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CONTINUUM DAMAGE MECHANICS FOR PLAIN, FIBRE-REINFORCED, AND REINFORCED CONCRETE MATERIALS AND STRUCTURES

by

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A thesis submitted to
the Faculty of Graduate Studies and Research

in partial fulfillment of
the requirements for the degree of
DOCTOR OF PHILOSOPHY (Ph.D.)[†]

Department of Civil and Environmental Engineering
Carleton University,
Ottawa, Canada
June, 1997

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[†] The doctor of philosophy in Civil and Environmental Engineering is a joint program with the University of Ottawa, administered by the Ottawa-Carleton Institute for Civil Engineering.

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in partial fulfillment of the requirements
for the degree of
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ABSTRACT

Concrete behaviour is often difficult to predict for all load scenarios using a single material model (e.g. elasticity, plasticity, fracture mechanics, etc.). This is because concrete nonlinearity is the result of two major energy dissipation mechanisms: the evolution of the internal microcracks, and plastic flow or yielding. Each mechanism may be triggered, depending on the nature and level of loading, the composition of the material, the shape, orientation and distribution of the internal microvoids and defects, as well as the interaction among these parameters. Therefore, a unified theory should appropriately acknowledge the complex heterogeneous nature of concrete material and trace the appropriate active energy mechanism(s) while accounting for all deformation histories. Such a model should also be able to switch from one mechanism to another without interference by the user. When the switch is made, the model should appreciate the history and type of deformations of the material. The new concept of micromechanical damage mechanics, which is further developed in this study, is capable of tracing the behaviour due to crack evolution. Classical theory of plasticity, on the other hand, is well suited and accepted to describe the behaviour of a material undergoing plastic flow.

Accordingly, this thesis combines a process micromechanical damage model and the general theory of anisotropic plasticity in order to obtain a unified plain concrete model. The model is simple and requires few unfitted material parameters. In the proposed model, the cement-paste material is assumed to be an homogeneous and isotropic matrix while the paste-aggregate interfacial properties are indirectly accounted for through the treatment of the mechanical evolution of the pre-existing microcracks. The proposed model is capable of tracing monotonic, cyclic and non-proportional loading. The 2-D version of the model is implemented in the finite element program *Nonlacs* and subsequently used to determine the full response of concrete elements and structures subjected to various stress regimes. The results of the proposed method of analysis compare well with available experimental data.

When fibres are introduced to strengthen concrete and to enhance its tensile carrying capabilities, they contribute more complexity to the previously mentioned mechanisms. This stems from the deformation, sliding, and/or pull or push-out of the fibres. They usually reinforce the crack and bridge its faces and hence reduce its opening and sliding displacements. Therefore, another elasto-plastic damage based model is formulated in this study to trace these nonlinear effects. The fibre deformation and crack bridging pressure and displacement functions are developed by employing the shear lag theory. The fibres are assumed to be randomly distributed and may experience de-bonding, sliding, fracture, and/or pull or push-out. However, the effect of chemical reaction between the fibres and surrounding concrete are neglected. The overall tangential elasto-plastic damage stiffness of a fibre-reinforced cracked concrete element is developed in a similar fashion to the first model. This model is also implemented in the general purpose finite element program *Nonlacs*. The results of this model are in good agreement with their experimental counterparts.

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1. INTRODUCTION

1.1 PROBLEM STATEMENT

Concrete is the most popular building material among others such as steel, masonry, wood, etc. It is widely used in construction because it is thought to be more economical than the other materials. As a result, concrete is used in most existing structures -not including one and two story residential buildings. Structures that require a high measure of safety, permanence, or durability (such as nuclear reactors, defense and bomb shelters, refractory ovens, water tanks and stations, sewage treatment plants, etc.) are primarily built from concrete (plain or reinforced) because of its good fire resistance and its ability to withstand harsh chemical and natural environments. Concrete usage in highway bridges has also proven to be more economical and advantageous compared to steel, at least in the medium to short span ranges. However, many of the existing concrete structures are experiencing deterioration and distress which means that they may not live their full design life unless measures are taken to rehabilitate them. Clearly, the restoration of the existing stock of concrete structures worldwide is a major and costly undertaking. Any improvements in the evaluation, analysis, design, and construction of concrete structures can have great economic and/or safety implications.

Most existing concrete structures, in particular, bridges and other infrastructure, have been built over the last sixty years. Their rehabilitation and upgrading to the increased demands of new loading and usage require proper evaluation of their true ultimate load carrying capacity, which will result in more efficient use of the available resources and will

lead to more appropriate methods of retrofit. A study by the National Materials Advisory Board (NMAB) of the National Research Council in the United States of America (ACI 1988) concludes that there exists a problem with regard to inadequate durability of concrete structures. It estimates that the current value of concrete infrastructure in the United States is at least \$6 trillion and that repair and replacement costs can be well over several hundred billion dollars. The cause of inadequate durability is identified as both *technical* and *institutional*.

On the technical side, one of the reasons cited is, “*inadequate knowledge of the mechanisms of development, and subsequent degradation, of concrete properties*”. Another reason is the “*difficulty of assessing the present condition and remaining service life of concrete structures*”. Among the recommendations made by the Board to ameliorate the current situation is: “*Develop in-depth understanding of the manner in which the composition and micro-structure of materials affect their performance. . .*”. The observations, conclusions and recommendations of this distinguished board apply not only to the United States but world-wide, including Canada. It may be argued that the concrete infrastructure in Canada is in better shape than that in the United States. If this is so, then it is due more to the institutional policies and the younger age of the structures in Canada than to superior technical know-how.

Clearly, the problem of durability is not just due to over-stressing of structures; a whole range of physio-chemical and mechanical processes, as well as improper design and construction practices are responsible. Regardless of the influence of the other effects, the ability of concrete to transfer loads, without experiencing distress, is dependent on the nature and magnitude of the applied stresses and the micro-structure of concrete. Present design practice relies heavily on essentially a single macroscopic property of concrete, namely, the compressive cylinder or cubic strength. Available empirical data that relate compressive strength to various physical parameters, such as water-cement ratio, curing temperature, time, etc., do not generally attempt to quantify the relative significance of each of those parameters on the micro-structure of concrete. Yet, in reality compressive

strength is the global manifestation of their collective effect on the micro-structure of concrete.

It is the opinion of the writer that concrete behaviour does not lie outside the bounds of theoretical solid mechanics. It is also believed that many physical observations concerning the mechanical behaviour of concrete, which at times appear anomalous, can be explained using the basic principles of mechanics and the computational power of modern computers. Although, as stated earlier, many complex phenomena contribute to the final micro-structure of concrete, at the present time it is expedient to investigate them individually, otherwise, the problem will become too complex and practically intractable.

1.2 BACKGROUND

Concrete is a non-homogeneous composite which comprises cement, water, fine and coarse aggregates, air (voids), and sometimes other chemical admixtures. The incompatible mechanical, hygroscopic (shrinkage and swelling) and thermal properties of maturing cement paste (matrix) and the coarse aggregates is the underlying cause of the microcracks that exist in concrete. The presence and propagation of the cracks under applied stresses is one of the principal mechanisms that determine the behaviour and strength of concrete (Mindess 1983; Zaitsev 1982; 1983; Horii and Nemat-Nasser 1985; 1986). When proper measures are taken to minimize these cracks (e.g. using lower water-cement ratio) or retard their propagation (e.g. by confinement), concrete exhibits superior strength, impermeability and durability. Here we do not deal with chemical processes such as alkali-aggregate reactivity or sulfate attack on concrete, but it is conceptually possible to treat those phenomenon as chemical processes that induce damage and thus the potentials driving the damage are chemical rather than mechanical.

The propagation of cracks in concrete is governed by the activation energy for fracture; which is the specific energy required to break bonds and create surfaces. This is the characteristic of a brittle material versus a ductile material for which the activation energy for fracture is the specific energy required to cause slip between planes of densest packing of atoms in the crystal lattices. Thus the key to understanding the micro-mechanics of

concrete is the manner in which fracture energy is related to the energy produced by the applied stresses. The theory of elasticity has been used with success to explain the behaviour of undamaged materials, materials for which the energy supplied is less than the activation energy for fracture, while the theory of plasticity has proven successful in explaining the behaviour of ductile crystalline materials. However, until recently existing theories have failed to adequately identify the behaviour of damaged brittle materials. Since concrete fractures under loading, it was realized that the relation between the fracture energy and the energy produced by the applied stresses is the key to understanding the micro-mechanics of concrete (Kunin 1983; Mindess 1983; Zaitsev 1982; 1983).

Classical fracture mechanics was originally developed to describe the behaviour of discrete macroscopic cracks, but its adaptation to an ensemble of randomly distributed microcracks is a relatively new development (Bazant 1988 a; b). This new area of mechanics is referred to as continuum damage mechanics (CDM). The development of CDM aims at explaining the link between the microscopic processes that a damaged material experiences under the influence of different stress fields and the macroscopic behaviour that is observed in physical experiments.

As will be subsequently described in detail, by properly considering the important relevant energy dissipation mechanisms, damage has proven to be a valuable theoretical tool for explaining the observed mechanical behaviour of concrete. To date, however, the vast majority of the works have focused on constitutive laws, with particular emphasis on plain concrete subjected to uniaxial stresses. Simulating the mechanical behaviour of a given brittle material under biaxial or triaxial loading has not been widely attempted. The majority of existing damage constitutive laws for the latter case are not general, and can apply to some limited biaxial stress ratios. It should also be mentioned that these attempts have focused on a material point, without considering the full structural ensemble. On the other hand, practically, all real concrete structures are reinforced with either steel or some other strong tensile materials, but the effect of the reinforcement on the response of

concrete, in the context of damage, has not been investigated in any detail. Presently, the reinforcement may also include steel and/or some organic fibres, hence one needs to investigate the applicability of CDM to fibre-reinforced concrete (FRC) structures. Since many concrete structures in practice may be subjected to multi-axial stress fields, one also needs to extend the existing principles to 3-D problems. These are some of the areas that need further in-depth investigation.

1.3 OBJECTIVES AND SCOPE

The principal objective of this thesis is to investigate, by using the micromechanical damage mechanics and the theory of plasticity, the effect of microcracks and their interaction with the applied stresses on the mechanical behaviour and ultimate strength of plain, reinforced and fibre-reinforced concrete elements and structures. Specifically, it is aimed at:

- [1] Improving and refining the existing two-dimensional micromechanical constitutive models of CDM for concrete and, applying the resulting models to actual concrete structures.
- [2] Investigating the effect of steel reinforcement on the CDM modelling of concrete structures.
- [3] Investigating the effect of fibres on the CDM modelling of plain, fibre-reinforced, or reinforced concrete structures, with or without conventional reinforcement.
- [4] Developing a general 2-D elasto-plastic micromechanical CDM model for the mechanical analysis of plain and fibre-reinforced concrete
- [5] Implementing the developed models in a general purpose nonlinear finite element program, Nonlacs, and testing their validity by comparing the results with available experimental data, and also with analytical results obtained from other types of constitutive laws.
- [6] The scope of the thesis will be limited to small deformation theory, static, non-time dependent loads, and stable systems. The emphasis will be principally on monotonic

versus cyclic loads. The nonlinear finite element technique will be coupled with CDM constitutive laws to develop appropriate computer programs for the analysis of concrete elements and structures.

1.4 THESIS STRUCTURE

Towards achieving the goals and tasks mentioned in the previous section, the thesis organization will be of the following format. Chapter 2 will survey the current literature on nonlinear analysis and constitutive modelling of concrete using damage mechanics.

Chapter 3 presents the general mathematical framework of damage models for weakened solids (e.g. plain concrete). The elasticity statement of a solid that contains either an embedded crack, micro-defect, or void, is addressed and the associated boundary value problem is solved. The stress-strain behaviour of a material point, subjected to uniaxial or biaxial stress fields, will be determined using the CDM and the theory of plasticity.

Chapter 4 will address the formulation of the effective stiffness matrix of a cracked finite element. The finite element equations will be developed using the constitutive relationships established in Chapter 3. Treatment of a fully bonded reinforcement will be addressed and presented. Finally, the numerical implementation of the different integration techniques used to evaluate the different integrals of the developed set of constitutive equations will also be presented. A comparison among the performances of these method will be included.

Chapter 5 will first test the model developed in Chapter 3 by capturing the mechanical behaviour of concrete cylinder and prism specimens subjected to either axial and/or biaxial monotonic and cyclic loading. The associated stress failure envelope of plain concrete in the biaxial case will also be shown. Secondly, a series of plain concrete beams and structures tested experimentally by others will be analyzed using the program developed in Chapter 4. The numerical results of the stresses and displacements of these structures will be plotted and compared with their experimental counterparts. Finally, the behaviour of real structures reinforced with steel reinforcement will be tested and the resulting response

will be analyzed. Chapter 5 shows the proposed elasto-plastic damage model capabilities, range of applications, and its limitations.

Chapter 6, on the other hand, summarizes the current methods of the analysis of the mechanical de-bonding and pull-out of an embedded fibre in a brittle matrix. Subsequently, the chapter presents an enhanced method for the analysis of fibre-reinforced concrete. The result of this analysis is then used in the damage model presented in Chapter 3 to account for the fibre presence. A discussion on the plasticity models of anisotropic materials implemented in the finite element model will be presented. This chapter will be concluded by presenting the combination of the anisotropic plasticity theory and the damage theory to obtain the elasto-plastic constitutive laws of fibre-reinforced concrete materials.

Chapter 7 will show some of the results of available experimental and analytical tests on fibre-reinforced concrete carried out by others. The numerical testing will be for fibre-reinforced concrete reinforced with either continuous steel or organic fibres. Structural components that are analyzed comprise beams, slabs, slabs-on-beams, box girders, cylindrical shells and skeleton towers. Some of the tested specimens will be notch sensitive.

The summary, conclusions and some recommendations based on the finding of the current study can be found in Chapter 8. Finally, the thesis is concluded by the list of references cited in the different sections.

2.

LITERATURE REVIEW

2.1 GENERAL

Review of the existing literature is essential in order to demonstrate the need for a new comprehensive unified theory for concrete materials. This part of the thesis will focus on investigating some of the relevant and more important existing concrete constitutive models developed to simulate its behaviour under static and/or non-proportional cyclic loading. A unified theory to capture the response of a given material subjected to general loading requires proper understanding of:

- [1] the type (ductile or brittle) and amount of deformation (large or small) that may take place,
- [2] the main mechanisms (i.e. slippage at crystalline planes or microcrack evolution) by which the material constituents rearrange themselves in the new deformed configuration of the body, and finally
- [3] the level (*scale*) (i.e. atomic, molecule, chains, grains, or structure) at which the material consumes most of the applied energy in order to deform.

It is well-known (Stroh 1957; Krajcinovic and Sumarac 1987) that almost all materials respond *elastically*, provided the internal stresses are less than or equal to the inter-atomic forces bonding the atoms together. However, their *ultimate mechanical strength* and *non-linear behaviour* are the macroscopic manifestation of the irreversible changes in the *non-linear behaviour* concentration, distribution, orientation, type, and arrangement of

defects in its body (e.g. Kunin 1983; Zaitsev 1982; 1983; Mindess 1983; Nemat-Nasser and Horii 1982; Ilankamban 1985; Krajcinovic 1988; etc.).

Under given loads, certain materials, for example rubber-like materials, may exhibit very large deformations while, most metals will deform substantially less. Brittle and semi-brittle materials, on the other hand, are most likely to undergo very small deformations before failure. Aluminum, iron, concrete, ceramic, and rocks are typical examples of the latter type. Since the aforementioned materials differ only in their internal composition, it is suggested that their deformation processes are dependent on the arrangement of their internal elements. For the sake of simplification, investigators (Krajcinovic 1985) have categorized the response of all materials into two types, namely; ductile and brittle, or sometimes semi-ductile semi-brittle responses. The latter type of deformation is typical for plain, low-strength, fibre-reinforced, and reinforced concrete, and high performance fibre composite materials. The specific response can be identified if the conditions and the associated characteristics displayed in Table 2.1 are met.

Table 2.1: Comparison between brittle and ductile response of solids.

Response Type	Ductile	Brittle
Mode of micro-structural changes	Slip, twinning, diffusions flow	Microcrack nucleation and growth
Location	Slip plane, grain boundary	Cleavage plane, grain boundary, inclusion
Failure mode	Localization into a shear band, necking	Localization into macro-crack or fault
Residual strain	Substantial	Minimal
History recording parameter	Inelastic strain (yielding)	Relative loss of cohesion (damage)

Table 2.1 clearly shows that in order to formulate an appropriate theory for a certain response, the theory should account for all, or at least the most dominant, modes of deformation and the associated energy dissipation mechanisms. To develop a unified theory, however, it must be formulated such that all energy dissipation mechanisms are recognized. Furthermore, it should be able to switch from one mode to another without intervention from the user. It is now clear that formulating a general theory for all types of materials and for all possible loading conditions is difficult and may be practically impossible. For example, concrete models must acknowledge the evolution and progress

of the pre-existing microcracks and micro-defects. These cracks are often located at the aggregate cement interfaces for virgin concrete and at joints and weak interfaces in rock masses. Other materials, such as metals may have cracks at the grain boundaries or at very weak planes. Therefore, prior to carrying out stress analyses for any material element, the following question often arises: *at what scale is the predominant defect located?* Subsequently, this prompts the twin question: *which constitutive law should be used to represent and simulate the mechanisms of the evolution of the defects?* For convenience, engineers and mechanist prefer to formulate their theories within the framework of continuum mechanics. It is known that the mathematical formalisms of continuum mechanics deal with continuous functions that describe entropies, energies, stresses, strains, elastic waves, thermal and electrical conductivity, etc. for a given material with smooth definite derivatives with respect to any of the listed variables. This, in turn, means that formulating a constitutive law for a material point will be valid everywhere with proper transformation procedures with respect to space.

In reality, however, the deformation of brittle materials often occurs as a result of the evolution and progress of defects which have physical dimensions and cannot be represented by a point. Hence, strictly speaking, point representation is not valid. Recognizing the type of defects in the material space (e.g. dislocation, microcracks, or shear band, etc.) and the mode (e.g. ductile, or brittle) by which deformation may take place, one can establish the different scales at which the various theoretical models can be developed and employed (e.g. Bazant 1984; Fanella and Krajcinovic 1988).

Table 2.2: Physical representative volume element (RVE) definition.

Scale	Material	Defect	Model
Micro	Atoms, molecule, and chains	Vacancies and dislocations	Physics and Material Sciences
Meso	Ensemble of grains	Microcracks, pores, and slips	Micromechanics
Macro	Test Specimen, structure, etc.	Macrocrack and shear band	Continuum theories and fracture mechanics

Using physical observations, (Hashin 1983; Krajcinovic and Selvaraj 1983; Wittmann 1983; Fanella 1986; Krajcinovic and Fanella 1988; Nemat-Nasser and Hori 1989) one can

postulate three levels at which materials may deform. These levels are labeled as (i) *Micro*, (ii) *Meso* and (iii) *Macro*-scales. Each of the scales and the corresponding type of models that can be used to obtain their constitutive relationships are shown in Table 2.2. In the current thesis the above definition of *scales* are adopted in order to formulate the elasto-plastic damage-based constitutive relations that will be developed and applied. In particular, the damage evolution laws are derived for the different concrete components (e.g. cement-paste, aggregate, and voids) at the *meso* scale. Subsequently, these relations are integrated over the representative volume element to obtain the global (*macro*) unified constitutive laws for concrete materials.

2.2 THE STATE OF THE ART

Although CDM in conjunction with the theory of plasticity will be utilized in the present thesis, nevertheless, we will summarize the reasons for which early models were developed, using different forms of both elasticity and plasticity theories. This will help show the shortcomings and inadequate representations of some of these models for describing the true behaviour of concrete materials. Since micromechanical-based damage constitutive models adopt, at some levels, the linear theory of fracture mechanics, we will briefly discuss the pros and cons of using the latter theory alone for the analysis of concrete elements. Subsequently, existing models of *continuum damage mechanics* are presented and discussed in a rather detailed fashion. The existing theories for analyzing fibre-reinforced concrete will be also summarized later at the beginning of Chapter 6.

2.3 MODELS BASED ON CONTINUUM MECHANICS

With reference to Figure 2.1 (Ziegeldorf 1983), we observe that the individual materials in a typical concrete mix (e.g. aggregate, sand, cement paste, etc.) basically behave in a linear fashion up to maximum stress while the whole mix (concrete) always shows nonlinear behaviour. For the sake of simplicity, early studies assumed that concrete was an isotropic material and that its mechanical response could be modelled using the linear or nonlinear elasticity and/or plasticity theories.

2.3.1 ELASTICITY AND PLASTICITY MODELS

Early developments of constitutive relationships were formulated using *Cauchy's* stress principles (Elwi and Murray 1979). In addition, the mechanical response of concrete materials was assumed to follow a linear stress-strain relationship. Of course, such an assumption does not reflect the physical nature of real concrete elements even at relatively early stages of loading, but it can be used to estimate their deformations at very low load levels. However, since the classical theory of elasticity does not account for the energy

Figure 2.1: Stress-strain curves for different materials.

dissipation mechanisms of concrete, it cannot be used to predict the overall deformation characteristics of a concrete structure in a complete loading program (i.e. from zero to failure load).

Experimental results (e.g. Kupfer and Gerstle 1973) of concrete always show that its macroscopic uniaxial compressive stress-

strain curve follows a nonlinear path. They also show that concrete elements subjected to 30% of their peak strength, or less, behave more or less in a linear manner. Figure 2.2 shows the different stages of deformation and the dominant energy dissipation mechanism at different stress levels. Recognizing the nonlinearities, investigators (Kupfer *et al.* 1969; Liu *et al.* 1972; Kupfer and Gerstle 1973; Gerstle 1981) assumed the material to be a *Cauchy elastic material* and they used the nonlinear elasticity theory in developing the corresponding constitutive relations. This model is partially successful since it can predict, to a limited degree, the response of the majority of planar concrete structures, provided

the loading is increased monotonically and the loading paths do not change drastically. Many current major finite element programs have adopted the aforementioned method for evaluating the overall response of concrete structures (Razaqpur and Nofal 1990). Other investigators have developed models using *hyper-elasticity* which treats the material as a *Cauchy* elastic material, but the existence of an elastic strain energy function is assumed (Elwi and Murray 1979; Desai and Siriwardane 1984). To distinguish this type of behaviour from the *Cauchy*

elastic material, sometimes it is labeled a *Green* elastic material. For the interested reader, a complete presentation of this concept is given in (Desai and Siriwardane 1984). Once again, these models do not consider the different constituents of the material and the micro-structure of concrete, hence they fail to give accurate results for all stress combinations (Elwi and Murray 1979). If a material is

Figure 2.2: Stress-strain curve characteristics for plain concrete.

considered elastic in the incremental sense, it is called a *hypo-elastic* material (Ju 1986).

Realizing that the nonlinear deformations are sometimes permanent, irreversible, and are of a macroscopic nature, many investigators (Chen 1990) have employed the classical theory of plasticity or elasto-plasticity to develop constitutive relationships. Some examples are the models of Kupfer and Gerstle (1978); Elwi and Murray (1979); Salami and Desai (1991). In their formulations, they have assumed the existence of an associative flow rule and have used the well-known von-Mises, Tresca, Drucker-Prager, or Mohr-

Coulomb failure criteria (Owen and Hinton 1988). Others have used a non-associative flow rule together with strain-softening (Dougill 1976; Dougill *et al.* 1977; Dragon and Mroz 1979; Bazant and Kim 1979; and Pietruszczak *et al.* 1988). Despite the sophistication and mathematical elegance of the models developed on the basis of the theory of plasticity, they all have basic shortcomings. The major problem with the theory of plasticity is the justification of the existence of yield strength and flow rule for concrete materials. Another problem is that the theory of plasticity assumes that slip at the crystal planes due to critical shear stresses is the principal energy dissipation mechanism which may be activated for a very small loading range.

Recently, Bazant (1986) and Onat *et al.* (1987) employed the theory of plasticity in conjunction with fracture/damage parameters to account for the degradation of concrete and other brittle materials. The estimation of the stiffness reduction was obtained as a result of shrinking the yield surface in the stress space. This was attributed to the increase of the applied loads and the resulting cracks volume. However, in a typical brittle heterogeneous material such as concrete this may not happen because the notion of a yield surface under general stress fields is not physically possible.

Some investigators (e.g. Wittmann 1983; Ziatsev 1982; 1983) assumed that the theory of visco-elasticity or visco-plasticity could be used for predicting concrete behaviour because cement paste is, in fact, a visco-elastic material (Wittmann 1979; 1983). These models have not achieved the intended success and acceptance within the solid mechanics community.

Finally, the so-called *endochronic plasticity theory* (Valanis 1985; Bazant 1980) was also used to describe the plasticity and visco-plasticity of concrete. It did not receive acceptance in the engineering community because of its tedious mathematical hypothesis and the associated difficulties in its numerical implementation in finite element computational schemes. Dafalias and Popov (1975; 1977) and Dafalias (1979) proposed a simplified version of the latter model and named it the *bounding surface model*. Hughes and Taylor (1984) and many others generalized this model and gave it a new title, the *cap*

model, while Ju (1986; 1989) showed its numerical implementation. The cap model accounts for plasticity and fracturing by introducing a function in the stress-strain space. This function is written in terms of the plastic and fracture strains. Many of the above models have not been implemented in finite element programs and as a result have not been fully verified. It is therefore difficult to ascertain whether they are capable of predicting the response of concrete structures under different load combinations. Many of the models derived from the previously mentioned theories fail to predict concrete behaviour for all load combinations because each recognizes one or a few of the energy dissipation mechanism(s) for a specific loading case. A realistic model should acknowledge the many energy dissipation mechanisms which may be a result of microcracks, voids, and/or inter-facial micro-defects (slip) that are present in the concrete element.

2.4 MODELS BASED ON FRACTURE MECHANICS

These types of models employ the well-known Griffith's criterion for brittle fracture (Cherpanov 1979). They use the crack tip stresses and displacements to determine whether a crack would propagate or would be arrested. If the material surrounding a given crack has no stress residual, then the material can be assumed to be linear elastic and linear fracture mechanics can be used to trace its deformations. Sometimes the material within this area may have experienced significant plastic strain and therefore the principles of linear fracture mechanics cannot be employed, hence some investigators (Sih 1965; Cherpanov 1979) have adopted nonlinear fracture mechanics.

2.4.1 LINEAR AND NONLINEAR FRACTURE MECHANICS

Assuming that the behaviour of a concrete element is dominated by a finite number of discrete cracks, linear (Shah 1980; Carpinteri 1984) or nonlinear (Bazant 1982; Shah 1980) fracture mechanics can be used for the analysis of their propagation. In principle, classical nonlinear fracture mechanics can be used to analyze concrete structures, but in practice it may be difficult and nearly impossible. First, microcracks in concrete exist prior to the application of any loads, and are of random size, orientation, and distribution.

Secondly, on the meso-scale they are neither discrete nor finite. In addition, in a finite element scheme, the treatment of propagating cracks requires re-meshing and changes in mesh topology at every load step or possibly at every iteration. This is an extremely cumbersome logistic problem which renders the method unmanageable and impractical.

Others (Hillerborg 1985; Carpinteri 1984; Bazant 1990) employed the laws of fracture mechanics in a global sense by assuming that many cracks will display the exact same behaviour at any instant. A well-known model that uses the latter argument is the Hillerborg-fictitious crack model (Hillerborg 1985). Integrating the energy required for crack propagation over a given specimen volume yields the desired constitutive relationships. Numerical results (Hillerborg *et al.* 1976; de Borst 1984 and 1987; Crisfield 1989) showed that for about 85% of the test cases it could predict the actual behaviour. Local fracture analysis along with some averaging techniques (Zaitsev 1982, 1983; Nemat-Nasser and Horii 1983 a, 1985 a, 1988) have shown acceptable agreement with experimental results for concrete, rocks, and other geophysical materials. For more details and references on fracture mechanics of concrete, the reader is referred to Wittmann (1979) and Mindess (1983 c). The second reference presents a collection of all abstracts (405 entries) of the work on fracture mechanics applied to concrete in the period 1928-1982.

At this point, it should be mentioned that there is still considerable debate in the literature about the applicability of fracture mechanics to concrete (Rice 1975; Krajcinovic 1983). For example, Wittmann (1983) questions whether the stress intensity factors or a J -integral are material dependent (parameters) while Kachanov (1991) argues the paradox of the existence of many cracks, and the toughening, and strengthening of the cracked solid according to the hypothesis of fracture mechanics. Mindess (1983 a, b), in his valuable synthesis discusses and compares, at great length, the different arguments about the applicability of fracture mechanics concepts (linear or nonlinear) to plain, fibre-reinforced, and polymer impregnated concrete. The main conclusions are (i) fracture mechanics can be used for the analysis of discrete cracks in plain concrete elements

provided the stress intensity factors of the cement paste, K_I^c , K_{II}^c , etc., can be measured from experiments. (ii) assuming that the cracks would not branch or propagate through aggregates, which is the case in fibre-reinforced concrete, the theory can be applied provided the hierarchy of stress intensity factors for aggregates, cement pastes, and interfaces (Wittmann 1983) is employed. This issue will be discussed in detail later. In principle, however, fracture mechanics can be employed for the analysis of crack propagation if and only if a uniform stress field exists in the material ahead of the crack tip. Undoubtedly, this phenomenon never exists in cementitious materials (Mindess 1983). Thus, tracing the evolution of a macro discrete crack in concrete materials using fracture mechanics is subject to the same restrictions as those in the continuum theory.

It was stated earlier that cracks with very small sizes exist in concrete in very large numbers. Furthermore, the material within the meso-scale can be considered homogeneous and the stress field can be assumed to be uniform. Thus, the mechanical behaviour of the meso-scale is indeed controlled by the crack propagation, which can be traced using the customary fracture mechanics theory. To obtain the macro-scale behaviour, however, we need to integrate the meso-scale stiffnesses, deformations, etc. over the whole body (structure) and obtain an overall response. Hence, it can be concluded that for a concrete-like material, an appropriate theory should employ the fracture mechanics principle at the meso-scale level to trace the material deformation responses. Continuum damage mechanics is a realistic and practical compromise between classical fracture mechanics and hypo-elasticity, and lends itself to implementation in finite element programs for the analysis of structures.

2.5 DAMAGE MECHANICS

A realistic constitutive law for concrete must recognize its principal energy dissipation mechanism(s), i.e. the nucleation, growth, and coalescence of the pre-existing microvoids. It should also be amenable to implementation in finite element computer programs in order to be tested and subsequently applied to analyze real structures. According to experimental evidence (Zaitsev 1982, 1983; Nemat-Nasser and Horii 1983), the

propagation of existing voids, cracks, defects, etc. is a function of both the direction and the type of applied loads. For example, it has been observed that a randomly oriented crack with respect to uniaxial compression loading finally aligns itself parallel to the load direction, and the specimen fails by splitting. It has been found also that the orthogonal direction suffers little damage. Consequently, it has been concluded that concrete, rocks, ceramic, etc. under general loading display load-induced anisotropy which is a function of the direction of the applied loads.

2.5.1 THE CONTINUUM DAMAGE MECHANICS (CDM) CONCEPT (L. KACHANOV MODEL 1958)

The above facts prompted L. Kachanov (1958) to suggest the novel idea of continuum damage mechanics and used it to determine the creep response of a uniaxially loaded bar. Since this thesis will employ CDM to analyze concrete structures, a brief description of Kachanov's pioneering idea will be given.

Without details, in principle, for the bar of Figure 2.3 subjected to an axial load, F_i , in the direction, i , the engineering and actual stresses are written, respectively, as

$$\sigma_i \equiv \frac{F_i}{A_i} \quad [2.1]$$

$$\bar{\sigma}_i \equiv \frac{F_i}{\bar{A}_i} \quad [2.2]$$

where σ_i is the resulting stress on the cross-sectional area A_i . The bar over symbols signifies effective or actual value of the variable. The effective area can be related to the net area by a variable, d_i , as

$$\bar{A}_i = (1 - d_i)A_i; \quad d_i = \frac{A_v}{A_i} \quad [2.3]$$

where A_v is the current void area. The parameter, d_i , can take any value between $0 \leq d_i \leq d_c$. A value of zero signifies no fracture (undamaged), while the value d_c indicates full fracture (damaged), with d_c being $0 \leq d_c \leq 1$. In fact, experiments show that for brittle materials, whose failure type is either “*Cleavage 1*” or “*Cleavage 2*” (Ashby 1979), d_c is

less than 1 and greater than 0.85, depending on the material type (Chaboche 1979). Using the above equation, we can write

$$\bar{\sigma}_i \equiv \frac{A_i}{\bar{A}_i} \sigma_i = \left[\frac{1}{1-d_i} \right] \sigma_i \quad [2.4]$$

Figure 2.3: 3-D cracked bar subjected to uniaxial force.

In the latter equation the effective stress, $\bar{\sigma}_i$, is related to the undamaged area A_i . Assuming the axial strain of the bar to be a result of the actual stress (Lemaitre 1985), the elastic strain of a damaged section can be written as

$$\varepsilon_i = \frac{\bar{\sigma}_i}{E} = \frac{\sigma_i}{(1-d_i)E} \quad [2.5]$$

which can be simplified as

$$\varepsilon_i = \frac{\sigma_i}{\bar{E}} \quad [2.6]$$

Equation [2.6] indicates that Hooke's law for a fractured section in the continuum sense has its customary form, with Young's modulus, E , replaced by the effective modulus, $\bar{E}$, i.e.

$$\bar{E} = (1 - d_i)E \quad [2.7]$$

Therefore, the effective elastic moduli are functions of the parameters d_i which will henceforth be called the *damage parameters*. The advantage of writing the damage parameter, d_i , as a function of the elastic moduli instead of the areas (see Equations [2.3] and [2.7]) is that the former values are measurable quantities from simple uniaxial tests (Lemaitre 1988). For example, the damage variable may be measured simply by comparing the unloading parts of the stress-strain curve with the initial slope of the stress-strain curve of the undamaged material (Krajcinovic 1989). Secondly, the constitutive equations for the damaged material are equivalent to those of the virgin material provided the effective stress tensor, $\bar{\sigma}$, is used instead of the nominal stress, σ .

The question now is: *how would the damage parameter evolve with the application of a non-proportional loading?*. The answer to which will indeed complete the solution. Many material scientists and specialists on micromechanics have formulated general two-dimensional (2-D) and three-dimensional (3-D) constitutive models for brittle materials (Krajcinovic 1984) and have attempted to answer that question. The early damage-based constitutive models were obtained for the description of the behaviour of materials subjected to creep loading (e.g. Kachanov 1985; Rabotnov 1963; Hult 1973; Broberg 1974; Murakami and Ohno 1981; Murakami 1988; etc.). Krajcinovic (1984) reviewed CDM application to fatigue damage. The early applications were successful because the damage evolution laws were relatively easy to postulate. The initial attempts were concerned with the development of constitutive laws for materials subjected to isothermal monotonic or cyclic loading, with concentration on phenomenological models. The literature on those models is overwhelming and indeed varies from simple one-dimensional models to models that incorporate eighth order tensor representations of the damage variables (Chaboche 1978; Dragon and Mroz 1979).

A brief, and admittedly, incomplete survey of the damage based constitutive models for static and cyclic isothermal loading will be given in the following. Other damage

models such as creep or fatigue damage will not be discussed. Some references, however, are listed and the interested reader can consult them for details.

In formulating damage constitutive laws, two avenues have been explored. They are dubbed the *phenomenological* and *micromechanical* models (Krajcinovic 1984; Lemaitre 1985), but as argued by Krajcinovic (1989), the outcome of the two models is practically the same. However, the former concept traces the damage via a macroscopic homogeneous damage parameter (Chaboche 1979). It assumes also that the deformations are dependent only on the current state of the material, i.e. the material has a *fading-memory* (Coleman and Mizel 1964). The choice of the set of internal variables identifying the damage are left up to the analyst's preferences (Krajcinovic 1984). Furthermore, the form of internal energy density and kinetic equations are arbitrarily assumed in order to fit physical experimental data (Lemaitre 1988).

The micromechanical damage models, however, employ the basic laws of continuum and fracture mechanics to monitor the process of opening and propagation of microcracks in concrete. The latter may be stationary *non-process* (Horii and Nemat-Nasser 1983; Krajcinovic and Fanella 1988) or evolutionary *process* (Krajcinovic and Sumarac 1989; Ju 1990) models. Non-process models express crack displacements in terms of the initial crack geometry and orientation while process models update the crack dimensions according to the level of applied load and the current composite properties. They do not require any fitted material parameters, since locally, they check whether a crack would or would not increase in size (i.e. *is it stable/unstable?*). However, a process model is more physically appealing for concrete, therefore such a model is adopted in the present study.

For the sake of brevity, a short review of the general form of the basic development of phenomenological CDM theories will be presented next while the micromechanical CDM models will be presented in the next chapter. The writer understands that any attempt to review all the current literature will be a major task because of the plethora of relevant research material. Therefore, no claim of completeness is made.

Before proceeding with the discussion of the existing CDM models, the material point definition employed in deriving these methods should be clarified. According to Table 2.1 and 2.2, a material point should represent all constituents of the material internal structure (e.g. atoms, molecule, chains, vacancies, dislocations, ensemble of grains, microcracks, etc.). This definition is also needed because a given damage model should be able to describe the behaviour of crack-like defects at a given point. This, once again, raises the question of scale of the so-called representative volume element (RVE). In the literature, many measures (Nemat-Nasser and Hori 1990) have been proposed to estimate the size of the RVE. For example, according to Krajcinovic and Fanella (1986), the RVE should contain approximately 30-100 aggregates for concrete. Bazant and Pijaudier-Cabot (1988), on the other hand, suggest that the characteristic length l_c of the RVE should be about three times the maximum aggregate size. The latter authors also introduced an additional parameter called, ω_c which represents a weighting function to estimate the minimum size of a concrete unit cell. It is obvious that the difference of opinion on this aspect is minimal, but there is no universal method to estimate the RVE size. In order to avoid this controversy, this study does not specify a physical dimension for its RVE size, but it requires that *“the RVE size should be large enough to account for all material constituents and small enough to be considered a point in the structure space”*. This means that the models derived in Chapters 3 and 6 of this study are formulated for the meso-scale as defined earlier in Table 2.2.

2.5.2 ELEMENTS OF DAMAGE MECHANICS

Prior to the review of the literature, let us identify the basic steps that are needed to formulate and employ CDM models. Chaboche (1981); Krajcinovic (1983); and Lemaitre (1984) have identified those steps and are listed in Table 2.3 for the sake of easy reference.

influence the *free energy*, σ . In addition, we further assume that the rate of change of the free energy is governed by differential equations in which strains appear (Coleman and Mizel 1967). It is clear from the above discussion that

[1] no one dissipation mechanism renders sufficient generality in order to include all the dissipative effects,

[2] each postulate has its own domain of application, limitations, and usefulness.

Coleman and Gurtin (1967) used the method of thermodynamics of continua (Coleman and Noll 1963; Coleman and Mizel 1964) together with techniques obtained from the stability theory of nonlinear differential equations in order to unify the *phenomenological* theories of dissipation.

They considered materials for which the basic mechanical and thermal variables are evaluated via *constitutive equations* when the following “*state variables*” are known: (1) the temperature tensor T , (2) the thermal flux gradient, $\text{grad } q$, (3) the deformation gradient, F (or the “strain”), and (4) a set of N internal or “hidden” state variables $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N)$. The local mechanical and thermal variables are (1) the stress tensor, σ , (2) the heat flux vector, q , (3) the specific Helmholtz free energy, ψ , and (4) the specific entropy, η , respectively. Furthermore, their formulation permits that the *rate of change of each* of the state variables, α_i , be governed by an arbitrary function f_i of q , F , $\text{grad } q$, and the complete internal state $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N)$, in mathematical formalism:

$$\dot{\alpha}_i = f_i(q, F, \text{grad } q, \alpha_1, \alpha_2, \alpha_3, \dots, \alpha_N), \quad i = 1, 2, 3, \dots, N. \quad [2.8]$$

where the superposed dot indicates the derivative of a given variable (function) with respect to time, t . Furthermore, they imposed the necessary restrictions placed on those theories by *the second law of thermodynamics* in the form of the Clausius-Duhem inequality. For a given dynamical stability of equilibrium, the solutions of the evolution equations of Equation [2.8] are determined by employing some of the thermodynamical

principles. These issues are discussed at length in the articles of Coleman and Noll (1963); Coleman and Mizel (1964); and Ilankamban (1985).

2.5.3.1 DAMAGE VARIABLE REPRESENTATION

It is now obvious that defining the state variables will depend on the material and the experience of the analyst with the type of internal state variables that consume most of the energy. Earlier, we noted that crack-like defects and voids are the main internal state variables that control the behaviour of concrete, rock, ceramic, and geophysical materials. It is also well known that micro-defects and voids produce local (i.e. at the meso-scale) stress concentrations in the material body and increase its thermal and electrical resistance. Strictly speaking, concrete contains many micro-defects, air voids, and vacancies, of different sizes and shapes, which are randomly distributed and oriented. Thus, if we assume that the shape of a crack may be of one or two geometries (e.g. circular, elliptical in 2-D configuration or spherical, cylindrical, in 3-D) this would allow us to describe the state of the material by a finite set of internal state variables (Krajcinovic 1984; Rice 1985; Murakami 1988).

In all existing phenomenological models, the internal state variables are the crack shape, dimensions, and orientation in space with respect to the loading. As will be discussed in the subsequent section, the mathematical representation of internal state variables in phenomenological CDM models is not quite clear and sometimes does not reflect the evolution of the cracks.

Of course, if one can specify the shape, initial size, orientation, and the kinematics law of evolution of every crack in the medium, the description of the material would be exact and accurate. That would limit the practicality of the application of such models. Therefore, one has to choose a limited number of parameters to represent and identify the majority of the cracks. Certain averaging procedures can be applied, which will eliminate some unnecessary and unimportant details. In practice, the crack dimensions are sufficiently small; they occur in large numbers in the solids; and are somewhat of similar

shape. This permits us to write their distribution function in a medium by an *appropriate continuum variable*.

One way to devise such a variable is to describe the microcracks distribution at a material point by the ratio of the area of the cracks to the total area in every plane, Figure 2.4, (Murakami 1981; Krajcinovic 1984). Ilankamban (1985) showed that this function must always be positive and definite, since it is simply a ratio of areas and it should admit the symmetry condition. Furthermore, if one assumes that the individual materials have cracks of similar shapes, the average relative displacement of given crack faces at the material point can be described by a vector valued function. There are many forms for expressing both quantities in terms of the material point and its normal (Krajcinovic 1985). However, almost all phenomenological damage models (Ilankamban 1985; Krajcinovic 1985) drop the dependency of the crack distribution and its displacement functions on its position. This assumption complies with the continuity postulate of the state variables defined by Equation [2.8].

The damage parameter (sometimes called crack density) can now be obtained by combining the two functions, in two possible ways (Krajcinovic 1985). The first option (Kachanov 1958) is to assume particular forms of those functions. In addition, the members of the assumed sets of functions can be evaluated as functions of the loading history. The form of those function may also depend on the problem dimension (e.g. one, two or three dimension) for which the model is formulated.

Another alternative is to define the damage measure (parameter) as a second, fourth, or eighth order tensor (Vakulenko and Kachanov 1971; Dragon 1985, Chaboche 1985). The components of the damage tensor are the crack densities in a given plane(s). The damage tensor must admit the same rule of transformation as those for stresses and strains which could be used to obtain the crack density in a particular plane(s).

Using thermodynamic arguments (Rice 1975; Krajcinovic 1984), it can be proved that (i) the damage (crack area) would be equivalent to a conjugate force acting on a particular plane, and (ii) the damage tensor components are equivalent to those of the associated

stress tensor. Depending on the amount of information that may be stored in the tensor and any symmetry it may display, the damage models are labeled as scalar; vectorial, or tensorial damage-models.

2.5.3.2 SCALAR REPRESENTATION

In the scalar models, the damage is assumed to be invariant with respect to the frame of reference. As described, the pioneering concept of L. Kachanov (1958), which uses the ratio of the effective area to the total area of the element as the damage measure is, in fact, a scalar model. In its early application, this model failed to predict tertiary creep since it assumed that the creep rate did not depend on the damage state. Therefore, Rabtanov (1968) utilized a kinematics law for the growth of the damage evolution equation in order to change the damage state of the material as the applied load increased. The newly introduced law was assumed to be a monotonically increasing function of the strains, ϵ . The introduction of this law was somewhat arbitrary since it was based neither on a theoretical hypothesis nor on experimental evidence. This clearly shows the importance of the correct choice of the damage evolution laws. At this juncture, it should be mentioned that the choice of a specific damage evolution law for any form of phenomenological damage models (e.g. scalar, vector, or tensor) is still unresolved (Krajcinovic 1984).

The basic definition of scalar damage models limits application to modelling one dimensional damage or isotropic damage. Thus, early models (Davison 1977; Krajcinovic 1981; 1984; 1988) evolved around the descriptions of the creep behaviour of ductile materials (metals) with dilute concentration of spheroid voids. For example, Lemaitre and Chaboche (1975, 1978, 1985) formulated an isotropic damage model for predicting the softening phenomena of a steel specimen subjected to uniaxial tensile loading. The damage evolution law was obtained from the difference in the elastic moduli of successive loading and unloading of a uniaxially loaded tensile steel specimen. Others, (e.g. Hult 1973; 1974, Broberg 1974) used Rabotnov's (1963) creep-damage model after employing different damage evolution laws. The laws used in the latter models were obtained from

experiments designed specifically to measure the amount of damage in specimens subjected to uniaxial creep loading. These models have been extended to predict the behaviour of other problems such as torsion-bending, creep-fatigue interaction, and creep rupture under non-proportional loading for metals (Krajcinovic 1984).

Later, Hanson and Hult (1977) (who apparently used the name *continuum damage mechanics*) extended the scalar damage models to analyze the behaviour of brittle materials. They applied them to study plain concrete bars subjected to tension. Once again, the damage evolution laws were obtained from experiments (e.g. split cylinder tests) with slight modifications. These modifications were concerned with measuring stiffnesses and compliance of the specimens for the loading; unloading; and reloading. Using the same principle, Krajcinovic (1979); Lemaitre and Mazars (1980); and Loland (1980) studied the bending of concrete beams. Grady and Kipp (1980) solved the one-dimensional geomechanical problems utilizing a scalar damage variable. However, micrograph analyses (Frenando 1987; Slate 1983) have shown that (i) the damage of a material is directional in nature, and (ii) cracks or micro-defects may change their mechanism when the material is subjected to multi-dimensional non-proportional, cyclic, or dynamical loading. Obviously, these situations cannot be modelled using the scalar representation of the damage. Therefore, it can be concluded that scalar damage models are restrictive and cannot be used in general applications.

2.5.3.3 VECTOR REPRESENTATION

At the same time, the vectorial nature of the degradation of materials prompted investigators to recast the Kachanov's hypothesis in a continuum sense (Davison and Stevens 1973; Krajcinovic 1981; Krajcinovic and Fonseka 1981) and to develop models with a vector representation of the damage variable. Fonseka and Krajcinovic (1981); Krajcinovic and Selvaraj (1983); Wu (1983); Krajcinovic (1984); Suaris and Shah (1984) used the vector representation of damage to analyze concrete materials. Krajcinovic and Selvaraj (1984) applied their model to creep problems while Talreja (1984) employed his model to analyze composite materials. All these models are only capable of depicting the

damage of initially isotropic materials and they have to retain a minimum of orthotropic symmetry for any loading.

To represent the general three dimensional case (Ilankamban 1985), three mutually perpendicular vectors can be utilized which are assumed to be enough to simulate all the directional degradation of the material. Depending on the smoothness of the proposed crack intensity function (the damage parameters), a sufficient amount of information can be stored within it to be applicable at specific finite number of orientations. The damage is assumed to evolve according to a postulate similar to Drucker's stability postulate (Drucker 1959) for rigid-plastic material. This assumption and consideration of thermodynamics, permits the establishment of a damage surface in the stress-strain spaces. The rate of change of the increment of damage is assumed to be normal to it. It is not clear how one can experimentally obtain this surface, and therefore, different surfaces have been employed to fit certain experimental data. It is the writer's opinion that new experimental techniques need to be developed and directed towards measuring the damage surfaces for different load conditions.

Fernando (1987) designed a specific testing machine for uniaxial loading to measure damage. This machine utilized ultrasonic wave transmission to monitor damage growth. The damage growth was inferred from the amplitude attenuation of the received waves and it was reported that this technique provides a better indication of damage than the more common pulse velocity method. The damage coefficient, d_c , was determined according to

$$d_c = \frac{|\bar{A}_o - A|}{|\bar{A}_o|} \quad [2.9]$$

where $\bar{A}_o$ and A are vectors with the amplitude of the waveform as components, received at the unstressed and stressed states, respectively.

