1

MICRO

National Library
of Canada

Bibliothèque nationale
du Canada

Canadian Theses Service

Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

Orderings on Division Rings and Normal Subgroup Structure of a Unitary Group

by

Parnesh Kumar Grover

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Ottawa-Carleton Institute of Mathematics and Statistics

Carleton University, Ottawa, Canada.

University of Ottawa, Ottawa, Canada.

August, 1989.

National Library
of Canada

Bibliothèque nationale
du Canada

Canadian Theses Service Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

The author has granted an irrevocable non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of his/her thesis by any means and in any form or format, making this thesis available to interested persons.

The author retains ownership of the copyright in his/her thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without his/her permission.

L'auteur a accordé une licence irrévocable et non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de sa thèse de quelque manière et sous quelque forme que ce soit pour mettre des exemplaires de cette thèse à la disposition des personnes intéressées.

L'auteur conserve la propriété du droit d'auteur qui protège sa thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou autrement reproduits sans son autorisation.

ISBN 0-315-54409-0

Canada

The undersigned hereby recommend to
The Faculty of Graduate Studies and Research
acceptance of the thesis

ORDERINGS ON DIVISION RINGS AND NORMAL
SUBGROUP STRUCTURE OF A UNITARY GROUP

submitted by

PARNESH KUMAR GROVER

in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

Chairman, Department of
Mathematics and Statistics

Thesis Supervisor

External Examiner

Abstract

This work consists of two independent topics, the first topic is with regard to the Normal and Subnormal subgroup structure of a unitary group of a non-isotropic unitary space under suitable restrictions on the unitary space. We will establish the main result which asserts that if N and M are normal subgroups of a unitary group U such that $[N, M]$ is contained in $Z(U)$, the centre of U , then either N is contained in $Z(U)$ or M is contained in $Z(U)$. Under a strong assumption it is shown that if N is an abelian subnormal subgroup of U such that the commutator subgroup $[N, M]$ is contained in $Z(U)$, then N itself is contained in $Z(U)$.

The second topic of this work is about Jordan orderings, q -orderings and Jordan q -orderings on division rings. A fairly striking result asserts that every Jordan ordering is a Hilbert ordering. A necessary and sufficient condition for a q -ordering to become a Hilbert ordering is determined. We shall also investigate in detail the order valuation induced by Jordan orderings and q -orderings and we will open the question to know whether this valuation exists in the case of Jordan q -ordering. We construct a power series division ring which is q -ordered but definitely not Hilbert ordered and we will compute in this instance all the various notions: order valuation, order subring and commutators. Finally, we will take a look at the space of all q -orderings and prove an extension theorem in which we extend a given q -ordering on a field F to the Hilbert division ring over F in one indeterminate.

Dedicated to my parents

Acknowledgements

I wish to express my sincere gratitude to my thesis supervisor Professor M. Chacron for suggesting the problem and for his most valuable guidance and encouragement. His very kind, friendly and helpful attitude have made my stay at Carleton University a rich and rewarding educational experience.

My deepest thanks go to my parents and my wife, Neelam, for their love, support and the faith they have always shown in me.

Finally, I express my thanks to the Canadian Commonwealth Scholarship and Fellowship Plan for the financial assistance and to the Doaba College, Jalandhar (India) for the study leave which enabled me to carry out my work here at Carleton University.

TABLE OF CONTENTS

	Page
Abstract	
Acknowledgements	
Introduction	1
Chapter 1 : Normal Subgroup Structure of a Unitary Group	
1. 1 Co-finitary Linear Transformations	5
1. 2 Normal Abelian Subgroups	18
1. 3 Normal and Subnormal Subgroups	26
Chapter 2 : Jordan orderings and q-orderings	
2. 1 Jordan orderings	36
2. 2 q-orderings	44
2. 3 Example of a q-ordering	54
Chapter 3 : q-orderings and Jordan q-orderings	
3. 1 Semisection	67
3. 2 Extension of a q-ordering	81
3. 3 Jordan q-orderings	84
References	91
Notations	93

Introduction

Recently, the unitary group of a non isotropic unitary space was investigated in [C 3] . There some results were announced without proofs concerning the normal subgroup structure of the considered unitary group. On a suggestion of Professor M. Chacron, I was encouraged to reorganize the material and to provide the necessary line of argument needed for the full justification of the announced results. This will occupy part of my thesis ; namely, Chapter 1 .

From a separate view point, I was asked by my thesis supervisor, Professor M. Chacron , to investigate certain ordered division rings where the involution was " forgotten " . The motivation was two-fold. Primo : The recent works done in [Id ; C 2] . Secondo : The commutative background as evidenced by Prestel's " Lectures on Formally Real Fields " . Thus the second part of this thesis is about Jordan orderings, q -orderings, and Jordan q -orderings on division rings to be studied in this order of succession in Chapters 2 and 3 .

In what will follow, we all first present some of the necessary ingredients for the reading of chapter 1 . We will next guide the reader for the reading of chapters 2 and 3 .

Concerning Chapter 1 : We assume the reader is aware

of the basic notion of non-isotropic unitary space V over a division ring with $(\ast)$, as in Kaplansky's "Linear Algebra and Geometry". We recall that the unitary group U of such a space V stands for the group of all invertible linear transformations $u : V \rightarrow V$ such that $u^\ast = u^{-1}$. We shall be concerned with the normal subgroup structure of U . For instance, can one say that if N and M are normal subgroups of U such that $N \cap M = Z \cap U$, the subgroup of the central unitaries, then, in fact, $N \subseteq Z \cap U$ or $M \subseteq Z \cap U$? There are exceptional cases where the answer to this question is in the negative. Our task will be to answer this question in the affirmative under reasonable restrictions (see Theorem 1.3.1.). We shall also attempt an investigation of the subnormal subgroup structure of U . The results are admittedly partial. Under a strong assumption we will be in a position to characterize the abelian subnormal subgroups of U (see Theorem 1.3.4.). As a closing restriction for Chapter 1, we shall always assume that the underlying division ring D is with characteristic not 2 to ensure enough unitaries via the Cayley parametrization

$$k \rightarrow (1 - k)(1 + k)^{-1}$$

where $k = -k^\ast \in L$.

Concerning Chapter 2 : There are two types of orderings to be investigated in this chapter : namely, a Jordan version of the usual Hilbert ordering on a division ring, and the non-commutative version of a q -ordered field, where the axioms are taken exactly as postulated in [P ; pp. 5]. Essentially, a Jordan ordering is one where the multiplicative property is relaxed by means of forming the Jordan product $a \bullet b = ab + ba$ instead of the usual product ab , where a, b are in the carrier division ring D . This type of generalization occurred first in [Id ; C 4]. We will show here a rather unexpected fact that a Jordan ordering is, in fact, a Hilbert ordering (see Theorem 2. 1. 10.). For the considered q -ordered structure D , we will be concerned with the existence of the order valuation. This will require the introduction of the order subring R attached to a given q -ordering. We will show among other things that R is a valuation ring in D , which is preserved under conjugation (see Theorem 2. 2. 10. and Corollary 2. 2. 12.). An example of an infinite dimensional q -ordered division ring D , will be given where the ordering is not Hilbert (see Theorem 2. 3. 1.). The order valuation will be determined and the nature of the valuation units will be also determined (see Theorems 2. 3. 2. and 2. 3. 3.).

Concerning Chapter 3 : More is to be said in this Chapter about q -orderings. Firstly, we will characterize the

situation in which for a given valued division ring D , every q -ordering is a Hilbert ordering (see Theorem 3. 1. 7.). We were motivated by the commutative case, where the nature of the space of all q -orderings is also determined ([P ; pp. 66.]. We will employ a similar tool ; namely, the notion of semisection attached to a given valuation on D (see Definition 3. 1. 1.). A second objective will be to extend a q -ordering on a field F to a q -ordering on the Hilbert division ring over F in one indeterminate (see Theorem 3. 2. 2.). A concluding objective will be a Jordan version of q -ordering. Although we can show that from Jordan q -orderability follows Hilbert orderability (see Theorem 3. 3. 2.), we do not know whether or not a Jordan q -ordering must be a q -ordering. However, we do have an order subring R attached to the given Jordan q -ordering (see Theorem 3. 3. 6.). Still, we cannot decide whether or not R is preserved under conjugation.

Finally, as a guide line for the reading of this thesis we should add that the reader could find at the beginning of each chapter a summary of the contained material as well as a summary of the sections of the chapter.

CHAPTER 1

Normal Subgroup Structure

of a Unitary Group :

The structure of the multiplicative group of a division ring was studied first in Scott [S ; pp. 426 - 440]. In this chapter, we will generalize some of the results from division rings to the unitary group U of a non-isotropic unitary space V . Among other things, we will show that if two normal subgroups of U intersect onto the centre of U then, in fact, one of these coincides with the centre of U provided some reasonable restrictions are made as to the nature of the considered vector space V (see Theorem 1. 3. 1.). Finally, we attempt a generalization of these results from normal subgroups to subnormal subgroups of U , where we assume that every subnormal subgroup of the unitary group which consists only of reflections must be central.

Section 1.1:

Co-finitary Linear Transformations

In this section, we assemble the technical facts about co-finitary linear transformations (see Definition 1. 1. 1.) to be

used at later stages of this chapter. The basic tools will be the notion of Cayley complement of a co-finitary linear transformation combined with the notion of Engel commutator (see Definitions 1. 1. 5. and 1. 1. 6.). Here is the setting of the unitary space we will be working with .

Let D denote a division ring with involution (denoted by $*$) having at least 7 elements, and let Z stand for the centre of D . We assume, throughout this chapter, that $\text{characteristic}(Z) \neq 2$. Let $(V, \langle \rangle)$ be a non-degenerate, non-isotropic, unitary space over D as in [C 3] . Specifically, we will exclude the following special cases :

- (1) V has dimension 1 over D , and D is non-normal quaternionic with $*$ of first kind ;
- (2) V is an orthogonal 2-dimensional space ;
- (3) V has dimension 1 over D , and D has dimension 16 over Z .

Let L be the full ring of linear transformations of V . We identify the centre of L with Z . In this chapter and this chapter only, for the sake of simplification, we will make the mappings in L act on the right side of elements of the space V .

Let U stand for the unitary group of V , and let $Z(U)$ be the centre of U . Let F denote the set of those

elements of $\mathbf{L}$ whose range is finite dimensional. In the sequel, we refer to the elements of $\mathbf{F}$ as finitary linear transformations ; for abbreviation, $\mathbf{FLT}$.

Definition 1.1.1: We call $a \in \mathbf{U}$, a co-finitary linear transformation (for abbreviation ; $\mathbf{CFLT}$), if $(a - 1)$ is $\mathbf{FLT}$.

The set of all unitary co-finitary linear transformations will be denoted by $\mathbf{CF}(\mathbf{U})$.

Lemma 1.1.2: $\mathbf{CF}(\mathbf{U})$ is a normal subgroup of $\mathbf{U}$.

Proof: Evidently, $1 \in \mathbf{CF}(\mathbf{U})$. Let a, b be in $\mathbf{CF}(\mathbf{U})$. Since $(a - 1)$ and $(b - 1)$ are in $\mathbf{F}$, $(a - 1)(b - 1) = (ab - 1) - (a - 1) - (b - 1)$ is in $\mathbf{F}$, which implies $(ab - 1)$ is in $\mathbf{F}$. Thus $ab \in \mathbf{CF}(\mathbf{U})$. Also $a^{-1} = a^*$ exists in $\mathbf{U}$. We have $a^{-1} - 1 = a^* - 1 = (a - 1)^* \in \mathbf{F}$, for the set $\mathbf{F}$ is closed under $*$. Thus $\mathbf{CF}(\mathbf{U})$ is a subgroup of $\mathbf{U}$.

Let $n \in \mathbf{U}$. Since $(a - 1)$ is in $\mathbf{F}$,

$n^{-1}(a - 1)$ is also in $\mathbf{F}$. Thus $(V n^{-1}(a - 1))n$ is finite dimensional. Thus $n^{-1}(a - 1)n$ is in $\mathbf{F}$, so $n^{-1}an \in CF(\mathbf{U})$.

Theorem 1.1.3: (1) *For every $k \in \mathbf{L}$, a skew symmetric element with respect to the adjoint involution, $(1 - k)$ is right invertible.*

(2) *If, in addition, k is also F L T, then $(1 - k)$ is invertible and $(1 - k)/(1 + k)$ is a well defined C F L T.*

Proof: (1) To begin with there is no $x \in \mathbf{V}$, $x \neq 0$, such that $xk = x$. For otherwise, we would have

$$\begin{aligned} \langle x, x \rangle &= \langle x, xk \rangle \\ &= \langle xk^*, x \rangle \\ &= -\langle xk, x \rangle \\ &= -\langle x, x \rangle \end{aligned}$$

so $\langle x, x \rangle = 0$ and $x = 0$.

In other words, $(1 - k)$ has zero kernel or $(1 - k)$ is one to one. By an elementary fact, $(1 - k)$ has a right inverse in $\mathbf{L}$.

(2) Since $(1 + k) = (1 - k)^*$ is right invertible, there exists $(1 + m) \in \mathbf{L}$ such that

$(1 + k)(1 + m) = 1$. Equivalently, $k + m + km = 0$,

which implies $\text{Range}(m) = \text{Range}(-k - km)$. Since k is

FLT so must be m . Also $(1 + m)^*$ exists and

$(1 + m)^*(1 + k)^* = 1$, which implies that

$(1 + k)^* = 1 - k$ is left invertible. Thus $(1 + k)$ and

$(1 - k)$ are both invertible in L . Since $2k/(1 + k)$

commutes with k , which is a FLT, we have $2k/(1 + k)$ is

also a FLT. Now $\frac{1 - k}{1 + k} - 1 = -\frac{2k}{1 + k}$, implies $\frac{1 - k}{1 + k}$

is a well defined CFLT.

Remark 1.1.4: For each $a \in CF(U)$,

$k = a - a^*$ is a FLT and

$\frac{1 + a - a^2}{1 - a - a^2}$ is a well defined CFLT.

Proof: Since $a \in CF(U)$ and $k = a - a^*$ is

a skew-symmetric element of L , $k = (a - 1) - (a - 1)^*$

is a FLT. Also by a trivial adaptation of Theorem 1.1.3.,

$$\frac{1 - k}{1 + k} = \frac{1 - a + a^*}{1 + a - a^*} = \frac{1 - a + \frac{1}{a}}{1 + a - \frac{1}{a}} = -\frac{1 + a - a^2}{1 - a - a^2}$$

is a well defined CFLT.

Definition 1.1.5: For every $a \in CF(\mathbf{U})$, we define the Cayley complement $\hat{a}$ of a by:

$$\hat{a} = - \frac{1 + a - a^2}{1 - a - a^2}.$$

Definition 1.1.6: For all invertible elements a, b in $\mathbf{L}$, we write $[a, b] = a^{-1}b^{-1}ab$ (the commutator of a and b). For invertible elements $a, b_1, b_2, \dots, b_n$ ($n \geq 2$), we define inductively higher commutators as follows:

$$(1) \quad [a, b_1, b_2, \dots, b_n] = [[a, b_1, \dots, b_{n-2}] b_{n-1}] b_n$$

Further when $b_1 = b_2 = \dots = b_n = b$, we write

$[a, b]_n = [a, b, \dots, b]$ ($n \geq 2$), (Engel commutator of a and b).

