Spherical Space Bessel-Legendre-Fourier Mode Solver for Electromagnetic Fields.

by

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A Thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

in

Ottawa-Carleton Institute for Electrical Engineering

Carleton University

Ottawa, Ontario, Canada

August, 2017

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Abstract

Controlling and confining light in three dimensions is of significant interest as it represents devices in their entirety with minimal approximations. Localized modes in resonator structures are commonly modelled using the supercell plane wave expansion method, finite element method, or the finite difference time domain. However, these standard techniques are expensive in terms of computational resources when applied to three-dimensional structures. The technique presented in this thesis is a new method for solving Maxwell's vector wave equations for localized modes in three-dimensional spherically resonator structures as well as its application to sensor configurations. The technique requires minimal implementation, provides normalized results, works for finite size and is computationally efficient. The method is such that the structures under test can be arbitrary shape, isotropic, anisotropic, lossless, or lossy.

For the structures examined, a modified basis set composed of spherical Bessel, Legendre and Fourier functions (BLF) are used to expand the electric, magnetic, and inverse relative permittivity. These expansions allow for Maxwell's wave equations to be cast as an eigenvalue problem from which the steady state localized modes (not propagating) can be determined from the eigen-frequencies and eigenvectors. This work applies to a number of spherically symmetric structures. Selections of structures whose resonator properties are reported in the literature are used to compare and verify the accuracy of the use of the BLF functions as an expansion basis.

Acknowledgement

I would like to begin by thanking Prof. Gauthier for the great support he has provided throughout my post-graduate studies. I appreciate everything he has done for me during my stay here as international student, I feel that I have found a good friend as well as a supervisor.

I would like to thank the Ministry of Higher Education in Saudi Arabia and Taibah University for the financial support.

To my family and my wife (Eman), your support has been limitless. There are no words to sufficiently thank you.

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1 Chapter: Introduction

An optical resonator is a powerful optical component in photonic device design. Geometries providing this functionality are vital in many applications. The optical resonator is one of the most commonly used for the generation or control of electromagnetic radiation. The resonator is a key component in lasers, waveguides, sensors, and filters devices [1-3]. They are photonic structures that allow a portion of the field to circulate in a closed path within the device such as a cavity. The resonant frequencies, frequencies that correspond to constructive interference, depend on features such as the size, geometry of the cavity, or the materials of the device. Optical resonators can be designed using two plane mirrors in air [4], macro- and micro-optic ring [3], disk [5], cylindrical structures [6,7], photonic crystal cavities [8-10] and spherical structures [11].

Spherical resonators have gained interest as they provide an environment for controlling and confining light in three dimensions and form the backdrop for a number of more advanced resonator applications [9, 12-17]. The spherical resonators will be the focus of this work.

Current computation techniques for calculating the resonator properties in the spherical structures are highly dependent on the physical size of the device features relative to the wavelength of the electromagnetic fields involved. For relatively large lossless objects a geometrical optics approach can lead to important insights on the modal properties using straight forward equations [15]. When the sphere's size, or internal structures, is comparable to the wavelength the full vector wave analysis is required for accurate results. The full vector wave approach is needed to model the diffraction effect which is vital to optical characteristics of these devices. In the specific case of a dielectric sphere, exact solutions to Maxwell's wave equation results in a characteristic equation whose roots provide access to the resonator wavelengths and field profiles [18, 19]. For spherical structures, consisting of single or multiple layers, the

scattering matrix method has been used to obtain the resonator properties [20]. When the spherical resonator includes material variations with respect to the angular coordinate, numerical and approximate methods are frequently required in determining the optical properties of the resonator. There are 3 dominant methods encountered throughout the literature; the plane-wave-method (PWEM) [21, 22], the finite-difference-time-domain method (FDTD) [23, 24] and the finite-element-method (FEM) [25]. These techniques have advantages and disadvantages, and the choice of which is preferred depends on the applications. Recently, an efficient numerical mode solver for cylindrically symmetric resonator configurations was developed by casting Maxwell's wave equation into an eigenvalue problem using the Fourier Bessel basis to expand the fields and inverse permittivity [26, 27]. That technique is reconfigured here such that it is applicable to spherical objects of arbitrary internal structure. When solved the eigenvalue problem yields resonator state frequencies or wavelengths (eigenvalues) and field profiles (reconstructed using the eigenvectors) [28].

The basis selected for this work is the Bessel-Legendre-Fourier (BLF) basis set where spherical Bessel functions are used in the radial direction (r), Legendre functions are used for the polar angle (θ) and the complex exponential functions (Fourier) are used for the azimuthal angle (φ). The BLF technique has been verified, as the results of computation matches those obtained using other numerical techniques.

1.1 Motivation

Current numerical techniques require large computer memory and long computation time when used to determine localized modes of 3-D structures. The BLF solver is a technique designed to provide information on the localized modes (frequency, wavelength, mode profile) for structures describable in spherical coordinate. This approach makes no approximations regarding the

uniformity of the dielectric, internal features or loss. The BLF technique solves the full vector electromagnetic wave equation. In addition, BLF technique can be optimised to target a specific mode order and enhance the technique further. In addition to its applications to spherical resonators, such as lasers, sensors, and filters devices, a number of naturally occurring objects and biological specimens can be approximated using spherically symmetric shapes; liquid droplet [29, 30], air bubble [31], viruses, biological cells and bacteria [32]. The potential exists to extend the technique to more complicated structures and in other areas of science such as the Schrödinger and acoustic wave equations.

1.2 Contributions

The significant contribution presented here is the use of modified spherical basis functions to develop a numerical technique (BLF) to study localized electromagnetic waves in three dimensions systems. The modified spherical basis functions greatly simplify the computer implementation. The BLF technique and its applications has been published by the author (as principle author) in a peer review journal paper (chapter 3 and 4) [28] and three conference contributions, 2 oral presentations (main theory of the technique and segmentations) [33, 34] and a poster presentation (slot channel sensors) [35]. At the time of writing this thesis two papers are in preparation, one about the new formulation in chapter 5 and the other one about application of the segmentations approach. All calculations were performed using in-house programs written in Matlab® by the author. In addition, work for this thesis contributed, but not included in the thesis, to two published journal papers (as co- author), which about the cylindrical solver FFB and slot channel design using the FFB technique [27, 36] and one conference presentation [37]. As a co-author the contributions were to review the equations used and citation needed.

1.3 Thesis organization

Chapter 2 is a literature review of commonly used numerical methods for obtaining the steady state resonator modes for the spherically symmetric structures selected to be under study. Chapter 3 provides the theoretical development of the BLF mode solver. Chapter 4 provides the numerical results obtained using the BLF technique and contrasts these results relative to other techniques. In chapter 5, a simpler theoretical approach to apply the numerical technique and to model anisotropic structure is provided. In chapter 6, the modified numerical technique is applied to sensing configurations. The thesis then concludes with final discussions and suggestions for future work.

2 Chapter: Literature review of calculation methods

While analytical solutions in closed form are appropriate for a very restricted number of cases the ability to solve Maxwell's equations is critical to the study of electromagnetic waves in complex structures. The advances in computing power have changed our ability to solve Maxwell's equations in a profound way. This chapter will explore the numerical and computational techniques that have allowed researchers in the field of electromagnetics to analyze and solve many interesting problems including phenomena such as scattering, guiding, localization, and propagation. These include the Plane Wave Expansion Method (PWEM), the Finite Difference Time Domain technique (FDTD) and Finite Element Method (FEM) [21-25]. FDTD and FEM are general methods that directly discretize and solve Maxwell's equations and can be applied to a wide range of dielectric structures. However, they require significant amounts of computational resources in terms of memory and time. PWEM is primarily used to calculate the band structure of propagating wave in photonic crystals (PhCs). It has symmetry restrictions and is primarily limited to structures that display a translationally symmetric lattice. For resonance calculations of PhCs containing defects, a supercell PWEM method that uses larger unit cell is needed. The supercell PWEM requires increased computational resources.

2.1 Basic methods for resonance calculations

Maxwell's equations in differential form are:

$$\nabla \cdot \vec{B}(r,t) = 0 \quad (2.1)$$

$$\nabla \cdot \vec{D}(r,t) = \rho \quad (2.2)$$

$$\vec{\nabla} \times \vec{E}(r,t) = - \frac{\partial \vec{B}(r,t)}{\partial t} \quad (2.3)$$

$$\vec{\nabla} \times \vec{H}(r,t) = \frac{\partial \vec{D}(r,t)}{\partial t} + \vec{J} \quad (2.4)$$

Where ε is the permittivity and μ is the permeability of the medium and the relations $\vec{D} = \varepsilon \vec{E}$ and $\vec{B} = \mu \vec{H}$ are used. Maxwell's equations for linear, lossless, source free ($\rho = 0, \vec{J} = 0$) and non-magnetic ($\mu = \mu_o$) materials are as follows:

$$\nabla \cdot \vec{H}(r,t) = 0 \quad (2.5)$$

$$\nabla \cdot \varepsilon(r) \vec{E}(r,t) = 0 \quad (2.6)$$

$$\vec{\nabla} \times \vec{E}(r,t) + \mu_o \frac{\partial \vec{H}(r,t)}{\partial t} = 0 \quad (2.7)$$

$$\vec{\nabla} \times \vec{H}(r,t) - \varepsilon(r) \frac{\partial \vec{E}(r,t)}{\partial t} = 0 \quad (2.8)$$

Maxwell's equations are commonly solved in Cartesian, cylindrical, or spherical coordinate systems. The symmetry properties of the structure under test usually dictate the choice of the coordinate system to utilize.

The calculations of the resonance properties in three dimensional systems are of considerable interest, as they are most suitable when studying real world resonators structures such as spherical structures [8, 9]. For a spherical cavity in spherical coordinates the electromagnetic wave can be classified as spherical TE and TM polarized modes [38, 39]. For the spherical TE modes, the field components' expressions are:

$$E_r = 0, E_\theta = -\frac{1}{\varepsilon} \frac{1}{r \sin \theta} \frac{\partial F_r}{\partial \varphi}, E_\varphi = \frac{1}{\varepsilon} \frac{1}{r} \frac{\partial F_r}{\partial \theta}, \quad (2.9)$$

$$H_r = \frac{1}{j\omega\mu\epsilon} \left(\frac{\partial^2}{\partial r^2} + k^2 \right) F_r, H_\theta = \frac{1}{j\omega\mu\epsilon} \frac{1}{r} \frac{\partial^2 F_r}{\partial r \partial \theta}, H_\phi = \frac{1}{j\omega\mu\epsilon} \frac{1}{r \sin \theta} \frac{\partial^2 F_r}{\partial r \partial \phi}, \quad (2.10)$$

where ω is the angular frequency, k is the wavenumber, and F_r is the solution that satisfies the scalar Helmholtz equation in spherical coordinates. The spherical TM the field components' expressions are:

$$H_r = 0, H_\theta = \frac{1}{\mu} \frac{1}{r \sin \theta} \frac{\partial F_r}{\partial \phi}, H_\phi = -\frac{1}{\mu} \frac{1}{r} \frac{\partial F_r}{\partial \theta}, \quad (2.11)$$

$$E_r = \frac{1}{j\omega\mu\epsilon} \left(\frac{\partial^2}{\partial r^2} + k^2 \right) F_r, E_\theta = \frac{1}{j\omega\mu\epsilon} \frac{1}{r} \frac{\partial^2 F_r}{\partial r \partial \theta}, E_\phi = \frac{1}{j\omega\mu\epsilon} \frac{1}{r \sin \theta} \frac{\partial^2 F_r}{\partial r \partial \phi}, \quad (2.12)$$

The function F_r has the general form:

$$F_r = j_n(kr) P_n^q(\cos \theta) e^{iq\phi}. \quad (2.13)$$

Where $j_n(kr)$ is the spherical Bessel function, $P_n^q(\cos \theta)$ is the Legendre functions and $e^{iq\phi}$ is the Fourier function. These expressions can be used to solve for resonance mode frequencies and profiles for a homogenous sphere bounded by applying zero boundary condition for the tangential electric components at the sphere surface (perfect electric conductor or PEC). For the TE polarization, the E_r field's component is zero, while the E_θ and E_ϕ components are:

$$E_\theta = -iq \frac{1}{\epsilon} \frac{1}{r \sin \theta} j_n(kr) P_n^q(\cos \theta) e^{iq\phi} \quad (2.14)$$

$$E_\phi = \frac{1}{\epsilon} \frac{1}{r} j_n(kr) \frac{\partial P_n^q(\cos \theta)}{\partial \theta} e^{iq\phi} \quad (2.15)$$

For monopole modes ($q=0$, $E_\theta = 0, E_\phi \neq 0$) the TE equations can be reduced to (2.15) and a characteristic equation can be obtained that can be solved for the resonant TE mode polarization. Applying the zero-boundary condition at the sphere radius R , $E_\phi|_{r=R} = 0$, results in a characteristic equation that can be solved for resonance scaled wavenumbers of the localized states. The characteristic equation is written as:

$$j_n(kR) = 0. \quad (2.16)$$

The results for the solution of (2.16) are shown in table 1.1. For each n value (Bessel order) there will be a number of zeros (wavenumber scaled by radius) solutions. The subscript N on R_N and θ_N are used to indicate the number of peaks in the radial and polar directions.

Table 1.1: Scaled resonance wavenumber in a cavity bounded by a PEC for the TE polarization, obtained from references [38, 39].

Radial Order Polar Order	R_1	R_2	R_3	R_4
θ_1	4.4934	7.7252	10.9041	14.0661
θ_2	5.7634	9.0950	12.3229	15.5146
θ_3	6.9879	10.4171	13.6980	16.9236
θ_4	8.1825	11.7049	15.03966	18.3012

A similar procedure is applied to the spherical TM polarization providing the results shown in table 1.2. This solution method is valid only for ideal dielectric sphere bounded by the PEC.

The solutions for the resonant modes for a dielectric sphere surrounded by air or immersed in a dielectric material will require applying boundary conditions for the tangential components of $\vec{E}$ or $\vec{H}$ and normal components of $\vec{D}$ and $\vec{B}$ at the sphere surface [18, 19].

Table 1.2: Scaled resonance wavenumber in a cavity bounded by PEC obtained from references [38, 39], for the TM polarization.

Radial Order Polar Order	R_1	R_2	R_3	R_4
θ_1	2.7437	6.1167	9.3166	12.4859
θ_2	3.8702	7.4430	10.7130	13.9205
θ_3	4.9734	8.7217	12.0635	15.3135
θ_4	6.0619	9.9675	13.3801	16.6741

For resonance modes, the field is expected to be oscillating inside the sphere and decaying outside the dielectric sphere. Therefore, spherical Bessel functions of the first kind, j_n , are used to represent the field inside the dielectric sphere and spherical Bessel functions of the third kind, h_n , are used to represent the field outside the sphere:

$$\psi(r, \theta, \varphi) = \begin{cases} A j_n(k_d r) P_n^q(\cos \theta) e^{iq\varphi} & r < a \\ B h_n(k_o r) P_n^q(\cos \theta) e^{iq\varphi} & r > a \end{cases}, \quad (2.17)$$

$$k_d = k_o \cdot \sqrt{\epsilon\mu} \quad . \quad (2.18)$$

Applying the continuity condition of $\psi(r, \theta, \varphi)$ $\nabla \psi(r, \theta, \varphi)$ at the boundary $r=a$ for the tangential components gives the characteristic equation for spherical TE mode as:

$$\frac{j_{n-1}(ka)}{j_n(ka)} = \frac{1}{\sqrt{\epsilon}} \frac{H_{n-1}(k_x a)}{H_n(k_x a)} \quad (2.19)$$

A similar procedure is used for calculating the spherical TM resonance modes [18, 19]. The resonance mode wavenumber is shown in tables 1.3 and 1.4.

Table 1.3: Scaled ka resonance wavenumber in dielectric sphere obtained from references [18, 19], for the TE polarization.

Radial Order Polar Order	R_1	R_2
θ_1	1.02	2.06
θ_2	1.46	2.53
θ_3	1.90	2.97
θ_4	2.32	3.44
θ_5	2.70	3.86

Table 1.3: Scaled ka resonance wavenumber in dielectric sphere obtained from references [18, 19], for the TM polarization.

Radial Order Polar Order	R_1	R_2
θ_1	1.46	2.58
θ_2	1.77	3.03
θ_3	2.24	3.38
θ_4	2.68	3.74
θ_5	3.08	4.17

Note that this method is only applicable for obtaining resonance states in a uniform dielectric sphere. For complex structures of arbitrary configurations such as three dimensions PhCs with defects to localize light in the structures, enhanced numerical solvers are required.

2.2 Advanced numerical techniques

Numerical methods can be classified as either time or frequency domain methods based on the domain in which they are used to solve Maxwell's equations. One of the most prolific time domain methods is FDTD [24]. It is based on the discretization of the problem in space and time making the technique suitably applicable for many different structures as it makes no assumptions regard the simulation space. FDTD includes the electric and magnetic fields across the simulation space at various time steps making the calculations computationally expensive. Frequency domain

methods are applied using the time independent Maxwell's equations and can be more efficient. FEM and PWEM are two well-known examples of this class of method [21, 25]. FEM requires meshing the problem in space and then applying approximate solution for each element to build a matrix and then solve for the appropriate boundary conditions. FEM does require large computer memory for 3D analysis [40]. PWEM is based on the expansion of the fields and the structure using basis functions over a unit cell that is assumed to repeat translationally through space. PWEM is mainly used to study propagated fields in periodic structures. Therefore, the PWEM time and memory requirements are increased when used for studying localized modes in structures of finite size or disordered configuration because this requires the use of a supercell [22]. The computational time and memory concerns of the numerical techniques are increased when applied to 3-D problems. When selecting the numerical technique to use it is critical to consider the structure being investigated, the properties to be determined, and the computing resources available.

2.2.1 Finite difference time domain (FDTD)

Implementation of the FDTD technique requires discretization in space and time of Maxwell's curl equation and the creation of update equations that include the electric and magnetic fields coupled in time and space. The update equations express the unknown future fields in term of known past and present fields.

Theoretical development of the FDTD equations for a simple one-dimension problem will be examined. Faraday's and Ampere's law for a system that has variation only in the x direction and electric field polarized in the z direction is written as:

$$\mu \frac{\partial H_y(x,t)}{\partial t} = -\frac{\partial E_z(x,t)}{\partial x}, \quad (2.20)$$

$$\varepsilon \frac{\partial E_z(x,t)}{\partial t} = \frac{\partial H_y(x,t)}{\partial x}. \quad (2.21)$$

The derivatives need to be expressed as discretized quantities in order to create the update equations. The derivative operation with respect to time and space for the magnetic and electric field in equation (2.20) and (2.21) will be replaced by 1st order finite difference expressions in time and space. E_z and H_y are as follows:

$$E_z(x,t) = E_z(m \Delta x, q \Delta t), \quad (2.22)$$

$$H_y(x,t) = H_y(m \Delta x, q \Delta t) \quad (2.23)$$

where m and q are the step indices in space and time, respectively. Δx is the spatial step and Δt is the temporally step. The equations (2.20) and (2.21) will have the following form:

$$\mu \frac{H_y^{q+\frac{1}{2}}[m+\frac{1}{2}] - H_y^{q-\frac{1}{2}}[m+\frac{1}{2}]}{\Delta t} = - \frac{E_z^q[m+1] - E_z^q[m]}{\Delta x} \quad (2.24)$$

$$\varepsilon \frac{E_z^{q+1}[m] - E_z^q[m]}{\Delta t} = \frac{H_y^{q+\frac{1}{2}}[m+\frac{1}{2}] - H_y^{q-\frac{1}{2}}[m-\frac{1}{2}]}{\Delta x} \quad (2.25)$$

The magnetic field and the electric field are offset in time and space by a half step in order to support a leap frog calculation along x. The equations (2.24) and (2.25) are rearranged as follows:

$$H_y^{q+\frac{1}{2}}[m+\frac{1}{2}] = H_y^{q-\frac{1}{2}}[m+\frac{1}{2}] + \frac{\Delta t}{\mu \Delta x} (E_z^q[m+1] - E_z^q[m]) \quad (2.26)$$

$$E_z^{q+1}[m] = E_z^q[m] + \frac{\Delta t}{\varepsilon \Delta x} \left(H_y^{q+\frac{1}{2}}[m+\frac{1}{2}] - H_y^{q-\frac{1}{2}}[m-\frac{1}{2}] \right) \quad (2.27)$$

To visualize the update equations, figure 2.1 shows the spatial offset between electric and magnetic fields, the time offset is indicated by the q indices.

Figure 2.1: Visualizing the computing method of the update equation in time and space for one dimension.

Due to the simplicity of using the technique and applicability to many structures, FDTD is widely used. However, it may require a significant amount of memory and long computational time for some structures. The FDTD is preferred for electromagnetic studies in random or complex media. The technique has also been used in [41] to study the relation between random lasing modes and localized modes, and the study of localized modes in photonic crystals structures of negative permittivity elements [42].

2.2.2 Finite element method (FEM)

FEM is one of the most powerful and flexible numerical technique for handling problems involving complex geometries and inhomogeneous media. The residual of the equations is weighted with an arbitrary function, then the weighted residual equation is written as:

$$\int_0^L w(x) R(x) dx = 0 \quad (2.28)$$

where $w(x)$ is the weight function and $R(x)$ is the residual equations. Equation 2.28 will be solved to compute the exact solution over the interval from 0 to L. The finite element analysis technique begins with the discretization of the problem region into a finite number of elements. Expansion

functions for a typical element in the solution region are then defined. The expansion functions are then substituted in the weighted residual equation and integrated to form the weak form equation that will be used to populate the matrix elements. The boundary condition terms of the weak form will be specified according to the problem under study. The final steps are the assembling of all elements in the solution region, and then solving the system of equations that is obtained. FEM has the capability to be applied to arbitrary structures but it becomes time consuming and sensitive to the boundary conditions.

FEM can be used to study the localized modes of resonators as demonstrated in [43] using commercially available software. However, it is limited as the azimuthal direction was restricted to being uniform to provide computational speed.

2.2.3 Plane wave expansion method (PWEM)

Plane wave expansion method (PWEM) is a frequency domain method that solves the time independent Maxwell equations, and can be used to solve for the E-field or the H-field separately. PWEM is primarily used to calculate the band structure of translationally symmetric photonic crystals (PhCs) [21, 22]. This method is based on the assumption that the structures under test are infinite. PWEM depends on the use of Fourier series to represent the medium and the Bloch wave theorem to represent the electromagnetic field propagating through it. These expansions are inserted in the electromagnetic wave equation to form an eigenvalue matrix equation. The eigenvalue matrix is then solved to obtain the frequencies (eigenvalues) and profiles (reconstructed using the eigenvectors) of propagated states. The one-dimension photonic crystal, shown in figure 2.2, is the easiest structure for studying and understanding the main properties of PWEM. Many of the properties employed in one-dimensional analysis can be applied in two or three dimensions.

As such a study of figure 2.2 with would be beneficial. Additionally, PWEM has many similarities to BLF in regards to general concept and implementation.

Figure 2.2: One dimensional photonic structure. Black represent low relative permittivity, ϵ_l , and white represent high relative permittivity, ϵ_h .

For a one dimensional periodic structure in figure 2.2 where the electric field is parallel to the z-direction and the layers' interfaces the wave equation is obtained by combining the curl equations (2.7) and (2.8) and can be written as:

$$\frac{1}{\epsilon_r(x)} \left(\frac{\partial^2 E_z}{\partial x^2} \right) = \frac{1}{c^2} \frac{\partial^2 E_z}{\partial t^2}. \quad (2.29)$$

The relative permittivity of this system is a position-dependent function $\epsilon_r(x)$. The structure is periodic in x with lattice constant a , therefore $\epsilon_r(x)$ must satisfy the relation:

$$\epsilon_r(x) = \epsilon_r(x + a). \quad (2.30)$$

The material expansion is obtained by expressing the inverse of the relative permittivity using the Fourier series of the form:

$$\frac{1}{\epsilon_r(x)} = \sum_{p=-\infty}^{\infty} \kappa_p^{\epsilon_r} e^{-jG_p \cdot x} \quad (2.31)$$

where $\kappa_p^{\epsilon_r}$ are the expansion coefficients and $G_\epsilon = 2\pi p/a$ is the reciprocal lattice. The variable p is an integer identifying a specific cell. Similarly, Bloch solutions are used to represent the field and can be written as:

$$E_z(x, t) = \sum_{n=-\infty}^{\infty} \kappa_n^{E_z} e^{-j(k_x + G_E) \cdot x} e^{-j\omega t} \quad (2.32)$$

where the wavenumber $k_x = 2\pi/\lambda$ and $G_E = 2\pi n/a$, n is an integer. Note the derivation assumes the field to be harmonic in time. All possible normal modes of the matrix equation can be obtained by letting k_x vary from $-\pi/a$ to π/a , the Brillouin zone for 1-D periodic system, and solving the matrix equations for eigenvalue and eigenvector.

Inserting equations (2.31) and (2.32) into the one dimension electromagnetic wave equation (2.29) provides the wave equation of the form:

$$\sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \kappa_p^{\epsilon_r} \kappa_n^{E_z} \left(\frac{2\pi n}{a} + k_x \right)^2 e^{-j(\frac{2\pi p}{a} + \frac{2\pi n}{a})x} e^{-jk_x x} e^{-j\omega t} = \left(\frac{\omega}{c} \right)^2 \sum_{n=-\infty}^{\infty} \kappa_n^{E_z} e^{-j(\frac{2\pi n}{a})x} e^{-jk_x x} e^{-j\omega t} \quad (2.33)$$

Multiplying this equation by the orthogonal function $e^{j\frac{2\pi m}{a}x}$, where m is an integer and then integrating each side over a unit cell from $x = -\frac{a}{2}$ to $\frac{a}{2}$ results in:

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \kappa_p^{\epsilon_r} \kappa_n^{E_z} \left(\frac{2\pi n}{a} + k_x \right)^2 \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j(\frac{2\pi p}{a} + \frac{2\pi n}{a} - \frac{2\pi m}{a})x} dx = \left(\frac{\omega}{c} \right)^2 \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \kappa_n^{E_z} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j(\frac{2\pi n}{a} - \frac{2\pi m}{a})x} dx \quad (2.34)$$

Using the orthogonality properties of the basis functions, the non-zero solutions are obtained if $n + p = m$ for the left integral and $n = m$ for the right integral. Therefore, the summations can be reduced from two to one in the right-hand side, and from three to two in the left-hand side.

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \kappa_{m-n}^{\epsilon_r} \kappa_n^{E_z} \left(\frac{2\pi n}{a} + k_x \right)^2 = \left(\frac{\omega}{c} \right)^2 \sum_{n=-\infty}^{\infty} \kappa_n^{E_z} \quad (2.35)$$

The summation over n and m are used to build the matrix. As a result, the expansion indices of the inverse relative permittivity have to be double the expansion indices of the field. To illustrate the process, the matrix for a small expansion number of expansions is shown in equation (2.36). The field expansion indices n and m will control the populating of the matrix elements' in columns and rows, respectively. For each element only one value of the dielectric expansion will be used, which is the one that satisfied the condition $p = m - n$.

$$\begin{bmatrix} Q(m,n)_{m=-2,n=-2}, & Q(m,n)_{m=-2,n=-1}, & \cdots \cdots \\ Q(m,n)_{m=-1,n=-2}, & & \\ Q(m,n)_{m=0,n=-2}, & & \\ \vdots & & \\ & \vdots & \\ & Q(m,n)_{m=1,n=2}, & \\ & \dots Q(m,n)_{m=2,n=1}, & Q(m,n)_{m=2,n=2} \end{bmatrix} \begin{bmatrix} \kappa_{n=-2}^{E_z} \\ \kappa_{n=-1}^{E_z} \\ \vdots \\ \vdots \\ \kappa_{n=1}^{E_z} \\ \kappa_{n=2}^{E_z} \end{bmatrix} = \left(\frac{\omega}{c} \right)^2 \begin{bmatrix} \kappa_{n=-2}^{E_z} \\ \kappa_{n=-1}^{E_z} \\ \vdots \\ \vdots \\ \kappa_{n=1}^{E_z} \\ \kappa_{n=2}^{E_z} \end{bmatrix} \quad (2.36)$$

where $Q(m,n) = \kappa_{m-n}^{\epsilon_r} \left(\frac{2\pi n}{a} + k_x \right)^2$, solving the matrix for the eigenvalues, the frequencies ω , that are permitted to propagate in the medium at each wave number are plotted to produce the band structure of the dielectric structures under test. The eigenvectors are obtained by reinserting individual eigenvalues in the matrix provides the dielectric expansion coefficients and then can be used to define the propagating mode profile. A band gap is defined as a range of frequencies for which there is no corresponding wave vector and thus the energy levels within the band gap are not available, see figure 2.3.

Figure 2.3: Band gap, indicated in grey, in a periodic material with high-dielectric-constant material ($\epsilon_h = 13$) of width $0.2a$ and low-dielectric-constant material ($\epsilon_h = 1$) of width $0.8a$ [45].

The presence of defects in the photonic crystal and intentional doping result in the modification of the band structure and under the right conditions can introduce localized modes within the original band gap. For localized modes analysis, a supercell PWEM technique is required. A supercell is a larger unit cell size with the defect in the centre of the supercell. The supercell must be large enough that it is sufficiently modelling the structure and the defects are not normally repeated. This requires more memory and computation time. From the derivation, it is apparent that the PWEM requires translation symmetry in the structure, therefore structural assumptions are imposed when dealing with non-translation symmetry or when geometries are finite in size.

