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Interacting Systems and Subordinated Systems in Time-varying and Random Environments

by

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A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics
Ottawa-Carleton Institute for Mathematics and Statistics

Carleton University
Ottawa, Ontario, Canada
September, 2005

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Abstract

In this thesis we develop two topics in the theory and application of stochastic systems, namely, 1) a class of interacting systems and 2) subordinated systems. We study in both cases the systems evolving in time-varying and random environments.

The first topic includes two subtopics: A) multiagent models and B) interacting agent feedback finance models. The multiagent models with multinomial sampling, which are related to the Wright-Fisher model, have one kind of interaction, multinomial sampling. The interacting agent feedback finance models have two kinds of interaction, i.e., the effect of the empirical distribution of the agent types and the feedback of market price on the switching of an agent's type.

In Chapter 2, we use various methods or criteria to analyze the asymptotic behavior of four kinds of multiagent models. Especially, we develop the weak convergence criteria for time inhomogeneous Markov processes to prove the weak convergence of the general multiagent models. The theory of *Volterra integral equations of second kind* is employed to prove the uniqueness of the solution of the martingale problem for the general multiagent models.

In Chapter 3, we discuss the interacting agent feedback finance models. The weak convergence criteria for time inhomogeneous Markov processes is used to analyze the asymptotics of these models. We also develop the theory of uniqueness of *nonlinear Volterra equations of second kind* to prove the uniqueness of the martingale problem. A detailed case study of a simplified example is given to illustrate the interacting agent feedback finance models.

The subordinated systems are discussed in Chapter 4. The main result is the establishment of several versions of weak convergence theorems for continuous processes and *càdlàg* processes under random time change, which provide a unified tool for proving weak convergence related to the subordinated systems. This is illustrated by an example of time inhomogeneous critical linear growth birth-death processes evolving in various random environments.

Acknowledgements

First of all, I would like to express my sincere gratitude to Professor Donald Dawson, my thesis supervisor, for his constant and patient guidance and especially for the invaluable research experience I learned in the research of this thesis. I also thank him for his supports, and concerns about my life and career during these years.

I would like to thank Professor Amitava Bose and Professor André Robert Dabrowski, who conducted the advanced comprehensive exam for my Ph. D. Program and taught me two probability courses. I also thank Professor Amitava Bose for his help and suggestions of many references.

I am indebted to Professor Shixue Cheng, my thesis supervisor for Master's Degree, for his critical comments on the topic of subordinated systems. I would like to thank Professor Zenghu Li for his comments on the weak convergence criteria. I would like to thank Professor C.K. Fong for his references on analysis and David Amundsen for his references on equations.

I would like to thank the School of Mathematics and Statistics, Carleton University, for providing the necessary research facilities for completing this thesis. My thanks also go to the Faculty members of the School, who provided the excellent academic environment with many interesting courses and seminars. Special thanks are given to Dr. Gang Li for his assistance with Latex.

I wish to thank my parents for their love and concerns. I wish to give my heartfelt thanks to my wife, Na Liu. She has been patiently supporting me for many years and just brought a new source of joy to my family, my daughter Hannalee. My sincere thanks are given to my parents-in-law, who help us with every respect of my family.

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Chapter 1

Introduction

The objective of this thesis is to develop two topics in the theory and application of stochastic systems, namely, 1) a class of interacting systems and 2) subordinated systems. In both cases we will consider the systems evolving in time-varying and random environments. Most processes arising from the first topic, whose parameters are functions of time $t \in [0, \infty)$ or stochastic processes, are time inhomogeneous Markov processes. The processes arising from the second topic have simpler structure than those processes arising from the first topic do. They are some basic stochastic processes, which are time homogeneous Markov processes, time changed by subordinators. That is to say, the parameters of these processes are related to the time scale of the basic processes.

The first topic is a huge one which includes two subtopics: A) multiagent models and B) interacting agent feedback finance models. We will discuss the multiagent models in Chapter 2. For the multiagent models, we will assume that there exist a closed system of N agents where each agent has an internal state from an internal

system which contains r internal states. These N agents are subdivided into r types corresponding to the internal states. Each agent can change his/her type by his/her observation of the performances of the internal states or by the relative frequency of different internal states among the others (The latter is being called multinomial sampling). The asymptotics as $N \rightarrow \infty$ of four kinds of multiagent models will be the main issues of Chapter 2. These four kinds of multiagent models are homogeneous multiagent models, evolutionary multiagent models, general multiagent models and evolutionary multiagent models in random environments. For each of these four kinds of models, two cases, i.e., the model with no multinomial sampling and the model with multinomial sampling, will be dealt with separately. There is only one kind of interaction, which is the effect of the empirical distribution of the types of agents on the choice of each agent for switching his/her type, among agents in the multiagent models with multinomial sampling. The first two kinds of multiagent models, i.e., homogeneous multiagent models and evolutionary multiagent models, are the special cases of the general multiagent models. The homogeneous multiagent models are solved by the classical convergence criteria for homogeneous Markov processes. The evolutionary multiagent model in absence of multinomial sampling is solved by the standard martingale problem method for proving weak convergence. For proving the weak convergence of the general multiagent models, we will develop in subsection 2.4.1 the weak convergence criteria for time inhomogeneous Markov processes. The evolutionary multiagent models in random environments are discussed, based on the results of evolutionary multiagent models in deterministic environments.

The interacting agent feedback finance models are discussed in Chapter 3. In addition to the assumptions on the multiagent models discussed in Chapter 2, we assume that there is a financial asset in a certain financial market whose price is

observable by all agents and there exists a recursive log price mechanism, that is, the log price at current time unit can be computed, given the log price at the previous time unit and the empirical distribution of the types of agents at current time unit. Each agent knows this market price mechanism and the change of types of agents will be based on the current empirical distribution of the types of agents as well as the current market price of the financial asset. In these models, the current market price has an influence, which is called *feedback*, on the change of the internal states of agents. We will see two kind of interactions among agents in these models. The first kind is the effect of the empirical distribution of the types of agents on the choice of each agent for switching his/her type. The second kind is the market price feedback, which also depends on the empirical distribution of types of agents, on the choice of each agent for switching his/her type.

The second topic, subordinated systems, will be discussed in Chapter 4. *Subordination* occurs when a basic process X is time changed by another nonnegative nondecreasing process Λ , which is called a subordinator, by the composition $X \circ \Lambda$. The basic processes are assumed to be continuous processes or *càdlàg* processes. One of the main results of the thesis is to establish the weak convergence of continuous processes and *càdlàg* processes under random time change. There are at least three versions of weak convergence related to the subordinated systems, e.g., quenched weak convergence, annealed weak convergence and joint weak convergence. Under rather general assumptions, we will prove weak convergence theorems for these three versions for the subordinated systems. We will also give an example of the applications of these weak convergence theorems.

In the rest of this introduction, we will explain the economic motivations of the

interacting systems and then describe the mathematical framework, methods, and results of interacting systems in time-varying and random environments in greater details. We will also do the same things for the subordinated systems, that is to say, we will explain the motivation of the subordinated systems and then describe the mathematical methods and results of the subordinated systems in time-varying and random environments.

The price trajectory of a risky asset in a financial market is usually described by a function on $[0, \infty)$ for each sample point in some probability space $(\Omega, \mathcal{F}, P)$. In 1900, Bachelier [BA00] first introduced Brownian Motion to model the price evolutions on the Paris stock exchange. Geometric Brownian motion as a model in mathematical finance was suggested by Samuelson [SA64] in 1964 and is now a classical model studied by many authors. The market reality indicates that there are huge number of active traders on the financial market whose demands directly generate the price of a given financial asset. Föllmer [FÖ74] considered an interacting agent model of endogenous preference formation. Black [BL86] classified traders into two groups: information traders and noise traders. In many recent references, information traders and noise traders are notionally equivalent to fundamentalists and chartists. Day and Huang [DH90] classified traders into three groups: α -investors, β -investors and market makers.

Stochastic models of interacting systems play an important role in population biology [KIN80]) and statistical physics [CH92]. In recent years a number of leading thinkers have expressed the need for developing economic models that incorporate interactions between agents and evolutionary mechanisms (see [PAH⁺94] and [FL99]). Before we construct such economic models, we are going to give an account of some

aspects of the works done by other authors, which are related to our construction.

Föllmer and Schweizer [FS93] in 1993 considered an interacting agent financial model in which they used Black's classification of traders: information trader and noise traders (The number of traders is assumed to be countable at all times). In their model, they introduced an individual excess demand function which takes the simple log-linear form:

$$e_{a,k}(p, \omega) = \alpha_{a,k} \log(\hat{S}_{a,k}(\omega)/p) + \delta_{a,k}(\omega), \quad (1.0.1)$$

where $a \in A$ and A is a finite set of agents, k is a period of time, ω is a sample point in some underlying probability space, $\alpha_{a,k}$ is a random coefficient, $\delta_{a,k}$ is viewed as liquidity demand and $\hat{S}_{a,k}$ denotes an individual reference level of agent a for period k . The stock price $S_k(\omega)$ in period k is determined by the equilibrium condition of zero excess demand

$$\sum_{a \in A} e_{a,k}(S_k(\omega), \omega) = 0. \quad (1.0.2)$$

Lux (see [LU95], [LU97] and [LU98]) divided the traders, whose number is assumed to be finite at all times, into two groups: chartists and fundamentalists. Lux also divided the chartists into two subgroups: optimists and pessimists. In Lux's model, the three types of traders can probabilistically change their types, according to the profitability of each type. In Lux's model, there are also new entrants into the market and exits of current traders from the market.

Horst (see [HO00], [HF01], [HO02] and [HO05]) kept some aspects of Föllmer and Schweizer's model. Horst considered interacting agent models with local and global interactions. He assumed that the set of active agents $\mathcal{A}$ is a countable set and there is a sequence of finite sets $\mathcal{A}_n$ satisfying $\lim_{n \rightarrow \infty} \mathcal{A}_n = \mathcal{A}$. In Horst's example, the

traders are divided into fundamental traders and noise traders. But Horst divided the fundamental traders into optimistic and pessimistic fundamental traders when he introduced the concept of individual mood into his models. At each period $t \geq 0$, each fundamental trader has his/her mood, e.g., $x_t^a = +1$ or $x_t^a = -1$, that is to say, the fundamental trader is a optimist or a pessimist. Let C be a fixed set of individual states, i.e., $x_t^a \in C$ for each $a \in A$ and $t \geq 0$. Let $x_t = \{x_t^a\}_{a \in A}$. Horst define the empirical distribution as follows:

$$\rho_t = \rho(x_t) = \lim_{n \rightarrow \infty} \frac{1}{|\mathcal{A}_n|} \sum_{a \in \mathcal{A}_n} \delta_{x_t^a}(\cdot). \quad (1.0.3)$$

Horst calls $\rho(x_t)$ the mood of the market at time t .

The market mood drives the market price in the following way:

$$p_{t+1} = f(\rho_{t+1}, p_t), \quad (1.0.4)$$

where p_t is the market price in period t and f is a certain function. Assuming a simple log-linear structure for the excess demand function, Horst got the recursive log price formula of the following form: for each period $t \geq 0$,

$$\log p_{t+1} = f(\rho_{t+1}) \log p_t + g(\rho_{t+1}). \quad (1.0.5)$$

Note that the function f in (1.0.4) is different from that one in (1.0.5). The mood for each individual $a \in A$ evolves as follows:

$$\pi_a(x_{t+1}^a = s | x_t, e_{t+1}, h_t) = \pi(x_{t+1}^a = s | x_t^a, e_{t+1}, h_t), \quad (1.0.6)$$

where $s \in C$, and $e_{t+1} \sim Q(\rho(x_t); \cdot)$ is the signal of the market mood $\rho(x_t)$ and h_t is some (exogenous) economic fundamentals revealed in period t . Therefore, the type of each trader can be changed during the evolution of the models and there are

interactions among traders in Horst's models. But there is no feedback of the price p_t on the evolution of individual state.

The common aspects of Lux's and Horst's models are as follows. There are countably many or finite agents which are grouped into certain types. The type of each agent can be changed during the evolution of the models. There are interactions among agents. We will keep these aspects of Lux's and Horst's models when we construct our models. We will assume that the agents are of fixed finite number N , and there are no new entrants into the market and exits of current agents from the market. Therefore, we will assume the change of types among agents in the way related to that used by Lux when we construct our multiagent models in the absence of multinomial sampling. But we will use Horst's idea to assume a recursive log price formula when we construct our interacting agent feedback finance model (IAFFM). When we give an example of IAFFM, we will use Föllmer and Schweizer's individual excess demand function (1.0.1) with some modifications.

In Chapter 2, we assume that there are N agents in a certain system, and agents are grouped into $r \geq 2$ types. There are no entries of new agents into the system or exits of current agents from the system. $\mathbf{n}^N(k) = (n_1^N(k), \dots, n_r^N(k))$ is the distribution of N agents among r types at each time unit $k \geq 0$. As shown in Horst's models, the evolution of the empirical distribution of the types $\frac{\mathbf{n}^N(k)}{N}$, which is similar to Horst's market mood, is of vital importance. We will discuss the asymptotics of $\{\mathbf{X}^N(t) = \frac{\mathbf{n}^N([Nt])}{N}, t \geq 0\}$ in Chapter 2. Before we describe the results related to it, we should specify the transition structure of $\{\mathbf{n}^N(k), k \geq 0\}$. Let $\{A_N(t) = (a_{N,i,j}(t))_{r \times r}, 0 \leq t < \infty\}$ and $\{A(t) = (a_{i,j}(t))_{r \times r}, 0 \leq t < \infty\}$ be $r \times r$ matrix-valued càdlàg functions on $[0, \infty)$ which satisfy the conditions specified at the beginning of

subsection 2.3.1 and $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$, where d_U is the uniform topology on $D_{R^r \times r}[0, \infty)$. Define

$$P_{N,k} = I + \frac{1}{N} A_N\left(\frac{k}{N}\right) \quad (1.0.7)$$

as the internal transition matrix of $\{\mathbf{n}^N(k), k \geq 0\}$. This kind of multiagent model is called a general multiagent model (GMAM). If $A(t)$ is a continuous matrix-valued function on $[0, \infty)$, we call the corresponding model an evolutionary multiagent model (EMAM). If $A_N(t) = A$ and $A(t) = A$, we call the corresponding model homogeneous multiagent model (HMAM). Assume that $\mathbf{X}^N(0) \Rightarrow \mathbf{x}_0$ as $N \rightarrow \infty$, then the limit process of $\{\mathbf{X}^N(t), t \geq 0\}$ is the solution of the following differential equations with initial condition $\mathbf{x}(0) = \mathbf{x}_0$:

$$\frac{d\mathbf{x}'}{dt} = A(t)' \mathbf{x}', \quad (1.0.8)$$

where $'$ denotes transpose.

There are no interactions among agents in the above constructed multiagent models. One way of introducing interactions is by multinomial sampling. This will make sense since each agent might change his/her type according to the relative popularity of the types, i.e., empirical distribution of the types $\frac{\mathbf{n}^N(k)}{N}$. We illustrate the general multiagent model with multinomial sampling (GMAMWMS) by the Figure 1.1. The

Figure 1.1: The structure of GMAMWMS

detailed description of motivations and formulations is in section 2.1, subsection 2.2.2 and 2.4.3.

Then based on the assumptions on $A(t)$, we have **1)** a general multiagent model with multinomial sampling (GMAMWMS) if $A(t)$ is a *càdlàg* function on $[0, \infty)$; **2)** an evolutionary multiagent model with multinomial sampling (EMAMWMS) if $A(t)$ is a continuous function on $[0, \infty)$; and **3)** a homogeneous multiagent model with multinomial sampling (HMAMWMS) if $A(t)$ is a constant matrix A on $[0, \infty)$. We note that HMAMWMS is closely related to the Wright-Fisher model [WR31] which arises in population biology. The type change determined by the matrix A is similar to the gene mutation in the Wright-Fisher model.

We will use classical convergence criteria for homogeneous Markov processes to prove the weak convergence of HMAM and HMAMWMS in subsection 2.2. We note that the convergence criteria used here permit the addition of two generators, which make it possible to first prove the weak convergence of HMAM and then use the results of HMAM and the Wright-Fisher model which has just multinomial sampling to prove the weak convergence of HMAMWMS. We note that Bernstein's theorem plays an important role in the proof of the weak convergence of HMAMWMS. We will use the standard martingale problem method to prove the weak convergence of EMAM in subsection 2.3.2 and find that it is very complicated to prove the weak convergence of EMAMWMS by the standard martingale problem method. Instead, we treat the weak convergence of EMAMWMS as a special case of GMAMWMS. Then, it is not surprising that we need to develop new tools for proving the weak convergence of GMAM and GMAMWMS. We will prove preliminary tools for time-dependent generators in subsection 2.4.1. Several weak convergence criteria for time inhomogeneous Markov processes are formulated there, which are the analogues of those criteria for time homogeneous Markov processes. Then, to prove the weak convergence of GMAM, we can easily verify the conditions for tightness based on the

weak convergence criteria in subsection 2.4.1. We will find that the weak convergence criteria used by GMAM also permits the addition of two generators. Then we can again use Bernstein's theorem, the result on the Wright-Fisher model which just has multinomial sampling, and the result from GMAM to verify the conditions for tightness for GMAMWMS. We note that the theory of *Volterra equations of the second kind* is employed to prove the uniqueness of the martingale problem for GMAM and GMAMWMS.

Next, we consider the evolutionary multiagent models in random environments. We assume that $\{A_N(t) = (a_{N,i,j}(t))_{r \times r}, 0 \leq t < \infty\}$ are $D_{R^{r \times r}}[0, \infty)$ -valued stochastic processes and $\{A(t) = (a_{i,j}(t))_{r \times r}, 0 \leq t < \infty\}$ is $C_{R^{r \times r}}[0, \infty)$ -valued stochastic processes and $A_N \Rightarrow A$. We can construct the evolutionary multiagent model in random environment (EMAMIRE) and evolutionary multiagent model with multinomial sampling in random environment (EMAMWMSIRE) by following the procedures in subsection 2.5.2. Since $A_N \Rightarrow A$, we can use the Skorohod Representation theorem to construct a sequence $\{\hat{A}_N\}$ and $\hat{A}$, which have the same probability laws as those of $\{A_N\}$ and A , on some probability space, satisfying $\lim_{N \rightarrow \infty} d(\hat{A}_N, \hat{A}) = 0$ a.s. Since A is a $C_{R^{r \times r}}[0, \infty)$ -valued process, so is $\hat{A}$. Then it is clear that $\lim_{N \rightarrow \infty} d_U(\hat{A}_N, \hat{A}) = 0$ a.s. Then we can use the results of EMAM and EMAMWMS in deterministic environment, i.e., $\{A_N\}$ and $\{A\}$ are deterministic functions, to obtain the convergence theorems of EMAMIRE and EMAMWMSIRE.

In Chapter 3, we describe the mathematical framework, methods, and results of the interacting agent feedback finance models (IAFFM). We note that we will not consider these models with multinomial sampling. We assume that $\{A_N(t, \mathbf{x}, q) = (a_{N,i,j}(t, \mathbf{x}, q))_{r \times r}, 0 \leq t < \infty\}$ and $\{A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}, 0 \leq t < \infty\}$ are $r \times r$

matrix-valued functions on $[0, \infty) \times K_N \times R$ which satisfy the conditions specified at the beginning of subsection 3.2.1. We need to assume some kind of uniform convergence of A_N to A as $N \rightarrow \infty$. One sufficient condition, which we will use in the example of IAFFM in random environment in section 3.3, is

$$\lim_{N \rightarrow \infty} \sup_{q \in R} \sup_{\mathbf{x} \in K_N} d_U(A_N(\cdot, \mathbf{x}, q), A(\cdot, \mathbf{x}, q)) = 0. \quad (1.0.9)$$

Define

$$P_{N,k}[\mathbf{x}, q] = I + \frac{1}{N} A_N\left(\frac{k}{N}, \mathbf{x}, q\right). \quad (1.0.10)$$

In addition to the multiagent systems, we need also to define the price mechanism for the financial asset. Define a real valued function $g_N(t, \mathbf{x}, q)$ on $[0, \infty) \times K_N \times R$ for each $N \geq 2$. We also define a real valued function $g(t, \mathbf{x}, q)$ on $[0, \infty) \times K \times R$. We assume that $g_N(t, \mathbf{x}, q)$, $g(t, \mathbf{x}, q)$ have the forms

$$g(t, \mathbf{x}, q) = \varphi(t, \mathbf{x}, q)q + \psi(t, \mathbf{x}, q), \quad (1.0.11)$$

$$g_N(t, \mathbf{x}, q) = \varphi_N(t, \mathbf{x}, q)q + \psi_N(t, \mathbf{x}, q), \quad (1.0.12)$$

where $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are functions on $[0, \infty) \times K \times R$, and $\varphi_N(t, \mathbf{x}, q)$ and $\psi_N(t, \mathbf{x}, q)$ are functions on $[0, \infty) \times K_N \times R$. Then, we can specify the price mechanism for IAFFM. By following Horst's idea, we will assume a log price mechanism. Fix N , for $k \geq 1$, assume that the empirical distribution $\frac{\mathbf{n}^N(k)}{N}$ and the log price $\tilde{q}^N(\frac{k-1}{N})$ of the previous period $k-1$ are given, then the current log price is determined by

$$\tilde{q}^N\left(\frac{k}{N}\right) = \tilde{q}^N\left(\frac{k-1}{N}\right) + \frac{1}{N} g_N\left(\frac{k}{N}, \frac{\mathbf{n}^N(k)}{N}, \tilde{q}^N\left(\frac{k-1}{N}\right)\right). \quad (1.0.13)$$

Then $P_{N,k}[\frac{\mathbf{n}^N(k)}{N}, \tilde{q}^N(\frac{k}{N})]$ is the internal transition matrix from $\mathbf{n}^N(k)$ to $\mathbf{n}^N(k+1)$. In this way, we can define $\{(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))\}_{k=0}^\infty$. The details of this formulation is in subsection 3.2.1.

We assume that $\mathbf{X}^N(t) = \frac{\mathbf{n}^N([Nt])}{N}$, $q^N(t) = \tilde{q}^N(\frac{[Nt]}{N})$ and $\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t))$. Assume that $\mathbf{Y}^N(0) \Rightarrow \mathbf{y}_0$. We need to also make some assumptions on φ_N and φ , ψ_N and ψ , $A(t, \mathbf{x}, q)$ and $g(t, \mathbf{x}, q)$, semi-Lipschitz conditions on $A(t, \mathbf{x}, q)$ and $g(t, \mathbf{x}, q)$ in the interval $[0, T]$ for any $T > 0$. Then we can expect that the limit process $\{\mathbf{y}(t), 0 \leq t < \infty\}$ of $\{\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t)), 0 \leq t < \infty\}$ is the solution of the following *nonlinear Volterra equation of the second kind*:

$$\mathbf{y}'(t) = \mathbf{y}'_0 + \int_0^t \mathbf{b}(s, \mathbf{y}(s))' ds, \quad t \geq 0, \quad (1.0.14)$$

where $\mathbf{b}(t, \mathbf{y})$ is defined as follows:

$$b_i(t, \mathbf{y}) = \sum_{j=1}^r x_j a_{ji}(t, \mathbf{y}), \quad 1 \leq i \leq r, \text{ and } \mathbf{y} = (x_1, \dots, x_r, q) \quad (1.0.15)$$

$$b_{r+1}(t, \mathbf{y}) = g(t, \mathbf{y}), \quad (1.0.16)$$

$$\mathbf{b}(t, \mathbf{y}) = (b_1(t, \mathbf{y}), \dots, b_{r+1}(t, \mathbf{y})). \quad (1.0.17)$$

The convergence of $\mathbf{Y}^N$ to $\mathbf{y}$ is proved by a weak convergence criteria of time inhomogeneous Markov processes developed in subsection 2.4.1. We will also develop the theory of uniqueness of the *nonlinear Volterra equations of the second kind* with semi-Lipschitz conditions to prove the uniqueness of the martingale problem for the process $\{\mathbf{y}(t), 0 \leq t < \infty\}$.

We will also give a complete case study of an example of IAFFM in section 3.3. We will use in that example Föllmer and Schweizer's individual excess demand function (1.0.1) to get the exact form of g_N , φ_N , and ψ_N . Then we can get the exact form of g , φ , and ψ . We can also consider the situation when some coefficients of (1.0.1) are evolving in a random environment. Finally, we can get a limit stochastic process $\{\mathbf{Y}(t), 0 \leq t < \infty\}$ on some probability space $(\Omega, \mathcal{F}, P)$, whose $r + 1$ -th

coordinate $Q(t)$ is the log price of a financial asset. But notice that for each sample point $\omega \in \Omega$, $Y(t, \omega)$ is the solution of *nonlinear Volterra equation of the second kind* of the form (1.0.14). This is exactly the case in which the price trajectory of a risky asset in a financial market is described by a function on $[0, \infty)$ for each sample point in some probability space.

In Chapter 4, we explain the motivation of the subordinated systems and then describe the mathematical methods and results of the subordinated systems in time-varying and random environment.

In applications, quite large families of stochastic processes with diffusion limits have a constant time scale parameter. The complicated features of natural phenomena recall the necessity of investigating stochastic processes with a time scale parameter evolving in a deterministic, or another independent random environment. This motivates us to consider weak convergence of the subordinated systems in time-varying and random environments.

Instead of following the standard routine of checking the tightness condition, and proving the uniqueness of martingale problem, we develop a new method which is based on the structure of the subordinated system. Briefly, we should first consider the convergence of time-changed sequence in C or D spaces. The time change is defined as follows: if $z \in C_E[0, \infty)$ or $D_E[0, \infty)$, and $x \in C_{R_+}[0, \infty)$ or $D_{R_+}[0, \infty)$, $z \circ x$ is called z time changed by x where $z \circ x$ is defined by $z \circ x(t) = z(x(t))$ for $0 \leq t < \infty$. We have the following important facts about time changes:

- Let $\{x_n\} \subset C_{R_+}[0, \infty)$, $x \in C_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$. Let $\{z_n\} \subset C_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$. Then we have that

$$\lim_{n \rightarrow \infty} d_U(z_n \circ x_n, z \circ x) = 0.$$

- Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Let $\{z_n\} \subset D_E[0, \infty)$, such that $\{z_n \circ x_n\} \subset D_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d(z_n, z) = 0$. Then we have that $\lim_{n \rightarrow \infty} d(z_n \circ x_n, z \circ x) = 0$.
- Let $\{x_n\} \subset C_{R_+}[0, \infty)$, $x \in C_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$. Let $\{z_n\} \subset C_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$. Then we have that $\lim_{n \rightarrow \infty} d_U((x_n, z_n \circ x_n), (x, z \circ x)) = 0$.
- Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Let $\{z_n\} \subset D_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d(z_n, z) = 0$ and such that $\{z_n \circ x_n\} \subset D_E[0, \infty)$. Then we have that $\lim_{n \rightarrow \infty} d((x_n, z_n \circ x_n), (x, z \circ x)) = 0$.

Based on these facts, we can prove three versions of weak convergence theorems for subordinated systems in random environment. For example, if we assume that $\{X_n\}$ and X are $C_{R_+}[0, \infty)$ -valued processes satisfying $X_n \Rightarrow X$ in the uniform topology on $C_{R_+}[0, \infty)$, and assume that $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, where X_n and Z_n , X and Z are defined on different probability spaces, satisfying $Z_n \Rightarrow Z$ in the uniform topology on $C_E[0, \infty)$, then we can prove $(X_n, Z_n \circ X_n) \Rightarrow (X, Z \circ X)$ in the uniform topology on $C_{R_+ \times E}[0, \infty)$. This is an example of joint weak convergence for subordinated systems. Skorohod's Representation theorem plays a vital role in this proof. We will illustrate these theorems by an example of a critical linear growth birth and death process evolving in random environments. The example shows that these theorems are efficient in considering the weak convergence of a subordinated system with the time scale parameter evolving in various random environments.

To summarize, the main results of the thesis are as follows: Theorem 2.4.12 for the general multiagent model, Theorem 2.4.16 for the general multiagent model with multinomial sampling, Theorem 3.2.14 for the interacting agent feedback finance model, Corollary 2.4.3 and Corollary 2.4.6 for weak convergence of time inhomogeneous Markov processes, Theorem 4.2.16 for quenched convergence, Theorem 4.2.18 for annealed convergence, Theorem 4.2.25 for joint convergence.

Chapter 2

Multiagent Models

In this chapter, we will mainly discuss weak convergence problems of four kinds of multiagent models, i.e. homogeneous multiagent models (HMAMS), evolutionary multiagent models (EMAMS), general multiagent models (GMAMS) and evolutionary multiagent models in random environments (EMAMSIRE). For each of these four kinds of models, two cases, i.e., the model with no multinomial sampling and the model with multinomial sampling, will be dealt with separately. The two cases of HMAMS are homogeneous multiagent with no multinomial sampling model, which is simply called homogeneous multiagent model (HMAM), and homogeneous multiagent model with multinomial sampling (HMAMWMS). Similarly, we have evolutionary multiagent model (EMAM) and evolutionary multiagent model with multinomial sampling (EMAMWMS), general multiagent model (GMAM) and general multiagent model with multinomial sampling (GMAMWMS), evolutionary multiagent model in random environment (EMAMIRE) and evolutionary multiagent model with multinomial sampling in random environment (EMAMWMSIRE).

Section 2.1 gives a brief description of multiagent models. Section 2.2 focuses on HMAMS. Classical convergence theorems for homogeneous Markov processes and classical theories of the Wright-Fisher model in population genetics are employed rather intuitively. Before the discussion of GMAMS, which requires a long preparation of preliminary tools about the convergence theorems for time inhomogeneous Markov processes, we discuss in Section 2.3 EMAM, a special case of GMAM. Standard martingale problem method for proving weak convergence can be used directly on this special case to deal with EMAM. Section 2.4 is devoted to the more complicated time inhomogeneous multiagent models, GMAMS. Several weak convergence criteria for time inhomogeneous Markov processes are formulated in subsection 2.4.1, which are the analogues of those criteria for time homogeneous Markov processes. Some of these criteria are used in chapter 3 when we study the interacting agent feedback finance models (IAFFM). We use the theory of *Volterra integral equations of second kind* to prove the uniqueness of the solution of the martingale problem both for GMAM and GMAMWMS. In section 2.5, we discuss the weak convergence problem for evolutionary multiagent models in random environments (EMAMSIRE).

We note here that some of the weak convergence criteria of time homogeneous Markov processes or of time inhomogeneous Markov processes permit the addition of two generators. Therefore, we can check the tightness and the uniqueness of solution of martingale problem and prove the weak convergence of HMAM or GMAM, and then based on what we did for the this situation check the tightness and uniqueness of solution of martingale problem and prove the weak convergence of HMAMWMS or GMAMWMS. The standard martingale problem method which is employed on EMAM cannot be employed easily on EMAMWMS.

2.1 A brief description of multiagent models

At the beginning, we describe the most general assumptions about the mechanism of the multiagent models:

1. The time is in nonnegative integer units, denoted by $k \geq 0$.
2. There are fixed $N \geq 2$ agents in the multiagent system at all times. There are no entries of new agents into the system or exits of current agents from the system.
3. There is an internal system with which all agents are concerned. The internal system has $r \geq 2$ states whose performances are observed by all agents.
4. The internal system will not change with time $k \geq 0$. That is to say, at any time $k \geq 0$, there is no new state added to the internal system and there is no existing state removed from it.
5. Each agent must have his/her preference on one and only one internal state at each time $k \geq 0$. Therefore, the N agents can be classified into r types by their own preferences of the r internal states at any time $k \geq 0$. $\mathbf{n}^N(k) = (n_1^N(k), \dots, n_r^N(k))$ is the distribution of N agents among r types at each time $k \geq 0$. By the second assumption, $n_1^N(k) + \dots + n_r^N(k) = N$ and $\frac{\mathbf{n}^N(k)}{N}$ is the empirical distribution of the types of agents at $k \geq 0$.
6. Immediately after each time unit $k \geq 0$, each agent can have a chance of probabilistically determining his/her preference for the next time unit $k + 1$. The characteristics of a choice is called a strategy or behavior. The strategies or behaviors are corresponded to the internal states. The agents' strategy or

- behavior is realized by keeping their preference or changing their preference, i.e. keeping their type or changing to a different type, for the next time unit.
7. At each time unit $k \geq 0$, each type of agent has a common strategy or behavior, i.e. they have the same probability of switching to the same preference for the next time unit $k + 1$.
 8. The state at time $k + 1$ of each type of agent is obtained in two stages from time k . At stage 1, each agent observes the performance of the internal states and determines the probabilities of switching to another type and switches his/her type accordingly. At stage 2, each agent observes the empirical distribution of the types of agents immediately after the stage 1, and switches his/her type independently, according to the empirical distribution.

Based on the items 1-7, and item 8 with only stage 1, we can construct the multiagent models without multinomial sampling. If we have all the items as our assumptions, we can construct multiagent model with multinomial samplings. The mathematical frameworks in subsections 2.2.1, 2.2.2, 2.3.1 and 2.4.3 will assume the above assumptions.

Voters in a country of parliamentary politics have the parties in their congress as an obvious internal system. The internal states would be the parties of all MP in the congress. Each voter can have one and only one favorite party in an election. Voters can observe the performance of each party and determine their preferences accordingly. Voters might not distinguish the performances of parties easily, they may make their decisions in an election based on the attitudes of their friends, neighbors, colleagues, i.e., they might be influenced by the relative popularity of the parties

and correspondingly vote according to the approximate empirical distribution of all voters. This is the multiagent models with multinomial sampling.

Another example of an internal system is a certain kind of merchandize with many brands. Customers of the merchandize can be viewed as agents in an multiagent system. Customers take different brands as different internal state of the merchandize and evaluate the utility of each brand. Customers can then change their choices in their next purchase by their evaluation of the brands or simply by the popularity of the brands among other customers. Therefore, the multiagent models are reasonable models to study the laws of customers' preferences on the brands of the merchandize.

2.2 Homogeneous multiagent models

In this section, we discuss homogeneous multiagent models (HMAMS). In subsection 2.2.1, we discuss the homogeneous multiagent model(HMAM), which is in absence of multinomial sampling. In subsection 2.2.2, we consider the homogeneous multiagent model with multinomial sampling (HMAMWMS).

2.2.1 Homogeneous multiagent model

Before we go to the main topic, we discuss the matrix power which will be employed later on. Let A be any $r \times r$ square matrix. The Jordan matrix decomposition of A is given by

$$A = QJQ^{-1}, \quad (2.2.1)$$

where Q is a nonsingular matrix, Q^{-1} is the matrix inverse of Q and J is a matrix of Jordan canonical form. We give J precisely as follows

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_b \end{bmatrix} \quad (2.2.2)$$

where the Jordan Block

$$J_l = \begin{bmatrix} \lambda_l & & & & \\ 1 & \lambda_l & & & \\ & 1 & \lambda_l & & \\ & & \ddots & \ddots & \\ & & & \ddots & \lambda_l \\ & & & & 1 & \lambda_l \end{bmatrix} \quad (l = 1, 2, \dots, b) \quad (2.2.3)$$

is a m_l by m_l matrix, and $m_1 + \dots + m_b = r$, b is the number of elementary factors of matrix $A - \lambda I$, I is the identity matrix of order r ; $\lambda_1, \dots, \lambda_b$ are roots of the characteristic equation $\det(A - \lambda I) = 0$; $\lambda_1, \dots, \lambda_b$ are not necessarily distinct.

For each $N \geq 1$, let

$$P_N = I + \frac{1}{N}A. \quad (2.2.4)$$

We have the following lemma about the form of P_N which is convenient for computing matrix powers.

Lemma 2.2.1. *Given the Jordan Matrix Decomposition of A by (2.2.1), (2.2.2) and (2.2.3), we have*

$$P_N = Q \tilde{J}_N Q^{-1} \quad (2.2.5)$$

where

$$\tilde{J}_N = \frac{1}{N}J + I = \begin{bmatrix} \frac{1}{N}J_1 + I_{m_1} & & & \\ & \frac{1}{N}J_2 + I_{m_2} & & \\ & & \ddots & \\ & & & \frac{1}{N}J_b + I_{m_b} \end{bmatrix}, \quad (2.2.6)$$

I_{m_l} is the identity matrix of order m_l , for $1 \leq l \leq b$.

Proof. This follows by (2.2.1), (2.2.2) and (2.2.4). $\square$

Then, we have the following lemma about the limit of the matrix powers of P_N .

Lemma 2.2.2. *Given the Jordan Matrix Decomposition of A by (2.2.1), (2.2.2) and (2.2.3), for any $t \geq 0$,*

$$\lim_{N \rightarrow \infty} P_N^{[Nt]} = Q \exp(Jt) Q^{-1} = \exp(At). \quad (2.2.7)$$

Proof. We denote $J_{N,l} = \frac{1}{N}J_l + I_{m_l}$ for $1 \leq l \leq b$. For any $k \geq m_l$, $J_{N,l}^k$ is equal to

$$\begin{bmatrix} (1 + \frac{\lambda_l}{N})^k & & & \\ \frac{\binom{k}{1}(1 + \frac{\lambda_l}{N})^{k-1}}{N} & (1 + \frac{\lambda_l}{N})^k & & \\ \vdots & \ddots & \ddots & \\ \frac{\binom{k}{m_l-2}(1 + \frac{\lambda_l}{N})^{k-m_l+2}}{N^{m_l-2}} & \frac{\binom{k}{m_l-3}(1 + \frac{\lambda_l}{N})^{k-m_l+3}}{N^{m_l-3}} & \ddots & (1 + \frac{\lambda_l}{N})^k \\ \frac{\binom{k}{m_l-1}(1 + \frac{\lambda_l}{N})^{k-m_l+1}}{N^{m_l-1}} & \frac{\binom{k}{m_l-2}(1 + \frac{\lambda_l}{N})^{k-m_l+2}}{N^{m_l-2}} & \dots & \frac{\binom{k}{1}(1 + \frac{\lambda_l}{N})^{k-1}}{N} & (1 + \frac{\lambda_l}{N})^k \end{bmatrix}.$$

Then for $t \geq 0$, and $1 \leq l \leq b$,

$$\lim_{N \rightarrow \infty} J_{N,l}^{[Nt]} = \begin{bmatrix} 1 & & & \\ t & 1 & & \\ \vdots & \ddots & \ddots & \\ \frac{t^{m_l-2}}{(m_l-2)!} & \frac{t^{m_l-3}}{(m_l-3)!} & \ddots & 1 \\ \frac{t^{m_l-1}}{(m_l-1)!} & \frac{t^{m_l-2}}{(m_l-2)!} & \dots & t & 1 \end{bmatrix} e^{\lambda_l t} = \exp(J_l t). \quad (2.2.8)$$

Since

$$\exp(Jt) = \begin{bmatrix} \exp(J_1 t) & & & \\ & \exp(J_2 t) & & \\ & & \ddots & \\ & & & \exp(J_b t) \end{bmatrix} \quad (2.2.9)$$

(2.2.7) then follows by (2.2.5) and (2.2.6). $\square$

We will use P_N as internal transition matrix in our models. Recall that a *transition matrix* is a stochastic matrix. This requires us putting some conditions on the matrix A when we construct our multiagent models. The matrix $A = (a_{ij})_{r \times r}$ is required to be a *Q-matrix* [AN91], that is to say, A satisfies the following two conditions:

- 1) $A\mathbf{e} = \mathbf{0}_{r \times 1}$, where $\mathbf{e} = [1, \dots, 1]'$ and “ $'$ ” denotes transpose;
- 2) For $1 \leq i, j \leq r$, $i \neq j$, $a_{ij} \geq 0$.

Therefore, if A is a *Q-matrix*, for large enough N , P_N will be a stochastic matrix.

Now we can specify the structure of HMAM. Let us assume that the number of agents in the market is the fixed integer N . For each time period $k \geq 0$, and $1 \leq j \leq r$, $n_j^N(k)$ is the number of agents of type j at period k . Of course, $n_1^N(k) + \dots + n_r^N(k) = N$. We define $\mathbf{n}^N(k) = (n_1^N(k), \dots, n_r^N(k))$. For $k \geq 0$, assume that $\mathbf{n}^N(k)$ is given. Then for each $1 \leq j \leq r$, we define the random vector $\Xi_{k,j} = (\xi_{k,j,1}, \dots, \xi_{k,j,r})$ on some probability space, which satisfies

$$\Xi_{k,j} \sim \text{multinomial}(n_j^N(k), P_{N,j}), \quad (2.2.10)$$

where $P_{N,j}$ is the j -th row of the *internal matrix* P_N . We assume that $\Xi_{k,1}, \dots, \Xi_{k,r}$

are independent. Then,

$$\mathbf{n}^N(k+1) \equiv \Xi_{k,1} + \cdots + \Xi_{k,r}. \quad (2.2.11)$$

The sequence $\{\mathbf{n}^N(k), k \geq 0\}$ defined this way is a homogeneous Markov chain. We call this model the homogeneous multiagent model (HMAM).

We note that P_N is called *internal transition matrix*, not *probability transition matrix*, because P_N is not the probability transition matrix of $\{\mathbf{n}^N(k), k \geq 0\}$, it is the matrix whose row vectors are used as probability mass functions when we generate the random vectors $\Xi_{k,j} = (\xi_{k,j,1}, \dots, \xi_{k,j,r})$ for $1 \leq j \leq r$, whose sum is $\mathbf{n}^N(k+1)$. The probability transition matrix of $\{\mathbf{n}^N(k), k \geq 0\}$, which is called *population transition matrix*, can be obtained by (2.2.10) and P_N , but it is of very complicated form. We will base all our discussions just on P_N .

We next give an important property of $\{\mathbf{n}^N(k), k \geq 0\}$. Define the positive vectors $\mathbf{V} = (v_1, \dots, v_r)'$ and $\mathbf{V}^{\frac{1}{N}} = (v_1^{\frac{1}{N}}, \dots, v_r^{\frac{1}{N}})'$. By (2.2.10), and the independence of $\Xi_{0,j}$'s, we have

$$E\left[\prod_{j=1}^r v_j^{n_j^{N(1)}}\right] = \prod_{j=1}^r (P_{N,j} \cdot \mathbf{V})^{n_j^{N(0)}}. \quad (2.2.12)$$

Then by the Markov property of $\{\mathbf{n}^N(k), k \geq 0\}$, we have for any $k \geq 1$,

$$E\left[\prod_{j=1}^r v_j^{n_j^{N(k)}}\right] = \prod_{j=1}^r (P_{N,j} \cdot P_N^{k-1} \mathbf{V})^{n_j^{N(0)}}. \quad (2.2.13)$$

Now we define $\mathbf{X}^N(t) = \frac{\mathbf{n}^N([Nt])}{N}$ and assume that $\mathbf{X}^N(0) = \frac{\mathbf{n}^N(0)}{N}$ approaches $\mathbf{x}^\infty(0)$ as N goes to ∞ . By (2.2.13), we have

$$E\left[\prod_{j=1}^r v_j^{X_j^{N(t)}}\right] = \prod_{j=1}^r (P_{N,j} \cdot P_N^{[Nt]-1} \mathbf{V}^{\frac{1}{N}})^{n_j^{N(0)}}. \quad (2.2.14)$$

Before we consider the weak convergence problem, we have the following lemmas which will be used later on.

Lemma 2.2.3. *If $\lim_{N \rightarrow \infty} a_N = a$, then $\lim_{N \rightarrow \infty} (1 + \frac{a_N}{N})^N = \exp(a)$.*

Lemma 2.2.4. *If matrix A is a Q-matrix, the internal transition matrix P_N is given by (2.2.4), and $\{\mathbf{X}^N(t), 0 \leq t < \infty\}_{N=1}^\infty$ is defined above, and $\lim_{N \rightarrow \infty} \mathbf{X}^N(0) = \mathbf{x}^\infty(0)$, then for each $0 \leq t < \infty$ and positive vectors $\mathbf{V} = (v_1, \dots, v_r)'$, we have*

$$\lim_{N \rightarrow \infty} E\left[\prod_{j=1}^r v_j^{X_j^N(t)}\right] = \exp\{\mathbf{x}^\infty(0) \exp(At)[\ln v_1, \dots, \ln v_r]'\}. \quad (2.2.15)$$

Proof. By (2.2.14), we need to consider $\lim_{N \rightarrow \infty} (P_{N,j} P_N^{[Nt]-1} \mathbf{V}^{\frac{1}{N}})^N$ for $1 \leq j \leq r$. Since $A\mathbf{e} = \mathbf{0}_{r \times 1}$, we have $P_N \mathbf{e} = (I + \frac{1}{N}A)\mathbf{e} = \mathbf{e}$. Then,

$$P_{N,j} P_N^{[Nt]-1} \mathbf{e} = (\mathbf{e}'_j + \frac{1}{N} A_{j,\cdot}) \mathbf{e} = \mathbf{e}'_j \mathbf{e} = 1, \quad (2.2.16)$$

where $\mathbf{e}_j = [0, \dots, 1, \dots, 0]'$, j -th coordinate being 1, other coordinates being 0, and $A_{j,\cdot}$ is the j -th row of matrix A .

By (2.2.16), we have

$$\begin{aligned} P_{N,j} P_N^{[Nt]-1} \mathbf{V}^{\frac{1}{N}} &= P_{N,j} P_N^{[Nt]-1} \mathbf{e} + P_{N,j} P_N^{[Nt]-1} (\mathbf{V}^{\frac{1}{N}} - \mathbf{e}) \\ &= 1 + P_{N,j} P_N^{[Nt]-1} (\mathbf{V}^{\frac{1}{N}} - \mathbf{e}). \end{aligned} \quad (2.2.17)$$

Using Lemma 2.2.2, we get

$$\begin{aligned} \lim_{N \rightarrow \infty} N P_{N,j} P_N^{[Nt]-1} (\mathbf{V}^{\frac{1}{N}} - \mathbf{e}) &= \lim_{N \rightarrow \infty} P_{N,j} P_N^{[Nt]-1} [N(\mathbf{V}^{\frac{1}{N}} - \mathbf{e})] \\ &= \mathbf{e}'_j \exp(At)[\ln v_1, \dots, \ln v_r]'. \end{aligned} \quad (2.2.18)$$

Then by Lemma 2.2.3, (2.2.17) and (2.2.18), we have

$$\lim_{N \rightarrow \infty} (P_{N,j} P_N^{[Nt]-1} \mathbf{V}^{\frac{1}{N}})^N = \exp\{\mathbf{e}'_j \exp(At)[\ln v_1, \dots, \ln v_r]'\}, \quad (2.2.19)$$

(2.2.15) follows by (2.2.14) and (2.2.19). $\square$

We can interpret Lemma 2.2.4 as the following form.

Lemma 2.2.5. *With the same assumptions as those in Lemma 2.2.4 of HMAM, for $t \geq 0$ and nonnegative vector $\mathbf{u} = [u_1, \dots, u_r]'$, we have*

$$\lim_{N \rightarrow \infty} E[e^{-\mathbf{X}^N(t)\mathbf{u}}] = \exp\{-\mathbf{x}^\infty(0) \exp(At)\mathbf{u}\}. \quad (2.2.20)$$

We will prove that the sequence $\{\mathbf{X}^N(t), 0 \leq t < \infty\}_{N=1}^\infty$ converges weakly to a limit process, which we denote by $\{\mathbf{X}^\infty(t), 0 \leq t < \infty\}$. By Lemma 2.2.5, $\mathbf{X}^\infty(t)$ is actually a deterministic process, and is given by the following equation:

$$\mathbf{X}^\infty(t) = \mathbf{x}^\infty(0) \exp(At), \text{ for } 0 \leq t < \infty. \quad (2.2.21)$$

Next, we introduce some new notations. We put $Z_+ = \{0, 1, \dots\}$,

$$K_N = \{N^{-1}\alpha : \alpha \in (Z_+)^r, \sum_{i=1}^r \alpha_i = N\}, \quad (2.2.22)$$

and

$$K = \{\alpha : \alpha \in (R_+)^r, \sum_{i=1}^r \alpha_i = 1\}. \quad (2.2.23)$$

We will now use $\mathbf{X}_{\mathbf{p}_N}^N$ to denote the process $\mathbf{X}^N$ satisfying $\mathbf{X}^N(0) = \mathbf{p}_N \in K_N$, and $\mathbf{X}_{\mathbf{p}}^\infty$ the process $\mathbf{X}^\infty$ satisfying $\mathbf{X}^\infty(0) = \mathbf{p} \in K$.

We define the linear contraction S_N related to the homogeneous Markov chain $\{\frac{\mathbf{n}^N(k)}{N}, k = 0, 1, \dots\}$ on $C(K_N)$ by

$$S_N f(\mathbf{p}) = E[f(\frac{\mathbf{n}^N(1)}{N}) | \frac{\mathbf{n}^N(0)}{N} = \mathbf{p}], \quad f \in C(K_N), \quad \mathbf{p} \in K_N, \quad (2.2.24)$$

and the semigroup $\{S(t)\}$ related to $\mathbf{X}^\infty$ on $C(K)$ by

$$S(t)f(\mathbf{p}) = E[f(\mathbf{X}_{\mathbf{p}}^\infty(t))], \quad f \in C(K), \quad \mathbf{p} \in K. \quad (2.2.25)$$

It is clear that $\{S(t)\}$ is a time-homogeneous strongly continuous contraction semi-group on $C(K)$. The generator G_A of $\{S(t)\}$ is

$$G_A f(\mathbf{p}) = \mathbf{p} A \frac{\partial f'}{\partial \mathbf{x}}, \text{ for } f \in \mathcal{D}(G_A) = C^1(K), \quad (2.2.26)$$

where $\frac{\partial f}{\partial \mathbf{x}} = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_r}]$.

Before we prove the weak convergence theorem for HMAM, we have the next lemma, which will be used also in the next subsection for the homogeneous multiagent model with multinomial sampling (HMAMWMS).

Lemma 2.2.6. *Under the above conditions, we have*

$$\lim_{N \rightarrow \infty} \sup_{\mathbf{p} \in K_N} |N(S_N - I)f(\mathbf{p}) - G_A f(\mathbf{p})| = 0, \quad (2.2.27)$$

and

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq t_0} \sup_{\mathbf{p} \in K_N} |S_N^{[Nt]} f(\mathbf{p}) - S(t)f(\mathbf{p})| = 0 \quad (2.2.28)$$

for every $f \in C^1(K)$ and $t_0 \geq 0$.

Proof. To prove (2.2.27) and (2.2.28), it suffices by Theorem 6.5 of Chapter 1 [EK86] to show that for any $t > 0$,

$$\lim_{N \rightarrow \infty} \sup_{\mathbf{p} \in K_N} |S_N^{[Nt]} f(\mathbf{p}) - S(t)f(\mathbf{p})| = 0, \quad (2.2.29)$$

i.e.

$$\lim_{N \rightarrow \infty} \sup_{\mathbf{p} \in K_N} |E[f(\mathbf{X}_{\mathbf{p}}^N(t))] - f(\mathbf{X}_{\mathbf{p}}^\infty(t))| = 0 \quad (2.2.30)$$

for every $f \in C(K)$.

We assume the negative of (2.2.30). Then there exists $\epsilon_0 > 0$, for any integer $k > 0$, there exists $N_k > k$, such that

$$\sup_{\mathbf{p} \in K_{N_k}} |E[f(\mathbf{X}_{\mathbf{p}}^{N_k}(t))] - f(\mathbf{X}_{\mathbf{p}}^\infty(t))| > \epsilon_0. \quad (2.2.31)$$

Then for each k , there exists $\mathbf{p}^{N_k} \in K_{N_k}$ with

$$|E[f(\mathbf{X}_{\mathbf{p}^{N_k}}^{N_k}(t))] - f(\mathbf{X}_{\mathbf{p}^{N_k}}^\infty(t))| > \epsilon_0. \quad (2.2.32)$$

Since $\{\mathbf{p}^{N_k}\} \subset K$ and K is compact, there exists a subsequence of $\{\mathbf{p}^{N_k}\}$, for convenience, we still denote it by $\{\mathbf{p}^{N_k}\}$ such that $\lim_{k \rightarrow \infty} \mathbf{p}^{N_k} = \mathbf{p}^0 \in K$.

