LARGE DEVIATIONS AND LAWS OF THE ITERATED LOGARITHM FOR MULTIDIMENSIONAL DIFFUSION PROCESSES WITH APPLICATIONS TO DIFFUSION PROCESSES WITH RANDOM COEFFICIENTS

by

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Large Deviations and Laws of the Iterated

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ABSTRACT

We use the martingale approach to study large deviations and laws of the iterated logarithm for certain multidimensional diffusion processes. The criteria for the validity of these properties are expressed in terms of averaging properties of the coefficients of the infinitesimal generator. Next we apply our results to diffusion processes with random coefficients. Doing so, we are led to study large deviations upper bounds for a class of stationary processes

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NOTATION

$\mathbb{R}^d$ ... d-dimensional Euclidian space

$< x, y > = \sum_{j=1}^{d} x_j y_j, \quad x^T = (x_1, \dots x_d), \quad y^T = (y_1, \dots y_d) \quad \in \mathbb{R}^d.$

$|x| = < x, x >^{\frac{1}{2}}, \quad x \in \mathbb{R}^d.$

S_d^+: set of symmetric and strictly positive definite $d \times d$ matrices.

$|A| = \sup_{\substack{x \in \mathbb{R}^d \\ |x|=1}} |Ax|, \quad A \in \mathbb{R}^{d \times d}.$

$C_0^\infty(\mathbb{R}^d)$ ... space of infinitely differentiable real-valued functions with compact support

$C(T; \mathbb{R}^d)$: space of continuous functions from a closed set $T \subset [0, \infty]$ into $\mathbb{R}^d$ with the topology of uniform convergence on compact subsets of T.

$\mathcal{B}_d$: Banach space of all $f \in C([0,1]; \mathbb{R}^d)$ such that $f(0) = 0$ with the norm $\|f\| = \sup_{t \in [0,1]} |f(t)|$

$\mathcal{B}(X)$: ... Borel σ-algebra on a topological space X.

$\overline{B(0,R)}$: closed ball of radius R in $\mathbb{R}^d$.

$C_b(X)$: space of continuous and bounded real-valued functions on a topological space X.

R: class of increasing and unbounded functions $\lambda : [1, \infty) \to [1, \infty)$ such that

$$\liminf_{t \to \infty} \lambda(\delta t)/\lambda(t) > 0 \text{ for every } \delta > 0.$$

$R_1 = \left\{ \lambda \in R; \inf\{\theta; \int_1^\infty \frac{e^{-\theta\lambda(t)}}{t} dt < \infty\} = 1 \text{ and } \lim_{t \to \infty} \frac{\lambda(t\delta)}{\lambda(t)} = 1 \quad \forall \delta > 0 \right\}.$

$\ll$ ... absolute continuity.

TABLE OF CONTENTS

INTRODUCTION

In a recent article ([10]), G. Papanicolaou and S.R.S. Varadhan studied weak convergence properties of diffusion processes $x(\cdot)$ with infinitesimal generator of the form

$$L_e f(x) = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x,e)\frac{\partial^2}{\partial x_j \partial x_j} f(x), \quad f \in C_0^\infty(\mathbb{R}^d),$$

where $a(x,e) = (a_{ij}(x,e))_{i,j=1}^d$ belongs to a class of ergodic stationary processes on some probability space $(\mathcal{E},\mathcal{M},\mu)$. They proved that for almost every $e \in \mathcal{E}$ with respect to μ, $x_n(\cdot)/\sqrt{n}$ converges weakly as $n \to \infty$ to a nondegenerate Gaussian process on $C([0,1];\mathbb{R}^d)$, where $x_s(t) = x(st), s \geq 1, t \in [0,1]$. Their proof was essentially based on an ergodic theorem for $\frac{1}{t}\int_0^t a(x(u),e)du$.

Motivated by their result and certain properties of Gaussian probability measures on Banach spaces, one can ask if we can prove "stronger" results concerning $x(\cdot)$, namely large deviations and/or laws of the iterated logarithm.

In this thesis, we find sufficient conditions which guarantee that for diffusion processes $y(\cdot)$ with infinitesimal generators of the form

$$(0.1) \qquad Lf(x) = \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} f(x), \quad f \in C_0^\infty(\mathbb{R}^d),$$

$\{y_s(\cdot)/\sqrt{s\lambda(s)}\}_{s\geq 1}$ (considered as $C([0,1];\mathbb{R}^d)$-valued random variables) has a large deviation property for some $\lambda \in R$ and/or $\{y_n(\cdot)/\sqrt{n\lambda(n)}\}_{n\geq 1}$ (resp. $\{y(n)/\sqrt{n\lambda(n)}\}_{n\geq 1}$) is relatively compact in $C([0,1];\mathbb{R}^d)$ (resp. $\mathbb{R}^d$) and the cluster set is a non-random compact convex subset of $C([0,1];\mathbb{R}^d)$ (resp. $\mathbb{R}^d$) for some $\lambda \in R_1$. Next we try to prove that these conditions are satisfied μ almost surely when $y(\cdot)$ is replaced by $x(\cdot) = x(\cdot,e)$. Doing so, we are led to study large deviations for a class of stationary processes.

To describe multidimensional diffusion processes, we will follow the martingale approach initiated by D. Stroock and S.R.S. Varadhan and developped in [13].

Let $a : \mathbb{R}^d \to S_d^+$ be bounded and measurable. We say that the martingale problem for $a(\cdot)$ is well-posed if for every $x \in \mathbb{R}^d$, there exists a unique probability measure P_x on $\mathcal{X} = C([0,\infty);\mathbb{R}^d)$ satisfying

i) $P_x(x(\cdot) \in \mathcal{X}; x(0) = x) = 1$.

ii) $\left(x(t) - \int_0^t Lf(x(u))du,\ \mathcal{F}_t = \sigma\{x(s); 0 \leq s \leq t\}\right)_{t\geq 0}$ is a martingale with respect to P_x for every $f \in C_0^\infty(\mathbb{R}^d)$, where Lf is defined as in (0.1).

Chapter One contains prelimary material on weak convergence and properties of solutions of martingale problems. We prove (Theorem 1.2.6) that if the martingale problem for $a(\cdot)$ is well-posed, where $a(\cdot) : \mathbb{R}^d \to S_d^+$ is bounded measurable and uniformly positive (i.e. $< a(x)\theta, \theta > \ \geq C|\theta|^2$ for every $x, \theta \in \mathbb{R}^d$ and some positive constant C), then $\left\{P_0 \circ (x_n/\sqrt{n})^{-1}\right\}_{n\geq 1}$ converges weakly as $n \to \infty$ to the solution $\mathcal{W}_{A,0}$ of the martingale problem for $A \in S_d^+$ whenever the following hypothesis is verified:

$$(H_0): \qquad \lim_{t\to\infty} P_0\left(\left|\frac{1}{t}\int_0^t a(x(u))du - A\right| > \epsilon\right) = 0 \qquad \forall\epsilon > 0.$$

In Chapter Two we study large deviations and laws of the iterated logarithm for solutions of martingale problems. Let $a : \mathbb{R}^d \to S_d^+$ be bounded, measurable and uniformly positive. Suppose that the martingale problem for $a(\cdot)$ is well-posed and set $P = P_0$. The hypotheses we need to prove our results are

$$(H_1): \qquad P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t a(x(u))du = A\right) = 1$$

and

$$(H_2(\lambda)): \qquad \lim_{t\to\infty}\sup\frac{1}{\lambda(t)}\log P\left(\left|\frac{1}{t}\int_0^t a(x(u))du - A\right| \geq \delta\right) = -\infty, \quad \forall\delta > 0,$$

where $\lambda \in R$.

Remark: Since $a \in S_d^+$ is bounded and uniformly positive, $A \in S_d^+$.

Before stating our main results, we need the following definitions:

$\mathcal{H}_A$: Hilbert space of $f \in B_d$ which are absolutely continuous and which satisfy $|f|_A^2 = \frac{1}{2}\int_0^1 < A^{-1}\dot{f}(t), \dot{f}(t) > dt < \infty$, where $\cdot$ denotes the derivative; further let

$$I_A(f) = |f|_A^2 \quad f \in \mathcal{H}_A, \quad I_A(f) = +\infty \quad f \in B_d/\mathcal{H}_A.$$

$$\mathcal{K}_A = \{f \in B_d; I_A(f) \le 1\} \text{ and } K_A = \{x \in \mathbb{R}^d; \frac{1}{2} < A^{-1}x, x > \le 1\}.$$

The first result (Theorem 2.1.1) concerns large deviations; assuming that (H_2) is verified for some $\lambda \in R$, we can prove that

$$\limsup_{s\to\infty} \frac{1}{\lambda(s)} \log P\left(x_s(\cdot)/\sqrt{s\lambda(s)} \in C\right) \le -\inf_{f\in C} I_A(f) \quad \text{for}$$

every closed subset C of B_d and

$$\liminf_{s\to\infty} \frac{1}{\lambda(s)} \log P\left(x_s(\cdot)/\sqrt{s\lambda(s)} \in \mathcal{O}\right) \ge -\inf_{f\in \mathcal{O}} I_A(f) \quad \text{for}$$

every open subset $\mathcal{O}$ of B_d, where we have made the convention that $\inf_{f\in\phi} I_A(f) = +\infty$.

The next results (Theorem 2.1.2 and Corallaries 2.1.3-4) concern laws of the iterated logarithm (we have kept the term "law of the iterated logarithm" even if the rate $\lambda \in R_1$ is different from $\ell(\ell(\cdot)) \in R_1$, where $\ell(t) = \max(1, \log t), t \ge 1$).

Under (H_1) we can prove

$$(0.2) \qquad P\left(\lim_{n\to\infty} d\left(x_n/\sqrt{n\lambda(n)}, \mathcal{K}_A\right) = 0\right) = 1, \quad \text{for every } \lambda \in R_1.$$

If (H_2) is verified for some $\lambda \in R_1$, then (H_1) holds true and

$$(0.3) \qquad P\left(\lim_{n\to\infty} C\left(x_n/\sqrt{n\lambda(n)}; n \ge 1\right) = \mathcal{K}_A\right) = 1,$$

where $d(x,y) = \inf_{y\in Y} \rho(x,y)$ for any subset Y of a metric space (X,ρ) and $C(y_n; n \ge 1)$ stands for the set of all limit points of a sequence $\{y_n\}_{n\ge1}$ in (X,ρ).