2.3 Recent techniques

Recently, new approaches have been developed to analyze structures that possess rotational symmetry [27, 28]. They are based on taking Maxwell's wave equation and expanding the

electromagnetic fields and the inverse relative permittivity of the dielectric profile using Fourier-Bessel (FB) basis functions in polar coordinates, and Fourier-Fourier-Bessel (FFB) basis functions in the cylindrical coordinates. FB and FFB were used to demonstrate a relationship between the rotational order of the structure and the rotational order of the localized modes [27, 28]. A significant reduction in the eigen-matrix order is possible using symmetry, which greatly reduces the computation time. In addition, the solver can be directed to compute a specific rotational mode order.

FB approach requires reasonable memory size and short computation time as a mode solver for 2-D plane PhCs, where the structure is assumed to be invariant along the axial direction perpendicular to the plane. The FFB technique assumes periodicity in z direction and can be configured to solve for hybrid steady states. A natural extension of the basis function based techniques is to consider spherically symmetric structures providing a means of solving for resonator states in a 3-D domain. Extending the solver to 3-D spherical space will enable the study of practical problems not previously solvable on a desktop PC [28].

The purpose of this work is to develop a technique that is suitable for analyzing localized modes in three dimensional spherical structures. Computation time and memory requirements are issues presented with many existing techniques when used to study localized modes for three dimensions structures. The BLF technique provides the researchers with a fast computation solver with low memory requirements for three dimensions spherical domain structures.

The achievement of using basis functions and boundary conditions that are more suitable to compute localized modes could open the path for using different basis functions and boundary conditions for studying different problems. The technique can be viewed more clearly when

compared to the PWEM. PWEM was designed mainly to study the propagated waves in periodic structures, while the BLF technique presented here is mainly designed to be used for localized modes computations. PWEM use periodic boundary condition of unit cells arranged in lattice configuration with the assumption of infinite structures and the use of exponential expansions functions in Cartesian coordinates. These boundary conditions and basis functions are more suitable for studying propagating waves in periodic structures. This makes the PWEM appropriate computations method for the band structures calculations. The BLF technique is applied to finite size structures with a zero-field value boundary condition in all directions. The expansion functions used are more suitable to represent localized modes in spherically symmetric structures. In addition, in the BLF there will be no need for translation symmetry restrictions. Users familiar with PWEM will be able to easily implement the BLF technique.

3 Chapter: Theoretical development of the BLF method

In this chapter, the details of the theoretical development of the BLF technique are presented. Fundamentally, the modified spherical basis functions BLF are first introduced, and are used to represent the inverse relative permittivity profile and the vector fields in Maxwell's H and E wave equations in spherical coordinates. The new expressions with BLF series representation are used to formulate an eigenvalue matrix. The eigenvalue matrix can be solved to obtain the localized mode's properties, such as angular frequency (eigenvalues), wavelength, field profile (eigenvectors).

3.1 Theory

The BLF mode solver is developed from the vector wave equation for either the electric or magnetic field in a charge free, current free, isotropic, nonmagnetic, and linear medium. Throughout the computation process, the time dependence takes the form of $e^{-j\omega t}$ where ω is the complex angular frequency, $\omega = \omega_r + j\omega_i$. The general starting wave equations, obtained from combining Maxwell's curl equations, are given in (3.1) and (3.2) for the H and E fields respectively, where the different dielectric regions can be represented by the spatial variations in the relative permittivity, ϵ_r . The product of the free space constitutive parameters is replaced by the speed of light in vacuum squared, c^2 .

$$\vec{\nabla} \times \frac{1}{\epsilon_r} \vec{\nabla} \times \vec{H} = \left(\frac{\omega}{c} \right)^2 \vec{H} \quad (3.1)$$

$$\frac{1}{\epsilon_r} \vec{\nabla} \times \vec{\nabla} \times \vec{E} = \left(\frac{\omega}{c} \right)^2 \vec{E} \quad (3.2)$$

The operator form of the equations in (3.1) and (3.2) differ only in the placement of the relative dielectric constant, which is in general a function of the coordinates. The operator form of the E field equation can be obtained from the H field equation when derivatives of the ϵ_r are set to zero. For this reason, it is sufficient to provide the mathematical steps for the H equation.

3.2 Dielectric profile and EM field representation in BLF

In spherical space, the fields and the inverse relative dielectric profile are expanded using basis functions of the form:

$$j_0(\rho_p \frac{r}{R}) P_n(\cos\theta) e^{jq\varphi} \quad (3.3)$$

where $j_0(\rho_p \frac{r}{R})$ is the zeroth order spherical Bessel function, ρ_p is the p^{th} root of the lowest order spherical Bessel function, and p is a positive integer. The root is scaled by the radius of the computational domain, R . The zeroth order and n^{th} degree associated Legendre function, $P_n(\cos\theta)$, is used for the dependence in polar angle (θ) and the complex exponential function $e^{jq\varphi}$ for the dependence in azimuthal angle (φ).

Restricting the radial basis to the lowest order spherical Bessel functions is a departure from the standard approach in the use of Bessel expansions where all orders are used. The lowest order Bessel function is also employed to represent the decaying field regions. This is made possible because any continuous radial function $f(r)$ on the interval from 0 to R can be written as a series expansion. Therefore any higher order spherical Bessel (or decaying function) can be accommodated without loss of accuracy using the expansion:

$$f(r) = \sum_p \kappa_p j_0(\rho_p \frac{r}{R}) \quad (3.4)$$

where κ_p are the coefficients of the expanding function. The benefit to using only the lowest order spherical Bessel function are two-fold in developing the eigen-matrix. The first is that the derivative along the radial coordinate will only involve the zero and first order spherical Bessel functions and therefore only two spherical Bessel functions are needed in building the matrix elements. The second is that in the matrix building process the integral of the product of three spherical Bessel functions (the field, the dielectric and the orthogonal field) is needed. Using only the lowest order spherical Bessel limits the number of different integrands to facilitate the normalization and tabulation for repeated use for all structures examined. The spherical Bessel functions have the following orthogonality property [46, 47]:

$$\int_0^R j_0(\rho_p \frac{r}{R}) j_0(\rho_{p^*} \frac{r}{R}) r^2 dr = \frac{R^3}{2} j_1(\rho_p)^2 \delta_{p,p^*} \quad (3.5)$$

Similarly, only the zeroth order of the associated Legendre functions $P_n(\cos\theta)$ is required as any continuous polar angular function between 0 and π can be series represented as:

$$f(\theta) = \sum_n \kappa_n P_n(\cos\theta) \quad (3.6)$$

where κ_n are the coefficients of the expanding function. The index of the Legendre polynomial is only limited to positive integer values and a series expansion on these functions can be used to represent any polar angle dependent function. This property greatly simplifies the expressions to be presented. In the matrix building process the integral of the product of three Legendre polynomials is present. Using a single unrestricted index limits the number of integrand to a

manageable set that can be tabulated for use in the analysis of all dielectric structures. The single index polynomial differs from the standard practice of using two indices and tying one of the indices to the azimuthal order though it works and is mathematically feasible. Using the single index Legendre polynomial simplifies the matrix building process. The zeroth order associated Legendre function satisfy the orthogonality conditions [48]:

$$\int_0^\pi P_n(\cos\theta)P_{n^*}(\cos\theta) \sin\theta d\theta = \frac{2}{(2n+1)}\delta_{n,n^*} \quad (3.7)$$

The known complex exponential function $e^{jq\varphi}$ related to the azimuthal angle (φ) can be used to represent any function in the azimuthal angle (φ) and is orthogonal when integrated over the interval $[0, 2\pi]$.

Maxwell's wave equation makes use of the inverse relative permittivity $\frac{1}{\epsilon_r} = \Omega$, and both the H and E fields. The inverse relative permittivity can be expressed using the BLF functions as:

$$\Omega = \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega j_0(\rho_{p_\Omega} \frac{r}{R}) P_{n_\Omega}(\cos\theta) e^{iq_\Omega\varphi} \quad (3.8)$$

The inverse relative permittivity expansion coefficients κ_Ω can be determined directly by the inverse Fourier transform of (3.8) once the details of the dielectric space are provided. See Matlab program in appendix E for details. Each component of the magnetic field or the electric field is also expanded using the spherical basis function as:

$$H_s = \sum_{p_s, n_s, q_s} \kappa_s j_0(\rho_{p_s} \frac{r}{R}) P_{n_s}(\cos\theta) e^{jq_s\varphi} \quad (3.9)$$

$$E_s = \sum_{p_s, n_s, q_s} \kappa_s j_0(\rho_{p_s} \frac{r}{R}) P_{n_s}(\cos\theta) e^{jq_s\varphi} \quad (3.10)$$

where $s=r, \theta, \varphi$ is used to distinguish between the field components and κ_s are the expansion coefficients. Using these series expansions, Maxwell's wave equations can be cast into an eigenvalue problem, which provides the steady state frequencies (eigenvalues) and modal profiles eigenvectors in spherical space (κ_s expansion coefficients).

3.3 Building the eigenvalue matrix

An eigenvalue formulation of the wave equations, suitable for providing information on the localized state complex frequencies and corresponding field profiles can be developed starting from either the magnetic (3.1) or electric (3.2) field equations. Treating the inverse of the relative permittivity as a scalar function of the coordinates, the expressions can be reformulated as:

$$\frac{1}{\varepsilon_r} \vec{\nabla} \times \vec{\nabla} \begin{bmatrix} \vec{0} \\ \vec{E} \end{bmatrix} + \vec{\nabla} \frac{1}{\varepsilon_r} \times \vec{\nabla} \times \begin{bmatrix} \vec{H} \\ 0 \end{bmatrix} = \left(\frac{\omega}{c}\right)^2 \begin{bmatrix} \vec{H} \\ \vec{E} \end{bmatrix} \quad (3.11)$$

In the wave equation of (3.11), the electric field equation uses terms with designation $\begin{bmatrix} \vec{0} \\ \vec{E} \end{bmatrix}$, the magnetic field equation uses terms with designation $\begin{bmatrix} \vec{H} \\ 0 \end{bmatrix}$, and a $\begin{bmatrix} \vec{0} \\ 0 \end{bmatrix}$ indicates that the term is not included.

Expressing the wave equations in spherical coordinates, operations $\left(\vec{\nabla} \times \frac{1}{\varepsilon_r(r, \theta, \varphi)} \vec{\nabla} \times\right)$ on the left-hand side of equation (3.1) results in a vector that can be defined as:

$$\vec{W} = W_r \hat{r} + W_\theta \hat{\theta} + W_\varphi \hat{\varphi} \quad (3.12)$$

The terms containing derivatives are collected into $\vec{W}$ and are provided in appendix A. For the E field, the inverse relative permittivity is external to the curl, resulting in simpler expressions for

$\bar{W}$. The complication for the H field is that the double curl application in (3.1) results in first and second derivatives of the basis function with respect to the spherical coordinates for the field components, and only first derivative of the basis function for the inverse relative permittivity.

The right-hand side of (3.1) is also written in spherical space as:

$$\left(\frac{\omega}{c}\right)^2 \bar{H} = \left(\frac{\omega}{c}\right)^2 (H_r \hat{r} + H_\theta \hat{\theta} + H_\phi \hat{\phi}) . \quad (3.13)$$

Using equations (3.12) and (3.13) in (3.1) gives the wave equation of the H field in spherical coordinate as:

$$W_r \hat{r} + W_\theta \hat{\theta} + W_\phi \hat{\phi} = \left(\frac{\omega}{c}\right)^2 (H_r \hat{r} + H_\theta \hat{\theta} + H_\phi \hat{\phi}) \quad (3.14)$$

Inserting the expansions, (3.8) and (3.9), into the expressions for W_r , W_θ , and W_ϕ on the left-hand side, and into the right-hand side of the equations (3.14). Performing the derivations and applying the orthogonally integrations over the spherical domain will result in the matrix populating functions, Appendices C and D. The derived expressions that can be collected into an eigenvalue matrix formulation,

$$A v = (eig) v \quad (3.15)$$

For convenience, the details of the procedure used are shown only for the right-hand side of (3.1). A similar procedure is followed for the left-hand side.

For the right-hand side of equation (3.14):

$$RHS = \left(\frac{\omega}{c}\right)^2 (H_r \hat{r} + H_\theta \hat{\theta} + H_\phi \hat{\phi}) \quad (3.16)$$

Substituting the expansion (3.9) for each component of H field gives:

$$RHS \text{ component}(s) = \left(\frac{\omega}{c}\right)^2 H_s = \left(\frac{\omega}{c}\right)^2 \sum_{p_s, n_s, q_s} \kappa_s j_0\left(\frac{\rho_{p_s}}{R} r\right) P_{n_s}(\cos\theta) e^{iq_s \phi} \quad (3.17)$$

Orthogonality is used by multiplying each expanded component by an expanded orthogonal function, and integrating over the volume of the sphere domain.

$$\begin{aligned} RHS \text{ component}(s) &= \left(\frac{\omega}{c}\right)^2 \iiint_{0,0,0}^{R,\pi,2\pi} \sum_s \kappa_s \\ &\left[j_0\left(\rho_{p_s} \frac{r}{R}\right) P_{n_s}(\mu) e^{-iq_s \phi} \right] \left[j_0\left(\rho_{p_s^*} \frac{r}{R}\right) P_{n_s^*}(\mu) e^{-iq_s^* \phi} \right] r^2 \sin\theta \, dr d\theta d\phi. \end{aligned} \quad (3.18)$$

Using the orthogonality relations for the three basis functions, it can be noted that non-zero values are obtained only when $(p_s, n_s, q_s) = (p_{s^*}, n_{s^*}, q_{s^*})$ resulting in:

$$RHS \text{ component}(s) = \left(\frac{\omega}{c}\right)^2 \kappa_s \left[\left(\frac{R^3}{2} [j_1(\rho_{p_s})]^2 \right) \left(\frac{2}{(2n_s+1)} \right) (2\pi) \right]. \quad (3.19)$$

The normalization term $\left[\left(\frac{R^3}{2} [j_1(\rho_{p_s} r)]^2 \right) \left(\frac{2}{(2n_s+1)} \right) (2\pi) \right]$ can be moved to the left-hand side.

Collecting all components for all indices $(p_{s^*}, n_{s^*}, q_{s^*})$ results in:

$$RHS = I \left(\frac{\omega}{c} \right)^2 \begin{bmatrix} R \\ \Theta \\ \Phi \end{bmatrix} \quad (3.20)$$

where I is the identity matrix, and (R, Θ, Φ) are the field expansion coefficients (κ_s) for the field components in the radial direction R , polar angle Θ and azimuthal angle Φ , respectively.

The left-hand side of the expanded wave equation is obtained using similar steps are applied for each term of each component of the W equations, see appendix C. This will result in three equations, an equation in each of the three spherical coordinates written as:

$$W_r = R_r R + \theta_r \Theta + \varphi_r \Phi , \quad (3.21)$$

$$W_\theta = R_\theta R + \theta_\theta \Theta + \varphi_\theta \Phi , \quad (3.22)$$

$$W_\varphi = R_\varphi R + \theta_\varphi \Theta + \varphi_\varphi \Phi . \quad (3.23)$$

These equations are functions of the three field components (R, Θ, Φ) . The nine final expressions for R_s, θ_s , and φ_s , are given in appendix D. This represents the left-hand side of the eigenvalue equations:

$$A v = \begin{bmatrix} R_r & \theta_r & \varphi_r \\ R_\theta & \theta_\theta & \varphi_\theta \\ R_\varphi & \theta_\varphi & \varphi_\varphi \end{bmatrix} \begin{bmatrix} R \\ \Theta \\ \Phi \end{bmatrix} \quad (3.24)$$

Providing an overall eigenvalue problem with the following final form:

$$\begin{bmatrix} R_r & \theta_r & \varphi_r \\ R_\theta & \theta_\theta & \varphi_\theta \\ R_\varphi & \theta_\varphi & \varphi_\varphi \end{bmatrix} \begin{bmatrix} R \\ \Theta \\ \Phi \end{bmatrix} = \left(\frac{\omega}{c} \right)^2 I \begin{bmatrix} R \\ \Theta \\ \Phi \end{bmatrix} \quad (3.25)$$

The eigenvalues are $\left(\frac{\omega}{c} \right)^2$ while the field expansion coefficients are the components of the eigenvector. The lower subscript of the elements in the 3 by 3 matrix represents the coordinate of the equation. It also serves to indicate the orthogonal field component that was applied prior to computation domain integration.

It is important to mention that the satisfaction of the orthogonality conditions for the three basis functions in the right-hand side of the equation does not ensure that they are satisfied in the left-hand side. This is due to the existence of the three basis terms within each integral. Therefore, the spherical Bessel and Legendre orthogonality integrals are calculated and saved in tables to reduce the amount of time computing as they are calculated independently from the structure profile

$$\xi_{abt} = \int_0^1 j_a(\rho_{p_\Omega} \zeta) j_b(\rho_p \zeta) j_0(\rho_{p^*} \zeta) \zeta^t d\zeta \text{ where } \zeta = r/R, \quad (3.26)$$

$$P_{c^d s^t ab} = \int_0^\pi c^d s^t P_{n_\Omega-a}(c) P_{n-b}(c) P_n^*(c) d\theta \quad (3.27)$$

where $c = \cos(\theta)$, $s = \sin(\theta)$. Also, in equation (3.27) the order index of the spherical Bessel and Legendre is altered. This is a consequence of the derivatives applied on the expanded fields and the inverse relative dielectric. The radius in the Bessel integral has been normalized, Appendix B. There are singularities at $r=0, \theta=0$ and $\theta=\pi$. To numerically deal with these singularities, these regions are excluded from the integrals. Therefore, the integral bounds in (3.26) changed to be from δr to one, where δr is a bound that is close to zero. For the integrals in (3.27) the bounds are changed to be from $\delta\theta_0$ to $\delta\theta_\pi$ where $\delta\theta_0$ is a bound that is close to zero and $\delta\theta_\pi$ is a bound that is close to π . The exclusion does not affect the validity of the solver as resonance fields of interest have zero value in these regions. However, these singularities are not present when using the new formulation for the technique in chapter 5.

For the exponential terms $e^{jq_s\varphi}$, the orthogonal integral gives

$$\int_0^{2\pi} e^{iq_\Omega\varphi} e^{iq_s\varphi} e^{-iq_{s^*}\varphi} d\varphi = 2\pi \delta_{(q_\Omega+q_s), q_{s^*}} \quad (3.28)$$

when $(q_\Omega + q_s) = q_{s^*}$, the integral will only result in a non-zero value. This gives the advantages of targeting specific modes in the φ direction and reducing the matrix order.

The eigenvalues from (3.25), $\left(\omega/c\right)^2$, are the wavenumbers squared, k^2 . The wavenumbers are normalized by R . Setting the dielectric sphere structure radius to unity, $R=1$, will result in normalized eigenvalues, $k_{normalized}$, for the valid localized modes of the structure. The localized mode wavenumber values are given by:

$$k = \frac{k_{normalized}}{R}. \quad (3.29)$$

The BLF technique is required to obtain $k_{normalized}$, then k values for sphere of different radius are calculated using the equation (3.29). The normalization helps to target and design resonators for specific wavelengths, similar normalization is common in PWEM [21, 22]. The equation (3.29) can be written as:

$$\frac{2\pi}{\lambda_o} = \frac{k_{normalized}}{R} \quad (3.30)$$

Using equation (3.30), it is possible to configure and design the dielectric sphere radius to satisfy any desired wavelength.

3.4 Solver steps

3.4.1 Defining the dielectric

The first step in using the numerical solver begins by defining the dielectric structure within the entire computation domain, and then determining the expansion coefficients. Therefore, the

coefficients of the inverse relative dielectric expansion (κ_Ω) have to be calculated from the inverse relative dielectric structure $\Omega(r, \theta, \varphi)$ using inverse Fourier transform of (3.8):

$$\kappa_\Omega = \iiint_{0,0,0}^{R,\pi,2\pi} \left[\Omega(r, \theta, \varphi) j_0 \left(\rho_{p_s^*} \frac{r}{R} \right) P_{n_s^*}(\mu) e^{-iq_s^* \varphi} \right] r^2 \sin \theta \, dr \, d\theta \, d\varphi \quad (3.31)$$

A zero-value boundary condition was used for the spherical Bessel function normalization, the right hand side of equations (3.5). This will tailor the technique to target the localized modes, modes that have no electric, $\bar{E}$, or magnetic, $\bar{H}$, field at the edges of the simulation domain. The extent of the simulation domain must be selected to ensure that the zero-value boundary condition at the edges for localized modes is satisfied, see figure 3.1.

Figure 3.1: Zero-value boundary condition located far away from the objective dielectric structure.

3.4.2 Populating the matrix

Each of the operators in the 3 by 3 matrix in equation (3.25), represents a square array (block) of numbers with an order equal to number of basis functions used in the field expansion series. The individual numbers for each operator can be generated using the expressions given in Appendix D. The general formulation requires two sets of field related indices (p, q, n) and (p^*, q^*, n^*) . The

elements in a block are collected for the field indices constant along a row and the orthogonal indices constant down a column. . Each individual matrix element is produced through a summation over the inverse relative permittivity indices, $(p_{\Omega}, q_{\Omega}, n_{\Omega})$.

3.4.3 Solving the matrix

The eigenvalues are the set of numbers $(\omega / c)^2$ returned using a numerical eigenvalue solver such as the eig() function in MATLAB® on the matrix in (3.25). The eigenvectors associated to each eigenvalue are conveniently obtained using the eig() function. The eigenvector (expansion coefficients) κ_s are used in (3.9) to produce the spatial field profile.

3.4.4 Localized mode profiles categories

When used to study dielectric profiles possessing spherical symmetry, the BLF method provides four different categories of modal fields. Localized states have high field strengths confined to the central region of the simulation domain and negligible field values along the edge of the computational domain. Edge states on the other hand show strong field components confined mostly along the computational edge of the simulation domain. These are highly dependent on the properties along the edge and are not normally states examined using the BLF technique. Changing the location of the computational edge will affect the edge state eigen-frequencies and field profiles, while the properties localized states remain unchanged. Super-states are states that encompass sizeable field values throughout the entire computation domain. These states are normally not the states under investigation when using the BLF technique. Interface states, which are states that show strong field confinement at the interface between dielectric regions. When the fields for these states decay gradually to zero at the computation radial edge, they represent localized BLF returned states. Such states can be returned when searching for WGMs, bottle,

SNAP, and ring resonator states for example [5] and [13]. In general, the mode solver returns one state for each basis function used to express the field component series. For instance, using 100 different basis functions to expand the field will return 300 eigenvalues and eigenvectors. The properties of the states can be filtered by examining the dominant field component in the eigenvector, the value of the computed eigen-wavelength compared to the structure size parameters and the dominant basis function expansion coefficients. The real or imaginary nature of the eigen-frequency and the relative magnitude of the real and imaginary parts can serve to filter returned states. An accurate measure of the modes' properties is determined by reconstructing the field using (3.9) and plotting the field in different planes.

3.5 Optimizing the technique

The orthogonality integration related to the angle φ plays an important part in reducing the eigen-matrix order and serves to direct the computation process to search for particular rotational orders. The integral is written in (3.32) and places a restriction on the azimuthal indices combinations acquired for non-zero values of the matrix elements.

$$\int_0^{2\pi} e^{j(q+q_\Omega-q^*)\varphi} d\varphi = 2\pi \delta_{q+q_\Omega, q^*} \quad (3.32)$$

For example, if the dielectric profile shows no φ variations, then $q_\Omega = 0$, the pairs (q, q^*) must be equal for the value of the matrix elements to be non-zero. The position of the non-zero matrix elements are such that matrix blocks composed of $(q=0, q^*=0)$ can be separated from $(q=1, q^*=1)$ and so on. In this way, a particular mode type in the azimuthal plane can be selected and solved for, using a much-reduced matrix order eigenvalue system. This feature is particularly useful when searching for high order WGMs in spherical resonators or for selectively searching for monopoles, dipole, quadruples or other order modes. In the table below the q values are only shown, the other

indices are hidden inside each element of the table. This means that every element in the table represent a matrix of size that is equal to the indices in p and n . In addition, this table shows the matrix elements for only the azimuthally uniform sphere structure $q_{\Omega} = 0$.

Table 3.1: The condition for non-zero matrix elements($q = q^*$).

	-2	-1	0	1	2	q
-2	$q = q^*$ Non zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	
-1	$q \neq q^*$ Zero matrix	$q = q^*$ Non zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	
0	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q = q^*$ Non zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	
1	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q = q^*$ Non zero matrix	$q \neq q^*$ Zero matrix	
2	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q \neq q^*$ Zero matrix	$q = q^*$ Non zero matrix	
						q^*

This table represent one of the 3x3 square arrays in left hand side of equations (3.25). For selecting modes, the (q, q^*) represent the order of the localized mode order, therefore, for monopole localized mode in the azimuthal direction only the matrices of the $(q=0, q^*=0)$ needs to be calculated and solved.

3.6 Segmentation of the dielectric profile

To decrease the computation time of the technique even further the material segmentation approach can be added. A dielectric segmentation approach has been presented as a supportive computation tool for the FFB technique in [49] and is applied here to the BLF technique. The

approach is extended here to facilitate the segmentation of the material profile into two or more separate profiles which when combined reproduce the original intended structure. The result of this is a system matrix that is the sum of the separate matrices for each material segment.

3.6.1 Offset-Residue material segmentation

In the general case, an optical resonator defined in the spherical coordinate system may have material variations along all three coordinate directions. These variations define the structures material profile which may be expanded using (3.8). Figure 3.2 top-left shows the variations in the relative permittivity of a representative resonator. The top-right figure shows the profile obtained from the inverse relative permittivity. The lower figures show that the inverse relative permittivity can be divided into a constant offset value (lower left, Ω_{off}) and a residue profile (lower right, Ω_{res}). The original inverse profile can be obtained from $\Omega = \Omega_{off} + \Omega_{res}$.

Figure 3.2: Segmentation process of inverse of the relative permittivity of the structure (top-right) is shown.

The wave equations for the E and H fields can now be written as a summation over two terms.

$$\Omega_{off} \vec{\nabla} \times \vec{\nabla} \times \vec{E} + \Omega_{res} \vec{\nabla} \times \vec{\nabla} \times \vec{E} = \left(\frac{\omega}{c} \right)^2 \vec{E} \quad (3.33)$$

$$\vec{\nabla} \times \Omega_{off} \vec{\nabla} \times \vec{H} + \vec{\nabla} \times \Omega_{res} \vec{\nabla} \times \vec{H} = \left(\frac{\omega}{c} \right)^2 \vec{H} \quad (3.34)$$

Applying the BLF technique derivations, as in previous sections, for (3.33) results in a matrix system that has the form:

$$\begin{bmatrix} R_{r\Omega_{off}} & \theta_{r\Omega_{off}} & \varphi_{r\Omega_{off}} \\ R_{\theta\Omega_{off}} & \theta_{\theta\Omega_{off}} & \varphi_{\theta\Omega_{off}} \\ R_{\varphi\Omega_{off}} & \theta_{\varphi\Omega_{off}} & \varphi_{\varphi\Omega_{off}} \end{bmatrix} \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} + \begin{bmatrix} R_{r\Omega_{res}} & \theta_{r\Omega_{res}} & \varphi_{r\Omega_{res}} \\ R_{\theta\Omega_{res}} & \theta_{\theta\Omega_{res}} & \varphi_{\theta\Omega_{res}} \\ R_{\varphi\Omega_{res}} & \theta_{\varphi\Omega_{res}} & \varphi_{\varphi\Omega_{res}} \end{bmatrix} \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} = \left(\frac{\omega}{c} \right)^2 \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} \quad (3.35)$$

The matrix elements of the square 3 X 3 matrices are populated using the expressions provided in Appendix D. In the case of the offset matrix, the inverse permittivity value Ω_{off} is a constant throughout the computation domain and does not require a series representation. Therefore, for the

process of generating the offset matrix elements the summation over the media is replaced by the offset value. In the case of the residue matrix, the series expansion form of (3.8) is required for the residue profile only and used in the matrix populating expressions.

It is easy to show that for two different media profiles in which one profile, designated in the equations as the original profile $\Omega = \Omega_{off} + \Omega_{res}$, is the scaled equivalent of the other $\Omega' = \Omega'_{off} + \Omega'_{res}$, the scaling factors related to the offset and the residue profiles are given by,

$$\Omega' = \Omega'_{off} + \Omega'_{res} = K_{off} \Omega_{off} + K_{res} \Omega_{res} :$$

$$K_{off} = \frac{\Omega'_{off}}{\Omega_{off}} \quad K_{res} = \frac{\Omega' - \Omega'_{off}}{\Omega - \Omega_{off}} \quad (3.36)$$

This reduces the computational burden in (3.35), because the scaling factors can be external multipliers to the matrix populating process and therefore the matrices generated for one structure can be used (rescaled) for the analysis of another structure which has the offset, residue or both profiles simply rescaled.

Figure 3.3 below illustrates that the residue profile can be further segmented into a number of profiles, one for each constituent material present in the resonator structure.

Figure 3.3: Segmentation of residue profile (Top Left) into 4 separated profiles, a profile for each constituent material present in the resonator structure.

In such a situation, the original inverse permittivity profile is given by $\Omega = \Omega_{off} + \sum_{i=1}^N \Omega_{res}^i$.