But by Lemma 2.2.5, we know that $\mathbf{X}_{\mathbf{p}^{N_k}}^{N_k}(t) \Rightarrow \mathbf{X}_{\mathbf{p}^0}^\infty(t)$. Then we have

$$\lim_{k \rightarrow \infty} E[f(\mathbf{X}_{\mathbf{p}^{N_k}}^{N_k}(t))] = f(\mathbf{X}_{\mathbf{p}^0}^\infty(t)). \quad (2.2.33)$$

By (2.2.21), $\mathbf{X}_{\mathbf{p}}^\infty(t)$ is continuous in $\mathbf{p}$, we have

$$\lim_{k \rightarrow \infty} f(\mathbf{X}_{\mathbf{p}^{N_k}}^\infty(t)) = f(\mathbf{X}_{\mathbf{p}^0}^\infty(t)). \quad (2.2.34)$$

(2.2.33) and (2.2.34) lead to a contradiction with (2.2.32). $\square$

We then conclude this subsection with the weak convergence theorem for HMAM.

Theorem 2.2.7. *If matrix A is a Q-matrix, the internal transition matrix P_N is given by (2.2.4), and let $\{\mathbf{n}^N(k), k \geq 0\}$ be the Markov chain defined above. Let $\mathbf{X}^N(t) = \frac{\mathbf{n}^N([Nt])}{N}$ and define $\mathbf{X}^\infty$ by (2.2.21). If $\lim_{N \rightarrow \infty} \mathbf{X}^N(0) = \mathbf{x}^\infty(0)$, then $\mathbf{X}^N \Rightarrow \mathbf{X}^\infty$.*

Proof. This follows from (2.2.28) and Corollary 8.9 of Chapter 4 [EK86]. $\square$

2.2.2 Homogeneous multiagent model with multinomial sampling

To begin with this subsection, we describe the homogeneous multiagent model with multinomial sampling (HMAMWMS). At each time period $k \geq 0$, the empirical

distributions $\frac{\mathbf{n}^N(k)}{N} = (\frac{n_1^N(k)}{N}, \dots, \frac{n_r^N(k)}{N})$ of the r types of agents switches to $\frac{\tilde{\mathbf{n}}^N(k)}{N} = (\frac{\tilde{n}_1^N(k)}{N}, \dots, \frac{\tilde{n}_r^N(k)}{N})$ by the internal dynamics specified in subsection 2.2.1, which is based on the internal transition matrix P_N ; then one multinomial sampling occurs which results in $\frac{\mathbf{n}^N(k+1)}{N} = (\frac{n_1^N(k+1)}{N}, \dots, \frac{n_r^N(k+1)}{N})$ for the next time period $k+1$. The system continues like the previous period and so on. We illustrate HMAMWMS by the Figure 2.1.

Figure 2.1: The structure of HMAMWMS

At first, we consider the system of multinomial sampling without the internal dynamics. This is illustrated by the Figure 2.2. This system is the simplest version

Figure 2.2: The structure of multinomial sampling

of the Wright-Fisher model in Chapter 10 [EK86]. We summarize some results of this system which will be used in this subsection and sections later on. Let

$$\mathbf{Z}^N(t) = \frac{\tilde{\mathbf{n}}^N([Nt])}{N} \quad (2.2.35)$$

We define a contraction T_N related to the homogeneous Markov chain $\{\frac{\tilde{\mathbf{n}}^N(k)}{N}, k =$

$0, 1, \dots\}$ on $C(K_N)$ as follows:

$$T_N f(\mathbf{p}) = E[f(\frac{\tilde{\mathbf{n}}^N(1)}{N}) | \frac{\tilde{\mathbf{n}}^N(0)}{N} = \mathbf{p}], \quad f \in C(K_N), \quad \mathbf{p} \in K_N. \quad (2.2.36)$$

We define the differential operator G_B on $C^2(K)$ by

$$G_B = \frac{1}{2} \sum_{i,j=1}^r b_{ij}(\mathbf{p}) \frac{\partial^2}{\partial x_i \partial x_j}, \quad (2.2.37)$$

where $b_{ij}(\mathbf{p}) = p_i(\delta_{ij} - p_j)$. Let $\{T(t)\}$ be the Feller semigroup on $C(K)$ generated by the closure of $\{(f, G_B f) : f \in C^2(K)\}$. We also use G_B denoting the generator of $\{T(t)\}$. Let $\mathbf{Z}$ be a diffusion process in K with generator G_B . Then we have the following result.

Proposition 2.2.8. *With the conditions above,*

$$\lim_{N \rightarrow \infty} \sup_{\mathbf{p} \in K_N} |N(T_N - I)f(\mathbf{p}) - G_B f(\mathbf{p})| = 0 \quad (2.2.38)$$

for every $f \in C^2(K)$. If $\mathbf{Z}^N(0) \Rightarrow \mathbf{Z}(0)$ in K , then $\mathbf{Z}^N \Rightarrow \mathbf{Z}$ in $D_K[0, \infty)$.

Next, we state the Bernstein's theorem on simplex.

Lemma 2.2.9. *Let K be the simplex defined by (2.2.23) and K_N be defined by (2.2.22). For any $f \in C(K)$, define $f_N(\mathbf{x})$ on K as follows: for $\mathbf{x} = (x_1, \dots, x_r) \in K$,*

$$f_N(\mathbf{x}) = \sum_{\alpha} \frac{N!}{\alpha_1! \dots \alpha_r!} (x_1)^{\alpha_1} \dots (x_r)^{\alpha_r} f\left(\frac{\alpha}{n}\right), \quad (2.2.39)$$

where $\alpha = (\alpha_1, \dots, \alpha_r)$ and $\frac{\alpha}{N} \in K_N$. Then f_N converges uniformly to f on K as $N \rightarrow \infty$.

Proof. A proof for $r = 2$, which can be generalized to the case $r \geq 2$, is in [BI95]. □

Now we go back to our case of homogeneous multiagent model with multinomial sampling (HMAMWMS). The notation $\frac{\mathbf{n}^N(k)}{N}$ is kept since it is clear from the context. But we use $\mathbf{Y}^N(t) = \frac{\mathbf{n}^N(\lfloor Nt \rfloor)}{N}$ for this new situation. We define the linear contraction U_N related to the new homogeneous Markov chain $\{\frac{\mathbf{n}^N(k)}{N}, k = 0, 1, \dots\}$ on $C(K_N)$ by

$$U_N f(\mathbf{p}) = E[f(\frac{\mathbf{n}^N(1)}{N}) | \frac{\mathbf{n}^N(0)}{N} = \mathbf{p}], \quad f \in C(K_N), \quad \mathbf{p} \in K_N. \quad (2.2.40)$$

It is clear from the Markov property of the internal dynamics and multinomial sampling that

$$U_N = T_N S_N. \quad (2.2.41)$$

Let us define a new generator $G_{AB} = G_A + G_B$, whose domain $\mathcal{D}(G_{AB}) = \mathcal{D}(G_A) \cap \mathcal{D}(G_B) = C^2(K)$. We denote by $\mathbf{Y}$ the diffusion process in K with generator G_{AB} , and $\{U(t)\}$ its corresponding Feller semigroup.

Then we have the following weak convergence theorem.

Theorem 2.2.10. *Under the conditions given above, we have*

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq t_0} \sup_{\mathbf{p} \in K_N} |U_N^{[Nt]} f(\mathbf{p}) - U(t)f(\mathbf{p})| = 0 \quad (2.2.42)$$

for every $f \in C^2(K)$ and $t_0 \geq 0$. If $\mathbf{Y}^N(0) \Rightarrow \mathbf{Y}(0)$ in K , then $\mathbf{Y}^N \Rightarrow \mathbf{Y}$ in $D_K[0, \infty)$.

Proof. To prove (2.2.42), it suffices by Theorem 6.5 of Chapter 1 [EK86] to prove that

$$\lim_{N \rightarrow \infty} \sup_{\mathbf{p} \in K_N} |N(U_N - I)f(\mathbf{p}) - G_{AB}f(\mathbf{p})| = 0 \quad (2.2.43)$$

for every $f \in C^2(K)$.

Notice that

$$\begin{aligned}
& |N(U_N - I)f(\mathbf{p}) - G_{AB}f(\mathbf{p})| \\
&= |T_N[N(S_N - I)]f(\mathbf{p}) - G_A f(\mathbf{p}) + N(T_N - I)f(\mathbf{p}) - G_B f(\mathbf{p})| \\
&\leq |T_N[N(S_N - I)]f(\mathbf{p}) - T_N G_A f(\mathbf{p})| + |T_N G_A f(\mathbf{p}) - G_A f(\mathbf{p})| \\
&\quad + |N(T_N - I)f(\mathbf{p}) - G_B f(\mathbf{p})| \\
&\leq \sup_{\mathbf{p} \in K_N} |N(S_N - I)f(\mathbf{p}) - G_A f(\mathbf{p})| + \sup_{\mathbf{p} \in K_N} |T_N G_A f(\mathbf{p}) - G_A f(\mathbf{p})| \\
&\quad + \sup_{\mathbf{p} \in K_N} |N(T_N - I)f(\mathbf{p}) - G_B f(\mathbf{p})|,
\end{aligned} \tag{2.2.44}$$

(2.2.43) is then proved by (2.2.27), Bernstein's theorem (see Lemma 2.2.9) and (2.2.38).

The second statement of the theorem results from (2.2.42) and Corollary 8.9 of Chapter 4 [EK86]. $\square$

Note that we can change the time structure of HMAMWMS in the Figure 2.1 to the structure in the Figure 2.3, where $0 < c < 1$ is a constant. With proper modi-

Figure 2.3: HMAMWMS with different time structure

fications on the definition of T_N , S_N , U_N and so on, we have the same convergence theorem.

2.3 Evolutionary multiagent model

In subsection 2.2.1, we proved the weak convergence of a sequence of time homogeneous stochastic processes to a deterministic process which is the solution of differential equations $\frac{dx}{dt} = A'x$ where A is a Q -matrix. As a special case of a general multiagent model (GMAM) in subsection 2.4.2, we consider in this section the evolutionary multiagent model (EMAM). We will prove the weak convergence of the sequence of evolutionary system to a limit which is the solution of differential equations $\frac{dx}{dt} = A(t)'x$ on any interval $[a, b] \subset [0, \infty)$, where $A(t)$ is a continuous matrix-valued function defined on $[0, \infty)$ and satisfies the condition that for fixed $t \geq 0$, $A(t)$ is a Q -matrix. The martingale problem method will be the basic method for establishing the results of this case. The evolutionary multiagent model with multinomial sampling (EMAMWMS) will be treated as a special case in subsection 2.4.3 with specific lemmas about martingale problem with time-dependent generator.

2.3.1 Formulation of the general multiagent model

Now we define the general multiagent model (GMAM) which is in absence of multinomial sampling. Fix $N \geq 1$, let $A_N(t) = (a_{N,i,j}(t))_{r \times r}$ be a $r \times r$ matrix-valued function on $[0, \infty)$, which satisfies the conditions

- 1) For each $t \geq 0$, $A_N(t)\mathbf{e} = \mathbf{0}_{r \times 1}$;
- 2) For each $t \geq 0$, $1 \leq i, j \leq r$, $i \neq j$, $a_{N,i,j}(t) \geq 0$.
- 3) A_N is a stepwise function, i.e., $A_N(t)$ is a constant on $[\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$.

Let $A(t) = (a_{i,j}(t))_{r \times r}$ be a $r \times r$ matrix-valued *càdlàg* function on $[0, \infty)$, which satisfies the conditions

- 1) For each $t \geq 0$, $A(t)\mathbf{e} = \mathbf{0}_{r \times 1}$;
- 2) For each $t \geq 0$, $1 \leq i, j \leq r$, $i \neq j$, $a_{i,j}(t) \geq 0$.

That is to say, $\{A_N(t)\}$ are *Q-matrix* valued stepwise functions and $A(t)$ is a *Q-matrix* valued function.

GMAM is constructed as follows. Let $R^{r \times r}$ be the Euclidean space corresponding to the $r \times r$ square matrix. For each $N \geq 1$, let $A_N \in D_{R^{r \times r}}[0, \infty)$ be matrix-valued function on $[0, \infty)$ satisfying the conditions above. For each $N \geq 1$, and $k \geq 0$, let

$$P_{N,k} = I + \frac{1}{N}A_N\left(\frac{k}{N}\right). \quad (2.3.1)$$

For each $k \geq 0$, we know that for large enough N , $P_{N,k}$ is a stochastic matrix, which we called internal transition matrix of GMAM.

As in the homogeneous case, we assume that the number of agents in the market is the fixed integer N . For each time period $k \geq 0$, let $\mathbf{n}^N(k) = (n_1^N(k), \dots, n_r^N(k))$, where for $1 \leq j \leq r$, $n_j^N(k)$ is the number of agents of type j at period k . For $k \geq 0$, the transition between $\mathbf{n}^N(k)$ and $\mathbf{n}^N(k+1)$ is given as follows: for each $1 \leq j \leq r$, we can define the random vector $\Xi_{N,k,j} = (\xi_{N,k,j,1}, \dots, \xi_{N,k,j,r})$ on some probability space, which satisfies

$$\Xi_{N,k,j} \sim \text{multinomial}(n_j^N(k), P_{N,k,j}), \quad (2.3.2)$$

where $P_{N,k,j}$ is the j -th row of matrix $P_{N,k}$. We assume that $\Xi_{N,k,1}, \dots, \Xi_{N,k,r}$ are independent. Then,

$$\mathbf{n}^N(k+1) \equiv \Xi_{N,k,1} + \dots + \Xi_{N,k,r}. \quad (2.3.3)$$

The sequence $\{\mathbf{n}^N(k), k \geq 0\}$ defined this way is a time inhomogeneous Markov chain. Without loss of generality, we assume that $\{\mathbf{n}^N\}_{N=1}^\infty$ are defined on the same probability space $(\Omega, \mathcal{F}, P)$. As usual, we define $\mathbf{X}^N(t) = \frac{\mathbf{n}^N([Nt])}{N}$.

We next consider some properties of $\{\mathbf{n}^N(k), k \geq 0\}$. Let $\mathbf{V} = (v_1, \dots, v_r)'$ be a positive vector. For any $k \geq 1$ and $m \geq 0$, by (2.3.2) and (2.3.3), the independence of $\Xi_{N,k,j}$'s, and the Markov property of $\mathbf{n}^N$, we have

$$E\left[\prod_{j=1}^r v_j^{n_j^N(m+k)} \middle| \mathbf{n}^N(m)\right] = \prod_{j=1}^r \left(P_{N,m,j} \cdot \prod_{l=1}^{k-1} P_{N,m+l} \mathbf{V} \right)^{n_j^N(m)}, \quad (2.3.4)$$

where $\prod_{l=1}^{k-1} P_{N,m+l} = P_{N,m+1} \times \dots \times P_{N,m+k-1}$. We make the convention that when we denote the product of a sequence of matrices by prod , we actually make the multiplication from the left to the right as the index increases its order.

Then it follows by (2.3.4) that

$$E[\mathbf{n}^N(m+k) | \mathbf{n}^N(m)] = \mathbf{n}^N(m) \prod_{l=m}^{m+k-1} P_{N,l}, \quad (2.3.5)$$

and for $1 \leq i \leq r$,

$$\begin{aligned} & E[n_i^N(m+k)(n_i^N(m+k) - 1) | \mathbf{n}^N(m)] \\ &= \left(\mathbf{n}^N(m) \left[\prod_{l=m}^{m+k-2} P_{N,l} \right] P_{N,m+k-1, \cdot, i} \right)^2 \\ & \quad - \sum_{j=1}^r \left(P_{N,m,j} \cdot \left[\prod_{l=m+1}^{m+k-2} P_{N,l} \right] P_{N,m+k-1, \cdot, i} \right)^2 n_j^N(m), \end{aligned} \quad (2.3.6)$$

where $P_{N,m+k-1, \cdot, i}$ is i -th column of matrix $P_{N,m+k-1}$. Then we can get

$$\begin{aligned} & E[(n_i^N(m+k) - n_i^N(m))^2 | \mathbf{n}^N(m)] \\ &= \left(\mathbf{n}^N(m) \left[\prod_{l=m}^{m+k-2} P_{N,l} \right] P_{N,m+k-1, \cdot, i} \right)^2 + \mathbf{n}^N(m) \left[\prod_{l=m}^{m+k-2} P_{N,l} \right] P_{N,m+k-1, \cdot, i} \end{aligned}$$

$$\begin{aligned}
& - \sum_{j=1}^r \left(P_{N,m,j}, \left[\prod_{l=m+1}^{m+k-2} P_{N,l} \right] P_{N,m+k-1,\cdot,i} \right)^2 n_j^N(m) \\
& - 2n^N(m) \left[\prod_{l=m}^{m+k-2} P_{N,l} \right] P_{N,m+k-1,\cdot,i} n_i^N(m) + (n_i^N(m))^2.
\end{aligned} \tag{2.3.7}$$

We next prove a proposition which will help us with setting up the martingale problem framework for our systems.

Proposition 2.3.1. *Let $(E, \bar{r})$ be a Polish space. Let $\{Y(k), k \geq 0\}$ be an E -valued time inhomogeneous Markov chain on some probability space with transition operators $\{T(k) : k \geq 0\}$ defined by*

$$T(k)f(Y(k)) = E[f(Y(k+1))|Y(k)], \text{ for } f \in B(E), \tag{2.3.8}$$

where $B(E)$ is the set of bounded Borel measurable functions on E . If $f \in B(E)$, then

$$M(k) = f(Y(k)) - \sum_{l=0}^{k-1} (T(l) - I)f(Y(l)) \tag{2.3.9}$$

is an $\{\mathcal{F}_k^Y = \sigma(Y(l), 0 \leq l \leq k), k \geq 0\}$ -martingale.

Proof. For any integers $n, m \geq 0$, by the Markov property of $\{Y(k), k \geq 0\}$,

$$\begin{aligned}
E[M(n+m)|\mathcal{F}_m^Y] &= E[f(Y(n+m))|\mathcal{F}_m^Y] - \sum_{l=m}^{n+m-1} E[(T(l) - I)f(Y(l))|\mathcal{F}_m^Y] \\
&\quad - \sum_{l=0}^{m-1} (T(l) - I)f(Y(l)) \\
&= \left[\prod_{l=m}^{n+m-1} T(l) \right] f(Y(m)) - \sum_{l=m}^{n+m-1} E[(T(l) - I)f(Y(l))|\mathcal{F}_m^Y] \\
&\quad - \sum_{l=0}^{m-1} (T(l) - I)f(Y(l)),
\end{aligned} \tag{2.3.10}$$

where $\left[\prod_{l=m}^{n+m-1} T(l)\right] = T(m) \cdots T(n+m-1)$. Then by the Markov property again, for $m \leq l \leq n+m-1$,

$$\begin{aligned} E[T(l)f(Y(l))|\mathcal{F}_m^Y] &= E[E[f(Y(l+1))|Y_l]|\mathcal{F}_m^Y] \\ &= E[E[f(Y(l+1))|\mathcal{F}_l^Y]|\mathcal{F}_m^Y] \\ &= E[f(Y(l+1))|\mathcal{F}_m^Y]. \end{aligned} \quad (2.3.11)$$

Therefore,

$$\begin{aligned} &E[M(n+m)|\mathcal{F}_m^Y] \\ &= \left[\prod_{l=m}^{n+m-1} T(l)\right]f(Y(m)) - \sum_{l=m+1}^{n+m-1} \left(\left[\prod_{i=m}^l T(i)\right]f(Y(m)) - \left[\prod_{i=m}^{l-1} T(i)\right]f(Y(m))\right) \\ &\quad - T(m)f(Y(m)) + f(Y(m)) - \sum_{l=0}^{m-1} (T(l) - I)f(Y(l)) \\ &= f(Y(m)) - \sum_{l=0}^{m-1} (T(l) - I)f(Y(l)) \\ &= M(m). \end{aligned} \quad (2.3.12)$$

□

For each $N \geq 1$, let us define the transition operators of $\mathbf{X}^N$ as follows:

$$S_{N,k}f(\mathbf{p}) = E[f(\mathbf{X}^N(\frac{k+1}{N}))|\mathbf{X}^N(\frac{k}{N}) = \mathbf{p}], \quad (2.3.13)$$

for each $k \geq 0$ and $f \in C(K_N)$, $\mathbf{p} \in K_N$. Then by Proposition 2.3.1, we know that

$$M_N(t) = f(\mathbf{X}^N(t)) - \sum_{k=0}^{\lfloor Nt \rfloor - 1} (S_{N,k} - I)f(\mathbf{X}^N(\frac{k}{N})) \quad (2.3.14)$$

is an $\{\mathcal{F}_t^{\mathbf{X}^N} = \sigma(\mathbf{X}^N(s), 0 \leq s \leq t), t \geq 0\}$ -martingale.

2.3.2 Weak convergence to an evolutionary system

Let $\{A_N\} \subset D_{R^r \times r}[0, \infty)$, $A \in C_{R^r \times r}[0, \infty)$ be matrix-valued functions on $[0, \infty)$ which satisfies the conditions specified in the previous subsection. We assume that $\lim_{N \rightarrow \infty} d(A_N, A) = 0$, where d is the Skorohod metric on $D_{R^r \times r}[0, \infty)$. We define by $\{\mathbf{X}^\infty(t)', 0 \leq t < \infty\}$ the solution of differential equations $\frac{d\mathbf{x}}{dt} = A(t)'\mathbf{x}$ on any bounded interval $[0, T]$ satisfying $\mathbf{X}^\infty(0)' = \mathbf{x}(0)$. We make the convention that a random vector is a row vector and the unknown vector functions of differential equations or integral equations are column vectors. We call this special case of GMAM evolutionary system because the system of the differential equations $\frac{d\mathbf{x}}{dt} = A(t)'\mathbf{x}$ with $A(t)$ being continuous on $[0, \infty)$ is called evolutionary system in the terminology of differential equations. The generator of the solution of the differential equations $\frac{d\mathbf{x}}{dt} = A(t)'\mathbf{x}$ satisfying $\mathbf{x}(s)' = \mathbf{p}$ is given by

$$G_A(s)f(\mathbf{p}) = \mathbf{p}A(s)\frac{\partial f'}{\partial \mathbf{x}}, \text{ for } f \in C^1(K). \quad (2.3.15)$$

It is clear that, for each $s \geq 0$, $\mathcal{D}(G_A(s)) = C^1(K)$.

We have the following lemma which will be used in the proof of relative compactness of the laws of $\{\mathbf{X}^N\}$.

Lemma 2.3.2. *Define ' $\geq$ ' on matrices as follows: for any two matrices $X = (x_{ij})_{m \times n}$ and $Y = (y_{ij})_{m \times n}$,*

$$X \geq Y \Leftrightarrow x_{ij} \geq y_{ij} \text{ for all } i = 1, \dots, m; j = 1, \dots, n. \quad (2.3.16)$$

This relation has the property that if $X = (x_{ij})_{m \times n} \geq Y = (y_{ij})_{m \times n}$, $W = (w_{ij})_{p \times m} \geq \mathbf{0}_{p \times m}$ and $Z = (z_{ij})_{n \times q} \geq \mathbf{0}_{n \times q}$, then

$$WX \geq WY, \text{ and } XZ \geq YZ. \quad (2.3.17)$$

Proof. This is obvious by matrix multiplication. $\square$

Now we consider the weak convergence of $\mathbf{X}^N$ to $\mathbf{X}^\infty$.

Lemma 2.3.3. *The laws of $\{\mathbf{X}^N\}$ defined in the last subsection are relatively compact in $D_K[0, \infty)$.*

Proof. By Problem 22, Chapter 3 [EK86], it suffices to prove that $\{X_i^N\}$ is relatively compact for each $1 \leq i \leq r$ and $\{X_i^N + X_j^N\}$ is relatively compact for each $1 \leq i, j \leq r$, $i \neq j$.

At first we will use Aldous conditions [AL78] for tightness to prove that $\{X_i^N\}$ is tight for each $1 \leq i \leq r$. Since $\mathbf{X}^N$ is vector-valued in the simplex K , Aldous' first condition for tightness is satisfied by $\{X_i^N\}$. Next we check that $\{X_i^N\}$ satisfies Aldous' second condition for tightness.

For given stopping times $\{\tau_N\}$ of $\{\mathcal{F}_t^{\mathbf{X}^N}\}$, bounded by $T > 0$, and $\{\delta_N\}_{N=1}^\infty$ satisfying $\lim_{N \rightarrow \infty} \delta_N = 0$. Without loss of generality, we can assume that $\tau_N + \delta_N$ is also bounded by T . We need to prove

$$\lim_{n \rightarrow \infty} P\{|X_i^N(\tau_N + \delta_N) - X_i^N(\tau_N)| > \epsilon\} = 0, \text{ for any } \epsilon > 0. \quad (2.3.18)$$

To prove (2.3.18), it suffices to prove for each $1 \leq i \leq r$,

$$\lim_{N \rightarrow \infty} E[(X_i^N(\tau_N + \delta_N) - X_i^N(\tau_N))^2] = 0. \quad (2.3.19)$$

By strong Markov property of $\mathbf{X}^N$, and (2.3.7), we have that

$$\begin{aligned} & E[(X_i^N(\tau_N + \delta_N) - X_i^N(\tau_N))^2] \\ &= \frac{1}{N^2} E[E[(n_i^N([N(\tau_N + \delta_N)]) - n_i^N([N\tau_N]))^2 | \mathbf{n}^N([N\tau_N])]] \\ &= \frac{1}{N^2} E\left[\left(\mathbf{n}^N([N\tau_N]) \left[\prod_{l=[N\tau_N]}^{[N(\tau_N + \delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N + \delta_N)]-1,i}\right)^2\right] \end{aligned}$$

$$\begin{aligned}
& + \mathbf{n}^N([N\tau_N]) \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} \\
& - \sum_{j'=1}^r \left(P_{N,[N\tau_N],j'}, \left[\prod_{l=[N\tau_N]+1}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} \right)^2 n_{j'}^N([N\tau_N]) \\
& - 2\mathbf{n}^N([N\tau_N]) \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} n_i^N([N\tau_N]) + (n_i^N([N\tau_N]))^2 \Big].
\end{aligned} \tag{2.3.20}$$

For $i \neq j$, let $M_{ij} = \max_{0 \leq s \leq T} a_{i,j}(s)$, $M_{ii} = 0$, $m_{ij} = 0$, $m_{ii} = -\sum_{j \neq i} M_{ij}$. Let $M_U = (M_{ij})_{r \times r}$, $M_L = (m_{ij})_{r \times r}$. For large enough N , by Lemma 2.3.2, we get

$$\begin{aligned}
(I + \frac{1}{N} M_L)^{[N\delta_N]-1} (\mathbf{e}'_i + \frac{1}{N} M_{L,\cdot,i}) & \leq \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} \\
& \leq (I + \frac{1}{N} M_U)^{[N\delta_N]-1} (\mathbf{e}'_i + \frac{1}{N} M_{U,\cdot,i}).
\end{aligned} \tag{2.3.21}$$

Then we get

$$\lim_{N \rightarrow \infty} \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} = \mathbf{e}'_i. \tag{2.3.22}$$

Observe that

$$\begin{aligned}
& \frac{1}{N} \left(n_i^N([N\tau_N]) - \mathbf{n}^N([N\tau_N]) \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} \right) \\
& = \frac{1}{N} \mathbf{n}^N([N\tau_N]) \left(\mathbf{e}'_i - \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} \right).
\end{aligned} \tag{2.3.23}$$

By (2.3.22), we have that the left hand side of (2.3.23) goes to 0 P -a.s. as $N \rightarrow \infty$, we also know that the left side of (2.3.23) is bounded above and below. Then by Lebesgue dominated convergence theorem, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} E \left[\left(\mathbf{n}^N([N\tau_N]) \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} \right)^2 \right]$$

$$- \mathbf{n}^N([N\tau_N]) \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} n_i^N([N\tau_N]) \Big] = 0, \quad (2.3.24)$$

and

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^2} E \Big[(n_i^N([N\tau_N]))^2 - \mathbf{n}^N([N\tau_N]) \left[\prod_{l=[N\tau_N]}^{[N(\tau_N+\delta_N)]-2} P_{N,l} \right] \\ \times P_{N,[N(\tau_N+\delta_N)]-1,\cdot,i} n_i^N([N\tau_N]) \Big] = 0. \end{aligned} \quad (2.3.25)$$

This proves that the expectation of the summation of the first, fourth, and fifth terms on the right hand side of (2.3.20) goes to 0 as $N \rightarrow \infty$. It is clear that the expectation of the other two terms also goes to 0 as $N \rightarrow \infty$. Then we proved (2.3.19) and the relative compactness of $\{X_i^N\}$ for each $1 \leq i \leq r$.

The relative compactness of $\{X_i^N + X_j^N\}$ for each $1 \leq i, j \leq r, i \neq j$ follows by the next inequality and the argument for the relative compactness of X_i^N

$$\begin{aligned} E[(X_i^N(\tau_N + \delta_N) + X_j^N(\tau_N + \delta_N) - X_i^N(\tau_N) - X_j^N(\tau_N))^2] \\ \leq 2E[(X_i^N(\tau_N + \delta_N) - X_i^N(\tau_N))^2] + 2E[(X_j^N(\tau_N + \delta_N) - X_j^N(\tau_N))^2]. \end{aligned} \quad (2.3.26)$$

□

Next, we prove a lemma which will be used in our martingale problem argument.

Lemma 2.3.4. (*Mapping Theorem*) Let X_n and X be defined on probability spaces $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$, taking values in a Polish space (S, d) . Suppose that $P_n X_n^{-1} \Rightarrow P X^{-1}$. Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of continuous functions from (S, d) to (S', d') converging uniformly to a continuous function $f : S \mapsto S'$. Then $P_n f_n(X_n)^{-1} \Rightarrow P f(X)^{-1}$.

Proof. Choose any uniformly continuous function h from $\bar{C}(S')$. We have

$$\begin{aligned}
& \left| \int h(f_n(X_n(\omega)))P_n(d\omega) - \int h(f(X(\omega)))P(d\omega) \right| \\
& \leq \int |h(f_n(X_n(\omega))) - h(f(X_n(\omega)))|P_n(d\omega) \\
& \quad + \left| \int h(f(X_n(\omega)))P_n(d\omega) - \int h(f(X(\omega)))P(d\omega) \right| \\
& < \sup_{x \in S} |h(f_n(x)) - h(f(x))| + \left| \int h(f(X_n(\omega)))P_n(d\omega) - \int h(f(X(\omega)))P(d\omega) \right|.
\end{aligned} \tag{2.3.27}$$

Then by the conditions in the lemma, $P_n f_n(X_n)^{-1} \Rightarrow P f(X)^{-1}$. $\square$

Remark 2.3.5. *Actually, we can just assume the following conditions: $\{f_n\}_{n=1}^\infty$ and f are measurable mappings from (S, d) to (S', d') , $\{f_n\}_{n=1}^\infty$ converges uniformly to f , $P\{X \in C_f\} = 1$, and $P_n\{X_n \in C_f\} = 1$ where C_f is the set of points of S at which f is continuous.*

Lemma 2.3.6. *We make the assumptions given above, and assume that $\{\mathbf{X}^N(0)\} = \{\frac{\mathbf{n}^N(0)}{N}\}$ has the limit distribution $\mu \in \mathcal{P}(K)$ as N goes to ∞ . If $\mathbf{Y}$ is a limit point of $\mathbf{X}^N$, where $\mathbf{Y}$ is defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, $\mathbf{Y}$ must be a solution of the martingale problem for (G_A, μ) : i.e.*

$$\begin{aligned}
& E[(f(\mathbf{Y}(t_{n+1})) - f(\mathbf{Y}(t_n))) \prod_{k=1}^n h_k(\mathbf{Y}(t_k))] \\
& = E\left[\int_{t_n}^{t_{n+1}} \mathbf{Y}(s)A(s) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{Y}(s))' ds \prod_{k=1}^n h_k(\mathbf{Y}(t_k))\right],
\end{aligned} \tag{2.3.28}$$

where $0 \leq t_1 < t_2 < \dots < t_{n+1}$, $f \in C^1(K)$, $\frac{\partial f}{\partial \mathbf{x}} = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_r}]$, and $h_1, \dots, h_n \in C(K)$.

Proof. We know that $\mathbf{Y}$ is a limit point of $\mathbf{X}^N$. Without loss of generality, we assume that $\mathbf{X}^N \Rightarrow \mathbf{Y}$. Let us define $D(\mathbf{Y}) \equiv \{t \geq 0 : \tilde{P}\{\mathbf{Y}(t) = \mathbf{Y}(t-)\} = 1\}$.

Firstly, we will prove (2.3.28) for any fixed $n \geq 1$, arbitrary irrational numbers $0 \leq t_1 < t_2 < \cdots < t_n < t_{n+1}$ such that $t_k \in D(\mathbf{Y})$ for $1 \leq k \leq n+1$, $f \in C^1(K)$ and $h_k \in C(K)$ for $1 \leq k \leq n$. The reason for choosing irrational numbers will be clear in context.

We have showed that for each $N \geq 1$, and any fixed $f \in C^1(K)$, $\{M_N(t), t \geq 0\}$ is an $\{\mathcal{F}_t^{\mathbf{X}^N}, t \geq 0\}$ -martingale. Then we have

$$\begin{aligned} E[(f(\mathbf{X}^N(t_{n+1})) - f(\mathbf{X}^N(t_n))) \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\ = \sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} E[(S_{N,l} - I)f(\mathbf{X}^N(\frac{l}{N})) \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))]. \end{aligned} \quad (2.3.29)$$

Therefore, to prove (2.3.28), it suffices to show that

$$\begin{aligned} \lim_{N \rightarrow \infty} E[(f(\mathbf{X}^N(t_{n+1})) - f(\mathbf{X}^N(t_n))) \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\ = E[(f(\mathbf{Y}(t_{n+1})) - f(\mathbf{Y}(t_n))) \prod_{k=1}^n h_k(\mathbf{Y}(t_k))], \end{aligned} \quad (2.3.30)$$

and

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} E[(S_{N,l} - I)f(\mathbf{X}^N(\frac{l}{N})) \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\ = E[\int_{t_n}^{t_{n+1}} \mathbf{Y}(s)A(s) \frac{\partial f}{\partial \mathbf{X}}(\mathbf{Y}(s))' ds \prod_{k=1}^n h_k(\mathbf{Y}(t_k))]. \end{aligned} \quad (2.3.31)$$

Since we assumed $\mathbf{X}^N \Rightarrow \mathbf{Y}$ and $t_k \in D(\mathbf{Y})$ for $1 \leq k \leq n+1$, by Theorem 7.8, Chapter 3 [EK86], $(\mathbf{X}^N(t_1), \dots, \mathbf{X}^N(t_{n+1})) \Rightarrow (\mathbf{Y}(t_1), \dots, \mathbf{Y}(t_{n+1}))$ and (2.3.30) is proved.

Next, we prove (2.3.31). Note that by the Markov property of $\mathbf{X}^N$ and Taylor's

expansion,

$$\begin{aligned}
& E[(S_{N,l} - I)f(\mathbf{X}^N(\frac{l}{N})) \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\
&= E[E[(f(\mathbf{X}^N(\frac{l+1}{N})) - f(\mathbf{X}^N(\frac{l}{N}))) | \mathbf{X}^N(\frac{l}{N})] \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\
&= E[E[(\mathbf{X}^N(\frac{l+1}{N}) - \mathbf{X}^N(\frac{l}{N})) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}_*^N(\frac{l}{N}))' | \mathcal{F}_{\frac{l}{N}}^{\mathbf{X}^N}] \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\
&= E[E[(\mathbf{X}^N(\frac{l+1}{N}) - \mathbf{X}^N(\frac{l}{N})) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}^N(\frac{l}{N}))' | \mathbf{X}^N(\frac{l}{N})] \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\
&\quad + E[(\mathbf{X}^N(\frac{l+1}{N}) - \mathbf{X}^N(\frac{l}{N})) \{ \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}_*^N(\frac{l}{N}))' - \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}^N(\frac{l}{N}))' \} \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))],
\end{aligned} \tag{2.3.32}$$

where $\mathbf{X}_*^N(\frac{l}{N}) = \mathbf{X}^N(\frac{l}{N}) + \theta_l^N(\mathbf{X}^N(\frac{l+1}{N}) - \mathbf{X}^N(\frac{l}{N}))$, for some $\theta_l^N \in (0, 1)$ for each $[Nt_n] \leq l \leq [Nt_{n+1}] - 1$. Actually, for each $\mathbf{z}_N \in D_{K_N}[0, \infty)$ such that $\mathbf{z}_N(t)$ is a constant on $[\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$, there exists a sequence $\{\theta_k^N, k \geq 0\}$ corresponding to $\mathbf{z}_N$, which is determined by the Taylor's expansion. By (2.3.5),

$$\begin{aligned}
& \sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} E[(S_{N,l} - I)f(\mathbf{X}^N(\frac{l}{N})) \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\
&= \frac{1}{N} \sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} E[\mathbf{X}^N(\frac{l}{N}) A_N(\frac{l}{N}) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}^N(\frac{l}{N}))' \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))] \\
&\quad + \sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} E[(\mathbf{X}^N(\frac{l+1}{N}) - \mathbf{X}^N(\frac{l}{N})) \{ \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}_*^N(\frac{l}{N}))' - \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}^N(\frac{l}{N}))' \} \\
&\quad \times \prod_{k=1}^n h_k(\mathbf{X}^N(t_k))].
\end{aligned} \tag{2.3.33}$$

We denote by $I_{N,1}$, $I_{N,2}$ the first term and second term of the right hand side of (2.3.33). To prove (2.3.31), it suffices to prove

$$\lim_{N \rightarrow \infty} I_{N,1} = E[\int_{t_n}^{t_{n+1}} \mathbf{Y}(s) A(s) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{Y}(s))' ds \prod_{k=1}^n h_k(\mathbf{Y}(t_k))], \tag{2.3.34}$$

and

$$\lim_{N \rightarrow \infty} I_{N,2} = 0. \quad (2.3.35)$$

We define the following functions on $D_K[0, \infty)$:

$$g_N(\mathbf{z}) = \int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} \mathbf{z}(s) A_N\left(\frac{[Ns]}{N}\right) \frac{\partial f}{\partial \mathbf{X}}(\mathbf{z}(s))' ds \prod_{k=1}^n h_k(\mathbf{z}(t_k)), \quad \mathbf{z} \in D_K[0, \infty), \quad N \geq 1, \quad (2.3.36)$$

$$g(\mathbf{z}) = \int_{t_n}^{t_{n+1}} \mathbf{z}(s) A(s) \frac{\partial f}{\partial \mathbf{X}}(\mathbf{z}(s))' ds \prod_{k=1}^n h_k(\mathbf{z}(t_k)), \quad \mathbf{z} \in D_K[0, \infty). \quad (2.3.37)$$

g is continuous at $\mathbf{z}$ if $\mathbf{z}$ is continuous at $t_1, \dots, t_n$, i.e. if $\lim_{l \rightarrow \infty} d(\mathbf{z}_l, \mathbf{z}) = 0$ where d is the Skorohod metric on $D_K[0, \infty)$, then $\lim_{l \rightarrow \infty} g(\mathbf{z}_l) = g(\mathbf{z})$. Since $t_1, \dots, t_{n+1}$ are irrational numbers, for each $N \geq 1$, each sample path of $\mathbf{X}^N$ is continuous at $t_1, \dots, t_n$, and it follows that g is continuous at each sample path of $\mathbf{X}^N$, then $P(\mathbf{X}^N \in C_g) = 1$, where C_g is the set of points in $D_K[0, \infty)$ at which g is continuous. Since $t_1, \dots, t_{n+1}$ belong to $D(\mathbf{Y})$, We have $\tilde{P}(\mathbf{Y} \in C_g) = 1$. Since $\lim_{N \rightarrow \infty} d(A_N, A) = 0$, where d is the Skorohod metric on $D_{R^r \times r}[0, \infty)$, A is continuous and K is compact, we have that g_N converges uniformly on $D_K[0, \infty)$ to g . Observing that

$$g_N(\mathbf{X}^N) = \frac{1}{N} \sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} \mathbf{X}^N\left(\frac{l}{N}\right) A_N\left(\frac{l}{N}\right) \frac{\partial f}{\partial \mathbf{X}}(\mathbf{X}^N\left(\frac{l}{N}\right))' \prod_{k=1}^n h_k(\mathbf{X}^N(t_k)), \quad (2.3.38)$$

$$g(\mathbf{Y}) = \int_{t_n}^{t_{n+1}} \mathbf{Y}(s) A(s) \frac{\partial f}{\partial \mathbf{X}}(\mathbf{Y}(s))' ds \prod_{k=1}^n h_k(\mathbf{Y}(t_k)), \quad (2.3.39)$$

by Remark 2.3.5, we conclude that $g_N(\mathbf{X}^N) \Rightarrow g(\mathbf{Y})$. Since $\{g_N(\mathbf{X}^N)\}$ is a bounded sequence, it is uniformly integrable. Then

$$\lim_{N \rightarrow \infty} E[g_N(\mathbf{X}^N)] = E[g(\mathbf{Y})], \quad (2.3.40)$$

i.e, we proved (2.3.34).

Now we prove (2.3.35). Let $C = \prod_{k=1}^n \|h_k\|$. By Jensen's inequality, we know that

$$\begin{aligned}
& (I_{N,2})^2 \\
& \leq C^2 E[(\sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} (\mathbf{X}^N(\frac{l+1}{N}) - \mathbf{X}^N(\frac{l}{N})) \{\frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}_*^N(\frac{l}{N}))' - \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}^N(\frac{l}{N}))'\})^2] \\
& = C^2 N^2 E[(\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} (\mathbf{X}^N(s + \frac{1}{N}) - \mathbf{X}^N(s)) \{\frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}_*^N(s))' - \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}^N(s))'\} ds)^2],
\end{aligned} \tag{2.3.41}$$

where $\mathbf{X}_*^N(s) = \mathbf{X}^N(s) + \tilde{\theta}_s^N(\mathbf{X}^N(s + \frac{1}{N}) - \mathbf{X}^N(s))$, and $\tilde{\theta}_s^N = \theta_l^N$ where θ_l^N is specified by the Taylor's expansion in (2.3.32), if $\frac{l}{N} \leq s < \frac{l+1}{N}$ for some l satisfying $[Nt_n] \leq l \leq [Nt_{n+1}] - 1$. By Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& (\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} (\mathbf{X}^N(s + \frac{1}{N}) - \mathbf{X}^N(s)) \{\frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}_*^N(s))' - \frac{\partial f}{\partial \mathbf{x}}(\mathbf{X}^N(s))'\} ds)^2 \\
& \leq r \sum_{i=1}^r (\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} (X_i^N(s + \frac{1}{N}) - X_i^N(s)) \{\frac{\partial f}{\partial x_i}(\mathbf{X}_*^N(s)) - \frac{\partial f}{\partial x_i}(\mathbf{X}^N(s))\} ds)^2 \\
& \leq r \sum_{i=1}^r \int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} (X_i^N(s + \frac{1}{N}) - X_i^N(s))^2 ds \\
& \quad \times \int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} \{\frac{\partial f}{\partial x_i}(\mathbf{X}_*^N(s)) - \frac{\partial f}{\partial x_i}(\mathbf{X}^N(s))\}^2 ds.
\end{aligned} \tag{2.3.42}$$

Then by Cauchy-Schwartz inequality again, we get

$$\begin{aligned}
(I_{N,2})^2 & \leq C^2 r N^2 \sum_{i=1}^r E[(\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} (X_i^N(s + \frac{1}{N}) - X_i^N(s))^2 ds)^2] \\
& \quad \times E[(\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} \{\frac{\partial f}{\partial x_i}(\mathbf{X}_*^N(s)) - \frac{\partial f}{\partial x_i}(\mathbf{X}^N(s))\}^2 ds)^2] \\
& = C^2 r \sum_{i=1}^r I_{N,3,i} I_{N,4,i},
\end{aligned} \tag{2.3.43}$$

where

$$I_{N,3,i} = N^2 E[(\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} (X_i^N(s + \frac{1}{N}) - X_i^N(s))^2 ds)^2],$$

and

$$I_{N,4,i} = E[(\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} \{\frac{\partial f}{\partial x_i}(\mathbf{X}_*^N(s)) - \frac{\partial f}{\partial x_i}(\mathbf{X}^N(s))\}^2 ds)^2].$$

Then, to prove (2.3.35), it suffices to prove $\{I_{N,3,i}\}$ is bounded and $\lim_{N \rightarrow \infty} I_{N,4,i} = 0$.

We consider here the boundedness of $\{I_{N,3,i}\}$. By $(X_i^N(s + \frac{1}{N}) - X_i^N(s))^2 \leq 1$, and (2.3.7)

$$\begin{aligned} I_{N,3,i} &= E[(\sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} (X_i^N(\frac{l+1}{N}) - X_i^N(\frac{l}{N}))^2)^2] \\ &\leq ([Nt_{n+1}] - [Nt_n]) E[\sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} (X_i^N(\frac{l+1}{N}) - X_i^N(\frac{l}{N}))^2] \\ &= \frac{([Nt_{n+1}] - [Nt_n])}{N^2} E[\sum_{l=[Nt_n]}^{[Nt_{n+1}]-1} \{\mathbf{X}^N(\frac{l}{N}) A_{N,\cdot,i}(\frac{l}{N}) + (\mathbf{X}^N(\frac{l}{N}) A_{N,\cdot,i}(\frac{l}{N}))^2 \\ &\quad - 2X_i^N(\frac{l}{N}) a_{N,i,i}(\frac{l}{N}) - \sum_{j=1}^r \frac{X_j^N(\frac{l}{N})}{N} a_{N,j,i}^2(\frac{l}{N})\}] \\ &\leq \frac{([Nt_{n+1}] - [Nt_n])}{N} E[\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} \mathbf{X}^N(s) A_{N,\cdot,i}(\frac{[Ns]}{N}) + (\mathbf{X}^N(s) A_{N,\cdot,i}(\frac{[Ns]}{N}))^2 \\ &\quad - 2X_i^N(s) a_{N,i,i}(\frac{[Ns]}{N}) - \sum_{j=1}^r \frac{X_j^N(s)}{N} a_{N,j,i}^2(\frac{[Ns]}{N}) ds] \end{aligned} \quad (2.3.44)$$

Therefore

$$\limsup_{N \rightarrow \infty} I_{N,3,i} \leq (t_{n+1} - t_n) E[\int_{t_n}^{t_{n+1}} \mathbf{Y}(s) A_{\cdot,i}(s) + (\mathbf{Y}(s) A_{\cdot,i}(s))^2 - 2Y_i(s) a_{i,i}(s) ds]. \quad (2.3.45)$$

We now prove $\lim_{N \rightarrow \infty} I_{N,4,i} = 0$. For $1 \leq i \leq r$, we have

$$I_{N,4,i} \leq 4 \left\| \frac{\partial f}{\partial x_i} \right\|^2 \frac{([Nt_{n+1}] - [Nt_n])}{N} E \left[\int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} \left\{ \frac{\partial f}{\partial x_i}(\mathbf{X}_*^N(s)) - \frac{\partial f}{\partial x_i}(\mathbf{X}^N(s)) \right\}^2 ds \right]. \quad (2.3.46)$$

Let $\tilde{D}_{K_N}[0, \infty) \equiv \{\mathbf{z}_N \in D_{K_N}[0, \infty) : \mathbf{z}_N(t) \text{ is a constant on } [\frac{k}{N}, \frac{k+1}{N}) \text{ for each } k \geq 0\}$. For $1 \leq i \leq r$, $N \geq 1$, we define the following function on $\tilde{D}_{K_N}[0, \infty)$:

$$g_{N,i}(\mathbf{z}_N) = \int_{\frac{[Nt_n]}{N}}^{\frac{[Nt_{n+1}]}{N}} \left\{ \frac{\partial f}{\partial x_i}(\mathbf{z}_{N,*}(s)) - \frac{\partial f}{\partial x_i}(\mathbf{z}_N(s)) \right\}^2 ds, \quad (2.3.47)$$

where $\mathbf{z}_{N,*}(s) = \mathbf{z}_N(s) + \tilde{\theta}_s^N(\mathbf{z}_N(s + \frac{1}{N}) - \mathbf{z}_N(s))$, and $\tilde{\theta}_s^N = \theta_k^N$ for $\frac{k}{N} \leq s < \frac{k+1}{N}$ and $\{\theta_k^N : k \geq 0\}$, determined by the Taylor's expansion, corresponds to $\mathbf{z}_N$. It is clear that $g_{N,i}$ is a bounded function. If $\mathbf{z}_N \in \tilde{D}_{K_N}[0, \infty)$, $\mathbf{z} \in D_K[0, \infty)$ and $\lim_{N \rightarrow \infty} d(\mathbf{z}_N, \mathbf{z}) = 0$, then $\lim_{N \rightarrow \infty} g_{N,i}(\mathbf{z}_N) = 0$. Since $\mathbf{X}^N \Rightarrow \mathbf{Y}$, by Skorohod Representation theorem, we have $\lim_{N \rightarrow \infty} E[g_{N,i}(\mathbf{X}_N)] = 0$. It follows that $\lim_{N \rightarrow \infty} I_{N,4,i} = 0$.

So far, we proved (2.3.28) for arbitrary irrational numbers $t_1, \dots, t_{n+1}$ in $D(\mathbf{Y})$. Since the complement of $D(\mathbf{Y})$ is at most countable, we can generalize this consequence to arbitrary $0 \leq t_1 < t_2 < \dots < t_n < t_{n+1}$ by the right continuity of $\mathbf{Y}$ and Lebesgue's dominated convergence theorem. $\square$

Theorem 2.3.7. *Under the assumptions made above, assuming that*

$\lim_{N \rightarrow \infty} \mathbf{X}^N(0) = \mathbf{x}(0)'$, we have that $\mathbf{X}^N \Rightarrow \mathbf{X}^\infty$.

Proof. The solution of the martingale problem in (2.3.28) with μ satisfying $\mu(\{\mathbf{x}(0)'\}) = 1$ must be a solution of the differential equations $\frac{d\mathbf{x}}{dt} = A(t)'\mathbf{x}$ satisfying $\mathbf{X}^\infty(0)' = \mathbf{x}(0)$ and vice versa. Since these differential equations with initial condition have a unique solution, so has the martingale problem in (2.3.28). By Lemma 2.3.3 and 2.3.6, the theorem follows. $\square$

2.4 General multiagent models

In this section, we will discuss the general multiagent models (GMAMS). The convergence theorem for the evolutionary multiagent model with multinomial sampling (EMAMWMS) is a by-product of this section. Updated tools for the martingale problem with time-dependent generator will be built up.

2.4.1 Preliminary tools for time-dependent generators

Before we formulate general multiagent models and prove the corresponding results, we set up the tools for the whole section, and also for Chapter 3. We use some of the notations and methods from [EK86]. For $n = 1, 2, \dots$, let $\{\mathcal{G}_t^n\}$ be a complete filtration, and let $\mathcal{L}_n$ be the space of real-valued $\{\mathcal{G}_t^n\}$ -progressive processes ξ satisfying

$$\sup_{0 \leq t \leq T} E[|\xi(t)|] < \infty$$

for each $T > 0$. Let $\hat{\mathcal{A}}_n$ be the collection of pairs $(\xi, \varphi) \in \mathcal{L}_n \times \mathcal{L}_n$ such that

$$\xi(t) - \int_0^t \varphi(s) ds$$

is a $\{\mathcal{G}_t^n\}$ -martingale.

