Representations by Quaternary Quadratic Forms with Coefficients 1, 2, 5 or 10

by

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Abstract

We determine explicit formulas for the number of representations of a positive integer n by the quaternary quadratic forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 5, 10\}$ which satisfy the simplifying assumptions $a_1 \leq a_2 \leq a_3 \leq a_4$ and $\gcd(a_1, a_2, a_3, a_4) = 1$. We use a modular form approach. We then extend our work to determine explicit formulas for the number of representations of n by the octonary quadratic forms $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+5(x_7^2+x_8^2)$, $x_1^2+x_2^2+x_3^2+x_4^2+5(x_5^2+x_6^2+x_7^2+x_8^2)$ and $x_1^2 + x_2^2 + 5(x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2)$.

Dedication

To my father ***Waslallah*** *and my mother* ***Moslaha*** *who never gave up, with massive thanks for everything.*

To my husband ***Rashad*** *who has always been there with love and appreciation.*

To my lovely children, ***Battal****,* ***Julia****,* ***Judi*** *and* ***Reman****.*

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Chapter 1

Introduction

Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers, rational numbers, real numbers and complex numbers respectively. Let $a_1, a_2, a_3, a_4 \in \mathbb{N}$, and $n \in \mathbb{N}_0$. We write $N(a_1, a_2, a_3, a_4; n)$ to denote the number of representations of n by the quaternary quadratic form $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$, that is

$$N(a_1, a_2, a_3, a_4; n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}.$$

Clearly $N(a_1, a_2, a_3, a_4; 0) = 1$. As $N(a_1, a_2, a_3, a_4; n)$ remains invariant under a permutation of a_1, a_2, a_3, a_4, we may suppose that

$$a_1 \le a_2 \le a_3 \le a_4. \tag{1.0.1}$$

If $\gcd(a_1, a_2, a_3, a_4) = d$, then $N(a_1, a_2, a_3, a_4; n) = N(a_1/d, a_2/d, a_3/d, a_4/d; n/d)$. So we may also suppose that

$$\gcd(a_1, a_2, a_3, a_4) = 1. \tag{1.0.2}$$

Our first aim in this thesis is to determine explicit formulas for $N(a_1, a_2, a_3, a_4; n)$, where $a_1, a_2, a_3, a_4 \in \{1, 2, 5, 10\}$ which satisfy the simplifying assumptions (1.0.1) and (1.0.2).

We then extend our work to determine the number of representations of n by the octonary quadratic forms

$$(x_1^2 + \cdots + x_i^2) + 5(x_{i+1}^2 + \cdots + x_8^2)$$

where i is an even integer. We write $N(1^i, 5^j; n)$ to denote the number of representations of n by the forms $(i, j) = (6, 2),\ (4, 4),\ (2, 6)$, that is

$$N(1^i, 5^j; n) := \text{card}\{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \mid n = (x_1^2 + \cdots + x_i^2) + 5(x_{i+1}^2 + \cdots + x_8^2)\}.$$

Lagrange [14] stated that any positive integer can be written as a sum of four squares. A famous formula for $N(1, 1, 1, 1; n)$ was given by Jacobi in 1828, that is

$$N(1, 1, 1, 1; n) = 8\sigma(n) - 32\sigma(n/4) = \begin{cases} 8\sigma(n) & \text{if } 4 \nmid n, \\ 24\sigma(n) & \text{if } 4 \mid n, \end{cases} \tag{1.0.3}$$

where $\sigma(n)$ is the sum of divisors function. See Jacobi [24, 25, 26]. Many proofs of (1.0.3) have been given, see for example [9], [16], [19], [20], [21], [22] and [54]. Since then, there has been an extensive study of other quadratic forms to determine a formula for the number of representations. See for example [4], [7], [15], [17], [18], [32], [45], [47] or [50].

Liouville stated many formulas through his series of eighteen papers. He [34], [35] gave a formula for $N(1, 1, 2, 2; n)$, $N(1, 1, 1, 2; n)$ and $N(1, 2, 2, 2; n)$. These formulas have been proved by Benz [6], Demuth [12], Pepin [44], Wild [56] and Petr [46]. Also, Liouville [38] deduced a formula for $N(1, 1, 5, 5; n)$.

A. Alaca, S. Alaca, M. F. Lemire and Kenneth S. Williams [2], [3] have recently given simple proofs of Liouville's formulas corresponding to the quaternary forms $(a_1, a_2, a_3, a_4) = (1,1,2,2)$, $(1,1,1,2)$ and $(1,2,2,2)$. They [1] have also given a simple proof of Liouville's formula for $N(1,1,5,5;n)$ as well as they have deduced a simpler formula for it. Their formula for $N(1,1,5,5;n)$ is given by

$$N(1,1,5,5;n) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(n/4) + \frac{20}{3}\sigma(n/5) - \frac{80}{3}\sigma(n/20) + \frac{8}{3}a_1(n),$$

where $a_1(n)$ is given by (4.2.1).

In Chapter 2 we present some basic properties of the modular group $SL_2(\mathbb{Z})$ and congruence subgroups. We also study the theory of modular forms. In Chapter 3 we state formulas for the dimensions of the spaces $M_k(\Gamma_0(N),\chi)$, $E_k(\Gamma_0(N),\chi)$ and $S_k(\Gamma_0(N),\chi)$. We present our research results in Chapter 4. We determine an explicit formula for $N(a_1,a_2,a_3,a_4;n)$ for each of the nineteen quaternary quadratic forms given by

$$\begin{aligned}(a_1,a_2,a_3,a_4) = \quad & (1,1,2,5),\ (1,2,2,5),\ (2,2,2,5),\ (1,2,5,5),\ (2,2,5,5),\\ & (2,5,5,5),\ (1,1,1,10),\ (1,1,2,10),\ (1,2,2,10),\ (1,1,5,10),\\ & (1,2,5,10),\ (2,2,5,10),\ (1,5,5,10),\ (2,5,5,10),\ (1,1,10,10),\\ & (1,2,10,10), (1,5,10,10),\ (2,5,10,10),\ (1,10,10,10).\end{aligned}$$

To the best of our knowledge, these are the only remaining diagonal quaternary quadratic forms for which explicit formulas for $N(a_1,a_2,a_3,a_4;n)$ have not been determined so far. Finally, in Chapter 5 we extend our work to determine explicit formulas for $N(1^6,5^2;n)$, $N(1^4,5^4;n)$ and $N(1^2,5^6;n)$. We conclude our thesis by mentioning some further directions for our research.

Chapter 2

Modular Forms

In this chapter we present some basic concepts regarding modular forms. This material can be found in [13], [30], [40], [43], [51] or [55].

2.1 The Congruence Subgroups

The modular group $SL_2(\mathbb{Z})$ is defined by

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\},$$

which acts on the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d} \quad \text{for } z \in \mathbb{H}.$$

It is known that the modular group $SL_2(\mathbb{Z})$ is generated by two elements

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \text{ see for example Apostol [5].}$$

A subgroup Γ of $SL_2(\mathbb{Z})$ is called a congruence subgroup if it contains $\Gamma(N)$ for some positive integer N, where

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\text{mod N}) \right\}.$$

We call $\Gamma(N)$ the principal congruence subgroup of level N. The important congruence subgroups are

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\text{mod N}) \right\},$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\text{mod N}) \right\},$$

where $*$ denotes any integer. Note that $\Gamma_0(1) = \Gamma_1(1) = SL_2(\mathbb{Z})$.

2.2 The Space $M_k(\Gamma_0(N), \chi)$ of Modular Forms

Definition 2.2.1. Let $N \in \mathbb{N}$. A Dirichlet character (mod N) is a function $\chi : \mathbb{Z} \to \mathbb{C}$ satisfying

(1) $\chi(ab) = \chi(a)\chi(b)$ for any $a, b \in \mathbb{Z}$,

(2) $\chi(a) = \chi(b)$ if $a \equiv b$ (mod N),

(3) $\chi(a) \neq 0$ if $\gcd(a, N) = 1$,

(4) $\chi(a) = 0$ if $\gcd(a, N) > 1$.

A character of modulus 1 is called the trivial character and is denoted by χ_0.

Definition 2.2.2. The conductor of a Dirichlet character χ is the smallest positive integer M dividing its modulus such that there exists a Dirichlet character ψ of modulus M with $\chi(a) = \psi(a)$ for all $a \in \mathbb{Z}$ with $(a, N) = 1$. We say that a Dirichlet character modulo N is primitive if its conductor equals its modulus.

Definition 2.2.3. Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. A weakly modular function of weight k for Γ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f(\gamma(z)) = (cz + d)^k f(z).$$

We define the "slash" operator $|_k$ on functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f|[\gamma]_k = (cz + d)^{-k} f(\gamma z) \quad \text{for } \gamma \in SL_2(z).$$

Definition 2.2.4. Let Γ be a congruence subgroup of level N and let k be an integer. We say that a function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of wegiht k for Γ if

(1) f is holomorphic on $\mathbb{H}$,

(2) f is weakly modular for Γ, i.e. $f|[\gamma]_k = f$ for all $\gamma \in \Gamma$,

(3) If $\alpha \in SL_2(z)$, then $f|[\alpha]_k$ has a Fourier expansion of the form

$$f|[\alpha]_k = \sum_{n=0}^{\infty} a_n q_N^n, \qquad q_N = e^{2\pi i z/N}.$$

In addition,

(4) If $a_0 = 0$ in the Fourier expansion of $f|[\alpha]_k$ for all $\alpha \in SL_2(z)$, then f is a cusp form of wegiht k with respect to Γ.

The space of all modular forms of weight k with respect to Γ is denoted $M_k(\Gamma)$, the subspace of all cusp forms $S_k(\Gamma)$.

Definition 2.2.5. Let χ be a Dirichlet character modulo N. Let $f \in M_k(\Gamma_1(N))$ and suppose further that f satisfies

$$f|[\gamma]_k = \chi(d)f(z) \quad \forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Then we say that f is a modular form of level N and character χ.

We write $M_k(\Gamma_0(N), \chi)$ to denote the space of modular forms of weight k and character χ, and $S_k(\Gamma_0(N), \chi)$ to denote the subspace of cusp forms of weight k and character χ.

We note that if $\chi = \chi_0$, then we denote $M_k(\Gamma_0(N), \chi_0)$ and $S_k(\Gamma_0(N), \chi_0)$ by $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$, respectively.

2.3 Eisenstein Series

For $n \in \mathbb{N}$ we define $\sigma_{(k-1,\chi,\psi)}(n)$ by

$$\sigma_{(k-1,\chi,\psi)}(n) := \sum_{1 \le m | n} \psi(m)\chi(n/m)m^{k-1}. \tag{2.3.1}$$

If $n \notin \mathbb{N}$ we set $\sigma_{(k-1,\chi,\psi)}(n) = 0$. If χ and ψ are trivial characters then $\sigma_{(k-1,\chi,\psi)}(n)$ coincides with the sum of divisors function

$$\sigma_{k-1}(n) = \sum_{1 \le m | n} m^{k-1}.$$

We write $\sigma(n)$ for $\sigma_1(n)$. We define the generalized Bernoulli numbers $\{B_{k,\psi}\}_{k\in\mathbb{N}}$ by their exponential generating function

$$\sum_{a=1}^{N} \frac{\psi(a)\cdot x\cdot e^{ax}}{e^{Nx}-1} = \sum_{k=0}^{\infty} B_{k,\psi}\cdot \frac{x^k}{k!}.$$

Suppose that χ and ψ are primitive Dirichlet characters with conductors L and M, respectively, and let k be a positive integer such that $\chi(-1)\psi(-1) = (-1)^k$. We define the Eisenstein series by

$$E_{k,\chi,\psi}(q) = c_0 + \sum_{n\geq 1}\Big(\sum_{m|n}\psi(m)\chi(n/m)m^{k-1}\Big)q^n,$$

where c_0 is given by

$$c_0 = \begin{cases} 0 & \text{if } L > 1, \\ -\frac{B_{k,\psi}}{2k} & \text{if } L = 1. \end{cases}$$

When χ and ψ are trivial characters, the Eisenstein series $E_{2,\chi_0,\chi_0}(q)$ and $E_{4,\chi_0,\chi_0}(q)$ become

$$L(q) := E_2(q) := E_{2,\chi_0,\chi_0}(q) = -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma(n)q^n,$$

and

$$E_4(q) := E_{4,\chi_0,\chi_0}(q) = \frac{1}{240} + \sum_{n=1}^{\infty}\sigma_3(n)q^n.$$

We use the following theorem to determine an explicit basis for the Eisenstein subspace. See [55, Section 5.3, p. 88].

Theorem 2.3.1. *Let χ and ψ be primitive Dirichlet characters with conductors L and M. Suppose that k is a positive integer such that $\chi(-1)\psi(-1) = (-1)^k$.*

(a) *If* $k = 2$ *and* χ, ψ *are trivial characters then* $L(q) - tL(q^t) \in M_2(\Gamma_0(t))$ *for* $t > 1$, t *integer .*

(b) *If* $k \geq 4$, *or* $k = 2$ *with* χ *and* ψ *nontrivial characters, then* $E_{k,\chi,\psi}(q^t) \in M_k(\Gamma_0(N), \chi)$.

Furthermore, the Eisenstein series $E_{k,\chi,\psi}(q^t)$ *such that* $\chi \cdot \psi = \varepsilon$ *and* $LMt \mid N$ *form a basis for the Eisenstein subspace* $E_k(\Gamma_0(N), \varepsilon)$.

Let $k \in \mathbb{Z}$. We write $E_k(\Gamma_0(N), \chi)$ to denote the subspace of Eisenstein forms. It is known (see for example [55, p.83]) that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi). \tag{2.3.2}$$

2.4 Eta Quotients and the Ligozat Theorem

The Dedekind eta function $\eta(z)$ is the holomorphic function defined on the upper half plane $\mathbb{H}$ by the product formula

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi i z}. \tag{2.4.1}$$

An eta quotient is defined to be a finite product of the form

$$f(z) = \prod_{\delta} \eta^{r_\delta}(\delta z), \tag{2.4.2}$$

where δ runs through a finite set of positive integers and the exponents r_δ are nonzero integers. By taking N to be the least common multiple of the δ's we can write the eta quotient (2.4.2) as

$$f(z) = \prod_{1 \leq \delta \mid N} \eta^{r_\delta}(\delta z), \tag{2.4.3}$$

where some exponents r_δ may be 0. When all exponents are non-negative, $f(z)$ is said to be an eta product. For $q \in \mathbb{C}$ with $|q| < 1$ we set

$$F(q) = \prod_{n=1}^{\infty}(1 - q^n). \tag{2.4.4}$$

Throughout the remainder of this work we take $q = q(z) := e^{2\pi i z}$ with $z \in \mathbb{H}$ and appealing to (2.4.4) we can express the eta function (2.4.1) as

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n) = q^{1/24} F(q).$$

We use the following theorem to determine if an eta quotient $f(z)$ given by (2.4.3) is in $M_k(\Gamma_0(N), \chi)$. See [30, Theorem 5.7, p. 99], [31, Corollary 2.3, p. 37] and [43, Theorem 1.64, p. 18].

Theorem 2.4.1. (Ligozat) *Let $N \in \mathbb{N}$ and let $f(z) = \prod_{1 \leq \delta \mid N} \eta^{r_\delta}(\delta z)$ be an eta quotient which satisfies the following conditions*

(L1) $\displaystyle\sum_{1 \leq \delta \mid N} \delta \cdot r_\delta \equiv 0 \pmod{24}$,

(L2) $\displaystyle\sum_{1 \leq \delta \mid N} \frac{N}{\delta} \cdot r_\delta \equiv 0 \pmod{24}$,

(L3) *for each $d \mid N$,* $\displaystyle\sum_{1 \leq \delta \mid N} \frac{\gcd(d, \delta)^2 \cdot r_\delta}{\delta} \geq 0.$

Then $f(z) \in M_k(\Gamma_0(N), \chi)$, *where* χ *is given by*

$$\chi(m) = \left(\frac{(-1)^k s}{m}\right)$$

with weight

$$k = \frac{1}{2} \sum_{1 \le \delta | N} r_\delta$$

and

$$s = \prod_{1 \le \delta | N} \delta^{r_\delta}.$$

In addition to the above conditions if $f(z)$ *also satisfies the following condition*

(L4) *for each* $d \mid N$, $\displaystyle\sum_{1 \le \delta | N} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta} > 0$,

then $f(z) \in S_k(\Gamma_0(N), \chi)$.

Ramanujan's theta function $\varphi(q)$ is defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C} \text{ with } |q| < 1.$$

We note that for quaternary forms $a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2$ we have

$$\sum_{n=0}^{\infty} N(a_1, a_2, a_3, a_4; n)q^n = \varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}), \tag{2.4.5}$$

and for octonary forms $a_1x_1^2 + \cdots + a_8x_8^2$ we have

$$\sum_{n=0}^{\infty} N(a_1, \ldots, a_8; n)q^n = \varphi(q^{a_1}) \cdots \varphi(q^{a_8}). \tag{2.4.6}$$

Ramanujan's theta function can be written [8] as

$$\varphi(q) = \frac{F^5(q^2)}{F^2(q)F^2(q^4)} = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}. \tag{2.4.7}$$

Chapter 3

Dimension Formulas

Let $N, k \in \mathbb{N}$ and χ a Dirichlet character $\pmod N$. In this chapter we state formulas for the dimensions of the spaces $M_k(\Gamma_0(N), \chi)$, $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$.