If in addition $A(x) = a_1(x)A_1$ where $A_1 \in S_d^+$ and $a_1 : \mathbb{R}^d \to (0,\infty)$ is bounded, measurable and uniformly postive, then (0.3) holds for every $\lambda \in R_1$, if we only assume that (H_1) is verified.

Furthermore, under (H_1), we have

$$P\left(\lim_{n\to\infty} d(x(n)/\sqrt{n\lambda(n)}, K_A) = 0\right) = 1$$

and

$$P\left(C\left(x(n)/\sqrt{n\lambda(n)}; n \geq 1\right) = K_A\right) = 1$$

for every $\lambda \in R_1$.

Chapter Three is devoted to the applications of the results of Chapter Two when $a(\cdot)$ is random. We consider three class of dynamical systems $S_i = ((\mathcal{E},\rho), \mathcal{B}(\mathcal{E}), \{\tau_x; x \in \mathbb{R}^d\}, \mu)$, $i = I, II$ or III, where $(\mathcal{E},\rho)$ is a complete and separable metric space, $\{\tau_x; x \in \mathbb{R}^d\}$ is a group $(\tau_x \circ \tau_y = \tau_{x+y})$ of jointly measurable mappings of $\mathcal{E}$ onto $\mathcal{E}$, and μ is an invariant ergodic (with respect to $\{\tau_x; x \in \mathbb{R}^d\}$) probability measure on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ with support $\mu = \mathcal{E}$. Corresponding to these dynamical systems, we define three classes $M_i, i = I, II$ or III, of continuous bounded functions $V : \mathcal{E} \to S_d^+$ with the property that the martingale problem for $V(\cdot, e) = V(\tau.e)$ is well-posed. Using a result of G. Papanicolaou and S.R.S Varadhan [10], we can prove that (H_0) and (H_1) are verified μ almost surely when $P = P_{e,0}$ is the solution of the martingale problem for $V(\cdot, e), V \in M_i, i = I, II$ or III (Theorem 3.1.1).

Finally, in Chapter Four, we find sufficient conditions depending on $S_i, V(\in M_i), i = I, II$ or III and $\lambda \in R$, for the validity of (H_2) for almost every $e \in \mathcal{E}$ with respect to μ. When S is of first type, we only study the case when $\mathcal{E}$ is compact. Using results of M.D. Donsker and S.R.S. Varadhan, we prove (Corollary 4.1.3) that (H_2) holds for every $e \in \mathcal{E}$ and $\lambda \in R$ such that $\lim_{t\to\infty} \lambda(t)/t = 0$ whenever there is a unique probability measure on $(\mathcal{E}, \mathcal{B}(\mathcal{E}))$ which is invariant with respect to the Markov semigroup S_t on $C_b(\mathcal{E})$, where $S_t f(e) = E^{P_{e,0}}(f(\tau_{x(t)}e), t \geq 0, f \in C_b(\mathcal{E}), e \in \mathcal{E}$. We also prove that the last condition is satisfied when μ is induced by a periodic function or μ is induced by an almost periodic function and $V = v_1 V_1$, where $V_1 \in S_d^+$ and $v_1 \in C_b(\mathcal{E})$ is uniformly positive. The case when S is of type II or III is studied in Section 2. In that case, we prove that when λ

increases sufficiently fast (i.e. $\theta\lambda \in R_1 \quad \forall\theta > 0$), we only need to study upper bounds for $\lim_{t\to\infty} \frac{1}{\lambda(t)} \log \mathcal{W} \otimes \mu \left(\left| \frac{1}{t} \int_0^t V(x(u)), e) du - E_\mu(V) \right| \geq \delta \right)$ where $V \in C_B(\mathcal{E})$. When the rate λ is given by $\lambda(t) = t$, we prove that the "free energy"

$$c(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log E^{\mathcal{W}\otimes\mu} \left(\exp \left(\lambda \int_0^t V(x(u), e) du \right) \right)$$

exists for every $\lambda \in \mathbb{R}$ and we find a necessary and sufficient condition for the differentiability of $c(\cdot)$ at 0, namely

$$\lim_{a\downarrow 0} \sup_{f\in\mathcal{C}_a} \sup_{e\in\mathcal{E}} \left| \int_{\mathbb{R}^d} V(x,e) f^2(x) dx - E_\mu(V) \right| = 0,$$

where

$$\mathcal{C}_a = \left\{ f \in C_0^\infty(\mathbb{R}^d); \int_{\mathbb{R}^d} \mathrm{f}^2(\mathrm{x}) \mathrm{dx} = 1 \text{ and } \frac{1}{2} \int_{\mathbb{R}^d} |\mathrm{grad\ f(x)}|^2 \mathrm{dx} \leq \mathrm{a} \right\}.$$

Therefore we expect that the phenomenon of phase transition occurs in almost every case of interest. Since we are interested in negative upper bounds, we develop a method to handle the case when $\lim_{t\to\infty} \frac{\lambda(t)}{t} = 0$ (the method is also valid when $\lambda(t) = t$). However, this method is too crude to find very good upper bounds but we believe that we can use it to find "critical rates" at least when $d = 1$. The main results of the first part of Section 2 are Lemma 4.2.7, Theorem 4.2.8, Corollary 4.2.9 and Theorem 4.2.14. The rest of the section is devoted to the study of " potentials" of the form $V(\cdot) = \xi_{[U+\cdot]}$, where U is uniformly distributed over $[0,1)^d$ and is independent of the i.i.d random variables $\{\xi_k; k \in Z^d\}$, which are assumed to be bounded. In that case, we prove that $\lambda(t) = t^{1/3}$ is a "critical rate" when $d = 1$ and we conjecture that $\lambda(t) = t^{d/d+2}$ is a "critical rate" when $d \geq 2$. Finally we also prove weak convergence theorems for some renormalization of $\int_0^t V(x(u))du$.

CHAPTER ONE

PRELIMINARIES

1.1 WEAK CONVERGENCE

Throughout this section, X is a Polish space, i.e. X is a complete and separable metrizable space.

DEFINITION 1.1.1 Let $\{P_n\}_{n\geq 1}, P$ be probability measure on $(X, \mathcal{B}(X))$. We say that P_n converges weakly to P as $n \to \infty$ (which will be denoted by $P_n \Longrightarrow P$) if for every $f \in C_b(X)$,

$$\lim_{n\to\infty} \int_X f(x)P_n(dx) = \int_X f(x)P(dx).$$

Note: From now on, $M(X)$ stands for the space of probability measures on $(X, \mathcal{B}(X))$ with the topology induced by weak convergence. It is easy to see that $M(X)$ is a Polish space, and if X is compact, then $M(X)$ is also compact.

THEOREM 1.1.2: A family $\mathcal{P} \subset M(X)$ is relatively compact iff $\mathcal{P}$ is tight i.e. for every $\epsilon > 0$ one can find a compact $K_\epsilon \subset X$ such that $\inf_{P\in\mathcal{P}} P(K_\epsilon) \geq 1 - \epsilon$.

Proof: See [13]. □

We will now restrict our attention to the case $X = C([0,\infty); \mathbb{R}^d)$. Using Arzela-Ascoli theorem, it is easy to prove

THEOREM 1.1.3 Let $\{P_n\}_{n\geq 1}, P \in M\left(C\left([0,\infty); \mathbb{R}^d\right)\right)$. Then $P_n \Longrightarrow P$ iff $\{P_n\}_{n\geq 1}$ is tight and the finite dimensional distributions of P_n converges weakly to those of P as $n \to \infty$. Moreover $\{P_n\}_{n\geq 1}$ is tight if $\lim_{\lambda\to\infty} \limsup_{n\to\infty} P_n(|x(0)| \geq \lambda) = 0$, and for every ϵ and $T > 0$

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P_n(w_T(x,\delta) \geq \epsilon) = 0,$$

$$\textit{where} \quad w_T(x,\delta) = \sup_{\substack{0\leq s\leq t\leq T \\ 0\leq t-s\leq\delta}} |x(t) - x(s)|, \quad w = w_1.$$

REMARK: One can easily prove that a sequence $\{P_n\}_{n\geq 1}$ in $M\left(C([0,1];\mathbb{R}^d)\right)$ is tight if

$$\lim_{\lambda\to\infty}\limsup_{n\to\infty} P_n\left(|x(0)|\geq\lambda\right)=0,$$

and for every $\epsilon>0, \lim_{\delta\downarrow 0}\limsup_{n\to\infty} P_n(w(x,\delta)\geq\epsilon)=0.$

Tightness is usually hard to prove directly. The next theorem is very useful; its proof can be found in [1].

THEOREM 1.1.4: Suppose that $P_n\in C\left([0,1];\mathbb{R}^d\right), n\geq 1$. If for every $\delta\in(0,1)$,

$$\sup_{\substack{0\leq s\leq t\leq 1\\ 0\leq t-s\leq\delta}} P_n\left(|x(t)-x(s)|\geq\lambda\right)\leq\frac{K_n}{\lambda^\alpha}|t-s|^{1+\beta},\quad \lambda>0,$$

where $\alpha\geq 0,\beta>0$ and K_n is a bounded sequence independent of δ, then one can find a constant $C=C(\alpha,\beta)$ so that $P_n(w(x,\delta)\geq\epsilon)\leq\dfrac{K_nC}{\epsilon^\alpha}\delta^\beta, \epsilon>0.$

We close this section with a result proved by C. Kipnis and S.R.S Varadhan ([9]).

THEOREM 1.1.5: Let $Y(t)$ be a Markov process, reversible with respect to a probability measure π, and let us suppose that the reversible stationary process $\mathbf{P}$ with π as invariant measure is ergodic. Further let V be a function on the state space in $L_2(\pi)$ satisfying $\int V d\pi=0$. Set $X(t)=\int_0^t V(Y(s))ds$. If $\lim_{t\to\infty}\frac{1}{t}E\left(X^2(t)\right)=\sigma^2<\infty$, then letting $X_n(t)=X(nt), t\in[0,1]$, we have $\mathbf{P}\mathrm{o}\left(X_n/\sigma\sqrt{n}\right)^{-1}\Longrightarrow\mathcal{W}=$ standard Wiener measure on $C([0,1];\mathbb{R})$.