Functions of the principal values of the damage tensor and of the normal and shear strains on the cracked planes were used to include the effects of crack growth on the strain

energy. The damage growth rate was obtained by using a bounding surface and a loading surface. These surfaces were defined in terms of the thermodynamic force conjugates of the principal values of the damage tensor.

From the experimental program Fernando (1987) drew some important conclusions. First, during uniaxial compression tests he monitored damage by transmitting ultrasonic waveforms perpendicular to the loading axis. He observed that the pulse velocity began to decrease slowly when the specimen stress reached 70% of its strength and continued to steadily decrease up to 90% of the peak stress. After 90%, the velocity reduction was quite rapid. It was also reported that the decrease in velocity is more a function of the crack opening than of the development of microcracks. The amplitude of the waves, on the other hand, was found to be more sensitive to the development of microcracks.

Finally, he reported that the use of pulse velocity equipment, without an oscilloscope gives misleading information, partly due to the very high attenuation of the nominal transducer frequency near failure. The experimentally obtained damage parameters were used in his damage model. However, this model's predictions for macroscopic quantities such as uniaxial stress-strain curves were not in accord with those obtained from the same experiments. He argued that the testing apparatus needs some improvements and enhancement but those improvements were not fully described. Once again, these observations indicate that experimental methods to evaluate the state of damage in a material under loading requires further study. It is also not easy to assume a damage surface that may be able to describe all loading cases and paths. Therefore, as mentioned earlier, the micromechanical concept was adopted here in order to avoid the need for a damage surface.

2.5.3.4 SECOND ORDER TENSOR REPRESENTATION

The aforementioned discussion on both scalar and vector damage models indicates clearly that such representations will not be able to capture the anisotropic nature of damage, but at most will capture an orthotropic damage situation. Therefore, Vakulenko and Kachanov (1971); Murakami and Ohno (1981); Dragon and Mroz (1979); and Batten

(1983) employed symmetric second order tensors in developing their models for brittle materials. They assumed that the damage variable is a monotonic function of the two basic quantities of a propagating crack. Therefore, the same change in the internal damage state of a material can be effected by changing either the crack area or the relative displacement of crack faces.

Earlier in this chapter, it was stated that monitoring of each individual crack will yield an exact representation of the propagation of damage. Thus, Lecki and Onat (1981) introduced an infinite set of even order crack intensity tensors to describe the internal state variables. They also introduced a similar set of infinite number of even order tensors to locate the sites of crack growth. It is obvious, however, that the proposed representation is impractical as recognized by the authors themselves, and therefore, they used only a second order tensor in their applications. This tensor was similar to the one proposed by Murakami and Ohno (1981). Once again, those authors did not show a practical procedure which can be employed to experimentally obtain the damage parameters needed for their models.

2.5.3.5 HIGHER ORDER TENSOR REPRESENTATION

Finally, fourth and eighth order tensor representations of the damage state variable have been introduced. For example, Chaboche (1978, 1979) proposed a phenomenological fourth order damage tensor. The main phenomenological hypothesis was based on the effective stress concept. Lemaitre and Chaboche (1978); Cordebois and Sidoroff (1982); Lemaitre (1984, 1985, 1986); Mazars (1984); Chow and Wang (1987 a); among others postulated elastic-damage potential energy based on the concept of effective stress. The results of those formulations were that the Poisson's ratio remained constant for both material phases; (i.e. virgin and damaged).

The latter outcome is very restrictive and, of course, contradicts experimental findings which show that for specimens subjected to uniaxial or biaxial tensile monotonically applied loading, the effective (apparent) Poisson's ratio decreases as the applied loading increases. On the contrary, when the applied loads become compressive in nature, the

Poisson's ratio increases. The limitations placed on this type of models stem from the fact that the effective stress concept is of one-dimensional nature. Although others (Lemaitre 1984; Chow and Wang 1987 a; b; Ju 1989; 1991) have attempted to remove this anomaly, we shall not pursue the matter further here. It can, however, be stated that phenomenological damage-based models are developed using elaborate procedures from thermodynamic and basic mechanics laws. Nevertheless, the choice of state variable (damage parameter) and the form of the free strain energy is arbitrary. Furthermore, those models are sometimes empirical and ambiguous in their evolution laws and often contradict one another.

As indicated earlier, experimental procedures are lacking which can be used to obtain the continuous functions of the state variable which represent the damage.

2.6 CLOSURE

The behaviour of concrete materials cannot be realistically described by continuous functions of a given state variable regardless of the complexity of the mathematical representation of damage. This is because generally concrete, composites, and some polycrystalline materials essentially deform through the process of evolution and nucleation of pre-existing cracks. Micromechanical-damage based models can be developed by employing the basic laws of mechanics at the meso-scale of the material which will eliminate much of the empiricism associated with other models, including the phenomenological-damage based models.

It is important at this juncture to mention that phenomenological-damage based models have not been implemented in general finite element programs. Only Lemaitre (1984, 1985, 1986) showed schematically the implementation of his models but with insufficient verification. Chow and Wang (1987 a, b) also implemented a phenomenological model in a finite element scheme. However, the application of the program is restricted to those classes of problems for which sufficient experimental data are available to be able to establish an empirically derived damage surface.

From the above discussion, it can be concluded that the propagation of cracks reduces the stiffness of concrete, which is manifested by the decreasing slope of the stress-strain curve. In addition, concrete shows plastic flow and yield-like response if it is subjected to overall compressive loading. This plastic flow and yield of a basically brittle composites does not occur in the same fashion as those of ductile metals. Experiments by Zaitsev (1983); Mindess (1984); and Horii and Nemat-Nasser (1985), among many others, have shown that concrete-like materials display transition from brittle to ductile behaviour if the principal compressive stress ratio exceeds a critical value.

The aforementioned discussion shows that a *unified concrete model* should account for the microcracks evolution and propagation, and the plasticity due to confinement by recognizing essentially the brittle nature of concrete. Attempts have been made in the past to combine the continuum damage mechanics and plasticity theories to obtain a unified concrete theory. Ortiz (1985); Simo and Ju (1987); Yazdani and Schreyer (1990), among others, combined either plasticity or visco-plasticity and phenomenological damage models in order to formulate proper constitutive laws for concrete. In those models, a damage and a yield criterion were empirically established in order to fit certain experimental data, but no experimental procedure was proposed to obtain the continuous functions of the state variables which represent damage. Instead, some data fitting to available experimental results was used to obtain the damage functions. Despite the mathematical elegance of phenomenological damage-based constitutive models, establishing the crack evolution criterion is a serious issue yet to be resolved (Krajcinovic 1984).

In view of the foregoing discussion, the present study combines a process micromechanical damage model and the theory of plasticity in order to obtain a unified concrete theory. The model is implemented in a nonlinear finite element scheme in order to determine the full response of concrete structures subjected to various stress combinations, including cyclic and non-proportional loading. The results of the analyses are compared with available experimental data.

3. CONCRETE DAMAGE MECHANICS

3.1 GENERAL

Damage of concrete materials due to the application of loads can be assessed by solving the equations of mechanics at the meso-scale, since the mechanisms that consume most of the applied energy are only activated at this level. However, the macro-scale behaviour is the manifestation of the collective effect of those mechanisms on the micro-structure of the material. As mentioned earlier, the mechanisms comprise the evolution of defects and the rearrangement of the different material constituents in the new deformed configuration (Ilankamban 1985; Krajcinovic 1988). To evaluate the global mechanical response of the macro-scale (i.e. laboratory specimens, structures, etc.), a mapping of the field variables (e.g. cracks, shear bands, flaws, etc.) can be used between the two levels (Kunin 1982); such a mapping requires adopting a proper homogenization procedure (Kunin 1982). To employ the principles of mechanics in deriving the required model, within the RVE, the governing variables can be assumed to be smooth and continuous up to their second derivatives with respect to their positions in the material space (Muskhelishvili 1963). This means that the elasticity and plasticity theories can be employed at this scale to find the mechanical response of a material due to the existence of a single crack or an ensemble of cracks (Wu 1985). It was discussed in Chapter 2 that the RVE has physical dimensions, which justifies the use of the theory of fracture mechanics. In this chapter, the mathematical derivation of the elasto-plastic damage constitutive laws will be presented. It is assumed that the nonlinearities of concrete are the result of (i) crack formation and propagation, and (ii) plasticity in some special loading cases. Therefore, the steps shown

later will employ the elasticity and plasticity theories, linear fracture mechanics, and continuum damage mechanics in deriving the necessary relations.

3.2 PROPOSED ELASTO-PLASTIC DAMAGE MODEL

The mathematical solution of the equations of the relevant mechanics at the meso-scale is the key to modelling the behaviour of a given heterogeneous material. Thus, in the following sections of this chapter, the solution to the pertinent equations of equilibrium and continuity for a two dimensional solid will be presented. This is effected by splitting the total strain into three components, viz. elastic, damage, and plastic strains. Subsequently, the incremental elasto-plastic damage mechanics constitutive relationships are derived for general 2-D applications. The proposed relationships are similar in structure and form to those proposed by others (e.g. Krajcinovic 1989; Ju 1990). However, previous phenomenological (e.g. Lemaitre 1984; Talreja 1984; Krajcinovic 1988) and micromechanical (e.g. Fanella and Krajcinovic 1988; Ju 1990) CDM theories have developed only the secant compliance and stiffness of the cracked composite. This limitation introduces some problems when implementing them in general purpose finite element programs since these programs generally use the tangential stiffness in deriving the finite element stiffness matrices. Therefore, this study presents a relationship between the secant and tangent stiffness, or compliance, and it gives the explicit form for that relation. It is worthwhile to mention that the latter relationship does not require any changes to the existing damage methods or to the procedures of the finite element programs, therefore, its numerical implementation is relatively easy.

In the subsequent sections, the derivation of the crack (damage) compliance under different load scenarios will be presented. The derivation accounts for loading, unloading, and reloading the cracked material. The exact amount of propagating and stable cracks - within the confines of the assumptions made- are calculated properly for the latter conditions. This means that the model is capable, for example, of tracing the nonlinear behaviour of an initially cracked concrete panel subjected to overall biaxial compression. These characteristics of the model allow the finite element program to trace the cyclic

behaviour of test specimens and structures. Another advantage of utilizing this model is that it shows clearly when damage mechanics cannot be used because the material exhibits yield-like deformation and the existing cracks do not influence its behaviour (Horii and Nemat-Nasser 1984). According to this model's prediction, for yielding to occur the concrete element may be subjected to biaxial compressive stresses and for all planes within the material, the applied sliding shear must be less than or equal to the frictional shear resistance.

3.2.1 GOVERNING EQUATIONS OF SOLID WEAKENED BY MICROCRACKS

Consider a representative volume element (RVE) of concrete including the microcracks Figure 3.1 in which the element behaviour is governed by the general requirements of equilibrium

Figure 3.1: Representative volume element (RVE) of concrete with microcracks.

$$\begin{aligned}\sigma_{ij,j} &= 0 \\ \sigma_{ij} \cdot n_j &= T_i\end{aligned}\tag{3.1}$$

and the strains are given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad [3.2]$$

where σ_{ij} ($i, j = 1, 2, 3$), is the homogeneous far field macro-stress tensor that is in equilibrium with the external applied tractions and n_j is the normal to the surface, A , of the RVE, ε_{ij} is the corresponding macro-strain field while u_i is the displacement vector of the body. The comma indicates the derivative of the function with respect to the coordinate argument that follows. As mentioned earlier, the size of the chosen element should be large enough to contain all the different components and small enough to be considered as a point within the material space. This is often called a *representative volume element* or unit cell. For concrete materials, the cracks and the aggregates are assumed to be uniformly distributed within this volume, and the cracks are dilute in concentration.

In Equations [3.1] and [3.2], it is assumed that (i) the behaviour is isothermal, and (ii) body forces are absent. The total strain tensor, ε_{ij} , of damaged (cracked) concrete is decomposed into three components, viz. an elastic strain, ε_{ij}^e , a damage strain, $\varepsilon_{ij}^d = \varepsilon_{ij}^{cr}$, where the superscript, *cr*, denotes crack, and a plastic strain, ε_{ij}^p , i.e.

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^{cr} + \varepsilon_{ij}^p \quad [3.3]$$

It is recognized that other forms of strain decomposition are possible. Furthermore, the crack strain will comprise both reversible and permanent strains. An existing stable crack that opens under increased load will close upon unloading, accordingly it will behave elastically, but an existing crack that propagates or a newly formed crack will introduce permanent and irreversible deformations. Thus the identification of crack or damage strains as elastic and permanent in a model may be a matter of reference, but in either case the proper energy dissipation mechanisms must be considered. In the proposed model we combine the crack strains with the elastic strains for the sake of convenience, but at no points the assumption is made that the crack strains are elastic. Subsequent explanations

in this article will show the model correctly accounts for both the elastic and inelastic components of the crack strains.

3.2.1.1 ELASTIC STRAIN CONTRIBUTION

The stresses, σ_{ij} , and the elastic strains, ϵ_{ij}^o , in the matrix can be related as

$$\sigma_{ij} = D_{ijmn}^o \epsilon_{mn}^o \quad \epsilon_{ij}^o = C_{ijmn}^o \sigma_{mn} \quad [3.4]$$

where D_{ijmn}^o and C_{ijmn}^o are the elastic stiffness and compliance tensors, respectively, reflecting all symmetries and other properties of the material. If elastic and damage strains are combined, Equation [3.3] can be written in incremental form as

$$d\epsilon_{ij} = d\epsilon_{ij}^t + d\epsilon_{ij}^p \quad [3.5]$$

where ϵ_{ij}^t is the sum of the elastic and crack strains. Let us relate the increments of the latter strains to the far-field stress increment, $d\sigma_{ij}$, according to

$$d\sigma_{ij} = \bar{D}_{ijmn} d\epsilon_{mn}^t = \bar{D}_{ijmn} (d\epsilon_{mn}^o + d\epsilon_{mn}^{cr}) \quad [3.6]$$

where $\bar{D}_{ijmn}$ is, as yet, the unknown effective stiffness that reflects the impact of crack on concrete mechanical properties. Making use of [3.5] gives

$$d\sigma_{ij} = \bar{D}_{ijmn} (d\epsilon_{mn} - d\epsilon_{mn}^p) \quad [3.7]$$

3.2.1.2 PLASTIC STRAIN

According to the theory of plasticity, the incremental plastic strain, $d\epsilon_{ij}^p$, can be obtained from a *plastic potential* function, $g(\sigma_{ij}, \epsilon_{ij}^p, \kappa)$, as

$$d\epsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}} \quad [3.8]$$

where $d\lambda$ is a scalar proportionality constant. Assuming a *yield* function, $f(\sigma_{ij}, \epsilon_{ij}^p, \kappa)$, that is

$$f(\sigma_{ij}, \epsilon_{ij}^p, \kappa) = 0 \quad [3.9]$$

where κ is the hardening parameter, the consistency condition requires that

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} + \frac{\partial f}{\partial \varepsilon_{ij}^p} d\varepsilon_{ij}^p + \frac{\partial f}{\partial \kappa} d\kappa \quad [3.10]$$

Notice that the yield function assumes *mixed-hardening rule* (Hodge 1957), but it can be reduced to either isotropic (anisotropic) or kinematic hardening.

Using the stress increment, $d\sigma_{ij}$ of Equation [3.7] and the flow rule given by Equation [3.8], Equation [3.10] gives

$$\frac{\partial f}{\partial \sigma_{ij}} \bar{D}_{ijmn} (d\varepsilon_{mn} - d\varepsilon_{mn}^p) + \frac{\partial f}{\partial \varepsilon_{ij}^p} d\lambda \frac{\partial g}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon_{ij}^p} d\lambda \frac{\partial g}{\partial \sigma_{ij}} = 0 \quad [3.11]$$

or

$$d\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} \bar{D}_{ijmn} d\varepsilon_{mn}}{h + \frac{\partial f}{\partial \sigma_{kl}} \bar{D}_{klpq} \frac{\partial g}{\partial \sigma_{pq}}} \quad [3.12]$$

where h is the hardening parameter given by

$$h = - \frac{\partial f}{\partial \varepsilon_{ij}^p} \frac{\partial g}{\partial \sigma_{ij}} - \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial \varepsilon_{ij}^p} \frac{\partial g}{\partial \sigma_{ij}} \quad [3.13]$$

Substituting for $d\lambda$ from Equation [3.12] into equation [3.8], yields

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial g}{\partial \sigma_{ij}} = \frac{\frac{\partial f}{\partial \sigma_{rs}} \bar{D}_{rsmn} d\varepsilon_{mn}}{h + \frac{\partial f}{\partial \sigma_{kl}} \bar{D}_{klpq} \frac{\partial g}{\partial \sigma_{pq}}} \frac{\partial g}{\partial \sigma_{ij}} \quad [3.14]$$

The stress-strain relations of a damaged work-hardening plastic solid can be now obtained from Equation [3.7] as

$$d\sigma_{ij} = D_{ijmn}^p d\varepsilon_{mn} = (\bar{D}_{ijmn} - D_{ijmn}^p) d\varepsilon_{mn} \quad [3.15]$$

in which the plastic stiffness tensor, D_{ijmn}^p , is

$$D_{ijmn}^p = \frac{\bar{D}_{ijtu} \frac{\partial f}{\partial \sigma_{rs}} \frac{\partial g}{\partial \sigma_{tu}} \bar{D}_{rsmn}}{h + \frac{\partial f}{\partial \sigma_{kl}} \bar{D}_{klpq} \frac{\partial g}{\partial \sigma_{pq}}} \quad [3.16]$$

where $\bar{D}_{ijmn}$ reflects the impact of damage on the plastic stiffness tensor.

3.2.1.3 YIELD CRITERION

The plastic stiffness tensor of a damaged material is similar in form to that of an undamaged material. The main difference is in the use of the effective damage tensor in lieu of the elasticity tensor in the conventional theory of plasticity. However, the determination of the yield function will require a special setup. Current data are obtained from experiments in which no attempt has been made to separate plastic strains from damage or crack strains. A possible experimental procedure for obtaining the yield function would be to suppress crack growth and formation by applying the necessary stress state. For instance, micromechanics reveals that for certain stress combinations the cracks will not propagate and thus a completely damage free response would be predicted as long as the stress ratios are maintained constant. However, experimental evidence reveals that the material does not remain elastic, hence the observed inelasticity is not due to crack evolution, but rather due to plastic flow. Accordingly, if the yield criterion is based on data obtained from such tests, it would not include the effect of crack evolution.

In the present treatment we recognize the lack of such data, but even if existing concrete yield functions are utilized, by considering the reduction in the elastic stiffness due to damage engendered by crack evolution, one still notices the significance of both crack evolution and flow on the total response of concrete elements. Clearly, in many situations either crack evolution or plasticity may be the dominant energy dissipation mechanism, but a general model which recognizes their simultaneous effect will better represent the behaviour of concrete structures under general loading conditions.

While as stated earlier, the development of a proper yield criterion for concrete has to occur yet, any of the available yield surfaces (see Chen 1988) can be used where in the applications reported in this thesis the von-Mises yield criterion is employed. Concrete is

assumed to be an elastic-plastic hardening material under overall compressive stress fields. Since the model shows that plastic strains exist only in some cases of applied biaxial compression, the concrete yielding stress is assumed to be a function of its compressive strength, f'_c . A value of $f_y = \sqrt{3}(0.85f'_c)$ is used for the example given later. This value was chosen so that the model's prediction of the concrete failure stress would coincide with the failure envelope obtained by Kupfer and Gerstle (1977) during their experimental studies. This yield stress gave excellent results for predicting other results. At the same time, this issue requires further verification and validation with new examples. It may be noticed that the proposed model accounts for the effect of damage on the stiffness of an elasto-plastic material. This is particularly important in the case of complicated load paths where the onset and propagation of damage may precede any plastification.

3.2.1.4 DAMAGE STRAIN CONTRIBUTION

The presence of an inclusion (i.e. in this case the crack) introduces disturbances into the elastic stress and strain fields of the matrix. The introduced stress and strain disturbances are functions of the geometry and mechanical properties of the inclusion (Eshelby 1957), and they must satisfy the general statement of Equations [3.1] and [3.2]. The mean stresses and strains, σ_{ij}^s and ε_{ij}^s , of an RVE can be written as

$$\sigma_{ij}^s \equiv \frac{1}{V} \int_{V'} \sigma_{ij} dV \quad \varepsilon_{ij}^s \equiv \frac{1}{V} \int_{V'} \varepsilon_{ij} dV \quad [3.17]$$

where the superscripts i and s take the symbol m or cr for the matrix and the inclusion (crack), respectively, and V is the total element volume. The strains in Equation[3.17] can be rewritten for a given crack using the divergence theorem (Horii and Nemat-Nasser 1985) as

$$\varepsilon_{ij}^{cr} = \frac{1}{V} \int_A \frac{1}{2} (u_i n_j + u_j n_i) dA_c \quad [3.18]$$

where A_c is the representative-surface area (RSA). The crack strains can be written in terms of the crack compliance, C_{ijkl}^{cr} , and applied far-field stresses, σ_{kl} , as

$$\varepsilon_{ij}^{cr} = \frac{1}{V} \int_{A_c} \frac{1}{2} (u_i n_j + u_j n_i) dA_c = C_{ijkl}^{cr} \sigma_{kl} \quad [3.19]$$

It follows that the elastic strains and the damage strains may be evaluated according to

$$\varepsilon_{ij}^t = \varepsilon_{ij}^o + \varepsilon_{ij}^{cr} = (C_{ijmn}^o + C_{ijmn}^{cr}) \sigma_{mn} = \bar{C}_{ijmn} \sigma_{mn} \quad [3.20]$$

where $\bar{C}_{ijmn}$ is the effective overall compliance of the damaged material. The evaluation of the effective elastic moduli, Equation [3.20], of the cracked solid should be performed using a trial and error procedure since the crack induced compliance, C_{ijmn}^{cr} , is a function of crack size and orientation, which vary with the level of loads and the as yet unknown current effective damage compliance, $\bar{C}_{ijmn}$. This can be performed using one of the effective elastic moduli methods explained later. In this study, the self-consistent model will be utilized to obtain, $\bar{C}_{ijmn}$, since it accounts for moderate microcrack interaction. For non-proportional or cyclic loadings, Equation[3.20] should be rewritten in incremental fashion (Kunin 1983; Krajcinovic 1989)

$$d\varepsilon_{ij}^t = \bar{C}_{ijmn} : d\sigma_{mn} + d\bar{C}_{ijmn} : \sigma_{mn} \quad [3.21]$$

or

$$d\varepsilon_{ij}^t = \left[\bar{C}_{ijmn} + \frac{\partial C_{ijmn}^{cr}}{\partial \sigma_{ij}} \sigma_{mn} \right] : d\sigma_{mn} \quad [3.22]$$

Noting that

$$\frac{\partial}{\partial \sigma_{ij}} [\bar{C}_{ijmn}] = \frac{\partial}{\partial \sigma_{ij}} [C_{ijmn}^{cr}] \quad [3.23]$$

because the elastic material compliance, C_{ijmn}^o , is not a function of stress.

Equation [3.22] reflects the impact of the presence of the crack on the overall response. It is significant to notice in Equation [3.22] that the crack (damage) strain depends on the total effective stress, σ_{ij} , and it could be an irreversible deformation process (Krajcinovic

1988). Employing matrix notation, Equation [3.22] can be rewritten in incremental form and in a notation more suitable for the finite element computation scheme

$$\Delta \boldsymbol{\varepsilon}' = \left[\bar{\mathbf{C}} + \frac{\Delta \bar{\mathbf{C}}}{\Delta \sigma} \sigma \right] \Delta \sigma \quad [3.24]$$

where the boldface symbols represent vectors and matrices corresponding to those on the left-hand side of Equation [3.22]. The compliance given in Equations [3.22] and [3.24] is, in fact, the tangential compliance.

Figure 3.2: Stress-strain curve (displacement control test).

To prove the latter assertion, let us first consider the typical compressive uniaxial stress-strain curve of a concrete cylinder shown in Figure 3.2. Such a stress-strain curve is often obtained by using *displacement controlled testing machines*. The strains corresponding to stresses, σ_i and σ_{i+1} , may be evaluated according to the following constitutive relationships

$$\varepsilon_i = C_i \sigma_i \quad [3.25]$$

and

$$\varepsilon_{i+1} = C_{i+1} \sigma_{i+1} \quad [3.26]$$

In the preceding two sets of equations and all subsequent equations within this section, repeated indices do not imply summation. Note that C_i and C_{i+1} are secant compliance corresponding to stresses, σ_i and σ_{i+1} .

Subtracting Equation [3.25] from Equation [3.26] yields

$$\varepsilon_{i+1} - \varepsilon_i = C_{i+1} \sigma_{i+1} - C_i \sigma_i \quad [3.27]$$

Equation [3.27] can be written as

$$\Delta \varepsilon = C_{i+1} \sigma_{i+1} - C_i \sigma_i \quad [3.28]$$

or

$$\Delta \varepsilon = C_{i+1} [\sigma_i + \Delta \sigma] - C_i \sigma_i \quad [3.29]$$

which can be rearranged to give

$$\Delta \varepsilon = C_{i+1} \Delta \sigma + [C_{i+1} - C_i] \sigma_i \quad [3.30]$$

Dividing the latter equation by $\Delta \sigma$ gives

$$C_i = \frac{\Delta \varepsilon}{\Delta \sigma} = C_{i+1} + \frac{\Delta C}{\Delta \sigma} \sigma_i \quad [3.31]$$

where C_{i+1} is the current secant compliance and ΔC represents the change of the compliance with respect to the applied incremental stresses, $\Delta \sigma$, and $\Delta \varepsilon$ is the incremental strain. Clearly, in the limit C_i would be the tangent compliance. By comparing Equation [3.6] and Equation [3.24] it is evident that the tangential form of the effective stiffness matrix $\bar{D}_{ijmn}$ is

$$D' = \left[\bar{C} + \frac{\Delta \bar{C}}{\Delta \sigma} \sigma \right]^{-1} \quad [3.32]$$

Substituting D' from Equation [3.32] and D_{ijmn}^p from Equation [3.16] into Equation [3.6], leads to the following constitutive relation for an elasto-plastic damaged material

$$\Delta\sigma = \left[D' - \frac{D' \frac{\partial f}{\partial \sigma} \left\{ \frac{\partial g}{\partial \sigma} \right\}^T D'}{h + \left\{ \frac{\partial f}{\partial \sigma} \right\}^T D' \frac{\partial g}{\partial \sigma}} \right] \Delta\epsilon \quad [3.33]$$

It is observed that all the quantities in Equation [3.33] are either known or can be calculated. For example, the increments of applied stress are known (selected) while the change of crack compliance, $[C^c]$ can be calculated at a given load level. It should be mentioned that writing the damage elasto-plastic constitutive relationships in the form of Equation [3.33] is very useful for implementation in finite element programs. Thus, the current study implements Equations [3.24] and [3.33] in the general purpose finite element program Nonlacs. It should be pointed out that the proposed elasto-plastic damage-based relationships can model pure elasto-plastic behaviour. The proposed model requires the calculation of the effective (damaged) compliance and stiffness of cracked concrete. This is achieved by utilizing micromechanics within the frame work of the self-consistent method (Kunin 1983) as described below. To simplify the presentation, in the following, the 2-D version of Equation [3.33] will be developed.

3.2.1.5 OVERALL COMPLIANCE OF A SOLID WITH A SINGLE CRACK

To obtain the damage induced compliance and stiffness of a solid containing a single crack, we must know the crack shape. Experiments (Murakami 1990) have shown that the shape of the crack (inclusion) has little impact on its behaviour, provided it has the same size and is equidistant from the external applied stresses. Therefore, we will allow the crack to be spherical or penny-shaped since closed form solutions of their deformations for different loading conditions exist. The displacement jump of spherical or penny-shaped inclusions is available in closed analytic form (Mura 1982) and therefore, one can employ the relevant Green's functions to obtain the compliance of the inclusion.

According to Green's theorem, the far field applied stress, σ_{ij} , is related to the inclusion eigenstrains (Mura 1982).

For the sake of completeness, let us write the procedures by which one can obtain the inelastic compliance of cracks through the application of Green's functions. Assume that the displacement component at a crack k in the x_i direction at point x due to an applied unit load at point ξ in direction x_j is $G_{ij}(x, \xi)$, then the displacement due to applied stress, σ_{jk} , can be written as

$$u_{i(x)} = \int_A G_{ij}(x, \xi) T_j dA = \sigma_{jk} \int_A G_{ij}(x, \xi) n_k(\xi) dA \quad [3.34]$$

Substituting the value of $u_i(x)$ Equation [3.19] yields

$$C_{ijmn}^{cr} = \frac{1}{2V} \int_A \left\{ \int_A \left[G_{im}(x, \xi) n_j(x) n_n(\xi) + G_{jm}(x, \xi) n_i(x) n_n(\xi) \right] dA \right\} dA. \quad [3.35]$$

The analytical evaluation of the Green's function, $G_{ij}(x, \xi)$, for a solid with an arbitrary crack-like defect is extremely difficult (Horii and Nemat-Nasser 1983). However, for some special shapes of cavities and remote stress combinations, Mura (1982) gives explicit expressions for such functions. At this juncture, it should be remembered that the current work evaluates the crack opening displacements, $u_i(x)$ and $u_i(y)$, as well as the inelastic damage compliance from the perspective of fracture mechanics. This is because of the simplicity of the method of fracture mechanics, and due to the fact that the crack evolution laws derived from it are clear, simple and render themselves to easy implementation in the finite element method.

3.2.1.6 COORDINATES TRANSFORMATION

The various quantities (stress, strain, compliance and stiffness) are calculated in the crack local coordinate system which must be subsequently transformed to the principal material coordinates. To establish the basic equations for transformation, let us assume that the crack shape is elliptical and therefore its geometrical specification within the material space can be completely defined via a vector of 8 scalars (Figure 3.3). Three

scalars define its position, three Euler's angles circumscribe its orientation, and two scalars characterize its size. Let the crack diameter be $2a$ and its local coordinate system be defined by the triad e'_1, e'_2, e'_3 such that e'_3 coincides with the unit normal of its surface, A_c , and e'_1 is parallel to the x_1, x_2 plane (Figure 3.3). This new unit vector, e'_i , can be related to the fixed vector e_i according to

Figure 3.3: Crack coordinate system.

$$e_i = T_{ij} e'_j \quad [3.36]$$

where the rotation tensor, T_{ij} , takes the value

$$[T] = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\cos \theta \sin \phi & \cos \theta \cos \phi & \sin \theta \\ \sin \theta \sin \phi & -\sin \theta \cos \phi & \cos \theta \end{bmatrix} \quad [3.37]$$

where $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq 2\pi$ are the Euler's angles (Figure 3.3). Thus, if we evaluate the inelastic compliance, C'^{cr}_{ijkl} , in the local crack coordinate system (primed), the corresponding quantity in the global coordinate system (unprimed) can be obtained as

$$C^{cr}_{ijkl} = T_{ip} T_{jq} T_{kr} T_{ls} C'^{cr}_{pqrs} \quad [3.38]$$

where the rotation tensor T_{ij} is given by Equation [3.37].

3.2.2 OVERALL COMPLIANCE OF A SOLID WITH AN ENSEMBLE OF CRACKS

In reality, the representative volume element (RVE) of volume V contains a very large number N of cracks each of volume $V_{\alpha}, (i = 1, 2, \dots, N)$. During their evolution (growth), they may or may not interact (Kachanov 1982). Theoretically, the exact solution can be obtained by solving the system of N coupled equations (Kunin 1983). In these equations, the external stress field, σ'_{ij} , of the i th inclusion (microdefect) of volume V_i within the RVE is written in terms of Green's function of the internal stresses and the stiffness tensors as well as the far-field macro-stress, σ_{ij} , and the eigenstrains (i.e. crack strains), ϵ''_{ij} , respectively.

These equations account for the interaction among the different cracks and assume that the macro-stress field is homogeneous within the RVE. Of course, such a technique is neither feasible nor desirable because of the complexity associated with recording the crack data associated with many cracks and their evolution. However, if instead of solving N number of equations, a single formula can be used within a given scheme that may account for the crack interaction, a reasonably accurate but not exact solution would be possible.

3.2.3 METHODS OF SOLVING THE RESPONSE EQUATIONS

Since the number of cracks is very large and they are randomly distributed, it is possible to obtain a solution by applying the principles of probability. This strategy inherently transforms the actual solid into an effective (equivalent) one and maps its field variables into the original state of the material. Typically for these type of models it is assumed that an effective continuum (EC) exists. Its general theory assumes that the external fields of each crack weakly depend on its exact position in the RVE, and is described fully by Kunin (1983). Therefore, these models assume weak or no microcrack interaction. The solution of the effective solid is often obtained via three main methodologies to yield the overall moduli of the effective solid. These techniques are: *Taylor Model* (TM), *Self-Consistent Method* (SC), and the *Differential Scheme* (DS). A brief descriptions of these methods,

their advantages and limitations are given below. For the sake of completeness, other methods that are not widely used in the literature will be also discussed and assessed.

3.2.3.1 DILUTE DISTRIBUTION OF INHOMOGENEITIES, *TAYLOR MODEL (TM)*

This model is due to Taylor (1938) and is similar to the slip theory of plasticity. The microcracks displacements and their surface tractions are typically expressed in terms of the mechanical properties of the virgin elastic matrix material, Wu (1986). Furthermore, no microcrack interaction is accounted for, and the cracks grow in a self-similar fashion. Therefore, as expected, this type of modelling always produces a stiff response (Nemat-Nasser and Hori 1990) by overestimating the stiffness. Krajcinovic and Fanella (1986); Fanella and Krajcinovic (1988) used this method in developing damage models for concrete subjected to uniaxial tension, and uniaxial and triaxial compression loadings. In summary, this method indicates that the crack compliance is a function of the initial virgin material properties throughout a given loading program.

3.2.3.2 SELF-CONSISTENT METHOD (SC)

This model is the simplest and the most popular of all models. It assumes that the external stress fields of all cracks are equal to the far field stresses. This would, of course, eliminate the need to solve the system of the integral equations (Kunin 1983). Therefore, if we obtain the effective elastic moduli of a solid containing a single inclusion (microdefect, microcrack, etc.) and subsequently apply a certain averaging procedure along with a statistical technique, one can find the overall average stiffness of the actual solid of many cracks. Budiansky and O'Connell (1976); Horii and Nemat-Nasser (1983) used this method to develop a non-process model for rocks while Sumarac and Krajcinovic (1987); and Ju (1990; 1991) employed it for a process (evolutionary) model for plain concrete. The method is completely explained in Hershey (1954); Kroner (1967); Willis (1981); and Mura (1982). It should be mentioned, however, that the method fails when the volume fraction of microdefects increases; but if it decreases, the method reverts to the Taylor model. Finally, this type of modelling usually produces a soft response and tends to give zero stiffness for a rather large microdefect volume fraction. In contrast to its predecessor, this method ensures that the pertinent equations describing the

crack evolution are functions of the current effective material properties. This indicates that the solution used in this method requires nonlinear techniques to obtain the desired results. Fortunately, it has a super convergence characteristic which explains its popularity.

3.2.3.3 DIFFERENTIAL SCHEME (DS)

This model was proposed by Roscoe (1952; 1973) to remedy the problems associated with the previous two techniques. In applying this model, the effective solid is assumed to have an initially homogeneous matrix with a small microcrack volume fraction. The latter is allowed to increase in volume as the loading increases. Therefore, the changes in the overall elastic moduli are due to the incremental increase of the inhomogeneity volume fraction. The mathematical details and its complete procedure are given by McLaughlin (1977) and Hashin (1988). To the writer's knowledge, the method has not been utilized in any of the existing micromechanical-based damage models. It has been used primarily for calculating the effective moduli of a unidirectional fiber composite material under uniaxial loading (e.g. Hashin 1988).

3.2.3.4 OTHER MODELS

It is important at this juncture to mention two other methods, namely; the Mori-Tanaka method (Mori and Tanaka 1973; Benveniste 1986), and the direct strong microcrack interaction model (Kachanov 1987; Ju 1991). These two methods are sophisticated and account for all kinds of interaction among the heterogeneous material constituents; unfortunately, this has a price. It comes at the expense of requiring additional parameters and special numerical tricks to locate the exact position of cracks. Therefore, they are neither economical nor feasible in finite element implementations, and are discussed no further.

Some comparisons and assessments of all of the preceding models were made by other investigators (e.g. Laws and Dvorak 1987; Horii and Sahasakmontri 1990; and Nemat-Nasser and Hori 1990). Based on those studies, there is no conclusive evidence to support the use of a single method against all others in general applications. Considering

economy and accuracy, it is generally settled upon that the self-consistent method will produce reasonable results when analyzing a brittle solid that contains a small or moderate volume fraction of microdefects. Therefore, in this study we will adopt the self-consistent model because concrete generally contains a small microcrack volume fraction. Furthermore, the self-consistent method can be reduced to the Taylor model (TM), which is useful in establishing bounds for the prediction of a given variable. We will present within the self-consistent method, the mathematical details of a 2-D constitutive model for plain concrete element. The kinetic equations are developed for all load combinations in any biaxial loading program. The loading program can be non-proportional, monotonic or cyclic. Some problems associated with the numerical integration of the governing equations will be described later.

3.3 SELF-CONSISTENT ELASTIC-DAMAGE COMPLIANCE

The change of elastic and damage strain and compliance can be simply obtained by summing the contributions of all cracks. Assuming no or weak crack interaction, the self-consistent model can be used to obtain the desired quantity (e. g strain, compliance, and stiffness). Equation [3.22] is general and can be used to obtain the constitutive relationships for 3-D and 2-D plane stress or strain problems. To simplify the presentation, in the following, the 2-D version of Equations [3.22] and [3.32] will be developed, and for clarity matrix notations will be employed.

3.3.1 DERIVATION OF 2-D DAMAGED CONCRETE COMPLIANCE

Imagine that the 2-D concrete element in Figure 3.4(a) can be replaced by an equivalent solid that contains a single elliptical crack of length, $2a$, Figure 3.4(b), with the axes of the ellipse being x' and y' . Axis x' of the ellipse is assumed to make an angle θ with the axis X , where the X and Y axes are the directions of the applied major and minor principal stresses, respectively. If this element is subjected to stresses, $\{\sigma\}$, in the x and y directions, then the elastic and damage strains, $\{\epsilon'\}$, can be determined from Equation [3.20], where according to Voigt's notation, the strains and stresses are defined as

$$\{\varepsilon\}^T = \{\varepsilon_x; \varepsilon_y; \gamma_{xy}\}; \quad \{\sigma\}^T = \{\sigma_x; \sigma_y; \tau_{xy}\} \quad [3.39]$$

Figure 3.4: Concrete material (a) typical structure, (b) simulated and (c) equivalent cracked solid.

It is convenient to write crack displacements and compliance in the crack coordinate system, which will require the usual transformations. By considering Equation [3.20], we can see that the effective compliance is the sum of the elastic compliance of the virgin concrete and the crack compliance. The elastic compliance is the usual elasticity compliance of the concrete while the crack compliance need to be determined by means of continuum damage mechanics.

3.3.2 SINGLE CRACK COMPLIANCE

The compliance of a given single crack can be determined from the relationships between the crack displacements and stresses, depending on whether the crack is *open* or *closed*. It should be remembered that, the compliance developed in the subsequent sections are calculated at the end of the application of a given load increment. This means

that the proposed damage model is applied when the cracks reach their expected final sizes and their diameters remain unchanged for the current loading increment until the application of a new increment. Thus, further studies can be carried out to enhance the model to be able to trace the dynamical behaviour of brittle cracked materials.

3.3.2.1 OPEN CRACK MODE-I

According to Lekhnitskii (1963), for an open crack of length, $2a$, in an anisotropic, homogeneous, and linearly elastic solid, the displacement discontinuities, $u_{x'}$ and $u_{y'}$, across the crack faces are given by

$$\begin{Bmatrix} u_{x'} \\ u_{y'} \end{Bmatrix} = 2\sqrt{a^2 - x'^2} \begin{Bmatrix} \bar{C}_{11}' [(\alpha_1 \beta_2 + \alpha_2 \beta_1) \sigma_{y'} + (\beta_1 + \beta_2) \tau_{xy'}] \\ \bar{C}_{22}' \left[\left(\frac{\beta_1}{\alpha_1^2 + \beta_1^2} + \frac{\beta_2}{\alpha_2^2 + \beta_2^2} \right) \sigma_{y'} + \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)} \tau_{xy'} \right] \end{Bmatrix} \quad [3.40]$$

in which α and β are the real and imaginary parts of the roots $\lambda_r = \alpha_r \pm i\beta_r$ of the characteristic equation of the governing strain compatibility of an anisotropic body

$$\bar{C}_{11}' \lambda^4 - (\bar{C}_{13}' + \bar{C}_{31}') \lambda^3 + (\bar{C}_{12}' + \bar{C}_{21}' + \bar{C}_{33}') \lambda^2 - (\bar{C}_{23}' + \bar{C}_{32}') \lambda + \bar{C}_{22}' = 0 \quad [3.41]$$

which, for isotropic materials or isotropic damage reduces to

$$\bar{C}_{11}' \lambda^4 - 2\bar{C}_{13}' \lambda^3 + (2\bar{C}_{12}' + \bar{C}_{33}') \lambda^2 - 2\bar{C}_{23}' \lambda + \bar{C}_{22}' = 0 \quad [3.42]$$

where $(r=1, 2)$ and $i = \sqrt{-1}$, with $\beta_1, \beta_2 > 0$; $\beta_1 \neq \beta_2$. Note that the characteristic equations are written in terms of the overall effective inelastic compliance in the primed (crack) coordinate system. This means that the solutions of these equations are required for each crack orientation. Fortunately, the latter solutions can be obtained by solving for their roots, λ_r , for each trial compliance, $[\bar{C}]^{(n)}$, at the orientation plane $\theta_{a_i} = 0$ only. For other orientations, (i.e. $\theta_{a_i} \neq 0$) according to Lekhnitskii (1963), the local roots, λ_r' , can be expressed in terms of the roots λ_r at $\theta_{a_i} = 0$ as follows:

$$\lambda_r' = \frac{\lambda_r \cos \theta - \sin \theta}{\cos \theta + \lambda_r \sin \theta}; \quad \bar{\lambda}_r' = \frac{\bar{\lambda}_r \cos \theta - \sin \theta}{\cos \theta + \bar{\lambda}_r \sin \theta} \quad [3.43]$$

The crack strains, $\{\varepsilon'\}^{\sigma}$, can now be obtained according to

$$A_c \begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \end{Bmatrix}^{\sigma} = \int_{-a}^a \begin{Bmatrix} [u_{x'}]n'_1 \\ [u_{y'}]n'_2 \\ \frac{1}{2}([u_{x'}]n'_2 + [u_{y'}]n'_1) \end{Bmatrix} dS \quad [3.44]$$

where A_c is the representative-surface area (RSA), the square brackets $[]$ indicate the jump of the displacement functions inside them, and n'_1 and n'_2 are the direction cosines of the normal to the crack surface, S . Note that for a slit (line crack), $n'_1 = 0$, $n'_2 = 1$. If we substitute for $u_{x'}$ and $u_{y'}$ from Equation [3.40] into Equation [3.44], and make use of the normal values as indicated, Equation [3.44] can be rewritten as

$$\{\varepsilon'\}^{\sigma} = \frac{\pi a^2}{A_c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & G_{11}^{\sigma} & G_{12}^{\sigma} \\ 0 & G_{21}^{\sigma} & G_{22}^{\sigma} \end{bmatrix} \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{Bmatrix} \quad [3.45]$$

or

$$\{\varepsilon'\}^{\sigma} = [C']_o^{\sigma} \{\sigma'\} \quad [3.46]$$

The elements of G_{ij}^{σ} in Equation [3.45] are given by

$$[G]^{\sigma} = \begin{bmatrix} \bar{C}_{22}' \left[\frac{\beta_1}{\alpha_1^2 + \beta_1^2} + \frac{\beta_2}{\alpha_2^2 + \beta_2^2} \right] & \bar{C}_{22}' \left[\frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{[\alpha_1^2 + \beta_1^2][\alpha_2^2 + \beta_2^2]} \right] \\ \bar{C}_{11}' [\alpha_1 \beta_2 + \alpha_2 \beta_1] & \bar{C}_{11}' [\beta_1 + \beta_2] \end{bmatrix} \quad [3.47]$$

3.3.2.2 CLOSED CRACK MODE-II

Equation [3.45] is valid for an open crack. In the case of a closed crack, $u_{y'} = 0$, and the crack is able to transmit both compression and shear. However, if the shear, $\tau_{x'y'}$, across the crack surfaces is less than the frictional shear resistance, $\mu \sigma_{y'}$, where μ is the coefficient of dry friction and $\sigma_{y'}$ is the normal compression on the crack, then no crack displacements can occur. This means that

$$[C']_c^{\sigma} = 0 \quad [3.48]$$

which according to Equation [3.46] clearly indicates that under these cases of overall compression, the inelastic damage strains, $\{\epsilon'\}^{\sigma}$ become zero. However, experiments on concrete (e.g. Mindess and Young 1981; Zaitsev 1983; Horii and Nemat-Nasser 1985, etc.) show nonlinear responses for these cases of loading, thus contradicting the above conclusion. This paradox has puzzled many investigators, but experimental work of Zaitsev (1983) and Horii and Nemat-Nasser (1985) has indicated that the heterogeneous brittle materials under these loading conditions ignore existing cracks, and they deform in a ductile manner. Therefore, the proposed model calculates the inelastic strain using the plasticity theory, assuming that the nonlinearity beyond this point is solely ascribed to plastic flow and yield-like deformations. It is obvious that the above stress distribution on the crack will occur when the applied principal external stress ratio, $s_r = \frac{\sigma_1}{\sigma_2}$, exceeds a certain critical value. This value will be termed, henceforth the *ductility parameter*, Δ_c , which separates the brittle and ductile behaviour (Horii and Nemat-Nasser 1985). It is interesting to note according to the above discussions and the results of the proposed equations that heterogeneous material behaviour is load dependent. In the subsequent derivation, an explicit relationship for the Δ_c value is given and will be considered a material parameter since it depends on the coefficient of friction of the material, μ , and the orientation of the existing critical cracks, θ , with respect to the major principal stress axes.

On the other hand, if $\tau_{c'v'}$ is greater than $\mu\sigma_{v'}$, then the crack will slide. The non-vanishing displacements and normal stress, respectively, can be written as (Horii and Nemat-Nasser 1983)

$$u_{c'} = 2\sqrt{a^2 - x'^2} \bar{C}_{11}' \left[(\alpha_1\beta_2 + \alpha_2\beta_1)(\sigma_{v'} - \sigma_{v'}^c) + (\beta_1 + \beta_2)(\tau_{c'v'} + \mu \text{sgn}(\tau_{c'v'})\sigma_{v'}^c) \right] \quad [3.49]$$

with

$$\sigma_{y'}^c = \frac{[\beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2)]\sigma_{y'} + (\alpha_1\beta_2 + \alpha_2\beta_1)\tau_{x'y'}}{\beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2) - \mu \text{sgn}(\tau_{x'y'}) (\alpha_1\beta_2 + \alpha_2\beta_1)} < 0 \quad [3.50]$$

where the symbol sgn may read as 1, 0, or -1, depending on whether its argument is positive, zero, or negative. This leads to

$$\{\epsilon'\}^{cr} = \frac{\pi a^2}{A_c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & G_{21}^c & G_{22}^c \end{bmatrix} \begin{Bmatrix} \sigma_{x'}^c \\ \sigma_{y'}^c \\ \tau_{x'y'}^c \end{Bmatrix}; \quad \text{or} \quad \{\epsilon'\}^{cr} = [C']_c^{cr} \{\sigma'\} \quad [3.51]$$

where

$$\begin{Bmatrix} G_{21}^c \\ G_{22}^c \end{Bmatrix} = \begin{Bmatrix} G_{21}^u \\ G_{22}^u \end{Bmatrix} - \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} \quad [3.52]$$

with the quantity R_i given by

$$\begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = C_c \begin{Bmatrix} \beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2) \\ (\alpha_1\beta_2 + \alpha_2\beta_1) \end{Bmatrix} \quad [3.53]$$

where the coefficient C_c is

$$C_c = \frac{[(\alpha_1\beta_2 + \alpha_2\beta_1) - \mu \text{sgn}(\tau_{x'y'}) (\beta_1 + \beta_2)]}{\beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2) - \mu \text{sgn}(\tau_{x'y'}) (\alpha_1\beta_2 + \alpha_2\beta_1)} \quad [3.54]$$

3.3.2.3 KINKED CRACK MIXED MODE-I/II

Experimentally, it has been observed that a closed crack may branch in the matrix due to increased applied compressive load (Zaitsev 1983; Horii and Nemat-Nasser 1985). The crack shows first curving for a finite length (referred to henceforth as the opening wing) and then propagates (snaps) in a straight line fashion (referred to as the sliding wing) until it gets arrested by the energy barrier of the matrix, Figure 3.5.