3.1 Dimension Formulas for $E_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$

For any prime p and any positive integer N. Let

$$\mu_0(N) = N \prod_{p|N} (1 + 1/p),$$

$$\mu_{0,2}(N) = \begin{cases} 0 & \text{if } 4|N, \\ \prod_{p|N}(1 + (\frac{-4}{p})) & \text{otherwise,} \end{cases}$$

$$\mu_{0,3}(N) = \begin{cases} 0 & \text{if } 2|N \text{ or } 9|N, \\ \prod_{p|N}(1 + (\frac{-3}{p})) & \text{otherwise,} \end{cases}$$

$$c(N) = \sum_{d|N} \phi(\gcd(d, N/d)),$$

where d runs through all the positive divisors of N and ϕ is the Euler totient function. Also, let

$$g(N) = 1 + \frac{\mu_0(N)}{12} - \frac{\mu_{0,2}(N)}{4} - \frac{\mu_{0,3}(N)}{4} - \frac{c(N)}{2}.$$

Then we have the following proposition which is taken from [55, Section 6.1, p. 93].

Proposition 3.1.1. *We have* $\dim S_2(\Gamma_0(N)) = g(N)$, *and for* $k \geq 4$ *even,*

$$\begin{aligned}\dim S_k(\Gamma_0(N)) = (k-1)\cdot(g(N)-1) + \left(\frac{k}{2}-1\right)\cdot c(N) \\ +\mu_{0,2}(N)\cdot\left\lfloor\frac{k}{4}\right\rfloor + \mu_{0,3}(N)\cdot\left\lfloor\frac{k}{3}\right\rfloor,\end{aligned}$$

where $\lfloor\ \rfloor$ *denotes the floor function. The dimension of the Eisenstein subspace is*

$$\dim E_k(\Gamma_0(N)) = \begin{cases} c(N)-1 & \text{if } k=2, \\ c(N) & \text{if } k \neq 2. \end{cases}$$

Example 3.1.1. Let $N = 40$. We have $\mu_0(40) = 72$, $\mu_{0,2}(40) = \mu_{0,3}(40) = 0$ and $c(40) = 8$. Hence

$$g(40) = 1 + \frac{72}{12} - \frac{8}{2} = 3.$$

Thus by Proposition 3.1.1 we have $\dim S_2(\Gamma_0(40)) = 3$ and $\dim E_2(\Gamma_0(40)) = 7$.

3.2 Dimension Formulas for $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$

Let $v_p(N)$ denote the largest $r \in \mathbb{N}_0$ such that $p^r \mid N$ and let c be the conductor of χ. We set

$$\lambda_{(p,N,v_p(c))} = \begin{cases} p^{\frac{r}{2}} + p^{\frac{r}{2}-1} & \text{if } 2 \cdot v_p(c) \leq r \text{ and } 2 \mid r, \\ 2 \cdot p^{\frac{r-1}{2}} & \text{if } 2 \cdot v_p(c) \leq r \text{ and } 2 \nmid r, \\ 2 \cdot p^{r-v_p(c)} & \text{if } 2 \cdot v_p(c) > r. \end{cases}$$

The rational numbers γ_4 and γ_3 are defined as follows

$$\gamma_4(k) = \begin{cases} -1/4 & \text{if } k \equiv 2 \pmod 4, \\ 0 & \text{if } k \text{ is odd}, \\ 1/4 & \text{if } k \equiv 0 \pmod 4). \end{cases}$$

$$\gamma_3(k) = \begin{cases} -1/3 & \text{if } k \equiv 2 \pmod 3, \\ 0 & \text{if } k \equiv 1 \pmod 3, \\ 1/3 & \text{if } k \equiv 0 \pmod 3. \end{cases}$$

Let χ be a Dirichlet character of modulus N for which $\chi(-1) = (-1)^k$, the formulas are taken from [55, Section 6.3, p. 98-100], then we have

$$\dim S_k(\Gamma_0(N), \chi) - \dim M_{2-k}(\Gamma_0(N), \chi) = \frac{k-1}{12} \cdot \mu_0(N) - \frac{1}{2} \cdot \prod_{p \mid N} \lambda(p, N, v_p(c))$$
$$+\gamma_4(k) \cdot \sum_{x \in A_4(N)} \chi(x) + \gamma_3(k) \cdot \sum_{x \in A_3(N)} \chi(x), \tag{3.2.1}$$

where

$$A_4(N) = \left\{x \in \mathbb{Z}/N\mathbb{Z} \mid x^2 + 1 = 0\right\} \text{ and } A_3(N) = \left\{x \in \mathbb{Z}/N\mathbb{Z} \mid x^2 + x + 1 = 0\right\}.$$

To compute $\dim M_k(\Gamma_0(N), \chi)$ for $k \geq 2$, we use the fact that $\dim S_k(\Gamma_0(N), \chi) = 0$ for $k \leq 0$. Then we have

$$\begin{aligned}\dim M_k(\Gamma_0(N), \chi) =& -\left(\dim S_{2-k}(\Gamma_0(N), \chi) - \dim M_k(\Gamma_0(N), \chi)\right) \\ =& -\Big(\frac{1-k}{12} \cdot \mu_0(N) - \frac{1}{2} \cdot \prod_{p|N} \lambda(p, N, v_p(c)) \\ &+ \gamma_4(2-k) \cdot \sum_{x \in A_4(N)} \chi(x) + \gamma_3(2-k) \cdot \sum_{x \in A_3(N)} \chi(x)\Big), \quad (3.2.2)\end{aligned}$$

and

$$\dim E_k(\Gamma_0(N), \chi) = \dim M_k(\Gamma_0(N), \chi) - \dim S_k(\Gamma_0(N), \chi). \quad (3.2.3)$$

Example 3.2.1. For $N = 40$, $k = 2$, $\chi_1(m) = \left(\frac{5}{m}\right)$, $\chi_2(m) = \left(\frac{8}{m}\right)$ and $\chi_3(m) = \left(\frac{40}{m}\right)$, we have

$$\sum_{x \in A_4(N)} \chi(x) = \sum_{x \in A_3(N)} \chi(x) = 0.$$

Also, we have

χ	χ_1	χ_2	χ_3
$\prod_{p\|40} \lambda(p, 40, v_p(c))$	8	4	4

Thus by (3.2.1)–(3.2.3) we obtain

χ	$\dim M_2(\Gamma_0(40),\chi)$	$\dim S_2(\Gamma_0(40),\chi)$	$\dim E_2(\Gamma_0(40),\chi)$
χ_1	10	2	8
χ_2	8	4	4
χ_3	8	4	4

Chapter 4

Representations by Quaternary Quadratic Forms with Coefficients 1, 2, 5 or 10

We present our research results in this chapter. In Section 4.1 we give a brief description of our research. We then state and prove our main results in Sections 4.2–4.5.

4.1 A Brief Description of our Research

We recall that, for $a_1, a_2, a_3, a_4 \in \mathbb{N}$ and $n \in \mathbb{N}_0$, we define $N(a_1, a_2, a_3, a_4; n)$ as

$$N(a_1, a_2, a_3, a_4; n) := \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2\}.$$

We also make the simplifying assumptions

$$a_1 \leq a_2 \leq a_3 \leq a_4, \tag{4.1.1}$$

and

$$\gcd(a_1, a_2, a_3, a_4) = 1. \tag{4.1.2}$$

We also recall that χ_0 denotes the trivial character. For $m \in \mathbb{Z}$ we define three characters by

$$\chi_1(m) = \left(\frac{5}{m}\right), \chi_2(m) = \left(\frac{8}{m}\right), \chi_3(m) = \left(\frac{40}{m}\right). \tag{4.1.3}$$

Under the simplifying assumptions (4.1.1) and (4.1.2) there are twenty-six quaternary quadratic forms (a_1, a_2, a_3, a_4) for which $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40), \chi)$ where $\chi \in \{\chi_0, \chi_1, \chi_2, \chi_3\}$. They are listed in Table 4.1.1.

Table 4.1.1

$M_2(\Gamma_0(40), \chi_0)$	$M_2(\Gamma_0(40), \chi_1)$	$M_2(\Gamma_0(40), \chi_2)$	$M_2(\Gamma_0(40), \chi_3)$
$(1, 1, 1, 1)$	$(1, 1, 1, 5)$	$(1, 1, 1, 2)$	$(1, 1, 2, 5)$
$(1, 1, 2, 2)$	$(1, 2, 2, 5)$	$(1, 2, 2, 2)$	$(2, 2, 2, 5)$
$(1, 1, 5, 5)$	$(1, 5, 5, 5)$	$(1, 2, 5, 5)$	$(2, 5, 5, 5)$
$(2, 2, 5, 5)$	$(1, 1, 2, 10)$	$(1, 1, 5, 10)$	$(1, 1, 1, 10)$
$(1, 2, 5, 10)$	$(2, 5, 5, 10)$	$(2, 2, 5, 10)$	$(1, 2, 2, 10)$
$(1, 1, 10, 10)$	$(1, 5, 10, 10)$	$(1, 2, 10, 10)$	$(1, 5, 5, 10)$
			$(2, 5, 10, 10)$
			$(1, 10, 10, 10)$

For the quaternary quadratic forms $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1),\ (1, 1, 2, 2),\ (1, 1, 5, 5),$ $(1, 1, 1, 5),\ (1, 5, 5, 5),\ (1, 1, 1, 2)$ and $(1, 2, 2, 2)$, formulas for $N(a_1, a_2, a_3, a_4; n)$ have appeared in the literature, see [1], [2], [29] and [57]. In this chapter we determine formulas for $N(a_1, a_2, a_3, a_4; n)$ for the remaining nineteen quaternary quadratic forms listed in Table 4.1.1. We will treat each space with the corresponding quadratic forms separately.

4.2 The Space $M_2(\Gamma_0(40), \chi_0)$

We determine formulas for $N(a_1, a_2, a_3, a_4; n)$ for the quaternary forms given by

$$(a_1, a_2, a_3, a_4) = (2, 2, 5, 5),\ (1, 2, 5, 10),\ (1, 1, 10, 10)$$

in terms of $\sigma(n)$, $\sigma(n/2)$, $\sigma(n/4)$, $\sigma(n/5)$, $\sigma(n/8)$, $\sigma(n/10)$, $\sigma(n/20)$, $\sigma(n/40)$, and $a_k(n)$ $(1 \le k \le 3)$ defined by

$$A_1(q) = \sum_{n=1}^{\infty} a_1(n) q^n = \eta^2(2z)\eta^2(10z), \tag{4.2.1}$$

$$A_2(q) = A_1(q^2) = \sum_{n=1}^{\infty} a_2(n) q^n = \eta^2(4z)\eta^2(20z), \tag{4.2.2}$$

$$A_3(q) = \sum_{n=1}^{\infty} a_3(n) q^n = \frac{\eta^5(4z)\eta(10z)\eta^2(40z)}{\eta(2z)\eta^2(8z)\eta(20z)}. \tag{4.2.3}$$

There is no linear relationship among the $A_k(q)$ $(1 \le k \le 3)$. The first forty values of $a_k(n)$ $(1 \le k \le 3)$, are given in Table 4.2.1.

Table 4.2.1

n	$a_1(n)$	$a_2(n)$	$a_3(n)$	n	$a_1(n)$	$a_2(n)$	$a_3(n)$
1	1	0	0	21	−4	0	2
2	0	1	0	22	0	0	0
3	−2	0	1	23	6	0	−1
4	0	0	0	24	0	0	0
5	−1	0	1	25	1	0	0
6	0	−2	0	26	0	2	0
7	2	0	−3	27	4	0	−2
8	0	0	0	28	0	0	0
9	1	0	−2	29	6	0	−4
10	0	−1	0	30	0	2	0
11	0	0	2	31	−4	0	−2
12	0	0	0	32	0	0	0
13	2	0	−2	33	0	0	0
14	0	2	0	34	0	−6	0
15	2	0	−1	35	−2	0	−1
16	0	0	0	36	0	0	0
17	−6	0	4	37	2	0	2
18	0	1	0	38	0	−4	0
19	−4	0	4	39	−4	0	2
20	0	0	0	40	0	0	0

Theorem 4.2.1. *For* $(a_1, a_2, a_3, a_4) = (2,2,5,5),\ (1,2,5,10),\ (1,1,10,10)$, *we have* $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40))$.

Proof. Appealing to (2.4.7) we have

$$\varphi^2(q^2)\varphi^2(q^5) = \frac{\eta^{10}(4z)\eta^{10}(10z)}{\eta^4(2z)\eta^4(5z)\eta^4(8z)\eta^4(20z)}, \tag{4.2.4}$$

$$\varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10}) = \frac{\eta^3(2z)\eta^3(4z)\eta^3(10z)\eta^3(20z)}{\eta^2(z)\eta^2(5z)\eta^2(8z)\eta^2(40z)}, \tag{4.2.5}$$

$$\varphi^2(q)\varphi^2(q^{10}) = \frac{\eta^{10}(2z)\eta^{10}(20z)}{\eta^4(z)\eta^4(4z)\eta^4(10z)\eta^4(40z)}. \tag{4.2.6}$$

We first consider $\varphi^2(q^2)\varphi^2(q^5)$. From (4.2.4), we have

Table 4.2.2(a)

δ	2	4	5	8	10	20
r_δ	-4	10	-4	-4	10	-4

Table 4.2.2(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1 \le \delta \mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12/5	24	0	0	60	24	0

It follows from Table 4.2.2(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.2.2(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^2(q^2)\varphi^2(q^5) \in M_2(\Gamma_0(40))$.

Next we consider $\varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10})$. From (4.2.5) we have

Table 4.2.3(a)

δ	1	2	4	5	8	10	20	40
r_δ	-2	3	3	-2	-2	3	3	-2

Table 4.2.3(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	36/5	72/5	0	0	36	72	0

It follows from Table 4.2.3(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.2.3(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10}) \in M_2(\Gamma_0(40))$.

Finally, we consider $\varphi^2(q)\varphi^2(q^{10})$. From (4.2.6), we have

Table 4.2.4(a)

δ	1	2	4	10	20	40
r_δ	-4	10	-4	-4	10	-4

Table 4.2.4(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12	24/5	0	0	12	120	0

It follows from Table 4.2.4(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.2.4(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^2(q)\varphi^2(q^{10}) \in M_2(\Gamma_0(40))$. ■

Theorem 4.2.2. *$A_k(q)$ $(1 \le k \le 3)$ given by* (4.2.1)–(4.2.3) *are in $S_2(\Gamma_0(40))$.*

Proof. We first consider

$$A_1(q) = \eta^2(2z)\eta^2(10z).$$

Table 4.2.5(a)

δ	2	10
r_δ	2	2

Table 4.2.5(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/5	24/5	24/5	6	24/5	24	24	24

It follows from Table 4.2.5(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.2.5(b), L4 is also satisfied for each positive divisor d of 40. Hence $A_1(q) \in S_2(\Gamma_0(40))$.

Next we consider

$$A_2(q) = \eta^2(4z)\eta^2(20z).$$

Table 4.2.6(a)

δ	4	20
r_δ	2	2

Table 4.2.6(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/5	12/5	48/5	3	48/5	12	48	48

It follows from Table 4.2.6(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.2.6(b), L4 is also satisfied for each positive divisor d of 40. Hence $A_2(q) \in S_2(\Gamma_0(40))$.

Finally, we consider

$$A_3(q) = \frac{\eta^5(4z)\eta(10z)\eta^2(40z)}{\eta(2z)\eta^2(8z)\eta(20z)}.$$

Table 4.2.7(a)

δ	2	4	8	10	20	40
r_δ	-1	5	-2	1	-1	2

Table 4.2.7(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\displaystyle\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/5	12/5	72/5	3	24/5	12	24	72

It follows from Table 4.2.7(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.2.7(b), L4 is also satisfied. Hence $A_3(q) \in S_2(\Gamma_0(40))$. Therefore, $A_k(q)$ $(1 \le k \le 3) \in S_2(\Gamma_0(40))$. ■

Theorem 4.2.3. **(a)** $\{A_1(q), A_2(q), A_3(q)\}$ *is a basis for* $S_2(\Gamma_0(40))$.

(b) $L(q) - tL(q^t)$ $(t = 2, 4, 5, 8, 10, 20, 40)$ *constitute a basis for* $E_2(\Gamma_0(40))$.

(c) $L(q) - tL(q^t)$ $(t = 2, 4, 5, 8, 10, 20, 40)$ *together with* $A_k(q)$ $(1 \le k \le 3)$ *constitute a basis for* $M_2(\Gamma_0(40))$.

Proof. **(a)** By Theorem 4.2.2, $A_k(q)$ $(1 \le k \le 3) \in S_2(\Gamma_0(40))$. They are linearly independent over $\mathbb{C}$. By Example 3.1.1, we have $\dim S_2(\Gamma_0(40)) = 3$. Thus $A_k(q)$ $(1 \le k \le 3)$ constitute a basis for $S_2(\Gamma_0(40))$.

(b) By Example 3.1.1, we have $\dim E_2(\Gamma_0(40)) = 7$. By Theorem 2.3.1(a), $L(q) - tL(q^t)$ $(t = 2, 4, 5, 8, 10, 20, 40)$ constitute a basis for $E_2(\Gamma_0(40))$.