1.2. MARTINGALE PROBLEMS.

Let $\mathcal{X}=C([0,\infty);\mathbb{R}^d), \mathcal{F}=\mathcal{B}(\mathcal{X})$, and let $\mathcal{F}_t=\sigma\{x(s);0\leq s\leq t\}$.

For a given $a(\cdot):\mathbb{R}^d\to S_d^+$ which is bounded and measurable, we define

$$Lf(x)=\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{n}a_{ij}(x)\frac{\partial^2}{\partial x_j\partial x_j}f(x),\quad f\in C_0^\infty(\mathbb{R}^d),\quad x\in\mathbb{R}^d.$$

We will say that the martingale problem is well-posed if for each $x\in\mathbb{R}^d$, there exists a unique $P_x\in M(\mathcal{X})$ satisfying

i) $P_x(x(\cdot) \in \mathcal{X}; x(0) = x) = 1$.

ii) $\left(f(x(t)) - \int_0^t Lf(x(u))du, \mathcal{F}_t\right)_{t \geq 0}$ is a P_x-martingale (i.e. a martingale under P_x) for every $f \in C_0^\infty(\mathbb{R}^d)$.

In what follows we state the basic properties of solutions of martingale problems needed in the proof of our results. For more general and/or precise results, the reader may consult [13]; the proofs of the Theorems 1.2.1-5 can also be found in [13].

THEOREM 1.2.1 Let $a(\cdot) : \mathbb{R}^d \to S_d^+$ be bounded and measurable. Let $P \in M(\mathcal{X})$ be given. Then the following are equivalent;

(2.1) $$\left(f(x(t)) - \int_0^t Lf(x(u))du, \mathcal{F}_t\right)_{t \geq 0}$$ is a P-martingale for every $f \in C_0^\infty(\mathbb{R}^d)$.

(2.2) $$\left(\exp(< \theta, x(t) - x(0) > -\frac{1}{2}\int_0^t < a(x(u))\theta, \theta > du), \mathcal{F}_t\right) t \geq 0$$

is a P-martingale for every $\theta \in \mathbb{R}^d$.

(2.3) $$\left(\exp(i < \theta, x(t) - x(0) > +\frac{1}{2}\int_0^t < a(x(u))\theta, \theta > du), \mathcal{F}_t\right)_{t \geq 0}$$

is a P-martingale for every $\theta \in \mathbb{R}^d$. Moreover if any of the above equivalent relations holds, then for each $t > s \geq 0$, and $\lambda > 0$,

(2.4) $$P\left(\sup_{s \leq u \leq t} |x(u) - x(s)| \geq \lambda\right) \leq 2d\exp\left(-\frac{\lambda^2}{2\Lambda d(t-s)}\right), \text{ where}$$

$$\Lambda = \sup_{x \in \mathbb{R}^d} \sup_{\substack{\theta \in \mathbb{R}^d \\ |\theta|=1}} < a(x)\theta, \theta > .$$

THEOREM 1.2.2: Suppose that $a(\cdot) : \mathbb{R}^d \to S_d^+$ is bounded and continuous. Then the martingale problem for $a(\cdot)$ is well-posed and the corresponding family of solutions $\{P_x; x \in \mathbb{R}^d\}$ is measurable and has the strong Markov property.

REMARK: When $a(x) \equiv A \in S_d^+$, Theorem 1.2.2 applies and the corresponding solutions $W_{A,x}$ are Gaussian; if $A = I = I_d, W_{I,0}$ is called the Wiener measure, and $x(t)$, the (canonical) Wiener process.

THEOREM 1.2.3: Let $a : \mathbb{R}^d \to S_d^+$ be bounded and measurable, and let $c : [0,\infty] \to \mathbb{R}^d$ be bounded and left (or right) continuous. Suppose that $P \in M(\mathcal{X})$ satisfies (2.1) (and (2.2), (2.3)), and define

$$R(t) = \exp\left(\int_0^t < c(u), dx_u > -\frac{1}{2}\int_0^t < a(x(u))c(u), c(u) > du\right), \quad t \geq 0.$$

Then there exists a unique $Q \in M(\mathcal{X})$ such that $Q \ll P$ on $\mathcal{F}_t$ and $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = R(t), t \geq 0$. Moreover $\left(\exp\left(< \theta, \bar{x}(t) > -\frac{1}{2}\int_0^t < a(x(u))\theta, \theta > du\right), \mathcal{F}_t\right)_{t\geq 0}$ is a Q-martingale for every $\theta \in \mathbb{R}^d$, and (2.4) holds for Q and $\bar{x}(\cdot)$ replacing P and $x(\cdot)$, where $\bar{x}(t) = x(t) - \int_0^t a(x(u))c(u)du, t \geq 0$.

The next theorem is very useful when we want to construct new solutions from existing ones.

THEOREM 1.2.4: Let $a(\cdot) : \mathbb{R}^d \to S_d^+$ be bounded and continuous. Suppose that $\phi(\cdot) : \mathbb{R}^d \to (0,\infty)$ is bounded, measurable and $\inf_{x\in\mathbb{R}^d}\phi(x) > 0$. For such a $\phi(\cdot)$, define $\tau_\phi(t, x(\cdot)) = \inf\{s > 0; \int_0^s 1/\phi(x(u))du > t\}, t \geq 0$, and let $S_\phi : \mathcal{X} \to \mathcal{X}$ be the map determined by $S_\phi x(t) = x(\tau_\phi(t, x(\cdot)))$. Then there is a one to one correspondence between solutions P_x for the martingale problem for $a(\cdot)$ starting from x and those for $\phi(\cdot)a(\cdot)$ starting from the same point. This correspondence sends P_x into $P_x \circ S_\phi^{-1}$. In particular the martingale problem for $\phi(\cdot)a(\cdot)$ is well-posed and the family of solutions is measurable and has the strong Markov property.

Let $\mathcal{H}_d^{loc}(\lambda_R, \Lambda_R, \delta_R)$ be the class of measurable $a : \mathbb{R}_d \to S_d^+$ such that for every $R > 0, \lambda_R|\theta|^2 \leq \; < a(x)\theta, \theta > \; \leq \Lambda_R|\theta|^2, \theta \in \mathbb{R}^d, x \in \overline{B(0,R)}$ and $|a(x) - a(y)| \leq \delta_R(|x-y|) \quad \forall x, y \in \overline{B(0,R)}$, for some constants $0 < \lambda_R \leq \Lambda_R < \infty$ and some non-increasing functions $\delta_R : (0,\infty) \to (0,\infty)$ satisfying $\lim_{\epsilon\downarrow 0}\delta_R(\epsilon) = 0$.

THEOREM 1.2.5 Let $\{a_n\}_{n\geq 1}, a \in \mathcal{H}_d^{loc}(\lambda_R, \Lambda_r, \delta_R)$. Then the martingale problem for a_n (resp. a) is well-posed and if $\{P_{n,x}; x \in \mathbb{R}^d\}$ (resp. $\{P_x; x \in \mathbb{R}^d\}$) is the corresponding family of solutions, then $P_{n,x}(x(t) \in \cdot)$ (resp. $P_x(x(t) \in \cdot)$) has a strictly positive density

$p_n(x,t,\cdot)$ (resp. $p(x,t,\cdot)$) with respect to Lebesgue measure for each $t > 0$. Moreover if for every $f \in C_0^\infty(\mathbb{R}^d)$ we have $\lim_{n\to\infty} \int_{\mathbb{R}^d} a_n(x) f(x) dx = \int_{\mathbb{R}^d} a(x) f(x) dx$, then $x_n \to x \in \mathbb{R}$, $t_n \to t, t > 0$ imply that $P_{n,x_n} \Longrightarrow P_x$ and $\lim_{n\to\infty} \int_{\mathbb{R}^d} |p_n(x_n,t_n,y) - p(x,t,y)| dy = 0$.

In closing this chapter, we prove a weak convergence theorem which is complementary to Theorem 1.2.5.

THEOREM 1.2.6: Suppose that $a(\cdot) : \mathbb{R}^d \to S_d^+$ is bounded, measurable and

$$\inf_{x \in \mathbb{R}^d} \inf_{\theta \in \mathbb{R}^d \setminus \{0\}} < a(x)\theta, \theta > / |\theta|^2 > 0;$$

further suppose that the martingale problem is well-posed and let P be solution starting from 0. Set $x_n(t) = x(nt), n \in \mathbb{N}, t \geq 0$. Then if $A \in S_d^+$ and

$$(H_0): \qquad \lim_{t\to\infty} P\left(\left| \frac{1}{t} \int_0^t a(x(u)) du - A \right| \geq \delta \right) = 0 \qquad \forall \delta > 0,$$

we have $P \circ (x_n/\sqrt{n})^{-1} \Longrightarrow \mathcal{W}_{A,0}$ on $\mathcal{X}$.

