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QUASI-HEREDITARY
ENDOMORPHISM ALGEBRAS

by

Peter Heath M.A.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics
Carleton University
Ottawa, Ontario
May 1994

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ABSTRACT

Quasi-hereditary algebras were introduced by Cline-Parshall-Scott (see [CPS] or [PS]) to deal with highest weight categories which occur in the study of semi-simple complex Lie algebras and algebraic groups. In fact, the quasi-hereditary algebras which appear in these applications enjoy a number of additional properties. The objective of this thesis is to describe a class of lean quasi-hereditary algebras [ADL] which possess such typical characteristics. The algebras in this class arise as certain endomorphism algebras over commutative local self-injective algebras. A study of these questions originated in the work of V. Dlab, in collaboration with C.M. Ringel (see [DR1]).

The undersigned hereby recommend to
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QUASI-HEREDITARY
ENDOMORPHISM ALGEBRAS

submitted by
PETER HEATH M.A.

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for the degree of Doctor of Philosophy

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S.D.G.

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INTRODUCTION

Quasi-hereditary algebras, in the sense of Cline/Parshall/Scott, provide a class of algebras whose bounded derived categories can be built up from the bounded derived categories of division rings, using the process of “recollement.” Their module categories are just the highest weight categories which arise in the representation theory of semi-simple complex Lie algebras. In fact, the importance of this concept is now well-established in many other areas of mathematics.

The quasi-hereditary algebras which appear in these applications enjoy some additional properties such as duality. Moreover, in some instances, these algebras have been shown to be endomorphism algebras of certain special types of modules. This fact provides a motivation for these investigations, which form an integral component of the overall contemporary program to represent certain important categories (for instance, the Bernstein/Gelfand/Gelfand category $\mathcal{O}$) as categories of representations of explicitly described finite-dimensional associative algebras.

In Chapter 1 of this thesis, we shall introduce the basic definitions and results needed for the rest of the thesis. In particular, path algebras over a quiver Q are introduced and illustrated through several examples. Quasi-hereditary algebras are also defined, and their basic properties are given, including the equivalence of the “heredity ideal” definition and the “standard module filtration” definition. Chapter 1 concludes with an example of an

endomorphism algebra of the sort which is to be the subject of the main theorem of this thesis.

In Chapter 2 we assume that the base algebra R is split, local, self-injective and commutative, and that $A = \text{End}_R X$, where X is the direct sum of a set $\mathcal{X}$ of local (right) ideals of R partially ordered by inclusion. The existence of a duality functor on the category $A\text{-mod}$ of left modules over such an endomorphism algebra A is demonstrated. The main theorem of the thesis is then presented and proved, giving necessary and sufficient conditions for such an algebra A to be quasi-hereditary. Certain other properties are described which any quasi-hereditary A enjoys. These properties concern: the socles of the standard modules; multiplicities of composition factors of the standard modules; the stability of the standard modules under refinement of the inclusion order; the leanness of A ; the structure of the quiver of A ; the dimension of the indecomposable projective modules; and the dominant dimension of A . Some of these are demonstrated within the proof of the main theorem, while the others are demonstrated in Chapter 3. Furthermore, a number of examples are presented in Chapter 3 which illustrate how the main theorem works, as well as where its limitations lie.

In the final chapter, we remove the assumption that R is commutative. Examples are given which demonstrate that the main theorem cannot be extended to the non-commutative case and that even in the best of circumstances not all of the properties enjoyed in the commutative case are carried through. For instance, the standard modules can be altered depending on the particular refinement of the inclusion order which is chosen; in Chapter 2 the standard modules are shown to be stable under refinement

in the commutative case. Furthermore, $A = \text{End}_R X$, with R non-commutative, can be quasi-hereditary without having the number of summands in X equal to the K -dimension of R ; this cannot occur when R is commutative. Finally, in Chapter 4 we see a class of endomorphism algebras over non-commutative R which provide a good approximation of the results obtained in Chapter 2, provided that a refinement of the inclusion order is chosen which is also compatible with the Loewy length of the right ideals of A . The quiver of such an endomorphism algebra is rather similar to that obtained for commutative R . Moreover, many of the properties of the standard modules discussed in Chapter 2 extend to this class of endomorphism algebras over non-commutative R .

CHAPTER I

THE PRELIMINARIES

1.1 - BASIC CONCEPTS

Throughout this thesis K will be a (commutative) field and we will consider all K -algebras to be finite dimensional, and modules over a K -algebra to be finite dimensional over K . For a K -algebra A we will denote by $A\text{-mod}$ (resp. $\text{mod-}A$) the category of all finite dimensional left (resp. right) A -modules. Unless otherwise stated, all modules will be left modules. Recall that the (*Jacobson*) *radical of A* , $\text{rad}A$, is the intersection of all maximal left ideals of A ; as a consequence, $\text{rad}A$ is an ideal of A . Moreover, $A/\text{rad}A$ is a semi-simple K -algebra, i.e. $\text{rad}(A/\text{rad}A) = 0$. The *socle of A* , $\text{soc}A$, is the left ideal of A generated by the set of all simple left ideals of A . In general, for an A -module X , the *radical of X* , $\text{rad}X$, is the intersection of all maximal submodules of X and the *socle of X* , $\text{soc}X$, is the submodule of X generated by the set of all simple submodules of X .

An element e of A is an *idempotent* if $e^2 = e$. Two idempotents e and f are said to be *orthogonal* if $ef = fe = 0$. An idempotent e is *primitive* if e cannot be decomposed into a sum of (non-zero) orthogonal idempotents. Now the (left) *regular module* ${}_AA$ can be decomposed as ${}_AA = \bigoplus_{\lambda \in \Lambda} Ae_\lambda$, where each Ae_λ is indecomposable, and $\sum_{\lambda \in \Lambda} e_\lambda = 1$. Thus these e_λ 's are primitive idempotents. In fact, we can write ${}_AA = \bigoplus_{i=1}^n \bigoplus_{\lambda_i \in \Lambda_i} Ae_{\lambda_i}$, where $\{\Lambda_i | 1 \leq i \leq n\}$ is a partition of Λ such that $Ae_\lambda \simeq Ae_\mu$ if and only if $\lambda, \mu \in \Lambda_i$ for some $1 \leq i \leq n$. Taking one representative λ_i from each Λ_i , we call the set $\{e_{\lambda_i} | 1 \leq i \leq n\}$

satisfying the preceding conditions a *complete set* of primitive orthogonal idempotents. Recall that A is said to be *basic* if $\sum_{i=1}^n e_{\lambda_i} = 1$; thus, A is basic if and only if $A/\text{rad} A$ is isomorphic to a product of division algebras.

The dual of A_A , i.e. the (left) injective A -module $D(A_A) = \text{Hom}_K(A_A, K)$ is called the (left) *co-regular module*. We shall also use the following notations: $P(\lambda) = Ae_\lambda$, the *projective module corresponding to e_λ* ; $S(\lambda) = Ae_\lambda/\text{rad } Ae_\lambda$, the *simple module corresponding to e_λ* ; and $Q(\lambda) = \text{Hom}_K(e_\lambda A, K)$, the *injective module corresponding to e_λ* . $P(\lambda)$ is the projective cover of $S(\lambda)$ and $Q(\lambda)$ the injective envelope of $S(\lambda)$.

1.2 - PATH ALGEBRAS

The results in this thesis will be illustrated in several examples. The algebras in these examples will be path algebras of a quiver $\mathcal{Q}$ over a field K (cf. [A],[DrK]). Recall that a *quiver* $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1)$ consists of a finite set of vertices in the plane, $\mathcal{Q}_0 = \{1, 2, \dots, n\}$, along with a finite set of arrows between vertices,

$$\mathcal{Q}_1 = \{(i_k, j_k) | 1 \leq i_k, j_k \leq n \text{ for all } 1 \leq k \leq m\}.$$

If $\alpha = (i, j)$ is in $\mathcal{Q}_1$, then i is the *tail* of α , j is the *head* of α and α is drawn as $i \xrightarrow{\alpha} j$.

A *path* σ in $\mathcal{Q}$ is a sequence of arrows $\sigma = \sigma_k \dots \sigma_2 \sigma_1$ such that the head of σ_t coincides with the tail of σ_{t+1} for $1 \leq t \leq k-1$, i.e. $\sigma = i_0 \xrightarrow{\sigma_1} i_1 \xrightarrow{\sigma_2} i_2 \xrightarrow{\sigma_3} \dots \xrightarrow{\sigma_{k-1}} i_{k-1} \xrightarrow{\sigma_k} i_k$.

The *tail* of σ is just the tail of σ_1 , while the *head* of σ is the head of σ_k . The *length* of σ is k , the number of arrows in σ .

The *path algebra* $K\mathcal{Q}$, as a K -vector space, has a basis consisting of all paths in $\mathcal{Q}$, including the vertices of $\mathcal{Q}$ considered as paths of length zero. The product of two paths,

ρ times σ , is written $\sigma\rho$ and is the natural composition of paths if the head of ρ and the tail of σ coincide; otherwise, it is zero. Note the following facts: First, the set $\{e_i | i \in Q_0\}$ of all paths of length zero is a complete set of primitive orthogonal idempotents of KQ . Secondly, KQ is finite dimensional if and only if Q contains no oriented cycles and no loops. Finally, if Q has no oriented cycles and no loops, then KQ is hereditary. (Recall that a K -algebra A is *hereditary* if every submodule of every projective (left) A -module is projective. Equivalent conditions to the definition are: every quotient of every injective (left) A -module is injective; and, every left ideal of A is projective. The K -duality $\text{Hom}_K(-, K) : \text{mod-}A \rightarrow A\text{-mod}$ eliminates the need to refer to A as right-hereditary or left-hereditary.)

A module M over KQ can be identified with a representation of Q , i.e. a set of K -vector spaces indexed by Q_0 along with a set of linear maps indexed by Q_1 . Thus $M = (M_i, \phi_\alpha | i \in Q_0, \alpha \in Q_1)$, where $\phi_\alpha \in \text{Hom}_K(M_i, M_j)$ if α is an arrow $i \xrightarrow{\alpha} j$.

If Q contains an oriented cycle or loop we can still obtain a finite dimensional algebra using a slight variation on the path algebra construction. A *relation* on Q is a linear combination of paths of length ≥ 2 . Given a set $\mathcal{R}$ of relations, consider the KQ -ideal I generated by $\mathcal{R}$. If $\mathcal{R}$ is chosen appropriately then KQ/I is finite dimensional. A module over KQ/I can be identified with a representation of Q , where the set of maps now satisfies the additional condition that the combinations of homomorphisms corresponding to the relations in $\mathcal{R}$ are zero. (Of course, one can also take a quotient of the path algebra modulo an ideal of relations for a quiver without oriented cycles or loops.)

It is known that if an algebra A is basic and K is algebraically closed, then $A \simeq KQ/I$

for some quiver $\mathcal{Q}$ and some ideal of relations I ([DrK]). Furthermore, every algebra is Morita equivalent to a basic algebra. If A is basic, it is not difficult to determine the quiver $\mathcal{Q}$ such that $A \simeq K\mathcal{Q}/I$. Indeed, consider ${}_AA = P(1) \oplus P(2) \oplus \cdots \oplus P(n)$ with $P(i) \simeq P(j)$ if and only if $i = j$. To $P(i)$ we assign the vertex i in the plane. Now $\text{rad}P(i)/\text{rad}^2P(i) \simeq \bigoplus_{j=1}^n t_{ij}S(j)$; thus we assign to our quiver t_{ij} arrows $i \rightarrow j$. Completing this process for $i = 1, \dots, n$ will yield the desired quiver $\mathcal{Q}$; to find the relations which generate I requires more specific information about the particular algebra, such as the submodule structure of the projective modules $P(i)$, $1 \leq i \leq n$.

Finally, to illustrate the notation for the regular module, suppose $A = K\mathcal{Q}$ for $\mathcal{Q}: 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ with idempotents $\{e_1, e_2, e_3\}$. Then $P(1)$ has K -basis $\{e_1, \alpha, \beta\alpha\}$, $P(2)$ has K -basis $\{e_2, \beta\}$, and $P(3)$ has K -basis $\{e_3\}$. Since $\alpha \in e_2Ae_1$, $\beta \in e_3Ae_2$, $\beta\alpha \in e_3Ae_1$, the left regular module is written as

$${}_AA = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \oplus \begin{matrix} 2 \\ 3 \end{matrix} \oplus 3.$$

1.3 - QUASI-HEREDITARY ALGEBRAS

Let $\underline{e} = \{e_\lambda | \lambda \in \Lambda\}$ be a complete set of primitive orthogonal idempotents of the algebra A . We now assume that $(\Lambda, \leq)$ is a partially ordered set. The *standard module* $\Delta(\lambda)$ is the largest quotient module of $P(\lambda)$ with composition factors only of the form $S(\mu)$, $\mu \leq \lambda$, i.e. $\Delta(\lambda) = P(\lambda) / \text{trace} \left(\bigoplus_{\nu \not\leq \lambda} P(\nu) \rightarrow P(\lambda) \right)$. Similarly, the *co-standard module* $\nabla(\lambda)$ is the largest submodule of $Q(\lambda)$ whose composition factors are only of the form $S(\mu)$, $\mu \leq \lambda$, i.e. $\nabla(\lambda) = \bigcap_{\nu \not\leq \lambda} \text{Ker}(Q(\lambda) \rightarrow Q(\nu))$. (If we wish to emphasize that $\Delta(\lambda)$ or $\nabla(\lambda)$ are modules over a particular algebra A , then we write $\Delta_A(\lambda)$ or $\nabla_A(\lambda)$.)

We will denote by Δ the set $\{\Delta(\lambda) | \lambda \in \Lambda\}$ and by ∇ the set $\{\nabla(\lambda) | \lambda \in \Lambda\}$. An A -module X is said to have a Δ -filtration (resp. ∇ -filtration) if there is a filtration

$$0 = X_{m+1} \subset X_m \subset \cdots \subset X_{t+1} \subset X_t \subset \cdots \subset X_1 = X$$

with $X_t/X_{t+1} \simeq \Delta(\lambda)$ (resp. $X_t/X_{t+1} \simeq \nabla(\lambda)$) for some $\lambda \in \Lambda$, for all $1 \leq t \leq m$. The set of all modules with a Δ -filtration is denoted $\mathcal{F}(\Delta)$; the set of all modules with a ∇ -filtration is denoted $\mathcal{F}(\nabla)$.

