ASSESSING TESTS FOR MULTIVARIATE NORMALITY

by

KATARZYNA NACZK, B.Sc.

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i. Abstract

The purpose of this thesis was to assess tests previously found to have a high power in detecting departures from multivariate normality. A review of literature prompted the selection of the Royston (1983b) test (an extension of the Shapiro-Wilk W (1965) test) and the Henze-Zirkler (1990) test; which is the only test in the thesis belonging to the class of invariant and consistent tests. In addition, the Doornik-Hansen (1994) test statistic, a relatively recent statistic based on measures of skewness and kurtosis was also chosen. A Monte Carlo simulation study was used to generate data sets consisting of various combinations of sample size, n and number of covariates, p . The nominal significance level chosen for the above tests was checked using multivariate normal data, and the powers of the tests were estimated for a variety of alternative distributions. These alternatives include multivariate normal mixtures, distributions belonging to the elliptically contoured family, such as the Pearson Type II, skewed distributions such as the chi-square, as well as the Khintchine and Generalized Exponential Power distributions. The preliminary results demonstrated an apparent weakness of the Royston (1983b) test statistic when comparing the empirical significance level to the nominal; in particular, the former turned out to be much lower. A search of literature uncovered a new Royston (1992) test statistic that is a revision of the Royston (1983b) test statistic. The Royston (1992) test was added to the study while the Royston (1983b) statistic was retained and used as a comparison against the other test statistics. The estimated powers did not provide sufficient evidence for the recommendation of any of the examined test statistics to be used as a superior test for assessing multivariate normality over one another, excluding the Royston (1983b), which is not recommended.

However, the Henze-Zirkler (1990) test statistic had the best estimated power in the case of the Pearson Type II family of distributions, as well as the Khintchine distribution. In addition, the Henze-Zirkler (1990) also possesses the desired properties of consistency and invariance, while the other test statistics do not. Therefore the Henze-Zirkler (1990) is the only test that at this point could be recommended, if one is specifically to be chosen, for the assessment of multivariate normality.

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I. Introduction

A large body of statistical inference methodologies is based on the assumption that data come from a normal distribution. Checking the assumption of normality for univariate data is common practice, but this is still not the case for multivariate data, despite the fact that there are many tests available. Among the best known, or alternatively, the most widely used, are the multivariate extensions of tests based on skewness and kurtosis, multivariate extensions of the powerful univariate Shapiro-Wilk W (1965) test, as well as graphical extensions of the chi-square plot. A number of reasons may be responsible for the absence of testing for multivariate normality including (i) the lack of knowledge of the tests, (ii) the limited access to computational programs, and (iii) the conflicting ideas of which test to use and when. However there has recently been an increased interest in assessing tests for multivariate normality and seeking an “omnibus” test that can readily be used.

The main objective of this thesis is to assess the power of some of the more generally accepted tests for multivariate normality. The tests of interest in this thesis are Royston’s (1983b) extension of the Shapiro-Wilk W (1965) test, a revision of the Royston (1983b) extension of the Shapiro-Wilk W (given in Royston 1992), the Henze-Zirkler (1990) test statistic, as well as the Doornik-Hansen (1994) test statistic. The Royston (1983b) extension of the Shapiro-Wilk W (1965) test as well as the Henze-Zirkler (1990)

test statistic have been chosen for this thesis as they have been deemed to be powerful tests for detecting departures from multivariate normality in previous studies (such as Mecklin 2000). Royston (1992) points to a problem with the original extension of the Shapiro-Wilk W (1965) in Royston (1982b), specifically in the calculation of the weights and coefficients that leads to an incorrect specification of the null distribution. This problem in Royston (1982b) affects the Royston (1983b) test statistic as it is a multivariate extension of this test statistic. Oddly, none of the previous studies discovered in the literature on the power of multivariate normality tests have acknowledged this error in Royston (1982b). Both the Royston (1992) and Royston (1983b) extensions of the Shapiro-Wilk W (1965) test are considered in this thesis; however the problem associated with the null distribution in Royston (1983b) should be kept in mind. The Doornik-Hansen (1994) test statistic was selected as little is known about its performance in terms of power but initial results show that it appears to be a competitive test.

The assessment of the tests will be based on computer simulations, as is the case in previous studies. The ability of the tests to achieve the nominal alpha level was assessed using multivariate normal data. The tests were compared on a number of alternatives to the multivariate normal that covered a range of possibilities, including: alternatives that consist of components that are multivariate normal but containing different means and/or variances, elliptically contoured alternatives to the multivariate normal, such as the Pearson Type II distributions, alternative distributions that are skewed, as well as distributions containing some properties belonging to the multivariate normal but which are not actually multivariate normal.

Ensuring proper generation of data sets is an important consideration in the assessment of the tests for detecting multivariate normality. It is important to carefully implement the multivariate pseudo-random number generating functions, in such computer programs as Splus, for a number of distributions. This is not trivial, as data sets that are generated improperly may produce an invalid estimated power of the test. Some situations that can affect the proper generation of data would include mistakes in published algorithms as well as mistakes in computer programs for creating the data. Therefore, it is extremely important to use measures to ensure that the data is being generated correctly. Such measures would include graphical inspection of the data as well as consideration of the marginal components where appropriate.

II. Literature Review

The assessment and testing of both univariate and multivariate normality has been a vital topic in statistical inference. Until recently, tests for univariate normality have been investigated in more depth than those for multivariate normality. Recently, however, a marked increase in studying tests for multivariate normality has been noted (such as Mecklin 2000). Most of this research has focused on comparative and/or simulation studies. For instance Ward (1988) assessed various tests of goodness-of-fit for detecting departures from multivariate normality. He concluded, however, that no single test could be chosen.

There is a variety of ways to classify tests for assessing multivariate normality. Cox and Small (1978), for instance, classified tests according to coordinate-dependence or affine-invariance, whereas Ward (1988) classified tests based on whether they were combinations of univariate procedures or tests which he believed to be truly multivariate. Romeu and Ozturk (1993) gave a more specified classification for grouping test statistics which divided existing methods into six categories having common characteristics. The categories of Romeu and Ozturk (1993) are: (i) essentially multivariate procedures, (ii) multivariate methods of marginal analysis, (iii) regression methods, (iv) methods based on the union-intersection principle, (v) geometrical methods, and (vi) projection methods. In what follows, this thesis will use the categories of Baringhaus, Danschke, and Henze (1989), as used by Mecklin (2000) which include: (i) graphical approaches to testing for

multivariate normality, (ii) multivariate goodness-of-fit methods, (iii) multivariate skewness and kurtosis methods, as well as, (iv) multivariate consistent and invariant methods.

Graphical Approaches to Testing For Multivariate Normality:

When working with univariate data, a graphical representation of the data is one of the initial steps for checking the assumption of normality. A histogram showing a “bell-shape”, or a normal probability plot exhibiting a straight line, indicate that the assumption of normality is expected to hold when more exact tests are applied to the data. However in the case of multivariate data, the approach for checking the assumption of normality is not as straightforward as for univariate normality.

Several studies on the graphing of multivariate data have been published (Healy, 1968; Cox and Small, 1978). Of these, Healy (1968) considered the extension of plotting in the univariate case to the multivariate and suggested to use the concept of “distances” of an observation from the sample mean. Healy (1968) proposed a statistic for the bivariate case, speculating that it is extendable to larger variate sizes, $p \geq 2$. Johnson and Wichern (1998) have mentioned that the test proposed by Healy (1968) could indeed be used for $p \geq 2$ and was also able to provide a visual account of what the data is doing. This particular test is referred to as the chi-square plot. It is expected that when the data is indeed multivariate normal and n and $(n - p)$ are greater than 30 then the calculated distances act like a chi-square random variable with p degrees of freedom.

These distances are calculated as:

$$D_i^2 = (X_i - \bar{X})' S^{-1} (X_i - \bar{X})$$

where the X_i 's represents a data vector for observation i ($i = 1, \dots, n$), $\bar{X}$ represents the mean vector, and S is the corresponding sample variance-covariance matrix. To create the chi-square plot it is necessary to first order the D_i^2 's from smallest to largest and then plot them against the $100(i - \frac{1}{2})/n$ quantile of the χ_p^2 . Departure from a straight line, having a slope of 1 and going through the origin, would suggest deviations from normality. It is also expected to find that approximately half of the D_i^2 's are less than or equal to $q_{c,p}(0.50)$, which is the 50th percentile of a χ_p^2 (Johnson and Wichern, 1998). A chi-square plot is a useful visual aid to detect observations that are potential outliers, thus indicating that further investigation of the data is necessary. As with situations dealing with univariate data, graphical techniques are not sufficient for assessing multivariate normality and must be used alongside more formal tests.

Since this thesis focuses on more formal tests of multivariate normality, there will not be as much emphasis placed on graphical techniques for checking multivariate normality. The purpose of this thesis was to find a test for assessing multivariate normality for a multitude of dimensions and sample size, in which case the use of some graphical procedures would become very difficult with an increase in the number of variates, as the number of necessary graphs would increase dramatically. However, graphical approaches are very useful as preliminary tools in the overall analysis of the multivariate data sets and may be very helpful in forming a decision on how to proceed with the testing the data.

Multivariate Goodness-of-fit Methods:

There are many goodness-of-fit tests for assessing univariate normality that have been found to be powerful; these include the Shapiro-Wilk W (1965) test statistic. It is only natural to extend such tests to multivariate normality and assess their power. Monte Carlo simulations are carried out in this thesis to estimate the power of the Royston (1983b, 1992) extensions of the Shapiro-Wilk W (1965) test statistic to the multivariate case. In this section, however, some of the existing goodness-of-fit tests for multivariate normality are summarized.

The test statistic proposed by Mudholkar, McDermott, and Srivastava (1992) is an extension and adaptation of the univariate Lin and Mudholkar (1980) z -test with a modification to the Wilson and Hilferty (1931) cube root transformation. If the data $X_1, \dots, X_n$ are i.i.d. normal p -vectors then Mudholkar et al. (1992) propose a multivariate extension given by $W_i = (D_i^2)^{1/h}$ with $h = 1/3 - 0.11/p$ for $i = 1, \dots, n$. The test statistic is obtained as:

$$Z_p = \tanh^{-1}(r) = \frac{1}{2}(\log\{(1+r)/(1-r)\})$$

where $r = r(W_1, \dots, W_n)$ is the correlation coefficient defined by $\text{corr}(W_i, u_i)$ where

$$u_i = \left[\sum_{j \neq i} W_j^2 - \left(\sum_{j \neq i} W_j \right)^2 / (n-1) \right]^{1/3}, i = 1, \dots, n.$$

A null distribution of this statistic is a refinement of the distribution for the z -test as given by Lin and Mudholkar (1980). Mudholkar, McDermott and Srivastava (1992) showed that rejection of multivariate normality was valid for large values of:

$$|T_{n,p}| = \frac{|Z_p - \mu_{n,p}|}{\sigma_{n,p}}, \text{ where } \mu_{n,p} = \{-1/(p - 0.52p)n\} - \{0.8p^2/n^2\} \text{ and}$$

$$\sigma_{n,p}^2 = \left[\left\{ 3 - \frac{1.67}{p} + \frac{0.52}{p^2} \right\} / n \right] - \left[\left\{ 1.8p - \frac{9.75}{p^2} \right\} / n^2 \right]$$

where $T_{n,p}$ is assumed to follow a standard normal distribution under the null hypothesis.

It was found through a Monte Carlo simulation that the above Z_p statistic is able to achieve the nominal level of significance for $2 \leq p \leq 6$, $n \geq 10$, and is also reasonably powerful against long-tailed alternatives.

The Royston (1983b) H test is a multivariate test based on the well known omnibus test for testing univariate normality, the Shapiro-Wilk W (1965). Srivastava and Hui (1987) comment that “Shapiro and Wilk’s W (1965) statistic has been found to be the best omnibus test for detecting departures from univariate normality”. Therefore, it would be interesting to see how the multivariate extension of this test relates to other tests used for assessing multivariate normality. Royston (1982b) initially extended the Shapiro and Wilk W (1965) from $3 \leq n \leq 50$ up to $n = 2000$ and gives an approximate normalizing transformation suitable for computer implementation. An algorithm to calculate this extension of the Shapiro-Wilk W (1965) test statistic is provided in Royston (1982a, 1983a). Following this, Royston (1983b) used this univariate extension of the Shapiro-Wilk W (1965) to create a multivariate version of this test statistic.

The procedure used by Royston (1983b) to create his multivariate extension of the Shapiro-Wilk W (1965) test statistic involves a number of steps. Specifically, suppose there is a random sample consisting of n cases of p variates (a multivariate data set with n observations having p variables), with sample covariance matrix S . The first step is to

calculate the univariate Shapiro-Wilk W test statistic for each of the p variates. This is done by obtaining the order statistics for each variate, such that $v_1 < v_2 < \dots < v_n$ within each of the individual p -variables. Thus we still have a matrix of $n \times p$ but the p variables have the observations ordered within themselves. The Shapiro-Wilk W (1965) statistic is then calculated as:

$$W = \left(\sum_i a_i v_i \right)^2 / \sum_i (v_i - \bar{v})^2$$

where $\bar{v} = \sum_i v_i / n$ and the $a = a_1, \dots, a_n$ are the best linear unbiased coefficients of Sarhan and Greenberg (1956).

Royston (1982b) showed how to transform this W to approximately standard normal with the condition that the unordered v 's have come from a normal distribution. This transformation is as follows:

$$z = \left\{ (1 - W)^\lambda - \mu \right\} / \sigma$$

with λ , μ , and σ as calculated in Royston (1982b). These transformed z values are then used to obtain the $(p \times 1)$ vector k , where:

$$k_i = \left\{ \Phi^{-1} \left[\frac{1}{2} \Phi(-z_i) \right] \right\}^2 \text{ for } i = 1, 2, \dots, p$$

Finally, letting $c_{ij} = \text{corr}(k_i, k_j)$ be the correlation matrix associated with k , we define

$G = \sum_1^p k_i / p$. Under the null hypothesis of multivariate normality, with k_i 's that are mutually independent, $G \sim \chi_p^2 / p$ and is a multivariate normal test statistic that is based

on combining transformed univariate W statistics. When the k_i 's are not independent, $G \sim \chi_e^2/e$, where e is considered the “equivalent degrees of freedom”. The equivalent degrees of freedom, e , are calculated as:

$$e = \frac{p}{1 + (p-1)\bar{c}}$$

with $\bar{c} = \sum_i \sum_j c_{ij} / (p^2 - p)$ and are constructed from the first two moments of G . The multivariate test statistic is then calculated as:

$$H = eG = e \sum_{i=1}^p k_i / p$$

This H statistic is approximately distributed as a χ_e^2 and thus the p -value can be calculated.

Royston's (1983b) H test has been criticized for a number of reasons. According to Srivastava and Hui (1987) a computation involving the need of a justification of certain approximations, such as the relationship between the correlation of the original variates and the transformed variates, is one of the weaknesses of Royston's (1983b) H test. Royston's (1983b) H test has also been criticized as being unable to deal with data involving ties. Royston (1988) addressed the issue of the inability of the Shapiro-Wilk W (1965) to function desirably when presented with data having ties by modifying his earlier version of the Royston (1982b) extension of the Shapiro-Wilk W (1965) test statistic. Royston (1988) used Monte Carlo simulation to demonstrate that this new modification, referred to as W_g , performed well when presented with ties. He also showed that the modified test outperformed the Pearson χ^2 test, which is typically used

when dealing with ties. Moreover, Royston (1986) provided an algorithm for assessing this new test statistic W_g to deal with ties.

Royston (1992) revised a procedure for finding the coefficients used to obtain the Shapiro-Wilk W (1965); he was motivated to do so by the fact that using an approximation rather than the exact values created a problem. A comparison of various approximations to a_n showed poor performance of both the Royston (1982b) test for $n > 50$, as well as the Shapiro-Wilk (1965) W for $n > 20$. On the other hand, the new approximation of the coefficients (Royston 1992) produced more accurate results. Royston (1992) provided the following approximation $\tilde{a}$ to the weights for the W test. Specifically, for $n \geq 4$, define:

$$\begin{aligned}\tilde{a}_n &= c_n + 0.221157u - 0.147981u^2 - 2.071190u^3 + 4.434685u^4 - 2.706056u^5 \\ \tilde{a}_{n-1} &= c_{n-1} + 0.042981u - 0.293762u^2 - 1.752461u^3 + 5.682633u^4 - 3.582633u^5\end{aligned}$$

where $u = n^{-1/2}$, $c = (\tilde{m}'\tilde{m})^{-1/2}\tilde{m}$, and $\tilde{m}_i = \Phi^{-1}\{(i - 3/8)/(n + 1/4)\}$, $i = 1, \dots, n$ where Φ^{-1} is the inverse cumulative distribution function. Then, normalizing the remaining $\tilde{m}_i$ by writing

$$\begin{aligned}\phi &= (\tilde{m}'\tilde{m} - 2\tilde{m}_n^2)/(1 - 2\tilde{a}_n^2) & \text{if } n \leq 5 \\ &= (\tilde{m}'\tilde{m} - 2\tilde{m}_n^2 - 2\tilde{m}_{n-1}^2)/(1 - 2\tilde{a}_n^2 - 2\tilde{a}_{n-1}^2) & \text{if } n > 5\end{aligned}$$

we have

$$\tilde{a}_i = \phi^{-1/2}\tilde{m}_i \text{ for } i = 2, \dots, n-1 \text{ (} n \leq 5 \text{) or } i = 3, \dots, n-2 \text{ (} n > 5 \text{)}.$$

In addition to calculating the weights differently it is also necessary to use different coefficients than in Royston (1982b). The coefficients are calculated as:

For $4 \leq n \leq 11$:

$$\begin{aligned} w &= -\ln[\gamma - \ln(1 - W)] \\ \gamma &= -2.273 + 0.459n \\ \mu &= 0.5440 - 0.39978n + 0.025054n^2 - 0.0006714n^3 \\ \sigma &= \exp(1.3822 - 0.77857n + 0.062767n^2 - 0.0020322n^3) \end{aligned}$$

For $12 \leq n \leq 2000$:

$$\begin{aligned} x &= \ln(n) \\ w &= \ln(1 - W) \\ \mu &= -1.5861 - 0.31082x - 0.083751x^2 + 0.0038915x^3 \\ \sigma &= \exp(-0.4803 - 0.082676x + 0.0030302x^2) \end{aligned}$$

where W is as calculated before. Royston (1993) stated “investigation revealed that Shapiro and Wilk’s (1965) approximation to the weights (a), used by Royston (1982a,b) was inadequate, therefore the earlier ‘ W test’ differed seriously from the true test.” This is illustrated by the extremely low empirical significance level achieved in this thesis, relative to the nominal α , for the Royston (1983b) test with multivariate normal data (See Table 1, Appendix A). Despite the findings of Royston (1992) many recently published simulation studies seem to have used the uncorrected test of Royston (1983b) (for example, see Mecklin (2000), and Mecklin and Mundfrom (2004)). Royston (1995) provides an algorithm that employs the new calculations for any n in the range $3 < n < 5000$ and makes modifications for dealing with censored, grouped data, and ties.

Another set of tests for assessing multivariate normality that can be considered as a generalization of the Shapiro-Wilk W (1965) test statistic are the tests based on principal components introduced by Srivastava and Hui (1987). The two test statistics, M_1 and M_2 , involve the Shapiro-Wilk W (1965) statistic where M_1 also involves using

a Johnson's (1949) S_B transformation. Srivastava and Hui (1987) showed that M_2 was similar to a test statistic proposed by Malkovich and Afifi (1973), however M_2 has the advantage that the asymptotic null distributions are known and there is no need for tables of critical values.

Beirlant, Mason and Vynckier (1999) constructed a test of goodness-of-fit for multivariate normality based on generalized chi-square quantiles. Their test statistic is based on the notion of multivariate quantiles in terms of the minimum volume ellipsoids as discussed in Einmahl and Mason (1992). In addition, this test is not directly related to Mahalanobis distances, which are usually associated with quantile plots. The new statistic uses the idea of minimum volume ellipsoids and generalizes their median quantiles, with normality satisfied when the quantile plot has a linear shape with an intercept of zero and slope of $\tau_p = c_p^{2/p}$ where $c_p := |\Sigma|^{1/2} \frac{2\pi^{p/2}}{p\Gamma(p/2)}$ and $|\Sigma|$ is the determinant of the covariance matrix, Σ . Beirlant, Mason, and Vynckier (1999) also use a Kolmogorov-Smirnov type of statistic, a Cramer-von Mises type of statistic, as well as a test statistic, J_n , of the same type as that introduced by Jackson (1967). The minimum volume ellipsoids can be calculated using various algorithms such as an adaptation of the elemental set-type algorithm (Rousseeuw, 1985), the exact calculation (Cook et al., 1993), and the feasible subset algorithm (Hawkins, 1993). The algorithm of elemental-set type provided approximations to the J_n values and was used to produce a table of critical values. Furthermore, using Monte Carlo simulation, Beirlant, Mason, and

Vynckier (1999) found that the J_n statistic possessed the strongest potential of the test statistics they assessed.

Romeu and Ozturk (1993) compared two implementations of the multivariate Q_n statistic proposed by Romeu and Ozturk (1992), which is an extension of the univariate Q_n test statistic. The Royston (1983b) test statistic was one of tests used by Romeu and Ozturk (1993) as comparisons to this new multivariate Q_n test and was ranked 5th out of the ten statistics compared. Mecklin (2000) and Mecklin and Mundfrom (2004) also included the Royston (1983b) test statistic in their comparison studies and found it to be a test possessing good estimated power. Therefore, this test statistic was chosen for this thesis. It is important, however, to reiterate the fact that Royston (1992) stated “that Royston (1982b) test for $n > 50$, as well as the Shapiro-Wilk (1965) W for $n > 20$ is lacking”. This directly affects the Royston (1983b) test statistic since it is an extension of the statistics proposed in Royston (1982b) and Shapiro-Wilk (1965). In this thesis, the Royston (1983b) test will be referred to as H.test. The new Royston (1992) test statistic will also be investigated in this thesis and will be referred to as H.new.

Multivariate Skewness and Kurtosis Methods:

Many authors, including Looney (1995) and D’Agostino, Belanger, and D’Agostino (1990), consider an examination of skewness and kurtosis as one of the most commonly used and informative tests for assessing univariate normality. A number of attempts have been carried out to establish measures of skewness and kurtosis for assessing multivariate normality.

