C^A \chi_C(x) \cdot \text{Ind}_D^B \chi_D(y) \\
&= \chi_A(x) \cdot \chi_B(y) \\
&= \chi_G(x, y).
\end{aligned}$$

Claims 1 and 2 imply that ρ is subnormally monomial. Since there were no restrictions on the choice of ρ , we have that G is subnormally monomial.

PART 2. Assume one of A or B is not subnormally monomial. Without loss of generality we assume B is not subnormally monomial. Then there exists an irreducible representation σ of B such that if $\sigma = \text{Ind}_E^B \tau$ where τ is a one-dimensional representation of a subgroup E and E is not subnormal in B .

Let $\rho = \tau_A \otimes \sigma$ where τ_A is the trivial representation of A . By Theorem 10 [Ser77], ρ is an irreducible representation of G . Since G is monomial, ρ is also. Let H be a subgroup of G and let $\psi : H \rightarrow GL(W)$ be any one-dimensional representation of H such that ρ is induced from ψ (so the vector space W is one-dimensional).

Assume $A \not\subseteq H$ so there exists $a \in A$ such that $a \notin H$. As ρ is induced from ψ we have that $\rho : G \rightarrow V$ and $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$ where the W_i are isomorphic spaces to W for $i = 1, \dots, k$. As $a \notin H$, $\rho(a)$ must permute these spaces. Without loss of generality we can set $W_1 = W$ thus for $w \in W_1$ we have that $\rho(a)w \notin W_1$. However, by the definition of ρ , it is clear that $A \subseteq \ker \rho$ (see Section 2). Thus $\rho(a)w = w \in W_1$, a contradiction. Hence $A \subseteq H$.

Since $G = A \times B$, we have that A is normal and so we can form the quotient group H/A . We show that the formula $\bar{\psi}(hA) = \psi(h)$ provides a well-defined representation $\bar{\psi} : H/A \rightarrow GL(W)$. Let $h, h' \in H$ with $hA = h'A$. Then there exist $a \in A$ such that $h = h'a$. Therefore $\bar{\psi}(hA) = \psi(h) = \psi(h'a_1) = \psi(h')\psi(a_1)$. However $A \subseteq \ker \rho$, which implies that $\psi(a) = 1$ for all $a \in A$. Thus $\psi(h')\psi(a_1) = \psi(h') = \bar{\psi}(h'A)$. Hence $\bar{\psi}(hA) = \bar{\psi}(h'A)$ and $\bar{\psi}$ is well-defined. We next define the representation $\bar{\rho}$ of G/A by $\bar{\rho}(gA) = \rho(g)$. Since $A \subseteq \ker \rho$, this is well-defined.

We claim that $\bar{\rho}$ is induced from $\bar{\psi}$. Define the characters of these various representations as follows: χ_A is the character for τ_A , χ_B for σ , and χ_H for ψ .

We then have that $\chi_G = \chi_A \cdot \chi_B$ is the character for ρ , $\chi_G = \text{Ind}_H^G \chi_H$, $\chi_A(a) = 1$ for all $a \in A$, and $\chi_H(1) = 1$. Let $\chi_{H/A}$ be the character for $\bar{\psi}$ and $\chi_{G/A}$ be the character for $\bar{\rho}$. For $sA \in H/A$ we compute (see Section 2)

$$\begin{aligned} \text{Ind}_{AH/A}^{G/A} \chi_{H/A}(sA) &= \frac{1}{|H/A|} \sum_{\substack{tA \in G/A \\ t^{-1}stA \in H/A}} \chi_{H/A}(t^{-1}stA). \end{aligned}$$

From our definition of $\bar{\psi}$, we have that $\chi_{H/A}(t^{-1}stA) = \text{Tr}(\bar{\psi}(t^{-1}stA)) = \text{Tr}(\psi(t^{-1}st)) = \chi_H(t^{-1}st)$. Let $\{h_1, h_2, \dots, h_k\}$ be a set of coset representatives

for H/A . Then we have

$$\begin{aligned}
\frac{1}{|H/A|} \sum_{\substack{tA \in G/A \\ t^{-1}stA \in H/A}} \chi_{H/A}(t^{-1}stA) &= \frac{|A|}{|H|} \sum_{i=1}^k \sum_{\substack{tA \in G/A \\ t^{-1}stA = h_i A}} \chi_{H/A}(t^{-1}stA) \\
&= \frac{|A|}{|H|} \frac{1}{|A|} \sum_{i=1}^k \sum_{\substack{t \in G \\ t^{-1}stA = h_i A}} \chi_{H/A}(t^{-1}stA) \\
&= \frac{1}{|H|} \sum_{i=1}^k \sum_{\substack{t \in G \\ t^{-1}st \in h_i A}} \chi_H(t^{-1}st) \\
&= \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}st \in H}} \chi_H(t^{-1}st) \\
&= \text{Ind}_H^G \chi_H(s) \\
&= \chi_G(s).
\end{aligned}$$

But $\chi_G(s) = \text{Tr}(\rho(s)) = \text{Tr}(\bar{\rho}(sA)) = \chi_{G/A}(sA)$. Hence $\bar{\psi}$ induces $\bar{\rho}$.

Clearly, $G/A \cong B$ as $G = A \times B$. We may let B be a set of representatives for G/A and define an isomorphism $\beta : G/A \rightarrow B$ by $\beta(bA) = b$ for all $b \in B$. We now claim that $\beta \circ \bar{\rho} = \sigma$. For $b \in B$ we have that $\beta \circ \bar{\rho}(bA) = \beta \circ \rho(1, b) = \beta \circ (\tau_A(1) \otimes \sigma(b)) = \beta \circ \sigma(b) = \sigma(b)$ as desired. Furthermore, we also have that σ is induced from $\beta \circ \bar{\psi}$, as these two groups are isomorphic. Since $\beta \circ \bar{\psi}$ is a

homomorphism into W it is one-dimensional. By our initial assumptions, this implies that the subgroup $\beta(H/A)$ of B is not subnormal in B .