In general, both the standard and co-standard modules will change if we refine the (partial) order of $(\Lambda, \leq)$. We shall therefore restrict our considerations to *adapted partial orders* of $(\Lambda, \leq)$ (cf. [DR]), i.e. those satisfying the following condition: For every A -module X with $X/\text{rad}X \simeq S(\lambda_1)$ and $\text{soc}X \simeq S(\lambda_2)$, where λ_1 and λ_2 are incomparable, there is $\mu \in \Lambda$ such that $\mu > \lambda_1$, $\mu > \lambda_2$ and $S(\mu)$ is a composition factor of X , i.e. $e_\mu X \neq 0$ ([DrK]). Then, the standard and co-standard modules do not change under any refinement of $(\Lambda, \leq)$, and thus we may, without loss of generality, assume that $(\Lambda, \leq)$ is a total order; in this case, we will simply replace $(\Lambda, \leq)$ by the index set $\{1, 2, \dots, n\}$ with its natural order. Hence, assume that $\underline{e} = (e_1, e_2, \dots, e_n)$ is a (totally ordered) complete sequence of primitive orthogonal idempotents of A .

Definition 1.3.1.

An algebra A is said to be *quasi-hereditary* (with respect to an adapted partial order $(\Lambda, \leq)$, or equivalently to a total order $\underline{e}$) if

$$(1) \quad {}_A A \in \mathcal{F}(\Delta) \text{ and}$$

$$(2) \quad \text{End}_A \Delta(i) \text{ is a division algebra (i.e. } \Delta(i) \text{ is Schurian) for } 1 \leq i \leq n.$$

In order to underline the fact that A is quasi-hereditary with respect to the order given by an ordered complete sequence $\underline{e} = (e_1, e_2, \dots, e_n)$ of primitive orthogonal idempotents we will often write $(A, \underline{e})$ instead of just A . Let us consider some of the basic properties of the quasi-hereditary algebras as given in Definition 1.3.1.

Quasi-hereditary algebras were introduced by Cline/Parshall/Scott ([S], [PS], [CPS]) in order to deal with highest weight categories as they appear in the representation theory of semi-simple complex Lie algebras and algebraic groups. Their original definition of these algebras was recursive. The basic properties of these algebras were described by Dlab/Ringel [DR1], who defined them in terms of the so-called heredity chains:

Definition 1.3.2.

An ideal J of A is an heredity ideal if $J^2 = J$, $J(\text{rad} A)J = 0$ and J , considered as an A -module ${}_A J$, is projective. A is quasi-hereditary if there is a chain

$$0 = J_{m+1} \subset J_m \subset \dots \subset J_{i+1} \subset J_i \subset \dots \subset J_1 = A$$

of ideals of A such that for $1 \leq i \leq m$, J_i/J_{i+1} is an heredity ideal of A/J_{i+1} . Such a chain of idempotent ideals is called an heredity chain.

Lemma 1.3.3.

The two definitions of a quasi-hereditary algebra are equivalent.

Proof. First, let

$$(*) \quad 0 = J_{n+1} \subset J_n \subset \dots \subset J_{i+1} \subset J_i \subset \dots \subset J_1 = A$$

be an heredity chain of A . We may assume that $(*)$ is saturated, i.e. that no proper refinement of $(*)$ is possible, since any idempotent ideal contained in an heredity ideal is

itself an heredity ideal. By a proposition in [D] the saturated heredity chain (*) gives rise to a fixed order $\underline{e} = (e_1, e_2, \dots, e_n)$ of the complete set of primitive orthogonal idempotents of A , such that $J_i = A(e_i + e_{i+1} + \dots + e_n)A$ for $1 \leq i \leq n$. We will show that $(A, \underline{e})$ is quasi-hereditary according to Definition 1.3.1.

Indeed, for the idempotent ideal J_i we have

$$J_i = A(e_i + e_{i+1} + \dots + e_n)A = \text{trace} \left(\bigoplus_{j \geq i} P(j) \rightarrow A \right), \quad 1 \leq i \leq n,$$

and thus the filtration (*) is just the trace filtration (cf. [D]) of the regular module ${}_A A$. The fact that J_i/J_{i+1} is a (left)projective A/J_{i+1} -module means that J_i/J_{i+1} is isomorphic to a finite direct sum of $\Delta(i)$'s and thus ${}_A A \in \mathcal{F}(\Delta)$.

For an arbitrary idempotent ideal J of A , $J = AeA$ for some idempotent $e \in A$, so $J(\text{rad} A)J = 0$ is equivalent to $\text{End}_A(Ae)$ being a product of division algebras. Since J_n is an heredity ideal of A , and ${}_A(J_n) \simeq t_n(Ae_n)$, for some $t_n > 0$, it follows that $\text{End}_A \Delta(n)$ is a division algebra. Similarly, since ${}_A(J_i/J_{i+1}) \simeq t_i \Delta(i)$, for some $t_i > 0$, $\Delta(i)$ must be Schurian for all i . Consequently, $(A, \underline{e})$ is quasi-hereditary by Definition 1.3.1.

The proof of the converse statement proceeds along similar lines. $\square$

As pointed out in the proof, the filtration (*) is the trace filtration of A . In general, for $X \in A\text{-mod}$, the *trace filtration of X* (with respect to $\underline{e}$) is defined to be

$$0 = X_{n+1} \subset X_n \subset \dots \subset X_{i+1} \subset X_i \subset \dots \subset X_1 = X,$$

where $X_i = A(e_i + \dots + e_n)X$ is the trace of $A(e_i + \dots + e_n)$ in X .

The definitions 1.3.1 and 1.3.2 are both given in terms of left A -modules. The following lemma from [DR 1] guarantees that there is no need to refer to A as "left quasi-hereditary"

i.e. that the same conditions are satisfied for the right regular module A_A and the right standard modules.

Lemma 1.3.4.

Let A be a K -algebra and let e be an idempotent of A . If the right module $(AeA)_A$ or the left module ${}_A(AeA)$ is projective, then the multiplication map

$$\mu : Ae \otimes_{eAe} eA \rightarrow AeA$$

is bijective. Conversely, assume that $e(\text{rad} A)e = 0$. If μ is bijective then both modules $(AeA)_A$ and ${}_A(AeA)$ are projective.

At this point, we would like to present some examples of quasi-hereditary algebras. First of all, every hereditary algebra is quasi-hereditary. In fact, a K -algebra A is hereditary if and only if every chain of idempotent ideals of A can be refined to an heredity chain of A , or equivalently, if $(A, \underline{e})$ is quasi-hereditary for every complete sequence $\underline{e}$ of primitive orthogonal idempotents of A ([DR 1], [D]).

While there are examples of K -algebras of global dimension 3 which are not quasi-hereditary, it is a result in [DR] that all K -algebras of global dimension 2 are quasi-hereditary. As a result, all Auslander algebras are quasi-hereditary ([DR2]). Recall that the Auslander algebras can be constructed as follows: let A' be a finite dimensional representation finite K -algebra and let ${}_A M$ be a finite dimensional A' -module such that all the indecomposable A' -modules are contained in $\text{add } M$. Then $A = \text{End}_{A'} M$ is an Auslander algebra.

Let us now present some explicit examples of quasi-hereditary algebras.

1. Let $\mathcal{Q}$ be the quiver

$$1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3,$$

and let us consider $A = K\mathcal{Q}$. Then

$${}_A A = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \oplus 3.$$

Of course, A is hereditary, so A is quasi-hereditary for any ordering $\underline{e}$ of the idempotents. Take, for instance, the order $\underline{e} = (e_1, e_2, e_3)$. Here, $\Delta(1) \simeq 1$, $\Delta(2) \simeq 2$, and $\Delta(3) \simeq 3$; hence $\Delta(i)$ is Schurian for $i = 1, 2, 3$. We also obtain the following trace filtrations of $P(1)$ and $P(2)$:

$$0 \subset P(1)_3 \subset P(1)_2 \subset P(1); \quad 0 \subset P(2)_3 \subset P(2).$$

where $P(1)_3, P(2)_3 \simeq P(3)$ and $P(1)_2 \simeq P(2)$. This shows that the filtration condition is satisfied for all projectives, so $(A, \underline{e})$ is quasi-hereditary.

2. Consider the quiver

$$\mathcal{Q}: \quad 1 \begin{smallmatrix} \xrightarrow{\alpha} \\ \xleftarrow{\delta} \end{smallmatrix} 2 \begin{smallmatrix} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{smallmatrix} 3.$$

Here, $K\mathcal{Q}$ is infinite dimensional. Consider the following two factor algebras of $K\mathcal{Q}$:

(a) $\mathcal{S} = K\mathcal{Q}/\langle \beta\alpha, \beta\gamma, \delta\gamma, \alpha\delta \rangle$; hence,

$${}_s \mathcal{S} = \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \oplus 1 \begin{smallmatrix} 2 \\ 3 \\ 2 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}.$$

In the usual order $\underline{e}$ we obtain

$$\Delta(1) \simeq 1, \quad \Delta(2) \simeq \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \quad \Delta(3) \simeq \begin{smallmatrix} 3 \\ 2 \end{smallmatrix}$$

and a simple check confirms that $(\mathcal{S}, \underline{e})$ is quasi-hereditary.

On the other hand, consider the order $\underline{e}' = (e_3, e_2, e_1)$. Now

$$\Delta(1) \simeq \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}, \Delta(2) \simeq \begin{smallmatrix} 2 \\ 3 \\ 2 \end{smallmatrix}, \Delta(3) \simeq 3.$$

Since neither $\Delta(1)$ nor $\Delta(2)$ are Schurian $(\mathcal{S}, \underline{e}')$ is not quasi-hereditary.

(b) $\mathcal{D} = KQ/\langle \beta\gamma, \alpha\delta \rangle$, and so

$${}_{\mathcal{D}}\mathcal{D} = \begin{smallmatrix} & 1 \\ & 2 \\ 1 & & 3 \\ & 2 \\ & 1 \end{smallmatrix} \oplus \begin{smallmatrix} & 2 \\ & 3 \\ 1 & & 3 \\ & 2 \\ & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}.$$

Again we begin with the usual order $\underline{e}$, in which case

$$\Delta(1) \simeq 1, \Delta(2) \simeq \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \Delta(3) \simeq \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix},$$

and $(\mathcal{D}, \underline{e})$ is quasi-hereditary.

On the other hand, in the order $\underline{e}' = (e_1, e_3, e_2)$, $P(3) \notin \mathcal{F}(\Delta)$. Thus $(\mathcal{D}, \underline{e}')$ is not quasi-hereditary.

The two algebras above were not chosen arbitrarily. $\mathcal{S} = \mathcal{S}(Q)$ is the shallow algebra on Q and $\mathcal{D} = \mathcal{D}(Q)$ is the deep algebra on Q , for Q as given (cf. [D]). The shallow and deep algebras are important for a variety of reasons. In particular, if $A = KQ/I$ is a quasi-hereditary algebra, then $\dim_K \mathcal{S}(Q) \leq \dim_K A \leq \dim_K \mathcal{D}(Q)$. For further results on shallow and deep algebras, see [D], [ADL], [DR6].

1.4 - STANDARD AND CO-STANDARD MODULES

Recall that we have defined $\nabla(\lambda) = \bigcap_{\nu \not\leq \lambda} \text{Ker}(Q(\lambda) \rightarrow Q(\nu))$ for all $\lambda \in \Lambda$. In fact $\nabla(\lambda) \simeq D\Delta^\circ(\lambda)$. ($\Delta^\circ(\lambda)$ is the appropriate quotient of the right projective module $e_\lambda A$.) This fact leads to the following proposition.

Proposition 1.4.1 ([DR4]).

Assume that Λ is ordered, and that all standard modules are Schurian. Then the following are equivalent:

- (i) ${}_A A \in \mathcal{F}(\Delta)$;
- (ii) $\mathcal{F}(\Delta) = \{X | \text{Ext}^1(X, \nabla) = 0\}$;
- (iii) $\mathcal{F}(\Delta) = \{X | \text{Ext}^i(X, \nabla) = 0, \text{ for all } i \geq 1\}$;
- (iv) $\text{Ext}^2(\Delta, \nabla) = 0$.

In particular, condition (iv) is self-dual, so the dual formulations of (i), (ii) & (iii) may be added:

- (i)* $D({}_A A) \in \mathcal{F}(\nabla)$;
- (ii)* $\mathcal{F}(\nabla) = \{Y | \text{Ext}^1(\Delta, Y) = 0\}$;
- (iii)* $\mathcal{F}(\nabla) = \{Y | \text{Ext}^i(\Delta, Y) = 0, \text{ for all } i \geq 1\}$.

Thus, condition (i)* says that, given Schurian standard modules, every injective A -module can be filtered by ∇ precisely when A is quasi-hereditary.

Returning to the standard modules let us note that, for $(A, \leq)$ quasi-hereditary, the extensions of the $\Delta(i)$'s in the filtration of $P(k)$ appear only in a certain order. Indeed, since $\text{Ext}^1(\Delta(j), \Delta(i)) = 0$ for $j \geq i$, we get the filtration

$$0 = I_{n+1} \subset I_n \subset \cdots \subset I_{i+1} \subset I_i \subset \cdots \subset I_k = P(k)$$

where I_i/I_{i+1} is isomorphic to a direct sum of copies of $\Delta(i)$, for all $k \leq i \leq n$. Dually, we get the following filtration of $Q(k)$:

$$0 = J_k \subset J_{k+1} \subset \cdots \subset J_{i-1} \subset J_i \subset \cdots \subset J_n = Q(k)$$

where J_i/J_{i-1} is isomorphic to a direct sum of copies of $\nabla(i)$, for $k \leq i \leq n$. In particular, $P(k)$ is filtered by $\{\Delta(i) | k \leq i \leq n\}$ and $Q(k)$ is filtered by $\{\nabla(i) | k \leq i \leq n\}$ [DR4].

1.5 - ENDOMORPHISM ALGEBRAS

Let R be a K -algebra and let X be a right module over R . Consider the *endomorphism algebra*, $\text{End}_R X = \text{Hom}_R(X, X)$, of X ; here, multiplication of endomorphisms is given by composition. The endomorphisms act on the left of X , and the composition of the endomorphisms ϕ and ψ will be written $\psi\phi$, i.e. $\psi\phi : X \xrightarrow{\phi} X \xrightarrow{\psi} X$. In the sequel, the algebra A will always be the endomorphism algebra of a right module X over a K -algebra R .