Mardia (1970) developed multivariate measures of skewness and kurtosis by broadening studies on the robustness of the t statistic by involving univariate measures of skewness and kurtosis. According to Mardia (1970, 1975b) we have:

$$\beta_{1,p} = E\left\{(X - \mu)' \Sigma^{-1}(Y - \mu)\right\}^3$$

for the population, where X and Y are i.i.d. random vectors with mean vector μ and covariance matrix Σ , and $b_{1,p} = \frac{1}{n^2} \sum_{i,j=1}^n r_{i,j}^3$ is its sample equivalence.

Similarly,

$$\beta_{2,p} = E\left\{(X - \mu)' \Sigma^{-1}(X - \mu)\right\}^2$$

for the population and $b_{2,p} = \frac{1}{n} \sum_{i=1}^n r_{ii}^2$ for the sample where $r_{i,j} = (X_i - \bar{X})' S^{-1}(X_j - \bar{X})$ with S denoting the covariance matrix and $\bar{X}$ denoting the sample mean vector of the random sample $X_1, \dots, X_n$. Here $b_{1,p}$ and $b_{2,p}$ denote the measures of multivariate skewness and kurtosis respectively. Mardia (1970) shows that, for large samples, normality is assessed by testing $\beta_{1,p} = 0$ and $\beta_{2,p} = p(p+2)$. It can be shown that for $p = 1$ the multivariate values are the same as those expected for a univariate data set that is normally distributed; that is, a skewness value of zero and a kurtosis value of three.

Later, Mardia (1975) relates the Mardia (1970) multivariate measures of skewness and kurtosis to the Mahalanobis distances and 'angles'. The Mahalanobis distances are calculated as $d_{ij}^2 = (X_i - X_j)' S^{-1}(X_i - X_j)$ and Mardia (1975) shows that $r_{i,j} = (X_i - \bar{X})' S^{-1}(X_j - \bar{X})$, as used by Mardia's (1970) measures of skewness and

kurtosis, and d_{ij}^2 are related such that $r_{ij} = \frac{1}{2}(r_{ii}^2 + r_{jj}^2 - d_{ij}^2)$. Mardia (1975) also demonstrated that the Mahalanobis 'angle', represented as θ_{ij} , is related to r_{ij} such that $\cos \theta_{ij} = r_{ij} / r_{ii} r_{jj}$. Thus the multivariate skewness and kurtosis can be estimated as $b_{1,p} = \frac{1}{n^2} \sum_{i,j=1}^n (r_{ii} r_{jj} \cos \theta_{ij})^3$ and $b_{2,p} = \frac{1}{n} \sum_{i=1}^n r_{ii}^4$.

Furthermore, Mardia (1970, 1975) showed that using these tests for assessing the normality of a multivariate data set required checking the multivariate skewness by calculating $A = \frac{1}{6} n b_{1,p}$ which has a $\chi_{(p(p+1)(p+2)/6)}^2$ distribution and multivariate kurtosis by calculating $B = (b_{2,p} - \beta_{2,p}) / (8p[p+2]/n)^{1/2}$, where $\beta_{2,p} = p(p+2)$ and B is distributed as $N(0,1)$.

Small (1980) also developed a test based on measures of skewness and kurtosis by taking into account the fact that multivariate normal data is necessarily marginally normal, although the opposite is not necessarily true. Small (1980) derived a joint null distribution that takes into account a collection of univariate problems created from one multivariate problem. Small (1980) generates a method of testing, with the use of an appropriate quadratic form, that is a combination of both marginal skewness and kurtosis. Specifically, new test statistics Q_1 and Q_2 are created through the application of Johnson's (1949) S_U transformation on the marginal values of skewness and kurtosis. These statistics are shown to be well approximated as χ_p^2 . In addition, a combination of the two statistics, possible since they are nearly independent, produces a test statistic $Q_1 + Q_2$ which is χ_{2p}^2 .

A new omnibus test for assessing multivariate normality that is based on skewness and kurtosis was proposed in a working paper by Doornik and Hansen (1994). This test is a multivariate version of a simplified omnibus univariate test statistic created by Shenton and Bowman (1977). Doornik and Hansen (1994) proposed this test by letting $X' = (X_1, \dots, X_n)$ be a $p \times n$ matrix of n observations on a p -dimensional vector with sample mean vector, $\bar{X} = n^{-1}(X_1, \dots, X_n)$ and covariance matrix, $S = n^{-1}\tilde{X}\tilde{X}'$, where $\tilde{X}' = (X_1 - \bar{X}, \dots, X_n - \bar{X})$.

First, a diagonal matrix is created with the reciprocals of the standard deviation on the diagonal:

$$V = \text{diag}(S_{11}^{-1/2}, \dots, S_{pp}^{-1/2})$$

Following this, the correlation matrix $C = VSV$ is formed and the $p \times n$ matrix of transformed observations is defined as:

$$R' = H\Lambda^{-1/2}H'V\tilde{X}'$$

with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ the matrix with the eigenvalues of C on the diagonal. The columns of H are the corresponding eigenvectors, such that $H'H = I_p$ and $\Lambda = H'CH$. Using population values for C and V , a multivariate normal can thus be transformed into independent standard normals, which are then approximated using sample values. The univariate skewness and kurtosis is then computed for each transformed n -vector of observations, and the proposed multivariate statistic is:

$$E_p = Z_1'Z_1 + Z_2'Z_2 \sim \chi^2_{(2p)}$$

where $Z_1' = (z_{11}, \dots, z_{1p})$ and $Z_2' = (z_{21}, \dots, z_{2p})$ are determined by:

$$B = \frac{3(n^2 + 27n - 70)(n+1)(n+3)}{(n-2)(n+5)(n+7)(n+9)}$$

$$\omega^2 = -1 + \{2(\beta - 1)\}^{1/2}$$

$$\delta = \frac{1}{\{\log(\sqrt{\omega^2})\}^{1/2}}$$

$$y = \sqrt{b_1} \left\{ \frac{\omega^2 - 1}{2} \frac{(n+1)(n+3)}{6(n-2)} \right\}$$

$$z_1 = \delta \log \{y + (y^2 + 1)^{1/2}\}$$

this transforms the skewness $\sqrt{b_1}$ into z_1 as in D'Agostino (1970). The kurtosis b_2 is transformed from a gamma distribution to a χ^2 and then to a standard normal z_2 using the Wilson-Hilferty cubed root transformation:

$$\delta = (n-3)(n+1)(n^2 + 15n - 4)$$

$$a = \frac{(n-2)(n+5)(n+7)(n^2 + 27n - 70)}{6\delta}$$

$$c = \frac{(n-7)(n+5)(n+7)(n^2 + 2n - 5)}{6\delta}$$

$$k = \frac{(n+5)(n+7)(n^3 + 37n^2 + 11n - 313)}{12\delta}$$

$$\alpha = a + b_1 c$$

$$\chi = (b_2 - 1 - b_1)2k$$

$$z_2 = \left\{ \left(\frac{\chi}{2\alpha} \right)^{1/3} - 1 + \frac{1}{9\alpha} \right\} (9\alpha)^{1/2}$$

Doornik and Hansen (1994) compared their new E_p test statistic against four other methods of assessing multivariate normality, including the multivariate extension of the Shapiro-Wilk W (1965) test statistic as described by Royston (1983b). Two other test statistics used in their study, also based on multivariate measures of skewness and kurtosis, were those created by Small (1980) and Mardia (1970). The fourth test statistic was a variation of a new test proposed by Mudholkar, McDermott and Srivastava (1992). The latter test uses the correlation between the normalized diagonal of D , where $D = (d_{ij}) = \tilde{X}S^{-1}\tilde{X}'$, with $\tilde{X}'$ and S as defined for the E_p test statistic, and estimates of its variance evaluated via a jackknife approach.

In their comparison study, Doornik and Hansen (1994) used simulations to investigate the power of the tests of interest. Their results showed that their new E_p test statistic was simple with a correct size and with good power properties.

Horswell (1990) used a Monte Carlo simulation study for evaluating multivariate measures of skewness and kurtosis. In his study, Horswell (1990) included Small's (1980) measures of skewness, Q_1 , and kurtosis, Q_2 , and an omnibus measure that is a combination of the two, $Q_3 = Q_1 + Q_2$, as well as Srivastava's (1984) principal component-based measures of skewness and kurtosis, $b_{1,p}$ and $b_{2,p}$. In addition, Horswell (1990) also tested the suitability of measures of skewness and kurtosis, $b_{1,p}$ and $b_{2,p}$ proposed by Mardia (1970), as well as Foster's (1981) $S_w^2 = W^2(b_{1,p}) + W^2(b_{2,p})$. Horswell (1990) demonstrated that tests based on measures of skewness and kurtosis are not "truly diagnostic; that is, they do not distinguish well between 'skewed' and 'non-skewed' distributions". He also indicated the existence of a dependence between

skewness and kurtosis by showing that the results of a skewness test were affected by the kurtosis of the data as well. These results were later confirmed and reinforced in a study conducted by Horswell and Looney (1992). In the latter study the measures of skewness and kurtosis were further investigated by grouping them as either coordinate-dependent or affine-invariant. The results of the Horswell and Looney (1992) study also showed that tests based on skewness and kurtosis are very poor diagnostic tests for assessing multivariate normality. Furthermore, these authors also reported that neither of the test types, either coordinate-dependent or affine-invariant, had an obvious advantage over the other when assessing data for multivariate normality.

Recently, Mecklin and Mundfrom (2000) have stated that “it is inconceivable that any comprehensive study of the performance of multivariate normality tests would not include Mardia’s skewness and kurtosis”. However, the results of Mecklin and Mundfrom’s (2004) study were similar to those of Horswell and Looney (1992) for the measures of skewness. Moreover, these authors also found that measures of kurtosis displayed low power in the case of testing alternative distributions with kurtosis equal to or close to the level of multivariate normal kurtosis, namely $p(p+1)$. Since the purpose of this thesis was to find a test statistic that is powerful against alternatives to multivariate normality, including those alternative distributions having similar properties to the multivariate normal, such as skewness and kurtosis, it was decided not to include Mardia’s (1970, 1975) tests here, as they have already been shown to have low power for these types of distributions.

Since the Doornik-Hansen (1994) test was found to out perform the Royston (1983b) test statistic, which has been deemed a fairly strong test in assessing multivariate

normality in other comparison studies (such as Mecklin, 2000), it has been included in this thesis to be compared with the more popular tests for assessing multivariate normality. It is important to note that in previous studies the E_p test was only compared with the Royston (1983b) test statistic, which has associated with it all of the problems described earlier. A more appropriate comparison would be with Royston (1992). To our best knowledge this test was only included in the comparative study of Doornik and Hansen (1994). The Doornik-Hansen (1994) statistic will be referred to as E_p test in this thesis.

Multivariate Consistent and Invariant Methods:

A desirable characteristic for a test used for assessing multivariate normality to have is that of affine invariance. This property holds when the statistic “remains unaltered under arbitrary affine transformations of the underlying data” (Cox and Small, 1978). An additional sought-after property of the test is that of consistency. Consistency is indicated by the power for a given distribution in the alternative hypothesis going to one as $n \rightarrow \infty$ (Bickel and Doksum, 1977).

Another category of tests for assessing multivariate normality falls into the class consisting of tests that are considered consistent and/or invariant. Koziol (1982) describes distribution theory related to a group of invariant procedures for assessing multivariate normality. In particular, Koziol (1982) examined and tabulated the asymptotic distribution of a Cramer-von Mises type statistic, J_n . Koziol (1982) found that a practical implementation of J_n suitable for computation was of the form

$$J_n = \sum_{i=1}^n \left\{ Z_{(i)} - \left(i - \frac{1}{2} \right) / n \right\}^2 + (12n)^{-1}$$

where $Z_{(i)}$ are ordered values calculated by $Z_i = F_{(k)}(Y_i)$ ($i = 1, \dots, n$), where $F_{(k)}(\cdot)$ is the cumulative distribution function of the $\chi^2_{(k)}$ distribution, and $Y_i = (X_i - \bar{X})' S^{-1} (X_i - \bar{X})$ with $\bar{X}$ and S representing the sample mean vector and the sample covariance matrix, respectively.

Epps and Pulley (1983) proposed a test for assessing univariate normality that is based on the empirical characteristic function, $\phi_n(t)$, from a distribution $F(x)$. They found this test to have high power against a variety of alternative hypotheses and thus categorize it as an omnibus test for assessing univariate normality. Epps and Pulley (1983) proposed the following test statistic:

$$T(\alpha) = n^{-2} \sum_{j=1}^n \sum_{k=1}^n \exp\left\{-\frac{1}{2}(X_j - X_k)^2 / (\alpha^2 S^2)\right\} - 2n^{-1} (1 + \alpha^{-2})^{-\frac{1}{2}} \sum_{j=1}^n \exp\left[-\frac{1}{2}(X_j - \bar{X})^2 / \{S^2(1 + \alpha^2)\}\right] + (1 + 2\alpha^{-2})^{-\frac{1}{2}}$$

and found $T(\alpha)$ to be invariant.

Epps and Pulley (1983) transformed this test statistic into $T^*(\alpha) = -\log\{nT(\alpha)\}$ suitable for use in a Monte Carlo simulation study. They concluded that the test based on $T^*(\alpha)$ can be considered an omnibus test of univariate normality; however they did not recommend it “if high power is required against distributions which are short-tailed and symmetric”.

Csörgő (1986) proposed a test based on a multivariate extension of an approach introduced by Murota and Takeuchi (1981). Csörgő (1989) found this test to be consistent against all alternatives with a finite variance. Furthermore, it has been classified as one of the first ‘genuine’ tests for assessing multivariate normality

(Baringhaus and Henze, 1988). In addition, Csörgő (1989) showed the consistency, against any fixed non-normal alternative distribution, of the test statistic proposed by Baringhaus and Henze (1988). The statistic proposed by Baringhaus and Henze (1988) is based on the empirical characteristic function and is an extension of the Epps and Pulley (1983) univariate test statistic to the multivariate case. They proposed the test statistic:

$$T_n = \frac{1}{n} \sum_{j,k=1}^n \exp\left(-\frac{1}{2} R_{jk}\right) - 2^{1-d/2} \sum_{j=1}^n \exp\left(-\frac{1}{4} R_j^2\right) + n 3^{-d/2}$$

where

$$R_{jk} = (X_j - X_k)' S_n^{-1} (X_j - X_k) = |Y_j - Y_k|^2$$

and

$$R_j^2 = (X_j - \bar{X}_n)' S_n^{-1} (X_j - \bar{X}_n) = |Y_j|^2, j, k = 1, \dots, n$$

which are the squared Mahalanobis distances. An advantage of this test statistic is that it is invariant to any non-singular affine transformation and is relatively easy to compute.

A test statistic proposed by Henze and Zirkler (1990) was found to be affine-invariant (consistent against any fixed non-normal alternative distribution) and valid for any dimension, p . This Henze-Zirkler (1990) test is an extension of the Epps-Pulley (1983) test to the multivariate case, and is based on earlier work by Baringhaus and Henze (1988). The Henze-Zirkler (1990) test statistic is a function of a smoothing parameter, β , where:

$$\beta = \beta_p(n) = \frac{1}{\sqrt{2}} \left(\frac{2p+1}{4} \right)^{1/4} n^{1/(p+4)}$$

The test statistic is calculated as:

$$T_\beta(p) = n \left[\frac{1}{n^2} \sum_{j,k=1}^n \exp\left(-\frac{\beta^2}{2} \|Y_j - Y_k\|^2\right) - 2(1 + \beta^2)^{-\frac{p}{2}} \frac{1}{n} \sum_{j=1}^n \exp\left(-\frac{\beta^2}{2(1+\beta^2)} \|Y_j\|^2\right) + (1 + 2\beta^2)^{-\frac{p}{2}} \right]$$

where we have

$$\|Y_j - Y_k\|^2 = (X_j - X_k)' S_n^{-1} (X_j - X_k) \text{ and } \|Y_j\|^2 = (X_j - \bar{X}_n)' S_n^{-1} (X_j - \bar{X}_n).$$

Note that the Henze-Zirkler (1990) test statistic reduces to the Baringhaus and Henze (1988) test statistic in the situation of $\beta = 1$. Using the first two moments of the test statistic, $T_\beta(p)$,

$$\begin{aligned} E[T_\beta(p)] &= 1 - (1 + 2\beta^2)^{-\frac{p}{2}} \left[1 + \frac{p\beta^2}{1+2\beta^2} + \frac{p(p+2)\beta^4}{2(1+2\beta^2)^2} \right] \\ \text{Var}[T_\beta(p)] &= 2(1 + 4\beta^2)^{-\frac{p}{2}} + 2(1 + 2\beta^2)^{-p} \left[1 + \frac{2p\beta^4}{(1+2\beta^2)^2} + \frac{3p(p+2)\beta^8}{4(1+2\beta^2)^4} \right] - \\ &\quad 4w(\beta)^{-\frac{p}{2}} \left[1 + \frac{3p\beta^4}{2w(\beta)} + \frac{p(p+2)\beta^8}{2w(\beta)^2} \right] \end{aligned}$$

$$\text{where } w(\beta) = (1 + \beta^2)(1 + 3\beta^2)$$

the $T_\beta(p)$ distribution can be approximated as a lognormal using:

$$q_{\beta,p}(\rho) = \mu_{\beta,p} \left(1 + \frac{\sigma_{\beta,p}^2}{\mu_{\beta,p}^2} \right)^{-\frac{1}{2}} \exp \left(\phi^{-1}(\rho) \sqrt{\log \left(1 + \frac{\sigma_{\beta,p}^2}{\mu_{\beta,p}^2} \right)} \right)$$

where $\mu_{\beta,p} = E[T_\beta(p)]$ and $\sigma_{\beta,p}^2 = \text{Var}[T_\beta(p)]$. In the case where S is singular, and not invertible, the test statistic becomes $T_\beta(p) = 4n$, as suggested by Csörgő (1989).

Henze and Zirkler (1990) compare, via Monte Carlo simulation, their test statistic $T_{n,\beta}$, using various values of β , against Mardia's (1970) multivariate measures of skewness and kurtosis, the Malkovich and Afifi (1973) test based on a variation of the Shapiro-Wilk W test (1965), and Fattorini's (1986) test, which is a variation of the Malkovich and Afifi (1973) test. A number of alternatives, such as distributions with independent marginals, mixtures of normal distributions, and spherically symmetric distributions were used to evaluate these tests.

Henze and Zirkler (1990) found that the performance of their test statistic, $T_{n,\beta}$, varied with the value of β , thus affecting the ability of this test to detect departures from normality. They also found that setting $\beta = 0.5$ produced a powerful test against alternative distributions with heavy tails. The Henze and Zirkler (1990) test statistic will be considered in this thesis, referred to as the HZ.test. Since the value of β slowly depends on the size of n and is straightforward to calculate, the equation $\beta = \beta_p(n) = \frac{1}{\sqrt{2}} \left(\frac{2p+1}{4} \right)^{1/(p+4)} n^{1/(p+4)}$ will be used to calculate this smoothing parameter rather than setting $\beta = 0.5$.

Henze and Wagner (1997) present a new approach to the multivariate tests of Baringhaus and Henze (1988) and Henze and Zirkler (1990) as these tests have the following “desirable properties of; (i) affine invariance, (ii) consistency against each fixed non-normal alternative distribution, (iii) asymptotic power against contiguous alternatives of order $n^{-1/2}$, and (iv) feasibility for any dimension and any sample size”. According to Henze and Wagner (1997) the latter property has not been seen in any other tests for assessing multivariate normality. Henze and Wagner (1997) proposed a “new representation of the limiting null distribution of $T_{n,\beta}$ (the Henze-Zirkler 1990 test statistic) in terms of the Gaussian process in $C(R^d)$, the joint limiting distribution of $T_{n,\beta}$ for several values of β , and the asymptotic power of the test based on $T_{n,\beta}$, against contiguous alternatives”.

Henze and Wagner (1997) also found that the Henze-Zirkler (1990) $T_{n,\beta}$ test statistic was virtually independent of the sample size, meaning that their critical values

converged rapidly to their analogous asymptotic values. Henze and Wagner (1997) showed that both $q_{\beta,d}(\alpha)$, the $(1-\alpha)$ -quantile of a lognormal distribution, and $q_{\beta,d}^+(\alpha)$, the $(1-\alpha)$ -quantile of a three parameter lognormal distribution, could be used to approximate critical values for an α -level test based on $T_{n,\beta}$. This is due to the fact that both $q_{\beta,d}(\alpha)$ and $q_{\beta,d}^+(\alpha)$ showed extremely good agreement with the empirical quantiles. Henze and Wagner (1997) also conducted further research into the affect of the smoothing parameter, β , on the power of the test statistic $T_{n,\beta}$. These authors reported that the power did not increase for a certain range of β for the sample sizes used in the study.

Henze (2002) critically reviewed the invariant and consistent test statistics for assessing multivariate normality. He indicated that the class of tests based on the empirical characteristic function, such as the Henze-Zirkler (1990), and the Bowman and Foster (1993) test statistic were able to indicate non-normality of a data set as $n \rightarrow \infty$. It should be pointed out that this critical review was not based on any simulation studies, however the author stated that there was an extensive simulation study in progress. Moreover, Mecklin (2000) and Mecklin and Mundfrom (2004) also reported the Henze-Zirkler (1990) test statistic performed well against a variety of alternatives to the multivariate normal distribution. For these reasons, the Henze-Zirkler (1990) test statistic has been included in this thesis and will be referred to as the HZ.test.

III. Data: Generation and Problems

The multivariate distributions examined in this thesis, as well as the dimensions, were chosen to replicate and/or extend those used in other studies, such as Mecklin (2000). This allows for a more “valid” comparison between the results obtained in this thesis with those reported in previous studies. This approach was especially significant in the case of the Royston (1983b) test statistic. The following distributions were studied:

Multivariate Normal:

Multivariate normal data sets were generated using Splus 6.2 software using the built-in functions `mnorm` and `rmvnorm`. There are two ways to generate multivariate data using Splus 6.2 software. One way involves creating arrays of data, using `mnorm`, that are randomly and marginally univariate normal and then combining these marginals together. Another way to generate the data involves the use of an internal Splus command for multivariate normal generation, `rmvnorm`, creating data that is $N_p(\mu, \Sigma)$. See Appendix C, Program A on how to generate multivariate normal data.