Proposition 2.4.1. *Let $(E, \tilde{r})$ be a Polish space. Let $\{G(s), 0 \leq s < \infty\}$ be a family of operators on $\bar{C}(E)$. Suppose that there exists a countable set $\Gamma_1 \subset [0, \infty)$, such that for each $s \notin \Gamma_1$, $\{G(s), 0 \leq s < \infty\}$ has a common domain denoted by $\mathcal{D}(G)$ and for each $f \in \mathcal{D}(G)$, $G(s)f \in \bar{C}(E)$ for $s \notin \Gamma_1$, and $\|G(s)f\|$ is bounded for $s \in [0, T] \setminus \Gamma_1$ for any $T > 0$. Suppose that there is an algebra C_a contained in the closure of $\mathcal{D}(G)$ (in the sup norm) which separates points. Suppose that the $D_E[0, \infty)$*

martingale problem for (G, ν) has at most one solution where $\nu \in \mathcal{P}(E)$. Suppose $X_n, n = 1, 2, \dots$ is a $\{\mathcal{G}_t^n\}$ -adapted process with sample paths in $D_E[0, \infty)$. Suppose $P(X_n(0))^{-1} \Rightarrow \nu$ and the compact containment condition holds. Suppose $M \subset \bar{C}(E)$ is separating. Then either condition (a) or (b) implies that there exists a solution X of the $D_E[0, \infty)$ martingale problem for (G, ν) , and $X_n \Rightarrow X$:

(a) *There exists a countable set $\Gamma_2 \subset [0, \infty)$ such that for each $f \in \mathcal{D}(G)$, and $T > 0$, there exists $(\xi_n, \varphi_n) \in \hat{\mathcal{A}}_n$, such that*

$$\sup_n \sup_{0 \leq s \leq T, s \notin \Gamma_1} E[|\xi_n(s)|] < \infty, \quad (2.4.1)$$

$$\sup_n \sup_{0 \leq s \leq T, s \notin \Gamma_1} E[|\varphi_n(s)|] < \infty, \quad (2.4.2)$$

$$\lim_{n \rightarrow \infty} E[(\xi_n(t) - f(X_n(t))) \prod_{i=1}^k h_i(X_n(t_i))] = 0, \quad (2.4.3)$$

$$\lim_{n \rightarrow \infty} E[(\varphi_n(t) - G(t)f(X_n(t))) \prod_{i=1}^k h_i(X_n(t_i))] = 0, \quad (2.4.4)$$

for all $k \geq 0, 0 \leq t_1 < t_2 < \dots < t_k \leq t \leq T$ with $t_i, t \notin \Gamma_1 \cup \Gamma_2$, and $h_1, \dots, h_k \in M$, and

$$\lim_{n \rightarrow \infty} E[\sup_{t \in \bar{Q} \cap [0, T]} |\xi_n(t) - f(X_n(t))|] = 0, \quad (2.4.5)$$

$$\sup_n E[\|\varphi_n\|_{p, T}] < \infty, \text{ for some } p \in (1, \infty], \quad (2.4.6)$$

where $\bar{Q}$ is a countable and dense subset of R , $\|h\|_{p, T} = [\int_0^T |h(t)|^p dt]^{1/p}$ if $p < \infty$; $\|h\|_{\infty, T} = \text{ess sup}_{0 \leq t \leq T} |h(t)|$.

(b) *If in condition (a) we assume that for each $f \in \mathcal{D}(G)$, there exists some $\lambda > 0$, such that*

$$\int_0^\infty e^{-\lambda s} \|G(s)f\| ds < \infty, \quad (2.4.7)$$

then as in Chapter 4, Theorem 8.2, and Theorem 8.10 [EK86], the conditions (2.4.2) and (2.4.4) can be relaxed. It suffices to have

$$\lim_{n \rightarrow \infty} \int_0^{T-t} \left| E \left[(\varphi_n(t+s) - G(t+s)f(X_n(t+s))) \prod_{i=1}^k h_i(X_n(t_i)) \right] \right| ds = 0, \quad (2.4.8)$$

for all $k \geq 0$, $0 \leq t_1 < t_2 < \dots < t_k \leq t \leq T$ with $t_i, t \notin \Gamma_1 \cup \Gamma_2$, and $h_1, \dots, h_k \in M$.

Proof. (a) Using the conditions (2.4.5) and (2.4.6), by Chapter 3, Theorem 9.4 and 9.1 [EK86], we know that $\{X_n\}$ is relatively compact and there exists a limit point of $\{X_n\}$ denoted by X . (Note that Q in Chapter 3, Theorem 9.4 [EK86] can be replaced by any countable dense subset $\tilde{Q}$ of R .)

Let $f \in \mathcal{D}(G)$, and $T > 0$, and let $\{(\xi_n, \varphi_n)\}$ satisfy the conditions (2.4.1)-(2.4.4). Let $k \geq 0$, $0 \leq t_1 < t_2 < \dots < t_k \leq t \leq T$ with $t_i, t \notin \Gamma_1 \cup \Gamma_2$, and $h_1, \dots, h_k \in M$. Since

$$E \left[\left(\xi_n(t+s) - \xi_n(t) - \int_t^{t+s} \varphi_n(u) du \right) \prod_{i=1}^k h_i(X_n(t_i)) \right] = 0, \quad (2.4.9)$$

we have that

$$\lim_{n \rightarrow \infty} E \left[\left(f(X_n(t+s)) - f(X_n(t)) - \int_t^{t+s} G(u)f(X_n(u)) du \right) \prod_{i=1}^k h_i(X_n(t_i)) \right] = 0, \quad (2.4.10)$$

and then

$$E \left[\left(f(X(t+s)) - f(X(t)) - \int_t^{t+s} G(u)f(X(u)) du \right) \prod_{i=1}^k h_i(X(t_i)) \right] = 0. \quad (2.4.11)$$

By the right continuity of X , (2.4.11) holds for all $0 \leq t_1 < \dots < t_k \leq t < t+s$, and hence X is a solution of the martingale problem for (G, ν) . Then $X_n \Rightarrow X$ follows by the assumption of the uniqueness of the martingale problem for (G, ν) .

(b) By the proof of part (a), it is enough to show that the conditions (2.4.1), (2.4.3), (2.4.7) and (2.4.8) imply that the limit point X is a solution of the martingale problem for (G, ν) . To do so, it is sufficient to prove that for each $f \in \mathcal{D}(G)$, there exists another $(\hat{\xi}_n, \hat{\varphi}_n) \in \hat{\mathcal{A}}_n$, such that $\{(\hat{\xi}_n, \hat{\varphi}_n)\}$ satisfies the conditions (2.4.1)-(2.4.4) for each $T > 0$.

Let $f \in \mathcal{D}(G)$, define $\hat{\xi}_n$ and $\hat{\varphi}_n$ by

$$\hat{\xi}_n(t) = \int_0^\infty e^{-\lambda s} E[\lambda f(X_n(t+s)) - G(t+s)f(X_n(t+s)) | \mathcal{G}_t^n] ds, \quad (2.4.12)$$

and

$$\hat{\varphi}_n(t) = \lambda \hat{\xi}_n(t) - \lambda f(X_n(t)) + G(t)f(X_n(t)). \quad (2.4.13)$$

The definition of $(\hat{\xi}_n, \hat{\varphi}_n)$ is justified by (2.4.7). Since (2.4.7) also justifies the interchange in the order of integration in the proof of Theorem 7.1 of Chapter 2 [EK86], $(\hat{\xi}_n, \hat{\varphi}_n) \in \hat{\mathcal{A}}_n$. It is clear that for any given $T > 0$, (2.4.1) and (2.4.2) hold for $(\hat{\xi}_n, \hat{\varphi}_n)$. Since $\lambda(\hat{\xi}_n - f \circ X_n) = \hat{\varphi}_n - Gf \circ X_n$, we only have to verify (2.4.3).

For the given f and fix $T > 0$ with $T \notin \Gamma_1 \cup \Gamma_2$, let $(\xi_n, \varphi_n) \in \hat{\mathcal{A}}_n$ satisfy (2.4.1), (2.4.3) and (2.4.8) for any $k \geq 0$, $0 \leq t_1 < t_2 < \dots < t_k \leq t \leq T$ with $t_i, t \notin \Gamma_1 \cup \Gamma_2$, and $h_1, \dots, h_k \in M$. By a similar proof as that for Lemma 3.2 of chapter 4 [EK86], we know that

$$e^{-\lambda t} \xi_n(t) + \int_0^t e^{-\lambda u} (\lambda \xi_n(u) - \varphi_n(u)) du \quad (2.4.14)$$

is a $\{\mathcal{G}_t^n\}$ -martingale. Then,

$$\xi_n(t) = e^{-\lambda(T-t)} E[\xi_n(T) | \mathcal{G}_t^n] + \int_t^T e^{-\lambda(u-t)} E[\lambda \xi_n(u) - \varphi_n(u) | \mathcal{G}_t^n] du. \quad (2.4.15)$$

Given $k \geq 0$, $0 \leq t_1 < t_2 < \dots < t_k \leq t \leq T$ with $t_i, t \notin \Gamma_1 \cup \Gamma_2$, and $h_1, \dots,$

$h_k \in M$, it follows that

$$\begin{aligned} & E \left[(\hat{\xi}_n(t) - f(X_n(t))) \prod_{i=1}^k h_i(X_n(t_i)) \right] \\ &= E \left[(\hat{\xi}_n(t) - \xi_n(t)) \prod_{i=1}^k h_i(X_n(t_i)) \right] + E \left[(\xi_n(t) - f(X_n(t))) \prod_{i=1}^k h_i(X_n(t_i)) \right]. \end{aligned} \quad (2.4.16)$$

Notice that the first term on the right of (2.4.16) is

$$\begin{aligned} & E \left[\int_{T-t}^{\infty} e^{-\lambda s} E[\lambda f(X_n(t+s)) - G(t+s)f(X_n(t+s)) | \mathcal{G}_t^n] ds \prod_{i=1}^k h_i(X_n(t_i)) \right] \\ & - E \left[e^{-\lambda(T-t)} (E[\xi_n(T) | \mathcal{G}_t^n] - f(X_n(T))) \prod_{i=1}^k h_i(X_n(t_i)) \right] \\ & - E \left[e^{-\lambda(T-t)} f(X_n(T)) \prod_{i=1}^k h_i(X_n(t_i)) \right] \\ & + E \left[\int_0^{T-t} e^{-\lambda s} (E[\lambda f(X_n(t+s)) - G(t+s)f(X_n(t+s)) | \mathcal{G}_t^n] \right. \\ & \quad \left. - E[\lambda \xi_n(t+s) - \varphi_n(t+s) | \mathcal{G}_t^n]) ds \prod_{i=1}^k h_i(X_n(t_i)) \right]. \end{aligned} \quad (2.4.17)$$

The second term on the right of (2.4.16), the second and the fourth terms in (2.4.17) go to zero as $n \rightarrow \infty$. The dominated convergence theorem is employed for the interchange of limits and integration in the fourth term. Therefore,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \left| E \left[(\hat{\xi}_n(t) - f(X_n(t))) \prod_{i=1}^k h_i(X_n(t_i)) \right] \right| \\ & \leq \left(2e^{-\lambda(T-t)} \|f\| + e^{\lambda t} \int_T^{\infty} e^{-\lambda s} \|G(s)f\| ds \right) \prod_{i=1}^k \|h_i\|. \end{aligned} \quad (2.4.18)$$

Since T is arbitrary, let $T \rightarrow \infty$, (2.4.3) follows for $\hat{\xi}_n$ by (2.4.7). $\square$

We have the following corollaries whose conditions are more convenient to be verified for our general multiagent models.

Corollary 2.4.2. *Suppose in Proposition 2.4.1 that for each $n \geq 1$, X_n is a $\{\mathcal{G}_t^n\}$ -adapted process with sample paths in $D_E[0, \infty)$ and generator $\{G_n(s), 0 \leq s < \infty\}$ on $\bar{C}(E)$. Suppose also for each $n \geq 1$ there exists a countable set $\Gamma^n \subset [0, \infty)$, such that for each $s \notin \Gamma^n$, $\{G_n(s), 0 \leq s < \infty\}$ has a common domain denoted by $\mathcal{D}(G_n)$. Then condition (c) implies that there exists a solution X of the $D_E[0, \infty)$ martingale problem for (G, ν) , and $X_n \Rightarrow X$:*

(c) *For each $n \geq 1$, $\mathcal{D}(G_n) = \mathcal{D}(G)$ and there exists a countable set $\Gamma_2 \subset [0, \infty)$ such that for each $f \in \mathcal{D}(G)$, and $T > 0$,*

$$\sup_n \sup_{0 \leq s \leq T, s \notin \Gamma_1 \cup \Gamma^n} \|G_n(s)f\| < \infty, \quad (2.4.19)$$

where $\|\cdot\|$ is the sup norm on $B(E)$, and

$$\lim_{n \rightarrow \infty} \sup_{q \in E} |G_n(t)f(q) - G(t)f(q)| = 0, \quad (2.4.20)$$

for any t satisfying $t \notin \Gamma_1 \cup \Gamma_2 \cup \bigcup_{n=1}^{\infty} \Gamma^n$ and $0 \leq t \leq T$.

Proof. Fix $f \in \mathcal{D}(G)$, let $\xi_n(t) = f(X_n(t))$ and $\varphi_n(t) = G_n(t)f(X_n(t))$ for $t \geq 0$. Then for any $T > 0$, $(\xi_n, \varphi_n) \in \mathcal{A}_n$ for each $n \geq 1$ and $\{(\xi_n, \varphi_n)\}$ satisfies the conditions (2.4.1)-(2.4.6) with $p = \infty$. $\square$

Corollary 2.4.3. *Suppose in Proposition 2.4.1 that $X_n = \eta_n(Y_n([\alpha_n \cdot]))$ and $\{\mathcal{G}_t^n\} = \{\mathcal{F}_{[\alpha_n t]}^{Y_n}\}$, where $\{Y_n(k), k = 0, 1, 2, \dots\}$ is a time inhomogeneous Markov chain in a metric space E_n with transition functions $\mu_{n,k}(x, \Gamma)$, $\eta_n : E_n \mapsto E$ is Borel measurable, and $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. Define $T_{n,k} : B(E_n) \mapsto B(E)$ by*

$$T_{n,k}f(x) = \int f(y) \mu_{n,k}(x, dy), \quad (2.4.21)$$

and let $G_{n,k} = \alpha_n(T_{n,k} - I)$. Then condition (d) implies that there exists a solution X of the $D_E[0, \infty)$ martingale problem for (G, ν) , and $X_n \Rightarrow X$:

(d) There exists a countable set $\Gamma_2 \subset [0, \infty)$ such that for each $f \in \mathcal{D}(G)$, and $T > 0$,

$$\sup_n \sup_{0 \leq t \leq T, t \notin \Gamma_1 \cup \Gamma_2} \sup_{q \in E_n} |G_{n, [\alpha_n t]} f \circ \eta_n(q)| < \infty, \quad (2.4.22)$$

and

$$\lim_{n \rightarrow \infty} \sup_{q \in E_n} |G_{n, [\alpha_n t]} f \circ \eta_n(q) - G(t) f \circ \eta_n(q)| = 0, \quad (2.4.23)$$

for any t satisfying $t \notin \Gamma_1 \cup \Gamma_2$ and $0 \leq t \leq T$.

Proof. Fix $f \in \mathcal{D}(G)$, let $\xi_n(t) = f \circ \eta_n(Y_n([\alpha_n t]))$ and $\varphi_n(t) = G_{n, [\alpha_n t]} f \circ \eta_n(Y_n([\alpha_n t]))$ for $t \geq 0$. Then by the conditions (2.4.22) and (2.4.23), $\{(\xi_n, \varphi_n)\}$ satisfies the conditions (2.4.1)-(2.4.6) with $p = \infty$, but $(\xi_n, \varphi_n) \notin \hat{\mathcal{A}}_n$ for $n \geq 1$.

Consequently, we define $(\hat{\xi}_n, \hat{\varphi}_n)$ by

$$\begin{aligned} \hat{\xi}_n(t) &= \alpha_n \int_0^{1/\alpha_n} E[f \circ \eta_n(Y_n([\alpha_n(t+s)])) | \mathcal{G}_t^n] ds \\ &= f \circ \eta_n(Y_n([\alpha_n t])) + \frac{\alpha_n t - [\alpha_n t]}{\alpha_n} G_{n, [\alpha_n t]} f \circ \eta_n(Y_n([\alpha_n t])) \end{aligned} \quad (2.4.24)$$

and

$$\begin{aligned} \hat{\varphi}_n(t) &= \alpha_n E[f \circ \eta_n(Y_n([\alpha_n t] + 1)) - f \circ \eta_n(Y_n([\alpha_n t])) | \mathcal{G}_t^n] \\ &= G_{n, [\alpha_n t]} f \circ \eta_n(Y_n([\alpha_n t])). \end{aligned} \quad (2.4.25)$$

By Proposition 7.5 of Chapter 2 [EK86], $(\hat{\xi}_n, \hat{\varphi}_n) \in \hat{\mathcal{A}}_n$.

Note that $\varphi_n(t) = \hat{\varphi}_n(t)$, and $E[|\xi_n(t) - \hat{\xi}_n(t)|] \leq E[|\varphi_n(t)|]/\alpha_n$, we have that (ξ_n, φ_n) satisfies the conditions (2.4.1)-(2.4.6) with $p = \infty$. Then condition (a) is satisfied and the conclusion follows. $\square$

Remark 2.4.4. 1) We can just assume in Proposition 2.4.1 and Corollary 2.4.2 that $\Gamma_1, \{\Gamma^n\}$ have Lebesgue measure 0. 2) Assume that $D \subset \mathcal{D}(G)$, and $C_a \subset D$ where C_a is a subalgebra of $\bar{C}(E)$ which separates points. If we can prove that the uniqueness

of $D_E[0, \infty)$ martingale problem for (G, ν) holds for functions in D instead of the common domain $\mathcal{D}(G)$, then we can replace $\mathcal{D}(G)$ by D in the conditions (a), (b), (c) and (d). 3) (2.4.7) is a quite general condition. If for each $f \in \mathcal{D}(G)$, $\|G(t)f\|$ grows at most exponentially, i.e., there exist constants $C, \mu > 0$, such that $\|G(t)f\| \leq Ce^{\mu t}$ for $t \geq 0$, then (2.4.7) is satisfied.

Remark 2.4.5. We can simplify the condition (d) of Corollary 2.4.3 into the following version: there exists a countable set $\Gamma_2 \subset [0, \infty)$ such that for each $f \in \mathcal{D}(G)$, and $T > 0$,

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T, t \notin \Gamma_1 \cup \Gamma_2} \sup_{q \in E_n} |G_{n, [\alpha_n t]} f \circ \eta_n(q) - G(t)f \circ \eta_n(q)| = 0. \quad (2.4.26)$$

This condition is stronger than the condition (d), since (2.4.26) implies (2.4.22) and (2.4.23).

We conclude this subsection for preliminary tools with the last corollary which will be of vital importance for the interacting agent feedback finance models in Chapter 3.

Corollary 2.4.6. Let $(E, \bar{r})$ be a Polish space. Let $\{G(s), 0 \leq s < \infty\}$ be a family of operators on $\bar{C}(E)$. Suppose that there exists a countable set $\Gamma_1 \subset [0, \infty)$, such that for each $s \notin \Gamma_1$, $\{G(s), 0 \leq s < \infty\}$ has a common domain denoted by $\mathcal{D}(G)$. Suppose for each $f \in \mathcal{D}(G)$, $G(s)f \in \bar{C}(E)$ for $s \notin \Gamma_1$, $\|G(s)f\|$ is bounded for $s \in [0, T] \setminus \Gamma_1$ for any $T > 0$, and there exists some $\lambda > 0$, such that (2.4.7) holds. Suppose that there is an algebra C_a contained in the closure of $\mathcal{D}(G)$ (in the sup norm) which separates points. Suppose that the $D_E[0, \infty)$ martingale problem for (G, ν) has at most one solution where $\nu \in \mathcal{P}(E)$. For each $n \geq 1$, suppose $\{Y_n(k), k = 0, 1, 2, \dots\}$ is a time inhomogeneous Markov chain in a metric space E_n

with transition functions $\mu_{n,k}(x, \Gamma)$. For each $n \geq 1$, suppose $\eta_n : E_n \mapsto E$ is Borel measurable. Define $T_{n,k} : B(E_n) \mapsto B(E)$ by (2.4.21) and let $G_{n,k} = \alpha_n(T_{n,k} - I)$. Let $X_n = \eta_n(Y_n([\alpha_n \cdot]))$ and $\{\mathcal{G}_t^n\} = \{\mathcal{F}_{[\alpha_n t]}^{Y_n}\}$. Suppose $P(X_n(0))^{-1} \Rightarrow \nu$ and the compact containment condition holds for $\{X_n\}$. Assume that $\alpha_n \rightarrow \infty$ as $n \rightarrow \infty$. Then the following condition (e) implies that there exists a solution X of the $D_E[0, \infty)$ martingale problem for (G, ν) , and $X_n \Rightarrow X$:

(e) There exists a countable set $\Gamma_2 \subset [0, \infty)$ such that for each $f \in \mathcal{D}(G)$, and $T > 0$, there exists measurable set $F_n \subset E_n$ such that $\{Y_n([\alpha_n t]) \in F_n, 0 \leq t \leq T\}$ are events satisfying

$$\lim_{n \rightarrow \infty} P\{Y_n([\alpha_n t]) \in F_n, 0 \leq t \leq T\} = 1 \quad (2.4.27)$$

and

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T, t \notin \Gamma_1 \cup \Gamma_2} \sup_{y \in F_n} |G_{n, [\alpha_n t]} f \circ \eta_n(y) - G(t) f \circ \eta_n(y)| = 0. \quad (2.4.28)$$

Note that for each $f \in \mathcal{D}(G)$, and corresponding $\lambda > 0$, (2.4.7) implies that $\|G(s)f\|$ is bounded a.e. in $[0, T]$ for any $T > 0$. For convenience, we still assume in Corollary 2.4.6 that $\|G(s)f\|$ is bounded for $s \in [0, T] \setminus \Gamma_1$.

Proof. We note that for any $f \in \mathcal{D}(G)$ and $T > 0$, by the assumption that $\|G(s)f\|$ is bounded for $s \in [0, T] \setminus (\Gamma_1 \cup \Gamma_2)$ and (2.4.28), we have

$$\sup_n \sup_{0 \leq t \leq T, t \notin \Gamma_1 \cup \Gamma_2} \sup_{y \in F_n} |G_{n, [\alpha_n t]} f \circ \eta_n(y)| < \infty. \quad (2.4.29)$$

For the fixed f and T , by (2.4.28) again, for $\epsilon = 1$, there exists N , such that for $n > N$, $y \in F_n$ and $0 \leq t \leq T$, $t \notin \Gamma_1 \cup \Gamma_2$,

$$-\|G(t)f\| - \epsilon < G_{n, [\alpha_n t]} f \circ \eta_n(y) < \|G(t)f\| + \epsilon. \quad (2.4.30)$$

which implies

$$|G_{n, [\alpha_n t]} f \circ \eta_n(y)|^2 < (\|G(t)f\| + 1)^2. \quad (2.4.31)$$

Define

$$\tau_n = \inf\{t > 0 : \int_0^t |G_{n, [\alpha_n s]} f \circ \eta_n(Y_n([\alpha_n s]))|^2 ds \geq \int_0^t (\|G(s)f\| + 1)^2 ds\}. \quad (2.4.32)$$

Let $C_n = \{Y_n([\alpha_n t]) \in F_n, 0 \leq t \leq T\}$. Fix $n > N$, for any $\omega \in C_n$, since $Y_n([\alpha_n t]) \in F_n, 0 \leq t \leq T$, we have

$$\int_0^t |G_{n, [\alpha_n s]} f \circ \eta_n(Y_n([\alpha_n s]))|^2 ds < \int_0^t (\|G(s)f\| + 1)^2 ds. \quad (2.4.33)$$

Then $\omega \in \{\tau_n \geq T\}$, i.e.

$$C_n \subset \{\tau_n \geq T\}. \quad (2.4.34)$$

Then by (2.4.27), we proved

$$\lim_{n \rightarrow \infty} P\{\tau_n < T\} = 0. \quad (2.4.35)$$

Define $(\hat{\xi}_n, \hat{\varphi}_n)$ by (2.4.24) and (2.4.25). Then $(\hat{\xi}_n, \hat{\varphi}_n) \in \hat{\mathcal{A}}_n$. We also have

$$\hat{\xi}_n(t) = f \circ \eta_n(Y_n([\alpha_n t])) + \frac{\alpha_n t - [\alpha_n t]}{\alpha_n} \hat{\varphi}_n(t). \quad (2.4.36)$$

Define (ξ_n, φ_n) by

$$\xi_n(t) = \hat{\xi}_n(t \wedge \tau_n), \quad (2.4.37)$$

$$\varphi_n(t) = \hat{\varphi}_n(t) \chi_{\{\tau_n > t\}}. \quad (2.4.38)$$

which implies $(\xi_n, \varphi_n) \in \hat{\mathcal{A}}_n$. To justify our conclusion, we are going to verify the condition (b) of Proposition 2.4.1, i.e. we need to verify that (ξ_n, φ_n) satisfy (2.4.1), (2.4.3), (2.4.5), (2.4.6), and (2.4.8).

Firstly, we verify (2.4.1). This is given by

$$\begin{aligned}
\sup_n \sup_{0 \leq t \leq T} E[|\xi_n(t)|] &= \sup_n \sup_{0 \leq t \leq T} E[|\hat{\xi}_n(t \wedge \tau_n)|] \\
&\leq \sup_n \sup_{0 \leq t \leq T} E[|f \circ \eta_n(Y_n([\alpha_n(t \wedge \tau_n)]))|] \\
&\quad + \sup_n \sup_{0 \leq t \leq T} \frac{1}{\alpha_n} E[|\hat{\varphi}_n(t \wedge \tau_n)|] \\
&\leq \|f\| + \sup_n \sup_{0 \leq t \leq T} E[|T_{n, [\alpha_n(t \wedge \tau_n)]} - I| f \circ \eta_n(Y_n([\alpha_n(t \wedge \tau_n)]))|] \\
&\leq 3\|f\|.
\end{aligned} \tag{2.4.39}$$

Secondly, we verify condition (2.4.5). Let $\tilde{Q}$ be a countable and dense subset of R such that $\tilde{Q} \cap (\Gamma_1 \cup \Gamma_2) = \emptyset$. For $n > N$,

$$\begin{aligned}
&E\left[\sup_{t \in \tilde{Q} \cap [0, T]} |\xi_n(t) - f \circ \eta_n(Y_n([\alpha_n t]))|\right] \\
&\leq E\left[\sup_{t \in \tilde{Q} \cap [0, T]} |f \circ \eta_n(Y_n([\alpha_n(t \wedge \tau_n)])) - f \circ \eta_n(Y_n([\alpha_n t]))| \right. \\
&\quad \left. + \frac{1}{\alpha_n} E\left[\sup_{t \in \tilde{Q} \cap [0, T]} |\hat{\varphi}_n(t \wedge \tau_n)|\right]\right].
\end{aligned} \tag{2.4.40}$$

We denote by $I_{n,1}$, $I_{n,2}$ the first and second terms on the right of (2.4.40). Then by (2.4.34)

$$\begin{aligned}
I_{n,1} &= E\left[\sup_{t \in \tilde{Q} \cap [0, T]} |f \circ \eta_n(Y_n([\alpha_n(t \wedge \tau_n)])) - f \circ \eta_n(Y_n([\alpha_n t]))| \chi_{C_n}\right] \\
&\quad + E\left[\sup_{t \in \tilde{Q} \cap [0, T]} |f \circ \eta_n(Y_n([\alpha_n(t \wedge \tau_n)])) - f \circ \eta_n(Y_n([\alpha_n t]))| \chi_{C_n^c}\right] \\
&\leq 0 + 2\|f\| E[\chi_{C_n^c}] \\
&= 2\|f\|(1 - P(C_n)).
\end{aligned} \tag{2.4.41}$$

and

$$\begin{aligned}
I_{n,2} &\leq \frac{1}{\alpha_n} E \left[\sup_{t \in \tilde{Q} \cap [0, T]} |G_{n, [\alpha_n t]} f \circ \eta_n(Y_n([\alpha_n t]))| \chi_{C_n} \right] \\
&\quad + \frac{1}{\alpha_n} E \left[\sup_{t \in \tilde{Q} \cap [0, T]} |G_{n, [\alpha_n t]} f \circ \eta_n(Y_n([\alpha_n t]))| \chi_{C_n^c} \right] \\
&\leq \frac{1}{\alpha_n} \sup_k \sup_{0 \leq t \leq T, t \notin \Gamma_1 \cup \Gamma_2} \sup_{y \in F_k} |G_{k, [\alpha_k t]} f \circ \eta_k(y)| \\
&\quad + E \left[\sup_{t \in \tilde{Q} \cap [0, T]} E[|f \circ \eta_n(Y_n([\alpha_n t] + 1)) - f \circ \eta_n(Y_n([\alpha_n t]))| \mid \mathcal{G}_t^n] \chi_{C_n^c} \right] \\
&\leq \frac{1}{\alpha_n} \sup_k \sup_{0 \leq t \leq T, t \notin \Gamma_1 \cup \Gamma_2} \sup_{y \in F_k} |G_{k, [\alpha_k t]} f \circ \eta_k(y)| + 2\|f\|(1 - P(C_n)).
\end{aligned} \tag{2.4.42}$$

By (2.4.27) and (2.4.29), (2.4.5) is verified.

Note that $\tilde{Q}$ is chosen to be any countable and dense subset of R which satisfies $\tilde{Q} \cap (\Gamma_1 \cup \Gamma_2) = \emptyset$. In the procedure of verifying (2.4.5), we actually have verified (2.4.3) for any $0 \leq t \leq T$ with $t \notin \Gamma_1 \cup \Gamma_2$.

Thirdly, (2.4.6) for $p = 2$ is verified by the following calculations. Notice that for $n > N$, by the definition of τ_n ,

$$\begin{aligned}
&E \left[\left(\int_0^T |G_{n, [\alpha_n s]} f \circ \eta_n(Y_n([\alpha_n s]))|^2 \chi_{\{\tau_n > s\}} ds \right)^{\frac{1}{2}} \right] \\
&= E \left[\left(\int_0^{T \wedge \tau_n} |G_{n, [\alpha_n s]} f \circ \eta_n(Y_n([\alpha_n s]))|^2 ds \right)^{\frac{1}{2}} \right] \\
&\leq E \left[\left(\int_0^{T \wedge \tau_n} (\|G(s)f\| + 1)^2 ds \right)^{\frac{1}{2}} \right] \\
&\leq \left(\int_0^T (\|G(s)f\| + 1)^2 ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.4.43}$$

Lastly, we verify (2.4.8). We will prove for $0 \leq t \leq T$,

$$\lim_{n \rightarrow \infty} \int_0^{T-t} E[|\varphi_n(t+s) - G(t+s)f \circ \eta_n(Y_n([\alpha_n(t+s)]))|] ds = 0. \tag{2.4.44}$$

Observe that

$$\begin{aligned}
& \int_0^{T-t} E[|\varphi_n(t+s) - G(t+s)f \circ \eta_n(Y_n([\alpha_n(t+s)]))|]ds \\
&= \int_0^{T-t} E[|G_{n, [\alpha_n(t+s)]}f \circ \eta_n(Y_n([\alpha_n(t+s)]))\chi_{\{\tau_n > t+s\}} \\
&\quad - G(t+s)f \circ \eta_n(Y_n([\alpha_n(t+s)]))|]ds \\
&\leq \int_t^T E[|G_{n, [\alpha_n u]}f \circ \eta_n(Y_n([\alpha_n u])) - G(u)f \circ \eta_n(Y_n([\alpha_n u]))|\chi_{\{\tau_n > u\}}]du \\
&\quad + \int_t^T E[|G(u)f \circ \eta_n(Y_n([\alpha_n u]))|\chi_{\{\tau_n \leq u\}}]du.
\end{aligned} \tag{2.4.45}$$

We denote by $I_{n,3}$, $I_{n,4}$ the first and second terms in the last line of (2.4.45). Then, by interchanging the order of integration, we get

$$\begin{aligned}
I_{n,4} &\leq E\left[\int_t^T |G(u)f \circ \eta_n(Y_n([\alpha_n(u)]))|\chi_{\{\tau_n \leq u\}}du\right] \\
&\leq E\left[\int_t^T |G(u)f \circ \eta_n(Y_n([\alpha_n(u)]))|du\chi_{\{\tau_n < T\}}\right] \\
&\leq T \cdot \text{ess sup}_{0 \leq u \leq T} \|G(u)f\|P\{\tau_n < T\},
\end{aligned} \tag{2.4.46}$$

and

$$\begin{aligned}
I_{n,3} &= E\left[\int_t^T |G_{n, [\alpha_n u]}f \circ \eta_n(Y_n([\alpha_n u])) - G(u)f \circ \eta_n(Y_n([\alpha_n u]))|\chi_{\{\tau_n > u\}}du\right] \\
&= E\left[\int_t^T |G_{n, [\alpha_n u]}f \circ \eta_n(Y_n([\alpha_n u])) - G(u)f \circ \eta_n(Y_n([\alpha_n u]))|\chi_{\{\tau_n > u\}}du\chi_{C_n}\right] \\
&\quad + E\left[\int_t^T |G_{n, [\alpha_n u]}f \circ \eta_n(Y_n([\alpha_n u])) - G(u)f \circ \eta_n(Y_n([\alpha_n u]))|\chi_{\{\tau_n > u\}}du\chi_{C_n^c}\right].
\end{aligned} \tag{2.4.47}$$

Denote by $I_{n,5}$, $I_{n,6}$ the first and second terms in the last equality of (2.4.47). For $n > N$, we have

$$I_{n,5} \leq T \sup_{0 \leq t \leq T, t \notin \Gamma_1 \cup \Gamma_2} \sup_{y \in F_n} |G_{n, [\alpha_n t]}f \circ \eta_n(y) - G(t)f \circ \eta_n(y)|, \tag{2.4.48}$$

and by Cauchy-Schwartz inequality,

$$\begin{aligned}
I_{n,6} &\leq E\left[\int_0^T |G_{n,[\alpha_n u]} f \circ \eta_n(Y_n([\alpha_n u]))| \chi_{\{\tau_n > u\}} du \chi_{C_n^c}\right] \\
&\quad + E\left[\int_0^T |G(u) f \circ \eta_n(Y_n([\alpha_n u]))| du \chi_{C_n^c}\right] \\
&\leq (TE\left[\int_0^{T \wedge \tau_n} |G_{n,[\alpha_n u]} f \circ \eta_n(Y_n([\alpha_n u]))|^2 du\right])^{\frac{1}{2}} (E[\chi_{C_n^c}])^{\frac{1}{2}} \\
&\quad + T \cdot \text{ess sup}_{0 \leq u \leq T} \|G(u) f\| (1 - P(C_n)) \\
&\leq \left(T \int_0^T (\|G(s) f\| + 1)^2 ds\right)^{\frac{1}{2}} (1 - P(C_n))^{\frac{1}{2}} + T \text{ess sup}_{0 \leq u \leq T} \|G(u) f\| (1 - P(C_n)).
\end{aligned} \tag{2.4.49}$$

Since $\lim_{n \rightarrow \infty} I_{n,4} = 0$ by (2.4.35), $\lim_{n \rightarrow \infty} I_{n,5} = 0$ by (2.4.28), and $\lim_{n \rightarrow \infty} I_{n,6} = 0$ by (2.4.27), we proved (2.4.44).

Then we verified that (ξ_n, φ_n) satisfies the condition (b) of Proposition 2.4.1. $\square$

2.4.2 General multiagent model

In this subsection, we use the formulation in subsection 2.3.1 for the sequence $\{\mathbf{X}^N\}$, but the formulation for the limit process will be different from that for evolutionary system in subsection 2.3.2. Precisely, let $\{A_N\} \subset D_{R^r \times r}[0, \infty)$ and $A \in D_{R^r \times r}[0, \infty)$ be matrix-valued functions on $[0, \infty)$ satisfying the conditions specified in subsection 2.3.1. Then we define $P_{N,k}$, $\mathbf{n}^N$ and $\mathbf{X}^N$, $S_{N,k}$ exactly as those in subsection 2.3.1. We determine the limit process $\mathbf{X}^\infty$ as follows. Define time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C(K)$: for each $f \in C^1(K)$ and $s \geq 0$,

$$G_A(s)f(\mathbf{p}) = \mathbf{p}A(s)\frac{\partial f'}{\partial \mathbf{x}}. \tag{2.4.50}$$

It is clear that, for each $s \geq 0$, $\mathcal{D}(G_A(s)) = C^1(K)$ and $\mathcal{D}(G_A) = C^1(K)$ is the common domain of the generator $\{G_A(s), 0 \leq s < \infty\}$, and $D = C^2(K)$ is a subalgebra contained in $\mathcal{D}(G_A)$.

Define the metric d_U on $D_{R^r \times r}[0, \infty)$ as follows:

$$d_U(\mathbf{x}, \mathbf{y}) = \int_0^\infty e^{-u} \sup_{0 \leq t \leq u} [\|\mathbf{x}(t) - \mathbf{y}(t)\|_r \wedge 1] du, \quad \mathbf{x}, \mathbf{y} \in D_{R^r \times r}[0, \infty). \quad (2.4.51)$$

We will show that if $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$, and $\mu \in \mathcal{P}(K)$, then there exists a unique process $\mathbf{X}^\infty$ as the solution of the martingale problem for (G_A, μ) with $P(\mathbf{X}^\infty(0))^{-1} = \mu$. It means that A defined here can determine a unique $D_K[0, \infty)$ -valued process $\mathbf{X}^\infty$.

Example 2.4.7. *If A defined above has only finitely many discontinuities on $[0, T]$ for any $T > 0$, then there exists a strictly increasing sequence $\{t_k\}$ such that A is piece-wise continuous on each segment $[t_{k-1}, t_k)$ and $A(t_k-)$ is finite for $k \geq 1$. Note that $t_0 = 0$ and $\{t_k\}$ may have finitely many points. Then $\{\mathbf{X}^\infty(t), 0 \leq t < \infty\}$ is determined as follows. For each $k \geq 0$, we define matrix-valued functions $B_k(t)$ on $[t_k, t_{k+1}]$ by $B_k(t) = A(t)$ on $[t_k, t_{k+1})$, and $B_k(t_{k+1}) = A(t_{k+1}-)$. $B_k(t)$ is a continuous function on $[t_k, t_{k+1}]$. Assume that $\mathbf{p}$ is given, the solution of the differential equations $\frac{d\mathbf{x}}{dt} = B_0(t)' \mathbf{x}$ on the interval $[t_0, t_1]$ satisfying $\mathbf{x}(t_0) = \mathbf{p}'$ is given by $\mathbf{x}_1(t)$. We define $\mathbf{X}^\infty(t) = \mathbf{x}_1(t)'$ on $[t_0, t_1]$. Let $\mathbf{p}_1 = \mathbf{x}_1(t_1)'$, the solution of the differential equations $\frac{d\mathbf{x}}{dt} = B_1(t)' \mathbf{x}$ on the interval $[t_1, t_2]$ satisfying $\mathbf{x}(t_1) = \mathbf{p}_1'$ is given by $\mathbf{x}_2(t)$. We define $\mathbf{X}^\infty(t) = \mathbf{x}_2(t)'$ on $(t_1, t_2]$. We can continue this procedure and get a definition of $\mathbf{X}^\infty$ denoted by $\mathbf{X}_{\mathbf{p}}^\infty$ for given A and $\mathbf{p}$.*

Lemma 2.4.8. *Assume that $\{A_N\} \subset D_{R^r \times r}[0, \infty)$, $A \in D_{R^r \times r}[0, \infty)$, satisfying the conditions specified at the beginning of subsection 2.3.1. Define $P_{N,k}$, $\mathbf{X}^N$, and $S_{N,k}$*

in subsection 2.3.1. If $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$, where d_U is the metric on $D_{R^r \times r}[0, \infty)$ defined by (2.4.51), then

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} |N[S_{N,[Nt]} - I]f(\mathbf{p}) - G_A(t)f(\mathbf{p})| = 0 \quad (2.4.52)$$

for any $f \in D = C^2(K)$.

Note that if $\{A_N\}$ and A in Lemma 2.4.8 satisfy $A \in C_{R^r \times r}[0, \infty)$ and $\lim_{N \rightarrow \infty} d(A_N, A) = 0$ where d is the Skorohod metric on $D_{R^r \times r}[0, \infty)$, then $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$.

Proof. We will use ideas used in the proof of Lemma 2.3.6. Let $f \in D = C^2(K)$, fix $T > 0$. For $0 \leq t \leq T$, and $\mathbf{p} \in K_N$, note that by Taylor's expansion and (2.3.5)

$$\begin{aligned} & [S_{N,[Nt]} - I]f(\mathbf{p}) \\ &= E[(\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p}) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{p})' | \mathbf{X}^N(t) = \mathbf{p}] \\ & \quad + E[(\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p}) \frac{\partial^2 f}{\partial \mathbf{x}^2}(\mathbf{X}_*^N(t)) (\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p})' | \mathbf{X}^N(t) = \mathbf{p}] \quad (2.4.53) \\ &= \frac{1}{N} \mathbf{p} A_N(\frac{[Nt]}{N}) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{p})' \\ & \quad + E[(\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p}) \frac{\partial^2 f}{\partial \mathbf{x}^2}(\mathbf{X}_*^N(t)) (\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p})' | \mathbf{X}^N(t) = \mathbf{p}], \end{aligned}$$

where $\mathbf{X}_*^N(t) = \mathbf{p} + \theta_t^N(\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p})$, for some $\theta_t^N \in (0, 1)$. Then

$$\begin{aligned} & |N[S_{N,[Nt]} - I]f(\mathbf{p}) - G_A(t)f(\mathbf{p})| \\ & \leq |\mathbf{p}[A_N(\frac{[Nt]}{N}) - A(t)] \frac{\partial f}{\partial \mathbf{x}}(\mathbf{p})'| \quad (2.4.54) \\ & \quad + N|E[(\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p}) \frac{\partial^2 f}{\partial \mathbf{x}^2}(\mathbf{X}_*^N(t)) (\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{p})' | \mathbf{X}^N(t) = \mathbf{p}]|. \end{aligned}$$

Denote by $I_1(N, \mathbf{p}, t)$ and $I_2(N, \mathbf{p}, t)$, the first and second term on the right hand

side of (2.4.54). Let $\|\frac{\partial^2 f}{\partial \mathbf{x}^2}\| = \max_{1 \leq i, j \leq r} \|\frac{\partial^2 f}{\partial x_i \partial x_j}\|$. By Hölder inequality,

$$\begin{aligned} I_2(N, \mathbf{p}, t) &\leq \|\frac{\partial^2 f}{\partial \mathbf{x}^2}\| N \sum_{i=1}^r \sum_{j=1}^r E[|(X_i^N(t + \frac{1}{N}) - p_i)(X_j^N(t + \frac{1}{N}) - p_j)| | \mathbf{X}^N(t) = \mathbf{p}] \\ &\leq \|\frac{\partial^2 f}{\partial \mathbf{x}^2}\| N \sum_{i=1}^r \sum_{j=1}^r \left(E[(X_i^N(t + \frac{1}{N}) - p_i)^2 | \mathbf{X}^N(t) = \mathbf{p}] \right. \\ &\quad \left. \times E[(X_j^N(t + \frac{1}{N}) - p_j)^2 | \mathbf{X}^N(t) = \mathbf{p}] \right)^{\frac{1}{2}}. \end{aligned} \quad (2.4.55)$$

Since $A_N(t)$ is a constant on $[\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$, for fixed $1 \leq i \leq r$, by (2.3.7), we get

$$\begin{aligned} &N^2 E[(X_i^N(t + \frac{1}{N}) - p_i)^2 | \mathbf{X}^N(t) = \mathbf{p}] \\ &= \mathbf{p} A_{N,i}(\frac{[Nt]}{N}) + (\mathbf{p} A_{N,i}(\frac{[Nt]}{N}))^2 - 2p_i a_{N,i,i}(\frac{[Nt]}{N}) - \sum_{k=1}^r \frac{p_k}{N} a_{N,k,i}^2(\frac{[Nt]}{N}) \\ &\leq \sum_{l=1}^r p_l |A_{N,l,i}(t)| + (\sum_{l=1}^r p_l |A_{N,l,i}(t)|)^2 + 2p_i |a_{N,i,i}(t)| \\ &\leq \sum_{l=1}^r |A_{N,l,i}(t)| + (\sum_{l=1}^r |A_{N,l,i}(t)|)^2 + 2|a_{N,i,i}(t)|. \end{aligned} \quad (2.4.56)$$

Since $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$ and A is bounded on $[0, T]$, there exists $C_T > 0$ such that

$$N^2 E[(X_i^N(t + \frac{1}{N}) - p_i)^2 | \mathbf{X}^N(t) = \mathbf{p}] \leq C_T \quad (2.4.57)$$

for any $N \geq 1$, $0 \leq t \leq T$, $\mathbf{p} \in K_N$, and $1 \leq i \leq r$. Then

$$\sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} I_2(N, \mathbf{p}, t) \leq \frac{1}{N} \|\frac{\partial^2 f}{\partial \mathbf{x}^2}\| r^2 C_T. \quad (2.4.58)$$

Thus $\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} I_2(N, \mathbf{p}, t) = 0$.

Before we consider $I_1(N, \mathbf{p}, t)$, we recall norms of matrices and vectors. For $r \times r$ real-valued matrix $B = (b_{i,j})_{r \times r}$ and real-valued r -dimensional vector $\mathbf{y} =$

$(y_1, \dots, y_r)'$. We define $\|B\| = \sum_{i,j=1}^r |b_{i,j}|$ and $\|\mathbf{y}\|$ the Euclidean norm on R^r of $\mathbf{y}$. Then $\|B\mathbf{y}\| \leq \|B\| \cdot \|\mathbf{y}\|$. Now we have

$$\begin{aligned} I_1(N, \mathbf{p}, t) &\leq \|\mathbf{p}'\| \cdot \|A_N(\frac{[Nt]}{N}) - A(t)\| \cdot \|\frac{\partial f}{\partial \mathbf{x}}(\mathbf{p})'\| \\ &\leq \|A_N(\frac{[Nt]}{N}) - A(t)\| \cdot \|\frac{\partial f}{\partial \mathbf{x}}(\mathbf{p})'\|. \end{aligned} \quad (2.4.59)$$

Notice that $A_N(\frac{[Nt]}{N}) = A_N(t)$, $t \geq 0$,

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} I_1(N, \mathbf{p}, t) = 0 \quad (2.4.60)$$

follows by (2.4.59) and $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$. Therefore (2.4.52) is proved. $\square$

Note that to have (2.4.58), it is enough to just have $\lim_{N \rightarrow \infty} d(A_N, A) = 0$, but we must have $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$ to guarantee (2.4.60).

Corollary 2.4.9. *With the same conditions as those in Lemma 2.4.8, we have*

$$\sup_N \sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} |N[S_{N,[Nt]} - I]f(\mathbf{p})| < \infty \quad (2.4.61)$$

for any $T > 0$ and $f \in C^2(K)$.

Proof. This follows from Lemma 2.4.8 immediately. $\square$

Note also that to have (2.4.61), we just need to assume that $\lim_{N \rightarrow \infty} d(A_N, A) = 0$. The conclusion can be made stronger than (2.4.61) under this weak condition, i.e. (2.4.61) is true for any $T > 0$ and $f \in C^1(K)$. A new proof which is essentially similar to that for Lemma 2.4.8 is required for this new conclusion. Actually, we just need to use Taylor's expansion of order 1 for the proof of this case.

Before we prove the uniqueness of the martingale problem for (G_A, μ) where G_A is defined by (2.4.50) and $\mu \in \mathcal{P}(K)$, we introduce some notations. For each $n \geq 1$, let $\mathbb{N}_n = \{\alpha : \alpha \in (Z_+)^r, \sum_{i=1}^r \alpha_i = n\}$. It is clear that $\mathbb{N}_n$ has $c_n = \binom{n+r-1}{r}$ elements.

Define an order ' $>$ ' on $\mathbb{N}_n$ as follows: let $\alpha, \alpha' \in \mathbb{N}_n$, $\alpha > \alpha' \Leftrightarrow \alpha_1 > \alpha'_1$ or there exists $2 \leq k \leq r$, such that $\alpha_l = \alpha'_l$ for $1 \leq l \leq k-1$ and $\alpha_k > \alpha'_k$. We use $\mathbb{N}_n$ as the index set when we define $D_{R^{c_n \times c_n}}[0, \infty)$ -valued matrix functions. For given $A \in D_{R^{r \times r}}[0, \infty)$, define $A^{(n)} = (a_{\alpha, \alpha'}^{(n)})_{\alpha, \alpha' \in \mathbb{N}_n} \in D_{R^{c_n \times c_n}}[0, \infty)$ as follows: for each $\alpha, \alpha' \in \mathbb{N}_n$ and $s \geq 0$,

$$a_{\alpha, \alpha'}^{(n)}(s) = \begin{cases} \sum_{j=1}^r \alpha_j a_{jj}(s), & \text{if } \alpha = \alpha'; \\ \alpha_j a_{kj}(s), & \text{if there exist } 1 \leq j, k \leq r, j \neq k, \text{ such that } \alpha_l = \alpha'_l, \\ & \text{for } 1 \leq l \leq r \text{ satisfying } l \neq j \text{ or } k, \text{ and } \alpha'_j = \alpha_j - 1, \\ & \alpha'_k = \alpha_k + 1; \\ 0, & \text{otherwise.} \end{cases} \quad (2.4.62)$$

We arrange the elements of $A^{(n)}$ along the rows and columns decreasingly by the order ' $>$ '.

Next, we introduce some background on *Volterra equations of the second kind* (see [SMI58] or [TR57]). Let $\mathbb{L}^2([a, b], R^m)$ be the space of measurable and square integrable functions (or equivalent classes) on the bounded interval $[a, b] \subset R_+$, with respect to Lebesgue measure on the m -dimensional space R^m . $V(s, t)$ is called an $\mathbb{L}^2([a, b], R^m)$ kernel if it satisfies the following three conditions:

- (a) $V(s, t)$ is a measurable function of (s, t) in $[a, b] \times [a, b]$, such that

$$\int_a^b \int_a^b \|V(s, t)\|^2 ds dt < \infty; \quad (2.4.63)$$

- (b) for each $s \in [a, b]$, $V(s, t)$ is a measurable function of t such that

$$\int_a^b \|V(s, t)\|^2 dt < \infty; \quad (2.4.64)$$

(c) for each $t \in [a, b]$, $V(s, t)$ is a measurable function of s such that

$$\int_a^b \|V(s, t)\|^2 ds < \infty. \quad (2.4.65)$$

The following equation is called a *Volterra equation of the second kind*,

$$\mathbf{x}(s) = \mathbf{f}(s) + \int_a^s V(s, t)\mathbf{x}(t)dt, \quad (a \leq s \leq b) \quad (2.4.66)$$

where $\mathbf{f}$ is an $\mathbb{L}^2([a, b], R^m)$ function.

Proposition 2.4.10. *For any $\mathbb{L}^2([a, b], R^m)$ kernel $V(s, t)$ and $\mathbb{L}^2([a, b], R^m)$ function $\mathbf{f}$, equation (2.4.66) has a unique $\mathbb{L}^2([a, b], R^m)$ solution.*

Note that we can refer to (see [SMI58] or [TR57]) for the case $m = 1$ of Proposition 2.4.10. The general case $m \geq 1$ can be obtained by the same method as that for $m = 1$.

Lemma 2.4.11. *Define the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C(K)$ by (2.4.50). Let $\mu \in \mathcal{P}(K)$. Then the $D_K[0, \infty)$ martingale problem for (G_A, μ) has at most one solution on $C^2(K)$.*

Proof. Assume that $\{\mathbf{X}(t) = (X_1(t), \dots, X_r(t)) : 0 \leq t < \infty\}$ is one solution of the martingale problem for (G_A, μ) and fix $n \geq 1$. For arbitrary $\alpha \in \mathbb{N}_n$, and $t \geq 0$, define $f_\alpha(x_1, \dots, x_r) = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ and $y_n^\alpha(t) = E[f_\alpha(\mathbf{X}(t))]$, then $f_\alpha \in C^2(K)$. Define the column vector $\mathbf{y}_n(t) = (y_n^\alpha(t))_{\alpha \in \mathbb{N}_n}$, where the elements of $\mathbf{y}_n(t)$ is arranged decreasingly by the order $' >'$.