We say that R *splits over* K , or that K is a *splitting field* for R , if

$$R/\text{rad}R \simeq M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_s}(K).$$

R is *local* if $\text{rad}R$ is the unique maximal (left) ideal of R . Hence, when R is local, $R/\text{rad}R$ is a simple R -module. In particular, every local K -algebra R is basic; thus $R = KQ/I$ for some quiver Q consisting of one vertex, with all arrows being loops. Furthermore, if R is also split, then every simple module over R is isomorphic to K (as a vector space.) We say that R is *self-injective* if ${}_R R$ is also an injective module over R . In our case, this is equivalent to ${}_R R \simeq D(R_R) = \text{Hom}_K(R_R, K)$.

For the remainder of this thesis R will be a local, self-injective K -algebra over a splitting field K .

1.6 - AN EXAMPLE

Let us conclude this chapter with an example.

Let $\mathcal{Q}$ be the quiver

$$\mathcal{Q}: \begin{array}{c} \circlearrowright \\ 1 \quad \alpha \end{array}$$

consisting of a single vertex 1, and a single loop α . Let K be a field and let $I = \langle \alpha^4 \rangle$. Consider $R = K\mathcal{Q}/I$. R has K -basis $\{\bar{1}, \bar{\alpha}, \bar{\alpha}^2, \bar{\alpha}^3\}$ where $\bar{1}$ is the equivalence class of 1 modulo I , etc. R is uniserial; its lattice of ideals is given below:

$$0 \subset \langle \bar{\alpha}^3 \rangle \subset \langle \bar{\alpha}^2 \rangle \subset \langle \bar{\alpha} \rangle \subset R.$$

It is clear that R is commutative, local and self-injective. (Since $D(R_R)$ has the usual K -basis $\{\bar{1}^*, \bar{\alpha}^*, \bar{\alpha}^{2*}, \bar{\alpha}^{3*}\}$, we define $\Phi: {}_R R \rightarrow D(R_R)$ to be the R -homomorphism induced by $\Phi(\bar{1}) = \bar{\alpha}^{3*}$, $\Phi(\bar{\alpha}) = \bar{\alpha}^{2*}$, $\Phi(\bar{\alpha}^2) = \bar{\alpha}^*$, $\Phi(\bar{\alpha}^3) = \bar{1}^*$. It is clear that Φ is an isomorphism.)

Now consider the following four ideals of R :

$$X(1) = R; \quad X(2) = \langle \bar{\alpha} \rangle; \quad X(3) = \langle \bar{\alpha}^2 \rangle; \quad X(4) = \langle \bar{\alpha}^3 \rangle.$$

Let $X = \bigoplus_{i=1}^4 X(i)$ and let $A = \text{End}_R X$. Then $A = K\mathcal{Q}'/I'$, where $\mathcal{Q}'$ is the quiver

$$\begin{array}{ccccc} & \alpha & & \beta & \\ 1 & \xleftrightarrow{\quad} & 2 & \xleftrightarrow{\quad} & 3 & \xleftrightarrow{\quad} & 4 \\ & \phi & & \varepsilon & & \delta \end{array},$$

and $I' = \langle \gamma\delta, \beta\varepsilon\delta, \alpha\phi\varepsilon\delta, \alpha\phi - \varepsilon\beta, \beta\varepsilon - \delta\gamma \rangle$.

The (left) regular module ${}_A A$ is

$${}_A A = P(1) \oplus P(2) \oplus P(3) \oplus P(4) = \begin{array}{c} \begin{array}{c} 1 \\ \swarrow \quad \searrow \\ 2 \quad 3 \\ \swarrow \quad \searrow \\ 1 \quad 2 \\ \swarrow \quad \searrow \\ 1 \quad 2 \end{array} \oplus \begin{array}{c} \begin{array}{c} 2 \\ \swarrow \quad \searrow \\ 1 \quad 3 \\ \swarrow \quad \searrow \\ 1 \quad 2 \end{array} \oplus \begin{array}{c} \begin{array}{c} 3 \\ \swarrow \quad \searrow \\ 2 \quad 4 \\ \swarrow \quad \searrow \\ 1 \quad 2 \end{array} \oplus \begin{array}{c} 4 \\ 3 \\ 2 \\ 1 \end{array} \end{array}$$

The (left) co-regular module is

$$D(A_A) = Q(1) \oplus Q(2) \oplus Q(3) \oplus Q(4) =$$

Hence, we have all the indecomposable projective and injective A -modules.

In the usual order,

$$\Delta = \left\{ 1, \begin{matrix} 2 \\ 1 \end{matrix}, \begin{matrix} 3 \\ 2 \end{matrix}, \begin{matrix} 4 \\ 3 \end{matrix} \right\}, \quad \nabla = \left\{ 1, \begin{matrix} 1 \\ 2 \end{matrix}, \begin{matrix} 1 \\ 3 \end{matrix}, \begin{matrix} 1 \\ 4 \end{matrix} \right\}$$

and it is easy to see that $(A, \underline{e})$ is quasi-hereditary. Let us point out that A is, in fact, the Auslander algebra over the algebra R .

CHAPTER II

ENDOMORPHISM ALGEBRAS OVER COMMUTATIVE R

2.1 - INTRODUCTION

Throughout this chapter, R will be a commutative local self-injective K -algebra over a splitting field K , and A will be the endomorphism algebra of a (finite) direct sum $X = \bigoplus_{\lambda \in \Lambda} X(\lambda)$ of pairwise non-isomorphic local-colocal right R -modules $X(\lambda)$, i.e. such that both $X(\lambda)/\text{rad}X(\lambda)$ and $\text{soc}X(\lambda)$ are simple. For each $\lambda \in \Lambda$, we define the idempotent $e_\lambda = m_\lambda p_\lambda$, where $p_\lambda : X \rightarrow X(\lambda)$ and $m_\lambda : X(\lambda) \rightarrow X$ are the canonical projection and embedding, respectively. As before, for all $\lambda \in \Lambda$, $S(\lambda) = Ae_\lambda/\text{rad}Ae_\lambda$ is the simple left A -module corresponding to e_λ , $P(\lambda) = Ae_\lambda$ is its projective cover and $Q(\lambda) = \text{Hom}_K(e_\lambda A, K)$ is its injective hull. We note that $S(i) \simeq S(j)$ if and only if $i = j$.

Observe that, for each $X(\lambda)$, there is a (unique) embedding into R . Since R is self-injective every R -homomorphism $f : X(\lambda) \rightarrow X(\kappa)$ is induced by multiplication by an element $r \in R$. Indeed, given f , there is an extension $\bar{f} : R \rightarrow R$ and every endomorphism of R_R is given by multiplication,

$$\begin{array}{ccccc} 0 & \longrightarrow & X(\lambda) & \longrightarrow & R_R \\ & & \downarrow f & & \downarrow \bar{f} \\ 0 & \longrightarrow & X(\kappa) & \longrightarrow & R_R. \end{array}$$

In particular the image, $Im f$, is a submodule of $X(\lambda)$. As a result, the following three statements, which will be used repeatedly, are equivalent:

- (a) $R \supseteq X(\kappa) \supseteq X(\lambda)$;
- (b) there is a monomorphism from $X(\lambda)$ to $X(\kappa)$;
- (c) there is an epimorphism from $X(\kappa)$ to $X(\lambda)$.

The commutativity of R yields a useful duality functor F on the category $A\text{-mod}$. The following four lemmas give the construction of F and some of its properties. For all $X(\lambda)$, let $DX(\lambda)$ denote the usual K -dual of $X(\lambda)$, i.e. $Hom_K(X(\lambda), K)$.

Lemma 2.1.1.

$DX(\lambda) \simeq X(\lambda)$ for each $\lambda \in \Lambda$.

Proof. Since $X(\lambda)$ is local, it is a factor module of R , i.e. $X(\lambda) \simeq R/I$ for some (right) ideal I of R . $X(\lambda)$ is also co-local, so R/I is a local co-local quotient algebra of R . Hence R/I is itself a self-injective algebra. Thus, $D(X(\lambda)_{R/I}) \simeq {}_{R/I}X(\lambda)$. Since $I = ann\ X(\lambda)$ it follows that $D(X(\lambda)_R) \simeq {}_RX(\lambda)$. $\square$

Lemma 2.1.2.

The map $*$: $A \rightarrow A$ defined for

$$f : X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{f_{\kappa\lambda}} X(\kappa) \xrightarrow{m_\kappa} X$$

by

$$f^* : X \xrightarrow{p_\kappa} X(\kappa) \simeq DX(\kappa) \xrightarrow{Df_{\kappa\lambda}} DX(\lambda) \simeq X(\lambda) \xrightarrow{m_\lambda} X$$

is an involution.

Proof. We must show that (i) $(fg)^* = g^*f^*$, (ii) $f^{**} = f$ and (iii) $e_\lambda^* = e_\lambda$ hold for all $f, g \in A$ and for all $\lambda \in \Lambda$.

(i) The only non-trivial case is when we have

$$f : X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{f_{\kappa\lambda}} X(\kappa) \xrightarrow{m_\kappa} X$$

and

$$g : X \xrightarrow{p_\mu} X(\mu) \xrightarrow{g_{\lambda\mu}} X(\lambda) \xrightarrow{m_\lambda} X,$$

so that

$$fg : X \xrightarrow{p_\mu} X(\mu) \xrightarrow{f_{\kappa\lambda}g_{\lambda\mu}} X(\kappa) \xrightarrow{m_\kappa} X.$$

Then

$$(fg)^* : X \xrightarrow{p_\kappa} DX(\kappa) \xrightarrow{D(f_{\kappa\lambda}g_{\lambda\mu})} DX(\mu) \xrightarrow{m_\mu} X,$$

and since D is the usual K -duality, $D(f_{\kappa\lambda}g_{\lambda\mu}) = D(g_{\lambda\mu})D(f_{\kappa\lambda})$ which implies that $(fg)^* = g^*f^*$.

(ii) For f as in (i), it is clear that applying $*$ twice yields

$$f^{**} : X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{D(Df_{\kappa\lambda})} X(\kappa) \xrightarrow{m_\kappa} X;$$

since all modules are finite dimensional over K , $D(Df_{\kappa\lambda}) = f_{\kappa\lambda}$ and $f^{**} = f$.

(iii) By definition, $e_\lambda : X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{1_{X(\lambda)}} X(\lambda) \xrightarrow{m_\lambda} X$. Since $DX(\lambda) \simeq X(\lambda)$, we have $e_\lambda^* : X \xrightarrow{p_\lambda} X(\lambda) \xrightarrow{\text{Hom}_K(1_{X(\lambda)}, K)} X(\lambda) \xrightarrow{m_\lambda} X$, and it follows that

$$\text{Hom}_K(1_{X(\lambda)}, K) = 1_{X(\lambda)}, \text{ i.e. } e_\lambda^* = e_\lambda.$$

Therefore, $*$ is an involution. $\square$

Lemma 2.1.3.

There is a duality functor $F : A\text{-mod} \rightarrow A\text{-mod}$ such that $FS(\lambda) \simeq S(\lambda)$ and $FP(\lambda) \simeq Q(\lambda)$ for all $\lambda \in \Lambda$.

Proof. Let ${}_A Y$ and ${}_A Z$ be left A -modules and let $\phi : {}_A Y \rightarrow {}_A Z$ be an A -homomorphism. For ${}_A Y$, we define the right module Y^*_A by setting $Y^* = Y$ as K -vector spaces and defining the right action of A on Y^* by using the involution $*$, i.e. by setting $ya = a^*y$ for all $y \in Y^*$, $a \in A$. Similarly we define the (right) A -module map $\phi^* : Y^*_A \rightarrow Z^*_A$ by setting $\phi^*y = y\phi$ for all $y \in Y^*$.

We define the functor F as follows: $FY = \text{Hom}_K(Y^*_A, K)$, for all $Y \in A\text{-mod}$; $F\phi : FZ \rightarrow FY$, $f \mapsto f\phi^* : Y^*_A \xrightarrow{\phi^*} Z^*_A \xrightarrow{f} K$. It is easy to see that F is a duality functor.

To complete the proof, let us first observe that $P(\lambda)^*_A$ is a (right) projective A -module for all $\lambda \in \Lambda$. Indeed, consider the map $\Psi_\lambda : P(\lambda)^*_A \rightarrow e_\lambda A$, $ae_\lambda \mapsto e_\lambda a^*$. Since $*$: $A \rightarrow A$ is bijective, so is Ψ_λ . Furthermore, Ψ_λ is clearly an A -homomorphism, i.e. Ψ_λ is an isomorphism. Thus $P(\lambda)^*_A$ is the right projective A -module corresponding to e_λ , and so

$$FP(\lambda) = \text{Hom}_K(P(\lambda)^*_A, K) \simeq Q(\lambda).$$

Finally, we note that F is exact, being the composition of $*$ and D . Thus, $F(P(\lambda)/\text{rad}P(\lambda)) \simeq \text{soc}Q(\lambda)$, that is, $FS(\lambda) = S(\lambda)$ for all $\lambda \in \Lambda$. $\square$

Corollary 2.1.4.

For all $\lambda \in \Lambda$, $F\Delta(\lambda) \simeq \nabla(\lambda)$.

Proof. Let $I(\lambda)$ be the ideal $A(\sum_{\mu \neq \lambda} e_\mu)A$. Then we have $\Delta(\lambda) = P(\lambda)/P(\lambda) \cap I(\lambda)$, and it is easy to see that $(P(\lambda)/P(\lambda) \cap I(\lambda))^*_A \simeq P(\lambda)^*_A/P(\lambda)^*_A \cap I(\lambda)^*$. Indeed, consider the

map

$$\Phi_\lambda : (P(\lambda)/P(\lambda) \cap I(\lambda))^*_A \rightarrow P(\lambda)^*_A / (P(\lambda)^*_A \cap I(\lambda)^*), \quad \overline{e_\lambda a} \mapsto \overline{a^* e_\lambda}.$$

Using an argument similar to that used in Lemma 2.1.3 for Ψ_λ , we see that Φ_λ is A -linear, and a simple dimension check confirms that Φ_λ is surjective. Finally, since $\ker \Phi_\lambda = P(\lambda)^*_A \cap I(\lambda)^*$, Φ_λ is injective, and thus Φ_λ is an isomorphism. Since $D\Delta^\circ(\lambda) \simeq \nabla(\lambda)$ (cf. §1.4), and $I(\lambda)^* = I(\lambda)$ (recall that $*$ fixes idempotents), we have $F\Delta(\lambda) \simeq \nabla(\lambda)$. $\square$

2.2 - THE MAIN THEOREM AND REMARKS

The main theorem of this chapter is the following:

THEOREM [DHM].