(c) It follows from (a), (b) and (2.3.2) that the dimension of $M_2(\Gamma_0(40))$ is 10 and therefore $L(q) - tL(q^t)$ $(t = 2, 4, 5, 8, 10, 20, 40)$ together with $A_k(q)$ $(1 \le k \le 3)$ constitute a basis for $M_2(\Gamma_0(40))$. ■

Theorem 4.2.4.

$$\textbf{(a)}\ \varphi^2(q^2)\varphi^2(q^5) = \frac{2}{3}L(q) - \frac{2}{3}L(q^2) + \frac{4}{3}L(q^4) + \frac{10}{3}L(q^5) - \frac{16}{3}L(q^8)$$

$$-\frac{10}{3}L(q^{10})+\frac{20}{3}L(q^{20})-\frac{80}{3}L(q^{40})-\frac{2}{3}A_1(q)+\frac{8}{3}A_2(q)-4A_3(q),$$

(**b**) $\varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10}) = L(q)-L(q^2)-2L(q^4)-5L(q^5)+8L(q^8)$

$$+5L(q^{10})+10L(q^{20})-40L(q^{40})+A_1(q)+2A_3(q),$$

(**c**) $\varphi^2(q)\varphi^2(q^{10}) = \frac{2}{3}L(q)-\frac{2}{3}L(q^2)+\frac{4}{3}L(q^4)+\frac{10}{3}L(q^5)-\frac{16}{3}L(q^8)$

$$-\frac{10}{3}L(q^{10})+\frac{20}{3}L(q^{20})-\frac{80}{3}L(q^{40})+\frac{10}{3}A_1(q)+\frac{8}{3}A_2(q)+4A_3(q).$$

Proof. By Theorem 4.2.1 we have $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40))$ for $(a_1,a_2,a_3,a_4) = (2,2,5,5)$, $(1,2,5,10)$ and $(1,1,10,10)$. Therefore, by Theorem 4.2.3(c), each $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $L(q)-tL(q^t)$ ($t=2,4,5,8,10,20,40$) and $A_k(q)$ ($1\le k\le 3$), namely

$$\begin{aligned}\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) &= x_1(L(q)-2L(q^2))+x_2(L(q)-4L(q^4))\\ &+x_3(L(q)-5L(q^5))+x_4(L(q)-8L(q^8))+x_5(L(q)-10L(q^{10}))\\ &+x_6(L(q)-20L(q^{20}))+x_7(L(q)-40L(q^{40}))+y_1A_1(q)\\ &+y_2A_2(q)+y_3A_3(q).\end{aligned}$$

Using MAPLE we equate the first few coefficients of q^n on both sides of the equation above to obtain the x_i ($1\le i\le 7$) and y_j ($1\le j\le 3$). Then we have

(**a**) $\varphi^2(q^2)\varphi^2(q^5) = \frac{1}{3}(L(q)-2L(q^2))-\frac{1}{3}(L(q)-4L(q^4))-\frac{2}{3}(L(q)-5L(q^5))$

$$+\frac{2}{3}(L(q)-8L(q^8))+\frac{1}{3}(L(q)-10L(q^{10}))-\frac{1}{3}(L(q)-20L(q^{20}))$$

$$+\frac{2}{3}(L(q)-40L(q^{40}))-\frac{2}{3}A_1(q)+\frac{8}{3}A_2(q)-4A_3(q),$$

(**b**) $\varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10}) = \frac{1}{2}(L(q)-2L(q^2))+\frac{1}{2}(L(q)-4L(q^4))$

$$+(L(q)-5L(q^5))-(L(q)-8L(q^8))-\frac{1}{2}(L(q)-10L(q^{10}))$$

$$-\frac{1}{2}(L(q)-20L(q^{20}))+(L(q)-40L(q^{40}))+A_1(q)+2A_3(q),$$

$$\textbf{(c)}\ \varphi^2(q)\varphi^2(q^{10})=\frac{1}{3}(L(q)-2L(q^2))-\frac{1}{3}(L(q)-4L(q^4))-\frac{2}{3}(L(q)-5L(q^5))$$
$$+\frac{2}{3}(L(q)-8L(q^8))+\frac{1}{3}(L(q)-10L(q^{10}))-\frac{1}{3}(L(q)-20L(q^{20}))$$
$$+\frac{2}{3}(L(q)-40L(q^{40}))+\frac{10}{3}A_1(q)+\frac{8}{3}A_2(q)+4A_3(q).$$

Simplifying the coefficients of equations (a)–(c) we obtain the asserted results. ■

Theorem 4.2.5. *Let $n \in \mathbb{N}$. Then*

$$\textbf{(a)}\ N(2,2,5,5;n)=\frac{2}{3}\sigma(n)-\frac{2}{3}\sigma(n/2)+\frac{4}{3}\sigma(n/4)+\frac{10}{3}\sigma(n/5)-\frac{16}{3}\sigma(n/8)$$
$$-\frac{10}{3}\sigma(n/10)+\frac{20}{3}\sigma(n/20)-\frac{80}{3}\sigma(n/40)-\frac{2}{3}a_1(n)$$
$$+\frac{8}{3}a_2(n)-4a_3(n),$$

$$\textbf{(b)}\ N(1,2,5,10;n)=\sigma(n)-\sigma(n/2)-2\sigma(n/4)-5\sigma(n/5)+8\sigma(n/8)+5\sigma(n/10)$$
$$+10\sigma(n/20)-40\sigma(n/40)+a_1(n)+2a_3(n),$$

$$\textbf{(c)}\ N(1,1,10,10;n)=\frac{2}{3}\sigma(n)-\frac{2}{3}\sigma(n/2)+\frac{4}{3}\sigma(n/4)+\frac{10}{3}\sigma(n/5)-\frac{16}{3}\sigma(n/8)$$
$$-\frac{10}{3}\sigma(n/10)+\frac{20}{3}\sigma(n/20)-\frac{80}{3}\sigma(n/40)+\frac{10}{3}a_1(n)$$
$$+\frac{8}{3}a_2(n)+4a_3(n).$$

Proof. Appealing to (2.4.5), we have by Theorem 4.2.4

$$\begin{aligned}
&\textbf{(a)}\ \sum_{n=0}^{\infty} N(2,2,5,5;n)q^n = \varphi^2(q^2)\varphi^2(q^5)\\
&= \frac{2}{3}L(q) - \frac{2}{3}L(q^2) + \frac{4}{3}L(q^4) + \frac{10}{3}L(q^5) - \frac{16}{3}L(q^8)\\
&\quad - \frac{10}{3}L(q^{10}) + \frac{20}{3}L(q^{20}) - \frac{80}{3}L(q^{40}) - \frac{2}{3}A_1(q) + \frac{8}{3}A_2(q) - 4A_3(q)\\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8)\\
&\quad - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) - \frac{2}{3}a_1(n) + \frac{8}{3}a_2(n) - 4a_3(n)\Big)q^n,
\end{aligned}$$

$$\begin{aligned}
&\textbf{(b)}\ \sum_{n=0}^{\infty} N(1,2,5,10;n)q^n = \varphi(q)\varphi(q^2)\varphi(q^5)\varphi(q^{10})\\
&= L(q) - L(q^2) - 2L(q^4) - 5L(q^5) + 8L(q^8) + 5L(q^{10}) + 10L(q^{20})\\
&\quad - 40L(q^{40}) + A_1(q) + 2A_3(q)\\
&= 1 + \sum_{n=1}^{\infty}\Big(\sigma(n) - \sigma(n/2) - 2\sigma(n/4) - 5\sigma(n/5) + 8\sigma(n/8) + 5\sigma(n/10)\\
&\quad + 10\sigma(n/20) - 40\sigma(n/40) + a_1(n) + 2a_3(n)\Big)q^n,
\end{aligned}$$

$$\begin{aligned}
&\textbf{(c)}\ \sum_{n=0}^{\infty} N(1,1,10,10;n)q^n = \varphi^2(q)\varphi^2(q^{10})\\
&= \frac{2}{3}L(q) - \frac{2}{3}L(q^2) + \frac{4}{3}L(q^4) + \frac{10}{3}L(q^5) - \frac{16}{3}L(q^8)\\
&\quad - \frac{10}{3}L(q^{10}) + \frac{20}{3}L(q^{20}) - \frac{80}{3}L(q^{40}) + \frac{10}{3}A_1(q) + \frac{8}{3}A_2(q) + 4A_3(q)\\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{2}{3}\sigma(n) - \frac{2}{3}\sigma(n/2) + \frac{4}{3}\sigma(n/4) + \frac{10}{3}\sigma(n/5) - \frac{16}{3}\sigma(n/8)\\
&\quad - \frac{10}{3}\sigma(n/10) + \frac{20}{3}\sigma(n/20) - \frac{80}{3}\sigma(n/40) + \frac{10}{3}a_1(n)\\
&\quad + \frac{8}{3}a_2(n) + 4a_3(n)\Big)q^n.
\end{aligned}$$

Equating the coefficients of q^n on both sides of equations (a)–(c), we deduce the asserted results. ∎

4.3 The Space $M_2(\Gamma_0(40), \chi_1)$

Let χ_1 be the Dirichlet character as in (4.1.3). We define the following Eisenstein series

$$E_{2,\chi_0,\chi_1}(q) = -\frac{1}{5} + \sum_{n=1}^{\infty} \sigma_{(\chi_0,\chi_1)}(n) q^n,$$
$$E_{2,\chi_1,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_1,\chi_0)}(n) q^n.$$

We determine formulas for $N(a_1, a_2, a_3, a_4; n)$ for the quaternary forms given by

$$(a_1, a_2, a_3, a_4) = (1, 2, 2, 5),\ (1, 1, 2, 10),\ (2, 5, 5, 10),\ (1, 5, 10, 10)$$

in terms of $\sigma_{(\chi,\psi)}(n)$ where $\chi, \psi \in \{\chi_0, \chi_1\}$ and $b_k(n)$ $(1 \le k \le 2)$ defined by

$$B_1(q) = \sum_{n=1}^{\infty} b_1(n) q^n = \frac{\eta(2z)\eta^4(20z)}{\eta(10z)}, \tag{4.3.1}$$
$$B_2(q) = \sum_{n=1}^{\infty} b_2(n) q^n = \frac{\eta^4(4z)\eta(10z)}{\eta(2z)}. \tag{4.3.2}$$

There is no linear relationship between $B_1(q)$ and $B_2(q)$. The first twenty values of $b_1(n)$ and $b_2(n)$ are given in Table 4.3.1.

Table 4.3.1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$b_1(n)$	0	0	1	0	-1	0	-1	0	0	0	0	0	2	0	-1	0	0	0	0	0
$b_2(n)$	1	0	1	0	-2	0	-1	0	-1	0	-4	0	2	0	3	0	0	0	4	0

Theorem 4.3.1. *For* $(a_1, a_2, a_3, a_4) = (1, 2, 2, 5),\ (1, 1, 2, 10),\ (2, 5, 5, 10),$ $(1, 5, 10, 10)$, *we have* $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40), \chi_1)$.

Proof. Appealing to (2.4.7) we have

$$\varphi(q)\varphi^2(q^2)\varphi(q^5) = \frac{\eta(2z)\eta^8(4z)\eta^5(10z)}{\eta^2(z)\eta^2(5z)\eta^4(8z)\eta^2(20z)}, \tag{4.3.3}$$

$$\varphi^2(q)\varphi(q^2)\varphi(q^{10}) = \frac{\eta^8(2z)\eta(4z)\eta^5(20z)}{\eta^4(z)\eta^2(8z)\eta^2(10z)\eta^2(40z)}, \tag{4.3.4}$$

$$\varphi(q^2)\varphi^2(q^5)\varphi(q^{10}) = \frac{\eta^5(4z)\eta^8(10z)\eta(20z)}{\eta^2(2z)\eta^4(5z)\eta^2(8z)\eta^2(40z)}, \tag{4.3.5}$$

$$\varphi(q)\varphi(q^5)\varphi^2(q^{10}) = \frac{\eta^5(2z)\eta(10z)\eta^8(20z)}{\eta^2(z)\eta^2(4z)\eta^2(5z)\eta^4(40z)}. \tag{4.3.6}$$

We first consider $\varphi(q)\varphi^2(q^2)\varphi(q^5)$. From (4.3.3), we have

Table 4.3.2(a)

δ	1	2	4	5	8	10	20
r_δ	-2	1	8	-2	-4	5	-2

Table 4.3.2(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1 \le \delta \mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	36/5	24	0	0	36	24	0

It follows from Table 4.3.2(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.3.2(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi^2(q^2)\varphi(q^5) \in M_2(\Gamma_0(40), \chi_1)$.

Next we consider $\varphi^2(q)\varphi(q^2)\varphi(q^{10})$. From (4.3.4), we have

Table 4.3.3(a)

δ	1	2	4	8	10	20	40
r_δ	-4	8	1	-2	-2	5	-2

Table 4.3.3(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12	72/5	0	0	12	72	0

It follows from Table 4.3.3(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.3.3(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^2(q)\varphi(q^2)\varphi(q^{10}) \in M_2(\Gamma_0(40), \chi_1)$.

Now we turn to $\varphi(q^2)\varphi^2(q^5)\varphi(q^{10})$. From (4.3.5), we have

Table 4.3.4(a)

δ	2	4	5	8	10	20	40
r_δ	-2	5	-4	-2	8	1	-2

Table 4.3.4(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	12/5	72/5	0	0	60	72	0

It follows from Table 4.3.4(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.3.4(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q^2)\varphi^2(q^5)\varphi(q^{10}) \in M_2(\Gamma_0(40), \chi_1)$.

Finally, we consider $\varphi(q)\varphi(q^5)\varphi^2(q^{10})$. From (4.3.6), we have

Table 4.3.5(a)

δ	1	2	4	5	10	20	40
r_δ	-2	5	-2	-2	1	8	-4

Table 4.3.5(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	36/5	24/5	0	0	36	120	0

It follows from Table 4.3.5(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.3.5(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi(q^5)\varphi^2(q^{10}) \in M_2(\Gamma_0(40), \chi_1)$. ■

Theorem 4.3.2. *$B_1(q)$ and $B_2(q)$ defined in (4.3.1) and (4.3.2) respectively are in $S_2(\Gamma_0(40), \chi_1)$.*

Proof. First we consider

$$B_1(q) = \frac{\eta(2z)\eta^4(20z)}{\eta(10z)}.$$

Table 4.3.6(a)

δ	2	10	20
r_δ	1	-1	4

Table 4.3.6(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/5	12/5	24/5	3	24/5	12	72	72

It follows from Table 4.3.6(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.3.6(b), L4 is also satisfied for each positive divisor d of 40. Thus $B_1(q) \in S_2(\Gamma_0(40), \chi_1)$.

Next we consider

$$B_2(q) = \frac{\eta^4(4z)\eta(10z)}{\eta(2z)}.$$

Table 4.3.7(a)

δ	2	4	10
r_δ	-1	4	1

Table 4.3.7(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40}\dfrac{\gcd(d,\delta)^2\cdot r_\delta}{\delta}$	3/5	12/5	72/5	3	72/5	12	24	24

It follows from Table 4.3.7(a) that the conditions L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.3.7(b), L4 is also satisfied for each positive divisor d of 40. Thus $B_2(q) \in S_2(\Gamma_0(40), \chi_1)$. Hence we conclude that $B_k(q)$ $(k = 1, 2) \in S_2(\Gamma_0(40), \chi_1)$. ∎

Theorem 4.3.3. **(a)** *$\{B_1(q), B_2(q)\}$ is a basis for $S_2(\Gamma_0(40), \chi_1)$.*
(b) *$\{E_{2,\chi_0,\chi_1}(q^t), E_{2,\chi_1,\chi_0}(q^t) \mid t = 1, 2, 4, 8\}$ is a basis for $E_2(\Gamma_0(40), \chi_1)$.*
(c) *$\{E_{2,\chi_0,\chi_1}(q^t), E_{2,\chi_1,\chi_0}(q^t) \mid t = 1, 2, 4, 8\}$ together with $B_1(q)$ and $B_2(q)$ constitute a basis for $M_2(\Gamma_0(40), \chi_1)$.*

Proof.(a) By Theorem 4.3.2, $B_k(q)$ $(k = 1, 2) \in S_2(\Gamma_0(40), \chi_1)$. They are linearly independent over $\mathbb{C}$. By Example 3.2.1, we have $\dim S_2(\Gamma_0(40), \chi_1) = 2$. Therefore, $\{B_1(q), B_2(q)\}$ is a basis for $S_2(\Gamma_0(40), \chi_1)$.

(b) By Example 3.2.1, we have $\dim E_2(\Gamma_0(40), \chi_1) = 8$. By Theorem 2.3.1(b), $\{E_{2,\chi_0,\chi_1}(q^t), E_{2,\chi_1,\chi_0}(q^t) \mid t = 1, 2, 4, 8\}$ constitute a basis for $E_2(\Gamma_0(40), \chi_1)$.