Assume that H is subnormal in G . By the Lattice Isomorphism Theorem [DF91] it follows that H/A is subnormal in G/A . But $G/A \cong B$ and so $\beta(H/A)$ is subnormal in B . But we showed that this is not the case and so H is not subnormal in G . Since no restrictions were put on H initially, this implies that ρ is not subnormally monomial and thus G is not subnormally monomial. $\square$

Corollary 5.3. *If $S_1, S_2, \dots, S_n$ are supersolvable groups then $S_1 \times S_2 \times \dots \times S_n$ is subnormally monomial if and only if all of $S_1, S_2, \dots, S_n$ are.*

Corollary 5.4. *If S is a supersolvable group and p is an odd prime then $G = S \times J_p$ is not subnormally monomial.*

Proof. This result follows directly from Theorems 4.10 and 5.2. $\square$

6. THE CLASSIFICATION OF GROUPS OF ORDER $2p^3$

We now classify all groups G with $|G| = 2p^3$, for p an odd prime, according to those that are subnormally monomial and those that are not. We first prove that all such groups are supersolvable which then implies, by Theorem 3.5, that they are all monomial. The major result of this section will be to prove that either $G \cong J_p$ or G is subnormally monomial.

We begin by proving that any group of order $2p^3$ must be the semidirect product of its unique p -Sylow subgroup and $\mathbb{Z}_2$. This allows us to break the problem into various sub-cases. We then prove some general theorems allowing us to deal with all the cases in which the semidirect product is trivial (and so is a direct product) and in which the p -Sylow subgroup is abelian. Some more technical

theorems then deal with the remaining cases, allowing us to reach the desired result.

In this section we will always take p to be an odd prime and G to be a group of order $2p^3$.

Lemma 6.1. *Let p be an odd prime and let G be a group with order $2p^3$. Then G is supersolvable and monomial.*

Proof. That G is supersolvable is a sub-case of Theorem 7.2.15 [Sco64]. That G is monomial follows from Theorem 3.5. $\square$

Theorem 6.2. *Let p be an odd prime. Let G be a group of order $2p^3$. Then G is isomorphic to a semidirect product of its unique p -Sylow subgroup and $\mathbb{Z}_2$.*

Proof. The Sylow Theorem tells us that the number of p -Sylow subgroups of G must be equivalent to 1 modulo p and must also divide 2. Hence G has a unique p -Sylow subgroup P of order p^3 and this subgroup is normal in G . The Sylow Theorem also tells us that G has at least one subgroup of order 2. Choose one such subgroup and call it Q . Clearly $P \cap Q = \{1\}$. Therefore $G \cong Q \rtimes P \cong \mathbb{Z}_2 \rtimes P$. $\square$

In general, a semidirect product of groups K and N requires a homomorphism from K into $\text{Aut}(N)$. In the specific cases we are dealing with, however, we have that $K \cong \mathbb{Z}_2$ and so we need only specify an involution in $\text{Aut}(N)$ which the non-identity element of K should be mapped to. In the remainder of this section we will usually label the 2-group as A so $G = A \rtimes P$, where P is the p -Sylow subgroup. Letting a be the non-identity element of A , we will define a semidirect product by specifying an automorphism of P to which a is mapped.

Theorem 6.3. *Let p be an odd prime. Up to isomorphism, there are exactly five distinct groups of order p^3*

$$\mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p, \text{ and } \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p).$$

Explicit presentations for the two nonabelian groups are

$$\begin{aligned} \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2} &= \langle b, c \mid b^p = c^{p^2} = 1, bcb^{-1} = c^{1+p} \rangle, \\ \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p) &= \langle b, c, d \mid b^p = c^p = d^p = 1, cd = dc, bcb^{-1} = cd, bdb^{-1} = d \rangle. \end{aligned}$$

Proof. This is directly from the final example of §5.5 [DF91]. $\square$

Notice that the last group of Theorem 6.3 is isomorphic to the p -Sylow subgroup of J_p .

Lemma 6.4. *The center of the group*

$$Q = \langle b, c \mid b^p = c^{p^2} = 1, bcb^{-1} = c^{1+p} \rangle$$

is $\langle c^p \rangle$ and the center of the group

$$R = \langle b, c, d \mid b^p = c^p = d^p = 1, cd = dc, bcb^{-1} = cd, bdb^{-1} = d \rangle$$

is $\langle d \rangle$.

Proof. Both Q and R are p -groups of order p^3 . Since both are nonabelian, neither has a center of order p^3 . Theorem 6.3.4 [Sco64] gives us that the order of the center is not $p^{3-1} = p^2$. By Theorem 1, §6.1 [DF91], p -groups have non-trivial centers. Therefore the order of the centers of each is p .

For the group Q it is clear that c^p commutes with c , so if it commutes with b (and so commutes with all the generators of the group), it must be in the center. Checking this, we have that

$$c^p b = c^{p-1} b c^{1+p} = c^{p-2} b c^{2(1+p)} = \dots = b c^{p(1+p)} = b c^{p+p^2} = b c^p,$$

as desired. Thus $c^p \in Z(Q)$ and so $\langle c^p \rangle \subseteq Z(Q)$. Since $|\langle c^p \rangle| = p$ we have that $Z(Q) = \langle c^p \rangle$.

For the group R it is clear that d commutes with everything and so is in the center. Hence, as d has order p , we have that $Z(R) = \langle d \rangle$. $\square$

Theorem 6.5. *Let p be an odd prime and let G be a group of order $2p^3$. If the p -Sylow subgroup of G is abelian, then G is subnormally monomial.*

Proof. From Theorem 6.2 we know that the p -Sylow subgroup of G is unique and so is normal. Label this subgroup P . If $P \subseteq Z(G)$ then we have that G is the direct product of P and one of its 2-Sylow subgroups. This implies that G is nilpotent and so subnormally monomial (Theorem 3, §6.1 [DF91]). Assume $P \not\subseteq Z(G)$. Then P is an abelian normal subgroup which is not contained in the center of G . Let ρ be an irreducible representation of G of dimension greater than 1. Following the proof of Theorem 3.5, we have that there exists a subgroup H with an irreducible representation σ such that $P \subseteq H \subsetneq G$ and $\rho = \text{Ind}_H^G \sigma$. Since P is maximal, we have that $H = P$. Since P is a p -group, we have that σ is subnormally monomial. Therefore ρ is subnormally monomial as P is normal in G . As no restrictions were placed on the choice of ρ , we have that G is subnormally monomial. $\square$

This theorem deals with all possible groups of order $2p^3$ with abelian p -Sylow subgroups. Theorem 5.2 deals with all possible groups where the semidirect product is trivial as the group is then the direct product of p -groups which are nilpotent and thus subnormally monomial. What remains are groups with non-abelian p -Sylow subgroups and where the semidirect product is nontrivial. Using the results of Section 3 we will show that all these groups are subnormally monomial except for the group J_p . To do so we divide these groups into those which are centerless and those which have a nontrivial center.