Let R be a commutative local selfinjective K -algebra over a splitting field K , $\dim_K R < \infty$. Let $\mathcal{X} = \{X(\lambda) | \lambda \in \Lambda\}$ be a set of local right ideals of R indexed by a finite partially ordered set Λ reflecting inclusions: $X(\lambda') \subset X(\lambda'')$ if and only if $\lambda' > \lambda''$. Let $R = X(\lambda_1)$ belong to $\mathcal{X}$. Then $A = \text{End}_R(\bigoplus_{\lambda \in \Lambda} X(\lambda))$ is a quasi-hereditary algebra with respect to Λ if and only if

(i) $\text{card}(\Lambda) = \dim_K R$ and

(ii) $\text{rad} X(\lambda) = \sum_{\lambda < \kappa} X(\kappa)$, for all $\lambda \in \Lambda$.

Remarks.

Recall that the standard module $\Delta(\lambda)$ is the maximal factor module of $P(\lambda)$ with composition factors only of the form $S(\kappa)$ for $\kappa \leq \lambda$, i.e.

$$\Delta(\lambda) = P(\lambda) / \text{trace} \left(\bigoplus_{\mu \not\leq \lambda} P(\mu) \rightarrow P(\lambda) \right).$$

We say that A is *quasi-hereditary with respect to the partially ordered set Λ* if there is a linear order $\lambda_1 < \lambda_2 < \dots < \lambda_n$ on Λ refining the given partial order and satisfying the following conditions:

for each $1 \leq i \leq n$,

(i) for the standard module defined above we also have

$$\Delta(\lambda_i) = P(\lambda_i) / \text{trace} \left(\bigoplus_{j > i} P(\lambda_j) \rightarrow P(\lambda_i) \right),$$

(ii) the endomorphism algebra of $\Delta(\lambda_i)$ is a division algebra and

(iii) $P(\lambda_i)$ can be filtered by $\{\Delta(\lambda_j) | j \geq i\}$.

The latter condition is equivalent to the fact that the factors

$$\text{trace} \left(\bigoplus_{j=k}^n P(\lambda_j) \rightarrow P(\lambda_i) \right) / \text{trace} \left(\bigoplus_{j=k+1}^n P(\lambda_j) \rightarrow P(\lambda_i) \right)$$

of the trace filtration of $P(\lambda_i)$ are direct sums of $\Delta(\lambda_k)$'s, for all $i \leq k \leq n$. [D]

Let us note that under the conditions of the theorem, we do indeed have the simple-fixing duality functor $F : A\text{-mod} \rightarrow A\text{-mod}$. In the course of proving the Theorem we will also prove that $[\Delta(\lambda) : S(\kappa)] \leq 1$ for all $\lambda, \kappa \in \Lambda$; in fact, we have $[\Delta(\lambda) : S(\kappa)] = 1$ if and only if $\kappa \leq \lambda$, and thus $\dim_K \Delta(\lambda) = \text{card}\{\kappa | \kappa \leq \lambda\}$. In addition, we will prove that every standard module $\Delta(\lambda)$ has a simple socle isomorphic to $S(\lambda_1)$ and that $R/\text{rad}R \simeq X(\lambda_n) \in \mathcal{X}$.

Finally, under the conditions of the Theorem, the following statements are also true and will be proved in Chapter 3:

- (1) the algebra A is lean (see [ADL]);
- (2) $\dim_K P(\kappa) = \sum_{\lambda \geq \kappa} \dim_K \Delta(\lambda)$ for all $\kappa \in \Lambda$ (so $\dim_K P(\lambda_n) = \dim_K R$),
and thus $\dim_K A = \sum_{\lambda \in \Lambda} (\dim_K \Delta(\lambda))^2$;
- (3) the dominant dimension of A is ≥ 2 (see [T]);

2.3 - PROOF OF SUFFICIENCY

The endomorphisms of $X = \bigoplus_{\lambda \in \Lambda} X(\lambda)$ will operate on the module from the left and we shall deal with the left regular module, ${}_A A$, of the (basic) K -algebra $A = \text{End}_R(X)$. Note that the set $\{S(\lambda) | \lambda \in \Lambda\}$ of all (left) simple A -modules is indexed by the partially ordered set Λ . For every $\lambda \in \Lambda$, we define $\Lambda(\lambda) = \{\mu \in \Lambda | X(\mu) \subsetneq X(\lambda)\}$. Note that, since every $X(\lambda)$ is local, there is some $x_\lambda \in R$ such that $X(\lambda) = x_\lambda R$.

For the remaining portion of this section we will assume that conditions (i) and (ii) of the Theorem hold. Let us remark that condition (ii) can be expressed in the form $\text{rad} X(\lambda) = \sum_{\mu \in \Lambda(\lambda)} X(\mu)$, $\lambda \in \Lambda$. Since $\dim_K R < \infty$, R has finite Loewy length, say m . By applying condition (ii) repeatedly we can obtain $\text{rad}^{m-2} R = \sum_{\lambda \in \Lambda'} X(\lambda)$ for some subset $\Lambda' \subset \Lambda$ where $\text{rad} X(\lambda) \subseteq \text{soc} R$, for all $\lambda \in \Lambda'$. Since R is co-local, $\text{rad} X(\lambda) = \text{soc} R$ (which is simple) for all $\lambda \in \Lambda'$. Condition (ii) then guarantees the existence of the largest element $\lambda_n \in \Lambda$ (i.e. $\lambda \leq \lambda_n$ for all $\lambda \in \Lambda$) and $X(\lambda_n)$ is the (unique) simple R -module. Hence, $R/\text{rad} R \simeq X(\lambda_n)$.

For a subset $\Lambda' \subset \Lambda$, we define $X(\Lambda')$ to be the ideal generated by $\{x_\lambda | \lambda \in \Lambda'\}$ and we define the set $\overline{\Lambda'} = \{\mu \in \Lambda | X(\mu) \subset X(\Lambda')\}$.

Lemma 2.3.1.

$\overline{\overline{\Lambda'}} = \overline{\Lambda'}$ and $X(\overline{\Lambda'}) = X(\Lambda')$.

Proof. Clearly $X(\Lambda') \subseteq X(\overline{\Lambda'})$. On the other hand, if $\mu \in \overline{\Lambda'}$, then $X(\mu) \subset X(\Lambda')$, which implies that $x_\mu \in X(\Lambda')$ and so $X(\overline{\Lambda'}) \subset X(\Lambda')$. Furthermore,

$$\overline{\overline{\Lambda'}} = \{\mu \in \Lambda | X(\mu) \subset X(\overline{\Lambda'})\} = \{\mu \in \Lambda | X(\mu) \subset X(\Lambda')\} = \overline{\Lambda'}. \quad \square$$

Lemma 2.3.2.

$\{x_\mu | \mu \in \overline{\Lambda'}\}$ is a K -basis for $X(\Lambda')$, for all $\Lambda' \subseteq \Lambda$.

Proof. If we show that $\{x_\mu | \mu \in \overline{\Lambda'}\}$ generates ${}_K X(\Lambda')$ for all $\Lambda' \subseteq \Lambda$, then in particular $\{x_\mu | \mu \in \Lambda\}$ generates ${}_K R$. Since $\dim_K R = \text{card} \Lambda$ (condition (i)) $\{x_\mu | \mu \in \Lambda\}$ will be linearly independent and so $\{x_\mu | \mu \in \overline{\Lambda'}\}$ will be a K -basis for $X(\Lambda')$.

To show that $\{x_\mu | \mu \in \overline{\Lambda'}\}$ generates $X(\Lambda')$ we use induction. Let $\Lambda' = \overline{\Lambda'} = \{\lambda_n\}$; clearly $\{x_{\lambda_n}\}$ generates ${}_K X(\lambda_n)$. Next, assume by induction that ${}_K \text{rad} X(\lambda)$ is generated by $\{x_\mu | \mu \in \Lambda(\lambda)\}$, for $\lambda \in \Lambda$. Since $\overline{\{\lambda\}} = \Lambda(\lambda) \cup \{\lambda\}$ and ${}_K X(\lambda) = Kx_\lambda + {}_K \text{rad} X(\lambda)$ (K is a splitting field) it follows that ${}_K X(\lambda)$ is generated by $\{x_\mu | \mu \in \overline{\{\lambda\}}\}$.

Finally, for $\Lambda' \subset \Lambda$, $X(\Lambda') = \sum_{\mu \in \Lambda'} X(\mu)$, so ${}_K X(\Lambda')$ is generated by $\bigcup_{\mu \in \Lambda'} \{x_\rho | \rho \in \overline{\{\mu\}}\}$ which is equal to $\{x_\rho | \rho \in \overline{\Lambda'}\}$. $\square$

Lemma 2.3.3.

The set of ideals $\{X(\Lambda') | \Lambda' \subseteq \Lambda\}$ forms a distributive lattice under intersection and addition.

Proof. By Lemma 2.3.1, it is sufficient to show that

$$[X(\overline{\Lambda_1}) + X(\overline{\Lambda_2})] \cap X(\overline{\Lambda_3}) = [X(\overline{\Lambda_1}) \cap X(\overline{\Lambda_3})] + [X(\overline{\Lambda_2}) \cap X(\overline{\Lambda_3})],$$

for all $\Lambda_1, \Lambda_2, \Lambda_3 \subseteq \Lambda$. Let $x \in [X(\overline{\Lambda_1}) + X(\overline{\Lambda_2})] \cap X(\overline{\Lambda_3})$. We can write

$$x = \sum_{\mu \in \overline{\Lambda_1}} \alpha_\mu x_\mu + \sum_{\rho \in \overline{\Lambda_2}} \beta_\rho x_\rho = \sum_{\sigma \in \overline{\Lambda_3}} \gamma_\sigma x_\sigma, \quad \text{with } \alpha_\mu, \beta_\rho, \gamma_\sigma \in K.$$

Since $\{x_\lambda | \lambda \in \Lambda\}$ is a K -basis of R we can write

$$x = \sum_{\mu \in \overline{\Lambda_1} \cap \overline{\Lambda_3}} \alpha_\mu x_\mu + \sum_{\rho \in \overline{\Lambda_2} \cap \overline{\Lambda_3}} \beta_\rho x_\rho$$

and therefore we have $x \in [X(\overline{\Lambda_1}) \cap X(\overline{\Lambda_3})] + [X(\overline{\Lambda_2}) \cap X(\overline{\Lambda_3})]$,

$$\text{i.e. } [X(\overline{\Lambda_1}) + X(\overline{\Lambda_2})] \cap X(\overline{\Lambda_3}) \subseteq [X(\overline{\Lambda_1}) \cap X(\overline{\Lambda_3})] + [X(\overline{\Lambda_2}) \cap X(\overline{\Lambda_3})].$$

The reverse inclusion is automatic. $\square$

Lemma 2.3.4.

Every R -homomorphism $f : X(\lambda) \rightarrow \sum_{\mu \in \Lambda'} X(\mu) \subseteq R$, for some $\Lambda' \subseteq \Lambda$, factors through the canonical (summation) map $p : \bigoplus_{\mu \in \Lambda'} X(\mu) \rightarrow \sum_{\mu \in \Lambda'} X(\mu)$. In particular, every R -homomorphism $f : X(\lambda) \rightarrow \text{rad}X(\kappa)$ factors through the canonical summation map

$$\bigoplus_{\mu \in \Lambda(\kappa)} X(\mu) \rightarrow \text{rad}X(\kappa).$$

Proof. Let $f : X(\lambda) \rightarrow \sum_{\mu \in \Lambda'} X(\mu)$ be such an R -homomorphism, for some $\Lambda' \subseteq \Lambda$. Since

f is induced by multiplication, there is $r \in R$ such that

$$f(x_\lambda) = x_\lambda r \in \left[\sum_{\mu \in \Lambda'} X(\mu) \right] \cap X(\lambda) = \sum_{\mu \in \Lambda'} [X(\mu) \cap X(\lambda)]$$

(by Lemma 2.3.3). Hence $x_\lambda r = \sum_{\mu \in \Lambda'} x_\lambda r_\mu$ with $x_\lambda r_\mu \in X(\mu) \cap X(\lambda)$, so we can write $f = pg$, with $g : X(\lambda) \rightarrow \bigoplus_{\mu \in \Lambda'} X(\mu)$, $x_\lambda s \mapsto \bigoplus_{\mu \in \Lambda'} x_\lambda s r_\mu$. Indeed, since g is well-defined (if $x_\lambda s = 0$ then $g(x_\lambda s) = \bigoplus_{\mu \in \Lambda'} x_\lambda s r_\mu = \bigoplus_{\mu \in \Lambda'} 0$), g is an R -homomorphism and the factorization is complete. $\square$

Lemma 2.3.5.

For every $\lambda \in \Lambda$,

$$(*) \quad \mathcal{B}(\lambda) = \{m_\kappa m_{\kappa\lambda} p_\lambda \mid X(\lambda) \subseteq X(\kappa)\}$$

is a K -basis for the (left) standard module $\Delta(\lambda)$, where $m_{\kappa\lambda}$ denotes the embedding $X(\lambda) \subseteq X(\kappa)$. In fact,

$$(**) \quad \Delta(\lambda) = P(\lambda) / \text{trace} \left(\bigoplus_{\mu \in \Lambda(\lambda)} P(\mu) \rightarrow P(\lambda) \right).$$

Proof. By definition, $\Delta(\lambda) = P(\lambda) / \text{trace} \left(\bigoplus_{X(\mu) \not\subseteq X(\lambda)} P(\mu) \rightarrow P(\lambda) \right)$. Thus to prove (**), it is sufficient to show that every R -homomorphism $f : X(\lambda) \rightarrow X(\mu)$ with incomparable λ and μ can be factored through the direct sum $\bigoplus_{\rho \in \Lambda(\lambda)} X(\rho)$. Since f cannot be a monomorphism, f factors through $\text{rad} X(\lambda) = \sum_{\rho \in \Lambda(\lambda)} X(\rho)$. By Lemma 2.3.4, f must also factor through $\bigoplus_{\rho \in \Lambda(\lambda)} X(\rho)$.

Regarding $\mathcal{B}(\lambda)$, it is clear that this is a linearly independent set, and that every monomorphism $X(\lambda) \hookrightarrow X$ is a linear combination of maps from $\mathcal{B}(\lambda)$. Since no monomorphism $f : X(\lambda) \rightarrow X(\kappa)$ can be factored through $\bigoplus_{\rho \in \Lambda(\lambda)} X(\rho)$, the K -span of $\mathcal{B}(\lambda)$ must sit inside $\Delta(\lambda)$. Finally, if $f : X(\lambda) \rightarrow X(\kappa)$ is not a monomorphism, then, as above, f factors through $\bigoplus_{\rho \in \Lambda(\lambda)} X(\rho)$. Therefore $\mathcal{B}(\lambda)$ is a K -basis for $\Delta(\lambda)$. $\square$

Remark.