In the opening wing no sliding occurs while in the sliding wing, there is no opening. This phenomenon is often called *kinked crack propagation* (Horii and Nemat-Nasser 1983). To date, however, there is no closed form solution even for the 2-D kinked crack problem (Fanella 1990). Many investigators (e.g. Horii and Nemat-Nasser 1985; Zaitsev

1983; Krajcinovic and Fanella 1988) have approximated the 3-D problem by projecting the actual 3-D kinked crack on a plane (section) and by assuming that the crack will kink only in that plane. Tada (1973) showed, based on the numerical analyses of the exact boundary value problem (Horii and Nemat-Nasser 1985), that if the average opening, $\bar{v}$, and sliding, $\bar{u}$, displacements, respectively, at any cross section make an angle, θ , with the $X-Y$ plane (Figure 3.5), then

Figure 3.5: 2-D kinked crack geometry and its stress distribution.

$$\begin{Bmatrix} \bar{v}(\psi) \\ \bar{u}(\psi) \end{Bmatrix} = \frac{4.8(1-\nu^2)}{\pi E} F_1(D_M, \theta, l_k) (\tau_{x'y'} - \mu \sigma_{y'}) \cos \psi \begin{Bmatrix} 1 \\ \sec \theta \end{Bmatrix} \quad [3.55]$$

where l_k is the kinked crack length (Figure 3.5), θ is the crack orientation angle with respect to the maximum applied compressive stress, E and ν are the elastic modulus and Poisson's ratio of the cement-paste, and D_M is the maximum original crack diameter. In the case of concrete, it is often assumed that this diameter is equal to the maximum aggregate size. The function, $F_1(D_M, \theta, l_k)$ is defined by Tada (1973) as

$$F_1(D_M, \theta, l_k) = \frac{D_M \cos \theta}{\bar{d}} \left\{ \sqrt{l_k(\bar{d} + l_k)} \left[\sin^{-1} \left(\frac{\bar{d}}{\bar{d} + 2l_k} \right) \right] + \frac{\bar{d}}{2} \ln \left(\frac{\bar{d} + 2l_k}{\bar{d}} \right) \right\} \quad [3.56]$$

where $\ln$ signifies natural logarithm and the distance $\bar{d} = \alpha_k D_M \sin \theta$. The coefficient α_k can be chosen such that the results of the approximate solution fit the exact numerical solution given by Horii and Nemat-Nasser (1985) over the entire domain of inclination angle, θ , for a given kink length, l_k . Fanella (1986) and Fanella and Krajcinovic (1988) have found out that a value of $\alpha_k = 0.25$ will yield results close to the numerical solution of Horii and Nemat-Nasser (1985). The same value will be utilized in the current study.

It follows that the approximate 3-D displacement functions can be obtained by integrating Equation [3.55] around the crack rim, i.e. for all cross sections $\left(-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}\right)$.

Assuming that all cross sections will kink at the same time once the section $\frac{\pi}{2}$ kinks, the integration of the kinked crack displacement function can be separated into three parts in order to cover the entire crack length. Part 1 will cover the original (parent) crack length (length D in Figure 3.5), which can be evaluated as previously described for Mode-I and II. The second integration will be carried out over the opening wing. If the opening displacement of this wing is $\bar{v}$, then its components in the kinked crack coordinate system (see Figure 3.5), i.e. v , and sliding, $v_{x'}$, $v_{y'}$, are given by

$$\begin{Bmatrix} v_{x'} \\ v_{y'} \end{Bmatrix} = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{v}(0) d\psi \begin{Bmatrix} -\cos \theta \\ -\sin \theta \end{Bmatrix} = \frac{9.6(1-\nu^2)}{\pi E} F_1(D_M, \theta, l_k) (\tau_{x'y'} - \mu \sigma_{y'}) \begin{Bmatrix} -\cos \theta \\ -\sin \theta \end{Bmatrix} \quad [3.57]$$

Finally for the sliding wing, assuming the sliding displacement to be u , its components in the original sliding crack coordinate system, i.e. x' and y' directions are given by

$$\begin{Bmatrix} u_{x'} \\ u_{y'} \end{Bmatrix} = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{u}(0) d\psi \begin{Bmatrix} \sin \theta \\ -\cos \theta \end{Bmatrix} = \frac{9.6(1-\nu^2)}{\pi E} F_1(D_M, \theta, l_k) (\tau_{x'y'} - \mu \sigma_{y'}) \begin{Bmatrix} -\tan \theta \\ -1 \end{Bmatrix} \quad [3.58]$$

Considering Mode-I and II fracture of the kinked crack of Figure 3.5, it can be shown that its length, l_k , may be computed using the following relationship

$$l_k = \frac{1}{\pi} \begin{cases} \frac{1}{4} \left[\frac{K_{IC}^c}{\sigma_t} + \sqrt{\left(\frac{K_{IC}^c}{\sigma_t} \right)^2 + \frac{4}{s_r} DG(s_r, \theta) \cos \theta} \right]; & |s_r| > 0 \\ \left[\left(\frac{\sigma_y}{K_{IC}^c} \right) DG(s_r = 0, \theta) \cos \theta \right]^2; & |s_r| = 0 \end{cases} \quad [3.59]$$

where the ratio, $s_r = \frac{\sigma_x}{\sigma_y}$, signifies the biaxiality of applied stresses in which σ_x and σ_y are the minor and major applied principal stresses, respectively. D , on the other hand, is the parent crack (without kink) diameter, K_{IC}^c is the mode-I fracture toughness of the cement-paste, and the stress transformation function, $G(s_r, \theta)$, can be obtained as a combination of two other stress transformation functions, $N(s_r, \theta)$ and $S(s_r, \theta)$ as

$$G(s_r, \theta) = S(s_r, \theta) - \mu N(s_r, \theta) \quad [3.60]$$

The $N(s_r, \theta)$ and $S(s_r, \theta)$ stress transformation functions are given by

$$\begin{cases} N(s_r, \theta) \\ S(s_r, \theta) \end{cases} = \begin{cases} \cos^2 \theta + s_r \sin^2 \theta \\ (1 - s_r) \sin \theta \cos \theta \end{cases} \quad [3.61]$$

The contribution of the opening wing to the crack strains can be obtained by inserting the displacement functions from Equation [3.57] into Equation [3.44] with the unit normal to the kinked crack wing being, $n_1'' = \sin \theta$ and $n_2'' = \cos \theta$, respectively. This will yield

$$\begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \end{Bmatrix}^{cr'} = B_k \begin{bmatrix} 0 & \mu \sin 2\theta & -\sin 2\theta \\ 0 & -\mu \sin 2\theta & \sin 2\theta \\ 0 & -2\mu \cos 2\theta & 2 \cos 2\theta \end{bmatrix} \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{Bmatrix} \quad \text{or} \quad \{\varepsilon'\}^{cr'} = [C']_k^{cr'} \{\sigma'\} \quad [3.62]$$

Similarly by using Equation [3.58] and the unit normal of the original sliding crack mentioned earlier, the contribution of the sliding wing can be obtained as

$$\begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \end{Bmatrix}^{cr'} = 2B_k \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu & -1 \\ 0 & -\mu \tan \theta & \tan \theta \end{bmatrix} \begin{Bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{Bmatrix} \quad \text{or} \quad \{\varepsilon'\}^{cr'} = [C']_k^{cr'} \{\sigma'\} \quad [3.63]$$

where

$$B_k = \frac{9.6(1-\nu^2)}{\pi EA_c} \left[l_k + \frac{\bar{d}}{2} \right] F_1(D_M, \theta, l_k) \quad [3.64]$$

Hence, for a kinked crack the total compliance is obtained by the summation of the compliance of the parent crack and the compliance given by Equations [3.62] and [3.63]. It may be observed that the kinked crack compliance are unsymmetrical, which signifies load induced anisotropy.

3.3.2.4 COMPLIANCE OF AN ELEMENT WITH AN ARRAY OF CRACKS

For a concrete element with N_c cracks, each with orientation θ_k , and initial diameter, a_k , the crack ensemble compliance, $[C]^{cr}$, can be written as

$$[C]^{cr} = \sum_{k=1}^N [T]_k^T [C]_k^{cr} [T]_k \quad [3.65]$$

where $[T]_k$ is the appropriate transformation matrix. Earlier discussion on the calculation of the crack compliance showed that $[C]^{cr}$ is both an explicit and implicit function of the as yet unknown overall compliance, $[\bar{C}]$. The solution can be obtained by utilizing the self-consistent model (Hershey 1954). In this work, the iterative process outlined by Horii and Nemat-Nasser (1983) is adopted. Subsequently, the overall nonlinear damage compliance for the $(n+1)th$ iteration can be written as

$$[\bar{C}]^{(n+1)} = [C] + \sum_{k=1}^N [T]_k^T \left[C'([\bar{C}]^{(n)}) \right]_k [T]_k \quad [3.66]$$

where $[\bar{C}]^{(n)}$ is the desired overall compliance which corresponds to the nth iteration. The process is terminated if one of the following convergence criteria are met;

$$\frac{|\bar{C}^{(n)} - \bar{C}^{(n+1)}|_2}{|\bar{C}^{(n+1)}|_\infty} \leq 10^{-6} \quad \text{or} \quad \frac{|\bar{C}^{(n)} - \bar{C}^{(n+1)}|_\infty}{|\bar{C}^{(n+1)}|_\infty} \leq 10^{-6} \quad [3.67]$$

which are based on L_2 , L_∞ norms (Bathe 1982), and the tolerance assumed is 10^{-6} .

Equation [3.65] is not, however, directly useful in numerical implementation because the number of cracks is a non-deterministic quantity. Therefore, we replace the summation by an integral relation, i.e.

$$[C]^{\sigma} = \int \int_{\theta, a_1} p(\theta, a) [T]^T [C']^{\sigma} [T] da d\theta \quad [3.68]$$

where $p(\theta, a)$ is the joint probability density function of the crack size and orientation.

Assuming the sizes of the existing cracks are independent of their orientations, the latter function can be decomposed into a product of two separate functions, i.e.

$$p(\theta, a) = p(\theta)p(a) = \frac{2}{\pi} p(a) \quad [3.69]$$

where the joint probability density function of microcrack orientation angle, $\left(0 \leq \theta \leq \frac{\pi}{2}\right)$, is assumed to be *random*, $p(\theta) = \frac{2}{\pi}$, while different forms for the function that describes the crack diameter variable can be utilized. These functions may represent homogeneous, triangular, or Poisson's distributions, e.g.

$$p(a) = \frac{1}{D_M} \begin{cases} 1 & \bullet \text{ homogeneous distribution} \\ 2\left(1 - \frac{D}{D_M}\right) & \bullet \text{ triangular distribution} \end{cases} \quad [3.70]$$

where D is the diameter of the crack at a given load step and D_M is the maximum aggregate size. Of course, these distributions may affect the final results of the predicted inelastic damage strain values but Krajcinovic (1989), found that the homogeneous distribution increases the strains by approximately 10% in comparison with the triangular distribution. The results of the current model support the latter observation. For all practical purposes, however, this difference can be neglected since the strain values are typically very small (e.g. $\epsilon_u \leq 0.006$). In carrying out the integration in Equation [3.68], we must distinguish between two types of cracks: the stable cracks and the propagating cracks. Thus, Equation [3.66] may be written as

$$[C]^{\sigma} = [C]_g^{\sigma} + [C]_a^{\sigma} \quad [3.71]$$

where $[C]_g^{\sigma}$ and $[C]_a^{\sigma}$ are the compliance contributions from the growing and arrested microcracks, respectively. Stable and propagating cracks can be identified by the critical crack orientation angle, θ_k , the magnitude of the applied stress ratio, s_r , and the mode-I and II fracture parameters. The arrested cracks are assumed to undergo elastic deformations while the growing cracks will cause inelastic deformation. To evaluate the critical angles, θ_k , the following kinetic algorithm and the associate damage evolution law are utilized.

3.3.3 CRACK EVOLUTION CRITERIA

It is sufficient to assume that failure of brittle materials is of a “*Cleavage 1*” or “*Cleavage 2*” deformation type and therefore linear fracture mechanics laws could be employed (Ashby 1979). It is likely that an open crack will experience mode-I type deformation; a closed crack will display mode-II deformation, while an open/closed crack will undergo mixed mode-I/II deformation. For mixed mode-I/II, the strain energy release rate governs the crack movements. The strain energy release rate, G' , along the weak plane for the mixed mode-I/II should include the contribution G'_I of mode-I and G'_2 of mode-II, thus according to Sih *et al.* (1965)

$$G' = G'_1 + G'_2 = \frac{1}{2} \left[G'_{11} K_I'^2 + (G'_{12} + G'_{21}) K_I' K_{II}' + G'_{22} K_{II}'^2 \right] \quad [3.72]$$

where K_I' and K_{II}' are the mode-I and mode-II fracture toughness, respectively, at a given orientation, θ . The coefficients G'_{ij} , ($i = j = 1, 2$) are given by Equations [3.47] and [3.52] for open and closed cracks, respectively. The calculated value of G' should be calibrated against a given critical strain energy release rate G_c to determine whether a microcrack is experiencing propagation or it has been arrested. The microcrack kinetic algorithm should be considered for (i) tension-tension (*mode-I*), (ii) compression-compression (*mode-II*), and finally (iii) tension-compression (*mixed mode-I/II*) stress regimes, which are discussed below.

3.3.3.1 UNIAXIAL AND BIAXIAL TENSION (MODE-I)

When a concrete element is subjected to either uniaxial tension, σ_y , or biaxial tension ($\sigma_y; \sigma_x = s_r \sigma_y$ with $0 \leq s_r \leq 1$), the embedded crack will experience mainly mode-I type of deformation. Therefore, mode-I critical stress intensity factors for this case of loading can be employed (Fanella 1990). Of course, it should be understood that for simplicity the mode-II effect is ignored, but it can be included in a straightforward manner, if necessary (refer to Equation [3.72]). According to the fracture criterion utilized, the fracture toughness, K_I and K_{II} , can be written as a function of the applied stress, σ_y , a given crack size, $2a$, and the crack orientation angle, θ , as

$$\begin{Bmatrix} K_I \\ K_{II} \end{Bmatrix} = \sigma_y \sqrt{\pi a} \begin{Bmatrix} N(s_r, \theta) \\ S(s_r, \theta) \end{Bmatrix} \quad [3.73]$$

where the two stress transformation functions, $N(s_r, \theta)$ and $S(s_r, \theta)$, are given by Equation [3.61].

The first pre-existing microcracks to propagate are those with minimum dimensions, $2a_0$, and are oriented at plane $\theta = 0$. These pre-existing microdefects are usually located at the aggregate-cement paste interfaces (Mindess and Young 1981; Zaitsev 1983). According to Equation [3.73], those cracks will snap if the applied stress, σ_y , reaches a critical value, f_t' , corresponding to the maximum intensity factor of mode-I of the interface, K_{IC}^I . The strength quantity, f_t' , is estimated by

$$f_t' = \frac{K_{IC}^I}{\rho} \sqrt{\frac{2}{\pi D_M}} \quad [3.74]$$

where D_M is the maximum aggregate diameter in the mix and, ρ is a scalar (less than one) that designates the fraction of the initially de-bonded interfaces. The magnitude of ρ depends on the concrete mix, the compacting method, and the curing procedure of the specimen, and it can be obtained via micrographics analysis of a typical concrete structure (Fanella 1990) or microscopic observation (Slate 1983). It can also be measured using radioactive techniques such as γ or x-ray radiography or the method of acoustic emission

(Fernando 1987). The ultrasonic wave techniques may also be employed to measure the amount of damage in concrete specimens and structures (Carlin 1960; Shah 1970; Rose 1983 and Krautkramer 1983). In the absence of exact theory for evaluating this function, Fanella (1986) proposed that it can be written in terms of the initial crack size, $2a_o$, and the mean diameter of the coarse aggregates, D_M , as

$$\rho = \sqrt{\frac{2a_o}{D_M}} \quad [3.75]$$

In deriving Equation [3.74], the hierarchical condition for the fracture toughness proposed by Mindess (1984) is used, which states that

$$K_{IC}^a \gg K_{IC}^c = 2K_{IC}^{if} = K_{IIC}^{if} \quad [3.76]$$

where the subscript IC and IIC , respectively, designate mode-I and II, while the superscripts, a , c and if stand for aggregate, cement paste, and aggregate-cement paste interface, respectively.

The determination of the effective stiffness of the cracked concrete is also required since the current model is not only concerned with the evaluation of concrete strength but also with its deformation characteristics under different loading conditions. This can be achieved by employing Equation [3.22], Equation [3.24], or Equation [3.33] which require the calculation of the cracks compliance. These compliance may be obtained by evaluating the integral of Equation [3.68] In order to carry out the computation of these integrals, the exact amount of arrested, η_a , and propagating, η_p , cracks, their sizes, $D's$, as well as their orientation, θ , must be evaluated *a priori*.

First of all, to distinguish propagating cracks from arrested ones, we compare the crack inclination with the critical angle, θ , given by

$$\theta_c = \tan^{-1} \sqrt{\frac{\sigma_r - f_t'}{f_t' - s_r \sigma_r}} \quad [3.77]$$

Whenever the crack inclination becomes equal to or greater than θ_c it will propagate otherwise it will remain stable. Equations [3.68] and [3.71] can now be completely computed if the different cracks diameters and the amount of each crack type (e.g. unstable and stable) are known. It should be mentioned that most of the existing micromechanical damage-based models utilize an approximate method to calculate the latter for monotonic loading programs. This study improves the approximate method and it also introduces an exact method for calculating the amount corresponding to the crack deformations subjected to static non-proportional monotonic and cyclic loadings.

To compute the amount of each crack type according to the above mentioned methods, the limits (i.e. a_k) of the second integral in Equation [3.68] need to be determined. These limits are the initial diameter, a_o , of a given crack prior to the application of the current load increment, and its final value, a_f , after deformation takes place. According to the assumption employed here, the expected crack diameter varies from $2a_o = D_{min}$ to only $2a_f = D_M$. Thus, the crack diameter, $D(\theta)$, at any orientation angle, θ , may be obtained by considering Equation [3.73] and Equation [3.74], i.e.

$$D(\theta) = \left[\frac{\rho f_t'}{\sigma_y N(s_r, \theta)} \right]^2 D_M \quad [3.78]$$

where the stress transformation function $N(s_r, \theta)$ is given by the first relation of Equation [3.61].

The minimum crack diameter can be obtained from Equation [3.74] by substituting $\theta = 0$, i.e.

$$\frac{D_{min}}{D_M} = \frac{D(0)}{D_M} = \left(\frac{\rho f_t'}{\sigma_y} \right)^2 = \gamma_t \quad [3.79]$$

where $\gamma_t \leq 1$ is the so-called sieve parameter grading that gives the measure of the spread of the aggregate sizes. Obviously, this parameter can be obtained from the customary sieve analysis of the aggregates used in the concrete mix.

Figure 3.6: Domain of cracks angle fan for Mode-I (A-S-K method).

Now, the propagating, $\eta'_s = \frac{N'_s}{N_c}$, and the arrested portions, η'_a , of the total number of existing cracks, N_c , must be determined. Sumarac and Krajcinovic (1987) have suggested that the portions of each crack type in a concrete specimen subjected to uniaxial tensile loading can be related to the crack orientation angle fan as shown for the current loading case, Figure 3.6. The approximate Sumarac and Krajcinovic (1987) evaluation of the cracks ratios will be labeled hereafter with the name A-S-K method.

A more accurate measure of these quantities can be obtained by relating the area enclosed by Equation [3.78] and the line D_M to the total area bounded by the lines $a = 0$, $a = \frac{D}{D_M}$, $\theta = 0$, and $\theta = \frac{\pi}{2}$, in $\left[\frac{D(\theta)}{D_M} - \theta \right]$ plane, as shown in Figure 3.7. Thus, the current propagating tensile cracks portion, $\eta'_s|_c$, can be estimated for an arbitrary load level, σ_r , according to

$$\eta'_s|_c = \frac{2}{\pi} \left\{ \theta_{t_1} - \left(\frac{\rho f'_r}{\sigma_r} \right)^2 \int_0^{\theta_{t_1}} \frac{d\theta}{N^2(s_r, \theta)} \right\} \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [3.80]$$

Figure 3.7: Domain of open cracks $\left[\frac{D(\theta)}{D_M} - \theta \right]$ plane for Mode-I.

which clearly shows that the A-S-K method will indicate softer structure compared to the exact method. The difference between the two methods is attributed to the assumptions used in both methods since the A-S-K method considers the ratios of arc lengths and the exact method uses the proportion of areas in the $\left[\frac{D(\theta)}{D_M} - \theta \right]$ plane. It can be observed from the latter equation that obtaining the closed form solution of its integral is relatively easy. This may not be the case for the other complex expressions developed here for the remaining loading situations (e.g. overall biaxial compression and/or biaxial tension-compression). Thus, those integrals will be evaluated numerically in the finite element program Nonlacs.

At this juncture, it should be mentioned that Fanella (1986) and Fanella and Krajcinovic (1988) have used similar procedures as those outlined above within the proposed exact method. However, they have assumed that the total number of cracks is represented by the area below the line represented by $\gamma_r = \frac{D_{min}}{D_M}$ (Figure 3.7) Nevertheless, their analysis has adopted a very small value for the parameter, $\gamma_r \approx 0.01$. The proposed exact method, on the other hand, acknowledges that there are numerous cracks in the concrete specimen

but with their diameters between 0 and D_{min} , they may not affect the overall behaviour of the specimen.

Using Equations [3.45] and [3.68] in Equation [3.71] and carrying out the necessary integration over the domains defined above, the propagating unstable cracks compliance will be given by

$$[C]_s^{cr} = \int_{\pm\theta_1}^{\pm\theta_2} p'_s(s_r, a) [T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_a^{cr} [T] da \quad [3.81]$$

and the same for the arrested stable cracks is

$$[C]_a^{cr} = \int_{\pm\theta_1}^{\pm\theta_2} p'_a(s_r, a) [T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_a^{cr} [T] da \quad [3.82]$$

where the functions, $p'_s(s_r, a)$ and $p'_a(s_r, a)$, can be determined, respectively, according to

$$\begin{Bmatrix} p'_s(s_r, a) \\ p'_a(s_r, a) \end{Bmatrix} = \left[\frac{\pi N_c}{A_c} \right] \begin{Bmatrix} \frac{2\eta'_s}{\pi} \int_{\pm\theta_1}^{\pm\theta_2} a^2 p(a) da \\ \frac{2(1-\eta'_s-\gamma_i)}{\pi} \int_{\pm\theta_1}^{\pm\theta_2} a^2 p(a) da \end{Bmatrix} \quad [3.83]$$

in which N_c is the total number of cracks per representative-surface area, A_c .

According to Equations [3.81], [3.82], and [3.83], the amount of damage, ω_i , at a given load level, i , is a function of the two coefficients, $p'_s(s_r, a)$ and $p'_a(s_r, a)$. This amount of damage is often equated to the non-dimensional microcrack area-concentration parameter per unit surface (Horii and Nemat-Nasser 1985; Ju 1991). According to Budiansky and O'Connell (1976) the damage parameter, ω_i , can be written as

$$\omega_i = \frac{\pi N_c \langle a^2 \rangle}{A_c} \quad [3.84]$$

where a is the mean value of radii of all existing cracks and $\langle . \rangle$ represents the expected value. As a result, the initial, ω_o , and final, ω_f , damage state parameters can be related according to

$$\omega_o = \rho^+ \omega_f = \frac{\pi N_c a_o^2}{A_c} \quad [3.85]$$

where all symbols were defined earlier. The only remaining undefined quantities are the integral limits, θ_i^c and θ_i^p of Equation [3.81]. These limits represent the angles delineating the fan of the cracks which have propagated due to the current and previous load steps, respectively, (see Figures 3.6 and 3.7).

The net amount of propagating cracks can be found by finding the difference between their current and previous values, i.e.

$$\eta_s^i = \eta_s^i|_c - \eta_s^i|_p \quad [3.86]$$

where $\eta_s^i|_c$ and $\eta_s^i|_p$ are the number of current and previously propagating cracks, respectively, and can be computed using Equation [3.80] for their corresponding load levels. These considerations permit modelling the mechanical and deformation behaviour of brittle concrete materials subjected to cyclic and non-proportional loading.

It can be observed according to Equations [3.81] and [3.82] and [3.83] that the proposed damage mechanics constitutive relationships allow the diameters of the crack to take their expected sizes in both the A-S-K and the exact methods. Therefore, the A-S-K method employed here is different from that used by Sumarac and Krajcinovic (1987); Sumarac (1987); and Ju (1990). This is because these studies assume that all propagating crack diameters will have a single value of D_M while the arrested crack diameters will be $2a_o$. Furthermore, these methods were only concerned with the evaluation of material behaviour under monotonically increasing static loadings whilst the current proposed method accounts for cyclic behaviour.

The process is terminated when the load level reaches a value that will link all microcracks into a single macro-crack. Considering the hierarchy of the critical fracture toughness given by Equation [3.76], the critical load can be calculated using the critical

fracture toughness of the cement-paste matrix, K_{IC}^c . The ultimate rupture (tensile) stress may be estimated as

$$f_t' = K_{IC}^c \sqrt{\frac{2}{\pi D_M}} \left\{ \sqrt{\frac{1}{\frac{4b}{\pi D_M} \tan\left(\frac{\pi D_M}{4b}\right)}} \right. \quad \begin{array}{l} \bullet \text{ no microcrack interaction} \\ \bullet \text{ microcrack interaction} \end{array} \quad [3.87]$$

where the mean free distance, $2b$, between two microcracks can be written according to Fanella (1990) as a function of the coarse aggregate volume fraction, V_a , and the maximum aggregate diameter, D_M , i.e.

$$2b = \frac{2D_M(1-V_a)}{3V_a} + D_M \quad [3.88]$$

where the principles of stereology (Underwood 1970) were employed in order to obtain the latter relationship.

3.3.3.2 UNIAXIAL AND BIAXIAL COMPRESSION (MODE-II)

Under uniaxial and biaxial compressive stresses, pre-existing cracks are mainly closed and may grow in a mode-II manner. Strictly speaking, the fracture criterion of Equation [3.72] should be employed, however, since the cracks will not grow in mode-I fashion, it is sufficient to employ mode-II fracture criterion only (Ju 1990). The effective shear stresses acting on the crack faces can be obtained from

$$\tau_s = \sigma_v G(s, \theta) \quad [3.89]$$

where $G(s, \theta)$ is given by Equation [3.60].

Assuming Coulomb's type friction, microcracks surfaces will slide relative to each other if the normal stress, $\sigma_n = \sigma_v < 0$ and the absolute net shear stresses $|\tau_s| \geq 0$. Hence, Equation [3.60] can be solved for the upper and lower bounds $[\pm\theta_c, \pm\theta_s]$ of microcrack orientation for the given stress combination

$$\theta_{s_r, s_1} = \pm \tan^{-1} \left[\frac{1 \pm \sqrt{1 - 4C_r(C_r + \mu)}}{2C_r} \right] \quad [3.90]$$

where $C_r = \frac{\mu \sigma_x}{(\sigma_y - \sigma_x)} = \frac{\mu s_r}{(1 - s_r)}$. It should be realized that cracks oriented at the critical angle given by Equation [3.90] are the only ones that will slide. The minimum value of θ required for sliding to occur are:

$$\theta_o = \pm \tan^{-1} \left[\mu + \sqrt{\mu^2 + 1} \right] \quad [3.91]$$

Alternatively, Equation [3.89] indicates that cracks will not slide if the ratio of the two applied principal stresses, (i.e., $\frac{\sigma_x}{\sigma_y}$) satisfy the following condition

$$s_r = \frac{\tan \theta - \mu}{\tan \theta [\mu \tan \theta + 1]} \quad [3.92]$$

which is graphically represented in Figure 3.8 for different coefficient of friction, μ values. According to this figure, any stress ratio which happens to be within the envelope for a given friction coefficient will cause the crack to slide while ratios outside the envelope will not cause it to slide. Pure micromechanical damage mechanics models will in the latter case predict an elastic response while in the proposed method we assume the material to undergo plastic deformations.

According to Equation [3.92] (Figure 3.8), it appears that the type of deformation (e.g. fracture and/or flow) of a brittle material under biaxial compressive stress regime is a function of the biaxial stress ratio, s_r , the critical angles, θ , of the existing defects (cracks) as well as the coefficient of friction, μ , of concrete. The corresponding critical axial compressive stress, f_c' , which indicates the onset of crack sliding, can be obtained for this case according to

$$f_c' = \frac{K_{IIc}^q}{\rho G(s_r, \theta_o)} \sqrt{\frac{2}{\pi D_M}} \quad [3.93]$$

which shows that the uniaxial threshold compressive stress is not only a function of the initial crack orientation angle, θ_o , which itself is a function of the coefficient of dry friction, but also depends on the lateral stress (confining pressure, σ_x). The specific stress ratio, s_r , is the ratio of the applied stresses (e.g. $\sigma'_x - f'_c$) at the onset of first fracture.

Figure 3.8: Fracture and plastic flow domains under biaxial compression.

As in the previous case, for the compressive loading situation, the crack diameter, $D(\theta)$, of any orientation, θ , can be estimated according to

$$D(\theta) = \left[\frac{\rho f'_c G(s_r, \theta_o)}{\sigma_y G(s_r, \theta)} \right]^2 D_M \quad [3.94]$$

where the latter equation can be used to estimate the minimum crack diameter, D_{min} , which occurs for cracks inclined at an angle equal to θ_o . Thus, the minimum and maximum crack diameters, resulting from this loading condition, can be related by the aggregate sieve grading parameter, γ_c , which can be computed using Equation [3.79] but replacing f'_i with f'_c .

Considering Equation [3.94], the critical orientation of the cracks that have reached their maximum size, D_M , can be calculated from

$$\theta_{f_z, f_t} = \pm \tan^{-1} \left[\frac{1 \pm \sqrt{1 - 4C_f(C_f + \mu)}}{2C_f} \right] \quad [3.95]$$

where $C_f = \frac{1}{(1-s_r)} \left[\mu s_r + \left(\frac{\rho f_c^r}{\sigma_y} \right) G(s_r, \theta_o) \right]$. Once again, the latter relationship shows that the critical orientation angle, θ_{f_z, f_t} , for this case is decreased for increased lateral stresses, σ_x .

Of course, increasing the applied principle stress, σ_y , beyond the value of Equation [3.93] will make some cracks kink into the matrix. The threshold stress, $\sigma_y = f_c^k$, corresponding to the onset of kinking (i.e. $l_k = 0$) can be calculated for those preferential kinked cracks whose diameters are $D = D_M$ and are also oriented at angle $\theta = \theta_o$ according to:

$$f_c^k = \frac{K_{IC}^c}{2G(s_r, \theta_o)} \sqrt{\frac{6}{\pi D_M}} \quad [3.96]$$

in which $K_{IIIC}^f = \frac{\sqrt{3}}{2} K_{IC}^c$. The crack diameter, $D(\theta)$, of this type of cracks can be obtained from

$$D(\theta) = \left[\frac{k_1 f_c^r G(s_r, \theta_o)}{\sigma_y G(s_r, \theta)} \right]^2 D_M \quad [3.97]$$

where k_1 is a dimensionless constant and equal to

$$k_1 = \frac{\sqrt{3} \rho K_{IC}^c}{2 K_{IIIC}^f} \quad [3.98]$$

For a typical virgin concrete material, $K_{IIIC}^f = 1.5 \rightarrow 1.75 K_{IC}^c$, $\rho = 0.60 \rightarrow 0.95$, (Mindess and Young 1981; Zaitsev 1983) which indicates that the constant k_1 varies from 1.235 to 1.

440. It follows that the threshold stress ratio, $\frac{f_c^k}{f_c^r}$, varies with the same ratio. The smallest defect and the critical kink angle, respectively, can be evaluated using

$$\frac{D_{min}^k}{D_M} = k_1^2 \gamma_c \quad [3.99]$$

and

$$\theta_{k_1, k_2} = \pm \tan^{-1} \left[\frac{1 \pm \sqrt{1 - 4C_k(C_k + \mu)}}{2C_k} \right] \quad [3.100]$$

in which $C_k = \frac{1}{(1-s_r)} \left[\mu s_r + \left(\frac{k_1 f_c^r}{\sigma_r} \right) G(s_r, \theta_o) \right]$.

Figure 3.9: Domain of cracks angles fan for Mode-II (A-S-K method).

At this stage, it is very important to estimate the portions of sliding, arrested, and kinking cracks in order to evaluate their inelastic compliance using Equations [3.62] and [3.63]. Once again, propagating and arrested cracks ratios for the current loading condition can be calculated utilizing the A-S-K method shown in Figure 3.9 or the exact method depicted by Figure 3.10. The kinking crack portion, η_k^c , can be evaluated according to

$$\eta_k^c = \frac{1}{(\theta_{s_2} - \theta_{s_1})} \left\{ \begin{aligned} & (\theta_{k_2} - \theta_{k_1}) \\ & (\theta_{k_2} - \theta_{k_1}) - \left(\frac{k_1 f_c^r}{\sigma_Y} \right)^2 G^2(s_r, \theta_o) \int_{\theta_{k_1}}^{\theta_{k_2}} \frac{d\theta}{G^2(s_r, \theta)} \end{aligned} \right. \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [3.101]$$

where the integral in the above equation can be evaluated using numerical integration. Subsequently, the portion of propagating cracks, η_s^c , can be obtained from

$$\eta_s^c = -\eta_k^c + \frac{1}{(\theta_{s_2} - \theta_{s_1})} \left\{ \begin{aligned} & (\theta_{f_2} - \theta_{f_1}) \\ & (\theta_{f_2} - \theta_{f_1}) - \left(\frac{p f_c^r}{\sigma_Y} \right)^2 G^2(s_r, \theta_o) \int_{\theta_{f_1}}^{\theta_{f_2}} \frac{d\theta}{G^2(s_r, \theta)} \end{aligned} \right. \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [3.102]$$

Figure 3.10: Domain of sliding, kinking, and arrested cracks $\left[\frac{D(\theta)}{D_M} - \theta \right]$ plane for Mode-II.

In the absence of a kinked crack regime, the coefficient, $\eta_k^c = 0$, and the propagating cracks ratio increases. Using similar procedure to that of open cracks, Equation [3.51] can be substituted in Equations [3.68] to yield the propagating sliding cracks inelastic compliance according to

$$[C]_s^{cr} = \int_{\pm \theta_h^s}^{\pm \theta_f^s} p_s^c(s_r, a) [T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_c^{cr} [T] d\theta \quad [3.103]$$

The same can be obtained for the arrested cracks i.e.

$$\begin{aligned}
[C]_a^{cr} = & \int_{\pm\theta_{\eta_1}^{cr}}^{\pm\theta_{\eta_2}^{cr}} p_a^c(s_r, a)[T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_c^{cr} [T] d\theta \\
& + \int_{\pm\theta_{\eta_2}^{cr}}^{\pm\theta_{\eta_1}^{cr}} p_s^{cr}(s_r, a)[T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_c^{cr} [T] d\theta
\end{aligned} \tag{3.104}$$

The contribution of the open and the sliding wings of kinked cracks to the overall inelastic compliance can be obtained by summing Equations [3.62] and [3.63] and substituting the results in Equation [3.68] to give

$$[C]_k^{cr} = \int_{\pm\theta_{\eta_1}^{cr}}^{\pm\theta_{\eta_2}^{cr}} p_k^c(s_r, l_k)[T]^T \left\{ \left[C^{(k)} \left(\theta, l_k \right) \right]_{k_-}^{cr} + \left[C^{(k)} \left(\theta, l_k \right) \right]_{k_+}^{cr} \right\} [T] d\theta \tag{3.105}$$

where the functions, $p_s^c(s_r, a)$, $p_a^c(s_r, a)$ and $p_k^c(s_r, l_k)$ $p_a^c(s_r, a)$, corresponding to sliding, arrested, and kinked cracks, respectively, due to this loading condition, can be obtained from

$$\begin{Bmatrix} p_s^c(s_r, a) \\ p_a^c(s_r, a) \\ p_k^c(s_r, l_k) \end{Bmatrix} = \left[\frac{\pi V_c}{A_c} \right] \begin{Bmatrix} \frac{2\eta_s^c}{\pi} \int_{a(\theta)}^{a_y} a^2 p(a) da \\ \frac{2(1-\eta_s^c - \eta_k^c - \gamma_r)}{\pi} \int_{a_m}^{a_y} a^2 p(a) da \\ \frac{9.6(1-\nu^2)\eta_k^c}{\pi^2 E} \int_{a(\theta)}^{a_y} \left[l_k + \frac{\bar{d}}{2} \right] F_1(D_M, \theta, l_k) p(a) da \end{Bmatrix} \tag{3.106}$$

where $\bar{d}$ is curved distance from the original (parent) crack to the kinked crack (see Figure 3.5). The length of the kinked crack, l_k , can be evaluated using Equation [3.59]. It is important to note that according to Equations [3.105] and [3.106] the current treatment of the kinked cracks deformations is approximate since their compliance depend only on the elastic properties of the virgin matrix. It shows also the paradox that kinked cracks can only form within an isotropic solid material. Of course, this aspect needs further study in order to obtain accurate estimation of the opening and sliding kinked crack displacements. It should be mentioned, however, that this drawback did not affect the results of the test problems given later in this thesis.

The failure stresses in the case of biaxial compression, depending on the amount of lateral stress ratio, s_r , will be either, (i) splitting, or (ii) shear fault. The critical angle, θ_{f_i} , of the cracks at the onset of failure may be calculated from maximizing the projected stress component in the direction of the opening wing of the kinked crack (i.e. $\tau'_n \cos \theta$) with respect to θ to give

$$\mu s_r \tan^3 \theta_{f_i} - 2(1-s_r) \tan^2 \theta_{f_i} + \mu(3-2s_r) \tan \theta_{f_i} + (1-s_r) = 0 \quad [3.107]$$

which has only one real positive definite root. The maximum kinked crack length at which the specimen will fail can be estimated according to

$$l_{k_u} = 4\rho^2 D_M \quad [3.108]$$

It follows that the failure stresses, ignoring microcrack interaction, can be written as

$$f'_c = \begin{cases} \frac{K_{IC}^c \sqrt{\pi l_{k_u}}}{\left[D_M G(s_r, \theta_{f_i}) \cos \theta_{f_i} \right] - s_r \pi l_{k_u}} & \bullet -1.000 \leq s_r \leq 0.050 \Rightarrow (\text{split failure}) \\ \frac{K_{IC}^c}{G(s_r, \theta_{f_i})} \sqrt{\frac{2}{\pi D_M}} & \bullet 0.050 \leq s_r \leq 0.306 \Rightarrow (\text{shear fault failure}) \end{cases} \quad [3.109]$$

while considering microcrack interaction, it may be obtained from

$$f'_c = \begin{cases} \frac{K_{IC}^c \sqrt{b \sin \left(\frac{\pi l_{k_u}}{b} \right)}}{\left[D_M G(s_r, \theta_{f_i}) \cos \theta_{f_i} \right] - s_r \left[\sqrt{2b^2 \sin \left(\frac{\pi l_{k_u}}{b} \right) \tan \left(\frac{\pi l_{k_u}}{2b} \right)} \right]} & \bullet -1.000 \leq s_r \leq 0.050 \\ \frac{K_{IC}^c}{G(s_r, \theta_{f_i})} \sqrt{\frac{2}{\pi D_M}} \frac{1}{\sqrt{\frac{4b}{\pi D_M} \tan \left(\frac{\pi l_{k_u}}{4b} \right)}} & \bullet 0.050 \leq s_r \leq 0.306 \end{cases} \quad [3.110]$$

where the distance b is evaluated with Equation [3.88].

Earlier discussion showed that some loading combinations within the current loading case will not produce any damage (i.e. crack displacements and strains) within the concrete material (see Equation [3.48]). It was also mentioned that yielding and flow will

prevail because the cracks will not feel the impact of the applied loads and/or respond to it in order to produce inelastic compliance. This is due to the basic hypothesis of damage mechanics. The current model utilizes the von Misses yield surface which can be written for the case of $\sigma_z = 0$ as (Chen 1982)

$$\sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 = f_y^2 = 3k^2 \quad [3.111]$$

where, f_y is the equivalent yield stress which can be related to the critical octahedral shear strength of concrete, k . A value of $k = 0.85f'_c$ may be used for practical applications. It follows that the elasto-plastic compliance of the concrete material subjected to these loading conditions can be found by employing Equation [3.33]. Finally, it should be mentioned that the proposed sets of equations recognize the effect of existing cracks on the plastic strains since Equation [3.33] uses the overall effective stiffness of the cracked material instead of the virgin one

3.3.3.3 BIAXIAL COMPRESSION-TENSION (MIXED MODE-I/II)

In the tension-compression stress regime, two possibilities of crack growth exist. Some cracks may open while others may close under this loading condition. Furthermore, open cracks may close, or closed cracks may open during the loading process. Open cracks grow in a mixed mode-I/II fashion while closed cracks grow in mode-II. To identify the domain of *open* cracks from the *closed* cracks, the normal stress at the crack site should be equated to zero; i.e.

$$\sigma_n = \sigma_{y'} = \sigma_y N(s_r, \theta) \quad [3.112]$$

the solution of which indicates that the *limit angle*, θ_l , is

$$\theta_l = \tan^{-1} \sqrt{\frac{1}{|s_r|}} \quad [3.113]$$

Thus, for a given load combination, mode-II fracture takes place for the cracks oriented at $\theta < \theta_l$, while cracks oriented at $\theta > \theta_l$ experience mixed mode-I/II. It should be noted that the upper and lower bounds for open cracks are $\frac{\pi}{2}$ and θ_l , respectively.

3.3.3.4 CLOSED CRACK KINEMATICS

For closed cracks (i.e. $\sigma_n < 0$), we evaluate the shear stress, τ , in a similar fashion as for the biaxial compression case, bearing in mind that the biaxial stress ratio, s , is negative. Hence, the bound of the sliding angle θ , can also be evaluated from Equation [3.90]. However, we require that $\theta > 0$, thus, there should be a single value of the angle which can be evaluated according to

$$\theta_s = \tan^{-1} \left[\frac{1 - \sqrt{1 - 4C_s(C_s + \mu)}}{2C_s} \right], \quad (C_s < 0) \quad [3.114]$$

The threshold angle, θ_{a_n} , at which cracks deform in mode-II fashion and its corresponding threshold stress, f'_{c_n} , can also be obtained from Equation [3.91] and Equation [3.93], respectively. The domain $[\theta_{f_1} - \theta_{f_2}]$ of unstable cracks can also be found in a similar fashion from Equation [3.95]. The latter equation is applicable if $\mu\sigma_x < \frac{2K_{IIc}^{II}}{\pi D_{min}}$, which is nearly always the case. However, when $\mu\sigma_x > \frac{2K_{IIc}^{II}}{\pi D_{min}}$, which is unlikely to occur, it has again only one root, i.e.

$$\theta_f = \tan^{-1} \left[\frac{1 - \sqrt{1 - 4C_f(C_f + \mu)}}{2C_f} \right] \quad [3.115]$$

It should be mentioned that, θ_{f_1} and θ_{f_2} , or if possible θ_f , should fall within the sliding range $[\theta, -\theta]$. With further increase in the external load (i.e. $\sigma_y > f'_{c_n}$), the domain of unstable cracks growing in mode-II fashion increases, i.e., the cracks in the range $[\theta, -\theta]$ increase their size from $2a_o$ to $D(\theta)$. Similarly, the kinked crack domain can be evaluated using Equation [3.97] to calculate their diameters and Equation [3.100] to obtain their orientations, $[\theta_{k_1} - \theta_{k_2}]$.

3.3.3.5 OPEN CRACK KINEMATICS

Unlike the biaxial tension case, an open crack subjected to the current loading condition will be subjected to normal tensile stresses and significant shear stress. This implies that the crack is experiencing both opening and sliding modes. Therefore, the mixed mode fracture criterion specified by Equation [3.72] must be applied in order to determine the microcracks stability. According to Kanninen and Popelar (1985), the governing fracture criterion can be written as

$$\zeta^2 K_I'^2 + K_{II}'^2 = K_{IIC}''^2 \quad [3.116]$$

where $\zeta = \frac{K_{II}'^2}{K_{IIC}''^2} > 1.0$. Substituting Equation [3.72] into the latter indicates that

$$\sigma_y = \frac{K_{IIC}''}{\rho} \sqrt{\frac{2}{\pi D}} \frac{1}{\sqrt{M(s_r, \theta, \zeta)}} \quad [3.117]$$

with

$$M(s_r, \theta, \zeta) = \zeta^2 [\cos^2 \theta + s_r \sin^2 \theta]^2 + [(1 - s_r) \sin \theta \cos \theta]^2 \quad [3.118]$$

It should be noticed that we dropped the cross term $(G_{21}' + G_{21}'') K_I' K_{II}'$ in order to simplify the analysis.

The threshold angle, θ , for the open crack domain can be obtained by maximizing the function $M(s_r, \theta, \zeta)$ with respect to θ , i.e

$$\theta = \begin{cases} \frac{\pi}{2} \\ \frac{1}{2} \cos^{-1} \frac{\zeta^2(1 + s_r)}{(1 - s_r)(1 - \zeta^2)} \end{cases} \quad [3.119]$$

It is interesting to observe that according to this model, no open crack domain exists for a specimen subjected to biaxial tension-compression with load ratio, $s_r > -0.285715$. This observation is in compliance with available experimental data (Mindess and Young 1981; Zaitsev 1983; Nemat-Nasser and Horii 1983). The threshold stress, f_m' , at which mixed

mode-I/II deformation takes place can be estimated by utilizing the critical angle θ_c in Equation [3.91], i.e.

$$f_m^r = \frac{K_{IIIC}^r}{\rho} \sqrt{\frac{2}{\pi D_M}} \frac{1}{\sqrt{M(s_r, \theta_o, \zeta_o)}} \quad [3.120]$$

while the critical crack diameter, $D(\theta)$, is

$$D(\theta) = \left[\left(\frac{\rho f_m^r}{\sigma_Y} \right)^2 \frac{M(s_r, \theta_o, \zeta_o)}{M(s_r, \theta, \zeta)} \right] D_M \quad [3.121]$$

and its orientation is

$$\theta_{t_2, t_1} = \tan^{-1} \pm \sqrt{\frac{-c_2 + \sqrt{c_2^2 - 4c_1c_3}}{2c_1}} \quad [3.122]$$

where the constants, c_i , ($i=1, 2$, and 3) are given by

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} \zeta^2 s_r^2 - \left(\frac{\rho f_m^r}{\sigma_Y} \right)^2 M(s_r, \theta_o, \zeta_o) \\ 2\zeta^2 s_r + (1-s_r)^2 - 2 \left(\frac{\rho f_m^r}{\sigma_Y} \right)^2 M(s_r, \theta_o, \zeta_o) \\ \zeta^2 - \left(\frac{\rho f_m^r}{\sigma_Y} \right)^2 M(s_r, \theta_o, \zeta_o) \end{Bmatrix} \quad [3.123]$$

The portions of unstable (*propagating*) and stable (*arrested*) cracks, can be obtained for the two possible scenarios of opening and sliding as before by considering the ratios of arc length (A-S-K method) or areas that are enclosed by Equations [3.94]; [3.97]; and [3.121] to the total area of the crack domain (Figures 3.11 and 3.12).

Under tension-compression loading, the kinked cracks portion η_k^m within the closed crack domain, can be calculated according to,

$$\eta_k^c = \frac{1}{\left(\frac{\pi}{2} - \theta_i \right)} \left\{ \left(\theta_{k_2} - \theta_{k_1} \right) - \left(\frac{k_1 f_c^r}{\sigma_Y} \right)^2 G^2(s_r, \theta_o) \int_{\theta_{k_1}}^{\theta_{k_2}} \frac{d\theta}{G^2(s_r, \theta)} \right\} \quad [3.124]$$

- A-S-K method
- exact method

and the propagating (sliding) closed cracks can be estimated according to

$$\eta_s^c = -\eta_k^c + \frac{1}{\left(\frac{\pi}{2} - \theta_s\right)} \left\{ \begin{aligned} & \left(\theta_{t_2} - \theta_{f_1} \right) \\ & \left(\theta_{t_2} - \theta_{f_1} \right) - \left(\frac{\rho f_c^r}{\sigma_y} \right)^2 G^2(s_r, \theta_o) \int_{\theta_{f_1}}^{\theta_{t_2}} \frac{d\theta}{G^2(s_r, \theta)} \end{aligned} \right. \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [3.125]$$

Figure 3.11: Domain of cracks angles fan for mixed Mode-I/II (A-S-K method).