Proof: It follows from Theorem 1.1.3 that we only have to prove tightness and weak convergence of the finite dimensional distributions. We begin with tightness. Let $T > 0$ be given. Then for every n in $\mathbb{N}$ and

$$f \in \mathcal{X}, w_T(f, T/n) \leq 3 \max_{0 \leq k \leq n-1} \sup_{\frac{kT}{n} \leq u \leq \frac{(k+1)T}{n}} \left| f(u) - f\left(\frac{kT}{n}\right) \right|.$$

Using the last inequality and (2.4) we get

$$(2.5) \qquad P\left(w_T(x_n/\sqrt{n}, T/n) \geq 3\epsilon\right) \leq 2nd \exp\left(-\frac{\epsilon^2}{2\Lambda d T}\right), \epsilon > 0, \text{ where}$$

$0 < \Lambda < \infty$ is such that $< a(x)\theta, \theta > \leq \Lambda |\theta|^2 \quad \forall \theta, x \in \mathbb{R}^d$. Clearly, (2.5) implies that

$$\left\{ P \circ \left(x_n/\sqrt{n}\right)^{-1} \right\}_{n \geq 1}$$

is tight. Next let $\theta_1, \ldots, \theta_m \in \mathbb{R}^d$, $0 = t_0 \leq t_1 \leq \cdots \leq t_m < \infty$ be given, $m \in \mathbb{N}$. Since $\mathcal{W}_{A,0}$ is Gaussian and $x(t)$ has independent increments under $\mathcal{W}_{A,0}$ we have

$$E^{\mathcal{W}_{A,0}}\left(\exp\left(i \sum_1^m < \theta, x(t_k) - x(t_{k-1}) > \right) \right) = \exp\left(-\frac{1}{2} \sum_1^m < A\theta_k, \theta_k > (t_k - t_{k-1}) \right).$$

Therefore to complete the proof, we only need to show

$$(2.6)\quad \begin{aligned} &\lim_{n\to\infty} E^P\left(\exp\left(i\sum_1^m < \theta_k, x(nt_k) - x(nt_{k-1}) > /\sqrt{n}\right)\right) \\ &= \exp\left(-\frac{1}{2}\sum_1^m < A\theta_k, \theta_k > (t_k - t_{k-1})\right). \end{aligned}$$

By repeated applications of (2.3) we get

(2.7)

$$E^P\left(\exp\left(i\sum_1^m \frac{< \theta_k, x(nt_k) - x(nt_{k-1}) >}{\sqrt{n}} + \frac{1}{2}\sum_{k=1}^m \frac{1}{n}\int_{nt_{k-1}}^{nt_k} < a(x(u))\theta_k, \theta_k > du\right)\right) = 1.$$

Since

$$0 \le \sum_1^m \frac{1}{n}\int_{nt_{k-1}}^{nt_k} < a(x(u))\theta_k, \theta_k > du \le \Lambda \sum_1^m |\theta_k|^2 (t_k - t_{k-1})$$

and

$$\frac{1}{n}\int_{ns}^{nt} < a(x(u))\theta, \theta > du \xrightarrow{P_r} < A\theta, \theta > (t-s) \text{ by } (H_0), \text{ for every } \theta \in \mathbb{R}^d \text{ and } \quad t \ge s \ge 0,$$

(2.6) follows from (2.7) and the convergence in probability of

$$\sum_1^n \frac{1}{n}\int_{nt_{k-1}}^{nt_k} < a(x(u))\theta_k, \theta_k > du \text{ to } \sum_1^n < A\theta_k, \theta_k > (t_k - t_{k-1}), \text{ as } n \to \infty. \qquad \square$$

CHAPTER TWO

Laws of the Iterated Logarithm and Large Deviations

2.0 Notations

Suppose that $(B, \|\ \ \|)$ is a separable Banach space (over the field $\mathbb{R}$); if $x \in B, y \in B, d(x,y)$ stands for $\inf_{y \in Y} \|x - y\|$; if $\{x_n\}_{n \geq 1}$ is a sequence in B, then $C(\{x_n\}_{n\geq 1})$ = the set of all limit points of $\{x_n\}_{n\geq 1}$. Note that $C(\{x_n\}_{n\geq 1}) = K \iff d(x_n, K) \to 0$ and $x_n \in O$ *i.o.* for every $O \in \mathcal{O}$ such that $O \cap K \neq \phi$ when $\mathcal{O}$ is a countable basis for the topology on B.

2.1 Statement of results

Let $A \in S_d^+$ be given, and let $\mathcal{H}_A$ be the real Hilbert space of all $f \in B_d$ which are absolutely continuous and $\int_0^1 < A_0^{-1} \dot{f}(t), \dot{f}(f) > dt < \infty$, with scalar product $(f,g)_A = \frac{1}{2} \int_0^1 < A^{-1} \dot{f}(t), \dot{g}(t) > dt$, where $\dot{f}$ stands for the derivative of f.

Suppose that $a : \mathbb{R}^d \to S_d^+$ is bounded, measurable and uniformly positive. Also suppose that the martingale problems for a is well-posed and let $\{P_x\}_{x \in \mathbb{R}^d}$ be the corresponding family of solutions; set $P = P_0$.

We shall now make two hypotheses:

$$(H_1) \qquad P\left(\lim_{t\to\infty} \frac{1}{t} \int_0^t a(x(u))du = A\right) = 1;$$

$$(H_2) \qquad \limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P\left(\left|\frac{1}{t}\int_0^t a(x(u))du - A\right| \geq \delta\right) = -\infty, \quad \forall \delta > 0,$$

where $\lambda \in R$.

Before stating the first result, we define $I_A(f) = |f|_A^2$ if $f \in \mathcal{H}_A, I_A(f) = +\infty$, $f \in B_d \backslash \mathcal{H}_A$. Let $\mathcal{K}_A = \{f \in B_d; I_A(f) \leq 1\}$ and $K_A = \{x \in \mathbb{R}^d; \frac{1}{2} < A^{-1}x, x > \leq 1\}$. Since $|f(t) - f(s)| \leq (2|t-s||A|I_A(f))^{1/2}$, $\forall f \in \mathcal{H}_A$, $0 \leq s, t \leq 1$, and $f(0) = 0 \quad \forall f \in B_d$, it follows from Arzela-Ascoli Theorem that $\{f \in B_d; I_A(f) \leq x\}$ is a non-empty compact convex subset of B_d for every $x \geq 0$. In particular $\mathcal{K}_A$ is a compact convex subset of B_d and K_A is a compact convex subset of $\mathbb{R}^d$.

Theorem 2.1.1 Suppose that (H_2) is verified for $\lambda \in R$. Then

$$(1.1) \qquad \limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P\left(x_t/\sqrt{t\lambda(t)} \in C\right) \leq -\inf_{f\in C} I_A(f)$$

for every closed subset C of B_d;

$$(1.2) \qquad \liminf_{t\to\infty} \frac{1}{\lambda(t)} \log P\left(x_t/\sqrt{t\lambda(t)} \in \mathcal{O}\right) \geq -\inf_{f\in \mathcal{O}} I_A(f)$$

for every open subset $\mathcal{O}$ of B_d.

Here we define $\inf_{f\in\phi} I_A(f) = +\infty$.

Theorem 2.1.2 Suppose (H_1) is verified. Then for every $\lambda \in R_1$,

$$(1.3) \qquad P\left(\lim_{n\to\infty} d\left(x_n/\sqrt{n\lambda(n)}, \mathcal{K}_A\right) = 0\right) = 1.$$

If (H_2) is verified for some $\lambda \in R_1$, then (H_1) holds true and

$$(1.4) \qquad P\left(C\left(\{x_n/\sqrt{n\lambda(n)}\}_{n\geq 1}\right) = \mathcal{K}_A\right) = 1.$$

Corollary 2.1.3 If a is of the form $a(x) = a_1(x)A_1$, where $a_1 : \mathbb{R}^d \to \mathbb{R}$ is measurable, $0 < c_1 \leq a_1(x) \leq c_2 < \infty$ $x \in \mathbb{R}^d$ for some constants c_1, c_2 and $A_1 \in S_d^+$, and if in addition (H_1) is verified, then for every $\lambda \in R_1$, (1.3) and (1.4) hold true.

Corollary 2.1.4 If (H_1) is verified, then for every $\lambda \in R_1$,

(1.5) $$P\left(\lim_{n\to\infty} d(x(n)/\sqrt{n\lambda(n)}, K_A) = 0\right) = 1, \quad \text{and}$$

(1.6) $$P\left(C\left(\{x(n)/\sqrt{n\lambda(n)}\}_{n\geq 1}\right) = K_A\right) = 1.$$

Before closing this section, let us remark that if $\ell(t) = \max(1, \log t), t \geq 1$, then $\lambda(t) = \ell(\ell(t)) \in R_1$. For this particular λ, the combination of (1.3) and (1.4) is usually called the functional law of the iterated logarithm, while the combination of (1.5) and (1.6) is called the law of the iterated logarithm.

2.2 Proof of the results

We begin by proving four fundamental lemmas; the first one is an adaptation of a result proved by J. Kuelbs [8].

Lemma 2.2.1 Let $(\Omega, \mathcal{M}, P)$ be a probability space and let $\{X_n\}_{n\geq 1}$ be a sequence of measurable functions from Ω into a separable Banach space $(B, \|\ \ \|)$ with scalar field $\mathbb{R}$. Further let K be a nonempty compact convex subset of B. Suppose that the event $A = \{\{X_n\}_{n\geq 1}$ is relatively compact in $B\}$ is measurable. Then the following are equivalent:

(2.1) $$P\left(\lim_{n\to\infty} d(X_n, K) = 0\right) = 1;$$

(2.2) $$P(A) = 1 \quad \text{and } P\left(\lim_{n\to\infty} \sup f(X_n) \leq \sup_{x\in K} f(x)\right) = 1 \quad \forall f \in B^*,$$

where B^* is the strong dual of B.

Proof: (2.1) $\Rightarrow$ (2.2) is obvious. So suppose that (2.2) holds, and let $\epsilon > 0$ be given. Since B is separable, $K_\epsilon^c = \{x \in B, d(x, K) > \epsilon\} = \cup_1^\infty O_n$, where each O_n is open and convex; therefore $\overline{O_n}$ is convex and $\overline{O}_n \cap K = \phi$. Using (2.2) we get

(2.3) $$P(X_n \in K_\epsilon^c \quad i.o.) = P(A \cap \{X_n \in K_\epsilon^c \quad i.o.\}) \leq \sum_{k=1}^{\infty} P(X_n \in \overline{O}_k \quad i.o.)$$

By the separation theorem, for each $k \geq 1$, one can find $f_k \in B^*$, so that $\sup_{x \in K} f_k(x) < \inf_{x \in \overline{O}_k} f_k(x)$. Next (2.2) and (2.3) yield

$$P(d(X_n, K) > \epsilon \quad i.o.) \leq \sum_{k=1}^{\infty} P\left(\lim_{n\to\infty} \sup f_k(X_n) \geq \inf_{x \in \overline{O}_k} f_k(x)\right) = 0 \quad \forall \epsilon > 0$$

which proves (2.1). ◻

Lemma 2.2.2 Suppose all hypotheses of Lemma 2.2.1 are satisfied. If in addition there exists a Hilbert space $(H, (\cdot,\cdot)) \subset B$ such that the injection $i : H \hookrightarrow B$ is dense and continuous, K is the closed unit ball of H and K is compact in B, then

$$P\left(\lim_{n\to\infty} d(X_n, K) = 0\right) = 1 \text{ and } P\left(\lim_{n\to\infty} \sup f(X_n) = \sup_{x \in K} f(x)\right) = 1$$

for some $f \in B^*$ together imply

$$P\left(\lim_{n\to\infty} \inf \|X_n - i^* f/|i^* f|\| = 0\right) = 1, \quad \text{where } i^* B^* \hookrightarrow H$$

is defined by $(i^* f, x) = f(x), \quad f \in H^*, \quad x \in H$.