Let us point out that Lemma 2.3.5 describes the structure of the standard modules: the factorizations $m_{\kappa\lambda} = m_{\kappa\rho}m_{\rho\lambda}$ correspond to the embeddings $X(\lambda) \subseteq X(\rho) \subseteq X(\kappa)$, so the lattice of submodules of $\Delta(\lambda)$ is anti-isomorphic to the lattice of ideals of R which contain $X(\lambda)$. In particular, every ideal $X(\lambda)$ is a subset of $X(\lambda_1) = R$. Thus the standard module $\Delta(\lambda)$ has a simple socle generated by $m_{\lambda_1\lambda}$, i.e. $\text{soc}\Delta(\lambda) \simeq S(\lambda_1)$ for all $\lambda \in \Lambda$.

Another immediate consequence of Lemma 2.3.5 is the fact that the standard modules $\Delta(\lambda)$ remain unchanged under any refinement of the partial order on Λ . Therefore, letting $\dim_K R = n$, we shall consider $\Lambda = \{1, 2, \dots, n\}$ with its natural order, keeping in mind that $X(j) \subset X(i)$ implies $i < j$. We shall also consider the complete sequence $\varepsilon = (e_1, e_2, \dots, e_n)$ of primitive orthogonal idempotents, $e_i : X \xrightarrow{p_i} X(i) \xrightarrow{m_i} X$. Note that $1_X = \sum_{i=1}^n e_i$; we will write $\varepsilon_t = \sum_{i=t}^n e_i$ for $1 \leq t \leq n$, $\varepsilon_{n+1} = 0$.

To complete the proof of sufficiency, we are going to show that $\text{End}_A(\Delta(i))$ is a division algebra, for $1 \leq i \leq n$, and that all factors $A\varepsilon_j A e_i / A\varepsilon_{j+1} A e_i$ of the trace filtration of $A e_i$ are direct sums of $\Delta(j)$'s, $i \leq j \leq n$. The first statement is an instant consequence of Lemma 2.3.5: the multiplicity $[\Delta(i) : S(k)] = 1$ if $X(i) \subseteq X(k)$ and $[\Delta(i) : S(k)] = 0$ otherwise. Since the identity map on $X(i)$ does not factor through $\bigoplus_{k=i+1}^n X(k)$, there is no non-zero map from $\Delta(i)$ into $\text{rad}\Delta(i)$. The second statement is established in the following lemma.

Lemma 2.3.6.

If $X(j) \subseteq X(i)$, then $A\varepsilon_j A e_i / A\varepsilon_{j+1} A e_i \simeq \Delta(j)$. If $X(j) \not\subseteq X(i)$, then $A\varepsilon_j A e_i = A\varepsilon_{j+1} A e_i$.

Proof. We know that $X(j) \subseteq X(i)$ if and only if there is a surjective R -homomorphism $f : X(i) \rightarrow X(j)$. We also know that the elements of $Ae_j Ae_i$ are sums of maps of the form $m\phi p_i$ with $\phi : X(i) \rightarrow \bigoplus_{t=j}^n X(t)$ and $m : \bigoplus_{t=j}^n X(t) \rightarrow X$. Furthermore, for $j+1 \leq t \leq n$ either $X(t) \subset X(j)$ or else $X(t)$ and $X(j)$ are incomparable. In either case there is no epimorphism $X(t) \rightarrow X(j)$ for $j+1 \leq t \leq n$; hence, the elements of $Ae_{j+1} Ae_i$ also are sums of maps of the form $m\phi p_i$ where the additional condition that $p_j m\phi : X(i) \rightarrow X(j)$ not be surjective is also satisfied. Therefore, for $X(j) \not\subseteq X(i)$, $Ae_j Ae_i = Ae_{j+1} Ae_i$.

On the other hand, if $X(j) \subseteq X(i)$, let p_{ji} be the surjective R -homomorphism from $X(i)$ to $X(j)$ which maps x_i to $x_j = x_i r$. Suppose that $f : X(i) \rightarrow X(j)$ is another surjective R -homomorphism. Then there is an automorphism g of $X(j)$ such that $f = gp_{ji}$. Hence, $Ae_j Ae_i / Ae_{j+1} Ae_i$ is isomorphic to a quotient of $P(j)$. Now if $h : X(j) \rightarrow X(k)$ is a monomorphism (for instance, m_k from Lemma 2.3.5), then $hp_{ji} : X(i) \rightarrow X(k)$ cannot be factored through $\bigoplus_{t=j+1}^n X(t)$. Indeed, there is no monomorphism $X(j) \rightarrow X(t)$ for $j+1 \leq t \leq n$. On the other hand, recall that if h is not a monomorphism then $Imh \subseteq radX(j) = \sum_{t \in \Lambda(j)} X(t)$ and so hp_{ji} factors through $\bigoplus_{t=j+1}^n X(t)$. As the elements of $\Delta(j)$ are all the monomorphisms $X(j) \rightarrow X$, it follows that $Ae_j Ae_i / Ae_{j+1} Ae_i \simeq \Delta(j)$. $\square$

This completes the proof of sufficiency of Theorem I.

2.4 - PROOF OF NECESSITY

We now assume that $A = End_R X$ is quasi-hereditary with respect to the partial order on Λ . Thus, we have $X = \bigoplus_{i=1}^n X(i)$, where the linear order of the index set is a refinement of the partial order given by inclusions (i.e. $X(j) \subseteq X(i) \subseteq R$ implies $i \leq j$) and $card\Lambda = n$.

Recall that $X(1) = R$ and $\Lambda(j) = \{i | X(i) \subsetneq X(j)\}$. Taking $\underline{e} = (e_1, e_2, \dots, e_n)$ to be the complete sequence of primitive orthogonal idempotents, e_i defined as in §2.1, $(A, \underline{e})$ is quasi-hereditary. Moreover, with respect to the original partial order, $\Delta(i)$ is the maximal factor module of $P(i)$ whose composition factors are only of the form $S(k)$, where $i \in \Lambda(k)$, i.e. $X(i) \subseteq X(k) \subseteq R$. Hence

$$\begin{aligned} \text{trace} \left(\bigoplus_{j=i+1}^n P(j) \rightarrow P(i) \right) &= \text{trace} \left(\bigoplus_{X(i) \subsetneq X(j)} P(j) \rightarrow P(i) \right) \\ &= \text{trace} \left(\bigoplus_{i \notin \Lambda(j)} P(j) \rightarrow P(i) \right). \end{aligned}$$

We will prove the necessity of conditions (i) and (ii) of Theorem I in Lemmas 2.4.2 and 2.4.3. First, let us establish an auxiliary result.

Lemma 2.4.1.

Let $f : X(i) \rightarrow X(k)$ be an R -homomorphism. If f is monic, then $m_k f p_i \notin A e_{i+1} A e_i$. If f is not monic, and A is quasi-hereditary, then $m_k f p_i \in A e_{i+1} A e_i$. Thus, if A is quasi-hereditary, then the multiplicity $[\Delta(i) : S(k)] = 1$ for $X(i) \subseteq X(k)$ and $[\Delta(i) : S(k)] = 0$ otherwise.

Proof. The image $\text{Im} f$ of a homomorphism $f : X(i) \rightarrow X(k)$ which factors through $\bigoplus_{j=i+1}^n X(j)$ is isomorphic to a submodule of $\text{rad} X(i)$, since f cannot be a monomorphism. However, if f is a monomorphism then $\text{Im} f \simeq X(i)$, and thus $m_k f p_i \notin A e_{i+1} A e_i$. Now, if f is not a monomorphism, then it induces a non-invertible endomorphism of $X(i)$. Under the assumption that A is quasi-hereditary, $m_k f p_i$ must belong to $A e_{i+1} A e_i$, since $[\Delta(i) : S(i)] = 1$ and $m_k \text{id}_{X(i)} p_i \notin A e_{i+1} A e_i$.

Since $\Delta(i) = A e_i / A e_{i+1} A e_i$, it follows that $[\Delta(i) : S(k)] \neq 0$ if and only if $X(i) \subseteq X(k)$.

Furthermore, in this case $[\Delta(i) : S(k)] = 1$. Indeed, if $f_1, f_2 : X(i) \rightarrow X(k)$ are both monomorphisms, then both maps are induced by multiplication by invertible elements of R and thus $f_2 = \alpha f_1$, for some invertible $\alpha \in R$. Now consider the map rf_1 where r is any non-invertible element of R . rf_1 is not monic, so $m_k(rf_1)p_i \in Ae_{i+1}Ae_i$, which implies that $m_k f_2 p_i - m_k(\bar{\alpha} f_1)p_i = m_k(f_2 - \bar{\alpha} f_1)p_i \in Ae_{i+1}Ae_i$ with $\alpha \in R/\text{rad}R \simeq K$. Therefore, modulo $Ae_{i+1}Ae_i$, f_2 is a K -multiple of f_1 and $[\Delta(i) : S(k)] = 1$ as claimed. $\square$

Lemma 2.4.2.

If A is quasi-hereditary, then $\text{card}\Lambda = \dim_K R$.

Proof. By Lemma 2.4.1, $[\Delta(i) : S(1)] = 1$ for all $1 \leq i \leq n$. In view of the duality $F : A\text{-mod} \rightarrow A\text{-mod}$ satisfying $F(S(i)) \simeq S(i)$, and $F\Delta(i) \simeq \nabla(i)$ (cf. §2.1) it follows that $[\nabla(i) : S(1)] = 1$ for all $1 \leq i \leq n$. Hence, the Bernstein-Gelfand-Gelfand reciprocity law yields $[P(1) : \Delta(i)] = [\nabla(i) : S(1)] = 1$ for all $1 \leq i \leq n$. Consequently,

$$\dim_K R = [P(1) : S(1)] = \sum_{i=1}^n [P(1) : \Delta(i)][\Delta(i) : S(1)] = n = \text{card}\Lambda. \quad \square$$

Lemma 2.4.3.

If A is quasi-hereditary, then $\text{rad}X(i) = \sum_{j \in \Lambda(i)} X(j)$, for all $1 \leq i \leq n$.

Proof. Clearly, $\sum_{j \in \Lambda(i)} X(j) \subseteq \text{rad}X(i)$ for $1 \leq i \leq n$. Now, recall that $X(i) = x_i R$. Thus, if $x \in \text{rad}X(i)$, there is $r \in \text{rad}R$ such that $x_i r = x$. Multiplication by r induces a non-invertible endomorphism f of $X(i)$ which must factor through $\bigoplus_{j \in \Lambda(i)} X(j)$, so that $f = \sum_{j \in \Lambda(i)} m_{ij} f_{ji}$, with $m_{ij} : X(j) \rightarrow X(i)$ being the canonical embedding and $f_{ji} : X(i) \rightarrow X(j)$, for all $j \in \Lambda(i)$. Therefore

$$x \in \sum_{j \in \Lambda(i)} m_{ij} f_{ji} X(i) \subseteq \sum_{j \in \Lambda(i)} m_{ij} X(j) \subseteq \sum_{j \in \Lambda(i)} X(j),$$

i.e. $\text{rad}X(i) \subseteq \sum_{j \in \Lambda(i)} X(j)$ and the lemma follows. $\square$

As an immediate consequence of Lemma 2.4.2, we state the following corollary.

Corollary 2.4.4.

If A is quasi-hereditary, then

$$[P(i) : \Delta(j)] = [\Delta(j) : S(i)] = \begin{cases} 1, & \text{if } X(j) \subseteq X(i); \\ 0, & \text{otherwise} \end{cases}.$$

This completes both the proof of Theorem I and this chapter.

CHAPTER III

APPLICATIONS AND EXAMPLES

As mentioned in section 2.2 there are a number of interesting corollaries to and consequences of the main theorem, which we will now establish. Unless otherwise specified, throughout this chapter we will assume that $(A, \underline{e})$ is quasi-hereditary where, as in the main theorem, $A = \text{End}_R X$, $X = \bigoplus_{i=1}^n X(i)$ is a direct sum of ideals of R and $\underline{e} = (e_1, e_2, \dots, e_n)$ is a complete sequence of primitive orthogonal idempotents of A . The linear order $1 < 2 < \dots < n$ is a refinement of the original partial order on Λ , i.e. $X(j) \subseteq X(i) \subseteq R$ implies $j \geq i$. Recall that $\varepsilon_i = \sum_{j=i}^n e_j$ for $1 \leq i \leq n$; $\varepsilon_{n+1} = 0$. Of course, $\dim_K R = n$.

3.1 - LEAN ALGEBRAS

For an algebra A' with radical N , and a complete sequence of primitive orthogonal idempotents $\underline{e}'$ we say that $(A', \underline{e}')$ is *lean* if $e_i N^2 e_j = e_i N \varepsilon_m N e_j$ for all i and j , where $m = \min\{i, j\}$ (cf. [ADL]). We say that $(A', \underline{e}')$ is *lean quasi-hereditary* if and only if A' is lean and quasi-hereditary with respect to the same order $\underline{e}'$. An embedding of an A' -module Y into an A' -module Z is a *top embedding*, denoted $Y \overset{t}{\subseteq} Z$, if it induces an

embedding of $\text{top}Y = Y/\text{rad}Y$ into $\text{top}Z = Z/\text{rad}Z$. Furthermore, a filtration

$$0 = Y_{m+1} \subseteq Y_m \subseteq \cdots \subseteq Y_i \subseteq \cdots \subseteq Y_2 \subseteq Y_1 = Y$$

of a module Y is called a *top filtration* if $Y_i \subseteq^t Y$ for every $2 \leq i \leq m$.

If we assume that $\Delta(i)$ is Schurian for all $1 \leq i \leq n$, then $(A', \underline{e}')$ is lean if and only if the trace filtrations of $\text{rad}\Delta(i)$ and of $\text{rad}\Delta^o(i)$ are top filtrations for $1 \leq i \leq n$.

Corollary 3.1.1.

$(A, \underline{e})$ is lean quasi-hereditary.