Allowing for the possibility of rescaling one or more regions, the system matrix obtained is:

$$K_{off} \begin{bmatrix} R_{r\Omega_{off}} & \theta_{r\Omega_{off}} & \varphi_{r\Omega_{off}} \\ R_{\theta\Omega_{off}} & \theta_{\theta\Omega_{off}} & \varphi_{\theta\Omega_{off}} \\ R_{\varphi\Omega_{off}} & \theta_{\varphi\Omega_{off}} & \varphi_{\varphi\Omega_{off}} \end{bmatrix} \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} + \sum_{i=1}^N \left\{ K_{res}^i \begin{bmatrix} R_{r\Omega_{res}^i} & \theta_{r\Omega_{res}^i} & \varphi_{r\Omega_{res}^i} \\ R_{\theta\Omega_{res}^i} & \theta_{\theta\Omega_{res}^i} & \varphi_{\theta\Omega_{res}^i} \\ R_{\varphi\Omega_{res}^i} & \theta_{\varphi\Omega_{res}^i} & \varphi_{\varphi\Omega_{res}^i} \end{bmatrix} \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} \right\} = \left(\frac{\omega}{c} \right)^2 \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix}. \quad (3.37)$$

This particular form of the system matrix is useful when exploring the structures response when one medium is changed such as in the analysis of resonant optical sensors [15]. All matrices are built once and rescaled when the sensing medium is changed.

The segmentation of the material properties into offset and residue profiles can also be applied in the analysis of structures which undergo a perturbation within one or more regions. The offset profile, which was initially uniform throughout the computation domain, is replaced by a reference structure which is composed based on the unperturbed structure. Figure 3.4 illustrates the process. The lower two figures show the unperturbed structure serving as the reference on the left and on the right a perturbation applied to the regions characterized by a perturbation profile. The material perturbation applied to the reference structure can be treated as the residue profile and when the matrix is populated provides the perturbation matrix $[dM]$. A change in the level of the applied perturbation is easily transformed into the rescaling of the matrix $[dM] \Rightarrow K_{pert}[dM]$, with K_{pert} a constant scaling factor.

Figure 3.4: Reference, large perturbation and small perturbation profiles.

If the perturbations applied to the reference structure are small and the states are non-degenerate the eigenvalues and eigenvectors of the perturbed system can be obtained directly through the following expressions [50]:

$$\xi_i = \xi_{o,i} + \langle \psi_{o,i} | dM | \psi_{o,i} \rangle, \quad (3.38)$$

$$|\psi_i\rangle = |\psi_{o,i}\rangle + \sum_{i \neq j} \frac{\langle \psi_{o,j} | dM | \psi_{o,i} \rangle}{\xi_{o,i} - \xi_{o,j}} |\psi_{o,j}\rangle \quad (3.39)$$

The matrix generated for the reference structure can be solved to yield the unperturbed eigenvalues $\{\xi_{o,i}\}$ and eigenvectors $\{|\psi_{o,i}\rangle\}$. When the perturbations are large, the first order analysis leading to equations (3.38) and (3.39) diverge from the exact solution. In such a case either higher order correction terms are added to the expressions or the full structure is examined using a non-perturbative approach. The material segmentation process presented does not rely on the perturbation being small, a large perturbation simply results in a large scaling factor applied to the perturbation matrix. The entire matrix can be rapidly populated using the following expression:

$$M \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} + dM \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} = \left(\frac{\omega}{c}\right)^2 \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} \quad (3.40)$$

$$\begin{bmatrix} R_{r\Omega_{ref}} & \theta_{r\Omega_{ref}} & \varphi_{r\Omega_{ref}} \\ R_{\theta\Omega_{ref}} & \theta_{\theta\Omega_{ref}} & \varphi_{\theta\Omega_{ref}} \\ R_{\varphi\Omega_{ref}} & \theta_{\varphi\Omega_{ref}} & \varphi_{\varphi\Omega_{ref}} \end{bmatrix} \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} + K_{per} \begin{bmatrix} R_{r\Omega_{per}} & \theta_{r\Omega_{per}} & \varphi_{r\Omega_{per}} \\ R_{\theta\Omega_{per}} & \theta_{\theta\Omega_{per}} & \varphi_{\theta\Omega_{per}} \\ R_{\varphi\Omega_{per}} & \theta_{\varphi\Omega_{per}} & \varphi_{\varphi\Omega_{per}} \end{bmatrix} \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} = \left(\frac{\omega}{c}\right)^2 \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\varphi \end{bmatrix} \quad (3.41)$$

and solved for the eigenvalues and eigenvectors.

As for any numerical technique a number of limitations for the BLF technique are present. One of the limitations is the edge state inclusion in the solutions. Adding a layer of imaginary relative permittivity at the solver domain edge can help to filter out those solutions as the imaginary eigenvalue of those states will be large. Another limitation of this technique is the singularities in the integrals (3.26) and (3.27). Dealing with these singularities is explained in section (3.3). In the equations the inverse of relative permittivity is used, this means very high relative permittivity will

have very small inverse relative permittivity, which makes the numerical computations harder. In addition, the technique needs to be improved to deal with frequency dependent media.

4 Chapter: Computation examples

To verify the accuracy of the BLF technique the computation results for different structures are examined. All computations examples shown in this chapter are performed using the matrix system introduced in chapter 3. The relative permeability of all examples performed in this thesis is constant $\mu_r = 1$, while the relative permittivity is defined from the proposed structure $\varepsilon_r(r, \theta, \varphi)$. However, the BLF technique can deal with structure profiles of changing relative permeability $\mu_r(r, \theta, \varphi)$ and relative permittivity $\varepsilon_r(r, \theta, \varphi)$. The structures under test are structures whose solutions are available from other techniques. In particular, the BLF computation approach is applied to 4 specific examples: the spherical cavity with a perfectly conducting layer placed on its outer surface; the monopole and WGM states for a high relative permittivity sphere in air; resonator states for periodic and aperiodic spherical radial layers; and a disc of uniform and non-uniform dielectric profile in the azimuthal direction.

4.1 Spherical cavity

Figure 4.1–left shows a cross section of a resonator composed of a perfect conducting spherical shell of radius R filled with homogeneous, and lossless dielectric material such as air with $\varepsilon_r = 1$. Due to the uniform nature of the dielectric medium the monopole states decouple into spherical TE, with components H_r, H_θ, E_φ , and spherical TM, with components E_r, E_θ, H_φ . In the series expansion of the inverse relative permittivity profile using (3.8), the expansion indices n_Ω for θ and q_Ω for φ contributions to the basis functions can be set to zero as the sphere is uniform in the polar and azimuthal directions. To properly resolve the dielectric radial profile, 100 spherical Bessel terms were included (p_Ω from 1 to 100). The reconstruction of the inverse relative permittivity profile is shown in the figure 1-Right, obtained using (3.8).

The matrix generating expressions in Appendix D would indicate that for monopole states ($q = 0$) in a dielectric medium showing no φ dependence ($q_\Omega = 0$) the E_φ component is decoupled from the other components. This field component and polarization state are selected as they match up with the results presented in [38, 39] that were computed directly from the solution of the Maxwell's equations.

Figure 4.1: (Left) Cross section of the spherical cavity structure. (Right) Reconstructed inverse relative permittivity profile using 100 Bessel terms. Small deviations in dielectric value are unresolved by the resonator states of interest.

To solve for the E_φ components, only the φ_φ block is required in the matrix calculations. The decoupling of fields for monopoles, the fundamental mode in the azimuthal direction, in the dielectric sphere environment is similar to the decoupling of the fields observed when the out of plane direction has no dielectric variation as observed in the plane wave analysis of photonic crystals. Population of the matrix elements, for the E field equation, using the generating expressions in Appendix D, results in 4 of the 9 elements blocks being zero padded, expression (4.1), and enables the E_φ matrix block to be solved independently for its eigenvalues and

eigenvectors. A reduction in the matrix order by a factor of 3, in addition to targeting a particular azimuthal mode order, greatly speeds up the computation process when dealing with large matrices.

$$\begin{bmatrix} R_r & \theta_r & 0 \\ R_\theta & \theta_\theta & 0 \\ 0 & 0 & \varphi_\varphi \end{bmatrix} \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\phi \end{bmatrix} = \left(\frac{\omega}{c} \right)^2 \begin{bmatrix} \kappa_r \\ \kappa_\theta \\ \kappa_\phi \end{bmatrix} \quad (4.1)$$

To populate the φ_φ block in the eigen-matrix of (4.1), fields were series expanded using (3.10) with 50 spherical Bessel terms ($p = 1$ to 50) and 51 Legendre terms ($n = 0$ to 50). The q index was set to zero which configures the eigen-solver for localized states that represent the monopole modes. Directing the BFL to target a different mode type with respect to the φ direction is also possible. The matrices produced for each of the different q field indices can be separated into individual sub matrices (see section 3.5). The eigenvalues and eigenvectors associated with each mode were computed using the `eig()` function in MATLAB[®]. A restricted range of the eigenvalues were selected, and converted to kR for comparison with references, for which the eigenvector is dominated by expansion coefficient for the E_φ field component are shown in Table 4.1.

Table 4.1: Comparison of the scaled resonance wavenumber obtained using BLF and from references in [38, 39]. Bold-field profiles plotted in figures 4.2 and 4.3 computed using the BLF technique with eigenvector dominated by the E_φ field component.

Radial Order Polar Order	R ₁		R ₂	
θ_1	4.4935	[4.4934]	7.7253	[7.7252]
θ_2	5.7639	[5.7634]	9.0954	[9.0950]
θ_3	6.9889	[6.9879]	10.4181	[10.4171]
θ_4	8.1845	[8.1825]	11.7070	[11.7049]
θ_5	9.3582	[9.3558]	12.9696	[12.9665]

Table 4.1 shows the comparison of the monopole states in the azimuthal direction computed using BLF and those quoted in [38, 39] using analytical solutions shown in square brackets. The comparison is made by examining states with similar field properties, indicating the number of radial zero crossings and the dominant Legendre index orders. Compared to analytical solution for the same structures and boundary conditions obtained from the references, the BLF technique wavenumbers computed match the published values. The eigenvalues with the bold text in table 4.1, correspond to localized states that are plotted in figures 4.2 and 4.3. Where the field profiles generated through equation (3.10) for the E_φ field component. The fields show a zero value at the computations radial limit due to the imposed of zero-boundary condition. The computing time for this example is about 5 minutes, using a computer of 16 GB RAM and processor speed of 2.93 GHz.

Figure 4.2: The modal profile for E_φ field component normalized to a maximum of 1, $R_1\theta_1\varphi_0$ mode order, for $kR = 4.4935$. Left – $(x, y) = (r, 90, \varphi)$ plane. Right – $(y, z) = (r, \theta, 90)$ plane. The black to white color represent the normalized field values from one to zero.

Figure 4.3: The modal profile for E_ϕ field component normalized to a maximum of 1, $R_2\theta_3\phi_0$ mode order, for $ka=10418$. Left – $(x, y) = (r, 90, \phi)$ plane. Right – $(y, z) = (r, \theta, 90)$ plane. The black to white color represent the normalized field values from one to zero.

The BLF numerical technique requires the truncation of the expansion series; therefore, convergence of the results needs to be addressed. Figure 4.4 shows convergences of one of the selected normalized eigenvalues (kR), obtained using BLF technique, the analytical solutions is the horizontal line. The figure shows that using 40 basis functions for the field expansion provide well converged results.

Figure 4.4: The convergence for mode $R_{1\theta_3\phi_0}$. The solid horizontal line labels the analytical solution [38, 39], the dots represent the numerical results obtain using BLF.

The convergence in figure 4.4 is shown as the curve is closely forming a horizontal line. The converged value of the normalized wavenumber 6.989 is accurate to third decimal point. The technique converges as the basis number used to expand the relative permittivity and the fields are increasing.

4.2 High relative permittivity spheres

A single high dielectric constant sphere placed in air in the second test for the accuracy of the BLF mode solving technique. Published and well established results are available for comparison using analytical solutions [18, 19]. The dielectric sphere is selected to have a radius a and relative permittivity $\epsilon_r = 9$. For computational purposes, the air region is extended to a radius $R=2a$ with the dielectric sphere placed at the centre, as shown in figure 4.5-Left. The air region extending to

2a is sufficient to isolate the sphere localized states from the computational edge influence, a requirement for BLF. States of the dielectric sphere have effectively a zero field component on the computation edge when well localized. Similar reasoning is also employed when searching for localized states in single defect containing photonic crystals using the plane wave expansion technique. For the geometry examined here, the relative dielectric profiles only possess radial variations, with an abrupt change at $r=a$. The series representation of the inverse relative permittivity profile can be represented by setting q_Ω and n_Ω both to zero and summing over p_Ω . The series expansion coefficients of the inverse relative permittivity obtained using (3.31) are plotted in figure 4.5-Right versus basis function index p_Ω . The expansion coefficients obtained contain only real parts as the mediums involved are lossless and of spherical symmetry. The dielectric decomposition is well converged as demonstrated by the coefficients rapidly approaching zero with increased orders. An alternative confirmation is the process of reconstructing the inverse dielectric using equation (3.8) and comparing to the original structure.

To populate the eigen-matrix (4.1), the field components of the series in (3.10) were expressed using 70 spherical Bessel terms and 71 Legendre terms. Note that the Legendre expansion terms are required for the field, even though the dielectric is uniform, in order to resolve the θ related profile properties. The q index was set to zero target monopole like localized states in the $(x, y) = (r, 90, \phi)$ plane. The number of basis functions was sufficient to ensure converged localized states for the dielectric sphere in air.

Figure 4.5:Left – Geometry of the solid-uniform, $\epsilon_r = 9$, dielectric sphere, radius a . The spherical computation domain is of radius $R = 2a$. Right – Real part of the dielectric series expansion coefficients plotted versus spherical Bessel basis function p_n index. Imaginary parts are all zero for this dielectric profile.

When monopole states are considered in a dielectric medium showing no φ dependence, the field components decouple into two groups; TE with components H_r, H_θ, E_φ ; and TM with components E_r, E_θ, H_φ .

The eigenvalues returned from (4.1) and converted to normalized resonance wavenumbers, ka , for several lower order monopoles are listed in table 4.2 for the $E_\varphi (TE)$ field component, along with corresponding values extracted from [18, 19]. The results indicate an excellent agreement between the two different techniques. The bold values correspond to localized states that are plotted in figures 4.6 and 4.7. Mode nomenclature refers to the number of field maximums observed along the unit vector directions. Field profiles also match those plotted in the cited reference.

Table 4.2: Scaled resonance wavenumbers determined using BLF and from references in [18, 19] for the E_φ , TE polarization. Bold entries correspond to field profiles shown in figures 4.6 and 4.7.

Radial Order Polar Order	R_1	R_2
θ_1	1.0604 [1.02]	2.0375 [2.06]
θ_2	1.4750 [1.46]	2.4955 [2.53]
θ_3	1.8908 [1.90]	2.9542 [2.97]
θ_4	2.3011 [2.32]	3.4080 [3.44]
θ_5	2.7040 [2.70]	3.8523 [3.86]

Figure 4.6: The $(x, y) = (r, 90, \varphi)$ plane modal profile for E_φ field component (Left), and the $(y, z) = (r, \theta, 90)$ plane modal profile for E_φ field component (Right). Field profiles with for the $R_1\theta_1\varphi_0$ localized state are listed in table 2, computed using BLF, $ka = 1.0604$.

Figure 4.7: The $(x, y) = (r, 90, \varphi)$ plane modal profile for E_φ field component (Left), and the $(y, z) = (r, \theta, 90)$ plane modal profile for E_φ field component (Right) for the $R_1\theta_5\varphi_0$ localized state listed in table 2, computed using BLF, $ka=2.7040$.

In a similar fashion the upper left 2 by 2 matrix element blocks of the matrix (4.1) can be solved for the TM polarization states containing (E_r, E_θ) field components. The range of scaled wavenumbers correspond to the lower set of localized states listed in table 4.3 along with those extracted from [18, 19] that were computed using analytical solutions.

The field profiles for the bold entries are plotted in figures 4.8 to 4.10 for the various field components and are in excellent agreement with the features of the profiles plotted in the cited references.

Table 4.3: Scaled resonance wavenumbers determined using BLF and from references [18, 19] for the E_r, E_θ, TM polarization. Bold entries correspond to field profiles shown in figures 4.8 to 4.10.

Radial Order Polar Order	R_1	R_2
θ_1	1.4749 [1.46]	2.4954 [2.58]
θ_2	1.8906 [1.77]	2.9539 [3.03]
θ_3	2.3007 [2.24]	3.4075 [3.38]
θ_4	2.7035 [2.68]	3.8517 [3.74]
θ_5	3.0993 [3.08]	4.2850 [4.17]

Figure 4.8: The $(x, y) = (r, 90, \phi)$ plane modal profile for E_θ field component (Left), and the $(y, z) = (r, \theta, 90)$ plane modal profile for E_θ field component (Right), for $R_2\theta_1\phi_0$ localized state listed in table 3, computed using BLF, $ka = 2.4954$. The black to white color represent the normalized field values from one to zero.

Figure 4.9: The $(x, y) = (r, 90, \phi)$ plane modal profile for E_θ field component (Left), and the $(y, z) = (r, \theta, 90)$ planemodal profile for E_θ field component (Right), for $R_2\theta_5\phi_0$ localized state listed in table 3, computed using BLF, $ka = 4.2850$. The black to white color represent the normalized field values from one to zero.

Figure 4.10: The $(y, z) = (r, \theta, 90)$ plane modal profile for E_r field component for $R_{2\theta_1\phi_0}$ localized state listed in table 3, computed using BLF, $ka = 2.4954$ (Left). The $(y, z) = (r, \theta, 90)$ plane modal profile for E_r field component for $R_{2\theta_5\phi_0}$ localized state listed in table 3, computed using BLF, $ka = 4.2850$ (Right). The $(x, y) = (r, 90, \phi)$ plane have zero E_r field values. The black to white color represent the normalized field values from one to zero.

4.2.1 Whispering Gallery Modes

The high dielectric sphere in air is known to support high azimuthal order states referred to as Whispering Gallery Modes (WGMs). The basic concept of (WGMs) could be explained using the ray approach and then matching the phase of the wave for resonance conditions. Rays that strike the concave surface of the sphere at an angle larger than the critical angle will experience total internal reflection. For a wave trapped close to the surface the phase matching condition is:

$$2\pi n r = m\lambda \quad (4.2)$$

Where m is the mode number, n is the effective refractive index, and λ is the wavelength. WGMs have many important properties such as small mode volume, high power density and narrow spectral line width. That is due to minimal reflection losses and possibility of using very low material absorption, which enhance the Q-factor to be very large values, providing an opportunity to improve sensor and laser applications [4, 15].

WGMs generally present a field profile highly confined at the interface between the high dielectric sphere and surrounding air region and a large number of field amplitude cycles within the azimuthal plane sphere's circumference [13]. WGM were calculated using BLF by populating the eigen-matrix using azimuthal order 20, and azimuthal mode order 40. The field components were represented using 60 spherical Bessel functions and 61 for Legendre polynomials. Since the dielectric expansion has non-zero expansion coefficients only for $q_\Omega = 0$, there is no mixing of the azimuthal mode orders and they can be separately solved, see figure 4.11. Note the figure is built using small basis numbers ($p = 1$ to 2) and ($n = 0$ to 2) and ($q = 0$ to 2) for illustration purposes. The red, orange, and yellow cells are the non-zero elements for the first, second, and third azimuthal mode orders respectively.

Figure 4.11: Sample eigen-matrix showing locations of zero and non-zero elements. The white elements are the zero elements. Each color represents different azimuthal order; the red, yellow, and orange cells are the non-zero elements for the first, second, and third azimuthal mode orders.

The entire populated eigen-matrix for each mode must be solved using all 9 generating expressions. The field profile of a WGM of order φ_{20} and $ka=8.3280$, shown in figure 4.12- top, displays 40 field extremes (20 field amplitude cycles) when observed in the (x, y) plane. In figure 4.12- bottom, the field profile of a WGM of order φ_{40} and $ka=15.470$, is shown, similar properties of the mode profile confinement are presented but in this case the profile displays 80 field extremes (40 field amplitude cycles) when observed in the (x, y) plane. The field is highly confined to the high dielectric side of the sphere-air interface with decaying field towards the air side.

Figure 4.12: Modal profile for the E_θ field component of the BLF computed WGMs of the solid sphere in air. Top pair – Azimuthal mode order 20 with $R_{101}\varphi_{20}$ and $ka=8.3280$. Bottom pair – Azimuthal mode order 40 with $R_{101}\varphi_{40}$ and $ka=15.4701$. Left – $(x, y) = (r, 90, \varphi)$ plane. Right – $(y, z) = (r, \theta, 90)$ plane. The black to white color represent the normalized field values from one to zero.

The computation time needed depends on many factors such as, the speed of the computer used, and the basis used to compute the fields and the dielectric structures. The computations time for

this section takes about 2 minutes if the accuracy is about 2 % using 20 basis functions for the spherical Bessel and Legendre. For accuracy of 1% using 50 basis functions for the spherical Bessel and Legendre, it takes about half an hour using a computer of 16 GB RAM and processor speed of 2.93 GHz. Therefore, the computing time and memory required always depends on the basis number used to represent the field and the dielectric. In addition, the codes were written in a way that makes it flexible to edit but with the cost of lower speed computations. More improvements for the codes speed are possible.

4.3 Spherical “Bragg” structures

When solid spheres consist of alternating layers of thin dielectric materials, several of the resonance states are caused by total/partial internal reflection at the shell layer region. The ability to control the resonance properties in three dimensions makes the spherical Bragg shell resonator of considerable interest and suitable for numerous applications. For a sphere coated by Bragg shells light can be confined strongly for a range of wavelengths. In figure 4.13, the radial profile of a microsphere placed in the center and coated with Bragg spherical shell of 24 layers is shown, as per the specifications of [20]. The radius of the microsphere is 1 μm with a relative dielectric constant of 2.25. The Bragg layers are formed from alternating regions with 0.301 μm ($\epsilon_r = 2.10$) and 0.213 μm ($\epsilon_r = 4.20$). The computation domain is extended to 5 μm beyond the outermost layer resulting in $R = 12.47 \mu\text{m}$. The results from [20], were studied using the scattering matrix method [51, 52] and will be compared to those from the BLF calculations.

The inverse dielectric was resolved using 400 spherical Bessel basis terms with n_Ω and q_Ω set to 0, as the dielectric medium has no φ or θ dependence. The radial dielectric expansion coefficients for the spherical shell structure are shown in figure 4.14.

The field components were represented using 60 spherical Bessels and 61 Legendre polynomials. The BLF computed state for the H_ϕ field component was at a frequency $f = 188 \text{ THz}$ and shown in figure 4.15. The modal profile and computed frequency match the cited reference; Only a 2% difference in frequency value is observed. The localized mode is highly confined around the center, with zero value at the edge of the computation domain.

Figure 4.13: Radial profile of sphere coated by dielectric shells. The shells acts as a quarter wave Bragg reflector. The dielectric has real values only.

Figure 4.14: Radial dielectric expansion coefficients for the spherical Bragg structure.

Figure 4.15: The modal profile for H_ϕ field component for $R_1\theta_1\phi_0$, computed using BLF.

The field profile and corresponding eigenvalue match those of reference [20]. Left – $(x, y) = (r, 90, \phi)$ plane. Right – $(y, z) = (r, \theta, 90)$ plane.

Figure 4.16: The modal profile for E_θ field component for $R_{1\theta_1\phi_{30}}$, computed using BLF, $f=134.12$ THz ($ka=2.8089$). Left – $(x, y) = (r, 90, \phi)$ plane. Right – $(y, z) = (r, \theta, 90)$ plane.

Figure 4.17: The modal profile for E_θ field component for $R_{1\theta_3\phi_{30}}$, computed using BLF, $f=142.09$ THz ($ka=2.9760$). Left – $(x, y) = (r, 90, \phi)$ plane. Right – $(y, z) = (r, \theta, 90)$ plane.

Results for other monopoles of different order and field component are also obtained, while in the referenced paper the solution shown was just for one mode. In figures 4.16 and 4.17 mode profiles for E_θ field component for $R_{1\theta_1\phi_{30}}$ at $f=134.12$ THz ($ka=2.8089$) and $R_{1\theta_3\phi_{30}}$, at

$f = 142.09$ THz ($ka = 2.9760$) are shown, respectively. The field components were represented using 60 spherical Bessel and 61 Legendre polynomials.

4.4 Aperiodic reflector resonators

The geometry of the aperiodic reflector is similar to the Bragg layer structure but designed to have a low dielectric fill factor. The high dielectric regions close to the center are thin and gradually increase in width as their radius increases. The outermost layer is designed to have a thickness that is $\frac{1}{4}$ wavelength at the design value. The parameters for the structure in figure 4.18 are from [53] and consisted of alternating spherical layers of air and nested alumina. The mode-matching technique was used for [53] calculations. The nested alumina has a complex dielectric value, $\epsilon_{r,\text{real}} = 9.8$, and $\epsilon_{r,\text{imaginary}} = 0.000098$. In figure 4.19 and 4.20, the real and imaginary expansion space decompositions are shown. Large numbers (500 spherical Bessel basis terms) are used to make accurate representation of the structure ($p_\Omega = 0$ to 500, $q_\Omega = 0$, and $n_\Omega = 0$).

In figure 4.21, the mode profile for the E_ϕ field components for $R_{101}\phi_0$ is plotted using 60 for the spherical Bessel and 61 for the Legendre polynomial indices. The localized mode profile matches the results obtained in the referenced paper, the field is decaying outside the center sphere due to the aperiodic Bragg shells. The frequency result also matches the referenced result. The resonance frequency has real and imaginary parts. The real part represents confined energy and the imaginary part represents energy loss due to absorption.

In [53] the Q factor was calculated using an analytical technique based on the electric filling factor and multiplying it by the absorption loss of the thin dielectric. For the given structure [53] determined the Q factor to be 2.5×10^7 . Using the BLF method, the losses of the material in ϵ_r are

included and the Q factor for a given mode is calculated using the real and imaginary components of the frequency values:

$$Q = \frac{Real(\omega)}{2*Imaginary(\omega)}. \quad (4.3)$$

The Q factor obtained using the BLF is calculated to be 2.01×10^7 . In this example the comparing was with results obtain using the mode-matching technique. The absorption losses are calculated using the electric filling factor for dielectric region multiplied by the loss tangent. The error percentage between the two numerical calculations is about 20%.

Figure 4.18: Relative dielectric constant profile along the radial direction for a sphere of air coated by aperiodic spherical shells.

Figure 4.19: Real radial inverse relative permittivity expansion coefficients for the aperiodic shell structure.

Figure 4.20: Imaginary radial dielectric expansion coefficients for the aperiodic shell structure.

Figure 4.21: The modal profile for E_ϕ field component for $R_1\theta_1\phi_0$, computed using BLF. The field profile and corresponding eigenvalue match those of reference [53]. Left – $(x, y) = (r, 90, \phi)$ plane. Right – $(y, z) = (r, 90, \phi)$ plane.

While the examples presented here have been limited to structures only showing radial variations this is a limitation of the available publish research on spherical resonator properties for comparison and not a limitation of the BLF technique. The technique presented here can be readily applied to other geometries within the spherical boundary, containing dielectric variations in any combination of the spherical coordinate directions.

4.5 Dielectric disc

The resonance mode wavelengths and profiles for a dielectric disk are presented here using the BLF technique. The disc is selected as it has well established results from ray optics. Figure 4.22 displays the structure under test along with the variation in polar angle and radial direction. The dielectric disc is tested to verify the accuracy of the technique as it is the simplest case that has variation in the polar angle plus the radial direction. The dielectric disc of $\epsilon_r = 9.0$ is assumed to have a width of $0.3a$. The domain radius is double the disc radius $R=2a$.

Figure 4.22: Dielectric disc is shown, where a is the radius of the disc. The structure is uniform in the azimuthal direction.

Equation (4.2) is used to compute the resonance wavelengths based on the ray optics approach. The use of this approach requires that the wavelength should be much smaller than the structure size. This is satisfied by examining the technique to study high order modes (from 50 to 100) for the E_ϕ components.

To configure the BLF technique to solve for the selected mode order, the condition

$$\int_0^{2\pi} e^{j(q+q_\Omega-q^*)\varphi} d\varphi = 2\pi \delta_{q+q_\Omega, q^*} \text{ is applied. The dielectric disc is uniform in the azimuthal angle}$$

($q_\Omega = 0$), which additionally means only ($q = 50, 70, \dots$ or 100) are needed for the field expansion in the azimuthal angle, as the integral will have zero value for other field's indices. As the mode order targeted for this computation is high (short wavelengths), field components were

represented using 80 basis functions for the spherical Bessel and 81 basis functions for Legendre polynomial index even though the structure is quite simple. The dielectric expansion of the disc uses 50 basis functions for the spherical Bessel and 50 basis functions for Legendre polynomial index are used. The expansion coefficients are shown in figure 4.23. It can be confirmed that the basis number used for the dielectric expansion is sufficient as the figure 4.23 shows that the coefficients become negligible for higher indices. The calculations of the resonance mode wavelengths are obtained normalized to the radius and together with the ray optics results are plotted in figure 4.24.

Figure 4.23: The expansion coefficients of the dielectric disc.

Figure 4.24: Scaled resonance wavelengths (λ/a) determined using BLF and using the ray optics for the E_ϕ field.

The BLF calculations were done with a relatively low number of basis expansions, but a comparison of the results those calculated using the ray optics approach shows good agreement. While the computed field eigenvalue and eigenvector is three dimensional, the selection ability of the technique reduces the field computation to what is equivalently a two dimensions computation in terms of the number of calculations.