Theorem 1.1.7: Let $a, b \in \mathbf{U}$ be such that $z = [a, b]$ commutes with both a and b . Let $r(t)$ be a rational function in the indeterminate t over the subalgebra generated by a, b over $\mathbf{Z}$ such that $r(a)$ is well defined. Then the Engel commutator

$$[r(a), b]_n = r(a)^{(-1)^n} r(za)^{(-1)^{n+1}} \binom{n}{1} \dots r(z^n a)^{(-1)^{2n}} \binom{n}{n},$$

where $\binom{n}{i}$ is the i -th binomial coefficient for $i = 0, 1, \dots, n$.

In particular, if $a \in CF(U)$, and $r(t) = -\frac{1+t-t^2}{1-t-t^2}$, then

$$[\hat{a}, b]_n = r((-1)^n a) r((-1)^{n+1} za)^{\binom{n}{1}} \dots r((-1)^{2n} z^n a)^{\binom{n}{n}}.$$

Proof: From Definition 1.1.6., it readily follows

that

$$[r(a), b] = r(a)^{-1} b^{-1} r(a) b = r(a)^{-1} r(za),$$

$$[r(a), b]_2 = [[r(a), b], b]$$

$$= [r(a)^{-1} r(za), b]$$

$$= r(a) r(za)^{-1} r(za)^{-1} r(z^2 a)$$

$$= r(a) r(za)^{-2} r(z^2 a) \quad \text{and}$$

$$[r(a), b]_3 = [[r(a), b]_2, b]$$

$$= [r(a) r(za)^{-2} r(z^2 a), b]$$

$$= r(a)^{-1} r(za)^2 r(z^2 a)^{-1} r(za) r(z^2 a)^{-2} r(z^3 a)$$

$$= r(a)^{-1} r(za)^3 r(z^2 a)^{-3} r(z^3 a).$$

Generally, one can show the following formula :

$$[r(a), b]_n = r(a)^{(-1)^n} r(za)^{(-1)^{n+1} \binom{n}{1}} \dots r(z^n a)^{(-1)^{2n} \binom{n}{n}},$$

where $\binom{n}{i}$ is the i -th binomial coefficient for $i = 0, 1, \dots, n$.

$$\text{In particular, let } r(t) = -\frac{1+t-t^2}{1-t-t^2}.$$

Since, $a \in \text{CF}(\mathbf{U})$, implies $\hat{a} \in \text{CF}(\mathbf{U})$, we have $r(a) = \hat{a}$ and

$$r(-a) = -\left(\frac{1 - a - a^2}{1 + a - a^2}\right) = r(a)^{-1}.$$

Now from the preceding, it can be readily checked that

$$(1) \quad [\hat{a}, b]_n = r((-1)^n a) r((-1)^{n+1} za)^{\binom{n}{1}} \dots r((-1)^{2n} z^n a)^{\binom{n}{n}}.$$

Definition 1.1.8: Let $L[t]$ denote the polynomial ring over L in the central indeterminate t and $z = z^* \in L$, $z \neq 1$. For each $n (\geq 2) \in \mathbb{Z}$, we define the polynomials

$$h_z^n(t) = (1 + t - t^2)(1 - zt - z^2 t^2)^{\binom{n}{1}} \dots (1 + (-1)^{i-1} z^i t - z^{2i} t^2)^{\binom{n}{i}} \dots$$

$$\dots (1 + (-1)^n z^n t - z^{2n} t^2); \text{ and}$$

$$C_z^n(t) = h_z^n(t) - h_z^n(-t).$$

Lemma 1.1.9: For each $n (\geq 2) \in \mathbb{Z}$, $C_z^n(t)$ is a non-trivial polynomial.

Proof: we will show that the coefficient of t in $C_z^n(t)$ is non-zero. Adding the coefficient of t in each quadratic factor of $h_z^n(t)$, we find the coefficient of t in $h_z^n(t)$ to be :

$$1 + \binom{n}{1}(-z) + \dots + \binom{n}{i}(-1)^i z^i + \dots + (-1)^n z^n = (1 - z)^n$$

Thus the coefficient of t in $C_z^n(t)$ is equal to $2(1 - z)^n$.

Since the space V is non-isotropic, for $x \in L$,

if x has an adjoint x^* , then $xx^* = 0$ implies $x = 0$.

Thus L has no symmetric nilpotents. Now $z = z^* \in L$,

$z \neq 1$ implies that $2(z - 1)^n \neq 0$, as desired.

Theorem 1.1.10: Let $a \in CF(U)$, and b be any unitary such that $z = [a, b]$ commutes with both a and b . Then $[\hat{a}, b]_n = 1$ if and only if a is a root of the polynomial $C_z^n(t)$.

Proof: Combining Theorem 1.1.7. equation (1) and Definition 1.1.8., we have

$$[\hat{a}, b]_n = \begin{cases} \frac{h_z^n(a)}{h_z^n(-a)} & \text{for } n \text{ is even} \\ \frac{h_z^n(-a)}{h_z^n(a)} & \text{for } n \text{ is odd} \end{cases},$$

where it is understood that the terms of the above fractions are invertible in L . Thus

$$[\hat{a}, b]_n = 1 \text{ iff } \frac{h_z^n(a)}{h_z^n(-a)} = 1 ; \text{ equivalently, still iff}$$

$$h_z^n(a) - h_z^n(-a) = 0, \text{ or, it is the same, } a \text{ is a root of the polynomial} \\ h_z^n(t) - h_z^n(-t) = C_z^n(t).$$

Theorem 1.1.11: *Let a and b be as in*

Theorem 1.1.10. and n be a natural number such that

$$[\hat{a}, b]_n = 1. \text{ Then } [a, b]^{2^n!} = 1.$$

Proof: Put $z = [a, b]$. The theorem is obvious for $z = 1$. Assume $z \neq 1$. Let A be the subring of L generated by a, b, a^{-1} and b^{-1} . Then A carries the same adjoint involution $*$ and for all $d \in A$, $dd^* = 0$ implies $d = 0$.

We will show that A is semiprime. Let I be a nilpotent ideal in A . Then I^* and $I + I^*$ are also nilpotent in A . If $e \in I + I^*$, then $ee^* \in I + I^*$ so for some natural number n , $(ee^*)^n = 0$, which implies $ee^* = 0$, equivalently, $e = 0$. Thus $I + I^* = 0$, which implies $I = 0$. Therefore, A is semiprime.

Since A is semiprime, A is a subdirect product of prime rings. For the sake of simplicity, we assume that A itself is prime. Since $C_z^n(t)$ is a non trivial polynomial (Lemma 1.1.9)

and has all its coefficients in the centre of A (which is an integral domain), these coefficients are non zero divisors. Thus we can write

$$C_z^n(t) = c_1 t + c_2 t^2 + \cdots + c_m t^m,$$

where the c_i 's are the coefficients of t^i 's in $C_z^n(t)$ and m is the degree of $C_z^n(t)$, $m \leq 2^n$.

By a trivial adaptation of Theorem 1.1.10.,

$C_z^n(a) = 0$, which implies

$$(1) \quad c_1 + c_2 a + \cdots + c_m a^{m-1} = 0.$$

Since $b^{-1}ab = za$, conjugating (1) with b , we have

$$(2) \quad c_1 + z c_2 a + \cdots + z^{m-1} c_m a^{m-1} = 0.$$

Multiplying (1) by z^{m-1} and subtracting it from (2), we get

$$c_1(1 - z^{m-1}) + \cdots + c_{m-1}(z^{m-2} - z^{m-1})a^{m-2} = 0.$$

Step by step, we can eliminate all the non constant terms in (1), which gives

$$c_1(1 - z^{m-1})(1 - z^{m-2}) \cdots (1 - z) = 0$$

Substituting for $c_1 = 2(1 - z)^n$, as being calculated in the proof of Lemma 1.1.9., we get

$$2(1 - z)^n(1 - z^{m-1})(1 - z^{m-2}) \cdots (1 - z) = 0.$$

so $(1 - z^i) = 0$ for some i , $2 \leq i \leq m - 1$, which

implies $1 - z^{2^n} = 0$.

Theorem 1.1.12: Let M and N be two normal subgroups of the unitary group U with $N \subseteq CF(U)$ and $N \cap M \subseteq Z(U)$ and suppose that the centralizer of $N \subseteq Z(U)$. For every $a \in N$ and every $b \in M$ with $[a, b] \neq 1$, there exists a polynomial $P_{a,b}$ of bounded positive degree over Z , not depending on a and b , which vanishes at a and $[a, b]$ is a root of unity of bounded order.

Proof: Let a, b be as in the statement. We have $1 \neq [a, b] = z \in N \cap M \subseteq Z(U)$, and so, z commutes with both a and b . Also for $a \in N \subseteq CF(U)$, $\hat{a}$ is a well defined CFLT.

Let $a_1 = [\hat{a}, b] \in CF(U) \cap M$. By construction a_1 is a rational expression in a . Thus, $\hat{a}_1 \in CF(U)$ is well defined. Let $a_2 = [\hat{a}_1, a']$ for some $a' \in N$ (to be fixed later). Clearly, a_2 is a rational expression in a and $a_2 \in N$. Thus

$$[a_2, b] \in N \cap M \subseteq Z(U) \text{ and} \\ (1) \quad [a_2, b, b] = 1.$$

Also $a_1 = [\hat{a}, b] = [r(a), b] = r(a)^{-1} r(za)$.

Write $z' = [a_1, a'] \in [M, N] \subseteq Z(U)$, then

$a_2 = [\widehat{a_1}, a'] = r(a_1)^{-1} r(z'a_1)$. Thus

$$\begin{aligned} [a_2, b] &= a_2^{-1} b^{-1} a_2 b = a_2^{-1} r(za_1)^{-1} r(zz'a_1) \\ &= r(a_1) r(z'a_1)^{-1} r(za_1)^{-1} r(zz'a_1) \quad \text{and} \end{aligned}$$

$$\begin{aligned} [a_2, b, b] &= [a_2, b]^{-1} r(za_1) r(zz'a_1)^{-1} r(z^2a_1)^{-1} r(z^2z'a_1) \\ &= r(a_1)^{-1} r(z'a_1) r(za_1) r(zz'a_1)^{-1} r(za_1) r(zz'a_1)^{-1} \times \\ &\quad r(z^2a_1)^{-1} r(z^2z'a_1) \\ &= r(a_1)^{-1} r(za_1)^2 r(z^2a_1)^{-1} r(z'a_1) r(zz'a_1)^{-2} r(z^2z'a_1) \\ &= 1. \end{aligned}$$

Two cases arise:

Case (i) : $z' \neq 1$, that is, we can find a' in N such that a_1 and a' do not commute.

Then the above equation is a non-trivial rational expression in a so it gives the required polynomial $P_{a,b}$ of bounded degree over Z which vanishes at a .

Case (ii) : $z' = 1$, that is, a_1 centralizes N .

By hypothesis, $1 = [a_1, b] = a_1^{-1} b^{-1} a_1 b$. Thus

$$r(a) r(za)^{-1} r(za)^{-1} r(z^2a) = 1.$$

The latter being a rational expression in a , of bounded degree, yields quickly a polynomial $P_{a,b}$ as desired.

In both cases the exploitation of the polynomials vanishing at a readily give that $z = [a, b]$ is a root of unity.

SECTION 1.2 : Normal Abelian Subgroups

In this section, we will initiate the promised analogy between the normal subgroup structure of a division ring and that of the considered unitary group U by showing that if N has a central commutator subgroup $[N, N]$, then, in fact, $N \subseteq Z$, ([S, Theorem 14.4.5]; Theorem 1.2.6). A central hypothesis needed throughout will be :

(H) : Every normal abelian subgroup of the considered unitary group U must be central.

For instance, if the considered unitary space V is finite dimensional then it was shown in [C1, Theorem 11], that every invariant subalgebra W of L (under unitary conjugation) must be trivial ; that is, $W = 0$, Z or L . Now if N is a normal subgroup of U , then the subalgebra $Z[N]$ spanned by N is clearly invariant. Should N be

abelian, we then readily get that $N \subseteq Z$, and so, hypothesis (H) is automatically verified in that instance. We proceed to :

Lemma 1.2.1: *Let N be a normal subgroup of the unitary group U of L with $[N, N] \subseteq Z$ such that $N \subseteq CF(U)$. Then for each $a \in N$, we have : $[\hat{a}, b]_3 = 1$, for all $b \in N$, where $\hat{a}$ is the Cayley complement of a .*

Proof: Since N is a normal subgroup of U , and the Cayley complement $\hat{a}$ of a is well defined (Remark 1.1.4.), we get $[\hat{a}, b] \in CF(U) \cap N$ and $[\hat{a}, b]_2 \in [N, N] \subseteq Z$. Thus $[\hat{a}, b]_3 = 1$, as desired.

Our next goal will be to show that for N as in the statement of Lemma 1.2.1., we have, in fact, $N \subseteq Z$. Deny this. We shall build up a contradiction in the series of Lemmas.

Lemma 1.2.2: *For N as in Lemma 1.2.1., $N \not\subseteq Z$, the centre of the group U is the two element subgroup.*

Proof: Assume, to the contrary, that $Z(U)$ is infinite. Since $[a, b] \in Z$ for all $a, b \in N$, $[a, b]$

commutes with both a and b . By Lemma 1.2.1., as in

Theorem 1.1.12., there exists a fixed polynomial $P (\neq 0)$

over Z with coefficients depending only on $[a, b]$ such that

$P(a) = 0$. Since for every central unitary z ,

$[za, b] = [a, b]$ and since

$$k = (za) - (za)^* = (z^* - z) + z(a - 1) - z^*(a^* - 1)$$

is a skew symmetric element of L , we can show exactly as in

Remark 1.1.4., that $\frac{(1 - k)}{(1 + k)} = \widehat{za}$ is well defined.

From Theorem 1.1.12., we get that za is a root of the polynomial P . For q the degree of P and arbitrary q

elements $z_i \in Z(U)$, we have $P(z_i a) = 0$ for all

$1 \leq i \leq q$. Expanding this, we get the system of equations :

$$c_0 + c_1 z_1 a + \dots + c_q z_1^q a^q = 0$$

$$\dots$$

$$c_0 + c_1 z_q a + \dots + c_q z_q^q a^q = 0$$

where $P(t) = c_0 + c_1 t + \dots + c_q t^q$.

The preceding system written in the matrix form is a Vander Monde system, whose determinant is non-zero since the z_i 's are distinct. Thus the above system of linear equations has only the

trivial solution $c_i a^i = 0$, $1 \leq i \leq q$. Since $a \in \mathbf{U}$, we get $c_i = 0$ for all i , so, $P = 0$, which contradicts the choice of P . This shows that $Z(\mathbf{U})$ is finite.

If now $Z(\mathbf{U})$ contains more than two elements, then Z contains a non-zero skew symmetric element. Hence the finite field Z has the subfield $Z^+ = \{a \in Z \mid a = a^*\}$ of index two. Since $\text{characteristic}(Z) \neq 2$, we have $|Z| = 2|Z^+| = 2p^n$, where p is an odd prime, which brings a contradiction.

Lemma 1.2.3: For N as before, we have

Characteristic $(Z) \neq 0$.

Proof: Let $a, b \in N$ be such that $[a, b] \neq 1$.

