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A FRICTIONAL THEORY OF PLASTICITY
FOR GRANULAR MATERIALS

by

Maciej B. Szymanski, M.Sc., D.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
DOCTOR OF PHILOSOPHY

Department of Civil Engineering

Carleton University

Ottawa, Ontario

July 1980

The undersigned recommend to the
Faculty of Graduate Studies and Research
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ABSTRACT

The objective of this study is to formulate an incremental theory of plasticity for granular materials behaviour. The theory is developed on the basis of an analogy with the behaviour of two flat bodies in frictional sliding. This theory is expected to be of both theoretical as well as practical significance.

No formalism of the classical theories of plasticity is employed in the formulation of the proposed theory. This theory, however, falls in the framework of the classical, incremental theories of plasticity as far as this framework is established by the following fundamental assumptions: i) dynamic, thermal and viscous effects are neglected, ii) elastic and plastic strains are additive and non-coupled, iii) material is isotropic in plastic behaviour in the sense of St. Venant's hypothesis.

In the formulation of the proposed theory two main questions are considered. Firstly, the formulation of the theory involves two fundamental postulates: a postulate for energy dissipation and a postulate for the mode of plastic deformation. By a linearization of the equations which follow the latter postulate, specific forms of a yield condition and an incremental stress-strain relationship are derived. The second involves the introduction of certain intuitive assumptions. These assumptions result in a set of limitations for the material functions and their derivatives. Therefore, from purely

theoretical considerations, there is available a considerable knowledge of the granular materials plastic properties. The abovementioned limitations are partly based on dynamic and kinematic restrictions which must be satisfied for any plastic deformation process of granular materials.

The thesis consists of two parts. In the first part the behaviour of a locally homogeneous granular material is investigated. In this part it is postulated that there exists an analogy between plastic behaviour of granular materials and the behaviour of two bodies in frictional sliding, as described by Amonton's friction law. This analogy is postulated for axisymmetric compression of granular materials.

Two possibilities of the generalization of the constitutive equations for axisymmetric compression are presented and the simplest specific forms of a yield function and an incremental stress-strain relationship are derived. In these equations the density of granular material satisfies the role of a hardening/softening parameter. This part also includes a "thermodynamic" interpretation of the proposed postulates.

In the second part of the thesis the behaviour of a locally nonhomogeneous granular material is examined. Here, the constitutive relations are based on the postulates which are applicable to the locally homogeneous material; the essential difference is the introduction of a new material function by which a new state parameter is defined. This material function attempts to describe the intensity of some local nonhomogeneities of granular materials. Again, the introduction of

such a function can be justified by considering an analogy with the behaviour of two bodies in frictional sliding, as described by Bowden and Tabor's theory of friction. Finally, the stress-strain-strength behaviour of the locally nonhomogeneous granular material is examined in detail under the cuboidal deformation conditions.

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CHAPTER ONE

INTRODUCTION

1.1 Introductory Remarks

In this study a theory of plastic flow (a set of constitutive relations) for granular materials is developed. The term "plastic flow" describes the time independent but history dependent behaviour of materials. Since this theory is intended to be of both theoretical and practical significance it seems to be instructive to outline at the outset certain general guidelines within which such a theory could be developed. These guidelines may be summarized as follows:

- i) Any proposed constitutive relation should not violate basic, commonly accepted laws of physics (e.g. the laws of thermodynamics).
- ii) The mechanical behaviour of real materials is, in general, extremely complex. Therefore, any theory is a mathematical model of real materials behaviour. In other words, a theory describes the behaviour of an ideal material. Such a material must be, possibly in physical terms, clearly defined.
- iii) Owing to the complexity of real material behaviour a theory exaggerates or neglects certain aspects of this behaviour. These aspects should be, if possible, emphasized.
- iv) It is desirable that a new theory incorporates well established concepts of more general theories. It does not, however,

mean that the concepts of any well-established theory may, without certain justification, be accepted. In other words, care should be taken to assess whether a well-established theory is more general or parallel to the theory being developed. The term parallel means developed for a different class of materials or for the description of phenomena which are different from those considered.

- v) Constitutive equations should be amenable to physical interpretations. Such physical interpretations help to introduce additional, reasonable simplifications in constitutive equations. Furthermore, they may help to establish proper limits for the validity of these equations.
- vi) Mathematical rigour, as well as mathematical convenience of a theory should stay in a reasonable relation to its reliability. Otherwise, even the most rigorous or convenient theory is useless, at least from the practitioner's perspective.
- vii) If a constitutive theory is to be considered a practically successful one then the material parameters or functions encountered in the theory should be derivable from relatively simple experimental techniques. The number of material parameters and functions should be as limited as possible without considerable loss in accuracy.

It should be emphasized that a theory does not need to satisfy all of these postulates to be useful, either from a theoretical or a

practical point of view. The postulates are presented here in order to outline a certain framework which we will attempt to follow in this study.

Current studies of the mechanical behaviour of granular materials indicate that constitutive relationships can be developed by adopting either the microscopic or phenomenological approach. The microscopic approach offers constitutive relations (being finally expressed in phenomenological terms) and a physical interpretation of these relations simultaneously. This is a considerable advantage of the microscopic approach. There can be, however, certain shortcomings if this approach is used. Consider the case of plastic deformation of granular materials. In microscopic theories for such materials it is usually assumed that the frictional slip at grain boundaries is responsible for plastic deformations. The slip is further assumed to be described by Amonton's friction law. As the crushing of grains can be neglected for low stress levels, one may feel that this assumption is more or less correct. Nevertheless, the philosophy underlying such an assumption can be open to objections. Since the final goal is to derive a set of constitutive equations (a phenomenological law), the basic laws employed on the microscopic level should have a certain physical superiority* over the phenomenological law to be derived. Such a superiority of Amonton's law over constitutive

* To obtain an example of what this superiority means, it is useful to consider the ideal gas phenomenological law which has been also derived by employing the Newton law on a microscopic level (see, for example, Feynman et al., 1963).

equations for plastic deformation of granular materials is doubtful since this law is phenomenological itself. Furthermore, this law has been explained, on a microscopic level, in terms of a phenomenological theory of plasticity (Bowden and Tabor, 1956; Green 1954). The aim of this discussion is not to criticize or reject the microscopic approach. On the contrary, it is believed that such an approach offers considerable advantages. These advantages, however, are rather limited to the physical interpretations of phenomenological equations, and so, to a justification of phenomenological postulates or hypotheses.

In the postulates set forward previously the need for the availability of the physical interpretations of constitutive equations has been emphasized. On the other hand one may notice, from the publications dealing with the mechanics of granular materials, that in recent years there has been a sharp increase in the development of purely mathematical constitutive models which do not usually offer physical interpretations (e.g. Davis and Mullenger, 1979; Prevost; 1979).

The mechanics of granular materials is of importance, in the first place, to Soil Mechanics and Foundation Engineering. One may classify the problems where engineering practice deals directly with constitutive equations as follows:

- design and calculation of earth structures and soil-foundation interactions,
- design of experiments and the interpretation of experimental data.

The advantage of having a physical interpretation of constitutive equations in the case of the first problem was pointed out previously in postulate "v". This advantage is even more obvious in the case of the second problem. For example, if a physical interpretation of a material parameter is available, then the choice of an experimental test to determine this parameter, as well as the interpretation of the test results, may be considerably improved.

1.2 Basic Experimental Shear Characteristic

The basic stress-strain characteristics of granular materials are obtained from the conventional axisymmetric tests which are performed under the mixed stress-strain loading program (also referred to as the "strain-controlled" program): $\dot{\epsilon}_1(t) > 0$; $\dot{\sigma}_2 = \dot{\sigma}_3 = 0$. Such a loading program allows to observe the material response resulting in either the increase or the decrease of axial stress σ_1 . There is also sometimes used the stress-controlled program: $\dot{\sigma}_1(t) > 0$, $\dot{\sigma}_2 = \dot{\sigma}_3 = 0$. It follows that no decrease of the axial stress can be observed in this case.

The behaviour of granular materials under the above mentioned mixed loading program is illustrated in Figure 1 (e.g., Taylor, 1948; Bishop, 1966; Ko and Scott, 1967; Lee and Ingles, 1968). σ_1 and σ_3 are the major and the minor principal stresses, respectively; ϵ_1 , ϵ_3 and v are the major, minor and volumetric strains, respectively. The following conclusions may be drawn from this figure:

- the behaviour of granular materials depends both qualitatively and quantitatively on their density. The terms "loose" and

Figure 1: Basic Stress-Strain Characteristics for Granular Materials

"dense" refer to the materials of relatively low and high densities, respectively,

- plastic (irreversible) deformation occurs far before the peak-stress (also referred to as the "peak-strength") is reached.

In the figure the peak-stress ratio and the irreversible part of axial strain are denoted Γ_p and ϵ_1^p , respectively. Note that in the case of loose materials no peak-stress ratio appears and the stress-strain characteristic approximates final stress ratio, denoted Γ_f in Figure 1,

- under monotonically increasing deformation the effects of hardening as well as softening may take place. These effects are defined, respectively, by the increase ($\dot{\Gamma} > 0$) or the decrease ($\dot{\Gamma} < 0$) of the stress ratio Γ ,
- under monotonically increasing deformation the volume changes can be either positive (contraction, $\dot{v} > 0$) or negative (dilatation, $\dot{v} < 0$).

These effects will be explained in the terms of the proposed theory.

From the viewpoint of engineering practice the point P, indicating maximum stress ratio Γ_p , is of particular importance. The angle of internal friction, being a basic design parameter for granular materials, is determined from this ratio. The physical interpretation of $\Gamma_p = \Gamma_{\max}$ will also be suggested in the following chapters.

1.3 Some Previous Studies

For the last thirty years many studies have been made concerning the stress-strain behaviour of granular materials. The advancement of the subject is such that it is possible to describe the mechanical behaviour of granular materials with good accuracy, regarding a considerable number of experimentally observed phenomena. This can be observed, for instance, in the theory developed by Lade and Duncan (1975) and later modified by Lade (1977). The research concerning the constitutive relationships for soils, and for granular materials in particular, is in a state of progressive development. There is a large number of constitutive relationships describing the mechanical behaviour of soils available at the present time. Some of them have been discussed and reviewed in a few articles, e.g., Christian and Desai (1977), Naylor (1978), Mogami (1977, 1979), Desai (1979). Aiming to indicate trends appearing in the development of the constitutive relationships for granular materials, some widely known theories and models are reviewed in this section.

In the field of microscopic approach, the most widely known theory is probably the stress-dilatancy theory proposed by Rowe (1962); see also Rowe (1971). Rowe derived constitutive relations from microscopic considerations which involved an examination of the regular assemblies of spheres and rods. The flow rule resulting from this theory, for the case of axisymmetric compression, is given as

$$\frac{\sigma_1}{\sigma_3} = \left(1 - \frac{dv}{d\epsilon_1^p}\right) \tan^2 \left(\frac{\pi}{4} + \frac{\phi_f}{2}\right) \quad 1.1$$

where ϕ_f is a half-empirical material parameter. A more general formulation of the stress-dilatancy theory, developed for irregular assemblies of spherical particles, was given by Horne (1965, 1969). Microscopic approach to the mechanics of granular materials has been discussed and reviewed in detail in the volume edited by Cowin and Satake (1978).

The phenomenological approach by which the constitutive relationships for granular materials are usually developed may be divided into four main groups (see, e.g., Christian and Desai, 1977; Desai, 1979):

- i) non-linear elasticity method
- ii) curve-fitting method
- iii) elastic-plastic theories
- iv) hypoelastic theories

These methods are used both separately, or more often, in some combinations. For example, the relations found from experiments by means of the curve-fitting method are often incorporated in "pseudo-elastic" or elastic-plastic theories.

The first method is not (physically) realistic in the case of granular materials although it may provide a powerful method for solving boundary value problems by the finite element analysis (see, e.g., Ko and Masson 1976). The combination of the first and the second methods leads to pseudo-elastic models. The most widely known of such models have been obtained by assuming the hyperbolic relation between stress and strain which, for conventional axisymmetric tests, is presented as

$$\sigma_1 - \sigma_3 = \frac{\epsilon_1}{a\epsilon_1 + b}$$

1.2

where a and b are material constants. This relation has been originally proposed by Kondner (1963) and Kondner and Zelasko (1963) from the analysis of experimental results. The models based on this relation have been frequently used in finite element analysis of boundary value problems (e.g., Clough and Woodward, 1967; Duncan and Chang, 1970). Other stress-strain relations which were obtained by means of curve-fitting (e.g. Desai and Wu, 1976) have been reviewed and discussed by Christian and Desai (1977).

Drucker and Prager (1952) first proposed a description of soil behaviour which involved the concepts of the classical incremental theories of plasticity. They assumed the flow rule associated with a Coulomb type yield condition, also referred to as "extended von Mises" or "Drucker-Prager" yield condition. Since for granular materials this assumption predicts no dissipated energy increments (e.g., Harkness, 1971), such a proposition cannot be accepted. In order to take into account both plastic deformation of soils in shear and under hydrostatic compression, Drucker et al. (1959) suggested the introduction of the flow rule associated with a "cap"-shaped yield function which intersects the hydrostatic axis if drawn in the principal stress space. This idea has been later adopted in the widely known critical state models developed by Roscoe and co-workers (Roscoe et al., 1963; Calladine, 1963; Roscoe and Burland, 1968; Schofield and Wroth, 1968). These models describe plastic hardening (or softening) of soils as a function of the void

ratio. The "bullet"-shaped yield function included in Critical State Models is presented, for the case of axisymmetric compression, in Figure 2. The mode of plastic deformation in this figure is indicated by the vectors denoted " $\dot{\epsilon}^P$ ". The idea to combine the two yield functions (i.e. the Coulomb-conical-type and the cap-type) for the description of soil behaviour has been set forward by Jenike and Shield (1959) and later developed by other researchers, e.g. DiMaggio and Sandler (1971), Lade (1977). Typical yield surface of the cap model, proposed by DiMaggio and Sandler (1971), is presented in Figure 2 with dotted lines. In this model the failure surface is assumed to be fixed and the motion of the cap is related to plastic strains. More mathematically sophisticated models, which are capable of describing both isotropic and anisotropic hardening (by means of Prager's concept of kinematic hardening), have been proposed by Mroz et al. (1978, 1979), Prevost (1979) and others.

Schofield and Wroth (1968) developed a Critical State Model (referred to as "Granta-Gravel") for the description of the plastic behaviour of granular materials. This model has been derived, for the case of axisymmetric compression, on the basis of the following postulates:

- material is rigid-plastic in response,
- the equation for the density of the rate of energy dissipation is:

$$\dot{D} = p M |\dot{\epsilon}_q|, \quad p > 0 \quad 1.3$$

- there exists a set of critical states which are defined by the empirical equations:

Figure 2: Examples of Yield Functions for Axisymmetric Compression

$$q = Mp \quad 1.4$$

$$v = A - B \ln p \quad 1.5$$

- associated with yield function plastic flow rule (the condition of normality) holds. In the above equations A, B and M are material constants and

$$q = \sigma_1 - \sigma_3, \quad p = \sigma_1 + 2\sigma_3, \quad \dot{\epsilon}_q = \frac{2}{3} (\dot{\epsilon}_1 - \dot{\epsilon}_3), \quad \dot{v} = \dot{\epsilon}_1 + 2\dot{\epsilon}_3$$

Schofield and Wroth (1968) also extended their model for a general stress state by retaining Eqs. (1.3)-(1.5) and the substitution, instead of q and p , the generalized variables

$$q^* = \frac{1}{\sqrt{2}} [(\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{\frac{1}{2}} \quad 1.6$$

$$p^* = \sigma_1 + \sigma_2 + \sigma_3 \quad 1.7$$

The yield function resulting from the above postulates is of a bullet-shape (see Figure 2) and it may expand (hardening) or contract (softening) depending on the void ratio. From experimental observations, however, it appears that such a function satisfies the role of plastic potential, rather than yield condition (Poorooshasb et al., 1966, 1967). It has also been found that in the case of granular materials the yield function for plastic shear can be adequately described by a Coulomb-type yield criterion (Poorooshasb et al., 1966, 1967; Lee, 1967; Barden et al., 1969; Roscoe, 1970; Tatsuoka, 1978).

A very comprehensive description of granular materials elastic-plastic behaviour has been reached in Lade's theory (Lade and Duncan, 1975; Lade, 1977). This theory has been developed on the basis of the results

of cubical triaxial tests. In the final formulation of the theory Lade (1977) assumed a generalized Hooke's law for elastic strains ($\underline{\epsilon}^e$) and divided plastic strains into two parts: plastic "collapse" strains ($\underline{\epsilon}^C$) and plastic "expansive" strains ($\underline{\epsilon}^P$). Hence

$$\dot{\underline{\epsilon}} = \dot{\underline{\epsilon}}^e + \dot{\underline{\epsilon}}^C + \dot{\underline{\epsilon}}^P \quad 1.8$$

Lade proposed to describe total plastic deformation of granular materials in terms of two yield functions. For plastic collapse strains he introduced a cap-shaped yield surface and associated flow rule. For plastic expansive strains the yield condition chosen was of a form similar to that of the failure criterion determined from experiments. This condition is described by a function of the first and the third principal invariants of stress tensor. The flow rule for plastic expansive strains is expressed with a plastic potential function which does not coincide with the yield condition. For the description of hardening-softening effects Lade followed Hill's hypothesis by which the plastic work is considered to be a parameter of yield function.

Alternatively to elastic-plastic theories it has been more recently proposed to describe soils behaviour by means of hypoelastic models (e.g. Stutz and Sawczuk, 1971; Romano, 1974; Davis and Mullenger, 1979). In this approach the concepts of hypoelasticity and critical states are combined and the constitutive equation takes the general form

$$\dot{\underline{\sigma}} = \underline{G}(\underline{\sigma}, \underline{D}, e) \quad 1.9$$

where $\dot{\underline{\sigma}}$ denotes co-rotational or Jauman stress-rate tensor, $\underline{\sigma}$ and $\underline{D}$ are stress and the rate of deformation tensors and e is the void ratio.

If it is assumed that the functional form of $\underline{G}$ is polynomial in $\underline{\sigma}$ and

linear in D then the most general form of equation (1.9) is given by a relation which contains twelve material functions of stress invariants and e (see, e.g., Davis and Mullenger, 1979). Some particular forms of this relation have been examined in the above mentioned literature. In the hypoelastic approach as applied to the description of the soils behaviour the existence of yield condition is postulated, as in elastic-plastic theories. Therefore, both these approaches, although arrived at by means of different mathematical formalisms, are capable of describing the elastic-plastic behaviour of materials. A review and comparison of both approaches has been recently presented by Mroz (1980).

As mentioned previously the description of the mechanical behaviour of soils by means of incremental theory of plasticity was first suggested by Drucker and Prager (1952). Prior to that Taylor (1948) considered the description of the irreversible deformation of granular materials by means of a frictional analogy. From his strength calculations, performed for the peak-strength conditions in the direct shear tests, the first frictional flow rule can be established. Similar analysis of axisymmetric compression tests data was presented by Bishop (1954). The considerations of Taylor and Bishop have been reexamined by Rowe et al. (1964). The use of a frictional analogy in the description of granular materials mechanical behaviour was first suggested by Coulomb (1776). This is, thus, the oldest approach to the study of the constitutive behaviour of such materials. Also in the present study a frictional analogy will form a basis for the formulation of an incremental

theory of plasticity for granular materials.

1.4 Fundamental Hypotheses Underlying Classical Incremental Theories of Plasticity

Incremental theories of plasticity (also referred to as "theories of plastic flow") discussed henceforth are capable of describing the plastic behaviour of the materials for which it can be assumed that

- i) the behaviour of material is not sensitive to time-scale changes, i.e., no viscous effects occur,
- ii) the behaviour of material is not sensitive to temperature changes.