Unlike the ray optics, BLF can examine modes whose wavelength is on the same order as the structural features. Figure 4.25 displays a plot of the normalized resonance wavenumbers for the 20th azimuthal mode order, including the mode profiles for a number of wavenumbers, for the disc shown in figure 4.22. It can be seen that modes that are affected by the domain edges are presented in the solutions. In identifying modes these modes are filtered out, as they do not represent a correct localized mode of the dielectric structure.

Even if the structure is not uniform in the azimuthal direction, the ability to select modes and reduce the matrix remains. If the structure has symmetric variation in the azimuthal direction then only the indices of dielectric coefficients that are non-zero plus or mins the target field mode order are needed in building the matrix.

Figure 4.25: The wavenumbers of modes with the 20th order azimuthal mode order are shown. The field profiles are plotted in the $(x, y) = (r, 90, \phi)$ plane (left) and in the $(y, z) = (r, 90, \phi)$ plane (right). The dielectric sphere radius a is half the solver domain sphere R . The two highest mode profiles are affected by the domain boundary.

In the following section a disc that is non-uniform in the azimuthal direction is studied. This structure is three-dimensional structure in the spherical coordinate. The computation time for this example was about four hours, in a computer of 32 GB RAM and processor speed of 4 GHz.

4.6 Dielectric disc with azimuthal dielectric change

For a dielectric disc with variation in the azimuthal direction, the BLF can be used to compute the resonance modes. The dielectric disc has a dielectric variation of the form of $\epsilon_r = 9.0 + \cos(\phi)$. The expansion coefficients of this structure are shown in figure 4.26 and 4.27. It is clear from the figures that only low indices of the dielectric expansions are needed for the azimuthal directions as higher indices have a zero value. To put this in context the expansion coefficients have a maximum value of 4.21 when the zero index of the dielectric in the azimuthal direction is used (see figure 4.26) while for higher indices the value reduces strongly to 6.57×10^{-6} at the third azimuthal index (figure 4.27, bottom). This simplifies the computation, as the field calculation will only use the first three indices in the azimuthal direction of the dielectric plus or minus the target field mode order.

Figure 4.26: The expansion coefficients for the zero index of the azimuthal direction $q_\Omega = 0$, and 50 indices for the polar and the radial directions.

Figure 4.27: Expansion coefficients for the dielectric disc with azimuthal variation **(Top)** The expansion coefficients for the first index of the azimuthal direction $q_\Omega = -1$ and 1 and 50 indices for the polar and the radial directions. **(Middle)** The expansion coefficients for the second index of the azimuthal direction $q_\Omega = -2$ and 2 and 50 indices for the polar and the radial directions. **(Bottom)** The expansion coefficients for the third index of the azimuthal direction $q_\Omega = -3$ and 3 and 50 indices for the polar and the radial directions.

The orthogonality condition is $\int_0^{2\pi} e^{j(q+q_\Omega-q^*)\varphi} d\varphi = 2\pi \delta_{q+q_\Omega,q^*}$, however, since the azimuthal angle has non-zero coefficients only for $q_\Omega = 0, \pm 1, \pm 2$, only these values are needed for the field expansion in the azimuthal angle. Figure 4.28 below show the results for, 10th and 20th, two selected mode orders in the azimuthal direction. A comparison of the wavenumber and mode profile with the 20th mode order in the previous section reveals agreement. The resulting wavenumbers and field profiles of the disc with cosine variation in the azimuthal direction are compared and matched to the uniform azimuthal dielectric disc. This result is expected, as the average refractive indices of the two structures are similar.

Figure 4.28: The 10th mode order in the azimuthal direction $k=4.81$ in the disc of azimuthal variation dielectric (left). The 20th mode order in the azimuthal direction $k=8.84$ in the disc of azimuthal variation dielectric (left).

Figure 4.29 displays other states for the 10th azimuthal mode order. Similar behavior is obtained for the 20th mode order. The field is plotted for the E_φ component, and similar results are obtained for other components of the field.

Figure 4.29: The wavenumbers of 10th order azimuthal mode are shown. The field profiles are plotted in the $(x, y) = (r, 90, \varphi)$ plane (left) and in the $(y, z) = (r, 90, \varphi)$ plane (right).

Good agreement has been shown between the results computed using BLF and other techniques. More complicated structure can be studied simply by ensuring the use of enough basis functions; this is a basic rule for any series-expansion calculations.

5 Chapter: Advanced formulation and basis functions

In this chapter, the BLF technique presented in chapter 3 will be re-examined to provide a more compact form. This step is needed to facilitate the mathematical work for the implementation of the technique for more practical structures such those that are Anisotropic. Instead of beginning with the wave equations the derivation will begin with Maxwell's time independent equations,

$$\frac{1}{\vec{\mu}_r} \vec{\nabla} \times \vec{E} = \frac{\omega}{c} \vec{\mathfrak{S}}, \quad (5.1)$$

$$\frac{1}{\vec{\epsilon}_r} \vec{\nabla} \times \vec{\mathfrak{S}} = \frac{\omega}{c} \vec{E}. \quad (5.2)$$

where $\vec{\mathfrak{S}} = jZ_o \vec{H}$, and $Z_o = \sqrt{\frac{\mu_o}{\epsilon_o}}$ is the free space impedance. The inverse relative permittivity and permeability tensor matrices, respectively, can be written as,

$$\frac{1}{\vec{\epsilon}_r} = \vec{\Omega} = \begin{bmatrix} \Omega_{rr} & 0 & 0 \\ 0 & \Omega_{\theta\theta} & 0 \\ 0 & 0 & \Omega_{\varphi\varphi} \end{bmatrix}, \frac{1}{\vec{\mu}_r} = \vec{\Lambda} = \begin{bmatrix} \Lambda_{rr} & 0 & 0 \\ 0 & \Lambda_{\theta\theta} & 0 \\ 0 & 0 & \Lambda_{\varphi\varphi} \end{bmatrix}. \quad (5.3)$$

Using (5.3), the left-hand side of (5.1), becomes:

$$\Lambda \vec{\nabla} \times \vec{E} = \begin{bmatrix} \Lambda_{rr} & 0 & 0 \\ 0 & \Lambda_{\theta\theta} & 0 \\ 0 & 0 & \Lambda_{\varphi\varphi} \end{bmatrix} \left(\frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (E_\varphi \sin \theta) - \frac{\partial E_\theta}{\partial \varphi} \right) \hat{r} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial E_r}{\partial \varphi} - \frac{\partial}{\partial r} (r E_\varphi) \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r E_\theta) - \frac{\partial E_r}{\partial \theta} \right) \hat{\varphi} \right), \quad (5.4)$$

$$= \Lambda_{rr} \frac{1}{r \sin \theta} \left(\sin \theta \frac{\partial E_\varphi}{\partial \theta} + E_\varphi \cos \theta - \frac{\partial E_\theta}{\partial \varphi} \right) \hat{r} + \Lambda_{\theta\theta} \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial E_r}{\partial \varphi} - E_\varphi - r \frac{\partial E_\varphi}{\partial r} \right) \hat{\theta} + \Lambda_{\varphi\varphi} \frac{1}{r} \left(E_\theta + r \frac{\partial E_\theta}{\partial r} - \frac{\partial E_r}{\partial \theta} \right) \hat{\varphi}.$$

$$(5.5)$$

Similarly, the right-hand side of (5.1) is written as,

$$\frac{\omega}{c} \tilde{\mathfrak{S}} = \frac{\omega}{c} \begin{bmatrix} \mathfrak{S}_r \\ \mathfrak{S}_\theta \\ \mathfrak{S}_\varphi \end{bmatrix}. \quad (5.6)$$

Using equation (5.5) and (5.6), equation (5.1) can be written in the following form:

$$\begin{bmatrix} \Lambda_{rr} \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (E_\varphi \sin \theta) - \frac{\partial E_\theta}{\partial \varphi} \right) \\ \Lambda_{\theta\theta} \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial E_r}{\partial \varphi} - \frac{\partial}{\partial r} (r E_\varphi) \right) \\ \Lambda_{\varphi\varphi} \frac{1}{r} \left(\frac{\partial}{\partial r} (r E_\theta) - \frac{\partial E_r}{\partial \theta} \right) \end{bmatrix} = \left(\frac{\omega}{c} \right) \begin{bmatrix} \mathfrak{S}_r \\ \mathfrak{S}_\theta \\ \mathfrak{S}_\varphi \end{bmatrix}. \quad (5.7)$$

Using the same process above, equation (5.2) becomes,

$$\begin{bmatrix} \Omega_{rr} \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (\mathfrak{S}_\varphi \sin \theta) - \frac{\partial \mathfrak{S}_\theta}{\partial \varphi} \right) \\ \Omega_{\theta\theta} \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial \mathfrak{S}_r}{\partial \varphi} - \frac{\partial}{\partial r} (r \mathfrak{S}_\varphi) \right) \\ \Omega_{\varphi\varphi} \frac{1}{r} \left(\frac{\partial}{\partial r} (r \mathfrak{S}_\theta) - \frac{\partial \mathfrak{S}_r}{\partial \theta} \right) \end{bmatrix} = \left(\frac{\omega}{c} \right) \begin{bmatrix} E_r \\ E_\theta \\ E_\varphi \end{bmatrix}. \quad (5.8)$$

The inverse relative permeability Λ_{ss} can be expanded using the spherical basis functions as:

$$\Lambda_{ss} = \sum_{p_\Lambda, n_\Lambda, q_\Lambda} \kappa_{\Lambda_{ss}} j_0(\rho_{p_\Lambda} \frac{r}{R}) P_{n_\Lambda}(\cos \theta) e^{iq_\Lambda \varphi}. \quad (5.9)$$

where ss is used to label the element of the permeability tensor matrix (5.3), ($ss = rr, \theta\theta, \text{ or } \varphi\varphi$).

Furthermore, each component of the $\vec{E}$ and $\vec{\mathfrak{T}}$ is also expanded using the spherical basis functions as follows:

$$E_s = \sum_{p_s, n_s, q_s} \kappa_s j_0(\rho_{p_s} \frac{r}{R}) P_{n_s}(\cos\theta) e^{iq_s\varphi}, \quad (5.10)$$

$$\mathfrak{T}_s = \sum_{p_s, n_s, q_s} \kappa_s j_0(\rho_{p_s} \frac{r}{R}) P_{n_s}(\cos\theta) e^{iq_s\varphi}. \quad (5.11)$$

In equation (5.7) the field derivatives can be replaced with the following expansions:

$$\frac{\partial E_s}{\partial r} = - \sum_{p_s, n_s, q_s} \kappa_s \left(\frac{\rho_{p_s}}{R} \right) j_1(\rho_{p_s} \frac{r}{R}) P_{n_s}(\cos\theta) e^{iq_s\varphi}, \quad (5.12)$$

$$\frac{\partial E_s}{\partial \theta} = \sum_{p_s, n_s, q_s} \kappa_s \frac{n_s \cos\theta}{\sin\theta} j_0(\rho_{p_s} \frac{r}{R}) P_{n_s}(\cos\theta) e^{iq_s\varphi} - \sum_{p_s, n_s, q_s} \kappa_s \frac{n_s}{\sin\theta} j_0(\rho_{p_s} \frac{r}{R}) P_{n_s-1}(\cos\theta) e^{iq_s\varphi}, \quad (5.13)$$

$$\frac{\partial E_s}{\partial \varphi} = \sum_{p_s, n_s, q_s} \kappa_s (iq_s) j_0(\rho_{p_s} \frac{r}{R}) P_{n_s}(\cos\theta) e^{iq_s\varphi}. \quad (5.14)$$

Putting equations (5.9) – (5.11), (5.13) and (5.14) back into the r components of equation (5.7)

results in the following expansion,

$$\Lambda_{rr} \frac{1}{r \sin\theta} \left(\sin\theta \frac{\partial E_\varphi}{\partial \theta} + E_\varphi \cos\theta - \frac{\partial E_\theta}{\partial \varphi} \right) \hat{r} =$$

$$\begin{aligned}
& \left(\sum_{p_{\Lambda_{rr}}, n_{\Lambda_{rr}}, q_{\Lambda_{rr}}} \sum_{p_{\varphi}, n_{\varphi}, q_{\varphi}} \kappa_{\varphi} \kappa_{\Lambda_{rr}} \frac{(n_{\varphi} \cos \theta + \cos \theta)}{r \sin \theta} j_0(\rho_{p_{\varphi}} \frac{r}{R}) P_{n_{\varphi}}(\cos \theta) e^{iq_{\varphi} \varphi} j_0(\rho_{p_{\Lambda_{rr}}} \frac{r}{R}) P_{n_{\Lambda_{rr}}}(\cos \theta) e^{iq_{\Lambda_{rr}} \varphi} \right. \\
& - \sum_{p_{\Lambda_{rr}}, n_{\Lambda_{rr}}, q_{\Lambda_{rr}}} \sum_{p_{\varphi}, n_{\varphi}, q_{\varphi}} \kappa_{\varphi} \kappa_{\Lambda_{rr}} \frac{n_{\varphi}}{r \sin \theta} j_0(\rho_{p_{\varphi}} \frac{r}{R}) P_{n_{\varphi}-1}(\cos \theta) e^{iq_{\varphi} \varphi} j_0(\rho_{p_{\Lambda_{rr}}} \frac{r}{R}) P_{n_{\Lambda_{rr}}}(\cos \theta) e^{iq_{\Lambda_{rr}} \varphi} \\
& \left. - \sum_{p_{\Lambda_{rr}}, n_{\Lambda_{rr}}, q_{\Lambda_{rr}}} \sum_{p_{\theta}, n_{\theta}, q_{\theta}} \kappa_{\theta} \kappa_{\Lambda_{rr}} \frac{(iq_{\theta})}{r \sin \theta} j_0(\rho_{p_{\theta}} \frac{r}{R}) P_{n_{\theta}}(\cos \theta) e^{iq_{\theta} \varphi} j_0(\rho_{p_{\Lambda_{rr}}} \frac{r}{R}) P_{n_{\Lambda_{rr}}}(\cos \theta) e^{iq_{\Lambda_{rr}} \varphi} \right) \\
& = \sum_{p_r, n_r, q_r} \kappa_r j_0(\rho_{p_r} \frac{r}{R}) P_{n_r}(\cos \theta) e^{iq_r \varphi}
\end{aligned} \tag{5.15}$$

Furthermore, the substitutions (5.9) - (5.14), for the θ and φ component of equation (5.7) give,

$$\begin{aligned}
& \Lambda_{\theta\theta} \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial E_r}{\partial \varphi} - E_{\varphi} - r \frac{\partial E_{\varphi}}{\partial r} \right) \hat{\theta} = \\
& \left(+ \sum_{p_{\Lambda_{\theta\theta}}, n_{\Lambda_{\theta\theta}}, q_{\Lambda_{\theta\theta}}} \sum_{p_r, n_r, q_r} \kappa_r \kappa_{\Lambda_{\theta\theta}} \frac{(iq_r)}{r \sin \theta} j_0(\rho_{p_r} \frac{r}{R}) P_{n_r}(\cos \theta) e^{iq_r \varphi} j_0(\rho_{p_{\Lambda_{\theta\theta}}} \frac{r}{R}) P_{n_{\Lambda_{\theta\theta}}}(\cos \theta) e^{iq_{\Lambda_{\theta\theta}} \varphi} \right. \\
& - \sum_{p_{\Lambda_{\theta\theta}}, n_{\Lambda_{\theta\theta}}, q_{\Lambda_{\theta\theta}}} \sum_{p_{\varphi}, n_{\varphi}, q_{\varphi}} \kappa_{\varphi} \kappa_{\Lambda_{\theta\theta}} \frac{1}{r} j_0(\rho_{p_{\varphi}} \frac{r}{R}) P_{n_{\varphi}}(\cos \theta) e^{iq_{\varphi} \varphi} j_0(\rho_{p_{\Lambda_{\theta\theta}}} \frac{r}{R}) P_{n_{\Lambda_{\theta\theta}}}(\cos \theta) e^{iq_{\Lambda_{\theta\theta}} \varphi} \\
& \left. + \sum_{p_{\Lambda_{\theta\theta}}, n_{\Lambda_{\theta\theta}}, q_{\Lambda_{\theta\theta}}} \sum_{p_{\varphi}, n_{\varphi}, q_{\varphi}} \kappa_{\varphi} \kappa_{\Lambda_{\theta\theta}} \left(\frac{\rho_{p_{\varphi}}}{R} \right) j_1(\rho_{p_{\varphi}} \frac{r}{R}) P_{n_{\varphi}}(\cos \theta) e^{iq_{\varphi} \varphi} j_0(\rho_{p_{\Lambda_{\theta\theta}}} \frac{r}{R}) P_{n_{\Lambda_{\theta\theta}}}(\cos \theta) e^{iq_{\Lambda_{\theta\theta}} \varphi} \right) , \\
& = \sum_{p_{\theta}, n_{\theta}, q_{\theta}} \kappa_{\theta} j_0(\rho_{p_{\theta}} \frac{r}{R}) P_{n_{\theta}}(\cos \theta) e^{iq_{\theta} \varphi}
\end{aligned} \tag{5.16}$$

$$\Lambda_{\varphi\varphi} \frac{1}{r} \left(E_{\theta} + r \frac{\partial E_{\theta}}{\partial r} - \frac{\partial E_r}{\partial \theta} \right) \hat{\varphi} =$$

$$\begin{aligned}
& \left(+ \sum_{p_{\Lambda\varphi\varphi}, n_{\Lambda\varphi\varphi}, q_{\Lambda\varphi\varphi}} \sum_{p_{\theta}, n_{\theta}, q_{\theta}} \kappa_{\theta} \kappa_{\Lambda\varphi\varphi} \frac{1}{r} j_0\left(\rho_{p_{\theta}} \frac{r}{R}\right) P_{n_{\theta}}(\cos\theta) e^{iq_{\theta}\varphi} j_0\left(\rho_{p_{\Lambda\varphi\varphi}} \frac{r}{R}\right) P_{n_{\Lambda\varphi\varphi}}(\cos\theta) e^{iq_{\Lambda\varphi\varphi}\varphi} \right. \\
& - \sum_{p_{\Lambda\varphi\varphi}, n_{\Lambda\varphi\varphi}, q_{\Lambda\varphi\varphi}} \sum_{p_{\theta}, n_{\theta}, q_{\theta}} \kappa_{\theta} \kappa_{\Lambda\varphi\varphi} \left(\frac{\rho_{p_{\theta}}}{R} \right) j_1\left(\rho_{p_{\theta}} \frac{r}{R}\right) P_{n_{\theta}}(\cos\theta) e^{iq_{\theta}\varphi} j_0\left(\rho_{p_{\Lambda\varphi\varphi}} \frac{r}{R}\right) P_{n_{\Lambda\varphi\varphi}}(\cos\theta) e^{iq_{\Lambda\varphi\varphi}\varphi} \\
& - \sum_{p_{\Lambda\varphi\varphi}, n_{\Lambda\varphi\varphi}, q_{\Lambda\varphi\varphi}} \sum_{p_r, n_r, q_r} \kappa_r \kappa_{\Lambda\varphi\varphi} \frac{n_r \cos\theta}{r \sin\theta} j_0\left(\rho_{p_r} \frac{r}{R}\right) P_{n_r}(\cos\theta) e^{iq_r\varphi} j_0\left(\rho_{p_{\Lambda\varphi\varphi}} \frac{r}{R}\right) P_{n_{\Lambda\varphi\varphi}}(\cos\theta) e^{iq_{\Lambda\varphi\varphi}\varphi} \\
& \left. + \sum_{p_{\Lambda\varphi\varphi}, n_{\Lambda\varphi\varphi}, q_{\Lambda\varphi\varphi}} \sum_{p_r, n_r, q_r} \kappa_r \kappa_{\Lambda\varphi\varphi} \frac{n_r}{r \sin\theta} j_0\left(\rho_{p_r} \frac{r}{R}\right) P_{n_r-1}(\cos\theta) e^{iq_r\varphi} j_0\left(\rho_{p_{\Lambda\varphi\varphi}} \frac{r}{R}\right) P_{n_{\Lambda\varphi\varphi}}(\cos\theta) e^{iq_{\Lambda\varphi\varphi}\varphi} \right) \\
& = \sum_{p_{\varphi}, n_{\varphi}, q_{\varphi}} \kappa_{\varphi} j_0\left(\rho_{p_{\varphi}} \frac{r}{R}\right) P_{n_{\varphi}}(\cos\theta) e^{iq_{\varphi}\varphi}
\end{aligned} \tag{5.17}$$

At this point the orthogonality relationship, as described in chapter 3, is applied to equation (5.15). Multiplying the orthogonal functions, $j_0\left(\rho_{p_r^*} \frac{r}{R}\right) P_{n_r^*}(\cos\theta) e^{iq_r^*\varphi}$ with the first term of equation (5.15) followed by integrating over the spherical domain gives:

$$\begin{aligned}
& \iiint_{0,0,0}^{R,\pi,2\pi} \left[\sum_{\Lambda_{rr} E_{\varphi}} \kappa_{\varphi} \kappa_{\Lambda_{rr}} \left(\frac{n_{\varphi} \cos\theta + \cos\theta}{r \sin\theta} \right) j_0\left(\rho_{p_{\varphi}} \frac{r}{R}\right) P_{n_{\varphi}}(\cos(\theta)) e^{iq_{\varphi}\varphi} j_0\left(\rho_{p_{\Lambda_{rr}}} \frac{r}{R}\right) P_{n_{\Lambda_{rr}}}(\cos(\theta)) e^{iq_{\Lambda_{rr}}\varphi} \right. \\
& \quad \left. j_0\left(\rho_{p_r^*} \frac{r}{R}\right) P_{n_r^*}(\cos(\theta)) e^{iq_r^*\varphi} r^2 \sin\theta \, dr \, d\theta \, d\varphi \right] \\
& = \sum_{\Lambda_{rr} E_{\varphi}} \kappa_{\varphi} \kappa_{\Lambda_{rr}} \left\{ \int_0^{\pi} \left(\frac{n_{\varphi} \cos\theta + \cos\theta}{\sin\theta} \right) P_{n_{\Lambda_{rr}}}(\cos(\theta)) P_{n_{\varphi}}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \sin\theta \, d\theta \right. \\
& \quad \left. \int_0^R j_0\left(\rho_{p_{\Lambda_{rr}}} \frac{r}{R}\right) j_0\left(\rho_{p_r} \frac{r}{R}\right) j_0\left(\rho_{p_r^*} \frac{r}{R}\right) r \, dr \int_0^{2\pi} e^{iq_r\varphi} e^{iq_{\Lambda}\varphi} e^{-iq_r^*\varphi} d\varphi \right\}.
\end{aligned} \tag{5.15a}$$

The first term of equation (5.15) is written now as:

$$= \sum_{\Lambda_{rr} E_{\varphi}} \kappa_{\varphi} \kappa_{\Lambda_{rr}} \left[(n_{\varphi} + 1) P_{c^1 s^0 00} \right] R^2 \xi_{001} 2\pi \delta_{q_{\varphi} + q_{\Lambda_{rr}}, q_r^*}$$

$$(5.16)$$

where ξ_{abt} and $P_{c^{d_s t} ab}$ are described in chapter 3.

The same orthogonality process is applied for the other terms of equation (5.15) and the results are written as follows:

$$R_r = 0, \quad (5.18)$$

$$\theta_r = -\sum_{\Lambda_{rr} E_\theta} \kappa_\theta \kappa_{\Lambda_{rr}} [(iq_\theta) P_{c^0 s^0 00}] R^2 \xi_{001} 2\pi \delta_{q_\theta + q_{\Lambda_{rr}}, q_r^*}, \quad (5.19)$$

$$\varphi_r = \sum_{\Lambda_{rr} E_\varphi} \kappa_\varphi \kappa_{\Lambda_{rr}} [(n_\varphi + 1) P_{c^1 s^0 00}]$$

$$R^2 \xi_{001} 2\pi \delta_{q_\varphi + q_{\Lambda_{rr}}, q_r^*} - \sum_{\Lambda_{rr} E_\varphi} \kappa_\varphi \kappa_{\Lambda_{rr}} [(n_\varphi) P_{c^0 s^0 01}] R^2 \xi_{001} 2\pi \delta_{q_\varphi + q_{\Lambda_{rr}}, q_r^*}. \quad (5.20)$$

Furthermore, the orthogonality relationship are also applied to all the terms of equations (5.16) and (5.17) giving expression,

$$R_\theta = \sum_{\Lambda_{\theta\theta} E_r} \kappa_r \kappa_{\Lambda_{\theta\theta}} [(iq_r) P_{c^0 s^0 00}] R^2 \xi_{001} 2\pi \delta_{q_r + q_{\Lambda_{\theta\theta}}, q_\theta^*} \quad (5.21)$$

$$\theta_\theta = 0, \quad (5.22)$$

$$\varphi_\theta = -\sum_{\Lambda_{\theta\theta} E_\varphi} \kappa_\varphi \kappa_{\Lambda_{\theta\theta}} [P_{c^0 s^1 00}] R^2 \xi_{001} 2\pi \delta_{q_\varphi + q_{\Lambda_{\theta\theta}}, q_\theta^*}$$

$$+ \sum_{\Lambda_{\theta\theta} E_\varphi} \kappa_\varphi \kappa_{\Lambda_{\theta\theta}} \rho_{p_\varphi} [P_{c^0 s^1 00}] R^2 \xi_{102} 2\pi \delta_{q_\varphi + q_{\Lambda_{\theta\theta}}, q_\theta^*} \quad , \quad (5.23)$$

$$R_\varphi = - \sum_{\Lambda_{\varphi\varphi} E_r} \kappa_r \kappa_{\Lambda_{\varphi\varphi}} n_r [P_{c^1 s^0 00}] R^2 \xi_{001} 2\pi \delta_{q_r + q_{\Lambda_{\varphi\varphi}}, q_\varphi^*} \\ + \sum_{\Lambda_{\varphi\varphi} E_r} \kappa_r \kappa_{\Lambda_{\varphi\varphi}} n_r [P_{c^0 s^0 01}] R^2 \xi_{001} 2\pi \delta_{q_r + q_{\Lambda_{\varphi\varphi}}, q_\varphi^*} \quad (5.24)$$

$$\theta_\varphi = \sum_{\Lambda_{\varphi\varphi} E_\theta} \kappa_\theta \kappa_{\Lambda_{\varphi\varphi}} [P_{c^0 s^1 00}] R^2 \xi_{001} 2\pi \delta_{q_\theta + q_{\Lambda_{\varphi\varphi}}, q_\varphi^*}, \\ - \sum_{\Lambda_{\varphi\varphi} E_\theta} \kappa_\theta \kappa_{\Lambda_{\varphi\varphi}} \rho_{p_\theta} [P_{c^0 s^1 00}] R^2 \xi_{102} 2\pi \delta_{q_\theta + q_{\Lambda_{\varphi\varphi}}, q_\varphi^*} \quad , \quad (5.25)$$

$$\varphi_\varphi = 0, \quad (5.26)$$

Including the above simplifications, the general matrices forms will be

$$\begin{bmatrix} R_{r\Lambda} & \theta_{r\Lambda} & \varphi_{r\Lambda} \\ R_{\theta\Lambda} & \theta_{\theta\Lambda} & \varphi_{\theta\Lambda} \\ R_{\varphi\Lambda} & \theta_{\varphi\Lambda} & \varphi_{\varphi\Lambda} \end{bmatrix} \begin{bmatrix} \kappa_E^r \\ \kappa_E^\theta \\ \kappa_E^\varphi \end{bmatrix} = \frac{\omega}{c} \begin{bmatrix} \kappa_\varphi^r \\ \kappa_\varphi^\theta \\ \kappa_\varphi^\varphi \end{bmatrix}, \quad (5.27)$$

$$\begin{bmatrix} R_{r\Omega} & \theta_{r\Omega} & \varphi_{r\Omega} \\ R_{\theta\Omega} & \theta_{\theta\Omega} & \varphi_{\theta\Omega} \\ R_{\varphi\Omega} & \theta_{\varphi\Omega} & \varphi_{\varphi\Omega} \end{bmatrix} \begin{bmatrix} \kappa_\varphi^r \\ \kappa_\varphi^\theta \\ \kappa_\varphi^\varphi \end{bmatrix} = \frac{\omega}{c} \begin{bmatrix} \kappa_E^r \\ \kappa_E^\theta \\ \kappa_E^\varphi \end{bmatrix}. \quad (5.28)$$

The general matrices forms can be now used to uncouple (5.1) and (5.2) as follow,

$$\begin{bmatrix} R_{r\Omega} & \theta_{r\Omega} & \varphi_{r\Omega} \\ R_{\theta\Omega} & \theta_{\theta\Omega} & \varphi_{\theta\Omega} \\ R_{\varphi\Omega} & \theta_{\varphi\Omega} & \varphi_{\varphi\Omega} \end{bmatrix} \begin{bmatrix} R_{r\Lambda} & \theta_{r\Lambda} & \varphi_{r\Lambda} \\ R_{\theta\Lambda} & \theta_{\theta\Lambda} & \varphi_{\theta\Lambda} \\ R_{\varphi\Lambda} & \theta_{\varphi\Lambda} & \varphi_{\varphi\Lambda} \end{bmatrix} \begin{bmatrix} \kappa_H^r \\ \kappa_H^\theta \\ \kappa_H^\varphi \end{bmatrix} = \frac{\omega^2}{c^2} \begin{bmatrix} \kappa_H^r \\ \kappa_H^\theta \\ \kappa_H^\varphi \end{bmatrix} \quad (5.29)$$

5.1 Boundary conditions

For the radial basis used in this work, the spherical Bessel functions satisfy the orthogonality relation

$$\int_0^\infty j_0(\rho_p r) j_0(\rho_{p^*} r) r^2 dr = \frac{\pi}{2\rho_p^2} \delta_{p,p^*} \quad (5.30)$$

This generally hold only for infinite space, while for finite interval $(0, R)$ the orthogonality is satisfied by

$$\int_0^R j_0(\rho_p r) j_0(\rho_{p^*} r) r^2 dr = N_p \delta_{p,p^*} \quad (5.31)$$

This means that the boundary conditions at $r = R$, need to be specified to have a set of orthogonal basis. Spherical Bessel functions with the zero-boundary condition have been used for all calculations. The normalized term that results when using the orthogonality integrals as in (5.31) and (3.19) is written as:

$$N_p = \frac{R^3}{2} j_1^2(\rho_p) \quad (5.30)$$

where R is the sphere radius for the calculation domain as previously stated and ρ_p are the zeros of the Bessel functions. This normalization imposes zero boundary condition, details are in [46].