Theorem 6.6. *Let p be an odd prime. Let G be a group of order $2p^3$ such that its p -Sylow subgroup is nonabelian and G is centerless. Then G is subnormally monomial.*

Proof. We have that $G = AP$ where P is the unique normal p -Sylow subgroup and $|A| = 2$. Let $Z = Z(P)$. We have that $Z \triangleleft G$ as the center of P is invariant under any automorphism and that $|Z| = p$ from Lemma 6.4. Since G is centerless, $Z \neq Z(G)$. Hence Z is a normal abelian subgroup of G which is not contained in the center of G . Looking back to Proposition 3.1, we can identify this Z with the normal subgroup A from that proof.

Let ρ be an irreducible representation of G . If ρ is not faithful then by Lemma 3.3, we can examine the factor group $G/\ker\rho$ and the representation $\bar{\rho}(x\ker\rho) = \rho(x)$. As ρ is not faithful, the factor group is smaller than G and so has order at most p^3 or $2p^2$. In the first case the factor group is a p -group and so $\bar{\rho}$ is subnormally monomial and, as seen in the proof of Lemma 3.3, so is ρ . In the second case, similar reasoning to Theorem 6.5, gives us that $\bar{\rho}$ is monomial and so is ρ as well. Thus we restrict ourselves to cases where ρ is faithful.

As Z is not contained in the center of G and ρ is faithful, we have that $\rho(Z)$ is not contained in the center of $\rho(G)$. Thus there exists $z \in Z$ such that $\rho(z)$ is not a homothety. As Z is abelian, this implies the restriction of ρ to Z is not isotypic. Therefore case (a) of Proposition 3.1 holds.

Recall from Proposition 3.1, that we have that $\rho_j : Z \rightarrow GL(V_j)$, defined by restricting ρ to Z and then to its action on V_j (identifying the subgroup Z with the subgroup A of the Proposition). We also have, as case (a) of Proposition 3.1 holds, a subgroup H of G with $Z \subseteq H \subsetneq G$ and an irreducible representation σ of H such that $\rho = \text{Ind}_H^G \sigma$. Finally, from Lemma 3.2, we have that $H = \{s \in G \mid \rho_j^s = \rho_j\}$. Let $z \in Z$ and let $A = \langle a \rangle$. Since G is centerless and Z is the center of P we have that $za \neq az$. But a defines an automorphism α of P by conjugation, satisfying $\alpha(\alpha(z)) = z$. Since z is in the center of P , and the center of a group is preserved under any isomorphism, this implies that $\alpha(z) = z^i$ for some $1 \leq i \leq p-1$. So $z^{i^2} = z$ implying $i = \pm 1 \pmod{p}$, but $i \neq 1$ from above, so we have that $za = az^{-1}$. Thus we have that

$$\rho_j(a^{-1}za) = \rho_j(z)^{-1},$$

but $Z \subseteq H \subseteq \{s \in G \mid \rho_j^s = \rho_j\}$ so, if $a \in H$, we have $\rho_j(a^{-1}za) = \rho_j(z)$, implying that

$$\rho_j(z) = \rho_j(z)^{-1},$$

which in turn implies that

$$\rho_j(z) = \pm I_m,$$

where I_m is the identity matrix for $GL(V_j)$. But z has order p , an odd prime, and so we have that $\rho_j(z) = I_m$ and ρ is the trivial representation of H . This implies that $H = G$ which was proven to not be possible in Proposition 3.1. Thus we have that $a \notin H$. By Sylow's Theorem, every element of order 2 in G is conjugate, so we can conclude that H does not contain any elements of order 2. Therefore we have that $Z \subseteq H \subseteq P$, and H is subnormal in G (if $H \neq Z$ and $H \neq P$ then $[P : H] = p$ and so H is normal in P). Hence ρ is induced from σ , an irreducible representation of H , which is subnormal. If σ is not one-dimensional, we can look at the group H which is a p -group and hence subnormally monomial. Thus σ is subnormally monomial implying that ρ is subnormally monomial by the transitivity of induction (Remark 3 after Proposition 18 [Ser77]). $\square$

We now examine the groups of order $2p^3$ with nonabelian p -Sylow subgroups and nontrivial centers.

Theorem 6.7. *Let p be an odd prime. Let*

$$P = \mathbb{Z}_p \ltimes \mathbb{Z}_{p^2} = \langle b, c \mid b^p = c^{p^2} = 1, cb = bc^{p+1} \rangle.$$

Let $A = \mathbb{Z}_2 = \langle a \mid a^2 = 1 \rangle$. Suppose $G = A \ltimes P$ and let $\varphi \in \text{Aut}(P)$ be the involution defined by a . If G has a nontrivial center then $G = A \times P$, i.e., φ is the trivial automorphism.

Proof. We prove this by contradiction. Suppose that $Z(G) \neq 1$ so either $\varphi(b) \neq b$ or $\varphi(c) \neq c$. This implies that $a \notin Z(G)$ and so $Z(G) \subseteq Z(P)$. From Lemma 6.4, $Z(P) = \langle c^p \rangle$. Hence $Z(G) = \langle c^p \rangle$ and so $\varphi(c^p) = c^p$.