Since $[a, b]$ is in $Z(\mathbf{U})$, it follows from Lemma 1.2.2., that

$[a, b] = -1$. Also for each $m \in \mathbb{Z}$,

$$\bar{a}_m = -\frac{m(a - a^*) - 1}{m(a - a^*) + 1} \text{ is a unitary.}$$

By Lemma 1.2.1. we get $[\bar{a}_m, b]_3 = 1$ for all $m \in \mathbb{Z}$. Since a anti-commutes with b , the commutator of $[\bar{a}_m, b]$ is precisely $(\bar{a}_m)^{-2}$. Because $[\bar{a}_m, b]_3 = 1$, we

have $(\bar{a}_m)^{-8} = 1$ or $(\bar{a}_m)^8 = 1$ for all $m \in \mathbb{Z}$. By a Vander Monde determinant argument, we readily get either $\text{Characteristic}(\mathbb{Z}) \neq 0$, in which case we are through or else, $a^2 = 1$ for all a not in the centre of N . In the later case, by a routine argument we can show that N is commutative, which is a contradiction to the choice of N .

Lemma 1.2.4: *For every $a \in N$, either $a^2 = 1$ or else, $a^2 = -1$.*

Proof: By hypothesis (H), we have $Z(N) = N \cap \mathbb{Z}$. Let $a, b \in N$ be such that $[a, b] \neq 1$. We have $[a, b] \in [N, N] \subset \mathbb{Z}$ so, $[a, b] \in Z(\mathbf{U}) = \langle -1 \rangle$ and so, $[a, b] = -1$. Thus for a not in the centre of N , $[a^2, b] = [a, b]^2 = 1$ so, a^2 centralizes N and, consequently, $a^2 \in Z(\mathbf{U})$. Thus either $a^2 = 1$ or $a^2 = -1$. Also for $a \in Z(N) \subset Z(\mathbf{U})$, we have $a = \pm 1$ so, $a^2 = 1$, as desired.

Lemma 1.2.5: *Any two elements in N squaring to -1 must commute.*

Proof: Let $a, b \in N$ be such that $a^2 = -1$, $b^2 = -1$. Deny the conclusion $ab = ba$. Then $[a, b] = -1$. Let then C be the subring of L generated by a and b over the prime subfield of Z . By Lemma 1. 2. 3., C is a finite subset of L . Since for all $x \in L$, $xx^* = 0$ implies $x = 0$, C is semiprime. Also C is $*$ closed. Thus C is a subdirect product of finite prime rings with involution, where each prime factor carries the anti-commuting invertible elements a and b corresponding to the elements a and b .

Let $\underline{C}$ be an arbitrary prime component of C . Since $\underline{C}$ cannot be a division ring, it follows that $\underline{C}$ must be a quaternionic central simple algebra over a finite field. In other words, $\underline{C}$ is a 2×2 matrix ring over a finite field. Since for all $x \in \underline{C}$, $xx^* = 0$ implies $x = 0$, the involution of $\underline{C}$ cannot be of symplectic type. Thus the involution is of the conjugate transpose type. Since $Z(\underline{C})$ the centre of $\underline{C}$ is a finite field with Characteristic not 2, the induced involution on $Z(\underline{C})$ is trivial so, the involution is the usual transpose involution.

We wish to classify all the orthogonal matrices in $\underline{C}$.

Let $\begin{bmatrix} u & v \\ x & y \end{bmatrix}$ be an orthogonal matrix,

where u, v, x, y are in the finite field. Then

$$\begin{bmatrix} u & v \\ x & y \end{bmatrix} \begin{bmatrix} u & x \\ v & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus $u^2 + v^2 = 1$, $x^2 + y^2 = 1$ and $ux + vy = 0$.

Which implies $\frac{u^2}{v^2} = \frac{y^2}{x^2}$ equivalently, $\frac{1 - v^2}{v^2} = \frac{1 - x^2}{x^2}$.

Therefore, $x = \pm v$. Equivalently, $y = \mp u$, so the orthogonal matrices are

$$\text{either } \begin{bmatrix} u & v \\ -v & u \end{bmatrix} \text{ (Rotation) or } \begin{bmatrix} u & v \\ v & -u \end{bmatrix} \text{ (Reflection),}$$

where u and v are any elements in the finite field such that

$$u^2 + v^2 = 1.$$

Since the images $\underline{a}$ and $\underline{b}$ of a and b in $\underline{C}$ cannot be reflections, which automatically square to -1 , they must be rotations and thus these commute, which is not the case. Hence any two elements in N squaring to -1 must commute.

Theorem 1.2.6: Let V be a unitary space as postulated (see p. 6) and suppose that the centralizer of $CF(\mathbf{U})$ is in $Z(\mathbf{U})$. Under the hypothesis (H), if N is a normal subgroup of the unitary group $\mathbf{U}$ such that $[N, N] \subseteq Z$, then N is contained in Z .

Proof: The proof is divided into two separate cases depending upon whether $N \subseteq CF(U)$ or $N \not\subseteq CF(U)$.

Case 1: $N \subseteq CF(U)$.

Let $a, b \in N$ be anti-commuting. If $a^2 = 1$ and $b^2 = 1$, then $(ab)^2 = -1$. Since all the conjugates of ab are in N , and since each of these elements square to -1 , all these conjugates of ab must commute (Lemma 1.2.5.). Thus they generate an abelian normal subgroup of U and so, ab itself is in the centre, a contradiction to the fact that ab and a are anti-commuting elements.

Case 2: $N \not\subseteq CF(U)$. Let $N_1 = N \cap CF(U)$.

By Case 1, N_1 is in the centre of U or, it is the same,

$N_1 = 1$. Thus N and $CF(U)$ commute, which implies N and F commute. Hence $N \subset Z$.

Corollary 1.2.7: *Every solvable normal subgroup of U is in the centre of U .*

Proof: Let N be a solvable normal subgroup of U , which is not in $Z(U)$. Define $N^{(0)} = 1$, $N^{(1)} = N$ and $N^{(i)} = [N^{(i-1)}, N^{(i-1)}]$ for each integer i , $i \geq 2$. Thus

there exists a least positive integer m such that $N^{(m)} = 1$. If $m \neq 1$, then $(m-1) \in \mathbb{Z}^+$ and $N^{(m-1)}$ is abelian. The latter being a normal abelian subgroup of U , it is in $Z(U)$. By a trivial adaptation of the Theorem 1.2.6., $N^{(m-2)}$ is in $Z(U)$. Thus $N^{(m-1)} = 1$, a contradiction to the choice of m . Hence N is abelian and so N must be central.

Corollary 1.2.8: The factor group $U / (U \cap Z)$ is centerless and every normal abelian subgroup of $U / (U \cap Z)$ is, in fact, the identity subgroup.

Proof: Since $Z(U) = (U \cap Z)$, $U / (U \cap Z)$ is centreless. If N is a normal abelian subgroup, then $[N, N] = (1)$, where (1) is the centre of $U / (U \cap Z)$. Theorem 1.2.6. implies that $N \subset Z(U / (U \cap Z)) = 1$, as desired.

Section 1.3 :

Normal and Subnormal Subgroups

While hypothesis (H) appearing at the beginning of

section 1. 2. , was the right assumption to make in order for the relation $[N, N] \subseteq Z$ to imply $N \subseteq Z$, in dealing with the situation $[N, M] \subseteq Z$, where N, M are normal subgroups of U , one needs something more than the preceding hypothesis, namely :

(H') : The centralizer in U of every non-central normal subgroup N of U must be contained in $Z(U)$.

Again, by [C 1 , Theorem 11], the hypothesis (H') is automatically true in the case V is finite dimensional (all other things unchanged).

Theorem 1. 3. 1 : Let V be as in the Theorem 1. 2. 6. and the hypothesis (H') holds true. If N and M are normal subgroups of the unitary group U such that $[N, M] \subseteq Z$, then either $N \subseteq Z$ or $M \subseteq Z$.

Proof : Let N, M be as in the statement. Deny the conclusion. Then the conclusion will fail for $N \cap CF(U)$, $M \cap CF(U)$. Thus we may assume that $N \subseteq CF(U)$ and $M \subseteq CF(U)$.

Since $N \not\subseteq Z(U)$ and $M \not\subseteq Z(U)$, there are

$a \in N \setminus Z(U)$ and $b \in M \setminus Z(U)$ such that $[a, b] \neq 1$.

By hypothesis $z = [a, b] \in Z(U)$. By Theorem 1.1.12., z is a root of unity with bounded order q and, hence, for all

$a \in N$, a^q is in the centre of L . Similarly, for all $b \in M$, b^q is in the centre of L .

Since for $a \in N \subset CF(U)$, $\hat{a}$ exists, the normality of M implies $[\hat{a}, b] \in M$ and so, $[\hat{a}, b]^q$ is in the centre of L .

Thus for $r(a) = \hat{a}$, we have

$$(1) \quad [\hat{a}, b]^q = r(a)^{-q} r(za)^q.$$

Conjugating with b , we get

$$(2) \quad b^{-1} [\hat{a}, b]^q b = r(za)^{-q} r(z^2a)^q.$$

From (1) and (2), we have

$$[\hat{a}, b]^{-q} b^{-1} [\hat{a}, b]^q b = r(a)^q r(za)^{-2q} r(z^2a)^q.$$

Since $[\hat{a}, b]^q$ is in $Z(U)$, we get

$$r(a)^q r(za)^{-2q} r(z^2a)^q = 1, \text{ which shows that } a$$

is a root of some non-trivial polynomial P with coefficients in

Z (these coefficients depend only on $z = [a, b]$). Since for

arbitrary $z' \in Z(U)$, $[z'a, b] = [a, b]$, we can replace a

with $z'a$ in the rational expression $[\hat{a}, b]$. By the same line of argument as in the proof of Lemma 1.2.2., it follows that $z'a$ also

satisfies the same polynomial P .

Following the same procedure as in the Lemmas 1. 2. 2. , 1. 2. 3., 1. 2. 4. and 1. 2. 5. , we can readily check that $Z(\mathbf{U})$ is a two element group, that characteristic $(Z) \neq 0$ and that every element of $\mathbf{N}$ has square 1 or -1 and that any two elements of $\mathbf{N}$ squaring to -1 must commute . By way of symmetry, every element of $\mathbf{M}$ has square 1 or -1 and that any two elements of $\mathbf{M}$ squaring to -1 must commute.

Since for each pair a, b with $a \in \mathbf{N} \setminus Z(\mathbf{U})$, $b \in \mathbf{M} \setminus Z(\mathbf{U})$, a and b square to 1 or -1 there is some $a \in \mathbf{N} \setminus Z(\mathbf{U})$ such that $a^2 = -1$ (for otherwise $\mathbf{N}$ will be abelian and so, $\mathbf{N}$ will be central) . By way of symmetry, there is $b \in \mathbf{M} \setminus Z(\mathbf{U})$ with $b^2 = -1$. By the same line of argument as in the Lemma 1. 2. 5. , we can then deduce that a and b must commute and so do their conjugates in the respective groups $\mathbf{N}$ and $\mathbf{M}$. This yields us two commuting non-central normal subgroups of $\mathbf{CF}(\mathbf{U})$, which contradicts the hypothesis (H') , as desired.

We now turn to the second objective for this section ; namely, the investigation of subnormal subgroups of $\mathbf{U}$. We will make the following extra assumption :

(H'') : Every subnormal subgroup N of U which is formed entirely by reflections must be, in fact, the two element subgroup (or the identity subgroup).

For N a given subnormal subgroup of U , if m is the smallest integer such that there is an increasing subnormal series $(U_i)_{i=0,1,\dots,m}$ of length $(m+1)$, beginning at N , and terminating at U :

$$N = U_0 \triangleleft U_1 \triangleleft U_2 \triangleleft \dots \triangleleft U_m = U,$$

then we call N a m -subnormal subgroup.

Lemma 1. 3. 2 : With the hypothesis and the notations in the preceding, for every abelian m -subnormal subgroup N of U we have :

$$(i) \quad a, b \in N; [\hat{a}, b]_{m+1} = 1;$$

$$(ii) \quad \text{For } u \in U_2, b^u = u^{-1}bu \text{ and}$$

$$z = [a, b^u], \text{ we have } z^2 = 1.$$

Proof : (i) Since $b \in U_{m-2} \triangleleft U_{m-1} < U$ and $\hat{a}$ is a unitary, we have $[\hat{a}, b] \in U_{m-1}$ so, $[\hat{a}, b]_2 \in U_{m-2}$. Continuing in this way, we get $[\hat{a}, b]_m \in U_0 = N$ so,

$[\hat{a}, b]_{m+1} = 1$ for N is abelian and N contains b .

(ii) For $z = 1$, the result is obvious. Let $z \neq 1$.

Since $b \in U_1$, $u \in U_2$ implies $b^u \in U_1$ so, $[a, b^u] \in N$.

Since N is abelian, $z = [a, b^u]$ commutes with a .

Repeating this process for $a^u, b^u \in N^u$ and $u^{-1} \in U_2 = U_2^u$, we get $[a, b^u] = [(a^u)^u, b^u]$ commutes with b^u .

By Theorem 1.1.12., $z = [a, b^u]$ is a root of unity of bounded order r and so, for all $a \in N$, a^r commutes with b^u for all $b \in N$ and $u \in U_2$. Thus the subgroup $(a^r)^{U_2}$, generated by all the conjugates of a^r with exponent in U_2 , is an abelian $(m-1)$ -subnormal subgroup of U and, hence, by the induction step, it must be central or, it is the same, $a^r \in Z(U)$. Following the same procedure as in the proof of Lemma 1.2.2., we can readily check that $Z(U)$ is a two element group and so, $a^{2r} = 1$. Similarly, $(b^u)^{2r} = 1$. Moreover, the characteristic of Z is not zero.

Let $C = K(a, b^u)$ be the subalgebra generated by a and b^u over K , the prime subfield of Z . Since the commutator $z = [a, b^u]$ commutes with both a and b^u , we have that C is a finite module over its own centre $Z(C)$. Moreover, $Z(C)$ is algebraic over K . Furthermore, C

carries the induced involution $(\cdot)^*$ as the generators a and b^u are unitaries. Now C decomposes as a direct product of central simple algebras over fields. In each factor algebra $\underline{C}$ of C the involution fixes all the central elements as established in Lemma 1. 2. 2. (owing to the fact that for each x in C , $xx^* = 0$ implies $x = 0$ and that any finite field of odd cardinality has no non-trivial involution). Since $z = [a, b^u]$ is in the centre of U , we then have $z = z^*$, and since z is a unitary, it follows that $z^2 = 1$ in $\underline{C}$. As this holds in each prime factor $\underline{C}$ of C , we deduce that $z^2 = 1$ in C and hence in L .

Lemma 1. 3. 3: *For N as in Lemma 1. 3. 2. and $N \not\subset Z$, every element in N is a reflection.*

Proof: Let $a, b \in N$. For $u \in U_2$, we have by Lemma 1. 3. 2., that if $z = [a, b^u]$, then $z^2 = 1$. Now $b^u \in U_1$ so, $[a, b^u] \in N$. Since N is abelian and since $a \in N$ we have $[a, b^u]$ commutes with a . Thus $1 = [a, b^u]^2 = (a^{-1}(b^u)^{-1}ab^u)^2 = a^{-2}(b^u)^{-1}a^2b^u = [a^2, b^u]$ or a^2 commutes with b^u for all $u \in U_2$. Since $N \not\subset Z$, there is $b \in N$, $b \notin Z$. Then the centralizer of $(b)^{U_2}$

must be, by the induction step, the centre of U so,

$$[a^2, b^{U_2}] = 1, \text{ which implies } a^2 \in Z.$$

If now some $a \in N \setminus Z(U)$ is squaring to -1 , there must be $u \in U_2$ such that $[a, a^u] = z \neq 1$. From $a^2 = -1$ we deduce $(a^u)^2 = -1$. Let C be the subalgebra generated by a and a^u . Assume, to simplify, that C is a prime algebra. We have $z = [a, a^u] \in Z(C)$ and since $z^2 = 1$, we get $z = -1$. As in Lemma 1.2.3., we show that $\text{characteristic}(Z) \neq 0$ and so, C is finite and, hence, C must be the 2×2 matrix algebra over a finite field where the involution is transpose involution. Now by the classification of the orthogonal group in C , we have both a and a^u must be rotations so a and a^u commute, a contradiction. This shows that $a^2 = 1$, as desired.