Similarly, for the open crack domain (see Figures 3.11 and 3.12), we may calculate the amount of evolving cracks according to

$$\eta_s^i = \frac{1}{\left(\frac{\pi}{2} - \theta_s\right)} \left\{ \begin{aligned} & \left(\theta_{t_2} - \theta_{t_1} \right) \\ & \left(\theta_{t_2} - \theta_{t_1} \right) - \left(\frac{\rho f_m^r}{\sigma_y} \right)^2 M(s_r, \theta_o, \zeta_o) \int_0^{\theta_{t_1}} \frac{d\theta}{M(s_r, \theta, \zeta)} \end{aligned} \right. \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [3.126]$$

It can be observed that when the open or closed crack domain vanishes, the amount of their corresponding cracks become zero. This renders identical results to those obtained for the biaxial tension and compression loads, respectively.

Finally, the compliance for the cracks that lie in the closed crack domain are:-

◆ the sliding (*propagating*) closed cracks

$$[C]_s^{cr} = \int_{\pm\theta_s^r}^{\pm\theta_s^r} p_s^c(s_r, a)[T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_c^{cr} [T] d\theta \quad [3.127]$$

Figure 3.12: Domain of sliding, kinking, and arrested cracks $\left[\frac{D(\theta)}{D_M} - \theta \right]$ plane for mixed Mode-I/II.

◆ the arrested (*stable*) closed cracks

$$[C]_a^{cr} = \int_{\pm\theta_a^r}^{\pm\theta_a^r} p_a^c(s_r, a)[T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_c^{cr} [T] d\theta + \int_{\pm\theta_a^r}^{\pm\theta_a^r} p_s^c(s_r, a)[T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_c^{cr} [T] d\theta \quad [3.128]$$

◆ the kinked closed cracks

$$[C]_k^{cr} = \int_{\pm\theta_{k1}}^{\pm\theta_{k2}} p_k^c(s_r, l_k)[T]^T \left\{ \left[C^{(k)} \left(\theta, l_k \right) \right]_{k_1}^{cr} + \left[C^{(k)} \left(\theta, l_k \right) \right]_{k_2}^{cr} \right\} [T] d\theta \quad [3.129]$$

where the functions, $p_s^c(s_r, a)$, $p_a^c(s_r, a)$ and $p_k^c(s_r, l_k)$, are given by Equations [3.106].

At the same time, open cracks will contribute the following growing and arrested compliance to the overall effective material compliance, respectively, i.e.

◆ the growing (propagating) open cracks

$$[C]_g^{cr} = \int_{\pm\theta_g^r}^{\pm\theta_g^r} p_g^c(s_r, a)[T]^T \left[C^{(k)} \left([\bar{C}], \lambda_r, \theta \right) \right]_o^{cr} [T] d\theta \quad [3.130]$$

♦ the arrested (stable) open cracks

$$\begin{aligned}
 [C]_a^{cr} = & \int_{\pm\theta_1'}^{\pm\theta_1''} p_a'(s_r, a) [T]^T \left[C^{(k)'}([\bar{C}], \lambda_r, \theta) \right]_a^{cr} [T] d\theta \\
 & + \int_{\pm\theta_2'}^{\pm\theta_2''} p_a'(s_r, a) [T]^T \left[C^{(k)'}([\bar{C}], \lambda_r, \theta) \right]_a^{cr} [T] d\theta
 \end{aligned}
 \tag{3.131}$$

where the functions, $p_a'(s_r, a)$, and $p_a(s_r, a)$ can be calculated using Equations [3.83].

This concludes the derivation of the proposed elasto-plastic constitutive damage-based model. The proposed unified model can be used to predict the behaviour of brittle and cracked materials as well as composites subjected to loadings. The model application is limited to small deformation and static or cyclic non-proportional loadings. As mentioned earlier, the proposed set of constitutive relationships are capable of tracing both flow and cracking type of deformations with confidence and accuracy without interference from the user. Furthermore, it does not require many material parameters and those that are required are obtainable from simple tests and procedures which eliminate the need to employ curve fitting. Thus, it can be concluded that this model lends itself to easy implementation in numerical techniques such as the finite element method (e.g. program Nonlacs). The basic steps for implementing the current proposed elasto-plastic micromechanical damage-base model in finite element method are shown by the flow chart in Figure 3.13. In fact, the actual procedure involves other sub-steps, but are omitted from the latter flow chart for the sake of brevity. Nevertheless, the model can stand alone to trace and replicate the results of standard uniaxial and biaxial stress experiments on concrete cylinders and cubes.

3.4 CONCRETE FAILURE ENVELOPE

At this juncture, it is very important to examine the model's ability to predict the failure stresses under different combinations of biaxial loading. Towards that end, the experimental results of Kupfer, Hilsdrof and Rusch (1969) for plain concrete panels tested under different stress ratios are compared with corresponding model predictions.

Figure 3.13: Flow chart of the procedure for implementing the proposed model.

The material parameters used to determine the different failure stresses were as follows:

(i) mode-I fracture toughness of the cement-paste matrix $K_{IC}^c = 0.533 \text{ MN} / \text{m}^{\frac{3}{2}}$, (ii) mode-II fracture toughness, on the other hand, may be estimated from K_{IC}^c as $K_{IIC}^c = 1.5 K_{IC}^c$, (Mindess and Young 1981; Zaitsev 1983), and (iii) the maximum and minimum aggregate diameters, D_M and D_{min} are 19.1 mm and 4.8 mm, respectively.

Figure 3.14: Concrete failure envelope.

The model results are depicted in Figure 3.14 which shows excellent agreement with their experimental counterparts for both tensile and compressive strengths. In obtaining the theoretical failure envelope, Equations [3.87] and [3.109] are employed when damage prevails, and Equation [3.111] is used in the case of plastic flow. According to the latter figure, concrete may exhibit three different modes of behaviour from brittle, semi-brittle to ductile. It is also clear that while most of the existing damage models cannot predict the semi-brittle and plastic portions, the current model predicts the whole envelope.

It is interesting to note that other damage models (e.g. Fanella and Krajcinovic 1988; Ju 1990; Fanella 1990) can only predict a small portion of this envelope.

3.5 CLOSURE

The previous equations and procedures complete the 2-D continuum damage mechanics formulation for concrete. The detailed derivation shows that the model is conceptually simple and requires very few material parameters and it lends itself to easy implementation in existing finite element programs. This is facilitated by the proof, presented in this chapter, that for a given material there exists a relationship between the tangential stiffness (compliance) and the secant stiffness (compliance). The proposed theory is capable of capturing the true behaviour of cracked concrete material under different loading configurations in 2-D problems. It is also evident that the model is based on sound theories of mechanics and it omits any empiricism associated with the phenomenological damage-based models (e.g. Lemaitre 1984; Talreja 1984; Krajcinovic 1989). Furthermore, the use of the theories of elasticity, plasticity, and fracture mechanics are clear and justified when applied.

The current model differs from similar micromechanical damage models (e.g. Horii and Nemat-Nasser 1984; Fanella and Krajcinovic 1988; Ju 1990) because it accounts for loading, unloading and reloading. In addition, the current model uniquely quantifies the amount of propagating and stationary (stable) microcracks at a given load level. This model also combines the theory of plasticity with the micromechanical damage model in order to evaluate the behaviour of brittle heterogeneous materials under loading situations where plastification and yielding are the dominant modes of deformation.

To the writer's knowledge, this is the first attempt to combine the theory of plasticity and the micromechanical damage model in order to obtain a unified theory of concrete mechanics. Others have attempted, however, to combine the theory of plasticity and phenomenological damage models (e.g. Ortiz 1985; Simo and Ju 1987a and 1978b; Yazdani and Schreyer 1990). As mentioned earlier, the damage surface and a yield criterion employed in those models were empirically established in order to fit certain

experimental data. The proposed elasto-plastic damage-base method, on the other hand, distinguishes the onset of plastification and fracture and allows the material to undergo both deformation mechanisms.

In the next chapter, the steps to implement the preceding relations in the finite element scheme are outlined. The general finite element equations will be given in order to highlight where the proposed sets of constitutive equations can fit in. The details of the finite elements formulation are omitted because they can be found in any classical finite element book (e.g. Bathe 1982; Cook *et al.* 1989). However, the numerical integration of the crack compliance and the solution of the characteristic equations of crack deformation in an anisotropic medium will be discussed in detail since they are new and were not implemented in the Nonlacs finite element program before. This program is subsequently used to trace crack evolution and the nonlinear response of concrete elements and structures which are shown in Chapters 5 and 7.

4

4. FINITE ELEMENT IMPLEMENTATION

4.1 GENERAL

Implementing the sets of equations of the proposed elasto-plastic damage-based model developed in Chapter 3 in a general purpose finite element program requires some basic mathematical operations. For example, one must find the roots of 3rd and 4th order polynomials (e.g. Equations. [3.40], [3.41], and [3.106]) in order to evaluate a single crack compliance, $[C']^c$ (e.g. Equations [3.45] and [3.50]).

On the other hand, numerical integration of the crack inelastic compliance, and some basic matrix operations (e.g. dot and cross product, matrix inversion and determinate calculations, etc.) are needed for the evaluation of the overall compliance and stiffness of the cracked brittle material. The computer implementation phase usually consumes many working hours because problems and difficulties often arise during this stage over the course of the study. As an example, during this phase of study, the computer answers for the complex roots of the non-linear equations given by Equations. [3.40] and [3.41] were not correct using customary procedures and subroutines available from different computer mathematical libraries (e.g. linpack, eispack, slatec, etc.). There was also not much success utilizing symbolic algebra computer programs (e.g. Mathematica, MathCad, Maple, etc.). It was found that the computer solutions of these types of polynomials are extremely sensitive to their coefficients. Thus, a special technique, which will be described later in this chapter, had to be utilized in order to filter and purify the roots of these equations from the computer round off errors (numerical noises).

Another problem, associated with developing a non-linear finite element program code, is testing segments of it and calibrating the program for full fledged non-linear analysis. This is due to the fact that a typical non-linear analysis program uses many hours of real time to capture the complete response of a structure up to failure. Thus, the code errors cannot be easily spotted and corrected. Of course, if the errors occur at a given iteration, the problems escalate and the task becomes very frustrating. Therefore, in this chapter we will bring the readers' attentions to these problems which they might face when implementing the proposed methods in their programs.

This chapter will also present the development of the overall constitutive relationships of concrete materials reinforced with steel bar and/or long organic fibres because one of the main objectives of this thesis is to apply the concept of damage mechanics in analyzing some real concrete structures, which may include reinforcement. In addition, the finite element program in which the proposed models are implemented will be described. The general finite element equations and the execution steps of the computer procedure will be given in the subsequent sections. The finite elements used within the Nonlacs program are discussed and their superior performance to other elements for the same applications are illustrated, but their developments are omitted since they can be found elsewhere (Razaqpur and Nofal 1990).

4.2 BASIC EQUATIONS OF FINITE ELEMENT

Some basic equations of the finite element method, as used in the program Nonlacs, will be presented. The finite elements used are 3-D axial bar; shear stud, in-plane; bending and facet shell elements. As mentioned earlier, their detailed formulation can be found elsewhere and therefore they will not be discussed further. However, the in-plane elements, RQD4 and QLC3 have a rotational degree of freedom, θ_z (Razaqpur and Nofal 1990; Ghoniem and Ghali 1983). As a result, the FAQN facet shell element nodes will have the full six degree of freedom which will allow the user to analyze 3-D structures using the computer program Nonlacs. Using the principles of virtual work, the following incremental equation of equilibrium can be written:

$$[k]\{\Delta d\} = \{\Delta R\} + \{\Delta F\}_b + \{\Delta F\}_s + \{\Delta F\}_{\bar{\epsilon}} + \{\Delta F\}_{\bar{\sigma}} \quad [4.1]$$

where $[k]$ is the total tangential structure stiffness matrix, while $\{\Delta d\}$ is the vector of its nodal displacements increment, $\{\Delta R\}$ is the vector of applied nodal forces increment, $\{\Delta F\}_b$ and $\{\Delta F\}_s$, are vectors of nodal forces due to body forces and surface tractions, respectively, $\{\Delta F\}_{\bar{\epsilon}}$ is vector of nodal forces due to initial strains and $\{\Delta F\}_{\bar{\sigma}}$ is the vector of nodal forces due to initial stresses.

The displacement field, $\{d\}$, inside an element can be related to the nodal displacements, $\{d\}^e$, as

$$\{d\} = [N]\{d\}^e \quad [4.2]$$

where $[N]$ is the shape functions matrix. Note that the superscript e is used to denote quantities at the element level. The strain $\{\epsilon\}^e$ and nodal displacements $\{d\}^e$ are related by the strain-displacement matrix, $[B]$ as:

$$\{\epsilon\} = [B]\{d\}^e \quad [4.3]$$

Using the constitutive matrix, $[D]$, (e.g. Equation [3.32]), the element stresses are computed from the element strains as follows:

$$\{\sigma\} = [D](\{\epsilon\} - \{\bar{\epsilon}\}) + \{\bar{\sigma}\} \quad [4.4]$$

where $\{\bar{\epsilon}\}$ and $\{\bar{\sigma}\}$ are vectors of initial (residual) strains and initial stresses, respectively.

By applying Equations [4.2] to [4.4], elements of Equation [4.1] can now be rewritten in more detail as:

$$[k] = \sum_e [k]^e \quad [4.5]$$

where

$$\begin{aligned}
[k]^e &= \int_V [B]^T [D] [B] dV^e \\
\{F\}_b &= \sum_e \int_S [N]^T \{g\} dS^e \\
\{F\}_s &= \sum_e \int_S [N]^T \{p\} dS^e \\
\{F\}_\epsilon &= \sum_e \int_V [B]^T [D] \{\bar{\epsilon}\} dV^e \\
\{F\}_\sigma &= - \sum_e \int_V [B]^T \{\bar{\sigma}\} dV^e
\end{aligned} \tag{4.6}$$

In the above $\{g\}$ and $\{p\}$ are vectors of body forces and surface tractions, respectively. These equations and those developed earlier in Chapter 3 of this thesis will be used in subsequent sections to explain the various aspects of the non-linear finite element model Nonlacs.

4.3 FINITE ELEMENT PROGRAM NONLACS

Since one of the main purposes of this study is to use the proposed damage model within a non-linear finite element program, a brief discussion and description of the program used in the numerical tests will be given. This program is a general purpose finite element program and is called Nonlacs (Non-linear Analysis of Concrete and Steel (Razaqpur and Nofal 1990). Nonlacs can model any general structure employing layering technique, provided that small deformations and thin plate theories are applicable. 2-D constitutive laws can only be used in formulating the elements stiffness matrices and the global stiffness matrix. In other words, Nonlacs can model 3-D (spatial) structures comprising planar elements but all finite elements in the program are 2-D.

The general purpose finite element program Nonlacs is written in structured FORTRAN-77 (90) and its code can be compiled with most FORTRAN compilers such as, Microsoft MS-FORTRAN 4.0 and FORTRAN PowerStation; UNIX f-77, any FORTRAN 90 compilers, etc.; as well as with gnu compilers g-77. The program code was translated to C and C++ using gnu **f2c** and after slight modification to the C-code, most available C compilers could compile the program. As a matter of fact, the C code looks almost the same as that of the original FORTRAN code. This was achieved by commenting the IO statement (e.g. read, write, and

print) before the translation process and were later inserted in the C code using C syntax. At this juncture, it should be mentioned that Nonlacs can be used on different machines including Intel 386/486 DX2 and Pentium based machines and IBM computers as well as SUN SPARC stations, HP-Apollo, workstations.

The program consists of different modules which are designed to perform specific tasks. These modules are called IO, mathematical, elements, material models, and solvers, libraries. For example, the IO library reads the input data and prints the results from and to disk files, while the library of the material models is concerned with the evaluation of the constitutive laws from different theories (e.g. elasticity, plasticity, damage, etc.). It should be mentioned that each library can be incorporated individually in any other commercial mathematical software packages (e.g. Matlab, Mathematica, Maple, etc.). This is important because one can utilize the power of these programs and make use of their graphical capabilities. As an exercise, the element library was linked and compiled with Matlab package on a Sun-OS machine and it was used to solve very simple problems. In the following, each library in the computer program Nonlacs will be explained and the basic equations used will be described and presented.

4.3.1 NONLACS ELEMENT LIBRARY

Nonlacs element library contains bar, plane, plate bending and facet shell finite elements. In addition, the library has a two node boundary element which is used to enforce certain boundary conditions such as prescribed displacement at a certain node. Bar elements consist of the typical 3-D two node uniaxial truss and a 3-D two node shear connector element shown in Figures. 4.1(a) and 4.1(b) (Nofal 1988).

The plane elements are a high order four node plane quadrilateral elements QLC3 of Ghoniem and Ghali (1983) and RQD4 of Razaqpur and Nofal (1990). It should also be mentioned that RQD4 element has a drilling degree of freedom, ω_i , at each node in addition to the in-plane displacement u_i and v_i , Figure 4.1(c). The drilling degree of freedom, ω_i , is obtained using the basic concept of Allman (1984; 1988). The element shape functions, N_i , were developed from those of the customary Q8 element through a simple

transformation. In the program, necessary transformations are performed at the level of calculating the strain-displacement matrix, $[B]$ and of the nodal forces according to

Figure 4.1: Finite elements in the element library of program Nonlacs.

$$\begin{aligned} [B]_{RQD4} &= [T]^T [B]_{Q8} [T] \\ \{F\}_{RQD4} &= [T] \{F\}_{Q8} \end{aligned} \quad [4.7]$$

where $[T]$ is a 12×16 transformation matrix where its coefficients are some multiples of the parent element 8 nodes coordinates, x_i, y_i , ($i = 1, 2, 3, \dots, 8$). This matrix is given explicitly in Nofal (1988). $[B]_{RQD4}$ and $\{F\}_{RQD4}$ are the strain-displacement matrix and nodal force vector of element RQD4, respectively, while $[B]_{Q8}$ and $\{F\}_{Q8}$ are the same for element Q8. Element RQD4 is economically and computationally efficient. It has a superior convergence compared to other elements, including Q8 (Razaqpur and Nofal 1990).

Therefore, in the analysis of the in-plane problems reported in Chapters 5, 7, and 8, the latter element is employed. Element QLC3 also has relatively good behaviour if it is used in modelling in-plane bending cases (e.g. beams, box girders, etc.), but since this element is geometrically biased, it does not perform as well in some situations. Plate bending elements in Nonlacs include the rectangular bending element RBE (Zienkiewicz and Talyor 1991) and the improved discrete Kirchoff quadrilateral IDKQ developed by Chinniah (1985).

Both elements are four node quadrilateral elements with three degrees of freedom at each node (Figure 4.1(d)). Formulation of element RBE can be found elsewhere (e.g. Ghonim and Ghali 1983) and therefore will not be discussed here. It should be mentioned, however, that numerical testing showed that this element gives good results provided its geometry has a square or rectangular shape with aspect ratio, $\frac{b}{a} \leq 1.5$. Element IDKQ, on the other hand, is developed based on the Mindlin plate theory where the nodal rotations are defined independently of the transverse displacements, w (Figure 4.1(d)). The element shape functions are derived as the least square polynomial version of the customary Lagrange functions. Kirchoff's hypothesis is only applied to the Lagrange element at the rotational degrees of freedom, θ_x and θ_y . This element's stiffness matrix is

evaluated in program Nonlacs employing the same procedure as that used for the in-plane element. A comparison between the original implementation proposed by Chinniah (1985) and the one used in this study shows that the latter saves tremendous amount of hard disk and memory. For the plates and slabs analyzed later in this thesis, element IDKQ is used for its superior behaviour compared to other plate bending finite elements. Note that layered plate and shell elements are used for out of plane bending problems.

The program allows the user to combine any in-plane element with any plate bending element to obtain a facet quadrilateral shell element. However, facet shell element, FAQN, is used in the analysis of the shell problems shown later in this thesis. Facet shell element, FAQN, is the result of combining the in-plane element RQD4, and plate bending element, IDKQ. Since the program accounts for material nonlinearity, we observe that the facet shell element is an anisotropic shell in which in-plane and bending actions are assumed to be coupled. The strain $\{\epsilon\}$ at a general point within an element can be written as

$$\{\epsilon\} = \{\epsilon_o\} - z\{\chi\} \quad [4.8]$$

where $\{\epsilon_o\}$ are the in-plane strains of the projection of a given point on the reference plane, $\{\chi\}$ is the vector of curvature due to bending and z is the local z coordinate (through the thickness) of the point under consideration with the axis z being normal to middle plane and its origin coinciding with the middle plane of the element. Since we can write,

$$\begin{Bmatrix} \{\epsilon\} \\ \{\chi\} \end{Bmatrix} = \begin{Bmatrix} [B]_p \{d\}_p \\ [B]_b \{d\}_b \end{Bmatrix} \quad [4.9]$$

where the subscript p and b stand for in-plane and bending components. Equation [4.8] can be rewritten as

$$\{\epsilon\} = \begin{bmatrix} [B]_p & -z[B]_b \end{bmatrix} \begin{Bmatrix} \{d\}_p \\ \{d\}_b \end{Bmatrix} = [B]\{d\} \quad [4.10]$$

The stiffness of the element is given by

$$[k] = \int_V [B]^T [D] [B] dV \quad [4.11]$$

Making use of Equations [4.10] and [4.11] can be rewritten as

$$[k] = \int_V \begin{bmatrix} [B]_p^T [D] [B]_p & -z [B]_p^T [D] [B]_b \\ -z [B]_b^T [D] [B]_p & z^2 [B]_b^T [D] [B]_b \end{bmatrix} dV = \begin{bmatrix} [k]_{pp} & [k]_{pb} \\ [k]_{bp} & [k]_{bb} \end{bmatrix} \quad [4.12]$$

Figure 4.2: Layered finite elements.

The diagonal terms are the stiffness of in-plane and bending elements, respectively. The off-diagonal terms are the coupling matrices that will be non zero if the material properties are not symmetric with respect to the middle surface of the shell. This happens when yielding and cracking takes place in some fibres above or below the middle surface and not in others. As mentioned earlier, plate and shell elements are divided into layers as shown in Figure 4.2. Each layer is assumed to be in a state of plane stress. Within the thickness of each layer, the stresses and material properties are assumed to be constant.

To evaluate the sub-matrices that comprise the elements of the stiffness matrix, the volume integral is converted to a summation through the thickness and an area integral over the surface of the element. For example,

$$[k]_{pp} = \int_V [B]_p^T [D] [B]_p dV = \int_V \int_x [B]_p^T \left(\int_z [D] dz \right) [B]_p dx dy \quad [4.13]$$

The line integral is denoted by $[D]_{pp}$ and is calculated in the following fashion:

$$[D]_{pp} = \int_z [D] dz = \sum_{i=1}^n (z_{i+1} - z_i) [D]_i \quad [4.14]$$

where z_{i+1} and z_i are the z coordinates of the top and bottom surfaces of layer i and n is the total number of layers in a given element. Of course, the constitutive matrix $[D]$, for each layer will depend upon whether the layer is concrete, smeared reinforcement or steel materials as well as the theory employed to obtain it. For example, in order to evaluate the damage state in a concrete structure, the constitutive relationships developed in the preceding chapter will be used to obtain the $[D]$ matrices of its layers.

The remaining sub-matrices in Equation [4.12] are computed in a similar fashion, and the line integral evaluation for the remaining terms is given by

$$[D]_{pb} = - \int_z z [D] dz = - \frac{1}{2} \sum_{i=1}^n (z_{i+1}^2 - z_i^2) [D]_i \quad [4.15]$$

$$[D]_{bb} = \int_z z^2 [D] dz = \frac{1}{3} \sum_{i=1}^n (z_{i+1}^3 - z_i^3) [D]_i$$

It must be mentioned that the equivalent nodal forces $\{F\}_{\bar{\epsilon}}$ and $\{F\}_{\bar{\sigma}}$ due to initial strain and stress are similarly calculated. Specifically,

$$\{F\}_{\bar{\epsilon}} = \int_V \int_x \begin{Bmatrix} [B]_p^T \{\sigma_{\bar{\epsilon}}\}_p \\ [B]_b^T \{\sigma_{\bar{\epsilon}}\}_b \end{Bmatrix} dx dy \quad [4.16]$$

$$\{F\}_{\bar{\sigma}} = - \int_V \int_x \begin{Bmatrix} [B]_p^T \{\sigma_{\bar{\sigma}}\}_p \\ [B]_b^T \{\sigma_{\bar{\sigma}}\}_b \end{Bmatrix} dx dy$$

where

$$\begin{aligned}
\{\sigma_{\bar{\epsilon}}\}_p &= \int_z [D]\{\bar{\epsilon}\} = \sum_{i=1}^n (z_{i+1} - z_i) [D]_i \{\bar{\epsilon}\}_i \\
\{\sigma_{\bar{\epsilon}}\}_b &= -\int_z z [D]\{\bar{\epsilon}\} = -\frac{1}{2} \sum_{i=1}^n (z_{i+1}^2 - z_i^2) [D]_i \{\bar{\epsilon}\}_i \\
\{\sigma_{\bar{\sigma}}\}_p &= \int_z \{\bar{\sigma}\} = \sum_{i=1}^n (z_{i+1} - z_i) \{\bar{\sigma}\}_i \\
\{\sigma_{\bar{\sigma}}\}_b &= -\int_z z \{\bar{\sigma}\} = -\frac{1}{2} \sum_{i=1}^n (z_{i+1}^2 - z_i^2) \{\bar{\sigma}\}_i
\end{aligned} \tag{4.17}$$

Once again the $[D]_i$ matrix is dependent upon the layer material type and the level of stress. While one of the methods of calculating the $[D]_i$ matrix for concrete was outlined by the procedures of Chapter 3, its format for the reinforcing steel bars, organic long fibres, as well as high performance plastics and composites will be given later in the following subsections. In addition, the other concrete models implemented in the program will be highlighted

4.3.2 NONLACS MATERIAL LIBRARY

In evaluating the non-linear constitutive matrix, $[D]_{ep}$, the material models used in the program are based on the principles of: [1] theory of elasticity, [2] theory of plasticity [3] fracture mechanics [4] theory of continuum damage mechanics and [5] axially loaded layers (e.g. reinforcement layers in concrete and composites). In the following, the material models, except the one explained in the previous chapter, are presented.

4.3.2.1 THEORY OF ELASTICITY BASED MODELS

Models derived considering linear and non-linear elasticity are widely used in the analysis of the non-linear behaviour of concrete structure. Nonlacs program is no exception where its material library houses the concrete model of Darwin and Pecknold (1977) which is a non-linear hypo-elastic model. This model has been extensively used in analyzing many beams, plates and some shell structures. Another model implemented in program Nonlacs by the author is the modified version of the compression field theory (Vecchio and Collins 1986). However, the Nonlacs version of this model is different from that originally proposed by Vecchio and Collins (1986). The difference is that the Nonlacs compression field model assumes that the direction of the effective inelastic principal strain

increment coincide with the direction of the total effective principal stress instead of the hypothesis of the original theory that the direction of the total principal stress coincides with the direction of the total principal strain. Results of the analyses of Vecchio and Collins (1986) panels using the Nonlacs model were in excellent agreement with the experimental behaviour of these panels. The details are omitted here for this aspect is outside the scope of this thesis.

4.3.2.2 THEORY OF PLASTICITY BASED MODELS

Furthermore, the program has a complete plasticity model that accounts for anisotropy. The anisotropic plasticity is a new addition to the program and therefore, will be discussed here in more detail. Realizing that anisotropic materials may have different strengths in the two principle material axes, some investigators were prompted to develop the appropriate anisotropic failure surfaces of laminate and composites (Echaabi *et al.* 1996; Rowlands 1985). The lamina failure surfaces proposed by many authors often take the following form

$$F_i \sigma_i + (F_{ij} \sigma_i \sigma_j)^\alpha = 1 \quad [4.18]$$

where $\alpha = 0, \frac{1}{2}, \text{ or } 1$, F_i and $F_{ij} (i, j = 1, 2, 3)$ are functions of the longitudinal and transverse strength as well as the shear strength of the orthotropic plies.

It should be noted that the failure surfaces of Equation [4.18] are written with respect to the axes of material symmetry such that the positive and negative shear strengths are the same, and the coefficient $F_3 = 0$. Observation of Equation [4.18] indicates that these theories account for interaction or mixed-mode failure. Isotropic failure theories (e.g. von Mises, Tresca, etc.) never involve this aspect in postulating their failure envelopes which are often written in terms of a single strength value (e.g. Equation [3.111]). The models listed in Table 4.1 are all implemented in the plasticity model of program Nonlacs material model library.

The quantities X and Y in Table 4.1 are the normal longitudinal and transverse tensile strengths of the composite, respectively, while X' and Y' are the compressive strengths in the two directions. S , on the other hand, represents the shear strength of the orthotropic plies. It is well known that $S \neq S'$ for general anisotropic lamina except with respect to axes of material symmetry.

Table 4.1: Lamina failure surfaces in Nonlacs material library.

#	Theory	Criterion statement	F_1	F_2	F_3	F_4	F_5	F_6
1	Ashkenazi	$\bar{F}_1 \sigma_1 \sigma_2 = 1$	—	—	$\frac{1}{X}$	$\frac{1}{2} \left(\frac{4}{J} - \frac{1}{X} - \frac{1}{Y} - \frac{1}{S} \right)$	$\frac{1}{Y}$	$\frac{1}{S}$
2	Chamis	$\bar{F}_1 \sigma_1 \sigma_2 = 1$	—	—	$\frac{1}{X}$	$-\frac{K_1 K_2^2}{2XY}$	$\frac{1}{Y}$	$\frac{1}{S}$
3	Cowin	$\bar{F}_1 \sigma_1 \sigma_2 + \bar{F}_3 \sigma_3 = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\sqrt{\bar{F}_1 \bar{F}_2} - \frac{1}{2S}$	$\frac{1}{YY'}$	$\frac{1}{S'}$
4	Fischer	$\bar{F}_1 \sigma_1 \sigma_2 = 1$	—	—	$\frac{1}{X}$	$-\frac{K^1}{2XY}$	$\frac{1}{Y}$	$\frac{1}{S}$
5	Gol'denblat-Kopnov	$\sqrt{\bar{F}_1 \sigma_1 \sigma_2 + \bar{F}_3 \sigma_3} = 1$	$\frac{1}{2} \left(\frac{1}{X} - \frac{1}{X'} \right)$	$\frac{1}{2} \left(\frac{1}{Y} - \frac{1}{Y'} \right)$	$\frac{1}{4} \left(\frac{1}{X} + \frac{1}{X'} \right)^2$	$\frac{1}{8} \left[\left(\frac{1}{X} + \frac{1}{X'} \right)^2 + \left(\frac{1}{Y} + \frac{1}{Y'} \right)^2 + \left(\frac{1}{S} + \frac{1}{S'} \right)^2 \right]$	$\frac{1}{4} \left(\frac{1}{Y} + \frac{1}{Y'} \right)^2$	$\frac{1}{S}$
6	Hofmann	$\bar{F}_1 \sigma_1 \sigma_2 + \bar{F}_3 \sigma_3 = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$-\frac{1}{2XX'}$	$\frac{1}{YY'}$	$\frac{1}{S'}$
7	Malmeister	$\bar{F}_1 \sigma_1 \sigma_2 + \bar{F}_3 \sigma_3 = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\frac{1}{2} \left[(\bar{F}_1 - \bar{F}_2) + \bar{F}_1 \bar{F}_2 - \left(\frac{1}{S} \right)^2 \right]$	$\frac{1}{YY'}$	$\frac{1}{S'}$
8	Marin	$\bar{F}_1 \sigma_1 \sigma_2 + \bar{F}_3 \sigma_3 = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\frac{1}{XX'} - \frac{XX' - S \left[\frac{X'}{X} - \frac{X}{X'} + \frac{X(Y)}{Y} + Y \right]}{2S XX'}$	$\frac{1}{XX'}$	—
9	Norris	$\bar{F}_1 \sigma_1 \sigma_2 = 1, \begin{cases} \sigma_1 = X \\ \sigma_2 = Y \end{cases}$	—	—	$\frac{1}{X}$	$-\frac{1}{2XY}$	$\frac{1}{Y}$	$\frac{1}{S}$
10	Tsai-Hill	$\bar{F}_1 \sigma_1 \sigma_2 = 1$	—	—	$\frac{1}{X}$	$-\frac{1}{2X}$	$\frac{1}{Y}$	$\frac{1}{S}$
11	Tsai-Wu	$\bar{F}_1 \sigma_1 \sigma_2 + \bar{F}_3 \sigma_3 = 1$	$\frac{1}{X} - \frac{1}{X'}$	$\frac{1}{Y} - \frac{1}{Y'}$	$\frac{1}{XX'}$	$\leq \pm \sqrt{\bar{F}_1 \bar{F}_2}$	$\frac{1}{YY'}$	$\frac{1}{S'}$

† K_1 is experimentally determined correlation coefficient

$$K_1 = \frac{(1 + 4\nu_{12} + \nu_{13})E_{22} + (1 + \nu_{22})E_{11}}{\sqrt{E_{11}E_{22}(2 + \nu_{12} + \nu_{13})(2 + \nu_{21} + \nu_{22})}}$$

$$K = \frac{(1 + \nu_{21})E_{11} + (1 + \nu_{12})E_{22}}{\sqrt{E_{11}E_{22}(1 + \nu_{12})(1 + \nu_{21})}}$$

• determined under biaxial stress

The anisotropic plasticity theory as described above will prove very useful, and it is needed for developing the elasto-plastic damage model of fibre-reinforced concrete material given in Chapter 6. This is because both plain and fibre-reinforced concrete materials often have different strength values in various directions (Van Meir 1986). It should be mentioned that these models were applied in the analysis of the non-linear behaviour of anisotropic shells. For example, Razaqpur *et al.* (1988) have used Nonlacs program utilizing the Tsai-Wu theory (Table 4.1) from its material library to analyze the non-linear response of advanced composite quadratic shells. It is interesting to mention

that the latter authors have utilized the same Tsai-Wu theory as well as the well known Huber-von Mises yield criterion in analyzing an isotropic quadratic shell. Both methods gave identical results for the case of isotropic shells which indicated the correct implementation of these theories in the program. Further analyses were conducted and their results agree with the available theoretical solutions and also match favourably with the experimental data

4.3.2.3 FRACTURE MECHANICS BASED MODELS

During the phase of implementing the damage model described in the preceding chapter, it became necessary to include linear fracture mechanics, since, the damage evolution laws and crack displacement functions evaluation can be obtained from fracture mechanics considerations. Thus, a separate module was written to model fracture process and estimate the fracture criterion given by Equation [3.70]. In addition, this module can be used to calculate the stress intensity factors, K_I , K_{II} , and K_{III} corresponding to modes-I, II and mixed mode-I/II, respectively (e.g. Equations [3.71]). The linear fracture model can be used alone to trace the behaviour of a macro crack in concrete structure. In this study, however, no attempt was made to use fracture mechanics by itself since its correct implementation will be ascertained during testing of the proposed damage model

4.3.2.4 AXIALLY LOADED LAYERS (REINFORCEMENTS) MODELS

Nonlacs simulates layer(s) that cannot resist shear stresses, as axially loaded layers. The program assumes that these layers are fully bonded and assigns them an elasto-plastic relationship, with or without strain hardening. These are often used in the program to model bar reinforcement, prestressing tendons, and/or organic fibres (see Figure 4.3). As a matter of fact, ordinary reinforcing bars, prestressing tendons and bars, as well as organic fibres, according to Nonlacs, are modelled in two different ways. If the reinforcements are distributed uniformly (Figure 4.3), they are considered to be a layer of thickness, t_r , that may resist stresses in the direction of bars only. The layer thickness is simply the steel bar diameter divided by the bar spacing, s , (Figure 4.3). On the other

hand, when the bars are placed far apart, or bundled, NonIacs treats them as bar element, which can resist the applied forces uniaxially. The bars are also modelled as an elasto-plastic strain-hardening material with full bond with the surrounding concrete material. A bilinear or multilinear stress strain relationship is used. For the case of unloading and reloading, the so-called Bauschinger effect, or the reduction in the yield stress of reinforcement due to cyclic loading, is considered. The reinforcement material unloads itself in a direction parallel to the initial elastic moduli (see Figure 4.4) which shows permanent plastic deformations.

Figure 4.3: Typical reinforcement layers orientation.

The material parameters, the yield stress, f_y , the initial elastic moduli, E_o , the ultimate strain, ϵ_u , must be input to define the stress-strain relationship. If a multilinear curve needs to be input, a set of points n , ($n \leq 10$) must be entered at any change of the curve.

The coordinate of these points are the stress, and the corresponding strain $(\sigma_i, \varepsilon_i)$ at each point (Figure 4.4).

Depending on the stress level and the shape of the reinforcement stress-strain curve, the layer constitutive relationships can be written as

$$[D]_{R_{FMT}} = \begin{bmatrix} E'_1 & 0 & 0 \\ 0 & E'_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [4.19]$$

where E'_1 and E'_2 are the axially loaded layer elastic moduli in the directions, 1 and 2, respectively. Their values depend on the level of the applied stresses and the existence of reinforcements in the given direction. Finally, it should be mentioned that sometimes reinforced steel and/or fibres may be inclined at an angle, β_i , with respect to the concrete principle material axis,

Figure 4.4: Typical reinforcement stress-strain curve.

Figure 4.3. In that case the different quantities associated with the steel reinforcement such as stresses, strains, etc., must be transformed to the appropriate directions.

4.4 REINFORCED CONCRETE MODELING

Concrete structures are practically always reinforced with steel and/or (long and/or short) organic fibres in order to carry tensile stress and to control cracking. Therefore, any numerical model must account for the presence of these material on the behaviour of concrete structures. The proposed elasto-plastic damage mechanics model that has been developed in the previous chapter is basically applicable to plain concrete. Thus, if we combine these constitutive relationships with the model described in the preceding section, the mechanical response of reinforced concrete elements can be examined and evaluated.

Towards this end, let us consider the typical reinforced concrete element in Figure 4.5, which is subjected to a general state of stresses, σ_x , σ_y , and τ_{xy} in the x - y plane.

(a) Typical reinforced concrete element with thickness,

(b) Typical concrete layer, t_c

(c) Typical reinforcement layer, t_s

Figure 4.5: Stress distribution on typical steel and concrete layers.

To simplify the analyses, we assume first that full bond between the reinforcement and concrete exists. Nemat-Nasser and Horii (1991) showed that the assumption of full bond between an elastic inclusion (e.g. in this case steel, prestress tendon, organic fibre, etc.) is valid provided the brittle matrix material (e.g. concrete) is subjected to stresses below the bend over point (BOP) on its equivalent uniaxial stress-strain curve.

Of course, we can account for the slip effect by introducing the additional force (stress) produced due to bond deterioration. The results presented here and elsewhere (Razaqpur

and Nofal 1990) (even for the crude models) indicate clearly the minimal impact of the latter on the overall behaviour of concrete material subjected to stresses within the ascending branch of the load deformation curve. This mechanism, however, can not be neglected in the softening regime in concrete material which is not considered in this study. However, the slip and bridging mechanisms of the reinforcement material can be included in a straightforward fashion, which will be shown in the Chapter 6 of this work. For any point the principal stresses σ_1 and σ_2 are calculated; these stresses are then used in conjunction with the stress intensity parameters to evaluate the crack orientation, the threshold stress of cracking, crushing, or yielding of a given concrete layer. The next step is to evaluate the crack compliance and add them to the virgin concrete compliance to obtain the overall concrete compliance. The detailed steps of evaluating such compliance were outlined in the previous chapter. Knowing the stresses, strains and the elasto-plastic damage stiffness, increments of stresses, $\Delta\sigma$ are related to increment of strains $\Delta\epsilon$ using Equations [3.23] and [3.32]. The evaluation of the total structure stiffness matrix can be performed using Equation [4.12].

The open, closed, and sliding cracks are monitored by the model as explained earlier. While the proposed method allows the crack to rotate properly as the load changes, it eliminates the need for the empirical constants used by many models. For example, the Darwin and Pecknold (1977) model requires the so-called β factor, which is used to account, in an approximate fashion, for the shear stiffness due to aggregate interlock and dowel action. Razaqpur (1982), showed that the device of shear retention gives acceptable results provided the concrete is not severely under-reinforced for either normal or shear stresses.

4.5 COMPUTATIONAL STRATEGY

It may be noted from Chapter 3 that the evaluation of the overall stiffness of cracked concrete materials requires the determination of the integrals of the crack compliance (e.g. Equation [3.68], or [3.105], etc.). These integrals are functions of the roots of the characteristic equation, Equation [3.41]. It was mentioned earlier, however, that the

computer answers of these roots are highly dependent on the coefficients of the characteristic equations. In the following, some procedures for eliminating some of the problems associated with the determination of the exact solutions of the characteristic equations will be discussed. Furthermore, the performance of the different quadrature methods used will be evaluated and some methods will be recommended for calculating the integrals of the damage (crack) compliance.

4.5.1 SOLUTION OF THE CHARACTERISTIC EQUATIONS

Equations [3.41] and [3.42] are fourth order polynomial of variable, λ , and can be rewritten as

$$a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \quad [4.42]$$

where its coefficients, $a_i (i=1, 2, \dots, 5)$, are functions of the overall compliance of the brittle material. If one considers, an isotropic material, Equation [4.45] becomes

$$\lambda^4 + 2\lambda^2 + 1 = 0 \quad [4.42]$$

The roots of the latter quartic equation should be complex and conjugate, $\lambda = \pm i$, where $i = \sqrt{-1}$ (Lekhnitskii 1963). If this equation is multiplied by a factor larger than $1+e$, where $e \leq 0.000001$ is any arbitrary small number, the computer solution for the roots of the equation will not be correct. Of course, such small numbers can be the result of round-off error introduced by the computer. This trend was discovered through the exhaustive testing of the available subroutines in many numerical analysis libraries as well as symbolic algebra computer programs. The solution of the inelastic compliance utilizing incorrect roots was not numerically stable, and it fluctuated around the correct answer without convergence. If the correct roots are used, the solution becomes stable and it converges rapidly. Therefore, it was decided to use a filtering technique in order to obtain the roots from these routines to be identical to those obtained by hand calculations. Towards this end, the typical procedure (Kurtz 1991) for solving a quartic equation was coded such that the program Nonlacs checks the leading figures of a decimal point of a given number and then chops all digits beyond the six digit. For example, if the factor

reads 1.000 009 999, the program will treat it as 1.000 000. This is acceptable since the maximum number of digits of a real number in single precision for most machines is six and a half digits (23 bits). Of course, the digits beyond the six and a half digits are the cause of the computational errors of the machine used when single precision is utilized. It should be mentioned that the above procedure is used because the real numbers are represented in single precision in the finite element program, Nonlacs. However, if this model is to be implemented employing double precision for evaluating real numbers, the chopping should start immediately after the fourteen digit (52 bits).

4.5.2 NUMERICAL INTEGRATION METHODS

The necessary integration must also be performed numerically due to the complicated nature of the integrand. The choice of the integration method will affect both the accuracy and the speed of the calculation. Consequently, several integration schemes were tried to find out the most suitable one. The schemes that were employed comprised [1] Legendre, [2] Chebyshev of the first and the second kind, [4] Jacob, [5] generalized Laguerre, and [6] Simpson's rule. All methods were employed to solve the mechanical behaviour of axially loaded concrete cylinder in tension utilizing the concrete model presented in Chapter 3. The material properties and loading sequences for this test case will be given later in Chapter 5.

Figure 4.6 shows the ratio of compliance, $\frac{S_{22}}{S_{11}}$, in the two principal directions of the material versus the normalized applied tensile stresses, $\frac{\sigma_r}{f'_r}$, and compares the results of the different methods of integration. The exact solution provided by Sumarac and Krajcinovic (1987) and Ju (1990) is also displayed in the same figure. In Figure 4.6, it can be observed that the Legendre, the Chebyshev quadrature of the first and second kind and the Simpson's quadratures were the successful candidates. They, however, require at least 201 integration points for a *process* model (Ju 1990) or 60 integration points for a *non-process* (Hori and Nemat-Nasser 1983) to obtain accurate results. These time-consuming methods diminish their appeal despite their accuracy. Thus, in the current study we have

implemented the composite type of quadrature (Carnhan *et al.* 1969), which requires only 3 integration points with much improved accuracy and efficiency. Using composite integration formulas reduces the error associated with low-order integration formulas. The method works by subdividing the interval of the integration, $[a, b]$, into smaller sub-intervals and then using the formula (low-order) separately on each sub-interval. It is well known, however, (Carnhan *et al.* 1969) that repeated application of low-order formulas is always preferred to a single application of high-order formula. This is because of the simplicity of low-order formulas and secondly because of computational difficulties with the high-order integration formulas. Thus, the reason for the failure of some of the above methods may be due to the preceding argument.

Figure 4.6: Effect of quadrature method on the prediction of inelastic compliance.

Quadratures may use open or closed formulas in evaluating certain integrals. However, closed formulas are especially attractive because, except for the base points $\zeta = a$ and $\zeta = b$, base points at the ends of subdivision are also base points for adjacent sub-intervals. Therefore, this procedure leads to the use of $n(m-1)+1$ evaluations of the functional for n repeated applications of an m point formula instead of $n \times m$ evaluations. This yields considerable saving, especially when m is small. As mentioned above, it was found out that a maximum of 3 integration points is needed in applying Gauss-Legendre quadratures or even Simpson's rule. The relative difference between the 3 points of the latter

procedure and 256 points of other techniques is of the order of 10^{-16} , which is basically the round-off error of the machines used. Thus, the composite Simpson's rule is recommended since it is very easy to implement and it does not require any computer storage of some data compared to Gauss-Legendre quadratures method. Interested readers can find the complete theoretical developments of this method in any standard applied numerical methods book. Finally, it should be mentioned that the errors associated with the composite formulas is very small compared to the other methods. To improve their prediction, the so-called *Richardson's extrapolation* was used which is known to reduce the error drastically.

The next chapter will present examples of the experimental verification of the results of the proposed model. The close agreement between the results of the analysis and the corresponding test data illustrate the validity of the adopted integration method.

5. PLAIN CONCRETE NUMERICAL TESTING

5.1 GENERAL

Numerical results for some plain and reinforced concrete test specimens and structures will be presented in this chapter. These tests are evaluated using the elasto-plastic damage-based constitutive equations developed in Chapter 3 within the finite element method (e.g. program Nonaics) described in Chapter 4. The set of the numerical tests, shown later in this chapter, (i) verifies program Nonaics so that it can be used with confidence to predict the behaviour of real concrete structures subjected to general monotonic or cyclic mechanical loading; (ii) demonstrates the performance and the range of application of the proposed elasto-plastic micromechanical damage-based model; (iii) investigates the impact of existing cracks on the overall behaviour of concrete structures; and finally (iv) quantifies the amount of damage accumulated in a structure subjected to mechanical loading. Towards this end, the following presentation will show the analytical results of numerical testing of plain concrete test specimens and shallow beams, as well as reinforced concrete shallow and deep beams, slabs, thin plates, and thin shells. These problems are analyzed as 3-D spatial structures composed of assemblages of membrane, plate bending, and thin facet shells elements. Layered plate and facet shell elements are used for out of plane bending problems. When any reinforcement is present, it is modeled as either a bar or a smeared layer within the facet shell element, depending on its physical arrangement in the tested specimens and structures. It should be mentioned that full bond between the concrete and reinforcement is assumed and in all the examples presented in this chapter. The experimental test results that are used to measure the accuracy of the

proposed model and to calibrate its predictions were obtained by others. Therefore, a brief description of each test and the original authors' remarks and their observations on the behaviour of their actual specimens will be given prior to performing the numerical testing.

As stated earlier the material parameters required by the model are few and can be obtained from simple standard tests. These material parameters are listed in Table 5.1 with the corresponding standard test typically used to evaluate each one. Table 5.1 shows clearly that there is no need for data fitting because the required material parameters are basic properties that can be measured accurately and systematically.

Table 5.1: Proposed elasto-plastic damage model material parameters.

	1	2	3	4	5	6					
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			Maximum	Minimum	Cement-paste		Interface	Initial de-bonded interfaces	Initial damage
							Mode-I	Mode-I	Mode-II		
Symbol	E_o^p MPa	E_o^n MPa	ν	μ	$D_{\max}$ mm	$D_{\min}$ mm	K_I^c MPa $\sqrt{m}$	K_I^{cf} MPa $\sqrt{m}$	K_{II}^{cf} MPa $\sqrt{m}$	ρ	ϵ_o
Testing methods											
	Test name	Uniaxial test [*]		Uniaxial test [*]	Push test [*]	Sieve analysis		Notched beam test [*]			Ultrasonic test ⁺

† with respect to the direction of casting

! C and I_f correspond to the cement-paste and interface, respectively

* non-destructive test is sufficient

+ other tests can be used such as: [1] Micrography analysis
[2] Radioactive analysis
[3] Acoustic emission

It is obvious from Table 5.1 that the fracture toughness for the various modes (e.g. K_I^c , K_I^{cf} , and K_{II}^{cf}) are essential ingredients of the proposed elasto-plastic damage model because the model uses these fracture toughness in tracing the crack evolution due to the application of mechanical loading. Subsequently, it calculates the different crack sizes, D 's, which are used to estimate the compliance, of the cracked concrete material.