By definition, $f_\alpha \in C^1(K)$ for each $\alpha \in \mathbb{N}_n$. For given $\alpha \in \mathbb{N}_n$ and $t > 0$, we have

$$E[f_\alpha(\mathbf{X}(t))] = E[f_\alpha(\mathbf{X}(0))] + \int_0^t E[G_A(s)f_\alpha(\mathbf{X}(s))]ds. \quad (2.4.67)$$

It follows by (2.4.62) that

$$\begin{aligned}
y_n^\alpha(t) &= y_n^\alpha(0) + \int_0^t E[\mathbf{X}(s)A(s)\frac{\partial f_\alpha}{\partial \mathbf{x}}(\mathbf{X}(s))']ds \\
&= y_n^\alpha(0) + \int_0^t \left[\sum_{j=1}^r \alpha_j a_{j,j}(s)y_n^\alpha(s) + \sum_{k=1}^r \sum_{j=1}^{k-1} \alpha_j a_{k,j}(s)y^{(\alpha_1, \dots, \alpha_j-1, \dots, \alpha_k+1, \dots, \alpha_r)}(s) \right. \\
&\quad \left. + \sum_{k=1}^r \sum_{j=k+1}^r \alpha_j a_{k,j}(s)y^{(\alpha_1, \dots, \alpha_k+1, \dots, \alpha_j-1, \dots, \alpha_r)}(s) \right] ds \\
&= y_n^\alpha(0) + \int_0^t A_{\alpha, \cdot}^{(n)}(s)y_n(s)ds,
\end{aligned} \tag{2.4.68}$$

where $A_{\alpha, \cdot}^{(n)}$ is the α -row of the matrix $A^{(n)}$. Then, for each $n \geq 1$ and $t > 0$, we have

$$\mathbf{y}_n(t) = \mathbf{y}_n(0) + \int_0^t A^{(n)}(s)\mathbf{y}_n(s)ds. \tag{2.4.69}$$

Define

$$V_n(s, t) = \begin{cases} A^{(n)}(t) & \text{if } 0 \leq t \leq s < \infty; \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.70}$$

Then, by (2.4.69), we know that $\mathbf{y}_n$ is a solution of the *Volterra equation* of the second kind

$$\mathbf{x}(s) = \mathbf{y}_n(0) + \int_0^s V_n(s, t)\mathbf{x}(t)dt, \quad (0 \leq s \leq T) \tag{2.4.71}$$

on the space $\mathbb{L}^2([0, T], R^{c_n})$ for any $T > 0$ and $n \geq 1$.

Since $A \in D_{R^r \times r}[0, \infty)$, we know that $V_n(s, t)$ is an $\mathbb{L}^2([0, T], R^{c_n})$ kernel for each $n \geq 1$, by Proposition 2.4.10, we know that all moments of the one-dimensional marginal distribution of any two solutions of the martingale problem for (G_A, μ) are the same. The uniqueness of the martingale problem for (G_A, μ) then follows by Theorem 4.2 of Chapter 4 [EK86]. $\square$

Theorem 2.4.12. *With the same conditions as those in Lemma 2.4.8 and $\mu \in \mathcal{P}(K)$, if $P(\mathbf{X}^N(0))^{-1} \Rightarrow \mu$, there exists a solution $\mathbf{X}^\infty$ of the $D_K[0, \infty)$ martingale problem for (G_A, μ) , and $\mathbf{X}^N \Rightarrow \mathbf{X}^\infty$.*

Proof. Note that the uniqueness for the $D_K[0, \infty)$ martingale problem for (G_A, μ) is proved in Lemma 2.4.11 where we restrict the common domain $\mathcal{D}(G_A) = C^1(K)$ to the subalgebra $C^2(K)$. Then the conclusion follows by Lemma 2.4.8, 2.4.11, Remark 2.4.4, and Remark 2.4.5. $\square$

Note that Corollary 2.4.9 is not necessary for the proof of Theorem 2.4.12, since we verified the stronger condition in Lemma 2.4.8. But this situation will be changed in next subsection.

Note also that Theorem 2.3.7 can follow immediately by Theorem 2.4.12.

2.4.3 General multiagent model with multinomial sampling

We define the general multiagent model with multinomial sampling (GMAMWMS) exactly same way as that for the homogeneous multiagent model with multinomial sampling (HMAMWMS). This is illustrated by the Figure 2.4.

Figure 2.4: The structure of GMAMWMS

Recall the definition of T_N and B , in (2.2.36) and (2.2.37) and keep these symbols

as well as G_B . Replace A by $\{A(s), 0 \leq s < \infty\}$, P_N in (2.2.4) by $\{P_{N,k}, k \geq 0\}$ in (2.3.1), S_N in (2.2.24) by $\{S_{N,k}, k \geq 0\}$ in (2.3.13), and U_N in (2.2.40), by $\{U_{N,k}, k \geq 0\}$ defined as follows

$$U_{N,k}f(\mathbf{p}) = E[f(\frac{\mathbf{n}^N(k+1)}{N}) | \frac{\mathbf{n}^N(k)}{N} = \mathbf{p}], \quad f \in C(K_N), \quad \mathbf{p} \in K_N \text{ for any } k \geq 0. \quad (2.4.72)$$

Then (2.2.41) is replaced by

$$U_{N,k} = T_N S_{N,k}. \quad (2.4.73)$$

We also replace G_A in (2.2.26) by $\{G_A(s), 0 \leq s < \infty\}$ in (2.4.50). Let us define the new generator $\{G_{AB}(s), 0 \leq s < \infty\}$ on $C(K)$ by

$$G_{AB}(s) = G_A(s) + G_B, \text{ for any } s \geq 0, \quad (2.4.74)$$

whose common domain is $\mathcal{D}(G_{AB}) = \mathcal{D}(G_A) \cap \mathcal{D}(G_B) = C^2(K)$.

We need two lemmas which can play the role as Lemma 2.4.8 does in proving weak convergence theorem.

Lemma 2.4.13. *Assume that $\{A_N\} \subset D_{R^r \times r}[0, \infty)$, $A \in D_{R^r \times r}[0, \infty)$, satisfying the conditions specified at the beginning of subsection 2.3.1. Assume also that $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$. With the definitions above, we have*

$$\lim_{N \rightarrow \infty} \sup_{\mathbf{p} \in K_N} |N[U_{N,[Nt]} - I]f(\mathbf{p}) - G_{AB}(t)f(\mathbf{p})| = 0 \quad (2.4.75)$$

for every $f \in C^2(K)$ and continuous point t of A .

Proof. Let $f \in D = C^2(K)$, fix $T > 0$. For $0 \leq t \leq T$, t being continuous point of

A , and $\mathbf{p} \in K_N$, notice that by (2.4.73)

$$\begin{aligned}
& |N[U_{N,[Nt]} - I]f(\mathbf{p}) - G_{AB}(t)f(\mathbf{p})| \\
&= |T_N[N(S_{N,[Nt]} - I)]f(\mathbf{p}) - G_A(t)f(\mathbf{p}) + N(T_N - I)f(\mathbf{p}) - G_Bf(\mathbf{p})| \\
&\leq |T_N[N(S_{N,[Nt]} - I)]f(\mathbf{p}) - T_NG_A(t)f(\mathbf{p})| + |T_NG_A(t)f(\mathbf{p}) - G_A(t)f(\mathbf{p})| \\
&\quad + |N(T_N - I)f(\mathbf{p}) - G_Bf(\mathbf{p})| \\
&\leq \sup_{\mathbf{p} \in K_N} |N(S_{N,[Nt]} - I)f(\mathbf{p}) - G_A(t)f(\mathbf{p})| + \sup_{\mathbf{p} \in K_N} |T_NG_A(t)f(\mathbf{p}) - G_A(t)f(\mathbf{p})| \\
&\quad + \sup_{\mathbf{p} \in K_N} |N(T_N - I)f(\mathbf{p}) - G_Bf(\mathbf{p})|.
\end{aligned} \tag{2.4.76}$$

(2.4.75) is then proved by (2.4.52), Bernstein's theorem and (2.2.38). $\square$

Note that we fail to have a conclusion like

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} |N[U_{N,[Nt]} - I]f(\mathbf{p}) - G_{AB}(t)f(\mathbf{p})| = 0 \tag{2.4.77}$$

for every $f \in C^2(K)$ and $T > 0$ because $\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} |T_NG_A(t)f(\mathbf{p}) - G_A(t)f(\mathbf{p})| = 0$ can not be guaranteed by Bernstein's theorem.

Lemma 2.4.14. *With the same conditions as those in Lemma 2.4.13, we have*

$$\sup_N \sup_{0 \leq t \leq T} \sup_{\mathbf{p} \in K_N} |N[U_{N,[Nt]} - I]f(\mathbf{p})| < \infty \tag{2.4.78}$$

for any $T > 0$ and $f \in C^2(K)$.

Proof. Let $f \in D = C^2(K)$, fix $T > 0$. For $0 \leq t \leq T$, and $\mathbf{p} \in K_N$, notice that

$$\begin{aligned}
\sup_{\mathbf{p} \in K_N} |N[U_{N,[Nt]} - I]f(\mathbf{p})| &= \sup_{\mathbf{p} \in K_N} |T_N[N(S_{N,[Nt]} - I)]f(\mathbf{p}) + N(T_N - I)f(\mathbf{p})| \\
&\leq \sup_{\mathbf{p} \in K_N} |N(S_{N,[Nt]} - I)f(\mathbf{p})| + \sup_{\mathbf{p} \in K_N} |N(T_N - I)f(\mathbf{p})|.
\end{aligned} \tag{2.4.79}$$

(2.4.78) follows by (2.4.61) and (2.2.38). $\square$

As in subsection 2.4.2, we make some preparations before we prove the uniqueness of the martingale problem for (G_{AB}, μ) where G_{AB} is defined by (2.4.74) and $\mu \in \mathcal{P}(K)$. For each $n \geq 1$, using $\mathbb{N}_n$ as the index set, we define a $c_n \times c_n$ matrix $B^{(n)} = (b_{\alpha, \alpha'}^{(n)})_{\alpha, \alpha' \in \mathbb{N}_n}$ as follows: for each $\alpha, \alpha' \in \mathbb{N}_n$,

$$b_{\alpha, \alpha'}^{(n)} = \begin{cases} \frac{1}{2}(n - n^2), & \text{if } \alpha = \alpha'; \\ 0, & \text{otherwise.} \end{cases} \quad (2.4.80)$$

For each $n \geq 2$, using $\mathbb{N}_n$ and $\mathbb{N}_{n-1}$ as the index sets, we define a $c_n \times c_{n-1}$ matrix $D^{(n)} = (d_{\alpha, \alpha'}^{(n)})_{\alpha \in \mathbb{N}_n, \alpha' \in \mathbb{N}_{n-1}}$ as follows: for each $\alpha \in \mathbb{N}_n, \alpha' \in \mathbb{N}_{n-1}$,

$$d_{\alpha, \alpha'}^{(n)} = \begin{cases} \frac{1}{2}\alpha_i(\alpha_i - 1), & \text{if } \alpha'_i = \alpha_i - 1 \text{ and } \alpha'_l = \alpha_l \text{ for } l \neq i, 1 \leq l \leq r; \\ 0, & \text{otherwise.} \end{cases} \quad (2.4.81)$$

We also arrange the elements of $B^{(n)}, D^{(n)}$ along the rows and columns decreasingly by the order $' >'$.

Lemma 2.4.15. *Define the time-dependent generator $\{G_{AB}(s), 0 \leq s < \infty\}$ on $C(K)$ by (2.4.74). Let $\mu \in \mathcal{P}(K)$. Then the $D_K[0, \infty)$ martingale problem for (G_{AB}, μ) has at most one solution.*

Proof. We use the same idea as that for the proof of Lemma 2.4.11. Assume that $\{\mathbf{X}(t) = (X_1(t), \dots, X_r(t)) : 0 \leq t < \infty\}$ is one solution of the martingale problem for (G_{AB}, μ) . Let $n \geq 1$, for arbitrary $\alpha \in \mathbb{N}_n$, and $t \geq 0$, define $f_\alpha(x_1, \dots, x_r) = x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ and $y_n^\alpha(t) = E[f_\alpha(\mathbf{X}(t))]$. Define the column vector $\mathbf{y}_n(t) = (y_n^\alpha(t))_{\alpha \in \mathbb{N}_n}$, where the elements of $\mathbf{y}_n(t)$ is arranged decreasingly by the order $' >'$.

Then, for $n \geq 1$ and given $\alpha \in \mathbb{N}_n$ and $t > 0$, we have

$$E[f_\alpha(\mathbf{X}(t))] = E[f_\alpha(\mathbf{X}(0))] + \int_0^t E[G_{AB}(s)f_\alpha(\mathbf{X}(s))]ds. \quad (2.4.82)$$

At first, for $n = 1$, since $G_B f_\alpha \equiv 0$ for any $\alpha \in \mathbb{N}_1$, we still have (2.4.67), which implies

$$\mathbf{y}_1(t) = \mathbf{y}_1(0) + \int_0^t A^{(1)}(s) \mathbf{y}_1(s) ds. \quad (2.4.83)$$

Next, we calculate $E[f_\alpha(\mathbf{X}(t))]$ for $n \geq 2$. By (2.2.37), (2.4.80) and (2.4.81), we can get

$$\begin{aligned} & E[G_B f_\alpha(\mathbf{X}(s))] \\ &= \frac{1}{2} E\left[\sum_{i=1}^r h_{ii}(\mathbf{X}(s)) \frac{\partial^2 f_\alpha(\mathbf{X}(s))}{\partial x_i^2}\right] + \frac{1}{2} E\left[\sum_{i=1}^r \sum_{j=1, j \neq i}^r h_{ij}(\mathbf{X}(s)) \frac{\partial^2 f_\alpha(\mathbf{X}(s))}{\partial x_i \partial x_j}\right] \\ &= \frac{1}{2} E\left[\sum_{i=1}^r b_{ii}(\mathbf{X}(s)) \frac{\partial^2 f_\alpha(\mathbf{X}(s))}{\partial x_i^2}\right] + \frac{1}{2} E\left[\sum_{i=1}^r \sum_{j=1, j \neq i}^r b_{ij}(\mathbf{X}(s)) \frac{\partial^2 f_\alpha(\mathbf{X}(s))}{\partial x_i \partial x_j}\right] \\ &= \frac{1}{2} \sum_{i=1}^r E[X_i(s)(1 - X_i(s))\alpha_i(\alpha_i - 1)X_1(s)^{\alpha_1-1} \cdots X_i(s)^{\alpha_i-2} \cdots X_r(s)^{\alpha_r}] \\ &\quad - \frac{1}{2} \sum_{i=1}^r \sum_{j=1, j \neq i}^r E[\alpha_i \alpha_j X_1(s)^{\alpha_1} \cdots X_r(s)^{\alpha_r}] \\ &= \frac{1}{2} \sum_{i=1}^r \alpha_i(\alpha_i - 1) E[X_1(s)^{\alpha_1} \cdots X_i(s)^{\alpha_i-1} \cdots X_r(s)^{\alpha_r}] \\ &\quad - \frac{1}{2} \left[\sum_{i=1}^r \alpha_i(\alpha_i - 1) + \sum_{i=1}^r \sum_{j=1, j \neq i}^r \alpha_i \alpha_j \right] E[X_1(s)^{\alpha_1} \cdots X_r(s)^{\alpha_r}] \\ &= D_{\alpha, \cdot}^{(n)} \mathbf{y}_{n-1}(s) - \frac{1}{2} \left[\sum_{i=1}^r \sum_{j=1}^r \alpha_i \alpha_j - \sum_{i=1}^r \alpha_i \right] E[X_1(s)^{\alpha_1} \cdots X_r(s)^{\alpha_r}] \\ &= D_{\alpha, \cdot}^{(n)} \mathbf{y}_{n-1}(s) - \frac{1}{2} (n^2 - n) E[X_1(s)^{\alpha_1} \cdots X_r(s)^{\alpha_r}] \\ &= D_{\alpha, \cdot}^{(n)} \mathbf{y}_{n-1}(s) + B_{\alpha, \cdot}^{(n)} \mathbf{y}_n(s), \end{aligned} \quad (2.4.84)$$

where $D_{\alpha, \cdot}^{(n)}$ is the α -row of the matrix $D^{(n)}$, $B_{\alpha, \cdot}^{(n)}$ the α -row of the matrix $B^{(n)}$. Since by (2.4.82) and (2.4.68) we obtain

$$\begin{aligned} y_n^\alpha(t) &= y_n^\alpha(0) + \int_0^t E[(G_A(s) + G_B) f_\alpha(\mathbf{X}(s))] ds \\ &= y_n^\alpha(0) + \int_0^t A_{\alpha, \cdot}^{(n)}(s) \mathbf{y}_n(s) ds + \int_0^t E[G_B f_\alpha(\mathbf{X}(s))] ds, \end{aligned} \quad (2.4.85)$$

it follows by (2.4.84) that for $n \geq 2$,

$$\mathbf{y}_n(t) = \mathbf{y}_n(0) + \int_0^t D^{(n)} \mathbf{y}_{n-1}(s) ds + \int_0^t [A^{(n)}(s) + B^{(n)}] \mathbf{y}_n(s) ds. \quad (2.4.86)$$

Define for $n = 1$,

$$V_1(s, t) = \begin{cases} A^{(1)}(t) & \text{if } 0 \leq t \leq s < \infty, \\ 0, & \text{otherwise;} \end{cases} \quad (2.4.87)$$

and for $n \geq 2$,

$$V_n(s, t) = \begin{cases} A^{(n)}(t) + B^{(n)} & \text{if } 0 \leq t \leq s < \infty, \\ 0, & \text{otherwise.} \end{cases} \quad (2.4.88)$$

Let $f_1(t) = \mathbf{y}_1(0)$, $0 \leq t < \infty$. Then, by (2.4.83) and Proposition 2.4.10, we know that $\mathbf{x}_1 = \mathbf{y}_1$ is the unique solution of the *Volterra equation* of the second kind

$$\mathbf{x}(s) = f_1(s) + \int_0^s V_1(s, t) \mathbf{x}(t) dt, \quad (0 \leq s \leq T) \quad (2.4.89)$$

on the space $\mathbb{L}^2([0, T], R^{c_1})$ for any $T > 0$. We can do this procedure recursively. Assume that we know that for $n \geq 1$ the unique solution $\mathbf{x}_n = \mathbf{y}_n$ is determined, then we define $f_{n+1}(t) = \mathbf{y}_{n+1}(0) + \int_0^t D^{(n+1)} \mathbf{x}_n(s) ds$. Then by (2.4.86) and Proposition 2.4.10, we know that $\mathbf{x}_{n+1} = \mathbf{y}_{n+1}$ is the unique solution of the *Volterra equation* of the second kind

$$\mathbf{x}(s) = f_{n+1}(s) + \int_0^s V_{n+1}(s, t) \mathbf{x}(t) dt, \quad (0 \leq s \leq T) \quad (2.4.90)$$

on the space $\mathbb{L}^2([0, T], R^{c_{n+1}})$ for any $T > 0$. Note that the construction of the *Volterra equations* does depend just on $\mathbf{y}_n(0)$ for $n \geq 1$, and does not depend on $\mathbf{y}_n(t)$ for $t > 0$, $n \geq 1$. Then we conclude that all moments of the one-dimensional marginal distribution of any two solutions of the martingale problem for (G_{AB}, μ) are the same. The uniqueness of the martingale problem for (G_{AB}, μ) then follows by Theorem 4.2 of Chapter 4 [EK86]. $\square$

Theorem 2.4.16. *With the same conditions as those in Lemma 2.4.13 and $\mu \in \mathcal{P}(K)$, if $P(\mathbf{X}^N(0))^{-1} \Rightarrow \mu$, there exists a solution $\mathbf{X}^\infty$ of the $D_K[0, \infty)$ martingale problem for (G_{AB}, μ) , and $\mathbf{X}^N \Rightarrow \mathbf{X}^\infty$.*

Proof. The conclusion follows by Lemma 2.4.13, 2.4.14, 2.4.15 and Corollary 2.4.3. $\square$

Note that in section 2.3, we didn't discuss the evolutionary model with multinomial sampling. But the corresponding result can follow by Theorem 2.4.16.

2.5 Evolutionary multiagent models in random environments

2.5.1 Measurability with respect to random environment

In Section 2.3 and 2.4, we defined the general multiagent models based on the $D_{R^r \times r}[0, \infty)$ -valued sequence $\{A_N\}$ and A satisfying the conditions specified at the beginning of subsection 2.3.1. We defined $\mathbf{X}^N$ by A_N for both the general multiagent model which is in absence of multinomial sampling (GMAM) and the general multiagent model with multinomial sampling (GMAMWMS). We conclude that if $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$, then $\mathbf{X}^N \Rightarrow \mathbf{X}_A^\infty$, where $\mathbf{X}_A^\infty$ is the unique solution of the martingale problem for the generator $\{G_A(s), 0 \leq s < \infty\}$ of GMAM, and $\mathbf{X}^N \Rightarrow \mathbf{X}_{AB}^\infty$, where $\mathbf{X}_{AB}^\infty$ is the unique solution for the martingale problem for the generator $\{G_{AB}(s), 0 \leq s < \infty\}$ of GMAMWMS and B denotes the differential operator on $C^2(K)$ defined by (2.2.37).

In this section, we will consider only the evolutionary multiagent models in random environment (EMAMSIRE). This because $\lim_{N \rightarrow \infty} d_U(A_N, A) = 0$ is too strong a condition on the space $D_{R^r \times r}[0, \infty)$. Instead, we assume that A is $C_{R^r \times r}[0, \infty)$ -valued, and $\lim_{N \rightarrow \infty} d(A_N, A) = 0$. We note that d_U and d are equivalent metrics on $C_{R^r \times r}[0, \infty)$.

Let $(\mathcal{L}, d)$ be the subspace of $(D_{R^r \times r}[0, \infty), d)$ such that each element of $\mathcal{L}$ satisfies the conditions at the beginning of subsection 2.3.1. Let $(\mathcal{L}_c, d_U)$ be the subspace of $(C_{R^r \times r}[0, \infty), d_U)$ such that each element $A \in \mathcal{L}$ satisfies the conditions at the beginning of subsection 2.3.1 and $A \in C_{R^r \times r}[0, \infty)$. Let $(\mathcal{P}(D_K[0, \infty)), \rho)$ be the space of probability measures on $D_K[0, \infty)$ where ρ is the Prohorov metric on $\mathcal{P}(D_K[0, \infty))$. The above argument shows that for any $A \in \mathcal{L}_c$, there exist unique $P_A \in \mathcal{P}(D_K[0, \infty))$ such that under P_A , the coordinate process Z on $D_K[0, \infty)$ is the unique solution of the martingale problem for the generator $\{G_A(s), 0 \leq s < \infty\}$ of EMAM and $P_{AB} \in \mathcal{P}(D_K[0, \infty))$ such that under P_{AB} , Z is the unique solution of the martingale problem for the generator $\{G_{AB}(s), 0 \leq s < \infty\}$ of EMAMWMS. We define $\Phi : \mathcal{L}_c \mapsto \mathcal{P}(D_K[0, \infty))$ by $\Phi(A) = P_A$ and $\Psi : \mathcal{L}_c \mapsto \mathcal{P}(D_K[0, \infty))$ by $\Psi(A) = P_{AB}$. We summarize the above argument in following theorem and show that Φ and Ψ are continuous mappings from $(\mathcal{L}_c, d_U)$ to $(\mathcal{P}(D_K[0, \infty)), \rho)$.

Theorem 2.5.1. *For any $A \in \mathcal{L}_c$, we define the generators $\{G_A(s), 0 \leq s < \infty\}$ on $C(K)$ by (2.4.50), and $\{G_{AB}(s), 0 \leq s < \infty\}$ on $C(K)$ by (2.4.74), then there exist mappings Φ and Ψ from $\mathcal{L}_c$ to $\mathcal{P}(D_K[0, \infty))$ such that under $\Phi(A)$ and $\Psi(A)$, the coordinate process Z on $D_K[0, \infty)$ is the unique solution of the martingale problem for $\{G_A(s), 0 \leq s < \infty\}$ of EMAM and $\{G_{AB}(s), 0 \leq s < \infty\}$ of EMAMWMS respectively. In addition, Φ and Ψ are continuous mappings.*

Proof. We just need to show that Φ and Ψ are continuous mappings. This is immediate by (2.4.50), (2.4.74) and Corollary 2.4.2. $\square$

Now we consider the measurability related to A if A is a $C_{R^r \times r}[0, \infty)$ -valued process in $\mathcal{L}_c$ defined on some probability space.

Lemma 2.5.2. *Assume that A is an $\mathcal{L}_c$ -valued process defined on some probability space $(\Omega, \mathcal{F}, Q)$. Then $(A, \Phi(A))$, $(A, \Psi(A))$ are $\mathcal{F}/[\mathcal{B}(\mathcal{L}_c) \otimes \mathcal{B}(\mathcal{P}(D_K[0, \infty)))]$ -measurable.*

Proof. Since K is separable, so are $D_K[0, \infty)$ and $\mathcal{P}(D_K[0, \infty))$. Since $\mathcal{L}_c$ is also separable, to prove that $(A, \Phi(A))$ is measurable, it suffices to prove that if $C \in \mathcal{B}(\mathcal{L}_c)$, $D \in \mathcal{B}(\mathcal{P}(D_K[0, \infty)))$, $(A, \Phi(A))^{-1}(C \times D) \in \mathcal{F}$. This is clear by Theorem 2.5.1, and $(A, \Phi(A))^{-1}(C \times D) = A^{-1}(C \cap \Phi^{-1}(D))$. Similarly, we can prove that $(A, \Psi(A))$ is measurable. $\square$

Next, we consider the measurability related to A_N if A_N is a $D_{R^r \times r}[0, \infty)$ -valued process in $\mathcal{L}$ defined on some probability space. But $\mathcal{L}$ is too big a space to guarantee the measurability which we need. The reason is when we define $\mathbf{X}^N$ related to A_N for EMAM or EMAMWMS, we assume that A_N is a constant on $t \in [\frac{k}{N}, \frac{k+1}{N})$ for $k \geq 0$. Therefore we put one restriction on $\mathcal{L}$. For each $N \geq 1$, let $(\mathcal{L}_N, d)$ be the subspace of $(\mathcal{L}, d)$ such that each element in $\mathcal{L}_N$, is a stepwise function, i.e., for each $A_N \in \mathcal{L}_N$, $A_N(t)$ is a constant on $t \in [\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$. Based on what we discussed previously, for a given $A_N \in \mathcal{L}_N$, there are unique probability measure $P_{A_N} \in \mathcal{P}(D_{K_N}[0, \infty))$ which is related to EMAM, and unique probability measure $P_{A_N B} \in \mathcal{P}(D_{K_N}[0, \infty))$ which is related to EMAMWMS. Under P_{A_N} or $P_{A_N B}$, the coordinate process Z_N on $D_{K_N}[0, \infty)$ has the same distributions as those defined for $\mathbf{X}^N$ in EMAM or EMAMWMS. Then we can define $\Phi_N : \mathcal{L}_N \mapsto \mathcal{P}(D_{K_N}[0, \infty))$ and

$\Psi_N : \mathcal{L}_N \mapsto \mathcal{P}(D_{K_N}[0, \infty))$ correspondingly. Since $\{A_N(\frac{k}{N}), k \geq 0\}$ determines the transition structure of a Markov chain by $\{S_{N,k}, k \geq 0\}$ in EMAM or by $\{S_{N,k}, k \geq 0\}$ and T_N in EMAMWMS, it is clear that Φ_N and Ψ_N are continuous mappings from $\mathcal{L}_N$ to $\mathcal{P}(D_{K_N}[0, \infty))$. It follows the next lemma for measurability related to A_N where A_N is a $D_{R^r \times r}[0, \infty)$ -valued process in $\mathcal{L}_N$ on some probability space.

Lemma 2.5.3. *Assume that A_N is an $\mathcal{L}_N$ -valued process defined on some probability space $(\Omega_N, \mathcal{F}_N, Q_N)$. Then $(A_N, \Phi_N(A_N)), (A_N, \Psi(A_N))$ are $\mathcal{F}/[\mathcal{B}(\mathcal{L}_N) \otimes \mathcal{B}(\mathcal{P}(D_{K_N}[0, \infty)))]$ -measurable.*

Remark 2.5.4. *Using the new notations defined in this subsection, we can formulate the convergence theorems of EMAM and EMAMWMS, which are in Section 2.3, 2.4. Let $\hat{A}_N \in \mathcal{L}_N$ for each $N \geq 1$, and $\hat{A} \in \mathcal{L}_c$. Let $\mu_N \in \mathcal{P}(K_N)$ for each $N \geq 1$ and $\mu \in \mathcal{P}(K)$.*

1) *For EMAM, define $\Phi_N(\hat{A}_N) \in \mathcal{P}(D_{K_N}[0, \infty))$ for each $N \geq 1$ and $\Phi(\hat{A}) \in \mathcal{P}(D_K[0, \infty))$ such that $\Phi_N(\hat{A}_N)Z_N(0)^{-1} = \mu_N$ and $\Phi(\hat{A})Z(0)^{-1} = \mu$. If $\lim_{N \rightarrow \infty} d(\hat{A}_N, \hat{A}) = 0$ and $\mu_N \Rightarrow \mu$, then $\lim_{N \rightarrow \infty} \rho(\Phi_N(\hat{A}_N), \Phi(\hat{A})) = 0$.*

2) *For EMAMWMS, define $\Psi_N(\hat{A}_N) \in \mathcal{P}(D_{K_N}[0, \infty))$ for each $N \geq 1$ and $\Psi(\hat{A}) \in \mathcal{P}(D_K[0, \infty))$ such that $\Psi_N(\hat{A}_N)Z_N(0)^{-1} = \mu_N$ and $\Psi(\hat{A})Z(0)^{-1} = \mu$. If $\lim_{N \rightarrow \infty} d(\hat{A}_N, \hat{A}) = 0$ and $\mu_N \Rightarrow \mu$, then $\lim_{N \rightarrow \infty} \rho(\Psi_N(\hat{A}_N), \Psi(\hat{A})) = 0$.*

2.5.2 Weak convergence of evolutionary multiagent models in random environment

We assume that for each $N \geq 1$, A_N is an $\mathcal{L}_N$ -valued process defined on some probability space $(\Omega_N, \mathcal{F}_N, Q_N)$, and A is an $\mathcal{L}_c$ -valued process defined on $(\Omega, \mathcal{F}, Q)$. Then

we can define evolutionary multiagent model in random environment (EMAMIRE) and evolutionary multiagent model with multinomial sampling in random environment (EMAMWMSIRE). There come the natural questions: do $\Phi_N(A_N) \Rightarrow \Phi(A)$, $(A_N, \Phi_N(A_N)) \Rightarrow (A, \Phi(A))$, $\Psi_N(A_N) \Rightarrow \Psi(A)$ and $(A_N, \Psi_N(A_N)) \Rightarrow (A, \Psi(A))$ if $A_N \Rightarrow A$ as $N \rightarrow \infty$? That is to say, when $A_N \Rightarrow A$, we wonder whether EMAMIRE and EMAMWMSIRE converge jointly and marginally when the random environments converge. We can even consider the annealed weak convergence of $\Phi_N(A_N)$ and $\Psi_N(A_N)$.

We prove two lemmas which are used in the proof of convergence theorems.

Lemma 2.5.5. *If $\hat{A} = (\hat{a}_{i,j})_{r \times r}$ is a $D_{R^{r \times r}}[0, \infty)$ -valued process satisfying the conditions specified at the beginning of subsection 2.3.1 a.s., defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$, and $\tilde{A} = (\tilde{a}_{i,j})_{r \times r}$ is another $D_{R^{r \times r}}[0, \infty)$ -valued process defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ satisfying $\hat{P}\hat{A}^{-1} = \tilde{P}\tilde{A}^{-1}$ on $\mathcal{B}(D_{R^{r \times r}}[0, \infty))$, then $\tilde{A}$ also satisfies the conditions at the beginning of subsection 2.3.1.*

Proof. Define $\pi_t : D_{R^{r \times r}}[0, \infty) \mapsto R^{r \times r}$ by $\pi_t(x) = x(t)$ for each $t \geq 0$ and $g : R^{r \times r} \mapsto R^r$ by $g(y) = y\mathbf{e}$. Both π_t and g are measurable mappings. Then for each $t \geq 0$, $\pi_t^{-1}g^{-1}\{\mathbf{0}_{r \times 1}\} \in \mathcal{B}(D_{R^{r \times r}}[0, \infty))$,

$$\begin{aligned} \hat{P}\{\hat{\omega} : \hat{A}(t, \hat{\omega})\mathbf{e} = \mathbf{0}_{r \times 1}\} &= \hat{P}\hat{A}^{-1}\{x \in D_{R^{r \times r}}[0, \infty) : x(t)\mathbf{e} = \mathbf{0}_{r \times 1}\} \\ &= \hat{P}\hat{A}^{-1}\pi_t^{-1}\{y \in R^{r \times r} : y\mathbf{e} = \mathbf{0}_{r \times 1}\} \\ &= \hat{P}\hat{A}^{-1}\pi_t^{-1}g^{-1}\{\mathbf{0}_{r \times 1}\}. \end{aligned} \tag{2.5.1}$$

By the same reasoning, we have $\tilde{P}\{\tilde{\omega} : \tilde{A}(t, \tilde{\omega})\mathbf{e} = \mathbf{0}_{r \times 1}\} = \tilde{P}\tilde{A}^{-1}\pi_t^{-1}g^{-1}\{\mathbf{0}_{r \times 1}\}$. Therefore, $\tilde{P}\{\tilde{\omega} : \tilde{A}(t, \tilde{\omega})\mathbf{e} = \mathbf{0}_{r \times 1}\} = \hat{P}\{\hat{\omega} : \hat{A}(t, \hat{\omega})\mathbf{e} = \mathbf{0}_{r \times 1}\}$.

Similarly, for each $i \neq j$ and $t \geq 0$, define $g_{i,j} : R^{r \times r} \mapsto R$ by $g_{i,j}(y) =$

$y_{i,j}$, then $\pi_t^{-1}g_{i,j}^{-1}\{[0, \infty)\} \in \mathcal{B}(D_{R^r \times r}[0, \infty))$. We get $\tilde{P}\{\tilde{\omega} : \tilde{a}_{i,j}(t, \tilde{\omega}) \geq 0\} = \tilde{P}\pi_t^{-1}g_{i,j}^{-1}\{[0, \infty)\} = \hat{P}\pi_t^{-1}g_{i,j}^{-1}\{[0, \infty)\} = \hat{P}\{\hat{\omega} : \hat{a}_{i,j}(t, \hat{\omega}) \geq 0\}$. $\square$

Lemma 2.5.5 indicates that the conditions specified at the beginning of subsection 2.3.1 for $\{A_N\}$ and A just depend on distributions of $\{A_N\}$ and A . Furthermore, we have Lemma 2.5.6.

Lemma 2.5.6. *Let $\{A_N\}$ be defined above. For each $N \geq 1$, if $\hat{A}_N$ is another $D_{R^r \times r}[0, \infty)$ -valued process defined on $(\hat{\Omega}_N, \hat{\mathcal{F}}_N, \hat{Q}_N)$, satisfying $Q_N A_N^{-1} = \hat{Q}_N \hat{A}_N^{-1}$, then $\hat{A}_N$ is also an $\mathcal{L}_N$ -valued process.*

Proof. We just need to prove that $\hat{A}_N(t)$ is constant on $[\frac{k}{N}, \frac{k+1}{N})$ for any $k \geq 0$. Fix $k \geq 0$, for any rational $t \in (\frac{k}{N}, \frac{k+1}{N})$, define $h_{k,t} : D_{R^r \times r}[0, \infty) \mapsto R^{r \times r}$ by $h_{k,t}(x) = x(t) - x(\frac{k}{N})$, $x \in D_{R^r \times r}[0, \infty)$. $h_{k,t}$ is a measurable function. The conclusion follows by the following equality

$$\hat{Q}_N\{\hat{A}_N(t) = \hat{A}_N(\frac{k}{N})\} = \hat{Q}_N\hat{A}_N^{-1}h_{k,t}^{-1}\{0\} = Q_N A_N^{-1}h_{k,t}^{-1}\{0\} = Q_N\{A_N(t) = A_N(\frac{k}{N})\} \quad (2.5.2)$$

and the right continuity of $\hat{A}_N$. $\square$

Theorem 2.5.7. (Joint and marginal convergence) *Let $\{A_N\}$ and A be defined above, $\mu_N \in \mathcal{P}(K_N)$ for each $N \geq 1$ and $\mu \in \mathcal{P}(K)$. 1) For EMAMIRE, define $\Phi_N(A_N(\omega_N)) \in \mathcal{P}(D_{K_N}[0, \infty))$ for each $\omega_N \in \Omega_N$ and $N \geq 1$, and $\Phi(A(\omega)) \in \mathcal{P}(D_K[0, \infty))$ for each $\omega \in \Omega$ such that $\Phi_N(A_N(\omega_N))(Z_N(0))^{-1} = \mu_N$ for each $\omega_N \in \Omega_N$ and $\Phi(A(\omega))(Z(0))^{-1} = \mu$ for each $\omega \in \Omega$. If $A_N \Rightarrow A$, and $\mu_N \Rightarrow \mu$, then $\Phi_N(A_N) \Rightarrow \Phi(A)$ and $(A_N, \Phi_N(A_N)) \Rightarrow (A, \Phi(A))$. 2) For EMAMWMSIRE, define $\Psi_N(A_N(\omega_N)) \in \mathcal{P}(D_{K_N}[0, \infty))$ for each $\omega_N \in \Omega_N$ and $N \geq 1$, and $\Psi(A(\omega)) \in \mathcal{P}(D_K[0, \infty))$ for each $\omega \in \Omega$, such that $\Psi_N(A_N(\omega_N))(Z_N(0))^{-1} = \mu_N$ for each*

$\omega_N \in \Omega_N$ and $\Psi(A(\omega))(Z(0))^{-1} = \mu$ for each $\omega \in \Omega$. If $A_N \Rightarrow A$, and $\mu_N \Rightarrow \mu$, then $\Psi_N(A_N) \Rightarrow \Psi(A)$ and $(A_N, \Psi_N(A_N)) \Rightarrow (A, \Psi(A))$.

Proof. We just prove the EMAMIRE case. The EMAMWMSIRE can be proved similarly. Since $\{A_N\}$ are $D_{R^r \times r}[0, \infty)$ -valued processes, A is a $C_{R^r \times r}[0, \infty)$ -valued process, and $A_N \Rightarrow A$, by Skorohod Representation theorem, there exist some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{Q})$ and a sequence of $D_{R^r \times r}[0, \infty)$ -valued processes $\{\hat{A}_N\}$ and a $C_{R^r \times r}[0, \infty)$ -valued process $\hat{A}$ satisfying $\lim_{N \rightarrow \infty} d(\hat{A}_N, \hat{A}) = 0$ $\hat{Q}$ -a.s., $\hat{Q}\hat{A}^{-1} = QA^{-1}$, and $\hat{Q}\hat{A}_N^{-1} = Q_N A_N^{-1}$ on $\mathcal{B}(D_{R^r \times r}[0, \infty))$ for each $N \geq 1$. By Lemma 2.5.5 and 2.5.6, $\hat{A}$ is an $\mathcal{L}_c$ -valued process, and $\hat{A}_N$ is an $\mathcal{L}_N$ -valued process for each $N \geq 1$. By Lemma 2.5.2 and 2.5.3, $(\hat{A}, \Phi(\hat{A}))$ and $(\hat{A}_N, \Phi_N(\hat{A}_N))$ are measurable. By Remark 2.5.4, we have

$$\lim_{N \rightarrow \infty} \rho(\Phi_N(\hat{A}_N(\hat{\omega})), \Phi(\hat{A}(\hat{\omega}))) = 0, \quad \hat{Q}\text{-a.s.} \quad (2.5.3)$$

$$\begin{aligned} \text{For each } f \in \quad & \lim_{N \rightarrow \infty} \int f(A_N(\omega_N))g(\Phi_N(A_N(\omega_N)))Q(d\omega_N) \quad \text{and convergence theorem} \\ & = \lim_{N \rightarrow \infty} \int f(\hat{A}_N(\hat{\omega}))g(\Phi_N(\hat{A}_N(\hat{\omega})))\hat{Q}(d\hat{\omega}) \quad (\omega_N) \\ & = \int f(\hat{A}(\hat{\omega}))g(\Phi(\hat{A}(\hat{\omega})))\hat{Q}(d\hat{\omega}) \\ & = \int f(\hat{A}(\hat{\omega}))g(\Phi(\hat{A}(\hat{\omega})))\hat{Q}(d\hat{\omega}) \\ & = \int f(A(\omega))g(\Phi(A(\omega)))Q(d\omega). \end{aligned} \quad (2.5.4)$$

Since $\mathcal{L}_c$ and $\mathcal{P}(D_K[0, \infty))$ are separable, $\bar{C}(\mathcal{L}_c)$ and $\bar{C}(\mathcal{P}(D_K[0, \infty)))$ are convergence determining on $(\mathcal{L}_c, d_U)$ and $(\mathcal{P}(D_K[0, \infty)), \rho)$ respectively, we actually proved that $(A_N, \Phi_N(A_N)) \Rightarrow (A, \Phi(A))$. $\Phi_N(A_N) \Rightarrow \Phi(A)$ has also been proved by taking $f \equiv 1$ in (2.5.4). $\square$

Furthermore, the corresponding annealed convergence theorem can be expected for the evolutionary multiagent models evolving in random environment.

Theorem 2.5.8. (*Annealed convergence*) *With the same conditions and definitions as those in Theorem 2.5.7, it follows that*

1) *for EMAMIRE, $\lim_{N \rightarrow \infty} \rho(\int \Phi(A_N(\omega_N))Q_N(d\omega_N), \int \Phi(A(\omega))Q(d\omega)) = 0$;*

2) *for EMAMWMSIRE, $\lim_{N \rightarrow \infty} \rho(\int \Psi(A_N(\omega_N))Q_N(d\omega_N), \int \Psi(A(\omega))Q(d\omega)) = 0$.*

Proof. By (2.5.3), for any open set $G \subset D_K[0, \infty)$, we have

$$\liminf_{N \rightarrow \infty} \Phi_N(\hat{A}_N(\hat{\omega}))(G) \geq \Phi(\hat{A}(\hat{\omega}))(G), \quad \hat{Q}\text{-a.s.} \quad (2.5.5)$$

Then by Fatou Lemma, we get

$$\liminf_{N \rightarrow \infty} \int \Phi_N(\hat{A}_N(\hat{\omega}))(G) \hat{Q}(d\hat{\omega}) \geq \int \Phi(\hat{A}(\hat{\omega}))(G) \hat{Q}(d\hat{\omega}), \quad (2.5.6)$$

which implies that

$$\liminf_{N \rightarrow \infty} \int \Phi_N(A_N(\omega_N))(G) Q_N(d\omega_N) \geq \int \Phi(A(\omega))(G) Q(d\omega). \quad (2.5.7)$$

Since $D_K[0, \infty)$ is separable, 1) is proved. 2) can be proved similarly. $\square$

Chapter 3

Interacting Agent Feedback Finance Models

In Chapter 2, we discussed multiagent models (MAMS) whose mathematical framework has close relation to the Wright-Fisher model in population genetics. Although the types of agents in (MAMS) are classified by internal states of an internal system, not by the intrinsic character of each agent itself as the genotypes are, we modeled the change of types among different types of agents in the multiagent models by the same way as the change of genotype in genetics, i.e. gene mutation or multinomial sampling. In this chapter, we will concentrate on the interacting agent feedback finance models (IAFFM). In this special kind of interacting system, we will not introduce the internal system serving as the standard for classifying the types of agents, that is to say, the types of agents are classified by the individual intrinsic character of each agent. We assume that there is a financial asset in a certain market whose price is observable by all agents and there exists a recursive log price mechanism, the

current log price $\log S_k$ can be computed, given the log price $\log S_{k-1}$ and the current empirical distribution of types $\frac{\mathbf{n}^N(k)}{N}$. This market price mechanism is known to all agents. Now the change of types of agents will be based on the current empirical distribution of the types of agents as well as the current market price of the financial asset. *Feedback* in these models simply means that the current market price has an impact on the change of the types of agents. This feature distinguishes IAFFM from many other interacting agent models.

In Section 3.1, we describe the interacting agent feedback finance models and give a simplified example of it. Section 3.2 is devoted to the discussion of the most general interacting agent feedback finance models. General weak convergence criterion for time-dependent generators in subsection 2.4.1 are used in proving the weak convergence of the IAFFM sequence. The theory of uniqueness of *nonlinear Volterra equations of second kind* is updated to the case with semi-Lipschitz condition and is used on IAFFM. In Section 3.3, we concentrate on the case study of the simplified example given in Section 3.1. The conditions given in Section 3.2 for general interacting agent feedback finance models are verified. We will also discuss the conditions for fixed points of this example. In addition, we consider the situation of this model of IAFFM evolving in random environment. We can then ask some questions of this model similar to those of the theory of the classical finance model developed by Black-Scholes-Merton have answered.

3.1 Description of IAFFM and a simplified example

3.1.1 A brief description of IAFFM

To construct the interacting agent feedback finance models (IAFFM), we make the following assumptions on the strategies of the agents and log price mechanism of the financial market.

1. The time is in nonnegative integer units, denoted by $k \geq 0$.
2. There is only one financial asset—stock in the market, whose price S_k varies with time k .
3. There are a fixed number of N agents in the market at all times. There are no entries of new agents into the market or exits of current agents from the market.
4. There are $r \geq 2$ types of agents which are classified by the individual intrinsic character of agents. $\mathbf{n}^N(k) = (n_1^N(k), \dots, n_r^N(k))$ is the distribution of N agents among r types at each time $k \geq 0$ with $n_1^N(k) + \dots + n_r^N(k) = N$ by the third assumption. $\frac{\mathbf{n}^N(k)}{N}$ is the empirical distribution of the types of agents at $k \geq 0$.
5. There exists a log price mechanism for the financial asset, such that at time $k \geq 1$, the current log price $\log S_k$ is determined by the previous log price $\log S_{k-1}$ and the current empirical distribution of the types of agents $\frac{\mathbf{n}^N(k)}{N}$.

That is to say, there exists a function g_N defined on $[0, \infty) \times K_N \times R$, such that $\log S_k = \log S_{k-1} + \frac{1}{N} g_N(\frac{k}{N}, \frac{\mathbf{n}^N(k)}{N}, \log S_{k-1})$.

6. Each type of agent has its common strategies or behaviors. The strategies or behaviors are realized by keeping their type or changing to a different type, for the next time period. Therefore the agents of the same type at time $k \geq 0$ have the same probability of being same type for the next time period $k + 1$.
7. Each type of agents can observe the empirical distribution of the types of agents $\frac{\mathbf{n}^N(k)}{N}$ as well as the asset price S_k and determine the probabilities of switching to another type based on $\frac{\mathbf{n}^N(k)}{N}$ and $\log S_k$.
8. Each agent switches independently to his/her type of the next time period, based on his/her current type.

Note that in the seventh assumption, the empirical distribution of the types of agents $\frac{\mathbf{n}^N(k)}{N}$ plays a role in determining the transition probabilities to the next type from the current type, which is the first source of interaction. Note also that the current log price $\log S_k$ affects the transition probabilities, which is the second source of interaction. It means that the current price of the financial asset has feedback on the agents' behavior of changing types, although the current log price is determined partially by the current empirical distribution of the types of agents $\frac{\mathbf{n}^N(k)}{N}$ by the fifth assumption.

Note also in IAFFM that we do not introduce the internal system which serves as the standard for classifying the types of agents as we do in the multiagent models. The types of agents are classified by the individual intrinsic character of agents by the fourth assumption.

For the simplified example in subsection 3.1.2, different types of agents have different excess demand functions which can lead to a log price mechanism by the market clearing condition of zero excess demand, assuming that the empirical distribution of the current types of agents and the market price of the last period are known. Then we will see that g_N has an exact form for this example.

The mathematical setup for the general interacting agent feedback finance models given in 3.2.1 will demonstrate how the assumptions made above can be employed.

3.1.2 A simplified example of IAFFM

Now we illustrate the assumptions of IAFFM made in subsection 3.1.1 by an example given as follows.

We assume that there are N agents of three types in the financial market. The three types are fundamentalist, optimist, pessimist [LU98]. Agents of first type are also called information traders. Agents of the second and third type are called noise traders, or chartists. At each time $k \geq 0$, the number of fundamentalists, optimists and pessimists are $n_1^N(k)$, $n_2^N(k)$ and $n_3^N(k)$ respectively. Since the interacting agent system is closed, i.e., there are no new entrants into the market or quits of current traders from the market. Therefore $n_1^N(k) + n_2^N(k) + n_3^N(k) = N$ for all $k \geq 0$. We denote by $\mathbf{n}^N(k) = (n_1^N(k), n_2^N(k), n_3^N(k))$ the vector of types and $\frac{\mathbf{n}^N(k)}{N}$ the vector of empirical distribution of types.

We assume that each type of agents have their own excess demand functions. We introduce the excess demand functions of the form suggested by Föllmer and Schweizer [FS93]. Assume that there exists a probability space $(\Omega, \mathcal{F}, P)$. At time

unit $k \geq 1$, for each $\omega \in \Omega$, and a proposed price p , each agent a has an excess demand function $e_a^N(k, p, \omega)$ which is given by

$$e_a^N(k, p, \omega) = \alpha_a^N\left(\frac{k}{N}, \omega\right) \log(\hat{S}_{a,k}^N(\omega)/p) + \frac{\delta_a^N\left(\frac{k}{N}, \omega\right)}{N}. \quad (3.1.1)$$

Here $\delta_a^N(\frac{k}{N}, \omega)$ can be viewed as the total liquidity demand observed by agent a at time $k \geq 0$ and $\frac{\delta_a^N(\frac{k}{N}, \omega)}{N}$ is the average liquidity demand, and $\hat{S}_{a,k}^N(\omega)$ denotes an individual reference level of agent a at time k . The time scale for α_a^N and δ_a^N is of $\frac{1}{N}$, instead of 1. Note that α_a^N and δ_a^N depend on the market size of agents N .

We assume that agents of different types have different excess demand function and agents of the same type have the same function. Accordingly, we assume that in (3.1.1), for fundamentalist, optimist, pessimist, the individual reference levels $\hat{S}_{a,k}^N(\omega)$, the coefficients $\alpha_a^N(\frac{k}{N}, \omega)$ and $\delta_a^N(\frac{k}{N}, \omega)$ are $\hat{S}_1^N(k, \omega)$, $\alpha_1^N(\frac{k}{N}, \omega)$ and $\delta_1^N(\frac{k}{N}, \omega)$; $\hat{S}_2^N(k, \omega)$, $\alpha_2^N(\frac{k}{N}, \omega)$ and $\delta_2^N(\frac{k}{N}, \omega)$; $\hat{S}_3^N(k, \omega)$, $\alpha_3^N(\frac{k}{N}, \omega)$ and $\delta_3^N(\frac{k}{N}, \omega)$ for type 1, 2, 3 respectively. We also assume the following three individual reference levels for these three types. For fundamentalists, $\hat{S}_1^N(k, \omega)$ has the form

$$\log \hat{S}_1^N(k, \omega) = \log S_{k-1} + \frac{\beta_1^N(\frac{k}{N}, \omega)}{N} (\log S_{k-1} - \log F^N(\frac{k}{N}, \omega)) \quad (3.1.2)$$

with random coefficients $\beta_1^N(\frac{k}{N}, \omega) \leq 0$, where $F^N(\frac{k}{N}, \omega)$ is the fundamental value of the asset at time k . For optimists, $\hat{S}_2^N(k, \omega)$ has the form

$$\log \hat{S}_2^N(k, \omega) = \log S_{k-1} + \beta_2^N\left(\frac{k}{N}, \omega\right) (\log S_{k-1} - \log p) \quad (3.1.3)$$

with random coefficients $\beta_2^N(\frac{k}{N}, \omega) \leq 0$. For pessimists, $\hat{S}_3^N(k, \omega)$ has the form

$$\log \hat{S}_3^N(k, \omega) = \log S_{k-1} + \beta_3^N\left(\frac{k}{N}, \omega\right) (\log S_{k-1} - \log p) \quad (3.1.4)$$

with random coefficients $\beta_3^N(\frac{k}{N}, \omega) \leq 0$. Note that only $\beta_1^N(\frac{k}{N}, \omega)$ in fundamentalists' reference level is divided by N , the market size of the agents. One possible

explanation for it is that fundamentalists know the evolution of fundamental value $F^N(\frac{k}{N}, \omega)$ of the asset, they value less in their excess demand functions the importance of the difference between log price $\log S_{k-1}$ of the last time period $k-1$ and the log fundamental value $\log F^N(\frac{k}{N}, \omega)$ at current time k . We would like to take it as an assumption. Note also that the time scale for $\beta_1^N, \beta_2^N, \beta_3^N$ and $\log F^N$ is also of $\frac{1}{N}$ and that $\beta_1^N, \beta_2^N, \beta_3^N$ and $\log F^N$ depend on the market size of agents N .