Proof. Consider the A -modules $U(i) = \text{rad}\Delta(i)$ and $U^o(i) = \text{rad}\Delta^o(i)$. We must show that the trace filtrations of $U(i)$ and $U^o(i)$ are top filtrations for all $1 \leq i \leq n$. Let us fix a particular i . The trace filtration of $U(i)$ is

$$0 = A\varepsilon_i U(i) \subseteq A\varepsilon_{i-1} U(i) \subseteq \cdots \subseteq A\varepsilon_k U(i) \subseteq \cdots \subseteq A\varepsilon_2 U(i) \subseteq A\varepsilon_1 U(i) = U(i),$$

where the elements of $A\varepsilon_k U(i)$ are sums of maps of the form

$$fgm_{ji}p_i : X \xrightarrow{p_i} X(i) \xrightarrow{m_{ji}} X(j) \xrightarrow{g} \bigoplus_{t=k}^i X(t) \xrightarrow{f} X,$$

with p_i and m_{ji} the canonical projection and embedding. Then for $1 \leq k \leq i$, $\text{top}(A\varepsilon_k U(i))$ is isomorphic to the semi-simple module generated by $\{m_{ji} | j \in \mathfrak{S}_{i,k}\}$ where $\mathfrak{S}_{i,k}$ is the set of all j , $1 \leq j \leq i$, which satisfy

- (i) $i \in \Lambda(j)$;
- (ii) m_{ji} has no proper factorization;
- (iii) there is a non-zero map $gm_{ji} : X(i) \xrightarrow{m_{ji}} X(j) \xrightarrow{g} \bigoplus_{t=k}^i X(t)$.

Clearly, $\mathfrak{S}_{i,i} \subseteq \mathfrak{S}_{i,i-1} \subseteq \cdots \subseteq \mathfrak{S}_{i,k} \subseteq \cdots \subseteq \mathfrak{S}_{i,1}$ so that $\text{top}(A\varepsilon_k U(i)) \subseteq \text{top} U(i)$; thus, the trace filtration of $U(i)$ is a top filtration. Dually, the trace filtration of $U^o(i)$ also is a top filtration, and so $(A, \underline{e})$ is lean quasi-hereditary. $\square$

3.2 - THE QUIVER OF A

Corollary 3.2.1.

The quiver $\mathcal{Q}_A$ of A has n vertices, and it has a pair of arrows $i \rightrightarrows j$ if and only if $X(i) \subsetneq X(j)$, where $X(i) \subseteq X(k) \subseteq X(j)$ implies either $i = k$ or $j = k$.

Proof. Since $X = \bigoplus_{i=1}^n X(i)$, $A = \text{End}_R X$ has n indecomposable projectives, and so $\mathcal{Q}_A$ has n vertices.

Regarding the arrows of $\mathcal{Q}_A$, the elements of $\text{rad} P(i)/\text{rad}^2 P(i)$ are sums of maps of the form $X(i) \xrightarrow{\phi} X(j)$, where ϕ does not factor. Thus if $X(i) \subsetneq X(j)$, satisfying the condition that $X(i) \subseteq X(k) \subseteq X(j)$ implies either $i = k$ or $j = k$, then the embedding $m_{ji} : X(i) \rightarrow X(j)$ is in $\text{rad} P(i)/\text{rad}^2 P(i)$, and the projection $p_{ij} : X(j) \rightarrow X(i)$ is in $\text{rad} P(j)/\text{rad}^2 P(j)$.

Now, if $X(i) \subsetneq X(k) \subsetneq X(j)$ then both the monomorphism m_{ji} and the epimorphism p_{ij} factor in an obvious manner, so neither of these maps give rise to an arrow between i and j . On the other hand, suppose that $\phi : X(i) \rightarrow X(j)$ is neither an epimorphism nor a monomorphism. Then ϕ factors non-trivially through $\text{Im} \phi$, hence through $\text{rad} X(i)$, and so ϕ does not yield an arrow $i \rightarrow j$. Similarly, if $\psi : X(j) \rightarrow X(i)$ is neither injective nor surjective then ψ does not yield an arrow $j \rightarrow i$. $\square$

Note that the lattice structure of the ideals in $\mathcal{X}$ determines the indecomposable pro-

jective modules $P(i)$, $1 \leq i \leq n$, and hence the relations which generate the ideal I such that $A \simeq KQ/I$.

3.3 - DIMENSION OF A -PROJECTIVES

Corollary 3.3.1.

For the projective A -module $P(k)$, we have

$$\dim_K P(k) = \sum_{i \in \Lambda(k) \cup \{k\}} \dim_K \Delta(i);$$

in particular, $\dim_K P(n) = \dim_K R$. Furthermore, $\dim_K A = \sum_{i=1}^n (\dim_K \Delta(i))^2$.

Proof. By Corollary 2.4.4, we know that $[P(k) : \Delta(i)] = [\Delta(i) : S(k)] = 1$ if $X(i) \subseteq X(k)$ (i.e. if $i \in \Lambda(k) \cup \{k\}$) and 0 otherwise. Since $P(k)$ is filtered by $\{\Delta(i) | k \leq i \leq n\}$ it follows that $\dim_K P(k) = \sum_{i \in \Lambda(k) \cup \{k\}} \dim_K \Delta(i)$. In the case $k = n$, we have $\Lambda(n) \cup \{n\} = \{n\}$ and $P(n) = \Delta(n)$ which has a K -basis $\mathcal{B}(n) = \{m_i m_{in} p_n | X(n) \subseteq X(i)\}$. Since $X(n) \subseteq X(i)$ for all $1 \leq i \leq n$, $\dim_K P(n) = n = \dim_K R$.

Finally, $\dim_K A = \sum_{k=1}^n \dim_K P(k) = \sum_{k=1}^n \left(\sum_{i \in \Lambda(k) \cup \{k\}} \dim_K \Delta(i) \right)$ where $\dim_K \Delta(i)$ appears in this sum once for every inclusion $X(i) \subseteq X(k) \subseteq R$. Since $\Delta(i)$ has K -basis $\mathcal{B}(i) = \{m_k m_{ki} p_k | X(i) \subseteq X(k) \subseteq R\}$, it follows that $\dim_K A = \sum_{k=1}^n (\dim_K \Delta(k))^2$. $\square$

3.4 - THE DOMINANT DIMENSION OF A

Suppose that a left module $A'M$ of an algebra A' has an exact sequence

$$0 \rightarrow A'M \rightarrow A'U_1 \rightarrow A'U_2 \rightarrow \cdots \rightarrow A'U_n,$$

where U_i is a left projective-injective A' -module for all $1 \leq i \leq n$. Then the *left dominant dimension* of M is said to be $\geq n$ (cf. [T]). If each U_i is a direct sum of copies of some projective-injective A' -module U , then we say that the *left U -dominant dimension* of M is $\geq n$. The *dominant dimension* of A' is the left dominant dimension of ${}_A A'$. In particular, if dominant dimension of A' is ≥ 1 , then A' is QF-3.

Proposition 3.4.1.

The dominant dimension of $A \geq 2$; thus A is QF-3.

We will use the following lemmas in proving Proposition 3.4.1.

Lemma 3.4.2.

${}_A P(1) \simeq {}_A Q(1)$, i.e. $P(1)$ is projective-injective.

Proof. Let $\phi \in P(1)$, $\phi : X \xrightarrow{p_1} X(1) \xrightarrow{\phi_{j1}} X(j) \xrightarrow{m_j} X$. Since $X(j) \subseteq X(1)$ for all $1 \leq j \leq n$, we may write

$$\phi : X \xrightarrow{p_1} X(1) \xrightarrow{\phi_{j1}} X(j) \xrightarrow{m_{1j}} X(1) \xrightarrow{m_1} X.$$

Now consider the map $\psi \in e_1 A$ given by

$$\psi : X \xrightarrow{p_1} X(1) \xrightarrow{p_{j1}} X(j) \xrightarrow{D\phi_{j1}} X(1) \xrightarrow{m_1} X.$$

Then ϕ is in $\text{Hom}_K(e_1 A, K) \simeq Q(1)$, since $D\psi = \phi$, and it follows immediately that $P(1) \simeq Q(1)$. $\square$

Lemma 3.4.3.

$P(j)$ is isomorphic to a submodule of $P(1)$ for all $1 \leq j \leq n$.

Proof. We may consider elements of $P(j)$ to be sums of R -homomorphisms $X(j) \xrightarrow{\phi} X(i)$, for some $1 \leq i \leq n$. Thus, we define the (additive) map

$$f_j : P(j) \rightarrow P(1), f_j(\phi) = \phi p_{j1} : X(1) \xrightarrow{p_{j1}} X(j) \xrightarrow{\phi} X(i).$$

It is clear that f_j is a monomorphism and so, for all $1 \leq j \leq n$, $P(j) \simeq \text{Im } f_j \subseteq P(1)$. $\square$

Lemma 3.4.4.

$P(1)/P(j)$ is isomorphic to a submodule of $P(1)$ for all $1 \leq j \leq n$.

Proof. Of course, this is true vacuously for $j = 1$. For $j \neq 1$, let us denote by $P_j(1)$ the quotient module $P(1)/P(j)$, so we have the short exact sequence

$$0 \rightarrow P(j) \rightarrow P(1) \rightarrow P_j(1) \rightarrow 0.$$

By applying the duality functor F we obtain the short exact sequence

$$0 \rightarrow FP_j(1) \rightarrow P(1) \rightarrow Q(j) \rightarrow 0$$

(recall that $FP(j) \simeq Q(j)$ for all $1 \leq j \leq n$ and that $Q(1) \simeq P(1)$). We will show that $FP_j(1) \simeq P_j(1)$, from which the result follows immediately. The elements of $P_j(1)$ are just sums of R -homomorphisms $X(1) \rightarrow X(i)$ which do not factor through $X(j)$ or, in fact, through $\bigoplus_{k \in \Lambda(j) \cup \{j\}} X(k)$. If we define $X_j = \bigoplus_{i \notin \Lambda(j) \cup \{j\}} X(i)$ and $A_j = \text{End}_R X_j$, then $P_j(1) = A_j e_1$ is projective over A_j . Similarly, if we define $F_j : A_j\text{-mod} \rightarrow A_j\text{-mod}$ to be the simple-fixing duality functor over A_j then, by Lemmas 2.1.3 and 3.4.2, $F_j P_j(1) \simeq P_j(1)$. Since $F : A\text{-mod} \rightarrow A\text{-mod}$ fixes all simple A -modules, $F_j = F|_{A_j\text{-mod}}$, and it follows that $FP_j(1) \simeq P_j(1)$. $\square$

Proof of Proposition 3.4.1.

We will prove that the $P(1)$ -dominant dimension of A is ≥ 2 . In [T] this is shown to be equivalent to A being the maximal quotient ring $Q_l(A)$ of A . Now $Q_l(A) = \bigcap_{\phi \in \Phi} \text{Ker} \phi$, where Φ is the set of endomorphisms ϕ of the injective hull $E({}_A A)$ of ${}_A A$ such that ${}_A A \subseteq \text{Ker} \phi$. Thus, it suffices to find $\phi \in \Phi$ with ${}_A A = \text{Ker} \phi$. Observe that $E({}_A A) = \bigoplus_{j=1}^n P(1)$, so let us define $\phi : \bigoplus_{j=1}^n P(1) \rightarrow \bigoplus_{j=1}^n P(1)$ as follows: $\phi = \bigoplus_{j=1}^n \phi_j$ where $\phi_j : P(1) \rightarrow P_j(1) \hookrightarrow P(1)$ for $2 \leq j \leq n$ and $\phi_1 = 0$. Then $\text{Ker} \phi = {}_A A$, and the proof is complete. $\square$

Let us finish this section with one final corollary to Lemma 3.4.2.

Corollary 3.4.5.

$A = \text{End}_R X$ has exactly one indecomposable projective-injective module, namely $P(1)$.

Proof. Consider $P(j)$, for $2 \leq j \leq n$, and observe that $\text{top} P(j) \simeq S(j)$. Recall also that $P(j)$ is a submodule of $P(1) \simeq Q(1)$. Since $\text{soc} P(1) \simeq S(1)$, we have $\text{soc} P(j) \simeq S(1)$ for all $1 \leq j \leq n$. Since A is basic, $P(j)$ is not injective. $\square$

3.5 - EXAMPLES

The following examples should serve as simple illustrations of the theorem, as well as indications of its limitations.

1. $R = K[x]/\langle x^t \rangle$, $t \geq 1$. There is a unique choice of $X = \bigoplus_{i=1}^t X(i)$ ($X(1) = R$, $X(i) = \langle \bar{x}^{i-1} \rangle$). We have all the indecomposable R -modules; thus A is the respective

Auslander algebra, and so A is quasi-hereditary. The quiver, $\mathcal{Q}_A$, of A is as follows:

$$1 \rightleftharpoons 2 \rightleftharpoons \dots \rightleftharpoons t-1 \rightleftharpoons t.$$

In particular, if $t = 4$ then A is precisely the endomorphism algebra given in section 1.6.

In the general case, it is not hard to show that

$$\Delta = \left\{ 1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}, \dots, \begin{smallmatrix} t \\ \vdots \\ 2 \\ 1 \end{smallmatrix} \right\}$$

and that A is not only lean, but also replete. (cf. [ADL])

2. $R = K[x, y]/\langle xy, x^t - y^t \rangle$, $t \geq 2$. For $t \geq 3$, we have several choices for X : indeed, let $t = 3$.