Theorem 1.3.4: *Let U be a unitary space which satisfies the hypothesis (H'') ,*

(H'') : Every subnormal subgroup N of U which is formed entirely by reflections must be, in fact, the two element subgroup (or the identity subgroup).

Then every abelian subnormal subgroup N of U must be contained in the centre of U .

Proof: For assume to the contrary that $N \not\subseteq Z$.

Pick $a_0 \in N$, $a_0 \notin Z$, we have $a_0^2 = 1$ using

Lemma 1.3.3. For general $a \in N$, $a \notin Z$, then again

$a^2 = 1$ by the same Lemma. If, on the other hand, $a \in Z$, then

$aa_0 \notin Z$ and $aa_0 \in N$ so, again $(aa_0)^2 = 1$. Since

$a \in Z$, $aa_0 = a_0a$ so, $a^2 = 1$. Thus for all $a \in N$,

$a^2 = 1$ and so, by (H'') , $N \subseteq Z$.

CHAPTER 2.

Jordan orderings and q-orderings

We suppose that the reader is familiar with the notion of Hilbert order relation on a division ring D . A natural question to ask is whether this ordering can be "captured" from the underlying Jordan structure of D (see, for example, [Id]). Specifically, if we replace the product in D with the Jordan product :

$$(J) : \quad a \bullet b = ab + ba ,$$

it is clear that, we have the following :

$$(J_1) : \quad a, b > 0 \text{ together imply } a \bullet b > 0 .$$

Conversely, if we are granted requirement (J_1) in the above, can one, in fact, establish the multiplicative axiom $a, b > 0$ implies $ab > 0$ (all other things unchanged).

In the course of the considered investigation we will come across a property which is familiar to the commutative case ; namely :

$$(Q) : \quad a > 0 \Rightarrow ab^2 > 0 .$$

This type of property occurred first in [P], where the ordering is referred to as a q-ordering. It is thus natural to investigate

property (Q) on its own account, which we will do in section 2.2. Among other things we show that this ordering does admit an order valuation.

Finally in section 2.3., we show that the notion of q -ordering is genuinely weaker than that of Jordan ordering.

Section 2.1: Jordan orderings

In this section, the division ring D is equipped with a Jordan ordering denoted by $\geq$ (see Definition 2.1.1.). We prove that the set of all bounded elements of D is a valuation ring in D (Theorem 2.1.7.). Of crucial importance will be Theorem 2.1.9., which asserts that the conjugate of a positive bounded element in D is again a positive bounded element in D . Making use of these preliminary results, we then show that every Jordan ordering on the division ring is, in fact, nothing but a Hilbert ordering (Theorem 2.1.10.). We shall adopt throughout the section and all of this chapter the following notation :

$$(E_1) : \quad a, b \in D ; a \bullet b = ab + ba .$$

Definition 2.1.1 : By a Jordan ordering on a division ring D , we mean a binary relation $\geq$ on D such that for all

$a, b, c \in D$:

- (1) $1 > 0$ and $a \geq a$;
- (2) $a \geq b, b \geq a \Rightarrow a = b$;
- (3) $a \geq b, b \geq c \Rightarrow a \geq c$;
- (4) $a \geq b$ or $b \geq a$;
- (5) $a \geq b \Rightarrow a + c \geq b + c$; and
- (6) $a, b \geq 0 \Rightarrow ab + ba \geq 0$.

As a preliminary observation, the above definition will imply, of course, that the division ring D has characteristic 0.

It is obvious from Definition 2. 1. 1., that every Hilbert ordering on D is also a Jordan ordering. As announced, we wish to establish the converse. We now fix the Jordan ordering on D and let P be the domain of positivity of this order relation ; that is,

$P = \{ a \in D \mid a > 0 \}$. We have :

- (i) $1 \in P$, and $0 \notin P$;
- (ii) $P + P \subseteq P$ (i. e. $a, b \in P$ together imply $a + b \in P$);
- (iii) $P \bullet P \subseteq P$ (i. e. $a, b \in P$ together imply $a \bullet b = ab + ba \in P$);

$$(iv) \quad P \cup -P = D^{\bullet}.$$

Refer to any subset P of D verifying (i) - (iv) as to a Jordan cone of D . Such a subset P induces an order relation on D defined by :

$$(E_2) : \quad a \geq b \Leftrightarrow a - b \in P \cup \{0\};$$

which is evidently a Jordan ordering.

Definition 2. 1. 2 : The core C of D is the set of sums of products of the non-zero squares in D .

Definition 2. 1. 3 : For $a \in D$, put :

$$|a| = \begin{cases} a, & \text{for } a \geq 0 \\ -a, & \text{for } a < 0 \end{cases}$$

As a corollary to the preceding definition, we have for all $a, b \in D$:

$$(E_3) : \quad |a + b| \leq |a| + |b| ;$$

$$(E_4) : \quad |a + b| = |a| + |b| \quad \text{if and only if, } a \text{ and } b \text{ have same sign.}$$

Lemma 2.1.4: For all $a, b \in P$, $x \in D^\bullet$, and

$q \in \mathbb{Q}^+ (= \{q_0 \in \mathbb{Q}, q_0 > 0 \text{ in } \mathbb{Q}\})$ we have:

- (1) $a^{-1} \in P$;
- (2) $qa \in P$;
- (3) $ax^2 \in P$;
- (4) $a < c$ ($c \in C$) $\Rightarrow a^{-1} > c^{-1}$;
- (5) $a > b \Rightarrow a^2 > b^2$.

Proof: (1) Evidently,

$a \cdot a^{-2} = a \cdot a^{-2} + a^{-2} \cdot a > 0$ or $2a^{-1} > 0$ which implies $a^{-1} > 0$.

(2) If $q = m/n$, where m and $n (\neq 0)$ are integers, then for $a \neq 0$;

$$qa = (m/n)a = (n(ma)^{-1})^{-1} > 0 .$$

(3) If $ax^2 < 0$ for some x , then $-ax^2 > 0$.

By Lemma 2.1.4. (1) ,

$(a^{-1}) \cdot (-ax^2) = -ax^2 \cdot a^{-1} = a^{-1} \cdot ax^2 > 0$, contrary to the fact that the sum of two squares is always positive.

(4) From $a < c$, we derive $ac^{-1} < cc^{-1} = 1$.

Thus $ca^{-1} > 1$, which implies $a^{-1} > c^{-1}$

(5) Clearly, $a - b, a + b > 0$. Thus

$$(a - b) \cdot (a + b) > 0 \text{ which implies } 2(a^2 - b^2) > 0 ,$$

which again implies $a^2 > b^2$.

Lemma 2.1.5: For all $a, b \in D^\bullet$ and $q \in \mathbb{Q}^+$:

$$(1) \quad (a \pm b)^2 \leq 2(a^2 + b^2) ;$$

$$(2) \quad a^2 \leq q^2 \Leftrightarrow -q \leq a \leq q .$$

Proof: (1) Since

$(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$, and since every non-zero square in D is always positive, we have

$$(a \pm b)^2 \leq 2(a^2 + b^2) .$$

(2) If $a^2 \leq q^2$, then

$$q \pm a = (1/2q)(q \pm a)^2 + (1/2q)(q^2 - a^2) > 0 .$$

Thus $-q \leq a \leq q$.

Conversely, If $-q \leq a \leq q$, then $q - a > 0$

and $q + a > 0$. Therefore,

$(q + a)^2 \cdot (q - a) + (q + a) \cdot (q - a)^2 > 0$, which implies $2q(q^2 - a^2) > 0$, which again implies $a^2 \leq q^2$, as desired.

Definition 2.1.6: Let D be a Jordan ordered division ring. An element $a \in D$ is called a bounded element if

there exists some $q \in \mathbb{Q}^+$ such that $|a| \leq q$. Further if $|a| < q$, for all $q \in \mathbb{Q}^+$, then we call a an infinitesimal.

Let R denote the set of all bounded elements in D , let J denote the set of all infinitesimals in D , and let U denote the set of all elements u of D such that

$$(E_5): \quad q_1 < |u| < q_2 \text{ where } q_1 \text{ and } q_2 \in \mathbb{Q}^+.$$

Theorem 2.1.7: Let D be a Jordan ordered division ring. Then :

- (1) R is a valuation ring in D ;
- (2) J is the unique maximal ideal in R ;
- (3) U is the set of all the invertible elements in R .

Proof: (1) Let $a, b \in R$. By (E_3) , we have $a \pm b \in R$, so, R is an additive subgroup of D and, in fact, by Lemma 2.1.4.(2), R is $\mathbb{Q}$ -permissible. By Lemma 2.1.5.(1), we have $(a \pm b)^2 \leq 2(a^2 + b^2)$ so,
 $(a \pm b)^2 = a^2 + b^2 \pm (ab + ba) \in R$. Thus
 $2(a \bullet b) \in R$, and so, $a \bullet b \in R$. Since
 $bab = (1/2)[b \bullet (b \bullet a) - b^2 \bullet a]$, we have,
 $bab \in R$. Similarly, $ba^2b, ab^2a \in R$. Thus

$$(ab)^2 + (ba)^2 = a(bab) + (bab)a = a \bullet (bab) \in R.$$

$$\text{Also from } (ab - ba)^2 = (ab)^2 + (ba)^2 - [ab^2a + ba^2b],$$

we derive, $(ab - ba)^2 \in R$, and, hence, $ab - ba \in R$

(Lemma 2.1.4.). Since both $ab + ba$ and $ab - ba$ are in

R , so must be $2ab = (ab + ba) + (ab - ba)$ and, hence,

$$ab \in R.$$

Finally, for given $c \in D^\bullet$, if $c \notin R$, then $|c| > q$ for all $q \in \mathbb{Q}^+$ (Lemma 2.1.4. (4)), and, thus, $|c^{-1}| < 1/q$, so $c^{-1} \in J \subset R$. Hence R is a valuation ring in D .

(2) We have

$$\begin{aligned} J &= \{a \in D \mid |a| < q, \text{ for all } q \in \mathbb{Q}^+\} \\ &= \{a \in R \mid a^{-1} \in R\} = \text{Jac.}(R), \end{aligned}$$

where $\text{Jac.}(R)$ is the Jacobson radical of R .

(3) We have $U = \{a \in R \mid a^{-1} \in R\}$, which is a multiplicative subgroup of the division ring D . Thus U is the group of invertible elements of R .

Lemma 2.1.8: If $a > 0$, then $xax > 0$, for all $x \in D^\bullet$.

Proof: If $xax < 0$ for some x , then $-xax > 0$.

Thus $(-xax) \bullet a = (-xax) \cdot a + a \cdot (-xax) > 0$ or
 $-(xa)^2 - (ax)^2 > 0$, contrary to the fact that the sum of two squares is always positive.

Theorem 2.1.9: For given $a \in R$, $a > 0$,
 and $x \in D^\bullet$, we have :

$$(1) \quad xax^{-1} > 0, \text{ and}$$

$$(2) \quad xax^{-1} \in R.$$

Proof: Let a, x be as in the statement of the lemma.

By Lemma 2.1.8., $xax > 0$, and evidently $x^{-2} > 0$.

$$(1) \quad \text{By Lemma 2.1.4. (3)., } xax^{-1} = (xax) \cdot x^{-2} > 0.$$

$$(2) \quad \text{By definition } (xax) \bullet x^{-2} > 0 \text{ or}$$

$$xax^{-1} + x^{-1}ax = (xax) \cdot x^{-2} + x^{-2} \cdot (xax) = (xax) \bullet x^{-2} > 0.$$

Now there exists some $q \in \mathbb{Q}^+$, such that $a < q$, and so,

$q - a > 0$. Replacing a with $(q - a)$ in the preceding, we

get $x(q - a)x^{-1} + x^{-1}(q - a)x > 0$ or

$$2q > xax^{-1} + x^{-1}ax > 0 \text{ so } xax^{-1} + x^{-1}ax \in R.$$

Since by part (1) xax^{-1} and $x^{-1}ax$ are both positive, it is then clear that both must be in R , so $xax^{-1} \in R$, as desired.

Theorem 2. 1. 10 : *Every Jordan ordering on D is, in fact, a Hilbert ordering .*

Proof: Let $a, b > 0$. We wish to show that $ab > 0$. Now if $ab < 0$, then by the preceding Theorem 2. 1. 9. (1) $ba = a^{-1}(ab)a < 0$. Thus $ab + ba = a \bullet b < 0$, a contradiction.

Section 2.2 : q-orderings

As shown in section 2. 1. (Lemma 2. 1. 4. (3)) every Jordan ordering is a q-ordering according to the following definition,

Definition 2. 2. 1 : *By a q-ordering on a division ring D , we mean a binary relation $\geq$ on D such that for all $a, b, c \in D$:*

- (1) $1 > 0$ and $a \geq a$;
- (2) $a \geq b, b \geq a \rightarrow a = b$;
- (3) $a \geq b, b \geq c \Rightarrow a \geq c$;

$$(4) \quad a \geq b \text{ or } b \geq a ;$$

$$(5) \quad a \geq b \Rightarrow a + c \geq b + c ; \text{ and}$$

$$(6) \quad a \geq 0 \Rightarrow ab^2 \geq 0 , \text{ for all } b \neq 0 .$$

We repeat that the above definition is exactly as in the commutative case ([P ; pp. 5]). Our goals are as announced to show that this type of ordering behaves essentially the same as in the commutative case. As for a Jordan ordering we set :

$$P = \{ a \in D \mid a > 0 \} .$$

We have evidently the following :

$$(i) \quad 1 \in P , \text{ and } 0 \notin P ;$$

$$(ii) \quad P + P \subseteq P \text{ (i. e. } a, b \in P \text{ together imply } a + b \in P) ;$$

$$(iii) \quad a \in P \Rightarrow ab^2 \in P , \text{ for all } b \in D^\bullet ;$$

$$(iv) \quad P \cup -P = D^\bullet .$$

Conversely, every subset P of D as in the above induces an order relation on D , defined by :

$$a \geq b \Leftrightarrow a - b \in P \cup \{ 0 \} ;$$

which is evidently a q-ordering.

Consistent with the notations adopted in section 2. 1. ,

we write R , for the subset of bounded elements ; J , for the subset of infinitesimals, and ; U , for the subset of all the doubly bounded elements u ; that is, $q_1 < |u| < q_2$ for some q_1 and $q_2 \in \mathbb{Q}^+$. We now proceed to the specifics .

Proposition 2.2.2 : *Let D be any q -ordered division ring. Then :*

- (1) $0 \notin C$, so D is Hilbert orderable ;
- (2) If $a > 0$ and $c \in C$, then $ac > 0$.