Mixtures of Multivariate Normal:

The mixtures of multivariate normal data sets have been generated according to Johnson (1987). Programs were created within Splus 6.2 to generate the various multivariate normal mixtures (See Appendix C, Programs B1-B5). In total fifteen different mixtures were created using two means, two correlation matrices, and three mixture contamination levels. The following levels of mixture contamination, namely (90%, 10%), (50%, 50%), and (78.8675%, 21.1325%) have been chosen to replicate the results of Mecklin (2000). The (78.8675%, 21.1325%) mixture has the interesting property in that it is a non-normal distribution with kurtosis that is normal. Thus it would be expected that tests dealing with kurtosis, such as the Doornik-Hansen (1994), would have low estimated power with this data set.

Mixture 1 consists of a contamination level of 0.90 such that the first component (i.e. 90% of the data) is created as $N_p(\mu_1, \Sigma_1)$ where μ_1 is a mean vector consisting of all zeros and Σ_1 is a correlation matrix with $\rho = 0.2$ for all off-diagonal elements. In this instance the remainder of the data (remaining 10%, component 2) is created as $N_p(\mu_2, \Sigma_2)$ where μ_2 is a mean vector consisting of all zeros and Σ_2 is a correlation matrix with $\rho = 0.5$ for all off-diagonal elements. The remaining fourteen mixtures were created in a similar manner. All combinations of means, correlations and contamination levels used to create each mixture are listed in Table A (Appendix A).

Multivariate χ^2 (df=1):

An example of a distribution that is skewed and has non-normal kurtosis is the multivariate $\chi^2_{(1)}$. Due to the skewness and non-normal kurtosis it would be expected that the tests in this thesis would all detect the non-normality very easily and thus have a high estimated power.

The multivariate $\chi^2_{(1)}$ data was created using Splus 6.2 and according to Johnson (1987). This is a distribution that Johnson (1987, p.23) indicates can be created using the transformation method. Since Splus has an internal command to create a $\chi^2_{(1)}$ distribution it was used to produce each marginal. These marginals were then combined to give the desired $X = (X_1, X_2, \dots, X_p)'$ which is multivariate $\chi^2_{(1)}$ (See Appendix C, Program C).

Multivariate Lognormal:

A departure from multivariate normality that does not belong to the family of elliptically contoured data sets is known as the multivariate lognormal. This is a skewed distribution and therefore it would be expected that all tests used in this thesis would perform well at detecting this deviation from multivariate normality.

The multivariate lognormal data can be created by applying the transformation method of Johnson (1987, p.23 and p.63) (See Appendix C, Program D). A number of distributions can be created using this procedure. Thus, in this situation, each marginal is initially created as a lognormal and then a combination of these marginals was used to generate a multivariate lognormal. The data can be generated as follows:

$$X_i = \lambda_i \exp(Y_i) + \xi_i \text{ where: } \lambda_i = 1/b_i \\ \xi_i = -a_i/b_i$$

$$x_i = \ln(b_i y_i + a_i) \text{ where: } a_i = \exp(\sigma_i^2/2) \\ b_i = [\exp(2\sigma_i^2) - \exp(\sigma_i^2)]^{1/2}.$$

According to Johnson (1987 p.63), the vector $X = (X_1, X_2, \dots, X_p)'$ results in a distribution belonging to the multivariate Johnson system.

Since the Splus software has an internal command, `rlnorm`, to generate a univariate lognormal distribution, this technique was used to create the resulting X_i 's. These X_i 's were then combined to give the desired X , a multivariate lognormal data set.

Khintchine:

From this set of distributions, an interesting distribution that possesses marginal normality but not overall multivariate normality can be constructed. We would expect this distribution to shed some light on the “true” power of the tests that are involved in this thesis. Due to the nature of the tests we would not expect them to respond well in detecting the multivariate non-normality of the data, specifically the test based on Royston (1992), H.new. This test in a sense is a multivariate extension of the Shapiro-Wilk W test, which is “well known to be a generally superior omnibus test of normality” (Mudholkar, Srivastava, and Lin, 1995),

The Khintchine distribution was generated according to Johnson (1987) (See Appendix C, Program E). To create the data set, Splus was used to construct the function. The data was generated as:

$$X_1 = R(2U_1 - 1), X_2 = R(2U_2 - 1), \dots, X_p = R(2U_p - 1)$$

where R is the same for all of the components; thus we are generating a Khintchine distribution with identical generators. Here, R represents the square root of a $Gamma(1.5, 2)$ (Mecklin 2000, p.92) that is equivalent to the square root of a $\chi^2_{(3)}$ (Johnson 1987, p.149). The U_i 's are then independently and identically distributed as $Uniform[0,1]$, for $i = 1, \dots, p$.

In this situation a separate function was written for each desired value of p . This minimizes the looping in Splus, which is beneficial since the use of a loop is not optimal in Splus. In addition, this allows for a greater number of simulations to be run and larger data sets to be created, while minimizing the amount of memory usage on the computer.

Generalized Exponential Power:

Yet another distribution that is expected to be a challenge for some tests for detecting multivariate normality, specifically those based on skewness and kurtosis, is the Generalized Exponential Power. This is based on the fact that the “generalized exponential power” family of distributions has skewness and kurtosis equivalent to the multivariate normal but is not multivariate normal. However, since only one of the tests, the test based on Doornik and Hansen (1994), uses measures of skewness and kurtosis,

we would expect the other tests to exhibit a high power in detecting the multivariate non-normality of the “generalized exponential power” data sets.

These data sets, used in this thesis, were generated according to Mecklin (2000, p.92) (See Appendix C, Program F) as follows:

$$X_1 = R_1 U_1, \quad X_2 = R_2 U_2, \dots, X_p = R_p U_p$$

where each X_i , $i = 1, \dots, p$, is randomly given a sign using a binomial to generate either +1 or -1. The R_i 's are generated independently and identically as $[Gamma(0.1663, 1)]^{0.125}$, and the U_i 's are independently and identically distributed as $Uniform[0, 1]$.

As with the Khintchine distribution, each variable has its own function for creating the particular generalized exponential power for various p 's. This is done to minimize the use of looping within Splus and thus minimize memory usage and maximize the number of simulations that can be conducted.

Pearson Type II :

Additional distributions of interest are those belonging to the family of elliptically contoured distributions. According to Johnson (1987, p.106) “Elliptically contoured distributions provide a useful class of distributions for assessing the robustness of statistical procedures to certain types of multivariate non-normality”; thus we expect to interpret the estimated power of the tests used in this thesis based on the results against these distributions. Such distributions are closely related to the normal distribution due to their symmetry.

A group in this family of distributions is the Pearson Type II distribution. This distribution was generated according to Johnson (1987 p.115). In this instance the first component X_1 is generated as:

1. Generate X_1 having $Beta(\frac{1}{2}, m + \frac{p}{2} + \frac{1}{2})$ distribution.
2. Set $X_1 = \pm X_1^{1/2}$ where “ $\pm$ ” denotes a random sign (equally likely positive or negative).

After k components $X_1, X_2, \dots, X_k$ have been generated, the $(k+1)^{st}$ component is created as follows:

1. Generate X_{k+1} , having a $Beta(\frac{1}{2}, m + \frac{p}{2} + \frac{1}{2} - \frac{k}{2})$ distribution
2. Set $X_{k+1} = \pm [X_{k+1} (1 - X_1^2 - \dots - X_k^2)]^{1/2}$

A separate function was written for each desired value of p . This minimized the amount of looping and in turn, allows the running of more simulations with larger sets while minimizing memory usage on the computer.

Pearson Type II generations were created using $m = -0.5, -0.25, 0, 0.5, 1, 2, 4$ and 10. It should be noted that the special case of $m = 0$ generates the multivariate uniform distribution (See Appendix C, Program G).

Pearson Type VII:

Another distribution belonging to the family of elliptically contoured distributions is the Pearson Type VII distribution. From this distribution we are able to generate both the multivariate t , (df =10 was used in this thesis), and the multivariate Cauchy. The

multivariate Cauchy is a special case of this distribution, where the degrees of freedom is one.

According to both Mecklin (2000, p.90) and Johnson (1987, p.118), the multivariate t can be created using the transformation:

$$X = (\sqrt{S}/\nu)^{-1} Z + \mu$$

where Z is $N_p(0, \Sigma)$ and is independent of the random variable S , which is $\chi^2_{(\nu)}$. In the case that ν is one, the distribution is called the multivariate Cauchy.

Taking a closer look at the formula provided by both Mecklin (2000) and Johnson (1987) it is seen that the formula does not simplify to the univariate case of a t distribution. This is due to the fact that the degrees of freedom, ν , is not included under the square root, as would have been expected. Graphical inspection of multivariate data generated using this formula, shows that the marginals are off by a scale, which upon closer examination is equal to the square root of the degrees of freedom, ν (See Figure 1a, Appendix B). Johnson (1987) refers to Johnson and Kotz (1972, p. 134) for the density function of the multivariate Pearson Type VII.

Johnson and Kotz (1972), however, give a slightly but significantly different formula for generating the multivariate Pearson Type VII family of distributions. The distribution provided by Johnson and Kotz (1972) is:

$$X = (\sqrt{S/\nu})^{-1} Z + \mu$$

It is important to note that the only difference between this formula and the one provided by Mecklin (2000) and Johnson (1987) is that the degrees of freedom, ν , is included within the square root. It is also important to point out that this formula does reduce to

the univariate t distribution. Graphical inspection of data generated using the Johnson and Kotz (1972) formula gave marginal distributions that were distributed as t and Cauchy (See Figure 1b, Appendix B). This indicates that the formula provided by Johnson and Kotz (1972) seems to be the proper one; hence it is the formula used to create the multivariate t and multivariate Cauchy data used in this thesis (See Appendix C, Program H).

Further inspection, involving multivariate data generation, using both the formula of Johnson (1987) (used by Mecklin 2000) and that of Johnson and Kotz (1972) produced estimates of power that were significantly different from those reported by Mecklin (2000). For instance for the multivariate Cauchy data,

TEST	Mecklin				Current Thesis			
	n=25				n=25			
	p=2	p=3	p=4	p=5	p=2	p=3	p=4	p=5
H.test	4.90	5.10	5.00	14.10	29.06	31.00	30.74	31.75
HZ.test	3.70	3.00	2.70	2.60	98.89	99.68	99.86	99.90
H.new					97.92	99.01	99.52	99.78

Note:

H.test – Royston (1983b) corresponding to Roy in Mecklin (2000, p.121)

HZ.test – Henze-Zirkler (1990) corresponding to H-Z in Mecklin (2000, p.121)

H.new – Royston (1992) with no corresponding test in Mecklin (2000)

As can be seen from the above table, the estimated powers obtained in this thesis, including the ones for the ‘incorrect’ Royston (1983b) test, differ from those reported by Mecklin (2000). The estimates for the multivariate Cauchy distribution obtained in this thesis are not surprising based on the power of univariate tests for this distribution.

While the reason for the difference in the results provided in the table above is not known, it should be pointed out that the data generation mechanisms have been carefully

verified here. Measures such as graphical displays of both the complete data set (See Appendix B, Figure 1c) and the marginals were taken here to ensure that the data used in this study were properly generated. This in turn lends a confidence to the results found in this thesis.

IV. Simulation Study

The Monte Carlo simulation technique was used to assess the power of the Royston (1983b), Royston (1992), Henze-Zirkler (1990), and Doornik-Hansen (1994) test statistics. In order to achieve this, various multivariate distributions were considered. These distributions included the multivariate normal, and mixtures of the multivariate normal consisting of various contamination levels, means and variances. Members of the elliptically contoured family of distributions were also used such as the Pearson Type II family of distributions, including the multivariate uniform. In addition, two members of the Pearson Type VII distributions, the multivariate Cauchy and the multivariate t with ten degrees of freedom, were also used. Alternatives that are heavily skewed, such as the multivariate chi-square with one degree of freedom and the multivariate lognormal were included as well. Distributions with properties belonging to the multivariate normal but that are not inherently normal, such as the Khintchine and the Generalized Exponential Power, were also investigated. The Khintchine distribution has marginals that are normally distributed but the distribution is not jointly multivariate normal, while the Generalized Exponential Power has the same values of multivariate skewness and kurtosis as the multivariate normal (as described by Mardia, 1970) but is not a multivariate normal distribution. A summary of the distributions considered in this thesis can be found on the next page.

Distribution	Property
Multivariate Normal	Symmetric and mesokurtic. Note: Null hypothesis is true.
Multivariate Normal Mixtures	Contamination Level 1 (90%, 10%): mildly contaminated, skewed and leptokurtic. Contamination Level 2 (78.8675%, 21.1325%): moderately contaminated, skewed, and mesokurtic. Contamination Level 3 (50%, 50%): severely contaminated, symmetric, and platykurtic.
Multivariate $\chi^2_{(1)}$	Heavily skewed.
Multivariate Cauchy	Symmetric and platykurtic.
Multivariate Lognormal	Heavily skewed.
Khintchine	Normal marginals but is not jointly multivariate normal.
Generalized Exponential Power	Distribution having Mardia's skewness and kurtosis values equal to the multivariate normal but is not a multivariate normal distribution.
Multivariate Uniform	Symmetric but highly platykurtic.
Pearson Type II	Deviations ranging from mild to severe depending on the shape parameter, m .
Multivariate t (df=10)	Symmetric and mildly leptokurtic.

Sample sizes of $n = 25, 50, 75, 100$, and 250 along with dimensions of $p = 2, 3, 4, 5$, and 10 were used to create the data sets. In general $10,000$ data sets, unless otherwise indicated, were created using combinations of these sample sizes and dimensions. There were instances where only $7,500$ or $2,500$ samples could be created because of limited computer resources available. Each situation with a number of replications other than $10,000$ was indicated in the tables given in the Appendix.

The p -values for each test were calculated and compared to an appropriate alpha level. The number of rejections of the null hypothesis of multivariate normality were tabulated and divided by the number of replications and thus the empirical power of each test statistic were determined.

Results and Discussion

The results obtained using data sets generated in this thesis will help in determining the ability of the test statistics to detect the lack of multivariate normality for a particular data set. In the case of the multivariate normal data it is expected that the tests should have empirical rejection rates that are close to the nominal significance level, which in this thesis is set at $\alpha = 0.05$. Severe departures from this nominal α -level would indicate problems that need to be further investigated. Possible reasons for the test statistics to have rejection rates that differ greatly from the nominal α -level may include: (i) an error in the programming of the calculation of the test statistic, (ii) improper generation of the multivariate normal data set, or (iii) improper specification of the null distribution.

The following table provides a representative summary of what is to be described next:

Power Estimates:

MVN	25	2	Ep.test(4.64) ~ H.new(4.54) ~ HZ.test(4.09) >> H.test(0.00)
<u>Distribution</u>	<u>n</u>	<u>p</u>	<u>Relative Power Performance</u>
Mixture 1	25	2	H.new(5.22) ~ Ep.test(5.02) ~ HZ.test(3.75) >> H.test(0.00)
Mixture 14	25	2	Ep.test(6.24) ~ H.new(5.68) ~ HZ.test(4.54) >> H.test(0.00)
M. Chi-Square (1 df)	25	2	H.new(100) ~ Ep.test(99.95) ~ HZ.test(99.71) >> H.test(0.43)
M. Lognormal	25	2	H.new(99.89) ~ Ep.test(99.71) ~ HZ.test(98.95) >> H.test(3.17)
Khinchine	75	4	HZ.test(82.13) >> H.new(5.39) ~ Ep.test(3.88) >> H.test(0.47)
Generalized Exponential Power	75	4	H.new(96.08) ~ Ep.test(95.98) > H.test(81.05) ~ HZ.test(78.42)
M. Uniform	75	3	HZ.test(90.97) >> H.new(57.23) > Ep.test(25.03) > H.test(10.97)
Pearson Type II ($m = -0.25$)	75	3	HZ.test(97.46) >> H.new(48.81) >> Ep.test(16.02) >> H.test(5.39)
Pearson Type II ($m = -0.25$)	75	3	HZ.test(5.38) >> H.new(2.70) > Ep.test(1.28) >> H.test(0.19)
M. Cauchy	25	2	HZ.test(98.89) ~ H.new(97.92) ~ Ep.test(97.50) >> H.test(29.06)
M t (10 df)	75	5	Ep.test(48.72) ~ H.new(42.14) ~ HZ.test(35.92) > H.test(21.61)

Note:

“ ~ ” – approximately equivalent

“ > ” – greater than

“ >> ” – notably greater than

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

Multivariate Normal

The rejection rates for the H.test test statistic (Royston, 1983b) ranged from 0.00 to 1.01% (Table 1; Appendix A). This indicates a severe departure from the nominal α -

level. Initial investigation of the reason for this departure showed that the data was being generated properly and that the test statistic was properly programmed. Thus the problem of very low rejection rates may be a result of the test itself. Royston (1992) indicated that “the earlier ‘W test’ (1983) differed seriously from the true test” and thus we see that the problem lies with the test statistic itself. As a consequence of these results showing that the Royston (1983b) test statistic achieves an empirical significance level well below nominal, the empirical power results for any of the alternative distributions will not be comparable to counterparts obtained for the other tests. The test statistic was maintained, however, in the thesis to emphasize the difference between results obtained using Royston (1983b) and Royston (1992). It is important to note that past studies have used the Royston (1983b) test statistic (see Mecklin 2000).

The other three test statistics used in this thesis performed similarly to one another, all having rejection rates around the nominal α -level of 0.05 for the multivariate normal data. However, more detailed inspection of the data showed that the rejection rates for the HZ.test (Henze-Zirkler, 1990) statistic were more conservative than those found for either the H.new (Royston, 1992) or the Ep.test (Doornik-Hansen, 1994). Despite this the rejection rate of 5.85%, found for the HZ.test statistic, was the highest among the three test statistics.

Multivariate Normal Mixture 1-3

Tables 2-4 (Appendix A) provide information for mixtures of multivariate normal data (See Table A, Appendix A) containing components 1 and 2 generated using the

same mean and variance distributed among the three mixtures at different contamination levels (90%|10%, 78.8675%|21.1325%, and 50%|50%). For instance, for the (90%, 10%) contamination level the data was created such that 90% of the data is based on component 1, which is distributed as a normal with a specified mean and variance, and the remaining 10% of the data was based on component 2, also having a specified mean and variance. The other two mixtures used the same values of mean and variance for the two components, but the percentage of each component used varied depending on the contamination level.

The estimated power for the H .test statistic was very low but increased slightly from Mixture 1 to Mixture 3, thus having the highest estimated power of 1.33% for Mixture 3 (50%, 50%). It is important to keep in mind that this estimate is particularly low as a result of the fact that the H .test statistic is using a significance level that is extremely low. As mentioned earlier, a direct comparison of these estimates with those obtained from the other tests is not really informative due to the difference in significance levels. The estimated power generally decreased for these mixtures with increasing p , with a few exceptions. On the other hand, the estimated power increased with n increasing up to 75, at which point a decline in the estimated power was noted.

A slight increase in the maximum estimated power of the H .new statistic from Mixture 1 to Mixture 3 was observed. For all three mixtures the estimated power generally increased with increasing p , while for Mixtures 2 and 3 the estimated power also tended to increase with increasing n . In the case of the H .new test statistic the maximum estimated power of 15.44% was obtained for Mixture 3 (50%, 50%); however the H .new statistic had the lowest estimated power of all the tests.

Similarly to H.new, the maximum estimated power for both the Ep.test statistic and the HZ.test statistic increased moderately from Mixture1 to Mixture 3. However, in the case of the HZ.test statistic, the estimated power tended to increase with increasing n , and for Mixture 3 they also generally increase with increasing p , with the exception at $n=25$. On the other hand, the Ep.test statistics estimated power tended to increase with increasing p , and generally also increased with increasing n , with the exception of Mixture 1 where the rates increased with increasing n for $p \geq 5$.

In summary the estimated power for the H.new test were very similar to those of the HZ.test statistic and Ep.test statistic. This makes it difficult to truly distinguish between the three test statistics based on the results obtained using these data set mixtures.

Multivariate Normal Mixtures 4-6

Tables 5-7 (Appendix A) provide information on Mixtures 4-6 of multivariate normal data (See Table A, Appendix A) composed of 2 components. These mixtures were generated in a similar manner to the mixtures described above with the exception that different combinations of mean and variance were used. The mean and variance were distributed among these mixtures at different contamination levels (90%|10%, 78.8675%|21.1325%, and 50%|50%).

The estimated powers for the H.test statistic were very low for all of the various dimensions, with the highest estimated power of 1.50% found for Mixture 4 (90%, 10%). The maximum estimated power decreased marginally from Mixture 4 to Mixture 6. This is opposite to the trend observed for the multivariate normal Mixtures 1-3. The estimated

power, for Mixtures 4-6, generally decreased with increasing p but increased with increasing n up to $n = 75$ and then began to decline. The latter trend was similar to that noted for Mixtures 1-3.

The maximum estimated power of the H.new statistic increased slightly from Mixture 4 to Mixture 5, but decreased greatly for Mixture 6. The maximum estimated power for Mixtures 4 and 5 were 11.76% and 12.36%, respectively, while for Mixture 6 the maximum estimated power dropped to 4.80%. Moreover, in the case of Mixtures 4 and 5 a general increase in estimated power with increasing p and increasing n was noted. The maximum estimated power of 12.36% for Mixture 5 (78.8675%, 21.1325%) was the highest estimated power observed for all the tests and mixtures involved, but overall the estimated powers for the H.new test were very similar to those obtained for both the HZ.test statistic and the Ep.test statistic.

Similar to the H.new test statistic, the maximum estimated power for the HZ.test statistic also tended to increase slightly from Mixture 4 to Mixture 5 and then decreased for Mixture 6. The maximum estimated power of 8.66% was achieved for Mixture 5 (78.8675%, 21.1325%). A noticeable increase in the estimated power was observed with increasing n . On the other hand, the estimated power for the Ep.test statistic decreased from Mixture 4 to 6 reaching the maximum estimated power of 6.29% for Mixture 4 (90%, 10%). In the case of Mixture 5 the estimated power increased with increasing n .

In summary, the situation involving Mixtures 4-6 had estimated powers for the H.new, the HZ.test and the Ep.test statistics that were very similar. Therefore it is difficult to distinguish between the three test statistics based on the results obtained using these mixtures.