Moreover, the considerations are limited to the "quasi-static" deformation process so that the inertia effects can be neglected. The fundamental assumptions required for the formulation of an incremental theory of plasticity (within the framework of the phenomenological approach) do not seem to be well established. Although there are available some monographs where such assumptions are discussed (e.g. Hill, 1950; Fung, 1965; Malvern, 1969), it is not clear which of these assumptions are equally general. For example, is the assumption on the existence of a plastic potential function as fundamental as the assumption on the coincidence of the principal directions of stress and plastic strain-rate tensors in isotropic materials? It may be noted that the fundamentals of incremental theories of plasticity are usually reviewed from the viewpoint of metals mechanics and some particular concepts or postulates, appropriate for typical metals behaviour, are taken as basic hypotheses.

It seems that of all commonly discussed rules on the plastic deformation of materials the following hypotheses might be both general enough to be applicable for a broad class of materials and strong enough to lead to simple (of practical significance) descriptions of plastic behaviour:

- iii) total strain-rates ($\dot{\epsilon}$) in an elasto-plastically deformed material are, in the general case, equal to the sum of elastic ($\dot{\epsilon}^e$) and plastic ($\dot{\epsilon}^p$) strain-rates

$$\dot{\epsilon} = \dot{\epsilon}^e + \dot{\epsilon}^p \quad 1.10$$

and the constitutive equations are not coupled, i.e., elastic and plastic material parameters are independent of plastic and elastic strains, respectively,

- iv) material is plastically isotropic and the principal directions of stress and plastic strain-rate tensors coincide (i.e. St. Venant hypothesis holds),
- v) there exists a set of stress states under which plastic deformation can occur. This is called a yield criterion (or yield condition) and can be expressed by the equation of general form.

$$f(\sigma_1, \sigma_2, \sigma_3) = 0 \quad 1.11$$

where σ_i are the principal stresses. In the general case, the yield function f contains certain parameters which depend on loading history. Owing to plastic isotropy of material the yield function must be a symmetric function of the three principal stresses,

vi) the mode of plastic deformation depends on the state of stress and not on stress increments. This may be expressed by the equation

$$\dot{\underline{\epsilon}}^P = \dot{\lambda} \underline{B} \quad 1.12$$

which presents a general form of plastic flow rule.

In this equation $\dot{\lambda}$ is a scalar parameter dependent on stress increments and $\underline{B}$ is a second order tensor dependent on the state of stress. In the general case, equation (1.12) depends on the history of loading.

1.5 The Influence of Mechanics of Metals on the Description of Granular Materials Behaviour

In the formulations of the constitutive equations for the plastic deformation of granular materials there is a strong influence of metal mechanics. There is a set of mathematical concepts (plastic potential, associated flow rule, the stress deviator, the mean stress) which have been found reasonable to be used for the description of some metals behaviour. These concepts are often accepted for the description of the behaviour of granular materials.

If for the description of constitutive behaviour of a particular class of materials some concepts established for different classes of materials are to be adopted, there should exist a certain reasoning for that, theoretical (i.e. physical, logical) or experimental; see also postulate "iv", Section 1.1.

The characteristic physical features of granular materials, i.e., the content of voids, the granular composition and the absence of intergranular forces, cause the strong sensitivity of these materials to large, irreversible changes in internal structure, even if subjected to relatively small loads. Such sensitivities are not observed in the behaviour of typical (polycrystalline) metals. Moreover, the fundamental mechanisms of plastic deformation on a microscopic scale are different in both types of material, i.e., frictional versus crystalline sliding. It seems, therefore, that there are no direct, physical suggestions to adopt without question the concepts of metal plasticity for the description of plastic behaviour of granular materials. It should be noted, however, that under high stress levels the mechanism of plastic deformation in granular materials is most likely to be similar to that occurring in the metal case. This is because under high stress levels the intensive grain crushing may take place.

If a constitutive relation is to be postulated then a set of stress, strain or strain-rate invariants (state variables) have to be chosen. Such invariants have also to be chosen if an experimental relationship, determined for some particular loading conditions, is to be generalized. Unfortunately, again, there is a strong influence of the mechanics of metals (linear elasticity, Huber-Mises yield condition, Prandtl-Reuss plastic flow rule) on the mechanics of granular materials in the choice of these invariants. In the conventional approach to the study of the constitutive relations for granular materials it is customary to decompose the stress tensor into the deviatoric and isotropic parts

$$\underline{\sigma} = \underline{s} + p\underline{\delta} \quad 1.13$$

and to choose as the variables entering constitutive relations the familiar invariants

$$I_1 = \text{tr} \underline{\sigma} = 3p = \sigma_1 + \sigma_2 + \sigma_3 \quad 1.14$$

$$I_2' = \text{tr} (\underline{s}\underline{s}) = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \quad 1.15$$

where $\underline{s}$ and $\underline{\delta}$ denote the stress deviator tensor and Kronecker's delta, I_1 is the first invariant of stress tensor, I_2' is the second invariant of stress deviator tensor and the quantity p is called the mean stress. Consider the physical interpretations of invariants I_1 and I_2' which follow some well-established constitutive relations. The first of these invariants derives its physical meaning if the behaviour of an isotropic, linearly elastic material is considered. Then, the constitutive relation holds

$$I_1 = Cv \quad 1.16$$

where C is a material constant and v denotes the volumetric strain. This is a strong relation for its simplicity, for it is satisfied under any loading condition and it includes the invariant v having obvious geometrical interpretation. Therefore, the considering of invariant I_1 as a state variable is well justified. Furthermore, the meaning of eq. (1.13) is clear as well.

Invariant I_2' derives its physical meaning if the plastic deformation of material takes place in agreement with the Huber-Mises yield condition

$$I_2' = k^2 \quad 1.17$$

where k is a material parameter. In the case of plastic deformation the meaning of Eq. (1.13) is derived if Prandtl-Reuss flow rule

$$\dot{\epsilon}^p = \dot{\lambda} s \quad 1.18$$

is satisfied. Therefore, a meaning for the quantity p , or invariant I_1 , appears as well. To the author's best knowledge, there are no available experimental evidences to confirm the adequacy of Eqs. (1.16), (1.17) or (1.18) for the description of plastic behaviour of granular materials. Therefore, it seems that there is no point to assume a priori that invariants I_1 and I_2 should be included (as basic state variables) in the constitutive relations for granular materials plastic behaviour, unless a certain theoretical reasoning is available.

1.6 Objective and Scope

The aim of this study is to develop a simple theory for the description of the elastic-plastic behaviour of granular materials. This theory is formulated by employing the phenomenological approach. Attention is focussed on the constitutive equations for plastic deformation. For the description of elastic behaviour the generalized Hooke's law is adopted. In searching for a basis to obtain the constitutive equations for plastic deformation of granular materials a frictional analogy has been chosen. The characteristic features of the theory may be summarized as follows:

- i) the fundamental link between the theory proposed herein and the classical incremental theories of plasticity is the acceptance of St. Venant's isotropy condition. The

terminology of the classical theories (e.g. "yield function", "flow rule") is conserved,

- ii) the choice of stress and strain-rate invariants, involved in the development of the theory, is given careful consideration,
- iii) a frictional analogy and some intuitive postulates form the basis of the theory,
- iv) physical interpretation of state parameters, consistent with the definitions of considered (ideal) materials, is suggested.

The presented work consists of two main parts included in Chapters 2 and 3. In the first part the plastic behaviour of locally homogeneous granular materials is considered. The term "local homogeneity" is explained in Section 2.2 where the definition of the investigated (ideal) material is also given. Some basic inequalities for the granular materials constitutive behaviour are presented in Section 2.3. In the next section the constitutive equations for axisymmetric compression are derived by considering an analogy to Amonton's friction law. A generalization of these equations to complex stress states (Section 2.7) is based on an additional postulate, referred to as postulate P3. In the two simplest cases this generalization leads either to the constitutive equations with Coulomb's yield condition or to another set of constitutive equations which are called the "simplified theory". In both cases the constitutive equations contain two material functions depending on the density of material. The general formulation of the theory is presented in Section 2.8. In this formulation the mentioned

postulate P3 is not employed and it leads to the set of constitutive equations containing three material functions.. Finally a concept of "structural entropy" is introduced in Section 2.10.

In Chapter 3 the constitutive equations derived for locally homogeneous materials are modified by introducing a new material function. It is suggested that this function describes the variable degree of some local homogeneities in plastically deformed material. Such a function has been introduced by Szymanski (1976) in order to explain the effects of hardening, softening and peak-strength behaviour which are observed in the conventional axisymmetric compression of granular materials. The constitutive equations for the plastic deformation of locally nonhomogeneous materials are presented in Section 3.6. These equations reduce, under special conditions, to those proposed for locally homogeneous materials. Finally, in Section 3.8, an example of stress-strain relations under the cuboidal loading conditions is presented.

It is to be noticed that the plastic deformations of granular materials due to purely hydrostatic loading will not be investigated in this study. It does not however mean that the ideal granular materials considered in the following chapters may not be plastic plastically deformed under this loading condition. Such deformations can be taken into account by introducing an additional constitutive relationship as has been proposed, for instance, by DiMaggio and Sandler (1971) or Lade (1977).

1.7 Preliminary Notions

In the following chapters the indicial notation is used in which $i, j = 1, 2, 3$ unless is specified otherwise. The summation convention is also adopted and therefore a letter subscript repeated twice in a term indicates summation. Compressive stresses and strains are taken positive. The notation for the principal stresses and strain rates is used so that the inequalities

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

1.19

$$\dot{\epsilon}_1 \geq \dot{\epsilon}_2 \geq \dot{\epsilon}_3$$

1.20

are satisfied. Therefore, the minor, intermediate and major stresses and strain-rates are always singled out. Stresses are considered to be effective. It will be assumed in the following that total strains in deformed material are equal to the sum of elastic and plastic strains

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p$$

1.21

where the indexes "e" and "p" denote "elastic" and "plastic", respectively. For elastic strains the generalized Hooke's law is adopted

$$\dot{\epsilon}_{ij}^e = A_{ijkl} \dot{\sigma}_{kl}$$

1.22

where tensor A_{ijkl} may depend on stress state. For elastically isotropic materials this equation may be written, in terms of principal stress and strains, as

$$\dot{\epsilon}_i^e = \frac{1}{E} v_{ij} \dot{\sigma}_j, \quad v_{ij} = \begin{cases} 1 & \text{if } i=j \\ -\nu & \text{if } i \neq j \end{cases} \quad 1.23$$

By using the mass conservation law: $\dot{(\rho V)} = 0$ one obtains

$$\dot{\rho} = \rho \dot{v} \quad 1.24$$

where $\dot{v} = \dot{\epsilon}_{ii}$ is the volumetric strain-rate and ρ denotes the (dry) density of material. Owing to assumption (1.21), the volumetric strain-rate can be divided into two parts, i.e., $\dot{v} = \dot{v}^e + \dot{v}^p$. Hence, it follows that the mass conservation law can be also expressed by the equations

$$\dot{\rho}^e = \rho \dot{v}^e, \quad \dot{\rho}^p = \rho \dot{v}^p \quad 1.25a,b$$

The changes of material density due to plastic deformation may be described by the "plastic density", ρ^p , defined as

$$\rho^p = \rho_0 + \int \dot{\rho}^p dt \quad 1.26$$

where ρ_0 is the initial density.

The direction of the vector $\dot{\epsilon}^p$ is the principal plastic strain-rate space can be described by the ratios

$$\phi_i = \frac{\dot{\epsilon}_i^p}{\dot{\epsilon}_1^p} \quad 1.27$$

where $\phi_1 \equiv 1$. It will be said that these ratios determine the "mode" of plastic deformation. In Section 2.3 it will be proved that for granular materials it must be $\dot{\epsilon}_1^p > 0$ under any plastic deformation process. Therefore, ratios ϕ_i are always determined. By means of these ratios the plastic volumetric strain-rate can be expressed as

$$\dot{\gamma}^p = (1 + \phi_2 + \phi_3) \dot{\epsilon}_1^p$$

1.28

To describe the state of stress the following invariants will be used

$$q = \sigma_1 - \sigma_3, \quad r = \frac{q}{\sigma_3}, \quad b = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}$$

1.29a,b,c

where, by definition, the limits for the stress invariant b are

$$0 \leq b \leq 1$$

1.30

CHAPTER TWO

LOCALLY HOMOGENEOUS MATERIALS

2.1 Amonton's Friction Law

Figure 3a shows a rigid body which moves in frictional contact with the curvilinear surface. The body is subjected to forces F_1 and F_3 and moves in the direction denoted " v^+ ". The slide surface is described by a continuous function $x_3 = x_3(x_1)$. Hence (index "s" refers to the case of the frictional system)

$$\psi^S = n^S(x_3) \quad 2.1$$

where $\psi^S = \dot{x}_3/\dot{x}_1$ describes the "mode" of displacement (i.e. the direction of sliding) and n^S is assumed to be a monotonic function. It is considered that the body slides in agreement with Amonton's friction law

$$T = \mu N \quad 2.2$$

where T and N are the forces tangential and normal to the slide surface (Figure 3c) and μ is a constant coefficient of friction. Hence, it can be found that the limits for ψ^S are

$$-\frac{1}{\mu} < \psi^S < \mu \quad 2.3$$

The equation for the rate of energy dissipation can be expressed as

$$\dot{D}^S = k^S(\psi^S) F_3 \dot{x}_1 \quad 2.4$$

where k^S is a function including the friction coefficient.

There are some important conclusions to be noticed:

Figure 3: Illustration of Amont's Friction Law

- i) Eq.(2.1) expresses the fact that the mode of displacement depends on the geometry of the frictional system. This geometry is measured by the slope of slide surface and is uniquely determined by a coordinate, say, x_3 . The geometry of the system does not depend on the magnitudes of applied forces. Therefore the mode of displacement does not depend on these forces either,
- ii) the rate of energy dissipation depends on the mode of displacement and it can be expressed as being proportional to one of the two, say F_3 , forces,
- iii) as indicated in Figure 3b, the force ratio F_1/F_3 changes with the change of the geometry of the frictional system,
- iv) the frictional system presented in Figure 3a exhibits a structural anisotropy which is understood as the change in the slide condition under the reverse displacement (the reverse displacement is denoted " v^- " in Figure 3a). The slide condition can be expressed as a relation between the ratio F_1/F_3 and x_3 ; see Eq.(2.5). Such an anisotropy is named "structural," in order to emphasize the fact that this anisotropy is related to the geometry of the frictional system and not to the properties of the surfaces (the coefficient of friction is assumed to be independent of the direction of sliding).

These conclusions will be considered in the following sections where an analogy between the behaviour of the frictional system and the behaviour of plastically deformed granular material will be postulated.

Combining Eqs.(2.1) and (2.4) one can find the relations

$$f^S(F_1, F_3) = \frac{F_1}{F_3} - m^S(x_3) = 0 \quad 2.5$$

$$\psi^S = \psi^S\left(\frac{F_1}{F_3}\right) \quad 2.6$$

where $m^S(x_3)$ is a function including the coefficient of friction.

These equations may be called a "slide condition" and a "sliding rule", respectively. It may be noticed that these concepts are entirely analogical to the basic concepts of the incremental theories of plasticity, i.e., to the yield condition and the plastic flow rule, respectively. The Figure 4 presents the geometrical interpretation of Eqs. (2.5) and (2.6). In terms of incremental theories of plasticity the effects of hardening and softening are, illustrated in Figures 4a and 4b, respectively. The displacement vectors are denoted " $\dot{x}$ " in these figures and one may observe that the directions of these vectors are constant along a certain line $f^S(F_1, F_3) = 0$. The parameter of function f^S , i.e. x_3 , satisfies the role of a hardening/softening parameter. The inherent property of the two bodies in frictional sliding may be expressed, in the local coordinates system (Figure 3c), by the

inequality

$$F'_3 = N > 0 \quad 2.7$$

Finally, it should be emphasized that the discussion presented in this section would be meaningless if a flat slide surface is considered or, if the slide condition and the sliding rule are presented in the local coordinates system. In these cases the concept of the geometry of the

Figure 4: Hardening and Softening Effects in the Case of the Frictional System

frictional system is trivial and this corresponds to the ideal plastic flow of continuum materials (no effects of hardening/softening can be observed).

2.2 Description of Material

As pointed out in Section 1.1 it is of prime importance to define the material for which a theory is to be formulated. In this chapter the ideal granular material is assumed to be continuous (as discussed, e.g., by Freudenthal and Geiringer, 1958) and be described by the following assumptions:

- A1: the behaviour of material is not sensitive to time-scale changes,
- A2: the behaviour of material is not sensitive to temperature changes,
- A3: elastic and plastic strains are additive and non-coupled,
- A4: material is isotropic with respect to its plastic properties (St. Venan't hypothesis holds),
- A5: material is locally homogeneous,
- A6: frictional slip at grain boundaries is regarded as the mechanism for plastic deformation.

Assumptions A1 to A4 are common to the classical theories of plastic flow (e.g. Hill, 1950). In Section 1.4 these assumptions have been included in points "i" to "iv". Assumption A5 will be clarified below. Since the phenomenological approach is followed in this study, assumption A6 could be omitted. It has been included, however, in order to justify the use of a frictional analogy. Therefore, the ideal granular material investigated in this chapter is distinguished from other ideal

materials (which are investigated in the framework of the incremental theories of plasticity) by assumption A6 or, as a consequence of this assumption, by the use of frictional analogy.

It seems to be useful to clarify assumption A5. The local homogeneity of a material with respect to plastic deformation is understood herein in the way that: 'constitutive equations describing plastic behaviour of a representative, macroscopically homogeneous element of material do not include any state parameter, say K , which could reflect the variable advancement of some local nonhomogeneities of the element. However, the material remains locally homogeneous if there do exist some local nonhomogeneities but the parameter K does not change during deformation. Then $\dot{K} = 0$ and K may be included in the constitutive equations as a material constant'. One may expect that any real material is locally nonhomogeneous and "...the relations defining the behaviour of the continuum represent statistical averages of actual behaviour over the locally inhomogeneous (in this study: 'nonhomogeneous') and anisotropic elements of the real materials" (Freudenthal and Geiringer, 1958; p.232). However, in agreement with the above definition of local homogeneity we are interested in local nonhomogeneities of material only if the advancement of such nonhomogeneities changes or, in other words, if the state parameters (the properties of material) related to these nonhomogeneities change during deformation. The density of material will be considered to be a state parameter which do not describe local nonhomogeneities of material.

For the sake of clarity the elastic strains are neglected in the present chapter. Therefore, plastic strains are assumed to be equal to total strains. The elastic strains can be taken into account in the same way as presented in Chapter 3. It follows that the plastic density (Eq.1.26) is equal to the total (dry) density of material. As an intuitive assumption resulting from the definition of the ideal granular material it is considered that the internal structure of such a material is uniquely described by its density. Therefore, in this part of the study the density of material will be considered as only state parameter with respect to plastic behaviour. Hence, as discussed above, since no local nonhomogeneities influence the plastic deformations, the ideal material investigated next is referred to as a locally homogeneous medium.

It is also to be noticed that the deformation process is considered to be isothermal and slow enough to neglect the increments of kinetic energy.

2.3 Basic Inequalities

Owing to the assumption on the coincidence of the principal directions of tensors σ_{ij} and $\dot{\epsilon}_{ij}^p$ the density of the rate of energy dissipation in a plastic deformation process can be expressed as

$$\dot{D} = \sigma_i \dot{\epsilon}_i^p = \sigma_1 \dot{\epsilon}_1^p + \sigma_2 \dot{\epsilon}_2^p + \sigma_3 \dot{\epsilon}_3^p \quad 2.8$$

The condition of the thermodynamic irreversibility in the case of plastic deformation of materials leads to the inequality

$$\dot{D} > 0 \quad 2.9$$

which follows the second law of thermodynamics.

Since no tensile stresses can be applied to granular (cohesionless) materials, the considered stress states have to be limited by the dynamic restriction

$$\sigma_3 \geq 0 \quad 2.10$$

which is analogous to the inequality (2.7) applicable to the frictional system. Owing to this restriction one may suggest that the natural state of a granular material is defined by its internal structure and the condition

$$\sigma_3 = 0 \quad 2.11$$

As a consequence all the σ_{ij} are also equal to zero.