Proof: Let us first note that $f(i^* f) = |i^* f|^2$ and $\sup_{x\in K} f(x) = |i^* f|$, $f \in B^*$. Set $\Omega' = \{\omega, \lim_{n\to\infty} d(X_n(\omega), K) = 0\} \cap \{\omega; \lim_{n\to\infty} \sup f(X_n(\omega)) = |i^* f|\}$. Then $\Omega' \in \mathcal{M}$, $P(\Omega') = 1$, and for any $\omega \in \Omega'$, there exists a subsequence $\{n_j\}_{j\geq 1}$ (depending on ω) such that $f(X_{n_j}(\omega)) \to |i^* f|$ and $\|X_{n_j}(\omega) - x(\omega)\| \to 0$ for some $x(\omega) \in K$. Now $|x(\omega) - i^* f/|i^* f||^2 = |x(\omega)|^2 - 2\frac{f(x(\omega))}{|i^* f|} + 1 = \|x(\omega)\|^2 - 1 \leq 0$. Hence $\lim_{n\to\infty} \inf \|X_n(\omega) - \frac{i^* f}{|i^* f|}\| = 0$ for every $\omega \in \Omega'$. ◻

Lemma 2.2.3 Let X be a Polish space, $\{Q_t\}_{t\geq 1}$ be a family of probability measures on $(X, \mathcal{B}(X))$ and $\lambda : [1,\infty) \to [1,\infty)$ be increasing and unbounded.

a) If there exists a function $I : X \to [0,\infty]$, $I \not\equiv \infty$ such that

$$\lim_{t\to\infty} \sup \frac{1}{\lambda(t)} \log Q_t(C) \leq - \inf_{x\in C} I(x) \text{ for every closed set } C \subset X$$

then $\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log\left(\int_X e^{\lambda(t)f(x)} Q_t(dx)\right) \le \sup_{x\in X}(f(x) - I(x))$

for every continuous $f : X \to \mathbb{R}$ which satisfies

$$(2.4) \qquad \lim_{M\to\infty} \limsup_{t\to\infty} \frac{1}{\lambda(t)} \log\left(\int_{\{x; f(x)>M\}} e^{\lambda(t)f(x)} Q_t(dx)\right) = -\infty$$

b) Suppose F is a set of continuous functions $f : X \to \mathbb{R}$ so that every $f \in F$ satisfies (2.4) and for each $f \in F$, there exists a function $I_f : \mathbb{R} \to [0,\infty]$, such that $I_f \not\equiv \infty$, $c(f) = \sup_{x\in\mathbb{R}}(x - I_f(x))$ is finite and

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t\left(f^{-1}(C)\right) \le - \inf_{x\in C} I_f(x) \text{ for every closed } C \subset \mathbb{R}$$

Then

$$(2.5) \qquad \limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(K) \le - \inf_{x\in K} I(x) \text{ for every compact } K \subset X,$$

where $I(x) = \max\left(0, \sup_{f\in F}(f(x) - c(f))\right)$.

If in addition there exists a sequence of compact sets $\{K_n\}_{n\ge 1}$ such that

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(K_n^c) \le -n \quad \forall n \ge 1,$$

then (2.5) holds for every closed set $C \subset X$.

Proof: We first prove a). Suppose that $f : X \to \mathbb{R}$ is continuous and satisfies (2.4). Let $m_1, m_2, n \in \mathbb{N}$ be given. Then

$$\int_X e^{\lambda(t)f(x)} Q_t(dx) \le$$

$$e^{-\lambda(t)m_1/n} + \sum_{k=-m_1}^{m_2} e^{\lambda(t)\frac{(k+1)}{n}} Q_t\left(f^{-1}\left(\left[\frac{k}{n}, \frac{k+1}{n}\right]\right)\right) + \int_{\{x; f(x)>\frac{m_2+1}{n}\}} e^{\lambda(t)f(x)} Q_t(dx).$$

Hence $\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log\left(\int_X e^{\lambda(t)f(x)} Q_t(dx)\right) \le$

$$\max\left(-\frac{m_1}{n}, \max_{-m_1\le k\le m_2}\left(\frac{k+1}{n} - \inf_{x\in f^{-1}([\frac{k}{n},\frac{k+1}{n}])} I(x)\right), A\left(\frac{m_2+1}{n}\right)\right), \text{where}$$

$$A(M) = \limsup_{t\to\infty} \frac{1}{\lambda(t)} \log\left(\int_{\{x; f(x) > M\}} e^{\lambda(t) f(x)} Q_t(dx)\right), \quad M \in \mathbb{R}.$$

Since $\frac{k+1}{n} - \inf_{x\in f^{-1}([\frac{k}{n}, \frac{k+1}{n}])} I(x) \leq \frac{1}{n} + \sup_{x\in\mathbb{R}} (f(x) - I(x)) = \frac{1}{n} + \alpha$, we get

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log\left(\int_X e^{\lambda(t) f(x)} Q_t(dx)\right) \leq \max\left(-\frac{m_1}{n}, \frac{1}{n} + \alpha, \quad A\left(\frac{m_2+1}{n}\right)\right).$$

Therefore the result follows by letting, m_1, m_2 and n go to ∞. We will now prove b). Let us first remark that a) implies

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log\left(\int_X e^{\lambda(t) f(x)} Q_t(dx)\right) \leq c(f) \quad \forall f \in F.$$

Let K be a compact subset of X. If K is empty, (2.5) is obvious; so suppose that K is nonempty, and set $L = \inf_{x\in K} I(x)$. If $L = 0$ (2.5) is trivial so assume that $L > 0$ and choose $0 < L_1 < L, L_1$ finite. Then for each $x \in K$, one can find $f_x \in F$ so that $f_x(x) - c(f) > L_1$. Set $\mathcal{O}_x = \{y \in X; f_x(y) - c(f) > L_1\}$. Clearly $\{\mathcal{O}_x\}_{x\in K}$ is an open covering of K. Since K is compact, there exist $x_1, \cdots x_m \in K$, such that $K \subset \cup_{k=1}^m \mathcal{O}_{x_k}$. It follows that

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(K) \leq \max_{1\leq k\leq m} \left(\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(\mathcal{O}x_k)\right).$$

Next for any open set $\mathcal{O} \subset X$,

$$Q_t(\mathcal{O}) \leq Q_t\left(x; f(x) \geq \inf_{y\in\mathcal{O}} f(y)\right) \leq e^{-\lambda(t)\inf_{y\in\mathcal{O}} f(y)} \int_X e^{\lambda(t) f(x)} Q_t(dx).$$

Therefore $\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(\mathcal{O}) \leq \inf_{f\in F}\left(c(f) - \inf_{y\in\mathcal{O}} f(y)\right)$; if we take $\mathcal{O} = \mathcal{O}_{x_k}$ we get $\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(\mathcal{O}_{x_k}) \leq -L_1, 1 \leq k \leq m$. Hence $\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(K) \leq -L_1$. for every $0 < L_1 < L$ which proves (2.5).

Next for every closed set $C \subset X, C\cap K_n$ is compact and $\inf_{x\in C\cap K_n} I(x) \geq \inf_{x\in C} I(x)$. Therefore

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t(C) \leq \inf_{n\geq 1} \max\left(-n, -\inf_{x\in C} I(x)\right) = -\inf_{x\in C} I(k)$$

if

$$\lim_{t\to\infty} \sup \frac{1}{\lambda(t)} \log Q_t(K_n^c) \leq -n, n \geq 1.$$ □

Before stating the next Lemma, we need the following definition.

Definition 2.2.4 Suppose X is a Polish space and P, Q are two probability measures on $(X, \mathcal{B}(X))$. We define the entropy $h(Q;P)$ of Q with respect to P as

$$h(Q;P) = \sup_{f\in B_b(X)} \left(\int_X f(x)Q(dx) - \log \left(\int e^{f(x)} P(dx) \right) \right),$$

where $B_b(X)$ is the set of bounded and measurable $f : X \to \mathbb{R}$.

The next Lemma is due to M. Donsker and S.R.S. Varadhan [4].

Lemma 2.2.5 Suppose X is a Polish space and P, Q are two probability measures on $(X, \mathcal{B}(X))$. Then

(2.6) $$0 \leq h(Q;P) \leq \infty \text{ and } h(Q;P) = 0 \quad iff \quad P = Q.$$

(2.7) If $\ell = h(Q;P) < \infty$, then for every $A \in \mathcal{B}(X), P(A) \geq e^{-\ell}$ if $Q(A) = 1$ and $P(A) \geq Q(A)e^{-\ell/Q(A)+Q(A^c)\log Q(A^c)/Q(A)}$ if $0 < Q(A) < 1$.

(2.8) $$h(Q;P) < \infty \quad \text{iff} \quad Q \ll P \text{ and } \int_X |\log \frac{dQ}{dP}(x)|Q(dx) < \infty.$$

Moreover $h(Q;P) = \int \log \frac{dQ}{dP}(x)Q(dx)$ in the sense that if one member is finite then the other is finite and they are equal.