(a) Let $X = \bigoplus_{i=1}^6 X(i)$, for $X(1) = R$, $X(2) = \langle \bar{y} \rangle$, $X(3) = \langle \bar{x} \rangle$, $X(4) = \langle \bar{y}^2 \rangle$, $X(5) = \langle \bar{x}^2 \rangle$, $X(6) = \langle \bar{x}^3 \rangle$. Under the usual order, $(A, \underline{e})$ is quasi-hereditary with

$$\Delta = \left\{ 1, \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 5 \\ 3 \\ 1 \end{smallmatrix}, \begin{smallmatrix} 6 \\ 4 \\ 3 \\ 2 \\ 1 \end{smallmatrix} \right\}.$$

Furthermore, the quiver $\mathcal{Q}_A$ has the form

$$\begin{array}{ccc} 2 & \rightleftharpoons & 4 \rightleftharpoons 6 \\ \updownarrow & & \updownarrow \\ 1 & \rightleftharpoons & 3 \rightleftharpoons 5. \end{array}$$

(b) If we make $X(3) = \langle \bar{x} + \bar{y} \rangle$ and $X(5) = \langle \bar{x}^2 + \bar{y}^2 \rangle$, keeping the other $X(i)$'s the same as in (a), then $(A, \underline{e})$ is still quasi-hereditary in the usual order, but now we

have $\Delta(4) = \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ 2 \quad 3 \\ \uparrow \quad \downarrow \\ 1 \end{array}$ and $\Delta(6) = \begin{array}{c} 6 \\ \swarrow \quad \searrow \\ 4 \quad 5 \\ \swarrow \quad \searrow \\ 2 \quad 3 \\ \uparrow \quad \downarrow \\ 1 \end{array}$. $\mathcal{Q}_A$ is also altered, giving us the quiver

$$\begin{array}{ccccc} 2 & \rightleftharpoons & 4 & \rightleftharpoons & 6 \\ \updownarrow & & \updownarrow & & \updownarrow \\ 1 & \rightleftharpoons & 3 & \rightleftharpoons & 5. \end{array}$$

In general, for $t \geq 3$ we can choose X in many different ways. For instance, we can obtain all possible quivers $\mathcal{Q}_A$ of the form:

$$\begin{array}{ccccccc} 2 & \rightleftharpoons & 4 & \rightleftharpoons & \dots & \rightleftharpoons & 2s & \rightleftharpoons & 2s+2 & \rightleftharpoons & 2s+4 & \rightleftharpoons & \dots & \rightleftharpoons & 2t \\ \updownarrow & & \updownarrow & & \dots & & \updownarrow & & & & & & \dots & & \updownarrow \\ 1 & \rightleftharpoons & 3 & \rightleftharpoons & \dots & \rightleftharpoons & 2s-1 & \rightleftharpoons & 2s+1 & \rightleftharpoons & 2s+3 & \rightleftharpoons & \dots & \rightleftharpoons & 2t-1, \end{array}$$

where $1 \leq s \leq t$.

3. For any choice of R , if $\text{soc} R$ is not a summand of X then $A = \text{End}_R X$ cannot be quasi-hereditary. Indeed, if $\underline{e} = (e_1, e_2, \dots, e_n)$, so that $\Delta(n) = P(n)$, then $[\Delta(n) : S(n)] \geq 2$, since the projection of $X(n)$ onto its socle is not invertible.

For R as in Example 1 above, if $A' = \text{End}_R(\bigoplus_{i=1}^3 X(i))$, then A' is the centralizer algebra $(e_1 + e_2 + e_3)A(e_1 + e_2 + e_3)$ and in the usual order $\Delta(3) = \begin{array}{c} 3 \\ \swarrow \quad \searrow \\ 2 \quad 2 \\ \uparrow \quad \downarrow \\ 1 \end{array}$.

For $R = K[x, y]/\langle x^2, y^2 \rangle$ and $X = \bigoplus_{i=1}^4 X(i)$, with $X(1) = R$, $X(2) = \langle \bar{x} \rangle$, $X(3) = \langle \bar{y} \rangle$, $X(4) = \langle \bar{x} + \bar{y} \rangle$ (so only condition (i) of the main Theorem is satisfied), $[P(i) : S(i)] = 2$ for $i = 2, 3, 4$. Furthermore, $P(1)$ is not filtered by Δ in any order. Hence, A is not quasi-hereditary.

4. Let $R = K[x, y]/\langle x^2, y^2 \rangle$ and let $X = \bigoplus_{i=1}^5 X(i)$, for $X(1) = R$, $X(2) = \langle \bar{x} \rangle$, $X(3) = \langle \bar{y} \rangle$, $X(4) = \langle \bar{x} + \bar{y} \rangle$, $X(5) = \langle \bar{xy} \rangle$ (so only condition (ii) of the main Theorem is satisfied). Then $A = \text{End}_R X$ is not quasi-hereditary since $[P(1) : S(1)] = 4$ while a Δ -filtration of $P(1)$ requires $[P(1) : S(1)] = 5$.

5. $R = K[x, y]/\langle x^2 - y^3, x^3 - y^4, x^4 \rangle$. Here, the algebra R is 8-dimensional. Consider $X = \bigoplus_{i=1}^8 X(i)$ for $X(1) = R$, $X(2) = \langle \bar{x} \rangle$, $X(3) = \langle \bar{y} \rangle$, $X(4) = \langle \bar{xy} \rangle$, $X(5) = \langle \bar{y}^2 \rangle$, $X(6) = \langle \bar{xy}^2 \rangle$, $X(7) = \langle \bar{x}^2 \rangle$, $X(8) = \langle \bar{x}^3 \rangle$. Then $A = \text{End}_R X$ is a 159-dimensional quasi-hereditary algebra whose quiver $\mathcal{Q}_A$ has the form

6. Consider again the 4-dimensional algebra $R = K[x, y]/\langle xy, x^2 - y^2 \rangle$. Taking

$$X = R \oplus R/\bar{x}^2 R \oplus \bar{x}^2 R$$

(thus only 3 direct summands, not all local-colocal), or

$$X' = R \oplus (\bar{x}R \oplus R/\bar{x}^2 R)/\langle \bar{x}^2 - (\bar{y} + \bar{x}^2 R) \rangle \oplus \bar{x}R \oplus \bar{x}^2 R$$

(thus not all direct summands are local-colocal), the respective endomorphism algebras are still quasi-hereditary. The first one $A = \text{End}_R X$ is a 19-dimensional algebra (without duality) whose quiver $\mathcal{Q}_A$ is

$$1 \rightleftarrows 2 \rightleftarrows 3$$

The algebra $A' = \text{End}_R X'$, a 39-dimensional algebra with duality, has quiver $Q_{A'}$ of the form

$$1 \rightleftharpoons 2 \rightleftharpoons 4$$

$$\updownarrow$$

$$3$$

The standard A' -modules are uniserial with socles isomorphic to $S(1)$, but $[\Delta(4) : S(2)] = [P(2) : \Delta(4)] = 2$. Indeed, the indecomposable projective A' -modules are as follows:

$${}_A A = \begin{array}{c} 1 \\ | \\ 2 \\ / \quad \backslash \\ 3 \quad 4 \\ | \quad | \\ 2 \quad 1 \\ / \quad \backslash \\ 3 \quad 2 \\ | \quad | \\ 1 \end{array} \oplus \begin{array}{c} 2 \\ / \quad \backslash \\ 3 \quad 1 \\ | \quad | \\ 2 \quad 4 \\ / \quad \backslash \\ 3 \quad 2 \\ | \quad | \\ 1 \end{array} \oplus \begin{array}{c} 3 \\ | \\ 2 \\ / \quad \backslash \\ 1 \quad 4 \\ | \quad | \\ 2 \quad 1 \\ | \\ 1 \end{array} \oplus \begin{array}{c} 4 \\ | \\ 2 \\ | \\ 3 \\ | \\ 2 \\ | \\ 1 \end{array}.$$

CHAPTER IV

ENDOMORPHISM ALGEBRAS OVER NON-COMMUTATIVE R

In this final chapter we wish to illustrate the importance of the commutativity assumption on the algebra R in the main theorem. Without that assumption, only very weak conclusions can be made regarding the endomorphism algebra. To start with, the duality which plays a fundamental role in the main theorem is lost, as are the applications to the Bernstein/Gelfand/Gelfand category $\mathcal{O}$. Furthermore, we will provide an example showing that the number of non-isomorphic direct summands of the R -module X no longer needs to be the K -dimension of R . We will also show that the standard modules of the algebra A depend on the refinement of the partial order given by inclusions of the direct summands of X .

Example 4.1.

Let us start with a simple example. Let K be a field and let $R = K\langle a, b, c \rangle / I$, for the ideal of relations $I = \langle a^2, ac, b^2, ba, c^2, cb, abc - bca, abc - cab \rangle$. Then R is an 8-dimensional local self-injective K -algebra; $\{\bar{1}, \bar{a}, \bar{b}, \bar{c}, \bar{a}\bar{b}, \bar{b}\bar{c}, \bar{c}\bar{a}, \bar{a}\bar{b}\bar{c}\}$ is a K -basis, where $\bar{a}$ denotes the equivalence class $a + I$, etc.

(a) Let $X = \bigoplus_{i=1}^8 X(i)$ where $X(1) = R_R$, $X(2) = \bar{a}R$, $X(3) = \bar{b}R$, $X(4) = \bar{c}R$, $X(5) = \bar{a}\bar{b}R$, $X(6) = \bar{b}\bar{c}R$, $X(7) = \bar{c}\bar{a}R$, $X(8) = \bar{a}\bar{b}\bar{c}R$. Observe that the chosen order is a refinement of the inclusion order (i.e. $X(i) \supseteq X(j)$ implies $i \leq j$), and that

conditions (i) and (ii) of the main theorem are satisfied. Then for $A = \text{End}_R X$ we have

$${}_A P(1) = \begin{array}{c} & & 1 & & \\ & 3 & \swarrow & 2 & \searrow & 4 & \\ 1 & 5 & & 7 & & 6 & 1 \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ 2 & & 8 & & 4 & & 3 \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ 1 & 7 & & 6 & & 5 & 1 \\ & 4 & & 3 & & 2 & \\ & & 1 & & & & \end{array} ;$$

all the other indecomposable projectives $P(2), \dots, P(8)$ are submodules of $P(1)$.

Furthermore, we can see that

$$\Delta = \left\{ 1, \begin{array}{c} 2 \\ 1 \end{array}, \begin{array}{c} 3 \\ 1 \end{array}, \begin{array}{c} 4 \\ 1 \end{array}, \begin{array}{c} 5 \\ 1 \end{array}, \begin{array}{c} 6 \\ 1 \end{array}, \begin{array}{c} 7 \\ 1 \end{array}, \begin{array}{c} 8 \\ 6 \\ 3 \\ 1 \end{array} \right\},$$

and A is quasi-hereditary with respect to this order.

(b) In sharp contrast to the commutative situation, we may choose only seven of the direct summands of X . Indeed, let $Y = X(1) \oplus X(2) \oplus X(3) \oplus X(4) \oplus X(5) \oplus X(7) \oplus X(8)$, and let $A' = \text{End}_R Y$. Then

$${}_{A'} P(1) = \begin{array}{c} & & 1 & & \\ & 3 & \swarrow & 2 & \searrow & 4 & \\ 1 & 5 & & 7 & & 6 & 1 \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ 2 & & 8 & & 4 & & 3 \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ 1 & 7 & & 6 & & 5 & 1 \\ & 4 & & 3 & & 2 & \\ & & 1 & & & & \end{array}.$$

(Again, the other indecomposable projective modules, $P(i)$, $i \neq 1$, are submodules of $P(1)$.) In the usual order A' is quasi-hereditary, with standard modules

$$\Delta = \left\{ 1, \begin{array}{c} 2 \\ 1 \end{array}, \begin{array}{c} 3 \\ 1 \end{array}, \begin{array}{c} 4 \\ 1 \end{array}, \begin{array}{c} 5 \\ 1 \end{array}, \begin{array}{c} 6 \\ 1 \end{array}, \begin{array}{c} 8 \\ 6 \\ 3 \\ 1 \end{array} \right\},$$

despite the fact that neither condition (i) nor condition (ii) of the main theorem is satisfied. Furthermore, we see that $\text{soc}\Delta(j) \simeq S(1)$ for all j , and that $[\Delta(j) : S(i)] \leq 1$ for all i, j . However, since $[P(1) : \Delta(4)] = 2$, there is no duality functor on $A\text{-mod}$. Finally, let us observe that the algebra A' is not lean.

Example 4.2.

Now let us consider the following generalization of the previous example to a non-commutative local self-injective K -algebra R of dimension $n = t_1 n_1 + t_2 n_2 + \cdots + t_s n_s + 2$, $t_i, n_i \geq 1$ and a choice of (uniserial) right ideals $X(i)$ of R so that $\text{End}_R(\bigoplus_{i=1}^n X(i))$ is quasi-hereditary. R will be a quotient of a polynomial algebra over K where the set of variables is partitioned into multiplicative cycles, i.e. multiplication of variables from different cycles is always zero and, in each cycle, for any variable t_i there is a unique variable t_j such that $t_i t_j \neq 0$. The number n_i will be the number of variables in the i^{th} multiplicative cycle and $t_i + 1$ will be the maximal degree of a non-zero product in this i^{th} multiplicative cycle.

Let $R = K\langle a, b, c, d, e, f, g, h \rangle / I$, where

$$I = \left\langle \begin{array}{l} \text{all products of two variables except } \{ab, bc, cd, da; e^2; fg, gh, hf\}, \\ e^6 - abcd, e^6 - bcda, e^6 - cdab, e^6 - dabc, e^6 - fgh, e^6 - ghf, e^6 - hfg \end{array} \right\rangle$$

For simplicity of notation, we shall use a to denote both the variable a and its equivalence class modulo I , etc. (there will be no confusion), and ω to denote the equivalence class of e^6 , i.e. ω generates $\text{soc}R$.

The canonical K -basis for R is

$$\begin{aligned} \mathcal{B} = \{1\} \cup \{a, b, c, d, ab, bc, cd, da, abc, bcd, cda, dab\} \\ \cup \{e, e^2, e^3, e^4, e^5\} \cup \{f, g, h, fg, gh, hf\} \cup \{\omega\}. \end{aligned}$$

From this we can see that $\{a, b, c, d\}$, $\{e\}$, $\{f, g, h\}$ form disjoint multiplicative cycles, and that $n_1 = 4$, $t_1 = 3$, $n_2 = 1$, $t_2 = 5$, $n_3 = 3$, and $t_3 = 2$.

The following diagram will be of assistance. In particular, it illustrates the multiplicative structure of R by showing the right multiplications of a basis element by a variable (all solid lines) as well as the left multiplications of a basis element by a variable (all dashed lines; also, all solid lines passing through 1 , ω or e^m for $1 \leq m \leq 5$).

Let X be the direct sum of the set of right ideals of R generated by all basis elements in B , i.e. $X = \bigoplus_{\beta \in B} X(\beta)$, where $X(\beta) = \beta R$. We will show, in some generality, that $A = \text{End}_R X$ is quasi-hereditary with respect to the following partial order reflecting Loewy length: $\beta > \gamma$ if and only if $LL(X(\beta)) < LL(X(\gamma))$ for all $\beta, \gamma \in B$. (Note that this Loewy length order is a refinement of the inclusion order.)