Owing to the inequalities (1.19) and (1.20) one concludes from (2.8), (2.9) and (2.10) that the kinematic restriction

$$\dot{\epsilon}_1^p \geq 0 \quad 2.12$$

must hold under any plastic deformation process. Therefore, the condition of thermodynamic irreversibility and the dynamic restriction (2.10) impose a new limitation on the properties of granular materials. Note, that the inequality (2.12) is the required proof if the ratios ϕ_1 , defined in (1.27), are to be used for the description of the mode of plastic deformation.

Owing to dynamic restriction (2.10) the concept of the loading history of a granular material has to be a priori subjected to the restriction which can be explained as follows. If a granular material is loaded (including unloading and reloading) in the way that the condition $\sigma_3 = 0$ is never reached it will be considered that this is always the same material in the sense that it has a certain, determined (not

necessarily known) loading history. If $\sigma_3 = 0$ is reached at a certain moment the loading history does not influence any properties of the granular material.

It should be noticed that the inequalities presented in this section remain valid for any, ideal or real, granular material.

2.4 Axisymmetric Compression

For axisymmetric compression loading conditions ($\sigma_1 > \sigma_2 = \sigma_3$, $\dot{\epsilon}_1^p > \dot{\epsilon}_2^p = \dot{\epsilon}_3^p$) the following constitutive relations are postulated:

Postulate 1^C (P1^C): During plastic deformation, the density of the rate of energy dissipation is proportional to a material function $k(\psi^C)$, the minor principal stress σ_3 and the axial plastic strain-rate $\dot{\epsilon}_1^p$. Therefore

$$\dot{D} = k(\psi^C) \sigma_3 \dot{\epsilon}_1^p \quad 2.13$$

where ratio $\psi^C \equiv \dot{v}^p / \dot{\epsilon}_1^p (= 1 + 2\phi_3)$ describes the mode of plastic deformation in the case of axisymmetric compression conditions.

These conditions are indicated by index "c".

Postulate 2^C (P2^C): The mode of plastic deformation is determined by the current mass density of material. Therefore

$$\psi^C = n(\rho) \quad 2.14$$

where $n(\rho)$ is assumed to be a monotonic, material function.

Postulate $P1^C$ and equation (2.8) lead to the relation

$$r = k (\psi^C) \quad 2.15$$

where stress ratio r is defined in Eq. (1.29b). Using postulate $P2^C$, one also obtains from Eq. (2.15) that

$$f(\sigma_1, \sigma_3, \rho) = r - m(\rho) = 0 \quad 2.16$$

where $m(\rho) \equiv k [n(\rho)] - n(\rho)$.

Equation (2.16) holds if postulates $P1$ and $P2$ are satisfied simultaneously. Hence it determines the condition for the occurrence of plastic deformation. In terms of incremental theories of plasticity this equation expresses a yield condition. The mass density of material, ρ , is a hardening-softening parameter related, by (1.24) and (1.26), to the volumetric strain and initial density.

Equation (2.15) expresses a plastic flow rule since it relates the mode of plastic deformation to the state of stress.

Postulates $P1^C$ and $P2^C$ have been introduced on the basis of the frictional analogy which is noticeable from the comparison of equations (2.13, 2.14) and equations (2.4, 2.1), respectively. Either in the case of the frictional system or in the case of the plastic behaviour of the ideal granular material the rate of energy dissipation depends on the current kinematics, that is, on the mode of displacement or on the mode of plastic deformation, respectively. The current kinematics is determined by the geometry of the frictional system (described by a function of x_3) in the first case; and by the geometry, or the internal structure, of the ideal granular material (described by material density ρ) in the second case.

In postulate P2^C the assumption about the monotonic character of function n has been included. Such an assumption cannot be concluded from the frictional analogy* and it has been introduced intuitively. In physical terms this assumption states that the mode of plastic deformation is uniquely determined by the density of material and different material densities determine different modes of plastic deformation.

Using the mass conservation law (1.24), an incremental stress-strain relationship may be found from equations (2.14) and (2.16) and it is given by the set of relations

$$\dot{\tau} = h^C(\rho) \dot{\epsilon}_1^P \quad 2.17$$

$$\dot{\epsilon}_3^P = \frac{1}{2} [n(\rho) - 1] \dot{\epsilon}_1^P \quad 2.18$$

$$\dot{\rho} = \rho n(\rho) \dot{\epsilon}_1 \quad 2.19$$

where $h^C(\rho) = \rho m'(\rho) n(\rho)$ is a hardening-softening function and "prime" denotes a derivative. In order to obtain a stress-strain relationship these equations may be integrated (except for $n=0$) when material functions n and m are known.

Two of the three material functions (k , n , m) included in the constitutive equations are independent. Regarding that $r > 0$ and using the basic inequalities (section 2.3), the following limits for material

*It is obvious that the function n^S in Eq.(2.1) need not be monotonic or, in other words, the slide surface from Figure 2a need not be smoothly monotonic.

functions result from equations (2.13) and (2.16)

$$k = m + n > 0 \quad 2.20$$

$$m = k - n > 0 \quad 2.21$$

whence, it also follows that

$$-m < n < k \quad 2.22$$

Since $n = \dot{\psi}^C$, the inequalities (2.22) determine the limits for the mode of plastic deformation and this is similar to the case of frictional system where inequalities (2.3) determine the limits for the mode of displacement.

Using (1.24) and (2.12) it also follows from postulate $P2^C$ that

$$n > 0 \text{ if } \dot{\rho} > 0 \quad (\text{contraction}) \quad 2.23a$$

$$n < 0 \text{ if } \dot{\rho} < 0 \quad (\text{dilatation}) \quad 2.23b$$

$$n = 0 \text{ if } \dot{\rho} = 0 \quad (\text{no volume changes}) \quad 2.23c$$

It may be noticed that the plastic flow rule expressed by Eq. (2.15) reduces to Rowe's flow rule (Eq.1.1) if

$$k_1 (\dot{\psi}^C) = -K\dot{\psi}^C + K - 1, \text{ where } K = \tan^2 \left(\frac{\pi}{4} + \frac{\phi_f}{2} \right)$$

Finally, it should be emphasized that, due to the omission of elastic strains, the density of material ρ instead of plastic density ρ^P has been introduced in the constitutive equations for plastic deformation. Hence, in the derivations of this section the equation of mass conservation (1.24) instead of equation (1.25b) has been used. If elastic strains are not neglected a certain approximation has to be introduced, as will be discussed in Section 3.2.

2.5 Limitations for the Derivatives of Material Functions

It follows from postulate $P2^C$ that the first derivative of material function n can be either positive or negative. Consider the case

$$n'(\rho) < 0 \quad 2.24$$

which is illustrated in Figure 5a. By the mass conservation equation the ratio ψ^C can be expressed as

$$\psi^C = \frac{1}{\rho} \frac{\dot{\rho}}{\dot{\epsilon}_1^p} \quad 2.25$$

and, from inequality (2.12), it follows that as ratio ψ^C is positive or negative the density of material increases or decreases, respectively. This is illustrated in Figure 5a. It also follows that inequality (2.24) implies the existence of the final density, denoted ρ_f in Figure 5a. By using (1.24), postulate $P2^C$ can be written in the form of a differential equation

$$\frac{d\rho}{\rho n(\rho)} = d\epsilon_1^p \quad 2.26$$

and one concludes thus that the concept of the final density is a consequence of the singularity ($n=0$) appearing in this equation. If inequality (2.24) is accepted it results that under continuous deformation the density of material approaches ρ_f which is defined by

$$\rho_f = n^{-1}(0) \quad 2.27$$

The meaning of the final density is similar to the meaning of the "critical density" which has been introduced, on the basis of experimental results, in Critical State Models (e.g. Schofield and Wroth, 1968). However, the final density does not depend on stress states whereas the critical density does. To emphasize this difference, the term "final"

Figure 5: Alternative Assumptions for the Derivative
of the Function n

instead of "critical" will be used henceforth.)

Plastic deformation under the condition $\rho = \rho_f$ can be described by means of equations (2.15) and (2.16) if $\rho = \rho_f$ and $\psi^C = 0$ is respectively inserted. Under the condition $\rho \neq \rho_f$ deformed material may be in a loose or in a dense state, where the terms "loose" and "dense" denote "contracting" and "dilating" material, as indicated in Figure 5a. The sign of functions n for loose and dense materials is given in (2.23a) and (2.23b), respectively. Ideal plastic flow takes place if density of material is equal to the final density and then (2.23c) is satisfied.

Figure 5b illustrates the alternative case in which $n'(\rho)$ is assumed to be positive. This assumption implies both unbounded density changes as well as the extremely different behaviour of the two materials with nearly the same initial densities (in Figure 5 the initial conditions are indicated by black dots). Such conclusions do not seem to be logical and therefore the condition (2.24) is accepted.

By postulate $P2^C$, the function k included in postulate $P1^C$ can be expressed as a function of material density: $k = k[n(\rho)]$. It seems to be justified by intuition that the density of the rate of energy dissipation increases or decreases with the increase or decrease of material density, respectively (σ_3 and $\dot{\epsilon}_1^D$ are taken constants in Eq. 2.13).

Therefore

$$k'(\rho) > 0 \quad 2.28$$

and since

$$k'(\rho) = k'(\psi^C)n'(\rho) \quad , \quad m'(\rho) = k'(\rho) - n'(\rho)$$

one also concludes, using (2.24), that

$$k'(\psi^C) < 0, \quad m'(\rho) > 0 \quad 2.29a,b$$

Therefore, the sign of hardening-softening function h^C , included in Eq.(2.19), can be determined and it follows that the deformation of loose and dense materials results in hardening ($\dot{\epsilon} > 0$ since $h^C > 0$) and softening ($\dot{\epsilon} < 0$ since $h^C < 0$), respectively.

2.6 Postulate P3

It is believed that the use of frictional analogy should be limited to axisymmetric loading of a continuous, isotropic material. That is for under such loading conditions only the plastic flow of the material is determined by four variables (four invariants of σ_{ij} and $\dot{\epsilon}_{ij}^P$, e.g. $\sigma_1, \sigma_3, \dot{\epsilon}_1^P, \dot{\epsilon}_3^P$) and this number matches the number of variables involved in the case of the frictional system ($F_1, F_3, \dot{x}_1, \dot{x}_3$). The frictional analogy, however, cannot be directly used to describe both axisymmetric compression and the reverse loading condition, i.e., axisymmetric extension: $\sigma_1 = \sigma_2 > \sigma_3$. The reason is that in the case of the frictional system there reveals the structural anisotropy under the application of the reverse displacement (see section 2.1). In order to obtain constitutive equations for axisymmetric extension a new postulate is introduced:

P3: The yield condition derived for axisymmetric compression does not change when the reverse loading is applied. This is illustrated in Figure 6.

Figure 6: Illustration of Postulate P3

It is found that the constitutive relations for axisymmetric extension, satisfying this postulate and being adequate for the constitutive relations postulated for axisymmetric compression (Eqs. 2.13 and 2.14), are

$$p_1^{ex}: \dot{D} = k (\psi^{ex}) \sigma_3 2\dot{\epsilon}_1^p \quad 2.30$$

$$p_2^{ex}: \psi^{ex} \equiv \frac{\dot{\gamma}^p}{2\dot{\epsilon}_1^p} = n(\rho) \quad 2.31$$

It should be emphasized that there are no theoretical suggestions to support postulate P3 unless it is assumed that the mechanism of deformation is due to the frictional macroslidings along the classical Coulomb's planes. Such an assumption, however, may not be consistent with the assumed isotropy of the plastic properties of the ideal granular material (compare also the discussion on the appearance of Lueder's bands in the case of the plastic deformations of metals by Freudenthal and Geininger, 1958).

2.7 A Generalization

In this section a generalization of the relations proposed previously for axisymmetric compression is presented. This generalization is carried out so that the basic postulates for complex stress states are adequate for those proposed for axisymmetric compression. Furthermore, postulate P3 is employed.

2.7.1 State Variables and Fundamental Postulates

Constitutive relations for complex stress states are assumed to be of the general forms

$$G_i(Y_1, \dots, Y_6, \rho) = 0 \quad i=1,2,3 \quad 2.32$$

where the six variables Y_k are the invariants (scalar-valued functions) of σ_{ij} and $\dot{\epsilon}_{ij}^P$, and G_i are time-independent functions. The variables Y_k may also be some combinations of the invariants of σ_{ij} and $\dot{\epsilon}_{ij}^P$. Basic constitutive relations for axisymmetric compression (Eqs. 2.13 and 2.14) express relations between the four variables: $\dot{D}$, σ_3 , $\dot{v}^P$, $\dot{\epsilon}_1^P$. Some physical interpretations of these variables are suggested next.

Energy dissipation is the most fundamental (inherent) property of any plastic deformation process and hence it is accepted that the physical interpretation of $\dot{D}$, as a variable entering constitutive relations for plastic deformation, is established.

Physical meaning of minor principal stress σ_3 seems to be clear, since granular materials can be considered as continuous media only if subjected to $\sigma_3 > 0$. Moreover, the rate of energy dissipation (and also elastic energy) must vanish as σ_3 approaches zero and hence the appearance of σ_3 as a state variable is well justified.

The third variable, $\dot{v}^P$, is proportional to $\dot{\rho}$ and therefore it describes the rate of internal structure changes, as discussed in Section 2.2. It follows that physical interpretation of variable $\dot{v}^P$ is clear.

The fourth variable, $\dot{\epsilon}_1^P$, does not seem to have direct physical interpretation. Nevertheless, it may be convenient to use $\dot{\epsilon}_1^P$ as a variable entering

constitutive relations for the simple physical limitation (2.12) exists. It is to be noticed that suggested interpretations of variables $\dot{D}$, σ_3 and $\dot{v}^P$ hold under any loading conditions

The generalized postulates are assumed to be of the forms similar to those proposed for axisymmetric compression (Eqs. 2.13 and 2.14) and owing to the foregoing interpretations it is postulated that

$$P1: \quad \dot{D} = k(\psi) \sigma_3 \dot{\xi} \quad 2.33$$

$$P2: \quad \psi \equiv \frac{\dot{v}^P}{\dot{\xi}} = n(\rho) \quad 2.34$$

where $\dot{\xi}$ is a function which may depend on invariants of σ_{ij} and $\dot{\epsilon}_{ij}^P$ as well as on density ρ . Therefore, the meanings of variables $\dot{D}$, σ_3 and $\dot{v}^P$ have been maintained in the generalized postulates (as compared with postulates $P1^C$ and $P2^C$) and variable $\dot{\epsilon}_1^P$ has been replaced with function $\dot{\xi}$. From the basic inequalities and condition (2.20) it follows that this function must be positive

$$\dot{\xi} > 0 \quad 2.35$$

As indicated in (2.32) a third equation is required to complete a set of constitutive relations. A careful analysis of postulates $P1$ and $P2$ indicates that the third equation is expected to contain an information on the mode of plastic deformation.

Under axisymmetric compression conditions function $\dot{\xi}$ and ratios ϕ_2 , ϕ_3 , defined in Eqs. (1.27), are given from postulates $P1^C$ and $P2^C$ as

$$\dot{\xi} = \dot{\epsilon}_1^P, \quad \phi_2 = \phi_3 = \frac{1}{2} [n(\rho) - 1] \quad 2.36a,b$$

We note, therefore, that the function ξ and the mode of plastic deformation (described by the ratios ϕ_i) do not depend on the state of stress under axisymmetric compression. In this case there are involved two independent stress invariants by means of which the level (magnitude) of shear stresses and the level of normal stresses can be always determined. Therefore, one concludes that in this case function ξ and ratios ϕ_i depend neither on shear stress nor normal stress levels. One may suppose thus that under general stress states ξ and ϕ_i can depend on a third stress invariant only. This invariant should describe the influence of the intermediate stress σ_2 . Consequently, the postulate for the mode of plastic deformation under complex stress states may be introduced in the general form.

$$\phi_i = g_i (Y_4, \rho) \quad i=2,3 \quad 2.37$$

where Y_4 is a stress invariant describing the influence of σ_2 . As such an invariant the stress invariant b (defined in Eq. 1.29c) will be chosen. Hence, it is postulated that the mode of plastic deformation is determined by the equations

$$P4: \phi_i = g_i (b, \rho) \quad i=2,3 \quad 2.38$$

and one of these equations may be considered as the third constitutive relation.

It follows from Eq. (2.36a) that function ξ depends neither on density ρ nor the three variables $\dot{D}$, σ_3 and $\dot{v}^P$ under axisymmetric compression. In agreement with Eqs. (2.32) and postulate P1 one concludes

thus that this function does not depend on more than three variables. Owing to the choice of the third stress invariant and to condition (2.36a) it is assumed that

$$\dot{\xi} = \dot{\xi}(\dot{\epsilon}_1^p, b, Y_6) \quad 2.39$$

where Y_6 is a sixth, unknown variable since

$$Y_1 = \dot{D}, \quad Y_2 = \sigma_3, \quad Y_3 = \dot{v}^p, \quad Y_4 = b, \quad Y_5 = \dot{\epsilon}_1^p$$

If, for instance, the first of Eqs. (2.38) is chosen as the third constitutive relation then $\dot{\epsilon}_2^p$ (or ϕ_2) is to be considered as the sixth variable. It does not however mean that function $\dot{\xi}$ must depend on all of the three arguments.

It is to be noticed that in the foregoing discussion we have not considered plastic strains as state parameters. This follows the definition of the ideal granular material by which it was concluded (Section 2.2) that the internal structure of such a material, and therefore the material properties, depend on the current density of material only.

2.7.2 Functions g_i and $\dot{\xi}$

In the remainder of this section it is considered that postulate P3 holds. Therefore, the basic constitutive relations for axisymmetric extension (Eqs. 2.30 and 2.31), adequate for generalized postulates P1 and P2, are available and the following conditions for function $\dot{\xi}$ and ratios ϕ_i can be found

$$b = 1: \quad \dot{\xi} = 2\dot{\epsilon}_1^p; \quad \phi_2 = 1, \quad \phi_3 = 2(n-1) \quad 2.40a$$

$$b = 0: \quad \dot{\xi} = \dot{\epsilon}_1^p; \quad \phi_2 = \phi_3 = \frac{1}{2}(n-1) \quad 2.40b$$

where, for convenience, conditions (2.36) have been written again.

Regarding (2.40) one may express postulate P4 as follows

$$\Phi_2 = g_2(b, \rho) = \frac{1}{2} [(3-n)b + n - 1 + (b-b^2)N_2] \quad 2.41a$$

$$\Phi_3 = g_3(b, \rho) = \frac{1}{2} [(3n-3)b + n - 1 + (b-b^2)N_3] \quad 2.41b$$

In these equations N_2 and N_3 are unknown functions of b and ρ . By using (1.28), one may find from postulate P2, as a consequence of Eqs. (2.41), a general form of function ξ

$$\xi = [1 + b + (b-b^2) \frac{N_2 + N_3}{2n}] \dot{\epsilon}_1^p \quad (n \neq 0) \quad 2.42$$

and we note that Eqs. (2.41) and (2.42) have been found assuming that postulates P1 and P2 reduce to the postulates introduced previously for axisymmetric loading conditions. Therefore, postulate P3 has been assumed to hold.