Unfortunately, experimental studies conducted prior to the 1980's neither evaluated the fracture toughness nor the fracture energy toughness (e.g. G_I^c , G_I^{ef} , and G_{II}^{ef}) for the test specimen(s) and/or structure(s). This creates a problem when analyzing some of these specimens using the current elasto-plastic damage model. To overcome this problem, we use Mindess and Young's (1981) approach who showed that the different fracture toughness (e.g. K_{IC}^c) can be related to the effective Young's modulus of elasticity, E , of the elastic brittle matrix material (e.g. virgin cement-past) according to

$$K_{IC}^c = \sqrt{EG_{IC}^c} \quad [5.1]$$

where G_{IC}^c is the mode-I fracture energy of concrete (Gopalaratnam and Shah 1985). The mode-I fracture energy (e.g. G_{IC}^c) can be determined from the area under the stress-crack opening displacement curve (Figure 5.1), i.e.

Figure 5.1: Tensile stress vs. (a) tensile strain (b) crack mouth opening displacements.

$$G_{IC}^c = \int_0^{u_{c2}} f(u) du \quad [5.2]$$

where $f(u)$ is equivalent to the applied stress, σ , and can be expressed as a function of the crack opening displacements, u . Note that u_{c2} represents the critical value of u which usually corresponds to the formation of a real (visible) crack. Thus, critical crack opening displacement can be obtained as a multiple of the strain values and it varies from

$u_{c_1} = 0.0157$ to $u_{c_2} = 0.0313 \text{ mm}$, Figure 5.1(b), (Backlund 1981). On the other hand, the order of the stress function, $f(u)$, beyond the bend over point, is often assumed to be constant, linear (Backlund 1981), or parabolic (Jenq and Shah 1986) (see Figure 5.1(b)). In the following examples the linear distribution is used to calculate the missing values of the mode-I fracture energy toughness, G_{IC}^c . This value can be used in conjunction with the hierarchical condition, Equation [3.75], to obtain the fracture toughness for all the modes. This approach gave acceptable results for some of the problems analyzed later, but other distributions may be equally suitable. This issue requires further investigation but will not be pursued further here. The aforementioned discussion shows that the proposed model material parameters can be estimated or obtained in a straightforward fashion.

5.2 PLAIN CONCRETE SPECIMENS AND STRUCTURES

As stated earlier, the model implementation must be tested against either theoretical solutions or experimental results. Therefore, in the following numerical tests on plain concrete, some examples analyzed by other investigators, using damage mechanics, will be solved and compared. These examples are used to ascertain and validate the implementation of the current damage models (e.g. Krajcinovic and Fanella 1988; Ju 1990). This is very important because both the model implementation and behaviour can be critically examined. Therefore, the first example will test the problems solved by Krajcinovic and Fanella (1988), and Ju (1990).

5.2.1 CONCRETE CYLINDER SUBJECTED TO UNIAXIAL AND BIAXIAL STRESSES

The latter authors solved numerically the experiments of Newman and Newman (1972) in order to calibrate the performance of their damage models. The specimens were standard concrete cylinders and were loaded in axial compression, σ_y , while being subjected to a constant lateral compression $\sigma_x = -3.45 \text{ MPa}$ or to a lateral tension $\sigma_x = 0.69 \text{ MPa}$. Two other specimens were subjected to uniaxial monotonically increasing tension, σ_y . The material parameters were obtained from Krajcinovic and Fanella (1988), and Ju (1990) and they are shown in Table 5.2.

This table indicates the coefficient of dry friction, μ , to be 0.6 which is typical for concrete materials (Horii and Nemat-Nasser 1985). It should be pointed out that the latter value for the coefficient of friction will be utilized in all the examples reported in this thesis and it will remain constant regardless of the load type and level. Of course, experiments (e.g. Horii and Nemat-Nasser 1985; Mindess and Young 1981, etc.) have shown that μ of concrete diminishes under repetitive cyclic loading. This issue requires thorough investigation but is beyond the scope of the current work. Nevertheless, the results of the test problems reported later in this thesis indicate that such an assumption does not impact the behaviour of structures subjected to monotonic loading. Numerical results for specimens and structures under cyclic loading, where the loads are only cycled either from zero to certain level of tension or compression, have affirmed that the constant friction coefficient assumption gives reasonable results for those situations (Horii and Nemat-Nasser 1985).

Table 5.2: Material properties of concrete cylinder test problem.

#	1		2	3	4		5			6	
Material properties	Elastic moduli		Poisson's ratio ν	Coefficient of dry friction μ	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel E_o^p (GPa)	Normal E_o^n (GPa)			D_M (mm)	D_{Min} (mm)	Cement-paste		Interface	Initial de-bonded interfaces ρ	Initial damage ω_o
							Mode-I K_I^C ($\text{MN/m}^{3/2}$)	Mode-I K_I^{II} ($\text{MN/m}^{3/2}$)	Mode-II K_{II}^C ($\text{MN/m}^{3/2}$)		
Value	40.00	40.00	0.20	0.6	16.00	4.80	0.578 [*]	0.165 [*]	0.324 [*]	0.85	0.2262 or 0.1131

The test cylinders will be modeled here with a single four node plane strain finite element illustrated in Figure 5.2(b). This element has three degree freedom per node (Razaqpur and Nofal 1989) which include in addition to the customary displacements, u_i , v_i , a drilling degree of freedom, ω_i . The numerically obtained stress-strain curves are shown in Figures 5.2(c) and 5.2(d) which also show the relevant experimental data. The variation of the axial and lateral strains, σ_y and ϵ_x , respectively, for the cases of lateral tension or compression are plotted against the applied major stress, σ_y in Figure 5.2(d). Good agreement between the analytical and experimental results can be observed in these figures. It should be mentioned that the current results are in complete agreement with

those reported by Krajcinovic and Fanella (1988). Hence, it is concluded that the damage mechanics in the proposed model are correctly implemented in the finite element program Nonalcs. In the coming examples, the model is further assessed by analyzing general cases of loading.

Figure 5.2: Plane stress test results.

Figure 5.3 illustrates that the accumulated amount of damage increases in the direction of applied load, σ_y . The stress-damage relationships are nonlinear and they depend not only on the applied major stress but also on the applied minor stress. For example, Figure 5.3 shows that compared to the uniaxial compression condition, the rate of damage increases when the element is subjected to axial compression and lateral tension. Note however, that at the earlier stage of loading in the latter case, the initial cracks induced by the applied tension are closed by the axial compressive stress. However, the specimen failed at a much lower axial compressive stress, σ_y , which is the direct result of the contribution of the open cracks due to lateral tension.

Figure 5.3: Accumulated damage, ω' , in concrete cylinders versus applied stress.

5.2.2 DELFT UNIVERSITY CUBES

At the University of Delft, Van Meir (1986) tested a number of 100mm concrete cubes under uniaxial and biaxial tensile and compressive stress combinations. The test specimens were divided into eight groups, five of which are used here to test the proposed model. Once again, the material properties used in the analysis are displayed in Table 5.3 where the missing parameters are either calculated from other available data or assumed to conform to the values typically used (e.g. ACI, CSA, etc.).

In analyzing this set of tests a single inplane element was used. The stress-strain behaviour predicted by the finite element model Nonalcs using the proposed model is compared with its experimental counterpart in Figures 5.4(a), (b), (c), and (d). Figures 5.4(a) and 5.4(b) show the tensile and compressive stress-strain curves, respectively, of specimens subjected to uniaxial loading, together with the corresponding experimental data. The experimental data shown in these figure are the average for the specimens labelled by the original author as Groups 5A and 3B.

It can also be observed in these figures that the experimental response parallel to the direction of specimen casting is different from that normal to direction of casting which is

also predicted by the proposed model. In the analysis these directions are distinguished by the difference in their basic mechanical properties. The program was terminated beyond the ultimate load (bend over point) because the finite element program Nonalcs cannot, at this stage, capture the softening response passed the peak load. This issue will be discussed in more detail in some of the following examples.

Table 5.3: Delft University cubes material properties.

#	1		2	3	4		5			6	
	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Min}	Cement-paste	Interface		Initial de-bonded interfaces	Initial damage
	E_p (GPa)	E_n (GPa)	ν	μ	(mm)	(mm)	Mode-I K_I^C ($\text{MN/m}^{3/2}$)	Mode-I K_I^H ($\text{MN/m}^{3/2}$)	Mode-II K_{II}^H ($\text{MN/m}^{3/2}$)	ρ	ω
Value	27.00	23.58	0.20	0.6	16.00	3.75	0.562*	0.251*	0.377*	0.85	0.1000

* Calculated using the hierarchical condition, Eq. [3.75].

! Assumed

Note:- Symbols * and ! in any subsequent similar table are used as above.

Figure 5.4(c), shows the impact of the lateral stress on the compressive peak stress and the associated overall response. The latter figure indicates that increased applied compressive lateral stresses, σ_x , results in an increase in the apparent plain concrete strength in the orthogonal direction. The nonlinearity of the high compressive lateral stresses, σ_x , is due to two sources: the evolution of microcracks and the plastic flow and yielding. Each response depends on the manner with which the loads were applied. For example, if the test starts by applying the loads in one direction to a certain constant value (e.g. $\sigma_x = q^*$) and subsequently the orthogonal direction is loaded up to failure, the behaviour in this case will be completely different from that due to proportional biaxial loading. This issue will be further investigated and explained in the following test problem. Figure 5.3(d), on the other hand, depicts the failure envelope of the specimens in Groups 5A, 5B, and 6A. In this figure, the lines represent the prediction of the proposed model and the symbols are the experimental values (Van Meir 1986). It should be pointed out that the solid lines are obtained by employing the proposed strength equations, Equations. [3.88], [3.109], and [3.111]. The results in Figure 5.3(d) show that

application of pure damage mechanics to the analysis and prediction of concrete behaviour and strength under some loading conditions will not give satisfactory results and the theory of plasticity needs to be employed in order to obtain the correct concrete response.

Figure 5.4: Concrete cube tests stress-strain curves and failure envelope.

The accumulated amount of damage, ω' , for the test specimens is depicted in Figures 5.5(a) and (b). In both figures the applied stresses are normalized to either the concrete tensile or to its compressive strength (i.e. f'_t, f'_c). The experimental data points were

computed from the stress-strain diagrams given by the original author and the analytical curves were obtained from the finite element program output.

Figure 5.5: Accumulated damage, ω' , for (a) axial and (b) biaxial loading.

Figure 5.5(a) shows the effect of the type of loading (e.g. tension versus compression) and the impact of casting direction on the amount of accumulated damage. The predicted results are shown in these figures up to the ultimate load of the different specimens. As mentioned earlier, the finite element program Nonalcs cannot handle softening phenomena which often occur beyond the bend over point on the stress-strain diagram (see Figures 5.4(a) and 5.4(b)). It can be observed from Figure 5.5(a) that tensile loading produces more damage than compressive loading. Results in Figures 5.4 and 5.5(a) also show that

the concrete specimens exhibit stiffer response when loaded in compression parallel to the casting direction which is the opposite to the tensile loading case. This phenomenon is attributed to the formation of cracks, especially those due to bleeding (Mindess and Young 1986). The majority of bleeding cracks are usually formed perpendicular to the direction of casting which grow and produce more damage in the concrete specimens subjected to tensile loading.

Figure 5.5(b), on the other hand, shows the impact of the lateral stress on damage evolution. Filled symbols are used to differentiate the lateral damage, ω'_x from the axial damage, ω'_y . The experimental values are indicated by the circular, rectangular, and diamond symbols and were obtained from experimental stress-strain curves. Finally, the latter figure indicates clearly that increasing lateral compressive stresses reduces the amount of damage due to cracking, which is manifested as increase in the strength of concrete.

5.2.3 CONCRETE RESPONSE TO DIFFERENT LOADING PATHS

During the development of the proposed model in Chapter 3, a few questions were raised regarding the application of pure damage mechanics to the analysis of concrete behaviour under specific loading conditions or combinations. However, some problems arise under certain loading conditions. The first problem arises when pure continuum damage mechanics equations predict zero inelastic damage strains for a concrete element subjected to some overall compressive loading. This often occurs when the proportional biaxial compressive loading ratio, $s_r = \frac{\sigma_x}{\sigma_y}$, becomes greater than certain values (see

Equation [3.81] and Figure 3.8). Another problem stems from the way in which loads are applied on the structures, e.g. loading, unloading, and reloading scenarios. For example, during the course of loading, a certain loading path may activate mainly the plastic flow energy dissipation mechanism but upon unloading and reloading along another path, the mechanism associated with the evolution of cracks may dominate the structural response. Therefore, the distinction between the loading paths that produce plastic deformation from

those which produce damage due to crack evolution must be established and understood. Consequently, it is important to investigate the behaviour of specimens subjected to loading conditions and combinations.

Ju (1986) reported the experimental results from some of the University of Colorado concrete tests. Those tests will be simulated here to assess the proposed model's performance. Unfortunately, the original reference could not be obtained and therefore all the data of the specimens geometric details, material properties, and the experimental results were taken directly from Ju (1986). According to Ju (1986), this experimental program is both comprehensive and well documented. The program consisted of six major series of non-conventional, multiaxial stress tests, comprising 67 specimens tested under different loading combinations and paths. The loading combinations were triaxial extension (TE), triaxial compression (TC), conventional triaxial extension (CTE), conventional triaxial compression (CTC), reduced triaxial extension (RTE), reduced triaxial compression (RTC), and simple shear (SS). These load combinations were applied either in proportional or non-proportional loading schemes. The loads were applied along the hydrostatic, deviatoric, hydrostatic-deviatoric, stair-case type loading, and/or circular paths. It is important to mention that the stair-case type loading used in the experimental program in order to examine convergence to the proportional path.

In the following, specimens 4-7 and 5-1 of the University of Colorado experimental program will be tested numerically using the proposed elasto-plastic damage-based model. The labels (e.g. 4-7, etc.) are used to distinguish the various tests. The first digit (e.g. 4) indicates the major series number and the second number (e.g. 7) identifies the specimen number within a given test series. Specimens 4-7 and 5-1 were selected from series 4 and 5, respectively. Both groups were subjected to an overall compressive loading, with different stress ratios and paths. Group 4 consisted of 22 specimens loaded with axisymmetric triaxial loading in order to explore the load path effect on their global behaviour. The loads were applied along the hydrostatic-deviatoric paths as well as staircase-type path. The latter loading condition was used, as mentioned earlier, to

explore the convergence to the proportional path, where the loads are applied as hydrostatic stress increments, with or without hydrostatic preloading. The staircase-type loading was also compared to nonproportional loading. Group 5, on the other hand, consisted of 6 specimens loaded with unsymmetrical triaxial loading schemes. This group was used primarily to investigate the impact of loading and unloading on the global behaviour of concrete.

Figure 5.6: Different loading paths on concrete cylinders.

Figure 5.6 displays the loading conditions for both sets of specimens. All University of Colorado test specimens were cylinders of diameters 310 mm and height of 500 mm. The material properties employed in the current analysis are shown in Table 5.4. Once again, some of the properties were either calculated or assumed. Since the test resembles a material point subjected to biaxial stress field, a single finite element was used in the analysis (Figure 5.6).

Table 5.4: University of Colorado concrete tests material properties (Ju 1986).

Material properties	1		2	3	4		5			6	
	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Min}	Cement-paste		Interface	Initial de-bonded interfaces	Initial damage
	E_p (GPa)	E_n (GPa)	ν	μ	(mm)	(mm)	Mode-I K_I^C (MN/m ^{3/2})	Mode-II K_{II}^C (MN/m ^{3/2})	Mode-III K_{III}^C (MN/m ^{3/2})		
Value	34.50	34.50	0.22	0.6	19.00	2.75	0.562*	0.288*	0.377*	0.85	0.0050

! Assumed

Figure 5.7: University of Colorado concrete test results of specimen 5-1.

Figure 5.7 compares the stress-strain relationships predicted by program Nonalcs with the experimental results of the University of Colorado test specimen, 5-1. Both the analytical and experimental results are shown for the principal stress and strain in the y direction. It can be observed from this figure that this loading scenario beyond point A triggers the crack evolution mechanism and hence the inelasticity is due to damage strains. This is because the stress-strain curve predicted by employing only the damage mechanics portion of the proposed model coincides with the experimental data points.

When the same specimen is analyzed by the same program but employing pure plasticity, it is obvious that the analytical results deviate from the observed behaviour. It should be pointed out that the plasticity solution shows a stiffer response compared with

the damage solution. Of course, one may argue that the plasticity prediction is acceptable, but in fact its prediction is based on an energy dissipation mechanism which was not active in this test. The damage mechanics, on the other hand, attributes this nonlinearity to the evolution of the existing cracks which in fact dominated the mechanical behaviour of the concrete specimen. The unloading path, however, was not significantly influenced by the type of material model used (see branch B-0, Figure 5.7).

It is very important at this stage to investigate the behaviour of a concrete specimen subjected to loading that may induce nonlinearity due to plastic deformation and damage (crack evolution) mechanisms. Therefore, Specimen 4-7 will be analyzed to investigate that condition. This specimen was subjected to the loading path 4 in Figure 5.6. The analytical and experimental responses are compared in Figure 5.8.

Figure 5.8: University of Colorado concrete test results of specimen 4-7.

We observe in the latter figure three analytical curves corresponding to different methods of modelling. The curves were obtained by employing (i) pure damage mechanics, (ii) plasticity and (iii) combined damage and plasticity constitutive equations, respectively. It is obvious from the damage only curve that the response remained elastic during proportional loading. During the phase of unloading and reloading, employing the equations of damage mechanics only produced a very stiff response because the damage

mechanics equations did not consider the inelastic deformations produced due to plastic flow in the first stage of loading. Utilizing pure plasticity, on the other hand, gave excellent agreement with the experimental results for proportional loading. It failed, however, to pick up the exact nonlinearity during the second phase of loading because the theory of plasticity traces a different mechanism from the actual crack propagation which took place during this loading scheme. The proposed combined model produced more favorable results for both loading conditions because it traced the nonlinearity due to plastic deformation at the first stage of loading and stored the plastic strain as permanent inelastic deformations. When the existing cracks started to evolve and grow under the second phase of loading, the model was also able to capture their behaviour and calculate the proper inelastic response. It should be pointed out that most existing damage models would not be able to predict the behaviour of this specimen under the prescribed loading conditions.

Figure 5.9 compares the responses of the latter concrete specimen due to staircase-type and proportional loading. It is clear from the analytical curves of this figure that the staircase-type loading response does not converge to the proportional loading response because a different mechanism dominates each of these loading scenarios. When the staircase-type loading is applied (see Figure 5.6) the first load step will cause some cracks to open and slide and hence introduce inelastic crack deformation (damage strains). Program Nonalcs showed that the energy required by this mechanism is very low compared with that needed for the plastic flow. Therefore, the inelastic response due to damage should be less than the plastic strain at the same load level (see Figure 5.9). Subsequent load application in the orthogonal direction will start closing the cracks and hence the crack compliance will be reduced leading to a stiffer material. The stiffening of the material did not reach its virgin state because of the permanent nature of the deformations. The latter trend and the softer response due the staircase-type loading are clearly shown in Figure 5.9. It should be mentioned, however, that there were no experimental data for the staircase-type loading and therefore the latter discussion is based on the difference

between the analytical stress-strain curves. Of course, any conclusion on whether the staircase-type converges to the proportional loading is subject to experimental confirmation which is beyond the objective of this work.

Figure 5.9: Convergence of staircase-type to proportional loading.

Figure 5.10 shows the damage evolution in the two specimens due to the applied loads. It should be understood that in Figures 5.10 damage evolution means the amount of induced nonlinearity due to the propagation of cracks only. As mentioned earlier, one might argue that plastic deformation is also an indication of damage because it is permanent and reduces the effective elastic material properties. This is true but considering the basic definition of damage in the current treatise (i.e. reduction in the resisting area of concrete in different directions) the two are not the same. It is clear that when the applied stress ratio, $\frac{\sigma_x}{\sigma_y}$, increases, the amount of damage due to cracking decreases while plastic deformation may increase. The latter figure also indicates clearly that damage due to cracking becomes zero when the aforementioned biaxial stress ratio exceeds a certain critical value. However, the mechanical response and deformation remain nonlinear and they have been obtained from employing theory of plasticity. This means that the loss of material stiffness is due to flow and plastification. This flow is due to the applied loading combinations. Therefore, it can be concluded that the concrete

behaviour is load dependent and the damage corresponding to the latter loading conditions is due to plastification and flow in the material (see Figure 5.5).

Figure 5.10: Damage evolution corresponding the considered different loading paths.

5.2.4 NOTCHED PLAIN CONCRETE PANELS AND BEAMS

Recently, notched specimens subjected to either axial or bending stress fields have been tested by many researchers (e.g. Mazars and Lemaitre 1985; Mazars 1985; Bazant 1986; Jenq and Shah 1986; Hordijk 1992; etc.). The purpose of these tests was to measure the different critical fracture toughness (e.g. K_I , etc.) for concrete materials. Some of those test specimens will be analyzed here utilizing the proposed model. The analysis will included notched panel specimens and notched beams.

5.2.4.1 NOTCHED PLAIN CONCRETE

Mazars and Lemaitre (1985) tested notched plain concrete specimens subjected to point loads normal to the notch, Figure 5.11(a). A typical specimen geometry and loading is shown in Figure 5.11(a). This test is designed such that when the applied load, P , increases, it creates a damage zone at the tip of the notch, creating a macrocrack in the middle of the specimen. The current analysis utilized two different finite element meshes which are shown in Figures 5.11(b) and 5.11(c). Both meshes comprise an ensemble of the four node quadrilateral inplane element. It can be noticed from these figures that only half the panel is modeled by the finite element due to the symmetry of the problem about

should be established prior to any assessment of its results. Therefore, it is very important to see to what extent this phenomenon influences the predicted behaviour using the proposed model. The coarse mesh-I, Figure 5.11(b), has a total of 98 nodes and 78 elements and the fine mesh-II, Figure 5.11(c), has 302 nodal points and 288 quadrilateral finite elements, respectively. It may be interesting to mention that the computer times needed to complete the full nonlinear analysis in the two cases were approximately, 5 and 18 minutes for the coarse and fine mesh, respectively. The hardware used was a 16 bits Intel 486/66 DX2 IBM compatible machine with 8 megabyte of random access memory (RAM).

The material parameters provided by the original authors and used in the current analysis are given in Table 5.5. Once again, any material parameter not reported in the test data was assumed.

Table 5.5 Material properties of notched concrete specimen (Mazars and Lemaitre 1985).

#	1		2	3	4		5			6	
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Min}	Cement-paste		Interface	Initial de-bonded interfaces ρ	Initial damage ω_0
	E_o^p (GPa)	E_o^n (GPa)	ν	μ	(mm)	(mm)	Mode-I K_I^c (MN/m ^{3/2})	Mode-I K_I^p (MN/m ^{3/2})	Mode-II K_{II}^p (MN/m ^{3/2})		
Value	27.00	23.58	0.20	0.6	16.00	3.75	0.562 [*]	0.251 [*]	0.377 [*]	0.85 [†]	0.1000 [†]

The results of the analysis are displayed in Figures 5.12 (a) and 5.12(b) along with the corresponding experimental values. Figure 5.12(a) shows the global load-notch opening displacement curve. It is clear from the latter figure that the analytical and experimental results are in full agreement from zero load up to the bend over point (see Figure 5.12(a)). The analytical results of Figure 5.12(a) reveals two distinct features. First, the mesh size has little impact on the global deformation behaviour in the ascending branch of the load deformation. Second, the degradation of the overall stiffness of the specimen in the ascending portion of its load-deformation curve is only the result of the evolution and propagation of the existing microcracks. These observations conform to the findings of other numerical studies (e.g. Mazars and Lemaitre 1985; Mazars 1985; Bazant 1985;

Bazant *et al.* 1990; etc.) where the finite element size seems to impact only the behaviour in the softening zone.

Figure 5.12: Results of notched concrete panel, (a) global load-deflection curve (b) damage propagation.

Thus, for example, Mazars and Lemaitre (1985) and Mazars (1985) postulated a hypothesis indicating that this type of behaviour is controlled by the progress of damage (i.e. crack evolution) up to the maximum load. They proposed, however, the use of linear elastic fracture mechanics to obtain the mechanical deformation behaviour of brittle materials within the descending branch (see Figure 5.12(a)). Linear elastic mechanics can evaluate the response of the macrocrack produced by the accumulation of damage due to loading up to the maximum load, P_{max} . Of course, mesh refinement ahead of the

macrocrack would be required in the latter analysis. It should be understood that fully damaged elements[†] ahead of the notch represent the formation of a macrocrack within the specimen.

Figure 5.12(b) shows the distribution of damage within the specimen due to increased loads. This figure depicts the damage contours obtained from the finite element computations and the path of the experimentally observed macrocrack. It can be noticed that the analytical and experimental results are in good agreement which indicates that the proposed model is capable of tracing the progress of the inelasticity up to ultimate loads. The softening (descending) branch may be obtained if one employs the principle of linear elastic fracture mechanics as indicated earlier.

There are many other notched panel test results available in the literature which can be analyzed using the proposed model in order to further assess its performance. However, it is very important to determine the ability of the proposed model in simulating the mechanical behaviour of concrete structures and elements under different stress fields. Therefore, in the following, four different notched beam specimens (see Figure 5.13) will be analyzed employing the procedures described in Chapters 3 and 4.

5.2.4.2 NOTCHED PLAIN CONCRETE BEAMS

At Northwestern University, Shah and his research team as well as Bazant and his associates tested many notched beam specimens (Figure 5.13) for different purposes. Similarly, researchers at Delft University investigated the mechanical behaviour of notched concrete beams under static, cyclic, and dynamic loads as part of an ongoing experimental program.

The main agenda of the latter work was to calibrate and to assess their finite element computer program DIANA. The former teams, on the other hand, used their experiments to evaluate the fracture properties and toughness of concrete material so they could be used in some of their proposed analytical models (e.g. Jenq and Shah 1986; Bazant *et al.*

[†] Damaged elements are those elements where the damage value, ω , at their Gauss points is more than the maximum value, $\omega_{\max} = 0.8$

1990). These tests were varied from three to four point bending tests with different arrangements of the notches, supports and applied loads. Figure 5.13 shows four different test setups and the geometrical proportion of the various notched plain concrete beams.

Figure 5.13: Different notched plain concrete beams dimension and loading.

5.2.4.2.1 THREE POINT NOTCHED PLAIN CONCRETE BEAM #1

The first beam, Beam # 1 (Figure 5.13(a)), represents three different plain concrete beams chosen from the experimental test program conducted by Jenq and Shah (1985) at

Northwestern University. This experimental program involved testing 40 beams with initial notches of different size which were designed to investigate the impact of size effect on the measured fracture toughness, K_{IC} , and ultimate load carrying capacity, P_u , of concrete, mortar, and cement paste materials subjected to bending stresses. It can be noticed in Figure 5.13(a) that these beams were tested under three point bending. All beams were manufactured using the same concrete mix in three different forms (sizes) (consult the table in Figure 5.13(a)); large, medium and small. The beams span, S , thickness, t , and the initial notch size, a , were expressed as a fraction of the selected cross section height, h , according to the following ratios, $\frac{S}{h} = 4$, $\frac{t}{h} = 0.375$, and $\frac{a}{h} = \frac{1}{3}$. The size of the initial notch of the small beams was reduced from the latter value to a value of $\frac{a}{h} = \frac{0.88}{3}$ while the other fractions were kept the same.

Table 5.6 Material properties of Jenq and Shah (1986) notched beams.

#	1	2	3	4	5	6						
Material properties	Elastic modulus		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters		
	Parallel	Normal			D_M	D_{Min}	Cement-paste		Interface		Initial de-bonded interfaces ρ	Initial damage ω_o
	E_o^p	E_o^n					Mode-I	Mode-I	Mode-II			
	(GPa)	(GPa)					K_I^C ($\text{MN/m}^{3/2}$)	K_I^C ($\text{MN/m}^{3/2}$)	K_{II}^C ($\text{MN/m}^{3/2}$)			
Value	33.60	33.60	0.22	0.6	19.00	5.00	0.347(L) 0.299(M) 0.278(S)	0.174(L) 0.150(M) 0.139(S)	0.347(L) 0.299(M) 0.278(S)	0.85	0.0050	

(L) large, (M) medium, and (S) small specimen respectively.

In the current analysis, only three concrete beams from this experimental program will be analyzed using the proposed constitutive model. These beams are labelled as C1L1, C1M2, and C1S3. The concrete mix was made of 1 part cement, 2.6 parts of fine and 2.6 parts of coarse aggregates as well as 0.65 part of water. A small amount of sand was added to reduce the probability of shrinkage cracking. The maximum aggregate size used was about 19 mm (0.75 in.). All concrete beams were tested after curing for approximately 90 days in an environment with 96% relative humidity and at 26.7°C. Two closed-loop servo-controlled testing machines, which maintained a constant rate of increase of the notch opening (horizontal) displacements, were used to test the beams.

The large (C1L1) and medium (C1M2) concrete beams were tested in a 534 *kN* (120 *kips*) machine while the small (C1S3) beams were loaded by a 89 *kN* (20 *kips*) machine. The mechanical properties were obtained by testing 3 concrete cylinders (75×150 *mm*) made of the same mixes (Jenq and Shah 1985). The relevant properties reported by the original authors and required by the current model are listed in Table 5.6.

Figure 5.14: Finite element meshes for Jenq and Shah(1985) notched beams (a) coarse (b) fine meshes.

Once again, in analyzing this set of beams, we utilized two different finite element meshes for each beam (Figures 5.14). The Mesh #1 and Mesh #2, shown in the latter figure, represent the coarse and fine mesh, respectively. The coarse mesh contains a total of 229 nodes and 199 inplane quadrilateral elements while the fine mesh consists of 347 nodes and 310 similar elements. It can be noticed that only half of the beam was idealized because of the symmetry of the problem about the beam centerline. The CPU time used to solve for the full nonlinear response of each beam up to its bend over point (ultimate load) for the coarse and fine mesh were about 10 and 18 minutes, respectively. Each ultimate

point load, $\frac{P_u'}{2}$, was applied over 15 load steps and each load level was given 10 iterations to converge to the desired solution for all numerical tests.

The analytical load-vertical displacement and load-notch opening displacements responses are plotted for the three beams with their experimental counterparts (Figures 5.15(a) and 5.15(b)). The notch opening displacements are often referred to in the literature by the term *crack mouth opening displacement* (CMOD). It is obvious from these figures that the proposed model can predict exactly the experimental mechanical response of all three beams up to the maximum load. Unfortunately, the model in its present format cannot trace the softening branch of a material or a structure. The analytical load-deformation curves based on the proposed method lead to some interesting observations and present some useful conclusions. First, the nonlinearity of the ascending branch of each deformation response in the three concrete beams (Figures 5.15(a) and 5.15(b)) is the result of the evolution of the existing microcracks. This is because the analytical results of the nonlinear finite element program, Nonaics, showed that all the inelastic strains were the results of the damage strain.

Now if the hypothesis[‡] of Mazars (1985) and Mazars and Lemaitre (1985) is accepted, the current damage model will prove useful in estimating the major macrocrack size (i.e. width and length). Figure. 5.15(c) shows the zones that are completely or partially damaged and are contained within the damage contours of 0.6 or higher. These figures also display the experimental crack at complete failure of the different beams. It is clear that the experimental macrocrack is contained within the damage contour of 0.8, which represents the analytical macrocrack formed after applying the full ultimate load, P_u . If a pure fracture mechanics analysis is undertaken for the beam with an initial macrocrack of

[‡] This hypothesis states that the mechanical behaviour of any strain softening material or structure beyond the ultimate load point until complete separation of any two adjacent parts (i.e. strain softening branch) is the result of the propagation of the major macrocrack developed due earlier loading on the ascending branch.

identical results and therefore the results of the coarse mesh was presented only. This conclusion, together with the finding of the previous example, indicates that the element size and mesh refinement (i.e. mesh objectivity) has no impact on the behaviour of strain softening materials and structures from zero to ultimate load points (the ascending branch, see Figures 5.15(a), and 5.15(b)); its impact on the descending branch needs further study.

Table 5.7: Comparison of results of Jenq and Shah (1986) notched beams.

Beam Name	Beam Size (mm)	Maximum Load, P, (kN)		Load Ratio $\frac{P_d}{P_e}$
	S x h x t x a	Analytical, P_d	Experimental, P_e	
Large C1L1	914 x 229 x 85.7 x 76.2	6.4200	6.6483	0.9664
Medium C1M2	609 x 152 x 57.2 x 50.8	2.2083	2.2161	0.9965
Small C1S3	305 x 76 x 28.6 x 22.4	0.8476	0.7921	1.0700

It is very important to compare the analytical and experimental ultimate loads of the three beams which are shown in Table 5.7. It is clear from the results shown in this table that the proposed method of analysis is capable of predicting the ultimate load for this type of structure.

5.2.4.2.2 FOUR POINT NOTCHED PLAIN CONCRETE BEAM # 2

This example represents another beam tested by Hordijk (1992) subjected to cyclic four point bending (see Fig 5.13(b)). The reason for analyzing this beam is that first the stress field in the middle part of the beam is different than that in the latter example. Second, the beam is subjected to cyclic loading which causes opening and closing of the pre-existing microcracks. As in the previous example, this test represents three beams with dimensions of $500 \times 100 \times 50 \text{ mm}^3$. The only difference is that each beam has a different initial notch depth. The initial notch depth is 10, 30, and 50 mm, respectively. Since only the experimental results for the beam with 30 mm notch is available, the current analysis solves that beam and compares the numerical results and the experimental data. The required concrete material properties are listed in Table 5.8 which were obtained from the standard tests (see Table 5.1) performed by Hordijk (1992).

Table 5.8: Material properties of Hordijk (1992) notched beams.

#	1		2	3	4		5			6	
	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Mitr}	Cement-paste		Interface	Initial de-bonded interfaces: ρ	Initial damage: ω_0
	E_o^p (GPa)	E_o^n (GPa)	ν	μ	(mm)	(mm)	Mode-I K_I^c (MN/m ^{3/2})	Mode-I K_I^p (MN/m ^{3/2})	Mode-II K_{II}^p (MN/m ^{3/2})		
Value	38.00	38.00	0.20	0.6	16.00	3.75	0.218*	0.109*	0.218*	0.85	0.0050

The beam was solved using the finite element mesh shown in Figure 5.16(a). It can be noted that no special elements were used at the notch and only the proper boundary conditions were applied. The beam thickness was represented by a single concrete layer of thickness equal to 50 mm as shown in the latter figure. Despite the fact that the element size and further refinement of the finite element mesh did not impact the results of the previous examples, a coarse and a fine mesh will be used here. This is motivated by the nature of the applied loads on the structure which is different from all the other examples. In the analysis the loads were applied in 20 load steps and a maximum of 10 iterations were allowed per load step.

The analytical and experimental load-deflection behaviours are plotted in Figures 5.17. This figure shows the favorable agreement between the two sets for the first four load cycles. When the actual beam experienced unloading, the corresponding analytical results deviated from the observed behaviour as illustrated. However, the analytical cyclic envelope curve coincides with the experimental one. The deviation at later load cycles may be explained by the fact that the current finite element program, Nonaics, cannot handle situations when the elements on the diagonal of the structure global stiffness matrix become less than or equal to zero. Therefore, during the solution process, it was insured that unloading would start when any diagonal element of the global stiffness matrix approached zero. This technique, i.e. retaining small positive stiffness values for the damaged zones allowed the solution to proceed and to give the results of Figure 5.17(a).

Figure 5.16: Finite element mesh of Delft University notched concrete beams (Hordijk (1992)).

Another good agreement between the analytical and experimental results is shown for the horizontal deformation distributions at the midspan section for three different load levels (Figure 5.17(a)). These points are labelled A, B, C, and D, respectively. It should be mentioned that points A and B were selected from the ascending branch of the cyclic envelope curve while point C was chosen from its descending part.

Finally, the damage contours shown in Figure 5.17(c) indicate that the current proposed damage model is capable of predicting the location and size of a major crack. Of course, the crack width will equal to the maximum aggregate size D_{max} and its length can be obtained from the contour line that indicates full damage. The propagation of these major crack(s) can be obtained by employing linear elastic fracture mechanics because their exact size(s) and location(s) are computed by the proposed method.

Figure 5.17: Results of Delft University notched concrete beams (Hordijk (1992)).

The preceding results show the capabilities of the program and the developed constitutive model to simulate the cyclic behaviour. This indicates that the current version of the program, Nonalcs, can be used to trace the behaviour of a strain softening structure under static loading. This can be achieved by unloading the structure from a load level close to that which produces failure stress and then reloading it until it reaches a new level of damage and continuing the process until complete collapse occurs. This can be further explained in a separate study to devise a method by which one can establish the basis for unloading loads from the softening branch.

5.2.4.2.3 FOUR POINT NOTCHED PLAIN CONCRETE BEAM # 3

Another cyclic load example (see Figure 5.13(c)) will be tested using the proposed model. This example is obtained from the experimental work of Kasperkiewicz (1986) who tested several notched beams. The material properties are shown in Table 5.9.

Table 5.9: Material properties of Kasperkiewicz (1986) notched beams.

Material properties	1		2	3	4		5			6	
	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Min}	Cement-paste		Interface	Initial de-bonded interfaces: ρ	Initial damage ω
	E_o^p (GPa)	E_o^n (GPa)	ν	μ	(mm)	(mm)	Mode-I K_I^c ($\text{MN/m}^{3/2}$)	Mode-II K_{II}^c ($\text{MN/m}^{3/2}$)	Mode-III K_{III}^c ($\text{MN/m}^{3/2}$)		
Value	38.00	38.00	0.20	0.6	16.00	3.75	0.218*	0.109*	0.218*	0.85	0.0050

Figure 5.18: Results of Delft University notched concrete beams Kasperkiewicz (1986).

This problem will be solved numerically utilizing a single finite element mesh as shown in Figure 5.18(a). Crack tip elements were not used at the notch location to monitor possible stress concentrations because macrocrack propagation at the notch was not

observed in the test and was not therefore the cause of nonlinearity. It should be pointed out that in the test up to the sixth load cycle no visible crack appeared at the end of the notch. Thus the observed nonlinearity was considered to be due to the evolution of the microcracks under cyclic loading. The preceding beam was analyzed for the first six cycles of loading and the results are shown in Figure 5.18(b). Notice that the maximum stress during any loading cycle does not equal to the failure stress and the descending shape of the loading envelope does not necessarily represent strain-softening.

It can be noticed that good agreement between the experimental and analytical results exist, however, the analytical results diverged after the fourth cycle because the system was found to have a negative stiffness along the diagonal of the global stiffness matrix. This aspect relates to the previously mentioned problem in capturing the softening response with the current proposed model.

5.2.4.2.4 FOUR POINT SINGLE EDGE NOTCHED PLAIN CONCRETE BEAM # 4

The results up to this point show that the loading mechanism and the type of the machine often dictate the behaviour of the notched beam. This is reflected by the shape of the dominant crack; all previous tests ensured that the crack was subjected to mode I deformations. In reality, however, the crack propagation may be governed by mode II or mixed mode I/II. Of course, such modes of crack propagation in heterogeneous medium may be affected by other factors such as aggregate interlock and sequence and rate of application of external mechanical loads (RILEM 1985; Schalngén 1993).

Therefore, Schalngén (1993) designed an experimental program for single edge notched plain concrete beams based on the proposed criteria for the round robin experiments of RILEM committee 89-FMT. The geometry, loading arrangement and structural details are shown for a typical beam in Figure 5.13(d). This loading mechanism forces the notch to be subjected to mixed mode-I/II fracture and the formed crack will have a curved shape. Schalngén (1993) cast the beams in two groups of large and small sizes. The moulds were removed after two days and the specimens were cured under water. At the age of 14 days the beams were cut to the required size. For example, the

final dimensions shown in Figure 5.13(d) are obtained from the original specimen of $440 \times 160 \times 100 \text{ mm}$. These beams were removed from the water tanks after 28 days and let to dry until the testing date. The overall material properties of the concrete material were obtained from tensile and compressive tests on standard test specimens made of the same mixes. These material properties are listed in Table 5.10.

Table 5.10: Material properties of Schalngén (1993) single edge notched beams.

#	1	2	3	4	5	6					
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Min}	Cement-paste		Interface		Initial de-bonded interfaces
	E_o^p (GPa)	E_o^n (GPa)	Mode-I K_I^c ($\text{MN/m}^{3/2}$)	Mode-I K_I^i ($\text{MN/m}^{3/2}$)			Mode-II K_{II}^i ($\text{MN/m}^{3/2}$)				
	Value	35.00	35.00	0.15	0.6	8.00	0.25	0.495*	0.248*	0.495*	0.85

A single beam was selected from this program to be analyzed here using the proposed finite element method. The full beam was modeled, because of lack of symmetry of loading, using the finite element mesh shown in Figure 5.19(a). A total number of 246 nodes and 262 quadrilateral inplane elements, RQD4, were used in the analysis. Nonalcs finite element program used about 124 kilo bytes and consumed 20 minutes of CPU time on the 486/66 DX2 IBM compatible machine described earlier. The load deflection curve of this analysis is shown in Figure 5.19(b) which shows favourable agreement between the numerical solution and experimental response. Once again, the program could not trace the behaviour in the softening regime because of the limitation mentioned before. The results of the ascending branch, on the other hand, show noticeable nonlinear deformation. This is demonstrated in the exploded portion of the ascending branch in the top corner of Figure 5.19(b). This kind of inelastic response is clearly due to the evolution and propagation of the microcracks ahead of the notch. The finite element results showed that only elastic and damage strains were present. This observation is consistent with the findings of the other problem examples and the experimental behaviour for this type of concrete structure. Finally, the damage distribution within the beam at maximum loads is

shown in Figure 5.19(c). The damage contours of 0.8 or more indicate that a major crack is formed along their path.

Figure 5.19: Results of Delft University notched concrete beams Schlangen (1993).

Thus, it can be concluded that the current damage model is capable of predicting the shape of a major macro crack (see the experimental crack in Figure 5.19(c)). Further evolution of this macrocrack can be obtained by employing the principle of linear elastic fracture machines (Mazars 1985; Mazars and Lemaitre 1985).

So far the results of the analyzed plain concrete structures and test specimens prove that the proposed damage model is quite competent in predicting their actual mechanical behaviour. The mechanical behaviour of these structures from zero to maximum load is

essentially nonlinear and the current numerical results suggest that they are due to damage. It should be remembered that damage means microcrack propagation and evolution under applied stresses. However, the model could not simulate the softening behaviour of these structures. This is not a shortcoming of the model because the current study is not concerned with the softening phenomena which need more studies. Nevertheless, when a cyclic load program was used to apply the static loads, the model was able to trace the softening branch reasonably close to its experimental counterpart. Clearly, the latter procedure presupposes knowledge of the softening response a priori which is not the case in practice.

5.3 REINFORCED CONCRETE SPECIMENS AND STRUCTURES

It should be pointed out that plain concrete can be used alone in some applications to build real structures but its use in test specimens has many serious problems and disadvantages. The catastrophic nature of the specimen failure just after the formation of a major macrocrack is a typical problem. This type of failure often happens suddenly and does not permit us to record all the events leading to the complete separation (failure) of the specimen.

Therefore, as was shown earlier, investigators have used very special machines (e.g. Hordijk 1992) and different experimental techniques (e.g. Jenq and Shah 1985) to avoid some of these problems. Yet, there is no universally acceptable experimental method that can be used for these types of specimens. Therefore, researchers concerned with evaluating concrete behaviour often introduce reinforcement in order to keep the tested specimen responding to loading after the formation of one or more major crack(s). Reinforcement is also widely used for all practical applications in order to reduce cracks, ensure structural integrity, and to increase its load carrying capacity.

Mechanically speaking, the introduction of reinforcement may change the crack pattern within the concrete structure because they change the stress field which eventually influence the direction of propagating cracks. Thus, it is extremely useful to use the current model to analyze some reinforced concrete structures and to compare the

difference between their behaviour with that of plain concrete structures. Therefore, in this section, the analysis of a number of reinforced concrete elements subjected to biaxial tensile and compressive loading will be presented. In addition, some simply supported shallow and deep beams, as well as continuous reinforced concrete beams, will be analyzed and the obtained numerical behaviour will be compared to those shown in experiments. In all the examples reported hereafter, full bond between the concrete and the reinforcement is assumed throughout the loading from zero to failure. This assumption seems to be acceptable and often yielded acceptable results.

5.3.1 CORNELL UNIVERSITY PANELS

The first numerical test of reinforced concrete elements will involve the specimens tested at Cornell University (Perdikaris *et al.* 1980). In this test program, a total of sixteen reinforced concrete panels were experimentally tested to investigate their deformation under combined biaxial tension and shear stress conditions. The specimens were designed, manufactured, and loaded as shown in Figure 5.20. While the tensile loading was applied by pulling the steel reinforcement, the shear forces were applied at the corner in the manner shown in Figure 5.20(b). The loading sequence was as follows: application of the tensile load first, held constant throughout the test, followed by the incrementing of 0.5MPa shear stress until failure. It must be mentioned that the lateral tensile stresses, σ , were applied as a fraction of the yield stress of the reinforcements for the different panels. They were $0.3f_y$, $0.6f_y$, and $0.9f_y$ where f_y is the steel yield stress. Because the tension was not applied directly to the panels, it was difficult to determine the actual stress distribution within the panels. Consequently, they decided to use linear finite element in order to evaluate the exact stress distribution in the panels. Nonalcs finite element procedure can model this type of loading as an initial stress in the reinforcements. The calculation of the equivalent nodal forces was shown in Chapter 4 of this thesis. Typically, the failure occurred at one of the tensioned corners. In reporting their results, they used the average shear stresses across the width of the panel. The finite element mesh

for these panels will consist of the customary inplane quadrilateral element RQD4 as shown in Figure 5.21(a).

Figure 5.20: Cornell panels (a) geometry and reinforcements (b) loading scheme.

According to the Figure 5.21 only half or a quarter of the panels is modelled due to the symmetry of the loading about the axes shown in the same figure. Figure 5.21(b), on the other hand, shows the modelling in the z direction (i.e. through panel thickness) where two concrete material layers were used to model concrete, and each reinforcement layer in a given direction was replaced by an equivalent steel layer of thickness, t_s . The smeared steel layer thickness is obtained by dividing the area of one reinforcing bar by the bars spacing. Finally, since two finite element meshes were employed, one can establish the impact of the element size on the accuracy of the finite element results. Since the original reference did not report most of the material properties, some assumptions had to be made in the present analysis, and the concrete material properties were taken as shown in Table 5.11. It should be remembered that the fracture toughness parameters were calculated

using Equation [5.1] where it was assumed that $G_{ic}^c = 5.8 \text{ N/m}$ (Gopalaratnam and Shah 1985) and the elastic modulus as shown in Table 5.11. The elastic properties of steel reinforcement materials are also shown in Table 5.12.

Figure 5.21: Comell panels (a) coarse and fine finite element meshes I and II (b) layering system.

Table 5.11: Concrete material properties used for Cornell University panels.

Material properties	1		2	3	4		5			6	
	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Min}	Cement-paste		Interface	Initial de-bonded interfaces	Initial damage
	E_o^p (GPa)	E_o^n (GPa)	ν	μ	(mm)	(mm)	Mode-I K_I^c (MN/m ^{3/2})	Mode-II K_{II}^c (MN/m ^{3/2})	Mode-III K_{III}^c (MN/m ^{3/2})		
Value	21.00	21.00	0.20	0.6	19.00	4.75	0.350	0.175	0.350	0.85	0.0100

The shear stress-shear strain curves of the panels obtained from the analysis, based on the proposed model, and from the tests are compared in Figure 5.22. The predictions of the compression field theory (Vecchio and Collins 1982) are also plotted. It should be

pointed out that a modified version of the latter theory was also implemented in program, Nonalcs.

Table 5.12: Steel material properties used for Cornell University panels.

#	ordinary reinforcing bars						prestress strands and/or bars					
	1		2	3	4		1		2	3	4	
	Elastic moduli		Poisson's ratio	yield stress	steel ratios		Elastic moduli		Poisson's ratio	strands strength	prestress ratios	
	x-axis	y-axis			x-axis	y-axis	x-axis	y-axis			x-axis	y-axis
	E_{sx} (GPa)	E_{sy} (GPa)	ν_s	f_y (MPa)	ρ_{sx} (%)	ρ_{sy} (%)	E_x (GPa)	E_y (GPa)	ν_s	f_{ps} (MPa)	ρ_x	ρ_y
Value	193.0	193.0	0.20	300.0	0.224	0.112	N/A	N/A	N/A	N/A	N/A	N/A

As mentioned earlier, the Nonlacs modified version of the latter theory assumes that the direction of the incremental inelastic principal strain, $d\epsilon_i$, coincides with the direction of the total stress, σ_i . The results of this model are omitted since the purpose is to demonstrate the difference between the proposed model and the original form of the modified compression field theory. The results clearly show that a good model that is based on sound theoretical foundation and which includes all the important parameters can be used to predict the response of such elements. It is obvious from the latter figure that the proposed elasto-plastic damage-based model traces the nonlinear behaviour of nearly all the panels accurately.