Recall that the standard module $\Delta(\beta)$ is the maximal factor module of $P(\beta)$ with composition factors only of the form $S(\alpha)$ for $\alpha \leq \beta$, i.e.

$$\Delta(\beta) = P(\beta) / \text{trace} \left(\bigoplus_{\gamma \not\leq \beta} P(\gamma) \rightarrow P(\beta) \right).$$

Recall also that A is *quasi-hereditary with respect to the partially ordered set $\mathcal{B}$* if there is a linear order $\beta_1 < \beta_2 < \cdots < \beta_n$ on $\mathcal{B}$ refining the given partial order and satisfying the following conditions:

for each $1 \leq i \leq n$,

(i) the standard module defined above also equals

$$\Delta(\beta_i) = P(\beta_i) / \text{trace} \left(\bigoplus_{j > i} P(\beta_j) \rightarrow P(\beta_i) \right),$$

(ii) the endomorphism algebra of $\Delta(\beta_i)$ is a division algebra and

(iii) $P(\beta_i)$ can be filtered by $\{\Delta(\beta_j) | j \geq i\}$.

Lemma 4.2.1.

R is self-injective.

Proof. For every $\beta \in \mathcal{B}$ there is a unique $\tilde{\beta} \in \mathcal{B}$ such that $\beta\tilde{\beta} = \omega$. For instance, if $\beta = a$, then $\tilde{\beta} = bcd$. Furthermore, $D({}_R R) = \text{Hom}_K({}_R R, K)$ has K -basis $\mathcal{B}^* = \{\beta^* | \beta \in \mathcal{B}\}$, where $\beta^*(\beta) = 1$ but $\beta^*(\gamma) = 0$ for all $\gamma \in \mathcal{B} \setminus \{\beta\}$. Define the map $\Phi : R_R \rightarrow D({}_R R)$, $\beta \mapsto (\tilde{\beta})^*$. It is easy to check that Φ is an isomorphism. $\square$

As in the commutative case, endomorphisms of X will operate on the module from the left. Since R is self-injective, every R -homomorphism $X(\beta) \rightarrow X(\gamma)$ is given by (left) multiplication by an element of R . We will denote by e_β the canonical primitive idempotent $e_\beta : X \xrightarrow{p_\beta} X(\beta) \xrightarrow{m_\beta} X$ for all $\beta \in \mathcal{B}$.

Lemma 4.2.2.

(i) For every $\beta \in \mathcal{B} \setminus \{1, \omega\}$ there is a unique $\beta' \in \mathcal{B}$, $\beta' > \beta$, such that every R -homomorphism $f : X(\beta) \rightarrow X(\gamma)$ given by multiplication by $r \in \text{rad} R$ factors through

$X(\beta')$.

(ii) Every R -homomorphism $f : X(1) \rightarrow X(\gamma)$ given by multiplication by $r \in \text{rad} R$ factors through $X(a) \oplus X(b) \oplus \cdots \oplus X(h)$.

Proof. (i) Consider, for instance, $\beta = a$. For any non-zero f given by multiplication by $r \in \text{rad} R$, we may consider $r = r'd$ for some $r' \in R$. Then it is clear that f factors through the canonical projection $X(a) \xrightarrow{p_{da,a}} X(da)$, i.e. $\beta' = da$. The other cases are similar.

(ii) Here, we must have f given by left multiplication by some $\beta \in \mathcal{B}$, $\beta = r_a a + \cdots + r_h h$. Then f factors through the projection $X(1) \rightarrow X(a) \oplus \cdots \oplus X(h)$. $\square$

Note that, for β' as defined above, $LL(X(\beta')) = LL(X(\beta)) - 1$.

Lemma 4.2.3.

For every $\beta \in \mathcal{B}$, $\mathcal{B}(\beta) = \{m_\gamma m_{\gamma\beta} p_\beta | X(\beta) \subseteq X(\gamma)\}$ is a K -basis for the left standard module $\Delta(\beta)$, where $m_{\gamma\beta}$ denotes the canonical embedding $X(\beta) \subseteq X(\gamma)$. In fact, for $\beta \in \mathcal{B} \setminus \{1, \omega\}$, $\Delta(\beta) = P(\beta)/P(\beta')$, where β' is as defined in Lemma 4.2.2; furthermore, $\Delta(\omega) = P(\omega)$ and

$$\Delta(1) = P(1) / \text{trace}((P(a) \oplus \cdots \oplus P(h)) \rightarrow P(1)) .$$

Proof. First, let $\beta \in \mathcal{B} \setminus \{1, \omega\}$ and note that if $f : X(\beta) \rightarrow X(\gamma)$ is not a monomorphism, then f is given by left multiplication by some element $r \in \text{rad} R$, and so by Lemma 4.2.2 f factors through $X(\beta')$.

By definition,

$$\Delta(\beta) = P(\beta) / \text{trace} \left(\bigoplus_{LL(X(\gamma)) < LL(X(\beta))} P(\gamma) \rightarrow P(\beta) \right) ;$$

if $LL(X(\gamma)) < LL(X(\beta))$, then any map $X(\beta) \rightarrow X(\gamma)$ is not a monomorphism, and thus factors through $X(\beta')$. Therefore, $\Delta(\beta) = P(\beta)/P(\beta')$.

To show that $\mathcal{B}(\beta)$ is a K -basis for $\Delta(\beta)$, observe that $\mathcal{B}(\beta)$ is linearly independent and that every monomorphism $X(\beta) \rightarrow X$ is a linear combination of elements of $\mathcal{B}(\beta)$. Furthermore, if $f : X(\beta) \rightarrow X(\alpha)$ is a monomorphism, then it cannot factor through $X(\beta')$ since $LL(X(\beta')) < LL(X(\beta))$. Thus the K -span of $\mathcal{B}(\beta)$ is contained in $\Delta(\beta)$, and $\mathcal{B}(\beta)$ is indeed a K -basis for $\Delta(\beta)$.

The case $\beta = 1$ follows in a similar manner.

Finally, since ω is maximal in the Loewy length order, $\Delta(\omega) = P(\omega)$ according to the definition. Moreover, every map $f : X(\omega) \rightarrow X(\gamma)$ is a monomorphism, so $\mathcal{B}(\omega)$ is a K -basis for $\Delta(\omega)$. $\square$

Remark.

Lemma 4.2.3 gives us the structure of $\Delta(\beta)$ in the same manner as Lemma 2.3.5 does in the commutative case. In particular, we see that, once again, $\text{soc}\Delta(\beta) \simeq S(1)$ for all $\beta \in \mathcal{B}$, and $[\Delta(\beta) : S(\gamma)] \leq 1$ for all $\beta, \gamma \in \mathcal{B}$. In fact, $[\Delta(\beta) : S(\gamma)] = 1$ whenever $X(\beta) \subseteq X(\gamma)$; otherwise, $[\Delta(\beta) : S(\gamma)] = 0$.

Furthermore, we also see that the $\Delta(\beta)$'s remain unchanged under any refinement of the Loewy length order. Therefore, let us consider the index set $\{1, 2, \dots, 25\}$, with its natural order, where $LL(X(j)) < LL(X(i))$ implies $j > i$ and $j > i$ implies $LL(X(j)) \leq LL(X(i))$; $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{25})$ is the complete sequence of primitive orthogonal idempotents and $1_X = \sum_{i=1}^{25} \varepsilon_i$. For $1 \leq s \leq 25$ we define $\varepsilon_s = \sum_{i=s}^{25} \varepsilon_i$; we also define $\varepsilon_{26} = 0$. Finally, for $i \neq 1, 25$, we will denote by $X(i')$ the unique ideal such that every non-monic map

$f : X(i) \rightarrow X(j)$ factors through $X(i')$.

Lemma 4.2.4.

$\Delta(i)$ is Schurian for all $1 \leq i \leq 25$.

Proof. Let $i \neq 1, 25$. By Lemma 4.2.3 we see that $[\Delta(i) : S(i)] = 1$; since the isomorphism m_{ii} does not factor through $X(i')$, every non-zero non-invertible map $X(i) \rightarrow X(i)$ must factor through $X(i')$. Hence, $\text{End}_A \Delta(i)$ is a division algebra. The cases $i = 1, 25$ are similar. $\square$

Lemma 4.2.5.

$A\varepsilon_j A e_i / A\varepsilon_{j+1} A e_i \simeq \Delta(j)$ if there is an epimorphism $X(i) \rightarrow X(j)$. Otherwise, $A\varepsilon_j A e_i = A\varepsilon_{j+1} A e_i$.

Proof. The elements of $A\varepsilon_j A e_i$ are sums of maps of the form

$$m\phi p_i : X \xrightarrow{p_i} X(i) \xrightarrow{\phi} \bigoplus_{s=j}^{25} X(s) \xrightarrow{m} X.$$

Since there is no epimorphism $X(s) \rightarrow X(j)$ for $j+1 \leq s \leq 25$, we can think of the elements of $A\varepsilon_{j+1} A e_i$ as sums of maps of the same form $m\phi p_i$ where the following additional condition is also satisfied: $p_j m\phi : X(i) \rightarrow X(j)$ is not surjective. From this, it is immediate that if there is no epimorphism $X(i) \rightarrow X(j)$ then we have $A\varepsilon_j A e_i = A\varepsilon_{j+1} A e_i$.

Now suppose that we have the surjective R -homomorphism $p_{ji} : X(i) \rightarrow X(j)$, $x_i \mapsto x_j = r x_i$ for some $r \in R$. If $f : X(i) \rightarrow X(j)$ is another epimorphism then there is an automorphism g of $X(j)$ such that $f = g p_{ji}$, since $X(i)$ is local and R splits over K . Also, if $h : X(j) \rightarrow X(k)$ is a monomorphism (for instance, $h = m_{kj}$) then $h p_{ji} : X(i) \rightarrow X(k)$ cannot be factored through $\bigoplus_{s=j+1}^{25} X(s)$. Finally, if h is not monic, then h is given by

h is given by left multiplication by an element of $rad R$ and so, by Lemma 4.2.3, h factors through $\bigoplus_{s=j+1}^{25} X(s)$. Therefore $A\varepsilon_j A e_i / A\varepsilon_{j+1} A e_i \simeq \Delta(j)$. $\square$

Hence, A is quasi-hereditary, with the following set of standard modules:

$$\Delta = \left\{ \begin{array}{cccccccc} 1, & e & e^2 & a & b & c & d & e^3 \\ & 1, & e & 1, & 1, & 1, & 1, & e^2 \\ & & 1 & & & & & e \\ & & & & & & & 1 \\ \\ ab & bc & cd & da & e^4 & & & \\ a, & b, & c, & d, & e^3 & f & g & h \\ 1 & 1 & 1 & 1 & e^2 & 1 & 1 & 1 \\ & & & & e & & & \\ & & & & 1 & & & \\ \\ abc & bcd & cda & dab & e^5 & & & \\ ab & bc & cd & da & e^4 & fg & gh & hf \\ a, & b, & c, & d, & e^3 & f & g & h \\ 1 & 1 & 1 & 1 & e^2 & 1 & 1 & 1 \\ & & & & e & & & \\ & & & & 1 & & & \end{array} \right\}.$$

Since every indecomposable projective module $P(\beta)$, $\beta \neq 1$, is a submodule of $P(1)$, we display only $P(1)$ on the following page:

Note that for $X(\gamma) \subsetneq X(\beta)$, $\Delta(\gamma) \supsetneq \Delta(\beta)$ (eg. let $\gamma = ab$, $\beta = a$, so that $X(ab) \subsetneq X(a)$ and $\Delta(ab) = \begin{smallmatrix} ab \\ a \\ 1 \end{smallmatrix} \supsetneq \begin{smallmatrix} a \\ 1 \end{smallmatrix} = \Delta(a)$).

For the quiver $\mathcal{Q}_A$ of A , recall that an arrow $\alpha \rightarrow \beta$ in $\mathcal{Q}_A$ corresponds to an unfactored map $f : X(\alpha) \rightarrow X(\beta)$. Since every $X(\beta)$ is generated by a canonical basis element of R , every unfactored map is either a monomorphism or an epimorphism. Thus, there are pairs of arrows $1 \rightrightarrows a$, $\dots$, $1 \rightrightarrows h$, where the arrow $1 \rightarrow a$ corresponds to the epimorphism $X(1) \xrightarrow{p_{a1}} X(a)$ and the arrow $1 \leftarrow a$ corresponds to the monomorphism $X(a) \xrightarrow{m_{1a}} X(1)$. Furthermore, for every $\beta \in \mathcal{B} \setminus \{1, \omega\}$ there is an arrow $\beta \rightarrow \beta'$ corresponding to the epimorphism $X(\beta) \rightarrow X(\beta')$. Finally, if we define β'' to satisfy $X(\beta'') = \text{rad}X(\beta)$ for $\beta \in \mathcal{B} \setminus \{1, \omega\}$ (for instance, if $\beta = a$ then $\beta'' = ab$) then there is also an arrow $\beta'' \rightarrow \beta$ which corresponds to the monomorphism $X(\beta'') \rightarrow X(\beta)$. We give the quiver $\mathcal{Q}_A$ of A below, noting that, for the sake of clarity in the diagram, pairs of arrows (eg. $1 \rightrightarrows a$) are represented by a single two-headed arrow (eg. $1 \leftrightarrow a$):

Remarks.

1. For this type of K -algebra R the right ideals $X(i)$ need not be generated by the canonical basis elements. Indeed, we obtain the same results for any set $\mathcal{X} = \{X(i) | 1 \leq i \leq n\}$, $n = \dim_K R$, of uniserial right ideals of R .
2. While the standard modules $\Delta(i)$ of A are fixed under any refinement of the Loewy length order, this is not the case if we take an arbitrary refinement of the inclusion order. For example, consider R as given in example 4.1, and the R -module X from part (a) of that example, that is, the direct sum of all right ideals generated by the canonical K -basis elements of R . The linear order $1 < 2 < 5 < 3 < 6 < 4 < 7 < 8$ is a refinement of the inclusion order on $\mathcal{X}$, and $A = \text{End}_R X$ is quasi-hereditary with respect to this order. However, we now have $\Delta(4) = \begin{smallmatrix} 4 \\ 6 \\ 1 \end{smallmatrix}$ and $\Delta(3) = \begin{smallmatrix} 3 \\ 1 \\ 2 \end{smallmatrix}$; with respect to the usual order, we had $\Delta(4) = \begin{smallmatrix} 4 \\ 1 \end{smallmatrix}$ and $\Delta(3) = \begin{smallmatrix} 3 \\ 1 \end{smallmatrix}$.

Note that the new $\Delta(4)$ is just an extension of the old $\Delta(4)$ by $\Delta(6)$, and the new $\Delta(3)$ is an extension of the old $\Delta(3)$ by $\Delta(5)$. Lastly, we observe that in this new order, neither $\Delta(5)$ nor $\Delta(6)$ appear in the Δ -filtration of $P(1)$ whereas, in the main theorem, every $\Delta(i)$, $i \neq 1$, appears exactly once in the Δ -filtration of $P(1)$.

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