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EXTENSIONS AND ANALOGUES OF THE CHOWLA-SELBERG FORMULA

by

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A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

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Extensions and Analogues of the Chowla-Selberg Formula

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Abstract

The Chowla-Selberg formula for classes is extended to arbitrary negative discriminants. This result is used to obtain expressions for Weber's functions at $\sqrt{-n}$ for an arbitrary positive integer n . Also obtained are analogues for positive discriminants of the Chowla-Selberg formula for genera and the Chowla-Selberg formula for classes. These analogues are used to explicitly evaluate certain definite integrals.

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Introduction

We use (a, b, c) to denote the primitive, integral, binary quadratic form $ax^2 + bxy + cy^2$. The discriminant of the form (a, b, c) is the integer $d = b^2 - 4ac$. The class of the form (a, b, c) is denoted by $[a, b, c]$. The classes of primitive, integral binary quadratic forms of discriminant d (only positive-definite forms are used if $d < 0$) form a finite abelian group $H(d)$ under composition. The cosets of the subgroup of squares in $H(d)$ are called genera, and we denote the group of genera of discriminant d by $G(d)$. The Dedekind eta function $\eta(z)$ is defined for all complex numbers z with $\text{Im}(z) > 0$ by

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty} (1 - e^{2\pi imz}).$$

If $[a, b, c] = [a_1, b_1, c_1] \in H(d)$ with $d < 0$, a simple calculation, using the basic properties of the Dedekind eta function (given for example in [25]) shows that

$$a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| = a_1^{-1/4} \left| \eta \left(\frac{b_1 + \sqrt{d}}{2a_1} \right) \right|,$$

so that

$$\prod_{[a,b,c] \in H(d)} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right|$$

is well defined. In 1949 Chowla and Selberg (see [3] and [24]) gave an explicit formula for

$$\prod_{[a,b,c] \in H(d)} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right|$$

when d is a negative fundamental discriminant. In the early 1990's this formula was extended to arbitrary negative discriminants by Kaneko [15], Nakajima and Taguchi [20], and Kaplan and Williams [16]. In 1993 Williams and Zhang [28] extended the Chowla-Selberg formula to genera, when d is a negative fundamental discriminant, by obtaining a formula for

$$\prod_{[a,b,c] \in G} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right|,$$

where $G \in G(d)$. In 1995 Huard, Kaplan and Williams [14] extended the Chowla-Selberg formula for genera to arbitrary negative discriminants. Recently van der Poorten and Williams [23] have extended the Chowla-Selberg formula to classes, when d is a negative fundamental discriminant, by obtaining a formula for each individual term $a^{-1/4} |\eta((b + \sqrt{d})/(2a))|$ in the Chowla-Selberg formula.

Weber's functions $f(z)$, $f_1(z)$ and $f_2(z)$ are defined for all complex numbers z with $\text{Im}(z) > 0$ as follows

$$\begin{aligned} f(z) &= \frac{e^{-\pi i/24} \eta\left(\frac{1+z}{2}\right)}{\eta(z)}, \\ f_1(z) &= \frac{\eta\left(\frac{z}{2}\right)}{\eta(z)}, \\ f_2(z) &= \sqrt{2} \frac{\eta(2z)}{\eta(z)}. \end{aligned}$$

In this thesis we extend the Chowla-Selberg formula for classes due to van der Poorten and Williams to arbitrary negative discriminants (see Theorem

6.3) and apply this result to obtain new expressions for $f(\sqrt{-n})$, $f_1(\sqrt{-n})$ and $f_2(\sqrt{-n})$, where n is a positive integer (see Theorem 6.8). We also obtain a formula valid for positive discriminants which is an analogue of the Chowla-Selberg formula for genera (see Theorem 3.3). We apply this formula to explicitly evaluate the definite integrals $\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt$ and $\int_0^1 \frac{\tan^{-1} t^\epsilon}{1+t^2} dt$ for certain quadratic units ϵ (see Chapter 4). Our results include those of Herglotz [13] who had determined $\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt$ for $\epsilon = 4 + \sqrt{15}$, $6 + \sqrt{35}$ and $12 + \sqrt{143}$. We also obtain an analogue for positive discriminants of the Chowla-Selberg formula for classes (see Theorem 7.3).

Chapter 1

Primary representations by binary quadratic forms of discriminant d

1.1 Introduction

A non-zero integer d is called a discriminant if $d \equiv 0$ or $1 \pmod{4}$. We set

$$d = \Delta(d)f(d)^2, \tag{1.1}$$

where $f(d)$ is the largest positive integer for which $\Delta(d) = d/f(d)^2$ is a discriminant. When convenient, we write $\Delta = \Delta(d)$ and $f = f(d)$. The discriminant d is called fundamental if $f(d) = 1$. A discriminant is fundamental if and only if it is not divisible by the square of any odd prime and is either odd or $\equiv 8$ or $12 \pmod{16}$. The discriminant $\Delta(d)$ is fundamental. If $d = \Delta' f'^2$, where Δ' is a fundamental discriminant and f' is a positive integer, then $\Delta(d) = \Delta'$ and $f(d) = f'$ (see [17], Theorem 213). Also, if d is a discriminant and k is a positive integer then dk^2 is also a discriminant and

$$\Delta(dk^2) = \Delta(d), \quad f(dk^2) = f(d)k. \tag{1.2}$$

If m is positive integer with $m \mid f$ then $m^2 \mid d$, d/m^2 is a discriminant and

$$\Delta\left(\frac{d}{m^2}\right) = \Delta, \quad f\left(\frac{d}{m^2}\right) = \frac{f}{m}. \quad (1.3)$$

For the rest of this chapter, d represents a non-square discriminant. For any positive integer m we use $(\frac{d}{m})$ to denote the Kronecker symbol. Occasionally we write (d/m) instead of $(\frac{d}{m})$. For integers a , b , and c , we use (a, b, c) to denote the integral binary quadratic form $ax^2 + bxy + cy^2$. A form (a, b, c) with $b^2 - 4ac = d$ is called a form of discriminant d . The form (a, b, c) is called primitive if $\gcd(a, b, c) = 1$. Otherwise (a, b, c) is called imprimitive. An integer n is said to be represented by the form (a, b, c) if there exist integers x and y with $ax^2 + bxy + cy^2 = n$. Such a pair (x, y) is called a representation of n by (a, b, c) . If such integers x and y exist with $\gcd(x, y) = 1$, we say that (a, b, c) properly represents n (sometimes $\gcd(x, y)$ is written as (x, y)). All forms of non-square discriminant are irreducible in $\mathbb{Z}[x, y]$ and can be classified as follows. Let d be the discriminant of (a, b, c) . If $d > 0$ then (a, b, c) represents both positive and negative integers. Such a form is called indefinite. If $d < 0$, $a > 0$ then (a, b, c) represents only non-negative integers and represents zero only for $x = y = 0$. Such a form is called positive-definite. If $d < 0$, $a < 0$ then (a, b, c) represents only non-positive integers and represents zero only for $x = y = 0$. Such a form is called negative-definite.

Two forms (a, b, c) and (a', b', c') are said to be equivalent if there exist integers $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma = 1$ such that the substitution

$$x = \alpha X + \beta Y, \quad y = \gamma X + \delta Y$$

transforms $ax^2 + bxy + cy^2$ into $a'X^2 + b'XY + c'Y^2$, in which case we have

$$\begin{aligned} a' &= a\alpha^2 + b\alpha\gamma + c\gamma^2, \\ b' &= 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta, \\ c' &= a\beta^2 + b\beta\delta + c\delta^2. \end{aligned}$$

If (a, b, c) and (a', b', c') are equivalent, we write $(a, b, c) \sim (a', b', c')$. We note that $(a, b, c) \sim (a, 2at+b, at^2+bt+c)$ for any integer t . Also $(a, b, c) \sim (c, -b, a)$. The relation $\sim$ is an equivalence relation. We denote the equivalence class of (a, b, c) by $[a, b, c]$. Two equivalent forms represent (properly represent) the same integers, have the same discriminant, are both primitive or both imprimitive and are both of the same type (i.e. indefinite, positive-definite or negative-definite). We denote the set of classes of primitive, integral, binary quadratic forms of discriminant d (only positive-definite forms are taken if $d < 0$) by $H(d)$. Composition of forms is defined as follows (see [2], Chapter 4). Let $K_1, K_2 \in H(d)$. Then we can choose integers a_1, a_2, B and C such that

$$K_1 = [a_1, B, a_2C], \quad K_2 = [a_2, B, a_1C].$$

The composition of K_1 and K_2 (denoted by K_1K_2) is defined by

$$K_1K_2 = [a_1a_2, B, C] \in H(d).$$

It is known that $H(d)$ is a finite abelian group under composition (see [2], Theorem 4.8). We denote the order of the group $H(d)$ by $h(d)$. The identity element of $H(d)$ is the principal class $I = I(d)$ given by

$$I = I(d) = \begin{cases} [1, 0, -d/4], & \text{if } d \equiv 0 \pmod{4}, \\ [1, 1, (1-d)/4], & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

For $K = [a, b, c] \in H(d)$, we have $K^{-1} = [a, -b, c]$.

For any $K \in H(d)$, we say that K represents (properly represents) the integer n if the forms belonging to K represent (properly represent) n . If K properly represents n , there exist integers b and c with $[n, b, c] = K$. The following is an important property of forms (see [2], Proposition 4.2).

Lemma 1.1 *Let $K \in H(d)$ and let m be a non-zero integer. Then there exist integers a, b, c with $\gcd(a, m) = 1$ and $K = [a, b, c]$.*

The cosets of the subgroup of squares in $H(d)$ are called genera and we denote the group of genera by $G(d)$. The identity element of $G(d)$ is called the principal genus.

If $d > 0$, let $(x_0, y_0) = (x_0(d), y_0(d))$ be the solution in positive integers to $x^2 - dy^2 = 4$ for which y_0 is least. We set

$$\epsilon = \epsilon(d) = \frac{x_0 + y_0\sqrt{d}}{2}. \quad (1.4)$$

For the rest of this chapter, n represents a positive integer. Let $[a, b, c] \in H(d)$. Let x and y be integers with $ax^2 + bxy + cy^2 = n$. The representation (x, y) is said to be primary if

1. $d < 0$ or
2. $d > 0$, $2ax + (b - \sqrt{d})y > 0$ and

$$1 \leq \left| \frac{2ax + (b + \sqrt{d})y}{2ax + (b - \sqrt{d})y} \right| < \epsilon^2. \quad (1.5)$$

We set

$$R_{(a,b,c)}(n, d) = |\{(x, y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n, (x, y) \text{ primary}\}|. \quad (1.6)$$

The number $R_{(a,b,c)}(n, d)$ is finite and $R_{(a,b,c)}(n, d) = R_{(a',b',c')}(n, d)$ if $(a, b, c) \sim (a', b', c')$ (see [17], pp 180-186). Thus we can define

$$R_{[a,b,c]}(n, d) = R_{(a,b,c)}(n, d). \quad (1.7)$$

For $G \in G(d)$, we set

$$R_G(n, d) = \sum_{K \in G} R_K(n, d). \quad (1.8)$$

Huard, Kaplan and Williams [14] have obtained a formula for $R_G(n, d)$ when $d < 0$. In this chapter, we extend their method to cover the case $d > 0$, thereby obtaining a formula valid for all discriminants d (see Theorem 1.5). In Sections 1.2–1.7, we develop the results necessary to prove Theorem 1.5.

1.2 Prime discriminants and generic characters

The prime discriminants are the integers $-4, 8, -8$ together with all integers of the form $p^* = (-1)^{(p-1)/2}p$, where p is an odd prime. The prime discriminants corresponding to d are the discriminants $p_1^*, \dots, p_{t+1}^*$, together with p_{t+2}^* if $d \equiv 0 \pmod{32}$, where $t = t(d)$, given as follows:

- $d \equiv 1 \pmod{4}$ or $d \equiv 4 \pmod{16}$

$p_1 < p_2 < \dots < p_{t+1}$ are the odd prime divisors of d .

- $d \equiv 12 \pmod{16}$ or $d \equiv 16 \pmod{32}$

$p_1 < p_2 < \dots < p_t$ are the odd prime divisors of d and $p_{t+1}^* = -4$.

- $d \equiv 8 \pmod{32}$

$p_1 < p_2 < \dots < p_t$ are the odd prime divisors of d and $p_{t+1}^* = 8$.

- $d \equiv 24 \pmod{32}$

$p_1 < p_2 < \dots < p_t$ are the odd prime divisors of d and $p_{t+1}^* = -8$.

- $d \equiv 0 \pmod{32}$

$p_1 < p_2 < \dots < p_{t-1}$ are the odd prime divisors of d , $p_t^* = -4$, $p_{t+1}^* = 8$,
 $p_{t+2}^* = -8$.

Following Huard, Kaplan and Williams [14], we denote the set of prime discriminants corresponding to d by $P(d)$. We denote the set of all products of pairwise coprime elements of $P(d)$ by $F(d)$. It is known that a fundamental discriminant d can be written uniquely as a product of pairwise coprime prime discriminants and that any such product is a fundamental discriminant (see [17], p. 221). It is easy to check that the prime discriminants occurring in such a decomposition are precisely the elements of $P(d)$. It is also known that $|G(d)| = 2^t$ (see [10], pp. 400-407 and pp. 409-413). It is easy to prove the following which is the same as Lemma 2.1 of [14].

Lemma 1.2 (a) $F(d) = \{d_1 : d_1 \text{ is a fundamental discriminant, } d_1 \mid d, \text{ and } d/d_1 \text{ is a discriminant}\}$.

(b) For any positive integer k , $P(d) \subseteq P(dk^2)$ and $F(d) \subseteq F(dk^2)$. Also

$$P(\Delta) \subseteq P(d), 1 \in F(d), \Delta \in F(d), |F(d)| = 2^{t(d)+1},$$

$$|P(d)| = \begin{cases} t(d) + 2, & \text{if } d \equiv 0 \pmod{32}, \\ t(d) + 1, & \text{otherwise.} \end{cases}$$

Lemma 1.3 (a) Let $d_1 \in F(d)$. Then

$$f(d) = f\left(\frac{d}{d_1}\right) f\left(d_1 \Delta\left(\frac{d}{d_1}\right)\right).$$

(b) Let m be a positive integer with $m \mid f$. Let $d_1 \in F(d/m^2)$. Then $f(d/m^2 d_1) \mid f/m$ and $m \mid f(d/d_1)$.

(c) Let m be a positive integer. Then

$$m \mid f(d), d_1 \in F(d/m^2) \iff d_1 \in F(d), m \mid f(d/d_1).$$

Proof: (a) We note that as $d_1 \in F(d)$, d_1 and d/d_1 are discriminants by Lemma 1.2 (a). Also $d_1 \Delta(d/d_1)$ is a discriminant as it is the product of two discriminants. We have

$$\begin{aligned} d &= d_1 \frac{d}{d_1} \\ &= d_1 \Delta\left(\frac{d}{d_1}\right) f\left(\frac{d}{d_1}\right)^2 \\ &= \Delta\left(d_1 \Delta\left(\frac{d}{d_1}\right)\right) \left(f\left(d_1 \Delta\left(\frac{d}{d_1}\right)\right) f\left(\frac{d}{d_1}\right)\right)^2. \end{aligned}$$

Hence

$$\Delta(d) = \Delta\left(d_1 \Delta\left(\frac{d}{d_1}\right)\right), \quad f(d) = f\left(d_1 \Delta\left(\frac{d}{d_1}\right)\right) f\left(\frac{d}{d_1}\right).$$

(b) As $m \mid f$, d/m^2 is a discriminant. By part (a), we have $f(d/m^2 d_1) \mid f(d/m^2)$ so that $f(d/m^2 d_1) \mid f/m$ by (1.3). Using

$$\frac{d}{d_1} = \frac{d}{m^2 d_1} m^2$$

together with (1.2), we have $f(d/d_1) = f(d/m^2 d_1) m$ so that $m \mid f(d/d_1)$.

(c) First let $m \mid f(d)$ and $d_1 \in F(d/m^2)$. By part (b), we have $m \mid f(d/d_1)$. Also Lemma 1.2 (b) gives $F(d/m^2) \subseteq F(d)$ so that $d_1 \in F(d)$.

Next let $d_1 \in F(d)$ and $m \mid f(d/d_1)$. By part (a), we have $m \mid f(d)$. As $d_1 \in F(d)$, d_1 is fundamental. Using $m \mid f(d/d_1)$ and $d/d_1 = \Delta(d/d_1)f(d/d_1)^2$, we deduce that $m^2 \mid d/d_1$ and $(d/d_1)/m^2$ is a discriminant. Thus $d_1 \mid d/m^2$ and $(d/m^2)/d_1$ is a discriminant. Hence $d_1 \in F(d/m^2)$ by Lemma 1.2 (a). $\square$

Next we recall the basic properties of generic characters. Let $K \in H(d)$ and $p^* \in P(d)$. For any positive integer k coprime with p^* represented by K , it is known that $(\frac{p^*}{k})$ has the same value, so that we can set

$$\gamma_{p^*}(K) = \left(\frac{p^*}{k}\right) = \pm 1. \quad (1.9)$$

Let $G \in G(d)$. It is known that for any $K \in G$, $\gamma_{p^*}(K)$ has the same value, so we can set $\gamma_{p^*}(G) = \gamma_{p^*}(K)$. Also

$$\gamma_{p^*}(G_1 G_2) = \gamma_{p^*}(G_1) \gamma_{p^*}(G_2), \quad (1.10)$$

for $G_1, G_2 \in G(d)$. An important result of genus theory is the following (see [12], equation 9).

Lemma 1.4 (a) *If $G \in G(d)$ then with $\Delta = \Delta(d)$,*

$$\prod_{p^* \in P(\Delta)} \gamma_{p^*}(G) = 1, \quad (1.11)$$

together with

$$\gamma_{-4}(G) \gamma_8(G) \gamma_{-8}(G) = 1, \text{ if } d \equiv 0 \pmod{32}. \quad (1.12)$$

(b) Moreover, if $\delta_{p^*} = \pm 1$ for each $p^* \in P(d)$ and

$$\prod_{p^* \in P(\Delta)} \delta_{p^*} = 1,$$

together with

$$\delta_{-4}\delta_8\delta_{-8} = 1, \text{ if } d \equiv 0 \pmod{32},$$

then there exists a unique genus $G \in G(d)$ with

$$\gamma_{p^*}(G) = \delta_{p^*}, \text{ for each } p^* \in P(d).$$

Next we extend the definition of $\gamma_{p^*}(G)$ ($p^* \in P(d)$) to $\gamma_{d_1}(G)$ for $d_1 \in F(d)$.

For $d_1 \in F(d)$, we set

$$\gamma_{d_1}(G) = \prod_{p^* \in P(d_1)} \gamma_{p^*}(G) = \pm 1. \quad (1.13)$$

1.3 The derived genus G_m of G

We let $v_p(n)$ denote the exponent of the highest power of the prime p dividing n . In this section, following Huard, Kaplan and Williams [14], we define the derived genus $G_m \in G(d/(m, f)^2)$ of $G \in G(d)$, where m is a positive integer all of whose prime factors p divide d and satisfy

$$p \nmid \Delta \Rightarrow v_p(m) \leq v_p(f). \quad (1.14)$$

We begin with the case when m is a prime.

Lemma 1.5 *Let p be a prime with $p \mid d$, and let $G \in G(d)$. Then there is a unique genus*

$$G_p \in \begin{cases} G(d/p^2), & \text{if } p \mid f, \\ G(d), & \text{if } p \nmid f, \end{cases} \quad (1.15)$$

such that in the case $p \mid f$,

$$\gamma_{q^*}(G_p) = \gamma_{q^*}(G), \text{ for all } q^* \in P(d/p^2), \quad (1.16)$$

and in the case $p \nmid f$,

$$\gamma_{q^*}(G_p) = \begin{cases} \left(\frac{q^*}{p}\right) \gamma_{q^*}(G), & \text{for every } q^* \in P(d) \text{ with } p \nmid q^*, \\ \left(\frac{d/q^*}{p}\right) \gamma_{q^*}(G) = \left(\frac{\Delta/q^*}{p}\right) \gamma_{q^*}(G), & \\ \text{for the unique } q^* \in P(d) \text{ with } p \mid q^*. \end{cases} \quad (1.17)$$

Proof: We first consider the case $p \mid f$. Then $P(d/p^2) \subseteq P(d)$ by Lemma 1.2. Hence $\gamma_{q^*}(G)$ is defined for every $q^* \in P(d/p^2)$. Also $\Delta(d/p^2) = \Delta(d) = \Delta$ (by (1.2)). By Lemma 1.4(a), we have

$$\prod_{q^* \in P(\Delta)} \gamma_{q^*}(G) = 1,$$

together with

$$\gamma_{-4}(G)\gamma_8(G)\gamma_{-8}(G) = 1, \text{ if } d \equiv 0 \pmod{32}.$$

Applying Lemma 1.4(b) with $\delta_{q^*} = \gamma_{q^*}(G)$, we conclude that there exists a unique genus $G_p \in G(d/p^2)$ satisfying (1.16).

Next we consider the case $p \nmid f$ (so $p \mid \Delta$). If $p \neq 2$ then $q^* = p^*$ is the unique $q^* \in P(d)$ such that $p \mid q^*$. If $p = 2$ then f is odd so that $d \not\equiv 0 \pmod{32}$. Since $2 \mid \Delta$, it follows that there is a unique $q^* \in P(d)$ with $2 \mid q^*$. In both cases, the unique $q^* \in P(d)$ with $p \mid q^*$ satisfies $q^* \mid \Delta$ and $p \nmid \Delta/q^*$ as Δ is fundamental. We denote this q^* by r^* (note that $r^* \in P(\Delta)$).

Thus

$$\left(\frac{\Delta/r^*}{p}\right) = \left(\frac{d/r^*}{p}\right) = \pm 1.$$

We set

$$\delta_{q^*} = \begin{cases} \left(\frac{q^*}{p}\right) \gamma_{q^*}(G) = \pm 1, & \text{for every } q^* \in P(d) \text{ with } p \nmid q^*, \\ \left(\frac{\Delta/q^*}{p}\right) \gamma_{q^*}(G) = \pm 1, & \text{for } q^* = r^*. \end{cases}$$

We have

$$\begin{aligned} \prod_{q^* \in P(\Delta)} \delta_{q^*} &= \left(\frac{\Delta/r^*}{p}\right) \gamma_{r^*}(G) \prod_{\substack{q^* \neq r^* \\ q^* \in P(\Delta)}} \left(\frac{q^*}{p}\right) \gamma_{q^*}(G) \\ &= \left(\frac{\Delta/r^*}{p}\right)^2 \prod_{q^* \in P(\Delta)} \gamma_{q^*}(G) = 1, \end{aligned}$$

since

$$\Delta = r^* \prod_{\substack{q^* \neq r^* \\ q^* \in P(\Delta)}} q^*.$$

If $d \equiv 0 \pmod{32}$, then $p \neq 2$ (see above) and

$$\begin{aligned} \delta_{-4} \delta_8 \delta_{-8} &= \left(\frac{-4}{p}\right) \gamma_{-4}(G) \left(\frac{8}{p}\right) \gamma_8(G) \left(\frac{-8}{p}\right) \gamma_{-8}(G) \\ &= \left(\frac{256}{p}\right) \gamma_{-4}(G) \gamma_8(G) \gamma_{-8}(G) = 1. \end{aligned}$$

Hence by Lemma 1.4(b), there exists a unique genus $G_p \in G(d)$ satisfying (1.17). $\square$

Next, we define G_{p^i} for $p \mid d$ and $i \geq 0$. We set $G_1 = G$. By (1.15), we define successively

$$G_{p^i} = (G_{p^{i-1}})_p \in G(d/p^{2^i}), \text{ for } i = 1, \dots, v_p(f). \quad (1.18)$$

If in addition $p \mid \Delta$, as $p \nmid f/p^{v_p(f)}$, we define successively

$$G_{p^i} = (G_{p^{i-1}})_p \in G(d/p^{2^{v_p(f)}}), i = 1 + v_p(f), \dots \quad (1.19)$$

Thus for any $p \mid d$, we have defined $G_{p^i} \in G(d/(p^i, f)^2)$ for any $i \geq 0$ if $p \mid \Delta$ and for $0 \leq i \leq v_p(f)$ if $p \nmid \Delta$. For $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ satisfying (1.14), we define

$$G_m = (\dots ((G_{p_1^{\alpha_1}})_{p_2^{\alpha_2}}) \dots)_{p_r^{\alpha_r}} \in G(d/(m, f)^2). \quad (1.20)$$

It is easily checked that the order of the p_i 's does not matter.

Lemma 1.6 (a) *Let p be a prime with $p \mid d$. Let $d_1 \in F(d/(p, f)^2)$. Then, for any $G \in G(d)$, we have*

$$\gamma_{d_1}(G_p) = \begin{cases} \gamma_{d_1}(G), & \text{if } p \mid f, \\ \left(\frac{d_1}{p}\right) \gamma_{d_1}(G), & \text{if } p \nmid f, p \nmid d_1, \\ \left(\frac{d/d_1}{p}\right) \gamma_{d_1}(G), & \text{if } p \nmid f, p \mid d_1. \end{cases}$$

(b) *If m is a positive integer with $m \mid f$, $G \in G(d)$ and $d_1 \in F(d/m^2)$ then $\gamma_{d_1}(G_m) = \gamma_{d_1}(G)$.*

Proof: If $p \mid f$ then $d_1 \in F(d/p^2)$ and $P(d_1) \subseteq P(d/p^2)$. We have

$$\gamma_{d_1}(G_p) = \prod_{q^* \in P(d_1)} \gamma_{q^*}(G_p) = \prod_{q^* \in P(d_1)} \gamma_{q^*}(G) = \gamma_{d_1}(G),$$

by (1.13) and (1.16). Next, we suppose that $p \nmid f$, so that $p \mid \Delta$, $d_1 \in F(d)$. If $p \nmid d_1$ then

$$\gamma_{d_1}(G_p) = \prod_{q^* \in P(d_1)} \gamma_{q^*}(G_p) = \prod_{q^* \in P(d_1)} \left(\frac{q^*}{p}\right) \gamma_{q^*}(G) = \left(\frac{d_1}{p}\right) \gamma_{d_1}(G),$$

by (1.13) and (1.17).

If $p \mid d_1$, then p divides exactly one element of $P(d_1)$, say r^* . Then

$$\gamma_{d_1}(G_p) = \prod_{q^* \in P(d_1)} \gamma_{q^*}(G_p)$$

$$\begin{aligned}
&= \left(\frac{d/r^*}{p} \right) \gamma_{r^*}(G) \prod_{\substack{q^* \neq r^* \\ q^* \in P(d_1)}} \left(\frac{q^*}{p} \right) \gamma_{q^*}(G) \\
&= \left(\frac{d/r^*}{p} \right) \left(\frac{d_1/r^*}{p} \right) \gamma_{d_1}(G) = \left(\frac{d/d_1}{p} \right) \gamma_{d_1}(G),
\end{aligned}$$

by (1.13) and (1.17).

(b) This follows by repeatedly applying part (a). $\square$

1.4 Null primes and the integers M , Q and U

A prime p is said to be a null prime relative to n and d if

$$v_p(n) \equiv 1 \pmod{2}, \quad v_p(n) < 2v_p(f). \quad (1.21)$$

We denote the set of all such null primes by $\text{Null}(n, d)$.

Lemma 1.7 *If $\text{Null}(n, d) \neq \emptyset$, then $R_K(n, d) = 0$ for each $K \in H(d)$.*

Proof: Let $\text{Null}(n, d) \neq \emptyset$ and $R_K(n, d) > 0$ for some $K = [a, b, c] \in H(d)$.

Let $p \in \text{Null}(n, d)$. By Lemma 1.1, we may assume $p \nmid a$. By our assumption, there exist integers x and y with $n = ax^2 + bxy + cy^2$. Thus

$$4an = X^2 - \Delta f^2 y^2, \text{ where } X = 2ax + by.$$

Let $m = v_p(n)$ so that m is odd with $p^{m+1} \mid f^2$ (by (1.21)). Since $p \nmid a$, $v_p(4an)$ is odd and so $y \neq 0$.

We first consider the case when $v_p(\Delta f^2 y^2)$ is odd. Then $v_p(4an) = v_p(\Delta f^2 y^2)$. If $p \neq 2$ then $v_p(4an) = m$ and $v_p(\Delta f^2 y^2) \geq m + 1$, a contradiction. If $p = 2$, $v_2(4an) = m + 2$. But $v_2(\Delta)$ is odd (since $v_2(\Delta f^2 y^2)$ is odd). Thus $v_2(\Delta) = 3$ as Δ is fundamental. Hence $v_2(\Delta f^2 y^2) \geq 3 + (m + 1)$, a contradiction.

Next let $v_p(\Delta f^2 y^2)$ be even. Then $X \neq 0$ and $v_p(X^2) = v_p(\Delta f^2 y^2)$. If $p \neq 2$, we have $v_p(X^2) = v_p(\Delta f^2 y^2) \geq m + 1$, giving $v_p(4an) \geq m + 1$ and this contradicts $v_p(4an) = m$. If $p = 2$ then $v_2(\Delta)$ is even and so $v_2(\Delta) = 0$ or 2 . If $v_2(\Delta) = 2$, we have $v_2(X^2) = v_2(\Delta f^2 y^2) \geq m + 3$ and so $v_2(4an) \geq m + 3$. If $v_2(\Delta) = 0$ then $\Delta \equiv 1 \pmod{4}$. We set $v_2(X^2) = v_2(\Delta f^2 y^2) = 2w$. Then

$$v_2((X/2^w)^2 - \Delta(fy/2^w)^2) \geq 2.$$

Thus $v_2(X^2 - \Delta f^2 y^2) \geq 2 + 2w \geq 2 + (m + 1)$. Both cases contradict $v_2(4an) = m + 2$. $\square$

We note that it follows that $R_G(n, d) = 0$ if $\text{Null}(n, d) \neq \emptyset$.

Next we introduce three positive integers.

$$M = M(n, d) \text{ is the largest integer such that } M^2 \mid n \text{ and } M \mid f, \quad (1.22)$$

$$U = U(n, d) = \prod_{\substack{p \mid d \\ p \nmid f}} p^{v_p(n)}, \quad (1.23)$$

$$Q = Q(n, d) = U(n/M^2, d/M^2) = \prod_{\substack{p \mid d/M^2 \\ p \nmid f/M}} p^{v_p(n/M^2)}. \quad (1.24)$$

Lemma 1.8 (a) *If $\text{Null}(n, d) = \emptyset$ then*

$$\gcd(n/M^2, f/M) = 1 \text{ and } \gcd(n/M^2 Q, d/M^2) = 1.$$

(b) *$\gcd(n, f) = 1$ if and only if $\text{Null}(n, d) = \emptyset$ and $M = 1$.*

Proof: (a) Let $\text{Null}(n, d) = \emptyset$ but $\gcd(n/M^2, f/M) > 1$. Then there is a prime p with $p \mid n/M^2$ and $p \mid f/M$. By choice of M , $p^2 \nmid n/M^2$. Hence

$$v_p(n) = v_p(n/M^2) + 2v_p(M) = 1 + 2v_p(M) < 2(1 + v_p(M)) \leq 2v_p(f).$$

Thus $p \in \text{Null}(n, d)$, a contradiction.

Next let us suppose that there is a prime p dividing both n/M^2Q and d/M^2 . Then $p \nmid f/M$ since $\gcd(n/M^2, f/M) = 1$. Thus $v_p(Q) = v_p(n/M^2)$ contradicting $p \mid n/M^2Q$.

(b) Suppose $\gcd(n, f) = 1$. Then $M = 1$. If $p \in \text{Null}(n, d)$ then $v_p(n)$ is odd. Hence $p \mid n$ and so $p \nmid f$ contradicting $v_p(n) < 2v_p(f)$. Thence $\text{Null}(n, d) = \emptyset$.

Conversely, let $M = 1$ and $\text{Null}(n, d) = \emptyset$. By part(a), $\gcd(n, f) = 1$. $\square$

1.5 The sum $S(n, d_1, d/d_1)$

For $d_1 \in F(d)$ and $\gcd(n, f) = 1$, we set

$$S(n, d_1, d/d_1) = \sum_{\mu\nu=n} \left(\frac{d_1}{\mu} \right) \left(\frac{d/d_1}{\nu} \right), \quad (1.25)$$

where μ and ν run through all positive integers with $\mu\nu = n$.

Lemma 1.9 *Let $\gcd(n, f) = 1$ and let p be a prime dividing both n and d .*

Then, for $G \in G(d)$, we have

$$\sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1) = \sum_{d_1 \in F(d)} \gamma_{d_1}(G_p) S(n/p, d_1, d/d_1).$$

Proof: We have

$$\begin{aligned} & \sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1) \\ &= \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \sum_{\mu\nu=n} \left(\frac{d_1}{\mu} \right) \left(\frac{d/d_1}{\nu} \right) \\ &= \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \left\{ \sum_{\substack{\mu\nu=n \\ p \mid \mu}} \left(\frac{d_1}{\mu} \right) \left(\frac{d/d_1}{\nu} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{\substack{\mu\nu=n \\ p|\nu}} \left(\frac{d_1}{\mu} \right) \left(\frac{d/d_1}{\nu} \right) - \sum_{\substack{\mu\nu=n \\ p|\mu, p|\nu}} \left(\frac{d_1}{\mu} \right) \left(\frac{d/d_1}{\nu} \right) \Big\} \\
& = \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \left(\frac{d_1}{p} \right) S(n/p, d_1, d/d_1) \\
& \quad + \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \left(\frac{d/d_1}{p} \right) S(n/p, d_1, d/d_1).
\end{aligned}$$

In the first sum, we need only sum over those d_1 with $p \nmid d_1$, and in the second sum over those d_1 with $p \mid d_1$. Using this together with Lemma 1.6(a) gives the required result. $\square$

Lemma 1.10 *Let $\gcd(n, f) = 1$. Then, for $G \in G(d)$, we have*

$$\sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1) = \sum_{d_1 \in F(d)} \gamma_{d_1}(G_U) S(n/U, d_1, d/d_1),$$

where U is defined in (1.23).

Proof: This follows by repeatedly applying Lemma 1.9 to all the primes dividing U . $\square$

1.6 First reduction formula

In this section we prove our first reduction formula (see Theorem 1.1). This formula relates $R_G(n, d)$ to $R_{G_U}(n/U, d)$, where U is defined in (1.23).