Multivariate Normal Mixtures 7-9

Tables 8-10 (Appendix A) provide information on mixtures of multivariate normal data (See Table A, Appendix A) containing components 1 and 2 generated using the same mean and variance distributed among these three mixtures at contamination levels of (90%, 10%), (78.8675%, 21.1325%), and (50%, 50%).

The estimated powers for the H.test statistic were very low and had the maximum estimated powers decrease from Mixture 7 to Mixture 9 with the maximum estimated power at 1.65% for Mixture 7 (90%, 10%). The estimated power for Mixtures 7-9 generally decreased with increasing p and increased with increasing n up to $n = 75$ and then began to decrease. Similar trends were observed for Mixtures 4-6.

The maximum estimated power for the H.new statistic increased slightly from 20.84% for Mixture 7 to 23.92% for Mixture 8 but decreased greatly to 5.60% for Mixture 9. For Mixtures 7 and 8 the estimated power generally increased with the increasing of both p and n . On the other hand, for Mixture 9 the estimated power tended to decrease with n increasing up to $n = 75$ and then begins to increase. It should also be pointed out that the maximum estimated power of 23.92% for Mixture 8 (78.8675%, 21.1325%) was the highest estimated power found for all the tests and any of the three mixtures used, but overall the estimated powers for the H.new test statistic were very similar to those obtained for both the HZ.test statistic and the Ep.test statistic. The difference between the H.new test and the other two test statistics was most apparent when $p = 10$, or when $n = 250$.

The maximum estimated power for the HZ.test statistic tended to increase slightly from Mixture 7 to Mixture 8 and then decreased again for Mixture 9, reaching a

maximum estimated power of 17.44% for Mixture 8 (78.8675%, 21.1325%). A similar trend was also observed in the case of the estimated power for the H_{new} test statistic. The estimated power for the H_Z test statistic tended to increase with increasing p , as well as with increasing n with the exception of Mixture 8 at $n = 25$.

The maximum estimated power for the E_p test statistic increased slightly from Mixture 7 to Mixture 8 and then decreased again for Mixture 9, reaching a maximum estimated power of 6.73% for Mixture 8 (78.8675%, 21.1325%). Similar changes in the estimated power were observed for the H_{new} test statistic. Moreover, the estimated power for the E_p test statistic showed a tendency to increase with increasing n and declined with increasing p for Mixture 8, but in the case of Mixture 9 the estimated power increased with the increasing of both n and p .

Multivariate Normal Mixtures 10-12

Tables 11-13 (Appendix A) provide information on mixtures of multivariate normal data containing components 1 and 2 generated using the same mean and variance distributed among these three mixtures with different contamination levels (90%|10%, 78.8675%|21.1325%, and 50%|50%) as indicated in Table A, Appendix A.

The estimated powers for the H test statistic were again very low and tend to increase from Mixture 10 to Mixture 12, with the largest estimated power obtained for Mixture 11 (2.48%) and for Mixture 10 (2.41%). This trend was different from that observed for Mixtures 4-9. As for previous mixtures, the estimated power for Mixtures 10 and 12 increased with n increasing up to $n = 75$ and then began to decrease. In addition, for Mixture 12 the estimated power also increased with increasing p . However,

in the case of Mixture 11 the estimated power increased with increasing p for $n \geq 75$ and tended also to increase with increasing n up to $n = 75$ for $p \leq 5$. However, unlike for any of the previous mixtures, the estimated powers for Mixture 11 at $p = 10$ increased with increasing n . Such behaviour was not detected for any of the other tested mixtures.

The maximum estimated power for the H.new statistic increased from Mixture 10 to Mixture 12, having maximum estimated powers of 24.08%, 39.92%, and 81.04% respectively. For all of the three mixtures involved, the estimated power increased with increasing n and p . The maximum estimate power of 81.04% for Mixture 12 (50%, 50%) was the highest estimated power of all the tests and for any of the three mixtures. Overall, the estimated powers for the H.new test are very similar to those obtained for both the HZ.test statistic and the Ep.test statistic. The difference between the H.new test and the other two test statistics was most apparent for $p = 10$.

The maximum estimated power for the HZ.test statistic increased from Mixture 10 to Mixture 12, reaching a maximum estimated power of 67.48% for Mixture 12 (50%, 50%). These changes in the estimated power were similar to those found for the H.new test statistic. The estimated power tended to increase with increasing n . The estimated power also increased with increasing p for Mixture 11 and 12, however for Mixture 11 this increase was noted for $n \geq 75$. It should be pointed out that the estimated power for the HZ.test statistic were always slightly better than those for the Ep.test statistic. Such a situation was not detected for any of the other multivariate normal mixtures.

The maximum estimated power for the Ep.test statistic increased from Mixture 10 to Mixture 12, reaching a maximum estimated power of 46.72% for Mixture 12 (50%,

50%). Furthermore, the estimated power increased with increasing n for all of the mixtures involved. In the case of Mixtures 11 and 12 the estimated power also increased with increasing p . A similar trend was found with the estimated power for the H.new and HZ.test statistics.

Multivariate Normal Mixtures 13-15

Tables 14-16 (Appendix A) provide information on mixtures of multivariate normal data containing components 1 and 2 and generated using the same mean and variance distributed among these three mixtures at different contamination levels (90%|10%, 78.8675%|21.1325%, and 50%|50%) as described in Table A, Appendix A.

For the H.test statistic, the estimated powers for Mixture 13 and 14 reached levels as high as 6.11 % and 8.19%, respectively, but for Mixture 15 was much lower reaching a level of 2.65%. The estimated power for Mixture 14 (78.8675%, 21.1325%) was the highest one obtained for the H.test statistic when testing any of the multivariate normal mixtures involved in this thesis. As with Mixtures 1-12, the estimated power for Mixtures 13-15 increased with increasing p , with the exception of $p = 10$ for Mixture 14 where the estimated power increased until $n = 100$ and then decreased at $n = 250$.

The maximum estimated power for the H.new statistic increased from 72.82% for Mixture 13 to 93.08% for Mixture 14, and then slightly decreased to 80.68% for Mixture 15. These maximum estimated powers for Mixtures 13-15 were higher than those obtained for any of the other multivariate normal mixtures used in this thesis. Moreover, the estimated power for Mixtures 13-15 increased with the increasing of both n and p . However, overall the estimated powers for the H.new test were very similar to those

obtained for both the HZ.test statistic and the Ep.test statistic. The difference between the H.new test and the other two test statistics was most apparent for $p = 10$.

The maximum estimated power for the HZ.test statistic also increased from Mixture 13 to Mixture 15, reaching a maximum estimated power of 68.28% for Mixture 15 (50%, 50%). The increase in the estimated power with increasing of both n and p was also observed. The overall estimated powers for the HZ.test statistic were slightly smaller than those found for both the H.new and Ep.test statistics. This is most evident for Mixture 13 at $p \geq 3$. In addition, better estimated powers were obtained for the HZ.test statistic than for the Ep.test statistic with both Mixture 14 and 15 for $n = 250$, $p = 10$.

The Ep.test test statistic maximum estimated power showed a trend similar to that of the maximum estimated power for the H.new test statistic. These estimated powers increased from 40.2% for Mixture 13 to 55.2% for Mixture 14, and then decreased to 45.64% for Mixture 15. In general, an increase in the estimated power with increasing n and p was noted for all of the mixtures involved. It is interesting to point out that the greatest difference in estimated power between the Ep.test statistic and the H.new statistic was found at $p = 10$, where the values obtained for the Ep.test statistic were approximately twice as small as those for the H.new statistic.

Multivariate χ^2 (df = 1)

The maximum estimated power of 100% was reached by all tests involved in this study very quickly ($n = 50$) (Table 17; Appendix A). The main difference between the test statistics was noticeable at $n = 25$, where all estimated powers for the H.test statistic

were much lower than for the other tests, reaching a maximum of 0.43% and decreasing with p . On the other hand, an estimated power of 100% was observed for the H.new test at all n and p included in the study. Also, in the case of the HZ.test statistic and the Ep.test statistic an estimated power of 100% was reached quickly. The difference in estimated power between the HZ.test statistic and both the H.new and Ep.test test statistics was only evident at $n = 25$ and $p = 10$, where the estimated power for the HZ.test statistic and other two statistics were 95.28%, 100% and 100%, respectively.

Multivariate Cauchy

Similarly to the multivariate χ^2 ($df=1$) results, the estimated power for the multivariate Cauchy distribution reached a maximum of 100 % very quickly ($n = 50$) for all tests (Table 18; Appendix A). The differences between tests were most noticeable at $n = 25$. At this n , the estimated powers for the H.test statistic were much lower than those for the other three test statistics. In this instance, the estimated power for the H.test statistic was as low as 29.06% and increased with increasing p to 34.18%. It is also interesting to note that at this n the estimated power for the Ep.test statistic showed a decreasing tendency with increasing p .

Multivariate Lognormal

A rapid increase in the maximum estimated power to 100% at $n = 50$ was noted for all test statistics (Table 19; Appendix A). A similar trend in estimated power was also observed for both the multivariate χ^2 ($df=1$) and the multivariate Cauchy. The difference

between the test statistics was most evident at $n = 25$. In this instance estimated power for the H.test statistic was significantly lower than those for the other test statistics, reaching a maximum estimated power of 3.17%. Furthermore, this estimated power decreased to 1.50% with increasing p . The estimated power at $n = 25$ for the H.new, HZ.test, and Ep.test test statistics, were all above 95%.

Khintchine

The results for this distribution, shown in Table 20 (Appendix A), are very interesting. The Khintchine distribution is a unique distribution with the property of having normally distributed marginals but not having joint multivariate normality. The failure to detect the multivariate non-normality of this data set would indicate a lack of power of the test in dealing with a variety of alternatives to the multivariate normal. Therefore, with this data set it is desirable to achieve a high estimated power.

Inspection of the estimated power for the H.test statistic indicated that the power of the test increased from 0.00% at $n = 25$ to 1.18% for $n = 75$ and $p = 2$. The estimated power generally decreased with increasing p and increased with increasing n up to $n = 75$ and then began to decrease. Not surprisingly, the estimated power of the H.test statistic remained lower than that of the other three test statistics.

The H.new test statistic was not a good indicator of the non-normality of this data set, as speculated in the data generation (See Section III). The estimated power ranged from 4.87% to 6.64%. The estimated power generally increased with increasing p .

The estimated powers for the HZ.test statistic differed significantly from those for the other test statistics and were consistently larger. The estimated power ranged from

9.20% to 100%, being the only test statistic reaching such a high estimated power. The estimated power increased with increased p and n .

The estimated power for the Ep.test statistic was also not a good indicator of the non-normality of this data set. The estimated power ranged from 0.36% to 4.90%. A decrease with increase in p , as well as an increase with increase in n , was noted.

From these results it is apparent that the HZ.test statistic is the best in indicating data that is not distributed as a multivariate normal, but has marginals that are normal. The H.test, the H.new, and the Ep.test statistics all had difficulty in distinguishing this data from data that is generated as a multivariate normal distribution.

Generalized Exponential Power

Inspection of the estimated power for the H.test indicated that the power of the test increased dramatically from 0.00% at $n = 25$ to 100% for $n = 50$ and $p = 5$ (Table 21; Appendix A). For $n \geq 50$, the estimated power of the H.test was very similar to that of the other three test statistics. The estimated power of all test statistics increased to 100% with increasing n . This was also the case for increasing p with an exception of the HZ.test statistic for which the estimated power decreased with increasing p .

Although, overall, all test statistics demonstrated similar estimated power, a significant difference in the estimated power was noted for $n = 25$. At this n value, the H.new test statistic showed evidence of higher estimated power over the other tests.

The estimated power for the H.test statistic increased with increasing n and p . Similar changes in estimated power were also observed for the HZ.test statistic, when n increases. Surprisingly for $n \geq 50$, the estimated powers for the H.test statistic exceeded

those of the HZ.test statistic. Such dramatic changes in the estimated power were not noted for any other multivariate distributions used in this study.

Multivariate Uniform

The estimated powers for the multivariate uniform data set are shown in Table 22 (Appendix A). The estimated powers for the H.test statistic had the lowest range of the all tests used in this study and were between 0.00% and 92.82%. The estimated powers for the H.new, HZ.test, and Ep.test statistics all achieved a maximum value of 100%, but the lowest estimated power was found to be 2.13% and calculated for the H.new test. In the case of the H.test, the H.new and the HZ.test statistics, a decline in the estimated power with increasing p was noticed. The estimated power increased with increasing n for $p \leq 4$ for the H.test statistic and for any p in the case of the H.new and the HZ.test statistics. All tests reach their maximum expected power at $n = 250$ and $p = 2$. Furthermore, the estimated powers for the H.new test were very similar to those obtained for the HZ.test test statistic when $p = 2$.

The estimated powers for the Ep.test statistic were generally lower than those obtained for the H.new and the HZ.test statistics and vary from 0.31% to 100%. On the other hand, the estimated powers for the HZ.test statistic range from 3.98% to 100% and increase more rapidly than those obtained for the H.new and the Ep.test statistics. This is most apparent at $p = 10$. However a noticeable decrease in the estimated power is observed with an increase in p but to a smaller extent for the HZ.test statistic than for the H.new and the Ep.test statistics.

A possible reason for the decrease of estimated power with increase in p is that at higher dimensions it becomes more difficult to distinguish between data that is distributed as multivariate normal and data that is distributed as multivariate uniform. This can be seen in the plots of multivariate normal data versus plots of multivariate uniform at $n = 25, p = 2$ and $n = 25, p = 10$ (See Appendix B, Figures 2a through 2c). In the case of plots in the third-dimension the difference between multivariate normal data and multivariate uniform data can be described as the ‘denseness’ of the cloud displayed by the data on the plot. Multivariate uniform data will typically appear to be less dispersed than multivariate normal data of the same dimension.

Pearson Type II ($m = -0.5, -0.25, 0.5, 1, 2, 4$, and 10)

It is important to reiterate here, that for $m = 0$ the Pearson Type II distribution becomes the multivariate uniform distribution. The estimated powers for the multivariate uniform have been discussed above and therefore the Pearson Type II distribution for $m = 0$ will not be reviewed in this section.

The estimated powers for the Pearson Type II distributions are shown in Tables 23-29 (Appendix A). Inspection of these data indicated that the maximum estimated power for the H.test statistic decreased from 100% to 0.43% with an increase in the shape parameter, m , of the distribution and the estimated power for each statistic test was zero at $n = 25$. In addition, a decrease in the estimated power with increasing p was also noted for all of the Pearson Type II distributions tested. The estimated power of the tests was also greatly reduced for $m = -0.5$, and -0.25 as the number of variables increased from 2 to 3.

In the case of the Pearson Type II with $m = -0.5$ the estimated power for the H.test statistic generally increased with increasing n and reached a maximum estimated power of 100% at $n = 250$, $p = 2$. In addition, the estimated power of the H.test statistic for this distribution was very similar to that of the other test statistics for $p = 2$ and $n \geq 75$. A similar trend for the Pearson Type II with $m = -0.25$ was also observed. On the other hand, much lower estimated powers of the H.test statistic were noted for distributions generated for any of the other m -values. Furthermore, for the distribution with $m = -0.25$ and at $p \leq 4$ the estimated power increased with increasing n , whereas for $p \geq 5$ the estimated power increased at $n = 75$ and then began to decrease. A similar pattern was also seen with the multivariate normal mixtures. In the case of distributions with $m = 2, 4$, and 10 , the estimated power of the test statistics increased up to $n = 75$ and then decreased. A similar increase in estimated power for the distribution with $m = 0.5$ occurred at $p \geq 3$. On the other hand, for the distribution with $m = 1$ at $p \leq 4$ the estimated power of the test statistics increased up to $n = 100$ and then began to decline.

The maximum estimated power of the H.new test statistic also decreased with increasing m -value. For instance, for a Pearson Type II distribution with $m = -0.5$, the H.new test reaches a maximum estimated power of 97.64% which decreased to 4.70% for the distribution with $m = 10$. Generally, for all Pearson Type II distributions a decrease in estimated power with increasing p and an increase with increasing n was observed. The distribution with $m = 10$ was an exception to this, as in this case the estimated power first decreased with increasing n up to $n = 75$ and then began to increase.

In the case of the Pearson Type II ($m = -0.5$ and -0.25) distributions the estimated power of the H.new statistic was very similar to that of the HZ.test test statistic at $p = 2$ and for $p = 3$ for $n \geq 75$. Otherwise, the estimated power of the H.new test statistic was generally lower than that for the HZ.test test statistic, but generally greater than that for the Ep.test test statistic. However, it should be pointed out that at $n = 250$ and $p \leq 5$ the estimated powers for the H.new, the HZ.test and the Ep.test statistics were very similar. In addition, the H.new test statistic and the HZ.test statistic displayed very similar estimated powers for the distribution with $m = 0.5$ at $n = 25$, as well as at $p = 2$. In the case of the Pearson Type II distributions with $m = 1, 2, 4$, and 10 , the estimated power of the H.new test statistic was similar to that of the HZ.test statistic for $p = 2$ or $n = 25$, but otherwise it was generally lower than that of the HZ.test statistic. The most evident differences can be seen for $n \geq 100$.

The maximum estimated power of the HZ.test test statistic for the Pearson Type II distributions, with various values for the shape parameter m , decreased in a similar manner to the other tests. However, much smaller differences in the estimated power for any given n and regardless of m were noted than for the other test statistics involved. Such a situation was not encountered in any of the other tests for these distributions.

The estimated power of the Ep.test statistic also decreased with increasing m -value of the distribution. A maximum estimated power of 100% was observed for the distribution with $m = -0.5$ while for the distribution with $m = 10$ the maximum estimated power reached only a value of 3.58%. In general, the estimated power of the Ep.test statistic for any of the Pearson Type II distributions decreased with increasing p and

increased with increasing n . Exceptions were the Pearson Type II distributions with $m = 4$ and $m = 10$. No real pattern of estimated power change with increasing n was detected for the distribution with $m = 4$, but in the case of the distribution with $m = 10$ the estimated power decreased with n increasing up to $n = 75$. It is important to note here, that for all m -values the estimated power of the Ep.test statistic was frequently lower than that for either the H.new or the HZ.test statistics.

Multivariate t ($df = 10$)

The estimated powers for the H.test statistic, shown in Table 30 (Appendix A) were consistently lower than those obtained for other test statistics used in this study. The estimated powers for the H.test statistic were as low as 0.00% and reached a maximum of 30.86%. The estimated powers were generally low for $n \geq 50$, and increased with both increasing n and p . However, in the case of $n = 250$ a slight decrease in the estimated powers was noted.

Inspection of the estimated power for the H.new test statistic indicated that these estimated powers were increasing with both n and p . The estimated powers for the H.new test were slightly higher than those for the HZ.test statistic, with a few exceptions, but also were slightly lower than those for the Ep.test statistic.

The estimated power for the HZ.test statistic increased with the increasing of both n and p . Generally, these estimated powers were a bit lower than those for the H.new and Ep.test statistics. Even so, the maximum estimated power for the H.new test statistic reached a value of 99.63%, while for the H.new and Ep.test test statistics this value was 96.37% and 98.48%, respectively.

V. Conclusions and Recommendations

The purpose of this thesis was to re-assess tests for multivariate normality that have been previously found to have a high power in detecting deviations from multivariate normality. In the course of this thesis a number of interesting situations arose. One of situations involved the discovery of problems with using the Royston (1983b) test statistic for assessing multivariate normality. This led to an investigation of possible causes responsible for creating these problems. A search of literature uncovered a new Royston (1992) test statistic that is a revision of the Royston (1983b) test statistic. To our knowledge, there has not been any published simulation study involving this new Royston (1992) test statistic. Therefore, to draw attention to the difference between these two tests, the Royston (1992) was added to the thesis while the Royston (1983b) statistic was retained and used as a comparison against the other test statistics.

An important consideration in the assessment of tests for multivariate normality is the significance of ensuring proper generation of data. With this in mind, the tests were assessed against the multivariate normal distribution and a variety of alternatives. Although previous studies (such as Romeu and Ozturk, 1993) suggested that finding a single superior test is improbable, the objective of this thesis was to attempt to find a test that would have a strong or appropriate estimated power for all distributions studied. A

test meeting such requirements could be considered as an “omnibus” test and used for assessing multivariate normality regardless of prior knowledge of the distribution of the data being analyzed. This is the situation typical for most circumstances.

The results of this thesis demonstrated that the Royston (1983b) test statistic, here referred to as the H.test, had a much lower empirical significance level than the other tests when α was set to 0.05 (Table 1, Appendix A). These findings were in contrast to those reported by Mecklin (2000)

	n = 25					n = 100		
	p=2	p=3	p=4	p=5	p=2	p=3	p=4	p=5
Roy	4.9	5.2	4.7	4.8	4.8	5.2	5.30	4.70
H.test	0.0	0.0	0.0	0.0	0.6	0.5	0.32	0.34

Note:

Roy – Royston (1983b) according to Mecklin (2000)

H.test – Royston (1983b) according to this thesis

However, it is important to keep in mind that Royston (1993) stated that the H.test statistic should not be used for assessing multivariate normality. The results of this thesis confirm and reinforce his findings.

The estimated powers of the other test statistics, namely Royston (1992), Henze-Zirkler (1990), and Doornik-Hansen (1994) were very similar to each other. However, the estimated power for the Royston (1992) test statistic, here referred to as H.new, was generally slightly higher for the majority of the distributions than that of the Henze-Zirkler (1990), and Doornik-Hansen (1994) statistics. The Royston (1992) test statistic is

an extension of the Shapiro-Wilk W (1965) test statistic to the multivariate case. The Shapiro-Wilk W (1965) has been shown to be a powerful indicator of deviations from univariate normality. Thus it is natural to expect that an extension of the Shapiro-Wilk W (1965), the Royston (1992) statistic, would also be a good indicator of non-normality in the multivariate case.

When prior knowledge of the data is available and the data is generated from a member of the Pearson Type II family of distributions, such as the Multivariate Uniform, then the Henze-Zirkler (1990) test statistic would be best for analysis of the data. In the case of the Pearson Type II distributions the Henze-Zirkler (1990) test statistic has higher power and is more consistent with increasing dimension. This test offers the additional advantage of being both invariant and consistent.