In Sections 2.7.4 and 2.7.5 there will be considered the two simplest, linear in b , forms of function ξ which are

$$\xi(\dot{\epsilon}_1^p, \dot{\epsilon}_2^p, b) = \dot{\epsilon}_1^p + b\dot{\epsilon}_2^p \quad 2.43$$

$$\xi(\dot{\epsilon}_1^p, b) = \dot{\epsilon}_1^p + \dot{\epsilon}_1^p b \quad 2.44$$

Since for $b=1$ there is $\dot{\epsilon}_1^p = \dot{\epsilon}_2^p$, both these forms satisfy conditions (2.40). It can be found that these forms lead to the following relations between functions N_2 and N_3 , respectively,

$$N_3 = (nb-1)N_2 + n(n-3) \quad 2.43a$$

$$N_3 = -N_2 \quad 2.44a$$

2.7.3 A Limitation for Material Function n

Henceforth the minor plastic strain-rate ($\dot{\epsilon}_3^p$) will be assumed to be negative under axisymmetric deformations of l o o s e materials (in the case of d e n s e materials $\dot{\epsilon}_3^p$ is negative "by definition"). This is illustrated in Figure 7 where the modes of plastic deformation excluded by this assumption are indicated in dotted lines. Owing to basic inequality (2.12) and conditions (2.40) it follows that in both cases the above assumption leads to the limitation for material function n

$$n(\rho) < 1 \quad 2.45$$

It also follows that the density of material is limited, so that

$$\rho > n^{-1}(1) \quad 2.46$$

The meaning of the foregoing limitation will be discussed at a later stage. By now, it should be emphasized that we do not suggest the minimum density, $\rho_{\min} = n^{-1}(1)$, be the limit for physically possible densities of granular materials. It is suggested, however, that the deformation of material with density less than the minimum density must result, at least for certain loading conditions, in an unstable behaviour of material. Therefore, the constitutive equations presented subsequently stand if condition (2.45) is satisfied.

The stability condition discussed in this study should not be mistaken with the concept of stability in Drucker's sense (Drucker, 1959). Drucker's stability condition is often accepted as a base for the "associated plasticity" developed for metals. It may be noted that the frictional system from Figure 5b as well as the plastic behaviour of

Figure 7: Illustration of Assumption $\dot{\epsilon}_3^p < 0$ for Axisymmetric Compression and Extension

dense granular materials (as described in the present chapter) are unstable from the viewpoint of Drucker's stability definition.

2.7.4 Constitutive Equations with Coulomb's Yield Condition

Function $\dot{\epsilon}$ is assumed to be of the form (2.43) and therefore postulates P1 and P3 (Eqs. 2.33 and 2.34) are specified. As the third constitutive relation the first of Eqs. (2.38) is chosen and so the basic postulates are

$$\dot{D} = k(\psi) \sigma_3 (1 + b \phi_2) \dot{\epsilon}_1^p \quad 2.47$$

$$\psi \equiv \frac{\dot{v}^p}{(1+b\phi_2)\dot{\epsilon}_1^p} = n(\rho) \quad 2.48$$

$$\phi_2 = g_2(b, \rho) \quad 2.49$$

By using (1.28) it may be found from (2.48) and (2.49) that

$$\phi_3 = g_3(b, \rho) = (nb-1) g_2 + n-1 \quad 2.50$$

and therefore the mode of plastic deformation is determined if function g_2 is known. A yield condition follows from (2.47) and (2.48) and it is of the form

$$f(q, \sigma_3, \rho) = r - m(\rho) = 0 \quad 2.51$$

where function m is defined as for axisymmetric compression (Eq. 2.15).

An incremental stress-strain relationship can be also found and this may be presented in the form

$$\dot{\epsilon}_i^p = \frac{1}{h\sigma_3} [g_1 \lambda_j] \dot{\sigma}_j \quad 2.52$$

where $g_1 \equiv 1$,

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -(r+1),$$

and the hardening-softening function is

$$h(b, \rho) = \rho m'(\rho) n(\rho) [1 + b g_2(b, \rho)]$$

The details of the derivation of the relationship (2.52) are the same as presented in Section 2.7.5.

Equation (2.51) is equivalent to Coulomb's yield condition which, for granular materials, may be expressed as

$$\frac{\sigma_1}{\sigma_3} = \frac{1 + \sin \phi}{1 - \sin \phi}$$

and the angle of internal friction ϕ is uniquely related to function m by the relation

$$m = \frac{2 \sin \phi}{1 - \sin \phi}$$

It seems that an inconsistency appears in the constitutive equations derived in this section since the yield condition does not depend on the intermediate principal stress σ_2 , whereas the mode of plastic deformation does. This is illustrated in Figure 8 where $\dot{\epsilon}^p$ denotes the projections of plastic strain-rate vectors on an octahedral ($\sigma_{ii} = \text{const.}$) plane. It has been previously suggested to limit the use of frictional analogy to the axisymmetric conditions. It should be emphasized, therefore, that the form of $\dot{\epsilon}$ considered above could also be obtained if the use of frictional analogy is extended for complex stress states, that is, if it is postulated (instead of postulate P3) that the equation expressing the yield condition for axisymmetric compression remains unchanged for complex stress states. It seems, however, that there are no suggestions to postulate such a

Figure 8: Illustration of Coulomb's Yield Condition
(Plastic Flow Rule Depends on σ_2)

frictional analogy extension.

It is also to be noticed that functions g_2 and g_3 are of different orders in b and ρ , as follows from (2.50). Consider, for instance, the form of function g_2 linear in b , i.e.,

$$g_2(b, \rho) = \frac{1}{2} [(3-n)b + n-1] \quad (N_2=0) \quad 2.53$$

By (2.50) one also obtains in this case

$$g_3(b, \rho) = \frac{1}{2} [(3n-n^2)b^2 + (n^2-3)b + n-1] \quad 2.54$$

One may note that by using limitation (2.45) and the condition $g_2 > g_3$, which follows (1.20), the following limits for the density of material may be found

$$\rho_{\min} = n^{-1}(1) \quad \text{since } n_{\max} = 1 \quad 2.55a$$

$$\rho_{\max} = n^{-1}(-3) \quad \text{since } n_{\min} = -3 \quad 2.55b$$

It can be shown that for these limits the form of function ξ considered in this section is always positive, as required by inequality (2.35).

Owing to the mentioned inconsistency the constitutive relations presented in this section are not examined in detail. Nevertheless, it may be noticed that these are the simplest of the constitutive relations presented in this study.

2.7.5 The Simplified Theory

In this section it is assumed that function ξ is of the form

$$\xi = (1+b) \dot{\epsilon}_1^p \quad 2.44$$

which, by inequalities (1.30) and (2.12), satisfies always condition (2.35). Assuming the first of Eqs. (2.38) as the third constitutive relation one obtains the basic postulates in the forms

$$\dot{D} = k(\psi) \sigma_3 (1+b) \dot{\epsilon}_1^p \quad 2.56$$

$$\psi \equiv \frac{\dot{V}^p}{(1+b)\dot{\epsilon}_1^p} = n(\rho) \quad 2.57$$

$$\phi_2 = g_2(b, \rho) \quad 2.58$$

From (2.57) and (2.58) one also finds that

$$\phi_3 = g_3(b, \rho) = -g_2 + nb + n-1 \quad 2.59$$

As a first approximation, the following equations will be derived under the assumption $N_2 = 0$, i.e., function g_2 is approximated to be linear in b . By (2.44a) also $N_3 = 0$ and the mode of plastic deformation is therefore determined by the equations

$$\phi_2 = g_2(b, \rho) = \frac{1}{2} [(3-n)b + n-1] \quad 2.60a$$

$$\phi_3 = g_3(b, \rho) = \frac{1}{2} [(3n-3)b + n-1] \quad 2.60b$$

According to the limitation (2.45) it follows that $\phi_3 < 0$. Therefore, by the basic inequality (2.12), one also obtains

$$\dot{\epsilon}_3^p < 0 \quad 2.61$$

By (2.8), Eq.(2.56) can be presented in the form

$$q(1+b\phi_2) \dot{\epsilon}_1^P + \sigma_3 \dot{v}^P = k(\Psi) \sigma_3 (1+b) \dot{\epsilon}_1^P$$

Hence, by using (2.57), the yield condition can be found

$$f(q, \sigma_3, b, \rho) = r \frac{1+bg_2(b, \rho)}{1+b} - m(\rho) = 0 \quad 2.62$$

Since $r > 0$ it follows, according to (2.21), that

$$1 + bg_2 > 0 \quad 2.63$$

and, if function g_2 given in (2.60a) is considered, this leads to the upper limitation for the density of material

$$\rho_{\max} = n^{-1} \left(\frac{12 - 7\sqrt{2}}{4 - 3\sqrt{2}} \right) \text{ since } n_{\min} = \frac{12 - 7\sqrt{2}}{4 - 3\sqrt{2}} \quad 2.64a$$

The lower limitation, resulting from (2.45), is

$$\rho_{\min} = n^{-1}(1) \text{ since } n_{\max} = 1 \quad 2.64b$$

It can be shown that for these limits the inequalities (1.20) are always satisfied. The limit (2.64a) describes the case in which stress ratio r goes to infinity (at $b = \sqrt{2} - 1$), according to yield condition (2.62). This also indicates a limit for the mode of plastic deformation determined by Eqs. (2.60).

Inserting the first of Eqs. (2.60) into (2.62) one obtains yield condition in the form

$$f(q, \sigma_3, b, \rho) = r\{1+b_1[n(\rho)^3-3]\} - m(\rho) = 0 \quad 2.65$$

where

$$b_1 = \frac{1}{2} (b-b^2)/(1+b)$$

If $n=3$ this equation reduces to Coulomb's yield condition but for the limit (2.64b) such a case is ruled out. Note, that even if the limitation

$n < 1$ is disregarded, Eq. (2.65) may not reduce to Coulomb's equation for condition $g_2 > g_3$ results, in limitation $n < \frac{3}{2}$.

By differentiation of the yield condition and after some rearrangements the incremental relationship may be obtained ("incremental" - since time increments dt can be eliminated)

$$\frac{1}{\sigma_3} \lambda_i \dot{\sigma}_i = [m'(\rho) - \Gamma b_1 n'(\rho)] \dot{\rho}$$

Hence, using relation (1.24) and postulate (2.57), the following equation is obtained:

$$\frac{1}{\sigma_3} \lambda_i \dot{\sigma}_i = h \dot{\epsilon}_1^p \quad 2.66$$

where

$$\lambda_1 = 1 + (b_1 - b b_2)(n-3)$$

$$\lambda_2 = b_2(n-3)$$

$$\lambda_3 = b_2(b-1)(n-3) - (\Gamma+1) [1+b_1(n-3)]$$

h is hardening-softening function given as

$$h(q, \sigma_3, b, \rho) = \rho n(\rho) [m'(\rho) - \Gamma b_1 n'(\rho)] (1+b)$$

and

$$b_2 = \frac{1}{2} (1-2b-b^2)/(1+2b+b^2).$$

From the conditions for the derivatives of function m and n , which have been introduced previously (see Section 2.5), one finds that function h is positive for loose materials ($n > 0$) and negative for dense materials ($n < 0$).

By (2.60) and (2.66) one obtains an incremental stress-strain relationship which may be presented in the form

$$\dot{\epsilon}_i^p = \frac{1}{h\sigma_3} [g_i \lambda_j] \dot{\sigma}_j \quad (g_1 \equiv 1) \quad 2.67$$

Consider the plane strain loading conditions. Owing to inequalities (2.12) and (2.61) one concludes that the direction in which the zero-strain condition can be imposed is the one of the intermediate principal strain. Then $\dot{\epsilon}_2^p = g_2 = 0$ and it leads to

$$b = \frac{1-n}{3-n} \quad \text{or} \quad \sigma_2 = \frac{1-n}{3-n} (\sigma_1 - \sigma_3) + \sigma_3 \quad 2.68$$

This reduces under perfectly plastic flow conditions ($\dot{\rho}=n=0$) to

$$b = \frac{1}{3} \quad \text{or} \quad \sigma_2 = \frac{1}{3} (\sigma_1 + 2\sigma_3) \quad 2.69$$

It may be concluded from (2.68) that the plane deformation of the very loose materials, having densities $\rho < n^{-1}(1)$, cannot be described by proposed equations. Such a case, however, has been previously excluded by limitation (2.45).

The mentioned conclusion seems to be interesting. Starting from limitation (2.45) we have come, through basic inequality (2.12) and the approximations of functions g_i with linear forms, to the same limitation under the plane deformation loading.

It may be noted that limitation (2.45) must hold for the oedometric (the uni-axial strain) condition, regardless of the forms of function g_i . In this case there are external constraints $\dot{\epsilon}_2^p = \dot{\epsilon}_3^p = 0$ whereas $n > 1$ results in $\dot{\epsilon}_2^p = \dot{\epsilon}_3^p > 0$, see Eqs. (2.41) for $b = 0$. Since no tensile stresses can be applied in granular materials, it may be observed that, under these conditions the deformation of material would have to result

in a dynamic jump.

Constitutive equations derived in this section are illustrated in Figures 9-12. The yield condition, if presented in the principal stress space, is given with a set of curvilinear surfaces encircling the Coulomb's failure surfaces. The sections of such surfaces with the axisymmetric and octahedral planes are presented in Figures 9 and 10, respectively, where the directions of plastic strain-rate vectors are also indicated. Coulomb's angles of internal friction are given for axisymmetric compression in Figure 10. Material functions m and n , by which the curves of Figure 10 have been obtained, are indicated in Figure 11. These functions were found from the experiments (Szymanski, 1976) the results of which will be presented in Chapter 3.

Figures 11 and 12 present the stress-strain relationships obtained by the numerical integration of Eqs. (2.67). These relationships indicate the influence of stress parameter b and initial density ρ_0 on stress-strain behaviour. Elastic strains have been regarded in the way as indicated in Chapter 3. The stress-strain relationships from Figure 11 have been obtained for the loading conditions which are given as follows

$$\dot{\epsilon}_1 = \dot{\epsilon}_1(t) > 0; \quad \sigma_3 = \text{const.}; \quad b = \text{const.}$$

In Figure 12 the influence of material density on the stress-strain behaviour is indicated for the conventional axisymmetric compression loading conditions.

As it is shown in the next section the constitutive relations presented in the foregoing can be considered as a special case of some

Figure 9: Illustration of the Yield Function and the Flow Rule for Axisymmetric Compression

Figure 10: Illustration of the Yield Function and the Flow Rule; Representation on an Octahedral Plane

Figure 11: Influence of the Stress Invariant b on Stress-Strain Behaviour

Figure 12: Influence of Initial Density on Stress-Strain Behaviour

more general relations. For this reason the constitutive relations presented in this section are called the "simplified theory". It is to be noticed that in this theory two material functions are included. These functions can be determined from the conventional axisymmetric compression tests, as shown in the example presented in Chapter 3.

2.8 Formulation of a General Theory

In the ensuing the postulate P3 will be disregarded.

The basic postulates proposed in the following are similar to, although more general than those used for the formulation of the simplified theory. Perhaps, they are also of more physically understandable forms.

We shall consider plastic deformation of the ideal granular material under the condition $\dot{\rho} \neq 0$. The fundamental material functions will be denoted by

$$\alpha = \alpha(\rho) \quad ; \quad \beta_a = \beta_a(\rho) \quad ; \quad \beta_b = \beta_b(\rho)$$

The dependence of material functions on the density of material only is consistent, as discussed previously, with the definition of the ideal granular material.

2.8.1 Constitutive Equations

As indicated in Eqs. (2.32), three constitutive relations are required to describe plastic flow of a material under general stress states. These relations are established by the two postulates set forward below.

The equation for energy dissipation:

Consider the constitutive relation

$$\dot{D} = \dot{\alpha} \sigma_3 \quad 2.70$$

where $\dot{\alpha}$ is a function of the density of material. This is the simplest relation which accounts for the fact that the rate of energy dissipation must vanish if σ_3 is zero.

By the mass conservation law this relation can also be presented in the form

$$\dot{D} = \bar{\alpha} \sigma_3 \dot{v}^p \quad 2.71$$

where

$$\bar{\alpha} \equiv \rho \alpha'(\rho)$$

It follows that the postulate (2.70) also expresses the simplest relation between the three invariants having the physical interpretations given in Section 2.7.

The equations for the mode of plastic deformation: Consider constitutive relations

$$\phi_i = g_i(b, \rho) \quad i = 2, 3 \quad 2.72$$

Although these equations were introduced and discussed previously (Section 2.7) we will repeat for clarity the reasoning for such a postulate. It is supposed that the mode of plastic deformation of the ideal granular material may depend on its internal structure and the state of stress. Hence: i) since the material is isotropic and locally homogeneous it is assumed that its internal structure can be described by the mass density only, ii) since the material is isotropic the mode of plastic deformation can depend on (three) stress invariants. By the use of

frictional analogy we postulate that the mode of plastic deformation of the ideal granular material does not depend on the magnitudes of shear and normal stresses. These magnitudes can be described by two invariants of stress tensor. Therefore, the mode of plastic deformation can depend only on one stress invariant which describes the influence of σ_2 , i.e., the position of a stress plane containing the hydrostatic axis ($\sigma_1 = \sigma_2 = \sigma_3$) in the principal stress space.

A certain reasoning for the choice of b as the third stress invariant is presented in Section 2.9.

Therefore, in the postulate for the energy dissipation no frictional analogy is invoked. Also, in the present formulation of the theory, the concept of function ξ is not introduced.

The constitutive equations will be derived assuming, as a first approximation, that functions g_i are linear in b . Regarding the conditions resulting from the isotropy of material

$$\phi_2 = \phi_3 \text{ if } b = 0; \quad \phi_2 = 1 \text{ if } b = 1 \quad 2.73 \text{ a, b}$$

one obtains the relations for the mode of plastic deformation in the forms

$$\phi_2 = g_2(b, \rho) = (1 - \beta_a) b + \beta_a \quad 2.74a$$

$$\phi_3 = g_3(b, \rho) = \beta_b b + \beta_a \quad 2.74b$$

which, in general, are different from those of the simplified theory.

Inserting Eqs. (2.74) into (1.28) one may obtain the relation

$$\frac{\dot{\gamma}^p}{[z(\rho) + b] \dot{\epsilon}_1^p} = n(\rho) \quad 2.75$$

where $n \equiv 1 - \beta_a + \beta_b$ and $z \equiv (1+2\beta_a)/n$.

By using (2.75) the equation for energy dissipation can be presented in the form

$$\dot{D} = \sigma_3 \bar{\alpha} n (z+b) \dot{\epsilon}_1^p = \sigma_3 (m+n)(z+b) \dot{\epsilon}_1^p \quad 2.76$$

where $m \equiv n (\bar{\alpha}-1)$.

In the case $z(\rho)=1$ only both Eq.(2.76) and (2.75) reduce, respectively, to the postulates (2.56) and (2.57) which set up the simplified theory.

The expression $(z+b)\dot{\epsilon}_1^p$ satisfies the role similar to that of function $\bar{\epsilon}$ Eq. (2.39). This expression, however, depends on the density of material.

From Eqs. (2.75), (2.76) and (2.74a) a yield condition follows and this may be presented in the form

$$f(q, \sigma_3, b, \rho) = \Gamma p_1 (b, \rho) - m(\rho) = 0 \quad 2.77$$

where

$$p_1 = \frac{1}{2} \frac{2(1+b) + (b-b^2)(nz-3)}{z+b}$$

In the same way as outlined in the previous section the incremental stress-strain relationship can be found to be of the form

$$\dot{\epsilon}_i^p = \frac{1}{h\sigma_3} [g_i \lambda_j] \dot{\sigma}_j \quad 2.78$$

where $g_1 \equiv 1$

$$\lambda_1 = p_1 \frac{z+2b}{z+b} - p_2 b$$

$$\lambda_2 = -p_1 \frac{1}{z+b} + p_2$$

$$\lambda_3 = p_1 \frac{1-b-(\Gamma+1)(z+b)}{z+b} + p_2(b-1)$$

$$\text{and } p_2 = \frac{1}{2} \frac{2+(1-2b)(nz-3)}{z+b}$$

The hardening-softening function is given as

$$h = \frac{1}{2} \rho n [2(z+b)m' - r(b-b^2)(zn' + nz') + 2rp_1 z']$$

where "primes" denote derivatives with respect to ρ .

The stress-strain relationships resulting from Eqs. (2.78) are qualitatively similar to those plotted in Figure 11. Hence, the graphical presentation of Eqs. (2.78) is neglected here. In Chapter 3 there will be plotted some more general stress-strain relationships which include those resulting from Eqs. (2.78) as a particular case.