Lemma 1.11 *Let p be a prime with $p \mid d$, $p \nmid f$. Let $K \in H(d)$. Then*

- (a) K contains a form (a, b, cp) with $p \nmid ac$, $p \mid b$;
- (b) the mapping $\phi_p : H(d) \rightarrow H(d)$ given by $\phi_p([a, b, cp]) = [ap, b, c]$, with (a, b, cp) as in part (a), is a bijection;
- (c) if $G \in G(d)$ and $K \in G$ then $\phi_p(K) \in G_p$.

Proof: (a) By Lemma 1.1, we can choose $(a, b, c) \in K$ with $p \nmid a$. We first consider the case $p \neq 2$. Then $p \nmid d$. We choose t such that $b' = 2at + b \equiv 0 \pmod{p}$. We set $c' = (at^2 + bt + c)/p$. Then c' is an integer as $4a(at^2 + bt + c) = (2at + b)^2 - d \equiv 0 \pmod{p}$. The form $(a, b', pc') \in K$ is of the required type.

Next we consider the case $p = 2$. Since $2 \mid d$, $2 \nmid f$, we have $2 \mid b$ and $d \equiv 8$ or $12 \pmod{16}$. If $c \equiv 2 \pmod{4}$, we set $c = 2c'$ and we are done. If $c \not\equiv 2 \pmod{4}$ then using

$$\left(\frac{b}{2}\right)^2 - ac \equiv \frac{d}{4} \equiv 2 \text{ or } 3 \pmod{4},$$

we easily obtain $c \equiv 1 \pmod{2}$ and $a + b + c \equiv 2 \pmod{4}$. Thus $(a, b + 2a, a + b + c) \in K$ is of the required type.

(b) If $[a, b, cp] \in H(d)$ with $p \nmid ac$, $p \mid b$ then (ap, b, c) is easily seen to be primitive so that $[ap, b, c] \in H(d)$. Next we show that ϕ_p is well defined. Let

$$[a, b, cp] = [a', b', c'p], \quad \text{with } p \nmid aca'c', p \mid b, p \mid b'.$$

Then there exist integers α, β, γ and δ with $\alpha\delta - \beta\gamma = 1$ and

$$a' = a\alpha^2 + b\alpha\gamma + cp\gamma^2,$$

$$b' = 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2cp\gamma\delta,$$

$$c'p = a\beta^2 + b\beta\delta + cp\delta^2.$$

Since $p \mid b$, we see that $p \mid a\beta^2$ and so $p \mid \beta$. Set $\beta = p\beta'$, $\gamma' = p\gamma$ so that $\alpha\delta - \beta'\gamma' = 1$ and

$$a'p = ap\alpha^2 + b\alpha\gamma' + c\gamma'^2,$$

$$b' = 2ap\alpha\beta' + b(\alpha\delta + \beta'\gamma') + 2c\gamma'\delta,$$

$$c' = ap\beta'^2 + b\beta'\delta + c\delta^2.$$

Thus $[ap, b, c] = [a'p, b', c']$ and so ϕ_p is well defined.

$$\text{Also } \phi_p^2([a, b, cp]) = \phi_p([ap, b, c]) = \phi_p([c, -b, ap]) = [cp, -b, a] = [a, b, cp].$$

Hence ϕ_p is a bijection.

(c) Let $G \in G(d)$ and $K = [a, b, cp] \in G$ with $p \nmid ac$, $p \mid b$. Suppose $\phi_p(K)$ belongs to the genus $\hat{G}$ of $G(d)$. We wish to show that $\hat{G} = G_p$. Let $q^* \in P(d)$ with $p \nmid q^*$. Let μ be a positive integer coprime with q^* which is represented by $(a, b, cp) \in K$. Then $p\mu$ is represented by $(ap, b, c) \in \phi_p(K)$. Thus

$$\gamma_{q^*}(\hat{G}) = \left(\frac{q^*}{p\mu}\right) = \left(\frac{q^*}{p}\right) \left(\frac{q^*}{\mu}\right) = \left(\frac{q^*}{p}\right) \gamma_{q^*}(G) = \gamma_{q^*}(G_p), \quad (1.26)$$

by (1.17).

Next let r^* be the unique $q^* \in P(d)$ with $p \mid q^*$ (see Lemma 1.5). Then $r^* \in P(\Delta)$. Thus

$$\gamma_{r^*}(\hat{G}) = \prod_{\substack{q^* \in P(\Delta) \\ q^* \neq r^*}} \gamma_{q^*}(\hat{G}) = \prod_{\substack{q^* \in P(\Delta) \\ q^* \neq r^*}} \gamma_{q^*}(G_p) = \gamma_{r^*}(G_p),$$

by Lemma 1.4 and (1.26). Thus $\gamma_{q^*}(\hat{G}) = \gamma_{q^*}(G_p)$ for all $q^* \in P(d)$. Hence $\hat{G} = G_p$. $\square$

Lemma 1.12 *Let p be a prime with $p \mid n$, $p \mid d$ and $p \nmid f$. Then, for $K \in H(d)$, we have $R_K(n, d) = R_{\phi_p(K)}(n/p, d)$.*

Proof: Let $(a, b, cp) \in K$ with $p \nmid ac$, $p \mid b$. Then $(ap, b, c) \in \phi_p(K)$. We set

$$S = \{(x, y) \in \mathbb{Z}^2 : ax^2 + bxy + cpy^2 = n, (x, y) \text{ is primary}\},$$

$$T = \{(X, Y) \in \mathbb{Z}^2 : apX^2 + bXY + cY^2 = n/p, (X, Y) \text{ is primary}\}.$$

It is easily seen that $(X, Y) \rightarrow (pX, Y)$ is a bijection from T to S . $\square$

Lemma 1.13 *Let p be a prime with $p \mid n$, $p \mid d$ and $p \nmid f$. Then, for $G \in G(d)$, we have $R_G(n, d) = R_{G_p}(n/p, d)$.*

Proof: We have

$$\begin{aligned}
 R_G(n, d) &= \sum_{K \in G} R_K(n, d) \\
 &= \sum_{K \in G} R_{\phi_p(K)}(n/p, d) \\
 &= \sum_{K' \in G_p} R_{K'}(n/p, d) \\
 &= R_{G_p}(n/p, d),
 \end{aligned}$$

by Lemma 1.12 and Lemma 1.11(b) and (c). □

We are now ready to prove the main result of this section.

Theorem 1.1 *For $G \in G(d)$, we have*

$$R_G(n, d) = R_{G_U}(n/U, d),$$

where $U = U(n, d)$ is defined in (1.23).

Proof: This follows from Lemma 1.13 by repeatedly applying it to all the primes dividing U . □

1.7 Second reduction formula

In this section we prove our second reduction formula (see Theorem 1.2) which relates $R_G(n, d)$ to $R_{G_M}(n/M^2, d/M^2)$, where M is defined in (1.22).

Lemma 1.14 *Let $p \mid f$, $K \in H(d)$ and let l be a positive integer. Then*

- (a) *K contains a form (a, b, c) with $p \mid b$, $p^2 \mid c$ and $\gcd(a, pl) = 1$;*
- (b) *the mapping $\theta_p : H(d) \rightarrow H(d/p^2)$ given by $\theta_p([a, b, c]) = [a, b/p, c/p^2]$, with (a, b, c) as in (a) with $l = 1$, is a surjective homomorphism;*
- (c) *if $G \in G(d)$ and $K \in G$ then $\theta_p(K) \in G_p$.*

Proof: (a) By Lemma 1.1, we can choose $(a, b, c) \in K$ with $\gcd(a, pl) = 1$. We first consider the case $p \neq 2$. We choose t so that $b' = 2at + b \equiv 0 \pmod{p}$. Let $c' = at^2 + bt + c$. Then $(a, b', c') \in K$ is of the required type.

Next let $p = 2$. We note that b is even and

$$c \not\equiv 2 \pmod{4}, \quad (1.27)$$

otherwise $(b/2)^2 - \Delta(f/2)^2 = ac$ gives a contradiction. If $c \equiv 0 \pmod{4}$ then (a, b, c) is of the required type. If c is odd then $(a, b + 2a, a + b + c) \in K$ is of the required type ($a + b + c$ is even and therefore $\equiv 0 \pmod{4}$ by (1.27)).

(b) The proof that θ_p is well defined is similar to the proof that ϕ_p in Lemma 1.11 is well defined. We omit the details. θ_p is clearly surjective. It remains to show that θ_p is a homomorphism. Let $K_1, K_2 \in H(d)$ with $K_1 = [a_1, b_1, c_1]$, $K_2 = [a_2, b_2, c_2]$ where $p \nmid a_1 a_2$, $p \mid b_1$, $p \mid b_2$, $p^2 \mid c_1$, $p^2 \mid c_2$ and $\gcd(a_1, a_2) = 1$. Let δ_1, δ_2 be integers such that

$$a_2 \delta_2 - a_1 \delta_1 = \frac{b_1 - b_2}{2}.$$

Set $t_1 = \delta_1 + a_2 k$, $t_2 = \delta_2 + a_1 k$, where the integer k is chosen so that $p \mid t_1$.

Then

$$a_2 t_2 - a_1 t_1 = \frac{b_1 - b_2}{2}.$$

Set

$$B = b_1 + 2a_1t_1 = b_2 + 2a_2t_2,$$

$$l_1 = a_1t_1^2 + b_1t_1 + c_1,$$

$$l_2 = a_2t_2^2 + b_2t_2 + c_2.$$

It is clear that $p \mid B$ and $p^2 \mid l_1$. We have

$$K_1 = [a_1, B, l_1], \quad K_2 = [a_2, B, l_2].$$

From $B^2 - 4a_1l_1 = B^2 - 4a_2l_2 = d$, we obtain $a_1l_1 = a_2l_2$. Thus $a_2 \mid a_1l_1$ so that $a_2 \mid l_1$. Let $l_1 = a_2C$. Then $l_2 = a_1C$ and $p^2 \mid C$. Hence

$$\begin{aligned} \theta_p(K_1) &= \theta_p([a_1, B, a_2C]) = \left[a_1, \frac{B}{p}, a_2 \frac{C}{p^2} \right], \\ \theta_p(K_2) &= \theta_p([a_2, B, a_1C]) = \left[a_2, \frac{B}{p}, a_1 \frac{C}{p^2} \right], \\ \theta_p(K_1K_2) &= \theta_p([a_1a_2, B, C]) = \left[a_1a_2, \frac{B}{p}, \frac{C}{p^2} \right] = \theta_p(K_1)\theta_p(K_2), \end{aligned}$$

as required.

(c) Let $q^* \in P(d/p^2)$, $G \in G(d)$, $K \in G$. We choose a, b, c with $K = [a, b, c]$, $\gcd(a, pq^*) = 1$, $p \mid b$ and $p^2 \mid c$. Then $\theta_p(K) = [a, b/p, c/p^2]$. Let $\theta_p(K) \in \hat{G} \in G(d/p^2)$. We wish to show $\hat{G} = G_p$. We have

$$\gamma_{q^*}(\hat{G}) = \left(\frac{q^*}{a} \right) = \gamma_{q^*}(G) = \gamma_{q^*}(G_p),$$

by (1.16). Thus $\hat{G} = G_p$. □

Lemma 1.15 *Let $d > 0$, $[a, b, c] \in H(d)$ and let m be a positive integer. Set*

$$S = \left\{ (x, y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n, 2ax + (b - \sqrt{d})y > 0, \right.$$

$$1 \leq \left\{ \left| \frac{2ax + (b + \sqrt{d})y}{2ax + (b - \sqrt{d})y} \right| < \epsilon^2 \right\},$$

$$T = \left\{ (X, Y) \in \mathbb{Z}^2 : aX^2 + bXY + cY^2 = n, 2aX + (b - \sqrt{d})Y > 0, \right.$$

$$\left. \epsilon^{2m} \leq \left| \frac{2aX + (b + \sqrt{d})Y}{2aX + (b - \sqrt{d})Y} \right| < \epsilon^{2m+2} \right\},$$

where $\epsilon = \epsilon(d)$ is defined in (1.4). Then $|S| = |T|$.

We first recall the following result.

Lemma 1.16 *Let $d > 0$. If x and y are integers satisfying $x^2 - dy^2 = 4$ then*

$$\frac{x + y\sqrt{d}}{2} = \pm \epsilon^m, \quad (1.28)$$

for some integer m . Also, if x and y are rational numbers satisfying (1.28) for some integer m then x and y are integers satisfying $x^2 - dy^2 = 4$.

Proof: See [17], Theorem 111. □

We are now ready to prove Lemma 1.15.

Proof of Lemma 1.15: Let

$$\epsilon^m = \frac{t + u\sqrt{d}}{2},$$

where t and u are rational numbers. By Lemma 1.16, t and u are integers satisfying $t^2 - du^2 = 4$. We define a map from S to T by $(x, y) \rightarrow (X, Y)$, where

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} (t - bu)/2 & -cu \\ au & (t + bu)/2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Short calculations show that

$$ax^2 + bxy + cy^2 = aX^2 + bXY + cY^2,$$

$$2aX + (b + \sqrt{d})Y = \epsilon^m(2ax + (b + \sqrt{d})y).$$

Thus

$$2aX + (b - \sqrt{d})Y = \epsilon'^m(2ax + (b - \sqrt{d})y),$$

where $\epsilon' = (x_0 - y_0\sqrt{d})/2 = 1/\epsilon$.

Using these facts, it is easily seen that the map $(x, y) \rightarrow (X, Y)$ is a bijection. $\square$

Lemma 1.17 *Let p be a prime with $p \mid M$, where M is defined in (1.22).*

Then, for any $K \in H(d)$, we have

$$R_K(n, d) = \begin{cases} R_{\theta_p(K)}(n/p^2, d/p^2), & \text{if } d < 0, \\ \frac{\log \epsilon(d)}{\log \epsilon(d/p^2)} R_{\theta_p(K)}(n/p^2, d/p^2), & \text{if } d > 0. \end{cases}$$

Proof: We only do the case $d > 0$. The case $d < 0$ is easier. We choose $(a, b, c) \in K$ with $p \nmid a$, $p \mid b$ and $p^2 \mid c$ so that $\theta_p(K) = [a, b/p, c/p^2]$. Let

$$\begin{aligned} S &= \{(x, y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n, 2ax + (b - \sqrt{d})y > 0, \\ &\quad 1 \leq \left| \frac{2ax + (b + \sqrt{d})y}{2ax + (b - \sqrt{d})y} \right| < \epsilon(d)^2\}, \\ T &= \{(X, Y) \in \mathbb{Z}^2 : \frac{n}{p^2} = aX^2 + \frac{bXY}{p} + \frac{cY^2}{p^2}, 2aX + (b - \sqrt{d})\frac{Y}{p} > 0, \\ &\quad 1 \leq \left| \frac{2aX + (b + \sqrt{d})Y/p}{2aX + (b - \sqrt{d})Y/p} \right| < \epsilon(d)^2\}, \\ U &= \{(X, Y) \in \mathbb{Z}^2 : \frac{n}{p^2} = aX^2 + \frac{bXY}{p} + \frac{cY^2}{p^2}, 2aX + (b - \sqrt{d})\frac{Y}{p} > 0, \\ &\quad 1 \leq \left| \frac{2aX + (b + \sqrt{d})Y/p}{2aX + (b - \sqrt{d})Y/p} \right| < \epsilon(d/p^2)^2\}. \end{aligned}$$

As $x_0^2 - dy_0^2 = 4$, $(x, y) = (x_0, py_0)$ is a solution in integers to $x^2 - dy^2/p^2 = 4$.

Thus

$$\frac{x + y\sqrt{d}/p}{2} = \pm \epsilon(d/p^2)^m,$$

for some integer m by Lemma 1.16. Hence $\epsilon(d) = \epsilon(d/p^2)^m$ for some positive integer m (since $\epsilon(d)$ and $\epsilon(d/p^2)$ are both > 1). The map from T to S given by $(X, Y) \rightarrow (pX, Y)$ is easily seen to be a bijection. Hence

$$\begin{aligned} R_K(n, d) &= |S| = |T| \\ &= m|U| \\ &= \frac{\log \epsilon(d)}{\log \epsilon(d/p^2)} R_{\theta_p(K)}(n/p^2, d/p^2), \end{aligned}$$

by Lemma 1.15. □

Lemma 1.18 *Let p be a prime with $p \mid M$. Then, for $G \in G(d)$, we have*

$$R_G(n, d) = j(d, p) \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}} R_{G_p}(n/p^2, d/p^2),$$

where

$$j(d, p) = \begin{cases} 1, & \text{if } d < 0, \\ \frac{\log \epsilon(d)}{\log \epsilon(d/p^2)}, & \text{if } d > 0. \end{cases}$$

Proof: Let $G \in G(d)$ and $L \in G_p$. We have $G = K_1 H^2(d)$ for some $K_1 \in H(d)$. Since $K_1 \in G$, we have $\theta_p(K_1) \in G_p$ by Lemma 1.14 (c). Hence $G_p = \theta_p(K_1) H^2(d/p^2)$. As $L \in G_p$, there exists a class $L_1 \in H(d/p^2)$ with $L = \theta_p(K_1) L_1^2$. Since θ_p is surjective, there exists a class $K_2 \in H(d)$ with $\theta_p(K_2) = L_1$. Set $A = K_1 K_2^2$ so that $A \in G$. Then $\theta_p(A) = L$ and $G = AH^2(d)$. Set

$$N_G(L) = \sum_{\substack{K \in G \\ \theta_p(K) = L}} 1.$$

Then

$$N_G(L) = |\{K \in G : \theta_p(K) = L\}|$$

$$\begin{aligned}
&= |\{K \in H(d) : \theta_p(K) = L\} \cap G| \\
&= |A \operatorname{Ker} \theta_p \cap AH^2(d)| \\
&= |A(\operatorname{Ker} \theta_p \cap H^2(d))| \\
&= |\operatorname{Ker} \theta_p \cap H^2(d)|.
\end{aligned}$$

Thus $N_G(L)$ is independent of G and L . Hence

$$|G| = \sum_{K \in G} 1 = \sum_{L \in G_p} N_G(L) = N_G(L) \sum_{L \in G_p} 1 = N_G(L)|G_p|,$$

so that

$$N_G(L) = \frac{|G|}{|G_p|} = \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}}.$$

Hence we have

$$\begin{aligned}
R_G(n, d) &= \sum_{K \in G} R_K(n, d) \\
&= j(d, p) \sum_{K \in G} R_{\theta_p(K)}(n/p^2, d/p^2) \quad (\text{by Lemma 1.17}) \\
&= j(d, p) \sum_{L \in G_p} R_L(n/p^2, d/p^2) N_G(L) \\
&= j(d, p) \frac{h(d)/2^{t(d)}}{h(d/p^2)/2^{t(d/p^2)}} R_{G_p}(n/p^2, d/p^2),
\end{aligned}$$

as required. □

We are now ready to prove our second reduction formula.

Theorem 1.2 *For $G \in G(d)$, we have*

$$R_G(n, d) = j(d, M) \frac{1}{2^{t(d)-t(d/M^2)}} \frac{h(d)}{h(d/M^2)} R_{G_M}(n/M^2, d/M^2),$$

where

$$j(d, M) = \begin{cases} 1, & \text{if } d < 0, \\ \frac{\log \epsilon(d)}{\log \epsilon(d/M^2)}, & \text{if } d > 0. \end{cases}$$

Proof: This follows from Lemma 1.18 by applying it to all the primes dividing M . $\square$

1.8 Final formula for $R_G(n, d)$

We set

$$N(n, d) = \sum_{K \in H(d)} R_K(n, d), \quad (1.29)$$

and

$$w(d) = \begin{cases} 6, & \text{if } d = -3, \\ 4, & \text{if } d = -4, \\ 2, & \text{if } d < -4, \\ 1, & \text{if } d > 0. \end{cases} \quad (1.30)$$

We recall the following result of Dirichlet (see [17], Theorem 204).

Theorem 1.3 *If $\gcd(n, d) = 1$ then*

$$N(n, d) = w(d) \sum_{\nu|n} \left(\frac{d}{\nu} \right).$$

We make use of Theorem 1.3 to prove our next theorem.

Theorem 1.4 *If $\gcd(n, d) = 1$ and $G \in G(d)$ then*

$$R_G(n, d) = \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1).$$

Proof: If $N(n, d) > 0$, then n is represented by at least one class in $H(d)$. Let $\hat{G}$ be a genus containing a class which represents n . Since $\gcd(n, d) = 1$, we have $\gcd(n, q^*) = 1$ for all $q^* \in P(d)$, so that

$$\gamma_{q^*}(\hat{G}) = \left(\frac{q^*}{n} \right) \quad \text{for all } q^* \in P(d).$$

Thus $\hat{G}$ is unique and we have

$$R_G(n, d) = \begin{cases} N(n, d), & \text{if } G = \hat{G}, \\ 0, & \text{otherwise,} \end{cases}$$

that is,

$$R_G(n, d) = N(n, d) \prod_{q^* \in P(d)} \frac{1}{2} \left(1 + \gamma_{q^*}(G) \left(\frac{q^*}{n} \right) \right). \quad (1.31)$$

If $N(n, d) = 0$, then (1.31) holds trivially.

By Theorem 1.3 and (1.31), we have

$$\begin{aligned} R_G(n, d) &= w(d) \sum_{\nu|n} \left(\frac{d}{\nu} \right) \prod_{q^* \in P(d)} \frac{1}{2} \left(1 + \gamma_{q^*}(G) \left(\frac{q^*}{n} \right) \right) \\ &= \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \left(\frac{d_1}{n} \right) \sum_{\mu\nu=n} \left(\frac{d}{\nu} \right). \end{aligned}$$

We note that in the case $d \equiv 0 \pmod{32}$ (recall that $|P(d)| = t(d) + 2$), each term in the development of

$$\prod_{q^* \in P(d)} \left(1 + \gamma_{q^*}(G) \left(\frac{q^*}{n} \right) \right)$$

is obtained exactly twice since $\gamma_{-4}(G)\gamma_8(G)\gamma_{-8}(G) = 1$. Hence we have

$$\begin{aligned} R_G(n, d) &= \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) \sum_{\mu\nu=n} \left(\frac{d_1}{\mu} \right) \left(\frac{d_1}{\nu} \right) \left(\frac{d_1}{\nu} \right) \left(\frac{d/d_1}{\nu} \right) \\ &= \frac{w(d)}{2^{t(d)+1}} \sum_{d_1 \in F(d)} \gamma_{d_1}(G) S(n, d_1, d/d_1), \end{aligned}$$

as required. □

We are now ready to prove our main result.

Theorem 1.5 *Let $G \in G(d)$. If $\text{Null}(n, d) \neq \emptyset$ then $R_G(n, d) = 0$. If $\text{Null}(n, d) = \emptyset$ then*

$$R_G(n, d) = j(d, M) \frac{h(d)}{h(d/M^2)} \frac{w(d/M^2)}{2^{t(d)+1}} \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G) S(n/M^2, d_1, d/M^2 d_1),$$

where $j(d, M)$ is defined in Theorem 1.2.

Proof: If $\text{Null}(n, d) \neq \emptyset$ then $R_G(n, d) = 0$ by Lemma 1.7. Let $\text{Null}(n, d) = \emptyset$.

By Theorems 1.2 and 1.1, we have

$$\begin{aligned} R_G(n, d) &= \frac{j(d, M)}{2^{t(d)-t(d/M^2)}} \frac{h(d)}{h(d/M^2)} R_{G_M}(n/M^2, d/M^2) \\ &= \frac{j(d, M)}{2^{t(d)-t(d/M^2)}} \frac{h(d)}{h(d/M^2)} R_{G_{MQ}}(n/M^2 Q, d/M^2), \end{aligned}$$

as $U(n/M^2, d/M^2) = Q$ by (1.24). Since $\text{Null}(n, d) = \emptyset$, we have

$$\gcd\left(\frac{n}{M^2 Q}, \frac{d}{M^2}\right) = 1 \quad \text{and} \quad \gcd\left(\frac{n}{M^2}, \frac{f}{M}\right) = 1$$

by Lemma 1.8, so that by Theorem 1.4, Lemma 1.10 and Lemma 1.6(b), we have

$$\begin{aligned} R_G(n, d) &= \frac{j(d, M)}{2^{t(d)-t(d/M^2)}} \frac{h(d)}{h(d/M^2)} \frac{w(d/M^2)}{2^{t(d/M^2)+1}} \\ &\quad \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G_{MQ}) S(n/M^2 Q, d_1, d/M^2 d_1) \\ &= \frac{j(d, M)}{2^{t(d)+1}} \frac{h(d)}{h(d/M^2)} w(d/M^2) \\ &\quad \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G_M) S(n/M^2, d_1, d/M^2 d_1) \\ &= \frac{j(d, M)}{2^{t(d)+1}} \frac{h(d)}{h(d/M^2)} w(d/M^2) \\ &\quad \times \sum_{d_1 \in F(d/M^2)} \gamma_{d_1}(G) S(n/M^2, d_1, d/M^2 d_1), \end{aligned}$$

as required.

Chapter 2

The restricted Epstein zeta function of an indefinite binary quadratic form

2.1 Introduction

The Epstein zeta function (see [11]) $Z_Q(s)$ of a positive-definite binary quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ is defined for $s > 1$ by

$$Z_Q(s) = \sum_{\substack{x, y = -\infty \\ (x, y) \neq (0, 0)}}^{\infty} \frac{1}{Q(x, y)^s}.$$

Let d be a positive non-square discriminant. Let a, b and c be integers with $b^2 - 4ac = d$ and $a > 0$. We set $Q(x, y) = ax^2 + bxy + cy^2$ so that Q is an indefinite binary quadratic form of discriminant d . Let $\epsilon = \epsilon(d)$ be given by (1.4). For $s > 1$, we define the restricted Epstein zeta function $Z_Q(s)$ by

$$Z_Q(s) = \sum_{\substack{x, y = -\infty \\ Q(x, y) > 0 \\ 2ax + (b - \sqrt{d})y > 0 \\ 1 \leq \left| \frac{2ax + (b + \sqrt{d})y}{2ax + (b - \sqrt{d})y} \right| < \epsilon^2}}^{\infty} \frac{1}{Q(x, y)^s}. \quad (2.1)$$

We remark that Zagier [29] has considered a different analogue of the Epstein zeta function. In Section 2.2, it is shown that the series defining $Z_Q(s)$ converges for $s > 1$. In Section 2.3, the first two terms of the Laurent expansion of $Z_Q(s)$ about $s = 1$ are obtained. In Sections 2.4–2.7, several different expressions are obtained for the constant term in the Laurent expansion of $Z_Q(s)$ about $s = 1$. The final expression for the constant term is given in Theorem 2.1, which is the main result of this chapter.

2.2 The convergence of $Z_Q(s)$

We show that the series for $Z_Q(s)$ converges for $s > 1$. We examine the three parts of the series defining $Z_Q(s)$ corresponding to $y = 0$, $y > 0$ and $y < 0$ and show that each converges for $s > 1$. The part corresponding to $y = 0$ is clearly

$$\sum_{x=1}^{\infty} \frac{1}{Q(x, 0)^s} = \sum_{x=1}^{\infty} \frac{1}{(ax^2)^s} = a^{-s} \zeta(2s) \quad (2.2)$$

for $s > 1/2$, where $\zeta(s)$ denotes the Riemann zeta function.

We set

$$E = 2ax + (b + \sqrt{d})y, \quad E' = 2ax + (b - \sqrt{d})y.$$

The summation conditions occurring in the definition of $Z_Q(s)$ are

$$EE' > 0, \quad E' > 0, \quad 1 \leq \left| \frac{E}{E'} \right| < \epsilon^2.$$

These are equivalent to

$$E > 0, \quad E' > 0, \quad E' \leq E < \epsilon^2 E'.$$

First let $y > 0$. We have $E > E'$ so that the summation conditions are equivalent to

$$E' > 0, E < \epsilon^2 E'.$$

The second of these conditions is equivalent to

$$2ax > \left(-b + \sqrt{d} + \frac{2\sqrt{d}}{\epsilon^2 - 1} \right) y.$$

If this holds, we have $2ax > (-b + \sqrt{d})y$ so that $E' > 0$. Thus, if $y > 0$, we have shown that the summation conditions in the definition of $Z_Q(s)$ are satisfied if and only if $2ax > \lambda y$, where

$$\lambda = -b + \sqrt{d} + \frac{2\sqrt{d}}{\epsilon^2 - 1}. \quad (2.3)$$

Hence the part corresponding to $y > 0$ is

$$\sum_{y=1}^{\infty} \sum_{x > \lambda_1 y} \frac{1}{Q(x, y)^s}, \quad (2.4)$$

where

$$\lambda_1 = \frac{\lambda}{2a}. \quad (2.5)$$

If $y < 0$ then $E < E'$ and so the summation conditions in the definition of $Z_Q(s)$ are never satisfied. Thus we must examine the convergence of

$$\sum_{y=1}^{\infty} \sum_{x > \lambda_1 y} \frac{1}{Q(x, y)^s} = \sum_{y=1}^{\infty} y^{-2s} \sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-s}. \quad (2.6)$$

To evaluate the inner sum in (2.6), we apply the Euler-Maclaurin summation formula. For $s > 1/2$, $y > 0$, we obtain

$$\sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-s}$$

$$\begin{aligned}
&= P(\lambda_1 y)Q(\lambda_1, 1)^{-s} + \int_{\lambda_1 y}^{\infty} Q(xy^{-1}, 1)^{-s} dx \\
&\quad + \int_{\lambda_1 y}^{\infty} (-s)Q(xy^{-1}, 1)^{-s-1}(2axy^{-2} + by^{-1})P(x)dx \\
&= y \int_{\lambda_1}^{\infty} Q(t, 1)^{-s} dt - s \int_{\lambda_1}^{\infty} Q(t, 1)^{-s-1}(2at + b)P(ty)dt \\
&\quad + P(\lambda_1 y)Q(\lambda_1, 1)^{-s},
\end{aligned} \tag{2.7}$$

where $P(x) = x - [x] - 1/2$. Thus, for $s > 1$, we have

$$\begin{aligned}
&\sum_{y=1}^{\infty} y^{-2s} \sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-s} \\
&= \zeta(2s-1) \int_{\lambda_1}^{\infty} Q(t, 1)^{-s} dt + Q(\lambda_1, 1)^{-s} \sum_{y=1}^{\infty} \frac{P(\lambda_1 y)}{y^{2s}} \\
&\quad - s \sum_{y=1}^{\infty} y^{-2s} \int_{\lambda_1}^{\infty} \frac{(2at + b)P(ty)}{Q(t, 1)^{s+1}} dt.
\end{aligned} \tag{2.8}$$

This shows that

$$\sum_{y=1}^{\infty} y^{-2s} \sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-s},$$

and thus the original series for $Z_Q(s)$ converges for $s > 1$. Putting together (2.1), (2.2) and (2.8), we obtain

Lemma 2.1 *For $s > 1$, we have*

$$\begin{aligned}
Z_Q(s) &= a^{-s} \zeta(2s) + \zeta(2s-1) \int_{\lambda_1}^{\infty} Q(t, 1)^{-s} dt + Q(\lambda_1, 1)^{-s} \sum_{y=1}^{\infty} \frac{P(\lambda_1 y)}{y^{2s}} \\
&\quad - s \sum_{y=1}^{\infty} y^{-2s} \int_{\lambda_1}^{\infty} \frac{(2at + b)P(ty)}{Q(t, 1)^{s+1}} dt.
\end{aligned} \tag{2.9}$$

2.3 The Laurent expansion of $Z_Q(s)$ at $s = 1$

In this section we determine the first two terms of the Laurent expansion of $Z_Q(s)$ about $s = 1$. All the series and integrals appearing in (2.9) except

the series for $\zeta(2s - 1)$ converge uniformly on compact subsets of the region $\text{Re}(s) > 1/2$. Thus they are analytic in this region. As $s \rightarrow 1^+$, we have

$$\int_{\lambda_1}^{\infty} Q(t, 1)^{-s} dt = \int_{\lambda_1}^{\infty} \frac{1}{Q(t, 1)} dt - \left(\int_{\lambda_1}^{\infty} \frac{\log Q(t, 1)}{Q(t, 1)} dt \right) (s - 1) + O((s - 1)^2). \quad (2.10)$$

We have $Q(t, 1) = a(t + t_1)(t + t_2)$, where

$$t_1 = \frac{b + \sqrt{d}}{2a}, \quad t_2 = \frac{b - \sqrt{d}}{2a}. \quad (2.11)$$

We note that by (2.3) and (2.5), $t + t_1$ and $t + t_2$ are positive for $t \geq \lambda_1$. Using these facts, it is easily shown that

$$\int_{\lambda_1}^{\infty} \frac{1}{Q(t, 1)} dt = \frac{2 \log \epsilon}{\sqrt{d}}. \quad (2.12)$$

Also

$$\begin{aligned} & \int_{\lambda_1}^{\infty} \frac{\log Q(t, 1)}{Q(t, 1)} dt \\ &= \frac{1}{a(t_1 - t_2)} \int_{\lambda_1}^{\infty} \log(a(t + t_1)(t + t_2)) \left(\frac{1}{t + t_2} - \frac{1}{t + t_1} \right) dt \\ &= \frac{1}{\sqrt{d}} \int_{\lambda_1}^{\infty} \log a \left(\frac{1}{t + t_2} - \frac{1}{t + t_1} \right) dt \\ &\quad + \frac{1}{\sqrt{d}} \int_{\lambda_1}^{\infty} \log((t + t_1)(t + t_2)) \left(\frac{1}{t + t_2} - \frac{1}{t + t_1} \right) dt \\ &= \frac{1}{\sqrt{d}} \int_0^{\infty} \log((t + \lambda_1 + t_1)(t + \lambda_1 + t_2)) \left(\frac{1}{t + \lambda_1 + t_2} - \frac{1}{t + \lambda_1 + t_1} \right) dt \\ &\quad + \frac{2 \log a \log \epsilon}{\sqrt{d}}. \end{aligned}$$

Let

$$\lambda_0 = \frac{\sqrt{d}}{a(\epsilon^2 - 1)}, \quad (2.13)$$

so that by (2.3), (2.5) and (2.11), we have

$$\lambda_1 + t_1 = \epsilon^2 \lambda_0, \quad \lambda_1 + t_2 = \lambda_0. \quad (2.14)$$

Hence

$$\begin{aligned} & \int_0^\infty \log((t + \lambda_1 + t_1)(t + \lambda_1 + t_2)) \left(\frac{1}{t + \lambda_1 + t_2} - \frac{1}{t + \lambda_1 + t_1} \right) dt \\ &= \int_0^\infty \log((t + \epsilon^2 \lambda_0)(t + \lambda_0)) \left(\frac{1}{t + \lambda_0} - \frac{1}{t + \epsilon^2 \lambda_0} \right) dt \\ &= \int_0^\infty \left(\frac{\log(t + \lambda_0)}{t + \lambda_0} - \frac{\log(t + \epsilon^2 \lambda_0)}{t + \epsilon^2 \lambda_0} \right) dt \\ &\quad + \int_0^\infty \left(\frac{\log(t + \epsilon^2 \lambda_0)}{t + \lambda_0} - \frac{\log(t + \lambda_0)}{t + \epsilon^2 \lambda_0} \right) dt \\ &= 2 \log \epsilon \log(\epsilon \lambda_0) + \int_0^\infty \left(\frac{\log(t + \epsilon^2 \lambda_0)}{t + \lambda_0} - \frac{\log(t + \lambda_0)}{t + \epsilon^2 \lambda_0} \right) dt \\ &= 2 \log \epsilon \log(\epsilon \lambda_0) + 2 \log \epsilon \log \lambda_0 + \int_0^\infty \left(\frac{\log(u + \epsilon^2)}{u + 1} - \frac{\log(u + 1)}{u + \epsilon^2} \right) du. \end{aligned}$$

Thus

$$\int_{\lambda_1}^\infty \frac{\log Q(t, 1)}{Q(t, 1)} dt = \frac{2 \log \epsilon \log(a \epsilon \lambda_0^2)}{\sqrt{d}} + \frac{1}{\sqrt{d}} \int_0^\infty \left(\frac{\log(u + \epsilon^2)}{u + 1} - \frac{\log(u + 1)}{u + \epsilon^2} \right) du. \quad (2.15)$$

Let γ denote Euler's constant defined by (see [18], p. 413)

$$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \right).$$

Using (2.10), (2.12) and (2.15) together with

$$\begin{aligned} a^{-s} \zeta(2s) &= \frac{\pi^2}{6a} + O(s - 1), \\ \zeta(2s - 1) &= \frac{1/2}{s - 1} + \gamma + O(s - 1), \\ Q(\lambda_1, 1)^{-s} \sum_{y=1}^\infty \frac{P(\lambda_1 y)}{y^{2s}} &= Q(\lambda_1, 1)^{-1} \sum_{y=1}^\infty \frac{P(\lambda_1 y)}{y^2} + O(s - 1), \end{aligned}$$

$$s \sum_{y=1}^{\infty} y^{-2s} \int_{\lambda_1}^{\infty} \frac{(2at+b)P(ty)}{Q(t,1)^{s+1}} dt = \sum_{y=1}^{\infty} y^{-2} \int_{\lambda_1}^{\infty} \frac{(2at+b)P(ty)}{Q(t,1)^2} dt + O(s-1),$$

as $s \rightarrow 1^+$ in (2.9), we obtain

Lemma 2.2 *As $s \rightarrow 1^+$, we have*

$$Z_Q(s) = \frac{\log \epsilon}{\sqrt{d}} \frac{1}{s-1} + C_Q + O(s-1), \quad (2.16)$$

where

$$\begin{aligned} C_Q = & \frac{\pi^2}{6a} + \frac{2\gamma \log \epsilon}{\sqrt{d}} - \frac{1}{2\sqrt{d}} \int_0^{\infty} \left(\frac{\log(u+\epsilon^2)}{u+1} - \frac{\log(u+1)}{u+\epsilon^2} \right) du \\ & - \frac{\log \epsilon \log(a\epsilon\lambda_0^2)}{\sqrt{d}} + Q(\lambda_1, 1)^{-1} \sum_{y=1}^{\infty} \frac{P(\lambda_1 y)}{y^2} \\ & - \sum_{y=1}^{\infty} y^{-2} \int_{\lambda_1}^{\infty} \frac{(2at+b)P(ty)}{Q(t,1)^2} dt. \end{aligned} \quad (2.17)$$

Set

$$K(d) = \frac{2\gamma \log \epsilon}{\sqrt{d}} - \frac{1}{2\sqrt{d}} \int_0^{\infty} \left(\frac{\log(u+\epsilon^2)}{u+1} - \frac{\log(u+1)}{u+\epsilon^2} \right) du, \quad (2.18)$$

so that

$$\begin{aligned} C_Q = & K(d) + \frac{\pi^2}{6a} - \frac{\log \epsilon \log(a\epsilon\lambda_0^2)}{\sqrt{d}} + \frac{1}{Q(\lambda_1, 1)} \sum_{y=1}^{\infty} \frac{P(\lambda_1 y)}{y^2} \\ & - \sum_{y=1}^{\infty} y^{-2} \int_{\lambda_1}^{\infty} \frac{(2at+b)P(ty)}{Q(t,1)^2} dt. \end{aligned} \quad (2.19)$$

We emphasize that $K(d)$ depends only on d and not on the form (a, b, c) .