For time period $k \geq 1$, if we assume the price S_{k-1} and $\mathbf{n}^N(k)$ are known, then the equilibrium log price $\log S_k$ in period k is determined by the market clearing condition of zero excess demand:

$$\sum_a e_a^N(k, S_k(\omega), \omega) = 0, \quad (3.1.5)$$

i.e.

$$\begin{aligned} n_1^N(k) [\alpha_1^N(\frac{k}{N})(\log \hat{S}_1^N(k) - \log S_k) + \frac{\delta_1^N(\frac{k}{N})}{N}] &+ n_2^N(k) [\alpha_2^N(\frac{k}{N})(\log \hat{S}_2^N(k) \\ &- \log S_k) + \frac{\delta_2^N(\frac{k}{N})}{N}] + n_3^N(k) [\alpha_3^N(\frac{k}{N})(\log \hat{S}_3^N(k) - \log S_k) + \frac{\delta_3^N(\frac{k}{N})}{N}] = 0. \end{aligned} \quad (3.1.6)$$

We have omitted ω in the random variables in the above equation and in the rest of this subsection. Substituting (3.1.2), (3.1.3), (3.1.4) into (3.1.6), and solving for $\log p$ as $\log S_k$, we get

$$\begin{aligned} &\log S_k - \log S_{k-1} \\ &= \frac{1}{N} \cdot \frac{n_1^N(k) \alpha_1^N(\frac{k}{N}) \beta_1^N(\frac{k}{N}) (\log S_{k-1} - \log F^N(\frac{k}{N}))}{n_1^N(k) \alpha_1^N(\frac{k}{N}) + n_2^N(k) \alpha_2^N(\frac{k}{N}) (1 + \beta_2^N(\frac{k}{N})) + n_3^N(k) \alpha_3^N(\frac{k}{N}) (1 + \beta_3^N(\frac{k}{N}))} \\ &+ \frac{1}{N} \cdot \frac{n_1^N(k) \delta_1^N(\frac{k}{N}) + n_2^N(k) \delta_2^N(\frac{k}{N}) + n_3^N(k) \delta_3^N(\frac{k}{N})}{n_1^N(k) \alpha_1^N(\frac{k}{N}) + n_2^N(k) \alpha_2^N(\frac{k}{N}) (1 + \beta_2^N(\frac{k}{N})) + n_3^N(k) \alpha_3^N(\frac{k}{N}) (1 + \beta_3^N(\frac{k}{N}))}. \end{aligned} \quad (3.1.7)$$

(3.1.7) is a recursive log price formula.

Let $K_N^3 = \{N^{-1}\mathbf{x} : \mathbf{x} \in (Z_+)^3, \sum_{i=1}^3 x_i = N\}$. Define g_N on $[0, \infty) \times K_N^3 \times R$ as

follows:

$$\begin{aligned}
& g_N(t, \mathbf{x}, q) \\
&= \frac{x_1 \alpha_1^N(\frac{[nt]}{N}) \beta_1^N(\frac{[nt]}{N}) q}{x_1 \alpha_1^N(\frac{[nt]}{N}) + x_2 \alpha_2^N(\frac{[nt]}{N})(1 + \beta_2^N(\frac{[nt]}{N})) + x_3 \alpha_3^N(\frac{[nt]}{N})(1 + \beta_3^N(\frac{[nt]}{N}))} \\
&\quad + \frac{x_1 \delta_1^N(\frac{[nt]}{N}) + x_2 \delta_2^N(\frac{[nt]}{N}) + x_3 \delta_3^N(\frac{[nt]}{N}) - x_1 \alpha_1^N(\frac{[nt]}{N}) \beta_1^N(\frac{[nt]}{N}) \log F^N(\frac{[nt]}{N})}{x_1 \alpha_1^N(\frac{[nt]}{N}) + x_2 \alpha_2^N(\frac{[nt]}{N})(1 + \beta_2^N(\frac{[nt]}{N})) + x_3 \alpha_3^N(\frac{[nt]}{N})(1 + \beta_3^N(\frac{[nt]}{N}))}.
\end{aligned} \tag{3.1.8}$$

Then (3.1.7) can be represented by $g_N(t, \mathbf{x}, q)$ as

$$\log S_k = \log S_{k-1} + \frac{1}{N} g_N\left(\frac{k}{N}, \frac{\mathbf{n}^N(k)}{N}, \log S_{k-1}\right). \tag{3.1.9}$$

Note that $g_N(t, \mathbf{x}, q)$ defined by (3.1.8) is a random function, since $\{\alpha_i(\frac{k}{N})\}_{k=0}^\infty$, $\{\beta_i(\frac{k}{N})\}_{k=0}^\infty$ and $\{\delta_i^N(\frac{k}{N})\}_{k=0}^\infty$ for $1 \leq i \leq 3$ and $\{\log F^N(\frac{k}{N})\}_{k=0}^\infty$ are random sequences define on some probability space $(\Omega, \mathcal{F}, P)$. We have actually made preparations for the IAFFM evolving in a random environment.

Before we specify the transition structure for the time inhomogeneous Markov chain $\{(\mathbf{n}^N(k), \log S_k)\}_{k=0}^\infty$, we define some matrices to be used later. For fixed $N \geq 1$, we assume that $A_N(t, \mathbf{x}, q) = (a_{N,i,j}(t, \mathbf{x}, q))_{3 \times 3}$ is a 3×3 matrix-valued function on $[0, \infty) \times K_N^3 \times R$, which satisfies the conditions:

- 1) for each $(\mathbf{x}, q) \in K_N^3 \times R$, $A_N(\cdot, \mathbf{x}, q) \in D_{R^{r \times r}}[0, \infty)$;
- 2) for each $(\mathbf{x}, q) \in K_N^3 \times R$, and $t \geq 0$, $A_N(t, \mathbf{x}, q) \mathbf{e} = \mathbf{0}_{3 \times 1}$ where $\mathbf{e} = [1, 1, 1]'$;
- 3) for each $(\mathbf{x}, q) \in K_N^3 \times R$, and $t \geq 0$, $1 \leq i, j \leq 3$, $i \neq j$, $a_{N,i,j}(t, \mathbf{x}, q) \geq 0$.
- 4) for each $(\mathbf{x}, q) \in K_N^3 \times R$, $A_N(t, \mathbf{x}, q)$ is a constant on $t \in [\frac{k}{N}, \frac{k+1}{N})$ for $k \geq 0$.

For $k \geq 0$, $(\mathbf{x}, q) \in K_N \times R$, let

$$P_{N,k}[\mathbf{x}, q] = I_3 + \frac{1}{N} A_N\left(\frac{k}{N}, \mathbf{x}, q\right), \tag{3.1.10}$$

where I_3 is the identity matrix of 3 by 3. $P_{N,k}$ is called internal transition matrix.

Now we define the transition structure for $\{(\mathbf{n}^N(k), \log S_k)\}_{k=0}^{\infty}$. Assume that for time $k \geq 0$, $(\mathbf{n}^N(k), \log S_k)$ are given. An agent of j -type ($1 \leq j \leq 3$) has the probability $P_{N,k,j,i}[\frac{\mathbf{n}^N(k)}{N}, \log S_k]$ of being i -th type ($1 \leq i \leq r$) at next time period $k+1$. Assume that the distribution vector of types arising from the j -th type of agents at time k , which is of number $n_j^N(k)$, is $\Xi_{N,k,j} = (\xi_{N,k,j,1}, \xi_{N,k,j,2}, \xi_{N,k,j,3})$, where $\xi_{N,k,j,i}$ is the number of agents at time $k+1$ arising from the j -th type of agents. Since each agent changes his/her type independently, it is clear that

$$\Xi_{N,k,j} \sim \text{multinomial}\left(n_j^N(k), P_{N,k,j}[\frac{\mathbf{n}^N(k)}{N}, \log S_k]\right), \quad (3.1.11)$$

where $P_{N,k,j}[\frac{\mathbf{n}^N(k)}{N}, \log S_k]$ is the j -th row of matrix $P_{N,k}[\frac{\mathbf{n}^N(k)}{N}, \log S_k]$. Then the distribution vector of types at time $k+1$ is

$$\mathbf{n}^N(k+1) = \Xi_{N,k,1} + \Xi_{N,k,2} + \Xi_{N,k,3}, \quad (3.1.12)$$

and $\log S_{k+1}$ is determined by the recursive log price formula (3.1.9) base on $\log S_k$ and $\frac{\mathbf{n}^N(k+1)}{N}$.

3.2 Interacting agent feedback finance models

3.2.1 Formulation of IAFFM

We now use the assumptions made in subsection 3.1.1 to construct the interacting agent feedback finance models (IAFFM).

For each time period $k \geq 0$, let $\mathbf{n}^N(k) = (n_1^N(k), \dots, n_r^N(k))$, where for $1 \leq j \leq r$,

$n_j^N(k)$ is the number of agents of type j at period k . Then $\frac{n^N(k)}{N}$ is the empirical distribution of the types of agents at time $k \geq 0$.

We define K_N and K by (2.2.22) and (2.2.23) of Chapter 2, i.e. $K_N = \{N^{-1}\alpha : \alpha \in (Z_+)^r, \sum_{i=1}^r \alpha_i = N\}$ and $K = \{\alpha : \alpha \in (R_+)^r, \sum_{i=1}^r \alpha_i = 1\}$.

We specify matrices which are concerned with the shift of types among all agents. For fixed $N \geq 1$, we assume that $A_N(t, \mathbf{x}, q) = (a_{N,i,j}(t, \mathbf{x}, q))_{r \times r}$ is a $r \times r$ matrix-valued function on $[0, \infty) \times K_N \times R$, which satisfies the conditions:

- 1) for each $(\mathbf{x}, q) \in K_N \times R$, $A_N(\cdot, \mathbf{x}, q) \in D_{R^{r \times r}}[0, \infty)$;
- 2) for each $(\mathbf{x}, q) \in K_N \times R$, and $t \geq 0$, $A_N(t, \mathbf{x}, q)\mathbf{e} = \mathbf{0}_{r \times 1}$ where $\mathbf{e} = [1, \dots, 1]'$;
- 3) for each $(\mathbf{x}, q) \in K_N \times R$, and $t \geq 0$, $1 \leq i, j \leq r$, $i \neq j$, $a_{N,i,j}(t, \mathbf{x}, q) \geq 0$.
- 4) for each $(\mathbf{x}, q) \in K_N \times R$, $A_N(t, \mathbf{x}, q)$ is a constant on $t \in [\frac{k}{N}, \frac{k+1}{N})$ for $k \geq 0$.

We also define a $r \times r$ matrix-valued *càdlàg* function $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}$ on $[0, \infty) \times K \times R$, which satisfies the following slightly different conditions:

- 1) for each $(\mathbf{x}, q) \in K \times R$, $A(\cdot, \mathbf{x}, q) \in D_{R^{r \times r}}[0, \infty)$;
- 2) for each $(\mathbf{x}, q) \in K \times R$, and $t \geq 0$, $A(t, \mathbf{x}, q)\mathbf{e} = \mathbf{0}_{r \times 1}$;
- 3) for each $(\mathbf{x}, q) \in K \times R$, and $t \geq 0$, $1 \leq i, j \leq r$, $i \neq j$, $a_{i,j}(t, \mathbf{x}, q) \geq 0$.

Note that in the above specifications, $\mathbf{x}$ is always the empirical distribution of the types of agents and q is log price, i.e. $q = \log p$ where p is the market price of the only one financial asset.

Next, we specify the functions which are related to the log price mechanism [HO05]. For each $N \geq 1$, we define a real valued function $g_N(t, \mathbf{x}, q)$ on $[0, \infty) \times K_N \times R$. We also define a real valued function $g(t, \mathbf{x}, q)$ on $[0, \infty) \times K \times R$. Here $\mathbf{x}$ also denotes the empirical distribution of the types of agents and q is also the log price of the financial asset. In this section we are only concerned with functions $g_N(t, \mathbf{x}, q)$, $g(t, \mathbf{x}, q)$ which are ‘linearly’ growing with q , i.e. $g_N(t, \mathbf{x}, q)$, $g(t, \mathbf{x}, q)$ have the forms

$$g(t, \mathbf{x}, q) = \varphi(t, \mathbf{x}, q)q + \psi(t, \mathbf{x}, q), \quad (3.2.1)$$

$$g_N(t, \mathbf{x}, q) = \varphi_N(t, \mathbf{x}, q)q + \psi_N(t, \mathbf{x}, q), \quad (3.2.2)$$

where $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are functions on $[0, \infty) \times K \times R$, and $\varphi_N(t, \mathbf{x}, q)$ and $\psi_N(t, \mathbf{x}, q)$ are functions on $[0, \infty) \times K_N \times R$.

Finally, we specify the price mechanism for the interacting agent feedback finance models (IAFFM). Fix N , for $k \geq 1$, assume that the empirical distribution $\frac{\mathbf{n}^N(k)}{N}$ and the log price $\tilde{q}^N(\frac{k-1}{N})$ of the previous period $k-1$ are given, then the current log price is determined by

$$\tilde{q}^N(\frac{k}{N}) = \tilde{q}^N(\frac{k-1}{N}) + \frac{1}{N}g_N(\frac{k}{N}, \frac{\mathbf{n}^N(k)}{N}, \tilde{q}^N(\frac{k-1}{N})). \quad (3.2.3)$$

We assume that $\mathbf{X}^N(t) = \frac{\mathbf{n}^N(\lfloor Nt \rfloor)}{N}$, $q^N(t) = \tilde{q}^N(\frac{\lfloor Nt \rfloor}{N})$ and $\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t))$.

Then (3.2.3) represented by the new notations is for $t \geq \frac{1}{N}$,

$$q^N(t) = q^N(t - \frac{1}{N}) + \frac{1}{N}g_N(\frac{\lfloor Nt \rfloor}{N}, \mathbf{X}^N(t), q^N(t - \frac{1}{N})). \quad (3.2.4)$$

Now we construct IAFFM as follows. Let $\{A_N(t, \mathbf{x}, q)\}$ be defined above. For $k \geq 0$, $(\mathbf{x}, q) \in K_N \times R$, let

$$P_{N,k}[\mathbf{x}, q] = I + \frac{1}{N}A_N(\frac{k}{N}, \mathbf{x}, q). \quad (3.2.5)$$

As in Chapter 2, for each fixed $k \geq 0$, $(\mathbf{x}, q) \in K_N \times R$, $P_{N,k}[\mathbf{x}, q]$ is a stochastic matrix for large enough N . At first, assume that $\mathbf{n}^N(0)$ and the log price $\tilde{q}^N(0)$ at time period 0 are given. We define the time inhomogeneous Markov chain $\{(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))\}_{k=0}^{\infty}$ by using mathematical induction. Assume that for $k \geq 0$, $(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))$ is defined or given, we want to define $(\mathbf{n}^N(k+1), \tilde{q}^N(\frac{k+1}{N}))$. For each $1 \leq j \leq r$, we can define the random vector $\Xi_{N,k,j} = (\xi_{N,k,j,1}, \dots, \xi_{N,k,j,r})$ on some probability space, which satisfies

$$\Xi_{N,k,j} \sim \text{multinomial}\left(n_j^N(k), P_{N,k,j}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})]\right), \quad (3.2.6)$$

where $P_{N,k,j}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})]$ is the j -th row of matrix $P_{N,k}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})]$. We assume that $\Xi_{N,k,1}, \dots, \Xi_{N,k,r}$ are independent. Then, we define

$$\mathbf{n}^N(k+1) \equiv \Xi_{N,k,1} + \dots + \Xi_{N,k,r}, \quad (3.2.7)$$

and $\tilde{q}^N(\frac{k+1}{N})$ by formula (3.2.3).

Note that in the transition structure of $\{(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))\}_{k=0}^{\infty}$ specified above, $\tilde{q}^N(\frac{k+1}{N})$ depends on $\mathbf{n}^N(k)$ through $\tilde{q}^N(\frac{k}{N})$ and $\mathbf{n}^N(k+1)$ in a deterministic way, (see (3.2.3)), and $\mathbf{n}^N(k+1)$ depends on $(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))$ by (3.2.6). Note also that g_N does depend on N , which means that the price mechanism does reflect the influence of the size of the market. We will discuss later the situation when g_N is a random function.

For each $N \geq 1$, we define the transition operators of $\{(\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})), k \geq 0\}$ as follows: for each $k \geq 0$ and $f \in \bar{C}(K_N \times R)$, the set of bounded continuous function on $K_N \times R$, $(\mathbf{x}, q) \in K_N \times R$,

$$S_{N,k}f(\mathbf{x}, q) = E[f(\mathbf{X}^N(\frac{k+1}{N}), \tilde{q}^N(\frac{k+1}{N})) | \mathbf{X}^N(\frac{k}{N}) = \mathbf{x}, \tilde{q}^N(\frac{k}{N}) = q]. \quad (3.2.8)$$

We next consider the moments of $\{\mathbf{n}^N(k), k \geq 0\}$. Let $\mathbf{V} = (v_1, \dots, v_r)'$ be a

positive vector. For any $k \geq 1$, by (3.2.6), (3.2.7), the independence of $\Xi_{N,k,j}$'s, we have

$$E \left[\prod_{j=1}^r v_j^{n_j^N(k+1)} \middle| (\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N})) \right] = \prod_{j=1}^r \left(P_{N,k,j}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] \mathbf{V} \right)^{n_j^N(k)}. \quad (3.2.9)$$

Then it follows by (3.2.9) that

$$E \left[\mathbf{n}^N(k+1) \middle| (\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N})) \right] = \mathbf{n}^N(k) P_{N,k}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})], \quad (3.2.10)$$

and for $1 \leq i \leq r$,

$$\begin{aligned} & E \left[n_i^N(k+1)(n_i^N(k+1) - 1) \middle| (\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N})) \right] \\ &= \left(\mathbf{n}^N(k) P_{N,k,i}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] \right)^2 \\ & \quad - \sum_{j=1}^r \left(P_{N,k,j}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] P_{N,k,i}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] \right)^2 n_j^N(k), \end{aligned} \quad (3.2.11)$$

where $P_{N,k,i}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})]$ is i -th column of matrix $P_{N,k}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})]$. Then we can get

$$\begin{aligned} & E[(n_i^N(k+1) - n_i^N(k))^2 \middle| (\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))] \\ &= \left(\mathbf{n}^N(k) P_{N,k,i}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] \right)^2 + \mathbf{n}^N(k) P_{N,k,i}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] \\ & \quad - \sum_{j=1}^r \left(P_{N,k,j}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] P_{N,k,i}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] \right)^2 n_j^N(k) \\ & \quad - 2\mathbf{n}^N(k) P_{N,k,i}[\mathbf{X}^N(\frac{k}{N}), \tilde{q}^N(\frac{k}{N})] n_i^N(k) + (n_i^N(k))^2. \end{aligned} \quad (3.2.12)$$

3.2.2 Convergence theorem of IAFFM

So far, we have set up the sequence $\{\mathbf{Y}^N = (\mathbf{X}^N, q^N)\}$ in subsection 3.2.1. We make the assumption that $P(\mathbf{Y}^N(0))^{-1} \Rightarrow \mu$ where $\mu \in \mathcal{P}(K \times R)$. We need also to make

some assumptions on $\{A_N\}$ and A , and $\{g_N\}$ and g such that $\{\mathbf{Y}^N\}$ can be relatively compact, and have a weak limit $\mathbf{Y} = (\mathbf{X}, Q)$ satisfying $P(\mathbf{Y}(0))^{-1} = \mu$. It is natural from the consequences in Chapter 2 to expect that the transpose $\mathbf{Y}'$ of any limit $\mathbf{Y}$ be a solution of the following differential equations form:

$$\frac{d\mathbf{x}(t)}{dt} = A(t, \mathbf{x}(t), q(t))'\mathbf{x}(t), \quad (3.2.13)$$

$$\frac{dq(t)}{dt} = g(t, \mathbf{x}(t), q(t)), \quad (3.2.14)$$

which satisfy the initial conditions $\mathbf{x}(0) = \mathbf{X}(0)'$ and $q(0) = Q(0)$.

Let $K \times R$ be state space for the limit process and denote by $\bar{C}(K \times R)$ the set of bounded continuous functions on $K \times R$, and define $C_c(K \times R)$, $C_c^1(K \times R)$, $C_c^2(K \times R)$ by

$$C_c(K \times R) = \{f \in \bar{C}(K \times R), f \text{ has compact support on } K \times R\}, \quad (3.2.15)$$

$$C_c^1(K \times R) = \{f \in C_c(K \times R), \frac{\partial f}{\partial \mathbf{x}} \text{ and } \frac{\partial f}{\partial q} \text{ are continuous on } K \times R\}, \quad (3.2.16)$$

$$C_c^2(K \times R) = \{f \in C_c^1(K \times R), \frac{\partial^2 f}{\partial \mathbf{x}^2}, \frac{\partial^2 f}{\partial \mathbf{x} \partial q}, \text{ and } \frac{\partial^2 f}{\partial q^2} \text{ are continuous on } K \times R\}, \quad (3.2.17)$$

where $\frac{\partial f}{\partial \mathbf{x}} = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_r}]$, $\frac{\partial^2 f}{\partial \mathbf{x}^2}$, $\frac{\partial^2 f}{\partial \mathbf{x} \partial q}$, $\frac{\partial^2 f}{\partial q^2}$ have their corresponding meanings. Assume that $A(t, \mathbf{x}, q)$ satisfies the conditions at the beginning of subsection 3.2.1. We define the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C_c^1(K \times R)$: for each $f \in C_c^1(K \times R)$, and $s \geq 0$,

$$G_A(s)f(\mathbf{x}, q) = \mathbf{x}A(s, \mathbf{x}, q)\frac{\partial f'}{\partial \mathbf{x}} + g(s, \mathbf{x}, q)\frac{\partial f}{\partial q}, \quad (\mathbf{x}, q) \in K \times R. \quad (3.2.18)$$

$\mathcal{D}(G_A) = C_c^1(K \times R)$ is the common domain of the generator $\{G_A(s), 0 \leq s < \infty\}$, and $D = C_c^2(K \times R)$, is a subalgebra contained in $\mathcal{D}(G_A)$.

Next, we prove that $\{\mathbf{Y}^N\}$ satisfies the compact containment condition under certain conditions on $\{\mathbf{Y}^N(0)\}$, $\{g_N\}$ and g .

Lemma 3.2.1. *Assume that $P(\mathbf{Y}^N(0))^{-1} \Rightarrow \mu$ for some $\mu \in \mathcal{P}(K \times R)$, and $g(t, \mathbf{x}, q)$ and $g_N(t, \mathbf{x}, q)$ satisfy (3.2.1) and (3.2.2). Assume also that for any $T > 0$, $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are bounded on $[0, T] \times K \times R$ and that*

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{(\mathbf{x}, q) \in K_N \times R} |\varphi_N(t, \mathbf{x}, q) - \varphi(t, \mathbf{x}, q)| = 0, \quad (3.2.19)$$

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{(\mathbf{x}, q) \in K_N \times R} |\psi_N(t, \mathbf{x}, q) - \psi(t, \mathbf{x}, q)| = 0. \quad (3.2.20)$$

Then $\{\mathbf{Y}^N\}$ satisfies the compact containment condition, i.e., for every $\eta > 0$ and $T > 0$, there exists a compact set $\tilde{K}_{\eta, T} \subset K \times R$ for which

$$\inf_N P\{\mathbf{Y}^N(t) \in \tilde{K}_{\eta, T}, \text{ for } 0 \leq t \leq T\} \geq 1 - \eta. \quad (3.2.21)$$

Proof. Since $P(\mathbf{Y}^N(0))^{-1} \Rightarrow \mu$, then $\{P(q^N(0))^{-1}\}$ is tight. For any $\eta > 0$, there exists $b > 0$ such that

$$\inf_N P\{q^N(0) \in I\} \geq 1 - \eta, \quad (3.2.22)$$

where $I = [-b, b]$.

Fix $T > 0$, let $C_T > 0$ be a bound of $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ on $[0, T] \times K \times R$. By (3.2.3), (3.2.1), (3.2.2), (3.2.19) and (3.2.20), there exists N_0 , such that for $N > N_0$, we have for any $0 \leq k \leq [NT] - 1$,

$$|\tilde{q}^N(\frac{k+1}{N}) - \tilde{q}^N(\frac{k}{N})| \leq \frac{1}{N} C_T [|\tilde{q}^N(\frac{k}{N})| + 1], \quad (3.2.23)$$

which implies that for $0 \leq k \leq [NT]$,

$$|\tilde{q}^N(\frac{k}{N})| \leq (1 + \frac{1}{N} C_T)^k [|\tilde{q}^N(0)| + 1] - 1. \quad (3.2.24)$$

Let $I_{\eta,T} = [-e^{TC_T}(b+1), e^{TC_T}(b+1)]$. Since $(1 + \frac{1}{N}C_T)^N$ increases as N does, with the limit e^{C_T} , we have for $N > N_0$

$$P\{q^N(t) \in I_{\eta,T}, 0 \leq t \leq T\} = P\{q^N(0) \in I\}. \quad (3.2.25)$$

Let $\tilde{K}_{\eta,T} = K \times I_{\eta,T}$, then for $N > N_0$

$$P\{\mathbf{Y}^N(t) \in \tilde{K}_{\eta,T}\} = P\{q^N(0) \in I\} \quad (3.2.26)$$

and (3.2.21) holds. $\square$

Next, we prove a lemma which verifies (2.4.28) in condition (e) of Chapter 2, corollary 2.4.6 for $\{\mathbf{Y}^N\}$.

Lemma 3.2.2. *Assume that $\{A_N(t, \mathbf{x}, q) = (a_{N,i,j}(t, \mathbf{x}, q))_{r \times r}\}$ is a sequence of $r \times r$ matrix-valued functions on $[0, \infty) \times K_N \times R$ and $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}$ a $r \times r$ matrix-valued function on $[0, \infty) \times K \times R$. $\{A_N(t, \mathbf{x}, q)\}$ and $A(t, \mathbf{x}, q)$ satisfy the conditions at the beginning of subsection 3.2.1. Assume that $A(t, \mathbf{x}, q)$ is bounded on $[0, T] \times K \times R$ for any $T > 0$. Assume that $g(t, \mathbf{x}, q)$ and $\{g_N(t, \mathbf{x}, q)\}$ satisfy (3.2.1) and (3.2.2). Assume that $\varphi, \psi, \frac{\partial \varphi}{\partial \mathbf{x}}, \frac{\partial \psi}{\partial \mathbf{x}}$ are continuous. Assume also that (3.2.19) and (3.2.20) hold for $\{\varphi_N\}, \varphi$ and $\{\psi_N\}, \psi$. For $k \geq 0, (\mathbf{x}, q) \in K_N \times R$, define $P_{N,k}[\mathbf{x}, q]$ by (3.2.5) and $\{(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))\}_{k=0}$ by the procedure in subsection 3.2.1. Let $\mathbf{X}^N(t) = \frac{\mathbf{n}^N([Nt])}{N}$, $q^N(t) = \tilde{q}^N(\frac{[Nt]}{N})$ and $\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t))$ and for $k \geq 0$, define the transition operators $S_{N,k}$ on $\bar{C}(K_N \times R)$ by (3.2.8) and the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C_c^1(K \times R)$ by (3.2.18). If for each $f \in C_c^2(K \times R)$ and $T > 0$ there exist measurable sets $\{F_N\} \subset R$ satisfying*

$$\lim_{N \rightarrow \infty} \sup_{q \in F_N} \sup_{\mathbf{x} \in K_N} d_U(A_N(\cdot, \mathbf{x}, q), A(\cdot, \mathbf{x}, q)) = 0, \quad (3.2.27)$$

then

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N} \sup_{\mathbf{x} \in K_N} |N[S_{N,[Nt]} - I]f(\mathbf{x}, q) - G_A(t)f(\mathbf{x}, q)| = 0. \quad (3.2.28)$$

Proof. Let $f \in C_c^2(K \times R)$ and fix $T > 0$. Assume that $\tilde{K}$ is the support of f . For $0 \leq t \leq T$, and $\mathbf{y} = (\mathbf{x}, q) \in K_N \times F_N$, by Taylor's expansion, (3.2.10) and (3.2.4)

$$\begin{aligned} & [S_{N,[Nt]} - I]f(\mathbf{y}) \\ &= E[(\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{x}) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{y})' | \mathbf{Y}^N(t) = \mathbf{y}] + E[(q^N(t + \frac{1}{N}) - q) \frac{\partial f}{\partial q}(\mathbf{y}) | \mathbf{Y}^N(t) = \mathbf{y}] \\ & \quad + E[(\mathbf{Y}^N(t + \frac{1}{N}) - \mathbf{y}) \frac{\partial^2 f}{\partial \mathbf{y}^2}(\mathbf{Y}_*^N(t))(\mathbf{Y}^N(t + \frac{1}{N}) - \mathbf{y})' | \mathbf{Y}^N(t) = \mathbf{y}] \\ &= \frac{1}{N} \mathbf{x} A_N(\frac{[Nt]}{N}, \mathbf{y}) \frac{\partial f}{\partial \mathbf{x}}(\mathbf{y})' + \frac{1}{N} E \left[g_N \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) \frac{\partial f}{\partial q}(\mathbf{y}) \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \\ & \quad + E[(\mathbf{Y}^N(t + \frac{1}{N}) - \mathbf{y}) \frac{\partial^2 f}{\partial \mathbf{y}^2}(\mathbf{Y}_*^N(t))(\mathbf{Y}^N(t + \frac{1}{N}) - \mathbf{y})' | \mathbf{Y}^N(t) = \mathbf{y}], \end{aligned} \quad (3.2.29)$$

where $\mathbf{Y}_*^N(t) = \mathbf{y} + \theta_t^N(\mathbf{Y}^N(t + \frac{1}{N}) - \mathbf{y})$, for some $\theta_t^N \in (0, 1)$.

Notice that

$$\begin{aligned} & g_N \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) - g(t, \mathbf{x}, q) \\ &= \left[g_N \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) - g \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) \right] \\ & \quad + \left[g \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) - g \left(\frac{[Nt] + 1}{N}, \mathbf{x}, q \right) \right] \\ & \quad + \left[g \left(\frac{[Nt] + 1}{N}, \mathbf{x}, q \right) - g(t, \mathbf{x}, q) \right] \end{aligned} \quad (3.2.30)$$

and

$$\begin{aligned} & g \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) - g \left(\frac{[Nt] + 1}{N}, \mathbf{x}, q \right) \\ &= (\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{x}) \frac{\partial g}{\partial \mathbf{x}} \left(\frac{[Nt] + 1}{N}, \mathbf{X}_*^N(t + \frac{1}{N}), q \right)', \end{aligned} \quad (3.2.31)$$

where $\frac{\partial \mathbf{g}}{\partial \mathbf{x}} = [\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_r}]$, $\mathbf{X}_*^N(t + \frac{1}{N}) = \mathbf{x} + \zeta_{t+\frac{1}{N}}^N(\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{x})$, for some $\zeta_{t+\frac{1}{N}}^N \in (0, 1)$.

Then it follows that

$$\begin{aligned}
& |N[S_{N,[Nt]} - I]f(\mathbf{y}) - G_A(t)f(\mathbf{y})| \\
& \leq \left| \mathbf{x}[A_N(\frac{[Nt]}{N}, \mathbf{y}) - A(t, \mathbf{y})] \frac{\partial f}{\partial \mathbf{x}}(\mathbf{y})' \right| + E \left[\left\| \left(g_N \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) \right. \right. \right. \\
& \quad \left. \left. \left. - g \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) \right) \frac{\partial f}{\partial q}(\mathbf{y}) \right\| \mathbf{Y}^N(t) = \mathbf{y} \right] \\
& \quad + E \left[\left\| (\mathbf{X}^N(t + \frac{1}{N}) - \mathbf{x}) \frac{\partial g}{\partial \mathbf{x}} \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right)' \frac{\partial f}{\partial q}(\mathbf{y}) \right\| \mathbf{Y}^N(t) = \mathbf{y} \right] \\
& \quad + E \left[\left\| \left(g \left(\frac{[Nt] + 1}{N}, \mathbf{x}, q \right) - g(t, \mathbf{x}, q) \right) \frac{\partial f}{\partial q}(\mathbf{y}) \right\| \mathbf{Y}^N(t) = \mathbf{y} \right] \\
& \quad + NE \left[\left\| (\mathbf{Y}^N(t + \frac{1}{N}) - \mathbf{y}) \frac{\partial^2 f}{\partial \mathbf{y}^2}(\mathbf{Y}_*^N(t)) (\mathbf{Y}^N(t + \frac{1}{N}) - \mathbf{y})' \right\| \mathbf{Y}^N(t) = \mathbf{y} \right] \\
& = I_1(N, \mathbf{y}, t) + I_2(N, \mathbf{y}, t) + I_3(N, \mathbf{y}, t) + I_4(N, \mathbf{y}, t) + I_5(N, \mathbf{y}, t).
\end{aligned} \tag{3.2.32}$$

Notice that for each $(\mathbf{x}, q) \in K_N \times R$, $A_N(t, \mathbf{x}, q)$ is a constant on $[\frac{k}{N}, \frac{k+1}{N})$ for each $k \geq 0$, we have $A_N(t, \mathbf{x}, q) = A_N(\frac{[Nt]}{N}, \mathbf{x}, q)$ for $t \geq 0$. Recall the norms of Matrices and vectors used in the proof of Chapter 2, Lemma 2.4.8. That is to say, for $r \times r$ real-valued matrix $B = (b_{i,j})_{r \times r}$ and real-valued r -dimensional vector $\mathbf{y} = (y_1, \dots, y_r)'$, we define $\|B\| = \sum_{i,j=1}^r |b_{i,j}|$ and $\|\mathbf{y}\|$ the Euclidean norm on R^r of $\mathbf{y}$, and it follows that $\|B\mathbf{y}\| \leq \|B\| \cdot \|\mathbf{y}\|$. Then similar to the proof of Chapter 2, Lemma 2.4.8, we have

$$I_1(N, \mathbf{y}, t) \leq \|A_N(t, \mathbf{y}) - A(t, \mathbf{y})\| \cdot \left\| \frac{\partial f}{\partial \mathbf{x}}(\mathbf{y})' \right\|. \tag{3.2.33}$$

Then it follows by (3.2.27) that

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N} \sup_{\mathbf{x} \in K_N} I_1(N, \mathbf{x}, q, t) = 0. \tag{3.2.34}$$

Next, we consider $I_5(N, \mathbf{y}, t)$. Let $\left\| \frac{\partial^2 f}{\partial \mathbf{y}^2} \right\| = \max_{1 \leq i, j \leq r+1} \left\| \frac{\partial^2 f}{\partial y_i \partial y_j} \right\|$ and define $g_{N, \tilde{K}}$ as follows: $g_{N, \tilde{K}}(t, \mathbf{y}) = g_N(t, \mathbf{y})$, if $(t, \mathbf{y}) \in [0, \infty) \times (\tilde{K} \cap (K_N \times R))$; $g_{N, \tilde{K}}(t, \mathbf{y}) = 0$

otherwise. By Hölder inequality and Cauchy-Schwartz inequality,

$$\begin{aligned}
& I_5(N, \mathbf{y}, t) \\
& \leq \left\| \frac{\partial^2 f}{\partial \mathbf{y}^2} \right\| N \left\{ \sum_{i=1}^r \sum_{j=1}^r E \left[\left| (X_i^N(t + \frac{1}{N}) - x_i)(X_j^N(t + \frac{1}{N}) - x_j) \right| \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \right. \\
& \quad + \frac{2}{N} \sum_{i=1}^r E \left[\left| (X_i^N(t + \frac{1}{N}) - x_i) g_{N, \tilde{K}} \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) \right| \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \\
& \quad \left. + \frac{1}{N^2} E \left[\left| g_{N, \tilde{K}} \left(\frac{[Nt] + 1}{N}, \mathbf{X}^N(t + \frac{1}{N}), q \right) \right|^2 \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \right\} \\
& \leq \left\| \frac{\partial^2 f}{\partial \mathbf{y}^2} \right\| N \left\{ \sum_{i=1}^r \sum_{j=1}^r \left(E \left[(X_i^N(t + \frac{1}{N}) - x_i)^2 \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \right. \right. \\
& \quad \times E \left[(X_j^N(t + \frac{1}{N}) - x_j)^2 \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \Big)^{\frac{1}{2}} \\
& \quad \left. + \frac{2}{N} \|g_{N, \tilde{K}}\|_{T+1} \sum_{i=1}^r \left(E \left[(X_i^N(t + \frac{1}{N}) - x_i)^2 \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \right)^{\frac{1}{2}} + \frac{1}{N^2} \|g_{N, \tilde{K}}\|_{T+1}^2 \right\}, \tag{3.2.35}
\end{aligned}$$

where $\|g_{N, \tilde{K}}\|_{T+1} = \sup_{0 \leq t \leq T+1} \sup_{\mathbf{y} \in \tilde{K} \cap (K_N \times R)} |g_N(t, \mathbf{y})|$. By (3.2.19), (3.2.20) and the assumption that $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are continuous on $[0, T] \times K \times R$ for any $T > 0$, it follows that

$$\sup_{N \geq 1} \|g_{N, \tilde{K}}\|_{T+1} < \infty. \tag{3.2.36}$$

For fixed $1 \leq i \leq r$, by (3.2.12), we get

$$\begin{aligned}
& N^2 E \left[(X_i^N(t + \frac{1}{N}) - x_i)^2 \middle| \mathbf{Y}^N(t) = \mathbf{y} \right] \\
& = \mathbf{x} A_{N, \cdot, i} \left(\frac{[Nt]}{N}, \mathbf{y} \right) + (\mathbf{x} A_{N, \cdot, i} \left(\frac{[Nt]}{N}, \mathbf{y} \right))^2 - 2x_i a_{N, i, i} \left(\frac{[Nt]}{N}, \mathbf{y} \right) - \sum_{k=1}^r \frac{x_k}{N} a_{N, k, i}^2 \left(\frac{[Nt]}{N}, \mathbf{y} \right) \\
& \leq \sum_{l=1}^r |A_{N, l, i}(t, \mathbf{y})| + \left(\sum_{l=1}^r |A_{N, l, i}(t, \mathbf{y})| \right)^2 + 2|a_{N, i, i}(t, \mathbf{y})|. \tag{3.2.37}
\end{aligned}$$

Then it follows by (3.2.35), (3.2.36), (3.2.37), (3.2.27) and the assumption that

$A(t, \mathbf{x}, q)$ is bounded on $[0, T] \times K \times R$, that

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N} \sup_{\mathbf{x} \in K_N} I_5(N, \mathbf{x}, q, t) = 0. \quad (3.2.38)$$

Similarly, by (3.2.19) and (3.2.20), we can get

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N} \sup_{\mathbf{x} \in K_N} I_2(N, \mathbf{x}, q, t) = 0, \quad (3.2.39)$$

and by (3.2.37), (3.2.27) and some calculations, we can get that

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N} \sup_{\mathbf{x} \in K_N} I_3(N, \mathbf{x}, q, t) = 0. \quad (3.2.40)$$

Notice that for any $(\mathbf{x}, q) \in K \times R$ and $0 \leq t \leq T$,

$$I_4(N, \mathbf{x}, q, t) \leq \sup_{0 \leq t \leq T} \sup_{(\mathbf{x}, q) \in \tilde{K}} \left| g\left(\frac{[Nt] + 1}{N}, \mathbf{x}, q\right) - g(t, \mathbf{x}, q) \right| \cdot \sup_{\mathbf{y} \in \tilde{K}} \left| \frac{\partial f}{\partial q}(\mathbf{y}) \right|, \quad (3.2.41)$$

then it follows by the uniform continuity of $g(t, \mathbf{x}, q)$ on the compact set $[0, T+1] \times \tilde{K}$ that

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{q \in F_N} \sup_{\mathbf{x} \in K_N} I_4(N, \mathbf{x}, q, t) = 0, \quad (3.2.42)$$

and (3.2.28) is proved. $\square$

Remark 3.2.3. As a special case, we can assume that $g_N(t, \mathbf{x}, q) = g(\frac{[Nt]}{N}, \mathbf{x}, q)$, i.e., $\varphi_N(t, \mathbf{x}, q) = \varphi(\frac{[Nt]}{N}, \mathbf{x}, q)$ and $\psi_N(t, \mathbf{x}, q) = \psi(\frac{[Nt]}{N}, \mathbf{x}, q)$, for $(t, \mathbf{x}, q) \in [0, \infty) \times K_N \times R$. Then we do not need to assume (3.2.19) and (3.2.20) to get the conclusions of Lemma 3.2.1 and Lemma 3.2.2. But (3.2.19) and (3.2.20) hold under the stronger condition that $\frac{\partial \varphi}{\partial t}$ and $\frac{\partial \psi}{\partial t}$ are continuous and bounded on $[0, T] \times K \times R$ for any $T > 0$.

Corollary 3.2.4. If we assume in Lemma 3.2.2 that $P(\mathbf{Y}^N(0))^{-1} \Rightarrow \mu$ for some $\mu \in \mathcal{P}(K \times R)$, and $g(t, \mathbf{x}, q)$ satisfy (3.2.1), and $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are bounded

on $[0, T] \times K \times R$ for any $T > 0$, and for each $f \in C_c^2(K \times R)$ and $T > 0$, there exist measurable sets $\{F_N\} \subset R$ satisfying

$$\lim_{N \rightarrow \infty} P\{q^N(t) \in F_N, 0 \leq t \leq T\} = 1 \quad (3.2.43)$$

and (3.2.27), then $\{\mathbf{Y}^N\}$ is relatively compact. If we assume also that for any compact subset $\tilde{K}$ of $K \times R$, there exist $C > 0$ and $\lambda > 0$, such that $A(t, \mathbf{x}, q)$, $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ satisfy that

$$\sup_{(\mathbf{x}, q) \in \tilde{K}} \|A(t, \mathbf{x}, q)\| \leq Ce^{\lambda t}, \quad (3.2.44)$$

$$\sup_{(\mathbf{x}, q) \in \tilde{K}} |\varphi(t, \mathbf{x}, q)| \leq Ce^{\lambda t}, \quad (3.2.45)$$

$$\sup_{(\mathbf{x}, q) \in \tilde{K}} |\psi(t, \mathbf{x}, q)| \leq Ce^{\lambda t}, \quad (3.2.46)$$

where $\|A(t, \mathbf{x}, q)\|$ is the matrix norm of $A(t, \mathbf{x}, q)$, then any limit point $\mathbf{Y}$ of $\mathbf{Y}^N$ is a solution of the $D_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$.

Proof. By Lemma 3.2.1, we know that $\{\mathbf{Y}^N\}$ satisfies the compact containment condition. Let $\tilde{F}_N = K_N \times F_N$, by (3.2.43) and Lemma 3.2.2, we know that (2.4.27) and (2.4.28) in condition (e) of Chapter 2, corollary 2.4.6 holds for $\{\tilde{F}_N\}$. Note that

$$\|G_A(t)f\| \leq \|\mathbf{x}A(t, \mathbf{x}, q)\frac{\partial f}{\partial \mathbf{x}}\| + \|\frac{\partial f}{\partial q}g(t, \mathbf{x}, q)\|, \quad (3.2.47)$$

where $\|\cdot\|$ above is the supnorm with respect to $(\mathbf{x}, q)$, and that $A(t, \mathbf{x}, q)$ is bounded on $[0, T] \times K \times R$ for any $T > 0$, g is continuous on $[0, \infty) \times K \times R$, and f has a compact support, we have that $\|G_A(t)f\|$ is bounded on $[0, T]$ for any $T > 0$. Then by the proof of corollary 2.4.6 of Chapter 2, we know that the conditions (2.4.5) and (2.4.6) of Chapter 2 are verified. Then it follows that $\{\mathbf{Y}^N\}$ is relatively compact.

Let $\tilde{K}$ be the support of f , then by (3.2.1) we get

$$\begin{aligned} \|G_A(t)f\| \leq & \max_{1 \leq i \leq r} \sup_{(\mathbf{x}, q) \in \tilde{K}} \left\| \frac{\partial f}{\partial x_i} \right\| \cdot \sup_{(\mathbf{x}, q) \in \tilde{K}} \|A(t, \mathbf{x}, q)\| + \sup_{(\mathbf{x}, q) \in \tilde{K}} \left| \frac{\partial f}{\partial q} q \right| \cdot \sup_{(\mathbf{x}, q) \in \tilde{K}} |\varphi(t, \mathbf{x}, q)| \\ & + \sup_{(\mathbf{x}, q) \in \tilde{K}} \left| \frac{\partial f}{\partial q} \right| \cdot \sup_{(\mathbf{x}, q) \in \tilde{K}} |\psi(t, \mathbf{x}, q)|, \end{aligned} \quad (3.2.48)$$

where $\|G_A(t)f\|$ is the supnorm of $G_A(t)f$ with respect to $(\mathbf{x}, q)$, and the norms on the right side of (3.2.48) are matrix norm and Euclidean norm used in the proof of Lemma 3.2.2. By (3.2.44), (3.2.45) and (3.2.46), we know that (2.4.7) of condition (b) in Proposition 2.4.1 of Chapter 2 is verified and then second conclusion follows by the proof of corollary 2.4.6 of Chapter 2. $\square$

Next, we will show that any limit point $\mathbf{Y}$ of $\mathbf{Y}^N$ is continuous almost surely. Before we start the job, we specify the following notations:

$$b_i(t, \mathbf{y}) = \sum_{j=1}^r x_j a_{ji}(t, \mathbf{y}), \quad 1 \leq i \leq r, \text{ and } \mathbf{y} = (x_1, \dots, x_r, q) \quad (3.2.49)$$

$$b_{r+1}(t, \mathbf{y}) = g(t, \mathbf{y}), \quad (3.2.50)$$

$$\mathbf{b}(t, \mathbf{y}) = (b_1(t, \mathbf{y}), \dots, b_{r+1}(t, \mathbf{y})) \text{ and } \hat{\mathbf{b}}(t, \mathbf{y}) = (b_1(t, \mathbf{y}), \dots, b_r(t, \mathbf{y})). \quad (3.2.51)$$

Lemma 3.2.5. *Assume that $\{A_N(t, \mathbf{x}, q) = (a_{N,i,j}(t, \mathbf{x}, q))_{r \times r}\}$ is a sequence of $r \times r$ matrix-valued functions on $[0, \infty) \times K_N \times R$ and $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}$ a $r \times r$ matrix-valued function on $[0, \infty) \times K \times R$. $\{A_N(t, \mathbf{x}, q)\}$ and $A(t, \mathbf{x}, q)$ satisfy the conditions at the beginning of subsection 3.2.1. Assume that $g(t, \mathbf{x}, q)$ and $\{g_N(t, \mathbf{x}, q)\}$ satisfy (3.2.1) and (3.2.2). Assume that $\varphi, \psi, \frac{\partial \varphi}{\partial \mathbf{x}}, \frac{\partial \psi}{\partial \mathbf{x}}$ are continuous. Assume that (3.2.19) and (3.2.20) hold for $\{\varphi_N\}, \varphi$ and $\{\psi_N\}, \psi$. Assume that $A(t, \mathbf{x}, q), \varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ satisfy (3.2.44), (3.2.45), and (3.2.46) and are bounded on $[0, T] \times K \times R$ for any $T > 0$. Define $\mathbf{b}(t, \mathbf{y})$ by (3.2.49), (3.2.50) and (3.2.51).*

For $k \geq 0$, $(\mathbf{x}, q) \in K_N \times R$, define $P_{N,k}[\mathbf{x}, q]$ by (3.2.5) and $\{(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))\}_{k=0}$ by the procedure in subsection 3.2.1. Let $\mathbf{X}^N(t) = \frac{\mathbf{n}^N(\lfloor Nt \rfloor)}{N}$, $q^N(t) = \tilde{q}^N(\frac{\lfloor Nt \rfloor}{N})$ and $\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t))$ and for $k \geq 0$, define the transition operators $S_{N,k}$ on $\bar{C}(K_N \times R)$ by (3.2.8) and the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C_c^1(K \times R)$ by (3.2.18). Suppose that for each $f \in C_c^2(K \times R)$ and $T > 0$ there exist measurable sets $\{F_N\} \subset R$ satisfying (3.2.43) and (3.2.27). Suppose that $P(\mathbf{Y}^N(0))^{-1} \Rightarrow \mu$ for some $\mu \in \mathcal{P}(K \times R)$, then any limit point $\mathbf{Y}$ of $\mathbf{Y}^N$ is a solution of the $C_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$.

Proof. Assume that $\mathbf{Y} = (\mathbf{X}, Q)$, where $\mathbf{X} = (X_1, \dots, X_r)$, is a limit point of $\mathbf{Y}^N$. By Corollary 3.2.4, $\mathbf{Y}$ is a solution of the $D_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$. We just need to show that $\mathbf{Y}$ is continuous almost surely.

Fix $T > 0$. Since $A(t, \mathbf{x}, q)$, $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are bounded on $[0, T] \times K \times R$, there exists $C_T > 0$, such that for $1 \leq i \leq r$,

$$\sup_{0 \leq t \leq T} \sup_{\mathbf{y} \in K \times R} |b_i(t, \mathbf{y})| \leq C_T, \quad \sup_{0 \leq t \leq T} \sup_{\mathbf{y} \in K \times R} |\varphi(t, \mathbf{y})| \leq C_T, \quad \sup_{0 \leq t \leq T} \sup_{\mathbf{y} \in K \times R} |\psi(t, \mathbf{y})| \leq C_T. \quad (3.2.52)$$

By (3.2.19) and (3.2.20), there exists N_0 such that for $N > N_0$, we have

$$\sup_{0 \leq t \leq T} \sup_{\mathbf{y} \in K_N \times R} |\varphi_N(t, \mathbf{y})| \leq C_T + 1, \quad \sup_{0 \leq t \leq T} \sup_{\mathbf{y} \in K_N \times R} |\psi_N(t, \mathbf{y})| \leq C_T + 1. \quad (3.2.53)$$

Let $f_i(\mathbf{x}, q) = x_i$, $1 \leq i \leq r$. It follows that $f_i^j \in C_c^2(K \times R)$ for $1 \leq i \leq r$ and $1 \leq j \leq 4$. We assume that $\{\mathcal{F}_s, 0 \leq s < \infty\}$ is the filtration to which the $D_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) referred. Then $X_i^j(t) - j \int_0^t (X_i(u))^{j-1} \mathbf{X}(u) \times A_{\cdot, i}(u, \mathbf{Y}(u)) du = X_i^j(t) - j \int_0^t (X_i(u))^{j-1} b_i(u, \mathbf{Y}(u)) du$ is an $\{\mathcal{F}_s\}$ -martingale for $1 \leq i \leq r$ and $1 \leq j \leq 4$. Let $0 \leq s < t \leq T$, and fix $1 \leq i \leq r$,

it follows that

$$E[(X_i(t) - X_i(s))^j] = jE\left[\int_s^t (X_i(u) - X_i(s))^{j-1} b_i(u, \mathbf{Y}(u)) du\right] \quad (3.2.54)$$

for $1 \leq j \leq 4$. Then by (3.2.54), for $j = 2$, we have

$$\begin{aligned} E[(X_i(t) - X_i(s))^2] &\leq 2E\left[\int_s^t |X_i(u) - X_i(s)| |b_i(u, \mathbf{Y}(u))| du\right] \\ &\leq 2C_T(t - s). \end{aligned} \quad (3.2.55)$$

By (3.2.54) and (3.2.55), for $j = 4$,

$$\begin{aligned} E[(X_i(t) - X_i(s))^4] &\leq 4C_TE\left[\int_s^t |X_i(u) - X_i(s)|^3 du\right] \\ &\leq 4C_TE\left[\int_s^t |X_i(u) - X_i(s)|^2 du\right] \\ &\leq 4C_T \int_s^t 2C_T(u - s) du \\ &\leq 4(C_T)^2(t - s)^2. \end{aligned} \quad (3.2.56)$$

Then by Kolmogorov's Criterion, we proved for $1 \leq i \leq r$ that X_i is continuous almost surely.