In conclusion, the results of this thesis did not provide sufficient evidence for the recommendation of any of the examined test statistics to be used as a superior test for assessing multivariate normality over one another, excluding the Royston (1983b). Therefore, it is recommended to use any of these three test statistics, namely the Royston (1992), the Henze-Zirkler (1990), or the Doornik-Hansen (1994), or a combination of them, to assess whether the data is multivariate normally distributed. It is important to note that the Henze-Zirkler test statistic was the only test statistic that had an acceptable estimated power when assessing the Khintchine distribution, a multivariate non-normal distribution having normal marginals. Thus, for this reason, if one statistic were to be chosen as a better test of multivariate normality, it would be the Henze-Zirkler test statistic.

Recommendations for further research, based on these findings, include:

- Further investigation into the reasons for the peculiar behaviour of the Henze-Zirkler (1990), Doornik-Hansen (1994) and the Royston (1992) test statistics in the case of some of the distributions, such as for the family of Pearson Type II distributions in which the estimated power for the Henze-Zirkler (1990) test statistic decreased with increasing p for distributions with $m \leq 0$ and increased with increasing p for distributions with $m > 0$, whereas the other test statistics had estimated power continually decrease with increasing p , regardless of the value of m .
- Extend the number of tests used to assess multivariate normality, specifically since the results of this thesis contradict some of the findings reported in other studies. It is recommended to include the J_n test statistic of Beirlant, Mason, and Vynckier (1999), and the Bowman-Foster (1993) test statistic in future studies.
- Use larger sample sizes, dimensions and replications. This should provide a better insight into the performance of these test statistics for assessing of multivariate normality of a given data set.

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Appendix A: Tables

Table A: Distribution of Normal Mixtures

Mixture	Component 1			Component 2		
	$\tilde{\rho}$	Mean	Variance	$\tilde{\rho}$	Mean	Variance
1	0.9	μ_1	Σ_1	0.1	μ_1	Σ_2
2	0.788675	μ_1	Σ_1	0.211325	μ_1	Σ_2
3	0.5	μ_1	Σ_1	0.5	μ_1	Σ_2
4	0.9	μ_1	Σ_1	0.1	μ_2	Σ_1
5	0.788675	μ_1	Σ_1	0.211325	μ_2	Σ_1
6	0.5	μ_1	Σ_1	0.5	μ_2	Σ_1
7	0.9	μ_1	Σ_2	0.1	μ_2	Σ_2
8	0.788675	μ_1	Σ_2	0.211325	μ_2	Σ_2
9	0.5	μ_1	Σ_2	0.5	μ_2	Σ_2
10	0.9	μ_1	Σ_1	0.1	μ_2	Σ_2
11	0.788675	μ_1	Σ_1	0.211325	μ_2	Σ_2
12	0.5	μ_1	Σ_1	0.5	μ_2	Σ_2
13	0.9	μ_1	Σ_2	0.1	μ_2	Σ_1
14	0.788675	μ_1	Σ_2	0.211325	μ_2	Σ_1
15	0.5	μ_1	Σ_2	0.5	μ_2	Σ_1

μ_1 : mean vector of all zeros

μ_2 : mean vector of all ones

Σ_1 : correlation matrix with $\rho = 0.2$ for all off-diagonal elements

Σ_2 : correlation matrix with $\rho = 0.5$ for all off-diagonal elements

Table 1: Multivariate Normal

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
<u>n=25</u>	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.54	5.17	5.07	5.11	4.93
	HZ.test	4.09	3.39	3.79	3.23	3.44
	Ep.test	4.64	5.21	5.39	5.06	4.25
<u>n=50</u>	H.test	0.62	0.28	0.32	0.16	0.03
	H.new	5.14	4.75	4.87	4.87	5.04
	HZ.test	4.80	4.53	4.44	4.65	4.17
	Ep.test	5.52	4.64	5.09	4.85	4.48
<u>n=75</u>	H.test	1.01	0.62	0.56	0.77	0.22
	H.new	4.62	4.61	4.77	5.11	5.23
	HZ.test	4.57	4.35	4.82	4.82	4.88
	Ep.test	5.03	4.66	4.78	5.54	4.89
<u>n=100</u>	H.test	0.60	0.50	0.32	0.34	0.09
	H.new	4.94	4.93	5.04	4.75	5.05
	HZ.test	4.95	5.13	5.09	4.99	4.55
	Ep.test	4.98	5.17	5.17	5.06	5.15
<u>n=250</u>	H.test	0.01	0.00	0.00	0.00	0.00
	H.new	5.06	4.97	5.26	5.16	5.12
	HZ.test	4.72	4.84	5.06	5.10	5.85
	Ep.test	5.26	5.41	5.60	5.46	5.57

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

Table 2: Multivariate Normal Mixture 1

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.22	5.41	5.61	5.38	6.33
	HZ.test	3.75	3.88	3.68	3.41	3.71
	Ep.test	5.02	5.39	5.49	5.20	5.14
n=50	H.test	0.57	0.31	0.19	0.27	0.07
	H.new	5.69	5.40	5.22	5.82	6.89
	HZ.test	4.83	4.89	4.53	4.66	4.29
	Ep.test	5.55	5.87	5.21	6.03	6.29
n=75	H.test	1.15	0.81	1.01	0.98	0.77
	H.new	5.57	5.12	5.51	5.75	7.92
	HZ.test	4.95	4.50	4.72	5.32	5.06
	Ep.test	5.65	5.75	6.26	6.13	7.40
n=100	H.test	0.69	0.62	0.55	0.56	0.51
	H.new	5.12	5.42	5.53	6.02	7.86
	HZ.test	5.14	5.01	4.82	4.84	5.34
	Ep.test	5.40	5.83	6.06	6.68	8.18
n=250	H.test	0.00	0.02	0.01	0.00*	0.12**
	H.new	4.91	5.32	6.01	6.51*	11.84**
	HZ.test	4.80	5.24	5.27	5.12*	5.96**
	Ep.test	5.33	5.79	6.79	7.44*	11.84**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 3: Multivariate Normal Mixture 2

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.76	5.45	5.55	5.70	7.11
	HZ.test	4.13	3.74	3.77	3.77	3.52
	Ep.test	4.80	5.24	5.67	5.73	6.02
n=50	H.test	0.40	0.37	0.33	0.30	0.20
	H.new	5.25	5.65	5.91	6.54	8.35
	HZ.test	4.67	4.94	4.99	4.90	5.02
	Ep.test	5.40	6.06	6.22	6.93	7.92
n=75	H.test	1.16	1.09	0.93	0.86	1.06
	H.new	5.52	5.66	5.65	6.56	9.00
	HZ.test	5.16	5.00	5.14	5.09	5.61
	Ep.test	5.69	6.45	6.34	7.02	8.96
n=100	H.test	0.87	0.64	0.56	0.62	0.77
	H.new	5.40	5.42	5.57	6.71	9.76
	HZ.test	5.33	5.27	4.82	5.18	6.18
	Ep.test	5.88	6.47	7.12	8.10	9.80
n=250	H.test	0.02	0.02	0.03	0.07*	0.28**
	H.new	5.56	6.25	6.82	7.89*	14.68**
	HZ.test	5.15	5.36	5.68	6.15*	8.16**
	Ep.test	6.38	7.60	8.53	8.84*	14.36**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 4: Multivariate Normal Mixture 3

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.16	5.74	5.97	6.31	7.74
	HZ.test	4.87	4.03	3.98	3.93	3.80
	Ep.test	5.39	5.81	6.23	6.18	6.40
n=50	H.test	0.56	0.42	0.37	0.31	0.20
	H.new	5.52	5.61	6.51	6.66	8.67
	HZ.test	4.93	4.79	5.43	5.62	5.79
	Ep.test	5.81	6.65	7.39	8.08	9.10
n=75	H.test	1.21	1.33	1.24	1.12	0.89
	H.new	5.44	5.86	6.49	6.70	10.05
	HZ.test	5.11	5.21	5.07	6.17	7.08
	Ep.test	5.93	7.25	8.48	8.97	11.93
n=100	H.test	0.57	0.68	0.72	0.74	0.56
	H.new	5.74	6.02	6.77	7.57	11.01
	HZ.test	5.14	5.13	5.59	5.78	7.55
	Ep.test	6.51	7.58	8.97	10.46	13.12
n=250	H.test	0.02	0.03	0.01	0.05*	0.04**
	H.new	5.31	6.03	7.46	8.99*	15.44**
	HZ.test	5.24	6.10	6.28	8.16*	13.44**
	Ep.test	6.96	9.44	11.49	14.47*	19.52**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 5: Multivariate Normal Mixture 4

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.27	5.36	5.73	5.60	5.65
	HZ.test	4.12	4.02	3.98	3.59	3.43
	Ep.test	4.74	5.10	4.96	4.66	4.24
n=50	H.test	0.53	0.45	0.32	0.26	0.09
	H.new	5.47	5.79	6.46	5.81	6.60
	HZ.test	4.97	5.17	4.57	4.87	4.70
	Ep.test	5.26	5.51	5.45	4.77	5.00
n=75	H.test	1.50	1.22	0.92	0.99	0.46
	H.new	5.95	6.23	6.15	6.50	7.24
	HZ.test	5.70	5.53	5.34	5.48	5.05
	Ep.test	5.86	6.10	5.24	5.55	4.97
n=100	H.test	0.89	0.73	0.63	0.55	0.22
	H.new	6.36	5.99	6.27	7.04	7.68
	HZ.test	5.73	5.67	5.73	5.95	5.28
	Ep.test	5.84	5.96	5.59	5.46	5.43
n=250	H.test	0.09	0.01	0.02	0.03*	0.00**
	H.new	7.56	7.37	8.23	8.36*	11.76**
	HZ.test	5.80	6.38	7.24	6.96*	7.48**
	Ep.test	6.23	5.79	6.05	6.29*	4.92**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 6: Multivariate Normal Mixture 5

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.04	5.13	4.89	5.55	6.40
	HZ.test	4.16	3.60	3.37	3.25	3.46
	Ep.test	4.78	5.09	4.56	4.46	4.70
n=50	H.test	0.46	0.39	0.32	0.15	0.01
	H.new	5.32	5.59	5.33	5.61	6.26
	HZ.test	4.92	5.05	5.41	5.37	4.63
	Ep.test	5.08	5.23	5.04	5.09	5.00
n=75	H.test	1.32	1.00	0.92	0.64	0.38
	H.new	5.76	5.98	5.77	6.21	6.53
	HZ.test	5.49	5.43	5.41	5.56	4.85
	Ep.test	5.13	5.30	5.01	5.26	5.01
n=100	H.test	0.87	0.71	0.67	0.52	0.18
	H.new	5.72	6.09	6.46	6.36	7.52
	HZ.test	5.81	6.16	6.26	6.35	5.72
	Ep.test	5.34	5.50	5.62	5.17	5.51
n=250	H.test	0.02	0.04	0.01	0.00*	0.00**
	H.new	7.33	8.01	8.57	8.87*	12.36**
	HZ.test	7.49	7.51	8.66	8.21*	8.44**
	Ep.test	5.70	5.72	5.68	5.33*	6.24**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 7: Multivariate Normal Mixture 6

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
<u>n=25</u>	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.55	4.35	4.23	4.33	4.80
	HZ.test	3.91	3.46	3.11	3.08	3.11
	Ep.test	4.62	5.01	4.83	4.62	4.68
<u>n=50</u>	H.test	0.34	0.21	0.18	0.11	0.01
	H.new	4.46	4.04	4.34	4.08	4.27
	HZ.test	4.54	4.17	4.57	4.26	4.35
	Ep.test	4.61	4.52	4.79	4.69	4.90
<u>n=75</u>	H.test	0.63	0.63	0.57	0.35	0.13
	H.new	4.20	3.85	4.27	4.22	3.84
	HZ.test	4.34	4.57	5.52	4.87	4.80
	Ep.test	4.44	4.53	4.57	4.70	4.90
<u>n=100</u>	H.test	0.45	0.28	0.19	0.15	0.04
	H.new	4.22	3.83	3.95	4.11	3.79
	HZ.test	5.17	4.91	5.22	5.10	4.91
	Ep.test	4.10	4.47	4.33	5.06	4.86
<u>n=250</u>	H.test	0.00	0.00	0.00	0.00*	0.00**
	H.new	4.61	4.22	4.19	3.96*	3.88**
	HZ.test	5.43	5.54	6.55	6.08*	5.80**
	Ep.test	4.22	4.29	4.85	4.60*	5.12**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 8: Multivariate Normal Mixture 7

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.30	5.32	6.01	6.48	6.93
	HZ.test	4.16	3.92	4.14	3.55	3.95
	Ep.test	5.09	4.85	4.80	5.07	4.46
n=50	H.test	0.59	0.59	0.37	0.38	0.19
	H.new	5.73	6.25	6.60	7.20	8.49
	HZ.test	4.75	4.75	4.67	5.19	5.53
	Ep.test	5.63	5.59	5.38	5.03	5.07
n=75	H.test	1.65	1.40	1.55	1.34	0.88
	H.new	6.58	6.85	6.90	7.97	10.09
	HZ.test	5.35	5.36	5.71	5.77	6.91
	Ep.test	6.00	5.47	5.43	5.43	5.19
n=100	H.test	0.89	0.83	0.93	0.64	0.57
	H.new	6.32	7.12	7.96	8.16	10.89
	HZ.test	5.25	5.80	5.99	5.86	7.28
	Ep.test	5.49	6.18	5.99	5.43	5.45
n=250	H.test	0.04	0.07	0.06	0.03*	0.00**
	H.new	8.43	9.85	11.39	12.16*	20.84**
	HZ.test	5.80	6.53	6.93	8.19*	13.24**
	Ep.test	6.37	6.33	6.73	6.73*	6.36**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 9: Multivariate Normal Mixture 8

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.35	5.52	5.87	6.23	6.87
	HZ.test	4.26	3.87	4.00	3.66	3.69
	Ep.test	4.96	4.50	4.64	4.71	4.62
n=50	H.test	0.54	0.48	0.31	0.33	0.07
	H.new	5.65	6.08	6.37	7.01	8.11
	HZ.test	4.79	4.73	5.21	5.34	5.40
	Ep.test	5.18	5.04	5.48	5.07	4.57
n=75	H.test	1.45	1.31	1.05	1.01	0.64
	H.new	6.02	6.57	7.09	7.50	9.80
	HZ.test	5.33	5.39	5.56	6.07	6.99
	Ep.test	4.99	5.50	5.06	5.39	4.99
n=100	H.test	0.92	0.72	0.73	0.67	0.45
	H.new	6.13	6.96	7.39	7.97	10.95
	HZ.test	5.63	5.70	6.26	7.04	8.01
	Ep.test	5.29	5.46	5.40	5.30	4.97
n=250	H.test	0.03	0.02	0.02	0.04*	0.00**
	H.new	8.68	10.72	11.58	13.64*	23.92**
	HZ.test	6.75	7.44	8.50	10.23*	17.44**
	Ep.test	5.74	6.00	6.23	5.61*	4.80**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 10: Multivariate Normal Mixture 9

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.52	4.43	4.78	4.34	4.66
	HZ.test	3.98	3.67	3.67	3.12	3.30
	Ep.test	4.27	4.43	4.95	4.48	5.13
n=50	H.test	0.27	0.24	0.12	0.07	0.03
	H.new	4.24	4.27	4.72	3.90	4.41
	HZ.test	4.58	4.39	4.41	4.53	4.62
	Ep.test	4.55	4.48	4.77	4.37	5.15
n=75	H.test	0.84	0.69	0.47	0.37	0.16
	H.new	4.32	3.81	4.02	3.92	4.09
	HZ.test	4.45	4.62	4.93	4.88	5.86
	Ep.test	4.14	4.42	4.24	4.71	4.76
n=100	H.test	0.49	0.36	0.17	0.18	0.06
	H.new	4.44	4.02	4.16	4.41	4.24
	HZ.test	5.06	5.04	5.27	5.23	6.10
	Ep.test	4.17	4.63	4.63	5.20	4.40
n=250	H.test	0.00	0.00	0.00	0.00*	0.00**
	H.new	4.85	4.46	4.81	5.01*	5.60**
	HZ.test	5.68	5.75	6.49	6.88*	9.76**
	Ep.test	4.39	4.36	4.49	4.11*	4.48**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 11: Multivariate Normal Mixture 10

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.58	5.45	5.81	5.31	7.18
	HZ.test	4.29	3.65	3.85	3.52	3.60
	Ep.test	4.93	4.58	4.91	4.56	4.42
n=50	H.test	0.83	0.57	0.74	0.52	0.47
	H.new	6.00	6.31	6.34	7.04	8.59
	HZ.test	5.37	4.99	5.16	5.07	4.90
	Ep.test	5.21	5.03	4.89	4.95	5.39
n=75	H.test	1.70	1.75	1.65	1.80	1.75
	H.new	6.06	7.27	7.63	7.89	9.66
	HZ.test	5.58	5.78	5.87	5.51	5.66
	Ep.test	5.11	5.41	5.04	5.36	5.74
n=100	H.test	1.04	0.99	1.15	1.50	1.65
	H.new	6.34	7.11	7.93	9.11	11.57
	HZ.test	5.39	6.23	6.29	5.95	6.13
	Ep.test	5.77	5.51	5.68	5.73	6.25
n=250	H.test	0.14	0.17	0.27	0.45*	1.72**
	H.new	8.50	10.38	12.55	14.95*	24.08**
	HZ.test	6.40	7.64	7.82	8.09*	8.28**
	Ep.test	6.45	6.75	6.55	7.16*	9.20**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 12: Multivariate Normal Mixture 11

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.53	5.63	5.68	6.70	7.10
	HZ.test	4.28	3.95	4.12	3.90	3.87
	Ep.test	4.20	4.68	4.45	4.76	5.12
n=50	H.test	0.68	0.66	0.67	0.59	0.48
	H.new	5.89	6.71	7.75	7.70	10.23
	HZ.test	5.89	5.60	5.52	5.54	5.88
	Ep.test	4.71	5.06	5.71	5.11	6.14
n=75	H.test	2.07	1.60	2.04	1.76	2.19
	H.new	6.84	7.11	8.44	8.95	12.54
	HZ.test	6.40	6.62	6.60	6.61	7.57
	Ep.test	5.90	5.32	5.70	5.83	7.76
n=100	H.test	1.03	1.20	1.38	1.51	2.37
	H.new	7.05	8.11	9.32	10.72	15.57
	HZ.test	7.06	7.18	7.35	7.40	9.18
	Ep.test	5.53	5.69	6.20	6.76	8.42
n=250	H.test	0.08	0.24	0.41	0.59*	2.48**
	H.new	10.18	14.04	19.07	23.09*	39.92**
	HZ.test	9.69	11.27	12.62	13.00*	18.68**
	Ep.test	6.56	8.27	9.42	10.71*	15.28**

H.test – Royston (1983b)

HZ.test – Henze-Zirkler (1990)

H.new – Royston (1992)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 13: Multivariate Normal Mixture 12

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.98	6.13	6.74	6.71	9.46
	HZ.test	4.50	4.35	4.62	4.41	5.58
	Ep.test	5.34	5.91	6.18	6.09	6.18
n=50	H.test	0.35	0.37	0.50	0.46	0.25
	H.new	5.68	6.82	8.08	10.23	15.89
	HZ.test	5.20	6.30	7.30	8.18	11.34
	Ep.test	5.97	7.15	8.12	8.98	10.26
n=75	H.test	1.41	1.70	1.80	2.30	2.41
	H.new	6.31	8.39	10.12	13.23	23.51
	HZ.test	6.37	7.79	8.96	11.42	16.73
	Ep.test	6.91	8.93	10.04	11.57	13.49
n=100	H.test	0.90	1.15	1.50	1.49	2.40
	H.new	6.73	9.91	13.46	15.95	33.09
	HZ.test	6.52	8.52	11.34	13.74	24.98
	Ep.test	7.16	10.49	12.82	13.84	17.41
n=250	H.test	0.03	0.14	0.14	0.16*	0.64**
	H.new	11.69	21.24	31.21	42.48*	81.04**
	HZ.test	10.15	16.44	24.87	32.53*	67.48**
	Ep.test	13.54	22.75	28.02	33.25*	46.72**

H.test – Royston (1983b)

HZ.test – Henze-Zirkler (1990)

H.new – Royston (1992)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 14: Multivariate Normal Mixture 13

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
<u>n=25</u>	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.93	7.08	7.56	8.63	13.46
	HZ.test	4.30	4.64	4.56	4.73	4.66
	Ep.test	6.57	7.19	7.14	6.51	5.79
<u>n=50</u>	H.test	0.66	1.12	1.27	1.13	1.12
	H.new	6.67	9.00	10.79	12.30	21.05
	HZ.test	5.11	5.59	6.53	7.10	7.04
	Ep.test	7.84	9.51	10.48	10.26	9.70
<u>n=75</u>	H.test	2.63	3.10	3.83	4.46	6.11
	H.new	8.26	10.08	13.17	15.38	27.77
	HZ.test	5.57	6.31	7.45	7.79	9.44
	Ep.test	9.44	11.36	12.64	13.93	13.92
<u>n=100</u>	H.test	1.64	2.42	2.81	3.65	5.62
	H.new	7.76	10.95	14.69	17.99	36.75
	HZ.test	5.82	6.73	7.66	9.85	12.24
	Ep.test	9.90	13.29	14.41	16.54	18.42
<u>n=250</u>	H.test	0.31	0.65	0.83	1.16*	1.44**
	H.new	13.65	21.42	30.38	39.57*	72.84**
	HZ.test	7.37	9.76	12.41	15.76*	28.96**
	Ep.test	16.03	22.99	27.96	32.25*	40.20**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 15: Multivariate Normal Mixture 14