(c) By Example 3.2.1, we have $\dim M_2(\Gamma_0(40), \chi_1) = 10$. Therefore, by (2.3.2) $\{E_{2,\chi_0,\chi_1}(q^t), E_{2,\chi_1,\chi_0}(q^t) \mid t = 1, 2, 4, 8\}$ together with $B_k(q)$ $(k = 1, 2)$ constitute a basis for $M_2(\Gamma_0(40), \chi_1)$. ∎

Theorem 4.3.4. *Let χ_0 be the trivial character and χ_1 be as in* (4.1.3)*. Then*

$$
\begin{aligned}
\textbf{(a)}\ \varphi(q)\varphi^2(q^2)\varphi(q^5) &= \frac{1}{2}E_{2,\chi_0,\chi_1}(q) - \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4) \\
&\quad -4E_{2,\chi_0,\chi_1}(q^8) + \frac{5}{2}E_{2,\chi_1,\chi_0}(q) + \frac{5}{2}E_{2,\chi_1,\chi_0}(q^2) - 5E_{2,\chi_1,\chi_0}(q^4) \\
&\quad +20E_{2,\chi_1,\chi_0}(q^8) + 5B_1(q) - B_2(q),
\end{aligned}
$$

$$\textbf{(b)}\ \varphi^2(q)\varphi(q^2)\varphi(q^{10}) = -\frac{1}{2}E_{2,\chi_0,\chi_1}(q) + \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4)$$
$$-4E_{2,\chi_0,\chi_1}(q^8) + \frac{5}{2}E_{2,\chi_1,\chi_0}(q) + \frac{5}{2}E_{2,\chi_1,\chi_0}(q^2) + 5E_{2,\chi_1,\chi_0}(q^4)$$
$$-20E_{2,\chi_1,\chi_0}(q^8) + 2B_2(q),$$

$$\textbf{(c)}\ \varphi(q^2)\varphi^2(q^5)\varphi(q^{10}) = -\frac{1}{2}E_{2,\chi_0,\chi_1}(q) + \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4)$$
$$-4E_{2,\chi_0,\chi_1}(q^8) + \frac{1}{2}E_{2,\chi_1,\chi_0}(q) + \frac{1}{2}E_{2,\chi_1,\chi_0}(q^2) + E_{2,\chi_1,\chi_0}(q^4)$$
$$-4E_{2,\chi_1,\chi_0}(q^8) - 2B_1(q),$$

$$\textbf{(d)}\ \varphi(q)\varphi(q^5)\varphi^2(q^{10}) = \frac{1}{2}E_{2,\chi_0,\chi_1}(q) - \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4)$$
$$-4E_{2,\chi_0,\chi_1}(q^8) + \frac{1}{2}E_{2,\chi_1,\chi_0}(q) + \frac{1}{2}E_{2,\chi_1,\chi_0}(q^2) - E_{2,\chi_1,\chi_0}(q^4)$$
$$+4E_{2,\chi_1,\chi_0}(q^8) - B_1(q) + B_2(q).$$

Proof. By Theorem 4.3.1 we have $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40), \chi_1)$ for $(a_1, a_2, a_3, a_4) = (1, 2, 2, 5)$, $(1, 1, 2, 10)$, $(2, 5, 5, 10)$ and $(1, 5, 10, 10)$. Therefore, by Theorem 4.3.3(c), each $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $\{E_{2,\chi_0,\chi_1}(q^t), E_{2,\chi_1,\chi_0}(q^t) \mid t = 1, 2, 4, 8\}$ and $B_k(q)$ $(k = 1, 2)$, namely

$$\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) = x_1E_{2,\chi_0,\chi_1}(q) + x_2E_{2,\chi_0,\chi_1}(q^2) + x_3E_{2,\chi_0,\chi_1}(q^4)$$
$$+x_4E_{2,\chi_0,\chi_1}(q^8) + x_5E_{2,\chi_1,\chi_0}(q) + x_6E_{2,\chi_1,\chi_0}(q^2) + x_7E_{2,\chi_1,\chi_0}(q^4)$$
$$+x_8E_{2,\chi_1,\chi_0}(q^8) + y_1B_1(q) + y_2B_2(q).$$

Using MAPLE we equate the first few coefficients of q^n on both sides of the equation above to obtain the x_i $(i = 1, \ldots, 8)$ and y_j $(j = 1, 2)$. ∎

Theorem 4.3.5. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in (2.3.1) for $i,j \in \{0,1\}$. Then*

$$\begin{aligned}(\mathbf{a})\ N(1,2,2,5;n) =& \frac{1}{2}\sigma_{\chi_0,\chi_1}(n) - \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)\\ &+ \frac{5}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n/2) - 5\sigma_{\chi_1,\chi_0}(n/4) + 20\sigma_{\chi_1,\chi_0}(n/8)\\ &+ 5b_1(n) - b_2(n),\end{aligned}$$

$$\begin{aligned}(\mathbf{b})\ N(1,1,2,10;n) =& -\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) + \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)\\ &+ \frac{5}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n/2) + 5\sigma_{\chi_1,\chi_0}(n/4)\\ &- 20\sigma_{\chi_1,\chi_0}(n/8) + 2b_2(n),\end{aligned}$$

$$\begin{aligned}(\mathbf{c})\ N(2,5,5,10;n) =& -\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) + \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)\\ &+ \frac{1}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n/2) + \sigma_{\chi_1,\chi_0}(n/4)\\ &- 4\sigma_{\chi_1,\chi_0}(n/8) - 2b_1(n),\end{aligned}$$

$$\begin{aligned}(\mathbf{d})\ N(1,5,10,10;n) =& \frac{1}{2}\sigma_{\chi_0,\chi_1}(n) - \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)\\ &+ \frac{1}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n/2) - \sigma_{\chi_1,\chi_0}(n/4) + 4\sigma_{\chi_1,\chi_0}(n/8)\\ &- b_1(n) + b_2(n).\end{aligned}$$

Proof. Appealing to (2.4.5), we have by Theorem 4.3.4

$$\begin{aligned}(\mathbf{a})\ &\sum_{n=0}^{\infty} N(1,2,2,5;n)q^n = \varphi(q)\varphi^2(q^2)\varphi(q^5)\\ &= \frac{1}{2}E_{2,\chi_0,\chi_1}(q) - \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4) - 4E_{2,\chi_0,\chi_1}(q^8) + \frac{5}{2}E_{2,\chi_1,\chi_0}(q)\\ &\quad + \frac{5}{2}E_{2,\chi_1,\chi_0}(q^2) - 5E_{2,\chi_1,\chi_0}(q^4) + 20E_{2,\chi_1,\chi_0}(q^8) + 5B_1(q) - B_2(q)\\ &= 1 + \sum_{n=1}^{\infty}\Big(\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) - \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)\\ &\quad + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n/2) - 5\sigma_{\chi_1,\chi_0}(n/4) + 20\sigma_{\chi_1,\chi_0}(n/8)\\ &\quad + 5b_1(n) - b_2(n)\Big)q^n,\end{aligned}$$

(b) $$\sum_{n=0}^{\infty} N(1,1,2,10;n)q^n = \varphi^2(q)\varphi(q^2)\varphi(q^{10})$$
$$= -\frac{1}{2}E_{2,\chi_0,\chi_1}(q) + \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4) - 4E_{2,\chi_0,\chi_1}(q^8) + \frac{5}{2}E_{2,\chi_1,\chi_0}(q)$$
$$+\frac{5}{2}E_{2,\chi_1,\chi_0}(q^2) + 5E_{2,\chi_1,\chi_0}(q^4) - 20E_{2,\chi_1,\chi_0}(q^8) + 2B_2(q)$$
$$= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) + \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)$$
$$+\frac{5}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{5}{2}\sigma_{\chi_1,\chi_0}(n/2) + 5\sigma_{\chi_1,\chi_0}(n/4) - 20\sigma_{\chi_1,\chi_0}(n/8) + 2b_2(n)\Big)q^n,$$

(c) $$\sum_{n=0}^{\infty} N(2,5,5,10;n)q^n = \varphi(q^2)\varphi^2(q^5)\varphi(q^{10})$$
$$= -\frac{1}{2}E_{2,\chi_0,\chi_1}(q) + \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4) - 4E_{2,\chi_0,\chi_1}(q^8) + \frac{1}{2}E_{2,\chi_1,\chi_0}(q)$$
$$+\frac{1}{2}E_{2,\chi_1,\chi_0}(q^2) + E_{2,\chi_1,\chi_0}(q^4) - 4E_{2,\chi_1,\chi_0}(q^8) - 2B_1(q)$$
$$= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) + \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)$$
$$+\frac{1}{2}E_{2,\chi_1,\chi_0}(n) + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n/2) + \sigma_{\chi_1,\chi_0}(n/4) - 4\sigma_{\chi_1,\chi_0}(n/8) - 2b_1(n)\Big)q^n,$$

(d) $$\sum_{n=0}^{\infty} N(1,5,10,10;n)q^n = \varphi(q)\varphi(q^5)\varphi^2(q^{10})$$
$$= \frac{1}{2}E_{2,\chi_0,\chi_1}(q) - \frac{1}{2}E_{2,\chi_0,\chi_1}(q^2) - E_{2,\chi_0,\chi_1}(q^4) - 4E_{2,\chi_0,\chi_1}(q^8) + \frac{1}{2}E_{2,\chi_1,\chi_0}(q)$$
$$+\frac{1}{2}E_{2,\chi_1,\chi_0}(q^2) - E_{2,\chi_1,\chi_0}(q^4) + 4E_{2,\chi_1,\chi_0}(q^8) - B_1(q) + B_2(q)$$
$$= 1 + \sum_{n=1}^{\infty}\Big(\frac{1}{2}\sigma_{\chi_0,\chi_1}(n) - \frac{1}{2}\sigma_{\chi_0,\chi_1}(n/2) - \sigma_{\chi_0,\chi_1}(n/4) - 4\sigma_{\chi_0,\chi_1}(n/8)$$
$$+\frac{1}{2}\sigma_{\chi_1,\chi_0}(n) + \frac{1}{2}\sigma_{\chi_1,\chi_0}(n/2) - \sigma_{\chi_1,\chi_0}(n/4) + 4\sigma_{\chi_1,\chi_0}(n/8)$$
$$-b_1(n) + b_2(n)\Big)q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(d), we deduce the asserted results. ∎

The following formulas have been given by Liouville [36], [37]

$$N(1,1,1,5;n) = \sum_{d|n}(-1)^{n+d}\Big(\frac{5}{d}\Big)d + 5\sum_{d|n}(-1)^{n+d}\Big(\frac{5}{n/d}\Big)d,$$

$$N(1,5,5,5;n) = \sum_{d|n}(-1)^{n+d}\Big(\frac{5}{d}\Big)d + \sum_{d|n}(-1)^{n+d}\Big(\frac{5}{n/d}\Big)d.$$

We give equivalent formulas in terms of $\sigma_{\chi_0,\chi_1}(n/k)$ and $\sigma_{\chi_1,\chi_0}(n/k)$ $(k = 1,2,4)$ in the following theorem.

Theorem 4.3.6. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in* (2.3.1) *for $i,j \in \{0,1\}$. Then*

$$\begin{aligned}(\mathbf{a})\ N(1,1,1,5;n) =& \sigma_{\chi_0,\chi_1}(n) - 2\sigma_{\chi_0,\chi_1}(n/2) - 4\sigma_{\chi_0,\chi_1}(n/4) + 5\sigma_{\chi_1,\chi_0}(n)\\ &+ 10\sigma_{\chi_1,\chi_0}(n/2) - 20\sigma_{\chi_1,\chi_0}(n/4),\\ (\mathbf{b})\ N(1,5,5,5;n) =& \sigma_{\chi_0,\chi_1}(n) - 2\sigma_{\chi_0,\chi_1}(n/2) - 4\sigma_{\chi_0,\chi_1}(n/4) + \sigma_{\chi_1,\chi_0}(n)\\ &+ 2\sigma_{\chi_1,\chi_0}(n/2) - 4\sigma_{\chi_1,\chi_0}(n/4).\end{aligned}$$

Proof. Appealing to (2.4.7) we have

$$\begin{aligned}\varphi^3(q)\varphi(q^5) &= \frac{\eta^{15}(2z)\eta^5(10z)}{\eta^6(z)\eta^6(4z)\eta^2(5z)\eta^2(20z)},\\ \varphi(q)\varphi^3(q^5) &= \frac{\eta^5(2z)\eta^{15}(10z)}{\eta^2(z)\eta^2(4z)\eta^6(5z)\eta^6(20z)}.\end{aligned}$$

By Theorem 2.4.1, we have $\varphi^3(q)\varphi(q^5)$ and $\varphi(q)\varphi^3(q^5) \in M_2(\Gamma_0(40),\chi_1)$. By Theorem 4.3.3 (c), we know that $\{E_{2,\chi_0,\chi_1}(q^t), E_{2,\chi_1,\chi_0}(q^t) \mid t = 1,2,4,8\}$ together with $B_k(q)$ $(k = 1,2)$ constitute a basis for $M_2(\Gamma_0(40),\chi_1)$. Thus $\varphi^3(q)\varphi(q^5)$ and $\varphi(q)\varphi^3(q^5)$ must be expressible as linear combinations of $\{E_{2,\chi_0,\chi_1}(q^t), E_{2,\chi_1,\chi_0}(q^t) \mid t = 1,2,4,8\}$ and $B_k(q)$ $(k = 1,2)$, namely

$$\varphi^3(q)\varphi(q^5) = E_{2,\chi_0,\chi_1}(q) - 2E_{2,\chi_0,\chi_1}(q^2) - 4E_{2,\chi_0,\chi_1}(q^4) + 5E_{2,\chi_1,\chi_0}(q) + 10E_{2,\chi_1,\chi_0}(q^2) - 20E_{2,\chi_1,\chi_0}(q^4), \tag{4.3.7}$$

and

$$\varphi(q)\varphi^3(q^5) = E_{2,\chi_0,\chi_1}(q) - 2E_{2,\chi_0,\chi_1}(q^2) - 4E_{2,\chi_0,\chi_1}(q^4) + E_{2,\chi_1,\chi_0}(q) + 2E_{2,\chi_1,\chi_0}(q^2) - 4E_{2,\chi_1,\chi_0}(q^4). \tag{4.3.8}$$

Appealing to (2.4.5), (4.3.7) and (4.3.8), we have

$$\begin{aligned}
&\textbf{(a)} \sum_{n=0}^{\infty} N(1,1,1,5;n)q^n = \varphi^3(q)\varphi(q^5)\\
&= E_{2,\chi_0,\chi_1}(q) - 2E_{2,\chi_0,\chi_1}(q^2) - 4E_{2,\chi_0,\chi_1}(q^4) + 5E_{2,\chi_1,\chi_0}(q)\\
&\quad + 10E_{2,\chi_1,\chi_0}(q^2) - 20E_{2,\chi_1,\chi_0}(q^4)\\
&= 1 + \sum_{n=1}^{\infty} \Big(\sigma_{\chi_0,\chi_1}(n) - 2\sigma_{\chi_0,\chi_1}(n/2) - 4\sigma_{\chi_0,\chi_1}(n/4) + 5\sigma_{\chi_1,\chi_0}(n)\\
&\quad + 10\sigma_{\chi_1,\chi_0}(n/2) - 20\sigma_{\chi_1,\chi_0}(n/4)\Big)q^n,\\
&\textbf{(b)} \sum_{n=0}^{\infty} N(1,5,5,5;n)q^n = \varphi(q)\varphi^3(q^5)\\
&= E_{2,\chi_0,\chi_1}(q) - 2E_{2,\chi_0,\chi_1}(q^2) - 4E_{2,\chi_0,\chi_1}(q^4) + E_{2,\chi_1,\chi_0}(q)\\
&\quad + 2E_{2,\chi_1,\chi_0}(q^2) - 4E_{2,\chi_1,\chi_0}(q^4) \qquad .\\
&= 1 + \sum_{n=1}^{\infty} \Big(\sigma_{\chi_0,\chi_1}(n) - 2\sigma_{\chi_0,\chi_1}(n/2) - 4\sigma_{\chi_0,\chi_1}(n/4) + \sigma_{\chi_1,\chi_0}(n)\\
&\quad + 2\sigma_{\chi_1,\chi_0}(n/2) - 4\sigma_{\chi_1,\chi_0}(n/4)\Big)q^n.
\end{aligned}$$

Equating the coefficients of q^n on both sides of equations (a) and (b), we deduce the asserted results. ∎

4.4 The Space $M_2(\Gamma_0(40), \chi_2)$

Let χ_2 be the Dirichlet character as in (4.1.3). We define the following Eisenstein series

$$E_{2,\chi_0,\chi_2}(q) = -\frac{1}{2} + \sum_{n=1}^{\infty} \sigma_{(\chi_0,\chi_2)}(n)q^n,$$
$$E_{2,\chi_2,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_2,\chi_0)}(n)q^n.$$

We give formulas for $N(a_1, a_2, a_3, a_4; n)$ for the quaternary quadratic forms given by

$$(a_1, a_2, a_3, a_4) = (1, 2, 5, 5),\ (1, 1, 5, 10),\ (2, 2, 5, 10),\ (1, 2, 10, 10)$$

in terms of $\sigma_{\chi,\psi}$ where $\chi, \psi \in \{\chi_0, \chi_2\}$ and $c_k(n)$ $(1 \leq k \leq 4)$ defined by

$$C_1(q) = \sum_{n=1}^{\infty} c_1(n)q^n = \frac{\eta^2(z)\eta(8z)\eta^2(10z)\eta(40z)}{\eta(2z)\eta(20z)}, \tag{4.4.1}$$
$$C_2(q) = \sum_{n=1}^{\infty} c_2(n)q^n = \frac{\eta(z)\eta(5z)\eta^2(8z)\eta^2(20z)}{\eta(4z)\eta(10z)}, \tag{4.4.2}$$
$$C_3(q) = \sum_{n=1}^{\infty} c_3(n)q^n = \frac{\eta^6(2z)\eta(10z)\eta^2(40z)}{\eta^2(z)\eta^2(4z)\eta(20z)}, \tag{4.4.3}$$
$$C_4(q) = \sum_{n=1}^{\infty} c_4(n)q^n = \frac{\eta^6(4z)\eta^2(5z)\eta(20z)}{\eta^2(2z)\eta^2(8z)\eta(10z)}. \tag{4.4.4}$$

There is no linear relationship among the $C_k(q)$ $(1 \leq k \leq 4)$. The first forty values of $c_k(n)$ $(1 \leq k \leq 4)$, are given in Table 4.4.1.