Examining the orders of the elements of P we see that it has a unique subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, the subgroup $\langle b, c^p \rangle$. Thus this subgroup is mapped to

itself under every automorphism of P . This implies that $\varphi(b) = b^l c^{kp}$ for some $0 \leq k, l \leq p-1$. Thus

$$b = \varphi^2(b) = \varphi(b^l c^{kp}) = \varphi(b)^l \varphi(c^p)^k = (b^l c^{kp})^l (c^p)^k = b^{l^2} c^{kpl+kp},$$

which implies that $l = \pm 1$ and, if $l = 1$ then $k = 0$.

Let $\varphi(c) = b^i c^j$ for some $i, j \in \mathbb{Z}$. Then we must have that

$$c = \varphi^2(c) = \varphi(b^i c^j) = \varphi(b)^i \varphi(c)^j = (b^{\pm 1} c^{kp})^i (b^i c^j)^j = b^{\pm i + ij} c^x,$$

where $x \in \mathbb{Z}$. Thus $0 = \pm i + ij = i(j \pm 1)$ and so either $i = 0$ or $j \pm 1 = 0$. If $i = 0$ then $c = \varphi^2(c) = c^{j^2}$ and so then $j = \pm 1$ in either case. Notice that if $i \neq 0$ then $j = -l$. Therefore, every possible such automorphism is determined by the choices for i, j, k and l .

Assume $i \neq 0$ and $j = 1$ implying that $l = -1$. So $\varphi(c) = b^i c$ and

$$c = \varphi^2(c) = \varphi(b^i c) = (b^{-1} c^{kp})^i b^i c = c^{ikp+1}$$

This then implies that $ikp = 0$ and, as $i \neq 0$, we have that $k = 0$. Thus $\varphi(b) = b^{-1}$. Since φ is an automorphism the following identities ensue

$$\begin{aligned} \varphi(cb) &= \varphi(bc^{p+1}), \\ \varphi(c)\varphi(b) &= \varphi(b)\varphi(c^p)\varphi(c), \\ b^i cb^{-1} &= b^{-1} c^p b^i c, \\ c &= c^{2p+1}. \end{aligned}$$

The last identity is impossible and so our assumption that $i \neq 0$ and $j = 1$ must be false.

Assume instead that $i \neq 0$ and $j = -1$ implying that $l = 1$. So $\varphi(c) = b^i c^{-1}$ and

$$c = \varphi^2(c) = \varphi(b^i c^{-1}) = \varphi(b)^i \varphi(c)^{-1} = (bc^{pk})^i (b^i c^{-1})^{-1} = b^i c^{pki} cb^{-i}.$$

This implies

$$\begin{aligned} cb^i &= b^i c^{pki+1} \\ b^i c^{(p+1)i} &= b^i c^{pki+1} \\ c^{pi+1} &= c^{pki+1} \end{aligned}$$

which in turn implies that $k = 1$. So we have that $j = -1$, $k = 1$, $l = 1$ and $i \neq 0$ giving that $\varphi(c) = b^i c^{-1}$ and $\varphi(b) = bc^p$. Since φ is an automorphism, it follows that

$$\begin{aligned} \varphi(cb) &= \varphi(bc^{p+1}) \\ \varphi(c)\varphi(b) &= \varphi(b)\varphi(c^p)\varphi(c) \\ b^i c^{-1} bc^p &= bc^p c^p b^i c^{-1} \\ b^i c^{p^2-1} b &= bc^p b^i c^{p^2-1} \\ b^{i+1} c^{(p^2-1)(p+1)} &= b^{i+1} c^{p(p+1)+p^2-1} \\ c^{-p-1} &= c^{p-1} \end{aligned}$$

which is impossible and so our assumption that $i \neq 0$ and $j = -1$ must be false. As $j = \pm 1$ we have that $i = 0$ and so $\varphi(c) = c^j$ with $j = \pm 1$, $l = \pm 1$ and, if $l = 1$ then $k = 0$.

Given that $i = 0$, assume now that $j = 1$ so $\varphi(c) = c$. Thus $\varphi(b) \neq b$ and so $l = -1$ giving $\varphi(b) = b^{-1}c^{pk}$. Again, as φ is an automorphism, we must have that

$$\begin{aligned}\varphi(cb) &= \varphi(bc^{p+1}) \\ \varphi(c)\varphi(b) &= \varphi(b)\varphi(c^p)\varphi(c) \\ cb^{-1}c^{pk} &= b^{-1}c^{pk}c^pc \\ bc^{pk+1} &= c^{pk+p+1}b \\ bc^{pk+1} &= bc^{(pk+p+1)(p+1)} \\ c^{pk+1} &= c^{pk+2p+1}\end{aligned}$$

which is impossible. Therefore we must have that $j = -1$ so $\varphi(c) = c^{-1}$. But we must have the equality

$$\begin{aligned}\varphi(c)^p &= \varphi(c^p) \\ c^{-p} &= c^p\end{aligned}$$

which is not possible. Thus all possible choices for i , j , k , and l lead to contradictions. Therefore our original assumption that φ is not the trivial automorphism of P must be false and the theorem has been proven. $\square$

With these last two theorems all possible groups of order $2p^3$ have been dealt with except when the p -Sylow subgroup is isomorphic to $R = \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$, the semidirect product is nontrivial and the group has a nontrivial center. We will now show that under these conditions the group is isomorphic to J_p . We first give two lemmas which will allow us to prove the desired results.

Lemma 6.8. *Up to conjugation, there are two nontrivial involutions in $GL_2(\mathbb{F}_p)$, namely*

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof. We have that every element in $GL_2(\mathbb{F}_p)$ is similar to a unique matrix in rational canonical form. Thus every matrix is similar to one of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

or of the form $\begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix}$, where $a, b, c, d \in \mathbb{F}_p$.

Assuming the second, we have that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & c \\ 1 & d \end{bmatrix}^2 = \begin{bmatrix} c & cd \\ d & c + d^2 \end{bmatrix}$.