2.4 A second expression for the constant term in the Laurent expansion of $Z_Q(s)$ at $s = 1$

For the rest of this chapter, we focus on the constant C_Q defined in (2.19). We obtain different expressions for C_Q culminating in the final expression given in

Theorem 2.1. It is this form of C_Q that is used in Chapter 3. By (2.7) and (2.12), we have, for $y > 0$,

$$\begin{aligned} & \sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-1} \\ &= y \int_{\lambda_1}^{\infty} \frac{1}{Q(t, 1)} dt - \int_{\lambda_1}^{\infty} \frac{(2at + b)P(ty)}{Q(t, 1)^2} dt + \frac{P(\lambda_1 y)}{Q(\lambda_1, 1)} \\ &= \frac{2y \log \epsilon}{\sqrt{d}} - \int_{\lambda_1}^{\infty} \frac{(2at + b)P(ty)}{Q(t, 1)^2} dt + \frac{P(\lambda_1 y)}{Q(\lambda_1, 1)}. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{y=1}^{\infty} y^{-2} \left(\sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-1} - \frac{2y \log \epsilon}{\sqrt{d}} \right) \\ &= \sum_{y=1}^{\infty} y^{-2} \left(\frac{P(\lambda_1 y)}{Q(\lambda_1, 1)} - \int_{\lambda_1}^{\infty} \frac{(2at + b)P(ty)}{Q(t, 1)^2} dt \right). \end{aligned}$$

Using this in (2.19), gives our second expression for C_Q .

Lemma 2.3

$$\begin{aligned} C_Q &= K(d) + \frac{\pi^2}{6a} - \frac{\log \epsilon \log(a\epsilon\lambda_0^2)}{\sqrt{d}} \\ &\quad + \sum_{y=1}^{\infty} y^{-2} \left(\sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-1} - \frac{2y \log \epsilon}{\sqrt{d}} \right). \end{aligned} \quad (2.20)$$

2.5 A third expression for the constant term in the Laurent expansion of $Z_Q(s)$ at $s = 1$

By (2.11), we have, for $y > 0$,

$$\begin{aligned} & \sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-1} \\ &= \sum_{x > \lambda_1 y} \frac{1}{a(xy^{-1} + t_1)(xy^{-1} + t_2)} \end{aligned}$$

$$\begin{aligned}
&= \frac{y}{\sqrt{d}} \sum_{x=1+[\lambda_1 y]}^{\infty} \left(\frac{1}{x+t_2 y} - \frac{1}{x+t_1 y} \right) \\
&= \frac{y}{\sqrt{d}} \sum_{m=0}^{\infty} \left(\frac{1}{m+1+[\lambda_1 y]+t_2 y} - \frac{1}{m+1+[\lambda_1 y]+t_1 y} \right).
\end{aligned} \tag{2.21}$$

Using (2.14), we note that $1 + [\lambda_1 y] + t_2 y > (\lambda_1 + t_2)y = \lambda_0 y > 0$ and $1 + [\lambda_1 y] + t_1 y > (\lambda_1 + t_1)y = \lambda_0 \epsilon^2 y > 0$. We recall the formula (valid for $x > 0$) (see [18], p. 413)

$$-\psi(x) = \frac{1}{x} + \gamma + \sum_{m=1}^{\infty} \left(\frac{1}{x+m} - \frac{1}{m} \right),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $\Gamma(x)$ denotes the gamma function defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt.$$

Hence for $x_1 > 0, x_2 > 0$, we have

$$\psi(x_1) - \psi(x_2) = \sum_{m=0}^{\infty} \left(\frac{1}{m+x_2} - \frac{1}{m+x_1} \right).$$

Using this in (2.21), we obtain, for $y > 0$,

$$\sum_{x > \lambda_1 y} Q(xy^{-1}, 1)^{-1} = \frac{y}{\sqrt{d}} (\psi(1 + [\lambda_1 y] + t_1 y) - \psi(1 + [\lambda_1 y] + t_2 y)). \tag{2.22}$$

Hence, by (2.20), we have our third expression for C_Q .

Lemma 2.4

$$\begin{aligned}
C_Q &= K(d) + \frac{\pi^2}{6a} - \frac{\log \epsilon \log(a\epsilon \lambda_0^2)}{\sqrt{d}} \\
&+ \frac{1}{\sqrt{d}} \sum_{y=1}^{\infty} \frac{1}{y} (\psi(1 + [\lambda_1 y] + t_1 y) - \psi(1 + [\lambda_1 y] + t_2 y) - \log \epsilon^2).
\end{aligned} \tag{2.23}$$

2.6 A fourth expression for the constant term in the Laurent expansion of $Z_Q(s)$ at $s = 1$

Let

$$\epsilon' = \frac{x_0 - y_0\sqrt{d}}{2}. \quad (2.24)$$

Then, by (1.4), we have

$$\epsilon\epsilon' = 1 \quad (2.25)$$

since $x_0^2 - dy_0^2 = 4$. Hence $\epsilon'(\epsilon^2 - 1) = \epsilon - \epsilon' = y_0\sqrt{d}$, so that (by (2.13)),

$$\lambda_0 = \frac{\sqrt{d}}{a(\epsilon^2 - 1)} = \frac{\epsilon'}{ay_0}, \quad (2.26)$$

and (by (2.3))

$$\lambda = -b + \sqrt{d} + \frac{2\sqrt{d}}{\epsilon^2 - 1} = -b + \frac{\epsilon - \epsilon'}{y_0} + \frac{2\epsilon'}{y_0} = -b + \frac{\epsilon + \epsilon'}{y_0}.$$

Hence

$$\lambda = -b + \frac{x_0}{y_0} \quad (2.27)$$

and (by (2.5))

$$\lambda_1 = \frac{\lambda}{2a} = \frac{\alpha}{g}, \quad (2.28)$$

where

$$g = g(a, d) = ay_0, \quad \alpha = \alpha(b, d) = \frac{x_0 - by_0}{2}. \quad (2.29)$$

Since $b^2 - 4ac = d$ and $x_0^2 - dy_0^2 = 4$, we have

$$x_0^2 - b^2y_0^2 = 4 - 4acy_0^2 \equiv 0 \pmod{4},$$

so that $x_0 \equiv by_0 \pmod{2}$. Hence α is an integer. Moreover

$$\gcd(\alpha, g) = 1, \quad (2.30)$$

since

$$\alpha \left(\frac{x_0 + by_0}{2} \right) + gcy_0 = 1.$$

We have (by (2.5), (2.11), (2.27), (1.4) and (2.29))

$$\lambda_1 + t_1 = \frac{\lambda + b + \sqrt{d}}{2a} = \frac{x_0/y_0 + \sqrt{d}}{2a} = \frac{\epsilon}{g} \quad (2.31)$$

and similarly

$$\lambda_1 + t_2 = \frac{\epsilon'}{g}. \quad (2.32)$$

For any positive integer y , we let

$$r_y = g(\lambda_1 y - [\lambda_1 y]). \quad (2.33)$$

We note that r_y is an integer since $r_y = \alpha y - g[\lambda_1 y]$ by (2.28). Also

$$0 \leq r_y < g. \quad (2.34)$$

We recall the standard formulae valid for $x > 0$ (see [18], p. 418)

$$\begin{aligned} \log x &= \int_0^\infty \frac{(e^{-u} - e^{-xu})}{u} du, \\ \psi(x) &= \int_0^\infty \left(\frac{e^{-u}}{u} - \frac{e^{-xu}}{1 - e^{-u}} \right) du. \end{aligned}$$

Hence for $x_1 > 0$, $x_2 > 0$, we have

$$\log x_1 - \log x_2 = \int_0^\infty \frac{(e^{-x_2 u} - e^{-x_1 u})}{u} du.$$

Using the substitution $t = e^{-u}$, we obtain

$$\log x_1 - \log x_2 = \int_0^1 \frac{t^{x_1} - t^{x_2}}{t \log t} dt. \quad (2.35)$$

Similarly, for $1 + x_1 > 0$, $1 + x_2 > 0$, we obtain

$$\psi(1 + x_1) - \psi(1 + x_2) = \int_0^1 \frac{t^{x_2} - t^{x_1}}{1 - t} dt. \quad (2.36)$$

Hence, by (2.31), (2.32), (2.33), (2.34), (2.25), (2.35) and (2.36), we obtain

$$\begin{aligned} & \sum_{y=1}^{\infty} \frac{1}{y} (\psi(1 + [\lambda_1 y] + t_1 y) - \psi(1 + [\lambda_1 y] + t_2 y) - \log \epsilon^2) \\ &= \sum_{y=1}^{\infty} \frac{1}{y} (\psi(1 + (\lambda_1 + t_1)y - (\lambda_1 y - [\lambda_1 y])) \\ &\quad - \psi(1 + (\lambda_1 + t_2)y - (\lambda_1 y - [\lambda_1 y])) - \log \epsilon^2) \\ &= \sum_{y=1}^{\infty} \frac{1}{y} \left(\psi \left(1 + \frac{\epsilon y}{g} - \frac{r_y}{g} \right) - \psi \left(1 + \frac{\epsilon' y}{g} - \frac{r_y}{g} \right) - (\log \epsilon y - \log \epsilon' y) \right) \\ &= \sum_{y=1}^{\infty} \frac{1}{y} \left(\int_0^1 \frac{t^{(\epsilon' y - r_y)/g} - t^{(\epsilon y - r_y)/g}}{1 - t} dt - \int_0^1 \frac{t^{\epsilon y} - t^{\epsilon' y}}{t \log t} dt \right) \\ &= \sum_{y=1}^{\infty} \frac{1}{y} \left(g \int_0^1 \frac{t^{\epsilon' y} - t^{\epsilon y}}{1 - t^g} t^{g-1-r_y} dt - \int_0^1 \frac{t^{\epsilon y} - t^{\epsilon' y}}{t \log t} dt \right) \\ &= \int_0^1 \sum_{y=1}^{\infty} \frac{t^{\epsilon y} - t^{\epsilon' y}}{t y} \left(-\frac{1}{\log t} + \frac{g t^{g-r_y}}{t^g - 1} \right) dt. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{y=1}^{\infty} \frac{1}{y} (\psi(1 + [\lambda_1 y] + t_1 y) - \psi(1 + [\lambda_1 y] + t_2 y) - \log \epsilon^2) \\ &= \int_0^1 \left(\frac{1}{t \log t} \log \left(\frac{1 - t^{\epsilon}}{1 - t^{\epsilon'}} \right) + g \sum_{y=1}^{\infty} \frac{t^{\epsilon y} - t^{\epsilon' y}}{y(t^g - 1)} t^{g-r_y-1} \right) dt. \quad (2.37) \end{aligned}$$

Let $0 < t < 1$. We have

$$\begin{aligned} \sum_{y=1}^{\infty} \frac{g}{y} \frac{t^{\epsilon y} - t^{\epsilon' y}}{t^g - 1} t^{g-r_y-1} &= \sum_{k=0}^{g-1} \sum_{r_y=k} \frac{t^{\epsilon y} - t^{\epsilon' y}}{y} \frac{g t^{g-1-k}}{t^g - 1} \\ &= \sum_{k=0}^{g-1} \sum_{\alpha y \equiv k \pmod{g}} \frac{t^{\epsilon y} - t^{\epsilon' y}}{y} \frac{g t^{g-k-1}}{t^g - 1}. \quad (2.38) \end{aligned}$$

Let

$$\theta = e^{2\pi i/g}. \quad (2.39)$$

Then

$$\frac{gt^{g-k-1}}{t^g - 1} = \frac{gt^{g-k-1}}{(t - \theta) \dots (t - \theta^g)} = \sum_{l=1}^g \frac{A_{l,k}}{t - \theta^l}, \quad (2.40)$$

where

$$A_{l,k} = \frac{g\theta^{-l(k+1)}}{\prod_{j \neq l} (\theta^l - \theta^j)}.$$

But, for $1 \leq l \leq g$, we have

$$\prod_{j \neq l} (\theta^l - \theta^j) = \theta^{l(g-1)} \prod_{j \neq l} (1 - \theta^{j-l}) = \theta^{-l} \prod_{i=1}^{g-1} (1 - \theta^i) = g\theta^{-l}.$$

Hence

$$A_{l,k} = \theta^{-lk}. \quad (2.41)$$

Thus, by (2.38), (2.40) and (2.41), we have

$$\begin{aligned} & \sum_{y=1}^{\infty} \frac{g}{y} \frac{t^{\epsilon y} - t^{\epsilon' y}}{t^g - 1} t^{g-r_y-1} \\ &= \sum_{k=0}^{g-1} \sum_{\alpha y \equiv k \pmod{g}} \frac{t^{\epsilon y} - t^{\epsilon' y}}{y} \sum_{l=1}^g \frac{\theta^{-lk}}{t - \theta^l} \\ &= \sum_{l=1}^g \frac{1}{t - \theta^l} \sum_{k=0}^{g-1} \sum_{\alpha y \equiv k \pmod{g}} \frac{t^{\epsilon y} - t^{\epsilon' y}}{y} \theta^{-lk} \\ &= \sum_{l=1}^g \frac{1}{t - \theta^l} \sum_{k=0}^{g-1} \sum_{\alpha y \equiv k \pmod{g}} \frac{t^{\epsilon y} - t^{\epsilon' y}}{y} \theta^{-l\alpha y} \\ &= \sum_{l=1}^g \frac{1}{t - \theta^l} \sum_{y=1}^{\infty} \frac{t^{\epsilon y} - t^{\epsilon' y}}{y} \theta^{-l\alpha y} \\ &= - \sum_{l=1}^g \frac{1}{t - \theta^l} (\log(1 - \theta^{-l\alpha} t^{\epsilon}) - \log(1 - \theta^{-l\alpha} t^{\epsilon'})), \end{aligned}$$

where the principal values of the logarithms are taken. Using this in (2.37) gives

$$\begin{aligned}
& \sum_{y=1}^{\infty} \frac{1}{y} (\psi(1 + [\lambda_1 y] + t_1 y) - \psi(1 + [\lambda_1 y] + t_2 y) - \log \epsilon^2) \\
&= \int_0^1 \left(\frac{1}{t \log t} \log \left(\frac{1 - t^\epsilon}{1 - t^{\epsilon'}} \right) \right. \\
&\quad \left. - \sum_{l=1}^g \frac{1}{t - \theta^l} (\log(1 - \theta^{-l\alpha} t^\epsilon) - \log(1 - \theta^{-l\alpha} t^{\epsilon'})) \right) dt \\
&= \int_0^1 \left(\frac{1}{t \log t} - \frac{1}{t - 1} \right) \log \left(\frac{1 - t^\epsilon}{1 - t^{\epsilon'}} \right) dt \\
&\quad - \int_0^1 \sum_{l=1}^{g-1} \frac{1}{t - \theta^l} (\log(1 - \theta^{-l\alpha} t^\epsilon) - \log(1 - \theta^{-l\alpha} t^{\epsilon'})) dt. \quad (2.42)
\end{aligned}$$

For $1 \leq l \leq g - 1$, $\theta^l \neq 1$ and $\theta^{-l\alpha} \neq 1$ (since $\gcd(\alpha, g) = 1$), and

$$\begin{aligned}
& \int_0^1 \frac{1}{t - \theta^l} \log(1 - \theta^{-l\alpha} t^{\epsilon'}) dt \\
&= -\theta^{-l} \int_0^1 \frac{1}{1 - \theta^{-l} t} \log(1 - \theta^{-l\alpha} t^{\epsilon'}) dt \\
&= \left[\log(1 - \theta^{-l\alpha} t^{\epsilon'}) \log(1 - \theta^{-l} t) \right]_0^1 - \int_0^1 \frac{\log(1 - \theta^{-l} t)}{1 - \theta^{-l\alpha} t^{\epsilon'}} (-\theta^{-l\alpha}) \epsilon' t^{\epsilon'-1} dt \\
&= \log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) - \int_0^1 \frac{\log(1 - \theta^{-l} t^\epsilon)}{t - \theta^{l\alpha}} dt. \quad (2.43)
\end{aligned}$$

Hence, by (2.43), (2.42), (2.23), (2.26) and (2.25), we obtain our fourth expression for C_Q .

Lemma 2.5

$$C_Q = V(d) + \frac{\pi^2}{6a} + \frac{\log \epsilon \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_Q, \quad (2.44)$$

where

$$W_Q = \sum_{l=1}^{g-1} \left(\int_0^1 \frac{\log(1 - \theta^{-l\alpha} t^\epsilon)}{t - \theta^l} dt + \int_0^1 \frac{\log(1 - \theta^{-l} t^\epsilon)}{t - \theta^{l\alpha}} dt \right)$$

$$-\sum_{l=1}^{g-1} \log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) \quad (2.45)$$

and

$$V(d) = K(d) + \frac{\log \epsilon \log(\epsilon y_0^2)}{\sqrt{d}} + \frac{1}{\sqrt{d}} \int_0^1 \left(\frac{1}{t \log t} - \frac{1}{t-1} \right) \log \left(\frac{1-t^\epsilon}{1-t^{\epsilon'}} \right) dt. \quad (2.46)$$

We emphasize that $V(d)$ depends only on d and not on the form (a, b, c) .

2.7 Final expression for the constant term in the Laurent expansion of $Z_Q(s)$ at $s = 1$

Since $\gcd(\alpha, g) = 1$, we may choose an integer α' such that

$$\alpha\alpha' \equiv 1 \pmod{g}. \quad (2.47)$$

Changing the variable from l to $l\alpha'$ in the first sum in (2.45), we obtain

$$\begin{aligned} W_Q &= \sum_{l=1}^{g-1} \int_0^1 \log(1 - \theta^{-l} t^\epsilon) \left(\frac{1}{t - \theta^{l\alpha'}} + \frac{1}{t - \theta^{l\alpha}} \right) dt \\ &\quad - \sum_{l=1}^{g-1} \log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) \\ &= S_1(\alpha) + S_1(\alpha') - \sum_{l=1}^{g-1} \log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}), \end{aligned} \quad (2.48)$$

where

$$S_1(\alpha) = \sum_{l=1}^{g-1} \int_0^1 \frac{\log(1 - \theta^{-l} t^\epsilon)}{t - \theta^{l\alpha}} dt. \quad (2.49)$$

We set

$$F(\alpha, l, t, g) = \frac{\log(1 - \theta^{-l} t^\epsilon)}{t - \theta^{l\alpha}} + \frac{\log(1 - \theta^l t^\epsilon)}{t - \theta^{-l\alpha}}. \quad (2.50)$$

We first consider the case when g is odd. Let $g = 2m + 1$ where $m \geq 1$. (We note that $W_Q = 0$ if $g = 1$). Then

$$\begin{aligned}
 S_1(\alpha) &= \sum_{l=1}^{2m} \int_0^1 \frac{\log(1 - \theta^{-l} t^\epsilon)}{t - \theta^{l\alpha}} dt \\
 &= \int_0^1 \left(\sum_{l=1}^m \frac{\log(1 - \theta^{-l} t^\epsilon)}{t - \theta^{l\alpha}} + \sum_{l=m+1}^{2m} \frac{\log(1 - \theta^{2m+1-l} t^\epsilon)}{t - \theta^{-(2m+1-l)\alpha}} \right) dt \\
 &= \int_0^1 \sum_{l=1}^m \left(\frac{\log(1 - \theta^{-l} t^\epsilon)}{t - \theta^{l\alpha}} + \frac{\log(1 - \theta^l t^\epsilon)}{t - \theta^{-l\alpha}} \right) dt \\
 &= \int_0^1 \sum_{l=1}^m F(\alpha, l, t, g) dt.
 \end{aligned}$$

Similarly

$$\begin{aligned}
 &\sum_{l=1}^{g-1} \log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) \\
 &= \sum_{l=1}^m (\log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) + \log(1 - \theta^{l\alpha}) \log(1 - \theta^l)).
 \end{aligned}$$

Hence, by (2.48), we have

$$\begin{aligned}
 W_Q &= \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g)) dt \\
 &\quad - \sum_{l=1}^{[(g-1)/2]} (\log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) + \log(1 - \theta^{l\alpha}) \log(1 - \theta^l)).
 \end{aligned}$$

Similarly, for g even, we obtain

$$\begin{aligned}
 W_Q &= \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g)) dt + 2 \int_0^1 \frac{\log(1 + t^\epsilon)}{1 + t} dt \\
 &\quad - \sum_{l=1}^{[(g-1)/2]} (\log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) + \log(1 - \theta^{l\alpha}) \log(1 - \theta^l)) \\
 &\quad - \log^2 2.
 \end{aligned}$$

Thus, for all $g \geq 1$, we have

$$\begin{aligned}
W_Q &= \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g)) dt \\
&\quad - \sum_{l=1}^{[(g-1)/2]} (\log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) + \log(1 - \theta^{l\alpha}) \log(1 - \theta^l)) \\
&\quad + \left(2 \int_0^1 \frac{\log(1 + t^\epsilon)}{1 + t} dt - \log^2 2 \right) \frac{1 + (-1)^g}{2}. \tag{2.51}
\end{aligned}$$

Explicitly calculating the logarithms occurring in (2.50), we obtain, after some simplification,

$$\begin{aligned}
&F(\alpha, l, t, g) \\
&= \frac{(t - \cos(2\pi l\alpha/g)) \log(1 - 2t^\epsilon \cos(2\pi l/g) + t^{2\epsilon})}{t^2 - 2t \cos(2\pi l\alpha/g) + 1} \\
&\quad - \frac{2 \sin(2\pi l\alpha/g) \tan^{-1} \left(\frac{t^\epsilon \sin(2\pi l/g)}{1 - t^\epsilon \cos(2\pi l/g)} \right)}{t^2 - 2t \cos(2\pi l\alpha/g) + 1}, \tag{2.52}
\end{aligned}$$

for $1 \leq l \leq [(g-1)/2]$, $0 \leq t \leq 1$. Similarly, for $1 \leq l \leq [(g-1)/2]$, we obtain

$$\begin{aligned}
&\log(1 - \theta^{-l\alpha}) \log(1 - \theta^{-l}) + \log(1 - \theta^{l\alpha}) \log(1 - \theta^l) \\
&= 2 \left(\log \left(2 \sin \frac{\pi l}{g} \right) \log \left(2 \left| \sin \frac{\pi l\alpha}{g} \right| \right) - \left(\frac{\pi}{2} - \frac{\pi l}{g} \right) \tan^{-1} \left(\cot \frac{\pi l\alpha}{g} \right) \right).
\end{aligned}$$

Let

$$l\alpha \equiv l^* \pmod{g}, \tag{2.53}$$

where $0 \leq l^* < g$. Then

$$\tan^{-1} \left(\cot \frac{\pi l\alpha}{g} \right) = \frac{\pi}{2} - \frac{\pi l^*}{g}.$$

Hence our final formula for W_Q is

$$W_Q = \int_0^1 \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g)) dt$$

$$\begin{aligned}
& -2 \sum_{l=1}^{[(g-1)/2]} \left(\log \left(2 \sin \frac{\pi l}{g} \right) \log \left(2 \left| \sin \frac{\pi l \alpha}{g} \right| \right) - \left(\frac{\pi}{2} - \frac{\pi l}{g} \right) \left(\frac{\pi}{2} - \frac{\pi l^*}{g} \right) \right) \\
& + \left(2 \int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt - \log^2 2 \right) \left(\frac{1+(-1)^g}{2} \right). \tag{2.54}
\end{aligned}$$

Thus we have proved

Theorem 2.1 *As $s \rightarrow 1^+$, we have*

$$Z_Q(s) = \frac{\log \epsilon}{\sqrt{d}} \frac{1}{s-1} + C_Q + O(s-1), \tag{2.55}$$

where

$$C_Q = V(d) + \frac{\pi^2}{6a} + \frac{\log \epsilon \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_Q, \tag{2.56}$$

where W_Q is given in (2.54) and $V(d)$ is given in (2.46).

Chapter 3

An analogue for positive discriminants of the Chowla-Selberg formula for genera

3.1 Introduction

Let d be a positive non-square discriminant. Let $K \in H(d)$ and let $Q = (a, b, c) \in K$ with $a > 0$. In Chapter 2, we showed that $Z_Q(s)$ (defined in (2.1)) converges for $s > 1$. Hence, for $s > 1$, we have

$$Z_Q(s) = \sum_{n=1}^{\infty} \frac{R_Q(n, d)}{n^s} = \sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s}. \quad (3.1)$$

Thus, for $G \in G(d)$,

$$\sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} = \sum_{K \in G} \sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s}$$

converges for $s > 1$. In Section 3.2, we evaluate the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} \quad (3.2)$$

explicitly using the formula for $R_G(n, d)$ obtained in Theorem 1.5. In Section 3.3, we use this explicit evaluation to obtain the first two terms of the Laurent expansion of the series (3.2) at $s = 1$ (see Theorem 3.2). By putting the information about the series (3.2) obtained in Theorem 3.2 together with Theorem 2.1, we obtain the main result of this Chapter in Section 3.4 (see Theorem 3.3). This result is an analogue for positive discriminants of the Chowla-Selberg formula for genera.

3.2 Evaluation of the Dirichlet series $\sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s}$

For $s > 1$ and any discriminant D , the Dirichlet L -series $L(s, D)$ is defined by

$$L(s, D) = \sum_{n=1}^{\infty} \frac{\left(\frac{D}{n}\right)}{n^s}.$$

If D is not a perfect square, the above series converges for $s > 0$.

Theorem 3.1 *Let $d > 0$, $G \in G(d)$. For $s > 1$, we have*

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} \\ &= \frac{h(d) \log \epsilon(d)}{2^{t(d)+1}} \sum_{m|f} \frac{1}{\log \epsilon(d/m^2) h(d/m^2)} \frac{1}{m^{2s}} \sum_{d_1 \in F(d/m^2)} \gamma_{d_1}(G) \\ & \quad \times \prod_{p|f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-s}\right) \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-s}\right) \\ & \quad \times L(s, d_1) L(s, \Delta(d/d_1)). \end{aligned}$$

Proof: Let $M(n, d)$ be as defined in (1.22). By Theorem 1.5, we have

$$\sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s}$$

$$\begin{aligned}
&= \sum_{\substack{n=1 \\ \text{Null}(n,d)=\emptyset}}^{\infty} \frac{1}{n^s} \frac{\log \epsilon(d)}{\log \epsilon(d/M(n,d)^2)} \frac{h(d)}{h(d/M(n,d)^2)} \frac{1}{2^{t(d)+1}} \\
&\quad \times \sum_{d_1 \in F(d/M(n,d)^2)} \gamma_{d_1}(G) S(n/M(n,d)^2, d_1, d/M(n,d)^2 d_1) \\
&= \frac{h(d) \log \epsilon(d)}{2^{t(d)+1}} \sum_{m|f} \frac{1}{\log \epsilon(d/m^2) h(d/m^2)} \sum_{d_1 \in F(d/m^2)} \gamma_{d_1}(G) \\
&\quad \times \sum_{\substack{n=1 \\ M(n,d)=m \\ \text{Null}(n,d)=\emptyset}}^{\infty} \frac{S(n/m^2, d_1, d/m^2 d_1)}{n^s}. \tag{3.3}
\end{aligned}$$

For $m \mid f$, $d_1 \in F(d/m^2)$, we have

$$\begin{aligned}
&\sum_{\substack{n=1 \\ M(n,d)=m \\ \text{Null}(n,d)=\emptyset}}^{\infty} \frac{S(n/m^2, d_1, d/m^2 d_1)}{n^s} \\
&= \sum_{\substack{n=1 \\ m^2 \mid n \\ \text{Null}(n/m^2, d/m^2)=\emptyset \\ M(n/m^2, d/m^2)=1}}^{\infty} \frac{S(n/m^2, d_1, d/m^2 d_1)}{n^s} \\
&= \sum_{\substack{n=1 \\ m^2 \mid n \\ (n/m^2, f/m)=1}}^{\infty} \frac{S(n/m^2, d_1, d/(m^2 d_1))}{n^s} \quad (\text{by Lemma 1.8(b)}) \\
&= \sum_{\substack{N=1 \\ (N, f/m)=1}}^{\infty} \frac{S(N, d_1, d/m^2 d_1)}{(m^2 N)^s} \\
&= m^{-2s} \sum_{\substack{N=1 \\ (N, f/m)=1}}^{\infty} \frac{1}{N^s} \sum_{\mu \nu = N} \left(\frac{d_1}{\mu} \right) \left(\frac{d/m^2 d_1}{\nu} \right) \\
&= m^{-2s} \sum_{(\mu, f/m)=1} \frac{1}{\mu^s} \left(\frac{d_1}{\mu} \right) \sum_{(\nu, f/m)=1} \frac{1}{\nu^s} \left(\frac{d/m^2 d_1}{\nu} \right).
\end{aligned}$$

As $m \mid f$ and $d_1 \in F(d/m^2)$, we have $f(d/m^2 d_1) \mid f/m$ by Lemma 1.3 (b).