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.68	7.44	8.24	9.43	15.03
	HZ.test	4.54	4.99	4.81	5.00	5.30
	Ep.test	6.24	7.41	7.14	7.40	6.78
n=50	H.test	0.95	1.12	1.17	1.31	1.28
	H.new	7.46	9.43	11.90	14.17	27.53
	HZ.test	5.40	6.65	8.28	8.77	10.56
	Ep.test	8.40	10.11	11.31	11.79	11.75
n=75	H.test	2.16	3.30	4.31	4.91	7.91
	H.new	8.09	12.01	16.04	20.30	39.60
	HZ.test	6.07	8.56	10.20	12.02	16.33
	Ep.test	9.74	12.69	15.07	15.91	17.28
n=100	H.test	2.00	2.80	3.91	4.28	8.19
	H.new	9.68	14.29	20.37	24.59	51.33
	HZ.test	7.20	9.35	11.57	14.97	23.02
	Ep.test	11.36	15.15	18.15	19.76	22.28
n=250	H.test	0.32	0.71	1.24	1.55*	3.16**
	H.new	17.53	31.05	45.32	58.20*	93.08**
	HZ.test	9.99	16.20	23.82	32.65*	64.56**
	Ep.test	20.24	31.70	38.48	43.35*	55.20**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 16: Multivariate Normal Mixture 15

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.96	5.55	6.54	7.14	9.63
	HZ.test	4.41	4.50	4.76	4.89	5.19
	Ep.test	4.88	6.04	6.14	6.99	6.45
n=50	H.test	0.47	0.56	0.35	0.48	0.24
	H.new	5.70	6.71	8.36	9.97	16.33
	HZ.test	5.75	6.23	7.61	8.44	10.87
	Ep.test	5.88	7.16	8.64	9.21	10.39
n=75	H.test	1.24	1.42	2.00	2.22	2.65
	H.new	6.25	8.04	10.87	12.24	24.05
	HZ.test	5.82	7.66	9.52	11.15	17.58
	Ep.test	6.75	8.50	10.06	11.05	13.45
n=100	H.test	1.01	1.42	1.37	1.60	2.14
	H.new	6.33	9.93	12.86	16.01	32.93
	HZ.test	6.58	9.23	11.45	13.83	24.20
	Ep.test	7.10	10.51	12.28	13.86	18.31
n=250	H.test	0.04	0.09	0.20	0.12*	0.60**
	H.new	11.71	20.01	31.18	42.61*	80.68**
	HZ.test	9.93	16.31	24.22	32.60*	68.28**
	Ep.test	13.66	21.73	28.63	32.53*	45.64**

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

**r = 2500

Table 17: Multivariate χ^2 ($df=1$)

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.43	0.36	0.26	0.15	0.04
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	99.71	99.87	99.93	99.88	95.28
	Ep.test	99.95	100.00	100.00	100.00	100.00
n=50	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=75	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=100	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=250	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

Table 18: Multivariate Cauchy (Pearson Type VII, $\nu=1$)

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
<u>n=25</u>	H.test	29.06	31.00	30.74	31.75	34.18
	H.new	97.92	99.01	99.52	99.78	99.96
	HZ.test	98.89	99.68	99.86	99.90	99.89
	Ep.test	97.50	98.58	98.20	97.37	79.63
<u>n=50</u>	H.test	99.91	100.00	99.99	100.00	100.00
	H.new	99.98	100.00	100.00	100.00	100.00
	HZ.test	99.99	100.00	100.00	100.00	100.00
	Ep.test	99.98	100.00	100.00	100.00	100.00
<u>n=75</u>	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
<u>n=100</u>	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
<u>n=250</u>	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

Table 19: Multivariate Lognormal

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	3.17	2.30	2.36	1.84	1.50
	H.new	99.89	100.00	100.00	100.00	100.00
	HZ.test	98.95	99.58	99.77	99.75	97.03
	Ep.test	99.71	99.97	100.00	100.00	100.00
n=50	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=75	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=100	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=250	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	100.00
	Ep.test	100.00	100.00	100.00	100.00	100.00

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

Table 20: *Khintchine*

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	5.17	5.17	5.02	5.80	6.17
	HZ.test	9.20	17.10	28.48	38.71	64.05
	Ep.test	4.38	2.90	2.31	1.84	0.36
n=50	H.test	0.44	0.37	0.45	0.27	0.15
	H.new	4.87	5.15	5.26	5.13	6.64
	HZ.test	14.32	35.02	60.39	78.09	98.92
	Ep.test	4.23	4.10	2.86	2.12	0.70
n=75	H.test	1.18	0.81	0.72	0.67	0.63
	H.new	5.28	5.08	5.39	4.87	6.44
	HZ.test	19.36	51.81	82.13	94.59	99.95
	Ep.test	4.75	3.88	3.88	2.56	1.02
n=100	H.test	0.72	0.44	0.47	0.35	0.16
	H.new	5.00	5.25	5.05	5.40	5.78
	HZ.test	24.16	67.88	92.83	98.87	100.00
	Ep.test	4.77	4.16	4.26	3.59	1.10
n=250	H.test	0.03	0.00	0.00	0.00	0.02
	H.new	5.02	4.92	5.33	5.42	6.35
	HZ.test	61.19	98.80	99.98	100.00	100.00
	Ep.test	4.78	4.90	4.83	4.78	3.30

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

Table 21: Generalized Exponential Power

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	90.12	97.49	99.28	99.81	100.00
	HZ.test	61.75	58.48	52.90	45.47	6.56
	Ep.test	71.67	82.38	88.19	91.16	94.07
n=50	H.test	97.57	99.75	99.97	100.00	100.00
	H.new	99.97	100.00	100.00	100.00	100.00
	HZ.test	95.83	96.47	95.08	93.22	64.95
	Ep.test	99.33	99.96	100.00	100.00	100.00
n=75	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	99.75	99.83	99.74	99.66	93.34
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=100	H.test	100.00	100.00	100.00	100.00	100.00
	H.new	100.00	100.00	100.00	100.00	100.00
	HZ.test	100.00	100.00	100.00	100.00	99.25
	Ep.test	100.00	100.00	100.00	100.00	100.00
n=250	H.test	100.00	100.00	100.00	100.00	100.00*
	H.new	100.00	100.00	100.00	100.00	100.00*
	HZ.test	100.00	100.00	100.00	100.00	100.00*
	Ep.test	100.00	100.00	100.00	100.00	100.00*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 22: Multivariate Uniform (Pearson Type II, $m = 0$)

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	13.10	7.73	5.02	3.95	2.13
	HZ.test	17.49	12.97	9.14	7.20	3.98
	Ep.test	3.36	1.35	0.83	0.75	1.75
n=50	H.test	4.11	0.65	0.07	0.02	0.00
	H.new	50.21	26.82	15.50	9.47	2.72
	HZ.test	64.65	62.93	58.44	54.08	32.34
	Ep.test	20.25	6.62	2.39	1.08	0.31
n=75	H.test	38.13	10.97	3.14	0.92	0.00
	H.new	84.28	57.23	34.72	21.64	4.74
	HZ.test	89.44	90.97	90.20	88.80	79.49
	Ep.test	53.53	25.03	10.25	4.52	0.35
n=100	H.test	55.42	15.27	3.21	0.75	0.01
	H.new	97.40	83.54	59.71	39.64	7.30
	HZ.test	97.54	98.39	98.30	98.24	97.33
	Ep.test	84.30	55.78	28.89	14.01	0.84
n=250	H.test	92.82	21.58	1.10	0.07	0.00*
	H.new	100.00	100.00	99.97	99.52	51.36*
	HZ.test	100.00	100.00	100.00	100.00	100.00*
	Ep.test	100.00	100.00	99.84	98.18	32.60*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 23: Pearson Type II ($m = -0.5$)

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	42.75	15.83	8.56	5.04	2.36
	HZ.test	54.24	34.66	20.52	18.83	4.64
	Ep.test	14.87	3.07	0.96	0.65	1.46
n=50	H.test	41.47	3.76	0.48	0.05	0.00
	H.new	96.90	65.07	33.08	17.45	3.54
	HZ.test	97.87	95.83	92.20	85.22	47.78
	Ep.test	70.20	23.44	6.72	2.22	0.37
n=75	H.test	98.24	50.44	11.66	3.00	0.02
	H.new	99.99	96.20	69.10	41.24	5.53
	HZ.test	99.97	99.93	99.85	99.55	94.71
	Ep.test	97.81	69.01	28.92	10.71	0.43
n=100	H.test	99.92	71.83	17.50	3.14	0.00
	H.new	100.00	99.91	92.37	69.00	10.38
	HZ.test	100.00	100.00	100.00	100.00	99.77
	Ep.test	99.93	95.13	65.45	32.10	1.24
n=250	H.test	100.00	99.07	25.24	0.96	0.00*
	H.new	100.00	100.00	100.00	100.00	66.03*
	HZ.test	100.00	100.00	100.00	100.00	100.00*
	Ep.test	100.00	100.00	100.00	99.99	46.83*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 24: Pearson Type II ($m = -0.25$)

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	22.43	10.21	6.09	4.61	2.14
	HZ.test	29.94	19.78	13.18	9.46	4.04
	Ep.test	6.55	1.81	0.88	0.65	1.43
n=50	H.test	12.00	1.17	0.21	0.05	0.00
	H.new	74.70	40.39	22.17	12.80	2.75
	HZ.test	84.47	81.63	76.56	69.55	38.88
	Ep.test	37.56	12.05	4.20	1.53	0.32
n=75	H.test	73.14	23.16	5.39	1.56	0.02
	H.new	97.93	78.70	48.81	29.51	5.40
	HZ.test	98.52	98.08	97.46	96.28	88.52
	Ep.test	80.82	43.17	16.02	6.74	0.39
n=100	H.test	90.74	34.86	7.06	1.40	0.01
	H.new	99.97	96.72	77.01	51.76	8.62
	HZ.test	99.91	99.91	99.81	99.80	99.08
	Ep.test	97.41	77.20	44.13	20.01	0.98
n=250	H.test	99.99	67.51	5.84	0.18	0.00*
	H.new	100.00	100.00	100.00	99.97	58.55*
	HZ.test	100.00	100.00	100.00	100.00	100.00*
	Ep.test	100.00	100.00	100.00	99.68	38.57*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 25: Pearson Type II ($m = 0.5$)

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	6.88	4.94	3.96	2.93	2.37
	HZ.test	8.42	6.40	5.75	4.83	3.48
	Ep.test	1.54	0.99	0.87	0.72	1.95
n=50	H.test	0.91	0.17	0.06	0.01	0.01
	H.new	20.33	13.28	8.80	6.45	2.65
	HZ.test	33.25	34.53	33.82	32.02	23.03
	Ep.test	6.36	2.47	1.39	0.86	0.54
n=75	H.test	9.61	3.37	1.05	0.54	0.02
	H.new	44.25	27.64	19.08	13.11	4.03
	HZ.test	57.92	62.95	65.72	65.89	61.55
	Ep.test	19.82	9.63	4.51	2.47	0.53
n=100	H.test	13.04	3.33	1.14	0.30	0.01
	H.new	68.00	49.04	33.85	23.11	6.41
	HZ.test	76.25	81.76	85.09	87.04	88.63
	Ep.test	42.37	24.41	12.89	6.82	0.82
n=250	H.test	17.74	1.41	0.12	0.00	0.00*
	H.new	100.00	99.89	98.20	92.18	38.44*
	HZ.test	99.87	99.97	99.99	100.00	100.00*
	Ep.test	99.83	99.18	95.86	86.29	21.71*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 26: Pearson Type II ($m = 1$)

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.57	3.99	3.00	2.77	2.05
	HZ.test	5.28	4.80	4.43	3.53	3.11
	Ep.test	1.28	1.20	0.81	0.88	2.15
n=50	H.test	0.32	0.10	0.04	0.01	0.00
	H.new	11.16	8.03	5.98	4.70	2.61
	HZ.test	18.84	20.36	21.56	22.11	18.19
	Ep.test	2.94	1.71	0.92	0.72	0.56
n=75	H.test	3.55	1.25	0.60	0.28	0.00
	H.new	22.35	16.25	11.30	8.37	3.49
	HZ.test	34.25	39.67	42.52	45.87	48.96
	Ep.test	8.62	4.57	2.51	1.72	0.43
n=100	H.test	3.66	1.31	0.36	0.15	0.01
	H.new	38.10	27.71	19.76	15.09	5.25
	HZ.test	49.02	57.91	64.20	68.00	76.19
	Ep.test	19.33	11.67	6.40	3.80	0.78
n=250	H.test	1.83	0.16	0.01	0.00	0.00*
	H.new	98.26	94.25	86.44	76.29	29.35*
	HZ.test	95.44	98.88	99.48	99.80	100.00*
	Ep.test	95.12	89.93	79.73	66.66	15.23*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 27: Pearson Type II ($m = 2$)

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	3.72	2.89	2.53	2.86	2.28
	HZ.test	3.64	3.46	2.95	2.90	3.09
	Ep.test	1.33	1.17	1.19	1.16	2.34
n=50	H.test	0.09	0.02	0.03	0.00	0.00
	H.new	5.72	4.62	3.98	3.86	2.55
	HZ.test	9.09	10.70	11.04	11.62	11.85
	Ep.test	1.40	1.03	0.89	0.77	0.66
n=75	H.test	1.01	0.52	0.40	0.15	0.03
	H.new	8.74	6.99	6.47	5.06	2.68
	HZ.test	15.30	19.13	21.74	24.32	30.33
	Ep.test	2.90	1.85	1.56	1.09	0.56
n=100	H.test	0.79	0.31	0.21	0.13	0.01
	H.new	13.81	17.70	9.48	7.66	3.83
	HZ.test	22.54	27.80	33.94	37.79	51.86
	Ep.test	5.76	3.82	2.79	2.06	0.55
n=250	H.test	0.07	0.00	0.00	0.00	0.00*
	H.new	62.25	57.68	49.51	42.47	17.75*
	HZ.test	60.96	76.95	86.32	92.21	99.07*
	Ep.test	53.58	48.93	39.59	32.18	8.12*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 28: Pearson Type II ($m = 4$)

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	3.31	2.78	2.92	2.82	2.27
	HZ.test	3.14	2.86	2.75	2.62	2.75
	Ep.test	1.66	1.47	1.78	1.55	2.51
n=50	H.test	0.06	0.11	0.02	0.02	0.00
	H.new	3.57	3.44	2.86	2.63	2.41
	HZ.test	5.50	5.78	5.75	6.20	8.27
	Ep.test	1.33	1.17	0.81	0.85	0.94
n=75	H.test	0.43	0.26	0.20	0.21	0.04
	H.new	4.06	3.84	3.99	3.18	2.53
	HZ.test	7.64	8.60	9.53	11.09	15.64
	Ep.test	1.41	1.21	1.02	1.03	0.71
n=100	H.test	0.27	0.18	0.10	0.06	0.00
	H.new	5.40	5.17	4.48	3.87	3.05
	HZ.test	9.49	11.38	14.38	16.49	25.16
	Ep.test	1.98	2.06	1.28	1.00	0.67
n=250	H.test	0.00	0.00	0.00	0.00	0.00*
	H.new	18.39	17.74	16.28	14.15	8.57*
	HZ.test	22.56	32.55	42.54	51.95	80.51*
	Ep.test	13.38	12.46	10.75	8.53	3.47*

H.test – Royston (1983b)

H.new – Royston (1992)

HZ.test – Henze-Zirkler (1990)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 29: Pearson Type II ($m = 10$)

		p=2	p=3	p=4	p=5	p=10
n=25	H.test	0.00	0.00	0.00	0.00	0.00
	H.new	4.05	3.68	3.40	3.39	3.03
	HZ.test	3.11	3.01	3.30	2.72	2.96
	Ep.test	2.98	2.82	2.39	2.94	3.58
n=50	H.test	0.12	0.08	0.08	0.03	0.00
	H.new	3.57	3.34	2.96	2.78	2.72
	HZ.test	4.22	4.21	4.11	4.45	5.19
	Ep.test	2.25	2.01	1.77	1.55	1.95
n=75	H.test	0.50	0.30	0.19	0.13	0.02
	H.new	3.27	3.10	2.70	2.85	2.39
	HZ.test	5.04	5.18	5.38	5.84	6.64
	Ep.test	1.75	1.59	1.28	1.41	1.34
n=100	H.test	0.23	0.12	0.04	0.08	0.04
	H.new	3.57	3.39	2.88	2.95	2.70
	HZ.test	5.03	6.22	6.21	6.44	9.13
	Ep.test	1.99	1.66	1.47	1.48	1.36
n=250	H.test	0.00	0.00	0.00	0.00	0.00*
	H.new	4.70	4.55	4.35	4.45	3.55*
	HZ.test	7.42	9.56	11.53	13.18	25.12*
	Ep.test	2.79	2.59	2.56	2.34	1.41*

H.test – Royston (1983b)

HZ.test – Henze-Zirkler (1990)

H.new – Royston (1992)

Ep.test – Doornik-Hansen (1994)

r = 10000

*r = 7500

Table 30: Multivariate t ($df=10$)

		<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=10</u>
<u>n=25</u>	H.test	0.000	0.000	0.000	0.000	0.000
	H.new	13.89	16.26	18.06	19.60	27.19
	HZ.test	10.28	11.51	12.37	12.40	10.98
	Ep.test	15.35	16.21	15.83	15.46	10.32
<u>n=50</u>	H.test	7.16	8.16	8.07	9.01	10.35
	H.new	20.88	25.22	27.97	32.13	46.24
	HZ.test	14.80	18.52	21.33	25.61	34.45
	Ep.test	26.53	30.35	32.42	34.25	34.59
<u>n=75</u>	H.test	14.99	17.66	19.90	21.61	27.99
	H.new	26.98	32.88	37.71	42.14	58.96
	HZ.test	18.03	23.83	30.25	35.92	56.64
	Ep.test	33.93	40.36	46.10	48.72	55.91
<u>n=100</u>	H.test	15.82	17.93	20.79	23.24	30.86
	H.new	33.33	39.44	46.41	52.48	69.65
	HZ.test	21.87	29.48	38.02	47.30	73.87
	Ep.test	42.04	49.31	55.80	61.22	71.51
<u>n=250</u>	H.test	11.97	14.72	17.16	18.33	24.96
	H.new	59.35	70.91	79.43	84.50	96.37
	HZ.test	40.62	60.26	75.51	86.22	99.63
	Ep.test	70.88	81.67	87.96	92.41	98.48

H.test – Royston (1983b)

HZ.test – Henze-Zirkler (1990)

H.new – Royston (1995)

Ep.test – Doornik-Hansen (1994)

 $r = 10000$

Appendix B: Graphs

Figure 1a:

Multivariate t ($df=10$) Marginal ($n=100$, $p=2$, $rep=10000$) – Generated Using Johnson (1987)

Univariate t ($df=10$) ($n=1000000$)

Figure 1b:

Multivariate t ($df=10$) Marginal ($n=100$, $p=2$, $rep=10000$) – Generated using Johnson and Kotz (1972)

Univariate t ($df=10$) ($n=1000000$)

Figure 1c:

Multivariate t ($df=10$, $n=100$, $p=2$)

Figure 2a:

Multivariate Uniform ($n=25, p=2$)

Multivariate Normal ($n=25, p=2$)

Figure 2b:*Multivariate Uniform ($n=25$, $p=10$)* *Multivariate Normal ($n=25$, $p=10$)*

Figure 2c:*Multivariate Uniform ($n=75, p=2$)* *Multivariate Normal ($n=75, p=2$)*

Appendix C: Computer Programs

Program 1: *H.test*

```

"H.test"<-function(data)
{
  ## This function finds the equivalent degrees of freedom and calculates the Royston (1983) test
  statistic
  ## data is the data set being analyzed

  d <- dim(data)
  n <- d[1]
  p <- d[2]
  a <- find.coeff(n)
  b <- test.shapiro(data, n, p)
  z <- r.trans( n, p, a, b)
  r <- Roy( n, p, z)
  u <- 0.715
  v <- 0.21364+0.015124*((log(n))^2)-0.0018034*((log(n))^3)
  la <- 5
  corr.data <- cor(data)
  new.corr <- ((corr.data^la) * (1 - (u * (1 - corr.data)^
    u)/v))
  total <- sum(new.corr) - p
  avg.corr <- total/(p^2 - p)
  est.e <- p/(1 + (p - 1) * avg.corr)
  ## equivalent degrees of freedom
  ans <- list(e = est.e)
  ans$H <- (est.e * (sum(r)))/p
  ans$r <- r
  ans$z <- z
  ans$b <- b
  ans$p.value <- 1 - (pchisq(ans$H, df = est.e))
  return(ans)
}

```

find.coeff

```

"find.coeff"<-function(n)
{
  ## This function finds the coefficients used to obtain the values after applying Royston's
  normalizing transformation.
  ## n is the number of observations
  ## p is the number of variables

  if(n <= 20) {
    x = log(n) - 3
  }
  else if(n >= 21 && n <= 2000) {
    x = log(n) - 5
  }
  else {
    stop("n is too large, 7 >= n <= 2000")
  }
  if(n >= 7 && n <= 20) {

```

```

        lambda = 0.118898 + (0.133414) * (x) + (0.327907) * (x^2)
    }
    else if(n >= 21 && n <= 2000) {
        lambda = 0.480385 + (0.318828) * (x) + (-0.0241665) * (x^3) +
            (0.00879701) * (x^4) + (0.002989646) * (x^5)
    }
    else {
        stop("n is not in the proper size range")
    }
    ret <- list(lambda = lambda)
    if(n >= 7 && n <= 20) {
        mu = exp((-0.37542) + (-0.492145) * (x) + (-1.124332) * (x^
            2) + (-0.199422) * (x^3))
    }
    else if(n >= 21 && n <= 2000) {
        mu = exp((-1.91487) + (-1.37888) * (x) + (-0.04183209) * (
            x^2) + (0.1066339) * (x^3) + (0.03513666) * (x^4) +
            (0.01504614) * (x^5))
    }
    else {
        stop("n is not in the proper size range")
    }
    ret$mu <- mu
    if(n >= 7 && n <= 20) {
        sigma = exp((-3.15805) + (0.729399) * (x) + (3.01855) * (x^
            2) + (1.558776) * (x^3))
    }
    else if(n >= 21 && n <= 2000) {
        sigma = exp((-3.73538) + (-1.015807) * (x) + (-0.331885) * (
            x^2) + (0.1773538) * (x^3) + (-0.01638782) * (x^4) +
            (-0.03215018) * (x^5) + (0.003852646) * (x^6))
    }
    else {
        stop("n is not in the proper size range")
    }
    ret$sigma <- sigma
    return(ret)
}

```

test.shapiro

```

"test.shapiro"<-function(data, n, p)
{
    ## This function finds the values of Zj after the Royston transformation is applied
    ## data is the data file being analysed, it is in matrix form
    ## n is the number of observations
    ## p is the number of variables

    W <- rep(0, p)

    for(i in 1:p) {
        W[i] <- shapiro.test(data[, i])$statistic
    }
    return(W)
}