(2.9) $$h(Q;P) = \sup_{f\in C_b(x)} \left(\int_X f(x)Q(dx) - \log \left(\int_X e^{f(x)} P(dx) \right) \right).$$

Proof: Taking $f \equiv 0$, we get $h(Q;P) \geq 0$. Hence $0 \leq H(Q;P) \leq \infty$. Clearly $h(P;P) = 0$. Now suppose that $\ell = h(Q;P) < \infty$. If $A \in \mathcal{B}(X)$, then $\lambda 1_A \in B_b(X)$ for $\lambda \geq 0$. Therefore $P(A) \geq \frac{e^{\lambda Q(A)-\ell}-1}{e^\lambda - 1}$ for every $\lambda > 0$. If $\ell = 0$, then letting $\lambda \downarrow 0$ we get $P(A) \geq$

$Q(A) \quad \forall A \in \mathcal{B}(X)$ i.e. $P = Q$. This proves (2.6). If $P(A) = 0$, then $\ell \geq \lambda Q(A) \quad \forall \lambda > 0 \Rightarrow Q(A) = 0$. Therefore $h(Q;P) < \infty \Rightarrow Q \ll P$. It follows from Lusin's Theorem that for every $f \in \mathcal{B}_b(X)$, one can find a sequence of continuous functions g_n with compact support such that $\sup_n \sup_{x \in X} |g_n(x)| \leq \sup_{x \in X} |f(x)|$ and $P(x; g_n(x) \neq f(x)) \to 0$. Since $Q \ll P$ we also have $Q(x; g_n(x) \neq f(x)) \to 0$. Hence (2.9) holds if $h(Q;P) < \infty$. Since $h(Q;P) \geq \sup_{f \in C_b(X)} \left(\int_X f(x) Q(bx) - \log \left(\int_X e^{f(x)} P(dx) \right) \right)$ (2.9) follows.

We will now prove (2.7). If $Q(A) = 1$, then $P(A) \geq \frac{e^{\lambda - \ell} - 1}{e^{\lambda} - 1}$ for every $\lambda > 0$. Letting $\lambda \uparrow \infty$ we obtain $P(A) \geq e^{-\ell}$. Next suppose that $Q(A) \in (0,1)$. Then

$$P(A) \geq \sup_{\lambda > 0} \left(e^{-\ell - \lambda Q(A^c)} - e^{-\lambda} \right) = Q(A) e^{-\frac{\ell}{Q(A)} + \frac{Q(A^c) \log Q(A^c)}{Q(A)}}$$

by elementary calculus. This proves (2.7). Next let

$$q(x) = \frac{dQ}{dP}(x), x \in X, \text{ and } q_n(x) = \min\left(n, \max\left(\frac{1}{n}, q(x)\right)\right), n \in \mathbb{N}, x \in X.$$

Then $\log q_n \in \mathcal{B}_b(X)$ and $\lim_{n \to \infty} \int_X |q(x) - q_n(x)| P(dx) = 0$. Thus $\lim_{n \to \infty} \sup \int \log q_n(x) Q(dx) \leq \ell < \infty$. Now $(\log q_n)^+ \uparrow (\log q)^+$ and $(\log q_n)^- \uparrow (\log q)^-$ as $n \to \infty$. By monotone convergence we have

$$\int_X (\log q_n)^-(x) Q(dx) \uparrow \int_X (\log q)^-(x) Q(dx) < \infty$$

and

$$\int_X (\log q_n)^+(x) Q(dx) \uparrow \int_X (\log q)^+(x) Q(dx) \leq \ell + \int_X (\log q)^-(x) Q(dx) < \infty, \text{ as } n \uparrow \infty,$$

which proves that when

$$h(Q;P) < \infty, \text{ then } Q \ll P, \int_X \left|\log \frac{dQ}{dP}(x)\right| Q(dx) < \infty \text{ and } \int_X \log \frac{dQ}{dP}(x) Q(dx) \leq h(Q;P).$$

Next suppose that $Q \ll P$ with density $q(x)$ such that $\int |\log q(x)| Q(dx) < \infty$ and set $\ell_1 = \int_X \log q(x) Q(dx)$. For any $f \in \mathcal{B}_d(X)$,

$$\log \left(\int_X e^{f(x)} P(dx) \right) \geq \log \left(\int_{\{q > 0\}} \left(\frac{e^{f(x)}}{q(x)} \right) q(x) P(dx) \right) =$$

$$\log \left(\int_{\{q > 0\}} \left(\frac{e^{f(x)}}{q(x)} \right) Q(dx) \right) \geq \int_X f(x) Q(dx) - \ell_1 \quad \text{by Jensen's inequality.}$$

Hence $\infty > \ell_1 \geq H(Q;P)$. This completes the proof of the Lemma. $\square$

We are now ready to prove Theorems 2.1.1 and 2.1.2.

From now on we suppose that $a : \mathbb{R}^d \to S_d^+$ is measurable and satisfies $\zeta < \theta,\theta > \leq < a(x)\theta,\theta > \leq \xi < \theta,\theta > \quad \forall(\theta,x) \in \mathbb{R}^d \times \mathbb{R}^d$ and some constants $0 < \zeta < \xi < \infty$; we also suppose that $\{P_x\}_{x\in\mathbb{R}^d}$ is the unique solution of the martingale problem for a, where we set $P = P_0$.

Let $\lambda \in R$ be given and set $\alpha(t) = (t\lambda(t))^{1/2}, t \geq 1$.

Proposition 2.2.6 Suppose that $\lambda \in R_1$ and set $A = \{x_n/\alpha(n)$ is rel. compact in $B_d\}$. Then A is measurable and $P(A) = 1$. Moreover for every $\epsilon > 0$ one can find $c_1 > 1$ such [that] for every $1 < c \leq c_1$

$$P\left(\limsup_{k\to\infty} \sup_{c^{k-1}\leq n\leq c^k} \left\|\frac{x_n}{\alpha(n)} - \frac{x_{c^k}}{\alpha(c^k)}\right\| \leq \epsilon\right) = 1.$$

If $F_n = \cap_{k=1}^{\infty}\left\{f \in B_d; w(f,2^{-k}) \leq 3 \times 2^{-k/2}(2dnk\xi)^{1/2}\right\}$ then the closure K_n of F_n is compact in B_d and

$$\text{(2.11)} \qquad \limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P\left(x_t/\alpha(t) \in K_n^c\right) \leq -n \text{ for every } n \in \mathbb{N} \text{ and } \lambda \in R.$$

Proof. By Arzela-Ascoli theorem, a sequence $\{f_n\}_{n\geq 1}$ in B_d is relatively compact

$$\iff \lim_{\delta\downarrow 0} \limsup_{n\to\infty} w(f_n,\delta) = 0.$$

Therefore A is measurable for any $\lambda \in R_1$, and $P(A) = 1$ if we can show that for every $\epsilon > 0$, one can find $m \in \mathbb{N}$ so that

$$\text{(2.12)} \qquad P\left(w(x_n,1/m) > \epsilon\alpha(n) \quad i.o\right) = 0.$$

By Borel-Cantelli Lemma, (2.12) holds if we prove that for some $c > 1$ and $m \in \mathbb{N}$, $c = c(\epsilon), m = m(\epsilon)$

$$(2.13) \qquad \sum_k P\left(\sup_{c^{k-1} \le n \le c^k} w(x_n, 1/m) > 3\epsilon\alpha(c^{k-1})\right) < \infty$$

Now $\sup_{c^{k-1} \le n \le c^k} w(x_n, 1/m) \le w(x_{c^k}, 1/n)$. Recalling that for every $f \in B_d$,

$$w(f, 1/m) \le 3 \max_{0 \le j \le m-1} \sup_{t \in [j/m, j+1/m]} |f(t) - f(j/m)|$$

we get

$$P\left(\sup_{c^{k-1} \le n \le c^k} w(x_n, 1/m) > 3\epsilon\alpha(c^{k-1})\right) \le$$
$$\sum_{j=0}^{m-1} P\left(\sup_{t \in [j/m, j+1/m]} |x(c^k t) - x(c^k j/m)| > \epsilon\alpha(c^{k-1})\right) \le 2md \exp\left\{-\frac{\epsilon^2 m \lambda(c^{k-1})}{2d\xi c}\right\}$$

by Theorem 1.2.1. Since

$$\sum_k \exp\{-\theta\lambda(c^k)\} < \infty \text{ for some } \theta > 0 \text{ and } c > 1 \iff \int_1^\infty \frac{e^{\theta\lambda(t)}}{t} dt < \infty,$$

we see that (2.13) holds if $\frac{\epsilon^2 m}{2d\xi c} > 1$, for every $\lambda \in R_1$. This proves that $P(A) = 1$.

To prove (2.10), let us remark that

$$\sup_{c^{k-1} \le n \le c^k} \left\| \frac{x_n}{\alpha(n)} - \frac{x_{c^k}}{\alpha(c^k)} \right\| \le$$
$$w\left(\frac{x_{c^k}}{\alpha(c^{k-1})}, c - 1\right) + \left(1 - \frac{\alpha(c^{k-1})}{\alpha(c^k)}\right) \left\| \frac{X_{c^k}}{\alpha(c^{k-1})} \right\|.$$

Since

$$P\left(w\left(\frac{x_{c^k}}{\alpha(c^{k-1})}, \frac{1}{m}\right) > \epsilon \quad i.o.\right) = 0 \text{ for every } \epsilon > \left(\frac{2d\xi c}{m}\right)^{1/2} \text{ we obtain}$$

$$\limsup_{k \to \infty} \left(1 - \frac{\alpha(c^{k-1})}{\alpha(c^k)}\right) \left\| \frac{x_{c^k}}{\alpha(c^{k-1})} \right\| \le 3\left(\sqrt{c} - 1\right)(2d\xi)^{1/2} \quad P.a.s.,$$

and

$$\limsup_{k \to \infty} w\left(\frac{x_{c^k}}{\alpha(c^{k-1})}, c - 1\right) \le 6\left(\frac{d\xi}{m}\right)^{1/2} \quad P.a.s., \text{ if } 1 < c \le \frac{m+1}{m}, m \ge 1.$$

Hence (2.10) holds if $1 < c \leq \frac{m+1}{m}, m > m_1(\epsilon)$.