Thus, if $(\nu, f/m) = 1$, we have $(\nu, f(d/m^2 d_1)) = 1$. Hence, for $(\nu, f/m) = 1$,

we have

$$\left(\frac{d/m^2 d_1}{\nu}\right) = \left(\frac{\Delta(d/m^2 d_1) f(d/m^2 d_1)^2}{\nu}\right) = \left(\frac{\Delta(d/m^2 d_1)}{\nu}\right) = \left(\frac{\Delta(d/d_1)}{\nu}\right).$$

Thus

$$\begin{aligned} & \sum_{\substack{n=1 \\ M(n,d)=m \\ Null(n,d)=\emptyset}}^{\infty} \frac{S(n/m^2, d_1, d/m^2 d_1)}{n^s} \\ &= m^{-2s} \sum_{(\mu, f/m)=1} \frac{1}{\mu^s} \left(\frac{d_1}{\mu}\right) \sum_{(\nu, f/m)=1} \frac{1}{\nu^s} \left(\frac{\Delta(d/d_1)}{\nu}\right) \\ &= m^{-2s} L(s, d_1) \prod_{p|f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-s}\right) \\ &\quad \times L(s, \Delta(d/d_1)) \prod_{p|f/m} \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-s}\right). \end{aligned}$$

The required result now follows on using (3.3). $\square$

3.3 The Laurent expansion of $\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s}$ at $s = 1$

Next we use the explicit expression for $\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s}$ obtained in Theorem 3.1 to obtain the first two terms of the Laurent expansion of $\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s}$ at $s = 1$.

Theorem 3.2 *Let $d > 0$, $G \in G(d)$. As $s \rightarrow 1^+$, we have*

$$\sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} = \frac{h(d) \log \epsilon(d)}{2^{t(d)} \sqrt{d}} \frac{1}{s-1} + B(d) + \beta(d, G) + O(s-1),$$

where $B(d)$ depends only on d and not on G (see (3.12)) and

$$\beta(d, G) = \frac{1}{2^{t(d)+1}} \sum_{\substack{d_1 \in F(d) \\ d_1 \notin \{1, \Delta\}}} \gamma_{d_1}(G) L(1, d_1) L(1, \Delta(d/d_1))$$

$$\begin{aligned}
& \times \sum_{m|f(d/d_1)} \frac{1}{m} \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \left(\frac{\Delta}{p}\right) p^{-1}\right) \prod_{p|f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-1}\right) \\
& \times \prod_{p|f/m} \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-1}\right).
\end{aligned}$$

Proof: By Theorem 3.1, we have

$$\sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} = S_1 + S_2, \quad (3.4)$$

where

$$\begin{aligned}
S_1 &= \frac{h(d) \log \epsilon(d)}{2^{t(d)+1}} \sum_{m|f} \frac{1}{\log \epsilon(d/m^2) h(d/m^2)} \frac{1}{m^{2s}} \\
&\quad \times 2\zeta(s) L(s, \Delta) \prod_{p|f/m} (1 - p^{-s}) \left(1 - \left(\frac{\Delta}{p}\right) p^{-s}\right), \quad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
S_2 &= \frac{h(d) \log \epsilon(d)}{2^{t(d)+1}} \sum_{m|f} \frac{1}{\log \epsilon(d/m^2) h(d/m^2)} \frac{1}{m^{2s}} \sum_{\substack{d_1 \in F(d/m^2) \\ d_1 \notin \{1, \Delta\}}} \gamma_{d_1}(G) \\
&\quad \times \prod_{p|f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-s}\right) \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-s}\right) \\
&\quad \times L(s, d_1) L(s, \Delta(d/d_1)). \quad (3.6)
\end{aligned}$$

We first deal with S_2 . We have

$$\begin{aligned}
S_2 &= \frac{1}{2^{t(d)+1}} \sum_{m|f} \frac{h(d) \log \epsilon(d)}{h(d/m^2) \log \epsilon(d/m^2)} \frac{1}{m^2} \sum_{\substack{d_1 \in F(d/m^2) \\ d_1 \notin \{1, \Delta\}}} \gamma_{d_1}(G) \\
&\quad \times \prod_{p|f/m} \left(1 - \left(\frac{d_1}{p}\right) p^{-1}\right) \left(1 - \left(\frac{\Delta(d/d_1)}{p}\right) p^{-1}\right) \\
&\quad \times L(1, d_1) L(1, \Delta(d/d_1)) + O(s-1).
\end{aligned}$$

We recall that if l is a non-square discriminant and $d = lm^2$ then (see [17], Theorem 214)

$$\frac{L(1, d)}{L(1, l)} = \prod_{p|m} \left(1 - \left(\frac{l}{p} \right) p^{-1} \right). \quad (3.7)$$

By (3.7) and Dirichlet's class number formula (see [17], Theorem 209), we have

$$\begin{aligned} \frac{h(d) \log \epsilon(d)}{h(d/m^2) \log \epsilon(d/m^2)} &= \frac{\sqrt{d} L(1, d)}{\sqrt{(d/m^2)} L(1, d/m^2)} \\ &= m \prod_{p|m} \left(1 - \left(\frac{d/m^2}{p} \right) p^{-1} \right) \\ &= m \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \left(\frac{\Delta}{p} \right) p^{-1} \right). \end{aligned} \quad (3.8)$$

Using (3.8) and Lemma 1.3 (c), we obtain

$$\begin{aligned} S_2 &= \frac{1}{2^{t(d)+1}} \sum_{m|f} \frac{1}{m} \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \left(\frac{\Delta}{p} \right) p^{-1} \right) \sum_{\substack{d_1 \in F(d/m^2) \\ d_1 \notin \{1, \Delta\}}} \gamma_{d_1}(G) \\ &\quad \times \prod_{p|f/m} \left(1 - \left(\frac{d_1}{p} \right) p^{-1} \right) \left(1 - \left(\frac{\Delta(d/d_1)}{p} \right) p^{-1} \right) \\ &\quad \times L(1, d_1) L(1, \Delta(d/d_1)) + O(s-1) \\ &= \frac{1}{2^{t(d)+1}} \sum_{\substack{d_1 \in F(d) \\ d_1 \notin \{1, \Delta\}}} \gamma_{d_1}(G) L(1, d_1) L(1, \Delta(d/d_1)) \sum_{m|f(d/d_1)} \frac{1}{m} \\ &\quad \times \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \left(\frac{\Delta}{p} \right) p^{-1} \right) \prod_{p|f/m} \left(1 - \left(\frac{d_1}{p} \right) p^{-1} \right) \\ &\quad \times \prod_{p|f/m} \left(1 - \left(\frac{\Delta(d/d_1)}{p} \right) p^{-1} \right) + O(s-1) \\ &= \beta(d, G) + O(s-1). \end{aligned} \quad (3.9)$$

By (3.5) and (3.8), we have

$$S_1 = \frac{\zeta(s)}{2^{t(d)}} A(s, d),$$

where

$$\begin{aligned} A(s, d) &= L(s, \Delta) \sum_{m|f} \frac{1}{m^{2s-1}} \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \left(\frac{\Delta}{p} \right) p^{-1} \right) \\ &\quad \times \prod_{p|f/m} (1 - p^{-s}) \left(1 - \left(\frac{\Delta}{p} \right) p^{-s} \right). \end{aligned} \quad (3.10)$$

Hence, we obtain

$$S_1 = \frac{A(1, d)}{2^{t(d)}} \frac{1}{s-1} + B(d) + O(s-1), \quad (3.11)$$

where

$$B(d) = (A'(1, d) + \gamma A(1, d)) / 2^{t(d)}. \quad (3.12)$$

We emphasize that $B(d)$ depends only on d and not on the genus G . By (3.7) and Dirichlet's class number formula, we obtain

$$\begin{aligned} L(1, \Delta) &= L(1, d) \prod_{p|f} \left(1 - \left(\frac{\Delta}{p} \right) p^{-1} \right)^{-1} \\ &= \frac{\log \epsilon(d) h(d)}{\sqrt{d}} \prod_{p|f} \left(1 - \left(\frac{\Delta}{p} \right) p^{-1} \right)^{-1}. \end{aligned} \quad (3.13)$$

From (3.10) with $s = 1$ and (3.13), we obtain after some simplification

$$A(1, d) = \frac{h(d) \log \epsilon(d)}{\sqrt{d}} \sum_{m|f} \frac{1}{m} \prod_{p|f/m} (1 - p^{-1}) = \frac{h(d) \log \epsilon(d)}{\sqrt{d}}. \quad (3.14)$$

Then, from (3.11) and (3.14), we deduce that

$$S_1 = \frac{h(d) \log \epsilon(d)}{2^{t(d)} \sqrt{d}} \frac{1}{s-1} + B(d) + O(s-1). \quad (3.15)$$

The required result follows from (3.15), (3.9) and (3.4). $\square$

3.4 Evaluation of a class of definite integrals: analogue of the Chowla-Selberg formula for genera

We are now ready to prove the main result of this chapter which is an analogue for positive discriminants of the Chowla-Selberg formula for genera.

Theorem 3.3 *Let $d > 0$, $G_1, G_2 \in G(d)$. Then,*

$$\begin{aligned} & \int_0^1 \left(\sum_{[a,b,c] \in G_1} E(a, b, c, t) - \sum_{[a,b,c] \in G_2} E(a, b, c, t) \right) dt \\ &= \sum_{[a,b,c] \in G_1} \left(\frac{\pi^2 \sqrt{d}}{6a} - J(a, b, c) + \log \epsilon(d) \log a \right) \\ & \quad - \sum_{[a,b,c] \in G_2} \left(\frac{\pi^2 \sqrt{d}}{6a} - J(a, b, c) + \log \epsilon(d) \log a \right) \\ & \quad - \sqrt{d}(\beta(d, G_1) - \beta(d, G_2)), \end{aligned}$$

where all the forms (a, b, c) are chosen so that $a > 0$,

$$E(a, b, c, t) = \sum_{l=1}^{[(g-1)/2]} (F(\alpha, l, t, g) + F(\alpha', l, t, g)) + (1 + (-1)^g) \frac{\log(1+t^g)}{1+t},$$

$$\begin{aligned} & J(a, b, c) \\ &= -2 \sum_{l=1}^{[(g-1)/2]} \left(\log \left(2 \sin \frac{\pi l}{g} \right) \log \left(2 \left| \sin \frac{\pi l \alpha}{g} \right| \right) - \left(\frac{\pi}{2} - \frac{\pi l}{g} \right) \left(\frac{\pi}{2} - \frac{\pi l^*}{g} \right) \right) \\ & \quad - \frac{1 + (-1)^g}{2} \log^2 2, \end{aligned}$$

and $\beta(d, G)$ is defined in Theorem 3.2.

Proof: In this proof, all forms (a, b, c) satisfy $a > 0$. We also write $\epsilon = \epsilon(d)$.

By Theorem 2.1, for $G \in G(d)$, we have

$$\begin{aligned}
 \sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} &= \sum_{[a,b,c] \in G} Z_{(a,b,c)}(s) \\
 &= \sum_{[a,b,c] \in G} \left(\frac{\log \epsilon}{\sqrt{d}} \frac{1}{s-1} + V(d) + \frac{\pi^2}{6a} \right. \\
 &\quad \left. + \frac{\log \epsilon \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right) + O(s-1) \\
 &= \frac{h(d) \log \epsilon}{2^{l(d)} \sqrt{d}} \frac{1}{s-1} + \frac{V(d) h(d)}{2^{l(d)}} \\
 &\quad + \sum_{[a,b,c] \in G} \left(\frac{\pi^2}{6a} + \frac{\log a \log \epsilon}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right) + O(s-1).
 \end{aligned}$$

By comparing with Theorem 3.2, we obtain

$$\begin{aligned}
 B(d) + \beta(d, G) &= \frac{V(d) h(d)}{2^{l(d)}} \\
 &\quad + \sum_{[a,b,c] \in G} \left(\frac{\pi^2}{6a} + \frac{\log \epsilon \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right). \quad (3.16)
 \end{aligned}$$

The expression for $B(d)$ contains $L'(1, \Delta)$ (see (3.12) and (3.10)) which is difficult to evaluate for $\Delta > 0$ (see [8]). Thus we eliminate $B(d)$ by applying (3.16) with $G = G_1$ and $G = G_2$ and subtracting the results. Thus for $G_1, G_2 \in G(d)$ we obtain

$$\begin{aligned}
 \beta(d, G_1) - \beta(d, G_2) &= \sum_{[a,b,c] \in G_1} \left(\frac{\pi^2}{6a} + \frac{\log \epsilon \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right) \\
 &\quad - \sum_{[a,b,c] \in G_2} \left(\frac{\pi^2}{6a} + \frac{\log \epsilon \log a}{\sqrt{d}} - \frac{1}{\sqrt{d}} W_{(a,b,c)} \right).
 \end{aligned}$$

The result follows on noting that

$$W_{(a,b,c)} = \int_0^1 E(a, b, c, t) dt + J(a, b, c) \quad (3.17)$$

and rearranging terms. $\square$

Chapter 4

Application to the evaluation of definite integrals of the type

$$\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt \text{ and } \int_0^1 \frac{\tan^{-1} t^\epsilon}{1+t^2} dt$$

4.1 Introduction

Herglotz [13] explicitly calculated $\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt$ for $\epsilon = 4 + \sqrt{15}$, $6 + \sqrt{35}$ and $12 + \sqrt{143}$. In this chapter, using Theorem 3.3, we obtain results which allow us to recover the results of Herglotz as well as obtain several new integral evaluations. It is unlikely that these integrals can be evaluated by conventional methods such as residue theory. A similar view has been expressed by Chowla [5].

Let d be a positive non-square discriminant. We set

$$D = dy_0(d)^2, \tag{4.1}$$

so that D is also a positive non-square discriminant. Then $\epsilon(D) = \epsilon(d) = \epsilon$ and $y_0(D) = 1$. Since $D + 4$ is a square, we have $D + 4 \equiv 0, 1, 4$, or $9 \pmod{16}$, so that $D \equiv 12, 13, 0$, or $5 \pmod{16}$. In the case $D \equiv 0 \pmod{16}$, we also

have $\epsilon(D/4) = \epsilon$ and $y_0(D/4) = 2$. In Section 4.5, we show that if D is a positive integer with $D \equiv 12 \pmod{16}$, $D + 4$ a square and $H(D)$ has one class per genus then we can explicitly evaluate $\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt$ (see Theorem 4.2). In Section 4.6, we show that if D is a positive integer with $D \equiv 0 \pmod{16}$, $D + 4$ a square, $D/4 \equiv 8 \pmod{16}$ and $H(D)$ has one class per genus, we can evaluate explicitly both $\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt$ and $\int_0^1 \frac{\tan^{-1} t^\epsilon}{1+t^2} dt$ (see Theorems 4.3 and 4.4 as well as the remark following Theorem 4.4).

4.2 Existence of primitive forms of type $(2, b, c)$ or $(4, b, c)$ with discriminant d

For convenience we completely determine the congruence classes of those d for which there exists a primitive form of type $(2, b, c)$ or $(4, b, c)$ with discriminant d . We first consider primitive forms of type $(2, b, c)$ with discriminant d .

1. $d \equiv 8 \pmod{16}$

In this case $(2, 0, -d/8)$ is a form of the required type.

2. $d \equiv 12 \pmod{16}$

In this case $(2, 2, (4 - d)/8)$ is of the required type.

3. $d \equiv 1 \pmod{8}$

$(2, 1, (1 - d)/8)$ is of the required type.

4. $d \equiv 0 \text{ or } 4 \pmod{16}$

Let us suppose such a form exists. Then $b^2 - 8c = d$ so that b is even

and c is odd. Let $b = 2l$. Then $l^2 = 2c + d/4 \equiv 2 \text{ or } 3 \pmod{4}$, which is impossible. Thus no such form exists.

5. $d \equiv 5 \pmod{8}$

In this case it is easily seen that such a form does not exist.

Next we consider primitive forms of type $(4, b, c)$ with discriminant d .

1. $d \equiv 20 \pmod{32}$

$(4, 2, (4 - d)/16)$ is of the required type.

2. $d \equiv 0 \pmod{32}$

$(4, 4, (16 - d)/16)$ is of the required type.

3. $d \equiv 16 \pmod{32}$

$(4, 0, -d/16)$ is of the required type.

4. $d \equiv 1 \pmod{16}$

$(4, 1, (1 - d)/16)$ is of the required type.

5. $d \equiv 9 \pmod{16}$

$(4, 3, (9 - d)/16)$ is of the required type.

6. $d \equiv 8 \text{ or } 12 \pmod{16}$, $d \equiv 5 \pmod{8}$ and $d \equiv 4 \pmod{32}$

In these cases, simple arguments show that such a form does not exist.

4.3 Values of $E(a, b, c, t)$ and $J(a, b, c)$ for $g = 1, 2$ and 4

If $g = 1$, we have $E(a, b, c, t) = J(a, b, c) = 0$.

If $g = 2$, we have

$$E(a, b, c, t) = 2 \frac{\log(1 + t^\epsilon)}{1 + t}, \quad J(a, b, c) = -\log^2 2.$$

If $g = 4$, we have

$$\begin{aligned} E(a, b, c, t) &= 2 \frac{t \log(1 + t^{2\epsilon}) - 2(-1)^{(\alpha-1)/2} \tan^{-1}(t^\epsilon)}{1 + t^2} + 2 \frac{\log(1 + t^\epsilon)}{1 + t}, \\ J(a, b, c) &= -\frac{3}{2} \log^2 2 + (-1)^{(\alpha-1)/2} \frac{\pi^2}{8}. \end{aligned}$$

4.4 The equation $x^2 - (k^2 - 1)y^2 = m$

The following result is useful in proving that certain form classes are not equal.

A similar result was proved by Chowla [4] who considered the equation $x^2 - (k^2 + 1)y^2 = m$. We use Chowla's method in our proof.

Theorem 4.1 *Let k, m be integers with $k > 1$, m not a square and*

$$-(2k - 2) < m < 2k + 2.$$

Then the equation

$$x^2 - (k^2 - 1)y^2 = m \tag{4.2}$$

has no solution in positive integers x and y .

Proof: We suppose that (4.2) has a solution in positive integers x and y . Let (x_1, y_1) be the solution in positive integers to (4.2) for which y_1 is least. Let

$$x_2 = |kx_1 - (k^2 - 1)y_1|, \quad y_2 = |x_1 - ky_1|.$$

Then $x_2^2 - (k^2 - 1)y_2^2 = x_1^2 - (k^2 - 1)y_1^2 = m$. If $y_2 = 0$ then $m = x_2^2$, a contradiction. Thus $y_2 \geq 1$. If $x_2 = 0$, we have

$$m = -(k^2 - 1)y_2^2 \leq -(k^2 - 1) \leq -(2k - 2),$$

a contradiction. Thus $x_2 \geq 1$. Hence, by the minimality of y_1 , we have $y_2 \geq y_1$.

Thus, either $x_1 - ky_1 \geq y_1$ or $x_1 - ky_1 \leq -y_1$. If $x_1 - ky_1 \geq y_1$, we have

$$\begin{aligned} m &= x_1^2 - (k^2 - 1)y_1^2 \\ &\geq ((k + 1)^2 - (k^2 - 1))y_1^2 \\ &= (2k + 2)y_1^2 \\ &\geq 2k + 2, \end{aligned}$$

a contradiction. Similarly if $x_1 - ky_1 \leq -y_1$, we have $m \leq -(2k - 2)$, a contradiction. $\square$

4.5 The case $D \equiv 12 \pmod{16}$

For a positive integer D , it is easily checked that $D \equiv 12 \pmod{16}$ with $D + 4$ a square if and only if $D = 4(4l^2 - 1)$ for some positive integer l .

Lemma 4.1 *Let $D = 4(4l^2 - 1)$ for some positive integer l . Then*

$$\left[1, 0, -\frac{D}{4}\right] \neq \left[2, 2, \frac{4 - D}{8}\right]$$

in $H(D)$.

Proof: If $[1, 0, -D/4] = [2, 2, (4 - D)/8]$, we have

$$2 = x^2 - \frac{D}{4}y^2 = x^2 - (4l^2 - 1)y^2 \quad (4.3)$$

for some positive integers x, y . But, by Theorem 4.1, equation (4.3) has no solution in positive integers since $2 < 2(2l) + 2$. $\square$

Theorem 4.2 *Let $D = 4(4l^2 - 1)$ for some positive integer l and suppose that $H(D)$ has one class per genus. Let G_1 be the genus containing $[1, 0, -D/4]$ (so that G_1 is the principal genus) and let G_2 be the genus containing $[2, 2, (4 - D)/8]$. Then*

$$\int_0^1 \frac{\log(1 + t^\epsilon)}{1 + t} dt = \frac{\sqrt{D}}{2}(\beta(D, G_1) - \beta(D, G_2)) - \frac{\pi^2 \sqrt{D}}{24} + \frac{\log 2 \log 2\epsilon}{2},$$

where $\epsilon = \epsilon(D) = 2l + \sqrt{4l^2 - 1}$ and $\beta(D, G)$ is defined in Theorem 3.2.

Proof: The result follows on using Theorem 3.3, noting that $y_0(D) = 1$, $g = 1$ for the form $(1, 0, -D/4)$, and $g = 2$ for the form $(2, 2, (4 - D)/8)$, and using the values of $E(a, b, c, t)$ and $J(a, b, c)$ given in Section 4.3. $\square$

The following are the first few cases where the conditions of Theorem 4.2 are satisfied so that we can calculate

$$\int_0^1 \frac{\log(1 + t^\epsilon)}{1 + t} dt.$$

1. $l = 1$, $D = 12$, $\epsilon = 2 + \sqrt{3}$

$$\int_0^1 \frac{\log(1 + t^{2+\sqrt{3}})}{1 + t} dt = \frac{\pi^2}{12}(1 - \sqrt{3}) + \log 2 \log(1 + \sqrt{3}).$$

2. $l = 2, D = 60, \epsilon = 4 + \sqrt{15}$

$$\begin{aligned} & \int_0^1 \frac{\log(1 + t^{4+\sqrt{15}})}{1+t} dt \\ &= \frac{\pi^2}{12}(2 - \sqrt{15}) + \log\left(\frac{1+\sqrt{5}}{2}\right) \log(2 + \sqrt{3}) \\ & \quad + \log 2 \log(\sqrt{3} + \sqrt{5}). \end{aligned}$$

3. $l = 3, D = 140, \epsilon = 6 + \sqrt{35}$

$$\begin{aligned} & \int_0^1 \frac{\log(1 + t^{6+\sqrt{35}})}{1+t} dt \\ &= \frac{\pi^2}{12}(3 - \sqrt{35}) + \log\left(\frac{1+\sqrt{5}}{2}\right) \log(8 + 3\sqrt{7}) \\ & \quad + \log 2 \log(\sqrt{5} + \sqrt{7}). \end{aligned}$$

4. $l = 4, D = 252, \epsilon = 8 + \sqrt{63}$

$$\begin{aligned} & \int_0^1 \frac{\log(1 + t^{8+\sqrt{63}})}{1+t} dt \\ &= \frac{\pi^2}{12}(4 - \sqrt{63}) + \log\left(\frac{5+\sqrt{21}}{2}\right) \log(2 + \sqrt{3}) \\ & \quad + \log 2 \log(3 + \sqrt{7}). \end{aligned}$$

5. $l = 6, D = 572, \epsilon = 12 + \sqrt{143}$

$$\begin{aligned} & \int_0^1 \frac{\log(1 + t^{12+\sqrt{143}})}{1+t} dt \\ &= \frac{\pi^2}{12}(6 - \sqrt{143}) + \log\left(\frac{3+\sqrt{13}}{2}\right) \log(10 + 3\sqrt{11}) \\ & \quad + \log 2 \log(\sqrt{11} + \sqrt{13}). \end{aligned}$$

6. $l = 7$, $D = 780$, $\epsilon = 14 + \sqrt{195}$

$$\begin{aligned} & \int_0^1 \frac{\log(1 + t^{14+\sqrt{195}})}{1+t} dt \\ &= \frac{\pi^2}{12}(7 - \sqrt{195}) + \log\left(\frac{1+\sqrt{5}}{2}\right) \log(25 + 4\sqrt{39}) \\ & \quad + \log\left(\frac{3+\sqrt{13}}{2}\right) \log(4 + \sqrt{15}) + \log 2 \log(\sqrt{15} + \sqrt{13}). \end{aligned}$$

Integrals 2, 3 and 5 above are due to Herglotz [13]. It is not known whether there are infinitely many positive discriminants D such that $H(D)$ has one class per genus.

4.6 The case $D \equiv 0 \pmod{16}$

The following is an easy exercise.

Lemma 4.2 *Let D be a positive integer. Then $D \equiv 0 \pmod{16}$ with $D+4$ a square if and only if $D = 16(l^2 + l)$ for some positive integer l . If D has this form, then*

$$\frac{D}{4} \equiv \begin{cases} 0 \pmod{16}, & \text{if } l \equiv 0 \text{ or } 3 \pmod{4}, \\ 8 \pmod{16}, & \text{if } l \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

Lemma 4.3 *Let $D = 16(l^2 + l)$ for some positive integer l with $l \equiv 1$ or $2 \pmod{4}$. Then $H(D)$ and $H(D/4)$ have the same number of classes per genus.*

Proof: Since $D+4$ is a square, we have $\epsilon(D) = \epsilon(D/4)$. Hence (3.8) gives

$$\frac{h(D)}{h(D/4)} = 2.$$

Since $D \equiv 0 \pmod{32}$, we have $t(D) = 1 + \text{number of odd prime divisors of } D$. Since $D/4 \equiv 8 \pmod{16}$, we have $t(D/4) = \text{number of odd prime divisors of } D/4$. Hence $t(D) = 1 + t(D/4)$. Thus

$$\frac{h(D/4)}{2^{t(D/4)}} = \frac{h(D)/2}{2^{t(D/4)}} = \frac{h(D)}{2^{t(D)}}$$

as required. $\square$

Lemma 4.4 *Let $D = 16(l^2 + l)$, where l is a positive integer. Then $[1, 0, -D/4] \neq [4, 4, (16 - D)/16]$ in $H(D)$.*

Proof: Suppose that $[1, 0, -D/4] = [4, 4, (16 - D)/16]$. Then there exist co-prime integers α, γ such that $\alpha^2 - D\gamma^2/4 = 4$. By Lemma 1.16, we have

$$\frac{\alpha + \gamma\sqrt{D/4}}{2} = \pm\epsilon(D/4)^n,$$

for some integer n . But this gives

$$\alpha + \gamma\sqrt{D/4} = \pm 2(2l + 1 + \sqrt{D/4})^n,$$

so that α and γ are both even, a contradiction. $\square$

Lemma 4.5 *Let $D = 16(l^2 + l)$, for some positive integer l with $l \equiv 1$ or $2 \pmod{4}$. Then $[1, 0, -D/16] \neq [2, 0, -D/32]$ in $H(D/4)$ except if $l = 1$.*

Proof: If $l = 1$, we have $[1, 0, -2] = [2, 0, -1]$ in $H(8)$. If $l > 1$ and $[1, 0, -D/16] = [2, 0, -D/32]$ in $H(D/4)$, then there exist positive integers x, y with $x^2 - Dy^2/16 = 2$. Hence,

$$8 = (2x)^2 - \frac{D}{4}y^2 = u^2 - ((2l + 1)^2 - 1)v^2 \quad (4.4)$$

for some positive integers u, v . But by Theorem 4.1, the equation (4.4) has no solution in positive integers since $8 < 2(2l + 1) + 2$. $\square$

Theorem 4.3 *Let $D = 16(l^2 + l)$ for some positive integer l and suppose that $H(D)$ has one class per genus. Let G_1 be the genus containing $[1, 0, -D/4]$ and let G_2 be the genus containing $[4, 4, (16 - D)/16]$. Then*

$$\begin{aligned} & \int_0^1 \frac{3t \log(1 + t^{2\epsilon}) + 2(-1)^l \tan^{-1}(t^\epsilon)}{1 + t^2} dt \\ &= \frac{\sqrt{D}}{2} (\beta(D, G_1) - \beta(D, G_2)) - \frac{\pi^2 \sqrt{D}}{16} + (-1)^l \frac{\pi^2}{16} + \log 2 \log(2^{3/4} \epsilon), \end{aligned}$$

where $\epsilon = \epsilon(D) = 2l + 1 + \sqrt{D/4} = 2l + 1 + \sqrt{4l^2 + 4l}$ and $\beta(D, G)$ is defined in Theorem 3.2.

Proof: We have $y_0(D) = 1$. For the form $(1, 0, -D/4)$, we have $g = 1$. For the form $(4, 4, (16 - D)/16)$, we have $g = 4$, $\alpha = 2l - 1$. Using these facts together with the values of $E(a, b, c, t)$ and $J(a, b, c)$ from Section 4.3 and the relation

$$\int_0^1 \frac{\log(1 + t^\epsilon)}{1 + t} dt = 2 \int_0^1 \frac{t \log(1 + t^{2\epsilon})}{1 + t^2} dt$$

in Theorem 3.3 gives the required result. $\square$

In a similar manner, we obtain

Theorem 4.4 *Let $D = 16(l^2 + l)$, for some positive integer l with $l \equiv 1$ or $2 \pmod{4}$ (so that $D/4 \equiv 8 \pmod{16}$) and suppose that $H(D)$ has one class per genus. Then $H(D/4)$ also has one class per genus by Lemma 4.3. In $H(D/4)$, let $\hat{G}_1$ be the genus containing $[1, 0, -D/16]$ and let $\hat{G}_2$ be the genus containing $[2, 0, -D/32]$. Then*

$$\begin{aligned} & \int_0^1 \frac{t \log(1 + t^{2\epsilon}) + 2(-1)^{l+1} \tan^{-1}(t^\epsilon)}{1 + t^2} dt \\ &= \frac{\sqrt{D}}{4} (\beta(D/4, \hat{G}_1) - \beta(D/4, \hat{G}_2)) - \frac{\pi^2 \sqrt{D}}{48} + (-1)^{l+1} \frac{\pi^2}{16} + \frac{\log 2 \log(\sqrt{2}\epsilon)}{2}, \end{aligned}$$

where $\epsilon = \epsilon(D) = \epsilon(D/4) = 2l + 1 + \sqrt{4l^2 + 4l}$ and $\beta(D, G)$ is defined in Theorem 3.2.

Thus, if $D = 16(l^2 + l)$, for some positive integer l with $l \equiv 1$ or $2 \pmod{4}$ and $H(D)$ has one class per genus, both Theorem 4.3 and Theorem 4.4 are applicable. Thus in this case we can calculate both of the integrals

$$\int_0^1 \frac{\log(1 + t^\epsilon)}{1 + t} dt = 2 \int_0^1 \frac{t \log(1 + t^{2\epsilon})}{1 + t^2} dt$$

and

$$\int_0^1 \frac{\tan^{-1}(t^\epsilon)}{1 + t^2} dt.$$

The following are the first few cases where both Theorem 4.3 and Theorem 4.4 are satisfied so that we can calculate both of the integrals

$$\int_0^1 \frac{\log(1 + t^\epsilon)}{1 + t} dt \quad \text{and} \quad \int_0^1 \frac{\tan^{-1}(t^\epsilon)}{1 + t^2} dt.$$

1. $l = 1, D = 32, \epsilon = 3 + \sqrt{8}$

$$\begin{aligned} \int_0^1 \frac{\log(1 + t^{3+\sqrt{8}})}{1 + t} dt &= \frac{\pi^2}{24}(3 - \sqrt{32}) + \frac{\log 2 \log(2(3 + \sqrt{8})^{3/2})}{2}, \\ \int_0^1 \frac{\tan^{-1}(t^{3+\sqrt{8}})}{1 + t^2} dt &= \frac{\log 2 \log(3 + \sqrt{8})}{16}. \end{aligned}$$

2. $l = 2, D = 96, \epsilon = 5 + \sqrt{24}$

$$\begin{aligned} &\int_0^1 \frac{\log(1 + t^{5+\sqrt{24}})}{1 + t} dt \\ &= \frac{\pi^2}{24}(5 - \sqrt{96}) + \frac{\log(1 + \sqrt{2}) \log(2 + \sqrt{3})}{2} \\ &\quad + \frac{\log 2 \log(2(5 + \sqrt{24})^{3/2})}{2}, \\ &\int_0^1 \frac{\tan^{-1}(t^{5+\sqrt{24}})}{1 + t^2} dt \\ &= \frac{\log(1 + \sqrt{2}) \log(2 + \sqrt{3})}{8} - \frac{\log 2 \log(5 + \sqrt{24})}{16}. \end{aligned}$$

3. $l = 5$, $D = 480$, $\epsilon = 11 + \sqrt{120}$

$$\begin{aligned}
& \int_0^1 \frac{\log(1 + t^{11+\sqrt{120}})}{1+t} dt \\
&= \frac{\pi^2}{24} (11 - \sqrt{480}) + \frac{\log(1 + \sqrt{2}) \log(4 + \sqrt{15})}{2} \\
&\quad + \frac{\log(2 + \sqrt{3}) \log(3 + \sqrt{10})}{2} + \frac{\log\left(\frac{1+\sqrt{5}}{2}\right) \log(5 + \sqrt{24})}{2} \\
&\quad + \frac{\log 2 \log(2(11 + \sqrt{120})^{3/2})}{2}, \\
& \int_0^1 \frac{\tan^{-1}(t^{11+\sqrt{120}})}{1+t^2} dt \\
&= -\frac{\log(1 + \sqrt{2}) \log(4 + \sqrt{15})}{8} - \frac{\log(2 + \sqrt{3}) \log(3 + \sqrt{10})}{8} \\
&\quad + \frac{3}{8} \log\left(\frac{1 + \sqrt{5}}{2}\right) \log(5 + \sqrt{24}) + \frac{\log 2 \log(11 + \sqrt{120})}{16}.
\end{aligned}$$

4. $l = 6$, $D = 672$, $\epsilon = 13 + \sqrt{168}$

$$\begin{aligned}
& \int_0^1 \frac{\log(1 + t^{13+\sqrt{168}})}{1+t} dt \\
&= \frac{\pi^2}{24} (13 - \sqrt{672}) + \frac{\log(1 + \sqrt{2}) \log\left(\frac{5+\sqrt{21}}{2}\right)}{2} \\
&\quad + \frac{\log(2 + \sqrt{3}) \log(15 + \sqrt{224})}{4} + \frac{\log(5 + \sqrt{24}) \log(8 + \sqrt{63})}{4} \\
&\quad + \frac{\log 2 \log(2(13 + \sqrt{168})^{3/2})}{2}, \\
& \int_0^1 \frac{\tan^{-1}(t^{13+\sqrt{168}})}{1+t^2} dt \\
&= -\frac{3}{8} \log(1 + \sqrt{2}) \log\left(\frac{5 + \sqrt{21}}{2}\right) + \frac{\log(2 + \sqrt{3}) \log(15 + \sqrt{224})}{16} \\
&\quad + \frac{\log(5 + \sqrt{24}) \log(8 + \sqrt{63})}{16} - \frac{\log 2 \log(13 + \sqrt{168})}{16}.
\end{aligned}$$

Chapter 5

An expression for $\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s}$

5.1 Introduction

Let d be a non-square (positive or negative) discriminant. In this chapter, we obtain an explicit expression for

$$\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s},$$

where $K \in H(d)$, valid for $s > 1$ (see Theorem 5.14). In Sections 5.2–5.8 we develop the results necessary to prove Theorem 5.14.