Table 4.4.1

n	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$	n	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
1	0	0	0	1	21	0	0	2	2
2	1	1	0	0	22	−2	0	0	4
3	−2	−1	1	2	23	0	3	−3	2
4	0	−1	2	0	24	4	0	0	−4

n	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$	n	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
5	0	0	−1	−1	25	0	0	0	−1
6	2	1	−2	−2	26	−2	−2	4	0
7	0	−1	1	−2	27	0	0	−4	−4
8	0	0	−2	−4	28	−2	0	−2	4
9	0	2	−2	1	29	−8	−4	0	4
10	−1	0	0	2	30	0	1	−2	−2
11	0	0	−2	−2	31	0	2	−2	0
12	−2	0	2	4	32	4	4	−4	0
13	4	2	0	−2	33	0	−2	2	0
14	−2	−1	−2	−2	34	−2	0	−4	−4
15	0	−1	1	0	35	2	1	1	0
16	−4	−2	0	4	36	4	3	−2	−8
17	0	−2	2	−2	37	0	0	−2	−2
18	1	−1	4	4	38	2	−2	4	−8
19	4	2	4	2	39	0	−2	2	4
20	2	1	0	0	40	0	0	2	0

Theorem 4.4.1. *For* $(a_1, a_2, a_3, a_4) = (1, 2, 5, 5),\ (1, 1, 5, 10),\ (2, 2, 5, 10),$ $(1, 2, 10, 10),$ *we have* $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40), \chi_2)$.

Proof. Appealing to (2.4.7) we have

$$\varphi(q)\varphi(q^2)\varphi^2(q^5) = \frac{\eta^3(2z)\eta^3(4z)\eta^{10}(10z)}{\eta^2(z)\eta^4(5z)\eta^2(8z)\eta^4(20z)}, \tag{4.4.5}$$

$$\varphi^2(q)\varphi(q^5)\varphi(q^{10}) = \frac{\eta^{10}(2z)\eta^3(10z)\eta^3(20z)}{\eta^4(z)\eta^4(4z)\eta^2(5z)\eta^2(40z)}, \tag{4.4.6}$$

$$\varphi^2(q^2)\varphi(q^5)\varphi(q^{10}) = \frac{\eta^{10}(4z)\eta^3(10z)\eta^3(20z)}{\eta^4(2z)\eta^2(5z)\eta^4(8z)\eta^2(40z)}, \tag{4.4.7}$$

$$\varphi(q)\varphi(q^2)\varphi^2(q^{10}) = \frac{\eta^3(2z)\eta^3(4z)\eta^{10}(20z)}{\eta^2(z)\eta^2(8z)\eta^4(10z)\eta^4(40z)}. \tag{4.4.8}$$

We first consider $\varphi(q)\varphi(q^2)\varphi^2(q^5)$. From (4.4.5), we have

Table 4.4.2(a)

δ	1	2	4	5	8	10	20
r_δ	−2	3	3	−4	−2	10	−4

Table 4.4.2(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\leq\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	42/5	12	0	0	66	12	0

It follows from Table 4.4.2(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.2(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi(q^2)\varphi^2(q^5) \in M_2(\Gamma_0(40), \chi_2)$.

Next we consider $\varphi^2(q)\varphi(q^5)\varphi(q^{10})$. From (4.4.6), we have

Table 4.4.3(a)

δ	1	2	4	5	10	20	40
r_δ	−4	10	−4	−2	3	3	−2

Table 4.4.3(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\leq\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	66/5	12/5	0	0	42	60	0

It follows from Table 4.4.3(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.3(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^2(q)\varphi(q^5)\varphi(q^{10}) \in M_2(\Gamma_0(40), \chi_2)$.

Now we consider $\varphi^2(q^2)\varphi(q^5)\varphi(q^{10})$. From (4.4.7), we have

Table 4.4.4(a)

δ	2	4	5	8	10	20	40
r_δ	−4	10	−2	−4	3	3	−2

Table 4.4.4(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\leq\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6/5	132/5	0	0	30	84	0

It follows from Table 4.4.4(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.4(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^2(q^2)\varphi(q^5)\varphi(q^{10}) \in M_2(\Gamma_0(40), \chi_2)$.

Finally, we consider $\varphi(q)\varphi(q^2)\varphi^2(q^{10})$. From (4.4.8), we have

Table 4.4.5(a)

δ	1	2	4	8	10	20	40
r_δ	-2	3	3	-2	-4	10	-4

Table 4.4.5(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6	84/5	0	0	6	132	0

It follows from Table 4.4.5(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.5(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi(q^2)\varphi^2(q^{10}) \in M_2(\Gamma_0(40), \chi_2)$.

Theorem 4.4.2. *$C_k(q)$ $(1 \le k \le 4)$ defined by (4.4.1)–(4.4.4) are in $S_2(\Gamma_0(40), \chi_2)$.*

Proof. First we consider

$$C_1(q) = \frac{\eta^2(z)\eta(8z)\eta^2(10z)\eta(40z)}{\eta(2z)\eta(20z)}.$$

Table 4.4.6(a)

δ	1	2	8	10	20	40
r_δ	2	-1	1	2	-1	1

Table 4.4.6(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	9/5	6/5	12/5	6	48/5	18	12	48

It follows from Table 4.4.6(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.6(b), L4 is also satisfied for each positive divisor d of 40. Thus $C_1(q) \in S_2(\Gamma_0(40), \chi_2)$.

Next we consider

$$C_2(q) = \frac{\eta(z)\eta(5z)\eta^2(8z)\eta^2(20z)}{\eta(4z)\eta(10z)}.$$

Table 4.4.7(a)

δ	1	4	5	8	10	20
r_δ	1	-1	1	2	-1	2

Table 4.4.7(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/5	6/5	12/5	6	72/5	6	36	48

It follows from Table 4.4.7(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.7(b), L4 is also satisfied for each positive divisor d of 40. Thus $C_2(q) \in S_2(\Gamma_0(40), \chi_2)$.

Now we turn to

$$C_3(q) = \frac{\eta^6(2z)\eta(10z)\eta^2(40z)}{\eta^2(z)\eta^2(4z)\eta(20z)}.$$

Table 4.4.8(a)

δ	1	2	4	10	20	40
r_δ	-2	6	-2	1	-1	2

Table 4.4.8(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\leq\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/5	42/5	12/5	3	24/5	18	12	72

It follows from Table 4.4.8(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.8(b), L4 is also satisfied for each positive divisor d of 40. Thus $C_3(q) \in S_2(\Gamma_0(40), \chi_2)$.

Finally, we consider

$$C_4(q) = \frac{\eta^6(4z)\eta^2(5z)\eta(20z)}{\eta^2(2z)\eta^2(8z)\eta(10z)}.$$

Table 4.4.9(a)

δ	2	4	5	8	10	20
r_δ	-2	6	2	-2	-1	1

Table 4.4.9(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\leq\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/5	6/5	84/5	9	24/5	6	36	24

It follows from Table 4.4.9(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.4.9(b), L4 is also satisfied for each positive divisor d of 40. Thus $C_4(q) \in S_2(\Gamma_0(40), \chi_2)$. Hence we deduce that $C_k(q)$ $(1 \leq k \leq 4) \in S_2(\Gamma_0(40), \chi_2)$. ∎

Theorem 4.4.3. (a) *$\{C_k(q) \mid k = 1, 2, 3, 4\}$ is a basis for $S_2(\Gamma_0(40), \chi_2)$.*

(b) *$\{E_{2,\chi_0,\chi_2}(q^t), E_{2,\chi_2,\chi_0}(q^t) \mid t = 1, 5\}$ is a basis for $E_2(\Gamma_0(40), \chi_2)$.*

(c) *$\{E_{2,\chi_0,\chi_2}(q^t), E_{2,\chi_2,\chi_0}(q^t) \mid t = 1, 5\}$ together with $C_k(q)$ $(1 \leq k \leq 4)$ constitute a basis for $M_2(\Gamma_0(40), \chi_2)$.*

Proof. **(a)** By Theorem 4.4.2, $C_k(q)$ $(1 \leq k \leq 4) \in S_2(\Gamma_0(40), \chi_2)$. They are linearly independent over $\mathbb{C}$. By Example 3.2.1, we have $\dim S_2(\Gamma_0(40), \chi_2) = 4$. Thus, $C_k(q)$ $(1 \leq k \leq 4)$ constitute a basis for $S_2(\Gamma_0(40), \chi_2)$.

(b) By Example 3.2.1, we have $\dim E_2(\Gamma_0(40), \chi_2) = 4$. By Theorem 2.3.1(b), $\{E_{2,\chi_0,\chi_2}(q^t), E_{2,\chi_2,\chi_0}(q^t) \mid t = 1, 5\}$ constitute a basis for $E_2(\Gamma_0(40), \chi_2)$.

(c) By Example 3.2.1, we have $\dim M_2(\Gamma_0(40), \chi_2) = 8$. Therefore, by (2.3.2), $\{E_{2,\chi_0,\chi_2}(q^t), E_{2,\chi_2,\chi_0}(q^t) \mid t = 1, 5\}$ together with $C_k(q)$ $(1 \leq k \leq 4)$ constitute a basis for $M_2(\Gamma_0(40), \chi_2)$. ∎

Theorem 4.4.4. *Let χ_0 be the trivial character and χ_2 be as in* (4.1.3). *Then*

$$\begin{aligned}
&\textbf{(a)}\ \varphi(q)\varphi(q^2)\varphi^2(q^5) = -\frac{6}{13}E_{2,\chi_0,\chi_2}(q) - \frac{20}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{24}{13}E_{2,\chi_2,\chi_0}(q) \\
&\qquad -\frac{80}{13}E_{2,\chi_2,\chi_0}(q^5) - \frac{16}{13}C_2(q) - \frac{40}{13}C_3(q) + \frac{8}{13}C_4(q), \\
&\textbf{(b)}\ \varphi^2(q)\varphi(q^5)\varphi(q^{10}) = \frac{4}{13}E_{2,\chi_0,\chi_2}(q) - \frac{30}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{16}{13}E_{2,\chi_2,\chi_0}(q) \\
&\qquad +\frac{120}{13}E_{2,\chi_2,\chi_0}(q^5) + \frac{48}{13}C_1(q) - \frac{32}{13}C_2(q) - \frac{24}{13}C_3(q) + \frac{32}{13}C_4(q), \\
&\textbf{(c)}\ \varphi^2(q^2)\varphi(q^5)\varphi(q^{10}) = \frac{4}{13}E_{2,\chi_0,\chi_2}(q) - \frac{30}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{8}{13}E_{2,\chi_2,\chi_0}(q) \\
&\qquad +\frac{60}{13}E_{2,\chi_2,\chi_0}(q^5) - \frac{16}{13}C_1(q) + \frac{48}{13}C_2(q) + \frac{32}{13}C_3(q) - \frac{12}{13}C_4(q), \\
&\textbf{(d)}\ \varphi(q)\varphi(q^2)\varphi^2(q^{10}) = -\frac{6}{13}E_{2,\chi_0,\chi_2}(q) - \frac{20}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{12}{13}E_{2,\chi_2,\chi_0}(q) \\
&\qquad -\frac{40}{13}E_{2,\chi_2,\chi_0}(q^5) + \frac{8}{13}C_1(q) - \frac{8}{13}C_3(q) + \frac{20}{13}C_4(q).
\end{aligned}$$

Proof. By Theorem 4.4.1 we have $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40), \chi_2)$ for $(a_1, a_2, a_3, a_4) = (1, 2, 5, 5)$, $(1, 1, 5, 10)$, $(2, 2, 5, 10)$ and $(1, 2, 10, 10)$. Therefore, by Theorem 4.4.3(c), each $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $\{E_{2,\chi_0,\chi_2}(q^t), E_{2,\chi_2,\chi_0}(q^t) \mid t = 1, 5\}$ and $C_k(q)$ $(1 \leq k \leq 4)$, namely

$$\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) = x_1 E_{2,\chi_0,\chi_2}(q) + x_2 E_{2,\chi_0,\chi_2}(q^5) + x_3 E_{2,\chi_2,\chi_0}(q)$$

$$+x_4 E_{2,\chi_2,\chi_0}(q^5) + y_1 C_1(q) + y_2 C_2(q) + y_3 C_3(q) + y_4 C_4(q).$$

Using MAPLE we equate the first few coefficients of q^n on both sides of the equation above to obtain the x_i and y_i $(1 \le i \le 4)$. ∎

Theorem 4.4.5. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in* (2.3.1) *for $i, j \in \{0, 2\}$. Then*

$$\begin{aligned}
(\mathbf{a})\ N(1,2,5,5;n) =& -\frac{6}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{20}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{24}{13}\sigma_{\chi_2,\chi_0}(n) - \frac{80}{13}\sigma_{\chi_2,\chi_0}(n/5)\\
&- \frac{16}{13}c_2(n) - \frac{40}{13}c_3(n) + \frac{8}{13}c_4(n),\\
(\mathbf{b})\ N(1,1,5,10;n) =& \frac{4}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{30}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{16}{13}\sigma_{\chi_2,\chi_0}(n) + \frac{120}{13}\sigma_{\chi_2,\chi_0}(n/5)\\
&+ \frac{48}{13}c_1(n) - \frac{32}{13}c_2(n) - \frac{24}{13}c_3(n) + \frac{32}{14}c_4(n),\\
(\mathbf{c})\ N(2,2,5,10;n) =& \frac{4}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{30}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{8}{13}\sigma_{\chi_2,\chi_0}(n) + \frac{60}{13}\sigma_{\chi_2,\chi_0}(n/5)\\
&- \frac{16}{13}c_1(n) + \frac{48}{13}c_2(n) + \frac{32}{13}c_3(n) - \frac{12}{13}c_4(n),\\
(\mathbf{d})\ N(1,2,10,10;n) =& -\frac{6}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{20}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{12}{13}\sigma_{\chi_2,\chi_0}(n)\\
&- \frac{40}{13}\sigma_{\chi_2,\chi_0}(n/5) + \frac{8}{13}c_1(n) - \frac{8}{13}c_3(n) + \frac{20}{13}c_4(n).
\end{aligned}$$

Proof. Appealing to (2.4.5), we have by Theorem 4.4.4

$$\begin{aligned}
(\mathbf{a})\ &\sum_{n=0}^{\infty} N(1,2,5,5;n)q^n = \varphi(q)\varphi(q^2)\varphi^2(q^5)\\
&= -\frac{6}{13}E_{2,\chi_0,\chi_2}(q) - \frac{20}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{24}{13}E_{2,\chi_2,\chi_0}(q) - \frac{80}{13}E_{2,\chi_2,\chi_0}(q^5)\\
&\quad -\frac{16}{13}C_2(q) - \frac{40}{13}C_3(q) + \frac{8}{13}C_4(q)\\
&= 1 + \sum_{n=1}^{\infty}\Big(-\frac{6}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{20}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{24}{13}\sigma_{\chi_2,\chi_0}(n) - \frac{80}{13}\sigma_{\chi_2,\chi_0}(n/5)\\
&\quad -\frac{16}{13}c_2(n) - \frac{40}{13}c_3(n) + \frac{8}{13}c_4(n)\Big)q^n,
\end{aligned}$$

$$
\begin{aligned}
\textbf{(b)}\ \sum_{n=0}^{\infty} N(1,1,5,10;n)q^n &= \varphi^2(q)\varphi(q^5)\varphi(q^{10}) \\
&= \frac{4}{13}E_{2,\chi_0,\chi_2}(q) - \frac{30}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{16}{13}E_{2,\chi_2,\chi_0}(q) + \frac{120}{13}E_{2,\chi_2,\chi_0}(q^5) \\
&\quad + \frac{48}{13}C_1(q) - \frac{32}{13}C_2(q) - \frac{24}{13}C_3(q) + \frac{32}{13}C_4(q) \\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{4}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{30}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{16}{3}\sigma_{\chi_2,\chi_0}(n) + \frac{120}{13}\sigma_{\chi_2,\chi_0}(n/5) \\
&\quad + \frac{48}{13}c_1(n) - \frac{32}{13}c_2(n) - \frac{24}{13}c_3(n) + \frac{32}{13}c_4(n)\Big)q^n,
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(c)}\ \sum_{n=0}^{\infty} N(2,2,5,10;n)q^n &= \varphi^2(q^2)\varphi(q^5)\varphi(q^{10}) \\
&= \frac{4}{13}E_{2,\chi_0,\chi_2}(q) - \frac{30}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{8}{13}E_{2,\chi_2,\chi_0}(q) + \frac{60}{13}E_{2,\chi_2,\chi_0}(q^5) - \frac{16}{13}C_1(q) \\
&\quad + \frac{48}{13}C_2(q) + \frac{32}{13}C_3(q) - \frac{12}{13}C_4(q) \\
&= 1 + \sum_{n=1}^{\infty}\Big(\frac{4}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{30}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{8}{13}\sigma_{\chi_2,\chi_0}(n) + \frac{60}{13}\sigma_{\chi_2,\chi_0}(n/5) \\
&\quad - \frac{16}{13}c_1(n) + \frac{48}{13}c_2(n) + \frac{32}{13}c_3(n) - \frac{12}{13}c_4(n)\Big)q^n,
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(d)}\ \sum_{n=0}^{\infty} N(1,2,10,10;n)q^n &= \varphi(q)\varphi(q^2)\varphi^2(q^{10}) \\
&= -\frac{6}{13}E_{2,\chi_0,\chi_2}(q) - \frac{20}{13}E_{2,\chi_0,\chi_2}(q^5) + \frac{12}{13}E_{2,\chi_2,\chi_0}(q) - \frac{40}{13}E_{2,\chi_2,\chi_0}(q^5) + \frac{8}{13}C_1(q) \\
&\quad - \frac{8}{13}C_3(q) + \frac{20}{13}C_4(q) \\
&= 1 + \sum_{n=1}^{\infty}\Big(-\frac{6}{13}\sigma_{\chi_0,\chi_2}(n) - \frac{20}{13}\sigma_{\chi_0,\chi_2}(n/5) + \frac{12}{13}\sigma_{\chi_2,\chi_0}(n) - \frac{40}{13}\sigma_{\chi_2,\chi_0}(n/5) \\
&\quad + \frac{8}{13}c_1(n) - \frac{8}{13}c_3(n) + \frac{20}{13}c_4(n)\Big)q^n.
\end{aligned}
$$