The use of this zero-boundary condition includes solutions that are affected by edge of the solver domain resulting in impractical solutions being included, which creates the issue of the many categories of mode profiles that can be found as was discussed in the section 3.4.4.

However, using spherical Bessel function with continued boundary condition enables the solver only solve for more practical solutions. The boundary conditions of the spherical Bessel function are determined by altering the normalized term when applying the orthogonality. The modified normalized term in (5.31) for continued boundary condition is written as [46]:

$$N_p = \frac{R^3}{2} j_0^2(\rho_p). \quad (5.32)$$

Calculations using the continued boundary condition is confirmed by matching results obtained using zero boundary condition. This supports the statement that true localized modes are not affected by the edges of the solver domain.

5.2 Computation Examples

The developed BLF technique derived from Faraday's and Ampere's laws is used in the following sections to study localized modes of various examples. Results obtained using this new matrix formulation is validated as it matches those obtained using the old matrix formulation (chapter 3) when used in section 4.2.

5.2.1 Anisotropic Material

The solutions of the electromagnetic wave in anisotropic spheres have been studied analytically and numerically [54-56]. The numerical analysis of anisotropic structure is applicable using the compact formalization of the BLF technique from section 5.1. The new formalization starts from the curl equation of Faradays and Amperes law, therefore has no derivatives on the dielectric

structure. This simplifies dealing with more complicated dielectric structures, such as tensor dielectric structures. In the following examples, the technique is used to study the resonance modes of anisotropic spherical structures. In anisotropic medium the electric and magnetic properties of the structure are dependent on the direction of the fields' vectors. In such a case, the permittivity and permeability of the medium are defined by matrix tensors. The computation examples here will cover only diagonal permittivity tensor, although the technique is capable of working with full anisotropy tensors. For diagonal permittivity or diagonal permeability, the tensor matrix is written as in (5.3).

In the following computation example, the dielectric will be designed to have diagonal permittivity tensor matrix, and the material is assumed to be non-magnetic and lossless. The anisotropic dielectric sphere is placed in air. The permittivity of the sphere is set to 9 in the azimuthal angle, 5 in the polar angle, and 6.5 in the radial direction.

The inverse relative permittivity of the tensor ($\tilde{\Omega}$) are expanded using 100 basis functions in the radial direction and only one basis for the polar and azimuthal directions as the sphere is uniform in the polar and azimuthal direction. The fields are expanded using 20 basis functions in radial and polar directions. The domain radius is $R=2a$, where $a=1$. The normalized wavenumber of computed azimuthal modes orders is shown in figure 5.1.

Figure 5.1: The mode profiles and wavenumbers for a number of different localized modes in the polar directions. Modes (a and b) have a normalized wavenumber of $ka=2.08$ and (c and d) have normalized wavenumber of $ka=2.96$. Both are plotted using the E_ϕ field. Modes (e) $ka= 3.64$ and (g) $ka=4.13$ are plotted using E_r field vector, (f) $ka=3.64$ and (h) $ka= 4.14$ are plotted using the E_θ field vector.

The resonance modes of the structure for higher mode order in the azimuthal direction are also calculated using the same procedures. The fields are expanded using 40 basis functions, as the purposed mode order is higher. Figure 5.2 shows the localized wavenumber for a number of selected resonance modes in the azimuthal direction.

Figure 5.2: The mode profiles and wavenumbers for a number of different localized modes in the azimuthal direction. The field profiles are for the E_θ field vector.

A perfectly matched layer (PML) is widely used to truncate the computation domain in numerical methods such as FDTD and FEM. The PML has an absorbing layer that is gradually increased from zero. Uniaxial PML (UPML) is an attractive developed method and is accomplished by using anisotropic absorber layer. In equation (5.3) the dielectric profile is represented by a tensor function. To include the UPML in the BLF technique the tensor function is written as:

$$\frac{1}{\varepsilon_r} = \Omega = \begin{bmatrix} \Omega_{rr} + i\Omega_{rr} & 0 & 0 \\ 0 & \Omega_{\theta\theta} & 0 \\ 0 & 0 & \Omega_{\varphi\varphi} \end{bmatrix}, \quad (5.33)$$

In this case, the normal components will undergo the absorption of the layer and reduce reflection from the domain boundary. In spherical 3D domain, the design of the UPML is much easier than

the 3D structure in other coordinates, as it has only one component that strikes the edges of the domain with a normal angle.

5.2.2 Dielectric spherical shell of 6-fold variation in φ

The resonance mode wavelengths and profiles for a spherical shell with 6-fold variation in φ are studied using the BLF technique. Figure 5.3 shows a cross section of the dielectric spherical shell of $\epsilon_r = 9.0 + \cos(6\varphi)$ in the (r, θ) plane. The spherical shell is designed to have a width of $a/3$, with a computation domain radius that is double the sphere radius $R=2a$. In figure 5.4 the relative permittivity variation in the azimuthal direction of the tested spherical shell is shown.

Figure 5.3: A cross section of a dielectric spherical shell is shown, where a is the radius of the spherical shell. The structure is 6-fold symmetric in the azimuthal direction.

For the dielectric spherical shell 50 spherical Bessel basis series expansions are used for the radial direction, one index for Legendre polynomial index in the polar direction, and 50 Fourier

basis functions in the azimuthal directions are used. The polar uses only one expansion as the shell is uniform in the polar direction.

The dielectric spherical shell has 6-fold symmetry in the azimuthal direction; therefore, only indices of $(q_\Omega = 0, \pm 6, \pm 12 \pm 18 \dots)$ are non-zero [27]. For the azimuthal mode, the technique can be configured to study specific mode order. Only the indices of the dielectric in the azimuthal direction plus or minus the targeted azimuthal mode order are needed for the field expansion in the azimuthal angle, as the integral will have zero value for other field's indices [27].

Other field's components were series expanded with 20 spherical Bessel terms and 21 Legendre terms. In Figure 5.5 the normalized resonance wavenumbers, and select mode profiles for the low order states having the 3rd azimuthal mode order are plotted.

Figure 5.4: Relative permittivity variation in the azimuthal direction of the spherical shell.

Figure 5.5: The wavenumbers of 3^{ed} order azimuthal mode are shown. The field profiles are plotted in both the (y, z) plane (left) and in the (x, y) plane (right).

As shown in the figure, the technique is confirmed to study the 3^{ed} azimuthal mode order. The mode order for the radial is fundamental, while for the polar several different orders are shown.

6 Chapter: Sensing application

In recent years, optical devices have gained in importance in the field of sensor systems. Optical sensors use light for measuring a wide variety of environment effects. Compared to their electronic counterparts, optical sensors can provide numerous advantages. For example, they are immune to electromagnetic interference, which make them suitable for operation in environments with high electric field [57]. They are small and light in weight. In general, an optical sensor consists of an optical source, modulators, and a detector. The general principle of such devices is the modification of the light's parameters (intensity, phase, or wavelength) in response to some environmental effects under investigation (temperature, mechanical strain, displacements, pressure, or other quantities). The modulator component modifies the incoming light by changing some physical quantity in response to the environment effect. Spherical resonators are of significant interest in sensing applications, due to the existence of WGMs with a number of important properties: small mode volume, high power density, and narrow spectral line width.

Light can be coupled to a resonator using a waveguide placed near the resonator. Resonance modes will be stimulated by the evanescent coupling of the waveguide light. The resonance wavelengths of the unloaded resonator can be detected from the transmission spectrum of the waveguide. When the quantities under investigation are loaded to the resonator the dips in the transmission spectrum will be shifted in response to the quantities under investigation. The shift of the wavelength is due to the change of the refractive index of the resonator or the size of the resonator because of the loaded quantities. The BLF technique results are used to obtain the resonance wavelengths of the resonators and the change of these wavelengths as the refractive index or size change.

6.1 Spherical dielectric shell

The resonance analysis of a sphere dielectric shell resonator is numerically computed for two sets of mode orders. The calculations focus on studying the localized modes of a number of different orders that vary in polar directions, the first set, or azimuthal directions, the second set. The curves of wavelength shift versus the refractive index change of the dielectric shell are examined and shown.

6.1.1 Results

The segmentation approach is used and the Ω_{off} and Ω_{res} of a dielectric spherical shell are shown in figure 6.1. The radius of the domain, R , is twice the length of the dielectric spherical shell radius a . Further, the dielectric shell thickness is designed to be equals to $a/6$. The changes in the shell dielectric value due to the absorption of imposed materials or temperature effects are modeled using the segmentation procedure. The resonance-mode wavelengths and profiles are calculated using the BLF with both segmentation and normalization, so as to ensure high computational efficiency. In this calculation, the normalization is applied by setting the computation domain radius ($R=2$) and using $a = 1$. Equations (3.35) is used to compute the normalized resonance wavelengths (λ_0/a) of the dielectric spherical shell as a function of the changes in the dielectric values. The matrixes in (3.35) are populated only once Then, by substituting (3.36) into (3.37), the changes in the dielectric values are modeled. This will permit the user to perform faster calculations as it is not needed to populate the matrix every time the dielectric values change.

The Ω_{off} is designed to have 1/12 as the reference value, while Ω_{off}' are changed in value from 1/12.25 to 1/13.25. The Ω_{res} also has a reference inverse relative permittivity of 1-1/12 and Ω_{res}' are changed in the inverse relative permittivity from 1-1/12.25 to 1-1/13.25. The λ_0/a results and

profiles for each dielectric value change are then calculated by solving the eigenvalue equation (3.37).

Figure 6.1: Residual dielectric function (Ω_{res} , left) and offset dielectric function (Ω_{off} , right) of dielectric spherical shell.

In this example a set of modes including the fundamental modes in φ and r with four different mode orders in θ for the E_φ field is the first set to be analyzed. The λ_0/a behavior and the sensitivity curves of the dielectric spherical shell for the first set of modes, as functions of the changes in the dielectric values, are shown in Fig. 6.2.

Another set of mode orders are also computed using the BLF technique with the segmentation process. In this second set, the fundamental mode is in θ and r , and there are four different mode orders in φ . The λ_0/a behavior and the sensitivity curves of the spherical shell for the second set of modes, as functions of the changes in the dielectric values, are shown in Fig. 6.3.

Figure 6.2: Sensitivity curves and mode profiles for first set of modes. The modes order are (Top): $R_1\theta_4\phi_0$, $R_1\theta_8\phi_0$, left to right, (Bottom): $R_1\theta_{18}\phi_0$, $R_1\theta_{30}\phi_0$, left to right.

The results shown in figures 6.2 and 6.3 indicate higher sensitivity for a lower mode order in the polar and azimuthal angles. Note that the sensitivity for a mode order greater than the fundamental in the radial direction and the fundamental order in φ and θ has been studied in [58].

Using (3.30), the designer can set a to target a specific λ_0 at a selected mode order. For example, if the targeted λ_0 is approximately 1550 nm and the targeted mode number is the 8th order, a is calculated to be 1 μm using (3.30). If the value of a is too small for practical fabrication, the mode order can then be increased to achieve a design for the same λ_0 range, but with a larger a . For the 18th mode order, a must be increased to 1.737 μm in order to target the same 1550-nm-wavelength sensing range.

Figure 6.3: Sensitivity curves and mode profiles for second set of modes. The modes order are (Top): $R_1\theta_1\phi_{10}$, $R_1\theta_1\phi_{20}$, left to right, (Bottom): $R_1\theta_1\phi_{40}$, $R_1\theta_1\phi_{60}$, left to right.

The sensitivity is higher for lower-order modes, which is due to the higher confinement of the higher-order modes. Further, the sensitivity increases if a is decreased and, therefore, the dielectric shell thickness decreases.

We next consider a sphere with a relative dielectric permittivity value $\epsilon_{r-low} = 4$, which is coated with a high dielectric layer of $\epsilon_{r-high} = 12$. The substrate and overlay of the structure can be combined as a reference matrix and the perturbation of the third part (the air region) is applied by changing the inverse relative permittivity from 1 to 0.4, see figure 6.4.

Figure 6.4: Reference and perturbation profiles of the inverse relative permittivity. The top figure shows the original structure. The two figures in the bottom show the segmentations of the original structure into reference (left) and large perturbation (right).

The structure can also be treating as a sum of separate multiple values of the relative permittivity, see figure 6.5. The wavelength shift versus the dielectric change of the ambient medium is studied using (3.37), and the resonance shifts of two selected modes, $R_1\theta_1\varphi_{10}$ and $R_1\theta_1\varphi_{20}$, are shown in figure 6.6. The calculation procedure used here is similar to that employed in previous sections.

The resultant shifts of λ_0 are normalized; therefore, this approach can be applied to designs for a large range of wavelengths. This versatility is a significant advantage for the developed BLF technique.

Figure 6.5: Segmentation into 3 separated profiles, a profile for each material constituent present in the resonator structure.

Figure 6.6: Sensitivity curves of ambient medium and mode profiles of E_θ for selected modes. The mode profile is shown in the (φ, r) plane. The radius length of the dielectric spherical shell is normalized.

For small perturbations, equations (3.38) and (3.39) are used to study the change of the air region with perturbation of 0.2 of the relative permittivity, as shown in figure 6.7. The results match the segmentation approach with 1% error.

Figure 6.7: Reference and perturbation inverse relative permittivity profiles. The top figure shows the original structure. The two figures in the bottom show the segmentations of the original structure into reference (left) and small perturbation (right).

Both segmentation and radius normalization were shown to be very effective methods of enhancing the BLF-technique computational process.

6.2 Slot WGMs channel design

Slot channel waveguides are made by having low dielectric material sandwiched between two higher dielectric regions. In this kind of geometry the field will be predominantly localized in the air region, making this kind of structure less affected by the material loss. The continuity condition of the normal component of the electric flux density D_{normal} between the dielectrics interface provide a large electric field in the low dielectric region. Continuity of the normal component of D gives

$$D_{normal, \epsilon_{low}} = D_{normal, \epsilon_{high}} \quad (5.1)$$

$$E_{normal, \epsilon_{low}} = \frac{\epsilon_{high}}{\epsilon_{low}} E_{normal, \epsilon_{high}} \quad (5.2)$$

This demonstrates a large discontinuity of the normal component of the electric field with $\frac{\epsilon_{high}}{\epsilon_{low}}$ times higher electric field in the low dielectric region than in the high dielectric region. For Si – air interface the slot channel will have electric field that is about 12 times larger than the Si region. When the guided modes in the high dielectric regions have a decaying length that is larger than the slot channel width the discontinuity of the field normal component will be gained. The theoretical computations for slot channel waveguide in a cylindrical structure have previously been presented in using the FFB technique [36].

Figure 6.8: Side view of the proposed WGMs slot waveguide structure. The sphere has high relative permittivity value with squared slot channel filled with air.

Figure 6.9: The (x, y) plane modal E_θ field component (Left), and the (y, z) plane modal E_θ field component (Right) for the $R_{1\theta_1\phi_0}$ localized state in the slot channel region. The field is high in the slot channel region. The bar on the left shows the normalized values for the resonance field.

In figure 6.8, a sphere slot channel configuration is shown. The type of slot and the size will depend on many factors. For large WGMs order, the size of the slot channel has to be small compared to the sphere size, figure 6.8. This has the advantage of targeting small wavelengths modes, such as the $1.55 \mu m$ in a large sphere size. In figure 6.9, the slot channel localized light is shown. The field are strongly localized in the slot region.

7 Chapter: Conclusion

7.1 Summary

A technique for solving electromagnetic problems is presented. The technique is based on BLF basis functions in spherical coordinate system. In this approach, the vector wave equations for the H and E fields, which were obtained from Maxwell's equations, are solved in spherical coordinates and the BLF functions are used as the basis of expansion for the fields and the dielectric. Modified spherical Bessel functions are used in the radial direction (r), modified Legendre functions are used for the polar angle (θ), and the complex exponential functions (Fourier) are used for azimuthal angle (φ). The expanded field and dielectric functions are inserted into the vector wave equation to form the BLF eigenvalue matrix expression. The solution of the BLF eigenvalue matrix returns the eigenvalues and eigenvectors, for localized modes. The eigenvalues are the squared normalized wavenumber, and the eigenvectors are the expansion coefficients for the mode profile. The validity of the technique is confirmed as results obtained using the BLF technique match those commonly available in textbooks and journal publications obtained using other different techniques. The BLF has been applied to a number of spherically symmetric structures and applications (spherical cavity, dielectric spheres, spherical Bragg shell, dielectric discs, anisotropic structure, lossy and lossless structure), and the steady state modes supported are reported.

The technique has the advantage of being a fast and low resource solver for localized electromagnetic modes in dielectric structures with spherical symmetry. The BLF technique can target a specific selected localized mode, by making use of the symmetry of the spherical structures, allowing reduction in the matrix size. In addition, segmentation and radius normalization were shown to be very effective methods of enhancing the BLF-technique

computational process. Basis used in this technique are very suitable to represent resonance modes and to target a specific mode order in the azimuthal direction.

The technique can also be written and developed in a more convenient manner and was presented here. The matrix formation and expression derivation were developed for the time independent differential form of Faraday's law and Ampere's law, by passing the derivatives of the material properties found in the wave equations. This formalization has many advantages as it is simpler for development and can deal with anisotropic structures.

7.2 Future work

Many interesting developing ideas can be added to further enhancement for the BLF technique. One potential for the use of BLF is to explore the nature of localized modes in disorder structures. It is also interesting to consider the use of the BLF method in an inverse approach, using a known localized mode of unknown structure to determine the profile of the dielectric structure. In addition, it is expected that future work will include the use of different basis functions.

Outside the field of optics the BLF technique can be applied to Schrodinger's equation to study energy states available for electrons in atoms where the basic geometry is spherical, or the wave function profile can be calculated for existing energies. The BLF technique can also be applied to the acoustic wave equation, to calculate localized acoustic wave frequencies and profiles.

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Appendix A: Wave Equation for H field

The numerical solver is based on the electromagnetic wave equations for either the H or E fields. The equation (3.14) in the text is the wave equation written in the spherical coordinate and the derivative details are shown here.

The wave equation for the $\bar{H}$ field is:

$$\bar{\nabla} \times \frac{1}{\varepsilon_r(r, \theta, \varphi)} \bar{\nabla} \times \bar{H} = \left(\frac{\omega}{c}\right)^2 \bar{H}. \quad (\text{A.1})$$

The curl operation in spherical coordinate for a vector $\bar{H}$ is:

$$\nabla \times \bar{H} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (H_\varphi \sin \theta) - \frac{\partial H_\theta}{\partial \varphi} \right) \hat{r} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial H_r}{\partial \varphi} - \frac{\partial}{\partial r} (r H_\varphi) \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{\partial \theta} \right) \hat{\varphi}. \quad (\text{A.2})$$

The curl of a vector is vector that can be defined as $\nabla \times \bar{H} = \bar{C}$ with:

$$\bar{C} = C_r \hat{r} + C_\theta \hat{\theta} + C_\varphi \hat{\varphi} \quad (\text{A.3})$$

$$C_r = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (H_\varphi \sin \theta) - \frac{\partial H_\theta}{\partial \varphi} \right) \quad (\text{A.4})$$

$$C_\theta = \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial H_r}{\partial \varphi} - \frac{\partial}{\partial r} (r H_\varphi) \right) \quad (\text{A.5})$$

$$C_\varphi = \frac{1}{r} \left(\frac{\partial}{\partial r} (r H_\theta) - \frac{\partial H_r}{\partial \theta} \right) \quad (\text{A.6})$$

For the $\bar{H}$ wave equation, (A1), the first curl operation is multiplied by the inverse of the relative permittivity $\frac{1}{\varepsilon_r(r, \theta, \varphi)} = \Omega$. Therefore the inverse of the relative permittivity, which is a scalar valued function, must be multiplied by the vector $\bar{C}$. The resulting expression is:

$$\bar{F} = \Omega C_r \hat{r} + \Omega C_\theta \hat{\theta} + \Omega C_\varphi \hat{\varphi}, \quad (\text{A.7})$$

$$\bar{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_\varphi \hat{\varphi}. \quad (\text{A.8})$$

The second curl operation is then applied to the vector $\bar{F}$, which gives

$$\nabla \times \bar{F} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (F_\varphi \sin \theta) - \frac{\partial F_\theta}{\partial \varphi} \right) \hat{r} + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial F_r}{\partial \varphi} - \frac{\partial}{\partial r} (r F_\varphi) \right) \hat{\theta} + \frac{1}{r} \left(\frac{\partial}{\partial r} (r F_\theta) - \frac{\partial F_r}{\partial \theta} \right) \hat{\varphi} \quad (\text{A.9})$$

Once again, the result defines a new vector $\nabla \times \bar{F} = \bar{W}$ with:

$$\bar{W} = W_r \hat{r} + W_\theta \hat{\theta} + W_\varphi \hat{\varphi} \quad (\text{A.10})$$

The components of $\bar{W}$ are given below. Performing the derivatives results in the matrix expression provided in appendix C. The component of $\bar{W}$ are:

$$\begin{aligned}
W_r = & \frac{\partial \Omega}{\partial \theta} \left[\frac{H_\theta}{r^2} + \frac{1}{r} \frac{\partial H_\theta}{\partial r} - \frac{1}{r^2} \frac{\partial H_r}{\partial \theta} \right] + \frac{\partial \Omega}{\partial \varphi} \left[-\frac{1}{r^2 (\sin \theta)^2} \frac{\partial H_r}{\partial \varphi} + \frac{H_\varphi}{r^2 \sin \theta} + \frac{1}{r \sin \theta} \frac{\partial H_\varphi}{\partial r} \right] \\
& + \frac{\Omega}{r^2} \left[\frac{H_\theta \cos \theta}{\sin \theta} - \frac{\cos \theta}{\sin \theta} \frac{\partial H_r}{\partial \theta} + \frac{\partial H_\theta}{\partial \theta} - \frac{\partial^2 H_r}{\partial \theta^2} - \frac{1}{(\sin \theta)^2} \frac{\partial^2 H_r}{\partial \varphi^2} + \frac{1}{\sin \theta} \frac{\partial H_\varphi}{\partial \varphi} \right] \\
& + \frac{\Omega}{r} \left[\frac{1}{\sin \theta} \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} + \frac{\cos \theta}{\sin \theta} \frac{\partial H_\theta}{\partial r} + \frac{\partial^2 H_\theta}{\partial \theta \partial r} \right]
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
W_\theta = & \frac{\partial \Omega}{\partial \varphi} \left[\frac{1}{r^2 \sin \theta} \frac{\partial H_\varphi}{\partial \theta} + \frac{H_\varphi}{r^2 (\sin \theta)^2} \cos \theta - \frac{1}{r^2 (\sin \theta)^2} \frac{\partial H_\theta}{\partial \varphi} \right] + \frac{\partial \Omega}{\partial r} \left[-\frac{H_\theta}{r} - \frac{\partial H_\theta}{\partial r} + \frac{1}{r} \frac{\partial H_r}{\partial \theta} \right] \\
& + \frac{\Omega}{r^2} \left[\frac{1}{\sin \theta} \frac{\partial^2 H_\varphi}{\partial \varphi \partial \theta} + \frac{1}{(\sin \theta)^2} \cos \theta \frac{\partial H_\varphi}{\partial \varphi} - \frac{1}{(\sin \theta)^2} \frac{\partial^2 H_\theta}{\partial \varphi^2} \right] \\
& + \frac{\Omega}{r} \left[-2 \frac{\partial H_\theta}{\partial r} + \frac{\partial^2 H_r}{\partial r \partial \theta} \right] - \Omega \left[\frac{\partial^2 H_\theta}{\partial r^2} \right]
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
W_\varphi = & \frac{\partial \Omega}{\partial r} \left[\frac{1}{r \sin \theta} \frac{\partial H_r}{\partial \varphi} - \frac{H_\varphi}{r} - \frac{\partial H_\varphi}{\partial r} \right] + \frac{\partial \Omega}{\partial \theta} \left[-\frac{1}{r^2} \frac{\partial H_\varphi}{\partial \theta} - \frac{1}{r^2} \frac{\cos \theta}{\sin \theta} H_\varphi + \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial H_\theta}{\partial \varphi} \right] \\
& + \frac{\Omega}{r^2} \left[\frac{(\cos \theta)^2}{(\sin \theta)^2} H_\varphi - \frac{\cos \theta}{(\sin \theta)^2} \frac{\partial H_\theta}{\partial \varphi} - \frac{\partial^2 H_\varphi}{\partial \theta^2} - \frac{\cos \theta}{\sin \theta} \frac{\partial H_\varphi}{\partial \theta} + H_\varphi + \frac{1}{\sin \theta} \frac{\partial^2 H_\theta}{\partial \theta \partial \varphi} \right] \\
& + \frac{\Omega}{r} \left[\frac{1}{\sin \theta} \frac{\partial^2 H_r}{\partial r \partial \varphi} - 2 \frac{\partial H_\varphi}{\partial r} \right] - \Omega \left[\frac{\partial^2 H_\varphi}{\partial r^2} \right]
\end{aligned} \tag{A.13}$$

Appendix B: Derivatives of the Basis Functions

The numerical solver uses series expansions for the fields and inverse relative permittivity. The basis functions of the series are composed of the product of spherical Bessel functions, complex exponentials and Legendre polynomials. Derivatives of the basis function are required and provided here.

For the spherical Bessel function, the first derivative of the lowest order spherical Bessel function in generic form is:

$$\frac{d}{dr}j_0(r) = -j_1(r) \quad (\text{B.1})$$

However, the spherical Bessel function in this analysis has the form of $j_0(Ar)$ where A is a constant, therefore a change of variable is required,

$$Ar = \acute{r}, A\partial r = \partial \acute{r}, \frac{d}{d\acute{r}}j_0(\acute{r}) = -j_1(\acute{r}) \quad (\text{B.2})$$

$$\frac{d}{dr}j_0(Ar) = -Aj_1(Ar) \quad (\text{B.3})$$

The second derivative of the lowest order spherical Bessel function in generic form is:

$$\frac{d}{dr}j_1(r) = j_0(r) - \frac{2j_1(r)}{r} \quad (\text{B.4})$$

Again using change of variable:

$$Ar = \acute{r}, A\partial r = \partial \acute{r}, \frac{d}{d\acute{r}}j_1(\acute{r}) = j_0(\acute{r}) - \frac{2j_1(\acute{r})}{\acute{r}} \quad (\text{B.5})$$

$$\frac{d}{dr}j_1(Ar) = Aj_0(Ar) - \frac{2j_1(Ar)}{r} \quad (\text{B.6})$$

The normalization benefits the BLF technique by ensuring that the spherical Bessel integral tables are independent of the structure radius. Therefore, a large portion of the computations need only be calculated once, and then can be used directly for different structures. The integrals from the derivation of the matrix populating equations are normalized as follow:

$$\int_0^R j_0\left(k_{p_\Omega} \frac{r}{R}\right) j_0\left(k_{p_{H_\theta}} \frac{r}{R}\right) j_0\left(k_{p_{H_\theta}^*} \frac{r}{R}\right) r^\nu dr, \quad (\text{B.7})$$

$$\frac{r}{R} = \zeta, \quad r = R\zeta, \quad \frac{dr}{d\zeta} = \frac{d(R\zeta)}{d\zeta} = R, \quad dr = R d\zeta, \quad (\text{B.8})$$

$$\int_0^1 j_0(k_{p_\Omega} \zeta) j_0(k_{p_{H_\theta}} \zeta) j_0(k_{p_{H_\theta^*}} \zeta) R^v \zeta^v R d\zeta, \quad (\text{B.9})$$

$$R^{v+1} \int_0^1 j_a(k_{p_\Omega} \zeta) j_b(k_p \zeta) j_c(k_{p^*} \zeta) \zeta^v d\zeta. \quad (\text{B.10})$$

The first derivative of spherical Legendre polynomials function is

$$\frac{dP_n(\mu)}{d\theta} = \frac{n\mu P_n(\mu)}{\sqrt{1-\mu^2}} - \frac{(n)P_{(n-1)}(\mu)}{\sqrt{1-\mu^2}}. \quad (\text{B.11})$$

From the spherical Legendre polynomials equation, the second derivative of spherical Legendre polynomials function is:

$$\frac{d^2P_n(\mu)}{d\theta^2} = -(n^2 + n)P_n(\mu) - \frac{\cos \theta}{\sin \theta} \frac{dP_n(\mu)}{d\theta} \quad (\text{B.12})$$

$$\frac{d^2P_n(\mu)}{d\theta^2} = -(n^2 + n)P_n(\mu) - \frac{\cos \theta}{\sin \theta} \left[\frac{n \cos \theta P_n(\mu)}{\sin \theta} - \frac{(n)P_{(n-1)}(\mu)}{\sin \theta} \right] \quad (\text{B.13})$$

$$\frac{d^2P_n(\mu)}{d\theta^2} = -(n^2 + n)P_n(\mu) - \frac{n(\cos \theta)^2 P_n(\mu)}{(\sin \theta)^2} + \frac{(n) \cos \theta P_{(n-1)}(\mu)}{(\sin \theta)^2} \quad (\text{B.14})$$

These derivatives are used for building the matrix expression. Note that as discussed in Chapter 3 only the zeroth order and first order spherical Bessel are required.