```

r.trans

```
"r.trans"<-function( n, p, a, b)
{
  ## This function normalizes the Shapiro-Wilks statistic under Royston's Transformation
  ## data is the data set used, in matrix form
  ## n is the number of observations
  ## p is the number of variables

  Z <- rep(0, p)
  for(i in 1:p) {
    Z[i] <- (((1 - b[i])^a$lambda) - a$mu)/a$sigma)
  }
  return(Z)
}
```

Roy

```
"Roy"<-function( n, p, z)
{
  ## This function finds the statistics to be compared for multivariate normality
  ## data is the data set to be analysed
  ## n is the number of observations
  ## p is the number of variables

  M <- rep(0,p)
  G <- rep(0, p)

  for (i in 1:p){
    M[i]<-pnorm( - z[i])

    if (M[i] == 0)
    {
      M[i]<- 0.00000000000000000001
    }
    else {
      M[i]<-M[i]
    }

    G[i] <- (qnorm((M[i])/2))^2
  }

  return(G)
}
```


Program 2: H.new

```

"H.new"<-function(data)
{
  ## This function finds the equivalent degrees of freedom and calculates the Royston (1992) test
  statistic
  ## data is the data set being analyzed

  d <- dim(data)
  n <- d[1]
  p <- d[2]
  a <- find.coeff.new(n)
  b <- test.shapiro(data, n, p)
  z <- r.trans.new( n, p, a, b)
  r <- Royston( n, p, z)
  u <- 0.715
  v <- 0.21364+0.015124*((log(n))^2)-0.0018034*((log(n))^3)
  la <- 5
  corr.data <- cor(data)
  new.corr <- ((corr.data^la) * (1 - (u * (1 - corr.data)^
    u)/v))
  total <- sum(new.corr) - p
  avg.corr <- total/(p^2 - p)
  est.e <- p/(1 + (p - 1) * avg.corr)
  ## equivalent degrees of freedom
  ans <- list(e = est.e)
  ans$H <- (est.e * (sum(r)))/p
  ans$r <- r
  ans$z <- z
  ans$b <- b
  ans$p.value <- 1 - (pchisq(ans$H, df = est.e))
  return(ans)
}

```

find.coeff.new

```

"find.coeff.new"<-function(n)
{
  ## This function finds the coefficients used to obtain the values after applying Royston's
  normalizing transformation.
  ## n is the number of observations
  ## p is the number of variables
  if(n <= 3) {
    stop("n is too small!")
  }
  else if(n >= 4 && n <= 11) {
    x = n
  }
  else if (n >= 12 && n <= 2000){
    x = log(n)
  }
  else{
    stop("n is too large!")
  }
}

```

```

if(n >= 4 && n <= 11) {
  gamma = -2.273 + 0.459 * (x)
}
else if(n >= 12 && n <= 2000) {
  gamma = 0

}
else {
  stop("n is not in the proper size range")
}
ret <- list(gamma = gamma)
if(n >= 4 && n <= 11) {
  mu = 0.5440 - 0.39978 * (x) + 0.025054 * (x^2) - 0.0006714 * (x^3)
}
else if(n >= 12 && n <= 2000) {
  mu = -1.5861 - 0.31082 * (x) - 0.083751 * (x^2) + 0.0038915 * (x^3)
}
else {
  stop("n is not in the proper size range")
}
ret$mu <- mu
if(n >= 4 && n <= 11) {
  sigma = exp(1.3822 - 0.77857 * (x) + 0.062767 * (x^2) - 0.0020322 * (x^3))
}
else if(n >= 12 && n <= 2000) {
  sigma = exp(-0.4803 - 0.082676 * (x) + 0.0030302 * (x^2))
}
else {
  stop("n is not in the proper size range")
}
ret$sigma <- sigma
return(ret)
}

```

test.shapiro

```

"test.shapiro"<-function(data, n, p)
{
  ## This function finds the values of Zj after the Royston transformation is applied
  ## data is the data file being analyzed, it is in matrix form
  ## n is the number of observations
  ## is the number of variables

  W <- rep(0, p)

  for(i in 1:p) {
    W[i] <- shapiro.test(data[, i])$statistic
  }
  return(W)
}

```

r.trans.new

```

"r.trans.new"<-function( n, p, a, b)
{
  ## This function normalized the Shapiro-Wilks statistic under Royston's Transformation
  ## data is the data set used, in matrix form
  ## n is the number of observations
  ## p is the number of variables

  Z <- rep(0, p)
  if (n >= 4 && n <= 11)
  {
    for(i in 1:p) {
      Z[i] <- ((-log(a$gamma-(log(1-b[i]))))-a$mu)/a$sigma)
    }
  }
  else if (n >= 12 && n <= 2000)
  {
    for (i in 1:p){
      Z[i] <- (((log(1-b[i])) + a$gamma - a$mu)/ a$sigma)
    }
  }
  else {
    stop ("WARNING : n is not in the proper range")
  }
  return(Z)
}

```

Royston

```

"Royston"<-function( n, p, z)
{
  ## This function finds the statistics to be compared for multivariate normality
  ## data is the data set to be analysed
  ## n is the number of observations
  ## p is the number of variables

  G <- rep(0, p)
  for(i in 1:p) {
    G[i] <- (qnorm((pnorm( - z[i]))/2))^2
  }
  return(G)
}

```

Program 3: HZ.test

```

"HZ.test"<-function(data, default.r=0.05)
{
  ## This function calculates the Henze-Zirkler test statistic for assessing multivariate normality of a
  ## data set to be analyzed.
  ## data is the data set to be analyzed.
  ## r is the desired alpha level of significance.

  r <- default.r
  d <- dim(data)
  n <- d[1]
  p <- d[2]

  Beta <- 1/(sqrt(2)) * ((2 * p + 1)/4)^(1/(p + 4)) * (n^(1/(p + 4)))

  a <- var(data)
  m <- dim(a)[1]
  ranka <- qr(a)$rank

  if( ranka == m){

    dyn.open('c:\\loop4.dll')
    s <- loop5(data)
    square <- Square(data)

    T.beta <- n * (1/(n^2) * (sum(exp( - (Beta^2)/2 * s))) - 2 * ((1 + (
      Beta^2))^( - p/2)) * (1/n) * (sum(exp( - ((Beta^2)/(2 * (1 +
      (Beta^2)))) * square))) + ((1 + (2 * (Beta^2)))^( - p/2)))
    }
    else
    {
      T.beta <- n * 4
    }
    ans <- list(T = T.beta)
    mu <- 1 - ((1 + 2 * (Beta^2))^( - p/2)) * (1 + (p * (Beta^2))/(1 + 2 *
      (Beta^2)) + (p * (p + 2) * (Beta^4))/(2 * (1 + 2 * (Beta^2))^
      2))

    w.beta <- (1 + (Beta^2)) * (1 + 3 * (Beta^2))
    sigma.sq <- 2 * (1 + 4 * (Beta^2))^( - p/2) + 2 * (1 + 2 * (Beta^2))^(
      - p) * (1 + (2 * p * (Beta^4))/(1 + (2 * (Beta^2)))^2 + (
      3 * p * (p + 2) * (Beta^8))/(4 * (1 + 2 * (Beta^2))^4)) - 4 *
      (w.beta^( - p/2)) * (1 + (3 * p * (Beta^4))/(2 * w.beta) + (
      p * (p + 2) * (Beta^8))/(2 * (w.beta)^2))

    q.beta <- mu * (1 + (sigma.sq/mu^2))^( - 1/2) * exp(qnorm(1 - r) * sqrt(
      log(1 + sigma.sq/(mu^2))))
    ans$q <- (q = q.beta)
    p.mu <- log(sqrt((mu^4)/(sigma.sq + (mu^2))))

    p.sigma <- sqrt(log((sigma.sq + (mu^2))/(mu^2)))
  }
}

```

```

ans$p.value<- 1 - (pnorm(T.beta,p.mu,p.sigma))
ans$k<-(log(T.beta)-p.mu)/p.sigma
ans$p.mu<-p.mu
ans$p.sigma<-p.sigma

return(ans)
}

```

loop5

```

"loop5" <-function(data)
{
  n<- dim(data)[1]
  x<-data
  s.inv <- solve(var(x))
  X<-X.2<-data %*% s.inv %*% t(data)
  diag(X) <- 0
  tmp <- .C("loop4", X=as.double(X),as.double(X.2), as.integer(n))
  X <- matrix(tmp$X, n, n)
  return(X)
}

```

Square

```

"Square"<-function(data)
{
  S <- var(data)
  S.inv <- solve(S)
  x <- data
  x.diff <- scale(x, scale = F)
  square <- diag(x.diff %*% S.inv %*% t(x.diff))
  return(square)
}

```

Program 4: Ep.test

```
"Ep.test"<-function(data)
{
  ## This function calculates the Doornik-Hansen (1994) test statistic
  ## data is the data set to be analyzed

  trans<-transform(data)
  d<-dim(trans)
  n<-d[1]
  p<-d[2]
  z1<-find.z1(n,trans)
  z2<-find.z2(n,trans)
  Ep<-(sum(t(z1)*(z1)))+(sum(t(z2)*(z2)))
  ans<-list(Ep = Ep)
  d.f<-2*p
  ans$p.value<- 1-(pchisq(Ep,df=d.f))
  return(ans)
}
```

transform

```
"transform"<-function(data)
{
  x<-scale(data, scale=F)
  x<-t(x)
  s<-var(data)
  v<-diag(s)^(-1/2)
  v <- diag(v)
  c.data<-v %*% s %*% v
  values<- (tmp <- eigen(c.data, symmetric=T))$values
  vectors<-tmp$vectors
  lambda<-diag(values^(-1/2))
  trans<- vectors %*% lambda %*% t(vectors) %*% v %*% x
  return(t(trans))
}
```

find.z1

```
"find.z1"<-function(n,trans)
{
  skew <-get.skew(trans)
  B<-(3.*((n^2)+27.*n-70.)*(n+1.)*(n+3.))/((n-2.)*(n+5.)*(n+7.)*(n+9.))
  w.2<-(-1+((2*(B-1))^(1/2)))
  delta<-1/((log(sqrt(w.2)))^(1/2))
  y<-skew * (((w.2-1)/2)*(((n+1)*(n+3))/(6*(n-2))))^(1/2))
  z1<-delta*log(y+(((y^2)+1)^(1/2)))
  return(z1)
}
```

get.skew

```
"get.skew"<-function(data)
{
  d<- dim(data)
  n<-d[1]
  p<-d[2]
  m2<-get.m2(data)
  m3<-get.m3(data)

  b1<-rep(0,p)
  for(i in 1:p)
  {
    b1[i]<-m3[i]/(m2[i]^(3/2))
  }
  return(b1)
}
```

get.m2

```
"get.m2"<-function(data)
{
  d<-dim(data)
  n<-d[1]
  p<-d[2]
  mean.data<-apply(data,2,mean)
  m2<-rep(0,p)
  for(i in 1:p)
  {
    m2[i]<-(sum((data[,i]-mean.data[i])^2))/n
  }
  return(m2)
}
```

get.m3

```
"get.m3"<-function(data)
{
  d<-dim(data)
  n<-d[1]
  p<-d[2]
  mean.data<-apply(data,2,mean)

  m3<-rep(0,p)
  for(i in 1:p)
  {
    m3[i]<-(sum((data[,i]-mean.data[i])^3))/n
  }
  return(m3)
}
```

find.z2

```
"find.z2"<-function(n,trans)
{
  skew<-get.skew(trans)
  kurt<-get.kurt(trans)
  delta<-(n-3)*(n+1)*((n^2)+(15*n)-4)
  a<-(((n-2)*(n+5)*(n+7)*((n^2)+(27*n)-70)))/(6*delta)
  c<-(((n-7)*(n+5)*(n+7)*((n^2)+(2*n)-5)))/(6*delta)
  k<-(((n+5)*(n+7)*((n^3)+(37*(n^2)))+(11*n)-313))/(12*delta)
  alpha<-a + (skew^2)*c
  chi<-(kurt-1-(skew^2))*2*k
  z2<-(((chi/(2*alpha))^(1/3))-1+(1/(9*alpha)))*((9*alpha)^(1/2)))
}
```

get.kurt

```
"get.kurt"<-function(data)
{
  d<- dim(data)
  n<-d[1]
  p<-d[2]
  m2<-get.m2(data)
  m4<-get.m4(data)
  b2<-rep(0,p)
  for(i in 1:p)
  {
    b2[i]<-m4[i]/(m2[i]^2)
  }
  return(b2)
}
```

get.m4

```
"get.m4"<-function(data)
{
  d<-dim(data)
  n<-d[1]
  p<-d[2]
  mean.data<-apply(data,2,mean)

  m4<-rep(0,p)
  for(i in 1:p)
  {
    m4[i]<-((sum((data[,i]-mean.data[i])^4))/n)
  }
  return(m4)
}
```


Program A: Generating Multivariate Normal Data

```
set.seed(seed)
data.set<-array(mnorm(n*p*rep), dim=c(n,p,rep))
```

→ this is repeated for each desired data set

for example,

```
mvn25.2<-array(mnorm(25*2*10000), dim=c(25,2,10000))
```

Program B: Generating Mixtures of Multivariate Normal Data**B1: Multivariate Normal Mixture 1-3 (depending on pp values)**

for p=2:

```
"mix1.2"<-function(n,k,pp, seed=123)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-c(1,0.2,0.2,1)
  sigma1<-matrix(sigma1, 2, 2, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(rnorm(value*2), value, 2)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0), rep(value,2)), value,2)

  sigma2<-c(1,0.5,0.5,1)
  sigma2<-matrix(sigma2, 2, 2, byrow=T)
  sigma2.root<- chol(sigma2)
  x2<-matrix(rnorm(value*2), value, 2)
  y2<- x2 %*% t(sigma2.root)
  N2.data<- y2 + matrix(rep(c(0,0), rep(value,2)), value,2)

  N.final<-array(rep(0,value), dim=c(value,2))
  for (i in 1:value)
  {
    if(N1[i] == 1)
    {
      N.final[i,]<-N1.data[i,]
    }

    else
```

```

    {
      N.final[i,]<-N2.data[i,]
    }
  }

  mix<-array(t(N.final), dim=c(2,n,k))
  mixture<-array(rep(0,n*2*k), dim=c(n,2,k))
  for(i in 1:k)
  {
    mixture[,i]<-t(mix[,i])
  }

  return(mixture)
}

```

for p=3:

```

"mix1.3"<-function(n,k,pp, seed=124)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-c(1,0.2,0.2,0.2,1,0.2,0.2,0.2,1)
  sigma1<-matrix(sigma1, 3, 3, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(rnorm(value*3), value, 3)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0,0), rep(value,3)), value,3)

  sigma2<-c(1,0.5,0.5,0.5,1,0.5,0.5,0.5,1)
  sigma2<-matrix(sigma2, 3, 3, byrow=T)
  sigma2.root<- chol(sigma2)
  x2<-matrix(rnorm(value*3), value,3)
  y2<- x2 %*% t(sigma2.root)
  N2.data<- y2 + matrix(rep(c(0,0,0), rep(value,3)), value,3)

  N.final<-array(rep(0,value), dim=c(value,3))
  for (i in 1:value)
  {
    if(N1[i] == 1)
    {
      N.final[i,]<-N1.data[i,]
    }

    else
    {

```

```

        N.final[i,]<-N2.data[i,]
    }
}

mix<-array(t(N.final), dim=c(3,n,k))
mixture<-array(rep(0,n*3*k), dim=c(n,3,k))
for(i in 1:k)
{
    mixture[,i]<-t(mix[,i])
}

return(mixture)

}

for p=4:

"mix1.4"<-function(n,k,pp, seed=125)
{
    set.seed(seed)
    value<-n*k
    unif<-runif(value)
    N1<-rep(0,value)
    for (i in 1:value)
    {
        if(unif[i] <= pp)
            N1[i]<-1
    }

    sigma1<-c(1,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,1)
    sigma1<-matrix(sigma1, 4, 4, byrow=T)
    sigma1.root<- chol(sigma1)
    x1<-matrix(rnorm(value*4), value, 4)
    y1<- x1 %*% t(sigma1.root)
    N1.data<- y1 + matrix(rep(c(0,0,0,0), rep(value,4)), value,4)

    sigma2<-c(1,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,1)
    sigma2<-matrix(sigma2, 4, 4, byrow=T)
    sigma2.root<- chol(sigma2)
    x2<-matrix(rnorm(value*4), value, 4)
    y2<- x2 %*% t(sigma2.root)
    N2.data<- y2 + matrix(rep(c(0,0,0,0), rep(value,4)), value,4)

    N.final<-array(rep(0,value), dim=c(value,4))
    for (i in 1:value)
    {
        if(N1[i] == 1)
        {
            N.final[i,]<-N1.data[i,]
        }

        else
        {
            N.final[i,]<-N2.data[i,]
        }
    }
}

```

```

}

mix<-array(t(N.final), dim=c(4,n,k))
mixture<-array(rep(0,n*4*k), dim=c(n,4,k))
for(i in 1:k)
{
  mixture[,i]<-t(mix[,i])
}

return(mixture)

}

for p=5:

"mix1.5"<-function(n,k,pp, seed=126)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-c(1,0.2,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,0.2,1)
  sigma1<-matrix(sigma1, 5, 5, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(rnorm(value*5), value, 5)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0,0,0,0), rep(value,5)), value,5)

  sigma2<-c(1,0.5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,1)
  sigma2<-matrix(sigma2, 5, 5, byrow=T)
  sigma2.root<- chol(sigma2)
  x2<-matrix(rnorm(value*5), value,5)
  y2<- x2 %*% t(sigma2.root)
  N2.data<- y2 + matrix(rep(c(0,0,0,0,0), rep(value,5)), value,5)

  N.final<-array(rep(0,value), dim=c(value,5))
  for (i in 1:value)
  {
    if(N1[i] == 1)
    {
      N.final[i,<-N1.data[i,<
    }

    else
    {
      N.final[i,<-N2.data[i,<
    }
  }
}

```

```

mix<-array(t(N.final), dim=c(5,n,k))
mixture<-array(rep(0,n*5*k), dim=c(n,5,k))
for(i in 1:k)
{
  mixture[,i]<-t(mix[,i])
}

return(mixture)

}

for p=10:

"mix1.10"<-function(n,k,pp, seed=127)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-
    c(1,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,0.
    2,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,1,0.2,0.
    2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,1,0.2,0.2,0.2,0.2,0.2,0.2,0.2,
    0.2,0.2,1,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,0.2,1)
  sigma1<-matrix(sigma1, 10, 10, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(rnorm(value*10), value, 10)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0,0,0,0,0,0,0,0,0), rep(value,10)), value,10)

  sigma2<-
    c(1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.
    5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,1,0.5,0.
    5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,
    0.5,0.5,1,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,0.5,1)
  sigma2<-matrix(sigma2, 10, 10, byrow=T)
  sigma2.root<- chol(sigma2)
  x2<-matrix(rnorm(value*10), value,10)
  y2<- x2 %*% t(sigma2.root)
  N2.data<- y2 + matrix(rep(c(0,0,0,0,0,0,0,0,0,0), rep(value,10)), value,10)

  N.final<-array(rep(0,value), dim=c(value,10))
  for (i in 1:value)
  {
    if(N1[i] == 1)
    {
      N.final[i,<-N1.data[i,<
    }
  }

```

```

else
{
  N.final[i,]<-N2.data[i,]
}
}

mix<-array(t(N.final), dim=c(10,n,k))
mixture<-array(rep(0,n*10*k), dim=c(n,10,k))
for(i in 1:k)
{
  mixture[,i]<-t(mix[,i])
}

return(mixture)
}

```

→ the remaining mixtures are generated similarly and so only the program for generating mixtures with $p=2$ will be presented.