Equating the coefficients of q^n on both sides of equations (a)–(d), we deduce the asserted results. ∎

4.5 The Space $M_2(\Gamma_0(40), \chi_3)$

Let χ_1, χ_2 and χ_3 be the Dirichlet characters as in (4.1.3). We define the following Eisenstein series

$$E_{2,\chi_0,\chi_3}(q) = -7 + \sum_{n=1}^{\infty} \sigma_{(\chi_0,\chi_3)}(n) q^n,$$
$$E_{2,\chi_1,\chi_2}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_1,\chi_2)}(n) q^n,$$
$$E_{2,\chi_2,\chi_1}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_2,\chi_1)}(n) q^n,$$
$$E_{2,\chi_3,\chi_0}(q) = \sum_{n=1}^{\infty} \sigma_{(\chi_3,\chi_0)}(n) q^n.$$

We give formulas for $N(a_1, a_2, a_3, a_4; n)$ for the quaternary quadratic forms given by

$$(a_1, a_2, a_3, a_4) = \ (1,1,2,5),\ (2,2,2,5),\ (2,5,5,5),\ (1,1,1,10),\ (1,2,2,10),$$
$$(1,5,5,10),\ (2,5,10,10),\ (1,10,10,10)$$

in terms of $\sigma_{\chi,\psi}$ where $\chi, \psi \in \{\chi_0, \chi_1, \chi_2, \chi_3\}$ and $d_k(n)$ $(1 \leq k \leq 4)$ defined by

$$D_1(q) = \sum_{n=1}^{\infty} d_1(n) q^n = \frac{\eta^2(z)\eta^6(4z)\eta(20z)}{\eta^3(2z)\eta^2(8z)}, \tag{4.5.1}$$
$$D_2(q) = \sum_{n=1}^{\infty} d_2(n) q^n = \frac{\eta^2(5z)\eta(8z)\eta(10z)\eta(40z)}{\eta(20z)}, \tag{4.5.2}$$
$$D_3(q) = \sum_{n=1}^{\infty} d_3(n) q^n = \frac{\eta(z)\eta(5z)\eta(20z)\eta^2(40z)}{\eta(10z)}, \tag{4.5.3}$$
$$D_4(q) = \sum_{n=1}^{\infty} d_4(n) q^n = \frac{\eta(z)\eta(4z)\eta(5z)\eta^2(8z)}{\eta(2z)}. \tag{4.5.4}$$

There is no linear relationship among the $D_k(q)$ $(1 \leq k \leq 4)$. The first forty values of $d_k(n)$ $(1 \leq k \leq 4)$, are given in Table 4.5.1.

Table 4.5.1

n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$	n	$d_1(n)$	$d_2(n)$	$d_3(n)$	$d_4(n)$
1	1	0	0	1	21	−6	0	2	0
2	−2	1	0	−1	22	4	2	0	−2
3	2	0	0	0	23	−2	2	0	−2
4	−4	0	1	−1	24	4	0	0	4
5	1	0	−1	0	25	3	−4	0	3
6	2	0	−1	−1	26	0	0	0	0
7	2	−2	0	2	27	−8	0	0	0
8	4	0	0	0	28	−8	2	0	−2
9	−1	0	0	−1	29	0	0	0	0
10	−4	−1	1	2	30	−2	−2	1	−1
11	−6	0	2	0	31	4	0	0	4
12	0	−2	0	2	32	0	−4	0	4
13	0	0	0	0	33	4	−4	0	4
14	6	0	−1	3	34	−12	0	2	−6
15	−4	2	0	−4	35	12	0	−2	0
16	4	0	−2	−2	36	4	0	−1	1
17	−4	4	0	−4	37	12	0	0	0
18	2	−1	0	1	38	−4	−2	0	2
19	6	0	−2	0	39	0	0	0	0
20	0	2	1	1	40	−8	4	0	−8

Theorem 4.5.1. *For* $(a_1, a_2, a_3, a_4) = (1, 1, 2, 5),\ (2, 2, 2, 5),\ (2, 5, 5, 5),$ $(1, 1, 1, 10),\ (1, 2, 2, 10),\ (1, 5, 5, 10),\ (2, 5, 10, 10),\ (1, 10, 10, 10),$ *we have* $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(40), \chi_3)$.

Proof. Appealing to (2.4.7) we have

$$\varphi^2(q)\varphi(q^2)\varphi(q^5) = \frac{\eta^8(2z)\eta(4z)\eta^5(10z)}{\eta^4(z)\eta^2(5z)\eta^2(8z)\eta^2(20z)}, \tag{4.5.5}$$

$$\varphi^3(q^2)\varphi(q^5) = \frac{\eta^{15}(4z)\eta^5(10z)}{\eta^6(2z)\eta^2(5z)\eta^6(8z)\eta^2(20z)}, \tag{4.5.6}$$

$$\varphi(q^2)\varphi^3(q^5) = \frac{\eta^5(4z)\eta^{15}(10z)}{\eta^2(2z)\eta^6(5z)\eta^2(8z)\eta^6(20z)}, \tag{4.5.7}$$

$$\varphi^3(q)\varphi(q^{10}) = \frac{\eta^{15}(2z)\eta^5(20z)}{\eta^6(z)\eta^6(4z)\eta^2(10z)\eta^2(40z)}, \tag{4.5.8}$$

$$\varphi(q)\varphi^2(q^2)\varphi(q^{10}) = \frac{\eta(2z)\eta^8(4z)\eta^5(20z)}{\eta^2(z)\eta^4(8z)\eta^2(10z)\eta^2(40z)}, \tag{4.5.9}$$

$$\varphi(q)\varphi^2(q^5)\varphi(q^{10}) = \frac{\eta^5(2z)\eta^8(10z)\eta(20z)}{\eta^2(z)\eta^2(4z)\eta^4(5z)\eta^2(40z)}, \tag{4.5.10}$$

$$\varphi(q^2)\varphi(q^5)\varphi^2(q^{10}) = \frac{\eta^5(4z)\eta(10z)\eta^8(20z)}{\eta^2(2z)\eta^2(5z)\eta^2(8z)\eta^4(40z)}, \tag{4.5.11}$$

$$\varphi(q)\varphi^3(q^{10}) = \frac{\eta^5(2z)\eta^{15}(20z)}{\eta^2(z)\eta^2(4z)\eta^6(10z)\eta^6(40z)}. \tag{4.5.12}$$

We first consider $\varphi^2(q)\varphi(q^2)\varphi(q^5)$. From (4.5.5), we have

Table 4.5.2(a)

δ	1	2	4	5	8	10	20
r_δ	-4	8	1	-2	-2	5	-2

Table 4.5.2(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	66/5	12	0	0	42	12	0

It follows from Table 4.5.2(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.2(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^2(q)\varphi(q^2)\varphi(q^5) \in M_2(\Gamma_0(40), \chi_3)$.

Next we consider $\varphi^3(q^2)\varphi(q^5)$. From (4.5.6), we have

Table 4.5.3(a)

δ	2	4	5	8	10	20
r_δ	-6	15	-2	-6	5	-2

Table 4.5.3(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6/5	36	0	0	30	36	0

It follows from Table 4.5.3(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.3(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^3(q^2)\varphi(q^5) \in M_2(\Gamma_0(40), \chi_3)$.

Next we consider $\varphi(q^2)\varphi^3(q^5)$. From (4.5.7), we have

Table 4.5.4(a)

δ	2	4	5	8	10	20
r_δ	-2	5	-6	-2	15	-6

Table 4.5.4(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	18/5	12	0	0	90	12	0

It follows from Table 4.5.4(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.4(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q^2)\varphi^3(q^5) \in M_2(\Gamma_0(40), \chi_3)$.

Further we consider $\varphi^3(q)\varphi(q^{10})$. From (4.5.8), we have

Table 4.5.5(a)

δ	1	2	4	10	20	40
r_δ	-6	15	-6	-2	5	-2

Table 4.5.5(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	18	12/5	0	0	18	60	0

It follows from Table 4.5.5(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.5(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi^3(q)\varphi(q^{10}) \in M_2(\Gamma_0(40), \chi_3)$.

Next we consider $\varphi(q)\varphi^2(q^2)\varphi(q^{10})$. From (4.5.9), we have

Table 4.5.6(a)

δ	1	2	4	8	10	20	40
r_δ	-2	1	8	-4	-2	5	-2

Table 4.5.6(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6	132/5	0	0	6	84	0

It follows from Table 4.5.6(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.6(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi^2(q^2)\varphi(q^{10}) \in M_2(\Gamma_0(40), \chi_3)$.

Consider also $\varphi(q)\varphi^2(q^5)\varphi(q^{10})$. From (4.5.10), we obtain

Table 4.5.7(a)

δ	1	2	4	5	10	20	40
r_δ	-2	5	-2	-4	8	1	-2

Table 4.5.7(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	42/5	12/5	0	0	66	60	0

It follows from Table 4.5.7(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.7(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi^2(q^5)\varphi(q^{10}) \in M_2(\Gamma_0(40), \chi_3)$.

Furthermore, consider $\varphi(q^2)\varphi(q^5)\varphi^2(q^{10})$. From (4.5.11), we have

Table 4.5.8(a)

δ	2	4	5	8	10	20	40
r_δ	-2	5	-2	-2	1	8	-4

Table 4.5.8(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6/5	84/5	0	0	30	132	0

It follows from Table 4.5.8(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.8(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q^2)\varphi(q^5)\varphi^2(q^{10}) \in M_2(\Gamma_0(40), \chi_3)$.

Finally, we consider $\varphi(q)\varphi^3(q^{10})$. From (4.5.12), we have

Table 4.5.9(a)

δ	1	2	4	10	20	40
r_δ	-2	5	-2	-6	15	-6

Table 4.5.9(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	6	36/5	0	0	6	180	0

It follows from Table 4.5.9(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.9(b), L3 is also satisfied for each positive divisor d of 40. Thus by Theorem 2.4.1, $\varphi(q)\varphi^3(q^{10}) \in M_2(\Gamma_0(40), \chi_3)$. ∎

Theorem 4.5.2. *$D_k(q)$ $(1 \le k \le 4)$ given by (4.5.1)-(4.5.4) are in $S_2(\Gamma_0(40), \chi_3)$.*

Proof. First we consider

$$D_1(q) = \frac{\eta^2(z)\eta^6(4z)\eta(20z)}{\eta^3(2z)\eta^2(8z)}.$$

Table 4.5.10(a)

δ	1	2	4	8	20
r_δ	2	-3	6	-2	1

Table 4.5.10(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	9/5	6/5	84/5	3	24/5	6	36	24

It follows from Table 4.5.10(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.10(b), L4 is also satisfied for each positive divisor d of 40. Thus $D_1(q) \in S_2(\Gamma_0(40), \chi_3)$.

Next consider

$$D_2(q) = \frac{\eta^2(5z)\eta(8z)\eta(10z)\eta(40z)}{\eta(20z)}.$$

Table 4.5.11(a)

δ	5	8	10	20	40
r_δ	2	1	1	-1	1

Table 4.5.11(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\le\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	3/5	6/5	12/5	12	48/5	18	12	48

It follows from Table 4.5.11(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.11(b), L4 is also satisfied for each positive divisor d of 40. Thus $D_2(q) \in S_2(\Gamma_0(40), \chi_3)$.

Now consider

$$D_3(q) = \frac{\eta(z)\eta(5z)\eta(20z)\eta^2(40z)}{\eta(10z)}.$$

Table 4.5.12(a)

δ	1	5	10	20	40
r_δ	1	1	-1	1	2

Table 4.5.12(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\leq\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/5	6/5	12/5	6	24/5	6	36	96

It follows from Table 4.5.12(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.12(b), L4 is also satisfied for each positive divisor d of 40. $D_3(q) \in S_2(\Gamma_0(40).\chi_3)$.

Finally, we consider

$$D_4(q) = \frac{\eta(z)\eta(4z)\eta(5z)\eta^2(8z)}{\eta(2z)}.$$

Table 4.5.13(a)

δ	1	2	4	5	8
r_δ	1	−1	1	1	2

Table 4.5.13(b)

$d \mid 40$	1	2	4	5	8	10	20	40
$\sum_{1\leq\delta\mid 40} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/5	6/5	36/5	6	96/5	6	12	24

It follows from Table 4.5.13(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 4.5.13(b), L4 is also satisfied for each positive divisor d of 40. Thus condition L4 is also satisfied and so $D_4(q) \in S_2(\Gamma_0(40), \chi_3)$. Therefore, $D_k(q)$ $(1 \leq k \leq 4)$ are in $S_2(\Gamma_0(40), \chi_3)$. ∎

Theorem 4.5.3. **(a)** $\{D_k(q) \mid k = 1, 2, 3, 4\}$ *is a basis for* $S_2(\Gamma_0(40), \chi_3)$.

(b) $\{E_{2,\chi_0,\chi_3}(q), E_{2,\chi_1,\chi_2}(q), E_{2,\chi_2,\chi_1}(q), E_{2,\chi_3,\chi_0}(q)\}$ *is a basis for* $E_2(\Gamma_0(40), \chi_3)$.

(c) $\{E_{2,\chi_0,\chi_3}(q), E_{2,\chi_1,\chi_2}(q), E_{2,\chi_2,\chi_1}(q), E_{2,\chi_3,\chi_0}(q)\}$ *together with* $D_k(q)$ $(1 \leq k \leq 4)$ *constitute a basis for* $M_2(\Gamma_0(40), \chi_3)$.

Proof. **(a)** By Theorem 4.5.2, $D_k(q)$ $(1 \le k \le 4) \in S_2(\Gamma_0(40), \chi_3)$. They are linearly independent over $\mathbb{C}$. By Example 3.2.1, we have $\dim S_2(\Gamma_0(40), \chi_3) = 4$. Thus, $D_k(q)$ $(1 \le k \le 4)$ constitute a basis for $S_2(\Gamma_0(40), \chi_3)$.

(b) By Example 3.2.1, we have $\dim E_2(\Gamma_0(40), \chi_3) = 4$. By Theorem 2.3.1(b), $\{E_{2,\chi_0,\chi_3}(q), E_{2,\chi_1,\chi_2}(q), E_{2,\chi_2,\chi_1}(q), E_{2,\chi_3,\chi_0}(q)\}$ constitute a basis for $E_2(\Gamma_0(40), \chi_3)$.

(c) By Example 3.2.1, we have $\dim M_2(\Gamma_0(40), \chi_3) = 8$. Therefore, by (2.3.2), $\{E_{2,\chi_0,\chi_3}(q), E_{2,\chi_1,\chi_2}(q), E_{2,\chi_2,\chi_1}(q), E_{2,\chi_3,\chi_0}(q)\}$ together with $D_k(q)$ $(1 \le k \le 4)$ constitute a basis for $M_2(\Gamma_0(40), \chi_3)$. ■

Theorem 4.5.4. *Let χ_0 be the trivial character and χ_1, χ_2, χ_3 be as in* (4.1.3). *Then*

$$\textbf{(a)}\ \varphi^2(q)\varphi(q^2)\varphi(q^5) = -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{5}{7}E_{2,\chi_1,\chi_2}(q) - \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{20}{7}E_{2,\chi_3,\chi_0}(q) - \frac{8}{7}D_1(q) + \frac{16}{7}D_4(q),$$

$$\textbf{(b)}\ \varphi^3(q^2)\varphi(q^5) = -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{5}{7}E_{2,\chi_1,\chi_2}(q) - \frac{2}{7}E_{2,\chi_2,\chi_1}(q) + \frac{10}{7}E_{2,\chi_3,\chi_0}(q) - \frac{12}{7}D_1(q),$$

$$\textbf{(c)}\ \varphi(q^2)\varphi^3(q^5) = -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{1}{7}E_{2,\chi_1,\chi_2}(q) - \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{4}{7}E_{2,\chi_3,\chi_0}(q) - \frac{12}{7}D_1(q) - \frac{12}{7}D_2(q) - \frac{36}{7}D_3(q) + \frac{12}{7}D_4(q),$$

$$\textbf{(d)}\ \varphi^3(q)\varphi(q^{10}) = -\frac{1}{7}E_{\chi_0,\chi_3}(q) - \frac{5}{7}E_{2,\chi_1,\chi_2}(q) + \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{20}{7}E_{2,\chi_3,\chi_0}(q) - \frac{12}{7}D_1(q) + \frac{60}{7}D_2(q) - \frac{60}{7}D_3(q) + \frac{36}{7}D_4(q),$$

$$\textbf{(e)}\ \varphi(q)\varphi^2(q^2)\varphi(q^{10}) = -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) - \frac{5}{7}E_{2,\chi_1,\chi_2}(q) + \frac{2}{7}E_{2,\chi_2,\chi_1}(q) + \frac{10}{7}E_{2,\chi_3,\chi_0}(q) + \frac{4}{7}D_1(q) + \frac{20}{7}D_2(q) + \frac{20}{7}D_3(q) + \frac{4}{7}D_4(q),$$

$$\textbf{(f)}\ \varphi(q)\varphi^2(q^5)\varphi(q^{10}) = -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) - \frac{1}{7}E_{2,\chi_1,\chi_2}(q) + \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{4}{7}E_{2,\chi_3,\chi_0}(q) + \frac{8}{7}D_2(q) - \frac{8}{7}D_3(q) + \frac{8}{7}D_4(q),$$

$$\textbf{(g)}\ \varphi(q^2)\varphi(q^5)\varphi^2(q^{10}) = -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{1}{7}E_{2,\chi_1,\chi_2}(q) - \frac{2}{7}E_{2,\chi_2,\chi_1}(q)$$

$$+\frac{2}{7}E_{2,\chi_3,\chi_0}(q) - \frac{4}{7}D_1(q) + \frac{4}{7}D_2(q) - \frac{12}{7}D_3(q) + \frac{4}{7}D_4(q),$$

$$\textbf{(h)}\ \varphi(q)\varphi^3(q^{10}) = -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) - \frac{1}{7}E_{\chi_1,\chi_2}(q) + \frac{2}{7}E_{2,\chi_2,\chi_1}(q) + \frac{2}{7}E_{2,\chi_3,\chi_0}(q)$$
$$+\frac{12}{7}D_2(q) + \frac{12}{7}D_3(q) + \frac{12}{7}D_4(q).$$