5.2 Some preliminary results

The following result is well known (see [9], Lemma 2, p. 134).

Lemma 5.1 *Let n_1 and n_2 be relatively prime positive integers.*

(i) Let B be an integer such that $0 \leq B < 2n_1n_2$, $B^2 \equiv d \pmod{4n_1n_2}$. Then there exist unique integers b_1 and b_2 such that

$$0 \leq b_1 < 2n_1, \quad b_1^2 \equiv d \pmod{4n_1}, \quad b_1 \equiv B \pmod{2n_1}, \quad (5.1)$$

$$0 \leq b_2 < 2n_2, \quad b_2^2 \equiv d \pmod{4n_2}, \quad b_2 \equiv B \pmod{2n_2}. \quad (5.2)$$

(ii) Let b_1 and b_2 be integers such that $0 \leq b_1 < 2n_1$, $b_1^2 \equiv d \pmod{4n_1}$, $0 \leq b_2 < 2n_2$, $b_2^2 \equiv d \pmod{4n_2}$. Then there exists a unique integer B with

$$\begin{aligned} 0 \leq B < 2n_1n_2, \quad B^2 &\equiv d \pmod{4n_1n_2}, \\ B &\equiv b_1 \pmod{2n_1}, \quad B \equiv b_2 \pmod{2n_2}. \end{aligned} \quad (5.3)$$

Proof: (i) Set $b_1 = B - 2n_1[B/2n_1]$, $b_2 = B - 2n_2[B/2n_2]$. Then it is immediate that b_1 and b_2 satisfy the required conditions. If b_1 and b'_1 are integers satisfying (5.1), we have $|b_1 - b'_1| < 2n_1$ and $b_1 - b'_1 \equiv 0 \pmod{2n_1}$ so that $b_1 = b'_1$. Thus b_1 is unique. Similarly b_2 is unique.

(ii) Clearly $b_1 \equiv b_2 \pmod{2}$. Thus there exist integers λ_1, λ_2 such that

$$n_1\lambda_1 - n_2\lambda_2 = \frac{b_2 - b_1}{2}.$$

Set $B_1 = b_1 + 2n_1\lambda_1 = b_2 + 2n_2\lambda_2$. Then

$$B_1 \equiv b_1 \pmod{2n_1}, \quad B_1 \equiv b_2 \pmod{2n_2},$$

and

$$B_1^2 \equiv b_1^2 \equiv d \pmod{4n_1}, \quad B_1^2 \equiv b_2^2 \equiv d \pmod{4n_2}.$$

Since $\text{lcm}(4n_1, 4n_2) = 4n_1n_2$, we have $B_1^2 \equiv d \pmod{4n_1n_2}$. Set $B = B_1 - 2n_1n_2[B_1/2n_1n_2]$. Then B satisfies the required conditions. Uniqueness follows easily as in part (i). $\square$

Lemma 5.2 *Let p be a prime which does not divide d . Let h_1 be an integer such that*

$$0 \leq h_1 < 2p, \quad h_1^2 \equiv d \pmod{4p}.$$

Then, for each positive integer n , there is a unique integer h_n such that

$$0 \leq h_n < 2p^n, \quad h_n \equiv h_1 \pmod{2p}, \quad h_n^2 \equiv d \pmod{4p^n}.$$

Proof: We note that $p \nmid h_1$ since $p \nmid d$. We use induction on n . The result is clearly true for $n = 1$. We assume that the result is true for $n = N$. Thus there is a unique integer h_N with

$$0 \leq h_N < 2p^N, \quad h_N \equiv h_1 \pmod{2p}, \quad h_N^2 \equiv d \pmod{4p^N}.$$

Note that $p \nmid h_N$. Hence there is a unique integer λ_N satisfying

$$0 \leq \lambda_N < p, \quad \lambda_N h_N \equiv -\frac{(h_N^2 - d)}{4p^N} \pmod{p}.$$

We set $h_{N+1} = h_N + 2\lambda_N p^N$. Then we have $0 \leq h_{N+1} \leq 2p^N - 1 + 2(p-1)p^N < 2p^{N+1}$ and $h_{N+1} \equiv h_N \equiv h_1 \pmod{2p}$. Also

$$h_{N+1}^2 = h_N^2 + 4\lambda_N h_N p^N + 4\lambda_N^2 p^{2N} \equiv h_N^2 - (h_N^2 - d) \pmod{4p^{N+1}} \equiv d \pmod{4p^{N+1}}.$$

To show that h_{N+1} is unique, let $h'_{N+1} \neq h_{N+1}$ be another integer such that

$$0 \leq h'_{N+1} < 2p^{N+1}, \quad h'_{N+1} \equiv h_1 \pmod{2p}, \quad h_{N+1}'^2 \equiv d \pmod{4p^{N+1}}.$$

We define non-zero integers u, v by

$$u = \frac{1}{2}(h'_{N+1} - h_{N+1}), \quad v = \frac{1}{2}(h'_{N+1} + h_{N+1}).$$

Then $u \equiv 0 \pmod{p}$, $uv \equiv 0 \pmod{p^{N+1}}$. Let $p^\alpha \parallel u$ and $p^\beta \parallel v$, so that $\alpha \geq 1$, $\alpha + \beta \geq N + 1$. If $\beta \geq 1$ then $p \mid u$ and $p \mid v$ so that $p \mid v - u = h_{N+1}$, a contradiction. Hence $\beta = 0$ and $\alpha \geq N + 1$. Thus

$$h'_{N+1} \equiv h_{N+1} \pmod{2p^{N+1}}, \quad 0 \leq h_{N+1}, h'_{N+1} < 2p^{N+1}.$$

This gives $h'_{N+1} = h_{N+1}$, a contradiction. Hence h_{N+1} is unique and the induction is complete. $\square$

Lemma 5.3 *Let a, b, c, n be integers with $a > 0$, $n > 0$, $[a^n, b, c] \in H(d)$ and $(a, b) = 1$. Then*

$$[a^n, b, c] = [a, b, a^{n-1}c]^n.$$

Proof: Note that $[a^{n-i}, b, a^i c] \in H(d)$ for $0 \leq i \leq n$. We have $[a, b, a^{n-1}c][a^{n-1}, b, a^1 c] = [a^{n-(1-1)}, b, a^{1-1}c]$ for $1 \leq i \leq n$. Thus,

$$\begin{aligned} [a^n, b, c] &= [a, b, a^{n-1}c][a^{n-1}, b, ac] \\ &= [a, b, a^{n-1}c]^2[a^{n-2}, b, a^2c] \\ &= [a, b, a^{n-1}c]^3[a^{n-3}, b, a^3c] \\ &= \dots \\ &= [a, b, a^{n-1}c]^{n-1}[a, b, a^{n-1}c] \\ &= [a, b, a^{n-1}c]^n, \end{aligned}$$

as required. $\square$

5.3 The integer $k(n, d)$

Definition 5.1 *We define $k(n, d)$ to be the number of integers h satisfying $0 \leq h < 2n$, $h^2 \equiv d \pmod{4n}$.*

The value of $k(n, d)$ when $\gcd(n, d) = 1$ is well known (see [17], Theorem 97).

Lemma 5.4 *Let n be a positive integer with $\gcd(n, d) = 1$. Then*

$$k(n, d) = \prod_{p|n} \left(1 + \left(\frac{d}{p} \right) \right).$$

Next we generalize the above result.

Lemma 5.5 *Let n be a positive integer with $\gcd(n, f) = 1$. Then*

$$k(n, d) = \begin{cases} 0, & \text{if there exists a prime } p \text{ with } p^2 \mid n \text{ and } p \mid d, \\ \prod_{p \mid n} \left(1 + \left(\frac{d}{p}\right)\right), & \text{otherwise.} \end{cases}$$

Proof: We first assume that there exists a prime p with $p^2 \mid n$ and $p \mid d$ and that there exists an integer h with $0 \leq h < 2n$, $h^2 \equiv d \pmod{4n}$. Then $p \mid h$ and so $p^2 \mid d$. Since $d = \Delta f^2$ and $p \nmid f$, we have $p^2 \mid \Delta$. If $p > 2$, this is a contradiction since Δ is fundamental. If $p = 2$, we have

$$\frac{\Delta}{4} \equiv \frac{\Delta f^2}{4} \equiv \frac{d}{4} \equiv \left(\frac{h}{2}\right)^2 \equiv 0 \text{ or } 1 \pmod{4},$$

which is again a contradiction as Δ is fundamental. Hence $k(n, d) = 0$.

Next, we assume that there is no prime p with $p^2 \mid n$ and $p \mid d$. Hence, if p is a prime with $p \mid n$ then $p \parallel n$ or $p^2 \mid n$, and $p \nmid d$. Thus, if $p > 2$, the number of solutions of the congruence $h^2 \equiv d \pmod{p^{v_p(n)}}$ is $1 + \left(\frac{d}{p}\right)$. If $p = 2$, the number of solutions of the congruence $h^2 \equiv d \pmod{2^{2+v_2(n)}}$ is

$$\begin{cases} 2, & \text{if } 2 \nmid n, \\ 2 \left(1 + \left(\frac{d}{2}\right)\right), & \text{if } 2 \parallel n, \\ 2 \left(1 + \left(\frac{d}{2}\right)\right), & \text{if } 2^2 \mid n, 2 \nmid d. \end{cases}$$

Hence, the total number of solutions of the congruence $h^2 \equiv d \pmod{4n}$ is $2 \prod_{p \mid n} (1 + \left(\frac{d}{p}\right))$. The result follows as $k(n, d)$ is half this number. $\square$

5.4 The integer $H(n, d)$

Definition 5.2 *We define $H(n, d)$ to be the number of integers h satisfying*

$$0 \leq h < 2n, \quad h^2 \equiv d \pmod{4n}, \quad \gcd\left(n, h, \frac{h^2 - d}{4n}\right) = 1.$$

We need to completely determine $H(p^j, d)$ if $p \mid f$ and $j \geq 1$. If $p \mid f$, $j \geq 1$, we have $H(p^j, d) =$ number of integers h satisfying

$$0 \leq h < 2p^j, \quad h^2 \equiv d \pmod{4p^j}, \quad p \nmid \frac{h^2 - d}{4p^j}.$$

Theorem 5.1 Let $\Delta = p^v s$, $f = p^u t$, $p \nmid st$, $u \geq 1$, and p odd, so that $v = 0$ or 1.

If $v = 0$ then

$$H(p^j, d) = \begin{cases} 0, & \text{if } j = 2l + 1, 0 \leq l < u, \\ p^l - p^{l-1}, & \text{if } j = 2l, 1 \leq l < u, \\ p^u - \left(1 + \left(\frac{s}{p}\right)\right) p^{u-1}, & \text{if } j = 2u, \\ \left(1 + \left(\frac{s}{p}\right)\right) (p^u - p^{u-1}), & \text{if } j > 2u. \end{cases}$$

If $v = 1$ then

$$H(p^j, d) = \begin{cases} 0, & \text{if } j = 2l + 1, 0 \leq l < u, \\ p^u, & \text{if } j = 2u + 1, \\ p^l - p^{l-1}, & \text{if } j = 2l, 1 \leq l \leq u, \\ 0, & \text{if } j > 2u + 1. \end{cases}$$

Proof: Since $p \mid f$, and p is odd, for $j \geq 1$, we have $H(p^j, d) =$ number of integers h satisfying

$$0 \leq h < 2p^j, \quad h^2 \equiv d \pmod{4p^j}, \quad p \nmid \frac{h^2 - d}{p^j}.$$

Also

$$h^2 \equiv d \pmod{4p^j} \iff h \equiv d \pmod{2} \quad \text{and} \quad h^2 \equiv d \pmod{p^j}.$$

Let $1 \leq j \leq 2u + v$. As $d = p^{2u+v} st^2 \equiv 0 \pmod{p^j}$, we have

$$h^2 \equiv d \pmod{4p^j} \iff h \equiv d \pmod{2} \quad \text{and} \quad h^2 \equiv 0 \pmod{p^j}. \quad (5.4)$$

(a) Let $j = 2l + 1$, $0 \leq l < u$. Then, by (5.4), we have

$$\begin{aligned} h^2 \equiv d \pmod{4p^j} &\iff h \equiv d \pmod{2} \quad \text{and} \quad h \equiv 0 \pmod{p^{l+1}} \\ &\iff h = \lambda p^{l+1} \quad \text{and} \quad \lambda \equiv d \pmod{2}. \end{aligned} \quad (5.5)$$

For h satisfying (5.5), we have

$$\frac{h^2 - d}{p^j} = p\lambda^2 - st^2 p^{2(u-l)+v-1} \equiv 0 \pmod{p}.$$

Thus $H(p^j, d) = 0$.

(b) Next let $j = 2u + 1$, $v = 1$. Then, by (5.4), we have

$$\begin{aligned} h^2 \equiv d \pmod{4p^j} &\iff h \equiv d \pmod{2} \quad \text{and} \quad h \equiv 0 \pmod{p^{u+1}} \\ &\iff h = \lambda p^{u+1} \quad \text{and} \quad \lambda \equiv d \pmod{2}. \end{aligned} \quad (5.6)$$

For h satisfying (5.6), we have

$$0 \leq h < 2p^j \iff 0 \leq \lambda < 2p^u$$

and

$$\frac{h^2 - d}{p^j} = \lambda^2 p - st^2 \not\equiv 0 \pmod{p}.$$

Hence $H(p^j, d) = p^u$.

(c) Next let $j = 2l$, $1 \leq l \leq u$. Then, by (5.4), we have

$$\begin{aligned} h^2 \equiv d \pmod{4p^j} &\iff h \equiv d \pmod{2} \quad \text{and} \quad h \equiv 0 \pmod{p^l} \\ &\iff h = \lambda p^l \quad \text{and} \quad \lambda \equiv d \pmod{2}. \end{aligned} \quad (5.7)$$

For h satisfying (5.7),

$$0 \leq h < 2p^j \iff 0 \leq \lambda < 2p^l,$$

and

$$\begin{aligned}\frac{h^2 - d}{p^j} &= \lambda^2 - p^{2(u-l)+v} st^2 \\ &\equiv \begin{cases} \lambda^2 \pmod{p}, & \text{if } 1 \leq l < u \text{ or } l = u, v = 1, \\ \lambda^2 - st^2 \pmod{p}, & \text{if } l = u, v = 0. \end{cases}\end{aligned}$$

Thus, for $1 \leq l < u$ or $l = u, v = 1$, we have

$$\begin{aligned}H(p^j, d) &= |\{\lambda : 0 \leq \lambda < 2p^l, \lambda \equiv d \pmod{2}, \lambda \not\equiv 0 \pmod{p}\}| \\ &= p^l - p^{l-1}.\end{aligned}$$

For $l = u, v = 0$, we have

$$\begin{aligned}H(p^j, d) &= |\{\lambda : 0 \leq \lambda < 2p^u, \lambda \equiv d \pmod{2}, \lambda^2 \not\equiv st^2 \pmod{p}\}| \\ &= \begin{cases} p^u, & \text{if } \left(\frac{s}{p}\right) = -1, \\ p^u - 2p^{u-1}, & \text{if } \left(\frac{s}{p}\right) = 1, \end{cases} \\ &= p^u - \left(1 + \left(\frac{s}{p}\right)\right) p^{u-1}.\end{aligned}$$

(d) Next let $j > 2u + 1, v = 1$. Then the congruence

$$h^2 \equiv d \equiv p^{2u+1} st^2 \pmod{4p^j}$$

has no solutions so that $H(p^j, d) = 0$.

(e) Finally let $j > 2u, v = 0$. Let h satisfy

$$h^2 \equiv d \equiv p^{2u} st^2 \pmod{4p^j}, 0 \leq h < 2p^j, p \nmid \frac{h^2 - d}{p^j}.$$

Then $p^u \mid h$. Let $h = p^u h_1$ so that

$$h_1^2 \equiv st^2 \pmod{4p^{j-2u}}, 0 \leq h_1 < 2p^{j-u}, p \nmid \frac{h_1^2 - st^2}{p^{j-2u}},$$

as $(h^2 - d)/p^j = (h_1^2 - st^2)/p^{j-2u}$. Thus

$$H(p^j, d) = \left| \left\{ h_1 : 0 \leq h_1 < 2p^{j-u}, h_1^2 \equiv st^2 \pmod{4p^{j-2u}}, p \nmid \frac{h_1^2 - st^2}{p^{j-2u}} \right\} \right|.$$

If $\left(\frac{s}{p}\right) = -1$, it is clear that

$$H(p^j, d) = 0 = \left(1 + \left(\frac{s}{p}\right)\right) (p^u - p^{u-1}).$$

Let $\left(\frac{s}{p}\right) = 1$. Then

$$\begin{aligned} h_1^2 \equiv st^2 \pmod{4p^{j-2u}} &\iff h_1^2 \equiv st^2 \equiv d \pmod{4} \quad \text{and} \quad h_1^2 \equiv st^2 \pmod{p^{j-2u}} \\ &\iff h_1 \equiv d \pmod{2} \quad \text{and} \quad h_1^2 \equiv st^2 \pmod{p^{j-2u}}. \end{aligned}$$

The congruence

$$h_1^2 \equiv st^2 \pmod{p^{j-2u}} \tag{5.8}$$

has two solutions satisfying $0 < h_1 < p^{j-2u}$ (see [17], Theorem 87). If x is one such solution then the other is $p^{j-2u} - x$. As these are of opposite parity, we may assume that $x \equiv d \pmod{2}$. Thus the solutions to (5.8) satisfying

$$0 \leq h_1 < 2p^{j-2u}, \quad h_1 \equiv d \pmod{2},$$

are

$$x + 2mp^{j-2u} \quad \text{and} \quad p^{j-2u} - x + (2m+1)p^{j-2u},$$

for $0 \leq m \leq p^u - 1$. For $h_1 = x + 2mp^{j-2u}$, we have

$$\frac{h_1^2 - st^2}{p^{j-2u}} \equiv 4mx + \frac{x^2 - st^2}{p^{j-2u}} \pmod{p},$$

so that

$$\frac{h_1^2 - st^2}{p^{j-2u}} \equiv 0 \pmod{p} \iff m \equiv -(4x)^{-1} \frac{x^2 - st^2}{p^{j-2u}} \pmod{p}.$$

Similarly we find for $h_1 = p^{j-2u} - x + (2m+1)p^{j-2u}$ that

$$\frac{h_1^2 - st^2}{p^{j-2u}} \equiv 0 \pmod{p} \iff m \equiv (4x)^{-1} \frac{x^2 - st^2}{p^{j-2u}} - 1 \pmod{p}.$$

Thus

$$H(p^j, d) = 2(p^u - p^{u-1}) = \left(1 + \left(\frac{s}{p}\right)\right) (p^u - p^{u-1}),$$

as required. $\square$

Lemma 5.6 *Let $n \equiv 1 \pmod{8}$ and let l and m be integers with $l \geq 3$ and $m > 0$. Then*

$$\left| \left\{ x : x^2 \equiv n \pmod{2^l}, 0 \leq x < 2^l m, \frac{x^2 - n}{2^l} \text{ odd} \right\} \right| = 2m.$$

Proof: The congruence $x^2 \equiv n \pmod{2^l}$ has four solutions satisfying $0 < x < 2^l$ (see [17], Theorem 87). Let x_0 be the least of these. Then $0 < x_0 < 2^{l-1}$ (otherwise $2^l - x_0$ would be a smaller solution). The other three solutions are given by $x_1 = 2^l - x_0$, $x_2 = x_0 + 2^{l-1}$, $x_3 = 2^{l-1} - x_0$. Then

$$\frac{x_1^2 - n}{2^l} \equiv \frac{x_0^2 - n}{2^l} \pmod{2},$$

and

$$\frac{x_2^2 - n}{2^l} \equiv \frac{x_3^2 - n}{2^l} \equiv 1 + \frac{x_0^2 - n}{2^l} \pmod{2}.$$

The solutions to $x^2 \equiv n \pmod{2^l}$ satisfying $0 \leq x < 2^l m$ are $x_i + 2^l r$ for $0 \leq r < m$, $0 \leq i \leq 3$. Also

$$\frac{(x_i + 2^l r)^2 - n}{2^l} \equiv \frac{x_i^2 - n}{2^l} \pmod{2}.$$

Thus the required number is $2m$. $\square$

Theorem 5.2 Let $\Delta = 2^v s$, $f = 2^u t$, $2 \nmid st$, $u \geq 1$, so that $v = 0, 2$ or 3 and

$$s \equiv \begin{cases} 1 \pmod{4}, & \text{if } v = 0, \\ 3 \pmod{4}, & \text{if } v = 2. \end{cases}$$

If $v = 2v_1$, where $v_1 = 0$ or 1 , then

$$H(2^j, d) = \begin{cases} 0, & \text{if } j = 2l + 1, 0 \leq l \leq u + v_1 - 2, \\ 2^{l-1}, & \text{if } j = 2l, 1 \leq l \leq u + v_1 - 1, \\ 0, & \text{if } j = 2u + 2v_1 - 1, s \equiv 1 \pmod{4}, \\ 2^{u+v_1-1}, & \text{if } j = 2u + 2v_1 - 1, s \equiv 3 \pmod{4}, \\ 0, & \text{if } j = 2u + 2v_1, s \not\equiv 5 \pmod{8}, \\ 2^{u+v_1}, & \text{if } j = 2u + 2v_1, s \equiv 5 \pmod{8}, \\ 0, & \text{if } j > 2u + 2v_1, s \not\equiv 1 \pmod{8}, \\ 2^{u+v_1}, & \text{if } j > 2u + 2v_1, s \equiv 1 \pmod{8}. \end{cases}$$

If $v = 3$ then

$$H(2^j, d) = \begin{cases} 0, & \text{if } j = 2l + 1, 0 \leq l < u, \\ 2^u, & \text{if } j = 2u + 1, \\ 2^{l-1}, & \text{if } j = 2l, 1 \leq l \leq u, \\ 0, & \text{if } j > 2u + 1. \end{cases}$$

Proof: For $j \geq 1$, we have

$$H(2^j, d) = \left| \left\{ h : 0 \leq h < 2^{j+1}, h^2 \equiv d \pmod{2^{j+2}}, \frac{h^2 - d}{2^{j+2}} \text{ is odd} \right\} \right|.$$

If $j + 2 \leq 2u + v$, we have

$$h^2 \equiv d \pmod{2^{j+2}} \iff h^2 \equiv 0 \pmod{2^{j+2}}.$$

(a) Let $j = 2l + 1$, $0 \leq l \leq u + (v - 3)/2$ (so that $0 \leq l \leq u$ if $v = 3$ and $0 \leq l \leq u + v_1 - 2$ if $v = 2v_1$ where $v_1 = 0$ or 1). Then

$$h^2 \equiv d \pmod{2^{j+2}} \iff h^2 \equiv 0 \pmod{2^{2l+3}} \iff h = \lambda 2^{l+2} \quad \text{for some integer } \lambda.$$

For any such h , we have

$$0 \leq h < 2^{j+1} = 2^{2l+2} \iff 0 \leq \lambda < 2^l,$$

and

$$\begin{aligned} \frac{h^2 - d}{2^{j+2}} &= 2\lambda^2 - st^2 2^{2u-2l+v-3} \\ &\equiv \begin{cases} 0 \pmod{2}, & \text{if } v = 3, l < u \text{ or } v = 2v_1, v_1 = 0 \text{ or } 1, \\ 1 \pmod{2}, & \text{if } v = 3, l = u. \end{cases} \end{aligned}$$

Thus

$$H(2^j, d) = \begin{cases} 0, & \text{if } v = 3, j = 2l + 1, 0 \leq l < u, \\ 0, & \text{if } v = 2v_1, v_1 = 0 \text{ or } 1, j = 2l + 1, 0 \leq l \leq u + v_1 - 2, \\ 2^u, & \text{if } v = 3, j = 2u + 1. \end{cases}$$

(b) Next let $j = 2l$, $1 \leq l \leq u + v/2 - 1$ (so that $1 \leq l \leq u$ if $v = 3$ and $1 \leq l \leq u + v_1 - 1$ if $v = 2v_1$, $v_1 = 0$ or 1). Then

$$h^2 \equiv d \pmod{2^{j+2}} \iff h^2 \equiv 0 \pmod{2^{2l+2}} \iff h = \lambda 2^{l+1} \text{ for some integer } \lambda.$$

For such h , we have

$$0 \leq h < 2^{j+1} = 2^{2l+1} \iff 0 \leq \lambda < 2^l,$$

and

$$\begin{aligned} \frac{h^2 - d}{2^{j+2}} &= \lambda^2 - 2^{2u-2l+v-2} st^2 \\ &\equiv \begin{cases} \lambda \pmod{2}, & \text{if } v = 3 \text{ or } v = 2v_1, v_1 = 0 \text{ or } 1, l < u + v_1 - 1, \\ \lambda + 1 \pmod{2}, & \text{if } v = 2v_1, v_1 = 0 \text{ or } 1, l = u + v_1 - 1. \end{cases} \end{aligned}$$

Thus, in all cases under consideration, $H(2^j, d) = 2^{l-1}$.

(c) Next let $v = 3$, $j > 2u + 1$. Then the congruence $h^2 \equiv d \equiv 2^{2u+3} st^2 \pmod{2^{j+2}}$ has no solutions so that $H(2^j, d) = 0$.

If $v = 2v_1$, $v_1 = 0$ or 1 and $j + 2 > 2u + v = 2u + 2v_1$, it is easily shown that

$$H(2^j, d) = \left| \left\{ h_1 : 0 \leq h_1 < 2^{j+1-u-v_1}, h_1^2 \equiv st^2 \pmod{2^{j+2-2u-2v_1}}, \frac{h_1^2 - st^2}{2^{j+2-2u-2v_1}} \text{ is odd} \right\} \right|.$$

(d) Let $j = 2u + 2v_1 - 1$, $v = 2v_1$, $v_1 = 0$ or 1 . Then

$$H(2^j, d) = \left| \left\{ h_1 : 0 \leq h_1 < 2^{u+v_1}, h_1^2 \equiv st^2 \pmod{2}, \frac{h_1^2 - st^2}{2} \text{ is odd} \right\} \right|.$$

We have

$$h_1^2 \equiv st^2 \pmod{2} \iff h_1 \text{ is odd.}$$

If h_1 is odd, $h_1^2 - st^2 \equiv 1 - s \pmod{4}$. Thus, for the cases under consideration, we have

$$H(2^j, d) = \begin{cases} 0, & \text{if } s \equiv 1 \pmod{4}, \\ 2^{u+v_1-1}, & \text{if } s \equiv 3 \pmod{4}. \end{cases}$$

(e) Next let $j = 2u + 2v_1$, $v = 2v_1$, $v_1 = 0$ or 1 . Then

$$H(2^j, d) = \left| \left\{ h_1 : 0 \leq h_1 < 2^{u+v_1+1}, h_1^2 \equiv st^2 \pmod{4}, \frac{h_1^2 - st^2}{4} \text{ is odd} \right\} \right|.$$

If $s \equiv 3 \pmod{4}$, the congruence $h_1^2 \equiv st^2 \equiv 3 \pmod{4}$ has no solutions so that $H(2^j, d) = 0$. If $s \equiv 1 \pmod{4}$ then $h_1^2 \equiv st^2 \pmod{4} \iff h_1$ is odd. If h_1 is odd, $h_1^2 - st^2 \equiv 1 - s \pmod{8}$. Thus

$$H(2^j, d) = \begin{cases} 0, & \text{if } s \equiv 1 \pmod{8}, \\ 2^{u+v_1}, & \text{if } s \equiv 5 \pmod{8}. \end{cases}$$

(f) Finally let $j > 2u + 2v_1$, $v = 2v_1$, $v_1 = 0$ or 1 . Then the congruence

$$h_1^2 \equiv st^2 \pmod{2^{j+2-2u-2v_1}}$$

has no solutions if $s \not\equiv 1 \pmod{8}$ so that $H(2^j, d) = 0$. If $s \equiv 1 \pmod{8}$ then $H(2^j, d) = 2(2^{u+v_1-1}) = 2^{u+v_1}$ by Lemma 5.6. $\square$

The following is an immediate consequence of Theorems 5.1 and 5.2.

Corollary 5.1 *If $p \mid f$ then $H(p^j, d) \leq 2f$ for all $j \geq 1$.*

5.5 The integer $H_K(n)$

Definition 5.3 *For a positive integer n and $K \in H(d)$, we define $H_K(n)$ to be the number of integers h satisfying*

$$0 \leq h < 2n, \quad h^2 \equiv d \pmod{4n}, \quad \left[n, h, \frac{h^2 - d}{4n} \right] = K.$$

The following is well known (see [22], p. 174).

Lemma 5.7 *Let $K \in H(d)$ and let n be a positive integer. Then*

$$R_K(n, d) = w(d) \sum_{m^2 \mid n} H_K\left(\frac{n}{m^2}\right).$$

The following is immediate from Definitions 5.1, 5.2 and 5.3.

Lemma 5.8 *Let n be a positive integer and let $K \in H(d)$. Then*

$$0 \leq H_K(n) \leq k(n, d),$$

$$\sum_{K \in H(d)} H_K(n) = H(n, d).$$

By Lemmas 5.5 and 5.8, we have

Lemma 5.9 *Let n be a positive integer and let $K \in H(d)$. If $\gcd(n, f) = 1$ then*

$$H_K(n) = 0, \quad \text{if there is a prime } p \text{ with } p^2 \mid n, p \mid d,$$

$$\text{and } 0 \leq H_K(n) \leq \prod_{p \mid n} \left(1 + \left(\frac{d}{p}\right)\right), \quad \text{otherwise.}$$

We recall that the principal class I of $H(d)$ is given by

$$I = I(d) = \begin{cases} [1, 0, -d/4], & \text{if } d \equiv 0 \pmod{4}, \\ [1, 1, (1-d)/4], & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$

Theorem 5.3 *Let $K \in H(d)$. Then*

$$H_K(1) = \begin{cases} 1, & \text{if } K = I, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: We have

$$H_K(1) = \sum_{\substack{h=0 \\ h^2 \equiv d \pmod{4} \\ [1, h, (h^2-d)/4] = K}}^1 1.$$

The only possibility for h in the sum is $h = 0$ if $d \equiv 0 \pmod{4}$ and $h = 1$ if $d \equiv 1 \pmod{4}$. Hence,

$$H_K(1) = \begin{cases} 0, & \text{if } K \neq \begin{cases} [1, 0, -d/4] \text{ if } d \equiv 0 \pmod{4}, \\ [1, 1, (1-d)/4] \text{ if } d \equiv 1 \pmod{4}, \end{cases} \\ 1, & \text{if } K = \begin{cases} [1, 0, -d/4] \text{ if } d \equiv 0 \pmod{4}, \\ [1, 1, (1-d)/4] \text{ if } d \equiv 1 \pmod{4}, \end{cases} \end{cases}$$

and the result follows. $\square$

If p is a prime with $(d/p) = 1$, by Lemma 5.4, $k(p, d) = 2$. Let h_1 and h_2 be the two solutions of $h^2 \equiv d \pmod{4p}$, $0 \leq h < 2p$, chosen so that $h_1 < h_2$. Since $p \nmid d$, we have $p \nmid h_1$ and $p \nmid h_2$ and $h_2 = 2p - h_1$. We have

$$\left[p, h_i, \frac{h_i^2 - d}{4p} \right] \in H(d), \quad (i = 1, 2).$$

Also

$$\begin{aligned} \left[p, h_2, \frac{h_2^2 - d}{4p} \right] &= \left[p, 2p - h_1, \frac{(2p - h_1)^2 - d}{4p} \right] \\ &= \left[p, -h_1, \frac{h_1^2 - d}{4p} \right] \\ &= \left[p, h_1, \frac{h_1^2 - d}{4p} \right]^{-1}. \end{aligned}$$

Motivated by these observations, we make the following definition.

Definition 5.4 *If p is a prime with $(d/p) = 1$, we let h_1 and h_2 be the solutions of $h^2 \equiv d \pmod{4p}$, $0 \leq h < 2p$ with $h_1 < h_2$. We define $K_p \in H(d)$ by*

$$K_p = \left[p, h_1, \frac{h_1^2 - d}{4p} \right],$$

so that

$$K_p^{-1} = \left[p, h_2, \frac{h_2^2 - d}{4p} \right].$$

If p is a prime with $(d/p) = 0$, $p \nmid f$, we define $K_p \in H(d)$ by

$$K_p = \begin{cases} [p, 0, -d/4p], & \text{if } p > 2, d \equiv 0 \pmod{4}, \\ [p, p, (p^2 - d)/4p], & \text{if } p > 2, d \equiv 1 \pmod{4}, \\ [2, 0, -d/8], & \text{if } p = 2, d \equiv 8 \pmod{16}, \\ [2, 2, (4 - d)/8], & \text{if } p = 2, d \equiv 12 \pmod{16}, \end{cases}$$

Clearly $K_p = K_p^{-1}$.

Theorem 5.4 *Let p be a prime and let $K \in H(d)$.*

- (a) *If $(d/p) = -1$ then $H_K(p) = 0$.*
- (b) *If $(d/p) = 1$ and $K_p \neq K_p^{-1}$ then p is represented by exactly two classes of $H(d)$, namely K_p and K_p^{-1} . If $(d/p) = 1$ and $K_p = K_p^{-1}$ then p is represented by exactly one class of $H(d)$ namely K_p . Moreover,*

$$H_K(p) = \begin{cases} 0, & \text{if } K \neq K_p, K_p^{-1}, \\ 1, & \text{if } K = K_p \neq K_p^{-1} \text{ or } K = K_p^{-1} \neq K_p, \\ 2, & \text{if } K = K_p = K_p^{-1}. \end{cases}$$

(c) If $(d/p) = 0$ and $p \mid f$ then $H_K(p) = 0$.