Appendix C: Developing the Matrix Expression

The expressions for the eigenvalue matrix are developed here. The wave equations in spherical space are used to obtain the eigenvalue matrix expression.

For the radial component of the equation of (A.1), making use of (A.11) gives

$$\begin{aligned}
& \frac{H_\theta}{r^2} \frac{\partial \Omega}{\partial \theta} + \frac{1}{r} \frac{\partial \Omega}{\partial \theta} \frac{\partial H_\theta}{\partial r} - \frac{1}{r^2} \frac{\partial \Omega}{\partial \theta} \frac{\partial H_r}{\partial \theta} + \frac{H_\theta \Omega \cos \theta}{r^2 \sin \theta} + \frac{\Omega \cos \theta}{r \sin \theta} \frac{\partial H_\theta}{\partial r} - \frac{\Omega \cos \theta}{r^2 \sin \theta} \frac{\partial H_r}{\partial \theta} \\
& + \frac{\Omega}{r^2} \frac{\partial H_\theta}{\partial \theta} + \frac{\Omega}{r} \frac{\partial^2 H_\theta}{\partial \theta \partial r} - \frac{\Omega}{r^2} \frac{\partial^2 H_r}{\partial \theta^2} - \frac{1}{r^2 (\sin \theta)^2} \frac{\partial \Omega}{\partial \varphi} \frac{\partial H_r}{\partial \varphi} + \frac{H_\varphi}{r^2 \sin \theta} \frac{\partial \Omega}{\partial \varphi} + \frac{1}{r \sin \theta} \frac{\partial \Omega}{\partial \varphi} \frac{\partial H_\varphi}{\partial r} \\
& - \frac{\Omega}{r^2 (\sin \theta)^2} \frac{\partial^2 H_r}{\partial \varphi^2} + \frac{\Omega}{r^2 \sin \theta} \frac{\partial H_\varphi}{\partial \varphi} + \frac{\Omega}{r \sin \theta} \frac{\partial^2 H_\varphi}{\partial \varphi \partial r} = \left(\frac{\omega}{c}\right)^2 H_r
\end{aligned} \tag{C.1}$$

The expansion series for H_r, H_θ, H_φ (3.8) and Ω (3.9) are then substituted into each term of the equation. The details will be provided for the first four terms, remaining terms are obtained using similar steps.

Term 1:

$$\begin{aligned}
& \frac{H_\theta}{r^2} \frac{\partial \Omega}{\partial \theta} \\
& = \frac{1}{r^2} \left[\sum_{p_\theta, n_\theta, q_\theta} \kappa_\theta j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \right] \left[\sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \frac{n_\Omega \cos \theta}{\sin \theta} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \right. \\
& \quad \left. - \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \frac{n_\Omega}{\sin \theta} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} \right]
\end{aligned} \tag{C.1.a}$$

$$\begin{aligned}
&= \sum_{p_\theta, n_\theta, q_\theta} \sum_{m_\Omega, n_\Omega, q_\Omega} \kappa_\theta \kappa_\Omega \frac{n_\Omega \cos \theta}{r^2 \sin \theta} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \\
&+ \sum_{p_\theta, n_\theta, q_\theta} \sum_{m_\Omega, n_\Omega, q_\Omega} \kappa_\theta \kappa_\Omega \left(-\frac{n_\Omega}{r^2 \sin \theta} \right) j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi}
\end{aligned}
\tag{C.1.a}$$

Term 2:

$$\begin{aligned}
&\frac{1}{r} \frac{\partial \Omega}{\partial \theta} \frac{\partial H_\theta}{\partial r} \\
&= \frac{1}{r} \left[\sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \frac{n_\Omega \cos \theta}{\sin \theta} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \right. \\
&\quad \left. - \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \frac{n_\Omega}{\sin \theta} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} \right] \left[\sum_{p_\theta, n_\theta, q_\theta} \kappa_\theta \left(-\frac{\rho_{p_\theta}}{R} \right) j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \right]
\end{aligned}
\tag{C.1.b}$$

$$\begin{aligned}
&= \sum_{p_\theta, n_\theta, q_\theta} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\theta \kappa_\Omega \left(-\frac{\rho_{p_\theta} n_\Omega \cos \theta}{r R \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
&+ \sum_{p_\theta, n_\theta, q_\theta} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\theta \kappa_\Omega \left(\frac{\rho_{p_\theta} n_\Omega}{r R \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi}
\end{aligned}
\tag{C.1.b}$$

Term 3:

$$\begin{aligned}
& -\frac{1}{r^2} \frac{\partial \Omega}{\partial \theta} \frac{\partial H_r}{\partial \theta} \\
& = -\frac{1}{r^2} \left[\sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \frac{n_\Omega \cos \theta}{\sin \theta} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \right. \\
& \quad - \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \frac{n_\Omega}{\sin \theta} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} \left. \right] \left[\sum_{p_r, n_r, q_r} \kappa_r \frac{n_r \cos \theta}{\sin \theta} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \right. \\
& \quad \left. - \sum_{p_r, n_r, q_r} \kappa_r \frac{n_r}{\sin \theta} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi} \right]
\end{aligned}
\tag{C.1.c}$$

$$\begin{aligned}
& = \sum_{p_r, n_r, q_r} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_r \kappa_\Omega \left(-\frac{n_r n_\Omega (\cos \theta)^2}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \\
& + \sum_{p_r, n_r, q_r} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_r \kappa_\Omega \left(\frac{n_r n_\Omega \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi} \\
& + \sum_{p_r, n_r, q_r} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_r \kappa_\Omega \left(\frac{n_r n_\Omega \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} \\
& + \sum_{p_r, n_r, q_r} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_r \kappa_\Omega \left(-\frac{n_r n_\Omega}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi}
\end{aligned}
\tag{C.1.c}$$

Similar procedures are applied for all other terms. Rearranging and collecting similar terms the full radial equation is:

$$\begin{aligned}
R_r = & \\
= & \sum_{\Omega H_r} \left(-\frac{\kappa_{H_r} \kappa_{\Omega} n_{\Omega} n_{H_r} (\cos \theta)^2}{r^2 (\sin \theta)^2} - \frac{\kappa^{\Omega} \kappa^{H_r} (\cos \theta)^2 n_{H_r}}{r^2 (\sin \theta)^2} + \frac{\kappa^{\Omega} \kappa^{H_r} (n_{H_r}^2 + n_{H_r})}{r^2} + \frac{\kappa^{\Omega} \kappa^{H_r} n_{H_r} (\cos \theta)^2}{r^2 (\sin \theta)^2} \right. \\
& + \frac{\kappa_{H_r} \kappa_{\Omega} (q_{\Omega}) (q_{H_r})}{r^2 (\sin \theta)^2} \\
& + \frac{\kappa_{H_r} \kappa_{\Omega} (q_{H_r}^2)}{r^2 (\sin \theta)^2} \Big) j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) P_{n_{\Omega}}(\cos(\theta)) e^{iq_{\Omega} \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \\
& + \sum_{\Omega H_r} \left(\frac{\kappa_{H_r} \kappa_{\Omega} (n_{\Omega} n_{H_r}) \cos \theta}{r^2 (\sin \theta)^2} + \frac{\kappa_{H_r} \kappa_{\Omega} \cos \theta (n_{H_r})}{r^2 (\sin \theta)^2} \right. \\
& - \frac{\kappa_{H_r} \kappa_{\Omega} (n_{H_r}) \cos \theta}{r^2 (\sin \theta)^2} \Big) j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) P_{n_{\Omega}}(\cos(\theta)) e^{iq_{\Omega} \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi} \\
& + \sum_{\Omega H_r} \left(\frac{\kappa_{H_r} \kappa_{\Omega} (n_{H_r} n_{\Omega}) \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) P_{n_{\Omega}-1}(\cos(\theta)) e^{iq_{\Omega} \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \\
& - \sum_{\Omega H_r} \frac{\kappa_{H_r} \kappa_{\Omega} (n_{\Omega}) (n_{H_r})}{r^2 (\sin \theta)^2} j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) P_{n_{\Omega}-1}(\cos(\theta)) e^{iq_{\Omega} \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi}
\end{aligned} \tag{C.2}$$

The polar and azimuthal components of (A1) can be obtained similarly. For the polar the expression is:

$$\theta_r =$$

$$\begin{aligned}
& \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left(\frac{n_\Omega \cos \theta}{r^2 \sin \theta} + \frac{\cos \theta}{r^2 \sin \theta} \right. \\
& \quad \left. + \frac{n_\theta \cos \theta}{r^2 \sin \theta} \right) j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \\
& + \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left(-\frac{n_\Omega \cos \theta k_{p_\theta}}{rR \sin \theta} - \frac{\cos \theta k_{p_\theta}}{rR \sin \theta} \right. \\
& \quad \left. - \frac{k_{p_\theta} n_\theta \cos \theta}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
& - \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left(\frac{n_\Omega}{r^2 \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
& + \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left(\frac{n_\Omega k_{p_{H_\theta}}}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
& - \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left(\frac{n_{H_\theta}}{r^2 \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta-1}(\cos(\theta)) e^{iq_\theta \varphi} \\
& + \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left(\frac{k_{p_{H_\theta}} n_{H_\theta}}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta-1}(\cos(\theta)) e^{iq_\theta \varphi}
\end{aligned} \tag{C.3}$$

and the azimuthal coordinate the expression is:

$$\varphi_r =$$

$$\begin{aligned}
& + \sum_{\Omega H_\varphi} \left(\frac{\kappa_{H_\varphi} \kappa_\Omega (iq_\Omega)}{r^2 \sin \theta} \right. \\
& \quad \left. + \frac{\kappa^\Omega \kappa^{H_\varphi} (iq_{H_\varphi})}{r^2 \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \\
& + \sum_{\Omega H_\varphi} \left(- \frac{\kappa_{H_\varphi} \kappa_\Omega (iq_\Omega) k_{p_{H_\varphi}}}{rR \sin \theta} \right. \\
& \quad \left. - \frac{\kappa_{H_\varphi} \kappa_\Omega (iq_{H_\varphi}) k_{p_{H_\varphi}}}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi}
\end{aligned} \tag{C.4}$$

$$R_r + \theta_r + \varphi_r = \left(\frac{\omega}{c} \right)^2 \sum_{H_r} \kappa_r j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \tag{C.5}$$

The left-hand side each term of the equation (C.5) has a summation over one of the three field components indices (R_r, θ_r, φ_r), and also a summation over the inverse relative permittivity indices. In the right-hand side it is only the summation over the radial field component indices. Each of the three field components summations on the left-hand side represents one of the blocks of the eigenvalue matrix. However, orthogonality needs to be applied in order to obtain the final eigenvalue matrix expression. Orthogonality is applied by multiplying the equations by orthogonal functions with a summation over the orthogonal radial indices, and then an integration over the area of the sphere domain. The details of this operation were shown in the text for the right-hand side. The steps in details are shown here for a few terms for the left-hand side as follow.

The first collected term (R_{r_a}) is written as:

$$R_{r_a} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{n_\Omega n_r (\cos \theta)^2}{r^2 (\sin \theta)^2} + \frac{(n_r^2 + n_r)}{r^2} \right. \\ \left. + \frac{(q_\Omega q_r) + (q_r^2)}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi}$$

Applying orthogonality:

$$R_{r_a} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [R_{r_a}] j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) P_{n_r^*}(\cos(\theta)) e^{iq_r^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi \\ \rightarrow \sum_{\Omega H_r} \kappa_r \kappa_\Omega \int_0^\pi \left(-\frac{n_\Omega n_r (\cos \theta)^2}{\sin \theta} + (n_r^2 + n_r) \sin \theta + \frac{(q_\Omega q_r) + (q_r^2)}{\sin \theta} \right) P_{n_\Omega}(\cos(\theta)) P_{n_r}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \, d\theta \\ \int_0^R j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_\Omega \varphi} e^{-iq_r^* \varphi} d\varphi$$

Using the normalized form of the integral (see appendix B)

$$\int_0^R j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) dr = R \int_0^1 j_0(\rho_{p_\Omega} \hat{r}) j_0(\rho_{p_r} \hat{r}) j_0(\rho_{p_r^*} \hat{r}) d\hat{r}$$

using $\xi_{000} = \int_0^1 j_0(\rho_{p_\Omega} \hat{r}) j_0(\rho_{p_r} \hat{r}) j_0(\rho_{p_r^*} \hat{r}) d\hat{r}$

and $P_{c^0 s^{-1} 00} = \int_0^\pi \frac{1}{\sin \theta} P_{n_\Omega}(\cos(\theta)) P_{n_r}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) d\theta$

$$R_{r_a} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega [-(n_\Omega n_r) P_{c^2 s^{-1} 00} + (n_r^2 + n_r) P_{c^0 s^{-1} 00} + (q_\Omega q_r + q_r^2) P_{c^0 s^{-1} 00}] R \xi_{000} \llbracket 2\pi \rrbracket_{q_r + q_\Omega = q_r^*}$$

The second collected term (R_{r_b}) is written as:

$$R_{r_b} = + \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(\frac{(n_\Omega n_r) \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi}$$

$$\begin{aligned}
R_{r_b} &\rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [R_{r_b}] j_0\left(\rho_{p_r^*} \frac{r}{R}\right) P_{n_r^*}(\cos(\theta)) e^{iq_r^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi \\
&\rightarrow \sum_{\Omega H_r} \kappa_r \kappa_\Omega \int_0^\pi \left(\frac{(n_\Omega n_r) \cos \theta}{\sin \theta} \right) P_{n_\Omega}(\cos(\theta)) P_{n_r-1}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \, d\theta \\
&\quad \int_0^R j_0\left(\rho_{p_\Omega} \frac{r}{R}\right) j_0\left(\rho_{p_r} \frac{r}{R}\right) j_0\left(\rho_{p_r^*} \frac{r}{R}\right) dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_\Omega \varphi} e^{-iq_r^* \varphi} d\varphi
\end{aligned}$$

using $\xi_{000} = \int_0^1 j_0(\rho_{p_\Omega} \dot{r}) j_0(\rho_{p_r} \dot{r}) j_0(\rho_{p_r^*} \dot{r}) \, d\dot{r}$

and $P_{c^1 s^{-1} 00} = \int_0^\pi \frac{\cos \theta}{\sin \theta} P_{n_\Omega}(\cos(\theta)) P_{n_r-1}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \, d\theta$

$$R_{r_b} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega [(n_\Omega n_r) P_{c^1 s^{-1} 01}] R \xi_{000} \llbracket 2\pi \rrbracket_{q_r+q_\Omega=q_r^*}$$

The third collected term (R_{r_c}) is written as:

$$\begin{aligned}
R_{r_c} &= + \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(\frac{(n_r n_\Omega) \cos \theta}{r^2 (\sin \theta)^2} \right) j_0\left(\rho_{p_\Omega} \frac{r}{R}\right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} j_0\left(\rho_{p_r} \frac{r}{R}\right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \\
R_{r_c} &\rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [R_{r_c}] j_0\left(\rho_{p_r^*} \frac{r}{R}\right) P_{n_r^*}(\cos(\theta)) e^{iq_r^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi \\
&\quad \sum_{\Omega H_r} \kappa_r \kappa_\Omega \int_0^\pi \left(\frac{(n_\Omega n_r) \cos \theta}{\sin \theta} \right) P_{n_\Omega-1}(\cos(\theta)) P_{n_r}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \, d\theta \\
&\quad \int_0^R j_0\left(\rho_{p_\Omega} \frac{r}{R}\right) j_0\left(\rho_{p_r} \frac{r}{R}\right) j_0\left(\rho_{p_r^*} \frac{r}{R}\right) dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_\Omega \varphi} e^{-iq_r^* \varphi} d\varphi \\
R_{r_c} &= \sum_{\Omega H_r} \kappa_r \kappa_\Omega [(n_\Omega n_r) P_{c^1 s^{-1} 10}] R \xi_{000} \llbracket 2\pi \rrbracket_{q_r+q_\Omega=q_r^*}
\end{aligned}$$

The fourth collected term (R_{r_d}) is written as:

$$R_{r-d} = - \sum_{\Omega H_r} \kappa_r \kappa_{\Omega} \frac{(n_{\Omega})(n_{H_r})}{r^2 (\sin \theta)^2} j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) P_{n_{\Omega}-1}(\cos(\theta)) e^{iq_{\Omega} \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi}$$

$$R_{r-d} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [R_{r-d}] j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) P_{n_r^*}(\cos(\theta)) e^{iq_r^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\sum_{\Omega H_r} \kappa_r \kappa_{\Omega} \int_0^{\pi} \left(\frac{(n_{\Omega} n_r)}{\sin \theta} \right) P_{n_{\Omega}-1}(\cos(\theta)) P_{n_r-1}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \, d\theta$$

$$\int_0^R j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_{\Omega} \varphi} e^{-iq_r^* \varphi} d\varphi$$

$$R_{r-d} = \sum_{\Omega H_r} \kappa_r \kappa_{\Omega} [-(n_{\Omega} n_r) P_{c^0 s^{-1} 11}] R \, \xi_{000} \llbracket 2\pi \rrbracket_{q_r+q_{\Omega}=q_r^*}$$

Collecting all of the above expressions gives

$$R_r =$$

$$\begin{aligned} \sum_{\Omega H_r} \kappa_r \kappa_{\Omega} [-(n_{\Omega} n_{H_r}) P_{c^2 s^{-1} 00} + (n_{H_r}^2 + n_{H_r}) P_{c^0 s^{-1} 00} + (q_{\Omega} q_{H_r} + q_{H_r}^2) P_{c^0 s^{-1} 00} + (n_{\Omega} n_{H_r}) P_{c^1 s^{-1} 01} \\ + (n_{\Omega} n_{H_r}) P_{c^1 s^{-1} 10} - (n_{\Omega} n_{H_r}) P_{c^0 s^{-1} 11}] R \, \xi_{000} \llbracket 2\pi \rrbracket_{q_r+q_{\Omega}=q_r^*} \end{aligned}$$

Similar derivations are applied for the polar coordinate terms. The first collected term ($R_{\theta-a}$) is written as:

$$\theta_{r-a} =$$

$$\sum_{H_{\theta} \Omega} \kappa_{\theta} \kappa_{\Omega} \left(\frac{n_{\Omega} \cos \theta}{r^2 \sin \theta} + \frac{\cos \theta}{r^2 \sin \theta} + \frac{n_{\theta} \cos \theta}{r^2 \sin \theta} \right) j_0 \left(\rho_{p_{\theta}} \frac{r}{R} \right) P_{n_{\theta}}(\cos(\theta)) e^{iq_{\theta} \varphi} j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) P_{n_{\Omega}}(\cos(\theta)) e^{iq_{\Omega} \varphi}$$

$$\sum_{H_{\theta} \Omega} \kappa_{\theta} \kappa_{\Omega} \left(\frac{(n_{\Omega} + n_{\theta} + 1) \cos \theta}{r^2 \sin \theta} \right) j_0 \left(\rho_{p_{\theta}} \frac{r}{R} \right) P_{n_{\theta}}(\cos(\theta)) e^{iq_{\theta} \varphi} j_0 \left(\rho_{p_{\Omega}} \frac{r}{R} \right) P_{n_{\Omega}}(\cos(\theta)) e^{iq_{\Omega} \varphi}$$

$$\theta_{r-a} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [R_{\theta-a}] j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) P_{n_r^*}(\cos(\theta)) e^{iq_r^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\rightarrow \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \int_0^\pi \left(\frac{(n_\Omega + n_{H_\theta} + 1) \cos \theta}{\sin \theta} \right) P_{n_\Omega}(\cos(\theta)) P_{n_\theta}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \sin \theta d\theta$$

$$\int_0^R j_0\left(\rho_{p_\Omega} \frac{r}{R}\right) j_0\left(\rho_{p_\theta} \frac{r}{R}\right) j_0\left(\rho_{p_r^*} \frac{r}{R}\right) dr \int_0^{2\pi} e^{iq_\theta \varphi} e^{iq_\Omega \varphi} e^{-iq_r^* \varphi} d\varphi$$

Using the normalized form of the integral

$$\int_0^R j_0\left(\rho_{p_\Omega} \frac{r}{R}\right) j_0\left(\rho_{p_\theta} \frac{r}{R}\right) j_0\left(\rho_{p_r^*} \frac{r}{R}\right) dr = R \int_0^1 j_0(\rho_{p_\Omega} \dot{r}) j_0(\rho_{p_\theta} \dot{r}) j_0(\rho_{p_r^*} \dot{r}) d\dot{r}$$

$$\sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left[(n_\Omega + n_{H_\theta} + 1) \int_0^\pi \frac{\cos \theta}{\sin \theta} P_{n_\Omega}(\cos(\theta)) P_{n_\theta}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \sin \theta d\theta \right]$$

$$R \int_0^1 j_0(\rho_{p_\Omega} \dot{r}) j_0(\rho_{p_\theta} \dot{r}) j_0(\rho_{p_r^*} \dot{r}) d\dot{r} \int_0^{2\pi} e^{iq_\theta \varphi} e^{iq_\Omega \varphi} e^{-iq_r^* \varphi} d\varphi$$

using $\xi_{000} = \int_0^1 j_0(\rho_{p_\Omega} \dot{r}) j_0(\rho_{p_\theta} \dot{r}) j_0(\rho_{p_r^*} \dot{r}) d\dot{r}$

and $P_{c^1 s^0 00} = \int_0^\pi \cos \theta P_{n_\Omega}(\cos(\theta)) P_{n_\theta}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) d\theta$

gives

$$\theta_{r-a} = \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega [(n_\Omega + n_{H_\theta} + 1) P_{c^1 s^0 00}] R \xi_{000} [[2\pi]_{q_\theta + q_\Omega = q_{r^*}}]$$

The second collected term ($R_{\theta-b}$) is written as:

$$\theta_{r-b} =$$

$$+ \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left(-\frac{n_\Omega \cos \theta k_{p_\theta}}{rR \sin \theta} - \frac{\cos \theta k_{p_\theta}}{rR \sin \theta} \right. \\ \left. - \frac{k_{p_\theta} n_\theta \cos \theta}{rR \sin \theta} \right) j_0\left(\rho_{p_\Omega} \frac{r}{R}\right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1\left(\rho_{p_\theta} \frac{r}{R}\right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi}$$

$$\sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left(-\frac{(n_\Omega + n_{H_\theta} + 1) k_{p_{H_\theta}} \cos \theta}{R r \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi}$$

$$R_{\theta-b} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [R_{\theta-b}] j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) P_{n_r^*}(\cos(\theta)) e^{iq_r^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \int_0^\pi \left(-\frac{(n_\Omega + n_{H_\theta} + 1) k_{p_{H_\theta}} \cos \theta}{R r \sin \theta} \right) P_{n_\Omega}(\cos(\theta)) P_{n_\theta}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \sin \theta \, d\theta$$

$$\int_0^R j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) r \, dr \int_0^{2\pi} e^{iq_\theta \varphi} e^{iq_\Omega \varphi} e^{-iq_r^* \varphi} \, d\varphi$$

Using the normalized form of the integral

$$\int_0^R j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) j_0 \left(\rho_{p_r^*} \frac{r}{R} \right) r \, dr = R^2 \int_0^1 j_0(\rho_{p_\Omega} \dot{r}) j_1(\rho_{p_\theta} \dot{r}) j_0(\rho_{p_r^*} \dot{r}) \dot{r} \, d\dot{r}$$

$$\text{Using } \xi_{011} = \int_0^1 j_0(\rho_{p_\Omega} \dot{r}) j_1(\rho_{p_\theta} \dot{r}) j_0(\rho_{p_r^*} \dot{r}) \dot{r} \, d\dot{r}$$

$$\text{And } P_{c^1 s^0 00} = \int_0^\pi \cos \theta P_{n_\theta}(\cos(\theta)) P_{n_\Omega}(\cos(\theta)) P_{n_r^*}(\cos(\theta)) \, d\theta$$

$$\theta_{r-b} = + \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega [-\rho_{p_\theta} (n_\Omega + n_\theta + 1) P_{c^1 s^0 00}] R \xi_{011} \llbracket 2\pi \rrbracket_{q_\theta + q_\Omega = q_{r^*}}$$

This application of orthogonality on the equations is applied similarly for all other term for R_θ and the results are gathered in the following equation:

$$\theta_{r-c} = \sum_{H_\theta \Omega} \kappa^{H_\theta} \kappa^\Omega (-n_\Omega) P_{c^0 s^0 10} R \xi_{000} \llbracket 2\pi \rrbracket_{q_\theta + q_\Omega = q_{r^*}}$$

$$\theta_{r-d} = \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega (n_\Omega) P_{c^0 s^0 10} \rho_{p_\theta} R \xi_{011} \llbracket 2\pi \rrbracket_{q_\theta + q_\Omega = q_{r^*}}$$

$$\theta_{r-e} = \sum_{H_\theta \Omega} \kappa^\Omega \kappa^{H_\theta} (-n_\theta) P_{c^0 s^0 01} R \xi_{000} \llbracket 2\pi \rrbracket_{q_\theta + q_\Omega = q_{r^*}}$$

$$\theta_{r-f} = \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \rho_{p_\theta} n_\theta P_{c^0 s^0 01} R \xi_{011} \llbracket 2\pi \rrbracket_{q_\theta + q_\Omega = q_{r^*}}$$

Collecting all of the above expressions gives

$$\theta_r = \sum_{H_\theta} \kappa_\theta \kappa_\Omega \left\{ \left(\rho_{p_\theta} [(n_\theta) P_{c^0 s^0 01} + (n_\Omega) P_{c^0 s^0 10} - (n_\Omega + n_\theta + 1) P_{c^1 s^0 00}] R \xi_{011} \right) \left[[2\pi]_{q_\theta + q_\Omega = q_{r^*}} \right] \right\}$$

Also for the φ_r similar manipulation is applied the result is shown as:

$$\begin{aligned} \varphi_{r-a} &= \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega [(iq_\Omega + iq_\varphi) P_{c^0 s^0 00}] R \xi_{000} \left[[2\pi]_{q_\varphi + q_\Omega = q_{r^*}} \right] \\ \varphi_{r-b} &= \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega [-(iq_\Omega + iq_\varphi) \rho_{p_\varphi} P_{c^0 s^0 00}] R \xi_{011} \left[[2\pi]_{q_\varphi + q_\Omega = q_{r^*}} \right] \end{aligned}$$

$$\varphi_r = \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left\{ \left([(iq_\Omega + iq_\varphi) P_{c^0 s^0 00}] R \xi_{000} - [(iq_\Omega + iq_\varphi) \rho_{p_\varphi} P_{c^0 s^0 00}] R \xi_{011} \right) \left[[2\pi]_{q_{H_\varphi} + q_{H_\Omega} = q_{H_r^*}} \right] \right\}$$

The three block expressions R_r , θ_r and φ_r in the radial wave equations are determined. For the polar component of the wave equation a similar procedure is performed. The details for a few terms are as follow:

$$\begin{aligned} W_\theta &= \frac{1}{r^2 \sin \theta} \frac{\partial \Omega}{\partial \varphi} \frac{\partial H_\varphi}{\partial \theta} + \frac{H_\varphi}{r^2 (\sin \theta)^2} \frac{\partial \Omega}{\partial \varphi} \cos \theta - \frac{1}{r^2 (\sin \theta)^2} \frac{\partial \Omega}{\partial \varphi} \frac{\partial H_\theta}{\partial \varphi} + \frac{\Omega}{r^2 \sin \theta} \frac{\partial^2 H_\varphi}{\partial \varphi \partial \theta} \\ &+ \frac{\Omega}{r^2 (\sin \theta)^2} \cos \theta \frac{\partial H_\varphi}{\partial \varphi} - \frac{\Omega}{r^2 (\sin \theta)^2} \frac{\partial^2 H_\theta}{\partial \varphi^2} - \frac{H_\theta}{r} \frac{\partial \Omega}{\partial r} - \frac{\partial \Omega}{\partial r} \frac{\partial H_\theta}{\partial r} + \frac{1}{r} \frac{\partial \Omega}{\partial r} \frac{\partial H_r}{\partial \theta} \\ &- \frac{\Omega}{r} \frac{\partial H_\theta}{\partial r} - \frac{\Omega}{r} \frac{\partial H_\theta}{\partial r} - \Omega \frac{\partial^2 H_\theta}{\partial r^2} + \frac{\Omega}{r} \frac{\partial^2 H_r}{\partial r \partial \theta} = \left(\frac{\omega}{c} \right)^2 H_\theta \end{aligned} \tag{C.6}$$

Term 1:

$$\frac{1}{r^2 \sin \theta} \frac{\partial \Omega}{\partial \varphi} \frac{\partial H_\varphi}{\partial \theta} = \frac{1}{r^2 \sin \theta} \left[\sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega (iq_\Omega) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \right]$$

$$\begin{aligned}
& \left[\sum_{p_\varphi, n_\varphi, q_\varphi} \kappa_\varphi \frac{n_\varphi \cos \theta}{\sin \theta} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \right. \\
& \quad \left. - \sum_{p_\varphi, n_\varphi, q_\varphi} \kappa_\varphi \frac{n_\varphi}{\sin \theta} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi-1}(\cos(\theta)) e^{iq_\varphi \varphi} \right] \\
&= \sum_{p_\varphi, n_\varphi, q_\varphi} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\varphi \kappa_\Omega \left(\frac{(iq_\Omega) n_\varphi \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \\
&+ \sum_{p_\varphi, n_\varphi, q_\varphi} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\varphi \kappa_\Omega \left(-\frac{(iq_\Omega) n_\varphi}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi-1}(\cos(\theta)) e^{iq_\varphi \varphi}
\end{aligned} \tag{C.6.a}$$