B2: Multivariate Normal Mixture 4-6

for $p=2$:

```

"mix2.2"<-function(n,k,pp, seed=123)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-c(1,0.2,0.2,1)
  sigma1<-matrix(sigma1, 2, 2, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(rnorm(value*2), value, 2)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0), rep(value,2)), value,2)

  sigma2<-c(1,0.2,0.2,1)
  sigma2<-matrix(sigma2, 2, 2, byrow=T)
  sigma2.root<- chol(sigma2)
  x2<-matrix(rnorm(value*2), value,2)
  y2<- x2 %*% t(sigma2.root)
  N2.data<- y2 + matrix(rep(c(1,1), rep(value,2)), value,2)

```

```

N.final<-array(rep(0,value), dim=c(value,2))
for (i in 1:value)
{
  if(N1[i] == 1)
  {
    N.final[i,]<-N1.data[i,]
  }

  else
  {
    N.final[i,]<-N2.data[i,]
  }
}

mix<-array(t(N.final), dim=c(2,n,k))
mixture<-array(rep(0,n*2*k), dim=c(n,2,k))
for(i in 1:k)
{
  mixture[,i]<-t(mix[,i])
}

return(mixture)
}

```

B3: Multivariate Normal Mixture 7-9

for p=2:

```

"mix3.2"<-function(n,k,pp, seed=123)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-c(1,0.5,0.5,1)
  sigma1<-matrix(sigma1, 2, 2, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(mnorm(value*2), value, 2)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0), rep(value,2)), value,2)

  sigma2<-c(1,0.5,0.5,1)
  sigma2<-matrix(sigma2, 2, 2, byrow=T)
  sigma2.root<- chol(sigma2)
  x2<-matrix(mnorm(value*2), value, 2)
  y2<- x2 %*% t(sigma2.root)
  N2.data<- y2 + matrix(rep(c(1,1), rep(value,2)), value,2)
}

```

```

N.final<-array(rep(0,value), dim=c(value,2))
for (i in 1:value)
{
  if(N1[i] == 1)
  {
    N.final[i,<-N1.data[i,]
  }

  else
  {
    N.final[i,<-N2.data[i,]
  }
}

mix<-array(t(N.final), dim=c(2,n,k))
mixture<-array(rep(0,n*2*k), dim=c(n,2,k))
for(i in 1:k)
{
  mixture[,i]<-t(mix[,i])
}

return(mixture)
}

```

B4: Multivariate Normal Mixture 10-12

for p=2:

```

"mix4.2"<-function(n,k,pp, seed=123)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-c(1,0.2,0.2,1)
  sigma1<-matrix(sigma1, 2, 2, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(mnorm(value*2), value, 2)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0), rep(value,2)), value,2)

  sigma2<-c(1,0.5,0.5,1)
  sigma2<-matrix(sigma2, 2, 2, byrow=T)
  sigma2.root<- chol(sigma2)
  x2<-matrix(mnorm(value*2), value,2)

```



```

y2<- x2 %*% t(sigma2.root)
N2.data<- y2 + matrix(rep(c(1,1), rep(value,2)), value,2)

N.final<-array(rep(0,value), dim=c(value,2))
for (i in 1:value)
{
  if(N1[i] == 1)
  {
    N.final[i,<-N1.data[i,]
  }

  else
  {
    N.final[i,<-N2.data[i,]
  }
}

mix<-array(t(N.final), dim=c(2,n,k))
mixture<-array(rep(0,n*2*k), dim=c(n,2,k))
for(i in 1:k)
{
  mixture[,i]<-t(mix[,i])
}

return(mixture)
}

```

B5: Multivariate Normal Mixture 13-15

for p=2:

```

"mix5.2"<-function(n,k,pp, seed=123)
{
  set.seed(seed)
  value<-n*k
  unif<-runif(value)
  N1<-rep(0,value)
  for (i in 1:value)
  {
    if(unif[i] <= pp)
      N1[i]<-1
  }

  sigma1<-c(1,0.5,0.5,1)
  sigma1<-matrix(sigma1, 2, 2, byrow=T)
  sigma1.root<- chol(sigma1)
  x1<-matrix(rnorm(value*2), value, 2)
  y1<- x1 %*% t(sigma1.root)
  N1.data<- y1 + matrix(rep(c(0,0), rep(value,2)), value,2)

  sigma2<-c(1,0.2,0.2,1)
  sigma2<-matrix(sigma2, 2, 2, byrow=T)
  sigma2.root<- chol(sigma2)

```

```

x2<-matrix(rnorm(value*2), value,2)
y2<- x2 %*%% t(sigma2.root)
N2.data<- y2 + matrix(rep(c(1,1), rep(value,2)), value,2)

N.final<-array(rep(0,value), dim=c(value,2))
for (i in 1:value)
{
  if(N1[i] == 1)
  {
    N.final[i,<-N1.data[i,]
  }

  else
  {
    N.final[i,<-N2.data[i,]
  }
}

mix<-array(t(N.final), dim=c(2,n,k))
mixture<-array(rep(0,n*2*k), dim=c(n,2,k))
for(i in 1:k)
{
  mixture[,i]<-t(mix[,i])
}

return(mixture)
}

```

Program C: Generating Multivariate Chi-square data

```

set.seed(seed)
data.set<-array(rchisq(n*p*rep,df=df), dim=c(n,p,rep))

```

→ this is repeated for each desired data set

for example,

```

mchisq25.2<-array(rchisq(25*2*10000,df=1), dim=c(25,2,10000))

```

Program D: Generating Multivariate Lognormal Data

```

set.seed(seed)
data.set<-array(rlnorm(n*p*rep), dim=c(n,p,rep))

```

→ this is repeated for each desired data set

for example,

```

mlog25.2<-array(rlnorm(25*2*10000), dim=c(25,2,10000))

```

Program E: Generating Khintchine Data**for p=2:**

```
"khintchine.2"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G<-rgamma(value,1.5,scale=2)
  R<-sqrt(G)
  U.1<-runif(value)
  U.2<-runif(value)
  X.1<-R * (2 * U.1 -1)
  X.2<-R * (2 * U.2 -1)
  X<-cbind(X.1,X.2)

  K2<-array(t(X), dim=c(2,n,k))
  khintchine<-array(rep(0,n*2*k), dim=c(n,2,k))
  for(i in 1:k)
  {
    khintchine[,i]<-t(K2[,i])
  }

  return(khintchine)
}
```

for p=3:

```
"khintchine.3"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G<-rgamma(value,1.5,scale=2)
  R<-sqrt(G)
  U.1<-runif(value)
  U.2<-runif(value)
  U.3<-runif(value)
  X.1<-R * (2 * U.1 -1)
  X.2<-R * (2 * U.2 -1)
  X.3<-R * (2 * U.3 -1)

  X<-cbind(X.1,X.2,X.3)

  K3<-array(t(X), dim=c(3,n,k))
  khintchine<-array(rep(0,n*3*k), dim=c(n,3,k))
  for(i in 1:k)
  {
    khintchine[,i]<-t(K3[,i])
  }

  return(khintchine)
}
```

for p=4:

```
"khintchine.4"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G<-rgamma(value,1.5,scale=2)
  R<-sqrt(G)
  U.1<-runif(value)
  U.2<-runif(value)
  U.3<-runif(value)
  U.4<-runif(value)
  X.1<-R * (2 * U.1 -1)
  X.2<-R * (2 * U.2 -1)
  X.3<-R * (2 * U.3 -1)
  X.4<-R * (2 * U.4 -1)

  X<-cbind(X.1,X.2,X.3,X.4)

  K4<-array(t(X), dim=c(4,n,k))
  khintchine<-array(rep(0,n*4*k), dim=c(n,4,k))
  for(i in 1:k)
  {
    khintchine[,i]<-t(K4[,i])
  }

  return(khintchine)
}
```

for p=5:

```
"khintchine.5"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G<-rgamma(value,1.5,scale=2)
  R<-sqrt(G)
  U.1<-runif(value)
  U.2<-runif(value)
  U.3<-runif(value)
  U.4<-runif(value)
  U.5<-runif(value)
  X.1<-R * (2 * U.1 -1)
  X.2<-R * (2 * U.2 -1)
  X.3<-R * (2 * U.3 -1)
  X.4<-R * (2 * U.4 -1)
  X.5<-R * (2 * U.5 -1)

  X<-cbind(X.1,X.2,X.3,X.4,X.5)

  K5<-array(t(X), dim=c(5,n,k))
  khintchine<-array(rep(0,n*5*k), dim=c(n,5,k))
  for(i in 1:k)
  {
    khintchine[,i]<-t(K5[,i])
  }
}
```

```

    }

    return(khintchine)
}

for p=10:

"khintchine.10"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G<-rgamma(value,1.5,scale=2)
  R<-sqrt(G)
  U.1<-runif(value)
  U.2<-runif(value)
  U.3<-runif(value)
  U.4<-runif(value)
  U.5<-runif(value)
  U.6<-runif(value)
  U.7<-runif(value)
  U.8<-runif(value)
  U.9<-runif(value)
  U.10<-runif(value)

  X.1<-R * (2 * U.1 -1)
  X.2<-R * (2 * U.2 -1)
  X.3<-R * (2 * U.3 -1)
  X.4<-R * (2 * U.4 -1)
  X.5<-R * (2 * U.5 -1)
  X.6<-R * (2 * U.6 -1)
  X.7<-R * (2 * U.7 -1)
  X.8<-R * (2 * U.8 -1)
  X.9<-R * (2 * U.9 -1)
  X.10<-R * (2 * U.10 -1)

  X<-cbind(X.1,X.2,X.3,X.4,X.5,X.6,X.7,X.8,X.9,X.10)

  K10<-array(t(X), dim=c(10,n,k))
  khintchine<-array(rep(0,n*10*k), dim=c(n,10,k))
  for(i in 1:k)
  {
    khintchine[,i]<-t(K10[,i])
  }

  return(khintchine)
}

```

Program F: Generating Generalized Exponential Power Data**for p=2:**

```
"gep.2"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G.1<-rgamma(value,0.1663, scale=1)
  R.1<-(G.1^0.125)
  U.1<-runif(value)
  a<-(2*(rbinom(1,1,0.5)))-1
  X.1<-a * R.1 * U.1
  G.2<-rgamma(value,0.1663, scale=1)
  R.2<-(G.2^0.125)
  U.2<-runif(value)
  b<-(2*(rbinom(1,1,0.5)))-1
  X.2<-b * R.2 * U.2

  X<-cbind(X.1,X.2)

  GEP2<-array(t(X), dim=c(2,n,k))
  generalized<-array(rep(0,n*2*k), dim=c(n,2,k))
  for(i in 1:k)
  {
    generalized[,i]<-t(GEP2[,i])
  }

  return(generalized)
}
```

for p=3:

```
"gep.3"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G.1<-rgamma(value,0.1663, scale=1)
  R.1<-(G.1^0.125)
  U.1<-runif(value)
  a<-(2*(rbinom(1,1,0.5)))-1
  X.1<-a * R.1 * U.1
  G.2<-rgamma(value,0.1663, scale=1)
  R.2<-(G.2^0.125)
  U.2<-runif(value)
  b<-(2*(rbinom(1,1,0.5)))-1
  X.2<-b * R.2 * U.2
  G.3<-rgamma(value,0.1663, scale=1)
  R.3<-(G.3^0.125)
  U.3<-runif(value)
  d<-(2*(rbinom(1,1,0.5)))-1
  X.3<-d * R.3 * U.3
```

```

X<-cbind(X.1,X.2,X.3)

GEP3<-array(t(X), dim=c(3,n,k))
generalized<-array(rep(0,n*3*k), dim=c(n,3,k))
for(i in 1:k)
{
  generalized[,i]<-t(GEP3[,i])
}

return(generalized)
}

for p=4:

"gep.4"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G.1<-rgamma(value,0.1663, scale=1)
  R.1<-(G.1^0.125)
  U.1<-runif(value)
  a<-(2*(rbinom(1,1,0.5)))-1
  X.1<-a * R.1 * U.1
  G.2<-rgamma(value,0.1663, scale=1)
  R.2<-(G.2^0.125)
  U.2<-runif(value)
  b<-(2*(rbinom(1,1,0.5)))-1
  X.2<-b * R.2 * U.2
  G.3<-rgamma(value,0.1663, scale=1)
  R.3<-(G.3^0.125)
  U.3<-runif(value)
  d<-(2*(rbinom(1,1,0.5)))-1
  X.3<-d * R.3 * U.3
  G.4<-rgamma(value,0.1663, scale=1)
  R.4<-(G.4^0.125)
  U.4<-runif(value)
  f<-(2*(rbinom(1,1,0.5)))-1
  X.4<-f * R.4 * U.4

  X<-cbind(X.1,X.2,X.3,X.4)

  GEP4<-array(t(X), dim=c(4,n,k))
  generalized<-array(rep(0,n*4*k), dim=c(n,4,k))
  for(i in 1:k)
  {
    generalized[,i]<-t(GEP4[,i])
  }

  return(generalized)
}

```

for p=5:

```
"gep.5"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G.1<-rgamma(value,0.1663, scale=1)
  R.1<-(G.1^0.125)
  U.1<-runif(value)
  a<-(2*(rbinom(1,1,0.5)))-1
  X.1<-a * R.1 * U.1
  G.2<-rgamma(value,0.1663, scale=1)
  R.2<-(G.2^0.125)
  U.2<-runif(value)
  b<-(2*(rbinom(1,1,0.5)))-1
  X.2<-b * R.2 * U.2
  G.3<-rgamma(value,0.1663, scale=1)
  R.3<-(G.3^0.125)
  U.3<-runif(value)
  d<-(2*(rbinom(1,1,0.5)))-1
  X.3<-d * R.3 * U.3
  G.4<-rgamma(value,0.1663, scale=1)
  R.4<-(G.4^0.125)
  U.4<-runif(value)
  f<-(2*(rbinom(1,1,0.5)))-1
  X.4<-f * R.4 * U.4
  G.5<-rgamma(value,0.1663, scale=1)
  R.5<-(G.5^0.125)
  U.5<-runif(value)
  g<-(2*(rbinom(1,1,0.5)))-1
  X.5<-g * R.5 * U.5

  X<-cbind(X.1,X.2,X.3,X.4,X.5)

  GEP5<-array(t(X), dim=c(5,n,k))
  generalized<-array(rep(0,n*5*k), dim=c(n,5,k))
  for(i in 1:k)
  {
    generalized[,i]<-t(GEP5[,i])
  }

  return(generalized)
}
```

for p=10:

```
"gep.10"<-function(n,k,seed=123)
{
  set.seed(seed)
  value<-n*k
  G.1<-rgamma(value,0.1663, scale=1)
  R.1<-(G.1^0.125)
  U.1<-runif(value)
  a<-(2*(rbinom(1,1,0.5)))-1
```



```

X.1<-a * R.1 * U.1
G.2<-rgamma(value,0.1663, scale=1)
R.2<-(G.2^0.125)
U.2<-runif(value)
b<-(2*(rbinom(1,1,0.5)))-1
X.2<-b * R.2 * U.2
G.3<-rgamma(value,0.1663, scale=1)
R.3<-(G.3^0.125)
U.3<-runif(value)
d<-(2*(rbinom(1,1,0.5)))-1
X.3<-d * R.3 * U.3
G.4<-rgamma(value,0.1663, scale=1)
R.4<-(G.4^0.125)
U.4<-runif(value)
f<-(2*(rbinom(1,1,0.5)))-1
X.4<-f * R.4 * U.4
G.5<-rgamma(value,0.1663, scale=1)
R.5<-(G.5^0.125)
U.5<-runif(value)
g<-(2*(rbinom(1,1,0.5)))-1
X.5<-g * R.5 * U.5
G.6<-rgamma(value,0.1663, scale=1)
R.6<-(G.6^0.125)
U.6<-runif(value)
g<-(2*(rbinom(1,1,0.5)))-1
X.6<-g * R.6 * U.6
G.7<-rgamma(value,0.1663, scale=1)
R.7<-(G.7^0.125)
U.7<-runif(value)
h<-(2*(rbinom(1,1,0.5)))-1
X.7<-h * R.7 * U.7
G.8<-rgamma(value,0.1663, scale=1)
R.8<-(G.8^0.125)
U.8<-runif(value)
j<-(2*(rbinom(1,1,0.5)))-1
X.8<-j * R.8 * U.8
G.9<-rgamma(value,0.1663, scale=1)
R.9<-(G.9^0.125)
U.9<-runif(value)
m<-(2*(rbinom(1,1,0.5)))-1
X.9<-m * R.9 * U.9
G.10<-rgamma(value,0.1663, scale=1)
R.10<-(G.10^0.125)
U.10<-runif(value)
q<-(2*(rbinom(1,1,0.5)))-1
X.10<-q * R.10 * U.10

X<-cbind(X.1,X.2,X.3,X.4,X.5,X.6,X.7,X.8,X.9,X.10)

GEP10<-array(t(X), dim=c(10,n,k))
generalized<-array(rep(0,n*10*k), dim=c(n,10,k))
for(i in 1:k)
{
  generalized[,i]<-t(GEP10[,i])
}

```

```

    return(generalized)
}

```

Program G: Generating Pearson Type II Data

→ Note: In the case of $m=0$ the Multivariate Uniform is generated

for p=2:

```

"P2.2"<-function(n,k, m, seed = 123)
{
  value<-n*k
  set.seed(seed)
  v1 <- as.matrix(rbeta(value, (1/2), (m + 1 + (1/2))))
  a <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x1 <- a * (sqrt(v1))
  v2 <- as.matrix(rbeta(value, (1/2), (m + 1 + (1/2) - (1/2))))
  b <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x2 <- b * (sqrt(v2 * (1 - (x1^2))))
  pearson2 <- matrix(c(x1, x2), value, 2)
  P2<-array(t(pearson2), dim=c(2,n,k))
  pearson<-array(rep(0,n*2*k), dim=c(n,2,k))
  for(i in 1:k)
  {
    pearson[,i]<-t(P2[,i])
  }

  return(pearson)
}

```

for p=3:

```

"P2.3"<-function(n,k, m,seed=123)
{
  set.seed(seed)
  value<-n*k
  v1 <- as.matrix(rbeta(value, (1/2), (m + (3/2) + (1/2))))
  a <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x1 <- a * (sqrt(v1))
  v2 <- as.matrix(rbeta(value, (1/2), (m + (3/2) + (1/2) - (1/2))))
  b <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x2 <- b * (sqrt(v2 * (1 - (x1^2))))
  v3 <- as.matrix(rbeta(value, (1/2), (m + (3/2) + (1/2) - (2/2))))
  d <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x3 <- d * (sqrt(v3 * (1 - (x1^2) - (x2^2))))
  pearson3 <- matrix(c(x1, x2, x3), value, 3)
  P3<-array(t(pearson3), dim=c(3,n,k))
  pearson<-array(rep(0,n*3*k), dim=c(n,3,k))
  for(i in 1:k)
  {
    pearson[,i]<-t(P3[,i])
  }
}

```

```

    }

    return(pearson)

}

for p=4:

"P2.4"<-function(n,k, m,seed=123)
{
  set.seed(seed)
  value<-n*k
  v1 <- as.matrix(rbeta(value, (1/2), (m + (4/2) + (1/2))))
  a <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x1 <- a * (sqrt(v1))
  v2 <- as.matrix(rbeta(value, (1/2), (m + (4/2) + (1/2) - (1/2))))
  b <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x2 <- b * (sqrt(v2 * (1 - (x1^2))))
  v3 <- as.matrix(rbeta(value, (1/2), (m + (4/2) + (1/2) - (2/2))))
  d <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x3 <- d * (sqrt(v3 * (1 - (x1^2) - (x2^2))))
  v4 <- as.matrix(rbeta(value, (1/2), (m + (4/2) + (1/2) - (3/2))))
  e <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x4 <- e * (sqrt(v4 * (1 - (x1^2) - (x2^2) - (x3^2))))
  pearson4 <- matrix(c(x1, x2, x3, x4), value, 4)
  P4<-array(t(pearson4), dim=c(4,n,k))
  pearson<-array(rep(0,n*4*k), dim=c(n,4,k))
  for(i in 1:k)
  {
    pearson[,i]<-t(P4[,i])
  }

  return(pearson)

}

```

for p=5:

```

"P2.5"<-function(n,k, m,seed=123)
{
  set.seed(seed)
  value<-n*k
  v1 <- as.matrix(rbeta(value, (1/2), (m + (5/2) + (1/2))))
  a <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x1 <- a * (sqrt(v1))
  v2 <- as.matrix(rbeta(value, (1/2), (m + (5/2) + (1/2) - (1/2))))
  b <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x2 <- b * (sqrt(v2 * (1 - (x1^2))))
  v3 <- as.matrix(rbeta(value, (1/2), (m + (5/2) + (1/2) - (2/2))))
  d <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x3 <- d * (sqrt(v3 * (1 - (x1^2) - (x2^2))))
  v4 <- as.matrix(rbeta(value, (1/2), (m + (5/2) + (1/2) - (3/2))))
  e <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x4 <- e * (sqrt(v4 * (1 - (x1^2) - (x2^2) - (x3^2))))
  v5 <- as.matrix(rbeta(value, (1/2), (m + (5/2) + (1/2) - (4/2))))

```

```

f <- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
x5 <- f * (sqrt(v5 * (1 - (x1^2) - (x2^2) - (x3^2) - (x4^2))))
pearson5 <- matrix(c(x1, x2, x3, x4, x5), value, 5)
P5<-array(t(pearson5), dim=c(5,n,k))
pearson<-array(rep(0,n*5*k), dim=c(n,5,k))
for(i in 1:k)
{
  pearson[,i]<-t(P5[,i])
}

return(pearson)
}

```

for p=10:

```

"P2.10"<-function(n,k, m, seed=123)
{
  set.seed(seed)
  value<-n*k
  v1<- as.matrix(rbeta(value,(1/2),(m + (10/2) + (1/2))))
  a<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x1<- a * (sqrt(v1))
  v2<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (1/2))))
  b<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x2<- b * (sqrt(v2 * (1 - (x1^2))))
  v3<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (2/2))))
  d<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x3<- d * (sqrt(v3 * (1 - (x1^2) - (x2^2))))
  v4<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (3/2))))
  e<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x4<- e * (sqrt(v4 * (1 - (x1^2) - (x2^2) - (x3^2))))
  v5<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (4/2))))
  f<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x5<- f * (sqrt(v5 * (1 - (x1^2) - (x2^2) - (x3^2) - (x4^2))))
  v6<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (5/2))))
  g<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x6<- g * (sqrt(v6 * (1 - (x1^2) - (x2^2) - (x3^2) - (x4^2) - (x5^2))))
  v7<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (6/2))))
  h<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x7<- h * (sqrt(v7 * (1 - (x1^2) - (x2^2) - (x3^2) - (x4^2) - (x5^2) - (x6^2))))
  v8<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (7/2))))
  l<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x8<- l * (sqrt(v8 * (1 - (x1^2) - (x2^2) - (x3^2) - (x4^2) - (x5^2) - (x6^2) - (x7^2))))
  v9<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (8/2))))
  q<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x9<- q * (sqrt(v9 * (1 - (x1^2) - (x2^2) - (x3^2) - (x4^2) - (x5^2) - (x6^2) - (x7^2) - (x8^2))))
  v10<- as.matrix(rbeta(value, (1/2), (m + (10/2) + (1/2) - (9/2))))
  r<- as.matrix(2 * (rbinom(value, 1, 0.5)) - 1)
  x10<- r * (sqrt(v10 * (1 - (x1^2) - (x2^2) - (x3^2) - (x4^2) - (x5^2) - (x6^2) - (x7^2) - (x8^2) - (x9^2))))
  pearson10<-matrix(c(x1,x2,x3,x4,x5,x6,x7,x8,x9,x10), value, 10)
  P10<-array(t(pearson10), dim=c(10,n,k))
  pearson<-array(rep(0,n*10*k), dim=c(n,10,k))
  for(i in 1:k)
  {

```

```

        pearson[,i]<-t(P10[,i])
    }

    return(pearson)
}

```

Program H: Generating Pearson Type VII Data

- Note: In the case $v=10$, the Multivariate t with $df=10$ is generated
- Note: In the case $v=1$, the Multivariate Cauchy is generated

```

"PTSQ7"<-function(n,p,k,v, seed=123)
{
    set.seed(seed)
    value<-n*p*k
    value2<-n*k
    Z<-matrix(rnorm(value), ncol=p)
    S<-rchisq(value2,df=v)
    s<-sqrt(v)/sqrt(S)
    pearson7<-s * Z

    P<-array(t(pearson7), dim=c(p,n,k))
    pearson<-array(rep(0,n*p*k), dim=c(n,p,k))
    for(i in 1:k)
    {
        pearson[,i]<-t(P[,i])
    }

    return(pearson)
}

```