Next, we prove that Q is also continuous almost surely. We introduce the notations in Chapter 3, Section 10 [EK86]. Let (E, r) be a metric space. For $x \in D_E[0, \infty)$, define

$$J(x) = \int_0^\infty e^{-u} [J(x, u) \wedge 1] du, \quad (3.2.57)$$

where $J(x, u)$ is defined by

$$J(x, u) = \sup_{0 \leq t \leq u} r(x(t), x(t-)). \quad (3.2.58)$$

Since $\mathbf{Y}$ is a limit point of $\mathbf{Y}^N$, we have that a subsequence $\{q^{N_k}\}$ of $\{q^N\}$ converges weakly to Q . To prove that Q is continuous almost surely, by Chapter 3, Theorem

10.2 [EK86], it suffices to prove that $J(q^{N_k}) \Rightarrow 0$ as $k \rightarrow \infty$. In this case $E = R$ and r is the Euclidean metric. It is enough to show that $\lim_{k \rightarrow \infty} E[J(q^{N_k})] = 0$.

By Lemma 3.2.1, $\{q^{N_k}\}$ satisfies the compact containment condition, i.e. for any $\eta > 0$ and $T > 0$, there exists $B_T > 0$, such that

$$\inf_k P\{|q^{N_k}(t)| \leq B_T, 0 \leq t \leq T\} \geq 1 - \eta. \quad (3.2.59)$$

By the construction of q^N , (3.2.3) and (3.2.53), for fixed $N_k \geq N_0$ and $0 \leq u \leq T$,

$$\begin{aligned} J(q^{N_k}, u) &= \max_{0 \leq j \leq \frac{[N_k u]}{N_k}} |\tilde{q}^{N_k}(\frac{j}{N_k}) - \tilde{q}^{N_k}(\frac{j-1}{N_k})| \\ &\leq \frac{1}{N_k} \max_{0 \leq j \leq \frac{[N_k u]}{N_k}} \left[|\varphi_{N_k}(\frac{j}{N_k}, \mathbf{X}(\frac{j}{N_k}), \tilde{q}^{N_k}(\frac{j-1}{N_k})) \tilde{q}^{N_k}(\frac{j-1}{N_k})| \right. \\ &\quad \left. + |\psi_{N_k}(\frac{j}{N_k}, \mathbf{X}(\frac{j}{N_k}), \tilde{q}^{N_k}(\frac{j-1}{N_k}))| \right] \\ &\leq \frac{1}{N_k} (C_T + 1)(B_T + 1), \end{aligned} \quad (3.2.60)$$

on the event $F_{k,T} = \{|q^{N_k}(t)| \leq B_T, 0 \leq t \leq T\}$. It follows by (3.2.59) and (3.2.60) that

$$\begin{aligned} E[J(q^{N_k})] &\leq e^{-T} + E\left[\int_0^T e^{-u} (J(q^{N_k}, u) \wedge 1) du\right] \\ &\leq e^{-T} + E\left[\int_0^T e^{-u} (J(q^{N_k}, u) \wedge 1) du \chi_{F_{k,T}}\right] + P(F_{k,T}^c) \\ &\leq e^{-T} + \frac{T}{N_k} (C_T + 1)(B_T + 1) + \eta. \end{aligned} \quad (3.2.61)$$

Let $k \rightarrow \infty$ and then let $T \rightarrow \infty$, $\eta \rightarrow 0$, we proved that $\lim_{k \rightarrow \infty} E[J(q^{N_k})] = 0$. $\square$

Next, we give a lemma which relates the solutions of the $C_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$ to the solutions of the following stochastic integral equation:

$$\mathbf{Y}(t) = \mathbf{Y}(0) + \int_0^t \mathbf{b}(s, \mathbf{Y}(s)) ds, \quad t \geq 0, \quad (3.2.62)$$

where $P(\mathbf{Y}(0))^{-1} = \mu$ and $\mathbf{b}(t, \mathbf{y})$ is defined by (3.2.51).

Lemma 3.2.6. *Assume that $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}$ is a $r \times r$ matrix-valued function on $[0, \infty) \times K \times R$ which is bounded on $[0, T] \times K \times R$ for any $T > 0$. Assume that $g(t, \mathbf{x}, q)$ satisfy (3.2.1). Assume also that $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are bounded on $[0, T] \times K \times R$ for any $T > 0$ and that $\varphi, \psi, \frac{\partial \varphi}{\partial \mathbf{x}}, \frac{\partial \psi}{\partial \mathbf{x}}$ are continuous. Define $\mathbf{b}(t, \mathbf{y})$ by (3.2.49), (3.2.50) and (3.2.51). Define the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C_c^1(K \times R)$ by (3.2.18). If $\mathbf{Y}$ is a solution of the $C_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$ and with respect to $\{\mathcal{F}_t\}$ on a probability space $(\Omega, \mathcal{F}, P)$, then $(\Omega, \bar{\mathcal{F}}, P, \{\bar{\mathcal{F}}_t\}, \mathbf{Y})$ is a solution of the stochastic integral equation (3.2.62), where $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}_t$ are P -completion of $\mathcal{F}$ and $\mathcal{F}_t$.*

Proof. Assume that $\mathbf{Y}$ is a solution of the $C_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$ and with respect to $\{\mathcal{F}_t\}$. For convenience, we use $\mathbf{Y} \equiv (Y_1, \dots, Y_{r+1})$ instead of $\mathbf{Y} \equiv (\mathbf{X}, Q)$ in this proof. It is clear that for $1 \leq i \leq r+1$,

$$M_i(t) = Y_i(t) - Y_i(0) - \int_0^t b_i(s, Y(s)) ds \quad (3.2.63)$$

are continuous local martingales with respect to $\{\mathcal{F}_t\}$. For $1 \leq i, j \leq r+1$,

$$M_{ij}(t) = Y_i(t)Y_j(t) - Y_i(0)Y_j(0) - \int_0^t b_i(s, \mathbf{Y}(s))Y_j(s)ds - \int_0^t b_j(s, \mathbf{Y}(s))Y_i(s)ds \quad (3.2.64)$$

are also continuous local martingales.

By the *integration by parts formula*, for $1 \leq i, j \leq r+1$, we have

$$\langle M_i, M_j \rangle_t = Y_i(t)Y_j(t) - Y_i(0)Y_j(0) - \int_0^t Y_j(s)dY_i(s) - \int_0^t Y_i(s)dY_j(s), \quad t \geq 0. \quad (3.2.65)$$

It follows from (3.2.63) and (3.2.64) that

$$\langle M_i, M_j \rangle_t = M_{ij}(t) - \int_0^t Y_j(s) dM_i(s) - \int_0^t Y_i(s) dM_j(s), \quad t \geq 0. \quad (3.2.66)$$

Since the right side of (3.2.66) is a continuous local martingale, we actually have

$$\langle M_i, M_j \rangle_t = 0, \quad t \geq 0. \quad (3.2.67)$$

Thus (3.2.62) holds for $(\Omega, \bar{\mathcal{F}}, P, \{\bar{\mathcal{F}}_t\}, \mathbf{Y})$. $\square$

Remark 3.2.7. *The stochastic integral equation (3.2.62) is actually defined pathwisely. Therefore, in Lemma 3.2.6, for P -a.s. $\omega \in \Omega$, $\mathbf{Y}(\omega, t)'$ is a solution of the following deterministic integral equation*

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t \mathbf{b}(s, \mathbf{y}(s))' ds, \quad t \geq 0 \quad (3.2.68)$$

with $\mathbf{y}(0) = \mathbf{Y}(0, \omega)'$. It is obvious that (3.2.68) is equivalent to the following integral equations

$$\mathbf{x}(t) = \mathbf{x}(0)' + \int_0^t \hat{\mathbf{b}}(s, \mathbf{x}(s), q(s))' ds, \quad (3.2.69)$$

$$q(t) = q(0) + \int_0^t g(s, \mathbf{x}(s), q(s)) ds. \quad (3.2.70)$$

where $\mathbf{y}(0) = (\mathbf{x}(0), q(0))$. We note that the above integral equations are equivalent to the differential equations system (3.2.13) and (3.2.14) with the initial conditions $\mathbf{x}(0) = \mathbf{X}(0)$ and $q(0) = Q(0)$.

Now it is clear that to make $\{\mathbf{Y}^N\}$ have unique limit point $\mathbf{Y}$, we just need to put some conditions on $A(t, \mathbf{x}, q)$ and $g(t, \mathbf{x}, q)$ so that the integral equation (3.2.68) has unique solution. Before we embark on this task and prove the convergence theorem, we need to introduce some terminologies about *nonlinear Volterra equations*. Let

$\mathbb{L}^2([a, b], R^m)$ be the space of measurable and square integrable functions (or equivalent classes) on the bounded interval $[a, b] \subset R_+$, with respect to Lebesgue measure on the m -dimensional space R^m . Let $V(s, t, \mathbf{y})$ be a measurable R^m -valued function on $[a, b] \times [a, b] \times R^m$ and f an $\mathbb{L}^2([a, b], R^m)$ function. The following equations are called *nonlinear Volterra equations of the second kind*

$$\mathbf{y}(s) = f(s) + \int_a^s V(s, t, \mathbf{y}(t)) dt, \quad a \leq s \leq b. \quad (3.2.71)$$

Basically, integral equations (3.2.69) and (3.2.70) are *nonlinear Volterra equations of the second kind*. We will put on $(b_1(t, \mathbf{x}, q), \dots, b_r(t, \mathbf{x}, q))' = A(t, \mathbf{x}, q)' \mathbf{x}$ the Lipschitz condition. But we will deal with two cases for $g(t, \mathbf{x}, q)$: **A**) $g(t, \mathbf{x}, q)$ satisfies the Lipschitz condition and **B**) $g(t, \mathbf{x}, q)$ does not satisfy the Lipschitz condition.

At first, we discuss the case **A** that $g(t, \mathbf{x}, q)$ satisfies the Lipschitz condition. we need to introduce the standard knowledge about the *nonlinear Volterra equations of the second kind*. We put the following conditions on $V(s, t, \mathbf{y})$:

- (a) There exists a nonnegative $\mathbb{L}^2([a, b] \times [a, b], R)$ function $V_0(s, t)$, such that for any $a \leq t < s \leq b$ and $\mathbf{y}_1, \mathbf{y}_2 \in R^m$,

$$\|V(s, t, \mathbf{y}_1) - V(s, t, \mathbf{y}_2)\| \leq V_0(s, t) \|\mathbf{y}_1 - \mathbf{y}_2\|. \quad (3.2.72)$$

- (b) For each $s \in [a, b]$, we have

$$\left\| \int_a^s V(s, t, f(t)) dt \right\| \leq n(s), \quad (3.2.73)$$

where $n(s)$ is a nonnegative $\mathbb{L}^2([a, b], R)$ function.

- (c) $V_0(s, t)$ and $n(s)$ satisfy that

$$\int_a^b \int_a^s V_0^2(s, t) dt ds < \infty, \quad (3.2.74)$$

$$\int_a^s n(t)^2 dt < \infty, \quad a \leq s \leq b. \quad (3.2.75)$$

Note that the norm $\|\cdot\|$ on R^m can be either $\|\mathbf{y}\| = \sqrt{y_1^2 + \cdots + y_m^2}$ or $\|\mathbf{y}\| = |y_1| + \cdots + |y_m|$ for $\mathbf{y} = (y_1, \dots, y_m)' \in R^m$.

Proposition 3.2.8. *Assume that f is an $\mathbb{L}^2([a, b], R^m)$ function and V a measurable R^m -valued function on $[a, b] \times [a, b] \times R^m$ satisfying the conditions (a)-(c) above. Then the nonlinear Volterra equations of the second kind (3.2.71) have one and only one $\mathbb{L}^2([a, b], R^m)$ solution.*

Note that we can refer to [TR57] for the case $m = 1$ of Proposition 3.2.8, but we can use the same method as that for the case $m = 1$ to prove the case $m > 1$.

Lemma 3.2.9. *Assume that $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}$ is a $r \times r$ matrix-valued function on $[0, \infty) \times K \times R$ which is bounded on $[0, T] \times K \times R$ for any $T > 0$. Assume that $g(t, \mathbf{x}, q)$ satisfy (3.2.1). Assume also that $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are bounded on $[0, T] \times K \times R$ for any $T > 0$ and that $\varphi, \psi, \frac{\partial \varphi}{\partial \mathbf{x}}, \frac{\partial \psi}{\partial \mathbf{x}}$ are continuous. Define $\mathbf{b}(t, \mathbf{y})$ by (3.2.49), (3.2.50) and (3.2.51). Assume that for any $T > 0$, there exists a nonnegative $\mathbb{L}^2([0, T], R)$ function $C_T(t)$, such that for any $0 \leq t \leq T$ and $\mathbf{y}_1 = (\mathbf{x}_1, q_1), \mathbf{y}_2 = (\mathbf{x}_2, q_2) \in K \times R$,*

$$\|\mathbf{b}(t, \mathbf{y}_1) - \mathbf{b}(t, \mathbf{y}_2)\| \leq C_T(t) \|\mathbf{y}_1 - \mathbf{y}_2\|. \quad (3.2.76)$$

Define the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $\bar{C}(K \times R)$ by (3.2.18). Then the $C_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$ has at most one solution.

Proof. By Lemma 3.2.6, we just need to prove that the deterministic integral

equation (3.2.68) has at most one solution. Define

$$V(s, t, \mathbf{y}) = \begin{cases} \mathbf{b}(t, \mathbf{y})' & \text{if } 0 \leq t \leq s < T, \\ 0, & \text{if } s, t \in [0, T] \text{ and } t > s. \end{cases} \quad (3.2.77)$$

Note that any solution of the integral equation (3.2.68) is continuous, and by condition (3.2.76) and Proposition 3.2.8 that *nonlinear Volterra equations of the second kind* (3.2.71) has just one continuous solution. It follows that (3.2.68) has only one continuous solution. $\square$

Remark 3.2.10. *If $A(t, \mathbf{y})$ satisfies the Lipschitz condition, i.e., for any $T > 0$, there exists a nonnegative $\mathbb{L}^2([0, T], R)$ function $C_T(t)$, such that for any $0 \leq t \leq T$ and $\mathbf{y}_1 = (\mathbf{x}_1, q_1), \mathbf{y}_2 = (\mathbf{x}_2, q_2) \in K \times R$,*

$$\|A(t, \mathbf{y}_1) - A(t, \mathbf{y}_2)\| \leq C_T(t) \|\mathbf{y}_1 - \mathbf{y}_2\|, \quad (3.2.78)$$

then $\hat{\mathbf{b}}(t, \mathbf{y}) = \mathbf{x}A(t, \mathbf{y})$ satisfies the Lipschitz condition under the condition that $A(t, \mathbf{y})$ is bounded on $[0, T] \times K \times R$ for any $T > 0$. But even if $\varphi(t, \mathbf{y})$ and $\psi(t, \mathbf{y})$ are assumed to be bounded on $[0, T] \times K \times R$ for any $T > 0$ and satisfy the Lipschitz condition, we can not claim that $b_{r+1}(t, \mathbf{y}) = g(t, \mathbf{y})$ satisfies the Lipschitz condition. This is the reason that we need to expand the standard theory of nonlinear Volterra equations of the second kind for $g(t, \mathbf{y})$ that does not satisfy the Lipschitz condition.

Next, we discuss a special type of *nonlinear Volterra equations of the second kind*. Let $m = n + 1$. Let $V(s, t, \mathbf{y})$, $f(s)$ and the equation (3.2.71) be defined as before. We give different notations for the first n components of $\mathbf{y}$, $V(s, t, \mathbf{y})$ and $f(s)$. For any vector $\mathbf{y} = (y_1, \dots, y_n, y_{n+1})' \in R^{n+1}$, we let $\mathbf{x} = (y_1, \dots, y_n)'$. Similarly, we let $\hat{V}(s, t, \mathbf{y}) = (V_1(s, t, \mathbf{y}), \dots, V_n(s, t, \mathbf{y}))'$ and $\hat{f}(s) = (f_1(s), \dots, f_n(s))'$. Then the

equation (3.2.71) can be represented by equations as follows: for $a \leq s \leq b$,

$$\mathbf{x}(s) = \hat{f}(s) + \int_a^s \hat{V}(s, t, \mathbf{x}(t), y_{n+1}(t)) dt \quad (3.2.79)$$

$$y_{n+1}(s) = f_{n+1}(s) + \int_a^s V_{n+1}(s, t, \mathbf{x}(t), y_{n+1}(t)) dt \quad (3.2.80)$$

Now we give the semi-Lipschitz conditions on $V(s, t, \mathbf{y})$ which are sufficient for the uniqueness of the nonlinear Volterra equations (3.2.79) and (3.2.80):

(I) If $(\tilde{\mathbf{x}}(t)', \tilde{y}_{m+1}(t))'$ is any solution of (3.2.79) and (3.2.80), then $\tilde{y}_{m+1}(t)$ is the unique solution of (3.2.80) if we plug into (3.2.80) $\mathbf{x}(t) = \tilde{\mathbf{x}}(t)$. That is to say, there is a dependence, which is based on (3.2.80), between the first n components and the last component of any solution. Then we can actually write $\tilde{y}_{m+1}(t) = \xi_{\tilde{\mathbf{x}}}(t)$.

(II) $\hat{V}(s, t, \mathbf{y})$ satisfies the Lipschitz condition, i.e. there exists a nonnegative $\mathbb{L}^2([a, b] \times [a, b], R)$ function $V_0(s, t)$, such that for any $a \leq t < s \leq b$ and $\mathbf{y}_1 = (\mathbf{x}'_1, y_{n+1}^{(1)})', \mathbf{y}_2 = (\mathbf{x}'_2, y_{n+1}^{(2)})' \in R^{n+1}$,

$$\|\hat{V}(s, t, \mathbf{y}_1) - \hat{V}(s, t, \mathbf{y}_2)\| \leq V_0(s, t) \sqrt{\|\mathbf{x}_1 - \mathbf{x}_2\|^2 + (y_{n+1}^{(1)} - y_{n+1}^{(2)})^2}. \quad (3.2.81)$$

(III) Based on (I), if $(\mathbf{x}(t)', \xi_{\mathbf{x}}(t))'$ and $(\tilde{\mathbf{x}}(t)', \xi_{\tilde{\mathbf{x}}}(t))'$ are two solutions of (3.2.79) and (3.2.80), then there exists a nonnegative measurable function $B(s)$ on $[a, b]$ such that

$$\int_a^s (\xi_{\mathbf{x}}(t) - \xi_{\tilde{\mathbf{x}}}(t))^2 dt \leq B(s) \int_a^s \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt \text{ for } a \leq s \leq b, \quad (3.2.82)$$

and

$$\int_a^b \left[\int_a^s V_0(s, t)^2 dt (1 + B(s)) \right] ds < \infty. \quad (3.2.83)$$

Proposition 3.2.11. *Assume that V is a measurable R^{n+1} -valued function on $[a, b] \times [a, b] \times R^{n+1}$ satisfying the semi-Lipschitz conditions (I)-(III) above. Then the nonlinear Volterra equations of the second kind (3.2.79) and (3.2.80) have at most one $\mathbb{L}^2([a, b], R^{n+1})$ solution.*

Proof. Assume that $V_0(s, t)$ is a nonnegative $\mathbb{L}^2([a, b] \times [a, b], R)$ function which satisfies (3.2.81). Assume that $(\mathbf{x}(t)', \xi_{\mathbf{x}}(t))'$ and $(\tilde{\mathbf{x}}(t)', \xi_{\tilde{\mathbf{x}}}(t))'$ are two solutions of (3.2.79) and (3.2.80) and $B(s)$ is a nonnegative measurable function on $[a, b]$ which satisfies (3.2.82) and (3.2.83). Let $\mathcal{A}(s)^2 = \int_a^s V_0(s, t)^2 dt (1 + B(s))$ for $a \leq s \leq b$ and let $h^2 = \mathcal{A}(b)^2$. Then by (3.2.81), Cauchy-Schwartz inequality and (3.2.83), we have for $a \leq s \leq b$,

$$\begin{aligned} \|\mathbf{x}(s) - \tilde{\mathbf{x}}(s)\|^2 &\leq \left\{ \int_a^s \|\hat{V}(s, t, \mathbf{x}(t), \xi_{\mathbf{x}}(t)) - \hat{V}(s, t, \tilde{\mathbf{x}}(t), \xi_{\tilde{\mathbf{x}}}(t))\| dt \right\}^2 \\ &\leq \int_a^s V_0^2(s, t) dt \int_a^s [\|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 + (\xi_{\mathbf{x}}(t) - \xi_{\tilde{\mathbf{x}}}(t))^2] dt \\ &\leq \int_a^s V_0^2(s, t) dt \left[\int_a^s \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt + B(s) \int_a^s \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt \right] \\ &= \mathcal{A}(s)^2 \int_a^s \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt. \end{aligned} \tag{3.2.84}$$

Put $k^2 = \int_a^b \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt$. By successive substitutions into (3.2.84), we can get

$$\int_a^s \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt \leq k^2 \frac{1}{l!} \left[\int_a^s \mathcal{A}(t)^2 dt \right]^l \leq k^2 \frac{h^{2l}}{l!} \tag{3.2.85}$$

for any integer $l \geq 1$ and $a \leq s \leq b$. Let $l \rightarrow \infty$, we conclude that $\mathbf{x}(s) \equiv \tilde{\mathbf{x}}(s)$ for $s \in [a, b]$ in the sense of $\mathbb{L}^2([a, b], R^n)$. By condition (I), we also get that $\xi_{\mathbf{x}}(s) \equiv \xi_{\tilde{\mathbf{x}}}(s)$ for $s \in [a, b]$ in the sense of $\mathbb{L}^2([a, b], R)$. $\square$

Lemma 3.2.12. *Assume that $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}$ is a $r \times r$ matrix-valued function on $[0, \infty) \times K \times R$ which is bounded on $[0, T] \times K \times R$ for any $T > 0$.*

Assume that $g(t, \mathbf{x}, q)$ satisfy (3.2.1). Assume also that $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ are bounded on $[0, T] \times K \times R$ for any $T > 0$ and that $\varphi, \psi, \frac{\partial \varphi}{\partial \mathbf{x}}, \frac{\partial \psi}{\partial \mathbf{x}}$ are continuous. Define $\hat{\mathbf{b}}(t, \mathbf{y})$ by (3.2.49), (3.2.50) and (3.2.51). Assume that for any $T > 0$, there exists a nonnegative $\mathbb{L}^2([0, T], R)$ function $C_T(t)$, such that for any $0 \leq t \leq T$ and $\mathbf{y}_1 = (\mathbf{x}_1, q_1), \mathbf{y}_2 = (\mathbf{x}_2, q_2) \in K \times R$,

$$\|\hat{\mathbf{b}}(t, \mathbf{y}_1) - \hat{\mathbf{b}}(t, \mathbf{y}_2)\| \leq C_T(t) \|\mathbf{y}_1 - \mathbf{y}_2\|. \quad (3.2.86)$$

Assume that g satisfies that for fixed $T > 0$ and any given continuous $\mathbf{x}(t)$ on $[0, T]$, (3.2.70) determines a unique solution $q_{\mathbf{x}}(t)$ on $[0, T]$ and for any two solutions $(\mathbf{x}(t), q_{\mathbf{x}}(t))$ and $(\tilde{\mathbf{x}}(t), q_{\tilde{\mathbf{x}}}(t))$ of (3.2.69) and (3.2.70) on $[0, T]$, there exists a nonnegative measurable function $B(s)$ on $[0, T]$, such that $q_{\mathbf{x}}(t)$ and $q_{\tilde{\mathbf{x}}}(t)$ satisfy

$$\int_0^s (q_{\mathbf{x}}(t) - q_{\tilde{\mathbf{x}}}(t))^2 dt \leq B(s) \int_0^s \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt \text{ for } 0 \leq s \leq T \quad (3.2.87)$$

and

$$\int_0^T \left[\int_0^s C_T(t)^2 dt (1 + B(s)) \right] ds < \infty. \quad (3.2.88)$$

Define the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $\bar{C}(K \times R)$ by (3.2.18). Then the $C_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$ has at most one solution.

Proof. This proof is similar to that for Lemma 3.2.9. By Lemma 3.2.6, we just need to prove that the deterministic integral equations (3.2.69) and (3.2.70) have at most one solution. Define

$$\hat{V}(s, t, \mathbf{y}) = \begin{cases} \hat{\mathbf{b}}(t, \mathbf{y})' & \text{if } 0 \leq t \leq s < T, \\ 0, & \text{if } s, t \in [0, T] \text{ and } t > s; \end{cases} \quad (3.2.89)$$

and $V_{r+1}(s, t, \mathbf{y}) = g(t, \mathbf{y})$, if $0 \leq t \leq s < T$, and 0, if $s, t \in [0, T]$ and $t > s$. Note that any solution of the integral equations (3.2.69) and (3.2.70) is continuous. We

can check easily that the semi-Lipschitz conditions **(I)-(III)** for $V(s, t, \mathbf{y})$ required by Proposition 3.2.11 are satisfied. Then the *nonlinear Volterra equations of the second kind* (3.2.69) and (3.2.70) have at most one continuous solution. $\square$

Remark 3.2.13. *The requirement on g that for fixed $T > 0$ and any given continuous $\mathbf{x}(t)$ on $[0, T]$, (3.2.70) determines a unique solution $q_{\mathbf{x}}(t)$ on $[0, T]$ looks like a awkward one, but it can be easily satisfied for certain kind of functions. For example, g satisfies the requirement if we assume that the variable q in $\varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ is a dummy variable.*

Theorem 3.2.14. *Assume that $\{A_N(t, \mathbf{x}, q) = (a_{N,i,j}(t, \mathbf{x}, q))_{r \times r}\}$ be a sequence of $r \times r$ matrix-valued functions on $[0, \infty) \times K_N \times R$ and $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{r \times r}$ a $r \times r$ matrix-valued function on $[0, \infty) \times K \times R$. $\{A_N(t, \mathbf{x}, q)\}$ and $A(t, \mathbf{x}, q)$ satisfy the conditions at the beginning of subsection 3.2.1. Assume that $g(t, \mathbf{x}, q)$ and $\{g_N(t, \mathbf{x}, q)\}$ satisfy (3.2.1) and (3.2.2). Assume that $\varphi, \psi, \frac{\partial \varphi}{\partial \mathbf{x}}, \frac{\partial \psi}{\partial \mathbf{x}}$ are continuous. Assume that (3.2.19) and (3.2.20) hold for $\{\varphi_N\}, \varphi$ and $\{\psi_N\}, \psi$. Assume that $A(t, \mathbf{x}, q), \varphi(t, \mathbf{x}, q)$ and $\psi(t, \mathbf{x}, q)$ satisfy (3.2.44), (3.2.45), and (3.2.46) and are bounded on $[0, T] \times K \times R$ for any $T > 0$. Define $\mathbf{b}(t, \mathbf{y})$ and $\hat{\mathbf{b}}(t, \mathbf{y})$ by (3.2.49), (3.2.50) and (3.2.51). Assume either that*

i) $\mathbf{b}(t, \mathbf{y})$ satisfies the Lipschitz condition (3.2.76);

or that

ii) $\hat{\mathbf{b}}(t, \mathbf{y})$ satisfies the Lipschitz condition (3.2.86) and that g satisfies that for fixed $T > 0$ and any given continuous $\mathbf{x}(t)$ on $[0, T]$, (3.2.70) determines a unique solution $q_{\mathbf{x}}(t)$ on $[0, T]$ and for any two solutions $(\mathbf{x}(t), q_{\mathbf{x}}(t))$ and $(\tilde{\mathbf{x}}(t), q_{\tilde{\mathbf{x}}}(t))$ of (3.2.69) and (3.2.70) on $[0, T]$, there exists a nonnegative measurable function $B(s)$ on $[0, T]$, which satisfies (3.2.88), such that $q_{\mathbf{x}}(t)$ and $q_{\tilde{\mathbf{x}}}(t)$ satisfy (3.2.87).

For $k \geq 0$, $(\mathbf{x}, q) \in K_N \times R$, define $P_{N,k}[\mathbf{x}, q]$ by (3.2.5) and $\{(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))\}_{k=0}$ by the procedure in subsection 3.2.1. Let $\mathbf{X}^N(t) = \frac{\mathbf{n}^N(\lfloor Nt \rfloor)}{N}$, $q^N(t) = \tilde{q}^N(\frac{\lfloor Nt \rfloor}{N})$ and $\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t))$ and for $k \geq 0$, define the transition operators $S_{N,k}$ on $\bar{C}(K_N \times R)$ by (3.2.8) and the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C_c^1(K \times R)$ by (3.2.18). Suppose that for each $f \in C_c^2(K \times R)$ and $T > 0$ there exist measurable sets $\{F_N\} \subset R$ satisfying (3.2.43) and (3.2.27). Suppose that $P(\mathbf{Y}^N(0))^{-1} \Rightarrow \mu$ for some $\mu \in \mathcal{P}(K \times R)$, then there exists on some probability space $(\Omega, \mathcal{F}, P)$ a stochastic process $\mathbf{Y}$ satisfying $P(\mathbf{Y}(0))^{-1} = \mu$, which is the unique solution of the $C_{K \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K \times R)$ and for P -a.s. $\omega \in \Omega$, $\mathbf{Y}(t, \omega)'$ is the unique solution of the integral equation (3.2.68), such that $\mathbf{Y}^N \Rightarrow \mathbf{Y}$.

Proof. The conclusion follows by Corollary 3.2.4, Lemma 3.2.5, Lemma 3.2.6 and Lemma 3.2.9 or Lemma 3.2.12. $\square$

3.3 Case study of the simplified example

3.3.1 Conditions and verifications of the simplified example

In this section, we give a complete study of the simplified example of interacting agent feedback finance models given in subsection 3.1.2. We know that there are $r = 3$ types of agents, i.e., fundamentalist, optimist, pessimist. Recall that $\mathbf{n}^N(k) = (n_1^N(k), n_2^N(k), n_3^N(k))$ the vector of the number of three types of agents at time period $k \geq 0$, where $n_1^N(k)$, $n_2^N(k)$ and $n_3^N(k)$ represent the number of fundamentalists, optimists and pessimists respectively. For $k \geq 0$, let $\tilde{q}^N(\frac{k}{N}) = \log S_k$ where in our

notation we omit the N dependence of $\log S_k$ (see (3.1.7)). Let $\mathbf{X}^N(t) = \frac{\mathbf{n}^N([Nt])}{N}$, $q^N(t) = \tilde{q}^N(\frac{[Nt]}{N}) = \log S_{[Nt]}$ and $\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t))$ for $t \geq 0$.

Assume that $\alpha_i^N(t)$, $\beta_i^N(t)$, and $\delta_i^N(t)$ for $1 \leq i \leq 3$ and $\log F^N(t)$ are real valued functions defined on $[0, \infty)$ which are constants on $t \in [\frac{k}{N}, \frac{k+1}{N})$ for any $k \geq 0$. Assume also that for any $\mathbf{x} \in K_N^3$ and $t \geq 0$, $x_1\alpha_1^N(t) + x_2\alpha_2^N(t)(1 + \beta_2^N(t)) + x_3\alpha_3^N(t)(1 + \beta_3^N(t)) \neq 0$. Then we can define a simple version of g_N on $[0, \infty) \times K_N^3 \times R$ similar to that one defined in (3.1.8):

$$g_N(t, \mathbf{x}, q) = \frac{x_1\alpha_1^N(t)\beta_1^N(t)q}{x_1\alpha_1^N(t) + x_2\alpha_2^N(t)(1 + \beta_2^N(t)) + x_3\alpha_3^N(t)(1 + \beta_3^N(t))} + \frac{x_1\delta_1^N(t) + x_2\delta_2^N(t) + x_3\delta_3^N(t) - x_1\alpha_1^N(t)\beta_1^N(t)\log F^N(t)}{x_1\alpha_1^N(t) + x_2\alpha_2^N(t)(1 + \beta_2^N(t)) + x_3\alpha_3^N(t)(1 + \beta_3^N(t))}. \quad (3.3.1)$$

Then (3.2.3) and (3.2.4) follows by (3.1.9). We also define real valued functions φ_N , ψ_N on $[0, \infty) \times K_N^3$ as follows:

$$\varphi_N(t, \mathbf{x}) = \frac{x_1\alpha_1^N(t)\beta_1^N(t)}{x_1\alpha_1^N(t) + x_2\alpha_2^N(t)(1 + \beta_2^N(t)) + x_3\alpha_3^N(t)(1 + \beta_3^N(t))}, \quad (3.3.2)$$

and

$$\psi_N(t, \mathbf{x}) = \frac{x_1\delta_1^N(t) + x_2\delta_2^N(t) + x_3\delta_3^N(t) - x_1\alpha_1^N(t)\beta_1^N(t)\log F^N(t)}{x_1\alpha_1^N(t) + x_2\alpha_2^N(t)(1 + \beta_2^N(t)) + x_3\alpha_3^N(t)(1 + \beta_3^N(t))}. \quad (3.3.3)$$

Then g_N , φ_N and ψ_N satisfy the relation (3.2.2). Recall that $A_N(t, \mathbf{x}, q) = (a_{N,i,j}(t, \mathbf{x}, q))_{3 \times 3}$ is a 3×3 matrix-valued function on $[0, \infty) \times K_N^3 \times R$, which satisfies the conditions in subsection 3.1.2 and recall the definition of $P_{N,k}[\mathbf{x}, q]$ in (3.1.10). Then we can define $\{(\mathbf{n}^N(k), \tilde{q}^N(\frac{k}{N}))\}_{k=0}^\infty$ by the procedures in subsection 3.1.2 or 3.2.1.

In the next three paragraphs, we make preparations to describe the limit process $\{\mathbf{y}(t) = (\mathbf{x}(t), q(t)), 0 \leq t < \infty\}$ of $\{\mathbf{Y}^N(t) = (\mathbf{X}^N(t), q^N(t)), 0 \leq t < \infty\}$. Define $K^3 = \{\mathbf{x} : \mathbf{x} \in (R_+)^3, \sum_{i=1}^3 x_i = 1\}$. Assume that $\alpha_i(t)$, $\beta_i(t)$, and $\delta_i(t)$ for $1 \leq i \leq 3$

and $F(t)$ are continuous real valued functions defined on $[0, \infty)$, where $F(t) > 0$ for any $t \geq 0$. Assume also that for any $\mathbf{x} \in K^3$ and $t \geq 0$, $x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t)) \neq 0$. Then we define a function g on $[0, \infty) \times K^3 \times R$ as follows:

$$g(t, \mathbf{x}, q) = \frac{x_1\alpha_1(t)\beta_1(t)q}{x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t))} + \frac{x_1\delta_1(t) + x_2\delta_2(t) + x_3\delta_3(t) - x_1\alpha_1(t)\beta_1(t) \log F(t)}{x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t))}. \quad (3.3.4)$$

We also define real valued functions φ, ψ on $[0, \infty) \times K^3$ as follows:

$$\varphi(t, \mathbf{x}) = \frac{x_1\alpha_1(t)\beta_1(t)}{x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t))}, \quad (3.3.5)$$

$$\psi(t, \mathbf{x}) = \frac{x_1\delta_1(t) + x_2\delta_2(t) + x_3\delta_3(t) - x_1\alpha_1(t)\beta_1(t) \log F(t)}{x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t))}. \quad (3.3.6)$$

Then g, φ and ψ satisfy the relation (3.2.1).

We make further assumptions on $\alpha_i(t)$, $\beta_i(t)$, and $\delta_i(t)$ for $1 \leq i \leq 3$ and $F(t)$ such that the conditions required by Theorem 3.2.14 can be verified for φ and ψ . We assume that $\beta_i(t)$ is bounded on $[0, \infty)$ for $1 \leq i \leq 3$. Since the numerators and denominators of φ in (3.3.5) and ψ in (3.3.6) are linear in α_i and δ_i for $1 \leq i \leq r$, we assume that $\alpha_i(t)$ is also bounded on $[0, \infty)$ for $1 \leq i \leq 3$, which are very restrictive conditions on α_i for $1 \leq i \leq 3$, but we assume the not so restrictive conditions on δ_i for $1 \leq i \leq 3$ and $\log F$. We assume that there exist constants $C > 0$ and $\lambda > 0$ such that for $t \in [0, \infty)$,

$$|\delta_i(t)| \leq Ce^{\lambda t} \text{ for } 1 \leq i \leq 3 \text{ and } |\log F(t)| \leq Ce^{\lambda t}. \quad (3.3.7)$$

It follows that (3.2.45) holds for φ and (3.2.46) holds for ψ . Note that (3.2.45) and (3.2.46) are conditions required by Corollary 3.2.4 or Theorem 3.2.14. As a by product, it also follows that $\varphi(t, \mathbf{x})$ and $\psi(t, \mathbf{x})$ are bounded for $(t, \mathbf{x}, q) \in [0, T] \times K^3 \times R$ for any $T > 0$, viewing q as a dummy variable of $\varphi(t, \mathbf{x})$ and $\psi(t, \mathbf{x})$.

Define the matrix-valued càdlàg function $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{3 \times 3}$ on $[0, \infty) \times K^3 \times R$, which satisfies the conditions specified at the beginning of subsection 3.2.1. Now that $K^3 \times R$ is the state space for the limit process $\{\mathbf{y}(t), 0 \leq t < \infty\}$, we denote by $\bar{C}(K^3 \times R)$ the set of bounded continuous functions on $K^3 \times R$, and define $C_c(K^3 \times R)$, $C_c^1(K^3 \times R)$, $C_c^2(K^3 \times R)$ respectively in same way as we do in (3.2.15), (3.2.16), (3.2.17). We define the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $C_c^1(K^3 \times R)$: for each $f \in C_c^1(K^3 \times R)$, and $s \geq 0$,

$$G_A(s)f(\mathbf{x}, q) = \mathbf{x}A(s, \mathbf{x}, q)\frac{\partial f'}{\partial \mathbf{x}} + g(s, \mathbf{x}, q)\frac{\partial f}{\partial q}, \quad (\mathbf{x}, q) \in K^3 \times R. \quad (3.3.8)$$

$\mathcal{D}(G_A) = C_c^1(K^3 \times R)$ is the common domain of the generator $\{G_A(s), 0 \leq s < \infty\}$, and $D = C_c^2(K^3 \times R)$, is a subalgebra contained in $\mathcal{D}(G_A)$. Assume that $A(t, \mathbf{x}, q)$ is bounded on $[0, T] \times K^3 \times R$ for any $T > 0$ and for any compact subset $\hat{K}$ of R , there exist $C > 0$ and $\lambda > 0$, such that

$$\sup_{(\mathbf{x}, q) \in K^3 \times \hat{K}} \|A(t, \mathbf{x}, q)\| \leq Ce^{\lambda t}. \quad (3.3.9)$$

Then (3.2.44) holds for $A(t, \mathbf{x}, q)$. Next, we define for $\mathbf{y} = (x_1, x_2, x_3, q) \in K^3 \times R$

$$b_i(t, \mathbf{y}) = \sum_{j=1}^3 x_j a_{ji}(t, \mathbf{y}), \quad 1 \leq i \leq 3, \quad (3.3.10)$$

$$b_4(t, \mathbf{y}) = g(t, \mathbf{y}), \quad (3.3.11)$$

$$\hat{\mathbf{b}}(t, \mathbf{y}) = (b_1(t, \mathbf{y}), b_2(t, \mathbf{y}), b_3(t, \mathbf{y})) \text{ and } \mathbf{b}(t, \mathbf{y}) = (\hat{\mathbf{b}}(t, \mathbf{y}), b_4(t, \mathbf{y})). \quad (3.3.12)$$

Now it is clear that any candidate for the limit process $\{\mathbf{y}(t), 0 \leq t < \infty\}$ is related to a solution of the following *nonlinear Volterra equations of the second kind*:

$$\mathbf{z}(t) = \mathbf{y}(0)' + \int_0^t \mathbf{b}(s, \mathbf{z}(s))' ds, \quad (3.3.13)$$

which is equivalent to (3.2.69) and (3.2.70) for $r = 3$. It is clear that (3.2.70) is equivalent to

$$\frac{dq(t)}{dt} = g(t, \mathbf{x}(t), q(t)) \quad (3.3.14)$$

with initial condition $q(0) = y_4(0)$. By the form of g in (3.3.4), fix $T > 0$, we know that for any given K^3 valued function $\mathbf{x}(t)$ on $[0, T]$, there exists a unique solution $q_{\mathbf{x}}(t)$ of (3.3.14) on $[0, T]$.

We assume that $A(t, \mathbf{y})$ satisfies the Lipschitz condition, i.e., for any $T > 0$, there exists a nonnegative $\mathbb{L}^2([0, T], R)$ function $C_T(t)$, such that for any $0 \leq t \leq T$ and $\mathbf{y}_1 = (\mathbf{x}_1, q_1), \mathbf{y}_2 = (\mathbf{x}_2, q_2) \in K^3 \times R$,

$$\|A(t, \mathbf{y}_1) - A(t, \mathbf{y}_2)\| \leq C_T(t) \|\mathbf{y}_1 - \mathbf{y}_2\|, \quad (3.3.15)$$

then by Remark 3.2.10, $\hat{\mathbf{b}}(t, \mathbf{y}) = \mathbf{x}A(t, \mathbf{y})$ satisfies the Lipschitz condition stated in (3.2.86) since $A(t, \mathbf{y})$ is bounded on $[0, T] \times K^3 \times R$ for any $T > 0$.

We summarize the assumptions made on $\alpha_i(t)$, $\beta_i(t)$, and $\delta_i(t)$ for $1 \leq i \leq 3$, $F(t)$ and $A(t, \mathbf{x}, q)$ above as follows: $\alpha_i(t)$, $\beta_i(t)$, and $\delta_i(t)$ for $1 \leq i \leq 3$ and $F(t)$ are continuous real valued functions defined on $[0, \infty)$, where $F(t) > 0$ for any $t \geq 0$. For any $\mathbf{x} \in K^3$ and $t \geq 0$, $x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t)) \neq 0$. $\alpha_i(t)$ and $\beta_i(t)$ are bounded on $[0, \infty)$ for $1 \leq i \leq 3$. There exist constants $C > 0$ and $\lambda > 0$ such that for $t \in [0, \infty)$, (3.3.7) holds. The matrix-valued *càdlàg* function $A(t, \mathbf{x}, q) = (a_{i,j}(t, \mathbf{x}, q))_{3 \times 3}$ on $[0, \infty) \times K^3 \times R$ satisfies the conditions specified at the beginning of subsection 3.2.1 and is bounded on $[0, T] \times K^3 \times R$ for any $T > 0$. For any compact subset $\hat{K}$ of R , there exist $C > 0$ and $\lambda > 0$ such that (3.3.9) holds for $A(t, \mathbf{x}, q)$. $A(t, \mathbf{y})$ also satisfies the Lipschitz condition (3.3.15).

We are going to prove in the next lemma that (3.2.87) and (3.2.88) are satisfied

and it follows the uniqueness of the $C_{K^3 \times R}[0, \infty)$ -martingale problem for (G_A, μ) .

Lemma 3.3.1. *We make the assumptions on $\alpha_i(t)$, $\beta_i(t)$, and $\delta_i(t)$ for $1 \leq i \leq 3$, $F(t)$ and $A(t, \mathbf{x}, q)$ summarized above. Define φ, ψ on $[0, \infty) \times K^3$ by (3.3.5) and (3.3.6). Define g on $[0, \infty) \times K^3 \times R$ by (3.3.4). Define $b_i(t, \mathbf{y})$ for $1 \leq i \leq 4$ by (3.3.10) and (3.3.11). Define $\hat{\mathbf{b}}(t, \mathbf{y})$ and $\mathbf{b}(t, \mathbf{y})$ by (3.3.12). For fixed $T > 0$ and any given continuous K^3 -valued function $\mathbf{x}(t)$ on $[0, T]$, we denote the unique solution of (3.3.14) by $q_{\mathbf{x}}(t)$. Then, for fixed $T > 0$ and any two solutions $(\mathbf{x}(t), q_{\mathbf{x}}(t))$, $(\tilde{\mathbf{x}}(t), q_{\tilde{\mathbf{x}}}(t))$ of (3.3.13) on $[0, T]$, there exists $M > 0$, such that $q_{\mathbf{x}}(t)$ and $q_{\tilde{\mathbf{x}}}(t)$ satisfy*

$$\int_0^s (q_{\mathbf{x}}(t) - q_{\tilde{\mathbf{x}}}(t))^2 dt \leq M \int_0^s \|\mathbf{x}(t) - \tilde{\mathbf{x}}(t)\|^2 dt \text{ for } 0 \leq s \leq T. \quad (3.3.16)$$

Define the time-dependent generator $\{G_A(s), 0 \leq s < \infty\}$ on $\bar{C}(K^3 \times R)$ by (3.3.8). Then the $C_{K^3 \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K^3 \times R)$ has at most one solution.

Proof. For fixed $T > 0$ and any given continuous K^3 -valued function $\mathbf{x}(t)$ on $[0, T]$, we define $P_{\mathbf{x}}(t)$ and $Q_{\mathbf{x}}(t)$ as follows:

$$P_{\mathbf{x}}(t) = \frac{x_1(t)\alpha_1(t)\beta_1(t)}{x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)(1 + \beta_2(t)) + x_3(t)\alpha_3(t)(1 + \beta_3(t))}, \quad (3.3.17)$$

$$Q_{\mathbf{x}}(t) = \frac{x_1(t)\delta_1(t) + x_2(t)\delta_2(t) + x_3(t)\delta_3(t) - x_1(t)\alpha_1(t)\beta_1(t) \log F(t)}{x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)(1 + \beta_2(t)) + x_3(t)\alpha_3(t)(1 + \beta_3(t))}. \quad (3.3.18)$$

Then (3.3.14) becomes

$$\frac{dq(t)}{dt} = P_{\mathbf{x}}(t)q + Q_{\mathbf{x}}(t). \quad (3.3.19)$$

The unique solution of (3.3.19) with initial condition $q(0) = q_{\mathbf{x}}(0)$ is given by

$$q_{\mathbf{x}}(t) = e^{\int_0^t P_{\mathbf{x}}(u) du} \left[\int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v) dv} du + q_{\mathbf{x}}(0) \right]. \quad (3.3.20)$$

Let $h_{\mathbf{x}}(t) = x_1(t)\alpha_1(t) + x_2(t)\alpha_2(t)(1 + \beta_2(t)) + x_3(t)\alpha_3(t)(1 + \beta_3(t))$, the denominator of $P_{\mathbf{x}}(t)$. Since $\alpha_i(t), \beta_i(t)$ ($1 \leq i \leq 3$) are continuous and for any $\mathbf{x} \in K^3$ and $t \geq 0$, $x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t)) \neq 0$, it follows that

$$B_L \leq |h_{\mathbf{x}}(t)| \leq B_U, \text{ for all } t \in [0, T], \quad (3.3.21)$$

where $B_L, B_U > 0$ do not depend on $\mathbf{x}(t)$.

We assume that $(\mathbf{x}(t), q_{\mathbf{x}}(t))$ and $(\tilde{\mathbf{x}}(t), q_{\tilde{\mathbf{x}}}(t))$ are two solutions of (3.3.13) on $[0, T]$. It follows that $\mathbf{x}(0) = \tilde{\mathbf{x}}(0)$ and $q_{\mathbf{x}}(0) = q_{\tilde{\mathbf{x}}}(0)$. Then it is clear from (3.3.20) that

$$\begin{aligned} q_{\mathbf{x}}(t) - q_{\tilde{\mathbf{x}}}(t) &= q_{\mathbf{x}}(0) \left[e^{\int_0^t P_{\mathbf{x}}(u) du} - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u) du} \right] + \left[e^{\int_0^t P_{\mathbf{x}}(u) du} \int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v) dv} du \right. \\ &\quad \left. - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u) du} \int_0^t Q_{\tilde{\mathbf{x}}}(u) e^{-\int_0^u P_{\tilde{\mathbf{x}}}(v) dv} du \right]. \end{aligned} \quad (3.3.22)$$

Let $M_1 = \sup_{0 \leq t \leq T} e^{\int_0^t P_{\mathbf{x}}(u) du} \vee e^{\int_0^t P_{\tilde{\mathbf{x}}}(u) du} < \infty$, then for $0 \leq t \leq T$,

$$|e^{\int_0^t P_{\mathbf{x}}(u) du} - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u) du}| \leq M_1 \left| \int_0^t P_{\mathbf{x}}(u) - P_{\tilde{\mathbf{x}}}(u) du \right|. \quad (3.3.23)$$

Observe that for $0 \leq u \leq T$,

$$\begin{aligned} &|P_{\mathbf{x}}(u) - P_{\tilde{\mathbf{x}}}(u)| \\ &= \left| \frac{x_1(u)\alpha_1(u)\beta_1(u)}{h_{\mathbf{x}}(u)} - \frac{\tilde{x}_1(u)\alpha_1(u)\beta_1(u)}{h_{\tilde{\mathbf{x}}}(u)} \right| \\ &= \left| \frac{\alpha_1(u)\alpha_2(u)\beta_1(u)(1 + \beta_2(u))[x_1(u)\tilde{x}_2(u) - \tilde{x}_1(u)x_2(u)]}{h_{\mathbf{x}}(u)h_{\tilde{\mathbf{x}}}(u)} \right. \\ &\quad \left. + \frac{\alpha_1(u)\alpha_3(u)\beta_1(u)(1 + \beta_3(u))[x_1(u)\tilde{x}_3(u) - \tilde{x}_1(u)x_3(u)]}{h_{\mathbf{x}}(u)h_{\tilde{\mathbf{x}}}(u)} \right| \\ &= \left| \frac{\alpha_1(u)\alpha_2(u)\beta_1(u)(1 + \beta_2(u))[x_1(u)(\tilde{x}_2(u) - x_2(u)) + (x_1(u) - \tilde{x}_1(u))x_2(u)]}{h_{\mathbf{x}}(u)h_{\tilde{\mathbf{x}}}(u)} \right. \\ &\quad \left. + \frac{\alpha_1(u)\alpha_3(u)\beta_1(u)(1 + \beta_3(u))[x_1(u)(\tilde{x}_3(u) - x_3(u)) + (x_1(u) - \tilde{x}_1(u))x_3(u)]}{h_{\mathbf{x}}(u)h_{\tilde{\mathbf{x}}}(u)} \right| \end{aligned} \quad (3.3.24)$$

$$\leq M_2 \|\mathbf{x}(u) - \tilde{\mathbf{x}}(u)\|,$$

where M_2 depends on B_L, B_U in (3.3.21) and α_i and β_i ($1 \leq i \leq 3$). Therefore, for $0 \leq s \leq T$,

$$\begin{aligned} \int_0^s |e^{\int_0^t P_{\mathbf{x}}(u)du} - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du}|^2 dt &\leq \int_0^s M_1^2 \left[\int_0^t |P_{\mathbf{x}}(u) - P_{\tilde{\mathbf{x}}}(u)| du \right]^2 dt \\ &\leq M_1^2 M_2^2 \int_0^s t \int_0^t \|\mathbf{x}(u) - \tilde{\mathbf{x}}(u)\|^2 du dt \\ &\leq \frac{M_1^2 M_2^2 T^2}{2} \int_0^s \|\mathbf{x}(u) - \tilde{\mathbf{x}}(u)\|^2 du. \end{aligned} \quad (3.3.25)$$

Next we consider $\int_0^s [e^{\int_0^t P_{\mathbf{x}}(u)du} \int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\tilde{\mathbf{x}}}(u) e^{-\int_0^u P_{\tilde{\mathbf{x}}}(v)dv} du]^2 dt$. Notice that

$$\begin{aligned} &\left| e^{\int_0^t P_{\mathbf{x}}(u)du} \int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\tilde{\mathbf{x}}}(u) e^{-\int_0^u P_{\tilde{\mathbf{x}}}(v)dv} du \right| \\ &\leq \left| e^{\int_0^t P_{\mathbf{x}}(u)du} \int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du \right| \\ &\quad + \left| e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\tilde{\mathbf{x}}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du \right| \\ &\quad + \left| e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\tilde{\mathbf{x}}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\tilde{\mathbf{x}}}(u) e^{-\int_0^u P_{\tilde{\mathbf{x}}}(v)dv} du \right| \\ &= I_1(t) + I_2(t) + I_3(t), \end{aligned} \quad (3.3.26)$$

where $I_i(t)$ denotes the i -term on the right hand side of the inequality in (3.3.26). Since $P_{\mathbf{x}}(t)$ and $Q_{\mathbf{x}}(t)$ are continuous functions, there exists $M_3 > 0$, such that for any $0 \leq t \leq T$, $|\int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du| \leq M_3$. Then it follows by (3.3.25) that for $0 \leq s \leq T$,

$$\begin{aligned} \int_0^s I_1(t)^2 dt &\leq M_3^2 \int_0^s |e^{\int_0^t P_{\mathbf{x}}(u)du} - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du}|^2 dt \\ &\leq \frac{M_1^2 M_2^2 M_3^2 T^2}{2} \int_0^s \|\mathbf{x}(u) - \tilde{\mathbf{x}}(u)\|^2 du. \end{aligned} \quad (3.3.27)$$

As to $I_2(t)$, we have

$$\begin{aligned} I_2(t)^2 &= e^{2 \int_0^t P_{\bar{\mathbf{x}}}(u) du} \left| \int_0^t [Q_{\mathbf{x}}(u) - Q_{\bar{\mathbf{x}}}(u)] e^{-\int_0^u P_{\mathbf{x}}(v) dv} du \right|^2 \\ &\leq M_4^4 t \int_0^t [Q_{\mathbf{x}}(u) - Q_{\bar{\mathbf{x}}}(u)]^2 du, \end{aligned} \quad (3.3.28)$$

where $M_4 > 0$ satisfies that for any $0 \leq t \leq T$, $e^{\int_0^t P_{\bar{\mathbf{x}}}(u) du} \leq M_4$ and $e^{-\int_0^t P_{\mathbf{x}}(u) du} \leq M_4$.