Term 2:

$$\begin{aligned}
& + \frac{H_\varphi}{r^2 (\sin \theta)^2} \frac{\partial \Omega}{\partial \varphi} \cos \theta \\
&= \frac{\cos \theta}{r^2 (\sin \theta)^2} \left[\sum_{p_\varphi, n_\varphi, q_\varphi} \kappa_\varphi j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \right] \left[\sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega (iq_\Omega) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \right] \\
&= \sum_{p_\varphi, n_\varphi, q_\varphi} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\varphi \kappa_\Omega \frac{(iq_\Omega) \cos \theta}{r^2 (\sin \theta)^2} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi}
\end{aligned} \tag{C.6.b}$$

Similar procedures are applied for all other terms, then by rearranging and collecting similar terms the results are written as:

$$\begin{aligned}
R_\theta &= + \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{\rho_{p_\Omega} n_{H_r} \cos \theta}{rR \sin \theta} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \\
&+ \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(\frac{\rho_{p_\Omega} n_{H_r}}{rR \sin \theta} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi} \\
&+ \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{\rho_{p_r} n_r \cos \theta}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \\
&+ \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(\frac{\rho_{p_r} n_r}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi}
\end{aligned} \tag{C.7}$$

$$\begin{aligned}
\theta_\theta &= + \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left(-\frac{(iq_\Omega)(iq_\theta)}{r^2(\sin \theta)^2} + \frac{(q_\theta)^2}{r^2(\sin \theta)^2} + \frac{(\rho_{p_\theta})^2}{R^2} \right) \\
&j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
&+ \sum_{H_\theta \Omega} \kappa_\theta \kappa_\Omega \left(\frac{\rho_{p_\Omega}}{rR} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
&+ \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left(-\frac{\rho_{p_\Omega} \rho_{p_\theta}}{R^2} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi}
\end{aligned} \tag{C.8}$$

$$\begin{aligned}
\varphi_\theta = & \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left(\frac{(iq_\Omega) n_\varphi \cos \theta}{r^2 (\sin \theta)^2} + \frac{\cos \theta (iq_\Omega)}{r^2 (\sin \theta)^2} + \frac{(iq_\varphi) n_\Omega \cos \theta}{r^2 (\sin \theta)^2} \right. \\
& \left. + \frac{\cos \theta (iq_\varphi)}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \\
& - \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left(\frac{(iq_\Omega) n_\varphi}{r^2 (\sin \theta)^2} \right. \\
& \left. + \frac{(iq_\varphi) n_\Omega}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi-1}(\cos(\theta)) e^{iq_\varphi \varphi} j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi}
\end{aligned} \tag{C.9}$$

$$R_\theta + \theta_\theta + \varphi_\theta = \left(\frac{\omega}{c} \right)^2 \sum_{H_\theta} \kappa_\theta j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \tag{C.10}$$

The orthogonality is applied by multiplying the equations by orthogonal functions with summation over the orthogonal polar indices, and then integrates over the area of the sphere domain. Few terms are shown as:

The first collected term θ_{r-a} is written as:

$$\theta_{r-a} = + \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{\rho_{p_\Omega} n_r \cos \theta}{rR} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi}$$

$$\theta_{r-a} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [\theta_{r-a}] j_0 \left(\rho_{p_\theta^*} \frac{r}{R} \right) P_{n_\theta^*}(\cos(\theta)) e^{iq_\theta^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{\rho_{p_\Omega} n_r}{R} \right) \int_0^\pi \cos \theta P_{n_\Omega}(\cos(\theta)) P_{n_r}(\cos(\theta)) P_{n_\theta^*}(\cos(\theta)) d\theta$$

$$\int_0^R j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) j_0 \left(\rho_{p_\theta^*} \frac{r}{R} \right) r \, dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_\Omega \varphi} e^{-iq_\theta^* \varphi} d\varphi$$

$$\theta_{r-a} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega (-\rho_{p_\Omega} n_r) P_{c^1 s^0 00} R \xi_{101} \left[[2\pi]_{q_r+q_\Omega=q_{\theta^*}} \right]$$

$$\theta_{r-b} = + \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(\frac{\rho_{p_\Omega} n_r}{rR} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r-1}(\cos(\theta)) e^{iq_r \varphi}$$

$$\theta_{r-b} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [\theta_{r-b}] j_0 \left(\rho_{p_\theta^*} \frac{r}{R} \right) P_{n_\theta^*}(\cos(\theta)) e^{iq_\theta^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(\frac{\rho_{p_\Omega} n_r}{R} \right) \int_0^\pi P_{n_\Omega}(\cos(\theta)) P_{n_r-1}(\cos(\theta)) P_{n_\theta^*}(\cos(\theta)) d\theta$$

$$\int_0^R j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) j_0 \left(\rho_{p_\theta^*} \frac{r}{R} \right) r \, dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_\Omega \varphi} e^{-iq_\theta^* \varphi} d\varphi$$

$$\theta_{r-b} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega (\rho_{p_\Omega} n_r) P_{c^0 s^0 01} R \xi_{101} \left[[2\pi]_{q_r+q_\Omega=q_{\theta^*}} \right]$$

This application of orthogonality on the equations is applied similarly for all other term for θ_r and the results are gathered in the following equation:

$$\begin{aligned}
\theta_{r-c} &= \sum_{\Omega H_r} \kappa_r \kappa_\Omega (-\rho_{p_r} n_r) P_{c^1 s^0 00} R \xi_{011} \llbracket 2\pi \rrbracket_{q_r+q_\Omega=q_{\theta^*}} \\
\theta_{r-d} &= \sum_{\Omega H_r} \kappa_r \kappa_\Omega (\rho_{p_r} n_r) P_{c^0 s^0 01} R \xi_{011} \llbracket 2\pi \rrbracket_{q_r+q_\Omega=q_{\theta^*}} \\
\theta_r &= \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left\{ \left(\left[(\rho_{p_\Omega} n_r) P_{c^0 s^0 01} - (\rho_{p_\Omega} n_r) P_{c^1 s^0 00} \right] R \xi_{101} \right) \llbracket 2\pi \rrbracket_{q_r+q_\Omega=q_{\theta^*}} \right\} \quad (C.11)
\end{aligned}$$

Also for the θ_θ and θ_φ similar manipulation is applied the result is shown as:

$$\begin{aligned}
\theta_{\theta-a} &= \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left[(-iq_\Omega)(iq_\theta) + (q_\theta^2) \right] P_{c^0 s^{-1} 00} R \xi_{000} + (\rho_{p_\theta})^2 P_{c^0 s^1 00} \xi_{002} \llbracket 2\pi \rrbracket_{q_\theta+q_\Omega=q_{\theta^*}} \\
\theta_{\theta-b} &= \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \rho_{p_\Omega} P_{c^0 s^1 00} R \xi_{101} \llbracket 2\pi \rrbracket_{q_\theta+q_\Omega=q_{\theta^*}} \\
\theta_{\theta-c} &= \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega (-\rho_{p_\Omega} \rho_{p_\theta}) P_{c^0 s^1 00} R \xi_{112} \llbracket 2\pi \rrbracket_{q_\theta+q_\Omega=q_{\theta^*}} \\
\theta_\theta &= \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left[\begin{aligned} &(-iq_\Omega)(iq_\theta) + (q_\theta^2) P_{c^0 s^{-1} 00} R \xi_{000} \\ &+ (\rho_{p_\theta})^2 P_{c^0 s^1 00} \xi_{002} + \rho_{p_\Omega} P_{c^0 s^1 00} R \xi_{101} + (-\rho_{p_\Omega} \rho_{p_\theta}) P_{c^0 s^1 00} R \xi_{112} \end{aligned} \right] \llbracket 2\pi \rrbracket_{q_\theta+q_\Omega=q_{\theta^*}} \quad (C.12)
\end{aligned}$$

$$\begin{aligned}
\theta_{\varphi-a} &= \sum_{H_\varphi \Omega} \kappa_\varphi \kappa_\Omega \left[(iq_\Omega + iq_\varphi) n_\varphi + (iq_\Omega + iq_\varphi) P_{c^1 s^{-1} 00} \right] R \xi_{000} \llbracket 2\pi \rrbracket_{q_\varphi+q_\Omega=q_{\theta^*}} \\
\theta_{\varphi-b} &= \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega [-n_\varphi (iq_\Omega + iq_\varphi) P_{c^0 s^{-1} 01}] R \xi_{000} \llbracket 2\pi \rrbracket_{q_\varphi+q_\Omega=q_{\theta^*}} \\
\theta_\varphi &= \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left[[(iq_\Omega + iq_\varphi)(n_\varphi + 1)] P_{c^1 s^{-1} 00} - n_\varphi (iq_\Omega + iq_\varphi) P_{c^0 s^{-1} 01} \right] R \xi_{000} \llbracket 2\pi \rrbracket_{q_\varphi+q_\Omega=q_{\theta^*}} \quad (C.13)
\end{aligned}$$

The three block expressions θ_r, θ_θ and θ_φ for the polar wave equation are accomplished. For the azimuthal component of the wave equation similar procedure are performed. The details for few terms are as follow:

$$\begin{aligned}
W_\varphi = & \frac{1}{r} \frac{1}{\sin \theta} \frac{\partial \Omega}{\partial r} \frac{\partial H_r}{\partial \varphi} - \frac{H_\varphi}{r} \frac{\partial \Omega}{\partial r} - \frac{\partial \Omega}{\partial r} \frac{\partial H_\varphi}{\partial r} + \frac{\Omega}{r} \frac{1}{\sin \theta} \frac{\partial^2 H_r}{\partial r \partial \varphi} - \frac{\Omega}{r} \frac{\partial H_\varphi}{\partial r} - \frac{\Omega}{r} \frac{\partial H_\varphi}{\partial r} - \Omega \frac{\partial^2 H_\varphi}{\partial r^2} \\
& - \frac{1}{r^2} \frac{\partial \Omega}{\partial \theta} \frac{\partial H_\varphi}{\partial \theta} - \frac{1}{r^2} \frac{\partial \Omega}{\partial \theta} \frac{\cos \theta}{\sin \theta} H_\varphi + \frac{1}{r^2} \frac{\partial \Omega}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial H_\theta}{\partial \varphi} + \frac{1}{r^2} \Omega \frac{\cos \theta}{\sin \theta} \frac{\partial H_\varphi}{\partial \theta} \\
& + \frac{1}{r^2} \Omega \frac{(\cos \theta)^2}{(\sin \theta)^2} H_\varphi - \frac{1}{r^2} \Omega \frac{\cos \theta}{(\sin \theta)^2} \frac{\partial H_\theta}{\partial \varphi} - \frac{1}{r^2} \Omega \frac{\partial^2 H_\varphi}{\partial \theta^2} + \frac{1}{r^2} \Omega H_\varphi \\
& + \frac{1}{r^2} \Omega \frac{1}{\sin \theta} \frac{\partial^2 H_\theta}{\partial \theta \partial \varphi} = \left(\frac{\omega}{c} \right)^2 H_\varphi
\end{aligned} \tag{C.14}$$

Term 1:

$$\begin{aligned}
& \frac{1}{r} \frac{1}{\sin \theta} \frac{\partial \Omega}{\partial r} \frac{\partial H_r}{\partial \varphi} \\
& = \frac{1}{r} \frac{1}{\sin \theta} \left[\sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \left(-\frac{\rho_{p_\Omega}}{R} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \right] \left[\sum_{p_r, n_r, q_r} \kappa_r (iq_r) j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \right] \\
& = \sum_{p_r, n_r, q_r} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_r \kappa_\Omega \left(-\frac{\rho_{p_\Omega}(iq_r)}{rR \sin \theta} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi}
\end{aligned} \tag{C.14.a}$$

Term 2:

$$\begin{aligned}
& -\frac{H_\varphi}{r} \frac{\partial \Omega}{\partial r} \\
& = -\frac{1}{r} \left[\sum_{p_\varphi, n_\varphi, q_\varphi} \kappa_\varphi j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \right] \left[\sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\Omega \left(-\frac{\rho_{p_\Omega}}{R} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \right] \\
& = \sum_{p_\varphi, n_\varphi, q_\varphi} \sum_{p_\Omega, n_\Omega, q_\Omega} \kappa_\varphi \kappa_\Omega \left(\frac{\rho_{p_\Omega}}{rR} \right) j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi}
\end{aligned} \tag{C.14.b}$$

Similar procedures are applied for all other terms, then by rearranging and summing similar terms the results are written as:

$$\begin{aligned}
R_\varphi & = \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{k_{p_\Omega}(iq_r)}{rR \sin \theta} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi} \\
& + \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{(iq_r)k_{p_r}}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi}
\end{aligned} \tag{C.15}$$

$$\begin{aligned}
\theta_\varphi & = + \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left(\frac{n_\Omega \cos \theta (iq_\theta)}{r^2 (\sin \theta)^2} - \frac{\cos \theta (iq_\theta)}{r^2 (\sin \theta)^2} \right. \\
& \quad \left. + \frac{(iq_\theta)n_\theta \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
& - \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left(\frac{(n_\Omega + q_\Omega)(iq_\theta)}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta}(\cos(\theta)) e^{iq_\theta \varphi} \\
& - \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left(\frac{(iq_\theta)(n_\theta)}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\theta} \frac{r}{R} \right) P_{n_\theta-1}(\cos(\theta)) e^{iq_\theta \varphi}
\end{aligned}$$

(C.16)

 φ_φ

$$\begin{aligned}
&= \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left(\frac{(k_{p_\varphi})^2}{R^2} - \frac{n_\Omega n_\varphi \cos \theta^2}{r^2 (\sin \theta)^2} - \frac{n_\Omega (\cos \theta)^2}{r^2 (\sin \theta)^2} + \frac{1}{r^2} \frac{(\cos \theta)^2}{(\sin \theta)^2} \right. \\
&\quad \left. + \frac{(n_\varphi^2 + n_\varphi)}{r^2} + \frac{n_\varphi (\cos \theta)^2}{r^2 (\sin \theta)^2} - \frac{n_\varphi (\cos \theta)^2}{r^2 (\sin \theta)^2} + \frac{1}{r^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} \\
&\quad j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \\
&+ \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left(\frac{2k_{p_\varphi}}{rR} - \frac{2k_{p_\Omega}}{rR} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \\
&+ \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left(+ \frac{n_\Omega \cos \theta (n_\varphi)}{r^2 (\sin \theta)^2} - \frac{(n_\varphi) \cos \theta}{r^2 (\sin \theta)^2} \right. \\
&\quad \left. + \frac{(n_\varphi) \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi-1}(\cos(\theta)) e^{iq_\varphi \varphi} \\
&+ \sum_{H_\varphi \Omega} \kappa_\varphi \kappa_\Omega \left(\frac{k_{p_\Omega}}{rR} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \\
&+ \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left(\frac{(n_\Omega) n_\varphi \cos \theta}{r^2 (\sin \theta)^2} \right. \\
&\quad \left. + \frac{(n_\Omega) \cos \theta}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \\
&+ \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left(- \frac{k_{p_\Omega} k_{p_\varphi}}{R^2} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi}(\cos(\theta)) e^{iq_\varphi \varphi} \\
&+ \sum_{\Omega H_\varphi} \left(- \frac{(n_\Omega)(n_\varphi)}{r^2 (\sin \theta)^2} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega-1}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_\varphi} \frac{r}{R} \right) P_{n_\varphi-1}(\cos(\theta)) e^{iq_\varphi \varphi}
\end{aligned}$$

(C.17)

The orthogonality is applied by multiplying the equations by orthogonal function with summation over the orthogonal azimuthal indices, and then integrates over the area of the sphere domain. Few terms are shown as:

$$\varphi_{r-a} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{k_{p_\Omega}(iq_r)}{rR \sin \theta} \right) j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_0 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi}$$

$$\varphi_{r-a} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [\varphi_{r-a}] j_0 \left(\rho_{p_\Phi}^* \frac{r}{R} \right) P_{n_\Phi}^*(\cos(\theta)) e^{iq_\Phi^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{\rho_{p_\Omega}(iq_r)}{R} \right) \int_0^\pi P_{n_\Omega}(\cos(\theta)) P_{n_r}(\cos(\theta)) P_{n_\Phi}^*(\cos(\theta)) d\theta$$

$$\int_0^R j_1 \left(\rho_{p_\Omega} \frac{r}{R} \right) j_0 \left(\rho_{p_r} \frac{r}{R} \right) j_0 \left(\rho_{p_\Phi}^* \frac{r}{R} \right) r \, dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_\Omega \varphi} e^{-iq_\Phi^* \varphi} d\varphi$$

$$\varphi_{r-a} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\rho_{p_\Omega}(iq_r) \right) P_{c^0 s^0 00} R \xi_{101} \left[[2\pi]_{q_r+q_\Omega=q_\Phi^*} \right]$$

$$\varphi_{r-b} = + \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{(iq_r)k_{p_r}}{rR \sin \theta} \right) j_0 \left(\rho_{p_\Omega} \frac{r}{R} \right) P_{n_\Omega}(\cos(\theta)) e^{iq_\Omega \varphi} j_1 \left(\rho_{p_r} \frac{r}{R} \right) P_{n_r}(\cos(\theta)) e^{iq_r \varphi}$$

$$\varphi_{r-b} \rightarrow \iiint_{0,0,0}^{R,\pi,2\pi} [\varphi_{r-b}] j_0 \left(\rho_{p_\Phi}^* \frac{r}{R} \right) P_{n_\Phi}^*(\cos(\theta)) e^{iq_\Phi^* \varphi} r^2 \sin \theta \, dr \, d\theta \, d\varphi$$

$$\sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-\frac{(iq_r)k_{p_r}}{R} \right) \int_0^\pi P_{n_\Omega}(\cos(\theta)) P_{n_r}(\cos(\theta)) P_{n_\Phi}^*(\cos(\theta)) d\theta$$

$$\int_0^R j_0\left(\rho_{p_\Omega} \frac{r}{R}\right) j_1\left(\rho_{p_r} \frac{r}{R}\right) j_0\left(\rho_{p_\Phi^*} \frac{r}{R}\right) r dr \int_0^{2\pi} e^{iq_r \varphi} e^{iq_\Omega \varphi} e^{-iq_\Phi^* \varphi} d\varphi$$

$$\varphi_{r-b} = \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left(-(iq_r) \rho_{p_r} \right) P_{c^0 s^0 00} R \xi_{011} \left[[2\pi]_{q_r+q_\Omega=q_\Phi^*} \right]$$

$$\varphi_r = \sum_{\Omega H_r} \kappa_r \kappa_\Omega \left[\left(-\rho_{p_r} (iq_r) \right) \xi_{011} - \left(\rho_{p_\Omega} (iq_r) \right) \xi_{101} \right] R P_{c^0 s^0 00} \left[[2\pi]_{q_r+q_\Omega=q_\Phi^*} \right]$$

Also for the φ_θ and φ_φ similar manipulation is applied the result is shown as:

$$\varphi_{\theta-a} = \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left[(iq_\theta (n_\Omega - 1 + n_\theta)) P_{c^1 s^{-1} 00} R \xi_{000} \right] \left[[2\pi]_{q_\theta+q_\Omega=q_\Phi^*} \right]$$

$$\varphi_{\theta-b} = \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left[-(n_\Omega) (iq_\theta) P_{c^0 s^{-1} 10} R \xi_{000} \right] \left[[2\pi]_{q_\theta+q_\Omega=q_\Phi^*} \right]$$

$$\varphi_{\theta-c} = \sum_{\Omega H_\theta} \kappa_\theta \kappa_\Omega \left[-(n_\theta) (iq_\theta) P_{c^0 s^{-1} 01} R \xi_{000} \right] \left[[2\pi]_{q_\theta+q_\Omega=q_\Phi^*} \right]$$

$$\begin{aligned} \varphi_\theta &= \sum_{\Omega} \kappa_\Omega \kappa_\theta i q \{ [(n-1) P_{c^1 s^{-1} 00} - n P_{c^0 s^{-1} 01}] \xi_{000} \} \left[[2\pi]_{q_\theta+q_\Omega=q_\Phi^*} \right] \\ &\quad + \sum_{\Omega} \kappa_\Omega \kappa_\theta i q n_\Omega \{ [P_{c^1 s^{-1} 00} - P_{c^0 s^{-1} 10}] \xi_{000} \} \left[[2\pi]_{q_\theta+q_\Omega=q_\Phi^*} \right] \end{aligned}$$

$$\begin{aligned} \varphi_{\varphi-a} &= \sum_{\Omega H_\varphi} \kappa_\varphi \kappa_\Omega \left[[(-n_\Omega n_\varphi - n_\Omega + 1) P_{c^2 s^{-1} 00} + (n_\varphi^2 + n_\varphi + 1) P_{c^0 s^1 00}] \xi_{000} \right. \\ &\quad \left. + (k_{p_\varphi})^2 P_{c^0 s^1 00} \xi_{002} \right] R \left[[2\pi]_{q_\varphi+q_\Omega=q_\Phi^*} \right] \end{aligned}$$

$$\begin{aligned}
\varphi_{\varphi-b} &= \sum_{\Omega H_{\varphi}} \kappa_{\varphi} \kappa_{\Omega} [(n_{\Omega} n_{\varphi}) P_{c^1 s^{-1} 01} R \xi_{000}] \llbracket 2\pi \rrbracket_{q_{\varphi}+q_{\Omega}=q_{\varphi}^*} \\
\varphi_{\varphi-c} &= \sum_{\Omega H_{\varphi}} \kappa_{\varphi} \kappa_{\Omega} [(\rho_{p_{\Omega}}) P_{c^0 s^1 00} R \xi_{101}] \llbracket 2\pi \rrbracket_{q_{\varphi}+q_{\Omega}=q_{\varphi}^*} \\
\varphi_{\varphi-d} &= \sum_{\Omega H_{\varphi}} \kappa_{\varphi} \kappa_{\Omega} [(n_{\Omega} n_{\varphi} + n_{\Omega}) P_{c^1 s^{-1} 10} R \xi_{000}] \llbracket 2\pi \rrbracket_{q_{\varphi}+q_{\Omega}=q_{\varphi}^*} \\
\varphi_{\varphi-e} &= \sum_{\Omega H_{\varphi}} \kappa_{\varphi} \kappa_{\Omega} [(-\rho_{p_{\Omega}} \rho_{p_{\varphi}}) P_{c^0 s^1 00} R \xi_{112}] \llbracket 2\pi \rrbracket_{q_{\varphi}+q_{\Omega}=q_{\varphi}^*} \\
\varphi_{\varphi-f} &= \sum_{\Omega H_{\varphi}} \kappa_{\varphi} \kappa_{\Omega} [(-n_{\Omega} n_{\varphi}) P_{c^0 s^{-1} 11} R \xi_{000}] \llbracket 2\pi \rrbracket_{q_{\varphi}+q_{\Omega}=q_{\varphi}^*} \\
\\
\varphi_{\varphi} &= \sum_{\Omega} \kappa_{\Omega} \kappa_{\varphi} \{[(n^2 + n + 1) P_{c^0 s^1 00} + P_{c^2 s^{-1} 00}] \xi_{000} + \rho_p^2 P_{c^0 s^1 00} \xi_{002}\} \llbracket 2\pi \rrbracket_{q_{\varphi}+q_{\Omega}=q_{\varphi}^*} \\
&\quad + C^{-1} \sum_{\Omega} \kappa_{\Omega} \{n_{\Omega} [n P_{c^1 s^{-1} 01} - n P_{c^0 s^{-1} 11} + (n+1) P_{c^1 s^{-1} 10} \\
&\quad - (n+1) P_{c^2 s^{-1} 00}] \xi_{000} \\
&\quad + \rho_{p_{\Omega}} P_{c^0 s^1 00} [\xi_{101} - \rho_p \xi_{112}]\} \{[(n^2 + n + 1) P_{c^0 s^1 00} + P_{c^2 s^{-1} 00}] \xi_{000} \\
&\quad + \rho_p^2 P_{c^0 s^1 00} \xi_{002}\} \llbracket 2\pi \rrbracket_{q_{\varphi}+q_{\Omega}=q_{\varphi}^*}
\end{aligned}$$

The derivations above are long but needed to be completed only one time, and then the expression can be used for any structures to study localized modes. In the future work a much simpler and shorter way is mentioned.

Appendix D: Matrix Expression

The matrix elements of (3.25) can be determined from the expression given below. The representation of the generating expressions is further simplified by adopting the following notation for integrals involving the product of three Bessel functions,

$\xi_{abt} = \int_0^1 j_a(\rho_{p\Omega}\zeta) j_b(\rho_p\zeta) j_0(\rho_{p^*}\zeta) \zeta^t d\zeta$ with $\zeta = r/R$ is the scaled radial extent, and the product of

three Legendre polynomials, $P_{c^d s^t ab} = \int_0^\pi c^d s^t P_{n\Omega-a}(c) P_{n-b}(c) P_{n^*}(c) d\theta$ where $c = \cos(\theta)$, $s = \sin(\theta)$. In the

expressions below constants are collected as $C = \frac{2\pi R^2}{2} [j_1(\rho_p, r)]^2 \frac{2n^*}{2n^*+1} \delta_{q+q_\Omega, q}$.

$$R_r \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} = C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ [(n^2 + n) P_{c^0 s^1 00} + q^2 P_{c^0 s^{-1} 00}] \xi_{000} \} \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} \\ + C^{-1} \sum_{\Omega} \kappa_{\Omega} \left\{ \left[(n n_{\Omega}) (P_{c^1 s^{-1} 01} + P_{c^1 s^{-1} 10} - P_{c^2 s^{-1} 00} - P_{c^0 s^{-1} 11}) \right] \xi_{000} \right\} \begin{bmatrix} \bar{H} \\ 0 \end{bmatrix} \quad (D.1)$$

$$R_{\theta} \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} = C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ \rho_p [n P_{c^0 s^0 01} - (n+1) P_{c^1 s^0 00}] \xi_{011} \\ + [(n+1) P_{c^1 s^0 00} - n P_{c^0 s^0 01}] \xi_{000} \} \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} \\ + C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ \rho_p n_{\Omega} [P_{c^0 s^0 10} - P_{c^1 s^0 00}] \xi_{101} + n_{\Omega} [(P_{c^1 s^0 00} - P_{c^0 s^0 10})] \xi_{000} \} \begin{bmatrix} \bar{H} \\ 0 \end{bmatrix} \quad (D.2)$$

$$R_{\varphi} \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} = j C^{-1} \sum_{\Omega} \kappa_{\Omega} q P_{c^0 s^0 00} \{ \xi_{000} - \rho_p \xi_{011} \} \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} \\ + j C^{-1} \sum_{\Omega} \kappa_{\Omega} q_{\Omega} P_{c^0 s^0 00} \{ \xi_{000} + \rho_p \xi_{011} \} \begin{bmatrix} \bar{H} \\ 0 \end{bmatrix} \quad (D.3)$$

$$\Theta_r \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} = C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ n \rho_p [P_{c^0 s^0 01} - P_{c^1 s^0 00}] \xi_{011} \} \begin{bmatrix} \bar{H} \\ \bar{E} \end{bmatrix} \\ + C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ n \rho_{p\Omega} [P_{c^0 s^0 01} - P_{c^1 s^0 00}] \xi_{101} \} \begin{bmatrix} \bar{H} \\ 0 \end{bmatrix} \quad (D.4)$$

$$\begin{aligned}
\Theta_{\theta} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} &= C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ q^2 P_{c^0 s^{-1} 00} \xi_{000} + \rho_p^2 P_{c^0 s^1 00} \xi_{002} \} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} \\
&+ C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ q q_{\Omega} P_{c^0 s^{-1} 00} \xi_{000} + \rho_{p\Omega} P_{c^0 s^1 00} [\xi_{101} - \rho_p \xi_{112}] \} \begin{bmatrix} \overline{H} \\ 0 \end{bmatrix}
\end{aligned} \tag{D.5}$$

$$\begin{aligned}
\Theta_{\varphi} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} &= jC^{-1} \sum_{\Omega} \kappa_{\Omega} q \{ [(n+1)P_{c^1 s^{-1} 00} - nP_{c^0 s^{-1} 01}] \xi_{000} \} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} \\
&+ jC^{-1} \sum_{\Omega} \kappa_{\Omega} q_{\Omega} \{ [(n+1)P_{c^1 s^{-1} 00} - nP_{c^0 s^{-1} 01}] \xi_{000} \} \begin{bmatrix} \overline{H} \\ 0 \end{bmatrix}
\end{aligned} \tag{D.6}$$

$$\begin{aligned}
\Phi_r \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} &= -jC^{-1} \sum_{\Omega} \kappa_{\Omega} \{ \rho_p q P_{c^0 s^0 00} \xi_{101} \} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} \\
&- jC^{-1} \sum_{\Omega} \kappa_{\Omega} \{ \rho_{p\Omega} q P_{c^0 s^0 00} \xi_{011} \} \begin{bmatrix} \overline{H} \\ 0 \end{bmatrix}
\end{aligned} \tag{D.7}$$

$$\begin{aligned}
\Phi_{\theta} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} &= jC^{-1} \sum_{\Omega} \kappa_{\Omega} q \{ [(n-1)P_{c^1 s^{-1} 00} - nP_{c^0 s^{-1} 10}] \xi_{000} \} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} \\
&+ jC^{-1} \sum_{\Omega} \kappa_{\Omega} q n_{\Omega} \{ [P_{c^1 s^{-1} 00} - P_{c^0 s^{-1} 01}] \xi_{000} \} \begin{bmatrix} \overline{H} \\ 0 \end{bmatrix}
\end{aligned} \tag{D.8}$$

$$\begin{aligned}
\Phi_{\varphi} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} &= C^{-1} \sum_{\Omega} \kappa_{\Omega} \{ [(n^2 + n + 1)P_{c^0 s^1 00} + P_{c^2 s^{-1} 00}] \xi_{000} + \rho_p^2 P_{c^0 s^1 00} \xi_{002} \} \begin{bmatrix} \overline{H} \\ \overline{E} \end{bmatrix} \\
&+ C^{-1} \sum_{\Omega} \kappa_{\Omega} \left\{ n_{\Omega} [n P_{c^1 s^{-1} 10} - n P_{c^0 s^{-1} 11} + (n+1)P_{c^1 s^{-1} 01} - (n+1)P_{c^2 s^{-1} 00}] \xi_{000} \right. \\
&\quad \left. + \rho_{p\Omega} P_{c^0 s^1 00} [\xi_{011} - \rho_p \xi_{112}] \right\} \begin{bmatrix} \overline{H} \\ 0 \end{bmatrix}
\end{aligned} \tag{D.9}$$

These expressions are used to populate the matrix to solve for any structure once the dielectric coefficients are defined.