This implies that $c = 1$ and $d = 0$, so the matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In the other

case, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^2 = \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix}$ implies that $a^2 = b^2 = 1$. Thus every nontrivial involution in $GL_2(\mathbb{F}_p)$ is similar to one of

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

However we have that

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, every nontrivial involution is similar to one of the two given in the lemma. $\square$

Lemma 6.9. *Let H and K be groups, σ be an automorphism of H and $\varphi : K \rightarrow \text{Aut}(H)$ be an isomorphism. Then $1 \times \sigma : K \ltimes_{\varphi} H \rightarrow K \ltimes_{\sigma\varphi\sigma^{-1}} H$ is an isomorphism.*

Proof. Let $k_1, k_2 \in K$ and $h_1, h_2 \in H$. Define $(1 \times \sigma)(k_1, h_1) = (k_1, \sigma(h_1))$. Then we have

$$\begin{aligned} ((1 \times \sigma)(k_1, h_1))((1 \times \sigma)(k_2, h_2)) &= (k_1, \sigma(h_1))(k_2, \sigma(h_2)) \\ &= (k_1 k_2, (\sigma\varphi(k_2)\sigma^{-1}\sigma(h_1))\sigma(h_2)) \\ &= (k_1 k_2, \sigma((\varphi(k_2)h_1)h_2)) \\ &= (1 \times \sigma)(k_1 k_2, (\varphi(k_2)h_1)h_2) \\ &= (1 \times \sigma)((k_1, h_1)(k_2, h_2)). \end{aligned}$$

Clearly $1 \times \sigma$ is one-to-one and onto and is thus an isomorphism. $\square$

Theorem 6.10. *Let p be an odd prime. Let*

$$P = \langle b, c, d \mid b^p = c^p = d^p = 1, cb = bcd, db = bd, dc = cd \rangle \cong \mathbb{Z}_p \ltimes (\mathbb{Z}_p \times \mathbb{Z}_p)$$

and $A = \langle a \mid a^2 = 1 \rangle$. If $G = A \ltimes P$ with $G \neq A \times P$ and $Z(G) \neq \{1\}$, then $G \cong J_p$.

Proof. Let $\alpha \in \text{Aut}(P)$ be any of the possible images of $a \in A$ defining a semidirect product. We have that $\alpha^2 = \text{id}_{\text{Aut}(P)} \neq \alpha$ as the product is not direct. Since $G \neq A \times P$ we have that $Z(G) \subseteq Z(P)$ and since $Z(G) \neq \{1\}$ we have that $Z(G) = \langle d \rangle$, the center of P . Thus $\alpha(d) = d$. Let $D = \langle d \rangle$.

Let $S = \{\sigma \in \text{Aut}(P) \mid \sigma(d) = d\}$. Thus $\alpha \in S$. Define a homomorphism $\psi : S \rightarrow \text{Aut}(P/D)$ by $\psi(\sigma)(xD) = \sigma(x)D$ for all $x \in P$. It is easily checked that this map is onto as $D \subseteq Z(D)$. Since $|P/D| = p^2$ and P/D does not contain an element of order p^2 , we have that $\text{Aut}(P/D) \cong \text{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p) \cong \text{GL}_2(\mathbb{F}_p)$. We examine the kernel of this homomorphism

$$\begin{aligned} \ker \psi &= \{\sigma \in S \mid \psi(\sigma) = \text{id}_{\text{Aut}(P/D)}\} \\ &= \{\sigma \in S \mid \psi(\sigma)(xD) = xD, \forall x \in P\} \\ &= \{\sigma \in S \mid \sigma(x)D = xD, \forall x \in P\} \\ &= \{\sigma \in S \mid \sigma(x) = xd_x, \forall x \in P, \text{ for some } d_x \in D\} \\ &\subseteq \{\sigma \in S \mid \sigma^p(x) = x, \forall x \in P\}. \end{aligned}$$

Thus if $\sigma \in \ker \psi$ then $|\sigma| \mid p$ and so $\alpha \notin \ker \psi$. So there exists nontrivial $\bar{\alpha} \in \text{Aut}(P/D)$ such that $\psi(\alpha) = \bar{\alpha}$. As ψ is a homomorphism, this implies that $\bar{\alpha}^2 = \text{id}_{\text{Aut}(P/D)}$. By Lemma 6.8, we have that every nontrivial involution in $\text{GL}_2(\mathbb{F}_p)$ is conjugate to either $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. These correspond to $\bar{\sigma}_1, \bar{\sigma}_2 \in \text{Aut}(P/D)$ defined by

$$\bar{\sigma}_1 : \begin{cases} bD \rightarrow bD \\ cD \rightarrow c^{-1}D \end{cases} \quad \bar{\sigma}_2 : \begin{cases} bD \rightarrow b^{-1}D \\ cD \rightarrow c^{-1}D \end{cases}.$$

These in turn correspond to $\sigma_1, \sigma_2 \in \text{Aut}(P)$ defined by

$$\sigma_1 : \begin{cases} b \rightarrow bd^w \\ c \rightarrow c^{-1}d^x \end{cases} \quad \sigma_2 : \begin{cases} b \rightarrow b^{-1}d^y \\ c \rightarrow c^{-1}d^z \end{cases}.$$

for some $w, x, y, z \in \mathbb{Z}$. Looking at σ_1 we have that

$$\begin{aligned} \sigma_1(d) &= \sigma_1(c^{-1}b^{-1}cb) \\ &= (c^{-1}d^x)^{-1}(bd^w)^{-1}(c^{-1}d^x)(bd^w) \\ &= cd^{-x}b^{-1}d^{-w}c^{-1}d^x bd^w \\ &= cb^{-1}c^{-1}b \\ &\neq d \end{aligned}$$

and so we have that $\alpha \neq \sigma_1$. Looking at σ_2 we have that

$$\begin{aligned} \sigma_2(d) &= \sigma_2(c^{-1}b^{-1}cb) \\ &= (c^{-1}d^z)^{-1}(b^{-1}d^y)^{-1}(c^{-1}d^z)(b^{-1}d^y) \\ &= cd^{-z}bd^{-y}c^{-1}d^zb^{-1}d^y \\ &= cbc^{-1}b^{-1} \\ &= [c^{-1}, b^{-1}]. \end{aligned}$$

From Theorem 3.4.4 [Sco64], we have that, since $[c, b] = d \in Z(P) \subseteq C_P(c) \cap C_P(b)$, $[c^{-1}, b^{-1}] = [c, b]^{(-1)(-1)} = [c, b] = d$. Therefore $\sigma_2 = \alpha$ for some specific values of w, x, y , and z and we have that $\overline{\sigma_2} = \overline{\alpha}$. Let $\alpha' \in \text{Aut}(P)$ be any other of the possible images of $a \in A$ with the same restrictions as α . Then for $\overline{\alpha'} \in \text{Aut}(P/D)$ there exists $\overline{\tau} \in \text{Aut}(P/D)$ such that $\overline{\tau}^{-1} \cdot \overline{\alpha'} \cdot \overline{\tau} = \overline{\alpha}$ as both are

conjugate to $\overline{\sigma_2} \in \text{Aut}(P/D)$. Let $\tau \in S$ be such that $\psi(\tau) = \overline{\tau}$ (which exists as the map is onto). We now use this information to show that all the possible α' belong to a single conjugacy class.