Notice that for $0 \leq u \leq T$,

$$\begin{aligned} &|Q_{\mathbf{x}}(u) - Q_{\bar{\mathbf{x}}}(u)| \\ &\leq \left| \frac{[\alpha_1(u)\delta_2(u) - \alpha_2(u)\delta_1(u)(1 + \beta_2(u))] + \alpha_1(u)\alpha_2(u)\beta_1(u)(1 + \beta_2(u)) \log F(u)}{h_{\mathbf{x}}(u)h_{\bar{\mathbf{x}}}(u)} \right| \\ &\quad \times |[x_2(u)\tilde{x}_1(u) - x_1(u)\tilde{x}_2(u)]| \\ &\quad + \left| \frac{[x_3(u)\tilde{x}_1(u) - x_1(u)\tilde{x}_3(u)][\alpha_1(u)\delta_3(u) - \alpha_3(u)\delta_1(u)(1 + \beta_3(u))]}{h_{\mathbf{x}}(u)h_{\bar{\mathbf{x}}}(u)} \right| \\ &\quad + \left| \frac{[x_3(u)\tilde{x}_2(u) - x_2(u)\tilde{x}_3(u)][\alpha_2(u)\delta_3(u)(1 + \beta_2(u)) - \alpha_3(u)\delta_2(u)(1 + \beta_3(u))]}{h_{\mathbf{x}}(u)h_{\bar{\mathbf{x}}}(u)} \right| \\ &\leq M_5 \|\mathbf{x}(u) - \bar{\mathbf{x}}(u)\|, \end{aligned} \quad (3.3.29)$$

where M_5 depends on B_L , B_U in (3.3.21) and α_i , β_i , δ_i ($1 \leq i \leq 3$) and $\log F$. Then it follows by (3.3.28) and (3.3.29) that for $0 \leq s \leq T$,

$$\begin{aligned} \int_0^s I_2(t)^2 dt &\leq M_4^2 M_5^2 \int_0^s t \int_0^t \|\mathbf{x}(u) - \bar{\mathbf{x}}(u)\|^2 du dt \\ &\leq \frac{M_4^4 M_5^2 T^2}{2} \int_0^s \|\mathbf{x}(u) - \bar{\mathbf{x}}(u)\|^2 du. \end{aligned} \quad (3.3.30)$$

As to $I_3(t)$, we have

$$\begin{aligned} I_3(t)^2 &= e^{2 \int_0^t P_{\bar{\mathbf{x}}}(u) du} \left| \int_0^t Q_{\bar{\mathbf{x}}}(u) [e^{-\int_0^u P_{\mathbf{x}}(v) dv} - e^{-\int_0^u P_{\bar{\mathbf{x}}}(v) dv}] du \right|^2 \\ &\leq M_4^2 M_6^2 t \int_0^t [e^{-\int_0^u P_{\mathbf{x}}(v) dv} - e^{-\int_0^u P_{\bar{\mathbf{x}}}(v) dv}]^2 du, \end{aligned} \quad (3.3.31)$$

where $M_6 > 0$ satisfies that for any $0 \leq t \leq T$, $|Q_{\tilde{\mathbf{x}}}(t)| \leq M_6$. Similar to (3.3.25), we can prove that there exists $M_7 > 0$, for any $0 \leq t \leq T$, such that

$$\int_0^t [e^{-\int_0^u P_{\mathbf{x}}(v)dv} - e^{-\int_0^u P_{\tilde{\mathbf{x}}}(v)dv}]^2 du \leq M_7^2 \int_0^t \|\mathbf{x}(u) - \tilde{\mathbf{x}}(u)\|^2 du. \quad (3.3.32)$$

Then it follows that for $0 \leq s \leq T$,

$$\int_0^s I_3(t)^2 dt \leq \frac{M_4^2 M_6^2 M_7^2 T^2}{2} \int_0^s \|\mathbf{x}(u) - \tilde{\mathbf{x}}(u)\|^2 du. \quad (3.3.33)$$

Thus, by (3.3.26), (3.3.27), (3.3.30), (3.3.33), we get

$$\begin{aligned} & \int_0^s \left| e^{\int_0^t P_{\mathbf{x}}(u)du} \int_0^t Q_{\mathbf{x}}(u) e^{-\int_0^u P_{\mathbf{x}}(v)dv} du - e^{\int_0^t P_{\tilde{\mathbf{x}}}(u)du} \int_0^t Q_{\tilde{\mathbf{x}}}(u) e^{-\int_0^u P_{\tilde{\mathbf{x}}}(v)dv} du \right|^2 dt \\ & \leq \frac{3(M_1^2 M_2^2 M_3^2 T^2 + M_4^4 M_5^2 T^2 + M_4^2 M_6^2 M_7^2 T^2)}{2} \int_0^s \|\mathbf{x}(u) - \tilde{\mathbf{x}}(u)\|^2 du. \end{aligned} \quad (3.3.34)$$

By (3.3.22), (3.3.25) and (3.3.34), (3.3.16) is true for some $M > 0$.

The uniqueness of the $C_{K^3 \times R}[0, \infty)$ -martingale problem for (G_A, μ) follows by (3.3.16) and Lemma 3.2.12. $\square$

Next we give in the next corollary the conditions which can guarantee the weak convergence of $\{\mathbf{Y}^N(t), 0 \leq t < \infty\}$ to $\{\mathbf{y}(t), 0 \leq t < \infty\}$.

Corollary 3.3.2. *In addition to the assumptions in Lemma 3.3.1, we assume that $\{\varphi_N\}$, φ and $\{\psi_N\}$, ψ satisfy that for any $T > 0$,*

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\mathbf{x} \in K_N^3} |\varphi_N(t, \mathbf{x}) - \varphi(t, \mathbf{x})| = 0, \quad (3.3.35)$$

$$\lim_{N \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\mathbf{x} \in K_N^3} |\psi_N(t, \mathbf{x}) - \psi(t, \mathbf{x})| = 0. \quad (3.3.36)$$

Suppose that for each $f \in C_c^2(K^3 \times R)$ and $T > 0$ there exist measurable sets $\{F_N\} \subset R$ satisfying (3.2.43) and (3.2.27). Suppose that $P(\mathbf{Y}^N(0))^{-1} \Rightarrow \mu$ for

some $\mu \in \mathcal{P}(K^3 \times R)$, then there exists on some probability space $(\Omega, \mathcal{F}, P)$ a stochastic process $\mathbf{Y}$ satisfying $P(\mathbf{Y}(0))^{-1} = \mu$, which is the unique solution of the $C_{K^3 \times R}[0, \infty)$ -martingale problem for (G_A, μ) restricted on $C_c^2(K^3 \times R)$ and for P -a.s. $\omega \in \Omega$, $\mathbf{Y}(t, \omega)'$ is the unique solution of the nonlinear Volterra equations of the second kind (3.3.13), such that $\mathbf{Y}^N \Rightarrow \mathbf{Y}$.

Proof. We just need to note that (3.3.35) and (3.3.36) imply (3.2.19) and (3.2.20). □

Remark 3.3.3. The two conditions (3.3.35) and (3.3.36) are rather general, but they are not clear in our example. Notice that $\varphi_N(t, \mathbf{x})$ and $\psi_N(t, \mathbf{x})$ are completely determined by $\alpha_i^N(t)$, $\beta_i^N(t)$, $\delta_i^N(t)$ ($1 \leq i \leq 3$) and $\log F^N(t)$ (see (3.3.2) and (3.3.3)) and that $\varphi(t, \mathbf{x})$ and $\psi(t, \mathbf{x})$ are completely determined by $\alpha_i(t)$, $\beta_i(t)$, $\delta_i(t)$ ($1 \leq i \leq 3$) and $\log F(t)$ (see (3.3.5) and (3.3.6)). One sufficient condition to guarantee (3.3.35) and (3.3.36) is to assume that for any $T > 0$, $\alpha_i^N(t)$ converges uniformly to $\alpha_i(t)$ on $[0, T]$, $\beta_i^N(t)$ converges uniformly to $\beta_i(t)$ on $[0, T]$, $\delta_i^N(t)$ converges uniformly to $\delta_i(t)$ on $[0, T]$ ($1 \leq i \leq 3$) and $F^N(t)$ converges uniformly to $F(t)$ on $[0, T]$.

3.3.2 Fixed points of a special simplified example

In this subsection, we consider the fixed point problem for a special simplified example. Assume that $A(\mathbf{x}, q) = (a_{ij}(\mathbf{x}, q))_{3 \times 3}$ is a matrix-valued càdlàg function defined on $K^3 \times R$ which satisfies the following conditions:

- 1) For any $(\mathbf{x}, q) \in K^3 \times R$, $A(\mathbf{x}, q)\mathbf{e} = \mathbf{0}_{3 \times 1}$, where $\mathbf{e} = [1, 1, 1]'$;
- 2) For any $(\mathbf{x}, q) \in K^3 \times R$ and $1 \leq i, j \leq 3$, $i \neq j$, $0 \leq a_{ij}(\mathbf{x}, q) \leq 1$;

3) For any $(\mathbf{x}, q) \in K^3 \times R$ and $1 \leq i \leq 3$, $1 - \sum_{j \neq i} a_{ij}(\mathbf{x}, q) \geq 0$.

For any $(\mathbf{x}, q) \in K^3 \times R$, the first and second assumptions imply that $A(\mathbf{x}, q)$ is a *Q-matrix* and all three assumptions imply that $E_3 + A(\mathbf{x}, q)$ is a stochastic matrix. We also assume that $A(\mathbf{x}, q)$ satisfies the Lipschitz condition and is bounded on $K^3 \times R$. Then all the conditions related to $A(\mathbf{x}, q)$ required by Lemma 3.3.1 are satisfied, if viewing $A(\mathbf{x}, q)$ as a matrix-valued function on $[0, \infty) \times K^3 \times R$.

Next, we assume that in (3.3.5) and (3.3.6), $\alpha_i(t)$, $\beta_i(t)$, and $\delta_i(t)$ ($1 \leq i \leq 3$) and $\log F(t)$ are constants which are denoted by α_i , β_i , δ_i ($1 \leq i \leq 3$) and $\log F$. Assume also that for any $\mathbf{x} \in K^3$, $x_1\alpha_1 + x_2\alpha_2(1 + \beta_2) + x_3\alpha_3(1 + \beta_3) \neq 0$. Then all the conditions related to $\varphi(\mathbf{x})$, $\psi(\mathbf{x})$ and $g(\mathbf{x}, q)$ required by Lemma 3.3.1 are satisfied.

To get the fixed points of the following system, which are equivalent to the integral form (3.3.13),

$$\frac{d\mathbf{x}(t)}{dt} = A(\mathbf{x}(t), q(t))'\mathbf{x}(t), \quad (3.3.37)$$

$$\frac{dq(t)}{dt} = g(\mathbf{x}(t), q(t)), \quad (3.3.38)$$

we need to solve the following equations:

$$A(\mathbf{x}, q)'\mathbf{x} = 0, \quad (3.3.39)$$

$$g(\mathbf{x}, q) = 0. \quad (3.3.40)$$

Note that we use $\mathbf{x} \in K^3$ as a column vector.

At first, we assume that $\alpha_1 \neq 0$, $\beta_1 \neq 0$, $\delta_2\delta_3 > 0$. We know that $g(\mathbf{x}, q)$ is of the following form:

$$g(\mathbf{x}, q) = \frac{x_1\alpha_1\beta_1q}{x_1\alpha_1 + x_2\alpha_2(1 + \beta_2) + x_3\alpha_3(1 + \beta_3)} + \frac{x_1\delta_1 + x_2\delta_2 + x_3\delta_3 - x_1\alpha_1\beta_1 \log F}{x_1\alpha_1 + x_2\alpha_2(1 + \beta_2) + x_3\alpha_3(1 + \beta_3)}. \quad (3.3.41)$$

Then (3.3.40) implies that for $\mathbf{x} \in K^3$ with $x_1 \neq 0$, we have

$$q_{\mathbf{x}} = \frac{x_1 \alpha_1 \beta_1 \log F - x_1 \delta_1 - x_2 \delta_2 - x_3 \delta_3}{x_1 \alpha_1 \beta_1}. \quad (3.3.42)$$

Note that for any $\mathbf{x} \in K^3$ with $x_1 = 0$, (3.3.40) implies that $x_2 = x_3 = 0$ by the assumption $\delta_2 \delta_3 > 0$, which contradicts with $x_1 + x_2 + x_3 = 1$. Therefore, any $\mathbf{x} \in K^3$ with $x_1 = 0$ does not solve (3.3.40).

(3.3.42) defines a function $q_{\mathbf{x}}: K^3 \setminus \{\mathbf{x} \in K^3, x_1 = 0\} \mapsto R$. It is clear that for any $\tilde{\mathbf{x}} \in \{\mathbf{x} \in K^3, x_1 = 0\}$, $\lim_{x_1 \neq 0, \mathbf{x} \rightarrow \tilde{\mathbf{x}}} q_{\mathbf{x}} = \infty$ or $-\infty$. We can actually define $q_{\mathbf{x}}: K^3 \mapsto R \cup \{-\infty, \infty\}$, where for $\tilde{\mathbf{x}} \in \{\mathbf{x} \in K^3, x_1 = 0\}$,

$$q_{\tilde{\mathbf{x}}} = \lim_{x_1 \neq 0, \mathbf{x} \rightarrow \tilde{\mathbf{x}}} q_{\mathbf{x}}. \quad (3.3.43)$$

The function $q_{\mathbf{x}}$ with the extended definition is a continuous function on K^3 .

Since $A(\mathbf{x}, q)$ satisfies the Lipschitz condition, $A(\mathbf{x}, q)$ is a continuous function on $K^3 \times R$. We make more assumptions on $A(\mathbf{x}, q)$, such that we can extend $A(\mathbf{x}, q)$ to $K^3 \times (R \cup \{-\infty, \infty\})$. Assume that for any $\tilde{\mathbf{x}} \in K^3$, $\lim_{\mathbf{x} \rightarrow \tilde{\mathbf{x}}, q \rightarrow \pm\infty} A(\mathbf{x}, q)$ exists. Then we can extend $A(\mathbf{x}, q)$ to $K^3 \times (R \cup \{-\infty, \infty\})$ by defining

$$A(\tilde{\mathbf{x}}, \pm\infty) = \lim_{\mathbf{x} \rightarrow \tilde{\mathbf{x}}, q \rightarrow \pm\infty} A(\mathbf{x}, q), \quad (3.3.44)$$

where $\tilde{\mathbf{x}} \in K^3$. The extended function $A(\mathbf{x}, q)$ is continuous on $K^3 \times (R \cup \{-\infty, \infty\})$.

Define for $\mathbf{x} \in K^3$,

$$T(\mathbf{x}) = A(\mathbf{x}, q_{\mathbf{x}})' \mathbf{x} + \mathbf{x}. \quad (3.3.45)$$

Since $E_3 + A(\mathbf{x}, q_{\mathbf{x}})$ is a stochastic matrix, $T(\mathbf{x})$ is a map from K^3 to K^3 . Since $q_{\mathbf{x}}$ is a continuous function from K^3 to $R \cup \{-\infty, \infty\}$ and $A(\mathbf{x}, q)$ is a continuous function from $K^3 \times (R \cup \{-\infty, \infty\})$ to $R^{3 \times 3}$, we have that $T(\mathbf{x})$ is a continuous map from K^3

to K^3 . By Brouwer's fixed point theorem [SMA74], there exists fixed point $\mathbf{x}^0 \in K^3$ such that $T(\mathbf{x}^0) = \mathbf{x}^0$. It then follows that

$$A(\mathbf{x}^0, q_{\mathbf{x}^0})' \mathbf{x}^0 = 0. \quad (3.3.46)$$

We have to make further assumptions on $A(\mathbf{x}, q)$ to exclude the case $x_1^0 = 0$. Any of the following three conditions will guarantee $x_1^0 > 0$:

1. for any $\tilde{\mathbf{x}} \in \{\mathbf{x} \in K^3, x_1 = 0\}$, $a_{21}(\tilde{\mathbf{x}}, \pm\infty) > 0$ and $a_{31}(\tilde{\mathbf{x}}, \pm\infty) > 0$;
2. for any $\tilde{\mathbf{x}} \in \{\mathbf{x} \in K^3, x_1 = 0\}$, $a_{21}(\tilde{\mathbf{x}}, \pm\infty) > 0$ and $a_{32}(\tilde{\mathbf{x}}, \pm\infty) > 0$;
3. for any $\tilde{\mathbf{x}} \in \{\mathbf{x} \in K^3, x_1 = 0\}$, $a_{31}(\tilde{\mathbf{x}}, \pm\infty) > 0$ and $a_{23}(\tilde{\mathbf{x}}, \pm\infty) > 0$.

The fixed point $\mathbf{x}^0$ determined by (3.3.45) with $x_1^0 > 0$ satisfies that $(\mathbf{x}^0, q_{\mathbf{x}^0})$ is the solution of the system (3.3.46) and (3.3.40), i.e. $(\mathbf{x}^0, q_{\mathbf{x}^0})$ is the fixed point of (3.3.37) and (3.3.38).

3.3.3 The simplified example in random environment

We assumed in subsection 3.3.1 that $\alpha_i^N(t)$, $\beta_i^N(t)$ ($1 \leq i \leq 3$) are real valued deterministic functions on $[0, \infty)$ which are constants on $t \in [\frac{k}{N}, \frac{k+1}{N})$ for any $k \geq 0$. We assumed also that for any $\mathbf{x} \in K_N^3$ and $t \geq 0$, $x_1\alpha_1^N(t) + x_2\alpha_2^N(t)(1 + \beta_2^N(t)) + x_3\alpha_3^N(t)(1 + \beta_3^N(t)) \neq 0$. Then we defined function $\varphi_N(t, \mathbf{x})$ on $[0, \infty) \times K_N^3$ by (3.3.2). We assumed in subsection 3.3.1 that $\alpha_i(t)$, $\beta_i(t)$ ($1 \leq i \leq 3$) are continuous real valued bounded deterministic functions defined on $[0, \infty)$ which satisfy that for any $\mathbf{x} \in K^3$ and $t \geq 0$, $x_1\alpha_1(t) + x_2\alpha_2(t)(1 + \beta_2(t)) + x_3\alpha_3(t)(1 + \beta_3(t)) \neq 0$. Then we defined $\varphi(t, \mathbf{x})$ on $[0, \infty) \times K^3$ by (3.3.5). Now we assume also that for any $T > 0$,

$\alpha_i^N(t)$ converges uniformly to $\alpha_i(t)$ on $[0, T]$, $\beta_i^N(t)$ converges uniformly to $\beta_i(t)$ on $[0, T]$.

But we assume that $\delta_i^N(t)$, δ_i ($1 \leq i \leq 3$), $F^N(t)$ and $F(t)$ are stochastic processes. Precisely, we assume that $\delta_i^N(t)$ ($1 \leq i \leq 3$) and $F^N(t)$ are stochastic processes defined on some probability space $(\Omega_N, \mathcal{F}_N, P_N)$, whose sample path are constants on $t \in [\frac{k}{N}, \frac{k+1}{N})$ for any $k \geq 0$, P_N -a.s. We assume that there exists a probability space $(\Omega, \mathcal{F}, P)$ such that $\delta_i(t)$ ($1 \leq i \leq 3$) are Brownian Motions and $F(t)$ is a geometric Brownian Motion. We assume also that $(\delta_1^N, \delta_2^N, \delta_3^N, F^N) \Rightarrow (\delta_1, \delta_2, \delta_3, F)$ in the sense of weak convergence. We can define $\psi_N(t, \mathbf{x}, \omega_N)$ for each $\omega_N \in \Omega_N$ by (3.3.3) and $\psi(t, \mathbf{x}, \omega)$ for each $\omega \in \Omega$ by (3.3.6).

As to $A_N(t, \mathbf{x}, q)$ and $A(t, \mathbf{x}, q)$, we replace (3.2.27) and (3.2.43) by assuming the stronger one

$$\lim_{N \rightarrow \infty} \sup_{q \in R} \sup_{\mathbf{x} \in K_N} d_U(A_N(\cdot, \mathbf{x}, q), A(\cdot, \mathbf{x}, q)) = 0. \quad (3.3.47)$$

We keep all other conditions of $A_N(t, \mathbf{x}, q)$ and $A(t, \mathbf{x}, q)$ used in subsection 3.3.1.

We can follow the procedures in subsection 3.1.2 and 3.3.1 to define $\{\mathbf{Y}^N(t, \omega_N), 0 \leq t < \infty\}$ and determine $\{\mathbf{Y}(t, \omega), 0 \leq t < \infty\}$. As to the determination of $\{\mathbf{Y}(t, \omega), 0 \leq t < \infty\}$ by the *nonlinear Volterra Equations of the second kind*, we note that the Brownian paths $\delta_i(t, \omega)$ ($1 \leq i \leq 3$) and $\log F(t, \omega)$ satisfy (3.3.7) almost surely by the Law of the iterated logarithm.

Then we can follow the method used in subsection 2.5.2 to prove that $\mathbf{Y}^N(\omega_N) \Rightarrow \mathbf{Y}(\omega)$ and $(\delta_1^N, \delta_2^N, \delta_3^N, F^N, \mathbf{Y}^N) \Rightarrow (\delta_1, \delta_2, \delta_3, F, \mathbf{Y})$. As usual, the Skorohod Representation theorem is the basic tool to establish this result. Therefore, without loss of generality, we can assume that $(\delta_1^N, \delta_2^N, \delta_3^N, F^N)$ and $(\delta_1, \delta_2, \delta_3, F)$ are defined in

the same probability space $(\Omega, \mathcal{F}, P)$ and $(\delta_1^N, \delta_2^N, \delta_3^N, F^N)$ converges to $(\delta_1, \delta_2, \delta_3, F)$ almost surely. We note that for $1 \leq i \leq 3$, $\lim_{N \rightarrow \infty} d(\delta_i^N(\omega), \delta_i(\omega)) = 0$ implies that $\lim_{N \rightarrow \infty} d_U(\delta_i^N(\omega), \delta_i(\omega)) = 0$ and hence $\delta_i^N(\omega)$ converges to $\delta_i(\omega)$ uniformly on $[0, T]$ for any $T > 0$. Same thing holds for $F^N(\omega)$ and $F(\omega)$. Then we actually assumed or verified all the conditions required by Remark 3.3.3.

We note that the log price function $q(t)$ of the simplified example in random environment is determined pathwisely by the *nonlinear Volterra Equations of the second kind*. That is say, for P -a.s any sample point $\omega \in \Omega$, the *nonlinear Volterra Equations of the second kind* determines a unique log price function $q(t, \omega)$ on $[0, \infty)$. This is similar to the classical assumption of the stock price formula which was suggested by Samuelson [SA64] in 1964: $\log S_t = \mu t + \sigma W_t$, where W_t is a Brownian Motion. Then one interesting question might be asked: can we develop option pricing theory based on this pathwise log price function which is the solution of *nonlinear Volterra Equations of the second kind*? We have good reason to believe that this is an interesting question since this system generalizes the classical formulation of stock price by incorporating the interaction between different types of agents and the interaction between the stock price and the empirical distribution of the types of agents.

Chapter 4

Subordinated Systems in Random Environment

We consider the weak convergence of continuous processes and *càdlàg* processes under random time change. Under rather general conditions, several versions of weak convergence theorems, e.g., quenched weak convergence, annealed weak convergence and joint weak convergence, for time-changed sequence of continuous processes or *càdlàg* processes are proved. As an example of the applications of these convergence theorems, we discuss the weak convergence of the time inhomogeneous critical linear growth birth-death processes evolving in random environments. The above three types of weak convergence of the scaled time inhomogeneous birth-death processes evolving in a stationary environment, stable environment, or catalytic environment follow by these weak convergence theorems.

4.1 Subordinator, subordination and time change

There are quite large families of stochastic processes, which have diffusion limits, with constant parameter related to the scale of time. Continuous time random walk, and birth-death processes are two simple examples of these families. To describe the complicated features of natural phenomena, we have to consider sometimes stochastic processes with parameter evolving in another independent random environment. For example, the parameter λ of a critical linear growth birth-death process $\{X(t), 0 \leq t < \infty\}$, which is the birth and death rate of the birth-death process, can be a stochastic process itself in another random environment which is independent of the birth-death process. Precisely, we can assume that $\{Y(t), 0 \leq t < \infty\}$ is a standard critical linear growth birth-death process, i.e. the birth and death rate is 1, and $\{\lambda(t), 0 \leq t < \infty\}$ is a nonnegative process which is independent of $\{Y(t), 0 \leq t < \infty\}$. We are concerned with the new process $\{X(t) = Y(\int_0^t \lambda(s) ds), 0 \leq t < \infty\}$. Then $\{X(t), 0 \leq t < \infty\}$ is actually a compound time inhomogeneous birth-death process whose birth death rate is the process λ . $\{\Lambda(t) = \int_0^t \lambda(s) ds, 0 \leq t < \infty\}$ is called **subordinator** and Y is called **basic process**. The transformation $Y \circ \Lambda$ of Y by Λ is called **subordination**. We call this transformation throughout the rest of this chapter **time change**. We also call X **the subordinated process**. Usually, the subordinated processes are not Markov processes themselves, but the space-time processes are Markov processes if the subordinators are.

Henceforth, there are several weak convergence problems related to these subordinated processes, including quenched weak convergence, annealed weak convergence, and joint weak convergence. A standard method to solve these problems is the **Martingale Problem method**, which involves Aldous conditions for tightness or

Kurtz's tightness criterion. Aldous conditions for tightness or Kurtz's tightness criterion are sometimes not easy to check. As shown by the critical linear growth birth death process example in last paragraph, the basic structure of these subordinated processes are some basic processes, i.e, the processes with time scale parameter being constant, time changed by other stochastic processes, i.e. subordinators, which are the integral of the time scale parameter processes. In many cases, the weak convergence of the basic processes and subordinators are well-known or easy to prove. We give a new method, which is based on the known facts of the weak convergence of basic processes and subordinators, of proving the quenched weak convergence, annealed weak convergence, and joint weak convergence for subordinated systems. Generally speaking, we will develop a theory of proving the weak convergence problems related to continuous processes or *càdlàg* processes time changed by other processes.

In section 4.2, we give a general weak convergence theory for stochastic processes time changed by other processes, which includes quenched weak convergence theorem, annealed weak convergence theorem, and joint weak convergence theorem. In section 4.3, as an example of the applications of these theorems we prove the weak convergence of critical linear growth birth death process with parameter in a stationary environment, stable environment, or catalytic environment. We will see that the weak convergence theorems related to time change are really efficient in proving weak convergence of this kind of problems, and that the checking of conditions for tightness is avoided.

4.2 Time change, weak convergence of subordinated stochastic processes

Let (E, r) be a separable metric space. We will consider time change on $C_E[0, \infty)$ and $D_E[0, \infty)$. We denote the metric on $C_E[0, \infty)$ by d_U defined by $d_U(x, y) = \int_0^\infty e^{-u} \sup_{0 \leq t \leq u} [r(x(t), y(t)) \wedge 1] du$ for any $x, y \in C_E[0, \infty)$. We also use the same symbol for the metric on $C_{R_+}[0, \infty)$, $C_{R_+ \times E}[0, \infty)$ defined in the same way, but with r in different meanings. This will cause no confusion in the context. Similarly, we use the notation d for the Skorohod metric on $D_{R_+}[0, \infty)$, $D_E[0, \infty)$ and $D_{R_+ \times E}[0, \infty)$. The definition of d can be found in Chapter 3 [EK86].

4.2.1 Continuous time change on $C_E[0, \infty)$, quenched convergence theorem

For convenience, we state the following frequently used lemma.

Lemma 4.2.1. *Let $\{z_n\} \subset C_E[0, \infty)$ and $z \in C_E[0, \infty)$, then $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$ if and only if whenever $\{t_n\} \subset [0, \infty)$, $t \geq 0$, and $\lim_{n \rightarrow \infty} t_n = t$, we have $\lim_{n \rightarrow \infty} r(z_n(t_n), z(t)) = 0$.*

Now we define time change on $C_E[0, \infty)$.

Definition 4.2.2. *If $z \in C_E[0, \infty)$ and $x \in C_{R_+}[0, \infty)$, $z \circ x$ is called z time changed by x where $z \circ x$ is defined by $z \circ x(t) = z(x(t))$ for $0 \leq t < \infty$.*

By definition, $z \circ x \in C_E[0, \infty)$ if $z \in C_E[0, \infty)$ and $x \in C_{R_+}[0, \infty)$.

Lemma 4.2.3. *Let $\{x_n\} \subset C_{R_+}[0, \infty)$, $x \in C_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$.*

Let $\{z_n\} \subset C_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$. Then we have that $\lim_{n \rightarrow \infty} d_U(z_n \circ x_n, z \circ x) = 0$.

Proof. If $\{t_n\} \subset [0, \infty)$, $t \geq 0$, and $\lim_{n \rightarrow \infty} t_n = t$, since $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$, we have by Lemma 4.2.1 $\lim_{n \rightarrow \infty} x_n(t_n) = x(t)$. By Lemma 4.2.1 again and $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$, we get that $\lim_{n \rightarrow \infty} r(z_n \circ x_n(t_n), z \circ x(t)) = 0$. Finally, by sufficiency of Lemma 4.2.1, we conclude that $\lim_{n \rightarrow \infty} d_U(z_n \circ x_n, z \circ x) = 0$. $\square$

For $x \in C_{R_+}[0, \infty)$, we define a function $f_x : C_E[0, \infty) \mapsto C_E[0, \infty)$ by $f_x(z) = z \circ x$, $z \in C_E[0, \infty)$. By Lemma 4.2.3, we know that f_x is a continuous function, hence measurable. Therefore, for any $\Gamma \in \mathcal{B}(C_E[0, \infty))$, $f_x^{-1}(\Gamma) \in \mathcal{B}(C_E[0, \infty))$.

Now we give the continuous time change theorem.

Theorem 4.2.4. Let $\{x_n\} \subset C_{R_+}[0, \infty)$, $x \in C_{R_+}[0, \infty)$ with $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$. Suppose that $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$. Then $Z_n \circ x_n \Rightarrow Z \circ x$.

Note that in the above assumption, instead of writing $\tilde{P}_n Z_n^{-1} \Rightarrow \tilde{P} Z^{-1}$, we wrote $Z_n \Rightarrow Z$. Similar abbreviations will be used when there is no confusion.

Proof. We use the Skorohod Representation theorem. Assume that there exists a probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and a sequence of $C_E[0, \infty)$ -valued random variables $\{Y_n\}$ and Y defined on it, such that $\hat{P} Y_n^{-1} = \tilde{P}_n Z_n^{-1}$, $\hat{P} Y^{-1} = \tilde{P} Z^{-1}$ and $\lim_{n \rightarrow \infty} d_U(Y_n, Y) = 0$ $\hat{P}$ -a.s. Then Lemma 4.2.3 implies that $\lim_{n \rightarrow \infty} d_U(Y_n \circ x_n, Y \circ x) = 0$ $\hat{P}$ -a.s. Therefore, $\hat{P}(Y_n \circ x_n)^{-1} \Rightarrow \hat{P}(Y \circ x)^{-1}$. This implies that $\tilde{P}_n(Z_n \circ x_n)^{-1} \Rightarrow \tilde{P}(Z \circ x)^{-1}$. $\square$

Note: $Y_n \circ x_n = f_{x_n}(Y_n)$, $Y \circ x = f_x(Y)$. Since f_{x_n}, f_x are continuous functions, $\hat{P}(Y_n \circ x_n)^{-1}$, $\hat{P}(Y \circ x)^{-1}$ and $\tilde{P}_n(Z_n \circ x_n)^{-1}$ make sense.

We conclude this subsection with the following quenched convergence theorem, which is immediate from Theorem 4.2.4.

Theorem 4.2.5. (*Quenched Convergence*) *Let $\{X_n\}$ and X be $C_{R_+}[0, \infty)$ -valued processes, defined on $(\Omega, \mathcal{F}, P)$, and $\lim_{n \rightarrow \infty} d_U(X_n, X) = 0$ P -a.s. Suppose that $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$. Then for P -a.s $\omega \in \Omega$, $Z_n \circ X_n(\omega) \Rightarrow Z \circ X(\omega)$.*

4.2.2 Càdlàg time change on $C_E[0, \infty)$ and $D_E[0, \infty)$, quenched and annealed convergence

In this subsection, we consider more general time change, i.e. càdlàg time change. We state the next lemma, which is the analogue on $(D_E[0, \infty), d)$ of Lemma 4.2.1 on $(C_E[0, \infty), d_U)$. Readers are referred to Proposition 6.5, chapter 3 of Ethier and Kurtz.

Lemma 4.2.6. *Let $\{z_n\} \subset D_E[0, \infty)$ and $z \in D_E[0, \infty)$. Then $\lim_{n \rightarrow \infty} d(z_n, z) = 0$ if and only if for any $\{t_n\} \subset [0, \infty)$, $t \geq 0$, and $\lim_{n \rightarrow \infty} t_n = t$, the following conditions hold:*

- (a) $\lim_{n \rightarrow \infty} r(z_n(t_n), z(t)) \wedge r(z_n(t_n), z(t-)) = 0$.
- (b) If $\lim_{n \rightarrow \infty} r(z_n(t_n), z(t)) = 0$, $s_n \geq t_n$ for each n , and $\lim_{n \rightarrow \infty} s_n = t$, then $\lim_{n \rightarrow \infty} r(z_n(s_n), z(t)) = 0$.
- (c) If $\lim_{n \rightarrow \infty} r(z_n(t_n), z(t-)) = 0$, $0 \leq s_n \leq t_n$ for each n , and $\lim_{n \rightarrow \infty} s_n = t$, then $\lim_{n \rightarrow \infty} r(z_n(s_n), z(t-)) = 0$.

We define time change on $D_E[0, \infty)$ same way as that in definition 4.2.2. But notice that we put one constraint on it.

Definition 4.2.7. *If $z \in D_E[0, \infty)$ and $x \in D_{R+}[0, \infty)$, satisfying $z \circ x \in D_E[0, \infty)$, $z \circ x$ is called z time changed by x where $z \circ x$ is defined by $z \circ x(t) = z(x(t))$ for $0 \leq t < \infty$.*

Remark 4.2.8. *We have the following useful observations: (1) If $z \in C_E[0, \infty)$ and $x \in D_{R+}[0, \infty)$, then $z \circ x \in D_E[0, \infty)$. (2) If $z \in D_E[0, \infty)$, $x \in D_E[0, \infty)$ and x is an increasing function on $[0, \infty)$, then $z \circ x \in D_E[0, \infty)$. Note that in case (1), x is not assumed to be an increasing function, that is to say, for $t_1 < t_2$, it is possible that $x(t_1) > x(t_2)$, which means that in the composition $z \circ x$, z can go back to the times in history. This is the time-reversibility property of time change on $C_E[0, \infty)$. The time change on $D_E[0, \infty)$ does not have this property.*

Next, we prove a lemma about càdlàg time change.

Lemma 4.2.9. *Let $\{x_n\} \subset D_{R+}[0, \infty)$, $x \in D_{R+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Let $\{z_n\} \subset D_E[0, \infty)$, such that $\{z_n \circ x_n\} \subset D_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d(z_n, z) = 0$. Then we have that $\lim_{n \rightarrow \infty} d(z_n \circ x_n, z \circ x) = 0$.*

Proof. We use Lemma 4.2.6 to prove this lemma. For any $\{t_n\} \subset [0, \infty)$, $t \geq 0$, and $\lim_{n \rightarrow \infty} t_n = t$, We need to check:

$$(a1) \lim_{n \rightarrow \infty} r(z_n \circ x_n(t_n), z \circ x(t)) \wedge r(z_n \circ x_n(t_n), z \circ x(t-)) = 0.$$

$$(b1) \text{ If } \lim_{n \rightarrow \infty} r(z_n \circ x_n(t_n), z \circ x(t)) = 0, s_n \geq t_n \text{ for each } n, \text{ and } \lim_{n \rightarrow \infty} s_n = t, \\ \text{then } \lim_{n \rightarrow \infty} r(z_n \circ x_n(s_n), z \circ x(t)) = 0.$$

$$(c1) \text{ If } \lim_{n \rightarrow \infty} r(z_n \circ x_n(t_n), z \circ x(t-)) = 0, 0 \leq s_n \leq t_n \text{ for each } n, \text{ and } \lim_{n \rightarrow \infty} s_n = \\ t, \text{ then } \lim_{n \rightarrow \infty} r(z_n \circ x_n(s_n), z \circ x(t-)) = 0.$$

At the very beginning of checking the above three conditions, we emphasize one frequently used fact in this proof, which will also be used in the proof of Lemma 4.2.22. By $\lim_{n \rightarrow \infty} d(z_n, z) = 0$, and $z \in C_E[0, \infty)$, and by Lemma 4.2.6 for the space $D_E[0, \infty)$, for any $\{y_n\} \subset [0, \infty)$, $\lim_{n \rightarrow \infty} y_n = y$, we have $\lim_{n \rightarrow \infty} r(z_n(y_n), z(y)) = 0$.

At first, we check (a1). Since $\lim_{n \rightarrow \infty} t_n = t$, we have by Lemma 4.2.6 for the space $D_{R+}[0, \infty)$, $\lim_{n \rightarrow \infty} |x_n(t_n) - x(t)| \wedge |x_n(t_n) - x(t-)| = 0$. Next, we can divide $\{x_n(t_n)\}$ into two disjoint convergent subsequences if needed. We denote these two infinite subsequences by $\{x_{n_r}(t_{n_r})\}$ and $\{x_{n_l}(t_{n_l})\}$ which satisfy $\lim_{r \rightarrow \infty} x_{n_r}(t_{n_r}) = x(t)$ and $\lim_{l \rightarrow \infty} x_{n_l}(t_{n_l}) = x(t-)$. It is possible that one of subsequences has only finite elements, we then don't have to talk about its convergence. If we have just one subsequence with infinite elements, it must be either the one with limit $x(t)$ or the one with limit $x(t-)$.

By $\lim_{n \rightarrow \infty} d(z_n, z) = 0$ and z is continuous, and the emphasized fact above, we have $\lim_{r \rightarrow \infty} r(z_{n_r} \circ x_{n_r}(t_{n_r}), z \circ x(t)) = 0$ and $\lim_{l \rightarrow \infty} r(z_{n_l} \circ x_{n_l}(t_{n_l}), z \circ x(t-)) = 0$. Then we have

$$\lim_{r \rightarrow \infty} r(z_{n_r} \circ x_{n_r}(t_{n_r}), z \circ x(t)) \wedge r(z_{n_r} \circ x_{n_r}(t_{n_r}), z \circ x(t-)) = 0, \quad (4.2.1)$$

and

$$\lim_{l \rightarrow \infty} r(z_{n_l} \circ x_{n_l}(t_{n_l}), z \circ x(t)) \wedge r(z_{n_l} \circ x_{n_l}(t_{n_l}), z \circ x(t-)) = 0. \quad (4.2.2)$$

Then condition (a1) is satisfied.

Next, we use the similar method to check (b1). Since $\lim_{n \rightarrow \infty} s_n = t$, by Lemma 4.2.6, we have $\lim_{n \rightarrow \infty} |x_n(s_n) - x(t)| \wedge |x_n(s_n) - x(t-)| = 0$. Then $\{x_n(s_n)\}$ has possibly two kinds of disjoint convergent subsequences $\{x_{n_r}(s_{n_r})\}$ with $\lim_{r \rightarrow \infty} x_{n_r}(s_{n_r}) = x(t)$ and $\{x_{n_l}(s_{n_l})\}$ with $\lim_{l \rightarrow \infty} x_{n_l}(s_{n_l}) = x(t-)$.

Now we consider the case when $\lim_{l \rightarrow \infty} x_{n_l}(s_{n_l}) = x(t-)$ exists. By the emphasized fact we then have

$$\lim_{l \rightarrow \infty} r(z_{n_l} \circ x_{n_l}(s_{n_l}), z \circ x(t-)) = 0. \quad (4.2.3)$$

Since $0 \leq t_{n_l} \leq s_{n_l}$, by Lemma 4.2.6 (c), interchanging the role of t_{n_l} and s_{n_l} , we have $\lim_{l \rightarrow \infty} x_{n_l}(t_{n_l}) = x(t-)$. Then by the emphasized fact again, we get

$$\lim_{l \rightarrow \infty} r(z_{n_l} \circ x_{n_l}(t_{n_l}), z \circ x(t-)) = 0. \quad (4.2.4)$$

By the assumption in condition (b1), we have $r(z \circ x(t), z \circ x(t-)) = 0$. Combining this with (4.2.3), we get

$$\lim_{l \rightarrow \infty} r(z_{n_l} \circ x_{n_l}(s_{n_l}), z \circ x(t)) = 0. \quad (4.2.5)$$

For the case $\lim_{r \rightarrow \infty} x_{n_r}(s_{n_r}) = x(t)$, we simply by the emphasized fact get

$$\lim_{r \rightarrow \infty} r(z_{n_r} \circ x_{n_r}(s_{n_r}), z \circ x(t)) = 0. \quad (4.2.6)$$

Condition (b1) is then checked.

Condition (c1) can be checked similarly. □

Base on Lemma 4.2.9 and Remak 4.2.8, we have the following two corollaries.

Corollary 4.2.10. *Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Let $\{z_n\} \subset C_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$. Then $\lim_{n \rightarrow \infty} d(z_n \circ x_n, z \circ x) = 0$.*

Corollary 4.2.11. *Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. $\{x_n\}$ and x are nondecreasing functions on $[0, \infty)$. Let $\{z_n\} \subset D_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d(z_n, z) = 0$. Then $\lim_{n \rightarrow \infty} d(z_n \circ x_n, z \circ x) = 0$.*

Note, in Corollary 4.2.11, we don't have to assume that x is a nondecreasing function, since we also make the assumption $z \in C_E[0, \infty)$, which can guarantee that $z \circ x \in D_E[0, \infty)$. But since $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and $\{x_n\}$ are nondecreasing functions on $[0, \infty)$, we must have that x is also a nondecreasing function. Similar situations occur very often in the theorems and corollaries of this paper. We just choose the way which makes our statements simple.

Remark 4.2.12. By Corollary 4.2.10, for $x \in D_{R+}[0, \infty)$, we can define a continuous function $f_x : C_E[0, \infty) \mapsto D_E[0, \infty)$ by $f_x(z) = z \circ x$, $z \in C_E[0, \infty)$. By Corollary 4.2.11, for nondecreasing function $x \in D_{R+}[0, \infty)$, we can define a continuous function $f_x : D_E[0, \infty) \mapsto D_E[0, \infty)$ by $f_x(z) = z \circ x$. In both cases, f_x is measurable.

Example 4.2.13. Let us define that for $n \geq 1$,

$$x_n(t) = \begin{cases} \frac{1}{2} - \frac{1}{n+1}, & 0 \leq t < 1 - \frac{1}{n}, \\ 1, & 1 - \frac{1}{n} \leq t; \end{cases} \quad (4.2.7)$$

$$x(t) = \begin{cases} \frac{1}{2}, & 0 \leq t < 1, \\ 1, & 1 \leq t; \end{cases} \quad (4.2.8)$$

and

$$z(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \leq t < 1, \\ 1, & 1 \leq t. \end{cases} \quad (4.2.9)$$

We also define $z_n = z$ for each $n \geq 1$. Then x_n , z_n , and x , z are nondecreasing functions such that $\lim_{n \rightarrow \infty} d(x_n, x) = 0$, $\lim_{n \rightarrow \infty} d(z_n, z) = 0$, and $\{z_n \circ x_n\} \subset D_{R+}[0, \infty)$, $z \circ x \in D_{R+}[0, \infty)$. We have also that $z \circ x = x$ and for each $n \geq 1$

$$z_n \circ x_n(t) = \begin{cases} 0, & 0 \leq t < 1 - \frac{1}{n}, \\ 1, & 1 - \frac{1}{n} \leq t. \end{cases} \quad (4.2.10)$$

But we don't have $\lim_{n \rightarrow \infty} d(z_n \circ x_n, z \circ x) = 0$.

This example illustrates that the condition $z \in C_E[0, \infty)$ in Lemma 4.2.9 is not so strong as it looks. Actually this example satisfies all the conditions of Corollary 4.2.11 except $z \in C_E[0, \infty)$. So we couldn't expect that under the assumption $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$; $\{z_n\} \subset D_E[0, \infty)$, $z \in D_E[0, \infty)$, such that $\{z_n \circ x_n\} \subset D_E[0, \infty)$ and $z \circ x \in D_E[0, \infty)$, and $\lim_{n \rightarrow \infty} d(z_n, z) = 0$; we could then conclude that $\lim_{n \rightarrow \infty} d(z_n \circ x_n, z \circ x) = 0$.

Similar to Theorem 4.2.4, we have the following càdlàg time change theorems.

Theorem 4.2.14. *Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$ with $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Suppose that $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$. Then $Z_n \circ x_n \Rightarrow Z \circ x$.*

Theorem 4.2.15. *Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$ with $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. $\{x_n\}$ and x are nondecreasing functions on $[0, \infty)$. Suppose that $\{Z_n\}$ are $D_E[0, \infty)$ -valued processes, and Z is $C_E[0, \infty)$ -valued process, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$. Then $Z_n \circ x_n \Rightarrow Z \circ x$.*

The proofs of these two theorems are similar to that of Theorem 4.2.4.

Now we give the quenched convergence theorem for these situations.

Theorem 4.2.16. (Quenched Convergence) *Let $\{X_n\}$ and X be $D_{R_+}[0, \infty)$ -valued processes, defined on $(\Omega, \mathcal{F}, P)$, and $\lim_{n \rightarrow \infty} d(X_n, X) = 0$ P -a.s. Suppose that*

- (1) $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$;

(2) $\{X_n\}$ and X are nondecreasing processes. $\{Z_n\}$ are $D_E[0, \infty)$ -valued processes and Z is a $C_E[0, \infty)$ -valued process, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$.

Under either (1) or (2), $Z_n \circ X_n(\omega) \Rightarrow Z \circ X(\omega)$ for P -a.s $\omega \in \Omega$.

Before we conclude this subsection with the annealed convergence theorem, we prove a lemma which will be used in the proof of this theorem.

Lemma 4.2.17. *Let X be a nondecreasing $D_{R_+}[0, \infty)$ -valued random variable defined on $(\Omega, \mathcal{F}, P)$, and Y be a $D_{R_+}[0, \infty)$ -valued random variable defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. If $PX^{-1} = \tilde{P}Y^{-1}$, then Y is also a nondecreasing process $\tilde{P}$ -a.s.*

Proof. For any $0 \leq t_1 < t_2 < \infty$, by Proposition 7.1, Chapter 3 [EK86], and the separability of R_+ , we know that $g(x) = (x(t_1), x(t_2)) : D_{R_+}[0, \infty) \mapsto R_+^2$ is measurable. It is obvious that $h(y_1, y_2) = y_1 - y_2 : R_+^2 \mapsto R$ is also measurable. Then

$$\begin{aligned} \tilde{P}\{Y(t_1) \leq Y(t_2)\} &= \tilde{P}Y^{-1}g^{-1}h^{-1}(-\infty, 0] \\ &= PX^{-1}g^{-1}h^{-1}(-\infty, 0] \\ &= P\{X(t_1) \leq X(t_2)\} \\ &= 1. \end{aligned} \tag{4.2.11}$$

Since Y is a càdlàg process, we actually have that Y is a nondecreasing process $\tilde{P}$ -a.s. □

Now we can state and prove the annealed convergence theorem, which is different from the quenched convergence theorem.

Theorem 4.2.18. (Annealed convergence) *Let $\{X_n\}$ and X be $D_{R_+}[0, \infty)$ -valued processes, defined on $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively, and $X_n \Rightarrow X$. Let*

(E, r) be separable. Suppose that

(1) $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$;

(2) $\{X_n\}$ and X are nondecreasing processes, $\{Z_n\}$ are $D_E[0, \infty)$ -valued processes and Z is a $C_E[0, \infty)$ -valued process, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$.

Under either (1) or (2), we have that $E^{\tilde{P}_n}[\tilde{P}_n(Z_n \circ X_n)^{-1}] \Rightarrow E^{\tilde{P}}[\tilde{P}(Z \circ X)^{-1}]$, where $E^{\tilde{P}}[\tilde{P}(Z \circ X)^{-1}] = \int \tilde{P}(Z \circ X)^{-1} d\tilde{P}$, and $E^{\tilde{P}_n}[\tilde{P}_n(Z_n \circ X_n)^{-1}]$ has similar interpretation.