Unlike the compression field theory, the proposed finite element damage model predictions practically coincide with the experimental curve. The mismatch at certain points on the shear stress-shear strain curves may be due to reinforcement slip which was not considered in the current modelling. In the following, we will show the same good agreement for a completely different set of experimental results obtained from other panels subjected to biaxial loading as well as shear stresses. In addition, the cyclic behaviour of the model will be also tested.

Figure 5.22: Cornell University panels results.

In Figure 5.23, the predicted damage (crack) patterns at failure are plotted for the four panels. It should be mentioned that in the zones where damage reached its maximum value, $\omega_f = 0.85$, a major crack was assumed to have formed. Unfortunately, the experimental crack patterns were not available which does not allow verification of the analytical results. However, the analytical damage distribution showed that the failure damage (crack) angle is indeed a function of both the reinforcement ratio and the applied lateral stresses, σ_x (see Figure 5.20(a)). Because of lack of verification of the analytical crack pattern, the discussion will not be pursued here further.

Figure 5.23: Analytical damage (crack) patterns of Cornell University panels.

5.3.2 UNIVERSITY OF TORONTO PANELS

Due to the fact that the test procedure employed above did not use a well defined stress state for the panels, Vecchio and Collins (1986); designed a special panel and shell element tester to eliminate the problem. Furthermore, the loading conditions of Cornell University tests were not simple. Thus, the design of University of Toronto tester was intended to eliminate the undefined stress condition and to apply simple and controlled loads. The machines were used to test concrete elements subjected to in-plane and out of plane loads. In the following section, attention will be focused on the behaviour of concrete elements subjected to in-plane load since the two-dimensional damage model described in Chapter 3 can be easily used to analyze such elements.

Figure 5.24(a) displays the layout of the machine used for testing panels subjected to combined biaxial and shear stresses as well as the method by which the load was applied. It can be seen that the specimen is placed in the machine after casting it. A special assembly, Figure 5.24(b), is used to cast the specimen and also to attach the shear keys

needed for load application (see the jack attachment in Figure 5.24(a)). The specimens orientation during the tests and their dimensions are illustrated in Figure 5.24(c).

Figure 5.24: University of Toronto tester and typical panel specimen casting assembly.

To apply the loads to the specimen through the shear keys, in the manner shown in the last figure, hydraulic pressure was applied through the jacks shown in Figure 5.25. In the latter figure, the rigid links are provided in order to keep the panels in a fixed frame i.e., to prevent any rigid body motion. The panels were 890 mm × 890 mm by 70 mm thick. Two layers of welded wire mesh were provided to reinforce them orthogonally.

Typically, the wires were labeled as “*longitudinal*” and “*transverse*”, where the steel ratio in the latter direction was always less than or equal the former. To limit failure to

the middle part of the specimens, where strain readings were taken, the concrete used for casting the specimen were made of two separate mixes. A relatively strong concrete was cast in a 100 mm band around the perimeter while a weaker one cast in the central area of the panels. The concrete mix used for the two regions were composed of the same materials, but with different proportions which are listed in Table 5.13.

Figure 5.25: Load application to the tested specimen.

Table 5.13: Mix components for University of Toronto panels.

Material	100 mm Outside zone	The central area
Sand	40 kg (fine)	90 kg (fine)
Gravel	32 kg (6 rounded)	70 kg (6 rounded)
Cement	16 kg (High early strength)	28 kg (normal)
Water	8 kg	16 kg
Water/Cement ratio w/c	0.50	0.57

In the present analyses and for the time being, we will use the material parameters of the weak concrete since the material parameters of the strong concrete were not available. The different material parameters obtained by direct testing as reported by the original authors are listed in Table 5.14. All the panels were analyzed using the proposed model.

Due to space limitation, we will show detailed results of some sample panels which are highlighted in Table 5.14.

Table 5.14: Material parameters for University of Toronto panels.

Specimen	Loading	Longitudinal Steel		Transverse Steel		Concrete	
		f_y	f_{yt} (MPa)	f_y	f_{yt} (MPa)	ρ_c	f'_c (MPa)
PV 1*	PS	0.01785	483	0.01680	483	0.00220	34.5
PV 2	PS	0.00183	428	0.00183	428	0.00225	23.5
PV 3	PS	0.00483	662	0.00483	662	0.00230	26.6
PV 4*	PS	0.01056	242	0.01056	242	0.00250	26.6
PV 5	PS	0.00742	621	0.00742	621	0.00250	28.3
PV 6	PS	0.01785	266	0.01785	266	0.0025	29.8
PV 7*	PS	0.01785	453	0.01785	453	0.00250	31.0
PV 8*	PS	0.02616	462	0.02616	462	0.00250	29.8
PV 9	PS	0.01785	455	0.01785	455	0.00280	11.6
PV10	PS	0.01785	276	0.00980	276	0.00270	14.5
PV11	PS	0.01785	235	0.01306	235	0.00260	15.6
PV12	PS	0.01785	460	0.00446	460	0.00250	16.0
PV13	PS	0.01785	248	---	---	0.00270	18.2
PV14*	PS	0.01785	455	0.01785	455	0.00223	20.4
PV15	UC	0.00740	255	0.00740	255	0.00200	21.7
PV16	PS	0.00740	255	0.00740	255	0.00200	21.7
PV17	UC	0.00740	255	0.00740	255	0.00200	18.6
PV18	PS	0.01785	431	0.00315	412	0.00220	18.5
PV19	PS	0.01785	358	0.00713	299	0.00215	19.0
PV20	PS	0.01785	460	0.00685	297	0.00180	18.6
PV21	PS	0.01785	458	0.01295	302	0.00180	18.5
PV22	PS	0.01785	458	0.01524	420	0.00200	19.6
PV23	SBC	0.01785	518	0.01785	518	0.00200	20.5
PV24	SBC	0.01785	482	0.01785	482	0.00180	23.8
PV25	SBC	0.01785	466	0.01785	466	0.00180	19.2
PV26	PS†	0.01785	456	0.01009	463	0.00185	21.3
PV27	PS	0.01785	442	0.01265	442	0.00190	20.5
PV28	SBT	0.01785	483	0.01785	483	0.00185	19.0
PV29	CLR	0.01785	441	0.00385	424	0.00180	21.7
PV30	RCS†	0.01785	437	0.01009	472	0.00180	18.1

Note:-

* Premature failure

† Pre-cracked in biaxial tension

Designation

A PV27

B PV25

C PV19

D PV29

E PV13

Key

PS: Pure Shear

UC: Uniaxial Compression

SBT: Combined Shear and Biaxial Tension

SBC: Combined Shear and Biaxial Compression

CLR: Changing Load Ratio

RCS: Reversed Cyclic Shear

Those panels were selected because of their interesting results and behaviour. Let us for the sake of convenience denote the panels as follows: A for panel PV27, B for panel PV25, C for panel PV19, D for panel PV29, and E for panel PV13. For the rest of the panels, only their shear stress-shear strain curves will be shown.

In order to avoid complicated finite element meshes and in order to apply the loading and the boundary conditions properly, we modeled the whole panel as shown in Figure 5.26. The employed finite element mesh in the latter figure consists of 6×6 quadrilateral finite elements, RQD4, with a total number of 49 nodes.

Figure 5.26: Finite element mesh of University of Toronto panels.

Table 5.15: University of Toronto panels concrete material properties.

#	1	2	3	4	5	6					
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D	D'	Cement-paste		Interface		Initial de-bonded interfaces: ρ
	E_o^p	E_o^n	Mode-I K_I^C	Mode-I K_I^C			Mode-II K_{II}^C				
	Value	(GPa)	(GPa)	ν	μ	(mm)	(mm)	(MN/m ^{3/2})	(MN/m ^{3/2})	(MN/m ^{3/2})	
	20.00	20.00	0.20	0.6	6.00	0.50	0.560*	0.260*	0.560*	0.90	0.0050

Since there was no available information for the fracture toughness of the concrete used, we used the compressive and tensile strengths given by the original authors together with the relevant equations from Chapter 3 to calculate the required parameters. It should be mentioned that we have used the mean values of both strengths for all panels. The mean value of the compressive strength of the 30 panels was 21.69 MPa, which yields for a maximum aggregate, $D_M = 6 \text{ mm}$, mode-II stress intensity factor, $K_{IIc}^c = 0.560 \text{ MN/m}^{3/2}$. The average tensile strength for 22 tested panels was 2.674 MPa and the corresponding stress intensity factor for Mode-I, $K_{Ic}^c = 0.260 \text{ MN/m}^{3/2}$. It is interesting to note that the fracture toughness for both modes of fracture are related by a factor of 2, which is close to the assumption stated in Chapter 3. In Table 5.15, the concrete material fracture and elastic

parameters used in the analyses are listed. The reinforcement steel had an elastic modulus of, $E_s = 200 \text{ GPa}$, and yield stress, σ_y , as shown in Table 5.14.

5.3.2.1 PANEL A

Panel A was reinforced in both directions with the same reinforcement ratio and with steel of the same strength. Therefore, this panel is considered to be isotropic. Furthermore, the applied loading was monotonic pure shear, which before steel yielding, will impose on the cracks a biaxial tension/compression stress field. The numerical results showed that the preferential cracks before steel yielding have an inclination of $\theta_c = \frac{\pi}{4} = 45^\circ$ with respect to the steel reinforcement. The numerical results indicated the onset of cracking at an applied shear stress, $\tau = 1.741 \text{ MPa}$. Subsequently, the steel reinforcement started to carry more stress and the biaxial load ratio for the critical crack then changed, resulting in critical crack orientation of $\theta_c = 0.2180\pi = 39^\circ 24'$ with respect to the x -axis of the specimen. The experimental crack orientation was $\theta = 0.2176\pi = 39^\circ 18'$, which is in good agreement with the analytical value. Figure 5.27(a) shows the variation of the orientation of the critical cracks, θ , against the applied shear stress.

Numerical analyses showed that when concrete cracked, steel stresses increased which subsequently affected the stress distribution on the concrete material as well as the biaxial ratio, $s_r = \frac{\sigma'_t}{\sigma'_v}$ for the cracked concrete. This explains the change in the orientation angle shown in Figure 5.27(a). The model showed that the concrete ability to carry load was completely exhausted when the longitudinal steel reinforcement yielded. The same results indicated that concrete was completely damaged at the corners and the middle of the specimen. Once again, it should be mentioned that the damage measure is an indication of the initiation and propagation of macrocracks within the concrete specimen. It is interesting to mention that the same failure mode was also observed experimentally. Both sets of results show the impact of reinforcements on the redistribution of principal stresses after cracking. Finally, we show the overall shear stress-strain relation for this panel in Figure 5.27(b), with good agreement between the analytical and experimental results.

These results show the impact of reinforcements on the ultimate load behaviour where the panels showed ductile response prior to complete collapse.

Figure 5.27: Results of Panel A.

5.3.2.2 PANEL B

Once again we choose an isotropic panel (in terms of concrete and reinforcement arrangement) which was loaded in combined shear and biaxial compressive stresses. The applied compressive stresses were taken as a fraction of the accompanying shear stresses, $\sigma_x = \sigma_y = -0.69\tau_{xy}$. Therefore, unlike the preceding case, where all cracks were in the open crack domain, all cracks for this loading case will lie in the closed crack domain because according to Equation [3.80], the limit angle for closed sliding cracks will be about, $\theta_c = 85^\circ$, which in turn indicates lack of open cracks. At the onset of sliding cracks, according to Equation [3.81] their orientation will be $\theta_c = \frac{25}{39}\pi = 36^\circ 43'$ which is very close to the experimentally measured value of $\theta = 36^\circ 42'$. This obviously shows that the model can be used to explain behavioural phenomena of plain and reinforced concrete structures. The crack orientation angle variation is displayed in Figure 5.28(a).

It can be seen that the angle changes rapidly after the cracks begin to propagate but remain in the vicinity of $\frac{\pi}{4}$ until failure, which is similar to the previous case. However, as

expected, Figure 5.28(b) shows that the failure shear stress is much larger than in the previous case. The ultimate load in this case was overestimated by the model by about 5% which may be attributed to ignoring the microcrack interaction with the steel reinforcement and their sliding and slippage during test. It is interesting to mention that in a competition held by the original authors most of the models failed to predict the results for this problem.

Figure 5.28: Results of Panel B.

5.3.2.3 PANEL C

This example deals with a panel that was reinforced with much stronger longitudinal reinforcing steel than the transverse one, i.e. the latter reinforcement ratio was half of the value of the former. Thus this panel is expected to demonstrate anisotropic behaviour due to the asymmetry of its reinforcement which is similar to the Cornell University panels analyzed earlier. The panel was subjected to monotonically increasing pure shear. At the beginning of the loading, the cracks were inclined at angle equal to the one obtained from panel A. As the load advanced, the difference in the steel reinforcements ratios influenced the orientation of the critical crack angles, as shown in Figure 5.29(a). The shear stress-strain relationship is shown in Figure 5.29(b). Once again, there is good agreement between the experimental and the analytical results. The model, as illustrated in the latter

figure, shows a soft response at the onset of cracking, and at yielding of the transverse steel reinforcement. The difference, however, is negligible. On the other hand, the ultimate shear stress obtained with the current model was practically equal to its experimental value.

Figure 5.29: Results of Panel C.

It is clear that the proposed concrete damage model is capable of recognizing the salient aspects of the deformation process in reinforced concrete elements subjected to plane stress conditions. To further examine the performance of the model, we present two more examples. The following problem will deal with an anisotropically reinforced panel, but the applied load was changed from pure shear to shear plus biaxial

5.3.2.4 PANEL D

The panel under consideration was reinforced anisotropically and loaded in pure shear up to 65% of its ultimate strength, and subsequently further increment in shear stress was accompanied by biaxial compressive stresses of the same magnitude. The crack orientation due to pure shear is almost the same as the isotropic case since the difference in the reinforcement ratio was not as large as in the previous example. The experiment showed that for shear stress, $\tau = 2.057 \text{ MPa}$, the cracks were oriented at an angle,

$\theta = 48^{\circ}02'$. The model as shown in Figure 5.30(a) predicts for the same stress level, the same angle to be $\theta = 42^{\circ}08'$.

Figure 5.30: Results of Panel D.

After applying the biaxial compression load, we notice that the angle decreased substantially according to the proposed model and the experiment. This can be explained by the fact that the preferential critical cracks will belong to the closed crack domain. The angle was influenced by the anisotropy of the reinforcement. The shear stress-shear strain relationship, Figure 5.30(b), for the panel shows the stiffening effect after the application of biaxial compression. It indicates that cracks in the open domain closed completely and only the sliding cracks affected the response of the specimen.

It should be mentioned that for this panel, at the ultimate load the solution stopped due to large deformation detected by the model. The shear stress-strain relationship is shown in Figure 5.30(b).

5.3.2.5 PANEL E

Finally, we solve an example in which the specimen is reinforced in one direction only. The proposed model showed excellent agreement with the corresponding experimental

data for the crack orientation angle for different load levels, Figure 5.31(a). The analytical results showed less ductility compared to the experimentally observed behaviour, Figure 5.31(a). The shear stress-shear strain relationship is shown in Figure 5.31(b). The analytical failure mode was due to concrete spalling which was also observed experimentally.

Figure 5.31: Results of Panel E.

Figures. 5.32 and 5.33 show the shear stress-shear strains for all the other panels. The experimental and analytical results show favorable agreement. This means that the current model is capable of predicting the type of behaviour and can capture the overall mechanical response of reinforced concrete materials. The analysis indicated that all panels had almost the same damage contour shape at the onset of cracking (see Figure 5.34). It should be remembered that the cracking loads were different for the panels depending on their reinforcement ratio.

Figure 5.32: Results of University of Toronto panels PV01 to PV12.

Figure 5.33: Results of University of Toronto panels PV13 to PV29.

According to the analysis, the slight difference in the damage distribution was due to the presence of reinforcement. This difference can be obtained for any panel by rotating

the damage contours by an angle, α , with respect to the vertical or Y -axis (see Figure 5.34). This angle never exceeded 5° which is negligible for all practical purposes.

Figure 5.34: Typical damage contours and experimental crack patterns of panels at cracking loads.

On the other hand, the amount and yield strength of reinforcements greatly affected the damage distribution at failure. This can be clearly observed from the different failure planes obtained in the experiment and in the analysis. Figure 5.35 shows the damage contours at failure for a typical panel and the special panels A, B, C, D, and E, respectively. According to the latter figure, the analysis and experiment showed that if the panel is reinforced isotropically, the panel will fail along a plane which passes through its diagonal. Isotropic reinforcement means that the reinforcements in any two orthogonal directions must have the same ratio, ρ and yield stress, σ_y . Contrary to the latter observation, panels A and C failed at surfaces that suggest that both panels were reinforced anisotropically. This may be due to many reasons which may have occurred during the test itself. The analytical results showed that both panels must separate along a diagonal plane (Figure 5.35). Panels C, D, and E failure surfaces show the impact of the reinforcement distribution. The panels failed in the weak direction where no reinforcement or very small amount of reinforcement had been used.

Finally, it should be noted that all the Cornell and Toronto panels contain cracks that are subjected to primarily biaxial tension/compression plane stress conditions. The ability of the model to replicate the response of these model is obvious. It is equally important to investigate the behaviour of other two-dimensional concrete structures subjected to different load types. In the following, a series of reinforced concrete beam bending problems will be further tested by the proposed analytical model. All reinforced concrete beams analyzed hereafter were tested by others.

Figure 5.35: Damage contours and experimental crack patterns of panels at failure loads.

5.3.3 TWO REINFORCED CONCRETE DEEP BEAMS

Leonhardt and Walther (1966) tested nine-single span deep beams. Among them we have chosen the beam WT3 because this particular beam failed due to concrete crushing near the supports as a result of excessive compressive stresses. Figure 5.36 shows the beam position in the machine and the applied loading. The latter figure indicates that the beam was loaded with increasing uniform load in increments of 100 kN. The capacity of the apparatus (Figure 5.36) used was 5 MN.

Figure 5.37: Deep beam WT3 (a) structural details (b) finite element mesh used.

Table 5.16: Material properties for deep beam WT3.

#	1		2	3	4		5			6	
	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D_M	D_{Min}	Cement-paste		Interface	Initial de-bonded interfaces	Initial damage
	E_o^p	E_o^n	ν	μ			Mode-I	Mode-II			
Value	(GPa)	(GPa)			(mm)	(mm)	K_I^c	K_{II}^c	K_{III}^c	ρ	ω_o
	24.00	24.00	0.20	0.6	19.00	5.00	0.450°	0.225°	0.560°	0.90	0.0050

The load-midspan deflection is shown in Figure 5.38, which shows reasonable agreement between the two sets of results albeit ultimate load is overestimated by the present analysis. While the proposed model is not unique in predicting the response of

thin beams accurately, it is important to note the consistency of the model in producing accurate results for different types of structures.

Figure 5.38: Load mid-span deflection for beam WT3.

Figure 5.39: Steel stress variation along the beam span.

The steel strains for two load levels, 1000 kN and 1100 kN are shown in Figure 5.39. Inspection of the latter figure indicates that none of the steel yielded at ultimate load. However, the program was stopped due to large deformation which indicated failure of

the structure. This clearly indicates that the structure failed by crushing of concrete rather than yielding of the reinforcement. Unfortunately, the original authors did not report stress distribution in the concrete which would have been of interest.

5.3.3.1 TWO SPAN CONTINUOUS DEEP BEAM

Within the same experimental program Leonhardt and Walther (1966) tested two other deep beams on three supports. One of them was designated as DWT2 whose failure was dominated by the evolution of shear cracks. It is believed that many existing concrete models may fail to capture the nonlinear behaviour of this beam. Therefore, it was considered an interesting and challenging problem from the writer's point of view. This is because the current model traces and recognizes the evolution of microcracks under tensile mode-I, shear sliding mode-II as well as the mixed mode mode-I/II. Furthermore, the stress distribution due to applied loads will be different from those of thin beams. The beam geometry and structural details as well as loading are depicted by Figure 5.40.

Figure 5.40: Details of deep beam DWT2 (Leonhardt and Walther 1966).

This figure shows that the beam is subjected to four point loads in order to produce constant shear force diagram (Leonhardt and Walther 1966). The machine used in order to apply the desired load configuration are displayed in Figure 5.41. The machine load was distributed to the continuous concrete beam through the set of rigid steel I-beams. The elastic constants and fracture parameters of the concrete material used in manufacturing these beams are listed in Table 5.17.

Figure 5.41: Experimental set-up deep beam DWT2 (Leonhardt and Walther 1966).

Table 5.17: Material properties of deep beam DWT2.

Material properties	1		2	3	4		5			6	
	Elastic moduli		Poisson's ratio ν	Coefficient of dry friction μ	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel E_o^p (GPa)	Normal E_o^n (GPa)			D_M (mm)	D_{Min} (mm)	Cement-paste			Initial de-bonded interfaces ρ	Initial damage ω_0
							Mode-I K_I^c ($\text{MN/m}^{3/2}$)	Mode-II K_{II}^c ($\text{MN/m}^{3/2}$)	Mode-III K_{III}^c ($\text{MN/m}^{3/2}$)		
Value	32.00	32.00	0.20	0.6	19.00	5.00	0.415*	0.208*	0.560*	0.90	0.0050

The quantities marked by an asterisk were estimated for the purpose of the present analysis on the basis of the given concrete properties. The reinforcement of these beams, on the other hand, had the same elastic constants as the previous beam with the exception of its yield stress. The yield stress had a value of 410 MPa. In the finite element discretization symmetry in the structures was exploited and thus only half of the beam was modeled. The load-deflection curve of the midspan is shown in Figure 5.42. It can be seen that during the evolution and progress of the microcracks, the model captured the nonlinear behaviour well. However, when the macro shear cracks developed, at a load

$P \cong 1.64 \text{ kN}$, the model showed a stiff response compared to its experimental counterpart. The experimental trend, on the other hand, indicates that some loss of bond may have taken place (slippage of the steel at the left and right supports) which are not accounted for in the present model. But the model shows its capabilities in predicting the same basic experimental response and ultimate strength. Since there were no further data provided by the original authors of that test, further comparison could not be made.

Figure 5.42: Load mid-span deflection for beam DWT2.

In summary, the analyses in this chapter, which included unreinforced notched beams and reinforced concrete panels and beams, have demonstrated the predictive capabilities of the proposed model. It was demonstrated that the damage evolution and ultimate strength of those structures could be predicted with a reasonable level of confidence.

6

6. FIBRE-REINFORCED CONCRETE

6.1 GENERAL

Fibres may be used in the concrete mix in order to improve its strength and deformation (Shah 1987; and Balaguru and Ezeldin 1987). It was established earlier that the opening, propagation, and evolution of microcracks cause the load-deformation response of concrete structures to become nonlinear at relatively low levels of load. It was also stated that the observed nonlinearity is a manifestation of damage. Damage occurs provided the brittle concrete composite material is not subjected to a biaxial or triaxial compression state of stress, with its minor to major applied stress ratio, $s_r = \frac{\sigma_x}{\sigma_y} \geq \Delta_c$. The nonlinearity and damage under the latter loading condition is the result of yielding and flow.

The addition of moderate to high percentages of fibres in fibre-reinforced concrete (FRC) seems to retard the crack propagation in concrete but does not completely eliminate it. The fibres can strengthen a given brittle matrix (e.g. concrete, ceramics, high performance composites etc.) through suppressing the extension of the cracks in the composite (Marshall *et al.* 1985; Mori and Mura 1984; Shoichi *et al.* 1989; Aveston and Kelly 1976; Rose 1987; Razaqpur and Nofal 1992). Also, the presence of fibres significantly improves the penetration, scabbing, and fragmentation resistance of concrete (Jenq and Shah 1985). These improvements enhance the tensile capacity of concrete and usually yield a more ductile cement-based composite (Balaguru and Shah 1992).

Due to these favorable attributes the use of fibre-reinforced concrete is gaining increasing acceptance in the structural engineering community. Among its many

applications, it is used in explosion and shock resistant structures (e.g. bomb shelters, military installations, etc.) and in cladding of high-rise buildings. Currently, FRC is primarily employed in the rehabilitation of important structures such as bridges, airport runways, parking garages, etc. Furthermore, its superior fire resistance indicates its promising potential for high-temperature applications (Prewo and Brennan 1980).

6.2 FIBRE MATERIALS AND GEOMETRY

Steel, carbon, glass, Kevlar, polypropylene, etc. comprise the fibre materials utilized in structural applications. Fibre materials can be generally categorized into *metallic*, *mineral*, *polymeric* or *organic* (naturally occurring) material. The metallic fibre materials (e.g. steel, alloys) often behave in a ductile manner and always have high elastic modulus and strength. Mineral fibres (e.g. glass) have moduli higher than concrete but lower than the metallic fibres. Elastic moduli of polymeric fibres, on the other hand, are lower than concrete but they are strong and ductile. Some polymeric fibres (e.g. carbon, Kevlar, etc.)

Figure 6.1: Various shapes of steel fibres used in fibre-reinforced concrete.

may have higher moduli. Finally, organic fibres are relatively strong but they have very poor bond characteristics. Some of these fibre materials can interact with chemical agent(s) which may lead to the degradation of the FRC components. This deficiency can

be eliminated by such techniques as coating the fibres with some latex materials to ensure full isolation from the attacking chemical agent(s).

The fibres used in concrete structures can be either short or long fibres. Short and long fibres are often introduced at the mix and casting times, respectively. Thus short fibres affect the workability of the concrete mix (Jamrozy and Olek 1987; Balaguru and Shah 1992) while long fibres may cause bleeding cracks (Young and Mindess 1989). The short fibres, see Figure 6.1, can be straight or deformed sheets, wires, machined chips and/or melt extract.

Long fibres, on the other hand, can be smooth or deformed bars with circular cross section. In practice long fibres are often referred to as reinforced bars when their diameter is greater than 5 mm. The choice of the diameter depends mainly on the type of application and the amount of loads. Sometimes these bars are prestressed in order to suppress cracks or eliminate them altogether (Balaguru and Shah 1992).

Short fibres are customarily distinguished from long fibres by the so called fibre length-diameter aspect ratio, ϕ ,

$$\phi = \frac{l_f}{2r_f} \quad [6.1]$$

where l_f is the fibre length and r_f is its radius. A typical short fibre may act as a long fibre if it is crossing a crack whose diameter, $2a$, is very small compared to the fibre length, l_f , Figure 6.2. In most current theories of composite mechanics, one assumes the presence of either short or long fibres, but sometimes in concrete structures both long and short fibres are used. It should be mentioned that regardless of the value of ϕ a typical crack will behave as if it is crossed by a long fibre if its deflected wing length, l_k (kinked crack), is less than the fibre length (i.e. $2l_k \leq l_f$), (Marshall and Evans 1985; Shoichi *et al.* 1989). Also, fibre end cracking might be initiated at the end of a long or short fibre, Figure 6.3.

Figure 6.2: Typical fibre-reinforced concrete element and its constituents.

Therefore, the aspect ratio assumption is restrictive and cannot be employed indiscriminately. A smart constitutive model may be able to decide on whether the fibre is performing as a long or a short fibre, depending on the stress field, associated crack evolution and fibre geometry. Such a model may also consider the effect of fractured fibres as well as the impact of the inactive fibres on the mechanical behaviour of fibre composite materials. The term inactive fibres refers to those fibres which exist in the concrete mix but do not cross or bridge any existing crack (Figure 6.2). In current practice, the so called *fibre-efficiency* parameter (Young and Mindess 1989) usually accounts for this factor. These issues are addressed later in this chapter during the development of the proposed model for FRC materials.

Figure 6.3: Cracks bridged by short, long, or fractured fibres.

To the best of the writer's knowledge, however, there is no comprehensive theory with the exception of the Razaqpur and Nofal (1992), model that can be used to predict the behaviour of brittle FRC composites that contain both long and short fibres. Some of the existing theories are developed based on linear fracture mechanics (Sih 1984) and others are obtained from data fitting of available experimental findings (e.g. Bazant 1994; Jenq and Shah 1986). Most models can be applied over a small range. Some of them may be valid for statically determinate structures only (Jenq and Shah 1986); in other words, the stress field is known *a priori*. Other models can be used only for predicting the ultimate strength of a composite or its initial elastic material properties (Hashin 1989).

It is worthwhile to mention that most existing complex models for fibre-reinforced composite materials deal with fibre-reinforced ceramic composites (FCC) and/or with either long or short fibres embedded in polymeric matrices. Only one or two methods

exist for the detailed analytical modelling of the mechanical behaviour of some simple FRC structures subjected to static loading, but the available solutions are applicable to limited loading and structural configurations. The following subsection presents the relevant aspects of the various models which deal with fibre-reinforced ceramic composite (FCC). Since the FRC constitutive relationship developed in the present investigation employs the damage mechanics theory of plain concrete, the Razaqpur and Nofal (1992) model is reviewed and presented. The latter model employs the concept of damage mechanics in conjunction with the rule of mixture to obtain the necessary constitutive relationship of an FRC composite. The model assumes full bond between the matrix and fibres and it makes rather simplified assumptions regarding the role of fibres in cement matrices. Nevertheless, its results appear to agree with available experimental data on small test specimens as well as larger fibre reinforced concrete beams.

6.3 FIBRE-REINFORCED CERAMIC COMPOSITES (FCC) MODELS

Investigation of the mechanical behaviour of ceramic matrix composites (Evans and Marshall 1989; Shoichi *et al.* 1989; etc.) has revealed that their deformational behaviour strongly depends on the mechanical properties of the fibre-matrix interface. Therefore, most of the analytical studies (e.g. Aveston and Kelly 1973; Baklund 1981; Nemat-Nasser 1983; Mori and Mura 1984; Marshall *et al.* 1985; Rose 1987; Evans and Marshall 1989; Charalambides and Evans 1989; Shoichi *et al.* 1989) have been concerned with modelling the interface effect on the composite constitutive relationships. In essence, all these studies present methods of evaluating the crack opening displacement bridged by either sliding or (de)bonding fibres. In calculating the crack deformations, different concepts have been employed. Among these approaches are standard fracture mechanics theory (Marshall *et al.* 1985), the shear lag analysis of Aveston and Kelly (1973), the spring model of Rose (1987); the Eshelby (1957) inclusion method of Mori and Mura (1984), the exact formulation of the elasticity boundary value problem of Muskhelishvili's (1953) (e.g. Nemat-Nasser 1983; Dollar and Steif 1988; Cui 1995), and finally the finite and/or

boundary element numerical solutions (e.g. Charalambides and Evans 1989; Backlund 1981; Murakami 1990).

At this stage of the discussion, it should be emphasized that the aforementioned methods deal with axially loaded composites subjected to tensile forces in the direction of the fibre. Only recently, Fleck and Budiansky (1991) and Budiansky and Fleck (1993) have attempted to evaluate fibre-ceramic composites (FCC) mechanical response subjected to axial compressive loading in the direction of the fibre.

6.4 EXISTING MODELS FOR FIBRE REINFORCED CONCRETE (FRC)

As far as FRC material is concerned, the number of available models are few and very primitive (Balaguru and Shah 1992). This is due to many reasons, among them:-

- [1] the use of short fibres in concrete structure is a relatively new practice (ACI SP-44 1974).
- [2] heavy reliance of the concrete industry on empiricism.
- [3] lack of available data for the physical material properties compared to the rich database compiled for normal plain concrete.
- [4] inadequate study of some basic mechanisms and lack of understanding of their impact at the macroscopic level on the mechanical behaviour of an FRC concrete structure.
- [5] and finally the difficulty associated with the mixing and workability of FRC concrete materials.

Therefore, the early work (e.g. Shah and Rangan 1971; Abdull-Ahad and Abbas 1989; and Theodorakopoulos and Swamy 1989) focused on understanding the mechanical behaviour of FRC through experimental studies. However, the civil engineering community was in dire need of a method of design and analysis of structures to be built using this new material. Of course, the trend was to enhance the formulae and procedures that were being used in the codes for conventional reinforced concrete and to apply them to fibre-reinforced concrete structures. For example, Abdull-Ahad and Abbas (1989);

Faza (1991); and Ghoniem (1992), based on the results of their experimental results, have modified the codes equations with some constant factors. These constants were formatted as functions of the fibre aspect ratio, ϕ , volume fraction, V_f , and concrete compressive strength, f'_c . Some factors in these expression were test dependent because they fitted only the particular test results. Thus the use of their equations in a general computational scheme will not yield satisfactory results, albeit, their use will not require further training for the practicing engineer.

This shortcoming has recently prompted a number of researchers in North America, Europe, and Japan to propose some theoretical models to simulate the mechanical behaviour of fibre composite concrete materials. However, most of the research work on the mechanics of FRC have utilized only linear and/or nonlinear fracture mechanics (ACI Committee 544 1982; Bazant 1990; Balaguru and Shah 1992). This concept was employed to solve the problem of a propagating macro crack in a notched specimen (Bazant 1990; Jenq and Shah 1986). Another phenomenological[†] model (e.g. Abdull-Ahad and Abbas 1989) was also proposed to trace the axial and/or multiaxial stress-strain curve of FRC concrete materials. For the sake of brevity, the fundamentals of the two methods will be discussed below as well as their usage in design and analysis of FRC structures.

6.4.1 LINEAR ELASTIC FRACTURE MECHANICS (LEFM) MODELS

Jenq and Shah (1985; 1986) have applied linear fracture mechanics to develop a plain concrete model. This model is a two parameter fracture model. The parameters are the critical crack opening displacement components, u_x and u_y , as well as the mode-I critical stress intensity factor, K_{IC} . Making use of the linear fracture mechanics solution of Tada *et al.* (1973), these quantities can be determined for an existing crack with original length, $2a_o$, according to

[†] The term, phenomenological, means that the model is based on curve fitting of experimental data and it should not be confused with the term phenomenological used previously in this thesis.

$$K_I = \frac{E\delta^* \sqrt{\pi a}}{4} F(\zeta) \quad [6.2]$$

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \delta^* V(\zeta) Z(\zeta, x) \begin{Bmatrix} 1 \\ \left(\frac{K_I}{K_{IC}^s} \right)^n \end{Bmatrix} \quad [6.3]$$

with $a = a_o + l_c$, $\zeta = \frac{a}{b}$, and b according to Figure 6.4 is either the axial panel half width or the notched beam depth. The value of the multiplier, δ^* , is a function of the test setup and crack dimensions while constant n and functions $F(\zeta)$, $V(\zeta)$, and $Z(\zeta, x)$, are functions of the crack dimensions at the end of a given load increment.

Figure 6.4: Typical notched FRC (a) panels and (b) beams specimens.

Later, Jenq and Shah (1986) extended this model to evaluate the behaviour of notched FRC beams and panels. To do so, they assumed that only the critical stress intensity factor may be obtained from FRC experimental data. The other material parameters such as Young's modulus, E , and Poisson's ratio, ν , of FRC concrete are the same as in plain

concrete. As well, the critical effective crack extensions, l_{ec} , can be obtained for plain concrete employing linear fracture mechanics. Their analysis of three and four point bending test of beams include the superposition, strain compatibility and equilibrium condition at the crack site. It should be mentioned, however, that this model does not explicitly show the impact of the fibre on the crack bridging. Thus it is essentially a phenomenological model.

It would be difficult to apply this model to general structures. The assumptions and the experimental data which constitute the formulation of the method, for example, the *a priori* presence of macrocrack(s), are rather restrictive and may not be attainable in general structures. Thus, the model cannot be employed indiscriminately to evaluate the mechanical behaviour of general structures that may not contain macro defects.

6.4.2 CONTINUUM FRC MODELS

To eliminate the need for knowledge of the initial crack dimension in the latter model, Abdull-Ahad and Abbas (1989) proposed empirical relations for the stress-strain behaviour of FRC materials,

$$\sigma = \begin{cases} \frac{\psi f'_c \left(\frac{\epsilon}{\epsilon_o} \right)}{(\psi - 1) \left(\frac{\epsilon}{\epsilon_o} \right)^\psi} & \bullet \text{ axial compressive loading} \\ f'_t e^{-\alpha \epsilon} & \bullet \text{ axial tensile loading} \end{cases} \quad [6.4]$$

where σ is the axial compressive or tensile stress corresponding to a strain value of ϵ , with f'_c and f'_t being, respectively, the compressive and tensile strength (peak stress) of the FRC material. These values may be obtained from the classical compressive and tensile testing on FRC cylinders or prisms where different load paths, e.g. axial and/or multi-axial loading can be used. The coefficient, α , depends on the proportion of materials used in the concrete mix and can be determined from experimental results and regression analysis. Strain, ϵ_o , on the other hand, corresponds to the peak stress and depends on the fibre volume fraction, V_f , its length, l_f , and its radius, r_f . It can be evaluated according to

$$\varepsilon_o = 0.002 + 0.5 \times 10^{-6} \psi \leq 0.003 \quad (mm/mm) \quad [6.5]$$

which indicates that the strain value corresponding to the peak stress of the plain concrete is about 0.002 (*mm/mm*). The material parameter, ψ , which is a function of the fibre and matrix material is given by

$$\psi = \begin{cases} 1.09 + 0.71\varphi^{-0.93} & \bullet \text{ for hooked fibres} \\ 1.09 + 7.50\varphi^{-1.39} & \bullet \text{ for straight fibres} \end{cases} \quad [6.6]$$

where φ is a function of the fibre dimensions given by Equation [6.1]. The above equations were developed by curve fitting the experimental results of Balaguru and Shah (1992); Fanella and Naaman (1985); and Abdull-Ahad and Abbas (1989). It can be seen from these equations that the presence of fibres affects the overall deformational behaviour of FRC material specimens. However, the manner with which an active fibre may influence the crack bridging mechanism is not clear. When the FRC material element is subjected to a biaxial state of stress, Abdull-Ahad and Abbas (1989) employed the axial stress given by Equation [6.4] in conjunction with the Kupfer-Gerstle failure envelope of plain concrete (Kupfer and Gerstle 1982) to calculate its load-deformation response. It should be noted that this set of empirical equations will predict the FRC concrete stress-strain behaviour with reasonable accuracy provided the fibre volume fraction is less than 1%.

Having indicated the limitation of the existing FCC and FRC models, it is obvious that there is a need for a comprehensive theory which could be used to analyze the mechanical behaviour of FRC materials and structures. The main shortcomings of the existing FCC models is that they are derived for uniaxially reinforced composite ceramics and not for cementitious matrices, and they assume the fibres to behave only as long elastic fibres. The latter assumption may be reasonable for ceramic composites because they often fail before fibre fracture, but the fibres used in concrete may be short, long active and/or inactive within the FRC concrete element. Our discussion of the available FRC methods showed that the fibres are modelled in an approximate fashion through globalization of their effect on the stress-strain curve. Randomness of the fibres and cracks have not been

dealt with in those models. Therefore, it can be concluded that developing a sound theory for the mechanics of FRC material is opportune. This theory may be incorporated in a general finite element program to obtain a powerful and fast tool in evaluating FRC structures performance under non-proportional static and cyclic loading.

The foregoing shortcomings prompted Razaqpur and Nofal (1992) to develop an FRC model so that it may be employed in analyzing FRC beams and slabs by the non-linear finite element method. They used the model presented in Chapter 3 with some modifications as follows. First the model employed the “rule of mixture” in conjunction with micro-mechanical damage mechanics models developed in Chapter 3. Subsequently, they formulated the basic constitutive relations for fibre-reinforced concrete. Thus, in reality the FRC material is replaced by an equivalent homogeneous cracked brittle matrix with composite properties. It should be mentioned that the numerical test results given by Razaqpur and Nofal (1992) compare well with their experimental counterparts.

6.5 PROPOSED FRC DAMAGE MECHANICS THEORY

It was stated that a typical FRC element may contain short or long fibres which are usually randomly distributed and are oriented in a similar fashion as the microcracks and voids. Experiments (e.g. Budiansky *et al.* 1986; Evans *et al.* 1988; Cox 1990; Cox *et al.* 1990 Wang and Chen 1993; etc.) have revealed that defects (e.g. cracks, voids, etc.), beside being the result of bleeding, may be initiated at either the end of the fibre, within the matrix, around a given aggregate, or around the fibre perimeter. Figure 6.5 schematically illustrates a typical saw cut in a polished FRC element.

Obtaining the constitutive equations for this element will require consideration of the impact of all its components. This means that, the matrix, fibre-end, bleeding, and radial cracks need to be modeled, as well as the cement matrix, aggregates, and fibres. However, some of these parameters may not affect the outcome of the model's prediction. The choice of dropping or selecting certain parameters often depends on the type of application, albeit the analyst's opinion and preference will influence the choice. The

current work is subjected to the same kind of bias but attempts are made to minimize the impact of the latter on the choice of the parameters.

Figure 6.5: Typical saw cut in a polished FRC element.

Of course, planar cracks evolution and fibre deformations would consume a large portion of the applied energy. However, crack deformation mechanisms in FRC materials depend greatly on the fibre presence. For example, any crack crossed by a relatively long and reasonably stiff fibre will exhibit much smaller displacements than one that is crossed by a short fibre. On the other hand, when fibres do not cross a crack, such a crack will have larger displacements. The latter comparison is among cracks that are of the same type and subject to the same load. Figure 6.6 depicts these cracks as well as the so-called radial cracks that emanate from inclusions (e.g. fibre cross-section, aggregate, etc.). It should be mentioned that the current formulation does not consider the evolution of radial cracks. This is because they form parallel to the cross-section of the fibre and are perpendicular to the applied loads which make them irrelevant in the considered 2-D setting. Therefore, it is believed that ignoring the modelling of these cracks will not introduce any serious error in the final results. Neglecting the contribution of the propagating radial crack around heavy reinforcement bars may not be justifiable.

However, the finite element model employed in this study, employs the facet layered shell element where the fibres are assumed to be in a plane parallel to the surface of the element.

Figure 6.6: Crack types and geometry in FRC material.

The reader may have noticed from the discussion in this chapter that current mechanical models assume that the fibres are long and continuous. These assumptions restrict their use in modelling FRC materials because in practice both short and long fibres are employed. The proposed model, however, will remedy this shortcoming and would allow for any reasonable length, diameter, and orientation of the fibres. The developed set of constitutive equations consider fibre fracture and can decide on whether a fibre is long or short. According to the proposed formulation, the fibre behaviour may change from long fibre behaviour to short fibre's at different load levels. Unlike the current models, all the cracks are not assumed to behave as if fibres are crossing them (e.g. Figure 6.2). This means that the model allows some cracks to evolve ignoring the existence of fibres while others will recognize their presence. A distinct feature of the current model is that the remote applied loads need not be aligned with the fibres. The model is expected to predict

the full load-deformation history of a specimen and its different modes of failure, such as crack propagation, fibres breakage, and/or plastic flow of the matrix.

6.6 MODELLING STRATEGY OF FRC ELEMENTS

As stated earlier, the fibres and cracks would be randomly distributed within the concrete matrix. Furthermore, cracks will be either un-bridged, partially or fully bridged (reinforced). Therefore, the concrete element of Figure 6.2 can be replaced by an equivalent matrix with a single crack crossed by a fibre which is illustrated by Figure 6.7. It should be noted that some cracks will not be bridged by fibres and they will behave according to the model given in Chapter 3.

Figure 6.7: Equivalent cracked composite element.

Let us assume that the crack at the start of the loading process has major and minor diameters of $(2a_0, 2b_0)$. Due to the variable spacing of the fibres, a crack can be crossed by a single or multiple fibres (Figure 6.2). To simulate the mechanical behaviour of the element illustrated in the latter figure, the equilibrium conditions and axial displacement of the fibre must be studied. This can be achieved using Elshelby's hypothesis of cutting out the portion ABCD of the element, subject to the representation of the proper tractions and boundary conditions. The tractions can be computed according to the equations given in Chapter 3 (e.g. Equation [3.89]) and the resulting displacements can be calculated.

Subsequently, we re-insert the deformed cut-out into the element and restore the displacement compatibility of the system.

Figure 6.8: Load distribution on element ABCD.

The resulting element geometry, material constituent, and loading configuration may be represented according to Figure 6.8. The new configuration adopts three sets of Cartesian coordinate axes. These correspond to the principal material axes, $(x-y)$, the crack axes $(x'-y')$, and the fibre axes, $(\bar{x}-\bar{y})$, respectively. It should also be mentioned that the $x(x'$ or $\bar{x})$ represents the major axis of the corresponding system. Figure 6.8 shows that the shear stresses, $\tau_{\bar{x}\bar{y}}$, are mainly resisted by the dowel action of the fibre and the shear resistance of the portion of the matrix above and below the crack faces. On the other hand, the normal stresses, $\sigma_{\bar{y}}$, are resisted by fibre extension, sliding, and slipping in the

matrix. The dowel effect may be generally ignored but the fibre pull and/or push-out, as well as the crack deformations, are considered in developing the constitutive equations.

6.6.1 FIBRE CLOSING PRESSURE

Assuming the dowel action of the fibres to have negligible impact on the behaviour of FRC, it can be shown that utilizing the shear lag theory (Marshall and Cox 1992), the fibre closing pressure, $p(\bar{y})$, in a continuum can be written as

$$p(\bar{y}) = V_f T = \frac{2V_f E_f}{l_d} u(\bar{y}) \quad [6.7]$$

which is used to smear out the bridging stress, T . It is obvious from the above equation that the fibre stress-displacement relationship $(p(\bar{y}) - u(\bar{y}))^\dagger$ is a function of the fibre volume fraction, V_f , elastic moduli, E_f , and its half de-bond length, $\frac{l_d}{2}$. Once again, according to Equation [6.7] the desired solution requires the evaluation of the de-bond length, l_d . The latter length can be evaluated from equilibrium and strain compatibility of the fibre and the surrounding matrix provided the shear stress distribution at the fibre-matrix interface is known. Of course, this solution should be obtained for either intact or fractured fibres because a given fibre may remain intact throughout the deformation process of an FRC structures or it may fracture when the applied load exceeds the material strength. Accordingly, the de-bond length of a fibre can be evaluated for constant shear distribution at the fibre-matrix interface according to Cox and Marshall (1991)

$$l_d = \begin{cases} \sqrt{\frac{u(\bar{y}) r_f E_f}{\tau_o (1 + \eta)}} & \bullet \text{ intact fibre} \\ \frac{1}{2\mu b_1} \ln \left[\frac{k_1^2 \sigma_{\bar{y}}}{(1 - k_1)(\sigma_u - k_1 \sigma_{\bar{y}})} \right] & \bullet \text{ fractured fibre} \end{cases} \quad [6.8]$$

[†] Note that the closing pressure, $p(\bar{y})$ has units of stress. It is also important to observe that Equation [6.7] uses, $u(\bar{y})$, rather than $\sqrt{u(\bar{y})}$, which was used by Marshall and Cox (1992) and Meda and Steif (1992).

where the fibre-matrix rigidity ratio, $\eta = \frac{E_f V_f}{E_m V_m}$, and the constant k_t , are functions of the fibre and matrix elastic moduli and volume fractions, respectively. τ_o is the shear strength of the fibre-matrix interface, and σ_u is the ultimate pull/push-out stresses of the fibre.

6.6.2 OVERALL COMPLIANCE OF A SOLID WITH A BRIDGED SINGLE CRACK

Having established that the closing pressure, $p(\bar{y})$ is a function of the crack opening displacements, $u(\bar{y})$, and de-bond length, l_d , will enable us to calculate the crack compliance. As a result, the overall compliance and stiffness of the composite solid can be determined and the mechanical deformational response to a given applied stress field can be estimated. Before doing so, let us see the possible scenarios for applying the closing pressure on the crack faces.

Figure 6.9: Different bridging effects of a fibre(s) crossing an existing crack.

According to Figure 6.9, there are four situations in which the bridging effect can be applied. First if a crack is crossed by multiple fibres the closing pressure will be uniformly distributed on the crack faces, Figure 6.9(a) or Figure 6.9(d). At the same time for the single fibre case, the distribution can be as shown in Figure 6.9(b) or Figure 6.9(c). These situations can be modelled individually but this study will concentrate only on the first case. This is because fibres are short and randomly distributed in the matrix and thus the first event, Figure 6.9(a), is the commonly occurring case in FRC structures, but the other three conditions may happen in composites with unidirectional continuous fibres or in specially designed specimens. It should be mentioned that Meda and Steif (1992) use theory of elasticity to model a crack bridged by a single fibre which correspond to the case in Figure 6.9(b). using the elasticity solution. They found that the closing pressure method underestimates the bridging effect of the fibre in this situation. However, Marshall *et al.* (1985); Cox *et al.* (1989) and Cox and Marshall (1991) have shown that the expression of the closing pressure given by Equation [6.7] gives excellent results for the case of multiple fibres. Once again, it should be remembered that these studies were concerned only with longitudinally cracked composites reinforced with continuous fibres.