Proof. By Theorem 4.5.1, $\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) \in M_2(\Gamma_0(N), \chi_3)$ for $(a_1, a_2, a_3, a_4) = (1, 1, 2, 5)$, $(2, 2, 2, 5)$, $(2, 5, 5, 5)$, $(1, 1, 1, 10)$, $(1, 2, 2, 10)$, $(1, 5, 5, 10)$, $(2, 5, 10, 10)$ and $(1, 10, 10, 10)$. Therefore, by Theorem 4.5.3(c), each $\varphi(q^{a_1})\varphi(q^{a_2})$ $\varphi(q^{a_3})\varphi(q^{a_4})$ must be a linear combination of $\{E_{2,\chi_0,\chi_3}(q), E_{2,\chi_1,\chi_2}(q), E_{2,\chi_2,\chi_1}(q),$ $E_{2,\chi_3,\chi_0}(q)\}$ and $D_k(q)$ $(1 \le k \le 4)$, namely

$$\varphi(q^{a_1})\varphi(q^{a_2})\varphi(q^{a_3})\varphi(q^{a_4}) = x_1E_{2,\chi_0,\chi_3}(q) + x_2E_{2,\chi_1,\chi_2}(q) + x_3E_{2,\chi_2,\chi_1}(q)$$
$$+ x_4E_{2,\chi_3,\chi_0}(q) + y_1D_1(q) + y_2D_2(q) + y_3D_3(q) + y_4D_4(q).$$

Using MAPLE we equate the first few coefficients of q^n on both sides of the equation above to obtain the x_i and y_i $(1 \le i \le 4)$. ∎

Theorem 4.5.5. *Let $n \in \mathbb{N}$. Let $\sigma_{\chi_i,\chi_j}(n)$ be as in (2.3.1) for $i, j \in \{0, 1, 2, 3\}$. Then*

$$\textbf{(a)}\ N(1,1,2,5;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{4}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{20}{7}\sigma_{\chi_3,\chi_0}(n)$$
$$-\frac{8}{7}d_1(n) + \frac{16}{7}d_4(n),$$

$$\textbf{(b)}\ N(2,2,2,5;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{2}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{10}{7}\sigma_{\chi_3,\chi_0}(n)$$
$$-\frac{12}{7}d_1(n),$$

$$\textbf{(c)}\ N(2,5,5,5;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{4}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{4}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{12}{7}d_1(n)$$
$$-\frac{12}{7}d_2(n) - \frac{36}{7}d_3(n) + \frac{12}{7}d_4(n),$$

$$\textbf{(d)}\ N(1,1,1,10;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{4}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{20}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{12}{7}d_1(n) + \frac{60}{7}d_2(n) - \frac{60}{7}d_3(n) + \frac{36}{7}d_4(n),$$

$$\textbf{(e)}\ N(1,2,2,10;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{2}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{10}{7}\sigma_{\chi_3,\chi_0}(n) + \frac{4}{7}d_1(n) + \frac{20}{7}d_2(n) + \frac{20}{7}d_3(n) + \frac{4}{7}d_4(n),$$

$$\textbf{(f)}\ N(1,5,5,10;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{4}{7}(n)\sigma_{\chi_2,\chi_1}(n) + \frac{4}{7}\sigma_{\chi_3,\chi_0}(n) + \frac{8}{7}d_2(n) - \frac{8}{7}d_3(n) + \frac{8}{7}d_4(n),$$

$$\textbf{(g)}\ N(2,5,10,10;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{2}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{2}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{4}{7}d_1(n) + \frac{4}{7}d_2(n) - \frac{12}{7}d_3(n) + \frac{4}{7}d_4(n),$$

$$\textbf{(h)}\ N(1,10,10,10;n) = -\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{2}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{2}{7}\sigma_{\chi_3,\chi_0}(n) + \frac{12}{7}d_2(n) + \frac{12}{7}d_3(n) + \frac{12}{7}d_4(n).$$

Proof. Appealing to (2.4.5), we have by Theorem 4.5.4

$$\begin{aligned}\textbf{(a)}\ &\sum_{n=0}^{\infty} N(1,1,2,5;n)q^n = \varphi^2(q)\varphi(q^2)\varphi(q^5)\\ &= -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{5}{7}E_{2,\chi_1,\chi_2}(q) - \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{20}{7}E_{2,\chi_3,\chi_0}(q) - \frac{8}{7}D_1(q)\\ &\quad + \frac{16}{7}D_4(q)\\ &= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{4}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{20}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{8}{7}d_1(n)\\ &\quad + \frac{16}{7}d_4(n)\Big)q^n,\end{aligned}$$

$$\begin{aligned}\textbf{(b)}\ &\sum_{n=0}^{\infty} N(2,2,2,5;n)q^n = \varphi^3(q^2)\varphi(q^5)\\ &= -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{5}{7}E_{2,\chi_1,\chi_2}(q) - \frac{2}{7}E_{2,\chi_2,\chi_1}(q) + \frac{10}{7}E_{2,\chi_3,\chi_0}(q) - \frac{12}{7}D_1(q)\\ &= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{2}{7}\sigma_{\chi_2,\chi_1}(n)\end{aligned}$$

$$+\frac{10}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{12}{7}d_1(n)\Big)q^n,$$

(c) $$\sum_{n=0}^{\infty} N(2,5,5,5;n)q^n = \varphi(q^2)\varphi^3(q^5)$$
$$= -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{1}{7}E_{2,\chi_1,\chi_2}(q) - \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{4}{7}E_{2,\chi_3,\chi_0}(q) - \frac{12}{7}D_1(q)$$
$$-\frac{12}{7}D_2(q) - \frac{36}{7}D_3(q) + \frac{12}{7}D_4(q)$$
$$= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{4}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{4}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{12}{7}d_1(n)$$
$$-\frac{12}{7}d_2(n) - \frac{36}{7}d_3(n) + \frac{12}{7}d_4(n)\Big)q^n,$$

(d) $$\sum_{n=0}^{\infty} N(1,1,1,10;n)q^n = \varphi^3(q)\varphi(q^{10})$$
$$= -\frac{1}{7}E_{\chi_0,\chi_3}(q) - \frac{5}{7}E_{2,\chi_1,\chi_2}(q) + \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{20}{7}E_{2,\chi_3,\chi_0}(q) - \frac{12}{7}D_1(q)$$
$$+\frac{60}{7}D_2(q) - \frac{60}{7}D_3(q) + \frac{36}{7}D_4(q)$$
$$= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{4}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{20}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{12}{7}d_1(n)$$
$$+\frac{60}{7}d_2(n) - \frac{60}{7}d_3(n) + \frac{36}{7}d_4(n)\Big)q^n,$$

(e) $$\sum_{n=0}^{\infty} N(1,2,2,10;n)q^n = \varphi(q)\varphi^2(q^2)\varphi(q^{10})$$
$$= -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) - \frac{5}{7}E_{2,\chi_1,\chi_2}(q) + \frac{2}{7}E_{2,\chi_2,\chi_1}(q) + \frac{10}{7}E_{2,\chi_3,\chi_0}(q)$$
$$+\frac{4}{7}D_1(q) + \frac{20}{7}D_2(q) + \frac{20}{7}D_3(q) + \frac{4}{7}D_4(q)$$
$$= 1 + \sum_{n=1}^{\infty}\Big(-\frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{5}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{2}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{10}{7}\sigma_{\chi_3,\chi_0}(n) + \frac{4}{7}d_1(n)$$
$$+\frac{20}{7}d_2(n) + \frac{20}{7}d_3(n) + \frac{4}{7}d_4(n)\Big)q^n,$$

(f) $$\sum_{n=0}^{\infty} N(1,5,5,10;n)q^n = \varphi(q)\varphi^2(q^5)\varphi(q^{10})$$
$$= -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) - \frac{1}{7}E_{2,\chi_1,\chi_2}(q) + \frac{4}{7}E_{2,\chi_2,\chi_1}(q) + \frac{4}{7}E_{2,\chi_3,\chi_0}(q)$$
$$+\frac{8}{7}D_2(q) - \frac{8}{7}D_3(q) + \frac{8}{7}D_4(q)$$

$$= 1 + \sum_{n=1}^{\infty} \Big(- \frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{4}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{4}{7}\sigma_{\chi_3,\chi_0}(n) + \frac{8}{7}d_2(n) - \frac{8}{7}d_3(n) + \frac{8}{7}d_4(n)\Big) q^n,$$

(g) $$\sum_{n=0}^{\infty} N(2,5,10,10;n)q^n = \varphi(q^2)\varphi(q^5)\varphi^2(q^{10})$$

$$= -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) + \frac{1}{7}E_{2,\chi_1,\chi_2}(q) - \frac{2}{7}E_{2,\chi_2,\chi_1}(q) + \frac{2}{7}E_{2,\chi_3,\chi_0}(q) - \frac{4}{7}D_1(q) + \frac{4}{7}D_2(q) - \frac{12}{7}D_3(q) + \frac{4}{7}D_4(q)$$

$$= 1 + \sum_{n=1}^{\infty} \Big(- \frac{1}{7}\sigma_{\chi_0,\chi_3}(n) + \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) - \frac{2}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{2}{7}\sigma_{\chi_3,\chi_0}(n) - \frac{4}{7}d_1(n) + \frac{4}{7}d_2(n) - \frac{12}{7}d_3(n) + \frac{4}{7}d_4(n)\Big) q^n,$$

(h) $$\sum_{n=0}^{\infty} N(1,10,10,10;n)q^n = \varphi(q)\varphi^3(q^{10})$$

$$= -\frac{1}{7}E_{2,\chi_0,\chi_3}(q) - \frac{1}{7}E_{\chi_1,\chi_2}(q) + \frac{2}{7}E_{2,\chi_2,\chi_1}(q) + \frac{2}{7}E_{2,\chi_3,\chi_0}(q) + \frac{12}{7}D_2(q) + \frac{12}{7}D_3(q) + \frac{12}{7}D_4(q)$$

$$= 1 + \sum_{n=1}^{\infty} \Big(- \frac{1}{7}\sigma_{\chi_0,\chi_3}(n) - \frac{1}{7}\sigma_{\chi_1,\chi_2}(n) + \frac{2}{7}\sigma_{\chi_2,\chi_1}(n) + \frac{2}{7}\sigma_{\chi_3,\chi_0}(n) + \frac{12}{7}d_2(n) + \frac{12}{7}d_3(n) + \frac{12}{7}d_4(n)\Big) q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(h), we deduce the asserted results. ■

Chapter 5

Future Work

In this chapter we show how to extend our work to the octonary quadratic forms, and develop some explicit formulas for the number of representations of n by some specific octonary quadratic forms. We then conclude with a discussion on some possible further research directions.

5.1 Representations by Octonary Quadratic Forms

We recall that for $n \in \mathbb{N}_0$, we define $N(1^i, 5^j; n)$ as

$$N(1^i, 5^j; n) := \text{card}\{(x_1, \ldots, x_8) \in \mathbb{Z}^8 \mid n = (x_1^2 + \cdots + x_i^2) + 5(x_{i+1}^2 + \cdots + x_8^2)\},$$

where i is an even integer. We determine formulas for $N(1^i, 5^j; n)$ for the octonary quadratic forms given by

$$(i, j) = (6, 2),\ (4, 4),\ (2, 6)$$

in terms of $\sigma_3(n)$, $\sigma_3(n/2)$, $\sigma_3(n/4)$, $\sigma_3(n/5)$, $\sigma_3(n/10)$, $\sigma_3(n/20)$ and $u_k(n)$ $(1 \leq k \leq 6)$ defined by

$$U_1(q) = \sum_{n=1}^{\infty} u_1(n)q^n = \frac{\eta^8(2z)\eta^2(5z)}{\eta^2(z)}, \tag{5.1.1}$$

$$U_2(q) = \sum_{n=1}^{\infty} u_2(n)q^n = \frac{\eta(z)\eta^5(2z)\eta^3(5z)\eta^2(20z)}{\eta^2(4z)\eta(10z)}, \tag{5.1.2}$$

$$U_3(q) = \sum_{n=1}^{\infty} u_3(n)q^n = \frac{\eta^5(z)\eta^3(4z)\eta^2(10z)\eta(20z)}{\eta^2(2z)\eta(5z)}, \tag{5.1.3}$$

$$U_4(q) = \sum_{n=1}^{\infty} u_4(n)q^n = \frac{\eta^7(2z)\eta^7(5z)\eta^2(20z)}{\eta^3(z)\eta^2(4z)\eta^3(10z)}, \tag{5.1.4}$$

$$U_5(q) = \sum_{n=1}^{\infty} u_5(n)q^n = \frac{\eta^2(z)\eta^3(4z)\eta^5(10z)\eta(20z)}{\eta(2z)\eta^2(5z)}, \tag{5.1.5}$$

$$U_6(q) = \sum_{n=1}^{\infty} u_6(n)q^n = \frac{\eta^2(z)\eta^4(2z)\eta^4(10z)\eta^2(20z)}{\eta^2(4z)\eta^2(5z)}. \tag{5.1.6}$$

There is no linear relationship among the $U_k(q)$ $(1 \leq k \leq 6)$. The first twenty values of $u_k(n)$ $(1 \leq k \leq 6)$, are given in Table 5.1.1.

Table 5.1.1

n	$u_1(n)$	$u_2(n)$	$u_3(n)$	$u_4(n)$	$u_5(n)$	$u_6(n)$
1	1	0	0	0	0	0
2	2	1	1	1	0	0
3	−3	−1	−5	3	1	1
4	−6	−6	7	2	−2	−2
5	0	5	0	1	0	−5
6	−6	12	−3	4	0	10
7	1	−9	5	−5	−1	9
8	28	−4	−26	−20	8	−12
9	7	14	20	−10	−4	−14
10	0	−15	0	−7	0	−10
11	22	−34	30	−26	−6	34
12	−32	8	4	8	−4	16
13	−48	38	−60	46	12	−38
14	2	16	1	8	0	10
15	−25	5	−25	17	5	−5
16	−24	24	68	56	−16	24
17	46	4	20	−12	−4	−4
18	14	−83	7	−35	0	−60
19	0	−52	−60	28	12	52
20	50	10	−25	−14	10	−10

Theorem 5.1.1. *For* $(i,j) = (6,2),\ (4,4),\ (2,6)$, *we have* $\varphi^i(q)\varphi^j(q^5) \in M_4(\Gamma_0(20))$.

Proof. Appealing to (2.4.7) we obtain

$$\varphi^6(q)\varphi^2(q^5) = \frac{\eta^{30}(2z)\eta^{10}(10z)}{\eta^{12}(z)\eta^{12}(4z)\eta^4(5z)\eta^4(20z)}, \tag{5.1.7}$$

$$\varphi^4(q)\varphi^4(q^5) = \frac{\eta^{20}(2z)\eta^{20}(10z)}{\eta^8(z)\eta^8(4z)\eta^8(5z)\eta^8(20z)}, \tag{5.1.8}$$

$$\varphi^2(q)\varphi^6(q^5) = \frac{\eta^{10}(2z)\eta^{30}(10z)}{\eta^4(z)\eta^4(4z)\eta^{12}(5z)\eta^{12}(20z)}. \tag{5.1.9}$$

We first consider $\varphi^6(q)\varphi^2(q^5)$. From (5.1.7), we have

Table 5.1.2(a)

δ	1	2	4	5	10	20
r_δ	-12	30	-12	-4	10	-4

Table 5.1.2(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	192/5	0	0	96	0

It follows from Table 5.1.2(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.2(b), L3 is also satisfied for each positive divisor d of 20. Thus by Theorem 2.4.1, $\varphi^6(q)\varphi^2(q^5) \in M_4(\Gamma_0(20))$.

Next we consider $\varphi^4(q)\varphi^4(q^5)$. From (5.1.8), we have

Table 5.1.3(a)

δ	1	2	4	5	10	20
r_δ	-8	20	-8	-8	20	-8

Table 5.1.3(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	144/5	0	0	144	0

It follows from Table 5.1.3(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.3(b), L3 is also satisfied for each positive divisor d of 20. Thus by Theorem 2.4.1, $\varphi^4(q)\varphi^4(q^5) \in M_4(\Gamma_0(20))$.

Finally, we consider $\varphi^2(q)\varphi^6(q^5)$. From (5.1.9), we have

Table 5.1.4(a)

δ	1	2	4	5	10	20
r_δ	-4	10	-4	-12	30	-12

Table 5.1.4(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	0	96/5	0	0	192	0

It follows from Table 5.1.4(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.4(b), L3 is also satisfied for each positive divisor d of 20. Thus by Theorem 2.4.1, $\varphi^2(q)\varphi^6(q^5) \in M_4(\Gamma_0(20))$. ∎

Theorem 5.1.2. *$U_k(q)$ $(1 \le k \le 6)$ given by (5.1.1)-(5.1.6) are in $S_4(\Gamma_0(20))$.*

Proof. First we consider

$$U_1(q) = \frac{\eta^8(2z)\eta^2(5z)}{\eta^2(z)}.$$

Table 5.1.5(a)

δ	1	2	5
r_δ	-2	8	2

Table 5.1.5(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	12/5	72/5	72/5	12	24	24

It follows from Table 5.1.5(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.5(b), L4 is also satisfied for each positive divisor d of 20. Thus $U_1(q) \in S_4(\Gamma_0(20))$.