(d) If $(d/p) = 0$ and $p \nmid f$ then p is represented by exactly one class in $H(d)$, namely K_p , and

$$H_K(p) = \begin{cases} 0, & \text{if } K \neq K_p, \\ 1, & \text{if } K = K_p. \end{cases}$$

Proof: (a) This follows immediately from Lemma 5.9.

(b) Let L be a class of $H(d)$ which represents p . Then there exist integers r and s such that $L = [p, r, s]$. Note that $r^2 - 4ps = d$ and $\gcd(p, r, s) = 1$. Let $r = 2pt + h$ with $0 \leq h < 2p$. We have $h^2 \equiv r^2 \equiv d \pmod{4p}$. Hence $h = h_1$ or h_2 . Thus

$$L = [p, 2pt + h, s] = \left[p, h, \frac{h^2 - d}{4p} \right] = K_p \quad \text{or} \quad K_p^{-1}.$$

Also

$$H_K(p) = \sum_{\substack{0 \leq h < 2p \\ h^2 \equiv d \pmod{4p} \\ [p, h, (h^2 - d)/4p] = K}} 1 = \sum_{\substack{h = h_1 \text{ or } h_2 \\ [p, h, (h^2 - d)/4p] = K}} 1.$$

The required result follows from this and the definition of K_p .

(c) If $p \mid f$, we have $H(p, d) = 0$ by Theorems 5.1 and 5.2. Thus $H_K(p) = 0$ by Lemma 5.8.

(d) Let L be a class of $H(d)$ that represents p . Then there exist integers r, s with $\gcd(p, r, s) = 1$, $r^2 - 4ps = d$ and $L = [p, r, s]$. Let $r = 2pt + h$, $0 \leq h < 2p$. Then $h^2 \equiv r^2 \equiv d \pmod{4p}$. By Lemma 5.5, $k(p, d) = 1$. The unique solution h of $h^2 \equiv d \pmod{4p}$, $0 \leq h < 2p$ is

$$h = \begin{cases} 0, & \text{if } p > 2, d \equiv 0 \pmod{4}, \\ p, & \text{if } p > 2, d \equiv 1 \pmod{4}, \\ 0, & \text{if } p = 2, d \equiv 8 \pmod{16}, \\ 2, & \text{if } p = 2, d \equiv 12 \pmod{16}. \end{cases}$$

Hence $L = [p, 2pt + h, s] = [p, h, (h^2 - d)/4p] = K_p$. The result for $H_K(p)$ follows at once. $\square$

Theorem 5.5 *Let $K \in H(d)$ and let k be an integer with $k \geq 2$. Then*

$$H_K(p^k) = \begin{cases} \sum_{L^k=K} H_L(p), & \text{if } p \nmid d, \\ 0, & \text{if } p \mid d, p \nmid f, \end{cases}$$

where the sum is taken over all $L \in H(d)$ with $L^k = K$.

Proof: First we consider the case $p \nmid d$. We have

$$H_K(p^k) = \sum_{\substack{0 \leq h < 2p^k \\ h^2 \equiv d \pmod{4p^k} \\ [p^k, h, (h^2 - d)/4p^k] = K}} 1.$$

For each h occurring in the sum, $p \nmid h$. Thus by Lemma 5.3, we have

$$\left[p^k, h, \frac{h^2 - d}{4p^k} \right] = \left[p, h, \frac{h^2 - d}{4p} \right]^k.$$

Hence

$$\begin{aligned} H_K(p^k) &= \sum_{\substack{0 \leq h < 2p^k \\ h^2 \equiv d \pmod{4p^k} \\ [p, h, (h^2 - d)/4p]^k = K}} 1 \\ &= \sum_{0 \leq h_1 < 2p} \sum_{\substack{0 \leq h < 2p^k \\ h \equiv h_1 \pmod{2p} \\ h^2 \equiv d \pmod{4p^k} \\ [p, h, (h^2 - d)/4p]^k = K}} 1 \\ &= \sum_{\substack{0 \leq h_1 < 2p \\ h_1^2 \equiv d \pmod{4p}}} \sum_{\substack{0 \leq h < 2p^k \\ h \equiv h_1 \pmod{2p} \\ h^2 \equiv d \pmod{4p^k} \\ [p, h, (h^2 - d)/4p]^k = K}} 1. \end{aligned}$$

For $h \equiv h_1 \pmod{2p}$, we have $[p, h, (h^2 - d)/4p] = [p, h_1, (h_1^2 - d)/4p]$. Thus

$$H_K(p^k) = \sum_{\substack{0 \leq h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p, h_1, (h_1^2 - d)/4p]^k = K}} \sum_{\substack{0 \leq h < 2p^k \\ h \equiv h_1 \pmod{2p} \\ h^2 \equiv d \pmod{4p^k}}} 1.$$

By Lemma 5.2, corresponding to each h_1 , there is a unique integer h satisfying the conditions of the inner sum. Thus

$$\begin{aligned}
 H_K(p^k) &= \sum_{\substack{0 \leq h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p, h_1, (h_1^2 - d)/4p]^k = K}} 1 = \sum_{L \in H(d)} \sum_{\substack{0 \leq h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p, h_1, (h_1^2 - d)/4p]^k = K \\ [p, h_1, (h_1^2 - d)/4p] = L}} 1 \\
 &= \sum_{\substack{L \in H(d) \\ L^k = K}} \sum_{\substack{0 \leq h_1 < 2p \\ h_1^2 \equiv d \pmod{4p} \\ [p, h_1, (h_1^2 - d)/4p] = L}} 1 = \sum_{\substack{L \in H(d) \\ L^k = K}} H_L(p).
 \end{aligned}$$

In the case $p \mid d$, $p \nmid f$, the result follows from Lemma 5.9. $\square$

We have not been able to find a similar formula for $H_K(p^k)$ when $p \mid f$.

Theorem 5.6 *Let n_1 and n_2 be relatively prime positive integers. Let $K \in H(d)$. Then*

$$H_K(n_1 n_2) = \sum_{K_1 K_2 = K} H_{K_1}(n_1) H_{K_2}(n_2),$$

where K_1, K_2 run through all classes in $H(d)$ whose product is K .

Proof: We have

$$H_K(n_1 n_2) = \sum_{\substack{0 \leq B < 2n_1 n_2 \\ B^2 \equiv d \pmod{4n_1 n_2} \\ [n_1 n_2, B, (B^2 - d)/4n_1 n_2] = K}} 1.$$

If B ($0 \leq B < 2n_1 n_2$, $B^2 \equiv d \pmod{4n_1 n_2}$) is such that $[n_1 n_2, B, (B^2 - d)/4n_1 n_2] = K$ then as $\gcd(n_1, n_2) = 1$, we have

$$\left[n_1, B, \frac{B^2 - d}{4n_1} \right] \in H(d) \text{ and } \left[n_2, B, \frac{B^2 - d}{4n_2} \right] \in H(d).$$

Hence

$$\left[n_1 n_2, B, \frac{B^2 - d}{4n_1 n_2} \right] = \left[n_1, B, \frac{B^2 - d}{4n_1} \right] \left[n_2, B, \frac{B^2 - d}{4n_2} \right].$$

Conversely, if B ($0 \leq B < 2n_1n_2$, $B^2 \equiv d \pmod{4n_1n_2}$) is such that

$$\left[n_1, B, \frac{B^2 - d}{4n_1} \right], \left[n_2, B, \frac{B^2 - d}{4n_2} \right] \in H(d)$$

and

$$\left[n_1, B, \frac{B^2 - d}{4n_1} \right] \left[n_2, B, \frac{B^2 - d}{4n_2} \right] = K,$$

then

$$\left[n_1n_2, B, \frac{B^2 - d}{4n_1n_2} \right] = K.$$

Thus

$$\begin{aligned} H_K(n_1n_2) &= \sum_{\substack{0 \leq B < 2n_1n_2 \\ B^2 \equiv d \pmod{4n_1n_2} \\ [n_1, B, (B^2-d)/4n_1] \in H(d) \\ [n_2, B, (B^2-d)/4n_2] \in H(d) \\ [n_1, B, (B^2-d)/4n_1][n_2, B, (B^2-d)/4n_2] = K}} 1 \\ &= \sum_{K_1 K_2 = K} \sum_{\substack{0 \leq B < 2n_1n_2 \\ B^2 \equiv d \pmod{4n_1n_2} \\ [n_1, B, (B^2-d)/4n_1] = K_1 \\ [n_2, B, (B^2-d)/4n_2] = K_2}} 1 \\ &= \sum_{K_1 K_2 = K} \sum_{\substack{0 \leq b_1 < 2n_1 \\ 0 \leq b_2 < 2n_2 \\ B^2 \equiv d \pmod{4n_1n_2} \\ B \equiv b_1 \pmod{2n_1} \\ B \equiv b_2 \pmod{2n_2} \\ [n_1, B, (B^2-d)/4n_1] = K_1 \\ [n_2, B, (B^2-d)/4n_2] = K_2}} \sum_{\substack{0 \leq B < 2n_1n_2 \\ B^2 \equiv d \pmod{4n_1n_2} \\ B \equiv b_1 \pmod{2n_1} \\ B \equiv b_2 \pmod{2n_2} \\ [n_1, B, (B^2-d)/4n_1] = K_1 \\ [n_2, B, (B^2-d)/4n_2] = K_2}} 1 \\ &= \sum_{K_1 K_2 = K} \sum_{\substack{0 \leq b_1 < 2n_1 \\ b_1^2 \equiv d \pmod{4n_1} \\ [n_1, b_1, (b_1^2-d)/4n_1] = K_1}} \sum_{\substack{0 \leq b_2 < 2n_2 \\ b_2^2 \equiv d \pmod{4n_2} \\ [n_2, b_2, (b_2^2-d)/4n_2] = K_2}} \sum_{\substack{0 \leq B < 2n_1n_2 \\ B^2 \equiv d \pmod{4n_1n_2} \\ B \equiv b_1 \pmod{2n_1} \\ B \equiv b_2 \pmod{2n_2} \\ [n_1, B, (B^2-d)/4n_1] = K_1 \\ [n_2, B, (B^2-d)/4n_2] = K_2}} 1 \\ &= \sum_{K_1 K_2 = K} \sum_{\substack{0 \leq b_1 < 2n_1 \\ b_1^2 \equiv d \pmod{4n_1} \\ [n_1, b_1, (b_1^2-d)/4n_1] = K_1}} \sum_{\substack{0 \leq b_2 < 2n_2 \\ b_2^2 \equiv d \pmod{4n_2} \\ [n_2, b_2, (b_2^2-d)/4n_2] = K_2}} \sum_{\substack{0 \leq B < 2n_1n_2 \\ B^2 \equiv d \pmod{4n_1n_2} \\ B \equiv b_1 \pmod{2n_1} \\ B \equiv b_2 \pmod{2n_2}}} 1 \quad (\text{by Lemma 5.1}) \\ &= \sum_{K_1 K_2 = K} \sum_{\substack{0 \leq b_1 < 2n_1 \\ b_1^2 \equiv d \pmod{4n_1} \\ [n_1, b_1, (b_1^2-d)/4n_1] = K_1}} \sum_{\substack{0 \leq b_2 < 2n_2 \\ b_2^2 \equiv d \pmod{4n_2} \\ [n_2, b_2, (b_2^2-d)/4n_2] = K_2}} 1 \end{aligned}$$

$$= \sum_{K_1 K_2 = K} H_{K_1}(n_1) H_{K_2}(n_2),$$

as required. $\square$

5.6 The quantities $[K, L]$ and $\chi(K, L)$

As $H(d)$ is a finite abelian group, there exist positive integers $h_1, h_2, \dots, h_\nu$ and $A_1, A_2, \dots, A_\nu \in H(d)$ such that

$$h_1 h_2 \dots h_\nu = h(d), \quad 1 < h_1 \mid h_2 \mid \dots \mid h_\nu, \quad \text{ord}(A_i) = h_i (i = 1, \dots, \nu),$$

and for each $K \in H(d)$, there exist unique integers $k_1, \dots, k_\nu$ with

$$K = A_1^{k_1} \dots A_\nu^{k_\nu} \quad (0 \leq k_j < h_j, \quad 1 \leq j \leq \nu).$$

We fix once and for all the generators $A_1, \dots, A_\nu$. We define $\text{ind}_{A_j}(K) = k_j$ ($1 \leq j \leq \nu$). For $K, L \in H(d)$, we set

$$[K, L] = \sum_{j=1}^{\nu} \frac{\text{ind}_{A_j}(K) \text{ind}_{A_j}(L)}{h_j}.$$

The following is immediate.

Lemma 5.10 *Let $K, L, M \in H(d)$. Then*

$$[K, L] = [L, K], \quad [K, I] = 0, \quad [KL, M] \equiv [K, M] + [L, M] \pmod{1},$$

and

$$[K^r, L^s] \equiv rs[K, L] \pmod{1} \quad \text{for integers } r, s.$$

Definition 5.5 *For $K, L \in H(d)$, we define $\chi(K, L) = e^{2\pi i [K, L]}$.*

The next lemma follows immediately from Lemma 5.10 and the above definition.

Lemma 5.11 *Let $K, L, M \in H(d)$. Let r and s be integers. Then*

$$\chi(K, L) = \chi(L, K), \quad \chi(K, I) = 1, \quad \chi(KL, M) = \chi(K, M)\chi(L, M)$$

and

$$\chi(K^r, L^s) = \chi(K, L)^{rs}.$$

Moreover

$$\sum_{U \in H(d)} \chi(K, U) = \begin{cases} h(d), & \text{if } K = I, \\ 0, & \text{if } K \neq I, \end{cases}$$

and

$$\sum_{U \in H(d)} \chi(K, U)\chi(L, U)^{-1} = \begin{cases} h(d), & \text{if } K = L, \\ 0, & \text{if } K \neq L. \end{cases}$$

5.7 The quantity $Y_K(n)$

Definition 5.6 *Let $K \in H(d)$ and n a positive integer. We set*

$$Y_K(n) = \sum_{L \in H(d)} \chi(K, L)H_L(n).$$

Lemma 5.12 *Let $K \in H(d)$ and n a positive integer. Then*

$$|Y_K(n)| \leq H(n, d).$$

Proof: We have

$$|Y_K(n)| \leq \sum_{L \in H(d)} H_L(n) = H(n, d),$$

by Lemma 5.8. □

Theorem 5.7 $Y_K(1) = 1$ for $K \in H(d)$.

Proof: We have

$$Y_K(1) = \sum_{L \in H(d)} \chi(K, L) H_L(1) = \chi(K, I) = 1$$

by Theorem 5.3 and Lemma 5.11. $\square$

Theorem 5.8 Let $K \in H(d)$ and let n_1 and n_2 be relatively prime positive integers. Then

$$Y_K(n_1 n_2) = Y_K(n_1) Y_K(n_2).$$

Proof: We have

$$\begin{aligned} Y_K(n_1 n_2) &= \sum_{L \in H(d)} \chi(K, L) H_L(n_1 n_2) \\ &= \sum_{L \in H(d)} \chi(K, L) \sum_{L_1 L_2 = L} H_{L_1}(n_1) H_{L_2}(n_2) \quad (\text{by Theorem 5.6}) \\ &= \sum_{L \in H(d)} \sum_{L_1 L_2 = L} \chi(K, L_1 L_2) H_{L_1}(n_1) H_{L_2}(n_2) \\ &= \sum_{L_1 \in H(d)} \sum_{L_2 \in H(d)} \chi(K, L_1) \chi(K, L_2) H_{L_1}(n_1) H_{L_2}(n_2) \quad (\text{by Lemma 5.11}) \\ &= \sum_{L_1 \in H(d)} \chi(K, L_1) H_{L_1}(n_1) \sum_{L_2 \in H(d)} \chi(K, L_2) H_{L_2}(n_2) \\ &= Y_K(n_1) Y_K(n_2), \end{aligned}$$

as required. $\square$

Theorem 5.9 Let $K \in H(d)$ and let p be a prime.

(a) If $(d/p) = -1$, $Y_K(p^\alpha) = 0$ for $\alpha \geq 1$.

(b) If $(d/p) = 1$, $Y_K(p^\alpha) = \chi(K, K_p)^\alpha + \chi(K, K_p)^{-\alpha}$ for $\alpha \geq 1$.

(c) If $p \mid d$, $p \nmid f$,

$$Y_K(p^\alpha) = \begin{cases} \chi(K, K_p), & \text{if } \alpha = 1, \\ 0, & \text{if } \alpha \geq 2. \end{cases}$$

Proof: If $p \nmid d$, we have by Theorem 5.5,

$$Y_K(p^\alpha) = \sum_{L \in H(d)} \chi(K, L) H_L(p^\alpha) = \sum_{L \in H(d)} \chi(K, L) \sum_{M^\alpha = L} H_M(p). \quad (5.9)$$

(a) This follows from (5.9) and Theorem 5.4(a).

(b) Let $(d/p) = 1$. If $K_p^\alpha \neq K_p^{-\alpha}$, we have, by (5.9) and Theorem 5.4(b),

$$\begin{aligned} Y_K(p^\alpha) &= \sum_{L \neq K_p^\alpha, K_p^{-\alpha}} \chi(K, L) \sum_{M^\alpha = L} H_M(p) + \chi(K, K_p^\alpha) \sum_{M^\alpha = K_p^\alpha} H_M(p) \\ &\quad + \chi(K, K_p^{-\alpha}) \sum_{M^\alpha = K_p^{-\alpha}} H_M(p) \\ &= \chi(K, K_p^\alpha) H_{K_p}(p) + \chi(K, K_p^{-\alpha}) H_{K_p^{-1}}(p) \\ &= \chi(K, K_p^\alpha) + \chi(K, K_p^{-\alpha}). \end{aligned}$$

Similarly, if $K_p^\alpha = K_p^{-\alpha}$, we have

$$\begin{aligned} Y_K(p^\alpha) &= \chi(K, K_p^\alpha) \sum_{M^\alpha = K_p^\alpha} H_M(p) \\ &= \begin{cases} \chi(K, K_p^\alpha) (H_{K_p}(p) + H_{K_p^{-1}}(p)), & \text{if } K_p \neq K_p^{-1}, \\ \chi(K, K_p^\alpha) H_{K_p}(p), & \text{if } K_p = K_p^{-1}, \end{cases} \\ &= 2\chi(K, K_p^\alpha) \\ &= \chi(K, K_p^\alpha) + \chi(K, K_p^{-\alpha}). \end{aligned}$$

Thus, in both cases, we have

$$Y_K(p^\alpha) = \chi(K, K_p^\alpha) + \chi(K, K_p^{-\alpha}) = \chi(K, K_p)^\alpha + \chi(K, K_p)^{-\alpha},$$

by Lemma 5.11.

(c) Let $p \mid d$, $p \nmid f$. By Theorem 5.5, we have

$$Y_K(p^\alpha) = \sum_{L \in H(d)} \chi(K, L) H_L(p^\alpha) = 0 \text{ if } \alpha \geq 2.$$

Also, by Theorem 5.4(d), we have

$$Y_K(p) = \sum_{L \in H(d)} \chi(K, L) H_L(p) = \chi(K, K_p),$$

as required. $\square$

Corollary 5.2 *Let $K \in H(d)$ and let p be a prime. Then, for all $j \geq 1$, we have*

$$|Y_K(p^j)| \leq 2f.$$

Proof: If $p \mid f$, the result follows from Lemma 5.12 and Corollary 5.1. Otherwise, Theorem 5.9 gives

$$|Y_K(p^j)| \leq 2 \leq 2f,$$

as required. $\square$

5.8 The series $\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s}$

Next we introduce the quantity $j(K, d)$ which will be needed in our analysis of the series $\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s}$.

Theorem 5.10 *Let $K \in H(d)$ with $K \neq I$. Then*

$$j(K, d) = \lim_{s \rightarrow 1^+} \prod_{\substack{p \\ \left(\frac{d}{p}\right)=1}} \left(1 - \frac{\chi(K, K_p)}{p^s}\right) \left(1 - \frac{\chi(K^{-1}, K_p)}{p^s}\right)$$

exists and is a non-zero real number satisfying $j(K, d) = j(K^{-1}, d)$.

Proof: The existence of the above limit and the fact that it is non-zero is proved in [1], pp. 36–69. A proof in the case when d is negative and fundamental is given in [23]. The fact that it is real follows easily since

$$\prod_{\substack{p \\ \left(\frac{d}{p}\right)=1}} \left(1 - \frac{\chi(K, K_p)}{p^s}\right) \left(1 - \frac{\chi(K^{-1}, K_p)}{p^s}\right)$$

is real for $s > 1$. The relation $j(K, d) = j(K^{-1}, d)$ is clear. $\square$

Theorem 5.11 *Let p be a prime and let $K \in H(d)$. Then $\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$ converges absolutely and uniformly for $s \geq 1$. Moreover, for $s \geq 1$, we have*

$$\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}} = \begin{cases} 1, & \text{if } \left(\frac{d}{p}\right) = -1, \\ \frac{1 - \frac{1}{p^{2s}}}{\left(1 - \frac{\chi(K, K_p)}{p^s}\right) \left(1 - \frac{\chi(K, K_p)^{-1}}{p^s}\right)}, & \text{if } \left(\frac{d}{p}\right) = 1, \\ 1 + \frac{\chi(K, K_p)}{p^s}, & \text{if } p \mid d, p \nmid f. \end{cases}$$

Proof: By Corollary 5.2, $\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$ converges absolutely and uniformly for $s \geq 1$. The values of $\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$ for the cases given in the statement of the Theorem follow immediately on using Theorems 5.7 and 5.9. $\square$

Theorem 5.12 *Let $K \in H(d)$. Then, for $s > 1$, we have*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s} &= \prod_{\left(\frac{d}{p}\right)=1} \frac{1 - \frac{1}{p^{2s}}}{\left(1 - \frac{\chi(K, K_p)}{p^s}\right) \left(1 - \frac{\chi(K, K_p)^{-1}}{p^s}\right)} \\ &\quad \times \prod_{\substack{p \mid d \\ p \nmid f}} \left(1 + \frac{\chi(K, K_p)}{p^s}\right) \prod_{p \nmid f} \sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}. \end{aligned}$$

Proof: Let $s > 1$. We first show that

$$\prod_p \sum_{j=0}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right|$$

converges. By Corollary 5.2, we have

$$\sum_{j=1}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right| \leq \sum_{j=1}^{\infty} \frac{2f}{p^{js}} = \frac{2f}{p^s - 1}.$$

Hence $\sum_p \sum_{j=1}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right|$ converges so that

$$\prod_p \left(1 + \sum_{j=1}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right| \right) = \prod_p \sum_{j=0}^{\infty} \left| \frac{Y_K(p^j)}{p^{js}} \right|$$

converges. Since $Y_K(n)$ is multiplicative (Theorem 5.8), it follows that

$$\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s} = \prod_p \sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}.$$

The result now follows on using Theorem 5.11. □

Theorem 5.13 *Let p be a prime and $K \in H(d)$ with $K \neq I$. Let*

$$\begin{aligned} t_1(d) &= \prod_{\left(\frac{d}{p}\right)=1} \left(1 - \frac{1}{p^2} \right), \\ A(K, d, p) &= \sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^j}, \\ l(K, d) &= \prod_{\substack{p|d \\ p \nmid f}} \left(1 + \frac{\chi(K, K_p)}{p} \right) \prod_{p \nmid f} A(K, d, p). \end{aligned}$$

Then

$$\sum_{n=1}^{\infty} \frac{Y_K(n)}{n^s} = \frac{t_1(d)}{j(K, d)} l(K, d) (1 + o(1))$$

as $s \rightarrow 1^+$.

Proof: By the uniform convergence of $\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}}$ for $s \geq 1$ (Theorem 5.11), we have

$$\sum_{j=0}^{\infty} \frac{Y_K(p^j)}{p^{js}} = A(K, d, p) (1 + o(1)), \quad (5.10)$$

as $s \rightarrow 1^+$. Also

$$\prod_{\left(\frac{d}{p}\right)=1} \left(1 - \frac{1}{p^{2s}}\right) = t_1(d)(1 + o(1)), \quad (5.11)$$

$$1 + \frac{\chi(K, K_p)}{p^s} = \left(1 + \frac{\chi(K, K_p)}{p}\right)(1 + o(1)) \quad (5.12)$$

as $s \rightarrow 1^+$. The required result follows from (5.10), (5.11), (5.12), Theorem 5.12 and Theorem 5.10. $\square$

Lemma 5.13 *Let $s > 1$. Then*

$$\zeta(2s) \sum_{n=1}^{\infty} \frac{Y_I(n)}{n^s} = \frac{1}{w(d)} \sum_{n=1}^{\infty} \frac{N(n, d)}{n^s}.$$

Proof: We have

$$\begin{aligned} & \zeta(2s) \sum_{n=1}^{\infty} \frac{Y_I(n)}{n^s} \\ &= \sum_{L \in H(d)} \sum_{m=1}^{\infty} \frac{1}{m^{2s}} \sum_{n=1}^{\infty} \frac{H_L(n)}{n^s} \\ &= \sum_{L \in H(d)} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{m^2 | l} H_L(l/m^2) \\ &= \frac{1}{w(d)} \sum_{L \in H(d)} \sum_{l=1}^{\infty} \frac{R_L(l, d)}{l^s} \\ &= \frac{1}{w(d)} \sum_{l=1}^{\infty} \frac{N(l, d)}{l^s}, \end{aligned}$$

by Lemma 5.7. $\square$

5.9 The series $\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s}$

We are now ready to prove Theorem 5.14.

Theorem 5.14 *Let $K \in H(d)$. Then for $s > 1$, we have*

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s} \\ &= \frac{w(d)}{h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \left(\zeta(2s) \sum_{n=1}^{\infty} \frac{Y_L(n)}{n^s} \right) + \frac{1}{h(d)} \sum_{n=1}^{\infty} \frac{N(n, d)}{n^s}. \end{aligned}$$

Proof: We have

$$\begin{aligned} & \sum_{L \in H(d)} \chi(L, K)^{-1} Y_L(n) \\ &= \sum_{L \in H(d)} \chi(L, K)^{-1} \sum_{M \in H(d)} \chi(L, M) H_M(n) \\ &= \sum_{M \in H(d)} H_M(n) \sum_{L \in H(d)} \chi(L, M) \chi(L, K)^{-1} \\ &= H_K(n) h(d), \end{aligned}$$

by Lemma 5.11. Thus

$$H_K(n) = \frac{1}{h(d)} \sum_{L \in H(d)} \chi(L, K)^{-1} Y_L(n).$$

Hence, for $s > 1$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s} &= w(d) \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{m^2 | n} H_K(n/m^2) \\ &= w(d) \zeta(2s) \sum_{n=1}^{\infty} \frac{H_K(n)}{n^s} \\ &= \frac{w(d) \zeta(2s)}{h(d)} \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{L \in H(d)} \chi(L, K)^{-1} Y_L(n) \\ &= \frac{w(d)}{h(d)} \sum_{L \in H(d)} \chi(L, K)^{-1} \left(\zeta(2s) \sum_{n=1}^{\infty} \frac{Y_L(n)}{n^s} \right) \\ &= \frac{w(d)}{h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \left(\zeta(2s) \sum_{n=1}^{\infty} \frac{Y_L(n)}{n^s} \right) \end{aligned}$$

$$+\frac{w(d)}{h(d)}\zeta(2s)\sum_{n=1}^{\infty}\frac{Y_I(n)}{n^s}.$$

The result follows on using Lemma 5.13.

□

Chapter 6

The Dedekind eta function and Weber's functions at quadratic irrationalities

6.1 Introduction

The Dedekind eta function $\eta(z)$ is defined for all complex numbers z with $\text{Im}(z) > 0$ by

$$\eta(z) = e^{\pi iz/12} \prod_{m=1}^{\infty} (1 - e^{2\pi imz}). \quad (6.1)$$

We note that $\eta(iy)$ and $e^{-\pi i/24} \eta(\frac{1+iz}{2})$ are positive numbers for $y > 0$. Weber's functions are defined for complex z with $\text{Im}(z) > 0$ as follows

$$f(z) = \frac{e^{-\pi i/24} \eta(\frac{1+z}{2})}{\eta(z)}, \quad (6.2)$$

$$f_1(z) = \frac{\eta(\frac{z}{2})}{\eta(z)}, \quad (6.3)$$

$$f_2(z) = \sqrt{2} \frac{\eta(2z)}{\eta(z)}. \quad (6.4)$$

For convenience we set $f_0(z) = f(z)$ so that $f_i(z)$ is defined for $i = 0, 1, 2$. We note that $f_i(\sqrt{-y})$ is positive for $y > 0$, $i = 0, 1, 2$. The Chowla-Selberg for-

mula (see [24], p. 110) asserts that if d is a negative fundamental discriminant then

$$\prod_{[a,b,c] \in H(d)} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| = (2\pi|d|)^{-h(d)/4} \left(\prod_{m=1}^{|d|} \Gamma(m/|d|)^{(\frac{d}{m})} \right)^{w(d)/8}. \quad (6.5)$$

This result has been extended to arbitrary negative discriminants by Kaneko [15], Nakajima and Taguchi [20], and Kaplan and Williams [16] (see Theorem 6.4). A formula for $a^{-1/4} |\eta((b + \sqrt{d})/(2a))|$ where d is a negative fundamental discriminant and $[a, b, c] \in H(d)$ has been obtained by van der Poorten and Williams [23] (see Theorem 6.5). In Section 6.3, using Theorem 5.14 and Kronecker's "Grenz-Formel" (see [25], Theorem 1, p. 14), we extend the result of van der Poorten and Williams to arbitrary negative discriminants (see Theorem 6.3). In Section 6.4 we apply Theorem 6.3 to deduce the Chowla-Selberg formula for genera which was obtained by Huard, Kaplan and Williams [14]. In Section 6.5 we apply Theorem 6.3 to obtain expressions for $|f_i((b + \sqrt{d})/(2a))|$, $i = 0, 1, 2$, where d is a negative discriminant and $[a, b, c] \in H(d)$ (see Theorem 6.7). We also obtain expressions for $f_i(\sqrt{-n})$, $i = 0, 1, 2$, where n is a positive integer (see Theorem 6.8). We conclude this chapter with some numerical examples illustrating the evaluation of $f_0(\sqrt{-n})$.

6.2 Behaviour of $\sum_{n=1}^{\infty} \frac{N(n,d)}{n^s}$ and $\sum_{n=1}^{\infty} \frac{R_K(n,d)}{n^s}$ as $s \rightarrow 1^+$ for $d < 0$

We begin by considering $\sum_{n=1}^{\infty} \frac{N(n,d)}{n^s}$, where $N(n, d)$ is defined in (1.29).

Theorem 6.1 *Let $d < 0$. Then*

$$\sum_{n=1}^{\infty} \frac{N(n, d)}{n^s} = \frac{2\pi h(d)}{\sqrt{|d|}} \frac{1}{s-1} + B(d) + O(s-1),$$

as $s \rightarrow 1^+$ where

$$\begin{aligned} B(d) &= \frac{2\pi h(d)}{\sqrt{|d|}} \log(2\pi) + \frac{4\pi\gamma h(d)}{\sqrt{|d|}} \\ &\quad - \frac{2\pi h(d)}{\sqrt{|d|}} \sum_{p|f} \alpha_p(\Delta, f) \log p \\ &\quad - \frac{\pi h(d)w(\Delta)}{\sqrt{|d|h(\Delta)}} \sum_{m=1}^{|\Delta|} \left(\frac{\Delta}{m}\right) \log \Gamma\left(\frac{m}{|\Delta|}\right), \\ \alpha_p(\Delta, f) &= \frac{(p^{v_p(f)} - 1) \left(1 - \left(\frac{\Delta}{p}\right)\right)}{p^{v_p(f)-1}(p-1) \left(p - \left(\frac{\Delta}{p}\right)\right)}, \end{aligned}$$

and γ is Euler's constant.

Proof: By Theorem 10.2 of [14], we have as $s \rightarrow 1^+$,

$$\sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} = \frac{\pi h(d)}{2^{t(d)-1} \sqrt{|d|}} \frac{1}{s-1} + B_G(d) + O(s-1), \quad (6.6)$$

where

$$\begin{aligned} B_G(d) &= \frac{1}{2^{t(d)}} B(d) - \frac{8\pi}{\sqrt{|d|}} \sum_{\substack{d_1 \in F(d) \\ d_1 > 1}} \beta(d_1, d, G) \log(\eta_{d_1}), \\ \beta(d_1, d, G) &= \frac{-w(d)\gamma_{d_1}(G)f(d/d_1)h^*(d_1)h(\Delta(d/d_1))}{w(\Delta(d/d_1))2^{t(d)+1}} \\ &\quad \times \sum_{m|f(d/d_1)} \frac{1}{m} \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right) \\ &\quad \times \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p}\right), \end{aligned}$$

η_{d_1} is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{d_1})$ and $h^*(d_1)$ is the class number of this field.

Hence, as $s \rightarrow 1^+$, we have

$$\begin{aligned}
 & \sum_{n=1}^{\infty} \frac{N(n, d)}{n^s} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{G \in G(d)} R_G(n, d) \\
 &= \sum_{G \in G(d)} \sum_{n=1}^{\infty} \frac{R_G(n, d)}{n^s} \\
 &= \sum_{G \in G(d)} \left(\frac{\pi h(d)}{2^{t(d)-1} \sqrt{|d|}} \frac{1}{s-1} + B_G(d) + O(s-1) \right) \\
 &= \frac{2\pi h(d)}{\sqrt{|d|}} \frac{1}{s-1} + \sum_{G \in G(d)} B_G(d) + O(s-1).
 \end{aligned}$$

But

$$\begin{aligned}
 \sum_{G \in G(d)} B_G(d) &= B(d) - \frac{8\pi}{\sqrt{|d|}} \sum_{G \in G(d)} \sum_{\substack{d_1 \in F(d) \\ d_1 > 1}} \beta(d_1, d, G) \log(\eta_{d_1}) \\
 &= B(d),
 \end{aligned}$$

since

$$\sum_{G \in G(d)} \gamma_{d_1}(G) = 0,$$

for $d_1 > 1$. □

We now turn to $\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s}$, where $R_K(n, d)$ is defined in (1.7).