6.6.3 EFFECTIVE FRC CONSTITUTIVE RELATIONSHIPS

The effective elastic compliance can be calculated by considering the basic requirement of equilibrium and compatibility. The increment of far field stress $\{\Delta\sigma\}$ is equilibrated by the increment of stress in the concrete, $\{\Delta\sigma\}^c$, and the increment of stress in the fibres, $\{\Delta\sigma\}^f$ as follows

$$\{\Delta\sigma\} = \{\Delta\sigma\}^c V_c + \{\Delta\sigma\}^f V_f \quad [6.9]$$

where V_f , $V_c = (1 - V_f)$ are the fibres and concrete matrix volume fractions, respectively. Assuming perfect bond between the fibres and the surrounding concrete beyond the de-bond zone

$$\{\Delta\epsilon\} = \{\Delta\epsilon\}^c = \{\Delta\epsilon\}^f \quad [6.10]$$

Considering Equation [6.10], Equation [6.9] may be rewritten as

$$\{\Delta\sigma\} = ([D]^c + [D]^f) \{\Delta\varepsilon\}^c \quad [6.11]$$

where $[D]^c$ and $[D]^f$ are the stiffness of the concrete and the fibres, respectively.

Equation [6.11] can be rewritten as

$$\{\Delta\varepsilon\}^c = ([D]^c + [D]^f)^{-1} \{\Delta\sigma\} \quad [6.12]$$

or

$$\{\Delta\varepsilon\}^c = \left[\frac{[C]^c}{1 + [C]^c ([C]^f)^{-1}} \right] \{\Delta\sigma\} \quad [6.13]$$

Now since

$$\{\Delta\varepsilon\}^{cr} = [C]^{cr} \{\Delta\sigma\} \quad [6.14]$$

combining Equations [6.13] and [6.14] and recognizing that $\{\Delta\varepsilon\}^t$ is the total non-plastic strain in the fibre reinforced concrete and the cracks yields

$$\{\Delta\varepsilon\}^t = \left\{ \left[\frac{[C]^c}{1 + [C]^c ([C]^f)^{-1}} \right] + [C]^{cr} \right\} \{\Delta\sigma\} \quad [6.15]$$

where the superscripts, c , f , and cr denote concrete, fibres, and cracks, respectively.

Comparing Equations [6.15] and [3.7], leads to

$$[\bar{D}] = [\bar{C}]^{-1} \quad [6.16]$$

where $[\bar{C}]$ is the first term on the right-hand side of Equation [6.15]. It will be shown later how to employ continuum damage, fracture, and statistical mechanics as well as theories of elasticity and plasticity to evaluate the latter compliance. It should be emphasized that the proposed elasto-plastic constitutive equations (e.g. Equations [3.32], [3.33], and [6.16]) of a damaged FRC material is similar in form to that of an undamaged

material. Once again, the main difference is in the use of the effective damage tensor in lieu of the elasticity tensor in the conventional theory of plasticity.

The evaluation of the concrete matrix compliance can be carried out in a straightforward fashion utilizing theory of elasticity. That of the preexisting microcracks, on the other hand, may be obtained using the procedures described later in this chapter which are similar to those presented in Chapter 3. The fibre contribution to the overall compliance of the cracked fibre-reinforced concrete element may be obtained from the inverse of its stiffness at a given load level. The procedure for deriving fibre compliance and stiffness are given below.

6.6.3.1 FIBRE COMPLIANCE

Assuming that the fibre-reinforced concrete representative volume element (RVE) has a total number of fibres, N_f ; each with orientation, ψ_k , with respect to the major axis of the matrix material (e.g. concrete) and fibre initial stiffness, $\left[\frac{A_f E_f}{l_f} \right]_k$, the fibre stiffness within the RVE can be written as

$$[D]^f = \frac{1}{V} \sum_{k=1}^{N_f} [T]^T [D(\psi, l, A, E)]_k [T] \quad [6.17]$$

where V is the total volume of the element and

$$[D(\psi, l, A, E)]_k = \frac{1}{V} \left[\frac{A_f E_f}{l_f} \right]_k \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [6.18]$$

and $[T]$ in terms of ψ is given by

$$[T] = \begin{bmatrix} \cos^2 \psi & \sin^2 \psi & 2 \sin \psi \cos \psi \\ \sin^2 \psi & \cos^2 \psi & -2 \sin \psi \cos \psi \\ -\sin \psi \cos \psi & \sin \psi \cos \psi & \cos^2 \psi - \sin^2 \psi \end{bmatrix} \quad [6.19]$$

Equation [6.17] is suitable for the case of long aligned fibres (e.g. the case of reinforced steel) but it is not useful for the case of randomly distributed short fibres. For the case of short fibres, the summation can be replaced by integration, i.e.

$$[D]' = \frac{1}{V} \int \int \int \int p(\psi, l, A, E) [T]^T [D(\psi, l, A, E)]_k [T] dE dA dl d\psi \quad [6.20]$$

where $p(\psi, l, A, E)$ is the joint probability density function of the fibre, orientation, size, and its elastic modulus, Assuming that these parameters are independent of each other, this function can be decomposed into the product of separate probability functions, i.e.

$$p(\psi, l, A, E) = p(\psi)p(l)p(A)p(E) = p(\psi) = \frac{2}{\pi} \quad [6.21]$$

where all fibres are assumed to have the same sizes (i.e. l, A) and elastic material properties (i.e. E, ν), respectively. However, the analysis of fibre pull or push-out from the matrix presented in Marshall and Cox (1992) showed that the fibre de-bonding characteristics are dictated by its orientation, ψ , with respect to the applied stress field. Fibres with initial orientation the same as those of the neighboring stress field will de-bond the most. Therefore, there appears to be a correlation between fibre orientation angle, $\psi, \left(-\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}\right)$ and de-bond length, l_d , but any effect arising from such a correlation may be assumed to be rather small because of the fibre lengths used in practice. Consequently, the fibre orientation, ψ , can be assumed to be independent of the fibre geometry, and to follow a totally *random* distribution, which will reduce Equation [6.19] to

$$[D]' = \frac{N_f l_f^2}{V} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\psi) [T]^T [D(\psi)]_k [T] d\psi \quad [6.22]$$

Making use of Equations [6.18], [6.19], and [6.19] in Equation [6.22] and integrating it with respect to the fibre orientation angle, ψ , will yield

$$[D]^f = \frac{(N_f l_f A_f) E_f}{4V} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad [6.23]$$

where $\frac{(N_f l_f A_f)}{V}$ represents the fibre volume fraction, V_f , which finally gives

$$[D]^f = \frac{V_f E_f}{4} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad [6.24]$$

Of course, the overall compliance is the inverse of the overall effective stiffness of the cracked fibre-reinforced concrete element. The effective elastic stiffness of the cracked fibre-reinforced concrete can be obtained by summing the stiffness of the cracked concrete and the fibres. The stiffness of the cracked concrete will depend on the effective stress acting on it.

6.6.3.2 BRIDGED CRACK EFFECTIVE STRESSES

It was shown earlier that the fibre effect can be replaced by a closing pressure (stresses), $\bar{\sigma}_v^f$, given by Equation [6.7]. This means that the applied normal, $\sigma_{v'}$, and shear, $\tau_{x'y'}$, stresses will be reduced on the crack faces. Thus, the new modified applied normal, $\bar{\sigma}_{v'}$, and shear, $\bar{\tau}_{x'y'}$, stresses due to the presence of the fibres can be computed from

$$\begin{Bmatrix} \bar{\sigma}_{v'} \\ \bar{\tau}_{x'y'} \end{Bmatrix} = \begin{Bmatrix} \sigma_{v'} - \int_{\psi_1}^{\psi_2} p(\psi) \bar{\sigma}_v^f \cos^2 \psi d\psi \\ \tau_{x'y'} - \int_{\psi_1}^{\psi_2} p(\psi) \bar{\sigma}_v^f \sin \psi \cos \psi d\psi \end{Bmatrix} \quad [6.25]$$

which for a totally random fibre orientation will give

$$\begin{Bmatrix} \bar{\sigma}_{v'} \\ \bar{\tau}_{x'y'} \end{Bmatrix} = \begin{Bmatrix} \sigma_{v'} \\ \tau_{x'y'} \end{Bmatrix} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2}{\pi} \bar{\sigma}_v^f \begin{Bmatrix} \cos^2 \psi \\ \sin \psi \cos \psi \end{Bmatrix} d\psi = \begin{Bmatrix} \sigma_{v'} - \bar{\sigma}_v^f \\ \tau_{x'y'} \end{Bmatrix} \quad [6.26]$$

where these stresses are used to estimate the crack displacements, crack inelastic damage compliance as well as the crack damage strains. In the following, the evolution of cracks bridged (reinforced) by fibres will be presented.

6.7 BRIDGED (REINFORCED) CRACK EVOLUTION CRITERIA

It was shown earlier that a crack will evolve and propagate if the applied stress exceeds the critical intensity factor, depending on the mode of deformation (i.e. mode-I, mode-II and/or mixed mode-I/II). In addition, it was also established that cracks start propagating at the weakest link which was identified by the hierarchical law of fracture given by Equation [3.77]. The same laws of unreinforced crack evolution (i.e. Equation [3.71]) can be applied to the current situation. The stress intensity factors, K_I and K_{II} , in terms of the modified applied stresses given by Equation [6.26] can be written as

$$\begin{Bmatrix} \bar{K}_I \\ \bar{K}_{II} \end{Bmatrix} = \sqrt{\pi a} \begin{Bmatrix} \sigma_{y'} - \bar{\sigma}_v^f \\ \tau_{xy'} \end{Bmatrix} \quad [6.27]$$

where $\sigma_{y'} = \sigma_y N(s_r, \theta)$, and $\tau_{xy'} = \sigma_y S(s_r, \theta)$. Equation [6.27] will be used in the following to determine material fracture toughness, and crack diameters and orientation angles, respectively. Once again, the microcrack kinetic for fibre-reinforced concrete materials will be traced for the three modes of deformation identified earlier. In the subsequent sections the pertinent evolution equations for each mode will be given.

6.7.1 UNIAXIAL AND BIAXIAL TENSION OF BRIDGED (REINFORCED) CRACKS

It was established that cracks subjected to axial or biaxial tensile stresses will propagate mainly under mode-I effects. Thus, the first equation of Equation [6.27] can be used to determine the stresses corresponding to the onset of first crack to, i.e.

$$f_t' = \frac{K_{IC}^f}{\rho} \sqrt{\frac{2}{\pi D_M}} + \bar{\sigma}_v^f \quad [6.28]$$

where those cracks are oriented at an angle, θ and the scalar, ρ , was defined earlier as the friction of the initially de-bonded interfaces. According to the fibre pull-out analysis, the closure pressure, $\bar{\sigma}_v^f$, is related to the crack opening displacement, u_v , at a given

location along the crack length. The crack displacements should be evaluated for the new set of applied stresses, $\bar{\sigma}_{y'}$, and $\bar{\tau}_{xy'}$, given by Equation [6.24]. Hence, Equation [3.40] will read as

$$\begin{Bmatrix} u_{x'} \\ u_{y'} \end{Bmatrix} = 2\sqrt{a^2 - x'^2} \left\{ \begin{aligned} &\bar{C}_{11}' [(\alpha_1\beta_2 + \alpha_2\beta_1)\bar{\sigma}_{y'} + (\beta_1 + \beta_2)\bar{\tau}_{xy'}] \\ &\bar{C}_{22}' \left[\left(\frac{\beta_1}{\alpha_1^2 + \beta_1^2} + \frac{\beta_2}{\alpha_2^2 + \beta_2^2} \right) \bar{\sigma}_{y'} + \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)} \bar{\tau}_{xy'} \right] \end{aligned} \right\} \quad [6.29]$$

which indicates that the fibre closing pressure for the overall tensile loading will vary from zero at the two edges of the crack to a maximum value at its center. However, the crack compliance is written in terms of uniform distribution. Thus, for the sake of simplicity, the fibre closing pressure, $\bar{\sigma}_u^f$ may be replaced by an average uniformly distributed pressure, $\bar{\sigma}_u^f$, along the crack length, i.e.

$$\bar{\sigma}_u^f = \frac{1}{2a} \int_{-a}^a \bar{\sigma}_{\frac{y}{2}}^f dx' \quad [6.30]$$

which with the aid of Equation [6.7], can be rewritten as

$$\bar{\sigma}_u^f = \frac{V_f E_f}{al_d} \int_{-a}^a u(\bar{y}) dx' = \frac{V_f E_f}{al_d} \bar{u}(\bar{y}) \quad [6.31]$$

The latter leads to

$$\bar{u}(\bar{y}) = \pi a^2 \bar{C}_{22}' \left[\left(\frac{\beta_1}{\alpha_1^2 + \beta_1^2} + \frac{\beta_2}{\alpha_2^2 + \beta_2^2} \right) \bar{\sigma}_{y'} + \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)} \bar{\tau}_{xy'} \right] \quad [6.32]$$

which gives

$$\bar{u}(\bar{y}) = \frac{\pi a^2 \bar{C}_{22}'}{C_{f_n}} \left[\left(\frac{\beta_1}{\alpha_1^2 + \beta_1^2} + \frac{\beta_2}{\alpha_2^2 + \beta_2^2} \right) \bar{\sigma}_{y'} + \frac{\alpha_1\beta_2 + \alpha_2\beta_1}{(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2)} \bar{\tau}_{xy'} \right] \quad [6.33]$$

where

$$C_{f_n} = 1 + \pi a^2 \bar{C}_{22}' \left[\left(\frac{V_f E_f}{al_d} \right) \left(\frac{\beta_1}{\alpha_1^2 + \beta_1^2} + \frac{\beta_2}{\alpha_2^2 + \beta_2^2} \right) \right] \quad [6.34]$$

The displacement given by Equation. [6.33] can be used to evaluate the equivalent fibre closing pressure, $\bar{\sigma}_u^f$. Subsequently, the first crack will start evolving and propagating at a load, f_t^r given by

$$f_t^r = \frac{K_{IC}^f}{\rho} \sqrt{\frac{2}{\pi D_M}} \left[1 + \bar{u}(\bar{y}) \left(\frac{V_f E_f}{a l_d} \right) \right] \quad [6.35]$$

Of course the roots of the characteristic equations (i.e. Equations [3.41] and [3.42]) should be determined *a priori* using the virgin elastic material properties in order to evaluate the crack opening displacement. In addition the fibre de-bonded length, l_d , must be determined from Equation [6.8]. If this calculated length is less than the actual fibre length used in the mix, the fibre presence will not improve the concrete tensile strength and the concrete will crack at the same cracking load of plain concrete. In the case of isotropic concrete mix, Equation [6.33] can be reduced to

$$\bar{u}(\bar{y}) = \frac{2\pi a^2 l_d f_t^r \bar{C}_{22}'}{l + 2\pi a V_f E_f \bar{C}_{22}'} \quad [6.36]$$

because $\alpha_1 = \alpha_2 = 0$ and $\beta_1 = \beta_2 = 1$ for all isotropic material. Thus, the isotropic cement-paste matrix first crack of diameter, $2a = 2a_o = \rho^2 D_M$, will appear due to a load of

$$f_t^r = \frac{K_{IC}^f}{\rho} \sqrt{\frac{2}{\pi D_M}} \left[1 + \pi \rho^2 D_M \bar{C}_{22}' \left(\frac{V_f E_f}{a l_d} \right) \right] \quad [6.37]$$

where all quantities are defined earlier.

Recalling from Chapter 3 that the crack stress-strain relationships can be obtained by integrating the crack displacements around the crack rim which will give

$$\{\epsilon'\}^{\sigma} = \frac{\pi a^2}{A_c} \begin{bmatrix} 0 & 0 & 0 \\ 0 & G_{11}^o & G_{12}^o \\ 0 & G_{21}^o & G_{22}^o \end{bmatrix} \begin{Bmatrix} \bar{\sigma}_{x'} \\ \bar{\sigma}_{y'} \\ \bar{\tau}_{x'y'} \end{Bmatrix} \quad \text{or} \quad \{\epsilon'\}^{\sigma} = [C']_o^{\sigma} \{\bar{\sigma}'\} \quad [6.38]$$

where the new set of net applied stress, $\{\bar{\sigma}'\}$, on the crack can be obtained from Equation [6.24]. Equation [6.38] shows that the compliance format of the bridged (reinforced)

crack is similar to the embedded crack in plain concrete. The only difference, however, is that the applied net stresses on the crack(s) surfaces are less than those acting on plain concrete by the amount of the closing pressure, $\bar{\sigma}_i^f$.

It should be pointed out that the crack compliance coefficients, $\bar{C}_{ij}^{cr}$, will also be different from those of the unreinforced cracks because in addition to the factors mentioned in Chapter 3, they will depend on the fibre deformational behaviour. Therefore, the steps to calculate the overall inelastic compliance will remain the same as the plain concrete model, but the proper crack evolution laws should account for the presence of the fibres. This issue and the pertinent formulations will be developed and discussed later in this study for all loading scenarios. Similarly, the constitutive relationships for the other closed (sliding) and kinked cracks can be obtained using Equations [3.48] and [3.51] for the former type and Equations [3.61] and [3.62] for the latter kind of cracks, respectively. However, the applied net stress of those equations should be replaced by those given by Equation [6.27].

First of all, the critical propagating tensile cracks can be identified from the arrested ones by their inclination angle, θ . This angle can be obtained from Equation [6.27] according to

$$\theta_{cr} = \tan^{-1} \sqrt{\frac{\sigma_y - (f_i' + \bar{\sigma}_u^f)}{(f_i' + \bar{\sigma}_u^f) - s_r \sigma_y}} \quad [6.39]$$

and the corresponding crack diameter, $D(\theta)$, at any orientation angle, θ , may be obtained by considering Equations [6.27] and [6.37], i.e.

$$D(\theta) = \left[\frac{\rho(f_i' - \bar{\sigma}_u^f)}{\sigma_y N(s_r, \theta) - \bar{\sigma}_u^f} \right]^2 D_M \quad [6.40]$$

where the transformation function $N(s_r, \theta)$ is given by Equation [3.61]. Once again the minimum crack diameter, $D(0)$, may be evaluated by substituting $\theta = 0$, in the latter equation, i.e.

$$\frac{D_{min}}{D_M} = \frac{D(0)}{D_M} = \left(\frac{\rho(f'_t - \bar{\sigma}_u^f)}{\sigma_r - \bar{\sigma}_u^f} \right)^2 = \gamma_t \quad [6.41]$$

where all variables were defined earlier.

The propagating and arrested cracks portions, η'_s , and, η'_a , of the total number of existing cracks, N_c , can be calculated using the same procedures described in Chapter 3. Thus, the current propagating tensile cracks portion, $\eta'_s|_c$, may obtained from the following equation

$$\eta'_s|_c = \frac{2}{\pi} \left\{ \theta_{t_1} - (\rho f'_t - \bar{\sigma}_u^f)^2 \int_0^{\theta_{t_1}} \frac{d\theta}{[\sigma_r N(s_r, \theta) - \bar{\sigma}_u^f]^2} \right\} \quad [6.42]$$

- A-S-K method
- exact method

making use of these equations in Equations [3.81], [3.82], and [3.83], the crack damage compliance can be fully determined for mode-I deformation.

The process is also terminated for this material if the load level reaches a critical value corresponding to the formation of a single macro-crack. Note that the model stops at the onset of material softening. According to the hierarchy of the critical stress intensity factors, the failure load will be given by

$$f'_t = \bar{\sigma}_u^f + K_{IC} \sqrt{\frac{2}{\pi D_M}} \left\{ \sqrt{\frac{1}{\frac{4b}{\pi D_M} \tan\left(\frac{\pi D_M}{4b}\right)}} \right\} \quad [6.43]$$

- no microcrack interaction
- microcrack interaction

where $\bar{\sigma}_u^f$ is the fibre contribution to the tensile strength of the material at the onset of complete failure.

6.7.1.1 UNIAXIAL AND BIAXIAL COMPRESSION OF BRIDGED (REINFORCED) CRACKS

It was found earlier in this thesis that preexisting microcracks in concrete materials subjected to uniaxial and biaxial compressive stresses will grow in mode-II manner. The modified non-vanishing crack displacement function of these cracks can be obtained by substituting the new effective stresses, $\{\bar{\sigma}'\}$, in Equation [3.49], i.e.,

$$u_{x'} = 2\sqrt{a^2 - x'^2} \bar{C}_{11}' [(\alpha_1 \beta_2 + \alpha_2 \beta_1) \bar{\sigma}_{y'}^* + (\beta_1 + \beta_2) \bar{\tau}_{x'y'}^*] \quad [6.44]$$

in which

$$\begin{Bmatrix} \bar{\sigma}_{y'}^* \\ \bar{\tau}_{x'y'}^* \end{Bmatrix} = \begin{Bmatrix} \bar{\sigma}_{y'} - \bar{\sigma}_{y'}^c \\ \bar{\tau}_{x'y'} + \mu \text{sgn}(\bar{\tau}_{x'y'}) \bar{\sigma}_{y'}^c \end{Bmatrix} \quad [6.45]$$

with

$$\bar{\sigma}_{y'}^c = \frac{[\beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2)] \bar{\sigma}_{y'} + (\alpha_1 \beta_2 + \alpha_2 \beta_1) \bar{\tau}_{x'y'}}{\beta_1(\alpha_2^2 + \beta_2^2) + \beta_2(\alpha_1^2 + \beta_1^2) - \mu \text{sgn}(\bar{\tau}_{x'y'}) (\alpha_1 \beta_2 + \alpha_2 \beta_1)} < 0 \quad [3.46]$$

Since the fibre closing pressure, $\bar{\sigma}_{\bar{y}}^f$, is written in terms of the crack opening displacement component, i.e. $u_{y'} = 0$, the presence of the fibre will not be felt by the material. This is because the fibre dowel effect was assumed to have very minimal impact on the fibre-reinforced concrete mechanical deformation processes. However, this study also revealed that some of these cracks will eventually kink at certain load levels and align themselves with the direction of maximum compression. These cracks will have the two opening and sliding components. The new modified kinked crack displacements in the kinked (double primes) and original (single prime) cracks directions can be, respectively, rewritten as

$$\begin{Bmatrix} v_{x''} \\ v_{y''} \end{Bmatrix} = \frac{9.6(1-\nu^2)}{\pi E} F_1(D_M, \theta, l_k) (\bar{\tau}_{x'y'} - \mu \bar{\sigma}_{y'}) \begin{Bmatrix} -\cos \theta \\ -\sin \theta \end{Bmatrix} \quad [6.47]$$

and

$$\begin{Bmatrix} u_{x''} \\ u_{y''} \end{Bmatrix} = \frac{9.6(1-\nu^2)}{\pi E} F_1(D_M, \theta, l_k) (\bar{\tau}_{x'y'} - \mu \bar{\sigma}_{y'}) \begin{Bmatrix} -\tan \theta \\ -1 \end{Bmatrix} \quad [6.48]$$

where the displacements used for evaluating the fibre closing pressure, $\bar{\sigma}_{\bar{y}}^f$, shall be computed from the first relation of Equation [3.55], i.e.

$$\bar{u}(\bar{y}) = \bar{v}(0) = \frac{4.8(1-\nu^2)}{\pi E} F_1(D_M, \theta, l_k) (\bar{\tau}_{x'y'} - \mu \bar{\sigma}_{y'}) \quad [6.49]$$

It follows that the fibre presence will reduce the effective shear stresses, $\bar{\tau}_s$, on the closed sliding crack faces. This effective shear stresses can be computed from

$$\bar{\tau}_s = \sigma_v \bar{G}(s_r, s_f, \theta) \quad [6.50]$$

where the new transformation function, $\bar{G}(s_r, s_f, \theta)$, is given by

$$\bar{G}(s_r, s_f, \theta) = S(s_r, \theta) - \mu[N(s_r, \theta) - s_f] \quad [6.51]$$

where $s_f = \frac{\bar{\sigma}_v^f}{\sigma_v}$ represents the ratio between the fibre closing pressure, $\bar{\sigma}_v^f$, and the applied major stress, σ_v .

Once again, assuming Coulomb's type friction, the solution of Equation [6.51], will give the upper and lower bounds of the orientation of the sliding cracks as

$$\theta_{s_r, s_f} = \pm \tan^{-1} \left[\frac{1 \pm \sqrt{1 - 4\bar{C}_s(\bar{C}_s + \mu)}}{2\bar{C}_s} \right] \quad [6.52]$$

where $\bar{C}_s = C_s = \frac{\mu s_f}{(1 - s_r)}$. The coefficient, C_s , will be the same as that of Chapter 3. The minimum value of θ required for sliding will be the same as the plain concrete case which is given by Equation [3.91]. This shows that the orientations of the first preexisting microcracks neither depend on the lateral stress, σ_x , nor on the fibre presence. Equation [6.51] indicates that cracks will not slide if the ratio of the two applied principle stresses, (i.e., $\frac{\sigma_x}{\sigma_v}$) satisfy the following condition

$$s_r = \frac{\tan \theta - \mu[1 - s_f \sec^2 \theta]}{\tan \theta [\mu \tan \theta + 1]} \quad [6.53]$$

which is plotted in Figure 6.10 for different ratios of fibre closing pressure to the applied major stresses (i.e., $\frac{\bar{\sigma}_v^f}{\sigma_v}$). Once again, any stress ratio which happens to be within the

envelope for a given friction coefficient will cause the crack to slide while ratios outside the envelope will not cause it to slide. It is very interesting to observe, the fibre presence changes the mechanism from plastic flow to fracture. This may explain the brittleness of reinforced concrete members that are overly reinforced.

Figure 6.10: FRC Fracture and plastic flow domains under biaxial compression.

The corresponding critical axial compressive stress, f'_c , for the FRC member at which the sliding cracks start to propagate can be obtained according to

$$f'_c = \frac{K_{frc}'}{\rho \bar{G}(s_r, s_f, \theta_o)} \sqrt{\frac{2}{\pi D_M}} \quad [6.54]$$

Note that the latter equation will give exactly the same stresses as that for plain concrete because the fibre closing pressure equals zero at this load level. The corresponding crack diameter, $D(\theta)$, of any orientation, θ , can be estimated according to

$$D(\theta) = \left[\frac{\rho f'_c \bar{G}(s_r, s_f, \theta_o)}{\sigma_r \bar{G}(s_r, s_f, \theta)} \right]^2 D_M \quad [6.55]$$

The corresponding critical orientation angle becomes

$$\theta_{f_2, f_1} = \pm \tan^{-1} \left[\frac{1 \pm \sqrt{1 - 4\bar{C}_f(\bar{C}_f + \mu)}}{2\bar{C}_f} \right] \quad [6.56]$$

$$\text{in which } \bar{C}_f = \frac{1}{(1-s_r)} \left[\mu(s_r - s_f) + \left(\frac{\rho f'_c}{\sigma_y} \right) \bar{G}(s_r, s_f, \theta_o) \right].$$

Considering the same assumptions of the plain concrete material, the kinked cracks will be initiated at a loading given by

$$f_c^k = \frac{K_{IC}^c}{2\bar{G}(s_r, s_f, \theta_o)} \sqrt{\frac{6}{\pi D_M}} \quad [6.57]$$

The kinked crack diameter, $D(\theta)$, of this type of cracks and material can be obtained from

$$D(\theta) = \left[\frac{k_1 f'_c \bar{G}(s_r, s_f, \theta_o)}{\sigma_y \bar{G}(s_r, s_f, \theta)} \right]^2 D_M \quad [6.58]$$

where all constants are defined earlier.

The critical angle of those cracks can be obtained from

$$\theta_{k_2, k_1} = \pm \tan^{-1} \left[\frac{1 \pm \sqrt{1 - 4\bar{C}_k(\bar{C}_k + \mu)}}{2\bar{C}_k} \right] \quad [6.59]$$

$$\text{where the factor, } \bar{C}_k = \frac{1}{(1-s_r)} \left[\mu(s_r - s_f) + \left(\frac{k_1 f'_c}{\sigma_y} \right) \bar{G}(s_r, s_f, \theta_o) \right].$$

The different portions of kinking, sliding, and arrested cracks can be estimated according to

$$\eta_k^c = \frac{1}{(\theta_{k_2} - \theta_{k_1})} \left\{ (\theta_{k_2} - \theta_{k_1}) - \left(\frac{k_1 f'_c}{\sigma_y} \right)^2 \bar{G}^2(s_r, s_f, \theta_o) \int_{\theta_{k_1}}^{\theta_{k_2}} \frac{d\theta}{\bar{G}^2(s_r, s_f, \theta)} \right\} \quad [6.60]$$

- A – S – K method
- exact method

where the integral in the above equation can be evaluated using numerical integration.

Subsequently, the portion of propagating cracks, η_k^c , can be obtained from

$$\eta_s^c = -\eta_k^c + \frac{1}{(\theta_{f_2} - \theta_{f_1})} \left\{ \begin{aligned} & \left(\theta_{f_2} - \theta_{f_1} \right) \cdot \left(\frac{\rho f_c'}{\sigma_y} \right)^2 \bar{G}^2(s_r, s_f, \theta_o) \int_{\theta_{f_1}}^{\theta_{f_2}} \frac{d\theta}{\bar{G}^2(s_r, s_f, \theta)} \end{aligned} \right. \quad \begin{aligned} & \bullet \text{ A-S-K method} \\ & \bullet \text{ exact method} \end{aligned} \quad [6.61]$$

These equations can be substituted into Equation [3.103] to Equation [3.106] in order to evaluate the crack compliance and hence, its stiffness.

Fibre-reinforced concrete RVE element will fail in a similar fashion as the case of plain concrete under different ultimate loads. To distinguish the splitting from the shear fault failure types, the critical angle must be determined from maximizing $(\tau'_n \cos \theta)$ with respect to θ , i.e.

$$\mu s_r \tan^3 \theta_{f_1} - 2(1 - s_r + \mu s_f) \tan^2 \theta_{f_1} + \mu(3 - 2s_r + 3s_f) \tan \theta_{f_1} + (1 - s_r + \mu s_f) = 0 \quad [6.62]$$

which has also only one real positive definite root. The maximum kinked crack length at FRC element failure can be estimated as the case of plain concrete. This indicates that the failure stresses may be obtained according to

$$f_c' = \begin{cases} \frac{K_{IC}^c \sqrt{\pi l_{k_m}}}{\left[D_M \bar{G}(s_r, s_f, \theta_{f_1}) \cos \theta_{f_1} \right] - s_r \pi l_{k_m}} & \bullet -1.000 \leq s_r \leq 0.050 \Rightarrow (\text{split failure}) \\ \frac{K_{IC}^c}{\bar{G}(s_r, s_f, \theta_{f_1})} \sqrt{\frac{2}{\pi D_M}} & \bullet 0.050 \leq s_r \leq 0.306 \Rightarrow (\text{shear fault failure}) \end{cases} \quad [6.63]$$

while considering microcrack interaction, it may be obtained from

$$f_c' = \begin{cases} \frac{K_{IC}^c \sqrt{b \sin \left(\frac{\pi l_{k_m}}{b} \right)}}{\left[D_M \bar{G}(s_r, s_f, \theta_{f_1}) \cos \theta_{f_1} \right] - s_r \left[\sqrt{2b^2 \sin \left(\frac{\pi l_{k_m}}{b} \right) \tan \left(\frac{\pi l_{k_m}}{2b} \right)} \right]} & \bullet -1.000 \leq s_r \leq 0.050 \\ \frac{K_{IC}^c}{\bar{G}(s_r, s_f, \theta_{f_1})} \sqrt{\frac{2}{\pi D_M}} \frac{1}{\sqrt{\frac{4b}{\pi D_M} \tan \left(\frac{\pi l_{k_m}}{4b} \right)}} & \bullet 0.050 \leq s_r \leq 0.306 \end{cases} \quad [6.64]$$

where the distance b is evaluated using Equation [3.88].

Once again, as in the previous material case, crack compliance will vanish if the external mechanical applied stresses produce zero displacements (see Equation [3.48]). This means that any inelastic behaviour is the result of yielding and flow of the material. It was also previously stated that theory of plasticity may be used to trace the material mechanical deformation process under these loading conditions. However, the fibres will share in resisting the applied external stresses which may cause redistribution of stresses in concrete. Assuming that fibres are perfectly random in distribution as well as the composite, fibres, and matrix strains are compatible, the total applied stress in a given direction (e.g. $i = X, Y$, and Z) can be written according to

$$\sigma_i = \sigma_i^m V_m + \sigma_i^f V_f \quad [6.65]$$

where σ_i , σ_i^m , and σ_i^f are applied external stresses, the matrix and the fibres stresses, respectively. V_m and V_f signify the volume fraction of the matrix and the fibres. Since the fibre may yield or fracture, the net stresses in the matrix in the considered direction may be rewritten as

$$\sigma_i^m = \frac{1}{V_m} [\sigma_i - \varepsilon_i^f E_f V_f] \quad [6.66]$$

which can be substituted in Equation [3.111] or Equation [4.18] if the theories of isotropic and anisotropic plasticity are respectively employed. The same procedures utilized in the former model can be used to evaluate the elasto-plastic compliance of the fibre-reinforced concrete material.

6.7.1.2 BIAxIAL COMPRESSION-TENSION OF BRIDGED (REINFORCED) CRACKS

As in the case of plain concrete, there exist two possibilities of crack growth for the biaxial compression-tension loading condition. The same assumption for crack growth used in the former model will be utilized to evaluate the crack evolution for the current situation. Since open and closed crack deformations exist, let us identify their domains by equating the normal stress at the crack site to zero; i.e.

$$\sigma_n = \sigma_y = \sigma_y \bar{N}(s_r, s_f, \theta) \quad [6.67]$$

Solving Equation [6.67] gives the limiting angle, θ_l , as

$$\theta_l = \tan^{-1} \sqrt{\frac{|1 - s_f|}{|s_r - s_f|}} \quad [6.68]$$

The same upper and lower bounds for open and closed cracks will be the same as the case of plain concrete.

The kinematic of closed cracks will be similar to the previous case while only one single angle, θ_s , shall be used for sliding cracks and another angle, θ_f , for propagating ones.

6.7.1.3 OPEN BRIDGED (REINFORCED) CRACK KINEMATICS

In this case, the same procedure will be used as the plain concrete after modifying, the transformation function, $M(s_r, \theta, \zeta)$, in order to reflect the impact of fibres. This function shall be evaluated according to

$$\bar{M}(s_r, s_f, \theta, \zeta) = \zeta^2 [\cos^2 \theta + s_r \sin^2 \theta - s_f]^2 + [(1 - s_r) \sin \theta \cos \theta]^2 \quad [6.69]$$

The solution of this equation will give the same threshold angles given by Equation [3.119]. On the other hand, the threshold stress, f'_m , at which mixed mode-I/II deformation takes place can be estimated by

$$f'_m = \frac{K_{IIc}}{\rho} \sqrt{\frac{2}{\pi D_M}} \frac{1}{\sqrt{\bar{M}(s_r, s_f, \theta_o, \zeta_o)}} \quad [6.70]$$

while the critical crack diameter, $D(\theta)$, is

$$D(\theta) = \left[\left(\frac{\rho f'_m}{\sigma_y} \right)^2 \frac{\bar{M}(s_r, s_f, \theta_o, \zeta_o)}{\bar{M}(s_r, s_f, \theta, \zeta)} \right] D_M \quad [6.71]$$

and its orientation is

$$\theta_{i_2, i_1} = \tan^{-1} \pm \sqrt{\frac{-\bar{c}_2 + \sqrt{\bar{c}_2^2 - 4\bar{c}_1\bar{c}_3}}{2\bar{c}_1}} \quad [6.72]$$

where the constants, $\bar{c}_i$, ($i=1, 2$, and 3) are given by

$$\begin{Bmatrix} \bar{c}_1 \\ \bar{c}_2 \\ \bar{c}_3 \end{Bmatrix} = \begin{Bmatrix} \zeta^2 (s_r - s_f)^2 - \left(\frac{\rho f_m'}{\sigma_Y} \right)^2 \bar{M}(s_r, s_f, \theta_o, \zeta_o) \\ 2\zeta^2 (1 - s_f)(s_r - s_f) + (1 - s_r)^2 - 2 \left(\frac{\rho f_m'}{\sigma_Y} \right)^2 \bar{M}(s_r, s_f, \theta_o, \zeta_o) \\ \zeta^2 (1 - s_f)^2 - \left(\frac{\rho f_m'}{\sigma_Y} \right)^2 \bar{M}(s_r, s_f, \theta_o, \zeta_o) \end{Bmatrix} \quad [6.73]$$

The portions of unstable (*propagating*) and stable (*arrested*) cracks, can be obtained for the two possible scenarios of opening and sliding as before for the two proposed methods. Under these loading conditions, the kinked cracks portion η_k'' within the closed crack domain, can be calculated according to,

$$\eta_k^c = \frac{1}{\left(\frac{\pi}{2} - \theta_s \right)} \left\{ \begin{array}{l} (\theta_{k_2} - \theta_{k_1}) \\ (\theta_{k_2} - \theta_{k_1}) - \left(\frac{k_1 f_c'}{\sigma_Y} \right)^2 \bar{G}^2(s_r, s_f, \theta_o) \int_{\theta_{k_1}}^{\theta_{k_2}} \frac{d\theta}{\bar{G}^2(s_r, s_f, \theta)} \end{array} \right. \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [6.74]$$

and the propagating (sliding) closed cracks can be estimated according to

$$\eta_s^c = -\eta_k^c + \frac{1}{\left(\frac{\pi}{2} - \theta_s \right)} \left\{ \begin{array}{l} (\theta_{i_2} - \theta_{i_1}) \\ (\theta_{i_2} - \theta_{i_1}) - \left(\frac{\rho f_m'}{\sigma_Y} \right)^2 \bar{G}^2(s_r, s_f, \theta_o) \int_{\theta_{i_1}}^{\theta_{i_2}} \frac{d\theta}{\bar{G}^2(s_r, s_f, \theta)} \end{array} \right. \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [6.75]$$

Similarly, for the open crack domain the amount of evolving cracks may be calculated according to

$$\eta_s^i = \frac{1}{\left(\frac{\pi}{2} - \theta_s \right)} \left\{ \begin{array}{l} (\theta_{i_2} - \theta_{i_1}) \\ (\theta_{i_2} - \theta_{i_1}) - \left(\frac{\rho f_m'}{\sigma_Y} \right)^2 \bar{M}(s_r, s_f, \theta_o, \zeta_o) \int_0^{\theta_{i_1}} \frac{d\theta}{\bar{M}(s_r, s_f, \theta, \zeta)} \end{array} \right. \quad \begin{array}{l} \bullet \text{ A-S-K method} \\ \bullet \text{ exact method} \end{array} \quad [6.76]$$

Figure 6.11: Flow chart of the procedure of implementing the proposed FRC model.

All the above quantities can be used in Equation [3.127] through Equation [3.131] in order to evaluate the bridged crack compliance subjected to biaxial tensile-compressive loading.

6.8 CLOSURE

It should be observed that the fibre closing pressure, $\bar{\sigma}_v^f$, is a function of the as yet unknown crack opening displacements component, u_v . Furthermore, the crack displacements are themselves functions of the fibre closing pressure. Thus, the desired displacements and associated closing pressure can be obtained by iteration, starting the process with the approximate values obtained in the previous load step and/or iteration. The model equations may be implemented according to the flow chart shown in Figure 6.11 which is in fact a modified version of Figure 3.13 used in executing the plain concrete elasto-plastic damage model. The fundamental difference between the two charts is that crack orientation angles will keep on changing until the fibre closing pressure and associated crack opening displacements achieve self-consistency.

In the case of plain concrete, it was shown that the crack orientation angle depends only on the level of loading and the effective material elastic properties of the previous load step. Since fibres affect the crack sizes and orientation and redistribute the stresses on the neighboring concrete, this produces different evolution and propagation mechanisms of the preexisting cracks in fibre-reinforced concrete.

7. FIBRE-REINFORCED CONCRETE NUMERICAL TESTING

7.1 GENERAL

Numerical simulations of the mechanical behaviour of some fibre-reinforced concrete specimens and structures using the proposed elasto-plastic damage-based constitutive equations will be given in this chapter. The analyzed specimens here have been tested in the laboratory by other investigators. The purpose of the current test cases is the same as that stated earlier in Chapter 5. The same restrictions and assumptions stipulated for the examples shown in the latter chapter apply here with the exception of full bond assumption. In the following analyses, as described earlier in Chapter 6, short and/or long fibre will be allowed to slide and slip in the concrete material. Each analysis will include a brief description of each test and the original investigators remarks and observations about the behaviour of their specimens, a comparison between the experimental and analytical performance of the specimens and some concluding remarks.

The material parameters required in the analysis will be the same as those used in the plain concrete model with some additional parameters specific to the fibres and the properties of fibre-matrix interface. The additional parameters are radius, r_f and length, l_f of fibres, their volume fraction, V_f , their elastic modulus E_f , Poisson's ratio, ν_f , yield stress, σ_{f_y} , and yield strain, ϵ_{f_y} , as well as the shear strength, τ_o , of the fibre-matrix interface. Once again, the latter material parameters can be obtained from standard tests and do not require any assumption or data fitting because they are basic properties that can be measured systematically. It should be pointed out that the fracture toughness

required by the current model are those of cement paste (K_{IC}^c , and K_{IIC}^c) and the aggregate-cement interface (K_{IC}^f , and K_{IIC}^f), respectively. This is advantageous since all other current models of fibre-reinforced concrete and ceramic-reinforced composites require values of these parameters for the composite materials. Of course, one can experimentally determine the fracture toughness of the cement paste matrix and subsequently use the hierarchy laws given earlier in this thesis to compute the other factors.

It should be mentioned that the proposed pure damage model can be used to estimate the change in the apparent stress intensity factors of the different modes due to the fibres presence. The proposed method is also capable of simulating the behaviour of the customary pull-out specimen test (ACI 1966; 1991; Somayaji and Shah 1981). However, the numerical tests presented in the following sections will focus on investigating the overall mechanical deformational behaviour of fibre-reinforced concrete specimen, including beam and slab structures. The analytical results for these cases automatically show the model performance in predicting the fibre composite apparent stress intensity factors and the bond-slip characteristics of short and long fibres. Finally, it can be observed that the current model does not account for any chemical interaction between fibres and concrete matrix. Of course, some organic fibres may interact chemically with the surrounding cement paste during the hydration stage or during service if the structure is subjected to chemically hostile environments. These issues are neglected because they require further experimental and analytical studies in order to understand the mechanism of damage evolution due to chemical reactions.

7.2 FIBRE-REINFORCED CONCRETE SPECIMENS AND STRUCTURES

The discussions presented in the previous chapter have made it clear that fibre-concrete strength is a function of the fibre mechanical and geometric properties and its volume ratio, as well as the strength of the fibre-matrix interface. Particularly, the fibre-reinforced concrete uniaxial tensile and compressive strengths were shown to depend greatly on the de-bond length l_f . Of course, the fibres in a given concrete mix have to have an actual

length, l_f , that is greater or equal to the maximum de-bond length l_d . Therefore, it is very useful to establish the required minimum fibre length for enhancing the tensile and compressive strength of concrete materials. This specific characteristic is often referred to as the *fibre efficiency* parameter.

7.2.1 FIBRE-REINFORCED CONCRETE CYLINDERS SUBJECTED TO UNIAXIAL STRESS

The test specimens of Swamy and Al-Ta'an (1981) will be analyzed using the proposed method. The objective of their experimental test program was to investigate the effect of crimped low carbon steel fibres on the behaviour of concrete beams reinforced with cold-worked high tensile steel. The primary objective, however, was to see whether short fibres presence will help to increase the ultimate concrete strain and hence allow the normal reinforcement to reach its yielding stress. The normal reinforcement were made of high strength steel with yield strength more than 700MPa . Since, the current treatment deals with studying the impact of crimped steel fibre presence on the strength of concrete, the beams structural and geometrical details will not be discussed here. However, their properties and all the pertinent data for the loading and their observed behaviour will be given later in this chapter.

Table 7.1: Concrete material properties of Swamy and Al-Ta'an(1981) test.

#	1	2	3	4	5	6					
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal			D	D_{Min}	Cement-paste		Interface		Initial de-bonded interfaces
	E_o^p	E_o^n	Mode-I	Mode-I			Mode-II				
	(GPa)	(GPa)	ν	μ	(mm)	(mm)	K_I^c	K_I^p	K_{II}^p	ρ	ω_o
Value	28.13	28.13	0.20	0.6	20.00	6.50	0.578 [*]	0.165 [*]	0.324 [*]	0.98	0.0800

The material properties of the plain concrete mix used in the current analysis, as provided by the original authors, are given in Table 7.1. It should be remembered that the different fracture toughness were calculated using Equation [5.1]. The concrete mix was made of the portions of 1.00 (cement + fly ash): 2.00 (sand): 2.50 (aggregate) with water-(cement-fly ash) ratio of 0.52, all by weight. The sand used in all mixes was a natural river sand with fineness modulus of 2.78 while the coarse aggregate was graded river gravel

with maximum aggregate size of 20mm . The shape of the coarse aggregate varied from round to angular, with fineness moduli of 6.52 to 6.65. The cement used in all the mixes was ordinary Portland cement while the fly ash had a specific surface of $369 \times 10^3 \text{mm}^2/\text{g}$. The selected fly ash ratio substituted 30% by weight of cement which is believed to insure that concrete material will have optimum strength and elastic properties.

On the other hand, the crimped low carbon steel fibre properties are shown in Table 7.2. This table shows that only two fibre volume fractions were used in the above concrete mix. To enhance the workability of the fibre-reinforced concrete mixes, a water-reducing agent was used with them at a dosage of 3.2 cc/kg of (cement-fly ash). The slump of plain and fibre-reinforced concrete, with 0.5% and 1% fibre volume fractions, were 130, 100 and 80 mm , respectively. It should be stated that the original authors designed these mixes in order to obtain compressive strength of 35 MPa for both concrete types.

Table 7.2: Fibre material properties of Swamy and Al-Ta'an (1981) test.

#	1	2	3	4	5	6	
Material properties	Elastic moduli E_f (GPa)	Yield or fracture stress σ_{fy} (MPa)	Poisson's ratio ν	Fibre volume fraction(s) V_f (%)	Fibre dimensions		Interface shear strength τ_f (MPa)
					r_f (mm)	l_f (mm)	
Value	210	1050	0.20	0.50	0.25	50.00	2.46 [†]
				1.00			

[†] Based on the lower tensile strength value of plain concrete provided by the original authors

The finite element method in conjunction with the proposed elasto-plastic damage-based model was employed to analyze the mechanical behaviour of plain and fibre-reinforced concrete cylinders tested by Swamy and Al-Ta'an (1981). The same finite element type and procedures described in Chapter 4 and Chapter 5 are used here. The analytical results showed that addition of fibres enhances the over all tensile and compressive resistance to applied mechanical loads, provided they have sufficient length to prevent pull-out. The analytical results also showed that the increase of the apparent concrete compressive strength was not as significant as the tensile strength of concrete.

The latter was increased by about 20% for a fibre volume fraction of 1% while the former gained a 2% increase.

The analytical prediction of apparent tensile strength is plotted and compared with its experimental counterpart in Figure 7.1. The results of this figure show the variation of the tensile strength as function of the fibre volume fraction, V_f . It can be noted that the

Figure 7.1: Tensile stress vs. fibre volume fraction, V_f .

shows that the proposed model is capable of predicting the ultimate strength of fibre-reinforced concrete materials with good accuracy for the fibre volume fraction used, regardless of the value of the shear strength of the matrix-fibre interface. The curves shown in that figure were normalized with respect to the plain concrete tensile strength $f_c' = 4.12 \text{ MPa}$. The normalized experimental values are also plotted on the same curve. Furthermore, the horizontal axis in Figure 7.1 ranges from zero to 5% fibre volume fraction content because 5% is the maximum practical steel fibre ratio that may be used in building actual fibre-concrete reinforced structures. Of course, using higher volume fraction will introduce difficulty in the workability and placing of the concrete mix in the desired form and it will significantly reduce the concrete slump. Therefore, more water will be needed in order to circumvent these problems, this will have an adverse effect because it would weaken concrete and increase the volume of the bleeding cracks in the

strength variations were drawn in the latter figure for different values of the shear interface strength, τ_i . This is because the experimental program did not measure this property. Thus, from the preceding results an appropriate value for the shear strength, τ_o , can be selected for use in the analyses of some of the beam structures tested in the same program. However, Figure 7.1

final product. It should be pointed out that these curves show nonlinear relationship between fibre volume fraction, V_f , and the tensile strength, f'_t , for a given shear strength of the matrix-fibre interface, τ_i . If these curves are extended to larger fibre volume fractions, their nonlinearity will be noticeable.