We have that

$$\begin{aligned}
 \overline{\tau}^{-1} \cdot \overline{\alpha'} \cdot \overline{\tau} = \overline{\alpha} &\Rightarrow \psi(\tau^{-1}\alpha'\tau) = \psi(\alpha) \\
 &\Rightarrow \psi(\tau^{-1}\alpha'\tau)(xD) = \psi(\alpha)(xD), \text{ for all } x \in P \\
 &\Rightarrow (\tau^{-1}\alpha'\tau)(x)D = \alpha(x)D \\
 &\Rightarrow (\tau^{-1}\alpha'\tau)(x) = \alpha(x)d_x, \text{ for some } d_x \in D.
 \end{aligned}$$

But $\tau^{-1}\alpha'\tau$ is an involution, so for all $x \in P$ we have that

$$\begin{aligned}
 x &= (\tau^{-1}\alpha'\tau)^2(x) \\
 &= (\tau^{-1}\alpha'\tau)(\alpha(x)d_x) \\
 &= \alpha(\alpha(x)d_x) \\
 &= \alpha^2(x)d_x \\
 &= xd_x,
 \end{aligned}$$

implying that $d_x = 1$. Therefore $\tau^{-1}\alpha'\tau = \alpha$ and so every possible image of $a \in A$ under the semidirect product of G is a member of the same conjugacy class in $\text{Aut}(P)$. By Lemma 6.9 this implies that all possible groups meeting the assumptions of this theorem are isomorphic. Noticing that J_p meets the hypotheses we have that all possible groups are all isomorphic to J_p . $\square$

Theorem 6.11. *Let p be an odd prime and let G be a group with order $2p^3$. Then G is supersolvable and either*

- (a) G is subnormally monomial; or
- (b) G is isomorphic to J_p (which is not subnormally monomial).

Proof. By Lemma 6.1, we have that G is supersolvable. By Theorem 6.2, G is a semidirect product of its unique p -Sylow subgroup and $\mathbb{Z}_2$. By Theorem 6.5, if the p -Sylow subgroup is abelian then G is subnormally monomial. Since J_p does not have an abelian p -Sylow subgroup, these groups all fall in case (a). If the semidirect product is trivial, and so is a direct product, then by Theorem 5.2, G is subnormally monomial if and only if all of its Sylow subgroups are also. But all its Sylow subgroups are nilpotent and so subnormally monomial thus G is. Since the semidirect product in J_p is nontrivial, these groups fall in case (a). By Theorem 6.6, if G has a nonabelian p -Sylow subgroup and is centerless then it is subnormally monomial. As J_p has a nontrivial center, these groups all fall in case (a). By Theorem 6.7, if the p -Sylow subgroup is isomorphic to $\mathbb{Z}_p \rtimes \mathbb{Z}_{p^2}$ and the center is nontrivial then the semidirect product is actually a direct product and so Theorem 5.2 applies again and we are in case (a). The remaining groups all have their p -Sylow subgroups isomorphic to $\mathbb{Z}_p \rtimes (\mathbb{Z}_p \times \mathbb{Z}_p)$ and have nontrivial centers. Theorem 6.10 proves that all such groups are isomorphic to J_p which is not subnormally monomial and so we are in case (b). $\square$

7. SELECTED GROUPS WHOSE ORDER IS NOT DIVISIBLE BY $2p^3$

We now show that certain classes of supersolvable groups whose orders are not divisible by $2p^3$ for any odd prime p are subnormally monomial. To do this we need to present two major properties of supersolvable groups upon which the results are based. The first property is that for supersolvable groups the commutator subgroup is nilpotent and thus a direct product of its Sylow subgroups.

The second is that all supersolvable groups possess the Sylow tower property which allows us to write them as a semidirect product of their Sylow subgroups. These two theorems combine to give us a standard way of writing a supersolvable group and finding a subgroup which is a direct product of p -groups (which is subnormally monomial). We now give these two theorems with proofs.

All groups in this section are assumed to be finite.

Theorem 7.1. (*Theorem 7.2.13 [Sco64]*) *If G is a supersolvable group, then the commutator subgroup G' is nilpotent.*

Proof. Let $\{1\} = A_0 \subset A_1 \subset \cdots \subset A_n = G$ be a composition series of G , and let $B_i = A_i \cap G'$ for $1 \leq i \leq n$. Then $\{1\} = B_0 \subset B_1 \subset \cdots \subset B_n = G'$ is a composition series of G' . Each $g \in G$ induces an automorphism of B_{i+1}/B_i , so

$$\frac{G}{M} \cong \text{Aut} \frac{B_{i+1}}{B_i},$$

where

$$\frac{M}{B_i} = C_{G/B_i} \left(\frac{B_{i+1}}{B_i} \right).$$

By Theorems 5.7.4 and 5.7.12 [Sco64], the automorphism group of a cyclic group is abelian. Hence G/M is abelian. Therefore $M \supset G'$, so that $B_{i+1}/B_i \subset Z(G'/B_i)$. But this means that $\{1\} = B_0 \subset B_1 \subset \cdots \subset B_n = G'$ is a central series for G' , and G' is nilpotent. $\square$

Definition 7.2. Let G be a group and let $p_1 > p_2 > \cdots > p_r$ be the distinct primes dividing $|G|$. Then G is said to satisfy the *Sylow tower property* if there exist $P_{p_1}, P_{p_2}, \dots, P_{p_r}$ such that each P_{p_i} is a p_i -Sylow subgroup of G , for $1 \leq i \leq r$, and $P_{p_1} P_{p_2} \cdots P_{p_k}$ is normal in G , for $0 \leq k \leq r$.