2.8.2 Limitations on Material Functions

The yield condition and stress-strain relationship derived in this section are expressed in terms of material functions m , n and z . These are the functions of material density and it will be assumed that they are monotonic functions.

By regarding the basic inequalities (2.9, 2.10, 2.12) and the range of allowable values of b (1.30) it follows from the equation of energy dissipation (2.76) that one of the two alternative conditions must hold

$$z > 0 \text{ if } \bar{\alpha} n > 0 \quad \text{or} \quad z < -1 \text{ if } \bar{\alpha} n < 0$$

Inserting $b=0$ and $b=1$ into the yield condition (2.77) one may present the comparison of stress ratios r under axisymmetric compression and extension as

$$\frac{r^{ex}}{r^c} = \frac{z+1}{2z}$$

2.79

which is plotted in Figure 13. From this figure it follows that if the second of the mentioned conditions (i.e., $z < -1$) is assumed then the present formulation of the theory does not include the simplified theory as a particular case. In other words, postulate P3, which leads to $z=1$, could not hold for any ρ . In the remaining part of the study the investigations will be carried out only for the first of the mentioned conditions, i.e.,

$$z > 0 \quad 2.80$$

It will be discussed in Section 2.11 that there are some available experimental evidences which support such a choice. The influence of the values of function z on the yield condition (Eq. 2.77) is presented in Figure 14, where yield stresses are given in terms of Coulomb's angle of internal friction. Such a presentation is common in the soil mechanics literature. Note that the graphs for $\rho = 1.7$ ($\rho > \rho_f$) present also the strength criterion (see, for a qualitative comparison, the peak-stress points presented in Figures 11 and 12).

Since condition (2.80) is established there are available some following conditions. From Eq. (2.76) and the basic inequalities it results that $z(m+n) > 0$. Therefore, it is also that

$$m + n > 0 \quad 2.81$$

From the yield condition (2.77) it can be found that $zm > 0$. By (2.80) this leads to

$$m > 0 \quad 2.82$$

From Eq. (2.75) it results that positive or negative sign of function n determines, respectively, whether material contracts (loose material) or dilates

Figure 13: Alternative Assumptions for the Function z .

Figure 14: Influence of the Value of z on the Yield Condition

(dense material). Therefore

$$n \equiv 1 - \beta_a + \beta_b > 0 \quad (\text{loose}) \quad 2.83a$$

$$n \equiv 1 - \beta_a + \beta_b < 0 \quad (\text{dense}) \quad 2.83b$$

Expressing equation (2.75) in the form

$$\dot{\gamma}^P = (1 + 2\beta_a + nb)\dot{\epsilon}_1^P$$

and setting $b=0$, one also finds

$$1 + 2\beta_a > 0 \quad (\text{loose}) \quad 2.84a$$

$$1 + 2\beta_a < 0 \quad (\text{dense}) \quad 2.84b$$

Furthermore, since for $0 < b < 1$ there has to be $1 > g_2 > g_3$, one obtains, by using Eqs. (2.74),

$$\beta_a < 1 \quad 2.85$$

$$1 - \beta_a - \beta_b > 0 \quad 2.86$$

In Figure 15 the limits resulting from inequalities (2.83)-(2.86) are plotted and the ranges of the allowable values of β_a and β_b are indicated with the shadow areas.

The same assumption as introduced in Section 2.7.3 (see also Figure 7) leads to the conditions

$$\beta_a < 0 \quad 2.87a$$

$$\beta_a + \beta_b < 0 \quad 2.87b$$

which determine the limits for the stability of the material behaviour under uniaxial-strain compression, as discussed for the case of the simplified theory. It follows from these conditions and Eq. (2.74b) that there is

$$\dot{\epsilon}_3^P < 0 \quad 2.88$$

Figure 15: Limits for the Material Functions β_a and β_b

under any loading program. Again, one concludes that the plane-strain loading is determined by the condition $\dot{\epsilon}_2^p = 0$ and this leads to

$$b = \frac{\beta_a}{\beta_a - 1}$$

Therefore, no problem of unstable behaviour appears in this case if condition (2.87a) is to be satisfied.

The range for the values of β_a and β_b , (2.87), indicating the unstable behaviour of material is presented in Figure 15 by the area shadowed in dotted lines.

It may be observed that the constitutive equations proposed in this section reduce, under axisymmetric compression conditions, to those proposed in Section 2.4, if, instead of the products mz and nz , functions m and n are respectively introduced. Hence, following the discussion presented in Section 2.5, one may suggest the limitations on the derivatives of material functions

$$\frac{d}{d\rho} (nz) < 0 \quad 2.89$$

$$\frac{d}{d\rho} (\bar{\alpha}nz) = \frac{d}{d\rho} (mz + nz) > 0 \quad 2.90$$

Presenting these inequalities in the forms

$$z \frac{dn}{d\rho} + n \frac{dz}{d\rho} < 0 \quad ; \quad \frac{d(mz)}{d\rho} + \frac{d(nz)}{d\rho} > 0$$

and regarding (2.80), (2.83), (2.89) one may deduce that also

$$\frac{dn}{d\rho} < 0 \quad 2.91$$

$$\frac{d(mz)}{d\rho} > 0 \quad 2.92$$

By the mass conservation law, Eq. (2.75) can be written as

$$\frac{\dot{\rho}}{\rho \epsilon_1} = n (z+b) \quad 2.93$$

This equation for $b=\text{const}$, according to (2.89) and (2.91), gives the condition

$$\frac{\partial}{\partial \rho} \left(\frac{\dot{\rho}}{\rho \epsilon_1} \right) = \frac{d(nz)}{d\rho} + b \frac{dn}{d\rho} < 0 \quad 2.94$$

which leads to the concept of the final density of material as explained, by plotting Eq. (2.93), in Figure 16. The final density does not depend on the state of stress though the way on which this density is to be reached does.

Therefore, the behaviour of material is, in general, stress-path dependent.

It can be shown that the limits for the density of material are in the considered case the same as in the case of the simplified theory; see Eqs. (2.64).

2.9 The Stress Invariant b

In Section 2.7 it has been suggested that the minor stress σ_3 has clear physical interpretation if the constitutive behaviour of granular materials is considered (this will be further emphasized in Section 2.10). Owing to this interpretation a new tensor is defined so that

$$x_{ij} = \sigma_{ij} - x_{ij}^0 \quad \text{where } x_{ij}^0 = \sigma_3 \delta_{ij} \quad 2.95$$

The interpretation of this equation is presented in Figure 17. Since,

Figure 16: Dependence of the Material Density Changes
on the Stress Invariant b

Figure 17: Interpretation of Equation (2.95)

as suggested in Section 2.3, the natural state of granular materials can be defined by $\sigma_3 = 0$, the stress σ_3 can be considered as a particular parameter which indicates how far a current stress state of material is from the natural one. The stress invariant b can be defined as the ratio of the additional stresses (the principal component of $\mathbf{x}_{ij}$) acting on the element of material, i.e.,

$$b = \frac{x_2}{x_1} \quad 2.96$$

It seems to be reasonable to assume that the invariant b , besides the density of material, influences the mode of plastic deformation. This was assumed in the previous sections where b was introduced in the constitutive equations as an invariant which describes neither the magnitude of shear nor normal stresses.

However, it does not mean that, owing to Eq. (2.95), the physical interpretation of invariant b is established just by definition (2.96). There are other functions of the principal components of tensor $\mathbf{x}_{ij}$ which might satisfy the role similar to that of invariant b . By choosing this particular invariant we have chosen the simplest of these functions.

2.10 A "Thermodynamic" Interpretation

In this section an interpretation of the postulate for energy dissipation (Eq. 2.70) will be presented. This interpretation is based on the concepts of the "structural entropy" and the "structural reversibility" of the plastic deformation process of granular materials.

Before these concepts are introduced it should be emphasized that now we are interested in the question of the reversibility of the changes in the internal structure (i.e., in the properties) of granular materials, and not in the question of the thermodynamic reversibility which, in the case of plastic deformations supplies informations on material properties only as far as the condition (2.9) can be used:

Henceforth, the deformation process is considered in which elastic strains are not neglected. The equation of energy conservation can be presented in the form

$$\dot{W} = \dot{Q} + \dot{U} \quad (\text{per unit mass}) \quad 2.97$$

where $\dot{W}$ is the rate of external work, $\dot{Q}$ is the rate of heat flow and $\dot{U}$ is the rate of internal energy changes. Since the considered deformation process is isothermal and the increments of kinetic energy are neglected, the changes in the internal energy are due the elastic work only and therefore

$$\dot{W} = \frac{1}{\rho} \sigma_{ij} \dot{\epsilon}_{ij}^e, \quad \dot{U} = \frac{1}{\rho} \sigma_{ij} \dot{\epsilon}_{ij}^e, \quad \dot{Q} = \frac{1}{\rho} \dot{D} = \frac{1}{\rho} \sigma_{ij} \dot{\epsilon}_{ij}^p \quad 2.89a,b,c$$

Since elastic strains are taken into account it will be considered that the material functions involved in the description of plastic behaviour of the ideal granular material depend on the plastic density of material (Eq. 1.26). In other words, plastic behaviour of the ideal granular material is influenced by the changes in its internal structure which are due to plastic deformations only. This is a consequence of assumption A3 (Section 2.2).

2.10.1 Structural Reversibility

In the following there are discussed the changes in the internal structure of material related to plastic deformations. Therefore, any changes in the dimensions of an element of material are referred to those measured by irreversible displacements.

The concept of the "structural reversibility" is explained in Figure 18a on a microscopic level. The two grains, denoted "G1" and "G2", are in contact at point A. These grains are separated from an element of material which is deformed homogeneously on macroscopic level and which has a certain prescribed internal structure. Such an element will be referred to as the "body B^0 ". Suppose that body B^0 is plastically deformed under the initial state of stress σ_i^0 so that the grain G1 moves in the direction v^+ , relatively to G2, and the state of stress changes to $\sigma_i^0 + d\sigma_i$. Also the internal structure of material changes so after the deformation the element is referred to as the "body B^* ". Now suppose that there exists a state of stress $|\sigma_i^*| = |\sigma_i^0 + d\sigma_i|$ (the magnitudes of the principal stresses are the same but their directions are different) under which body B^* can be plastically deformed so that G1 moves in the direction v^- , relatively to G2. If such a state of stress exists it may be said that the plastic deformation of body B^0 is "structurally reversible at point A". Although the concept of the structural reversibility at a point will not be used, it will be helpful in understanding the following, phenomenological concepts of structural reversibility.

Figure 18: Illustration of Structural Reversibility

"Structural reversibility in direction" will be named the "r e v e r s i b i l i t y in L". This concept is explained as follows.

Suppose that body B^0 is plastically deformed under the initial state of stress σ_i^0 so that a linear element L_1 and a linear element L_2 is lengthened and shortened, respectively, and the state of stress changes to $\sigma_i^0 + d\sigma_i$. After the deformation, the element of material is referred to as the "body B^* ". Now suppose that there exist the states of stress $|\sigma_i^{1*}| = |\sigma_i^0 + d\sigma_i|$ and $|\sigma_i^{2*}| = |\sigma_i^0 + d\sigma_i|$ under which body B^* can be plastically deformed so that element L_1 is shortened and element L_2 is lengthened.

If both such states of stress exist it will be said that the plastic deformation of body B^0 is r e v e r s i b l e in L. Owing to the basic inequality (2.12) and the isotropy of plastic properties of the ideal granular material (the response of material does not depend on the rotation of the principal stress directions) one may observe that a plastic deformation process is reversible in L if the condition

$$\dot{\epsilon}_3^p < 0 \quad 2.99$$

is continuously satisfied during deformation. One also observes that the deformation process of d e n s e (dilatating materials) is always reversible in L.

Consider that $\dot{\epsilon}_3^p > 0$. In this case all linear elements of body B^0 are shortened under a primary deformation process and, due to the isotropy of material, after any rotation of the principal stress directions all linear elements of body B^0 will also be shortened. Therefore, in such a case the deformation process is irreversible in L.

A chain of grains (Figure 18c) may be imagined as a microscopic model of a linear element L.

"Structural reversibility in plastic density" will be named "reversibility in PD". This concept is explained as follows. Suppose that at a certain moment body B^P is plastically deformed so that $\dot{v}^P > 0$ or $\dot{v}^P < 0$. If during all subsequent states of plastic deformation of the same material there is always $\dot{v}^P > 0$ or $\dot{v}^P < 0$, respectively, it will be said that the plastic deformation process is irreversible in PD. In other words, the irreversibility in PD of a deformation process means that the plastic density can only increase or, alternatively, only decrease.

2.10.2 The Minor Principal Stress σ_3

In the following there are compared the constitutive relation of the ideal gas under isothermal volume changes and the constitutive relation for the energy dissipation in plastically deformed ideal granular material (Eq. 2.70). The constitutive relation for the isothermal volume changes of ideal gas may be presented in the form (e.g., Feynman et. al., 1963)

$$\dot{Q} = - R \frac{T}{V} \dot{V} \quad 2.100$$

where V denotes volume, T is the absolute thermodynamic temperature and R is the gas constant.

Using (1.25b) and (2.98c) one can present the postulate for energy dissipation (Eq. 2.70) in the form

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$$\begin{aligned}\dot{Q} &= \frac{1}{\rho} \dot{\alpha} \sigma_3 = \frac{1}{\rho} \alpha' \dot{\sigma}_3 \dot{\rho}^p = \alpha' \sigma_3 \dot{V}^p \\ &= -\alpha' \frac{\sigma_3}{V} \dot{V}^p\end{aligned}$$

2.101

where $\dot{V}^p$ denotes the rate of volume changes due to plastic deformations. One may, thus, observe a similarity between Eqs. (2.100) and (2.101). This similarity, however, may not be considered as an analogy since the two processes, described by (2.100) and (2.101) are uncomparable. This will be discussed further on. It follows, nevertheless, that T and σ_3 satisfy similar roles in both relations. Moreover, an analogy between these two quantities appears since

GRANULAR MATERIALS:

$$U > 0 \quad \text{if} \quad \sigma_3 > 0$$

$$U \rightarrow 0 \quad \text{if} \quad \sigma_3 \rightarrow 0$$

$$\sigma_3 < 0 - \text{inadmissible}$$

GAS:

$$U > 0 \quad \text{if} \quad T > 0$$

$$U \rightarrow 0 \quad \text{if} \quad T \rightarrow 0$$

$$T \leq 0 - \text{inadmissible}$$

where, in the case of granular materials, U refers to the elastic energy. The postulate for energy dissipation, Eqs. (2.70) or (2.101), has been introduced by the argument that the rate of energy dissipation must vanish if the stress σ_3 goes to zero. In the foregoing comparison of σ_3 and T , the importance of the stress σ_3 is further emphasized.

2.10.3 Structural Entropy

A new quantity, η , will now be introduced for the description of the plastic behaviour of granular materials. This will be named the "structural entropy" and is defined by the incremental (rate)

equation as

$$\dot{\eta} \equiv \frac{\dot{Q}}{\sigma_3} = \frac{1}{\rho} \frac{\dot{\rho}}{\sigma_3} \quad 2.102$$

The condition of thermodynamic irreversibility (2.9) and the dynamic restriction (2.10) lead to the condition

$$\dot{\eta} \geq 0 \quad 2.103$$

where the equality sign refers to purely elastic (thermodynamically reversible) deformation. The inequality (2.103) is equivalent to the condition of thermodynamic irreversibility of the plastic deformations of granular materials.

By using (2.102), postulate (2.70) can be expressed by the equation

$$\dot{\eta} = \frac{1}{\rho} \dot{\alpha} \quad 2.104$$

Since $\dot{\alpha} = \alpha'(\rho^P) \dot{\rho}^P$ and $\dot{\rho}^P = \rho \dot{v}^P$, this equation can also be presented in the form

$$\dot{\eta} = \alpha' \dot{v}^P > 0 \quad 2.104a$$

Assume that α is a monotonic function of ρ^P if a deformation process of "the same" material is considered (for the explanation of the concept of "the same" material refer to Section 2.3). Therefore, for any plastic deformation process, one of the two alternative conditions, must be satisfied

$$\alpha' > 0 \quad \text{and} \quad \dot{v}^P > 0 \quad 2.105a,b$$

or, alternatively,

$$\alpha' < 0 \quad \text{and} \quad \dot{v}^P < 0 \quad 2.106a,b$$

It follows, therefore, that the assumption about a monotonic form of material function α and the condition for the structural entropy increments (2.103) lead to the irreversibility in PD of any plastic deformation process.[†]

One may note, therefore, that the condition of the thermodynamic irreversibility of plastic deformations ($\dot{D} > 0$) and the dynamic restriction $\sigma_3 > 0$ have been helpful to find some limits for plastic properties of the ideal granular material. On one hand the inequality $\dot{\epsilon}_1^p > 0$ has been established and, on the other hand, the irreversibility of plastic density changes can be reasonably suggested.

2.10.4 Local Structural Entropies

In the case of a thermodynamically reversible process the (thermodynamic) entropy is defined as:

$$\dot{S} = \frac{\dot{Q}}{T} \quad 2.107$$

Hence, the constitutive equation for the isothermal volume changes of ideal gas can be presented as

$$\dot{S} \approx R\dot{v}, \quad \dot{S} > 0 \quad \text{or} \quad \dot{S} < 0 \quad 2.108$$

where $\dot{v} = -\dot{V}/V$. In spite of the similarity between Eqs. (2.104a) and (2.108) the processes described by these equations cannot be considered to be analogous. The first process is thermodynamically irreversible

[†] In the considerations of the previous sections the irreversibility in PD of any plastic deformation process followed the assumption about a monotonic form of function n ; see Figure 5, Section 2.5.

and, as discussed previously, the changes in the plastic density of the ideal granular material are irreversible as well. The second process is thermodynamically reversible and the changes in the density of ideal gas are also reversible. Therefore, no analogy between these two processes can be ever suggested.

On the other hand these two processes do not seem to be comparable for a d i f f e r e n t reason. The process of plastic deformation of the ideal granular material is three-dimensional in the sense that the three "pressures" ($\sigma_1, \sigma_2, \sigma_3$) determine the state of stress and the three kinematic variables ($\dot{\epsilon}_1^P, \dot{\epsilon}_2^P, \dot{\epsilon}_3^P$) are involved. The process of ideal gas volume changes is one-dimensional in the sense that only one quantity (the pressure P) describes the "state of stress" and only one kinematic quantity ($\dot{v}$) is involved. It seems, therefore, to be interesting to present the constitutive equation for plastic deformation of the ideal granular material as the sum of some t h r e e equations.

A cartesian coordinates system (x, y, z) is considered in which the normal plastic strain-rates are denoted $\dot{\epsilon}_i^P$ ($i=x, y, z$). This system is fixed in the relation to certain physical directions in material. Since

$$\dot{\epsilon}_{ii}^P = \dot{v}^P$$

the constitutive equation (2.104a) can be presented as follows

$$\dot{\eta}_i = \alpha' \dot{\epsilon}_i^P, \quad \dot{\eta}_{ii} = \dot{\eta} > 0 \quad (i=x, y, z) \quad 2.109$$

The quantities η_i will be named the "local structural entropies".

Assuming that plastic deformation of the ideal granular material is reversible in L (see Section 2.10.1) one may observe that for all of the three local entropies it is

$$\dot{\eta}_i = \alpha' \dot{\epsilon}_i^p, \quad \dot{\eta}_i > 0 \quad \text{or} \quad \dot{\eta}_i < 0 \quad 2.110$$

Each of these equations seems to be comparable with the relation for ideal gases (Eq. 2.108) in the sense that in both cases one-dimensional processes are described. The possibility of the change in the sign of the rates of local structural entropies means that the change of the length of any linear element is reversible. Such a reverse process is to be reached by a rotation of the principal stress directions in the relation to the coordinates system (x, y, z) .