In order to establish the fibre efficiency parameter, i.e. $\frac{l_f}{r_f}$, let us examine the variation

of the de-bond length, l_d , versus the shear strength of the matrix-fibre interface, τ_i . Figure 7.2 shows this relationship which indicates that for weak interfaces, the fibres must be long enough in order for adequate shear resistance to be developed. It should be remembered that the curves

Figure 7.2: Fibre deformation relationship ($l_d - \tau_i$).

shown in Figure 7.2 was calculated for the fibre diameter of 0.5 mm given in Table 7.2. The figure also shows that the utilized fibres will be efficient in strengthening and toughening the concrete matrix if the fibre-matrix interface shear strength, τ_i is about 2.46 MPa. However, one might argue that the shear strength of concrete should be of the order of $\frac{f'_t}{2}$. It is interesting to mention that the very weak plain concrete will produce a value close to the results of the latter expression of shear strength. Therefore, it can be concluded that these low carbon steel fibres will improve the strength characteristics of the concrete materials provided the fibre-matrix interface shear strength, τ_i is not less than 2.46 MPa.

It was stated earlier that an appropriate theoretical model should not only be able to predict the ultimate strength of the material but it should also capture its overall

deformational response under mechanical loading. It is well known that the deformation of any concrete material depends greatly on the initial elastic stiffness of the virgin concrete. Fortunately, Swamy and Al-Ta'an (1981) measured and reported the actual values of the moduli of elasticity of the virgin plain and fibre-reinforced concrete mixes. Table 7.3 compares the prediction of the proposed model for the modulus of elasticity with the experimental values as well as with those obtained by the rule of mixture. The results of this table shows that the proposed model and the rule of mixture give good estimation of the elastic moduli, but the proposed model gives better prediction than those obtained from the rule of mixture. It was mentioned earlier that Razaqpur and Nofal (1992) used the rule of mixture to obtain the initial elastic properties of fibre-reinforced concrete. Subsequently, they employed the plain concrete elasto-plastic damage-based model given in Chapter 3 in order to trace the full load-deformation characteristics of some fibre-reinforced concrete beams and slabs. Their analytical predictions were in good agreement with the corresponding experimental results. This may be attributed to the observation made with respect to the results of Table 7.3 that showed that the rule of mixture may give approximate answers to some of the less important material parameters.

Table 7.3: Comparison between analytical and experimental elastic modulus.

Fibre volume fraction (%)	Virgin Concrete Elastic Modulus (GPa)			predicted / experimental	
	Proposed Model	Rule of Mixture	Experimental	Proposed Model	Rule of Mixture
0.0	28.130	28.130	28.130	1.000	1.000
0.5	28.777	29.040	28.350	1.015	1.024
1.0	29.424	29.950	29.820	0.987	1.004

The above discussions and considerations shows that the proposed model is promising and fibres must have a minimum length, l_{min} , to be effective in strengthening and toughening the concrete matrix. If the fibres used have a length less than the minimum length, they will be ineffective and will not contribute to the strength resistance of fibre-reinforced concrete materials. In the following sections, analyses of fibre-reinforced concrete cubes, beams and slabs, using the proposed technique will be presented and

discussed. The slab structures are selected to test the computer program, Nonlacs, capabilities in modeling plate bending problems since all problems tested earlier in this thesis were plane stress problems.

7.2.2 FIBRE-REINFORCED CONCRETE CUBES

Razaqpur and Nofal (1992) used the experimental results of Abdull-Ahad and Abbas (1989) to test the performance of their pure fibre-reinforced concrete damage model. The same problem will be analyzed here using the proposed elasto-plastic damage-based model for FRC materials in order to establish its capabilities of tracing the full nonlinear stress-strain responses of fibre-reinforced concrete materials. This experimental program investigated the behaviour of steel fibre-reinforced concrete square panels subjected to axial and biaxial stress fields.

Table 7.4: Concrete material properties of Abdull-Ahad and Abbas(1989) panels.

#	1	2	3	4	5	6						
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters		
	Parallel	Normal	ν		μ	D_M	D_{Min}	Cement-paste		Interface		Initial de-bonded interfaces: ρ
	E_o^p (GPa)	E_o^n (GPa)		Mode-I K_I^C ($\text{MN/m}^{3/2}$)				Mode-II K_{II}^C ($\text{MN/m}^{3/2}$)	Mode-III K_{III}^C ($\text{MN/m}^{3/2}$)			
	Value	20.00	20.00	0.30	0.6	20.00	6.50	0.525*	0.160*	0.320*	0.85	0.075

The specimens were manufactured from plain and fibre-reinforced concrete with mix proportions of 1:1.5:2.5 cement : sand : aggregate by volume. Three different steel-fibres volume fraction, $V_f = 0.5\%$, 1.0% , 1.5% and two aspect ratios (fibre length/diameter, $l_f/d_f = 60, 100$) were employed. The concrete and steel fibres material properties are shown in Table 7.4 and Table 7.5, respectively.

The tested specimens were constructed as square panels with dimension $300 \times 300 \times 75 \text{ mm}$. The geometrical shapes and dimensions of the test specimens are shown in Figure 7.3. It can be observed from Figure 7.3(a) that the tensile specimens were stiffened at the edges where the tensile loads are applied in order to ensure that the cracks and their evolution are confined to the middle portion of the panels. It should be

mentioned that the reported results of the experimental program were only for concrete that had fibre volume fraction, $V_f = 1.5$ and fibre-diameter ratio, $l_f / d_f = 100$. Therefore, the present analysis will be restricted to that case.

Table 7.5: Fibre material properties of Abdull-Ahad and Abbas(1989) test

#	1	2	3	4	5	6
Material properties	Elastic moduli	Yield or fracture stress	Poisson's ratio	Fibre volume fraction (%)	Fibre dimensions:	
	E_f (GPa)	σ_{fy} (MPa)	ν	V_f (%)	r_f (mm)	l_f (mm)
Value	200	400	0.30	0.50 [*]	0.25	30.00 [*]
				1.00 [*]		50.00 [*]
				1.50 [*]		50.00 [*]
						Interface shear strength τ_f (MPa)
						2.00 [†]

† assumed

* different values are used

The panels, subjected to overall compressive loading, were modelled by a single finite element mesh. For panels loaded by any tensile loading, a typical quarter of the panel shown in Figure 7.3(a) was divided into several plane RQD4 finite elements (Razaqpur and Nofal 1990). This strategy is adopted in order to properly model the stiffened edges of these panels. Figures 7.4 compares the stress-strain curves for three different stress states and various stress ratios from the proposed analysis to the corresponding experimental curves. In addition, Figure 7.4(d) compares the predicted failure envelope of the panels with the corresponding experimental values. A good agreement between the two sets of results for all the considered load cases can be observed.

It is interesting to mention that Razaqpur and Nofal (1992) also analyzed these panels using their damage model described earlier in Chapter 6. The Razaqpur and Nofal's (1992) damage model differs from the proposed theory in that it assumes full bond between the fibres and the concrete matrix and neglects the closing effects of bridging fibres on the cracks opening displacements. Nevertheless, their results were in good agreement with the corresponding experimental data. Therefore, they concluded that fibre presence did not impact the global deformational behaviour within the ascending branch of the stress-strain curve. The current analysis showed that the fibres were short and were able to produce very small closing pressure. Therefore, the computed crack compliances were close to those of plain concrete. This in return indicated that the current model is also

capable of tracing the behaviour of the embedded ineffective short fibres. The numerical results also showed that fibre stresses never exceeded their yield strength and hence their contributions to the overall stiffnesses of the cracked panels remained constant throughout the test up to

Figure 7.3: Dimensions and topology of panel specimens (Abdull-Ahad and Abbas 1989).

failure. This indicates that all the observed nonlinearity was due to the crack evolution and propagation. Finally, it should be mentioned that when damage mechanics could not trace the inelastic behaviour of these panels under certain biaxial compression ratio (e.g. $\sigma_1 > \sigma_2 > 0$) with $\sigma_1 \geq 0.3\sigma_2$, the theory of plasticity was employed to evaluate the nonlinear response.

Figure 7.4: Results of fibre-reinforced concrete panels (Abdull-Ahad and Abbas 1989).

The accumulated amount of damage, ω' , due to the application of loads are depicted in Figure 7.5. Once again, the applied stresses were normalized to either the concrete tensile, f'_t , or compressive, f'_c , strength, respectively. The experimental data points were computed from the stress-strain diagrams given by the original authors while the analytical curves were obtained from the finite element program, Nontacs. Figure 7.5 shows the effect of the type of loading (e.g. tension versus compression stress) and the impact of

fibre volume fraction, V_f , on the amount of damage due to the applied loading. The predicted results are shown in these figures up to the ultimate load of the different specimens because the current theoretical treatment cannot simulate strain softening phenomena.

Figure 7.5: Accumulated damage, ω' , versus applied stresses.

It can be observed from Figure 7.5 that tensile loading produces more damage than compressive loading as in the case of plain concrete. Furthermore, the latter figure illustrates clearly that the introduction of fibres reduces the damage accumulation and propagation in the tested specimens. This shows that fibres indeed suppress the crack displacements and evolution, improve the deformation processes, and reduce the damage due to applied loads. It should be mentioned that the proposed model can be used to establish the optimum fibre lengths and diameters as well as its volume fraction for a specific concrete mix. These optimum values may ensure better performance of the fibre-reinforced concrete structures made of such a mix. However, it should be clear that these optimum values will be valid for a given loading configuration and sequence. These issues require further study and more refinements in the proposed constitutive elasto-plastic damage-based model of fibre-reinforced concrete materials.

Of course, one set of experimental results for the stress-strain is not sufficient to test the proposed constitutive equations and validate the procedures given in the previous chapter. However, testing whole structure(s) will also provide critical assessment of the accuracy of the equations that describe the stress-strain relationship. In the following sections some fibre-reinforced concrete beams and slabs will be tested using the proposed elasto-plastic damage-based model for fibre-reinforced concrete materials. The concrete materials of some of the following structures were reinforced by pitch-based carbon fibres while steel fibres were used in the rest. The steel fibres were made of low-carbon and normal steel extracts. Once again, the analytical results given later in this chapter will not consider any chemical interaction.

7.3 FIBRE-REINFORCED CONCRETE STRUCTURES

Existing old and/or damaged concrete structures require rehabilitation as an alternative to demolition and replacement of the entire structure. It is common practice to repair structural concrete elements damaged by earthquake or hurricane. Recent experimental studies on repaired simply supported reinforced concrete beams (e.g. Andrews and Sharma 1988) have shown the effectiveness of using ferrocement in repairing beams failing in flexure. On the other hand, the results of Andrews and Sharma's (1990) study showed substantial improvement in the behaviour of beams failing in shear when steel fibres in combination with conventional reinforcing steel bars were used in repairing these beams. Therefore, it is very important to investigate the behaviour of structures made of fibre-reinforced concrete.

7.3.1 FIBRE-REINFORCED CONCRETE BEAMS

Most recent experimental work has dealt with testing the behaviour of simple or continuous beams subjected to either static or dynamic loads. This is due to the relative simplicity of executing the tests and analyzing the results of such beams. Moreover, the stress field of these beam structures can be evaluated using the customary engineering beam theory where cracks and loss of stiffnesses are accounted for. Thus, the following

sections will be concerned with modelling the mechanical behaviour of beams made of fibre-reinforced concrete. The fibres types used in these beams are pitch-carbon or steel.

7.3.1.1 SIMPLY SUPPORTED STEEL FIBRE-REINFORCED CONCRETE BEAMS

Fibre-reinforced concrete was employed in its earlier application in repairing existing residential apartment buildings. It is customary to use beams and columns as the main structural elements in these types of structures. Therefore, early experimental investigations (e.g. Swamy and Al-Ta'an 1981; Andrews and Sharma 1988, etc.) were concerned with testing beams subjected to static and dynamic bending loads. These beams were reinforced with conventional steel tensile reinforcements. Recently, however, investigators have tested fibre-reinforced concrete beams with or without steel reinforcement.

The reinforcement materials are made of newly developed high performance composite materials. Due to space limits, in the following the beams from three experimental programs will be modelled analytically using the proposed elasto-plastic damage-based constitutive model. The first two test sets are for fibre-reinforced concrete beams reinforced with normal steel reinforcement. One of the objectives of the tests of Swamy and Al-Ta'an (1981) was to investigate suitability of using fibres in the tensile zone of beams. The third experiment is devoted to fibre-reinforced concrete beams without conventional steel. The latter tests were focused on the performance of fibres made from materials other than steel.

7.3.1.1.1 SHEFFIELD UNIVERSITY STEEL FIBRE-REINFORCED CONCRETE BEAMS

The experimental work of Swamy and Al-Ta'an (1981) comprised 15 beams made of plain or fibre-reinforced concrete. These beams were size $130 \times 203 \times 2500 \text{ mm}$ and were loaded with a central concentrated load (see Figure 7.6). The test variables were the type and amount of tensile reinforcement as well as the volume fraction, V_f , of the added low-carbon steel fibres. These short fibres were provided either over the whole depth of the beam or in the effective tension zone only. The reinforced concrete beams were divided

into three groups, each comprising five beams, with different type and amount of tensile steel reinforcement (see Figure 7.6). Within each group, three types of concrete was used, with one beam, called A, made of plain concrete, two beams, named B, made of fibre-reinforced concrete and the remaining two beams, C, containing fibres in the effective tension zone only (see Figure 7.6).

Figure 7.6: Simply supported fibre-reinforced concrete beam (Swamy and Al-Ta'an 1981).

Table 7.6: Steel material properties used for Swamy and Al-Ta'an (1981) beams.

Material properties	ordinary reinforcing bars (torsteel)						ordinary reinforcing bars (high tensile steel)					
	1		2	3	4		1		2	3	4	
	Elastic moduli		Poisson's ratio	yield stress	steel ratios		Elastic moduli		Poisson's ratio	strands strength	prestress ratios	
	x-axis E_{sx} (GPa)	y-axis E_{sy} (GPa)	ν_s	f_y (MPa)	x-axis ρ_{sx} (%)	y-axis ρ_{sy} (%)	x-axis E_x (GPa)	y-axis E_y (GPa)	ν_s	f_{ps} (MPa)	x-axis ρ_x (%)	y-axis ρ_y (%)
Value	200.0	200.0	0.20	460.0	0.99 [*] 1.78 [†]	N/A	200.0	200.0	0.20	617.0	0.94 [‡]	N/A

* Group I tensile steel reinforcement ratio

† Group II tensile steel reinforcement ratio

‡ Group III tensile steel reinforcement ratio

The concrete material properties for these beams are given in Table 7.1 and Table 7.2. The material properties of the two types of tensile steels are given in Table 7.6 where they

differ only in their yield stresses, σ_y . It should be mentioned that provided reinforcement was about 50% of the balanced steel ratio which ensured tension failure of the tested beams. All beams were cast in steel molds and were cured under uncontrolled internal environment. The actual testing of these beam specimens was executed after 28 days.

Prior to investigating the impact of the fibre presence on the behaviour of these beam, let us examine the effect of the two longitudinal high strength and torsteel reinforcing bars. Of course, the earlier model developed in Chapter 3 and tested in Chapter 5 can be used to analyze the plain reinforced concrete beams. But it is essential to test the proposed elastoplastic damage-based model performance for plain concrete behaviour, i.e when fibre content, V_f , is 0%.

Figure 7.7: Deformation characteristics of concrete beams DR10-DR14.

Figure 7.7 shows a comparison between the load-deflection and moment-end support rotation curves for beams DR10, DR20, and DR30, respectively. These comparisons show that the predicted deformational behaviour of beam DR10 is a little stiffer than its experimental counterpart. On the other hand, the analytical and experimental responses of beams DR20, and DR30 are in good agreement. It is obvious that the use of high tensile steel increased the ultimate strength of the beams. It should be mentioned that the

experimental results for beam DR30 were not completely reported. Consequently, the analytical behaviour of the latter beam was not compared with any experimental values with the exception of the points corresponding to the first cracking, service, and failure loads, respectively. These are identified by the solid triangular symbol in Figure 7.7. From the comparisons in Figure 7.7 one can conclude that the proposed elasto-plastic damage-based model for an FRC material is indeed identical, when the fibre volume fraction is zero, to the other model proposed for plain concrete earlier in this thesis.

Table 7.7: Analytical and experimental deformation characteristic of beams at service and ultimate loads.

Beam #	Response	DR10		DR20		DR30	
		Analysis	Exp.	Analysis	Exp.	Analysis	Exp.
Deflection (mm)	service	4.25	4.36	5.10	5.15	4.75	4.79
	Failure	4.36	36.87	32.00	32.53	39.00	39.15
Total Rotation (rad) $\times 10^3$	service	11.00	10.85	13.30	13.34	13.15	13.18
	Failure	78.00	78.24	65.00	65.16	57.50	57.60
Curvature (rad) $\times 10^3$	service	12.86	12.77	16.25	16.38	13.50	13.69
	Failure	187.50	186.74	148.00	148.30	166.67	167.72
Steel strain (mm/mm) $\times 10^6$	service	1120	1110	1040	1070	1200	N.A. [!]
	Failure	9500	9300	10000	12050	9400	N.A. [!]
Concrete strain (mm/mm) $\times 10^6$	service	666	660	1200	1210	715	720
	Failure	5400	5380	5500	5650	5500	5640

* All beams made of concrete type, A and have 0% fibres.

! Strain gauges were defective and inoperative during actual experiment

The analytical and experimental deformational behaviour at service and ultimate loads for the latter three beams are compared in Table 7.7. This comparison shows that the proposed analytical procedure is capable of producing results that are in good agreement with the actual behaviour of typical beam structures. The maximum difference between the experimental and analytical predictions is about 4% which is acceptable for all practical purposes. The important observation is that the current model predicted the behaviour of these beams immediately after the cracking and contrary to most models did not show a sudden loss in the stiffness. This is because the current model does not reduce the cracked

concrete stiffness to zero as most other concrete models, rather it allows gradual degradation of stiffness with increasing applied external loads.

In the following the impact of fibres on the behaviour of similar beams will be analyzed and the performance of the proposed model will be examined. It should be mentioned that the original reference reported the results of groups, DR1n, and DR3n, where $n = 1, 2, 3$, and 4, respectively. The results of the other group, DR2n, was not reported and therefore, the discussion given below is confined to the two former groups.

Table 7.8 compares the results at first crack loading for group, DR1n, while Figures 7.8 show the deformation characteristics of the same beams. The two illustrations show clearly that the proposed analytical model is capable of predicting the deformational response and the ultimate load capacities of FRC beams reinforced with main tensile reinforcement. The prediction of the model at early stages of loading up to the onset of cracking is rather excellent as shown in Table 7.8. This table shows that the analytical results obtained from employing the proposed elasto-plastic damage-based model are almost identical to the corresponding experimental values.

Table 7.8: Analytical and experimental deformation characteristics of Swamy and Al-Ta'an (1981) beams.

Beam # (V_f)	DR10(0.0%)		DR11(0.5%)		DR12(1.0%)		DR13(0.5%)		DR14(1.0%)	
	Analysis	Exp.	Analysis	Exp.	Analysis	Exp.	Analysis	Exp.	Analysis	Exp.
Deflection (mm)	1.17	1.17	0.84	0.84	0.51	0.54	0.80	0.82	0.75	0.76
Total Rotation (rad) $\times 10^3$	2.95	2.99	1.75	1.75	1.15	1.16	2.02	2.04	1.45	1.45
Curvature (rad) $\times 10^3$	2.52	2.53	1.35	1.31	1.29	1.33	2.09	2.14	2.10	2.10
Steel strain (mm/mm) $\times 10^6$	262	262	75	71	58	62	155	160	N.A. [!]	N.A. [!]
Concrete strain (mm/mm) $\times 10^6$	178	180	104	100	108	110	158	160	150	160

* fibre added in the effective tension zone only

! Strain gauges were defective and inoperative during actual experiment

The same conclusion is further supported by observing the good agreement over the whole load-deformation curves given in Figure 7.8. According to the results of this specific group, the introduction of fibres improves the overall response and it increased the ultimate loading carrying capacity of the beams. Addition of short low-carbon steel fibres

to the concrete skin and the effective tension zone also proved to be beneficial in retarding the propagation of the tensile cracks. This behaviour is very useful because rehabilitation and/or restoration of an existing damaged (i.e. cracked) structure(s) can be feasible and economical.

Figure 7.8: Deformation characteristics of concrete beams DR10-DR14.

Similar results were obtained for the other group, DR3x, which are shown graphically in Figures 7.9 while Table 7.9 displays some specific values of the beams deformation characteristics corresponding to their service and ultimate load levels, respectively. Once again, the same good agreement between the two sets of experimental and analytical behaviours can be observed. Figure 7.10 compares the load-concrete maximum compressive strain curves for the two groups, DR10-14 and DR10-34, respectively. Figure 7.11, on the other hand, shows typical load-steel strain behaviour of those beams in the same groups which contain fibre-reinforcement and steel bars. The agreement between the two sets of analytical and experimental results is clear which demonstrates the

performance of the proposed elasto-plastic damage-based model in capturing the overall response of similar structures subjected to mechanical loads.

Table 7.9: Analytical and experimental maximum deformation of Swamy and Al-Ta'an (1981) beams

Beam # (V_f)	Response	DR31(0.5%)		DR32(1.0%)		DR33(0.5%)		DR34(1.0%)	
		Analysis	Exp.	Analysis	Exp.	Analysis	Exp.	Analysis	Exp.
Deflection (mm)	service	4.15	4.16	3.95	3.98	4.35	4.32	4.10	4.11
	Failure	44.00	44.00	31.32	31.41	38.90	38.66	38.90	38.84
Total Rotation (rad) $\times 10^3$	service	9.50	9.50	8.90	8.97	10.00	9.98	9.20	9.16
	Failure	70.40	70.39	66.28	66.32	61.75	61.67	61.00	60.21
Curvature (rad) $\times 10^3$	service	11.75	11.77	7.60	7.74	14.00	13.89	12.60	12.68
	Failure	132.00	132.05	155.00	154.68	151.80	151.55	143.50	143.34
Steel strain (mm/mm) $\times 10^6$	service	1410	1400	1280	N.A.	1820	1805	1620	1600
	Failure	10000	6000	7500	N.A.	8000	7200	7500	7300
Concrete strain (mm/mm) $\times 10^6$	service	750	750	545	540	718	720	716	720
	Failure	5225	5240	6000	5970	5400	5420	5400	5420

* fibre added in the effective tension zone only

! Strain gauges were defective and inoperative during actual experiment

Figure 7.9: Deformation characteristics of concrete beams DR30-DR34.

Figure 7.10: Load-concrete compressive strain curves of (a) beams DR10-DR14 (b) beams DR30-DR34

Figure 7.11: Load-steel tensile strain curves of (a) beams DR10-DR14 (b) beams DR30-DR34

Figure 7.12: Comparison between the current and Razaqpur and Nofal (1990) damage models

Finally, we compare the results of the proposed model with that of Razaqpur and Nofal (1990) in Figure 7.12. It should be remembered that the latter model is the exact model presented in Chapter 3 except that the virgin elastic modulus of concrete is replaced by the effective elastic modulus of fibre-reinforced concrete. The effective elastic modulus can be calculated by employing the rule of mixture. This method was also used to obtain the apparent fracture intensity factors for the different modes of deformations. Observe that the Razaqpur and Nofal's model shows stiffer behaviour than the experimental and the current model from zero up to the first cracking load. Beyond this load, the former model shows softer response which is the result of ignoring the effect of the fibre closing pressure on reducing the crack opening displacements and hence their compliance. The current model, however, is capable of capturing the response at all loading stages because it calculates the effects of fibre closing pressure for the propagating cracks.

7.3.1.1.2 SEOUL UNIVERSITY STEEL FIBRE-REINFORCED CONCRETE BEAMS

Oh *et al.* (1989) tested a number of rectangular beams simply supported and loaded with third point loads (see Figure 7.13). The beams, as shown in Figure 7.13, were either singly or doubly reinforced with conventional mild steel bars in addition to up to 2% steel fibres. Diagonal tension reinforcement was also provided within the shear span. Typical

structural and load arrangement details are shown in Figure 7.13. The details of cross-sections, reinforcement arrangements and ratios, as well as the fibre volume fractions used in these beams are shown in Table 7.10

Figure 7.13: Seoul National University reinforced concrete-fibre steel beams.

Table 7.10: Soul National University beam cross section details (Oh *et al.* 1989)

Series	Test series 1 (S1VN)	Test series 2 (S2VN)	Test series 3 (S3VN)
Cross sections & reinforcements			
Reinforcements ratio (%)	$\rho = 0.40 \rho_b$	$\rho = 0.65 \rho_b$	$\rho = 0.90 \rho_b$, $\rho' = 0.0085$
Fibre contents (%)	0% (normal concrete), 1%, and 2%		

The concrete material properties are listed in Table 7.11 while Table 7.12 shows the fibres and fibre-concrete interface properties. It should be remembered that the fibres used were rather long with length of 40 mm and radius of 0.35 mm. The Young's modulus,

Poisson's ratio, and yield strength of the main steel reinforcement material are 200 *GPa*, 0.3, and 420 *MPa*, respectively. Beyond yield, the reinforcement is assumed to have a hardening modulus of 20 *GPa*. The reinforcements ratios and details were given earlier in Figure 7.13 and Table 7.10.

Table 7.11: Concrete material properties of Oh et al. (1989)

#	1	2	3	4	5	6						
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction		Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal	ν	μ	D M	D Mitr	Cement-paste		Interface		Initial de-bonded interfaces ρ	Initial damage ω _o
	E _o ^p	E _o ⁿ					Mode-I	Mode-I	Mode-II			
	(GPa)	(GPa)					K _I ^a (MN/m ^{3/2})	K _I ^a (MN/m ^{3/2})	K _{II} ^a (MN/m ^{3/2})			
Value	18.94	18.94	0.20	0.6	15.00	7.50	0.575 [*]	0.165 [*]	0.324 [*]	0.98	0.0800	

Table 7.12: Fibre and fibre-concrete interface material properties of Oh et al. (1989)

#	1	2	3	4	5	6	
Material properties	Elastic moduli	Yield or fracture stress	Poisson's ratio	Fibre volume fraction(s)	Fibre dimensions		Interface shear strength
	E_f	σ_{fy}	ν	V_f	r_f	l_f	τ_i
	(GPa)	(MPa)		(%)	(mm)	(mm)	(MPa)
Value	200	1303	0.30	1.00	0.70	40.00	2.50 ¹
				2.00			

! Based on the lower tensile strength value of plain concrete provided by the original authors

Due to symmetry, half of each beam was divided into 10 finite elements along the span and three along the depth (see Figure 7.14). The longitudinal reinforcement was modelled as bar elements as shown in the latter figure while the shear reinforcement was smeared into a layer that is fully bonded.

The load midspan-deflection curves of the reinforced beams are shown in Figures 7.15. We observe the close agreement between the experimental and numerical results for the two groups. Due to lack of experimental data, we are unable to compare other quantities of interest, such as strain and curvature and damage. The results for the doubly reinforced beams compared equally well as shown in Figure 7.16.

Figure 7.14: Finite element idealization for Seoul National University beams.

Figure 7.15: Load-deflection curves of Seoul National University beams.

According to the previous results, the inclusion of the fibres enhanced the overall response, increased the ductility of the beams, and increased their ultimate load carrying capacity. This in return indicates the usefulness of using fibre-reinforced concrete with high strength longitudinal reinforcement in beam structures. It should be remembered that

the above conclusions are drawn for fibre-reinforced concrete manufactured with low-carbon steel fibres. In the subsequent section, analyses of un-reinforced carbon fibre-reinforced concrete will be presented. It should be understood that the term un-reinforced here refers to absence of longitudinal steel reinforcement.

Figure 7.16: Load-deflection curves of Seoul National University series 3 (S3VN) beams

7.3.1.1.3 LAVAL UNIVERSITY CARBON FIBRE-REINFORCED CONCRETE BEAMS

Ohama *et al.* (1985); Banthia and Sheng (1990) and Banthia (1992) constructed a number of rectangular concrete prisms reinforced with pitch-based carbon fibres. Fibre contents of 1%, 3%, and 5% were used, while some control beams had no fibres. The beams were simply supported and tested under a central point load. Their dimensions and loading are shown in Figures 7.17.

Using symmetry, half of each beam was modelled by eighteen 4-node plane stress elements, eight along the span and two through the height. Each element had three nodal degrees of freedom, two inplane displacements and a drilling rotation. The finite element mesh is given by Figure 7.17(c). The plain concrete material properties as input in the

numerical model are given in Table 7.13 while the fibres and fibre-concrete matrix interface properties are shown in Table 7.14.

Figure 7.17: Laval University CFRC Carbon fibers beams (a) Ohama *et al.* (1985) (b) Banthia and Sheng (1990) and (c) utilized finite element mesh.

Table 7.13: Concrete material properties of Laval University beams.

#	1	2	3	4	5	6					
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction	Aggregate size		Fracture toughness			Initial damage parameters	
	Parallel	Normal	ν	μ	D _M	D _{Min}	Cement-paste	Interface		Initial de-bonded interfaces ρ	Initial damage ω
	E _o ^p	E _o ⁿ					Mode-I	Mode-I	Mode-II		
	(GPa)	(GPa)					K _I ³² (MN/m ^{3/2})	K _I ³² (MN/m ^{3/2})	K _{II} ³² (MN/m ^{3/2})		
Value	16.60	16.60	0.20	0.6	18.00	7.0	0.550 [*]	0.275 [*]	0.380 [*]	0.975	0.075

Table 7.14: Fibre and fibre-concrete interface material properties of Laval University beams.

#	1	2	3	4	5	6
Material properties	Elastic moduli	Yield or fracture stress	Poisson's ratio	Fibre volume fraction(s)	Fibre dimensions	
	E_f (GPa)	σ_{fy} (MPa)	ν	V_f (%)	r_f (mm)	l_f (mm)
Value	38	800	0.15	0.00	0.70	40.00
				1.00		
				3.00		
				5.00		
						τ_f (MPa)
						2.50 [†]

[†] Based on the lower tensile strength value of plain concrete provided by the original authors

The results of the analyses are compared with the available experimental data in Figure 7.18. It can be observed that except for one of the beams with 3% fibre, Figure 7.18, the numerical and experimental results agree well in the other cases. As mentioned earlier, the proposed elasto-plastic damage-based method does not consider strain softening. The analysis showed that the provided fibres enhanced the crack behaviour but because of their low strength they failed at rather early load stages.

Figure 7.18: Results of Laval University CFRC Carbon fibers beams (Banthia and Sheng 1990).

Finally, in the case of the one beam, for which the agreement is not as good, the experimental results appear to be anomalous, for the beam with 3% fibre content appears

to be stiffer than the one with 5% fibre. The same is not revealed by the experimental data of Ohama *et al.* (1985) in Figure 7.19. In the latter case, the beams behaved as expected because increasing carbon fibre volume fraction accordingly enhanced the behaviour of the beams. Because there were no repeat specimens for those beams, we ascribe the above lack of agreement to experimental variability rather than to the shortcoming of the model.

Figure 7.19: Results of Laval University CFRC Carbon fibers beams (Ohama *et al.* (1985).

7.3.2 FIBRE-REINFORCED CONCRETE SLABS

So far throughout this thesis, all structures and specimens analyzed were subjected to loads that produce plane stress field. Thus, it is very important to examine the proposed model(s) in simulating plate bending stress field. Therefore, we examine numerically the work of Theodorakopoulos and Swamy (1989) who tested a number of square slabs built monolithically with a central stub column and simply supported around the periphery. The slabs were made of lightweight concrete and were reinforced with an orthogonal mesh of mild steel reinforcement near the top and bottom faces. The stub columns also had steel reinforcement. The slabs were loaded with a central point load over the column. Figure

7.20 shows the complete details of a typical test specimen, including its dimensions and reinforcement arrangement.

Figure 7.20: Geometry and structural details of University of Sheffield fibre steel slabs.

Table 7.15: Concrete material properties of University of Sheffield slabs.

#	1	2	3	4	5	6						
Material properties	Elastic moduli		Poisson's ratio	Coefficient of dry friction		Aggregate size	Fracture toughness			Initial damage parameters		
	Parallel E_o^p (GPa)	Normal E_o^n (GPa)	v	μ	D_M (mm)	D_{Min} (mm)	Cement-paste		Interface		Initial de-bonded interfaces ρ	Initial damage ω_o
	E_o^p (GPa)	E_o^n (GPa)					Mode-I K_I^C ($\text{MN/m}^{3/2}$)	Mode-I K_I^T ($\text{MN/m}^{3/2}$)	Mode-II K_{II}^T ($\text{MN/m}^{3/2}$)			
Value	17.35	17.35	0.15	0.6	14.00	3.0	0.525 [*]	0.225 [*]	0.300 [*]	0.980	0.060	

The concrete mix comprised cement, fly ash, sand, and lightweight aggregates known by the trade name Lytag. These ingredients were mixed in the ratio 287 : 123 : 560 : 696 kg/m^3 , respectively, with a water/binder ratio of 0.40. The maximum aggregate size was

14 mm. The mix also contained steel fibres (length = 50 mm, diameter = 0.5 mm) with fibre contents of 0%, 0.5% and 1%. The concrete, fibres, and the interface of fibre-concrete material properties, as used in the analysis, are shown in Table 7.15 and Table 7.16.

Table 7.16: Fibre and fibre-concrete interface material properties of University of Sheffield slabs.

#	1	2	3	4	5	6
Material properties	Elastic moduli	Yield or fracture stress	Poisson's ratio	Fibre volume fraction(s)	Fibre dimensions	
	E_f (GPa)	σ_{fy} (MPa)	ν	V_f (%)	r_f (mm)	l_f (mm)
Value	200	460	0.30	0.00	0.50	50.00
				0.50		
				1.00		
						τ_f (MPa)
						2.46 [†]

[†] Based on the lower tensile strength value of plain concrete provided by the original authors

Figure 7.21: Finite Element mesh of University of Sheffield fibre steel slabs.

Due to symmetry one quarter of the slab was modelled by an 8×8 mesh of flat shell elements. The inplane component of the shell element was the same as the plane stress element described earlier. The bending part is a 4-node quadrilateral element with three degrees of freedom per node. The element is based on the discrete Kirchhoff theory of plate bending (Razaqpur and Nofal 1990).

Because the purpose of the stub column was to transfer loads to the slab without causing punching shear, the column was not modelled by finite elements. The load was applied uniformly over the elements occupying an area at the center of the slab equal to

the column cross-section. The slab reinforcement was modelled as smeared layers while the slab was divided into 5 concrete layers through its thickness. The finite element mesh and the loading are shown in Figure 7.21.

Figure 7.22: Results of deformation of University of Sheffield fibre steel slabs.

Figure 7.23: Results of the University of Sheffield fibre steel slabs.

Results of the analysis are compared in Figures 7.22 to 24. Figure 7.22(a) shows the load-central deflection of the slabs while Figure 7.22(b) depicts the load versus the edge rotations. Figures 7.23 show, respectively, the load versus strain variation in the bottom and top steel layers directly below the applied load. Figures 7.24 shows the concrete strain in the top surface of the slab at a point halfway between the center and the edge of the slab.

Figure 7.24: Concrete compressive strains of the University of Sheffield fibre steel slabs

Generally there is good agreement between the calculated and the measured values for the various quantities. There is some discrepancy between the two sets of results for the strains in the top steel layer in the slab with 0.5% fibre content. Since all other strains compare favourably, we speculate that the above difference may be due to some punching. The finite element program utilized in the current study does not consider through thickness deformations in the slab and is consequently incapable of accounting for punching shear.

7.4 CLOSURE

The analyses and examples presented in this chapter demonstrate the ability of the proposed damage model to capture the nonlinear response of fibre-reinforced concrete structures up to their maximum capacity. Although the mechanisms of fibre-matrix interaction are generally complex, the proposed model appears to be adequately representing them up to the point of maximum load.

Since some of the mechanisms tend to dominate fibre-reinforced concrete behaviour in the strain softening or post-peak state, it is not possible, given the limitation of the computer program *Nonlacs*, to test the ability of the model in this regard. Further work would be needed to modify the program before such a test can be carried out.

8. SUMMARY AND CONCLUSIONS

8.1 SUMMARY

The behaviour and deformations of concrete and its structures due to applied mechanical loading are generally nonlinear and extremely difficult to predict. The nonlinearity of concrete often starts at very low load levels. Recognizing the above facts, many studies have attempted to model the nonlinearity using both the theory of elasticity and plasticity as well as fracture mechanics. Unfortunately, these models could not trace the concrete nonlinear response for all possible loading conditions. Consequently, none of these models have emerged as a universally accepted method for the analysis of general concrete materials, specimens, and structures. This is because they fail to account rigorously for the presence and evolution of microcracks in concrete which is the most important energy dissipation mechanism.

However, results of many recent experiments have revealed that concrete nonlinear deformations under mechanical loading are the result and manifestation of the evolution and propagation of microcracks. Concrete materials also undergo plastic flow and show yield-like behaviour when they are subjected to overall compression stress field. Yielding under the latter condition often starts at the locations of preexisting microcracks in the material space. These micro defects are an inherent part of any concrete structure due to the very nature of concrete materials and its manufacturing processes. The microcracks develop due to bleeding and hydration of the cement paste during the maturation of fresh concrete.

It was established earlier that classical fracture mechanics is suited to describing the behaviour of discrete macroscopic cracks, but its adaptation to an ensemble of randomly distributed microcracks is a relatively new development. This new area of mechanics is referred to as continuum damage mechanics (CDM). Continuum damage mechanics aims at explaining the link between the microscopic processes that a damaged brittle material experiences under the influence of different mechanical stress fields and the macroscopic behaviour that is observed in physical experiments. Therefore, the continuum damage mechanics concept has proved to be a valuable theoretical tool for explaining the true observed mechanical behaviour of concrete when the energy dissipation mechanism is the microcrack evolution and propagation. Consequently, many recent and current studies have been devoted to developing *phenomenological* and *micromechanical* damage-based constitutive equations for brittle materials. The phenomenological damage-based models assume that crack evolution follows a continuous function which is differentiable with respect to the spatial material coordinate system $(x - y - z)$. The format of such functions is similar to the customary yield functions. The micromechanical damage-based models, on the other hand, consider a crack to be a unique entity and trace its deformations by employing the basic laws of continuum and fracture mechanics.

Unfortunately, when the mechanism of crack evolution is suppressed by the configuration of applied external stresses, pure damage mechanics fails to predict the inelastic response and shows elastic response. Thus, some current studies have attempted to combine the classical theory of plasticity with phenomenological damage-based models in order to trace concrete inelastic behaviour for all load scenarios. Of course, these models suffer from the empiricism of the classical nonlinear elasticity and plasticity because the continuous functions of damage are empirical, and are often subject to the developer's bias. In the case of phenomenological damage models the situation may be worse because despite the vast amount of experimental data in the literature, there is no study devoted to measuring and quantifying crack growth damage under mechanical loading. Consequently, part of this thesis was concerned with combining an improved

micromechanical damage-based model with theory of plasticity to obtain a set of constitutive equations for plain concrete elements and structures subject to applied external mechanical loading.

Of course, introducing short or long fibres to the concrete mix suppresses some of the microcrack propagation mechanisms. This is because fibres tend to bridge the crack faces and in so doing reduce its opening and sliding displacements. However, the presence of fibres complicates the crack propagation mechanisms and introduces new ones. Among these new mechanisms are (i) fibres deformations, (ii) fibres slipping and sliding in the concrete matrix, and (iii) the interaction between the fibres and the cracks.

Employing the principles of shear lag theory, a new method is proposed in this thesis to evaluate fibre stresses, strains and their sliding in the concrete body during a given loading program. This method is derived from sound theories and involves very few assumptions. Chemical interaction between fibres and concrete components as well as fibre dowel action were ignored in obtaining the fibre-reinforced concrete constitutive equations. This is because actual experiments have revealed that these phenomena have minimal impact on the overall deformational behaviour of fibre-reinforced concrete subjected to external mechanical loads. The fibre deformations are not only functions of the fibre and matrix material properties, but they also depend greatly on the microcrack size and displacements. Therefore, new crack opening displacement functions that recognize the fibres presence were formulated. These displacements are used in the developed micromechanical damage-based model to evaluate the bridged (reinforced) crack compliance. Subsequently, the overall elasto-plastic damage-based constitutive equations of fibre-reinforced concrete were obtained by combining the proposed micromechanical damage-based model with the general theory of plasticity.

The proposed elasto-plastic damage-based constitutive models were implemented in the finite element program Nonlacs, described in Chapter 4, to obtain useful numerical tools to trace the inelastic response of concrete structures subjected to general mechanical loading. The models implementation can be achieved in a similar fashion to any other general

purpose finite element programs. The program Nonlacs was then employed to analyze the global deformation response of concrete structures under the influence of externally applied loads. The observations made, and conclusions drawn and the recommendations for future as a result of the present study are presented next.

8.2 CONCLUSIONS

Based on the reported results of the current study, it can be concluded that the new concept of damage mechanics is a very useful method for the analysis of brittle concrete materials containing micro-defects, voids, and cracks. Constitutive models derived from the principles of micromechanical damage mechanics are very simple, require few basic material parameters yet give accurate results for the test problems presented in this thesis. More specifically, micromechanical damage-based models in conjunction with the theory of plasticity are capable of capturing the complete stress-strain characteristics and failure envelope of plain, fibre-reinforced, and reinforced concrete materials subjected to different mechanical loading combinations. The required material properties can be obtained from simple essentially non-destructive tests.

The tracing of the evolution of the overall nonlinear behaviour of concrete material due to mechanical loads revealed that concrete will behave as an anisotropic material under certain loading combinations. This is due to the fact that crack evolution is directional in nature, depending on the applied load direction. Therefore, concrete material can be called a load induced anisotropic material. The degree of anisotropy evolves with the level, type and combination of loads. For example, axial tension weakens concrete in the plane perpendicular to the applied load. Axial compression, on the other hand, shows that concrete is an anisotropic material with weak planes parallel to it. The load induced anisotropy in the latter case is the result of the kinking phenomena of the sliding cracks. The proposed approximate analytical treatment of the kinked crack shows that the crack compliance are unsymmetric.

The study also showed that the nonlinear finite element approach is capable of producing results that agree well with experimental data for the tested concrete structure,

especially when the nonlinear constitutive relationships are derived using the proposed elasto-plastic damage-based models. The finite element model, Nonlacs, is very fast and economical compared with expensive experimental work. For example, a numerical test of a concrete beam with a very fine finite element mesh usually takes a total of 2 working hours; 45 minutes input data preparation, 12 minutes for execution, and 75 minutes for processing the results and producing the necessary details and graphs. Of course, this time saving cannot be compared to the time and resources that would be consumed by even a modest experimental testing program of the same structure. This, in return, indicates that more parameters and more variables can be examined by the analytical model which indeed can give an optimum design.

The numerical testing of the concrete structures presented in this thesis showed that their nonlinear behaviour was the manifestation of the evolution and propagation of the preexisting microcracks and defects. Damage mechanics utilizes the fundamental laws of mechanics in tracing the behaviour of cracks and defects, thus it is advantageous to use damage mechanics in establishing the concrete materials nonlinear response. The procedures given in this thesis are free of the usual empiricism associated with many existing stress-strain relationships and failure criteria for plain concrete, fibre-reinforced, and reinforced concrete materials. The proposed plain and fibre-reinforced concrete elasto-plastic damage-based constitutive models are also capable of predicting the behaviour of general concrete structures with or without conventional mild steel reinforcement.

The assumption of full bond between the concrete and long fibres (e.g. mild steel reinforcements) within the range of zero to maximum load appears to be reasonable. In other words, for the normal reinforcement types and contents investigated herein, steel pull-out seems to have minimal effect on the strength and stiffness of the reinforced structures analyzed. On the other hand, slippage and sliding of fibres may affect the softening response and energy absorption capacity of concrete structures. Currently, the computer program used does not account for softening behaviour of structures.

The fibre slippage and sliding behaviour was found to be a function of the fibre and matrix material properties, level and configuration of the mechanical loading, and the mechanics of the preexisting microcracks. The results presented in this study showed that fibres can enhance significantly the stiffness properties of concrete. On the other hand, their impact on the strength was found to be dependent on their effective length. Short fibres may increase the fibre-reinforced concrete tensile strength by about 2% to 5% over plain concrete. The results of this study showed that neglecting the fibres dowel action will not impact the prediction of the proposed fibre-reinforced concrete elasto-plastic micromechanical damage-based model.

Based on the results of the current study, the two proposed constitutive models can be used to trace concrete behaviour subjected to static monotonic and cyclic loading. The concept of damage mechanics could be used, in conjunction with the theory of plasticity, to evaluate the behaviour of most brittle materials. This concept should be further developed and integrated in design procedures and codes of practice. However, more in depth studies should be conducted to completely understand the subject and to establish its range of application. These issues are further elaborated on as follows.

8.3 PROPOSAL FOR RECOMMENDATION FOR FUTURE WORK

According to the work presented herein and the above conclusions, the current study shows the need for further work in order to fully understand the proposed models. The purpose of the test cases presented here was to establish the correct implementation of the proposed models in the finite element method. Further tasks to be undertaken include (i) parametric studies, (ii) theoretical developments, and (iii) experimental investigations.

8.3.1 PARAMETRIC STUDY

This part of the future work should emphasize establishing the impact of some material parameters variation on the model's results. For example, all test cases used a value of 0.6 for the coefficient of dry friction, μ , of concrete material. More numerical testing is needed to find the impact of the coefficient of friction, μ , on the prediction of the models. This coefficient may be permitted to evolve with the load application especially for the

cyclic loading conditions. Other parameters that require further verification are the initial de-bonded interfaces (damage) parameter, ρ and the initial crack diameters, a_o . Moreover, the model sensitivity to the probability functions used to describe the randomness of the cracks and fibres diameters and their orientation need to be examined.

More examples should also be analyzed for different types of structures, other than those tested here, i.e., shells, shear walls, etc. The same is needed for prestressed and continuous structures that allow stress redistribution. The result of these analyses will critically evaluate the proposed models and establish their range of application as well as their limitations. In addition, more analyses are needed to establish the minimum length and volume fraction ratio of short and/or long fibres that can increase both stiffness and strength of fibre-reinforced concrete structures.

8.3.2 THEORETICAL DEVELOPMENTS

On the theoretical development front, the following issues must be addressed for the proposed 2-D elasto-plastic damage-based constitutive models.

- ◆ Exact solution of the kinked crack systems should be developed.
- ◆ Different types of cracks must be considered instead of the planar and elliptic cracks used in the current investigation.
- ◆ Where appropriate, damage due to chemical reactions and the cement hydration processes must be included in the formulation of the damage equations.
- ◆ The damage formulation should account for the degradation in the coefficient of friction, μ , due to applied static or cyclic mechanical loading.
- ◆ The damage equations must account for the phase changes of the material because concrete transforms from semi liquid to a solid over a few days after casting the concrete mix.

- ◆ The effect of creep and viscoelasticity on the behaviour of the concrete matrix and the attendant modelling of those effects would extend the range of application of damage based models to concrete long term behaviour.
- ◆ The final stage in the theoretical developments of the model is to formulate the damage equations for 3-D application. These equations can be utilized in the general elasto-plastic damage-based constitutive equations developed in this thesis since they have been derived for the general 3-D state of stresses.
- ◆ The finite element model, Nonlacs, on the other hand, requires some improvements such as;
 - ◆ The program must include geometric nonlinearity.
 - ◆ Program Nonlacs should have the capability of tracing the softening response of structures.
 - ◆ More finite elements should be included in the program such as triangular facet shell elements, beam elements, as well as 3-D solid elements.
 - ◆ The bond between the reinforcement and the concrete should be properly modelled and bond degradation should be accounted for.

8.3.3 EXPERIMENTAL INVESTIGATIONS

Experimental studies should be designed and conducted specifically to measure the required parameters. For example,

- ◆ An experimental method should be developed to measure the initial de-bond parameter, and the minimum crack size.
- ◆ The evolution of damage should also be quantified experimentally in order to verify the proposed method of analysis. The ultrasonic, micrography, radioactive, and/or acoustic emission techniques can be used to evaluate the latter parameter.

- ◆ More experiments should be conducted to determine the different stress intensity factors for the various crack opening displacements modes. Of course, a notch specimen can be designed to provide a comprehensive understanding of concrete behaviour in order to obtain accurate values for these parameters. Size effect on fracture toughness parameters needs further experimental investigation.
- ◆ Another set of experiments are required to determine the impact of chemical reaction and cement hydration on initiation of damage.
- ◆ Finally, monitoring damage evolution for existing structure should be conducted in order to evaluate long term damage and its evolution with age under sustained loads.

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IMAGE EVALUATION TEST TARGET (QA-3)

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