Next we consider

$$U_2(q) = \frac{\eta(z)\eta^5(2z)\eta^3(5z)\eta^2(20z)}{\eta^2(4z)\eta(10z)}.$$

Table 5.1.6(a)

δ	1	2	4	5	10	20
r_δ	1	5	-2	3	-1	2

Table 5.1.6(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	18/5	48/5	24/5	18	24	48

It follows from Table 5.1.6(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.6(b), L4 is also satisfied for each positive divisor d of 20. Thus $U_2(q) \in S_4(\Gamma_0(20))$.

Now we consider

$$U_3(q) = \frac{\eta^5(z)\eta^3(4z)\eta^2(10z)\eta(20z)}{\eta^2(2z)\eta(5z)}.$$

Table 5.1.7(a)

δ	1	2	4	5	10	20
r_δ	5	-2	3	-1	2	1

Table 5.1.7(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	24/5	24/5	72/5	6	24	48

It follows from Table 5.1.7(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.7(b), L4 is also satisfied for each positive divisor d of 20. Thus $U_3(q) \in S_4(\Gamma_0(20))$.

Also,

$$U_4(q) = \frac{\eta^7(2z)\eta^7(5z)\eta^2(20z)}{\eta^3(z)\eta^2(4z)\eta^3(10z)}.$$

Table 5.1.8(a)

δ	1	2	4	5	10	20
r_δ	-3	7	-2	7	-3	2

Table 5.1.8(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	6/5	48/5	24/5	30	24	48

It follows from Table 5.1.8(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.8(b), L4 is also satisfied for each positive divisor d of 20. Thus $U_4(q) \in S_4(\Gamma_0(20))$.

Now consider

$$U_5(q) = \frac{\eta^2(z)\eta^3(4z)\eta^5(10z)\eta(20z)}{\eta(2z)\eta^2(5z)}.$$

Table 5.1.9(a)

δ	1	2	4	5	10	20
r_δ	2	-1	3	-2	5	1

Table 5.1.9(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	12/5	24/5	72/5	6	48	72

It follows from Table 5.1.9(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.9(b), L4 is also satisfied for each positive divisor d of 20. Thus $U_5(q) \in S_4(\Gamma_0(20))$.

Finally, we consider

$$U_6(q) = \frac{\eta^2(z)\eta^4(2z)\eta^4(10z)\eta^2(20z)}{\eta^2(4z)\eta^2(5z)}.$$

Table 5.1.10(a)

δ	1	2	4	5	10	20
r_δ	2	4	-2	-2	4	2

Table 5.1.10(b)

$d \mid 20$	1	2	4	5	10	20
$\sum_{1\le\delta\mid 20} \frac{\gcd(d,\delta)^2 \cdot r_\delta}{\delta}$	18/5	48/5	24/5	6	48	72

It follows from Table 5.1.10(a) that L1 and L2 of Theorem 2.4.1 are satisfied. From Table 5.1.10(b), L4 is also satisfied for each positive divisor d of 20. Thus $U_6(q) \in S_4(\Gamma_0(20))$. ∎

Theorem 5.1.3. (a) $\{U_k(q) \mid k = 1,2,3,4,5,6\}$ *is a basis for* $S_4(\Gamma_0(20))$.

(b) $E_4(q^k)$ $(k = 1,2,4,5,10,20)$ *constitute a basis for* $E_4(\Gamma_0(20))$.

(c) *$E_4(q^k)$ $(k = 1, 2, 4, 5, 10, 20)$ together with $U_k(q)$ $(1 \leq k \leq 6)$ constitute a basis for $M_4(\Gamma_0(20))$.*

Proof. **(a)** By Theorem 5.1.2, $U_k(q)$ $(1 \leq k \leq 6) \in S_4(\Gamma_0(20))$. They are linearly independent over $\mathbb{C}$. By Proposition 3.1.1, we have $\dim S_4(\Gamma_0(20)) = 6$. Thus $U_k(q)$ $(1 \leq k \leq 6)$ constitute a basis for $S_4(\Gamma_0(20))$.

(b) By Proposition 3.1.1, we have $\dim E_4(\Gamma_0(20)) = 6$. By Theorem 2.3.1(b), $E_4(q^k)$ $(k = 1, 2, 4, 5, 10, 20)$ constitute a basis for $E_4(\Gamma_0(20))$.

(c) It follows from (a), (b) and (2.3.2) that the dimension of $M_4(\Gamma_0(20))$ is 12 and therefore $E_4(q^k)$ $(k = 1, 2, 4, 5, 10, 20)$ together with $U_k(q)$ $(1 \leq k \leq 6)$ constitute a basis for $M_4(\Gamma_0(20))$. ∎

Theorem 5.1.4.

$$\begin{aligned}
&\textbf{(a)}\ \varphi^6(q)\varphi^2(q^5) = \frac{124}{39}E_4(q) - \frac{248}{39}E_4(q^2) + \frac{1984}{39}E_4(q^4) + \frac{500}{39}E_4(q^5) \\
&\quad -\frac{1000}{39}E_4(q^{10}) + \frac{8000}{39}E_4(q^{20}) + \frac{344}{39}U_1(q) - \frac{518}{39}U_2(q) + \frac{352}{39}U_3(q) \\
&\quad +\frac{950}{39}U_4(q) + \frac{1760}{39}U_5(q) + \frac{144}{13}U_6(q), \\
&\textbf{(b)}\ \varphi^4(q)\varphi^4(q^5) = \frac{8}{13}E_4(q) - \frac{16}{13}E_4(q^2) + \frac{128}{13}E_4(q^4) + \frac{200}{13}E_4(q^5) \\
&\quad -\frac{400}{13}E_4(q^{10}) + \frac{3200}{13}E_4(q^{20}) + \frac{96}{13}U_1(q) - \frac{152}{65}U_2(q) - \frac{128}{65}U_3(q) \\
&\quad +\frac{120}{13}U_4(q) - \frac{128}{13}U_5(q) + \frac{448}{65}U_6(q), \\
&\textbf{(c)}\ \varphi^2(q)\varphi^6(q^5) = \frac{4}{39}E_4(q) - \frac{8}{39}E_4(q^2) + \frac{64}{39}E_4(q^4) + \frac{620}{39}E_4(q^5) \\
&\quad -\frac{1240}{39}E_4(q^{10}) + \frac{9920}{39}E_4(q^{20}) + \frac{152}{39}U_1(q) + \frac{2}{195}U_2(q) - \frac{1312}{195}U_3(q) \\
&\quad +\frac{86}{39}U_4(q) - \frac{1312}{39}U_5(q) + \frac{144}{65}U_6(q).
\end{aligned}$$

Proof. By Theorem 5.1.1, we have $\varphi^i(q)\varphi^j(q^5) \in M_4(\Gamma_0(20))$ for $(i, j) = (6, 2)$, $(4, 4)$ and $(2, 6)$. Therefore, by Theorem 5.1.3 (c) $\varphi^i(q)\varphi^j(q^5)$ must be a linear combination of $E_4(q^k)$ $(k = 1, 2, 4, 5, 10, 20)$ and $U_k(q)$ $(1 \le k \le 6)$, namely

$$\varphi^i(q)\varphi^j(q^5) = x_1E_4(q) + x_2E_4(q^2) + x_3E_4(q^4) + x_4E_4(q^5) + x_5E_4(q^{10}) + x_6E_4(q^{20})$$
$$+y_1U_1(q) + y_2U_2(q) + y_3U_3(q) + y_4U_4(q) + y_5U_5(q) + y_6U_6(q).$$

Using MAPLE we equate the first few coefficients of q^n on both sides of the equation above to obtain the x_i and y_i $(1 \le i \le 6)$. ∎

Theorem 5.1.5. *Let $n \in \mathbb{N}$. Then*

$$\textbf{(a)}\ N(1^6, 5^2; n) = \frac{124}{39}\sigma_3(n) - \frac{248}{39}\sigma_3(n/2) + \frac{1984}{39}\sigma_3(n/4) + \frac{500}{39}\sigma_3(n/5) - \frac{1000}{39}\sigma_3(n/10) + \frac{8000}{39}\sigma_3(n/20) + \frac{344}{39}u_1(n) - \frac{518}{39}u_2(n) + \frac{352}{39}u_3(n) + \frac{950}{39}u_4(n) + \frac{1760}{39}u_5(n) + \frac{144}{13}u_6(n),$$

$$\textbf{(b)}\ N(1^4, 5^4; n) = \frac{8}{13}\sigma_3(n) - \frac{16}{13}\sigma_3(n/2) + \frac{128}{13}\sigma_3(n/4) + \frac{200}{13}\sigma_3(n/5) - \frac{400}{13}\sigma_3(n/10) + \frac{3200}{13}\sigma_3(n/20) + \frac{96}{13}u_1(n) - \frac{152}{65}u_2(n) - \frac{128}{65}u_3(n) + \frac{120}{13}u_4(n) - \frac{128}{13}u_5(n) + \frac{448}{65}u_6(n),$$

$$\textbf{(c)}\ N(1^2, 5^6; n) = \frac{4}{39}\sigma_3(n) - \frac{8}{39}\sigma_3(n/2) + \frac{64}{39}\sigma_3(n/4) + \frac{620}{39}\sigma_3(n/5) - \frac{1240}{39}\sigma_3(n/10) + \frac{9920}{39}\sigma_3(n/20) + \frac{152}{39}u_1(n) + \frac{2}{195}u_2(n) - \frac{1312}{195}u_3(n) + \frac{86}{39}u_4(n) - \frac{1312}{39}u_5(n) + \frac{144}{65}u_6(n).$$

Proof. Appealing to (2.4.6), we have by Theorem 5.1.4

$$\textbf{(a)}\ \sum_{n=0}^{\infty} N(1^6, 5^2; n)q^n = \varphi^6(q)\varphi^2(q^5)$$
$$= \frac{124}{39}E_4(q) - \frac{248}{39}E_4(q^2) + \frac{1984}{39}E_4(q^4) + \frac{500}{39}E_4(q^5) - \frac{1000}{39}E_4(q^{10}) + \frac{8000}{39}E_4(q^{20}) + \frac{344}{39}U_1(q) - \frac{518}{39}U_2(q) + \frac{352}{39}U_3(q) + \frac{950}{39}U_4(q)$$

$$+\frac{1760}{39}U_5(q)+\frac{144}{13}U_6(q)$$
$$=1+\sum_{n=1}^{\infty}\Big(\frac{124}{39}\sigma_3(n)-\frac{248}{39}\sigma_3(n/2)+\frac{1984}{39}\sigma_3(n/4)+\frac{500}{39}\sigma_3(n/5)$$
$$-\frac{1000}{39}\sigma_3(n/10)+\frac{8000}{39}\sigma_3(n/20)+\frac{344}{39}u_1(n)-\frac{518}{39}u_2(n)+\frac{352}{39}u_3(n)$$
$$+\frac{950}{39}u_4(n)+\frac{1760}{39}u_5(n)+\frac{144}{13}u_6(n)\Big)q^n,$$

(b) $\sum_{n=0}^{\infty}N(1^4,5^4;n)q^n=\varphi^4(q)\varphi^4(q^5)$

$$=\frac{8}{13}E_4(q)-\frac{16}{13}E_4(q^2)+\frac{128}{13}E_4(q^4)+\frac{200}{13}E_4(q^5)-\frac{400}{13}E_4(q^{10})$$
$$+\frac{3200}{13}E_4(q^{20})+\frac{96}{13}U_1(q)-\frac{152}{65}U_2(q)-\frac{128}{65}U_3(q)+\frac{120}{13}U_4(q)$$
$$-\frac{128}{13}U_5(q)+\frac{448}{65}U_6(q)$$
$$=1+\sum_{n=1}^{\infty}\Big(\frac{8}{13}\sigma_3(n)-\frac{16}{13}\sigma_3(n/2)+\frac{128}{13}\sigma_3(n/4)+\frac{200}{13}\sigma_3(n/5)-\frac{400}{13}\sigma_3(n/10)$$
$$+\frac{3200}{13}\sigma_3(n/20)+\frac{96}{13}u_1(n)-\frac{152}{65}u_2(n)-\frac{128}{65}u_3(n)+\frac{120}{13}u_4(n)$$
$$-\frac{128}{13}u_5(n)+\frac{448}{65}u_6(n)\Big)q^n,$$

(c) $\sum_{n=0}^{\infty}N(1^2,5^6;n)q^n=\varphi^2(q)\varphi^6(q^5)$

$$=\frac{4}{39}E_4(q)-\frac{8}{39}E_4(q^2)+\frac{64}{39}E_4(q^4)+\frac{620}{39}E_4(q^5)-\frac{1240}{39}E_4(q^{10})$$
$$+\frac{9920}{39}E_4(q^{20})+\frac{152}{39}U_1(q)+\frac{2}{195}U_2(q)-\frac{1312}{195}U_3(q)+\frac{86}{39}U_4(q)$$
$$-\frac{1312}{39}U_5(q)+\frac{144}{65}U_6(q)$$
$$=1+\sum_{n=1}^{\infty}\Big(\frac{4}{39}\sigma_3(n)-\frac{8}{39}\sigma_3(n/2)+\frac{64}{39}\sigma_3(n/4)+\frac{620}{39}\sigma_3(n/5)-\frac{1240}{39}\sigma_3(n/10)$$
$$+\frac{9920}{39}\sigma_3(n/20)+\frac{152}{39}u_1(n)+\frac{2}{195}u_2(n)-\frac{1312}{195}u_3(n)+\frac{86}{39}u_4(n)$$
$$-\frac{1312}{39}u_5(n)+\frac{144}{65}u_6(n)\Big)q^n.$$

Equating the coefficients of q^n on both sides of equations (a)–(c), we deduce the asserted results. ∎

5.2 Conclusion and Future Work

In this dissertation we have presented formulas for $N(a_1, a_2, a_3, a_4; n)$ for the number of representations of a positive integer n by the quaternary quadratic forms

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2,$$

where $a_1, a_2, a_3, a_4 \in \{1, 2, 5, 10\}$ using a modular form approach. Also, we have determined explicit formulas for the number of representations of n by the octonary quadratic forms

$$(x_1^2 + \cdots + x_i^2) + 5(x_{i+1}^2 + \cdots + x_8^2)$$

where i is an even integer. We can develop our research in several directions. One possible direction is to determine formulas for $N(1^i, 2^j, 5^k, 10^l; n)$ such that $i+j+k+l = 8$; in other words, one can determine formulas for the number of representations of a positive integer n by octonary quadratic forms with coefficients $1, 2, 5$ or 10. Another interesting direction is the extension of the Ramanujan-Mordell formula. Ramanujan [48] gave without proof a formula for the number of representations of n by the quadratic forms

$$x_1^2 + x_2^2 + \cdots + x_{2m}^2$$

which we denote by $N(1^{2m}; n)$ $(m \in \mathbb{N})$. His formula was proved later by Mordell [41]. Mathieu Lemire [33] extended the Ramanujan-Mordell formula with coefficients 1, 2 and 4 under some conditions. Our interest is to extend the Ramanujan-Mordell formula to the representation of a positive integer n by the form

$$(x_1^2 + \cdots + x_i^2) + 5(x_{i+1}^2 + \cdots + x_{i+j}^2),$$

that is, we are interested in determining a formula for the quantity

$$\operatorname{card}\{(x_1, \ldots, x_{i+j}) \in \mathbb{Z}^{i+j} \mid n = (x_1^2 + \cdots + x_i^2) + 5(x_{i+1}^2 + \cdots + x_{i+j}^2)\},$$

such that $i + j = 2m$.

Furthermore, it is an interesting direction to determine a general formula for the number of representations of a positive integer n by the quaternary quadratic forms $x_1^2 + 2x_2^2 + px_3^2 + 2px_4^2$, where p is a prime number.

Moreover, working with eta quotients is another interesting direction to consider in the future. For example, we have 755 eta quotients which lie in the space $M_2(\Gamma_0(40), \chi)$. There are 277 eta quotients in $M_2(\Gamma_0(40), \chi_0)$, 278 eta quotients in $M_2(\Gamma_0(40), \chi_1)$, 100 eta quotients in $M_2(\Gamma_0(40), \chi_2)$ and 100 eta quotients in $M_2(\Gamma_0(40), \chi_3)$.

We determined all the eta quotients in $M_2(\Gamma_0(40), \chi)$ by investigating all the eta quotients with the exponents within the bound given by [52]

$$\sum_{\delta|N} |r_\delta| \leq 2k \prod_{p|N} \left(\frac{p+1}{p-1}\right)^{\min\{2, ord_p(N)\}},$$

where p runs through the prime divisors of N with $N = 40$ and $k = 2$. We plan to make a search algorithm in MAPLE using the above bound in order to determine the eta quotients in other modular form spaces, and therefore we can develop explicit formulas for the representations of integers by other quadratic forms.

People have been investigating the properties of Fourier coefficients of cusp forms, see for example [10], [27], [28] or [59] . They have also been studying the sign change of those Fourier coefficients, and find out if there is any pattern. See for example [11]. It would be interesting to investigate if there is any intersting properties of the Fourier coefficients of the cusp forms that we worked with in this thesis.

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