Theorem 7.3. (*Theorem 1.8 [DV82]*) *Supersolvable groups satisfy the Sylow tower property.*

Proof. Let G be supersolvable. It is enough to show that if p is the largest prime dividing $|G|$, then P_p , a p -Sylow subgroup of G , is normal in G , for the theorem will then follow by induction on the number of distinct prime divisors of $|G|$. We now proceed inductively on the order G . Let N be a minimal normal subgroup of G . By induction NP_p/N is a normal subgroup of G/N and hence $NP_p \triangleleft G$. Let $|N| = q$ where q is some prime (possible as G is supersolvable). If $p = q$, then $NP_p = P_p$ and we are done; hence we may assume that $p > q$. By Theorem 1.4 [DV82], $G/C_G(N)$ is abelian of exponent dividing $q - 1$. Since $p > q$, we conclude that $C_G(N) \geq P_p$. It now follows that P_p is a normal subgroup of NP_p and, as P_p is also a Sylow subgroup of NP_p , that P_p is mapped to itself under every automorphism of NP_p . This and $NP_p \triangleleft G$ imply $P_p \triangleleft G$ which establishes the theorem. $\square$

A direct result of Theorem 7.3 is that we can write the supersolvable group G as

$$P_{p_1} \rtimes P_{p_2} \rtimes \cdots \rtimes P_{p_r}.$$

Let G be a supersolvable group with order $p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ for distinct primes $p_1 > p_2 > \cdots > p_k$. We will now prove that if $m_i \leq 2$ for $1 \leq i \leq m - 1$ then G is subnormally monomial. We first give a proof for groups whose order is cubefree and then prove the more general result.

Theorem 7.4. *Let G be a supersolvable group whose order is cubefree. Then G is subnormally monomial.*

Proof. Since G is supersolvable, Theorem 7.1 gives us that G' is nilpotent and so is the direct product of its Sylow subgroups. Since the order of G is cubefree, the order of any of its Sylow subgroups must also be cubefree. Since the Sylow subgroups of G' must be subgroups of the Sylow subgroups of G we have that G' is the direct product of Sylow subgroups whose orders are cubefree. Since every p -group of cubefree order is abelian, we have that G' is the direct product of abelian groups. By Theorem 7.1, as G' is abelian and so nilpotent, this implies that G is subnormally monomial. $\square$

Theorem 7.5. *Let G be a supersolvable group with order $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} p^m$ with distinct primes $p_1 > p_2 > \cdots > p_k > p$, $m_1, m_2, \dots, m_k \in \{1, 2\}$ and $m \in \mathbb{Z}^+$. Then G is subnormally monomial.*

Proof. We prove this by induction on k . When $k = 1$ we have that G is a p -group and so is subnormally monomial, which provides our base case.

By Theorem 7.1, G' is nilpotent. Let Q_i be a p_i -Sylow subgroup of G' for $i = 1, \dots, k$ and let Q be a p -Sylow subgroup of G' . Since G' is nilpotent we have

$$G' \cong Q_1 \times Q_2 \times \cdots \times Q_k \times Q.$$

As these p -groups are also subgroups of G , they must each be contained in a corresponding Sylow subgroup of G . Let P_i be a p_i -Sylow subgroup of G for $i = 1, \dots, k$ and let P be a p -Sylow subgroup of G such that $Q_i \subseteq P_i$ and $Q \subseteq P$. By Theorem 7.3, G possesses the Sylow tower property

$$G \cong P_1 \rtimes P_2 \rtimes \cdots \rtimes P_k \rtimes P.$$

Notice that since $m_i \leq 2$ for $i = 1, \dots, k$ every P_i and Q_i is abelian. If $P_i \subseteq Z(G)$ for some $1 \leq i \leq k$, then G is the direct product of P_i and $P_1 \rtimes \dots \rtimes P_{i-1} \rtimes P_{i+1} \rtimes \dots \rtimes P_k \rtimes P$. Therefore by our induction hypothesis and Theorem 5.2, we have that G is subnormally monomial. Hence we can assume that $P_i \not\subseteq Z(G)$ for $i = 1, \dots, k$.

We divide the problem into multiple cases based on the intersection of the center with various subgroups. The purpose of this is to find a subgroup of G which is normal, abelian and is not contained in the center as required in the proof of Theorem 3.5. In keeping with the notation of that theorem we will label this subgroup A . We start with the subgroup

$$Z(G) \cap Q.$$

We have two possibilities: (a) $Z(G) \cap Q \subsetneq Z(Q)$ and (b) $Z(G) \cap Q = Z(Q)$.

Case (a): $Z(G) \cap Q \subsetneq Z(Q)$. Let

$$\begin{aligned} A &= Q_1 Q_2 \dots Q_k Z(Q) \\ &\cong Q_1 \times Q_2 \times \dots \times Q_k \times Z(Q). \end{aligned}$$

It is clear that A is abelian and not contained in the center of G . We must show that A is normal in G . Let $a_1 a_2 \dots a_k a \in A$ where $a_i \in Q_i$ for $i = 1, \dots, k$ and $a \in Z(Q)$ and let $x \in G$. Thus

$$x \cdot a_1 a_2 \dots a_k a \cdot x^{-1} = (x a_1 a_2 \dots a_k x^{-1}) x a x^{-1}.$$