Proof: (1) and (2) : Let $0 \neq a_i \in D$ and $a > 0$ in D . We have $a \cdot a_1^2 > 0$, so $a \cdot a_1^2 \cdot a_2^2 > 0$. Repeating enough times we get $a \cdot a_1^2 \cdot a_2^2 \dots a_n^2 > 0$. This then readily gives $ac > 0$ whenever $c \in C$. Thus $0 \notin C$ and so, D is Hilbert orderable (see [Se]).

Although a q -ordered division ring is Hilbert orderable there is nothing which gurantees that the underlying q -ordering will be a Hilbert ordering for this even fails in the commutative case. In keeping with the commutative case, can one say that the considered ordering is conducive to an order valuation ? In preparation for the answer to this question we then show the :

Lemma 2.2.3: For all $a, b \in P$, $x \in D^\bullet$, and

$q \in \mathbb{Q}^+$:

- (1) $a^{-1} \in P$;
- (2) $qa \in P$;
- (3) $a < c$ ($c \in C$) $\implies a^{-1} > c^{-1}$;
- (4) $a > b \implies a^2 > b^2$, provided a or b is in C .

Proof: (1) , (2) and (3) .

By a trivial adaptation of Lemma 2.1.4.

(4) Let $b \in C$. From $a > b$, we derive

$ab^{-1} > bb^{-1} = 1$. Thus $ba^{-1} < 1$, which implies $a^{-1} < b^{-1}$

or it is the same, $b^{-1} = a^{-2} \cdot a^2b \cdot b^{-2} > a^{-2} \cdot ab^2 \cdot b^{-2} = a^{-1}$.

From the preceding inequality we readily obtain $a^2b > ab^2$. Since

$b \in C$, we get $a^2 > ab$. Also $a > b$ implies $ab > b^2$.

Hence $a^2 > b^2$.

Similarly, for $a \in C$, we get $a^2 > b^2$.

Remark 2.2.4: For $a, b \in P$ and $c \in C$:

if $a > c > b$, then $a^2 > c^2 > b^2$.

Proof: By Lemma 2.2.3., applied to $c \in C$ and a ,

$a > c$ implies $a^2 > c^2$. By the same Lemma applied to

$c \in C$ and $b, c > 0$ implies $c^2 > b^2$ so, $a^2 > c^2 > b^2$.

Theorem 2.2.5: Every q -ordering on D which verifies the property :

$$a > b > 0 \implies a^2 > b^2$$

must be, in fact, a Hilbert ordering (and conversely).

Proof: For a, b as in the statement,

$$(a + b) > a > (a - b) > 0. \text{ Thus}$$

$$(a + b)^2 > a^2 > (a - b)^2. \text{ Expanding the preceding}$$

inequalities, we get $4(ab + ba) > 0$ and hence

$$a \bullet b = (ab + ba) > 0 \text{ (Lemma 2.2.3. (2)) ; this for all}$$

$a, b \in D, a, b > 0$. Thus D is a Jordan ordered division ring

and, hence, by Theorem 2.1.10., D is a Hilbert ordered division ring.

Proposition 2.2.6: Every algebraic element a in D over the centre Z of D is, in fact, in Z , where D is any q -ordered division ring.

Proof: By a trivial adaptation of a theorem of Albert

([H 1 ; Proposition p. 103.]), $a \in \mathbb{Z}$ follows.

Definition 2.2.7 : We call the q -ordering $(\geq)$ on a division ring D archimedean if for every $a \in D$, $a > 0$ there is some natural number n such that $a < n$.

Theorem 2.2.8 : A q -ordering $\geq$ on a division ring D is archimedean if and only if $\mathbb{Q}$ is dense in D with respect to $\geq$.

Proof: If $\mathbb{Q}$ is dense in D , then for every $a \in D$ there exists some $q \in \mathbb{Q}$ such that $a < q < a + 1$. Thus there exists a natural number n such that $a < n$.

Conversely, assume that $a > b > 0$. Thus $(a - b)^{-1} > 0$. Choose a natural number n such that $(a - b)^{-1} < n$. By Lemma 2.2.3. (3), $1/n < (a - b)$. Let m be the first natural number such that $nb < m$. Then $m < na$ for otherwise $n(a - b) = na - nb \leq 1$. Hence $nb < m < na$, which implies $b < m/n < a$.

Along the lines of the proof of [P ; Theorem 1 20.], we show the :

Theorem 2.2.9: *Every archimedean q -ordered division ring D is an ordered field.*

Proof: For by a well-known result of Hilbert, every archimedean Hilbert ordered division ring D is an ordered field. Hence to prove the result it suffices to show that an archimedean q -ordered division ring D is Hilbert ordered.

Now let $a > b > 0 \in D$. Then

$a + b > a - b > 0$. By Theorem 2.2.8., there is some

$q \in \mathbb{Q}$ such that $(a + b) > q > (a - b)$. Since

$q \in \mathbb{Q}^+ \subset \mathbb{C}$, by Remark 2.2.4.,

$(a + b)^2 > q^2 > (a - b)^2$, which implies

$2(ab + ba) > 0$ or $ab + ba = a \bullet b > 0$. By

Theorem 2.1.10., $ab > 0$. Thus $\geq$ is a Hilbert ordering on D .

Theorem 2.2.10: *Let D be a division ring equipped with a q -ordering denoted by $\geq$. Then :*

- (1) R is a valuation ring in D ;
- (2) J is the unique maximal ideal in R ;
- (3) U is the set of all the invertible elements in R .

Proof: By a trivial adaptation of Theorem 2.1.7.

Theorem 2.2.11: For D as usual a q -ordered division ring, the set U of all the order units, is preserved under conjugation.

Proof: If $a > 0$, then for all $x \in D^\bullet$ we have $xax^{-1} = a \cdot (a^{-1})^2 \cdot (ax)^2 \cdot (x^{-1})^2 > 0$. If $a \in U$, then there exist $p, q \in \mathbb{Q}^+$ such that $p \leq a \leq q$. Pick $r \in \mathbb{Q}$ such that, $r > q$, then $a < r$, that is, $r - a > 0$. By the above $x(r - a)x^{-1} > 0$, for all $x \in D^\bullet$, which implies $r - xax^{-1} > 0$ or $xax^{-1} < r$, for all $x \in D^\bullet$.

Similarly, if we choose $s \in \mathbb{Q}$, $s < p$, we get $s < xax^{-1}$. Hence $s < xax^{-1} < r$, so $xax^{-1} \in U$.

Corollary 2.2.12: R is preserved under conjugation in D .

Proposition 2.2.13: For all $a \in U$ and $x \in D^\bullet$, $xax^{-1}a^{-1}$ is a multiplicative infinitesimal.

Proof: For let $a \in U$ be positive and let $x \in D^\bullet$.

As shown in the course of Theorem 2.2.12., we have for all

$q \in \mathbb{Q}^+$:

$$a < q \Rightarrow xax^{-1} < q;$$

$$q < a \Rightarrow q < xax^{-1}.$$

By an argument of [C 2, Theorem 5.], $xax^{-1} - a \in J$. Since

$a^{-1} \in U$, we get $xax^{-1}a^{-1} - 1 \in J$, as desired.

Proposition 2.2.14: Let D be any q -ordered division ring, $\bar{D} = R/J$ be the residue division ring and let

$$P = \{a + J \mid a \in U, a > 0 \text{ in } D\}.$$

Define

$$\bar{a} = a + J > \bar{b} = b + J \iff \bar{a} - \bar{b} \in \bar{P},$$

then this is an order relation on $\bar{D}$, which is an archimedean

q -ordering, so, $\bar{D}$ is an archimedean ordered field.

Proof: (i) By construction, $\bar{1} = 1 + J \in \bar{P}$, and, since $J \cap U = \emptyset$, we have $\bar{0} \notin \bar{P}$.

(ii) Let $\bar{a}, \bar{b} \in \bar{D}$. write $\bar{a} = a + J$, $\bar{b} = b + J$, where $a, b \in U$ and $a, b > 0$. Now $\bar{a} + \bar{b} = (a + b) + J$. Here, $(a + b) \in U$ for evidently

$a + b$ is doubly bounded and $a + b > 0$ so, $\bar{a} + b \in P$.

(iii) Let $\bar{a} = a + J$ with $a > 0$ in D ,

$a \in U$ and let $\bar{b} = b + J, b \notin J$. Then $\bar{a}\bar{b}^2 = ab^2 + J$.

Since $a, b \notin J$, we have $ab^2 \notin J$. Also $ab^2 > 0$, so,

$$\bar{a}\bar{b}^2 > \bar{0}.$$

The remaining axiom of the q -ordering is straight forward to check.

Finally, for $\bar{a} = a + J, a \in U, a > 0$ in D , we have $q_1 < a < q_2$ for some $q_1, q_2 \in \mathbb{Q}^+$, where q_i can be taken to be such that $a - q_1 \in U$ and $q_2 - a \in U$, so, $\bar{q}_1 = q_1 + J < \bar{a} < \bar{q}_2 = q_2 + J$. $\bar{D}$ is an archimedean q -ordered division ring and, hence, by Theorem 2.2.9., $\bar{L}$ is an archimedean ordered field.

We conclude this section with some questions revolving about commutators. Are they valuation units? This is not always the case in the stronger situation in which the ordering is a Hilbert ordering. Also, if $a^{-1}b^{-1}ab \equiv 1 \pmod{J}$ for a fixed $a \in D$ and for all $b \in D^\bullet$, does it follow that $a \equiv z \pmod{J}$ for some $z \in Z$? The answer to this question is in the

Thus $c_1u_1x + c_3u_3xt \in P$ iff $\bar{u}_1 > 0$. Since $c_1u_1x \in P$,

we have $\bar{u}_1 > 0$, so $c_1u_1x + c_3u_3xt \in P$.

$$\text{Case (7): } c_1u_1x + c_3u_3xt = c_3u_3xt(1 + \varepsilon) = c_3u_3(1 + \varepsilon')xt \\ (\varepsilon, \varepsilon' \in J)$$

For $(2m + 1, 2n + 1) = v(c_3u_3xt)$

$$< v(c_1u_1x) = (2m', 2n' + 1),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_1u_1x + c_3u_3xt \in P$ iff $-\bar{u}_3 > 0$. Since $c_3u_3x \in P$,

we have $-\bar{u}_3 > 0$, so $c_1u_1x + c_3u_3xt \in P$.

$$\text{Case (8): } c_2u_2t + c_3u_3xt = c_3u_3xt(1 + \varepsilon) = c_3u_3(1 + \varepsilon')xt \\ (\varepsilon, \varepsilon' \in J)$$

For $(2m + 1, 2n + 1) = v(c_3u_3xt)$

$$< v(c_2u_2t) = (2m' + 1, 2n'),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_2u_2t + c_3u_3xt \in P$ iff $-\bar{u}_3 > 0$. Since $c_3u_3x \in P$,

we have $-\bar{u}_3 > 0$, so $c_2u_2t + c_3u_3xt \in P$.

$$\text{Case (9): } c_2u_2t + c_3u_3xt = c_3u_3t(1 + \varepsilon) = c_2u_2(1 + \varepsilon')t \\ (\varepsilon, \varepsilon' \in J)$$

$$v(a + b) \geq \min(v(a), v(b)).$$

Refer to v as to be the order valuation associated to D .

We now turn focus of our attention to an example of a q -ordered infinite dimensional division ring, which is not a Hilbert ordered division ring.

Start off with the usual ordered field $\mathbb{R}$ of real numbers. Let $\mathbb{R}((x))$ stand for the usual formal Laurent power series field over $\mathbb{R}$. Let $\theta : \mathbb{R}((x)) \rightarrow \mathbb{R}((x))$ be defined by :

$$\alpha = \sum_{\substack{i \geq m \\ a_m \neq 0}} a_i x^i \rightarrow \theta(\alpha) = \sum_{\substack{i \geq m \\ a_m \neq 0}} a_i (2x)^i$$

Write $D = \mathbb{R}((x))((t, \theta))$ for the division ring of Laurent power series in t over $\mathbb{R}((x))$ subjected to the following commutation rule and its consequences :

$$\alpha \in \mathbb{R}((x)); \quad t\alpha = \theta(\alpha)t.$$

Theorem 2.3.1 : *With the notation as in the above :*

(I) The centre Z of D is precisely $\mathbb{R}$, and D carries a q -ordering extending that in $\mathbb{R}$.

(II) The ordering on D is not Hilbert.

(III) The valuation v associated to this ordering is compatible with the ordering on D , that is, if $\alpha > \beta > 0$

in D , then $v(\alpha) \leq v(\beta)$.

Proof: (I) Let $f(x, t) \in Z$, the centre of D .

Now if $f(x, t) = \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^i t^j$ ($a_{ij} \in \mathbb{R}$), then

$$\sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^i t^j x = f(x, t) x = x f(x, t) = \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^{i+1} t^j$$

and

$$\sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^i t^{j+1} = f(x, t) t = t f(x, t) = \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} t x^i t^j$$

Using the basic commutation rule $t \cdot x = 2xt$, the above equalities imply that $i = 0$ and $j = 0$. Thus $Z \subset \mathbb{R}$, as desired.

Consider the additive abelian group $\mathbb{Z} \times \mathbb{Z}$ under the ordering $(n, m) > (n', m')$ iff $n > n'$ or $n = n'$, $m > m'$ (lexicographic ordering from left to right).

Define $v : D \rightarrow \mathbb{Z} \times \mathbb{Z}$ as follows :

$$v(a) = 0, \text{ for all } 0 \neq a \in \mathbb{R};$$

$$v(0) = \infty \text{ and}$$

$$v(f(x, t)) = v(ax^m t^n) = (n, m);$$

where $ax^m t^n$ is the monomial of the smallest value in the

series $f(x, t) \in D$.

It follows readily that

$$v(f \cdot g) = v(f) + v(g) \text{ and}$$

$$v(f + g) \geq \min\{v(f), v(g)\} \text{ for all } f, g \in D.$$

Therefore, v is a valuation on D with residue field $\mathbb{R}$.

For any $f(x, t) \in D^\bullet$,

$$f(x, t) = \sum_{i \geq n} b_i t^i = (b_n + \sum_{i > 1} b_{n+i} t^i) t^n$$

$b_n \neq 0$

where b_i 's $\in \mathbb{R}((x))$. Thus

$$(A) \quad f(x, t) = (b_n + \varepsilon) t^n \text{ (for } t \in J)$$

Further from $b_n \in \mathbb{R}((x))$, we get,

$$b_n = \sum_{j \geq m} a_j x^j = (a_m + \sum_{j > 1} a_{m+j} x^j) x^m$$

$a_m \neq 0$

where a_j 's $\in \mathbb{R}$. Thus

$$b_n = (a_m + \varepsilon') x^m \text{ (for } x \in J).$$

Substituting for b_n in (A), we get

$$f(x, t) = [(a_m + \varepsilon') x^m + \varepsilon] t^n = a_m x^m t^n (1 + \varepsilon'')$$

where $\varepsilon, \varepsilon', \varepsilon'' \in J$.