Theorem 6.2 *Let $d < 0$, $K \in H(d)$. Then*

$$\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s} = \frac{2\pi}{\sqrt{|d|}} \frac{1}{s-1} + A(K, d) + o(1),$$

as $s \rightarrow 1^+$, where

$$A(K, d) = \frac{B(d)}{h(d)} + \frac{\pi^2 w(d)}{6h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d).$$

Proof: The required result follows on using Theorem 5.14, Theorem 5.13 and Theorem 6.1. $\square$

Lemma 6.1 *Let $d < 0$, $K = [a, b, c] \in H(d)$. Then*

$$\sum_{n=1}^{\infty} \frac{R_K(n, d)}{n^s} = \frac{2\pi}{\sqrt{|d|}} \frac{1}{s-1} + B(a, b, c) + o(1),$$

as $s \rightarrow 1^+$, where

$$B(a, b, c) = \frac{4\pi\gamma}{\sqrt{|d|}} - \frac{2\pi \log(|d|)}{\sqrt{|d|}} - \frac{8\pi}{\sqrt{|d|}} \log \left(a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| \right).$$

Proof: This is essentially Kronecker's "Grenz Formel". See [14], p. 300. $\square$

6.3 Evaluation of $|\eta((b + \sqrt{d})/2a)|$ for $d < 0$

We are now ready to extend the Chowla-Selberg formula for classes due to van der Poorten and Williams [23] to arbitrary negative discriminants.

Theorem 6.3 *Let $d < 0$, $K = [a, b, c] \in H(d)$. Then*

$$\begin{aligned} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| &= (2\pi|d|)^{-1/4} \prod_{p|f} p^{\alpha_p(\Delta, f)/4} \\ &\quad \times \left(\prod_{m=1}^{|\Delta|} \Gamma \left(\frac{m}{|\Delta|} \right)^{\left(\frac{\Delta}{m} \right)} \right)^{\frac{w(\Delta)}{8h(\Delta)}} e^{-E(K, d)}, \end{aligned}$$

where

$$E(K, d) = \frac{\pi \sqrt{|d|} w(d)}{48h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d).$$

Remark: If $p \nmid f$, $v_p(f) = 0$ so that $\alpha_p(\Delta, f) = 0$. Thus $\prod_{p|f} p^{\alpha_p(\Delta, f)/4}$ can be replaced by $\prod_p p^{\alpha_p(\Delta, f)/4}$.

Proof: By Theorem 6.2 and Lemma 6.1, we have

$$A(K, d) = B(a, b, c).$$

Simplifying and exponentiating gives the required result. $\square$

Next, we use Theorem 6.3 to deduce the following result of Kaneko [15], Nakajima and Taguchi [20], and Kaplan and Williams [16].

Theorem 6.4 *Let $d < 0$. Then*

$$\prod_{[a,b,c] \in H(d)} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| = (2\pi|d|)^{-h(d)/4} \left(\prod_{p|f} p^{\alpha_p(\Delta, f)} \right)^{h(d)/4} \\ \times \left(\prod_{m=1}^{|\Delta|} \Gamma \left(\frac{m}{|\Delta|} \right)^{\left(\frac{\Delta}{m} \right)} \right)^{\frac{w(\Delta)h(d)}{8h(\Delta)}}.$$

Proof: The result follows on multiplying the result of Theorem 6.3 over all the $h(d)$ classes of $H(d)$ and the observation that

$$\sum_{K \in H(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d) \\ = \sum_{\substack{L \in H(d) \\ L \neq I}} \frac{t_1(d) l(L, d)}{j(L, d)} \sum_{K \in H(d)} \chi(L^{-1}, K) \\ = 0,$$

by Lemma 5.11. $\square$

Next, we use Theorem 6.3 to deduce the Chowla-Selberg formula for classes for negative fundamental discriminants due to van der Poorten and Williams [23].

Theorem 6.5 *Let $d < 0$ with d fundamental. Let $K = [a, b, c] \in H(d)$. Then*

$$\begin{aligned} & a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| \\ &= (2\pi|d|)^{-1/4} \left(\prod_{m=1}^{|d|} \Gamma \left(\frac{m}{|d|} \right)^{\left(\frac{d}{m} \right)} \right)^{\frac{w(d)}{8h(d)}} \\ & \times \exp \left(-\frac{\pi w(d) \sqrt{|d|}}{48h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K) \frac{t_1(d)}{j(L, d)} l(L, d) \right). \end{aligned}$$

Proof: Since d is fundamental, we have $\Delta = d$ and $f = 1$. Also

$$l(L, d) = \prod_{p|d} \left(1 + \frac{\chi(L, K_p)}{p} \right).$$

Thus

$$\begin{aligned} l(L^{-1}, d) &= \prod_{p|d} \left(1 + \frac{\chi(L^{-1}, K_p)}{p} \right) \\ &= \prod_{p|d} \left(1 + \frac{\chi(L, K_p^{-1})}{p} \right) \\ &= l(L, d), \end{aligned}$$

since $K_p = K_p^{-1}$ if $p | d$ and $p \nmid f$. Also $j(L, d) = j(L^{-1}, d)$ for $L \neq I$. Hence

$$\begin{aligned} & \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d) \\ &= \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L^{-1}, K) \frac{t_1(d)}{j(L, d)} l(L, d) \\ &= \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K) \frac{t_1(d)}{j(L^{-1}, d)} l(L^{-1}, d) \\ &= \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K) \frac{t_1(d)}{j(L, d)} l(L, d). \end{aligned}$$

The result now follows on using Theorem 6.3. □

6.4 Deduction of the Chowla-Selberg formula for genera

In this section, we use Theorem 6.3 to deduce the Chowla-Selberg formula for genera which was proved by Huard, Kaplan and Williams [14].

Theorem 6.6 *Let $G \in G(d)$ where $d < 0$. Then*

$$\begin{aligned} \prod_{[a,b,c] \in G} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| &= (2\pi|d|)^{-\frac{h(d)}{2t(d)+2}} \left(\prod_{p|f} p^{\alpha_p(\Delta, f)} \right)^{\frac{h(d)}{2t(d)+2}} \\ &\quad \times \left(\prod_{m=1}^{|\Delta|} \Gamma(m/|\Delta|)^{\left(\frac{\Delta}{m}\right)} \right)^{\frac{w(\Delta)h(d)}{2t(d)+3h(\Delta)}} \\ &\quad \times \prod_{\substack{d_1 \in F(d) \\ d_1 > 1}} \eta_{d_1}^{\beta(d_1, d, G)}, \end{aligned}$$

where η_{d_1} is the fundamental unit of the field $\mathbb{Q}(\sqrt{d_1})$ and

$$\begin{aligned} \beta(d_1, d, G) &= -\frac{w(d)\gamma_{d_1}(G)f(d/d_1)h^*(d_1)h(\Delta(d/d_1))}{2^{t(d)+1}w(\Delta(d/d_1))} \\ &\quad \times \sum_{m|f(d/d_1)} \frac{1}{m} \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p} \right) \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p} \right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p} \right), \end{aligned}$$

where $h^*(d_1)$ is the class number of $\mathbb{Q}(\sqrt{d_1})$.

Proof: By Theorem 6.3,

$$\begin{aligned} \prod_{[a,b,c] \in G} a^{-1/4} \left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right| &= (2\pi|d|)^{-\frac{h(d)}{2t(d)+2}} \left(\prod_{p|f} p^{\alpha_p(\Delta, f)} \right)^{\frac{h(d)}{2t(d)+2}} \\ &\quad \times \left(\prod_{m=1}^{|\Delta|} \Gamma(m/|\Delta|)^{\left(\frac{\Delta}{m}\right)} \right)^{\frac{w(\Delta)h(d)}{2t(d)+3h(\Delta)}} e^{-\sum_{K \in G} E(K, d)}. \end{aligned}$$

Thus it suffices to show

$$e^{-\sum_{K \in G} E(K, d)} = \prod_{\substack{d_1 \in F(d) \\ d_1 > 1}} \eta_{d_1}^{\beta(d_1, d, G)}. \quad (6.7)$$

We have

$$\begin{aligned} \sum_{K \in G} E(K, d) &= \frac{\pi \sqrt{|d|} w(d)}{48h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \frac{t_1(d) l(L, d)}{j(L, d)} \sum_{K \in G} \chi(L^{-1}, K) \\ &= \frac{\pi \sqrt{|d|} w(d)}{48h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \frac{t_1(d) l(L^{-1}, d)}{j(L, d)} \sum_{K \in G} \chi(L, K). \end{aligned}$$

Let $h_1, \dots, h_m$ be odd and $h_{m+1}, \dots, h_\nu$ be even (see Section 5.6). Then the distinct genera of discriminant d are given by

$$A_{m+1}^{r_{m+1}} \dots A_\nu^{r_\nu} H^2(d),$$

where each $r_i = 0$ or 1 . Thus there are $2^{\nu-m}$ genera so that $t(d) = \nu - m$. Let G be given by

$$G = A_{m+1}^{\delta_{m+1}} \dots A_\nu^{\delta_\nu} H^2(d),$$

where each $\delta_i = 0$ or 1 . The classes K of G are given by

$$K = A_1^{k_1} \dots A_m^{k_m} A_{m+1}^{2k_{m+1} + \delta_{m+1}} \dots A_\nu^{2k_\nu + \delta_\nu},$$

where

$$k_j = \begin{cases} 0, 1, \dots, h_j - 1, & j = 1, \dots, m, \\ 0, 1, \dots, \frac{h_j}{2} - 1, & j = m+1, \dots, \nu. \end{cases}$$

Let

$$L = A_1^{t_1} \dots A_\nu^{t_\nu}, \quad 0 \leq t_j < h_j.$$

Thus

$$[L, K] = \sum_{j=1}^m \frac{t_j k_j}{h_j} + \sum_{j=m+1}^\nu \frac{t_j (2k_j + \delta_j)}{h_j},$$

so that

$$\begin{aligned}
& \sum_{K \in G} \chi(L, K) \\
&= \sum_{K \in G} e^{2\pi i [L, K]} \\
&= \prod_{j=1}^m \left(\sum_{k_j=0}^{h_j-1} e^{\frac{2\pi i t_j k_j}{h_j}} \right) \prod_{j=m+1}^{\nu} \left(\sum_{k_j=0}^{h_j/2-1} e^{\frac{2\pi i t_j (2k_j + \delta_j)}{h_j}} \right) \\
&= \prod_{j=1}^m \left\{ \begin{array}{ll} h_j, & \text{if } t_j = 0 \\ 0, & \text{if } t_j \neq 0 \end{array} \right\} \prod_{j=m+1}^{\nu} \left\{ \begin{array}{ll} \frac{h_j}{2}, & \text{if } t_j = 0 \text{ or } \frac{h_j}{2} \\ 0, & \text{if } t_j \neq 0, \frac{h_j}{2} \end{array} \right\} \prod_{j=m+1}^{\nu} e^{\frac{2\pi i t_j \delta_j}{h_j}} \\
&= \begin{cases} \frac{h(d)}{2^{t(d)}} e^{2\pi i \sum_{j=m+1}^{\nu} \frac{t_j \delta_j}{h_j}}, & \text{if } t_j = 0 \text{ for } 1 \leq j \leq m, \\ & t_j = 0 \text{ or } \frac{h_j}{2} \text{ for } m+1 \leq j \leq \nu, \\ 0, & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{h(d)}{2^{t(d)}} (-1)^{\sum_{j=m+1}^{\nu} x_j \delta_j}, & \text{if } L = \prod_{j=m+1}^{\nu} A_j^{h_j x_j / 2}, \\ & \text{where each } x_j = 0 \text{ or } 1, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
& \sum_{K \in G} E(K, d) \\
&= \frac{\pi \sqrt{|d|} w(d)}{48 \times 2^{t(d)}} \sum_{\substack{x_{m+1}, \dots, x_{\nu}=0 \\ (x_{m+1}, \dots, x_{\nu}) \neq (0, \dots, 0)}}^1 \frac{t_1(d) l \left(\prod_{i=m+1}^{\nu} A_i^{h_i x_i / 2}, d \right)}{j \left(\prod_{i=m+1}^{\nu} A_i^{h_i x_i / 2}, d \right)} (-1)^{\sum_{i=m+1}^{\nu} x_i \delta_i}.
\end{aligned}$$

Let $F_+(d) = \{d_1 \in F(d) : d_1 > 0\}$, $F_-(d) = \{d_1 \in F(d) : d_1 < 0\}$. The mapping $d_1 \mapsto \Delta(d/d_1)$ is a bijection on $F(d)$ (see [14], p. 279). As $d < 0$, the mapping $F_+(d) \mapsto F_-(d)$ given by $d_1 \mapsto \Delta(d/d_1)$ is a bijection so that $|F_+(d)| = |F_-(d)| = 2^{t(d)}$. We define $\lambda : F_+(d) \mapsto \{0, 1\}^{\nu-m} = \{0, 1\}^{t(d)}$ by $\lambda(d_1) = (x_{m+1}, \dots, x_{\nu})$ where $\gamma_{d_1}(A_i) = (-1)^{x_i}$, $x_i = 0$ or 1 for $i = m+1, \dots, \nu$.

We show that λ is a bijection. Since $|F_+(d)| = |\{0, 1\}^{t(d)}| = 2^{t(d)}$, it suffices to show that λ is one to one.

We recall that the set $F(d)$ is a group under the operation $\circ$ defined by $d_1 \circ d_2 = \Delta(d_1 d_2)$. Also, if $\hat{G}(d)$ is the group of characters of $G(d)$, the mapping $\phi : F(d) \mapsto \hat{G}(d)$ given by $\phi(d_1) = \gamma_{d_1}$ is a surjective homomorphism with kernel $\{1, \Delta\}$ (see [14], p. 279).

Let $d_1, d'_1 \in F_+(d)$ with $\lambda(d_1) = \lambda(d'_1)$. Then $\gamma_{d_1}(A_i) = \gamma_{d'_1}(A_i)$ for $i = m+1, \dots, \nu$. Thus $\gamma_{d_1}(G_1) = \gamma_{d'_1}(G_1)$ for all $G_1 \in G(d)$. We have $\phi(d_1 \circ d'_1) = \phi(d_1)\phi(d'_1) = \gamma_{d_1}\gamma_{d'_1}$. But $\gamma_{d_1}\gamma_{d'_1}(G_1) = \gamma_{d_1}(G_1)\gamma_{d'_1}(G_1) = 1$ for all $G_1 \in G(d)$. Hence $\Delta(d_1 d'_1) = d_1 \circ d'_1 \in \text{Ker } \phi = \{1, \Delta\}$. As $d_1 > 0, d'_1 > 0, \Delta < 0$, we have $\Delta(d_1 d'_1) = 1$. Thus $d_1 = d'_1$. Hence λ is a bijection. (Note that $\lambda(1) = (0, \dots, 0)$.)

Thus $\lambda^{-1} : \{0, 1\}^{\nu-m} \mapsto F_+(d)$ is a bijection which associates each $(\nu - m)$ tuple $(x_{m+1}, \dots, x_\nu)$ (each $x_i = 0$ or 1) with a unique $d_1 \in F_+(d)$ such that $\gamma_{d_1}(A_i) = (-1)^{x_i}$ for $i = m+1, \dots, \nu$ and $\lambda^{-1}(0, \dots, 0) = 1$. For the rest of this proof, $(x_{m+1}, \dots, x_\nu) \in \{0, 1\}^{\nu-m}$ with $(x_{m+1}, \dots, x_\nu) \neq (0, \dots, 0)$, $d_1 = \lambda^{-1}(x_{m+1}, \dots, x_\nu)$, and $L = \prod_{i=m+1}^{\nu} A_i^{h_i x_i/2}$. We have

$$(-1)^{\sum_{i=m+1}^{\nu} x_i \delta_i} = \prod_{i=m+1}^{\nu} \gamma_{d_1}(A_i)^{\delta_i} = \gamma_{d_1} \left(\prod_{i=m+1}^{\nu} A_i^{\delta_i} \right) = \gamma_{d_1}(G).$$

Also, it is easily shown that $\chi(L, M) = \gamma_{d_1}(M)$ for all $M \in H(d)$. Let $p \mid d$, $p \nmid f$. Then $K_p = K_p^{-1}$ so that

$$K_p = \prod_{i=m+1}^{\nu} A_i^{h_i u_i/2}, \quad \text{where each } u_i = 0 \text{ or } 1.$$

Using $p \mid d$, $p \nmid f$, it is easily shown that p divides exactly one of d_1 and $\Delta(d/d_1)$. If $p \nmid d_1$ then $\gamma_{d_1}(K_p) = (\frac{d_1}{p})$. If $p \mid d_1$ then $p \nmid \Delta(d/d_1)$ so that

$$\gamma_{d_1}(K_p) = \gamma_{\Delta(d/d_1)}(K_p) = \left(\frac{\Delta(d/d_1)}{p} \right).$$

Thus

$$\begin{aligned} l(L, d) &= \prod_{\substack{p \mid d \\ p \nmid f}} \left(1 + \frac{\chi(L, K_p)}{p} \right) \prod_{p \mid f} A(L, d, p) \\ &= \prod_{\substack{p \mid d \\ p \nmid f \\ p \nmid d_1}} \left(1 + \frac{\left(\frac{d_1}{p} \right)}{p} \right) \prod_{\substack{p \mid d \\ p \nmid f \\ p \mid d_1}} \left(1 + \frac{\left(\frac{\Delta(d/d_1)}{p} \right)}{p} \right) \prod_{p \mid f} A(L, d, p) \\ &= \prod_{\substack{p \mid d \\ p \nmid f}} \left(1 + \frac{\left(\frac{d_1}{p} \right)}{p} \right) \left(1 + \frac{\left(\frac{\Delta(d/d_1)}{p} \right)}{p} \right) \prod_{p \mid f} A(L, d, p) \\ &= \lim_{s \rightarrow 1^+} \prod_{\substack{p \mid d \\ p \nmid f}} \frac{\left(1 - \frac{1}{p^s} \right) \left(1 + \frac{1}{p^s} \right)}{\left(1 - \frac{\left(\frac{d_1}{p} \right)}{p^s} \right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p} \right)}{p^s} \right)} \prod_{p \mid f} A(L, d, p). \end{aligned}$$

Next let p be a prime with $\left(\frac{d}{p} \right) = 1$. Then

$$\chi(L, K_p) = \gamma_{d_1}(K_p) = \left(\frac{d_1}{p} \right) = \left(\frac{\Delta(d/d_1)}{p} \right).$$

Thus

$$\begin{aligned} \frac{t_1(d)}{j(L, d)} &= \lim_{s \rightarrow 1^+} \prod_{\left(\frac{d}{p} \right) = 1} \frac{\left(1 - \frac{1}{p^s} \right) \left(1 + \frac{1}{p^s} \right)}{\left(1 - \frac{\left(\frac{d_1}{p} \right)}{p^s} \right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p} \right)}{p^s} \right)} \\ &= \lim_{s \rightarrow 1^+} \prod_{p \nmid d} \frac{\left(1 - \frac{1}{p^s} \right) \left(1 + \frac{1}{p^s} \right)}{\left(1 - \frac{\left(\frac{d_1}{p} \right)}{p^s} \right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p} \right)}{p^s} \right)}, \end{aligned}$$

as

$$\left(\frac{d}{p}\right) = \left(\frac{d_1}{p}\right) \left(\frac{\Delta(d/d_1)}{p}\right),$$

if $p \nmid d$. Then

$$\begin{aligned} & \frac{t_1(d)}{j(L, d)} l(L, d) \\ &= \lim_{s \rightarrow 1^+} \prod_p \frac{\left(1 - \frac{1}{p^{2s}}\right)}{\left(1 - \frac{\left(\frac{d_1}{p}\right)}{p^s}\right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p^s}\right)} \prod_{p|f} \frac{\left(1 - \frac{\left(\frac{d_1}{p}\right)}{p^s}\right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p^s}\right)}{\left(1 - \frac{1}{p^{2s}}\right)} A(L, d, p) \\ &= \frac{L(1, d_1) L(1, \Delta(d/d_1))}{\zeta(2)} \prod_{p|f} \frac{\left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right)}{\left(1 - \frac{1}{p^2}\right)} A(L, d, p) \\ &= \frac{2h^*(d_1) \log \eta_{d_1}}{\sqrt{d_1}} \frac{2\pi h(\Delta(d/d_1))}{w(\Delta(d/d_1)) \sqrt{|\Delta(d/d_1)|}} \frac{6}{\pi^2} \prod_{p|f} B(L, d, d_1, p) \\ &= \frac{24 h^*(d_1) h(\Delta(d/d_1)) f(d/d_1) \log \eta_{d_1}}{\pi w(\Delta(d/d_1)) \sqrt{|d|}} \prod_{p|f} B(L, d, d_1, p), \end{aligned}$$

where

$$B(L, d, d_1, p) = \frac{\left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right)}{\left(1 - \frac{1}{p^2}\right)} A(L, d, p).$$

Hence

$$\begin{aligned} & \sum_{K \in G} E(K, d) \\ &= \frac{1}{2^{t(d)+1}} \sum_{\substack{d_1 \in F(d) \\ d_1 > 1}} \frac{\gamma_{d_1}(G) w(d) h^*(d_1) h(\Delta(d/d_1)) f(d/d_1)}{w(\Delta(d/d_1))} \log \eta_{d_1} \prod_{p|f} B(L, d, d_1, p). \end{aligned}$$

Thus it suffices to prove

$$\prod_{p|f} B(L, d, d_1, p)$$

$$= \sum_{m|f(d/d_1)} \frac{1}{m} \prod_{\substack{p|m \\ p \nmid f/m}} \left(1 - \frac{\left(\frac{\Delta}{p}\right)}{p}\right) \prod_{p|f/m} \left(1 - \frac{\left(\frac{d_1}{p}\right)}{p}\right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{p}\right)}{p}\right),$$

where $L = \prod_{i=m+1}^{\nu} A_i^{x_i h_i/2}$, each $x_i = 0$ or 1 , and $d_1 = \lambda^{-1}(x_{m+1}, \dots, x_{\nu})$. To prove this, it suffices to prove that for each $p \mid f$,

$$B(L, d, d_1, p) = \sum_{m|p^{v_p(f(d/d_1))}} \frac{1}{m} \prod_{\substack{q|m \\ q \nmid \frac{p^{v_p(f)}}{m}}} \left(1 - \frac{\left(\frac{\Delta}{q}\right)}{q}\right) \prod_{q|\frac{p^{v_p(f)}}{m}} \left(1 - \frac{\left(\frac{d_1}{q}\right)}{q}\right) \left(1 - \frac{\left(\frac{\Delta(d/d_1)}{q}\right)}{q}\right). \quad (6.8)$$

Let $p \mid f$. As in Theorems 5.1 and 5.2, we write $\Delta = p^v s$, $f = p^u t$, $p \nmid st$. Note that $u \geq 1$. As $\chi(L, M) = \gamma_{d_1}(M)$, we have

$$Y_L(p^j) = \sum_{M \in H(d)} \gamma_{d_1}(M) H_M(p^j).$$

For $j \geq 1$, we have

$$\begin{aligned} Y_L(p^j) &= \sum_{M \in H(d)} \gamma_{d_1}(M) \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ [p^j, h, (h^2-d)/4p^j] = M}} 1 \\ &= \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ \gcd(p^j, h, (h^2-d)/4p^j) = 1}} \gamma_{d_1} \left(\left[p^j, h, \frac{h^2-d}{4p^j} \right] \right) \\ &= \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ p \nmid (h^2-d)/4p^j}} \gamma_{d_1} \left(\left[p^j, h, \frac{h^2-d}{4p^j} \right] \right) \\ &= \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ p \nmid (h^2-d)/4p^j}} \gamma_{\Delta(d/d_1)} \left(\left[p^j, h, \frac{h^2-d}{4p^j} \right] \right), \end{aligned}$$

as $\gamma_{d_1} = \gamma_{\Delta(d/d_1)}$.

We give the details when p is odd. In this case

$$\begin{aligned} Y_L(p^j) &= \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ p \nmid (h^2 - d)/p^j}} \gamma_{d_1} \left(\left[p^j, h, \frac{h^2 - d}{4p^j} \right] \right) \\ &= \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ p \nmid (h^2 - d)/p^j}} \gamma_{\Delta(d/d_1)} \left(\left[p^j, h, \frac{h^2 - d}{4p^j} \right] \right). \end{aligned}$$

If $p \nmid d_1$, we have

$$Y_L(p^j) = \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ p \nmid (h^2 - d)/p^j}} \left(\frac{d_1}{p^j} \right) = \left(\frac{d_1}{p^j} \right) H(p^j, d). \quad (6.9)$$

Thus, using Theorem 5.1, we can calculate $Y_L(p^j)$ and hence $A(L, d, p)$. We consider two cases according as $p \nmid \Delta$ or $p \mid \Delta$.

(a) $p \nmid d_1$, $p \nmid \Delta$ (so $v = 0$). We easily deduce $v_p(f(d/d_1)) = u$, $p \nmid \Delta(d/d_1)$ and

$$\left(\frac{d_1}{p} \right) \left(\frac{\Delta(d/d_1)}{p} \right) = \left(\frac{\Delta}{p} \right).$$

Using Theorem 5.1 and (6.9), we obtain

$$A(L, d, p) = \begin{cases} 1 + \frac{1}{p}, & \text{if } \left(\frac{\Delta}{p} \right) = -1, \\ 1 + \frac{1}{p}, & \text{if } \left(\frac{\Delta}{p} \right) = 1, \left(\frac{d_1}{p} \right) = 1, \\ 1 + \frac{1}{p} - \frac{4}{p^{u+1}} \frac{1}{1+1/p}, & \text{if } \left(\frac{\Delta}{p} \right) = 1, \left(\frac{d_1}{p} \right) = -1. \end{cases}$$

It is now easy to verify (6.8). Each side of (6.8) is $1 + 1/p$ if $\left(\frac{\Delta}{p} \right) = -1$, $1 - 1/p$ if $\left(\frac{\Delta}{p} \right) = 1$, $\left(\frac{d_1}{p} \right) = 1$ and $((1 + 1/p)^2 - 4/p^{u+1})/(1 - 1/p)$ if $\left(\frac{\Delta}{p} \right) = 1$, $\left(\frac{d_1}{p} \right) = -1$.

(b) $p \nmid d_1$, $p \mid \Delta$ (so $v = 1$). We easily deduce that $v_p(f(d/d_1)) = u$, $p \mid \Delta(d/d_1)$

and

$$A(L, d, p) = 1 + \frac{1}{p} + \frac{\left(\frac{d_1}{p} \right) - 1}{p^{u+1}}.$$

Using these, it is easily checked that each side of (6.8) is 1 if $(\frac{d_1}{p}) = 1$, and $(1 + 1/p - 2/p^{u+1})/(1 - 1/p)$ if $(\frac{d_1}{p}) = -1$.

If $p \mid d_1$, there exists a unique $q^* \in P(d_1)$ with $p \mid q^*$. Since p is odd, we have $q^* = p^*$. Thus

$$\begin{aligned}
 Y_L(p^j) &= \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ p \nmid (h^2 - d)/p^j}} \left(\frac{p^*}{(h^2 - d)/p^j} \right) \left(\frac{d_1/p^*}{p^j} \right) \\
 &= \left(\frac{d_1/p^*}{p^j} \right) \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j} \\ p \nmid (h^2 - d)/p^j}} \left(\frac{(h^2 - d)/p^j}{p} \right) \\
 &= \left(\frac{d_1/p^*}{p^j} \right) \sum_{\substack{0 \leq h < 2p^j \\ h^2 \equiv d \pmod{4p^j}}} \left(\frac{(h^2 - d)/p^j}{p} \right). \quad (6.10)
 \end{aligned}$$

To use (6.10), we need the solutions to the congruence

$$h^2 \equiv d \pmod{4p^j}. \quad (6.11)$$

These are recorded in the proof of Theorem 5.1. If we also have $p \nmid \Delta(d/d_1)$ then

$$Y_L(p^j) = \left(\frac{\Delta(d/d_1)}{p^j} \right) H(p^j, d). \quad (6.12)$$

We now consider two cases according as $p \nmid \Delta$ or $p \mid \Delta$.

(c) $p \mid d_1$, $p \nmid \Delta$ (so $v=0$). We obtain $p \mid \Delta(d/d_1)$ and $v_p(f(d/d_1)) = u - 1$.

Next, we calculate $Y_L(p^j)$ using (6.10).

If $j = 2l + 1$, $0 \leq l < u$ then $Y_L(p^j) = 0$ as $H(p^j, d) = 0$.

Next let $j = 2l$, $1 \leq l \leq u$. From the proof of Theorem 5.1, we know that the solutions to (6.11) satisfying $0 \leq h < 2p^j$ are given by

$$h = \lambda p^l, \quad 0 \leq \lambda < 2p^l, \quad \lambda \equiv d \pmod{2}.$$

Moreover, for such h , we have

$$\frac{h^2 - d}{p^j} = \lambda^2 - st^2 p^{2(u-l)}.$$

Thus

$$Y_L(p^j) = \sum_{\substack{0 \leq \lambda < 2p^l \\ \lambda \equiv d \pmod{2}}} \left(\frac{\lambda^2 - p^{2(u-l)} st^2}{p} \right).$$

If $l < u$ then

$$Y_L(p^j) = \sum_{\substack{0 \leq \lambda < 2p^l \\ \lambda \equiv d \pmod{2}}} \left(\frac{\lambda^2}{p} \right) = p^{l-1}(p-1).$$

If $l = u$, we have

$$\begin{aligned} Y_L(p^j) &= \sum_{\substack{0 \leq \lambda < 2p^l \\ \lambda \equiv d \pmod{2}}} \left(\frac{\lambda^2 - st^2}{p} \right) \\ &= \sum_{\nu=0}^{p-1} \sum_{\substack{0 \leq \lambda < 2p^l \\ \lambda \equiv d \pmod{2} \\ \lambda \equiv \nu \pmod{p}}} \left(\frac{\lambda^2 - st^2}{p} \right) \\ &= \sum_{\nu=0}^{p-1} \sum_{\substack{\mu \equiv d+\nu \pmod{2} \\ 0 \leq \mu < 2p^{l-1}}} \left(\frac{(p\mu + \nu)^2 - st^2}{p} \right) \\ &= \sum_{\nu=0}^{p-1} \sum_{\substack{\mu \equiv d+\nu \pmod{2} \\ 0 \leq \mu < 2p^{l-1}}} \left(\frac{\nu^2 - st^2}{p} \right) \\ &= \sum_{\nu=0}^{p-1} p^{l-1} \left(\frac{\nu^2 - st^2}{p} \right) \\ &= -p^{l-1}. \end{aligned}$$

Next let $j > 2u$. From Theorem 5.1, $Y_L(p^j) = 0$ if $(\frac{\Delta}{p}) = -1$. If $(\frac{\Delta}{p}) = 1$, from the solutions to (6.11) given in the proof of Theorem 5.1, we have

$$Y_L(p^j) = \left(\frac{d_1/p^*}{p^j} \right) \sum_{\substack{0 \leq h_1 < 2p^{j-u} \\ h_1^2 \equiv st^2 \pmod{p^{j-2u}} \\ h_1 \equiv d \pmod{2}}} \left(\frac{(h_1^2 - st^2)/p^{j-2u}}{p} \right)$$

$$\begin{aligned}
&= \sum_{m=0}^{p^u-1} \left(\frac{\frac{(x+2mp^{j-2u})^2-st^2}{p^{j-2u}}}{p} \right) + \sum_{m=0}^{p^u-1} \left(\frac{\frac{(p^{j-2u}-x+(2m+1)p^{j-2u})^2-st^2}{p^{j-2u}}}{p} \right) \\
&= \sum_{m=0}^{p^u-1} \left(\frac{4mx + \frac{x^2-st^2}{p^{j-2u}}}{p} \right) + \sum_{m=0}^{p^u-1} \left(\frac{-4x(m+1) + \frac{x^2-st^2}{p^{j-2u}}}{p} \right) \\
&= 0.
\end{aligned}$$

Thus we have obtained

$$Y_L(p^j) = \begin{cases} 0, & \text{if } j = 2l + 1, 0 \leq l < u, \\ 0, & \text{if } j > 2u, \\ p^l - p^{l-1}, & \text{if } j = 2l, 1 \leq l < u, \\ -p^{u-1}, & \text{if } j = 2u. \end{cases}$$

It is easily checked that $A(L, d, p) = (1 + 1/p)(1 - 1/p^u)$ and that each side of (6.8) is $(1 - 1/p^u)/(1 - 1/p)$.

(d) $p \mid d_1$, $p \mid \Delta$ (so $v = 1$). We have $p \nmid \Delta(d/d_1)$, $v_p(f(d/d_1)) = u$. Thus, we can use (6.12) together with Theorem 5.1 to calculate $Y_L(p^j)$ and hence $A(L, d, p)$. We obtain

$$A(L, d, p) = 1 + \frac{1}{p} + \frac{\left(\frac{\Delta(d/d_1)}{p}\right) - 1}{p^{u+1}}.$$

Finally, we easily check that each side of (6.8) is 1 if $(\frac{\Delta(d/d_1)}{p}) = 1$ and $(1 + 1/p - 2/p^{u+1})/(1 - 1/p)$ if $(\frac{\Delta(d/d_1)}{p}) = -1$.

This completes the deduction of (6.8) for p odd. The case $p = 2$ is dealt with similarly using Theorem 5.2 and its proof. We omit the details. $\square$

6.5 Weber's functions at quadratic irrationalities

We are now ready to apply Theorem 6.3 to evaluate $|f_i((b + \sqrt{d})/(2a))|$ for $i = 0, 1, 2$ where $[a, b, c] \in H(d)$ and $d < 0$.

Theorem 6.7 Let $d < 0$ and $K = [a, b, c] \in H(d)$. Set $q_0 = a + b + c$, $q_1 = c$,

$q_2 = a$,

$$\lambda_i = \begin{cases} 1, & \text{if } q_i \equiv 2 \pmod{4}, \\ 1, & \text{if } q_i \equiv 0 \pmod{4}, b \equiv 1 \pmod{2}, \\ \frac{1}{2}, & \text{if } q_i \equiv 0 \pmod{4}, b \equiv 0 \pmod{2}, \\ 2, & \text{if } q_i \equiv 1 \pmod{2}, \end{cases}$$

for $i = 0, 1, 2$,

$$M_0 = \left[2a\lambda_0, \lambda_0(b + 2a), \frac{\lambda_0}{2}(a + b + c) \right] \in H(\lambda_0^2 d),$$

$$M_1 = \left[2a\lambda_1, \lambda_1 b, \frac{\lambda_1}{2}c \right] \in H(\lambda_1^2 d),$$

$$M_2 = \left[\frac{\lambda_2}{2}a, \lambda_2 b, 2\lambda_2 c \right] \in H(\lambda_2^2 d),$$

$$m_i = 2 - 2^{1-v_2(\lambda_i)} = \begin{cases} 0, & \text{if } \lambda_i = 1, \\ 1, & \text{if } \lambda_i = 2, \\ -2, & \text{if } \lambda_i = 1/2, \end{cases}$$

for $i = 0, 1, 2$. Then

$$\left| f_i \left(\frac{b + \sqrt{d}}{2a} \right) \right| = \left(\frac{2}{\lambda_i} \right)^{1/4} e^{E(K, d) - E(M_i, \lambda_i^2 d)} 2^{m_i \frac{1 - (\frac{\lambda_i}{2})}{2 - (\frac{\lambda_i}{2})} 2^{-2-v_2(f)}}$$

for $i = 0, 1, 2$.