Appendix E: Mode Solver MATLAB© Programs

The MATLAB© codes used for implementing the technique is shown in this appendix. A couple routine codes can be used for the entire BLF matrices coding. The code presented below shows how the tables for the spherical Bessel integrals can be created. This code can be used to create all the integral tables with only minor changes.

```
% performs Bessel integrals files
clear;
clc;
tic;
disp('0 ++ Program start.');
```



```
disp('0 -- Program start.');
```



```
disp('1 ++ Reading the control file.');
```

```
    H_N_Bessel = 20;
    Bessel_Number_Field = 20;
    Bessel_Number_Dielectric = 50;
    Bessel_Number_Orthogonal_Field = 20;
    Radial_Number = 1000;
    Structure_Radius_Normalized = 1;
    v=0;
    AA=0;
    BB=0;
disp('1 -- control file read.');
```



```
disp('2 ++ Calculating the Bessel (p_s) table');
```

```
    p_number = max([Bessel_Number_Field, Bessel_Number_Dielectric,
    Bessel_Number_Orthogonal_Field]);
    j0_table = zeros(p_number, Radial_Number);
    j1_table = zeros(p_number, Radial_Number);
    for I = 1: 1: Radial_Number;
        R = Structure_Radius_Normalized * I / Radial_Number;
        for p = 1: 1: p_number;
            A = p * pi * R / Structure_Radius_Normalized;
            B = p * pi;
            j0_table(p,I) = sin(A) / A;
            j1_table(p,I) = (sin(B) / B^2 - cos(B) / B)^2;
        end
    end
disp('2 -- Completed the calculating of the Bessel (p_s) table')
```

```

jA_Dielectric=0;
jB_Field=0;
disp('3 ++ Calculating the Bessel integrals');
D_R = Structure_Radius_Normalized / Radial_Number;
j_TABLE_SvABC = zeros(Bessel_Number_Field, Bessel_Number_Dielectric ,
    Bessel_Number_Orthogonal_Field);
for B_N_Field = 1%: 1: Bessel_Number_Field;
    for B_N_Dielectric = 1%: 1: Bessel_Number_Dielectric ;
        for B_N_Orthg = 1%: 1: Bessel_Number_Orthogonal_Field ;
            Sum = 0;
            for I = 1: 1: Radial_Number;
                R = Structure_Radius_Normalized * I / Radial_Number;
                if AA==0
                    jA_Dielectric = j0_table(B_N_Dielectric,I) ;
                elseif AA==1
                    jA_Dielectric = j1_table(B_N_Dielectric,I) ;
                end
                if BB==0
                    jB_Field = j0_table(B_N_Field,I) ;
                elseif BB==1
                    jB_Field = j1_table(B_N_Field,I) ;
                end
                j0_Orth_Field = j0_table(B_N_Orthg,I);
                Sum = Sum + jA_Dielectric * jB_Field * j0_Orth_Field * R^v;
            end
            j_TABLE_SvABC(B_N_Field , B_N_Dielectric, B_N_Orthg ) = Sum * D_R ;
        end
    end
end
disp('3 -- Completed the calculating of the Bessel integrals')

```

```

disp('4 ++ saving the Bessel integrals file SvAB');
DT = fopen(sprintf('C:\\MATLAB\\Bessel integrals\\Bessel integrals_SvAB
%i.txt',H_N_Bessel),'w');
for B_N_Field = 1: 1: Bessel_Number_Field;
    for B_N_Dielectric = 1: 1: Bessel_Number_Dielectric ;
        for B_N_Orthg = 1: 1: Bessel_Number_Orthogonal_Field ;
            fprintf(DT,'%10.10f\r\n', real(j_TABLE_SvABC(B_N_Field , B_N_Dielectric,
B_N_Orthg )));
            fprintf(DT,'%10.10f\r\n', imag(j_TABLE_SSvABC(B_N_Field , B_N_Dielectric,
B_N_Orthg )));
        end
    end
end

```

```

    end
end
fclose('all');
disp('4 -- The Bessel integrals file SvAB is saved ');

```

The code below shows how the tables for the Legendre integrals tables can be created. This code can be used to create all the Legendre integrals tables with only minor changes.

```

% performs Legendre integrals file P_1
clear;
clc;
tic;
disp('0 ++ Program start. ');

disp('0 -- Program start. ');

disp('1 ++ Reading the control file. ');
H_N_Legendre = 20;
Legendre_Number_Field = 20;
Legendre_Number_Dielectric = 0;
Legendre_Number_Orthogonal_Field = 20;
Polar_Number = 10000;
AA=0;
BB=0;
disp('1 -- control file read. ');

disp('2 ++ Calculating the Legendre (n_s) table ');
n_number = max([Legendre_Number_Field, Legendre_Number_Dielectric,
Legendre_Number_Orthogonal_Field]);
Pn_table = zeros(n_number+1,Polar_Number);
for K=1: 1: Polar_Number;
    AV = 1e-3;
    Theta = ((pi-2 * AV) * (K - 1) / (Polar_Number - 1)) + AV;
    X =cos(Theta);
    for n=0: 1: n_number;
        Pn = legendre(n,X);
        Pn_table(n+1,K) = Pn(1);
    end
end

disp('2 -- Completed the calculating of the Legendre (n_s) table ')

D_Theta = ((pi - 2 * AV) / (Polar_Number - 1));

```



```

P_Dielectric=0;
P_Field=0;
disp('3 ++ Calculating the Legendre integrals');
PN_TABLE_P_1 = zeros(Legendre_Number_Field + 1, Legendre_Number_Dielectric + 1,
Legendre_Number_Orthogonal_Field + 1);
for L_N_Field = 0: 1: Legendre_Number_Field;
    if L_N_Field == 0;
        L_N_Field_d = 1;
    elseif L_N_Field ~= 0;
        L_N_Field_d = L_N_Field;
    end

    for L_N_Dielectric = 0: 1: Legendre_Number_Dielectric;
        if L_N_Dielectric == 0;
            L_N_Dielectric_d = 1;
        elseif L_N_Dielectric ~= 0;
            L_N_Dielectric_d = L_N_Dielectric;
        end
        for L_N_Orthg = 0: 1: Legendre_Number_Orthogonal_Field;
            Sum = 0;
            for K = 1: 1: Polar_Number;
                Theta = ((pi-2 * AV) * (K - 1) / (Polar_Number - 1)) + AV;
                if AA == 0;
                    P_Dielectric = Pn_table(L_N_Dielectric + 1, K);
                elseif AA == 1;
                    P_Dielectric = Pn_table((L_N_Dielectric_d - 1) + 1, K) ;
                end
                if BB == 0;
                    P_Field = Pn_table(L_N_Field + 1, K);
                elseif BB == 1;
                    P_Field = Pn_table((L_N_Field_d - 1) + 1, K) ;
                end
                P_Orth_Field = Pn_table(L_N_Orthg + 1, K);
                Sum = Sum + P_Field * P_Dielectric * P_Orth_Field;
            end
            PN_TABLE_P_1(L_N_Field + 1, L_N_Dielectric + 1, L_N_Orthg + 1) = Sum *
D_Theta;
        end
    end
end
disp('3 -- Completed the calculating of the Legendre integrals')

disp('4 ++ saving the Legendre integrals file P_AB');
DT = fopen(sprintf('C:\\MATLAB\\Legendre integrals\\Legendre integrals_P_AB
%i.txt',H_N_Legendre),'w');
for L_N_Field = 0: 1: Legendre_Number_Field;

```

```

for L_N_Dielectric = 0: 1: Legendre_Number_Dielectric;
    for L_N_Orthg = 0: 1: Legendre_Number_Orthogonal_Field;
        fprintf(DT,'%10.10f\r\n', real(PN_TABLE_P_1(L_N_Field + 1, L_N_Dielectric + 1,
            L_N_Orthg + 1)));
        %fprintf(DT,'%10.10f\r\n', imag(PN_TABLE_P_1(L_N_Field + 1, L_N_Dielectric + 1,
            L_N_Orthg + 1)));
    end
end
end
fclose('all');
disp('4 -- The Legendre integrals file P_AB is saved ');
toc;

```

The code below shows how the dielectric decomposition can be created.

```

function Dielectric_Decomposition
% program created Nov. 13, 2013
% part of BSH program collection
% performs dielectric decomposition
clear;
clc;
tic;
disp('0 ++ Program start. ');

disp('0 -- Program start. ');

disp('1 ++ Reading the decomposition control file. ');
Bessel_Number_Dielectric = 100;
Legendre_Number_Dielectric = 0;
Exponential_Number_Dielectric = 0;
Structure_Radius = 2;
Radial_Number = 1000;
Reconstruction_Radial_Number = 500;
Polar_Number = 1;
Azimuthal_Number = 1;

disp('1 -- Decomposition control file read. ');
Dielectric = (9) * ones(Radial_Number, Polar_Number, Azimuthal_Number);
Dielectric(501:1000, :, :) = 1;

figure (1)
plot(1./Dielectric);
disp('3 ++ Starting to compute expansion coefficients');
D_R = Structure_Radius / Radial_Number;
D_x = 2 / (Polar_Number);

```

```

D_Phi = 2 * pi / Azimuthal_Number;
Kappa = zeros(Bessel_Number_Dielectric, Legendre_Number_Dielectric + 1, 2 *
Exponential_Number_Dielectric + 1);
Int_B = 1 / (2 * pi);
for B_N = 1: 1: Bessel_Number_Dielectric;
    Rho_p = B_N * pi;
    B = Rho_p;
    Int_R = 2 / ((Structure_Radius)^3*(sin(B) / B^2 - cos(B) / B)^2);
    for L_N = 0: 1: Legendre_Number_Dielectric;
        Int_T = 1 / (2 / (2 * L_N + 1));
        for F_N = -Exponential_Number_Dielectric: 1: Exponential_Number_Dielectric;
            Sum = 0;
            for L = 1: 1: Radial_Number;
                r = (L - .5) * D_R;
                A = Rho_p * r / Structure_Radius;
                R_Part = sin(A) / A;
                for M = 1: 1: Polar_Number;
                    x = cos((M - .5) * pi / Polar_Number);
                    L1 = legendre(L_N, x);
                    Theta_Part = L1(1);
                    for N = 1: 1: Azimuthal_Number;
                        Phi = (N - .5) * D_Phi;
                        Phi_Part = exp(-1i * F_N * Phi);
                        Sum = Sum + 1 / Dielectric(L, M, N) * R_Part * Theta_Part * Phi_Part * r^2 *
D_R * D_x * D_Phi;
                    end
                end
            end
            Kappa(B_N, L_N + 1, F_N + Exponential_Number_Dielectric + 1) = Sum * Int_R *
Int_T * Int_B;
        end
    end
end
disp('3 -- Completed computation of expansion coefficients');

disp('4 ++ Saving the Kappa file');
DT = fopen(sprintf('C:\\MATLAB\\Kappa\\Kappa %i.txt',2),'w');
for B_N = 1: 1: Bessel_Number_Dielectric;
    for L_N = 0: 1: Legendre_Number_Dielectric;
        for F_N = -Exponential_Number_Dielectric: 1: Exponential_Number_Dielectric;
            fprintf(DT,'%10.10f\r\n', real(Kappa(B_N, L_N + 1, F_N +
Exponential_Number_Dielectric + 1)));
            fprintf(DT,'%10.10f\r\n', imag(Kappa(B_N, L_N + 1, F_N +
Exponential_Number_Dielectric + 1)));
        end
    end
end

```

```

end
fclose('all');
disp('4 -- The Kappa file is saved ');
figure(2)
plot((Kappa),'.')
end

```

The code below shows how the solver's matrix can be populated. This code can be used to create all the solver's matrix blocks with a few changes in the expression lines.

```

function Solver_E_PHY
% program created Dec. 10, 2013
% part of BSH program collection
% performs Eign-matrix blocks
clear;
clc;
tic;
disp('0 ++ Program start. ');

disp('0 -- Program start. ');

disp('1 ++ Reading the control file. ');
H_N_Bessel = 20;
H_N_Legendre = 20;
Bessel_Number_Orthogonal_Field = 20;
Legendre_Number_Orthogonal_Field = 20;
Exponential_Number_Orthogonal_Field = 0;
Bessel_Number_Field = 20;
Legendre_Number_Field = 20;
Exponential_Number_Field = 0;
Bessel_Number_Dielectric = 100;
Legendre_Number_Dielectric = 0;
Exponential_Number_Dielectric = 0;
Structure_Radius = 2;
disp('1 -- control file read. ');

disp('2 ++ Reading the Bessel file. ');
j_TABLE_S00 = zeros(Bessel_Number_Field, Bessel_Number_Dielectric,
Bessel_Number_Orthogonal_Field);
DT = fopen(sprintf('C:\\MATLAB\\Bessel integrals\\Bessel integrals_S00
%i.txt', H_N_Bessel), 'r');
for B_N_Field = 1: 1: H_N_Bessel;

```

```

for B_N_Dielectric = 1: 1: Bessel_Number_Dielectric;
    for B_N_Orthg = 1: 1: H_N_Bessel;
        j_TABLE_S00(B_N_Field, B_N_Dielectric, B_N_Orthg) = fscanf(DT, '%f', 1);
    end
end
fclose('all');
disp('2 -- Bessel file read.');
```

```

disp('2 ++ Reading the Bessel file.');
```

```

j_TABLE_S02 = zeros(Bessel_Number_Field, Bessel_Number_Dielectric,
    Bessel_Number_Orthogonal_Field);
DT = fopen(sprintf('C:\\MATLAB\\Bessel integrals\\Bessel integrals_S02
%i.txt', H_N_Bessel), 'r');
for B_N_Field = 1: 1: H_N_Bessel;
    for B_N_Dielectric = 1: 1: Bessel_Number_Dielectric;
        for B_N_Orthg = 1: 1: H_N_Bessel;
            j_TABLE_S02(B_N_Field, B_N_Dielectric, B_N_Orthg) = fscanf(DT, '%f', 1);
        end
    end
end
fclose('all');
disp('2 -- Bessel file read.');
```

```

disp('3 ++ Reading the Legendre file.');
```

```

PN_TABLE_Ps = zeros(Legendre_Number_Field + 1, Legendre_Number_Dielectric + 1,
    Legendre_Number_Orthogonal_Field + 1);
DT = fopen(sprintf('C:\\MATLAB\\Legendre integrals\\Legendre integrals_Ps
%i.txt', H_N_Legendre), 'r');
for L_N_Field = 0: 1: H_N_Legendre;
    for L_N_Dielectric = 0: 1: Legendre_Number_Dielectric;
        for L_N_Orthg = 0: 1: H_N_Legendre;
            PN_TABLE_Ps(L_N_Field + 1, L_N_Dielectric + 1, L_N_Orthg + 1) = fscanf(DT,
'%f', 1);
        end
    end
end
fclose('all');
disp('3 -- Legendre file read.');
```

```

disp('3 ++ Reading the Legendre file.');
```

```

PN_TABLE_Pc2_s = zeros(Legendre_Number_Field + 1, Legendre_Number_Dielectric + 1,
    Legendre_Number_Orthogonal_Field + 1);
DT = fopen(sprintf('C:\\MATLAB\\Legendre integrals\\Legendre integrals_Pc2_s
%i.txt', H_N_Legendre), 'r');
for L_N_Field = 0: 1: H_N_Legendre;
```

```

    for L_N_Dielectric = 0: 1: Legendre_Number_Dielectric;
        for L_N_Orthg = 0: 1: H_N_Legendre;
            PN_TABLE_Pc2_s(L_N_Field + 1, L_N_Dielectric + 1, L_N_Orthg + 1) = fscanf(DT,
'%f', 1);
        end
    end
end
fclose('all');
disp('3 -- Legendre file read.');
```

disp('4 ++ Reading the Kappa file');

```

    Kappa = zeros(Bessel_Number_Dielectric, Legendre_Number_Dielectric + 1, 2 *
Exponential_Number_Dielectric + 1);
    DT = fopen(sprintf('C:\\MATLAB\\Kappa\\Kappa %i.txt',2),'r');
    for B_N_Dielectric = 1: 1: Bessel_Number_Dielectric;
        for L_N_Dielectric = 0: 1: Legendre_Number_Dielectric;
            for F_N_Dielectric = -Exponential_Number_Dielectric: 1:
Exponential_Number_Dielectric;
                kappa_real = fscanf(DT, '%f', 1);
                kappa_imaginary = fscanf(DT, '%f', 1) ;
                Kappa(B_N_Dielectric, L_N_Dielectric + 1, F_N_Dielectric +
Exponential_Number_Dielectric + 1) = kappa_real + 1i * kappa_imaginary ;
            end
        end
    end
    fclose('all');
    disp('4 -- The Kappa file read ');
```

disp('5 ++ Calculating the elements of Phy_phy_eign-matrix block.');

```

    Phy_phy_Eign_matrix_block = zeros(Bessel_Number_Orthogonal_Field *
(Legendre_Number_Orthogonal_Field + 1) * (2 * Exponential_Number_Orthogonal_Field
+ 1), Bessel_Number_Field * ( Legendre_Number_Field + 1 ) * (2 *
Exponential_Number_Field + 1));
    OrthgFL = 0;
    for B_N_Orthg = 1: 1: Bessel_Number_Orthogonal_Field;
        disp(sprintf('Bessel orthogonal %i.', B_N_Orthg));
        for L_N_Orthg = 0: 1: Legendre_Number_Orthogonal_Field ;
            for F_N_Orthg = -Exponential_Number_Orthogonal_Field: 1:
Exponential_Number_Orthogonal_Field;
                OrthgFL = OrthgFL + 1;
                Rho_p = B_N_Orthg * pi;
                B = Rho_p;
                Radial_Normalization_Orthg = (((Structure_Radius)^3/2)*(sin(B) / B^2 - cos(B) /
B)^2);
                Theta_Normalization_Orthg = (2 / (2 * L_N_Orthg + 1));
                Phy_Normalization = 2 * pi;
```

```

        NORMALZ = (1 / Phy_Normalization) * (1 / Theta_Normalization_Orthg) * (1 /
        Radial_Normalization_Orthg);
        F = 0;
    for B_N_Field = 1: 1: Bessel_Number_Field;
        Rho_p = B_N_Field * pi;
        for L_N_Field = 0: 1: Legendre_Number_Field;
            for F_N_Field = -Exponential_Number_Field: 1: Exponential_Number_Field;
                F = F + 1;
                Sum = 0;
                for B_N_Dielectric = 1: 1: Bessel_Number_Dielectric;
                    for L_N_Dielectric = 0: 1: Legendre_Number_Dielectric;
                        for F_N_Dielectric = -Exponential_Number_Dielectric: 1:
                            Exponential_Number_Dielectric;
                            if (F_N_Field + F_N_Dielectric == F_N_Orthg);
                                Q = 2 * pi;
                                K_om = Kappa(B_N_Dielectric, L_N_Dielectric + 1,
                                F_N_Dielectric + Exponential_Number_Dielectric + 1);
                                S00 = j_TABLE_S00(B_N_Field, B_N_Dielectric, B_N_Orthg );
                                S02 = j_TABLE_S02(B_N_Field, B_N_Dielectric, B_N_Orthg );
                                Ps = PN_TABLE_Ps(L_N_Field + 1, L_N_Dielectric + 1,
                                L_N_Orthg + 1);
                                Pc2_s = PN_TABLE_Pc2_s(L_N_Field + 1, L_N_Dielectric + 1,
                                L_N_Orthg + 1);
                                Part1 = ((Rho_p /Structure_Radius)^2 * Ps)* (Structure_Radius)^3 *
S02;
                                Part2 = ( Pc2_s + (L_N_Field^2 + L_N_Field + 1) * Ps) *
(Structure_Radius) * S00;
                                Sum = Sum + K_om * (Part1 + Part2) * Q;
                            end
                        end
                    end
                end
            end
        end
        Phy_phy_Eign_matrix_block(OrthgFL , F) = Sum * NORMALZ;
    end
end
end
end
end
end
end
disp('5 ++ Completed the calculating of the elements of Phy_phy_eign-matrix block.');
```

```

MA = [Phy_phy_Eign_matrix_block ];

[EigVec,EigVa]=eig(MA);
EignValf = sum(EigVa);

```

```

a= real (EignValf);

b= imag (EignValf);
bb=[];
for vc=1:1:length(b)
    if b(vc) > 0
        bb(vc)=1;
    end
    if b(vc) < 0
        bb(vc)=-1;
    end
    if b(vc) == 0
        bb(vc)=0;
    end
end

EignVal_imag_imag = (bb/sqrt(2)).* sqrt(sqrt(a.^2+b.^2)-a);
EignVal_real_real = (1/sqrt(2)).*sqrt(a+sqrt(a.^2 + b.^2));
dss=EignVal_real_real + i*EignVal_imag_imag;
fzm1=find(abs(imag(dss))<1e-15 & real(dss)<20 & real(dss)>0);
K_Value = dss(fzm1);
K_Value = sort(K_Value')

end

```

The simplicity of creating the code demonstrates the advantage for the BLF technique. Only a couple of blocks of coding are created used and repeated with only minor small changes to complete the coding for the entire BLF technique solver.

The code below shows how the mode profile can be created.

```

tic
disp('0 ++ Program start. ');

disp('0 -- Program start. ');

disp('1 ++ Reading the Reconstruction control file. ');
    Bessel_Number_Orthogonal_Field_1 = 40;
    Legendre_Number_Orthogonal_Field_1 = 40;
    Exponential_Number_Orthogonal_Field_1 = 0;

```



```

Structure_Radius = 1;
Structure_Radius_toplot = 1;
lx_Number = 10;
ly_Number = 10;
lz_Number = 10;
SY_PHY = 0;
disp('1 -- the Reconstruction control file Read. ');

disp('5 ++ The Reconstruction is started ');
RST = (Structure_Radius)^3/2;
SPI=sqrt(1 / (2 * pi));

for tr =3
    Field=zeros(2*lx_Number+1,2*ly_Number+1,2*lz_Number+1);
    ModeEig = K_Value(tr);
    fzm1 = find(abs(imag(dss))<1e-1 & real(dss)< ModeEig+ModeEig*0.00001 &
    real(dss)>ModeEig-ModeEig*0.00001 & real(dss)>0);
    fzm1;
    EigVect =
    EigVec(1:Bessel_Number_Orthogonal_Field_1*(Legendre_Number_Orthogonal_Field_1
    +1),fzm1(1));
    K_Value(tr)
    EignVal_real_real(fzm1(1));
    freq=((EignVal_real_real(fzm1(1)))*3e8)/(2*pi);
    Kof_Er = real(EigVect);
    figure(tr+5000);
    plot(real(Kof_Er));
    Orth_N = 0;
    Orth_N_1 = 0;
    Bessel_Number_Orthogonal_Field = 40;
    Legendre_Number_Orthogonal_Field= 40;
    for B_N_Orthg = 1:1:Bessel_Number_Orthogonal_Field;
        for L_N_Orthg = 0:1:Legendre_Number_Orthogonal_Field ;
            Orth_N = Orth_N_1 + L_N_Orthg + 1;
            Kof_Profile_19(B_N_Orthg,L_N_Orthg + 1) = Kof_Er(Orth_N, 1);
        end
        Orth_N_1 = Orth_N_1 + (Bessel_Number_Orthogonal_Field_1)+1;
    end
    Kof_Er = Kof_Profile_19;
    figure(tr+1000);
    plot(real(Kof_Er),'.-');
    for lx = 1: 1: 2 * lx_Number + 1;
        for ly = 1: 1: 2 * ly_Number + 1;
            for lz = lz_Number + 1;

```

```

        x=((lx - lx_Number - 1) /lx_Number)*Structure_Radius_toplot;
        y=((ly - ly_Number - 1) /ly_Number)*Structure_Radius_toplot;
        z=((lz - lz_Number - 1) /lz_Number)*Structure_Radius_toplot;
        R = sqrt(x.^2+y.^2+z.^2);
    if R <= Structure_Radius_toplot
        Theta = acos(z/(R));
        Phi = atan2(y,x);
        Sum = 0;
        OrthgFL=0;
    for B_N_Orthg = 1: 1: Bessel_Number_Orthogonal_Field;
        Rho_p = B_N_Orthg * pi;
        A = Rho_p * R / Structure_Radius;
        Radial_Part = sin(A) / A;
    for L_N_Orthg = 0: 1: Legendre_Number_Orthogonal_Field ;
        Theta_Part = legendre(L_N_Orthg, cos(Theta));
        Theta_Part = Theta_Part( 1, 1);
    for F_N_Orthg = SY_PHY;
        OrthgFL = OrthgFL + 1;
        Phi_Part = exp(1i * F_N_Orthg * Phi) ;
        Sum = Sum + Kof_Er (B_N_Orthg,L_N_Orthg + 1) *Radial_Part *
            Theta_Part * Phi_Part;
    end
end
end
    Field ( lx,ly,lz) = (Sum);
end
end
end
end
end

for lx = 1: 1: 2 * lx_Number + 1;
    for ly = ly_Number + 1;
        for lz = 1: 1: 2 * lz_Number + 1;
            x=((lx - lx_Number - 1) /lx_Number)*Structure_Radius_toplot;
            y=((ly - ly_Number - 1) /ly_Number)*Structure_Radius_toplot;
            z=((lz - lz_Number - 1) /lz_Number)*Structure_Radius_toplot;
            R = sqrt(x.^2+y.^2+z.^2);
        if R <= Structure_Radius_toplot
            Theta = acos(z/(R));
            Phi = atan2(y,x);
            Sum = 0;
            OrthgFL=0;
        for B_N_Orthg = 1: 1: Bessel_Number_Orthogonal_Field;
            Rho_p = B_N_Orthg * pi;
            A = Rho_p * R / Structure_Radius;
            Radial_Part = sin(A) / A;

```

```

for L_N_Orthg = 0: 1: Legendre_Number_Orthogonal_Field ;
    Theta_Part = legendre(L_N_Orthg, cos(Theta));
    for F_N_Orthg = SY_PHY;
        OrthgFL = OrthgFL + 1;
        Theta_Part = Theta_Part( 1, 1);
        Phi_Part = exp(1i * F_N_Orthg * Phi) ;
        Sum = Sum + Kof_Er (B_N_Orthg,L_N_Orthg + 1) * Radial_Part *
            Theta_Part * Phi_Part;
    end
end
end
Field ( lx,ly,lz) = (Sum);
end
end
end
end
end

for lx = lx_Number + 1;
    for ly = 1: 1: 2 * ly_Number + 1;
        for lz = 1: 1: 2 * lz_Number + 1;
            x = ((lx - lx_Number - 1) / lx_Number) * Structure_Radius_toplot;
            y = ((ly - ly_Number - 1) / ly_Number) * Structure_Radius_toplot;
            z = ((lz - lz_Number - 1) / lz_Number) * Structure_Radius_toplot;
            R = sqrt(x.^2+y.^2+z.^2);
            if R <= Structure_Radius_toplot
                Theta = acos(z/(R));
                Phi = atan2(y,x);
                Sum = 0;
                OrthgFL=0;
                for B_N_Orthg = 1: 1: Bessel_Number_Orthogonal_Field;
                    Rho_p = B_N_Orthg * pi;
                    A = Rho_p * R / Structure_Radius;
                    Radial_Part = sin(A) / A;
                    for L_N_Orthg = 0: 1: Legendre_Number_Orthogonal_Field ;
                        Theta_Part = legendre(L_N_Orthg, cos(Theta));
                        Theta_Part = Theta_Part( 1, 1);
                        for F_N_Orthg = SY_PHY;
                            OrthgFL = OrthgFL + 1;
                            Phi_Part = exp(1i * F_N_Orthg * Phi) ;
                            Sum = Sum + Kof_Er (B_N_Orthg,L_N_Orthg + 1) * Radial_Part *
                                Theta_Part * Phi_Part;
                        end
                    end
                end
                Field ( lx,ly,lz) = (Sum);
            end
        end
    end
end

```

```
        end
    end
end
end
figure(tr+10000)
slice(abs(real(Field)),lx_Number+1,ly_Number+1,lz_Number+1)
end
disp('5--completed the reconstruction of the field profile')
toc
```