Relations (2.110) can be interpreted as follows. There are always three coupled, local processes at any point (a representative volume element) of plastically deformed ideal granular material. These processes are described in a cartesian coordinates system (x, y, z) which is fixed in the relation to certain physical directions in material. For any of the local processes a quantity, called "local structural entropy", is prescribed. The increments of local structural entropies are to be calculated by the constitutive relations (2.110). These increments can be negative, positive, or equal to zero depending on the position of the principal stress directions, in the relation to the coordinates system (x, y, z) . The sum of these increments is always positive.

If a plastic deformation process is irreversible in L ($\dot{\epsilon}_3^p \geq 0$) then all of the three increments of the local structural entropies must be

nonnegative, i.e.,

$$\dot{\eta}_i \geq 0$$

2.111

In this case, therefore, the suggested comparability of relations (2.108) and (2.110) fails since $\dot{S}$ can be both positive and negative. For the interpretation of such a failure consider a case in which the constitutive relation (2.108) fails. This would be if at a certain moment the internal structure of a gas is strongly nonhomogeneous, as indicated in Figure 19a. Then the total change in the entropy of the gas (the change resulting from the heat flow through the walls of the vessel, plus the change due to the entropy production inside of the vessel) will be always positive, if the internal nonhomogeneity is strong enough. This is a situation similar to that which can be expected if a granular materials is in a very loose state. Then, there are expected to exist strong local nonhomogeneities of the type indicated by the unstable structures in Figure 19b. One may further expect that the existence of such structures must lead to the unstable behaviour of the material in the sense discussed previously (Section 2.7.5). Therefore, the concept of the unstable behaviour of very loose granular materials and the concept of the reversibility in L of a plastic deformation process seem to have a common physical base.

It may be noted that from the viewpoint of the condition of the thermodynamic irreversibility ($\dot{D} > 0$) the unstable deformation process characterized by $\dot{\epsilon}_3^p > 0$, is correct. On the other hand, if the

[a]

GAS

[b]

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Figure 19: Nonhomogeneities of the Internal
Structures of a Gas and a Very Loose
Granular Material

reversibility in L of a plastic deformation process is postulated then an additional property of granular materials can be recognized, i.e., very loose materials may not be deformed in a stable way under any state of stress different than hydrostatic (it is a separate question whether such materials can exist at all).

2.10.5 The Stationary State

As follows from (2.105) and (2.106), function α can be assumed to be monotonic only if dense and loose materials are considered separately. This is illustrated in Figures 20a and 20b. Applying the reasoning identical to that presented in Section 2.5 one finds that the expected properties of function α are those indicated in Figure 20b. Therefore, again, under a plastic deformation process the final plastic density is approached and then $\rho^P \rightarrow 0$ and $\eta \rightarrow \infty$. In other words, during a plastic deformation process there is approached the stationary state which is characterized by the condition $\rho^P = \rho_f^P$ or $\eta = \infty$. Let us also introduce the second of the limitations suggested previously in Section 2.5. By using (2.72), the equation (2.104a) can be presented in the form

$$\frac{d\eta}{d\epsilon_1^P} = \gamma (h, \rho^P) \quad 2.113$$

where $\gamma = \alpha'(1+g_2+g_3)$. It has been suggested in Section 2.5 that the increment of energy dissipation increases if the density of material increases (for $\epsilon_1^P = \text{const.}$ and $\sigma_3 = \text{const.}$). Assuming now the same

Figure 20: Alternative Properties of the Function α

(for $\dot{\epsilon}_1^p = \text{const.}$, $\sigma_3 = \text{const.}$ and $b = \text{const.}$) one finds from (2.102) and (2.113) the inequality

$$\frac{\partial \gamma}{\partial \rho^p} > 0 \quad 2.114$$

which is equivalent to the given previously inequalities (2.28)

or (2.90). Therefore

$$\frac{\partial}{\partial \rho^p} \left(\frac{d\eta}{d\epsilon_1^p} \right) > 0 \quad 2.115$$

Let us call the ratio $d\eta/d\epsilon_1^p$ the "velocity of structural entropy changes". It follows, that for l o o s e materials ($\rho^p > 0$) the velocity of structural entropy changes i n c r e a s e s as the stationary state is approached. For d e n s e materials the velocity of structural entropy changes d e c r e a s e s. Therefore, as the stationary state is approached the velocity of structural entropy changes reaches maximum and minimum for loose and dense materials, respectively. One may note that, from this point of view, the process of the plastic deformation of d e n s e materials is similar to the processes described by the classical thermodynamics of irreversible processes (e.g. De Groot and Mazur, 1962).

2.11 Conclusions

In this chapter a theory for plastic deformation of an ideal granular material has been proposed. The internal structure of this material is characterized by the material density (or the plastic material density) only. Hence, the investigated granular material has

been referred to as locally homogeneous one.

The presented theory is based, in principle, on a frictional analogy and has been formulated by means of the phenomenological approach. Two formulations of the theory have been presented.

In the formulation presented in Section 2.7 the postulate P3 has been accepted. This postulate states that the yield conditions for axisymmetric compression and extension are the same in the sense that $r^c = r^{ex}$ for any density of material. Two, the simplest cases resulting from this formulation have been discussed. The first one leads to Coulomb's yield condition. The second case, referred to as the simplified theory, leads to the yield condition which is presented, in the principal stress space, by a set of surfaces encircling Coulomb's pyramids. In both cases the yield conditions and the incremental stress-strain relationships have been derived assuming that the functions of b included in basic postulates are linear with respect to this invariant. This leads, in both cases, to the constitutive equations which contain two material functions:

In the general formulation of the theory the postulate P3 has been disregarded. This leads to the yield condition which, in general, does not encircle Coulomb's pyramids. The yield condition and the stress-strain relationship have been derived assuming that the mode of plastic deformation (the ratios ϕ_i) changes linearly with the stress invariant b . This leads to the constitutive equations which contain three material functions. Therefore, although the general theory is more capable describing the behaviour of granular materials than the simplified theory,

it also leads to more complex constitutive equations.

Suppose that experimentally obtained strength surfaces are, more or less, of the shape of yield surfaces. If so, the results of experiments by, e.g., Sutherland and Mesdary (1969) seem to confirm the yield condition resulting from the simplified theory. On the contrary, the results of the experiments by, e.g., Lade and Duncan (1973) indicate the need to use the general theory (for a comprehensive review of such experimental results refer to the article by Ladd et.al., 1978).

It may also be noted that from the mentioned experimental results one could suggest that, for a certain $\rho = \text{const.}$, there is $r^{\text{ex}} \geq r^{\text{c}}$. Therefore, according to Eq. (2.79), it is also that $z \leq 1$. Hence, the alternative case for function z (i.e. $z < -1$) could be eliminated. Note also that there are no available experimental data which could be considered to support the adequacy of Coulomb's yield condition.

In the presented theory a set of inequalities has been introduced. The basic inequalities have been derived on rigorous grounds. The other inequalities have been obtained by some intuitive assumptions. It should be emphasized that to derive the basic constitutive equations, i.e. a yield condition and an incremental stress-strain relationship, no such inequalities are required. On the other hand, however, a number of informations about the properties of material have been obtained by means of those inequalities, for instance, the existence of the final density has been concluded.

The most important feature of the proposed constitutive relations is that no "pre-peak" plastic strains can be predicted; see, e.g., Figure 12. This is a consequence of the use of the frictional analogy based on Amonton's friction law. As indicated in Figure 3, no "pre-peak" irreversible (plastic) displacements can be predicted for the frictional system if Amonton's friction law is assumed to hold. Therefore, the theory presented in this chapter is capable of describing the behaviour of real granular material in the range of advanced deformations only. This may be suggested from the comparison of Figures 1 and 12. It also seems that the proposed yield criterion could be successfully used for the description of the strength properties of real granular materials.

CHAPTER THREE

LOCALLY NONHOMOGENEOUS MATERIALS

3.1 Description of Material

The ideal granular material considered in this chapter is described by the following fundamental assumptions:

- A1: the behaviour of material is not sensitive to time-scale changes;
- A2: the behaviour of material is not sensitive to temperature changes;
- A3: elastic and plastic strains are additive and non-coupled;
- A4: material is isotropic with respect to its plastic properties (St. Venant's hypothesis holds),
- A6: frictional slip at grain boundaries is regarded as the mechanism for plastic deformation.

Therefore, the ideal granular material considered henceforth differs from the material considered previously by the omission of the assumption A5. That means that now the ideal granular material can be locally nonhomogeneous.

In other words, in the description of plastic behaviour of the ideal granular material there can appear parameters describing variable intensities of some local nonhomogeneities. These intensities are to be referred to the volume of a representative element of material. Such an

element must be large enough to be macroscopically homogeneous in response and to include a large enough number of local nonhomogeneities so that average properties of material can be adequately reflected by the properties of the element.

Due to assumed isotropy of plastic behaviour of the ideal granular material, the mentioned intensities of local nonhomogeneities are not referred to the physical directions of material and the directions of applied loads. The local nonhomogeneity of granular materials, which will be discussed in this chapter, was suggested to be related to the strain anisotropy, i.e., the anisotropy which develops under a shear deformation process (Szymanski, 1976). In this study such a hypothesis is not considered in order to obtain simple (of practical significance) constitutive relations. Nevertheless, it is believed that a strain anisotropy develops in plastically deformed granular materials (see also the experimental evidence for such an anisotropy presented by Arthur et al., 1977). Therefore, it is suggested that the ideal granular material defined above differs from real granular materials mainly by the assumed isotropy of its plastic properties.

It is considered that the plastic behaviour of granular materials depends on their internal structure. Although elastic strains will be taken into account in the following, the influence of these strains on the changes in the internal structure of material will be neglected. Therefore, plastic behaviour of material will not depend on its elastic properties. This is a consequence of assumption A3. The

Using this condition as well as the conditions introduced in Sections 3.2 and 3.3 one finds that, depending the plastic density of material, it is that

$$\dot{\epsilon}_1 \geq 0 \quad \dot{\epsilon}_2 > 0 \quad (\text{loose: } \rho_0 < \rho^P < \rho_f^P) \quad 3.25a$$

$$\dot{\epsilon}_1 \geq 0 \quad \dot{\epsilon}_2 < 0 \quad (\text{dense: } \rho_0 > \rho^P > \rho_f^P) \quad 3.25b$$

$$\dot{\epsilon}_1 \geq 0 \quad \dot{\epsilon}_2 = 0 \quad (\rho_0 = \rho_f^P \text{ or } \rho^P = \rho_f^P) \quad 3.25c$$

From these conditions and Eq. (3.22) the following conclusions may be drawn (the effects of hardening/softening related to the changes of parameters ρ^P and κ are referred to as the " ρ^P - effect" and the " κ - effect", respectively):

- i) the total effect of hardening ($\dot{\epsilon} > 0$) or softening ($\dot{\epsilon} < 0$) depends on the sum of the ρ^P - and κ - effects. The ρ^P -effect can result in hardening ($n > 0 \rightarrow \dot{\epsilon}_2 > 0$) or softening ($n < 0 \rightarrow \dot{\epsilon}_2 < 0$), depending on the initial density of material. The κ -effect can result in hardening only,
- ii) in the case of dense materials the total effect can result in hardening or in softening or in the plastic flow under the condition $\dot{\epsilon} = 0$ if

$$\dot{\epsilon}_1 = -\dot{\epsilon}_2 \neq 0 \quad (\dot{\rho}^P \neq 0, \dot{\kappa} \neq 0) \quad 3.26$$

This equation explains what happens as the peak-stress (point P in Figure 1) appears; then the positive rate of $\dot{\epsilon}$ changes due to κ -effect is equal to the negative rate of $\dot{\epsilon}$ changes due to ρ^P -effect,

has been put forward for locally homogeneous materials, is modified so that

$$\dot{D} = \kappa \dot{\alpha} \sigma_3 \quad 3.7$$

where κ is a material (scalar-valued) function of plastic strain tensor, i.e.,

$$\kappa = \kappa(\epsilon_1^p, \epsilon_2^p, \epsilon_3^p) \quad 3.1a$$

Further, equation (3.1) is approximated as follows

$$\dot{D} = \kappa \sigma_3 \dot{\alpha} \dot{\rho}^p = \kappa \sigma_3 \dot{\alpha} \dot{\rho} \dot{V}^p \approx \kappa \sigma_3 \dot{\alpha} \dot{\rho}^p \dot{V}^p \quad 3.2$$

In other words, the relation (1.25b) has been replaced by $\dot{\rho}^p = \dot{\rho} \dot{V}^p$.

Due to this approximation the plastic properties of material are not coupled with the elastic properties, as required by assumption A3.

The postulate for the mode of plastic deformation introduced for locally homogeneous materials is not changed and this, for axisymmetric compression, can be expressed by the equation

$$\frac{\dot{V}^p}{\dot{\epsilon}_1^p} = z n \quad 3.3$$

It follows from (3.1a) that

$$\kappa^0 \equiv \kappa(0) = \text{const.} \quad 3.4$$

Therefore, since under the initial conditions function κ does not depend on the density of material, one concludes that this function does not depend on the plastic density of material, as follows from the definition of ρ^p (Eq. 1.26). The interpretation of function κ will be presented in the next section. It will also be shown that κ satisfies the role

of a hardening parameter. Hence, this function will be referred to as "parameter κ ".

Owing to the above mentioned interpretation it is assumed that κ may not be negative, i.e.,

$$\kappa \geq 0 \quad 3.5$$

From Eqs. (3.1), (3.3) one finds the equation

$$\dot{D} = \kappa \sigma_3 \alpha' \rho^p n z \dot{\epsilon}_1^p = \kappa \sigma_3 (m+n) z \dot{\epsilon}_1^p \quad 3.6$$

As for locally homogeneous materials, the case

$$z > 0 \quad 3.7$$

will be investigated. The alternative case, i.e. $z < -1$, will not be considered here. Note, that to prove the existence of these two alternative cases one should first consider the constitute equations for complex stress states. These equations will be given further on; nevertheless, the condition (3.7) has been already introduced in order to simplify the following discussions.

Owing to assumption (3.5) the limitations analogous to (2.81), (2.82) and (2.83) can be concluded in the same way as before. Therefore

$$m + n > 0 \quad 3.8$$

$$m > 0 \quad 3.9$$

and it is determined by the sign of function n whether material dilatates or contracts, i.e.,

$$n > 0 \quad (\text{loose}) \quad 3.10a$$

$$n < 0 \quad (\text{dense}) \quad 3.10b$$

Using the same arguments as presented in Section 2.5 one may also suggest that the inequalities analogous to (2.89) and (2.90) are satisfied, i.e.,

$$\frac{d}{d\rho^P} (nz) < 0 \quad 3.11$$

$$\frac{d}{d\rho^P} [(m+n)z] > 0 \quad 3.12$$

The considerations similar to those given previously in Sections 2.5 and 2.8 (see also Figures 5 and 16) lead to the concept of the final plastic density of material (ρ_f^P). During an elastic-plastic deformation process the plastic densities of both loose materials ($\rho_o < \rho^P < \rho_f^P$) and dense materials ($\rho_o \rightarrow \rho_f^P > \rho_f^P$) approach the final plastic density and then

$$n \approx 0; \quad m \approx m(\rho_f^P); \quad z \approx z(\rho_f^P)$$

Note that since elastic strains are not neglected now the density of material (ρ) depends on the elastic compressibility of material.

Therefore, as ρ^P approaches ρ_f^P , the density of material depends on the state of stress unless the material is elastically incompressible.

3.3, Material Function r and Interpretation of Parameter κ

A representative element of granular material, of the volume V_t , is considered. The hypothesis is suggested that, during elastic-plastic deformations of the element, the plastic deformations do not occur

in the whole volume of the element but only through some "plastic regions". In these regions both elastic and plastic deformations can take place. The remaining part of the volume of the element is elastically deformed only. The plastic regions are arranged after such a manner that the element is deformed homogeneously from the macroscopic point of view. In other words, the distribution of the plastic regions is homogeneous. Consequently, the arrangement of the plastic regions must change quickly during an elastic-plastic deformation process for considerable strain gradients not to appear inside the element. Then the macroscopic homogeneity with local nonhomogeneities can be considered to characterize such a deformation process. The intensity of the plastic regions can be determined by the ratio

$$r = \frac{V_{pl}}{V_e} \quad 3.13$$

where V_{pl} is the total volume of the plastic regions and V_e is the remaining part of the volume of the element. The foregoing hypothesis implies that one has to account for the ratio r if a description of plastic behaviour of granular materials is considered. This has to be regarded in order to make different plastic states of the material element comparable one to another. Therefore, ratio r should be included in the constitutive equations which describe plastic deformation of granular materials.

It is also suggested that the intensity of plastic regions may increase or be constant under elastic-plastic shear deformation, i.e.,

$$\dot{r} \geq 0 \quad 3.14$$

This condition constitutes the second part of the hypothesis. As the third part of this hypothesis it is assumed that the changes of r are determined by a function of plastic strains which may depend on the properties of material. Therefore, r is a material function of the general form

$$r = r(\epsilon_1^p, \epsilon_2^p, \epsilon_3^p) \quad 3.15$$

Now the parameter κ , included in constitutive equation (3.1), is defined by the equation

$$\kappa = \frac{r}{1+r} = \frac{V_{pl}}{V_t} \quad 3.16$$

From (3.13) and (3.14) it also follows that

$$0 \leq \kappa \leq 1 \quad 3.17$$

$$\dot{\kappa} \geq 0 \quad 3.18$$

Therefore, parameter κ describes the density of plastic regions.

Equation (3.16) explains the physical meaning of the appearance of this parameter in postulate (3.1): the rate of energy dissipation is proportional to the volume of the plastically deformed regions and not to the total volume of deformed material.

At the initial state there is $r(0)=r^0=\text{const.}$ It should be emphasized, however, that for real granular materials the "initial" state is usually unknown in the sense that the plastic strains under

this state are impossible to estimate. The reason is that it is practically not possible to describe in terms of plastic strains, for instance, the geological history of a granular soil or the history of the preparation of a sample of granular material. Therefore, r^0 (or κ^0) should not be considered to be a material constant if the description of the behaviour of a real granular material is investigated. In such a case r^0 is a constant which may depend on the previous (unknown) deformation history of material.

If there are no changes in the total volume of plastic regions then $\dot{r} = \dot{\kappa} = 0$. In this case parameter κ does not need to be explicitly included in Eq.(3.1) since this can be presented in the equivalent forms

$$\dot{D} = \kappa \dot{\alpha}_a(\rho^p) \sigma_3 = \dot{\alpha}_b(\rho^p) \sigma_3 \quad (\kappa = \text{const.})$$

One concludes, thus, that the locally homogeneous material investigated in Chapter 2 can be considered as a particular case of a locally nonhomogeneous material in which the density of plastic regions does not change under any elastic-plastic deformation process (compare the definition of the local homogeneity given in Section 2.2).

It follows from the foregoing hypothesis and postulate (3.1) that both the plastic density of material (ρ^p) and the density of plastic regions (κ) describe average plastic "properties" of a representative element of material. However, the phenomenological meanings of these two parameters are different. Assume that the influence of elastic strains on the change of the density of material can be neglected. Then ρ^p can be directly determined by the measurements of the weight and

the volume of the element. This can be carried out regardless whether plastic deformations of material take place or not. Contrary to ρ^p , parameter κ can be (theoretically) determined by its definition only if plastic deformations take place. Therefore, parameter κ is to be referred to the "plastic behaviour" and not to "plastic properties" of granular materials, unless an interpretation of the differences between the "plastic" and "elastic" regions is available.

Constitutive equation (3.1) and the hypothesis discussed above have been introduced by Szymanski (1976) in order to explain the behaviour of granular materials observed in the conventional axisymmetric tests. It has been suggested that the plastic regions are characterized by a certain preference of grain contact directions. The existence of the regions with directional structures implies anisotropic properties of the material. Consequently, the variable value of the density of plastic regions describes a variable anisotropy of material. This suggestion, however, is not considered in the present work and the plastic isotropy of the ideal granular material is assumed.