Proof: We note that $\Delta(\lambda_i^2 d) = \Delta(d) = \Delta$ and $f(\lambda_i^2 d) = \lambda_i f(d) = \lambda_i f$. Applying Theorem 6.3 to the classes K , M_0 , M_1 and M_2 , we obtain expressions for

$$\left| \eta \left(\frac{b + \sqrt{d}}{2a} \right) \right|, \left| \eta \left(\frac{1 + \frac{b + \sqrt{d}}{2a}}{2} \right) \right|, \left| \eta \left(\frac{1}{2} \frac{b + \sqrt{d}}{2a} \right) \right| \text{ and } \left| \eta \left(2 \frac{b + \sqrt{d}}{2a} \right) \right|.$$

Using these expressions in (6.2), (6.3) and (6.4), we obtain

$$\left| f_i \left(\frac{b + \sqrt{d}}{2a} \right) \right| = \left(\frac{2}{\lambda_i} \right)^{1/4} e^{E(K, d) - E(M_i, \lambda_i^2 d)} \prod_p p^{(\alpha_p(\Delta, \lambda_i f) - \alpha_p(\Delta, f))/4}$$

for $i = 0, 1, 2$.

If p is odd, $\alpha_p(\Delta, \lambda_i f) = \alpha_p(\Delta, f)$ as $v_p(\lambda_i f) = v_p(f)$. Thus

$$\prod_p p^{(\alpha_p(\Delta, \lambda_i f) - \alpha_p(\Delta, f))/4} = 2^{(\alpha_2(\Delta, \lambda_i f) - \alpha_2(\Delta, f))/4} = 2^{m_i \frac{1 - (\frac{\Delta}{4})}{2 - (\frac{\Delta}{4})} 2^{-2 - v_2(f)}}$$

as required. $\square$

Let n be a positive integer. Taking $K = [1, 0, n] \in H(-4n)$ in Theorem 6.7, we obtain expressions for $|f_i(\sqrt{-n})| = f_i(\sqrt{-n})$ for $i = 0, 1, 2$.

Theorem 6.8 *Let n be a positive integer and let $d = -4n$. Let $K = [1, 0, n] \in H(d)$. Set $M_2 = [1, 0, 4n] \in H(4d)$.*

(a) $n \equiv 0 \pmod{4}$. Set

$$\begin{aligned} M_0 &= [4, 4, n+1] \in H(4d), \\ M_1 &= \left[1, 0, \frac{n}{4}\right] \in H\left(\frac{d}{4}\right). \end{aligned}$$

Let $n = 4^\alpha \mu$, where α is a positive integer and $4 \nmid \mu$.

(i) $\mu \equiv 1$ or $2 \pmod{4}$ (so that Δ is even and $v_2(f) = \alpha$). We have

$$\begin{aligned} f_0(\sqrt{-n}) &= 2^{\frac{1}{2^\alpha+3}} e^{E(K, d) - E(M_0, 4d)}, \\ f_1(\sqrt{-n}) &= 2^{\frac{2^\alpha+1}{2^\alpha+2}} e^{E(K, d) - E(M_1, d/4)}, \\ f_2(\sqrt{-n}) &= 2^{\frac{1}{2^\alpha+3}} e^{E(K, d) - E(M_2, 4d)}. \end{aligned}$$

(ii) $\mu \equiv 3 \pmod{4}$ (so that $\Delta \equiv -\mu \pmod{8}$, $v_2(f) = \alpha + 1$).

If $\mu \equiv 3 \pmod{8}$, we have

$$\begin{aligned} f_0(\sqrt{-n}) &= 2^{\frac{1}{3 \cdot 2^\alpha+2}} e^{E(K, d) - E(M_0, 4d)}, \\ f_1(\sqrt{-n}) &= 2^{\frac{3 \cdot 2^\alpha-1}{3 \cdot 2^\alpha+1}} e^{E(K, d) - E(M_1, d/4)}, \\ f_2(\sqrt{-n}) &= 2^{\frac{1}{3 \cdot 2^\alpha+2}} e^{E(K, d) - E(M_2, 4d)}. \end{aligned}$$

If $\mu \equiv 7 \pmod{8}$, we have

$$\begin{aligned} f_0(\sqrt{-n}) &= e^{E(K,d)-E(M_0,4d)}, \\ f_1(\sqrt{-n}) &= \sqrt{2}e^{E(K,d)-E(M_1,d/4)}, \\ f_2(\sqrt{-n}) &= e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

(b) $n \equiv 1 \pmod{4}$ (so that Δ is even, f is odd). Set

$$\begin{aligned} M_0 &= \left[2, 2, \frac{n+1}{2}\right] \in H(d), \\ M_1 &= [4, 0, n] \in H(4d). \end{aligned}$$

Then

$$\begin{aligned} f_0(\sqrt{-n}) &= 2^{1/4}e^{E(K,d)-E(M_0,d)}, \\ f_1(\sqrt{-n}) &= 2^{1/8}e^{E(K,d)-E(M_1,4d)}, \\ f_2(\sqrt{-n}) &= 2^{1/8}e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

(c) $n \equiv 2 \pmod{4}$ (so that Δ is even, f is odd). Set

$$\begin{aligned} M_0 &= [4, 4, n+1] \in H(4d), \\ M_1 &= \left[2, 0, \frac{n}{2}\right] \in H(d). \end{aligned}$$

Then

$$\begin{aligned} f_0(\sqrt{-n}) &= 2^{1/8}e^{E(K,d)-E(M_0,4d)}, \\ f_1(\sqrt{-n}) &= 2^{1/4}e^{E(K,d)-E(M_1,d)}, \\ f_2(\sqrt{-n}) &= 2^{1/8}e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

(d) $n \equiv 3 \pmod{4}$ (so that $v_2(f) = 1$, $\Delta \equiv -n \pmod{8}$). Set

$$\begin{aligned} M_0 &= \left[1, 1, \frac{n+1}{4}\right] \in H\left(\frac{d}{4}\right), \\ M_1 &= [4, 0, n] \in H(4d). \end{aligned}$$

Then for $n \equiv 3 \pmod{8}$, we have

$$\begin{aligned} f_0(\sqrt{-n}) &= 2^{1/3} e^{E(K,d)-E(M_0,d/4)}, \\ f_1(\sqrt{-n}) &= 2^{1/12} e^{E(K,d)-E(M_1,4d)}, \\ f_2(\sqrt{-n}) &= 2^{1/12} e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

For $n \equiv 7 \pmod{8}$, we have

$$\begin{aligned} f_0(\sqrt{-n}) &= \sqrt{2} e^{E(K,d)-E(M_0,d/4)}, \\ f_1(\sqrt{-n}) &= e^{E(K,d)-E(M_1,4d)}, \\ f_2(\sqrt{-n}) &= e^{E(K,d)-E(M_2,4d)}. \end{aligned}$$

Example 6.1 $n = 1$

We take $K = [1, 0, 1] \in H(-4)$, $d = -4$. As $h(-4) = 1$, we have $E(K, d) = E(M_0, d) = 0$. Thus Theorem 6.8 gives $f_0(\sqrt{-1}) = 2^{1/4}$, in agreement with [27], Vol III, p. 721.

Example 6.2 $n = 5$

We take $K = [1, 0, 5] \in H(-20)$, $d = -20$. We have $\Delta = -20$, $f = 1$, $M_0 = [2, 2, 3]$, $H(-20) = \{K, M_0\}$. Thus

$$E(K, d) = \frac{\pi\sqrt{5}}{24} \frac{t_1(-20)}{j(M_0, -20)} l(M_0, -20).$$

We have $\nu = 1$, $h_1 = 2$, $\chi(M_0, M_0) = -1$. Since $H(-20)$ has one class per genus, it follows from genus theory that if $(-20/p) = 1$ then

$$\begin{aligned} K_p = K &\iff \left(\frac{5}{p}\right) = \left(\frac{-4}{p}\right) = 1, \\ K_p = M_0 &\iff \left(\frac{5}{p}\right) = \left(\frac{-4}{p}\right) = -1. \end{aligned}$$

Also we have

$$\frac{\prod_{\left(\frac{-20}{p}\right)=-1} \left(1 - \frac{1}{p^{2s}}\right)}{\prod_{\left(\frac{-20}{p}\right)=-1} \left(1 - \frac{\left(\frac{5}{p}\right)}{p^s}\right) \left(1 - \frac{\left(\frac{-4}{p}\right)}{p^s}\right)} = 1.$$

Thus

$$\begin{aligned} \frac{t_1(-20)}{j(M_0, -20)} &= \lim_{s \rightarrow 1^+} \frac{\prod_{\left(\frac{-20}{p}\right)=1} \left(1 - \frac{1}{p^{2s}}\right)}{\prod_{\left(\frac{-20}{p}\right)=1} \left(1 - \frac{\chi(M_0, K_p)}{p^s}\right)^2} \\ &= \lim_{s \rightarrow 1^+} \frac{\prod_{\left(\frac{-20}{p}\right)=1} \left(1 - \frac{1}{p^{2s}}\right)}{\prod_{\left(\frac{-20}{p}\right)=1} \left(1 - \frac{\left(\frac{5}{p}\right)}{p^s}\right) \left(1 - \frac{\left(\frac{-4}{p}\right)}{p^s}\right)} \\ &= \lim_{s \rightarrow 1^+} \frac{\prod_{p \neq 2, 5} \left(1 - \frac{1}{p^{2s}}\right)}{\prod_{p \neq 2} \left(1 - \frac{\left(\frac{5}{p}\right)}{p^s}\right) \prod_{p \neq 5} \left(1 - \frac{\left(\frac{-4}{p}\right)}{p^s}\right)} \\ &= \lim_{s \rightarrow 1^+} \frac{\left(1 - \frac{\left(\frac{5}{2}\right)}{2^s}\right) \left(1 - \frac{\left(\frac{-4}{5}\right)}{5^s}\right)}{\left(1 - \frac{1}{2^{2s}}\right) \left(1 - \frac{1}{5^{2s}}\right)} \frac{L(s, 5)L(s, -4)}{\zeta(2s)} \\ &= \frac{10}{\pi^2} L(1, 5)L(1, -4) = \frac{\sqrt{5}}{\pi} \log \left(\frac{1 + \sqrt{5}}{2} \right), \end{aligned}$$

as $L(1, -4) = \frac{\pi}{4}$ and $L(1, 5) = \frac{2}{\sqrt{5}} \log\left(\frac{1+\sqrt{5}}{2}\right)$. Also

$$l(M_0, -20) = \prod_{p=2,5} \left(1 + \frac{\chi(M_0, K_p)}{p}\right) = \frac{3}{5}.$$

Thus

$$E(K, d) = \frac{1}{8} \log\left(\frac{1+\sqrt{5}}{2}\right).$$

Also

$$E(M_0, d) = E(M_0, -20) = -E(K, d) = -\frac{1}{8} \log\left(\frac{1+\sqrt{5}}{2}\right).$$

Thus, using Theorem 6.8, we obtain

$$f_0(\sqrt{-5}) = 2^{1/4} e^{(1/4) \log((1+\sqrt{5})/2)} = (1 + \sqrt{5})^{1/4},$$

in agreement with [27], Vol III, p.721.

Example 6.3 $n = 19$

We take $K = [1, 0, 19] \in H(-76)$, $d = -76$. We have $\Delta = -19$, $f = 2$, $M_0 = [1, 1, 5] \in H(-19)$. Since $h(-19) = 1$, we have $E(M_0, -19) = 0$. Also $H(-76) = \{K, A, A^{-1}\}$, where $A = [4, 2, 5]$. We have $\nu = 1$, $h_1 = 3$, $\chi(A, A) = e^{2\pi i/3}$, $\chi(A, A^{-1}) = e^{-2\pi i/3}$, $\chi(A^{-1}, A^{-1}) = e^{2\pi i/3}$. Thus, using Theorem 6.8, we obtain

$$f_0(\sqrt{-19}) = 2^{1/3} e^{E(K, -76)}. \quad (6.13)$$

We have

$$E(K, -76) = \frac{\pi\sqrt{19}}{36} \frac{t_1(-76)}{j(A, -76)} (l(A, -76) + l(A^{-1}, -76)),$$

since $j(A, -76) = j(A^{-1}, -76)$.

For $L = A, A^{-1}$, we have

$$l(L, -76) = \left(1 + \frac{\chi(L, K_{19})}{19}\right) A(L, -76, 2) = \frac{20}{19} A(L, -76, 2).$$

We have

$$A(L, -76, 2) = 1 + \sum_{j=1}^{\infty} \frac{Y_L(2^j)}{2^j} = 1 + \frac{Y_L(4)}{4} = \frac{3}{4},$$

since $Y_L(2^j) = 0$ for $j = 1$ and $j > 2$ by Theorem 5.2 and Lemma 5.12. Thus

$l(L, -76) = 15/19$ for $L = A, A^{-1}$ so that

$$E(K, -76) = \frac{5\pi}{6\sqrt{19}} \frac{t_1(-76)}{j(A, -76)}. \quad (6.14)$$

We have

$$\begin{aligned} j(A, -76) &= \lim_{s \rightarrow 1^+} \prod_{\substack{(\frac{-76}{p})=1 \\ K_p=K}} \left(1 - \frac{1}{p^s}\right)^2 \\ &\quad \times \prod_{\substack{(\frac{-76}{p})=1 \\ K_p \neq K}} \left(1 - \frac{\chi(A, K_p)}{p^s}\right) \left(1 - \frac{\chi(A^{-1}, K_p)}{p^s}\right) \\ &= \lim_{s \rightarrow 1^+} \prod_{\substack{(\frac{-76}{p})=1 \\ K_p=K}} \left(1 - \frac{1}{p^s}\right)^2 \prod_{\substack{(\frac{-76}{p})=1 \\ K_p \neq K}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}}\right). \end{aligned}$$

Applying the main Theorem of [26] to the irreducible polynomial $x^3 - 2x - 2$

we deduce that if $(-76/p) = 1$ then

$$x^3 - 2x - 2 \equiv 0 \pmod{p} \quad \text{is solvable}$$

$$\iff p \quad \text{is represented by } K = [1, 0, 19]$$

$$\iff K_p = K.$$

Hence

$$\begin{aligned}
 j(A, -76) &= \lim_{s \rightarrow 1^+} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \text{ solvable}}} \left(1 - \frac{1}{p^s}\right)^2 \\
 &\times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \text{ not solvable}}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}}\right). \quad (6.15)
 \end{aligned}$$

Let θ be the unique real root of $x^3 - 2x - 2$. Then θ is positive. We set $F = \mathbb{Q}(\theta)$. Next, we link $E(K, -76)$ to the Dedekind zeta function $\zeta_F(s)$ of the number field F . By Theorem 1 of [19], we find that the rational primes decompose into prime ideals in the field F as follows

$$2 = P^3, \quad N(P) = 2,$$

$$19 = PQ^2, \quad N(P) = N(Q) = 19.$$

If $(-76/p) = 1$ and $x^3 - 2x - 2 \equiv 0 \pmod{p}$ is solvable then

$$p = PQR, \quad N(P) = N(Q) = N(R) = p.$$

If $(-76/p) = 1$ and $x^3 - 2x - 2 \equiv 0 \pmod{p}$ is not solvable then

$$p = P, \quad N(P) = p^3.$$

If $(-76/p) = -1$ then

$$p = PQ, \quad N(P) = p, \quad N(Q) = p^2.$$

Thus

$$\zeta_F(s) = \prod_P (1 - (N(P))^{-s})^{-1}$$

$$\begin{aligned}
&= (1 - 2^{-s})^{-1} (1 - 19^{-s})^{-2} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \\ \text{solvable}}} (1 - p^{-s})^{-3} \\
&\quad \times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \\ \text{not solvable}}} (1 - p^{-3s})^{-1} \prod_{\left(\frac{-76}{p}\right)=-1} (1 - p^{-s})^{-1} (1 - p^{-2s})^{-1} \\
&= \zeta(s) (1 - 19^{-s})^{-1} \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \\ \text{solvable}}} (1 - p^{-s})^{-2} \\
&\quad \times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \\ \text{not solvable}}} (1 + p^{-s} + p^{-2s})^{-1} \prod_{\left(\frac{-76}{p}\right)=-1} (1 - p^{-2s})^{-1} \\
&= \zeta(s) \zeta(2s) (1 - 2^{-2s}) (1 - 19^{-s})^{-1} (1 - 19^{-2s}) \\
&\quad \prod_{\left(\frac{-76}{p}\right)=1} (1 - p^{-2s}) \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \\ \text{solvable}}} (1 - p^{-s})^{-2} \\
&\quad \times \prod_{\substack{\left(\frac{-76}{p}\right)=1 \\ x^3-2x-2 \equiv 0 \pmod{p} \\ \text{not solvable}}} (1 + p^{-s} + p^{-2s})^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{s \rightarrow 1^+} (s-1) \zeta_F(s) &= \frac{\pi^2}{6} \left(1 - \frac{1}{4}\right) \left(1 + \frac{1}{19}\right) \frac{t_1(-76)}{j(A, -76)} \\
&= \frac{3\pi}{\sqrt{19}} E(K, -76),
\end{aligned} \tag{6.16}$$

by (6.15) and (6.14). It is well known that (see [21], p. 326)

$$\lim_{s \rightarrow 1^+} (s-1) \zeta_F(s) = \frac{2^{s+t} \pi^t R_F h_F}{W_F |d_F|^{1/2}},$$

where s = number of real embeddings of F ,

$2t$ = number of imaginary embeddings of F ,

h_F = class number of F ,

W_F = number of roots of unity in F ,

d_F = discriminant of F ,

R_F = regulator of F .

We have $s = 1$, $t = 1$, $h_F = 1$, $W_F = 2$, $d_F = -76$ and $R_F = \log(1 + \theta)$ since $1 + \theta$ is the fundamental unit of F (see [7], p. 519). Hence

$$\lim_{s \rightarrow 1^+} (s - 1)\zeta_F(s) = \frac{\pi}{\sqrt{19}} \log(1 + \theta). \quad (6.17)$$

By (6.16) and (6.17), we obtain

$$E(K, -76) = \frac{1}{3} \log(1 + \theta) = \log \left(\frac{\theta}{2^{1/3}} \right),$$

since $\theta^3 = 2\theta + 2$. Thus, using (6.13), we obtain

$$f_0(\sqrt{-19}) = 2^{1/3} e^{E(K, -76)} = \theta,$$

in agreement with [27], Vol III, p. 721.

Chapter 7

An analogue for positive discriminants of the Chowla-Selberg formula for classes

7.1 Introduction

Theorem 3.3 gives the evaluation of a class of definite integrals. This theorem is an analogue for positive discriminants of the Chowla-Selberg formula for genera. In this chapter, we use Theorem 5.14 together with Theorem 2.1 to derive an extension of Theorem 3.3 (see Theorem 7.3) with the genera replaced by classes. Our result is an analogue for positive discriminants of the Chowla-Selberg formula for classes given in Theorem 6.3. Then we use Theorem 7.3 to derive extensions of Theorems 4.2, 4.3 and 4.4 (see Theorems 7.4 and 7.5).

7.2 Behaviour of $\sum_{n=1}^{\infty} \frac{N(n,d)}{n^s}$ and $\sum_{n=1}^{\infty} \frac{R_K(n,d)}{n^s}$ as $s \rightarrow 1^+$ for $d > 0$

We begin by considering $\sum_{n=1}^{\infty} \frac{N(n,d)}{n^s}$.

Theorem 7.1 *Let $d > 0$. Then*

$$\sum_{n=1}^{\infty} \frac{N(n,d)}{n^s} = \frac{h(d) \log(\epsilon(d))}{\sqrt{d}} \frac{1}{s-1} + 2^{t(d)} B(d) + O(s-1),$$

as $s \rightarrow 1^+$, where $B(d)$ is the same as in Theorem 3.2.

Proof: By Theorem 3.2, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{N(n,d)}{n^s} \\ &= \sum_{G \in G(d)} \sum_{n=1}^{\infty} \frac{R_G(n,d)}{n^s} \\ &= \sum_{G \in G(d)} \left(\frac{h(d) \log(\epsilon(d))}{2^{t(d)} \sqrt{d}} \frac{1}{s-1} + B(d) + \beta(d, G) + O(s-1) \right) \\ &= \frac{h(d) \log(\epsilon(d))}{\sqrt{d}} \frac{1}{s-1} + 2^{t(d)} B(d) + \sum_{G \in G(d)} \beta(d, G) + O(s-1). \end{aligned}$$

The result follows since

$$\sum_{G \in G(d)} \gamma_{d_1}(G) = 0,$$

for $d_1 \notin \{1, \Delta\}$. □

We now turn to $\sum_{n=1}^{\infty} \frac{R_K(n,d)}{n^s}$.

Theorem 7.2 *Let $d > 0$ and let $K \in H(d)$. Then*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{R_K(n,d)}{n^s} &= \frac{\log(\epsilon(d))}{\sqrt{d}} \frac{1}{s-1} + \frac{2^{t(d)} B(d)}{h(d)} \\ &\quad + \frac{\pi^2}{6h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d) + o(1), \end{aligned}$$

as $s \rightarrow 1^+$.

Proof: The result follows on using Theorems 5.14, 5.13 and 7.1. $\square$

7.3 Evaluation of a class of definite integrals: analogue of the Chowla-Selberg formula for classes

We are now ready to prove our main result which extends Theorem 3.3 and is an analogue for positive discriminants of Theorem 6.3.

Theorem 7.3 *Let $d > 0$, $K_1 = [a_1, b_1, c_1] \in H(d)$, $K_2 = [a_2, b_2, c_2] \in H(d)$ with $a_1 > 0$, $a_2 > 0$. Then*

$$\begin{aligned} & \int_0^1 (E(a_1, b_1, c_1, t) - E(a_2, b_2, c_2, t)) dt \\ &= \frac{\pi^2 \sqrt{d}}{6} \left(\frac{1}{a_1} - \frac{1}{a_2} \right) + \log \epsilon(d) \log \frac{a_1}{a_2} - (J(a_1, b_1, c_1) - J(a_2, b_2, c_2)) \\ & - \frac{\pi^2 \sqrt{d}}{6h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \frac{t_1(d)l(L, d)}{j(L, d)} (\chi(L, K_1)^{-1} - \chi(L, K_2)^{-1}). \end{aligned}$$

Proof: Let $Q_i = (a_i, b_i, c_i)$ for $i = 1, 2$. By Theorem 7.2, (3.1) and Theorem 2.1, we have for $i = 1, 2$,

$$C_{Q_i} = \frac{2^{t(d)} B(d)}{h(d)} + \frac{\pi^2}{6h(d)} \sum_{\substack{L \in H(d) \\ L \neq I}} \chi(L, K_i)^{-1} \frac{t_1(d)}{j(L, d)} l(L, d).$$

Subtracting the result for $i = 2$ from the result for $i = 1$ and rearranging using (3.17) gives the required result. $\square$

In the same way that Theorem 4.2 was deduced from Theorem 3.3, we deduce the following result from Theorem 7.3.

Theorem 7.4 *Let $D = 4(4m^2 - 1)$ for some positive integer m . Let $K = [2, 2, (4 - D)/8] \in H(D)$. Then*

$$\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt = \frac{\pi^2 \sqrt{D}}{12h(D)} \sum_{\substack{L \in H(D) \\ L \neq I}} \frac{t_1(D)l(L, D)}{j(L, D)} (1 - \chi(L, K)^{-1}) \\ - \frac{\pi^2 \sqrt{D}}{24} + \frac{\log 2 \log 2\epsilon}{2},$$

where $\epsilon = \epsilon(D) = 2m + \sqrt{4m^2 - 1}$.

We can use Theorem 7.4 to re-evaluate the integrals following Theorem 4.2. We illustrate this with the example $D = 12$. We have $H(D) = \{I, K\}$. It is easily checked that $\chi(K, K) = -1$. Thus

$$\sum_{\substack{L \in H(D) \\ L \neq I}} \frac{t_1(D)l(L, D)}{j(L, D)} (1 - \chi(L, K)^{-1}) = 2 \frac{t_1(D)l(K, D)}{j(K, D)}.$$

We have $K_2 = K$, $K_3 = K$. Thus

$$l(K, D) = \prod_{p=2,3} \left(1 + \frac{\chi(K, K_p)}{p} \right) = \frac{1}{3}.$$

$H(12)$ has one class per genus. Hence, if p is a prime with $(\frac{12}{p}) = 1$, we have (by genus theory)

$$K_p = I \iff \left(\frac{-4}{p} \right) = \left(\frac{-3}{p} \right) = 1, \\ K_p = K \iff \left(\frac{-4}{p} \right) = \left(\frac{-3}{p} \right) = -1.$$

Thus, if $(\frac{12}{p}) = 1$, we have

$$\left(1 - \frac{\chi(K, K_p)}{p^s} \right)^2 = \left(1 - \frac{\left(\frac{-4}{p} \right)}{p^s} \right) \left(1 - \frac{\left(\frac{-3}{p} \right)}{p^s} \right).$$

Also

$$\prod_{\left(\frac{12}{p}\right)=-1} \frac{\left(1 - \frac{1}{p^{2s}}\right)}{\left(1 - \frac{\left(\frac{-4}{p}\right)}{p^s}\right) \left(1 - \frac{\left(\frac{-3}{p}\right)}{p^s}\right)} = 1.$$

Thus

$$\begin{aligned} \frac{t_1(D)}{j(K, D)} &= \lim_{s \rightarrow 1^+} \prod_{\left(\frac{12}{p}\right)=1} \frac{\left(1 - \frac{1}{p^{2s}}\right)}{\left(1 - \frac{\chi(K, K_p)}{p^s}\right)^2} \\ &= \lim_{s \rightarrow 1^+} \frac{\left(1 - \frac{\left(\frac{-4}{3}\right)}{3^s}\right) \left(1 - \frac{\left(\frac{-3}{2}\right)}{2^s}\right)}{\left(1 - \frac{1}{2^{2s}}\right) \left(1 - \frac{1}{3^{2s}}\right)} \prod_p \frac{\left(1 - \frac{1}{p^{2s}}\right)}{\left(1 - \frac{\left(\frac{-4}{p}\right)}{p^s}\right) \left(1 - \frac{\left(\frac{-3}{p}\right)}{p^s}\right)} \\ &= \frac{3}{\zeta(2)} L(1, -4) L(1, -3) \\ &= \frac{\sqrt{3}}{2}, \end{aligned}$$

as $L(1, -4) = \frac{\pi}{4}$ and $L(1, -3) = \frac{\pi}{3\sqrt{3}}$. Theorem 7.4 now gives

$$\begin{aligned} \int_0^1 \frac{\log(1 + t^{2+\sqrt{3}})}{1+t} dt &= -\frac{\pi^2 \sqrt{12}}{24} + \frac{\log 2 \log 2(2 + \sqrt{3})}{2} \\ &\quad + \frac{\pi^2 \sqrt{12}}{24} \times 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{3} \\ &= \frac{\pi^2}{12} (1 - \sqrt{3}) + \log 2 \log(1 + \sqrt{3}), \end{aligned}$$

in agreement with the result obtained in Section 4.5.

Next we consider the case $D = 396$ (corresponding to $m = 5$). Since $H(396)$ has two classes per genus, Theorem 4.2 is not applicable. Theorem 7.4 gives

$$\begin{aligned} \int_0^1 \frac{\log(1 + t^{10+\sqrt{99}})}{1+t} dt &= -\frac{\pi^2 \sqrt{11}}{4} + \log 2 \log(3 + \sqrt{11}) \\ &\quad + \frac{\pi^2 \sqrt{11}}{16} \sum_{\substack{L \in H(396) \\ L \neq I}} \frac{t_1(396)}{j(L, 396)} l(L, 396) (1 - \chi(L, K)^{-1}), \end{aligned}$$

where $K = [2, 2, -49]$. At the present time we are unable to simplify the expressions $j(L, 396)$ ($L \in H(396)$) occuring above.

In the same way that Theorems 4.3 and 4.4 were derived from Theorem 3.3, we obtain the following results from Theorem 7.3

Theorem 7.5 *Let $D = 16(l^2 + l)$ for some positive integer l . Let $K = [4, 4, (16 - D)/16]$. Then*

$$\begin{aligned} & \int_0^1 \frac{3t \log(1 + t^{2\epsilon}) + 2(-1)^l \tan^{-1}(t^\epsilon)}{1 + t^2} dt \\ &= \frac{\pi^2 \sqrt{D}}{12h(D)} \sum_{\substack{L \in H(D) \\ L \neq I}} \frac{t_1(D)l(L, D)}{j(L, D)} (1 - \chi(L, K)^{-1}) + \log 2 \log(2^{3/4}\epsilon) \\ & \quad + \frac{(-1)^l \pi^2}{16} - \frac{\pi^2 \sqrt{D}}{16}, \end{aligned}$$

where $\epsilon = \epsilon(D) = \epsilon(D/4) = 2l + 1 + \sqrt{4l^2 + 4l}$.

If $l \equiv 1$ or $2 \pmod{4}$ (so that $D/4 \equiv 8 \pmod{16}$), we have

$$\begin{aligned} & \int_0^1 \frac{t \log(1 + t^{2\epsilon}) + 2(-1)^{l+1} \tan^{-1}(t^\epsilon)}{1 + t^2} dt \\ &= \frac{\pi^2 \sqrt{D}}{24h(D/4)} \sum_{\substack{L \in H(D/4) \\ L \neq I}} \frac{t_1(D/4)l(L, D/4)}{j(L, D/4)} (1 - \chi(L, \hat{K})^{-1}) + \frac{\log 2 \log(\sqrt{2}\epsilon)}{2} \\ & \quad + (-1)^{l+1} \frac{\pi^2}{16} - \frac{\pi^2 \sqrt{D}}{48}, \end{aligned}$$

where $\hat{K} = [2, 0, -D/32] \in H(D/4)$.

Conclusion

In this thesis we extended the Chowla-Selberg formula for classes for negative fundamental discriminants to arbitrary negative discriminants. Using this extension we obtained complicated expressions for the Weber functions at $\sqrt{-n}$ for an arbitrary positive integer n . For certain special values of n we used these expressions to reprove some known evaluations of $f(\sqrt{-n})$. Future work will involve a systematic study of our expressions to see if they lead to further new results. We also obtained analogues for positive discriminants of the Chowla-Selberg formula for genera and the Chowla-Selberg formula for classes. Using the analogue of the Chowla-Selberg formula for genera we explicitly evaluated the definite integrals $\int_0^1 \frac{\log(1+t^\epsilon)}{1+t} dt$ and $\int_0^1 \frac{\tan^{-1} t^\epsilon}{1+t^2} dt$ for several different quadratic units ϵ . Most of these evaluations are new. Future work will involve studying the analogue of the Chowla-Selberg formula for classes to see if it leads to further new integral evaluations.

Notation

d	p. 4
$\Delta = \Delta(d)$	p. 4
$f = f(d)$	p. 4
$(\frac{d}{m})$	p. 5
(a, b, c)	p. 5
$\sim$	p. 6
$[a, b, c]$	p. 6
$K_1 K_2$ (composition of K_1 and K_2)	p. 6
$H(d)$	p. 6
$h(d)$	p. 6
$I(d)$	p. 6
$G(d)$	p. 7
$x_0 = x_0(d), y_0 = y_0(d)$	p. 7
$\epsilon = \epsilon(d)$	p. 7
$R_K(n, d)$	p. 8
$R_G(n, d)$	p. 8
$t(d)$	p. 8
$P(d)$	p. 9

$F(d)$	p. 9
$\gamma_{p^*}(G)$	p. 11
$\gamma_{d_1}(G)$	p. 12
$v_p(n)$	p. 12
G_p	p. 12
G_m	p. 15
$\text{Null}(n, d)$	p. 16
$M(n, d)$	p. 17
$U(n, d)$	p. 17
$Q(n, d)$	p. 17
$S(n, d_1, d/d_1)$	p. 18
ϕ_p	p. 19
θ_p	p. 23
$j(d, M)$	p. 28
$N(n, d)$	p. 29
$w(d)$	p. 29
$Z_Q(s)$	p. 33
λ	p. 35
λ_1	p. 35
t_1, t_2	p. 37
λ_0	p. 37
γ	p. 38
C_Q	pp. 39, 50
$K(d)$	p. 39

$\Gamma(x)$	p. 41
$\psi(x)$	p. 41
ϵ'	p. 42
g	p. 42
α	p. 42
r_y	p. 43
θ	p. 45
W_Q	pp. 46, 47, 49, 50
$V(d)$	p. 47
α'	p. 47
$F(\alpha, l, t, g)$	pp. 47, 49
$L(s, D)$	p. 52
$\beta(d, G)$	pp. 54, 55
$A(s, d)$	p. 57
$B(d)$ ($d > 0$)	p. 57
$E(a, b, c, t)$	p. 58
$J(a, b, c)$	p. 58
D	p. 60
$k(n, d)$	p. 75
$H(n, d)$	p. 76
$H_K(n)$	p. 85
K_p	p. 87
$h_1, \dots, h_\nu, A_1, \dots, A_\nu$	p. 92
$[K, L]$	p. 92

$\chi(K, L)$	p. 92
$Y_K(n)$	p. 93
$j(K, d)$	p. 96
$t_1(d)$	p. 98
$A(K, d, p)$	p. 98
$l(K, d)$	p. 98
$\eta(z)$	p. 102
$f(z), f_0(z), f_1(z), f_2(z)$	p. 102
$B(d) \ (d < 0)$	p. 104
$\alpha_p(\Delta, f)$	p. 104
$E(K, d)$	p. 106
λ_i	p. 120
M_0, M_1, M_2	p. 120
m_i	p. 120

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