Since $a_1 a_2 \dots a_k \in G'$ and G' is a normal subgroup of G , we have that $x a_1 a_2 \dots a_k x^{-1} \in G'$. Since $a_1 a_2 \dots a_k \in P_1 P_2 \dots P_k$, which is a normal subgroup of G , we have

that $xa_1a_2\cdots a_kx^{-1} \in P_1P_2\cdots P_k$. Thus $xa_1a_2\cdots a_kx^{-1} \in G' \cap (P_1P_2\cdots P_k) = Q_1Q_2\cdots Q_k$. Emphasizing the product with the p -Sylow subgroup, let $x = (y, z)$, where $y \in P_1 \rtimes P_2 \rtimes \cdots \rtimes P_k$ and $z \in P$. As $a \in Z(Q) \subseteq Q \subseteq P$, we have $a = (1, a)$ in this notation. Computing, we have

$$\begin{aligned}
xax^{-1} &= (y, z) (1, a) (y, z)^{-1} \\
&= (y, z) (1, a) (z^{-1} \circ y^{-1}, z^{-1}) \\
&= (y \cdot z \circ 1, za) (z^{-1} \circ y^{-1}, z^{-1}) \\
&= (y, za) (z^{-1} \circ y^{-1}, z^{-1}) \\
&= (y \cdot (zaz^{-1} \circ y^{-1}), zaz^{-1}) \\
&= \left(y (zaz^{-1}) y^{-1} (zaz^{-1})^{-1}, zaz^{-1} \right),
\end{aligned}$$

where the action of conjugation defined by the semidirect product is given by $\circ$. Since G' is fixed under any automorphism of G , Q is fixed under any automorphism of G' and $Z(Q)$ is fixed under any automorphism of Q , we have that $Z(Q)$ is fixed under the action of conjugation and $zaz^{-1} \in Z(Q)$. Since $y \in G$ we have that $y(zaz^{-1})y^{-1}(zaz^{-1})^{-1} \in G'$ (by definition). But $P_1 \rtimes P_2 \rtimes \cdots \rtimes P_k$ is normal in G so $y(zaz^{-1})y^{-1}(zaz^{-1})^{-1} \in P_1 \rtimes P_2 \rtimes \cdots \rtimes P_k$. This implies that

$$y(zaz^{-1})y^{-1}(zaz^{-1})^{-1} \in G' \cap P_1P_2\cdots P_k = Q_1Q_2\cdots Q_k.$$

Therefore A is normal in G .

Let ρ be an irreducible representation of G . Lemma 3.3 allows us to assume that ρ is faithful under our induction hypothesis. As A has the requisite properties, Proposition 3.1 tells us that there exists a subgroup H of G containing A and a

representation σ of H such that $\rho = \text{Ind}_H^G \sigma$. By Lemma 3.2, $s^{-1}as = a$ for all $s \in H$ and for all $a \in A$. Consequently $G' \subseteq H$ and H is normal in G . Since H is smaller than G , our induction hypothesis gives us that G is subnormally monomial.

Case (b): $Z(G) \cap Q = Z(Q)$. We have two sub-possibilities: either (c) there exists $i \in \{1, \dots, k\}$ such that $Z(G) \cap Q_i \subsetneq Q_i$, or (d) for all $i \in \{1, \dots, k\}$ we have that $Z(G) \cap Q_i = Q_i$.

Case (c): Letting $A = Q_1 Q_2 \cdots Q_k Z(Q)$ again, the same argument used in case (a) proves that G is subnormally monomial.

Case (d): In this case, as every $Q_i \in Z(G)$ we have that $Z = Q_1 Q_2 \cdots Q_k Z(Q) \subseteq Z(G)$. As noted earlier, $P_1 \not\subseteq Z(G)$ which implies $Q_1 \subsetneq P_1$. Let

$$A = P_1 Q_2 \cdots Q_k Z(Q).$$

We first show that A is a subgroup of G by showing that each of the products above are actually direct. We have that P_1 and $X = Q_2 \cdots Q_k Z(Q)$ are subgroups of G . If we can show that P_1 and X normalize each other and that their intersection is trivial then, by Theorem 9 [DF91], their product is a direct product. It is clear that they normalize each other since $X \subseteq Z(G)$. It is obvious from the orders of P_1 and X that their intersection is trivial. Thus A is a subgroup of G .

It is clear that A is abelian and not contained in the center of G . We must show that A is normal in G . Let $a_1 a_2 \cdots a_k a \in A$ where $a_1 \in P_1$, $a_i \in Q_i$ for

$i = 2, \dots, k$ and $a \in Z(Q)$ and let $x \in G$. Thus

$$\begin{aligned} x \cdot a_1 a_2 \cdots a_k a \cdot x^{-1} &= x a_1 x^{-1} x a_2 x^{-1} \cdots x a_k x^{-1} x a x^{-1} \\ &= x a_1 x^{-1} (x a_2 \cdots a_k x^{-1}) x a x^{-1}. \end{aligned}$$

Since $a_2 \cdots a_k \in G'$ and G' is a normal subgroup of G , we have that $x a_2 \cdots a_k x^{-1} \in G'$. Since $a_2 \cdots a_k \in P_1 P_2 \cdots P_k$, which is a normal subgroup of G , we have that $x a_2 \cdots a_k x^{-1} \in P_1 P_2 \cdots P_k$. Thus

$$\begin{aligned} x a_2 \cdots a_k x^{-1} &\in G' \cap P_1 P_2 \cdots P_k \\ &= Q_1 Q_2 \cdots Q_k \\ &\subset P_1 Q_2 \cdots Q_k \\ &\subset A. \end{aligned}$$

Since P_1 is normal in G we have that $x a_1 x^{-1} \in P_1 \subset A$. Using the same argument as in case (a) above we have that $x a x^{-1} \in A$ implying that A is normal in G . As shown above, this implies that G is subnormally monomial. $\square$

8. FUTURE RESEARCH

The obvious direction for further work would be to complete the classification of finite supersolvable subnormally monomial groups. The fact that all supersolvable groups have the Sylow tower property (Theorem 7.3) provides an obvious inductive structure for this problem based on the number of different prime factors of the group's order. Combining this with research into representations of semidirect products of p -groups should allow one to complete the classification.

In our initial exploration of subnormally monomial groups with GAP, we also examined solvable groups which were not supersolvable. The symmetric group S_4 on four elements is an example of such a group which is not monomial. The semidirect product $\mathbb{Z}_3 \ltimes Q_8$, where Q_8 is the quaternions, is monomial but is not subnormally monomial. The alternating group A_4 on four elements is an example which is subnormally monomial. The classification of all finite solvable groups into these three sets: not monomial, monomial, and subnormally monomial; is another area for further research.

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