By using the concept of the structural entropy, constitutive equation (3.1) can be also presented in the form

$$\dot{\eta} = \kappa \frac{1}{\rho} \dot{\alpha} \quad 3.19$$

Therefore, κ satisfies the role of a coefficient of proportionality between the increments of function α and the increments of structural entropy. From the hypothesis presented above one can conclude that

the correct form of equation (3.19) would be

$$\dot{\eta} = \frac{1}{\rho V_t} \sum_{p1} V_{p1}^i \dot{\alpha}^i \quad (i=1,2,\dots,N) \quad 3.20$$

where V_{p1}^i is the volume of a single plastic region, $\dot{\alpha}^i$ is the rate of the change of function α in this region and N is the number of plastic regions in the volume V_t . Therefore, equation (3.19) is actually based on the simplification

$$\dot{\alpha}^1(\rho^P) = \dot{\alpha}^2(\rho^P) = \dots \dot{\alpha}^N(\rho^P) \text{ or } \rho_1^P = \rho_2^P = \dots \rho_N^P$$

which follows the first part of the hypothesis by which no considerable strain gradients, and hence no considerable ρ^P gradients, appear in the volume of a representative element of material.

3.4 Frictional Analogy

Postulate (3.1) has been introduced by the modification of the postulate (2.70) which has been previously proposed for locally homogeneous granular materials. As shown in Section 2.4, in the case of axisymmetric compression the latter postulate can be suggested by the frictional analogy involving Amonton's friction law. The above mentioned modification, which means the introduction of the new parameter κ into Eq.(2.70), can also be suggested by an analogy to the more advanced theory of friction by McFarlane and Tabor (1950); see also Bowden and Tabor (1956), Courtney-Pratt and Eisner (1957), Tabor (1959). Courtney-Pratt and Eisner have experimentally shown that before the macroscopic displacement between two (metallic) bodies in frictional sliding occurs, there appears a region of elastic-plastic displacements,

as indicated in Figure 21. This seems to be analogous to the hardening-effect in the case of elastic-plastic deformations of continuous materials. In the mentioned works it has been found that the effect of hardening in the friction case is related to the variable area of the real contact between the two bodies (the mechanism of friction, as explained by Bowden-Tabor's theory, is related to the shearing of the junctions which create the area of the real contact between the two bodies). The area of real contact smoothly increases with continuous displacement until an upper limit is reached. The friction force (T), and hence energy dissipation, are both proportional to this area.

One may suggest, therefore, an analogy between the behaviour of two bodies in frictional sliding and the plastic behaviour of granular materials as outlined in the previous section. The variable area of real contact is analogous to the variable volume of plastic regions. Both the area of real contact and the volume of plastic regions smoothly increase during the two processes. This causes proportional increases of the rates of energy dissipation in both cases as well.

One may consider, thus, that friction analogy has been employed twice in the two postulates presented in Section 3.2. On one hand, the effect of plastic density changes is suggested to be analogous to the frictional motion of a body along a curvilinear surface. On the other hand, the changes in the density of plastic regions are suggested to be analogous to the changes in the area of real contact in the friction case.

Figure 21: T/N vs. x Relationship for Two Bodies in Frictional Sliding (After Courtney-Pratt and Eisner, 1959; simplified)

3.5 Hardening, Softening and Peak Strength Behaviour

Using Eq.(2.8) one obtains from Eqs. (3.3) and (3.6) the yield condition

$$f(q, \sigma_3, \kappa, \rho^p) = \Gamma - \kappa (m+n)z + nz = 0 \quad 3.21$$

Therefore, in the case of locally nonhomogeneous materials there appear two hardening/softening parameters, i.e., ρ^p and κ . Under plastic deformation it has to be that $\dot{f} = 0$ (the consistency condition) and hence.

$$\dot{f} = \dot{f}_1 + \dot{f}_2 \quad 3.22$$

where

$$\dot{f}_1 = - \frac{\partial f / \partial \kappa}{\partial f / \partial \Gamma} \dot{\kappa} ; \quad \dot{f}_2 = - \frac{\partial f / \partial \rho^p}{\partial f / \partial \Gamma} \dot{\rho}^p$$

Eq.(3.22) indicates the two effects of hardening/softening dependent on the changes of the density of plastic regions and the plastic density of material. By differentiation of the yield condition and using the mass conservation law one obtains

$$\dot{f}_1 = z(m+n) \frac{d\kappa}{d\epsilon_1^p} \dot{\epsilon}_1^p = h_{\kappa} \dot{\epsilon}_1^p \quad 3.23a$$

$$\dot{f}_2 = \rho n z \left\{ \kappa \frac{d}{d\rho^p} [(m+n)z] - \frac{d}{d\rho^p} (nz) \right\} \dot{\epsilon}_1^p = h_{\rho} \dot{\epsilon}_1^p \quad 3.23b$$

From (2.12) and (3.18) it follows that derivative $d\kappa/d\epsilon_1^p$ is nonnegative, i.e.,

$$\frac{d\kappa}{d\epsilon_1^p} = \frac{\partial \kappa}{\partial \epsilon_1^p} + \frac{\partial \kappa}{\partial \epsilon_2^p} \frac{\partial \epsilon_2^p}{\partial \epsilon_1^p} + \frac{\partial \kappa}{\partial \epsilon_3^p} \frac{\partial \epsilon_3^p}{\partial \epsilon_1^p} \geq 0 \quad 3.24$$

Using this condition as well as the conditions introduced in Sections 3.2 and 3.3 one finds that, depending the plastic density of material, it is that

$$\dot{r}_1 \geq 0 \quad \dot{r}_2 > 0 \quad (\text{loose: } \rho_0 < \rho^P < \rho_f^P) \quad 3.25a$$

$$\dot{r}_1 \geq 0 \quad \dot{r}_2 < 0 \quad (\text{dense: } \rho_0 > \rho^P > \rho_f^P) \quad 3.25b$$

$$\dot{r}_1 \geq 0 \quad \dot{r}_2 = 0 \quad (\rho_0 = \rho_f^P \text{ or } \rho^P = \rho_f^P) \quad 3.25c$$

From these conditions and Eq. (3.22) the following conclusions may be drawn (the effects of hardening/softening related to the changes of parameters ρ^P and κ are referred to as the " ρ^P - effect" and the " κ - effect", respectively):

- i) the total effect of hardening ($\dot{r}_1 > 0$) or softening ($\dot{r}_1 < 0$) depends on the sum of the ρ^P - and κ - effects. The ρ^P -effect can result in hardening ($n > 0 \rightarrow \dot{r}_2 > 0$) or softening ($n < 0 \rightarrow \dot{r}_2 < 0$), depending on the initial density of material. The κ -effect can result in hardening only,
- ii) in the case of dense materials the total effect can result in hardening or in softening or in the plastic flow under the condition $\dot{r} = 0$ if

$$\dot{r}_1 = -\dot{r}_2 \neq 0 \quad (\rho^P \neq 0, \kappa \neq 0) \quad 3.26$$

This equation explains what happens as the peak-stress (point P in Figure 1) appears; then the positive rate of r changes due to κ -effect is equal to the negative rate of r changes due to ρ^P -effect,

- iii) in the case of loose materials the total effect can result in hardening only,
- iv) if the plastic density of materials is close to the final plastic density, i.e., $\rho^p = \rho_f^p$, or if $\rho_0^p = \rho_f^p$, then the total effect can result in hardening or in perfectly plastic flow which is described by the conditions

$$\dot{\epsilon}_1 = \dot{\epsilon}_2 = 0 \quad 3.27$$

Since $r \geq 0$, one obtains from the yield condition that

$$\kappa \geq \frac{n}{m+n} \quad 3.28$$

Owing to ((3.5), (3.8) and (3.10b)), this inequality is always satisfied for dense materials. In the case of loose materials, it follows from (3.8) and (3.10a) that the value of κ may not be close to zero. At the beginning of shear loading ($r=0$) the initial value of κ must satisfy the condition

$$\kappa^0 \geq \frac{n(\rho_0)}{m(\rho_0)+n(\rho_0)} \quad (3.29)$$

Therefore, since in the case of l o o s e materials the right side of this inequality is greater than zero it follows that such materials may not be sheared if κ^0 is too small. That means, regarding the interpretation of κ , that in loose materials there have to exist some local nonhomogeneities which can satisfy the role of the plastic regions. Otherwise, loose materials may not be sheared in a stable deformation process. The minimum volume of plastic regions required for a stable shear deformation is given by the equality (3.29). Herein,

the term "unstable shear deformation" means that under any state of stress (different than hydrostatic) the yield condition (3.21) is violated.

3.6 General Loading Conditions

The postulate (2.70) introduced for locally homogeneous materials is modified, for complex stress states, in the same way as for axisymmetric compression, i.e.,

$$\dot{D} = \kappa \dot{\sigma}_3 = \kappa \sigma_3 \alpha' \rho^P \dot{\nu}^P \quad 3.30$$

The postulate (2.72) is modified by regarding elastic deformations only i.e., as an argument of functions g_i the plastic density of material, instead of the density ρ , is introduced. Therefore

$$\phi_i = g_i(b, \rho^P) \quad i=2,3 \quad 3.31$$

Similarly as for locally homogeneous materials, the following constitutive equations will be derived assuming that functions g_i are linear in b , i.e.,

$$\phi_2 = g_2(b, \rho^P) = (1 - \beta_a) b + \beta_a \quad 3.32a$$

$$\phi_3 = g_3(b, \rho^P) = \beta_b b + \beta_a \quad 3.32b$$

Using Eqs. (3.22) one may present (Eq. 3.30) in the form

$$\dot{D} = \sigma_3 \kappa (m+n) (z+b) \dot{\epsilon}_1^P \quad 3.33$$

The limits for the material functions and their derivatives have been presented previously in Section 2.8.2. They are not changed by the introduction of the new parameter since κ is nonnegative. A comprehensive review of these limitations will be presented in Section 3.8.2.

A yield condition and an incremental stress-strain relationship can be obtained by means of the technique similar to that outlined in Section 2.7.5. Since now the constitutive equations are considerably more complex this technique is outlined below in more detail.

From Eqs. (2.8), (3.32) and (3.33) one obtains the equations

$$q (1+b\phi_2) \dot{\epsilon}_1^p + \sigma_3 \dot{v}^p = \sigma_3^{\leftarrow \kappa} (m+n) (z+b) \dot{\epsilon}_1^p$$

$$\dot{v}^p = n (z+b) \dot{\epsilon}_1^p \quad 3.34$$

from which the yield condition follows

$$f(q, \sigma_3, b, \kappa, \rho^p) = \Gamma p_1 - \kappa (m+n) + n = 0 \quad 3.35$$

Note that functions $p_1, p_2, \lambda_1, \lambda_2, \lambda_3$ are the same as defined previously in Section 2.8. By differentiation of the yield condition it is that

$$p_1 \dot{\Gamma} + \Gamma \left(\frac{\partial p_1}{\partial b} \dot{b} + \frac{\partial p_1}{\partial n} \dot{n} + \frac{\partial p_1}{\partial z} \dot{z} \right) = \kappa \left(\frac{\dot{\kappa}}{m+n} \right) + (m+n) \dot{\kappa} - \dot{n}$$

where

$$\dot{\Gamma} = \frac{1}{\sigma_3^2} (\sigma_3 \dot{\sigma}_1 - \sigma_1 \dot{\sigma}_3)$$

$$\dot{b} = \frac{1}{(\sigma_1 - \sigma_3)^2} [(\sigma_3 - \sigma_2) \dot{\sigma}_1 + (\sigma_1 - \sigma_3) \dot{\sigma}_2 + (\sigma_2 - \sigma_1) \dot{\sigma}_3]$$

By the mass conservation law and Eq. (3.34) it can be found that

$$\dot{m} = m' \dot{\rho}^p = m' \rho n(z+b) \dot{\epsilon}_1^p$$

and analogous expressions result for $\dot{n}$ and $\dot{z}$. Using these relations

and inserting $\dot{\kappa} = \kappa'(\epsilon_1^p)\dot{\epsilon}_1^p$ one finds the incremental equation

$$\frac{1}{\sigma_3} \lambda_i \dot{\sigma}_i = h \dot{\epsilon}_1^p \quad 3.36$$

where the hardening/softening function is given as

$$h = (m+n)\kappa'(\epsilon_1^p) + \frac{1}{2}n\rho[2(z+b)(\kappa m' + \kappa n' - n') - \Gamma(b-b^2)(nz' + zn') + 2rp_1 z']$$

An incremental stress-plastic strain relationship follows from Eqs. (3.32) and (3.36) and this can be presented in the form

$$\dot{\epsilon}_i^p = \frac{1}{h\sigma_3} [g_i \lambda_j] \dot{\sigma}_j \quad 3.37$$

where $g_1 \equiv 1$. Assuming for elastic strains the generalized Hooke's law (Eqs. 1.23) and using Eq. (1.21) one obtains incremental stress-strain relationship in the form

$$\dot{\epsilon}_i = \left[\frac{1}{h\sigma_3} g_i \lambda_j + \frac{1}{E} \nu_{ij} \right] \dot{\sigma}_j \quad 3.38$$

Therefore, elastic-plastic behaviour of the locally nonhomogeneous granular material is described by constitutive equations (3.35) and (3.38). In these equations there are included four material functions (r, m, n, z or $r, \alpha, \beta_a, \beta_b$) involved in the description of plastic behaviour of the material. For the description of elastic deformations the modulus of elasticity and Poisson's ratio must be known. An example of the behaviour of granular material described by Eqs. (3.35) and (3.38) will be given in Section 3.8, after a simplification is introduced in the next section.

One may note that if $\kappa' = \text{const.}$ (i.e., $r = \text{const.}$) then the

constitutive equations derived in this section are equivalent to those following the general formulation of the theory for locally homogeneous materials; in the case $\kappa = 1$ they are identical.

3.7 Additional Postulate

One of the basic suggestions appearing in previous considerations is that the changes of the internal structure of material depend, in a certain way, on this structure itself. In other words $\dot{\rho}^P$, or $\dot{v}^P$, depends on ρ^P . The relation $\dot{v}^P - \rho^P$ derived in previous sections results from the postulate for the mode of plastic deformation, for instance, relation (3.34) follows Eqs. (3.32).

Let us introduce an additional postulate by which relation $\dot{v}^P - \rho^P$ is of the simplest form, i.e.,

$$\dot{v}^P = \dot{\mu} \beta (\rho^P) \quad 3.39$$

where $\dot{\mu}$ is a coefficient which does not depend on ρ^P . If it is assumed that this relation is an additional postulate, in the sense that this holds regardless of postulate (3.30) and approximated by Eqs. (3.32) postulate (3.31), one finds by comparison (3.34) and (3.39) the following equation

$$n(z+b)\dot{\epsilon}_1^P = \dot{\mu} \beta$$

Since β does not depend on b and $\dot{\mu}$ does not depend on ρ^P one can conclude that

$$\dot{\mu} = \frac{1}{C} (z+b)\dot{\epsilon}_1^P, \quad \beta = Cn \quad 3.40a,b$$

where $C \neq 0$ is an arbitrary constant and

$$z = \frac{1+2\beta_a}{1-\beta_a+\beta_b} = \text{const.} \quad 3.41$$

Therefore, equation (3.39) is a simplifying postulate which leads to an additional relation between material functions β_a and β_b . Now z is not a material function but it is a constant which may depend on the properties of material.

One can observe that by the acceptance of the additional postulate (3.39) the constitutive equations derived in the previous section are considerably simplified since it gives $z'=0$ in the expression for the hardening/softening function (Eq.3.36). It will be shown in the next section that the acceptance of this postulate leads to a simple procedure to find material functions m and n .

Consider the condition

$$\dot{\epsilon}_3^p < 0 \quad 3.42$$

which has been suggested previously in Chapter 2. By Eq.(3.32b) this leads to

$$\phi_3 = \beta_b b + \beta_a = \frac{1}{2} [(nz+2n-3)b + nz-1] < 0 \quad 3.43$$

where from, for axisymmetric conditions, the inequalities result

$$b = 0: \quad zn < 1 \quad 3.44a$$

$$b = 1: \quad zn < 2-n \quad 3.44b$$

These inequalities have to be satisfied for any value of function n .

Assume that the postulate (3.39) holds, i.e. $z = \text{const.}$, and consider the maximum value of the product zn , as permitted by (3.44a), i.e.,

$$zn_{\max} = 1 \quad 3.45$$

From (3.44b) and (3.45) it follows then that

$$zn_{\max} = 1 < 2 - n_{\max}$$

Hence

$$n_{\max} < 1 \quad \text{and} \quad \rho^p > n^{-1}(1) \quad 3.46$$

Therefore an upper limit for function n results.

Now assume that function n can reach this upper limit, i.e. $n_{\max} = 1$, and plastic deformation process takes place so that condition (3.42) is satisfied. Then it follows from (3.44a) that

$$z \leq 1 \quad 3.47$$

Therefore an upper limit for the material constant z results.

This is an interesting conclusion since, as discussed previously in Section 2.11, there are available experimental data which seem to support limitation (3.47). However, there are no available theoretical suggestions which could justify the assumption that for $n = 1$ material can be plastically deformed in agreement with (3.42).

3.8 An Example of Stress-Strain Behaviour

In this section some examples of elastic-plastic stress-strain behaviour of the ideal granular material will be presented. The description of this behaviour follows the constitutive equations

proposed for locally nonhomogeneous granular materials. The additional postulate, introduced in the previous section, is assumed to hold.

3.8.1 Material Functions

It has been assumed that material function r depends on plastic strains. However, it is not known how many invariants of ϵ_{ij}^p should be included. Neither it is known which of these invariants are (theoretically) justified to be considered. As an example, it will be assumed that r is a function of plastic shear strain, i.e.,

$$r = \bar{r}(\epsilon_q^p) \quad 3.48$$

where

$$\epsilon_q^p = \int \dot{\epsilon}_q^p dt \quad \text{and}$$

$$\begin{aligned} \dot{\epsilon}_q^p &= \left\{ \frac{2}{3} [(\dot{\epsilon}_1^p - \dot{\epsilon}_2^p)^2 + (\dot{\epsilon}_2^p - \dot{\epsilon}_3^p)^2 + (\dot{\epsilon}_3^p - \dot{\epsilon}_1^p)^2] \right\}^{\frac{1}{2}} \\ &= \left\{ \frac{1}{3} [1 + \phi_2^2 + \phi_3^2 + \phi_2 + \phi_2\phi_3 + \phi_3] \right\}^{\frac{1}{2}} \dot{\epsilon}_1^p \end{aligned} \quad 3.48a$$

In order to determine the material functions m , n and r the results of conventional axisymmetric ("triaxial") tests are used (Szymanski, 1976). These tests have been performed using fine quartz sand from Biala Gora. The sand is uniform and it is composed of subangular, partly subrounded grains; the specific gravity is 2.65. Minimum and maximum (by vibration) densities are 1.41 and 1.69, respectively (the densities are given here always in G/cm^3).

The samples were prepared by damping and the dry density before consolidation was equal to 1.71. The results of four tests are presented in Figure 22. These tests were performed at $\sigma_3 = 1.5, 2.0, 2.5$ and 3.0 (stresses are given here always in kg/cm^2).

Before the material functions are determined one should note the two remarks: i) as follows from postulate (3.30) only relative values of parameter κ (and function r) can be determined from phenomenological experiments. The real values of κ must remain unknown. It is sufficient, however, to know relative values of κ if the phenomenological description of material is intended; ii) from axisymmetric compression tests function r as well as products z_n and z_m can be determined. However, it is not possible to determine the material parameter z . This can be done only if the results of experiments performed under different loading conditions are available, for instance, if the results of axisymmetric compression and axisymmetric extension tests are known. Then, if z is assumed to be a material constant, this can be found from equation (2.79), where r^{ex} and r^{c} are both taken for the same value of r (or κ). If z is assumed to be a material function then r^{ex} and r^{c} in Eq.(2.79) must be taken for the same values of κ and ρ_n^{p} . Hence a value $z = z(\rho_n^{\text{p}})$ can be determined. Any of these procedures is convenient to be used for advanced deformations where elastic strains can be neglected and the values of κ are expected to remain approximately unchanged, as will be shown below.