Since $(1 + J) \subset U$, we have $f(x, t) = u x^m t^n$,

Where $m, n \in \mathbb{Z}$ and $u \in U$. From the basic commutation

rule $t \cdot x = 2xt$ and its immediate consequence

$x^2 t^2 = [(1/\sqrt{2}) xt]^2$, we deduce that $f(x, t)$ is of one of the following types :

$$f(x, t) \in U(D^\bullet)^2, U_x(D^\bullet)^2, U_t(D^\bullet)^2 \text{ or } U_{xt}(D^\bullet)^2.$$

Let P denote the set of all elements $\alpha \in D^\bullet$ such that α is of one of the following types :

$$(1) \quad \alpha = u, \bar{u} = u + J > \bar{0} \text{ in } \bar{D};$$

$$(2) \quad \alpha = ux, \bar{u} > \bar{0};$$

$$(3) \quad \alpha = ut, \bar{u} > \bar{0};$$

$$(4) \quad \alpha = uxt, -\bar{u} > \bar{0}; \text{ or}$$

(5) $\alpha = ab$, where $b \in (D^\bullet)^2$ and a is as in one of the preceding types.

Then P induces as usual a binary relation on D defined by:

$$\alpha, \beta \in D, \alpha > \beta \iff \alpha - \beta \in P.$$

It is to be checked that this is, in fact, an order relation on D , which is a q -ordering.

(1) To check that P is closed under arithmetical sums :

$$\text{Assume that } cu, c_1u_1x, c_2u_2t, c_3u_3xt \in P,$$

$$\text{where } u, u_1, u_2, u_3 \in U \text{ and } c, c_1, c_2, c_3 \in (D^\bullet)^2.$$

Case (1): $cu + c_1 u_1 x = cu(1 + \varepsilon) \ (\varepsilon \in J)$

Since $(2m, 2n + 1) = v(c_1 u_1 x)$

$$> v(cu) = (0, 0),$$

where $m, n \in \mathbb{Z}$.

Thus $cu + c_1 u_1 x \in P$ iff $\bar{u} > 0$. Since $cu \in P$, we have

$\bar{u} > 0$, so $cu + c_1 u_1 x \in P$.

Case (2): $cu + c_2 u_2 t = cu(1 + \varepsilon) \ (\varepsilon \in J)$

Since $(2m + 1, 2n) = v(c_2 u_2 t)$

$$> v(cu) = (0, 0),$$

where $m, n \in \mathbb{Z}$.

Thus $cu + c_2 u_2 t \in P$ iff $\bar{u} > 0$. Since $cu \in P$, we have

$\bar{u} > 0$, so $cu + c_2 u_2 t \in P$.

Case (3): $cu + c_3 u_3 xt = cu(1 + \varepsilon) \ (\varepsilon \in J)$

Since $(2m + 1, 2n + 1) = v(c_3 u_3 xt)$

$$> v(cu) = (0, 0),$$

where $m, n \in \mathbb{Z}$.

Thus $cu + c_3 u_3 xt \in P$ iff $\bar{u} > 0$. Since $cu \in P$, we have

$\bar{u} > 0$, so $cu + c_3 u_3 xt \in P$.

Case (4): $c_1u_1x + c_2u_2t = c_1u_1x(1 + \varepsilon) = c_1u_1(1 + \varepsilon')x$
 $(\varepsilon, \varepsilon' \in J)$

For $(2m + 1, 2n) = v(c_2u_2t)$

$$> v(c_1u_1x) = (2m', 2n' + 1),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_1u_1x + c_2u_2t \in P$ if and only if $\bar{u}_1 > 0$. Since

$c_1u_1x \in P$, we have $\bar{u}_1 > 0$, so $c_1u_1x + c_2u_2t \in P$.

Case (5): $c_1u_1x + c_2u_2t = c_2u_2t(1 + \varepsilon) = c_2u_2(1 + \varepsilon')t$
 $(\varepsilon, \varepsilon' \in J)$

For $(2m + 1, 2n) = v(c_2u_2t)$

$$< v(c_1u_1x) = (2m', 2n' + 1),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_1u_1x + c_2u_2t \in P$ iff $\bar{u}_2 > 0$. Since $c_2u_2x \in P$,

we have $u_2 > 0$, so $c_1u_1x + c_2u_2t \in P$.

Case (6): $c_1u_1x + c_3u_3xt = c_1u_1x(1 + \varepsilon) = c_1u_1(1 + \varepsilon')x$
 $(\varepsilon, \varepsilon' \in J)$

For $(2m + 1, 2n + 1) = v(c_3u_3xt)$

$$> v(c_1u_1x) = (2m', 2n' + 1),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_1u_1x + c_3u_3xt \in P$ iff $\bar{u}_1 > 0$. Since $c_1u_1x \in P$,

we have $\bar{u}_1 > 0$, so $c_1u_1x + c_3u_3xt \in P$.

$$\text{Case (7): } c_1u_1x + c_3u_3xt = c_3u_3xt(1 + \varepsilon) = c_3u_3(1 + \varepsilon')xt \\ (\varepsilon, \varepsilon' \in J)$$

$$\text{For } (2m + 1, 2n + 1) = v(c_3u_3xt)$$

$$< v(c_1u_1x) = (2m', 2n' + 1),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_1u_1x + c_3u_3xt \in P$ iff $-\bar{u}_3 > 0$. Since $c_3u_3x \in P$,

we have $-\bar{u}_3 > 0$, so $c_1u_1x + c_3u_3xt \in P$.

$$\text{Case (8): } c_2u_2t + c_3u_3xt = c_3u_3xt(1 + \varepsilon) = c_3u_3(1 + \varepsilon')xt \\ (\varepsilon, \varepsilon' \in J)$$

$$\text{For } (2m + 1, 2n + 1) = v(c_3u_3xt)$$

$$< v(c_2u_2t) = (2m' + 1, 2n'),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_2u_2t + c_3u_3xt \in P$ iff $-\bar{u}_3 > 0$. Since $c_3u_3x \in P$,

we have $-\bar{u}_3 > 0$, so $c_2u_2t + c_3u_3xt \in P$.

$$\text{Case (9): } c_2u_2t + c_3u_3xt = c_3u_3t(1 + \varepsilon) = c_2u_2(1 + \varepsilon')t \\ (\varepsilon, \varepsilon' \in J)$$

For $(2m + 1, 2n + 1) = v(c_3 u_3 x t)$

$$> v(c_2 u_2 t) = (2m' + 1, 2n'),$$

where $m, n, m', n' \in \mathbb{Z}$.

Thus $c_2 u_2 t + c_3 u_3 x t \in P$ iff $\bar{u}_2 > 0$. Since $c_2 u_2 x \in P$,

we have $\bar{u}_2 > 0$, so $c_2 u_2 t + c_3 u_3 x t \in P$.

Hence P is closed under arithmetical sums.

As a corollary to the definition of P , we have :

(2) For every $a \in D^\bullet$, $a \in P$ or $-a \in P$;

(3) For every $a \in D^\bullet$, $a^2 \in P$;

(4) Every positive element in $\mathbb{R}$ is in P ;

(5) P is preserved under multiplication by non-zero squares in D .

Thus P induces a q -ordering on D , which extends the orderings of its centre, the field $\mathbb{R}$. Therefore, the ordering on D is as desired.

(II) We have $x > 0, t > 0$; but $xt < 0$.

(III) To say that v is compatible with the ordering is to say that if $\alpha, \beta \in D$ are such that $\alpha > 0$, $v(\alpha) < v(\beta)$, then $\alpha > \beta$. Write

$$\alpha = \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^i t^j > 0 \quad \text{and} \quad \beta = \sum_{\substack{i' \geq m', j' \geq n' \\ b_{m'n'} \neq 0}} b_{i'j'} x^{i'} t^{j'}$$

Now $v(\beta) - v(\alpha) > 0$ implies

$(n', m') - (n, m) > 0$, which means either $n' > n$ or $n' = n, m' > m$.

In case $n' > n$, $\alpha - \beta = \sum_{\substack{i \geq m, j \geq n \\ c_{mn} \neq 0}} c_{ij} x^i t^j > 0$

(for $c_{mn} = a_{mn}$ and $\alpha > 0$.) and if $n' = n$ and $m' > m$, then

$$\alpha - \beta = \sum_{\substack{i \geq m, j \geq n \\ d_{mn} \neq 0}} d_{ij} x^i t^j > 0 \quad (\text{for } d_{mn} = a_{mn} \text{ and } \alpha > 0).$$

With this the proof is complete.

For the record, we should say that the commutative version of this theorem was kindly suggested to us by Professor A. Wadsworth.

Theorem 2.3.2: For D and v as in

Theorem 2.3.1, v is, in fact, the order valuation on D .

Proof: It suffices to show that $a \in D$ is an order

infinitesimal if and only if $v(a) > 0$.

Let then $a > 0$ with $v(a) > 0$. Write

$$a = \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^i t^j \in D.$$

Since $v(a) > 0$, either $j > 0$ or $j = 0$ and $i > 0$. To say that a is an order infinitesimal in D is to say that $-q < a < q$ for all $q \in \mathbb{Q}^+$. Now

$$\begin{aligned} q \pm a &= q \pm \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^i t^j = (1 \pm \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a'_{ij} x^i t^j) q \\ &= (1 \pm \varepsilon) q, \quad (\varepsilon \in J) \end{aligned}$$

Since $(1 \pm \varepsilon)$ is a positive unit, we have $(1 \pm \varepsilon) q > 0$.

Thus $q \pm a > 0$. In other words $-q < a < q$ and hence a is an order infinitesimal.

Conversely, we will show that if a is an order infinitesimal then $v(a^2) > 0$. Suppose $v(a^2) \leq 0$.

Since a^2 has for the first term $a_{mn} x^{2m} t^{2n}$, for certain m, n such that $a_{mn} \neq 0$, we get either $n < 0$ or if $n = 0$ then $m < 0$.

In either case we then get $a^2 > q$ for all $q \in \mathbb{Q}^+$, which contradicts the choice of a .

Theorem 2.3.3: Let D be as in Theorem 3.3.1.

Then :

(1) Every commutator $a^{-1}b^{-1}ab$ in D is a valuation unit.

(2) If $a^{-1}b^{-1}ab \equiv 1 \pmod{J}$ for a fixed $a \in D$ and for all $b \in D^*$, then $a \equiv z \pmod{J}$ for some $z \in Z$.

Proof: Let $a, b \in D$. Write

$$a = \sum_{\substack{i \geq m, j \geq n \\ a_{ij} \neq 0}} a_{ij} x^i t^j \quad \text{and} \quad b = \sum_{\substack{k \geq p, l \geq q \\ b_{kl} \neq 0}} b_{kl} x^k t^l.$$

We have

$$a \equiv a_{mn} x^m t^n \pmod{1+J},$$

$$b \equiv b_{pq} x^p t^q \pmod{1+J},$$

$$a^{-1} \equiv a_{mn}^{-1} 2^{mn} x^{-m} t^{-n} \pmod{1+J} \quad \text{and}$$

$$b^{-1} \equiv b_{pq}^{-1} 2^{pq} x^{-p} t^{-q} \pmod{1+J}.$$

Thus $a^{-1}b^{-1}ab \equiv 2^{np-mq} \pmod{1+J}$, so $a^{-1}b^{-1}ab$ is a valuation unit.

(2) For a and b as in (1) if

$a^{-1}b^{-1}ab \equiv 1 \pmod{J}$ then $np - mq = 0$. If then b is arbitrary in D , this means $np = mq$ for all p and q , which

means $n = m = 0$ or $a \equiv a_{00} \pmod{J}$, as desired.

CHAPTER 3.

q-orderings and Jordan q-orderings

In this chapter, we wish to stress the analogy between q -ordered fields and q -ordered division rings from the point of view of S. A. P. fields; namely a field is S. A. P. if every q -ordering of the field is, in fact, a field ordering. The topological setting for this question shall also be investigated. Furthermore, we give a method for extending a q -ordering from a field F to a Hilbert division ring over F in one indeterminate. Finally, the closing attempt will be to investigate the Jordan version of a q -ordering.

Section 3.1: Semisection

The goal for this section is a characterization patterned after the commutative case of a valued division ring D for which every q -ordering is a Hilbert ordering. We will use a line of argument, which is inspired by the commutative case and we assume that for every infinitesimal ε , $(1 + \varepsilon)$ is a square in

D . Also, we require the residue division ring $\bar{D}$ to be a formally real field.

Definition 3.1.1: Let D be a valued division ring with valuation v . By semisection s of the valuation $v : D^\bullet \rightarrow G$, we mean a map $s : G \rightarrow D^\bullet$ such that for all $g, h \in G$:

- (1) $s(0) = 1$;
- (2) $v(s(g)) = g$; and
- (3) $s(g + h) \equiv s(g) \cdot s(h) \pmod{C}$.

Note that in requirement (3), we require something less restrictive than in the commutative case, namely, we have congruence modulo C instead of congruence modulo the group of squares of the field.

Theorem 3.1.2: For every valuation $v : D^\bullet \rightarrow G$, there exists a semisection s of v .

Proof: Let Σ be a transversal for the valuation v , that is, a complete subset of non equivalent elements in $D^\bullet$ with respect to v . Given $g \in G$, there is a unique element a_g in

Also $ab\theta(b) > 0$ implies
 $a\theta^{-1}(b)\theta(\theta^{-1}(b)) > 0$ or $a\theta^{-1}(b)b = ab\theta^{-1}(b) > 0$.

The rest of the axioms needed for T to be a group can be readily checked.

Theorem 3.2.2: *Let F be a field equipped with any linear order relation $\geq$, which is additive and let $D = F((t, \theta))$ be as in the above. The following requirements are equivalent:*

- (1) *The ordering on F extends to the q -ordering on D obtained by declaring a series positive if, and only if, its lowest coefficient is positive.*
- (2) *The ordering on F is a q -ordering such that for all $a > 0$ and $b \neq 0$, $ab\theta(b) > 0$.*

Further this q -ordering on D is Hilbert if, and only if, the ordering on F is itself a field ordering.

Proof: (2) $\Rightarrow$ (1).

Let P be the set of all series

$\sum_{i \geq n} a_i t^i$, where the lowest coefficient a_n is positive in F .

Obviously, P contains the domain of positivity of F and

As a simplifying device for the considered investigation, in what will follow, we will identify a q -ordering with its domain of positivity. Let X (respectively, Y) denote the set of all Hilbert orderings (respectively, the set of all q -orderings) on D , and $X_{\bar{D}}$ (respectively, $Y_{\bar{D}}$) the set of all Hilbert orderings (respectively, all q -orderings) on $\bar{D}$. With the notations as in the above, we now show the :

Theorem 3.1.3: *Let $v : D^{\bullet} \rightarrow G$, be a valuation on D with group of units U and maximal ideal J . Let $s : G \rightarrow D^{\bullet}$ be a semisection of v . For each $P \in Y$ there is a unique pair of functions*

$$\alpha_P : G/2G \rightarrow Y_{\bar{D}} \text{ and } \beta_P : G/2G \rightarrow \{1, -1\}$$

such that for all $g \in G$ and $b \in U$,

$\beta_P(g) \cdot s(g) \in P$ and further $b + J \in \alpha_P(\bar{g})$ if, and only if, $b s(g) \beta_P(\bar{g}) \in P$. In particular, $\beta_P(\bar{0}) = 1$.

Proof: β_P is well defined.

Since $g + 2G = g' + 2G$ implies $s(g) \equiv s(g') \pmod{C}$, therefore, β_P is well defined.

α_P is well defined.