Next suppose that $\lambda \in R$. Then $P\left(w(x_t/\alpha(t), 2^{-k}) > 3 \times 2^{-k/2}(2dnk\xi)^{1/2}\right) \leq 2^{k+1}de^{-nk\lambda(t)}$ by Theorem 1.2.1. Hence

$$P(x_t/\alpha(t) \in K_n^c) \leq \frac{4d}{e^{n\lambda(t)} - 2}, \text{ if } e^{n\lambda(t)} > 2.$$

Therefore

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P(x_t/\alpha(t) \in K_n^c) \leq -n \quad \forall n \in \mathbb{N},$$

which proves (2.11). □

Let $B^* =$ the set of all signed measures on [0,1]. By Riesz's Theorem, B_d^* can be identified with $(B^*)^d$ in the sense that every $\nu \in (B^*)^d$ defines a unique $\Lambda \in B_d^*$ by setting $\Lambda f = < f, \nu >$ so that $< f, \nu > = \int < f(t), \nu(dt) >$, and for every $\Lambda \in B_d^*$, there exists a unique $\nu \in (B^*)^d$ so that $< f, \nu > = \Lambda f$ for every $f \in B_d$. From now on we set $B_d^* = (B^*)^d$. If $A \in S_d^{+*}$, then the canonical injection $i : \mathcal{H}_A \hookrightarrow B_d$ is dense and continuous, and $i^* : B_d^* \to \mathcal{H}_A$ is given by $i^*\nu(t) = 2A\int_0^t \nu([u,1])du, t \in [0,1]$, and $\frac{1}{2}|i^*\nu|_A^2 = \int_0^1 < A\nu([u,1]), \nu([u,1]) > du = \int (s \wedge t) < A\nu(ds), \nu(dt) >$, where $x \wedge y = \min(x,y), x, y \in \mathbb{R}$.

Proposition 2.2.7

a) If (H_1) holds, then

$$P\left(\lim_{t\to\infty} \sup_{s\in[0,1]} \left|\frac{1}{t}\int_0^{st} a(x(u))du - As\right| = 0\right) = 1. \text{ In particular,}$$

$$P\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t < a(x(u))\nu\left(\left[\frac{u}{t},1\right]\right), \nu\left(\left[\frac{u}{t},1\right]\right) > du = \frac{1}{2}|i^*\nu|_A^2\right) = 1$$

for every $\nu \in B_d^*$.

b) If (H_2) holds for $\lambda \in R$, then

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P\left(\sup_{s\in[0,1]} \left|\frac{1}{t}\int_0^{st} a(x(u))du - As\right| \geq \delta\right) = -\infty \quad \forall \delta > 0.$$

In particular

$$\lim_{t\to\infty} \sup \frac{1}{\lambda(t)} \log P\left(\left|\frac{1}{t}\int_0^t < a(x(u))\nu\left(\left[\frac{\dot{u}}{t},1\right]\right), \nu\left(\left[\frac{u}{t},1\right]\right) > du - \frac{1}{2}|i^*\nu|_A^2\right| \geq \delta\right) = -\infty,$$

for every $\delta > 0$ and $\nu \in B_d^*$.

c) If (H_2) holds for some $\lambda \in R_1$, then (H_1) holds.

Proof: a) and b) follows from $\sup_s \left|\frac{1}{t}\int_0^{ts} a(x(u))du - As\right| \leq$

$$\frac{2\xi}{n} + \max_{1\leq k\leq n}\left|\frac{1}{t\frac{k}{n}}\int_0^{tk/n} a(x(u))du - A\right| \text{ and } \lim_{t\to\infty}\inf \lambda(t\delta)/\lambda(t) > 0 \text{ for}$$

every $\delta \in (0,1)$ and $\lambda \in R$ (note that $\zeta < \theta, \theta > \ \leq < A\theta, \theta > \leq \ \xi < \theta, \theta > \ \forall \theta \in \mathbb{R}^d$).

To prove c) we only need to show that for every $\beta > 1$,

$$\sum_m P\left(\left|\frac{1}{\beta^m}\int_0^{\beta^m} a(x(u))du - A\right| \geq \delta\right) < \infty \quad \forall \delta > 0,$$

since

$$\sup_{\beta^m \leq t \leq \beta^{m+1}}\left|\frac{1}{t}\int_0^t a(x(u))du - \frac{1}{\beta^m}\int_0^{\beta^m} a(x(u))du\right| \leq 2\xi(\beta - 1), \quad \beta > 1.$$

If (H_2) holds for $\lambda \in R_1$, then $P\left(\left|\frac{1}{\beta^m}\int_0^{\beta^m} a(x(u))du - A\right| \geq \delta\right) \leq e^{-2\lambda(\beta^m)}$ for every $m > m_1(\delta)$, which completes the proof. □

Lemma 2.2.8 Let $\nu \neq 0 \in B_d^*$ be given, and set $\alpha(t) = (t\lambda(t))^{1/2}, t \geq 1, \lambda \in R$.

(2.14) Under (H_1), $\lim_{n\to\infty}\sup < \frac{x_n}{\alpha(n)}, \nu > \leq |i^*\nu|$ $P.a.s.$ $\forall \lambda \in R_1$.

If (H_2) holds for $\lambda \in R$, then

(2.15) $$\lim_{t\to\infty}\sup \frac{1}{\lambda(t)} \log P\left(< \frac{x_t}{\alpha(t)}, \nu > \geq x\right) \leq -\frac{x^2}{|i^*\nu|_A^2}, x \geq 0.$$

Proof: Using (2.10) we see that (2.14) holds true if under (H_1)

(2.16) $$P\left(< \frac{x_{c^m}}{\alpha(c^m)}, \nu > \ \geq x \text{ i.o.}\right) = 0, \forall c > 1, \quad \forall x > |i^*\nu|, \lambda \in R_1.$$

Set $A(t,x) = \left\{ < \frac{x_t}{\alpha(t)}, \nu > \quad \geq x \right\}$ and $C(t,\epsilon) = \{|Y_t - \ell| \geq \epsilon\}$,

where $\ell = \frac{1}{2}|i^*\nu|_A^2$ and $Y_t = \frac{1}{t}\int_0^t < a(x(u))\nu\left(\left[\frac{u}{t},1\right]\right), \nu\left(\left[\frac{u}{t},1\right]\right) > du, \quad t \geq 1$.

By Theorem 1.2.3, $E\left\{\exp\left\{\theta < x_t, \nu > -\frac{\theta^2}{2}tY_t\right\}\right\} = 1 \quad \forall\theta \in \mathbb{R}, \quad \forall t \geq 1$.

Hence $P(A(t,x) \cap C(t,\epsilon)^c) \leq P\left(\theta < x_t, \nu > -\frac{\theta^2}{2}tY_t \geq \theta x\alpha(t) - \frac{\theta^2}{2}t(\epsilon + \ell)\right)$

$\leq \exp\left\{-\theta x\alpha(t) + \frac{\theta^2}{2}t(\epsilon + \ell)\right\}, \quad \forall\theta > 0, x \geq 0, t \geq 1, \epsilon > 0.$

Thus $P(A(t,x) \cap C(t,\epsilon)^c) \leq \exp\left\{-\frac{x^2\lambda(t)}{2(\epsilon + \ell)}\right\}, \lambda \in R, x \geq 0, \epsilon > 0.$

We will now prove (2.16). Under (H_1), $P(C(c^m,\epsilon) \quad i.o.) = 0 \quad \forall c > 1, \epsilon > 0, \lambda \in R_1$. Hence

$$P(A(c^m,x)\ i.o.) = P(A(c^m,x) \cap C(c^m,\epsilon)^c \quad i.o.) = 0$$

if $x^2 > 2(\epsilon + \ell) = 2\epsilon + |i^*\nu|_A^2$ since $\sum_m e^{-\theta\lambda(c^m)} < \infty, \quad \forall c > 1, \theta > 1$ when $\lambda \in R_1$. Since ϵ is arbitrary, (2.16) holds.

If (H_2) holds for $\lambda \in R_1$, then $\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P(C(t,\epsilon)) = -\infty \quad \forall\epsilon > 0$ by Proposition 2.2.7 b). Thus $\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P(A(t,x)) \leq \frac{-x^2}{(2\epsilon + |i^*\nu|_A^2)}, \quad \forall\epsilon > 0, x > 0$. Letting $\epsilon \downarrow 0$ we obtain (2.15). □

Proof of (1.1): Suppose that (H_2) holds for $\lambda \in R$. Set $Q_t = P \circ (x_t/\alpha(t))^{-1}$, where $\alpha(t) = (t\lambda(t))^{1/2}$. Applying (2.15) to ν and $-\nu, \nu \in B_d^*$ we see that for any closed set C of $\mathbb{R}$,

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log Q_t\left(\Lambda_\nu^{-1}(C)\right) \leq -\inf_{x\in C} \frac{x^2}{|i^*\nu|_A^2}, \nu \neq 0,$$

where $\Lambda_\nu(f) = < f, \nu > \quad \forall f \in B_d$. Next

$$\int_{\{\Lambda_\nu > M\}} e^{\lambda(t)\Lambda_\nu(f)} Q_t(df) \leq .$$

$$e^{-M\lambda(t)} E\left\{\exp\left\{2\left(\frac{\lambda(t)}{t}\right)^{1/2} < x_t, \nu >\right\}\right\} \leq \exp(-M\lambda(t)) \exp\{\lambda(t)\gamma(\nu)\}$$

where

$$\gamma(\nu) = 2\xi \int_0^1 < \nu([u,1]), \nu([u,1]) > du.$$

Hence Λ_ν satisfies (2.4) $\forall \nu \in B_d^*$. Moreover $c(\nu) = \sup_{x\in\mathbb{R}} \left(x - \frac{x^2}{|i^*\nu|_A^2}\right) = \frac{1}{4}|i^*\nu|_A^2$. Next

$$J_A(f) = \sup_{\nu \in B_d^*} < f, \nu > - \frac{1}{4}|i^*\nu|_A^2.$$

Clearly $J_A(f) \geq 0 \quad \forall f \in B_d$. If $f \in \mathcal{H}_A$, then $< f, \nu > - \frac{1}{4}|i^*\nu|_A^2 = |f|_A^2 - |\frac{i^*\nu}{2} - f|_A^2$. Thus $J_A(f) = |f|_A^2$. Next $J_A(f) \geq$

$$\frac{1}{2}\sum_1^m < A^{-1}\frac{(f(t_k) - f(t_{k-1}))}{t_k - t_{k-1}}, f(t_k) - f(t_{k-1}) > \quad \forall 0 \leq t, < t, < \ldots < t_m \leq 1.$$

It follows easily that $J_A(f) < \infty$ iff $f \in \mathcal{H}_A$. Hence $J_A(f) = I_A(f) \quad \forall f \in B_d$.