Figure 22: Experimental (Szymanski, 1976) and Theoretical Strain-Density Relationship; Conventional Axisymmetric Compression Tests

It has been assumed that the elastic moduli can be estimated from the very beginning of the stress-strain relations presented in Figure 22. Hence, it has been found that E and ν can be approximated as follows.

$$E = 387 \sigma_3, \quad \nu = 0.25 \quad 3.49$$

It can be shown that the presented results of experimental tests are not sufficient to determine function $r(\epsilon_q^P)$. The reason is that all of these tests have been performed at nearly the same densities after consolidation; see Figure 22 for $\epsilon_1 = 0$. In the following the simplest, linear in b , form of function r will be investigated. Therefore, regarding (3.13) and (3.14),

$$r(\epsilon_q^P) = A\epsilon_q^P + B \quad (A > 0, B > 0 \text{ or } B = 0) \quad 3.50$$

It will be shown that this relation fits the experimental data with good accuracy. Using Eq.(3.16) one also finds that the hardening parameter κ is given by the equation

$$\kappa = \frac{A\epsilon_q^P + B}{1 + A\epsilon_q^P + B} \quad 3.51$$

It should be noted that this equation establishes the absolute values of parameter κ since

$$\kappa(\epsilon_q^P \rightarrow \infty) = 1 \quad 3.52$$

and any value of $\kappa(\epsilon_q^P)$ can be related to $\kappa(\infty)$.

From Eq.(3.35) the yield condition for axisymmetric compression

can be presented in the form

$$r = \kappa(\epsilon_q^p) \cdot [zm(\rho^p) + zn(\rho^p)] - zn(\rho^p) \quad 3.53$$

Inserting into this equation the relation $\rho^p = n^{-1} \{ (1/z) (\dot{v}^p / \dot{\epsilon}_1^p) \}$, which follows (3.34), one obtains

$$r = \kappa(\epsilon_q^p) w \left(\frac{\dot{v}^p}{\dot{\epsilon}_1^p} \right) - \frac{\dot{v}^p}{\dot{\epsilon}_1^p} \quad 3.54$$

Hence, one observes that if a unique relation $r - \dot{v}^p / \dot{\epsilon}_1^p$ exists then $\kappa(\epsilon_q^p) = \text{const.}$ From the experimental results presented in Figure 22 it has been found that such a relation exists for $\epsilon_1 > \epsilon_1^* = 0.035$. This is presented in Figure 23a (note that for advanced deformations the influence of elastic strains on stress-strain behaviour can be neglected and hence $\dot{v}^p / \dot{\epsilon}_1^p = \dot{v} / \dot{\epsilon}_1$). However, in agreement with Eq. (3.52) the condition $\kappa = \text{const.}$ can be satisfied approximately only. Hence, regarding (3.51), it results that

$$\text{for } \epsilon_1 > \epsilon_1^* : \quad \kappa = 1, \quad \dot{\kappa} \rightarrow 0; \text{ also} \quad 3.55a$$

$$\text{for } \epsilon_1 = \epsilon_1^* : \quad \kappa = 1 \quad 3.55b$$

It has been found from the experiments that for $\epsilon_1 = \epsilon_1^*$ there is $\epsilon_q^p = 0.092$.

In the following examples it will be assumed that

$$\kappa^* = \kappa(0.092) = 0.95 = \text{assumed value} \quad 3.56$$

The influence of an assumed value of κ^* on stress-strain relations will be investigated separately. The second equation, which is needed to determine constants A and B included in (3.51), is chosen to be

$$\kappa(0) = \kappa^0 = \text{an assumed value} \quad 3.57$$

In the following examples it will be assumed $\kappa^0 = 0$ or, if the behaviour of loose materials will be considered, then κ^0 will be assumed to be of such a value that ratio r at the elastic limit, denoted " r^0 ", is equal to zero (see also condition (3.29) and the related discussion). The influence of an assumed value of κ^0 will be investigated separately.

Owing to (3.55a), material functions m and n can be approximately determined from the equations

$$r = zm(\rho^P) \quad (\epsilon_1 > \epsilon_1^* : \kappa = 1) \quad 3.58$$

$$\frac{\dot{v}}{\dot{\epsilon}_1} = zn(\rho^P) \quad (\epsilon_1 > \epsilon_1^* : \frac{\dot{v}^P}{\dot{\epsilon}_1^P} \approx \frac{\dot{v}}{\dot{\epsilon}_1}) \quad 3.59$$

which follows Eqs. (3.53) and (3.34), respectively. The relations $r - \rho^P$ and $\dot{v}/\dot{\epsilon}_1 - \rho^P$ are presented in Figure 23c and 23b, respectively.

These relations have been approximated with the linear functions

$$m(\rho^P) = \frac{1}{z} (17.21 \rho^P - 24.71) \quad 3.60$$

$$n(\rho^P) = \frac{1}{z} (-8.02 \rho^P + 12.93) \quad 3.61$$

The material constant z is unknown. In the following the two examples of stress-strain relations will be presented in which $z=0.6$ and $z=1.0$ (as discussed in Section 2.11, the experimental data seem to suggest that $z \leq 1$).

3.8.2 Limitations on Material Functions

It is interesting to find how the experimentally determined functions m and n satisfy the limitations on material functions and

Figure 23: Experimental Relationships: $\Gamma - \dot{v}/\dot{\epsilon}_1$,
 $\dot{v}/\dot{\epsilon}_1 - \rho^p$, $\Gamma - \rho^p$

their derivatives which have been discussed in previous sections.

This will be investigated below.

The following calculations will be performed assuming that $z \leq 1$, unless it will be specified otherwise. Material functions m and n are given in Eqs. (3.60) and (3.61), respectively.

1: The postulate for energy dissipation and the basic inequalities lead to the condition (2.20), or (2.81), or (3.8). By Eqs. (3.60) and (3.61) these conditions are satisfied if

$$\rho^P > 1.282 \quad (m+n > 0)$$

2: The yield conditions lead to the inequalities (2.21), or (2.82), or (3.9). By Eq. (3.60) these inequalities are satisfied if

$$\rho^P > 1.436 \quad (m > 0)$$

3: It has been assumed that n is a monotonic function and hence it has been suggested (Section 2.5) that the derivative of n is negative. The same reasoning leads to the suggestion that the derivative of the product nz is negative. This has been regarded in the condition (2.24), or (2.89) or (3.11). By Eq. (3.61) these conditions are always satisfied since

$$\frac{d}{d\rho^P} (nz) = -8.02 < 0$$

Note that this leads to the concept of the final plastic density (see Figures 5 and 16) which, by Eq. (3.61), is

$$\rho_f^P = 1.612 \quad (n = 0)$$

4: It has been suggested that the rate of energy dissipation increases as the density (or the plastic density) of material increases, at $\sigma_3 = \text{const.}$; $b = \text{const.}$; $\dot{\epsilon}_1^p = \text{const.}$; $\kappa = \text{const.}$ This leads to the condition (2.28), or (2.90) or (3.12). By Eqs. (3.60) and (3.61) these conditions are always satisfied since

$$\frac{d}{d\rho^p} (mz + nz) = 9.19 > 0$$

5: If functions g_i are assumed to be linear in b then the inequalities $1 > g_2 > g_3$ lead to conditions (2.85) and (2.86), respectively. By Eq. (3.61) these conditions are satisfied if

$$\begin{aligned} \rho^p &> 1.238 & (\beta_a < 1) \\ \rho^p &> 1.425 & (1 - \beta_a - \beta_b > 0) \end{aligned}$$

6: It has been suggested that a plastic deformation process can be considered to be stable if $\dot{\epsilon}_3^p < 0$. Under the assumption that functions g_i are linear in b this leads to conditions (2.87a) and (2.87b). By Eq. (3.61) these conditions are satisfied if

$$\begin{aligned} \rho^p &> 1.487 & (\beta_a < 0) \\ \rho^p &\times 1.487 & (\beta_a + \beta_b < 0) \end{aligned}$$

It is to be noticed that the limitations presented in points "1"-"4" are valid for any value of z ; in these cases z can be considered to be whether a constant or a function of ρ^p . The limitations "5" and "6" have been given for $0 < z \leq 1$. If $z > 0$ is considered it can be shown that the limitations "5" and "6" lead to the condition

$$\rho^p \geq \rho_f^p$$

In this case, therefore, loose materials are to be excluded from the considerations. This conclusion is independent of the form of material function n .

3.8.3 Examples of Stress-Strain Relations

In this section some examples of the stress-strain behaviour of the ideal granular material are presented. These examples have been obtained by numerical integration of Eqs. (3.38). The material functions included in these equations have been previously determined in Section 3.8.1.

The following examples will be given for the two values of constant z : $z = 0.6$ and $z = 1.0$.

In Figure 22 there are plotted the relations $\Gamma - \epsilon_1$ and $\rho - \epsilon_1$ for axisymmetric compression. The loading program is defined by

$$\dot{\epsilon}_1^p(t) > 0, \quad \sigma_2 = \sigma_3 = \text{const.}, \quad \rho_0 = 1.716 \quad 3.62$$

Note that the value of stress $\sigma_2 (= \sigma_3)$ does not influence $\Gamma - \epsilon_1$ relations. There can be observed a fair agreement with the results of experiments. This indicates that the behaviour of real granular materials can be well approximated by the proposed theory. That also indicates that the assumptions for the argument and the form of material function n (Eqs. 3.48 and 3.50) are reasonable, at least for the presented experimental data. It should be noticed that considerably

better agreement can be achieved by assuming another value for κ^* (see Eq.3.56). This will be discussed further on.

In Figure 24 there is presented the influence of the initial density of material for the loading program given in (3.62). The behaviour of the material for which $\rho_0 = \rho_f^p$ is indicated by dotted lines. The elastic limits (r^0) for loose materials are equal to zero, as assumed previously in Section 3.8.1. The symbol $\{ \}_p$ denotes the peak-stress ratio conditions. The relation ρ_0 vs. $\{\dot{v}/\dot{\epsilon}_1\}_p$ vs. $\{r\}_p$ is presented in Figure 25.

The influence of stress invariant b on stress-strain-density relations is presented in Figure 26, where $z = 1.0$ has been assumed. The loading program for any of these relations is

$$\dot{\epsilon}_1(t) > 0, \quad b = \text{const.}, \quad \sigma_3 = \text{const.}, \quad \rho_0 = 1.716$$

Similar results for $z=0.6$ are presented in Figure 27. In this case the graphs $\rho - \epsilon_1^p$ are not plotted since they are very similar to those of Figure 26. Instead, the relations between principal total strains are presented. The "failure line" indicates the principal strains under the peak-stress conditions. The curves from Figure 27 are denoted in the same way as in Figure 26. The peak-stress envelopes from the results of these two figures are plotted in Figure 28. One may note that the presented stress-strain relations are in good agreement with the results of experiments by Lade and Duncan (1973), if $z < 1$ is assumed.

Figure 24: Influence of Initial Density on Stress-Strain-Density Behaviour

Figure 25: Stress-Initial Density-Rate of Dilatation
Relationship for the Peak-Stress Conditions

Figure 26: Influence of the Stress Invariant b on Stress-Strain-Density Behaviour ($z=1.0$)

Figure 27: Influence of the Stress Invariant b' on Stress-Strain Behaviour ($z=0.6$)

Figure 28: Strength Envelopes

The influence of the hardening/softening parameters (ρ^P and κ) on the yield function can be investigated using Eq. (3.35). This is presented by the graphs of yield function on an octahedral plane in Figure 29. In this figure there are also presented the projections of the plastic strain-rate vectors.

The influence of the hardening/softening parameters on the stress-strain behaviour of material can be investigated by an analysis of the hardening function. This will be done for axisymmetric compression by means of Eqs. (3.22) and (3.23) which lead to

$$\dot{\epsilon} = h \dot{\epsilon}_1^P = (h_\kappa + h_\rho) \dot{\epsilon}_1^P$$

where h_κ and h_ρ describe the "amount" of stress rate due to the changes of parameters κ and ρ^P , respectively. In Figure 30 there are plotted the graphs of h_κ and h_ρ versus axial strain. These graphs have been obtained for the same conditions as the stress-strain relation presented in Figure 22.

One may observe, therefore, that the influence of the changes of parameter ρ^P is negligible in the range of small deformations. This is an important result. It should not be, however, misunderstood. It does not mean that ρ^P does not influence stress-strain relations. On the contrary, as shown in Figure 24, ρ^P influences strongly these relations but the changes of ρ^P during a loading process can be neglected as far as small deformations are concerned.

Figure 29: Influence of the Hardening/Softening Parameters (ρ^p and κ) on the Yield Function

Figure 30: Influence of the Changes of ρ^D and κ on the Hardening/Softening Function h

Figure 31: Influence of Initial Value of κ on the Stress-Strain Behaviour

Figure 32: Influence of the "Parameter" κ^* on the Stress-Strain Behaviour.

Finally, in Figures 31 and 32 the influence of the values of κ^0 and κ^* on axisymmetric compression stress-strain relations is indicated. It follows that an assumed value of κ^0 does not significantly influence stress-strain relations, although, elastic limits can be considerably changed. It also follows that, unless function r (or parameter κ) is experimentally determined, the "parameter" κ^* can be used for fitting theoretical relations with experimental data.

3.9 Conclusions

The theory proposed for locally homogeneous granular materials has been modified by introducing of the new material function $r(\epsilon_{ij}^p)$. By means of this function the new hardening parameter κ has been defined. Therefore, the theory proposed for locally nonhomogeneous materials contains two hardening/softening parameters, i.e., σ^p and κ . The presented interpretation of κ seems to indicate an analogy between the plastic behaviour of locally nonhomogeneous granular materials and the behaviour of two bodies in frictional sliding. As discussed in Section 3.5 the proposed theory is capable to explain the basic phenomena known from experimental tests, i.e., hardening, softening, and peak-strength behaviour.

The additional postulate, introduced in Section 3.7, seems to be of particular importance since it considerably simplifies both the constitutive equations as well as the procedure to estimate material functions m and n . However, there is no available theoretical

suggestions by which this postulate could be supported. On the other hand, as discussed previously in Section 2.11, there is some experimental evidence which indicate that the case $z = \text{const.} = 1$ could be assumed. If so, the additional postulate would be satisfied as well.

It is to be noted that there are difficulties in the description of the plastic behaviour of loose, locally nonhomogeneous materials. It results from the inequality (3.29) that there exists a limitation for parameter κ . This limitation depends on the initial (plastic) density of material in spite of that κ , as a function of ϵ_{ij}^p , is independent of the plastic density of material. It is interesting to note that if it is supposed that κ describes some anisotropic properties of material (as discussed in Section 3.3) then it would follow that loose materials have to be anisotropic to be deformed in a stable way under the loading conditions different than purely hydrostatical. The abovementioned result must be considered carefully if the behaviour of real granular materials is considered. There are some limitations for the use of the theory which are discussed next.

SUMMARY AND DISCUSSION

The formulation of the theory. An incremental theory of plasticity for granular materials has been developed by means of the phenomenological approach. This theory is capable of describing the plastic deformations of granular materials under the loading conditions different from purely hydrostatical. The theory has been formulated without employing the formalism of the classical incremental theories of plasticity. A frictional analogy and some intuitive assumptions constitute the basis of the proposed theory. These can be summarized as follows:

- i) the postulate for energy dissipation (Eq.2.70) and its modification for locally nonhomogeneous materials (Eq.3.30); the physical interpretation of this modification (Section 3.3) allows to suggest a frictional analogy (Section 3.4),
- ii) the postulate for the mode of plastic deformation (Eqs. 2.72, or 3.31) introduced by an analogy to the frictional sliding of a rigid body along a curvilinear surface,
- iii) the limitations on material functions established by means of the thermodynamic (2.9), the dynamic (2.10) and the kinematic (2.12) restrictions for the properties of granular materials,
- iv) the limitations on the derivatives of material functions established by means of intuitive assumptions (Sections 2.5 and 3.3).

It should be emphasized that the limitations on material functions β_a and β_b have been established for the linear in b forms of the functions g_i . As an additional restriction, $\dot{\epsilon}_3^p$ has been assumed to be negative for axisymmetric compression and extension; this leads to some further limitations for β_a and β_b (Section 2.8.2).

The frictional analogy. In the constitutive equations describing the plastic behaviour of locally nonhomogeneous materials there are included two hardening/softening parameters i.e., the plastic density of material (ρ^p) and the density of plastic regions (κ). It has been suggested that the plastic behaviour of the ideal granular material, as governed by these two parameters, is analogous to the behaviour of two bodies in friction sliding. The ρ^p - effect has been suggested to be analogous to the frictional motion of a rigid body along a curvilinear surface. In this case the behaviour of the frictional system is governed by Amonton's friction law (which provides a force equation) and by the variable kinematics resulting from the curvature of the slide surface. The κ - effect has been suggested to be analogous to the behaviour of two bodies in frictional sliding in the range of small displacements, as explained by Bowden-Tabor's theory of friction. The phenomenological meanings of the parameters κ and ρ^p are different (Section 3.3). This is analogous to the case of the frictional system where the slope of

the slide surface can be measured by means of a phenomenological method and no knowledge of the loading history if needed to predict the response of the system. Similarly, in the case of granular materials, ρ^P can be measured by means of a phenomenological method. Otherwise, the area of real contact in the friction case and the density of plastic regions in the case of granular materials are to be determined (on the phenomenological level) by means of the histories of loading and constitutive assumptions.

It should be however emphasized that both the slope of the slide surface and the parameter ρ^P can also be determined by means of the histories of loading and constitutive assumptions.

If the influence of the κ -effect on the behaviour of granular materials is negligible ($\kappa = \text{const.}$), the constitutive equations for plastic deformations of locally nonhomogeneous materials reduce to those for locally homogeneous materials.

Stress-strain behaviour. The description of the plastic deformations by means of the two hardening/softening parameters leads to the simple explanations of the total effects of hardening, softening and the peak-stress behaviour. The existence of the final state results from: i) the assumption that the material functions (the material properties) change monotonically during a plastic deformation process ii) the intuitive assumptions for the derivatives of material functions (i.e., $d(nz)/d\rho^P < 0$, $\dot{\kappa} \geq 0$).

Limitations of the theory. It seems that the assumption on the isotropy of the plastic properties of granular materials is the strongest simplification if the behaviour of real materials is to be described. Therefore, the theory is expected to give proper results only for the loading conditions where no considerable rotation of stress (or strain-rate) directions occurs. The theory, in its present form, is not capable of describing the plastic deformations due to the increase of (purely) normal stresses (that is a consequence of the use of the frictional analogy: the theories of friction do not, in principle, deal with the effect of irreversible displacements due to the application of pure normal forces). Therefore, the theory may give worse results if applied for the description of the behaviour of loose materials; in this case the plastic deformations due to the increase in the level of normal stresses are expected to be larger. It does not however mean that these deformations cannot be taken into account. This could be done by the introduction of an additional constitutive law which describes such deformations, as has been proposed, for instance, in Lade's theory. The theory, in its present form, is incapable of describing the effect of grain crushing (non-frictional mechanism of plastic deformations). Therefore, the use of the theory is to be limited to the stress levels for which this effect can be neglected. If grains crushing occurs the material functions are expected to depend on the stress level.

✓
Some open questions. The main questions of an importance to the presented theory, which have been unanswered in this study, are: 1. Is there any theoretical reasoning for the restriction $\dot{\epsilon}_3^p < 0$ under general loading conditions? 2. Is there any theoretical reasoning to eliminate the case $z < -1$ or support the alternative case, $z > 1$? 3. Are there any theoretical suggestions to find the arguments (the invariants of plastic strain tensor) of material function r ? 4. Is there any theoretical reasoning to justify the simplifying postulate introduced in Section 3.7? 5. Is it possible to answer the questions 1, 2, 3 and 4 by means of phenomenological considerations only?

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