Let $b, b' \in U$ such that $b' = b + j$ for some $j \in J$,

$$\begin{aligned} \text{then } v(b' s(g) \beta_P(\bar{g})) &= v(b') + g + 0 \\ &= g < v(j) + g = v(j s(g) \beta_P(\bar{g})). \end{aligned}$$

Assuming $b' s(g) \beta_P(\bar{g}) \in P$, we get

$$(b' - j) s(g) \beta_P(\bar{g}) \in P.$$

Thus $\alpha_P(\bar{g})$ is independent of the choice of its representative and, hence, $\alpha_P(\bar{g})$ is well defined.

To show that $\alpha_P(\bar{g})$ is a g-ordering on D .

$$(i) \quad 1 \in \alpha_P(\bar{g}), \text{ and } 0 \notin \alpha_P(\bar{g}).$$

This follows readily from the definition of $\alpha_P(\bar{g})$.

$$(ii) \quad \alpha_P(\bar{g}) + \alpha_P(\bar{g}) \subseteq \alpha_P(\bar{g}).$$

Let $a + J, b + J \in \alpha_P(\bar{g})$ for some $a, b \in D^*$.

Then $a s(g) \beta_P(\bar{g}), b s(g) \beta_P(\bar{g}) \in P$ which implies

$$(a + b) s(g) \beta_P(\bar{g}) \in P. \text{ Equivalently,}$$

$$(a + b) + J \in \alpha_P(\bar{g}).$$

$$(iii) \quad (D^*)^2 \cap \alpha_P(\bar{g}) \subseteq \alpha_P(\bar{g}).$$

Let $a + J \in \alpha_P(\bar{g}), b + J \in D$ for some

$a \in D^*$ and $b \in U$. Thus $a s(g) \beta_P(\bar{g}) \in P$ which implies

$b^2 a s(g) \beta_P(\bar{g}) \in P$. Equivalently, $b^2 a + J \in \alpha_P(\bar{g})$.

$$(iv) \quad \alpha_P(\bar{g}) \cup -\alpha_P(\bar{g}) = \bar{D}^*.$$

For any $a + J \in \bar{D}$, $a + J \in \alpha_P(\bar{g})$ means $a s(g) \beta_P(\bar{g}) \in P$, that is, $-a s(g) \beta_P(\bar{g}) \in P$. Equivalently, $(-a) + J \in \alpha_P(\bar{g})$, as desired.

Theorem 3.1.4: Let v, s, P, α_P and β_P be as in the preceding Theorem 3.1.3. Then each pair of functions α_P and β_P induce a q -ordering $f_{\alpha_P \beta_P}$ on D , defined as follows:
 $a \in f_{\alpha_P \beta_P}$ if, and only if,
 $a / \{s(v(a))\} \beta_P(v(a)) + J \in \alpha_P(\overline{v(a)}),$
 where $a/b = ab^{-1}$.

Proof: To prove $f_{\alpha_P \beta_P}$ is a q -ordering on D .

$$(i) \quad 1 \in f_{\alpha_P \beta_P}, \text{ and } 0 \notin f_{\alpha_P \beta_P}.$$

Since $s(0) = 1$ and $\beta_P(\bar{0}) = 1$, thus $1 \in f_{\alpha_P \beta_P}$. $0 \notin f_{\alpha_P \beta_P}$ follows readily from the definition of $f_{\alpha_P \beta_P}$.

$$(ii) \quad f_{\alpha_P \beta_P} + f_{\alpha_P \beta_P} \subseteq f_{\alpha_P \beta_P}.$$

Let $a, b \in f_{\alpha_P \beta_P}$. Thus

$$(A) \dots \quad a / s(v(a)) \beta_P(\overline{v(a)}) + J \in \alpha_P(\overline{v(a)}) \quad \text{and}$$

$$(B) \dots \quad b/s(v(b)) \beta_P(\overline{v(\bar{b})}) + J \in \alpha_P(\overline{v(\bar{b})})$$

Three cases arise :

Case (1): $v(a) = v(b)$.

Adding (B) to (A), we get

$$(a + b)/s(v(a)) \beta_P(\overline{v(\bar{a})}) + J \in \alpha_P(\overline{v(\bar{a})}), \text{ which}$$

implies $(a + b)/s(v(a)) \beta_P(\overline{v(\bar{a})}) \in U$. Since

$v(a/b) = v(a) - v(b)$ and $v(\text{unit}) = 0$, therefore,

$v(a + b) = v(s(v(a))) = v(a)$. Thus

$$(a + b)/s(v(a)) \beta_P(\overline{v(\bar{a})}) + J \in \alpha_P(\overline{v(\bar{a})}) \text{ and so,}$$

$$(a + b)/s(v(a + b)) \beta_P(\overline{v(\bar{a} + \bar{b})}) \in \alpha_P(\overline{v(\bar{a} + \bar{b})}).$$

Equivalently, $a + b \in f_{\alpha\beta}$.

Case (2): $v(a) < v(b)$.

Since $v(b) - v(s(v(a))) = v(b) - v(a) > 0$,

thus $v(a + b) = v(a)$ and $b/s(v(a)) \in J$. From (A),

we get

$$(a + b)/s(v(a + b)) \beta_P(\overline{v(\bar{a} + \bar{b})}) + J \in \alpha_P(\overline{v(\bar{a} + \bar{b})})$$

which implies $a + b \in f_{\alpha\beta}$.

Case (3): $v(b) < v(a)$.

Since $v(a + b) = v(b)$ and $a/s(v(b)) \in J$.

From (B), we get

$$(b + a) / s(v(b + a)) \beta_P(\overline{v(\bar{b} + \bar{a})}) + J \in \alpha_P(\overline{v(\bar{a} + \bar{b})}),$$

which implies $a + b \in f_{\alpha \beta}$.

Hence in all the three cases $f_{\alpha \beta}$ is closed under addition.

$$(iii) \quad (D^\bullet)^2 f_{\alpha \beta} \subseteq f_{\alpha \beta}.$$

By definition $a \in f_{\alpha \beta}$ if, and only if,

$$a / \{s(v(a))\} \beta_P(\overline{v(\bar{a})}) + J \in \alpha_P(\overline{v(\bar{a})}), \text{ equivalently,}$$

$$a / \{s(v(a))\} \beta_P(\overline{v(\bar{a})}) \in P, \text{ since}$$

$$v[a / \{s(v(a))\} \beta_P(\overline{v(\bar{a})})] = 0.$$

Further since for all $b \in D^\bullet$,

$$s(v(a)) = s(v(ab^2)) \pmod{C} \text{ and } \overline{v(\overline{ab^2})} = \overline{v(\bar{a})}, \text{ we}$$

have $ab^2 / \{s(v(ab^2))\} \beta_P(\overline{v(\overline{ab^2})}) \in P$, which implies

$$ab^2 \in f_{\alpha \beta}, \text{ as desired.}$$

$$(iv) \quad f_{\alpha \beta} \cup -f_{\alpha \beta} = D^\bullet.$$

For any $a \in D^\bullet$, $a \in f_{\alpha \beta}$ means

$$a / s(v(a)) \beta_P(\overline{v(\bar{a})}) + J \in \alpha_P(\overline{v(\bar{a})}), \text{ that is,}$$

$$-a / s(v(a)) \beta_P(\overline{v(\bar{a})}) + J \in \alpha_P(\overline{v(\bar{a})}). \text{ Equivalently,}$$

$$-a \in f_{\alpha \beta}.$$

Theorem 3.1.5: Let $v, s, P, \alpha_P, \beta_P$ and f_{α_P} be defined as in Theorem 3.1.4. Then there is a 1-1 correspondence between the set of q -orderings on D and the set $\{\alpha_P \mid \alpha_P : G/2G \rightarrow Y_{\bar{e}}\} \times \{\beta_P \mid \beta_P : G/2G \rightarrow \{1, -1\}, \beta_P(\bar{0}) = 1\}$ such that $(f)_{\alpha_P} = P$ and $(\alpha_{f_{\alpha_P}}, \beta_{f_{\alpha_P}}) = (\alpha_P, \beta_P)$.

Proof: Given $a \in f_{\alpha_P}$ we have $a / \{s(v(a))\beta_P(\bar{v}(\bar{a}))\} + J \in \alpha_P(\bar{v}(\bar{a}))$ which implies $[a / \{s(v(a))\beta_P(\bar{v}(\bar{a}))\}]s(v(a))\beta_P(\bar{v}(\bar{a})) \in P$. Equivalently $a \in P$. Therefore, $(f)_{\alpha_P} = P$.

Further for given α_P, β_P and f_{α_P} and for all $g \in G$, by definition of f_{α_P} , $s(g)\beta_P(\bar{g}) \in f_{\alpha_P}$, which implies, $\beta_{f_{\alpha_P}} = \beta_P$.

Also for all $g \in G$ and $b \in U$, $b + J \in \alpha_{f_{\alpha_P}}(\bar{g})$ iff $b s(g)\beta_{f_{\alpha_P}} \in \alpha_P$, which implies $b s(g)\beta_P(\bar{g}) \in \alpha_P$. Equivalently, $b s(g) / \{s(g)\}(\beta_P(\bar{g}))^2 + J \in \alpha_P(\bar{g})$ or, it is the same,

$b + J \in \alpha_P(\bar{g})$. Hence $\alpha_{f_{\alpha\beta}} = f_{\alpha\beta}$.

Theorem 3.1.6: We have $P \in X$ if, and only if,

$\alpha_P : G/2G \rightarrow X_{\bar{D}}$ is a constant, and the associated

$\beta_P : G/2G \rightarrow \{1, -1\}$ is a character of the group $G/2G$.

Proof: Let $P \in X$. Then for all $g_1, g_2 \in G$,

$\beta_P(g_1 + g_2) = 1$, implies $s(g_1 + g_2) \in P$. Equivalently,

$s(g_1) \cdot s(g_2) \in P$, which implies either

$s(g_1), s(g_2) \in P$ or $s(g_1), s(g_2) \notin P$. In both cases, we

get $\beta_P(g_1) \beta_P(g_2) = 1$. Hence β_P is a character of the group

$G/2G$.

Further, for all $b \in U$ and $b + J \in \alpha_P(\bar{g})$, we

have $b s(g) \beta_P(\bar{g}) \in P$ and $b + J \in \alpha_P(\bar{0})$, if and only if

$b \in P$. Since $s(g) \beta_P(\bar{g}) \in P$ and since P is an ordering

on D , these requirements are equivalent. Hence α_P is a constant

function. By the preceding $\alpha_P(\bar{0})$ is an ordering on $\bar{D}$.

Conversely, assume $\alpha_P : G/2G \rightarrow X_{\bar{D}}$ is a

constant map and let β_P be a character of the group $G/2G$. Let

$a, b \in P$. By construction, we get

$$a / \{s(v(a))\} \beta_P(\overline{v(a)}) + J \in \alpha_P(\overline{v(a)}) = \alpha_P(0) \text{ and}$$

$$b / \{s(v(b))\} \beta_P(\overline{v(b)}) + J \in \alpha_P(\overline{v(b)}) = \alpha_P(0).$$

Multiplying the preceding relations term by term, we get

$$ab / \{s(v(a))s(v(b))\} \beta_P(\overline{v(a)})\beta_P(\overline{v(b)}) + J \in \alpha_P(0).$$

$$\text{Equivalently, } ab / \{s(v(ab))\} \beta_P(\overline{v(ab)}) + J \in \alpha_P(0) \text{ and}$$

so, $ab \in P$. Hence P is an ordering on D .

Theorem 3.1.7: *Let $v : D \rightarrow G$ be a valuation on D . Then $X = Y$ if, and only if, either G is divisible by 2 and $X_{\overline{v}} = Y_{\overline{v}}$ or $|G/2G| = 2$ and $|X| = 1$.*

Proof: By Theorem 3.1.6.,

$X = Y$ iff (i) Every map $\beta_P : G/2G \rightarrow \{1, -1\}$ such that $\beta_P(0) = 1$ is a character of the group $G/2G$ and (ii) $\alpha_P : G/2G \rightarrow X_{\overline{v}}$ is a constant map.

Now (i) holds true if and only if $|G/2G| \leq 2$.

When $|G/2G| = 2$ (ii) says that $|X_{\overline{v}}| = 1$ and if

$|G/2G| = 1$, then (ii) says $X_{\overline{v}} = Y_{\overline{v}}$, as desired.

As in the commutative case, we introduce a topology on the sets X and Y (Definition 3. 1. 8.). Under this topology, Theorem 3. 1. 12. will assert that Y is a totally disconnected compact space in which X is a closed subspace.

Definition 3. 1. 8.: For all $a \in D^\bullet$, define $H(a) = \{P \in Y \mid a \in P\}$. Let $H = \{H(a) \mid a \in D^\bullet\}$ be a sub basis for the topology on Y . Obviously, $H(1) = Y$, $H(-1) = \emptyset$ and $H(-a) = Y \setminus H(a)$ and every element $H(a)$ of the sub basis is a clopen set (i. e. , a closed as well as an open set).

On $X \subseteq Y$ we have the induced topology, which has subbasis $H_X = \{H_X(a) \mid a \in D^\bullet\}$ where $H_X(a) = H(a) \cap X$.

In general, H_X is not closed under intersection, but it is closed under symmetric difference Δ , where

$A \Delta B = (A - B) \cup (B - A)$. We have the following :

Lemma 3.1.9: For all $a, b \in D^*$,

$H_X(-ab) = H_X(-a) \Delta H_X(-b)$ and H_X is a sub group of $P(X)$ with respect to Δ , Where $P(X)$ is the power set of X .

Proof: By a trivial adaptation of [P ; Proposition 6.2].

We now change the focus of attention to determine the elements in D which are totally positive, that is, those which remain positive at all q -orderings on D .

Theorem 3.1.10: $a \in D$ is totally positive if, and only if, $a \in C$.

Proof: (i) $a \in C$ implies a is totally positive.

This is evident.

(ii) a is totally positive implies $a \in C$. Suppose $a \notin C$. Consider the set $W = (-a)C \cup C \cup (C - aC)$ (See [C 2, Theorem 1.]). Clearly, $0 \notin W$ and $1 \in W$. By Zorn's lemma, we can find a maximal subset P of D .

$W \subseteq P$, which is C -additive. By construction, we have

(i) $1 \in P$, and $0 \notin P$;

- (ii) $P + P \subseteq P$;
- (iii) $P \bullet (D^\bullet)^2 \subseteq P$;
- (iv) $P \cup -P = D^\bullet$.

Thus P is a q -ordering on D which contains $-a$.

Since a is totally positive, we get $(-a) + (a) = 0 \in P$,
a contradiction.

Theorem 3. 1. 11 : *The group $(D^\bullet / C, \cdot)$ is isomorphic to the group (H_X, Δ) , where $a \in C \rightarrow H_X(-a)$.*

Proof: By Lemma 3. 1. 9. , the map $a \rightarrow H_X(-a)$ is a homomorphism from $D^\bullet$ onto H_X and $H_X(-a) = \emptyset$ iff $H_X(a) = X$ or equivalently, a is a totally positive element, which implies C is the kernel of this map.

Theorem 3. 1. 12 : *The space Y is a totally disconnected compact space in which X is a closed subspace.*

Proof: By a trivial adaptation of [P ; Theorem 6. 5.].

Section 3.2 : Extension of a q-ordering

We are given a q -ordered field F , and an isotone automorphism θ on F . We let as before $D = F((t, \theta))$ stand for the Hilbert division ring over F relative to the automorphism θ . What we wish to do is to extend the q -ordering on F to all of D , by declaring a series in D to be positive if, and only if, its lowest coefficient is positive in the given field F . We begin with :

Lemma 3.2.1: *Let F be a q -ordered field, and let T be the set of all isotone automorphisms θ on F such that for all $a, b \in F$:*

$$a > 0, \text{ and } b \neq 0 \text{ together imply } ab\theta(b) > 0.$$

Then T is a group under composition of mappings.