Proof. Since $X_n \Rightarrow X$, by Skorohod representation theorem, there exists a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ and a sequence of $D_{R_+}[0, \infty)$ -valued random variables $\{\bar{X}_n\}$, and $\bar{X}$ such that $\bar{P}\bar{X}_n^{-1} = P_n X_n^{-1}$, $\bar{P}\bar{X}^{-1} = P X^{-1}$ and $\lim_{n \rightarrow \infty} d(\bar{X}_n, \bar{X}) = 0$ $\bar{P}$ -a.s. When $\{X_n\}$ and X are nondecreasing processes, so are $\{\bar{X}_n\}$, and $\bar{X}$, by Lemma 4.2.17.

Then by Theorem 4.2.14 and Theorem 4.2.15, for $\bar{P}$ -a.s. $\bar{\omega}$, $\tilde{P}_n(Z_n \circ \bar{X}_n(\bar{\omega}))^{-1} \Rightarrow \tilde{P}(Z \circ \bar{X}(\bar{\omega}))^{-1}$ as $n \rightarrow \infty$. Thus, for any open set $G \subset D_E[0, \infty)$, we have

$$\liminf_{n \rightarrow \infty} \tilde{P}_n\{Z_n \circ \bar{X}_n(\bar{\omega}) \in G\} \geq \tilde{P}\{Z \circ \bar{X}(\bar{\omega}) \in G\}, \quad \bar{P}\text{-a.s.} \quad (4.2.12)$$

Then, by Fatou Lemma, we have

$$\liminf_{n \rightarrow \infty} E^{\tilde{P}_n}[\tilde{P}_n\{Z_n \circ \bar{X}_n(\bar{\omega}) \in G\}] \geq E^{\tilde{P}}[\tilde{P}\{Z \circ \bar{X}(\bar{\omega}) \in G\}], \quad (4.2.13)$$

i.e., $\liminf_{n \rightarrow \infty} E^{\tilde{P}_n}[\tilde{P}_n\{Z_n \circ X_n \in G\}] \geq E^{\tilde{P}}[\tilde{P}\{Z \circ X \in G\}]$. Therefore, $E^{\tilde{P}_n}[\tilde{P}_n(Z_n \circ X_n)^{-1}] \Rightarrow E^{\tilde{P}}[\tilde{P}(Z \circ X)^{-1}]$. $\square$

4.2.3 Time change on $C_{R_+ \times E}[0, \infty)$ and $D_{R_+ \times E}[0, \infty)$, joint convergence

Let (E, r) be a separable metric space, $(R_+, |\cdot|)$ be a space with Euclidean metric on R_+ . We can define a metric $\tilde{r}$ on the product space $R_+ \times E$ as follows: for any pair $(x_1, z_1), (x_2, z_2) \in R_+ \times E$,

$$\tilde{r}((x_1, z_1), (x_2, z_2)) = \sqrt{(x_1 - x_2)^2 + r(z_1, z_2)^2}. \quad (4.2.14)$$

It is easy to check that $\tilde{r}$ is a metric on $R_+ \times E$. Since (E, r) is separable, so is $R_+ \times E$; and if (E, r) is complete, so is $R_+ \times E$. It's obvious that for $\{(x_n, z_n)\} \subset R_+ \times E$, $(x, z) \in R_+ \times E$,

$$\lim_{n \rightarrow \infty} \tilde{r}((x_n, z_n), (x, z)) = 0 \iff \lim_{n \rightarrow \infty} x_n = x \text{ and } \lim_{n \rightarrow \infty} r(z_n, z) = 0. \quad (4.2.15)$$

Then,

$$(x, z) \in C_{R_+ \times E}[0, \infty) \iff x \in C_{R_+}[0, \infty), \text{ and } z \in C_E[0, \infty). \quad (4.2.16)$$

$$(x, z) \in D_{R_+ \times E}[0, \infty) \iff x \in D_{R_+}[0, \infty), \text{ and } z \in D_E[0, \infty). \quad (4.2.17)$$

i.e. $C_{R_+ \times E}[0, \infty) = C_{R_+}[0, \infty) \times C_E[0, \infty)$, $D_{R_+ \times E}[0, \infty) = D_{R_+}[0, \infty) \times D_E[0, \infty)$.

Since (E, r) is separable, so is $C_{R_+ \times E}[0, \infty)$ and $D_{R_+ \times E}[0, \infty)$.

Similar to Lemma 4.2.3, we have the following lemma.

Lemma 4.2.19. *Let $\{x_n\} \subset C_{R_+}[0, \infty)$, $x \in C_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$. Let $\{z_n\} \subset C_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$. Then we have that $\lim_{n \rightarrow \infty} d_U((x_n, z_n \circ x_n), (x, z \circ x)) = 0$.*

Proof. For any $\{t_n\} \subset [0, \infty)$, $t \geq 0$, and $\lim_{n \rightarrow \infty} t_n = t$, since $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$, we have by Lemma 4.2.1 $\lim_{n \rightarrow \infty} x_n(t_n) = x(t)$. By Lemma 4.2.3, we have

$\lim_{n \rightarrow \infty} d_U(z_n \circ x_n, z \circ x) = 0$, we get that $\lim_{n \rightarrow \infty} r(z_n \circ x_n(t_n), z \circ x(t)) = 0$. Then by (4.2.15), we have $\lim_{n \rightarrow \infty} \bar{r}((x_n(t_n), z_n \circ x_n(t_n)), (x(t), z \circ x(t))) = 0$. Finally, by Lemma 4.2.1 for the space $C_{R_+ \times E}[0, \infty)$, we conclude that $\lim_{n \rightarrow \infty} d_U((x_n, z_n \circ x_n), (x, z \circ x)) = 0$. $\square$

Before we give a theorem stronger than Theorem 4.2.4, we have the next two lemmas.

Lemma 4.2.20. *Let us assume that 1) X and Y are $C_{R_+}[0, \infty)$ -valued random variables, defined on $(\Omega, \mathcal{F}, P)$ and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ respectively, Z and W are $C_E[0, \infty)$ -valued random variables, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ respectively; or 2) X and Y are $D_{R_+}[0, \infty)$ -valued random variables, defined on $(\Omega, \mathcal{F}, P)$ and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ respectively, Z and W are $D_E[0, \infty)$ -valued random variables, defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ respectively, such that $Z \circ X$ and $W \circ Y$ are $D_E[0, \infty)$ -valued. We also assume that (E, r) is separable. If $PX^{-1} = \bar{P}Y^{-1}$ and $\tilde{P}Z^{-1} = \hat{P}W^{-1}$, then $P \times \tilde{P}(X, Z \circ X)^{-1} = \bar{P} \times \hat{P}(Y, W \circ Y)^{-1}$ and $P \times \tilde{P}(Z \circ X)^{-1} = \bar{P} \times \hat{P}(W \circ Y)^{-1}$.*

Proof. We just prove the conclusion under condition 1). Since $C_{R_+ \times E}[0, \infty) = C_{R_+}[0, \infty) \times C_E[0, \infty)$, and $C_E[0, \infty)$ is separable, $\mathcal{B}(C_{R_+ \times E}[0, \infty)) = \mathcal{B}(C_{R_+}[0, \infty)) \otimes \mathcal{B}(C_E[0, \infty))$. For any $A \times B \in \mathcal{B}(C_{R_+ \times E}[0, \infty))$, we have

$$\begin{aligned}
 P \times \tilde{P}\{(X, Z \circ X) \in A \times B\} &= \int \int I_{A \times B}(X, Z \circ X) \tilde{P}(d\tilde{\omega}) P(d\omega) \\
 &= \int \int I_A(X) I_B(Z \circ X) \tilde{P}(d\tilde{\omega}) P(d\omega) \\
 &= \int I_A(X) \tilde{P}\{Z \circ X \in B\} P(d\omega) \quad (4.2.18) \\
 &= \int I_A(x) \tilde{P}\{Z \circ x \in B\} P X^{-1}(dx) \\
 &= \int I_A(x) \hat{P}\{W \circ x \in B\} P X^{-1}(dx)
 \end{aligned}$$

$$\begin{aligned}
&= \int I_A(y) \hat{P}\{W \circ y \in B\} \bar{P}Y^{-1}(dy) \\
&= \bar{P} \times \hat{P}\{(Y, W \circ Y) \in A \times B\}.
\end{aligned}$$

Therefore, $P \times \tilde{P}(X, Z \circ X)^{-1} = \bar{P} \times \hat{P}(Y, W \circ Y)^{-1}$. Similarly, $P \times \tilde{P}(Z \circ X)^{-1} = \bar{P} \times \hat{P}(W \circ Y)^{-1}$. $\square$

Theorem 4.2.21. (Joint and marginal convergence) *Let $\{X_n\}$ and X be $C_{R_+}[0, \infty)$ -valued processes, defined on $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively, and $X_n \Rightarrow X$. Let (E, r) be separable. Suppose that $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$. Then $(X_n, Z_n \circ X_n) \Rightarrow (X, Z \circ X)$ and $Z_n \circ X_n \Rightarrow Z \circ X$.*

Proof. Use Skorohod representation theorem and Lemma 4.2.19 and Lemma 4.2.20. $\square$

Notice that although we have (4.2.15), we don't have that if $\{(x_n, z_n)\} \subset D_{R_+ \times E}[0, \infty)$, $(x, z) \in D_{R_+ \times E}[0, \infty)$,

$$\lim_{n \rightarrow \infty} d((x_n, z_n), (x, z)) = 0 \iff \lim_{n \rightarrow \infty} d(z_n, z) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_n, x) = 0. \quad (4.2.19)$$

Actually, it fails in both directions. Then for $x \in D_{R_+}[0, \infty)$, $f_x : D_E[0, \infty) \mapsto D_{R_+ \times E}[0, \infty)$, where $f_x(z) = (x, z \circ x)$ for $z \in D_E[0, \infty)$, might not be a continuous function.

Fortunately, we still have

Lemma 4.2.22. *Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Let $\{z_n\} \subset D_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d(z_n, z) = 0$ and such that $\{z_n \circ x_n\} \subset D_E[0, \infty)$. Then we have that $\lim_{n \rightarrow \infty} d((x_n, z_n \circ x_n), (x, z \circ x)) = 0$.*

Proof. We use a proof similar to that for Lemma 4.2.9. Let $\{t_n\} \subset [0, \infty)$, $t \geq 0$,

and $\lim_{n \rightarrow \infty} t_n = t$. Since $\lim_{n \rightarrow \infty} d(x_n, x) = 0$, we have that $\lim_{n \rightarrow \infty} |x_n(t_n) - x(t)| \wedge |x_n(t_n) - x(t-)| = 0$. $\{x_n(t_n)\}$ has possibly two kinds of disjoint convergent subsequences $\{x_{n_r}(t_{n_r})\}$ with $\lim_{r \rightarrow \infty} x_{n_r}(t_{n_r}) = x(t)$ and $\{x_{n_l}(t_{n_l})\}$ with $\lim_{l \rightarrow \infty} x_{n_l}(t_{n_l}) = x(t-)$.

By the emphasized fact in the proof of Lemma 4.2.9, we have $\lim_{r \rightarrow \infty} r(z_{n_r} \circ x_{n_r}(t_{n_r}), z \circ x(t)) = 0$ and $\lim_{l \rightarrow \infty} r(z_{n_l} \circ x_{n_l}(t_{n_l}), z \circ x(t-)) = 0$. Then we have by (4.2.15)

$$\lim_{r \rightarrow \infty} \tilde{r}((x_{n_r}(t_{n_r}), z_{n_r} \circ x_{n_r}(t_{n_r})), (x(t), z \circ x(t))) = 0, \quad (4.2.20)$$

and

$$\lim_{l \rightarrow \infty} \tilde{r}((x_{n_l}(t_{n_l}), z_{n_l} \circ x_{n_l}(t_{n_l})), (x(t-), z \circ x(t-))) = 0. \quad (4.2.21)$$

Then condition (a) of Lemma 4.2.6 is satisfied.

Now we check condition (b) of Lemma 4.2.6. If $\lim_{n \rightarrow \infty} \tilde{r}((x_n(t_n), z_n \circ x_n(t_n)), (x(t), z \circ x(t))) = 0$, we must have $\lim_{n \rightarrow \infty} x_n(t_n) = x(t)$. For any $s_n \geq t_n$ such that $\lim_{n \rightarrow \infty} s_n = t$, we have $\lim_{n \rightarrow \infty} x_n(s_n) = x(t)$ by $\lim_{n \rightarrow \infty} d(x_n, x) = 0$ and Lemma 4.2.6 (b). Then by the emphasized fact, $\lim_{n \rightarrow \infty} r(z_n \circ x_n(s_n), z \circ x(t)) = 0$. Then by (4.2.15) $\lim_{n \rightarrow \infty} \tilde{r}((x_n(s_n), z_n \circ x_n(s_n)), (x(t), z \circ x(t))) = 0$. (b) is then satisfied.

(c) can be checked similarly. Therefore, the conclusion holds. $\square$

We have the following two corollaries.

Corollary 4.2.23. *Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. Let $\{z_n\} \subset C_E[0, \infty)$, $z \in C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d_U(z_n, z) = 0$. Then $\lim_{n \rightarrow \infty} d((x_n, z_n \circ x_n), (x, z \circ x)) = 0$.*

Corollary 4.2.24. *Let $\{x_n\} \subset D_{R_+}[0, \infty)$, $x \in D_{R_+}[0, \infty)$, and $\lim_{n \rightarrow \infty} d(x_n, x) = 0$. $\{x_n\}$ and x are increasing functions on $[0, \infty)$. Let $\{z_n\} \subset D_E[0, \infty)$, $z \in$*

$C_E[0, \infty)$ and $\lim_{n \rightarrow \infty} d(z_n, z) = 0$. Then we have that $\lim_{n \rightarrow \infty} d((x_n, z_n \circ x_n), (x, z \circ x)) = 0$.

By Corollary 4.2.23, and 4.2.24 we know that 1) For $x \in D_{R_+}[0, \infty)$, $f_x : C_E[0, \infty) \mapsto D_{R_+ \times E}[0, \infty)$, where $f_x(z) = (x, z \circ x)$ for $z \in C_E[0, \infty)$, is a continuous function; 2) for a nondecreasing function $x \in D_{R_+}[0, \infty)$, $f_x : D_E[0, \infty) \mapsto D_{R_+ \times E}[0, \infty)$, where $f_x(z) = (x, z \circ x)$ for $z \in D_E[0, \infty)$, is a continuous function.

Then, we have the following convergence theorem.

Theorem 4.2.25. (Joint and marginal convergence) Let $\{X_n\}$ and X be $D_{R_+}[0, \infty)$ -valued processes, defined on $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively, and $X_n \Rightarrow X$. Let (E, r) be separable. Suppose that

- (1) $\{Z_n\}$ and Z are $C_E[0, \infty)$ -valued processes, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$;
- (2) $\{X_n\}$ and X are nondecreasing processes, $\{Z_n\}$ are $D_E[0, \infty)$ -valued processes and Z is a $C_E[0, \infty)$ -valued process, defined on $(\tilde{\Omega}_n, \tilde{\mathcal{F}}_n, \tilde{P}_n)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, and satisfy $Z_n \Rightarrow Z$.

Under either (1) or (2), we have that $(X_n, Z_n \circ X_n) \Rightarrow (X, Z \circ X)$ and $Z_n \circ X_n \Rightarrow Z \circ X$.

Remark 4.2.26. The time change on $C_{R_+ \times E}[0, \infty)$ and $D_{R_+ \times E}[0, \infty)$, or the joint convergence theorems in this subsection depend on the relation (4.2.15), which resorts to the definition of the metric $\tilde{r}$ in (4.2.14). Actually, for any definition of the metric on the product space $R_+ \times E$ which can guarantee the relation (4.2.15), we also have the same time change on $C_{R_+ \times E}[0, \infty)$ and $D_{R_+ \times E}[0, \infty)$, or the joint convergence theorems with different meanings.

4.3 Time inhomogeneous birth and death process in random environments

Now we come to the applications of the weak convergence theorems in section 4.2 on the example of time inhomogeneous linear growth birth death process.

4.3.1 The construction of the time inhomogeneous critical linear growth birth and death family

At first, let us define a standard critical linear growth birth and death family.

Definition 4.3.1. *A standard critical linear growth birth and death family is a process $\{\tilde{X}(t), t \geq 0\}$ on measurable space $(\Omega_2, \mathcal{F}_2)$, and a family of probability measures $\{P_2^x\}_{x \in R_+}$, such that*

- (1) *for each $F \in \mathcal{F}_2$, the mapping $x \mapsto P_2^x\{F\}$ is universally measurable;*
- (2) *for each $x \in R_+$, $P_2^x\{\tilde{X}(0) = x\} = 1$;*
- (3) *under each P_2^x , the process $\tilde{X}$ is a standard critical linear growth birth and death process, i.e., a linear growth birth and death process with parameters $\lambda = \mu = 1$, starting at x .*

We assume that $\lambda(\cdot)$ and $\mu(\cdot)$ are two nonnegative functions on $R_+ = [0, \infty)$, which are integrable on each bounded subinterval of R_+ . We are just interested in the case $\lambda(t) \equiv \mu(t)$ for each $t \in R_+$. We define a time inhomogeneous critical linear growth birth and death family driven by the rate function $\lambda(\cdot)$ as follows:

Definition 4.3.2. *The process X on the measurable space $(\Omega_2, \mathcal{F}_2)$, defined by $X(t) = \tilde{X}(\int_0^t \lambda(u) du)$ for $0 \leq t < \infty$, and the family of probability measures $\{P_2^x\}_{x \in R_+}$ specified in Definition 4.3.1 is called a time inhomogeneous critical linear growth birth and death family driven by the rate function $\lambda(\cdot)$.*

4.3.2 Birth and death process evolving in a stationary environment

A) Birth and death rates being stationary processes

We assume that $(\Omega_1, \mathcal{F}_1, P_1)$ is a probability space, $\{\lambda(t)\}_{t \in R_+}$, $\{\mu(t)\}_{t \in R_+}$ are nonnegative strong stationary processes. We assume that $E^{P_1}[\lambda(t)] < \infty$, $E^{P_1}[\mu(t)] < \infty$ for all $t \in R_+$. We will concentrate on the critical case $\lambda(t) = \mu(t)$, $0 \leq t < \infty$ P_1 -a.s.

We assume that $\{I_n\}_{n=1}^\infty$ is a sequence of integer-valued nonnegative r.v.s and I nonnegative r.v. defined on $(\Omega_1, \mathcal{F}_1, P_1)$ such that $\lim_{n \rightarrow \infty} \frac{I_n}{n} = I$ P_1 -a.s. We also assume that $\lambda_n(t) = \mu_n(t) = \lambda(t)$, $0 \leq t < \infty$ P_1 -a.s for each $n \geq 1$. For each $n \geq 1$, let $\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}$ be a sequence of standard critical linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$ together with the family of probability measures $\{P_{2,n}^x\}_{x \in R_+}$. Then we can define a sequence of processes on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$ as follows: $\hat{X}^{(n)}(t, \omega_1, \omega_2) = \tilde{X}^{(n)}(\int_0^t \lambda_n(u, \omega_1) du, \omega_2)$. For $n \geq 1$ and fixed $\omega_1 \in \Omega_1$, $\hat{X}^{(n)}(\omega_1)$ is a time inhomogeneous linear growth birth and death process on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$, starting at $I_n(\omega_1)$ under the probability measure $P_{2,n}^{I_n(\omega_1)}$, with instantaneous birth and death rate $\hat{X}^{(n)}(t, \omega_1) \lambda_n(\omega_1)$ at time $t > 0$.

Now we define the scaled sequence for the time inhomogeneous sequence. For each $n \geq 1$, and $0 \leq t < \infty$, let $X^{(n)}(t) = \frac{\hat{X}^{(n)}(nt)}{n}$, i.e.

$$X^{(n)}(t) = \frac{\tilde{X}^{(n)}(\int_0^{nt} \lambda_n(u) du)}{n} = \frac{\tilde{X}^{(n)}(\int_0^{nt} \lambda(u) du)}{n}. \quad (4.3.1)$$

We define the scaled sequence for the standard processes. For each $n \geq 1$, and $0 \leq t < \infty$, let

$$Z_n(t) = \frac{\tilde{X}^{(n)}(nt)}{n}. \quad (4.3.2)$$

We define for each $n \geq 1$, and $0 \leq t < \infty$,

$$Y_n(t) = \frac{1}{n} \int_0^{nt} \lambda_n(u) du = \frac{1}{n} \int_0^{nt} \lambda(u) du. \quad (4.3.3)$$

Then we have the relation, for each $n \geq 1$ and $0 \leq t < \infty$,

$$X^{(n)}(t) = Z_n \circ Y_n(t), \quad (4.3.4)$$

i.e. the scaled time inhomogeneous sequence $\{X^{(n)}\}$ is the scaled standard sequence $\{Z_n\}$ time changed by the sequence $\{Y_n\}$.

B) Convergence theorems when birth and death rates are stationary processes

Before we give the convergence theorems for the time inhomogeneous birth and death process, we give some related lemmas.

Lemma 4.3.3. *Let $\{x_n\}$ be a sequence of nonnegative integers, $x \geq 0$, and $\lim_{n \rightarrow \infty} \frac{x_n}{n} = x$. Then $P_{2,n}^{x_n} Z_n^{-1} \Rightarrow \tilde{P}^x Z^{-1}$. Here Z is a Feller Branching process defined on some measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ together with a family of probability measures $\{\tilde{P}^x\}_{x \in R_+}$, such that $(\tilde{\Omega}, \tilde{\mathcal{F}}, Z, \{\tilde{P}^x\}_{x \in R_+})$ is a Feller Branching family.*

Lemma 4.3.4. *We assume that $\lambda(\cdot)$ is a nonnegative function defined on $[0, \infty)$. $\lambda(\cdot)$ is integrable on bounded intervals, and satisfies $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \lambda(s) ds = \bar{\lambda}$. We define $x_n(t) = \frac{1}{n} \int_0^{nt} \lambda(s) ds$, and $x(t) = \bar{\lambda}t$ on $[0, \infty)$. Then $\{x_n\} \subset C_{R_+}[0, \infty)$, $x \in C_{R_+}[0, \infty)$, and satisfies $\lim_{n \rightarrow \infty} d_U(x_n, x) = 0$.*

Proof. By Lemma 4.2.1 of section 4.2, it suffices to prove that if $\{t_n\} \subset [0, \infty)$, $t \geq 0$, and $\lim_{n \rightarrow \infty} t_n = t$, then $\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{nt_n} \lambda(s) ds = \bar{\lambda}t$.

If $t > 0$, then $\lim_{n \rightarrow \infty} nt_n = \infty$. We have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{nt_n} \lambda(s) ds = \lim_{n \rightarrow \infty} \frac{1}{nt_n} \int_0^{nt_n} \lambda(s) ds \cdot \lim_{n \rightarrow \infty} t_n = \bar{\lambda}t. \quad (4.3.5)$$

If $t = 0$, we still have that $\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^{nt_n} \lambda(s) ds = 0 = \bar{\lambda}t$. Because for any subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\{n_k t_{n_k}\}$ is bounded, bounded by A , say, we have

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \int_0^{n_k t_{n_k}} \lambda(s) ds \leq \lim_{k \rightarrow \infty} \frac{1}{n_k} \int_0^A \lambda(s) ds = 0. \quad (4.3.6)$$

For any subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $\lim_{k \rightarrow \infty} n_k t_{n_k} = \infty$, we have

$$\lim_{k \rightarrow \infty} \frac{1}{n_k} \int_0^{n_k t_{n_k}} \lambda(s) ds = \lim_{k \rightarrow \infty} \frac{1}{n_k t_{n_k}} \int_0^{n_k t_{n_k}} \lambda(s) ds \cdot \lim_{k \rightarrow \infty} t_{n_k} = \bar{\lambda}0. \quad (4.3.7)$$

□

Now we have several convergence theorems for the time inhomogeneous birth and death process when parameters are strong stationary processes.

Theorem 4.3.5. (Quenched Convergence) *Let birth rate $\{\lambda(t)\}_{t \in R_+}$, defined on $(\Omega_1, \mathcal{F}_1, P_1)$, be a nonnegative strong stationary process satisfying $E^{P_1}[\lambda(t)] < \infty$ for all $t \in R_+$. Let $\{I_n\}_{n=1}^\infty$ be a sequence of integer-valued nonnegative r.v.s and I nonnegative r.v. defined on $(\Omega_1, \mathcal{F}_1, P_1)$ such that $\lim_{n \rightarrow \infty} \frac{I_n}{n} = I$ P_1 -a.s. Let*

$\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}_{n=1}^\infty$ be a sequence of standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$, together with the sequence of families of probability measures $\{\{P_{2,n}^x\}_{x \in R_+}\}_{n=1}^\infty$. Let $X^{(n)}, Z_n, Y_n$ be defined by (4.3.1), (4.3.2) and (4.3.3), Z be defined in Lemma 4.3.3, then $P_{2,n}^{I_n}(X^{(n)})^{-1} = P_{2,n}^{I_n}(Z_n \circ Y_n)^{-1} \Rightarrow \tilde{P}^I(Z \circ Y)^{-1}$ P_1 -a.s. where Y is a process on $(\Omega_1, \mathcal{F}_1, P_1)$ defined by $Y(t) = \bar{\lambda}t$ for $0 \leq t < \infty$.

Proof. By Ergodic theorem, Lemma 4.3.3, Lemma 4.3.4 and Theorem 4.2.16 of Section 4.2. $\square$

Theorem 4.3.6. (Joint and marginal Convergence) *With the same conditions as those for Theorem 4.3.5, we have $(Y_n, X_n) \Rightarrow (Y, Z \circ Y)$ and $X_n \Rightarrow Z \circ Y$.*

Proof. By Ergodic theorem, Lemma 4.3.3, Lemma 4.3.4 and Theorem 4.2.25 of Section 4.2. $\square$

Theorem 4.3.7. (Annealed Convergence) *With the same conditions as those for Theorem 4.3.5, we have $E^{P_1}[P_{2,n}^{I_n}X_n^{-1}] \Rightarrow E^{P_1}[\tilde{P}^I(Z \circ Y)^{-1}]$.*

Proof. By Ergodic theorem, Lemma 4.3.3, Lemma 4.3.4 and Theorem 4.2.18 of Section 4.2. $\square$

4.3.3 The stable environment for the birth and death rates

A) Birth and death rates defined by a sequence in DOA

We still concentrate on the time inhomogeneous critical linear growth birth and death processes with birth rates evolving in a random environment.

Let $\xi, \{\xi_n\}_{n=1}^\infty$ be nonnegative i.i.d r.v.s, defined on some probability space $(\Omega_1, \mathcal{F}_1, P_1)$, and ξ is in the domain of attraction of Stable- α law with distribution

function F , where $0 < \alpha < 1$. We assume that $\{a_n\}_{n=1}^\infty$ satisfies

$$\lim_{n \rightarrow \infty} F^{*n}(a_n x) = G_\alpha(x), \quad (4.3.8)$$

for $x \geq 0$. F has Laplace transform φ and G_α has Laplace transform $\gamma_\alpha(s) = e^{-s^\alpha}$.

We define λ on $(\Omega_1, \mathcal{F}_1, P_1)$: for $0 \leq t < \infty$, if $k-1 \leq t < k$ for some integer $k \geq 1$,

$$\lambda(t) = \xi_k. \quad (4.3.9)$$

Then we define the sequence of the birth rates $\{\lambda_n\}$ on $(\Omega_1, \mathcal{F}_1, P_1)$: for each $n \geq 1$,

$$\lambda_n(t) = \frac{n}{a_n} \lambda. \quad (4.3.10)$$

We still assume that $\{I_n\}_{n=1}^\infty$ and I , defined on $(\Omega_1, \mathcal{F}_1, P_1)$, satisfy $\lim_{n \rightarrow \infty} \frac{I_n}{n} = I$ P_1 -a.s. For each $n \geq 1$, we assume that $\mu_n(t) = \lambda_n(t)$, $0 \leq t < \infty$ P_1 -a.s and $\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}$ be a standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$ together with the family of probability measures $\{P_{2,n}^x\}_{x \in R_+}$. We then define a sequence of processes on the product space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2)$: $\hat{X}^{(n)}(t, \omega_1, \omega_2) = \tilde{X}^{(n)}(\int_0^{[t]} \lambda_n(u, \omega_1) du, \omega_2)$. Notice that the upper limit for the integral of λ_n is $[t]$ instead of t . The reason for this will be clear in the next subsection.

Then we can define the scaled sequence as follows. For each $n \geq 1$, and $0 \leq t < \infty$,

$$X^{(n)}(t) = \frac{\tilde{X}^{(n)}(\int_0^{[nt]} \lambda_n(u) du)}{n} = \frac{\tilde{X}^{(n)}(\frac{n}{a_n} \int_0^{[nt]} \lambda(u) du)}{n}. \quad (4.3.11)$$

We define for each $n \geq 1$, and $0 \leq t < \infty$,

$$Y_n(t) = \frac{1}{n} \int_0^{[nt]} \lambda_n(u) du = \frac{1}{a_n} \int_0^{[nt]} \lambda(u) du. \quad (4.3.12)$$

Then we have the relation, for each $n \geq 1$ and $0 \leq t < \infty$,

$$X^{(n)}(t) = Z_n \circ Y_n(t), \quad (4.3.13)$$

i.e. the scaled time inhomogeneous sequence $\{X^{(n)}\}$ is the scaled standard sequence $\{Z_n\}$ time changed by the sequence $\{Y_n\}$.

Before we prove the convergence theorems for the scaled sequence $\{X^{(n)}\}$, we will consider the weak convergence of $\{Y_n\}$.

B) Stable analogue of invariance principle

Let $S_0 = 0$, and $S_k = \sum_{j=1}^k \xi_j$, for $k \geq 1$. Then, for $0 \leq t < \infty$ and $n \geq 1$,

$$Y_n(t) = a_n^{-1} S_{[nt]}. \quad (4.3.14)$$

By similarity with invariance principle, $Y_n \Rightarrow S_\alpha$, where S_α is a stable process defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$. For each $0 < t < \infty$, $S_\alpha(t)$ has the distribution function $Q_t(x) = G_\alpha(t^{-\frac{1}{\alpha}}x)$.

We will prove the statement by two steps. At first, we prove the following proposition.

Proposition 4.3.8. *With $\{Y_n\}$ satisfying (4.3.14), $d \geq 1$ and $0 \leq t_1 < \dots < t_d < \infty$, we have*

$$(Y_n(t_1), \dots, Y_n(t_d)) \Rightarrow (S_\alpha(t_1), \dots, S_\alpha(t_d)) \quad (4.3.15)$$

as $n \rightarrow \infty$, where S_α is a α -stable process.

Proof. We just consider the case $d = 2$, the other cases can be proved similarly. Fix $0 \leq t_1 < t_2$. We wish to prove that as $n \rightarrow \infty$,

$$(Y_n(t_1), Y_n(t_2)) \Rightarrow (S_\alpha(t_1), S_\alpha(t_2)), \quad (4.3.16)$$

i.e.

$$a_n^{-1}(S_{[nt_1]}, S_{[nt_2]} - S_{[nt_1]}) \Rightarrow (S_\alpha(t_1), S_\alpha(t_2) - S_\alpha(t_1)). \quad (4.3.17)$$

We consider the following Laplace transform. For $u, v \geq 0$,

$$\begin{aligned} E[\exp\{-ua_n^{-1}S_{[nt_1]} - va_n^{-1}(S_{[nt_2]} - S_{[nt_1]})\}] \\ = E[\exp\{-ua_n^{-1}S_{[nt_1]}\}]E[\exp\{-va_n^{-1}(S_{[nt_2]} - S_{[nt_1]})\}]. \end{aligned} \quad (4.3.18)$$

We will prove that

$$\lim_{n \rightarrow \infty} E[\exp\{-ua_n^{-1}S_{[nt_1]}\}] = e^{-t_1 u^\alpha}, \quad (4.3.19)$$

and

$$\lim_{n \rightarrow \infty} E[\exp\{-va_n^{-1}(S_{[nt_2]} - S_{[nt_1]})\}] = e^{-(t_2 - t_1)v^\alpha}. \quad (4.3.20)$$

We have the condition (4.3.8) which is equivalent to

$$\lim_{n \rightarrow \infty} -n \log \varphi(a_n^{-1}s) = -\log \gamma_\alpha(s) = s^\alpha. \quad (4.3.21)$$

Now we prove (4.3.19) which is equivalent to

$$\lim_{n \rightarrow \infty} [\varphi(a_n^{-1}u)]^{[nt_1]} = e^{-t_1 u^\alpha}, \quad (4.3.22)$$

and

$$\lim_{n \rightarrow \infty} -[nt_1] \log \varphi(a_n^{-1}u) = t_1 u^\alpha. \quad (4.3.23)$$

By (4.3.21), We immediately have (4.3.23). (4.3.19) is then proved.

Similarly, (4.3.20) is equivalent to

$$\lim_{n \rightarrow \infty} [\varphi(a_n^{-1}v)]^{[nt_2] - [nt_1]} = e^{-(t_2 - t_1)v^\alpha}, \quad (4.3.24)$$

and

$$\lim_{n \rightarrow \infty} -([nt_2] - [nt_1]) \log \varphi(a_n^{-1}v) = (t_2 - t_1)v^\alpha. \quad (4.3.25)$$

This is immediate by (4.3.21). □

Remark 4.3.9. *If we define $\tilde{Y}_n(t)$ being continuous as follows:*

$$\tilde{Y}(t) = S_{[t]} + (t - [t])\xi_{[t]+1}, \text{ for } t \geq 0, \quad (4.3.26)$$

$$\tilde{Y}_n(t) = a_n^{-1}\tilde{Y}(nt). \quad (4.3.27)$$

Then we can still prove proposition 4.3.8 for this situation:

$$(\tilde{Y}_n(t_1), \dots, \tilde{Y}_n(t_d)) \Rightarrow (S_\alpha(t_1), \dots, S_\alpha(t_d)). \quad (4.3.28)$$

But we need one more step. Since $|\tilde{Y}_n(t) - Y_n(t)| \leq a_n^{-1}\xi_{[nt]+1}$,

$$P\{|\tilde{Y}_n(t) - Y_n(t)| > \epsilon\} \leq P\{\xi_{[nt]+1} > a_n\epsilon\} = 1 - F(a_n\epsilon). \quad (4.3.29)$$

By Theorem 2 page 448, Feller, we have

$$1 - F(a_n\epsilon) \sim \frac{L(a_n\epsilon)}{(a_n\epsilon)^\alpha \Gamma(1-\alpha)} = \frac{L(a_n\epsilon)}{L(a_n)} \frac{nL(a_n)}{a_n^\alpha} \frac{1}{n\epsilon^\alpha \Gamma(1-\alpha)} \quad (4.3.30)$$

where L is a slowly varying function. By Theorem 2 again, $\lim_{n \rightarrow \infty} \frac{nL(a_n)}{a_n^\alpha} = 1$. By the definition of slowly varying function, $\lim_{n \rightarrow \infty} \frac{L(a_n\epsilon)}{L(a_n)} = 1$. Therefore

$$\lim_{n \rightarrow \infty} 1 - F(a_n\epsilon) = \lim_{n \rightarrow \infty} \frac{L(a_n\epsilon)}{(a_n\epsilon)^\alpha \Gamma(1-\alpha)} = 0. \quad (4.3.31)$$

i.e. $\tilde{Y}_n(t) - Y_n(t) \rightarrow 0$ in probability as $n \rightarrow \infty$. It is then clear that

$$\|(\tilde{Y}_n(t_1), \tilde{Y}_n(t_2)) - (Y_n(t_1), Y_n(t_2))\| \rightarrow 0 \text{ in probability as } n \rightarrow \infty. \quad (4.3.32)$$

Then we can follow the steps in the proof of proposition 4.3.8 to prove our new statement.

Proposition 4.3.10. *$\{Y_n\}$ defined above is a tight sequence. So by proposition 4.3.8, $Y_n \Rightarrow S_\alpha$.*

Proof. We just need to check Aldous condition for tightness.

By proposition 4.3.8, $Y_n(t) \Rightarrow S_\alpha(t)$ for each $0 \leq t < \infty$. Then Aldous first condition is satisfied.

Now we check Aldous second condition. For given stopping times $\{\tau_n\}$ bounded by $T > 0$ and $\{\delta_n\}$ satisfying $\lim_{n \rightarrow \infty} \delta_n = 0$. Then we have to prove that $Y_n(\tau_n + \delta_n) - Y_n(\tau_n)$ converges to 0 in probability. This is equivalent to prove that

$$\lim_{n \rightarrow \infty} E[\exp\{-va_n^{-1}(S_{[n(\tau_n + \delta_n)]} - S_{[n\tau_n]})\}] = 1, \text{ for any } v \geq 0. \quad (4.3.33)$$

Since

$$\begin{aligned} E[\exp\{-va_n^{-1}(S_{[n(\tau_n + \delta_n)]} - S_{[n\tau_n]})\}] &= E[E[\exp\{-va_n^{-1}(S_{[n(\tau_n + \delta_n)]} - S_{[n\tau_n]})\} | \mathcal{F}_{\tau_n}^Y]] \\ &= E[E[\exp\{-va_n^{-1}(S_{[n(\tau_n + \delta_n)]} - S_{[n\tau_n]})\} | S_{[n\tau_n]}]] \end{aligned} \quad (4.3.34)$$

We then prove that

$$\lim_{n \rightarrow \infty} E[\exp\{-va_n^{-1}(S_{[n(\tau_n + \delta_n)]} - S_{[n\tau_n]})\} | S_{[n\tau_n]}] = 1, \text{ a.s.} \quad (4.3.35)$$

instead. Again, it is equivalent to

$$\lim_{n \rightarrow \infty} -([n(\tau_n + \delta_n)] - [n\tau_n]) \log \varphi(a_n^{-1}v) = 0. \quad (4.3.36)$$

Since F is in the domain of attraction, by (4.3.21) we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} -([n(\tau_n + \delta_n)] - [n\tau_n]) \log \varphi(a_n^{-1}v) \\ &= \lim_{n \rightarrow \infty} \frac{[n(\tau_n + \delta_n)] - [n\tau_n]}{n} [-n \log \varphi(a_n^{-1}v)] \\ &= \lim_{n \rightarrow \infty} \frac{[n(\tau_n + \delta_n)] - [n\tau_n]}{n} \lim_{n \rightarrow \infty} -n \log \varphi(a_n^{-1}v) \\ &= \lim_{n \rightarrow \infty} \frac{[n(\tau_n + \delta_n)] - [n\tau_n]}{n} v^\alpha \end{aligned} \quad (4.3.37)$$

By the following inequality

$$\frac{n\delta_n - 1}{n} < \frac{[n(\tau_n + \delta_n)] - [n\tau_n]}{n} < \frac{n\delta_n + 1}{n}, \quad (4.3.38)$$

It's obvious that $\lim_{n \rightarrow \infty} \frac{[n(\tau_n + \delta_n)] - [n\tau_n]}{n} = 0$. Therefore, $Y_n(\tau_n + \delta_n) - Y_n(\tau_n)$ converges to 0 in probability. $\square$

Remark 4.3.11. $\{\tilde{Y}_n\}$ defined by (4.3.26) and (4.3.27) is not a tight sequence. Otherwise, we would have $\tilde{Y}_n \Rightarrow S_\alpha$, which contradicts with the fact that S_α is not a continuous process. This is the reason we use $[nt]$ as the upper limit for the integral of λ_n instead of nt in (4.3.11).

C) Joint, marginal, and annealed convergence for birth and death rates defined by a sequence in DOA

Now we can give the joint, marginal, and annealed weak convergence theorems for birth and death processes with parameters evolving in stable environment, but we can not give the quenched weak convergence theorem for this case since Y_n can not converge P_1 -a.s.

Theorem 4.3.12. (Joint and marginal Convergence) Let $\{\lambda(t)\}_{t \in R_+}$ be defined on $(\Omega_1, \mathcal{F}_1, P_1)$ by (4.3.9) and $\{\lambda_n\}$ be defined by (4.3.10). Let $\{I_n\}_{n=1}^\infty$ be a sequence of integer-valued nonnegative r.v.s defined on $(\Omega_1, \mathcal{F}_1, P_1)$ and nonnegative r.v. I , S_α be defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that $\frac{I_n}{n} \Rightarrow I$. Let $\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}_{n=1}^\infty$ be a sequence of standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$, together with the sequence of families of probability measures $\{\{P_{2,n}^x\}_{x \in R_+}\}_{n=1}^\infty$. Let $X^{(n)}$ and Y_n be defined by (4.3.12) and (4.3.13), Z_n be defined by (4.3.2), Z be defined in Lemma 4.3.3, then $(Y_n, X_n) \Rightarrow (S_\alpha, Z \circ S_\alpha)$ and $X_n \Rightarrow Z \circ S_\alpha$.

Proof. By Lemma 4.3.3, Proposition 4.3.10 and Theorem 4.2.25 of Section 4.2. $\square$

Theorem 4.3.13. (Annealed Convergence) With the same conditions as those for Theorem 4.3.12, we have $E^{P_1}[P_{2,n}^{I_n} X_n^{-1}] \Rightarrow E^{\tilde{P}}[\tilde{P}^I (Z \circ S_\alpha)^{-1}]$.

Proof. By Lemma 4.3.3 , Proposition 4.3.10 and Theorem 4.2.18 of Section 4.2. $\square$

D) Quenched convergence for birth and death rates defined by a sequence in DOA

We do have quenched weak convergence theorem for birth and death processes with parameters evolving in certain stable environment. We will give a lemma which can be proved by Skorohod representation theorem and then give the quenched convergence theorem.

Lemma 4.3.14. *There exists on certain probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ a sequence of random variables $\{\xi_{n,k}\}_{n,k=1}^{\infty}$ and a stable process S_{α} such that for each $n \geq 1$, $\{\xi_{n,k}\}_{k=1}^{\infty}$ are independent and identically distributed with distribution function F and if for each $n \geq 1$, we define $\lambda_n(t) = \frac{n}{a_n} \xi_{n,k}$ for $k-1 \leq t < k$, and $Y_n(t) = \frac{1}{n} \int_0^{[nt]} \lambda_n(u) du$, we have $\lim_{n \rightarrow \infty} d(Y_n, S_{\alpha}) = 0$, $\hat{P}$ -a.s.*

Theorem 4.3.15. (Quenched Convergence) *Let $\{\lambda_n\}$ be defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ as in Lemma 4.3.14. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of integer-valued nonnegative r.v.s and I nonnegative r.v. defined on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ such that $\lim_{n \rightarrow \infty} \frac{I_n}{n} = I$ $\hat{P}$ -a.s. Let $\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}_{n=1}^{\infty}$ be a sequence of standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$, together with the sequence of families of probability measures $\{\{P_{2,n}^x\}_{x \in R_+}\}_{n=1}^{\infty}$. Let Y_n be defined in Lemma 4.3.14 and $X^{(n)}$ be defined by $X^{(n)}(t) = \frac{\tilde{X}^{(n)}(\int_0^{[nt]} \lambda_n(u) du)}{n}$ for $0 \leq t < \infty$, Z_n be defined by (4.3.2), Z be defined in Lemma 4.3.3, then $P_{2,n}^{I_n}(X^{(n)})^{-1} = P_{2,n}^{I_n}(Z_n \circ Y_n)^{-1} \Rightarrow \tilde{P}^I(Z \circ S_{\alpha})^{-1}$ $\hat{P}$ -a.s.*

Proof. By Lemma 4.3.3, Lemma 4.3.14 and Theorem 4.2.16 of Section 4.2. $\square$

4.3.4 Weak convergence of $X^{(n)}$ when $\frac{\lambda_n}{n} \Rightarrow \lambda$

A) Convergence theorems for $X^{(n)}$ under general assumptions on $\{\lambda_n\}$ and λ

In subsection 4.3.2, we define λ on probability space $(\Omega_1, \mathcal{F}_1, P_1)$, and define $\lambda_n = \mu_n = \lambda$ for each $n \geq 1$. In subsection 4.3.3, we define λ by a sequence $\{\xi_n\}$ in DOA on probability space $(\Omega_1, \mathcal{F}_1, P_1)$ and then define $\lambda_n = \mu_n = \frac{n}{a_n} \lambda$. Now instead of specifying structures on $\{\lambda_n\}$, we consider a general assumption on $\{\lambda_n\}$. We assume that $D_{R_+}[0, \infty)$ valued process λ_n on probability space $(\Omega_n, \mathcal{F}_n, P_n)$ satisfy $\frac{\lambda_n}{n} \Rightarrow \lambda$, where λ is a $D_{R_+}[0, \infty)$ valued process defined on $(\Omega, \mathcal{F}, P)$. It is clear that λ_n is integrable over each bounded subinterval of R_+ P_n -a.s. and λ is integrable over each bounded subinterval of R_+ P -a.s.

We assume that $\{I_n\}_{n=1}^\infty$ and I , defined on $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively, satisfy $P_n(\frac{I_n}{n})^{-1} \Rightarrow PI^{-1}$. We still assume that $\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}$ be a standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$ together with the family of probability measures $\{P_{2,n}^x\}_{x \in R_+}$. Then we can define the scaled sequence. For each $n \geq 1$, and $0 \leq t < \infty$,

$$X^{(n)}(t) = \frac{\tilde{X}^{(n)}(\int_0^t \lambda_n(u) du)}{n}. \quad (4.3.39)$$

We define for each $n \geq 1$, and $0 \leq t < \infty$,

$$Y_n(t) = \frac{1}{n} \int_0^t \lambda_n(u) du. \quad (4.3.40)$$

Then we have the relation, for each $n \geq 1$ and $0 \leq t < \infty$,

$$X^{(n)}(t) = Z_n \circ Y_n(t). \quad (4.3.41)$$

Lemma 4.3.16. *Let $Y(t) = \int_0^t \lambda(u)du$ for each $0 \leq t < \infty$, then $Y_n \Rightarrow Y$.*

Proof. We define $f : D_{R_+}[0, \infty) \mapsto D_{R_+}[0, \infty)$ by $f(x)(t) = \int_0^t x(u)du$ for $x \in D_{R_+}[0, \infty)$. f is a continuous function by Problem 26 of Chapter 3 [EK86]. Since $\frac{\lambda_n}{n} \Rightarrow \lambda$, $f(\frac{\lambda_n}{n}) \Rightarrow f(\lambda)$, i.e. $Y_n \Rightarrow Y$. $\square$

We then give the joint, marginal, and annealed weak convergence theorems for this situation.

Theorem 4.3.17. (Joint and marginal Convergence) *Let $D_{R_+}[0, \infty)$ valued process λ_n on probability space $(\Omega_n, \mathcal{F}_n, P_n)$ satisfy $\frac{\lambda_n}{n} \Rightarrow \lambda$, where λ is a $D_{R_+}[0, \infty)$ valued process defined on $(\Omega, \mathcal{F}, P)$. Let $\{I_n\}_{n=1}^\infty$ and I , defined on $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively, satisfy $P_n(\frac{I_n}{n})^{-1} \Rightarrow P I^{-1}$. Let $\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}_{n=1}^\infty$ be a sequence of standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$, together with the sequence of families of probability measures $\{\{P_{2,n}^x\}_{x \in R_+}\}_{n=1}^\infty$. Let $X^{(n)}$ and Y_n be defined by (4.3.39) and (4.3.40), Z_n be defined by (4.3.2), Z be defined in Lemma 4.3.3, then $(Y_n, X_n) \Rightarrow (Y, Z \circ Y)$ and $X_n \Rightarrow Z \circ Y$.*

Proof. By Lemma 4.3.3, Lemma 4.3.16 and Theorem 4.2.25 of Section 4.2. $\square$

Theorem 4.3.18. (Annealed Convergence) *With the same conditions as those for Theorem 4.3.17, we have $E^{P_n}[P_{2,n}^{I_n} X_n^{-1}] \Rightarrow E^P[\tilde{P}^I(Z \circ Y)^{-1}]$.*

Proof. By Lemma 4.3.3, Lemma 4.3.16 and Theorem 4.2.18 of Section 4.2. $\square$

B) Time Slowing-down case

We have an interesting special case called **time-slowng down**. Let λ be defined on $(\Omega_1, \mathcal{F}_1, P_1)$ such that λ is integrable over each bounded subinterval of R_+ P_1 -a.s. Note that we don't need to assume that λ is $D_{R_+}[0, \infty)$ valued. Let $\lambda_n =$

$n\lambda$ for each $n \geq 1$. We have actually $\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \lambda$ for each $\omega_1 \in \Omega_1$. Then we have the above two convergence theorems as well as the quenched convergence theorem for this special case. This case is important because we have some interesting explanations. Let us adopt the notations and assumptions on $\{\tilde{X}^{(n)}\}$. We define the time inhomogeneous critical linear growth birth and death sequence $\{\hat{X}^{(n)}\}$ by

$$\hat{X}^{(n)}(t) = \tilde{X}^{(n)}\left(\int_0^t \lambda\left(\frac{u}{n}\right) du\right). \quad (4.3.42)$$

Then we have

$$X^{(n)}(t) = \frac{\tilde{X}^{(n)}\left(\int_0^t n\lambda(u) du\right)}{n} = \frac{\tilde{X}^{(n)}\left(\int_0^{nt} \lambda\left(\frac{u}{n}\right) du\right)}{n} = \frac{\hat{X}^{(n)}(nt)}{n}. \quad (4.3.43)$$

So $\{X^{(n)}\}$ is the scaled sequence of $\{\hat{X}^{(n)}\}$. But notice that the birth rate of $\hat{X}^{(n)}$ is $\lambda(\frac{t}{n})$ at time t . This is exactly like the time being slowing down proportionately.

C) Critical catalytic Branching case

Under this case, we assume that for each $n \geq 1$, C_n is a critical linear growth birth and death process with birth and death rate γ , defined on probability space $(\Omega_n, \mathcal{F}_n, P_n)$, starting at nonnegative integer c_n satisfying $\lim_{n \rightarrow \infty} \frac{c_n}{n} = c$. We also assume that $\tilde{X}^{(n)}$ is a standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$, together with a family of probability measures $\{P_{2,n}^x\}_{x \in R_+}$. Then we can define a sequence of birth and death processes with birth and death rates given by a second birth and death process.

For $n \geq 1$, let $\lambda_n(t) = C_n(nt)$ for $0 \leq t < \infty$. Thus we have $\frac{\lambda_n}{n} \Rightarrow \lambda$, where λ is a Feller Branching process with branching rate γ , starting at c , defined on some probability space $(\Omega, \mathcal{F}, P)$. We define $X^{(n)}$ by (4.3.39), Y_n by (4.3.40) and Y as that in Lemma 4.3.16. Then by Theorem 4.3.17 and 4.3.18, we have the following

two corollaries.

Corollary 4.3.19. (*Joint and marginal Convergence*) For each $n \geq 1$, let C_n be a critical linear growth birth and death process with birth and death rate γ , defined on probability space $(\Omega_n, \mathcal{F}_n, P_n)$, starting at nonnegative integer c_n satisfying $\lim_{n \rightarrow \infty} \frac{c_n}{n} = c$. Let λ_n and λ be defined as above. Let $\{I_n\}_{n=1}^\infty$ and I , defined on $(\Omega_n, \mathcal{F}_n, P_n)$ and $(\Omega, \mathcal{F}, P)$ respectively, satisfy $P_n(\frac{I_n}{n})^{-1} \Rightarrow PI^{-1}$. Let $\{\tilde{X}^{(n)}(t), 0 \leq t < \infty\}_{n=1}^\infty$ be a sequence of standard linear growth birth and death processes defined on $(\Omega_2, \mathcal{F}_2)$, together with the sequence of families of probability measures $\{\{P_{2,n}^x\}_{x \in R_+}\}_{n=1}^\infty$. Let $X^{(n)}$ and Y_n be defined by (4.3.39) and (4.3.40), Z_n be defined by (4.3.2), Z be defined in Lemma 4.3.3, then $(Y_n, X_n) \Rightarrow (Y, Z \circ Y)$ and $X_n \Rightarrow Z \circ Y$.

Corollary 4.3.20. (*Annealed Convergence*) With the same conditions as those for Corollary 4.3.19, we have $E^{P_n}[P_{2,n}^{I_n} X_n^{-1}] \Rightarrow E^P[\tilde{P}^I(Z \circ Y)^{-1}]$.

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