Proof: Let $a > 0$. If θ and θ' are both in T , then $ab\theta(b) > 0$ and $ab\theta'(b) > 0$. Now, $ab\theta'(b) > 0$ implies $ab\theta'(b) \cdot [\theta'(b)]^{-2} > 0$, which implies $ab\theta'(b)[\theta'(b)]^{-2}\theta'(b)\theta(\theta'(b)) = ab\theta(\theta'(b)) > 0$. Thus $ab\theta(\theta'(b)) > 0$.

Also $ab \theta(b) > 0$ implies
 $a \theta^{-1}(b) \theta(\theta^{-1}(b)) > 0$ or $a \theta^{-1}(b) b = ab \theta^{-1}(b) > 0$.

The rest of the axioms needed for T to be a group can be readily checked.

Theorem 3.2.2: *Let F be a field equipped with any linear order relation $\geq$, which is additive and let $D = F((t, \theta))$ be as in the above. The following requirements are equivalent:*

(1) *The ordering on F extends to the q -ordering on D obtained by declaring a series positive if, and only if, its lowest coefficient is positive.*

(2) *The ordering on F is a q -ordering such that for all $a > 0$ and $b \neq 0$, $ab \theta(b) > 0$.*

Further this q -ordering on D is Hilbert if, and only if, the ordering on F is itself a field ordering.

Proof: (2) $\Rightarrow$ (1).

Let P be the set of all series

$\sum_{i \geq n} a_i t^i$, where the lowest coefficient a_n is positive in F .

Obviously, P contains the domain of positivity of F and

$$(i) \quad 1 \in P, \text{ and } 0 \notin P,$$

$$(ii) \quad P + P \subseteq P, \text{ and}$$

$$(iii) \quad P \cup -P = D^*.$$

It is to be checked that for all $a \in P$, and $x \in D^*$,
 $ax^2 \in P$.

For $a = \sum_{i \geq n} a_i t^i$ as before and $x = \sum_{j \geq m} b_j t^j$, we have
 $a_n > 0$ $b_m \neq 0$

$$ax^2 = a_n t^n b_m t^m b_m t^m + \text{terms of higher powers in } t.$$

$$\begin{aligned} \text{Now } a_n t^n b_m t^m b_m t^m &= a_n \theta^n(b_m) t^{n+m} b_m t^m \\ &= a_n \theta^n(b_m) \theta^{n+m}(b_m) t^{n+2m}. \end{aligned}$$

Since $a_n > 0$, Lemma 3.2.1. implies that

$a_n \theta^n(b_m) \theta^{n+m}(b_m) > 0$. Thus $a_n t^n \cdot b_m t^m b_m t^m > 0$, which
 implies $ax^2 > 0$. Hence P is a q-ordering on D .

$$(1) \Rightarrow (2).$$

Take $a = a_0 t^0$, where $a_0 \in F$, and let $x = b_1 t$,

where $b_1 \in F, b_1 \neq 0$. By hypothesis, $ax^2 > 0$ in D .

Since $ax^2 = a_0 b_1 t b_1 t = a_0 b_1 \theta(b_1) t^2$, we have

$a_0 b_1 \theta(b_1) > 0$ in F , as desired.

Finally, let $a, b \in D^*$ be positive, write

$$a = \sum_{\substack{i \geq n \\ a_n \neq 0}} a_i t^i \text{ and } b = \sum_{\substack{j \geq m \\ b_m \neq 0}} b_j t^j, \text{ then } ab = \sum_{i \geq n, j \geq m} a_i b_j t^{i+j}.$$

Now ab is positive in D iff its lowest coefficient $a_n b_m$ is positive in F , equivalently iff $a_n b_m$ is positive in F , as desired.

Remark 3.2.3: If we take the field F to be $\mathbb{R}((x, y))$ with a q -ordering as in Theorem 2.3.1. and let $\theta : F \rightarrow F$ be defined by

$$\theta \left(\sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} x^i y^j \right) = \sum_{\substack{i \geq m, j \geq n \\ a_{mn} \neq 0}} a_{ij} (2x)^i y^j.$$

Then θ is an isotone automorphism of F such that for all $a > 0$ and $b \neq 0$ in F , we have $ab \theta(b) > 0$.

Section 3.3 : Jordan q -orderings

Definition 3.3.1: By a Jordan q -ordering $\geq$ on a division ring D , we mean a binary relation $\geq$ on D such that

for all $a, b, c \in D$:

- (1) $1 > 0$ and $a \geq a$;
- (2) $a \geq b, b \geq a \Rightarrow a = b$;
- (3) $a \geq b, b \geq c \Rightarrow a \geq c$;
- (4) $a \geq b$ or $b \geq a$;
- (5) $a \geq b \Rightarrow a + c \geq b + c$; and
- (6) $a > 0 \Rightarrow ab^2 + b^2a > 0$ for all $b \neq 0$.

Let $P = \{a \in D \mid a > 0\}$. We have:

- (i) $1 \in P$, and $0 \notin P$;
- (ii) $P + P \subseteq P$ (i.e., $a, b \in P$ together imply $a + b \in P$);
- (iii) $P \bullet (D^\bullet)^2 \subseteq P$ (i.e., $a \in P$ and $b \in D^\bullet$ together imply $a \bullet b^2 = ab^2 + b^2a \in P$);
- (iv) $P \cup -P = D^\bullet$.

Refer to any subset P of D verifying (i) - (iv) as to a Jordan q -cone of D . Such a subset P induces an order relation on D defined by:

$$a \geq b \Leftrightarrow a - b \in P \cup \{0\}.$$

which is evidently a Jordan q -ordering.

Evidently, every Hilbert ordered division ring is q -ordered and every q -ordered division ring is Jordan q -ordered . Although we do not know whether or not every Jordan q -ordering must be a q -ordering, we wish to establish here that (i) the order valuation of any Jordan q -ordering does exist ; and (ii) D is, at the very least, Hilbert orderable. This is the :

Theorem 3.3.2 : *Every Jordan q -ordered division ring D is Hilbert orderable.*

Proof: We prove that the product of non-zero squares in D is positive. Let $a, b \neq 0$. If $a^2b^2 < 0$, then $-a^2b^2 > 0$. Taking the Jordan product of $-a^2b^2$ and a^{-2} , we get $-a^2b^2a^{-2} - b^2 > 0$. Equivalently, $(a^2ba^{-2})^2 + b^2 < 0$, contrary to the fact that the sum of two squares is always positive. Thus for all $a, b \in D^*$, $a^2b^2 > 0$.

Let $a, b, c \neq 0$. If $a^2b^2c^2 < 0$, then $-a^2b^2c^2 > 0$. Taking the Jordan product of $-a^2b^2c^2$ and a^{-2} , we get $-a^2b^2c^2a^{-2} - b^2c^2 > 0$, which implies $-[a^2b^2a^{-2}a^2c^2a^{-2} + b^2c^2] > 0$. Equivalently, $[(a^2ba^{-2})^2(a^2ca^{-2})^2 + b^2c^2] < 0$, contrary to the fact that the

sum of products of squares is always positive. Thus for all $a, b, c \in D^*$, we have $a^2b^2c^2 > 0$.

Hence, by induction, the product of any number of non-zero squares in D is always positive. Therefore, $C \subset P$ and so $0 \notin C$ (i. e. , 0 cannot be expressed as a sum of products of non-zero squares in D), as desired.

Lemma 3.3.3: *For all $a, b \in P, x \in D^*$, and $q \in \mathbb{Q}^+$ we have :*

- (1) $a^{-1} \in P$;
- (2) $qa \in P$;
- (3) $x^2 \in P$;
- (4) $a > q \Rightarrow a^{-1} < q^{-1}$.

Proof: (1) and (2) . By a trivial adaptation of Lemma 2. 1. 4.

(3) Since $1 \cdot x^2 + x^2 \cdot 1 = 2x^2 \in P$, part (2) implies $x^2 \in P$.

(4) Since $a > q$, $q^{-1}a > 1$. Equivalently, $a^{-1}q < 1$, which implies $a^{-1} < q^{-1}$, for $q \in C$.

Lemma 3.3.4: For all $a, b \in D^*$ and $q \in \mathbb{Q}^+$:

$$(1) (a \pm b)^2 \leq 2(a^2 + b^2) ;$$

$$(2) a^2 \leq q^2 \Leftrightarrow -q \leq a \leq q .$$

Proof: (1) Same as in Lemma 2.1.5. (1) .

(2) If $a^2 \leq q^2$, then

$$q \pm a = (1/2q)(q \pm a)^2 + (1/2q)(q^2 - a^2) > 0 .$$

Thus $-q \leq a \leq q$.

Conversely, If $-q \leq a \leq q$, then $q - a > 0$

and $q + a > 0$. Therefore,

$$(q + a)^2 \cdot (q - a) + (q + a) \cdot (q - a)^2 > 0 ,$$

which implies $4q(q^2 - a^2) > 0$, which in turn implies

$a^2 \leq q^2$, as desired.

Definition 3.3.5: Let D be a Jordan q -ordered division ring. An element $a \in D$ is called a bounded element if there exists some $q \in \mathbb{Q}^+$ such that $|a| \leq q$. Further if $|a| < q$, for all $q \in \mathbb{Q}^+$, then we call a an infinitesimal .

Let R denote the set of all bounded elements in D , let J denote the set of all infinitesimals in D , and let U denote the set of all elements u of D such that

$q_1 < |u| < q_2$ where q_1 and $q_2 \in \mathbb{Q}^+$.

Theorem 3.3.6: *Let D be a Jordan ς -ordered division ring. Then :*

- (1) R is a valuation ring in D ;
- (2) J is the unique maximal ideal in R ;
- (3) U is the set of all the invertible elements in R .

Proof: (1) Let $a, b \in R$. By Lemma 3.3.4., we have $a \pm b \in R$, so, R is an additive subgroup of D and, in fact, by Lemma 3.3.3. (2), R is $\mathbb{Q}$ -permissible. By Lemma 3.3.4. (1), we have $(a \pm b)^2 \leq 2(a^2 + b^2)$, so, $(a \pm b)^2 = a^2 + b^2 \pm (ab + ba) \in R$. Thus $2(a \bullet b) \in R$, and so, $a \bullet b \in R$.

Since $bab = (1/2)[b \bullet (b \bullet a) - b^2 \bullet a]$, we have, $bab \in R$. Similarly, $ba^2b, ab^2a \in R$. Thus $(ab)^2 + (ba)^2 = a(bab) + (bab)a = a \bullet (bab) \in R$. Also from $(ab - ba)^2 = (ab)^2 + (ba)^2 - [ab^2a + ba^2b]$, we derive, $(ab - ba)^2 \in R$, and, hence, $ab - ba \in R$ (Lemma 3.3.4.). Since both $ab + ba$ and $ab - ba$ are in

National Library
of Canada

Bibliothèque nationale
du Canada

Canadian Theses Service

Service des thèses canadiennes

Ottawa, Canada
K1A 0N4

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

2

OF/DE

2

NEW LINEAR
MICRO
A BILLY IRVING

R , so must be $2ab = (ab + ba) + (ab - ba)$ and, hence, $ab \in R$.

Finally, for given $c \in D^\circ$, if $c \in R$, then $|c| > q$ for all $q \in \mathbb{Q}^+$. Lemma 3.3.3. (4), implies that $|c^{-1}| < 1/q$, so $c^{-1} \in J \subset R$. Hence R is a valuation ring in D .

(2) and (3). Same as in Theorem 2.1.7.

We will conclude this thesis with some open questions.

Concerning the unitary group U of a non-isotropic unitary space one could wonder whether the subnormal subgroups of U behave as for the normal subgroups. In this connection, one would like to carry over the results in [C 1] from invariant subalgebra to subinvariant subalgebra.

With regard to orderings, there are some natural questions to ask about a Jordan q -ordering. For instance,

Is it true that the associated bounded subring is preserved under conjugation?

Better still, can one show that every Jordan q -ordering is, in fact, a q -ordering?

Finally, when can one say that every Jordan q -ordering of a valued division ring is a q -ordering?

REFERENCES

- [C 1] M. Chacron, Unitaries in simple artinian rings, Can. J. Math. , Vol. xxi , No. 3 (1979) , pp. 512 - 527 .
- [C 2] M. Chacron, C-orderable division rings with involution, J. of Algebra, Vol. 75 , No. 2 (1982) , pp. 495 - 521 .
- [C 3] M. Chacron, Non-isotropic Unitary spaces and Modules with Cauchy-Schwarz inequalities, Pacific J. of Math. , Vol. 119, No. 1 (1985) , pp. 1 - 87 .
- [C 4] M. Chacron, A Symmetric version of the notion of C- ordered division ring, To appear in Communications in Algebra.
- [D] J. Dauns, A Concrete Approach to Division Rings, Heldermann Verlag Berlin (1982).
- [Di] J. Dieudonne, On the structure of the unitary groups, Trans. Amer. Math. Soc. , (1952), pp. 367-385.
- [Dr] P. K. Draxl, Skewfields, London Math. Soc. Lecture Note Series, 81, 1983 .
- [E] O. Endler, Valuation Theory, Springer-Verlag (1972).
- [H 1] I. N. Herstein, Noncommutative Rings, The Math. Assoc. of America, John Wiley.
- [H 2] I. N. Herstein, Rings with involution, University of Chicago Press (1976), Chicago.
- [Id] I. M. Idris, * valued Division Rings, Orderings and Elliptic Hermitian Spaces, Ph. D. thesis, Carleton University (1986).

- [K] I. Kaplansky, Linear Algebra and Geometry, Allyn and Bacon Inc.
- [L] T. Y. Lam, Orderings, Valuations and Quadratic forms, Amer. Math. Soc. Publ. Vol. 52 (1983) .
- [M] K. Mathiak, Valuations of Skew Fields and Projective Hjelmslev Spaces, Lecture Notes in Math. Vol. 1175 , Springer-Verlag (1986) .
- [P] A. Prestel, Lectures on Formally Real Fields, Lecture Notes in Math. Vol. 1093 , Springer-Verlag (1984) .
- [S] W. Scott, Group Theory, Prentice- Hall, Inc, New Jersey.
- [Se] T. Szele, On ordered skew fields, Proc. Amer. Math. Soc. 3 (1952), pp. 410 - 413 .

NOTATIONS

Here is a short list of the notations used in this thesis.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$	the set of natural numbers, the set of integers, the set of rationals and the set of real numbers, respectively
$\{x, y, \dots\}$	the set consisting of elements $x, y, \dots$
$[x, y]$	the commutator $x^{-1}y^{-1}xy$ of x and y
$a \bullet b$	$ab + ba$
a/b	ab^{-1}
$\langle , \rangle$	the inner product
$C_S(N)$	the centralizer in S of the subset N of S
$\triangleleft , \trianglelefteq$	subgroup, normal subgroup
V	the unitary space
L	the full ring of linear transformations of V
U	the unitary group of V
$D, D^\bullet$	the division ring, the multiplicative subgroup of D
P	the positive cone of the ordering
R	the order valuation ring
U	the set of valuation units
J	the set of infinitesimals
$\bar{D} = R/J$	

END

17-09-90

FIN