Applying Lemma 2.2.3 to $\{Q_t\}_{t\geq 1}, F = B_d^*$ and $\{K_n\}_{n\geq 1}$ (as defined in Proposition 2.2.6), we obtain (1.1) since $J_A = I_A$. □

Proof of (1.3): It follows from Proposition 2.2.6 and (2.14) that under (H_1)

$$\left\{\{X_n/\sqrt{n\lambda(n)}\}_{n\geq 1} \text{ is rel. compact in } B_d\right\}$$

is measurable and has P-measure 1, and

$$\lim_{n\to\infty} \sup < \frac{x_n}{\sqrt{n\lambda(n)}}, \nu > \leq \sup_{f\in\mathcal{K}_A} < f, \nu > \qquad P.a.s.$$

By Lemma 2.2.1 we can conclude that (1.3) holds i.e.

$$P\left(\lim_{n\to\infty} d\left(x_n/\sqrt{n\lambda(n)}, \mathcal{K}_A\right) = 0\right) = 1 \text{ for every } \lambda \in R_1. \qquad \square$$

Proof of (1.2): Suppose that (H_2) holds for $\lambda \in R$. Since $i^*(B_d^*)$ is dense in B_d, to prove (1.2) we just have to show that for every $\epsilon > 0$ and $\nu \in B_d^*$,

$$\text{(2.17)} \qquad \liminf_{t\to\infty} \frac{1}{\lambda(t)} \log P\left(\|x_t/\sqrt{t\lambda(t)} - i^*\nu\| < \epsilon\right) \geq -|i^*\nu|_A^2$$

Using Proposition 2.2.7 b) we see that (2.17) holds if

(2.18)

$$\liminf_{t\to\infty} \frac{1}{\lambda(t)} \log P\left(\sup_{0\le s\le t}\left|\frac{x(s)}{(t\lambda(t))^{1/2}} - \frac{2}{t}\int_0^s a(x(u))\nu\left(\left[\frac{u}{t},1\right]\right)du\right| < \epsilon\right) \ge -|i^*\nu|_A^2,$$

$\forall\epsilon > 0, \quad \forall\nu \in B_d^*$.

Set $f_t(u) = \nu([u/t, 1])$, $u \ge 0$. We define a probability measure Q_t on $\mathcal{X}$ by letting $\left.\frac{dQ_t}{dP}\right|_{\mathcal{F}_t}(x) =$

$$\exp\left\{2\left(\frac{\lambda(t)}{t}\right)^{1/2}\int_0^s < f_t(u), dx_u > -2\frac{\lambda(t)}{t}\int_0^s < a(x(u))f_t(u), f_t(u) > du\right\}.$$

Set $\bar{x}_t(s) = x(s) - 2\left(\frac{\lambda(t)}{t}\right)^{1/2}\int_0^s a(x(u))f_t(u)du$.

By Theorem 1.2.3 $\lim_{t\to\infty} Q_t\left(\sup_{0\le s\le t}\frac{|\bar{x}_t(s)|}{(t\lambda(t))^{1/2}} \le \epsilon\right) = 1 \quad \forall\epsilon > 0$. Since $\left.\frac{dQ_t}{dP}\right|_{\mathcal{F}_t}(x) =$ $\exp\left\{2\left(\frac{\lambda(t)}{t}\right)^{1/2}\int_0^s < f_t(u), d\bar{x}_t(u) > +2\frac{\lambda(t)}{t}\int_0^s < a(x(u))f_t(u), f_t(u) > du\right\}$ Q_t a.s. and $f_t(u) = 0, u > t$, it follows from Lemma 2.2.5 that

$$h(Q_t; P) = E^{Q_t}\left\{2\frac{\lambda(t)}{t}\int_0^t < a(x(u))f_t(u), f_t(u) > du\right\} \le$$

$$\log E\left\{\exp\left\{2\left(\frac{\lambda(t)}{t}\right)^{1/2}\int_0^t < f_t(u), dx_u >\right\}\right\} =$$

$$\log E\left\{\exp\left\{2\left(\frac{\lambda(t)}{t}\right)^{1/2} < x_t, \nu >\right\}\right\} \text{ since } < x_t, \nu > = \int_0^t < f_t(u), dx_u > P.a.s.$$

It follows from the proof of (2.1) that $\limsup_{t\to\infty}\frac{1}{\lambda(t)}h(Q_t; P) \le |i^*\nu|_A^2$. Using (2.7) we get $\liminf_{t\to\infty}\frac{1}{\lambda(t)}\log P\left(\sup_{0\le s\le t}\frac{|\bar{x}_t(s)|}{(t\lambda(t))^{1/2}} \le \epsilon\right) \ge -\limsup_{t\to\infty}\frac{1}{\lambda(t)}h(Q_t; P) \ge -|i^*\nu|_A^2$ proving (2.18). □

Before proving (1.4) we need the following Proposition which is a consequence of the convergence theorem for martingales.

Proposition 2.2.9 Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and let $\mathcal{F}_m$ be a non-decreasing sequence of sub σ-algebras of $\mathcal{F}$. If $A_m \in \mathcal{F}_m \quad \forall m \ge 1$, then

$$P(A_m \text{ i.o}) = P\left(\sum_{m:1}^{\infty} P(A_{m+1}|\mathcal{F}_m) = +\infty\right).$$

Proof of (1.4): Suppose that (H_2) holds for $\lambda \in R_1$. By Proposition 2.2.7 c), (H_1) also holds. Hence $P(\lim_{n\to\infty} d\left(\frac{x_n}{\alpha(n)}, \mathcal{K}_A\right) = 0\Big) = 1$, where $\alpha(t) = (t\lambda(t))^{1/2}, t \geq 1$. Therefore (1.4) holds if for every $\epsilon > 0$, we can prove

$$(2.19) \qquad P\left(\|\frac{x_n}{\alpha(n)} - f\| < \epsilon \quad i.o.\right) = 1 \quad \forall f \in \mathcal{H}_A, |f|_A < 1.$$

Since $i^*(B_d^*)$ is dense in $\mathcal{H}_A$, we just have to show that(2.19) holds for every $\nu \in B_d^*$ such that $|i^*\nu|_A^2 < 1$. Clearly (2.19) holds if for every $\nu \in B_d^*, |i^*\nu|_A^2 < 1$,

$$(2.20) \qquad \forall \epsilon > 0, \exists m \geq 2, \ s.t. \ P\left(\|\frac{x_{m^n}}{\alpha(m^n)} - i^*\nu\| < \epsilon \quad i.o.\right) = 1.$$

By Proposition 2.2.7 a), $P\left(\lim_{t\to\infty} \sup_{0\leq s\leq 1}\left|\frac{2}{t}\int_0^{st} a(x(u))f_t(u)du - i^*\nu(s)\right| = 0\right) = 1$, where $f_t(u) = \nu([u/t, 1]), u \geq 0, t \geq 1$. Therefore (2.20) holds true if $\forall \epsilon > 0, \exists m \geq 2, m \in$ IN so that

$$(2.21) \qquad P\left(\sup_{0\leq s\leq m^n}\left|\frac{x(s)}{\alpha(m^n)} - \frac{2}{m^n}\int_0^s a(x(u))f_{m^n}(u)du\right| < \epsilon \quad i.o.\right) = 1.$$

Since $\lim_{n\to\infty} \alpha(m^{n-1})/\alpha(m^n) = \frac{1}{\sqrt{m}}$ and (1.1) holds, we see that for every $\epsilon > 0$, one can find $m_1 \in$ IN so that $\forall m > m_1$,

$$\limsup_{n\to\infty}\left(\sup_{0\leq s\leq m^{n-1}}\frac{|x(s)|}{\alpha(m^n)} + \sup_{0\leq s\leq m^{n-1}}\left|\frac{2}{m^n}\int_0^s a(x(u))f_{m^n}(u)du\right|\right) \leq \epsilon \quad P.a.s.$$

Hence (2.21) holds if for every $\epsilon > 0$, there exist $m_2 \in$ IN so that for every $m > m_2, m \in$ IN,

$$(2.22) \quad P\left(\sup_{m^{n-1}\leq s\leq m^n}\left|\frac{x(s) - x(m^{n-1})}{\alpha(m^n)} - \frac{2}{m^n}\int_{m^{n-1}}^s a(x(u))f_{m^n}(u)dx\right| < \epsilon \quad i.o.\right) = 1.$$

Let $m \in$ IN, $m > 1$ be given and set

$$A_n = \left\{\sup_{m^{n-1}\leq s\leq m^n}\left|\frac{x(s) - x(m^{n-1})}{\alpha(m^n)} - 2\int_{m^{n-1}}^s a(x(u))f_{m^n}(u)du\right| < \epsilon\right\}.$$

then $A_n \in \mathcal{F}_{m^n}$ and $P\left(A_n \middle| \mathcal{F}_{m^{n-1}}\right) = P_{x(m^{n-1})}(D_n)$, where

$$D_n = \left\{\sup_{0\leq s\leq m^{n-1}(m-1)}\left|\frac{x(s) - x(0)}{\alpha(m^n)} - 2\int_0^s a(x(u))f_{m^n}(u + m^{n-1})du\right| < \epsilon\right\}.$$

Repeating the arguments we used to prove (1.2) we see that

$$\liminf_{n\to\infty} \frac{1}{\lambda(m^n)} \log P_{x(m^{n-1})}(D_n) \geq$$

$$-\limsup_{n\to\infty} \frac{1}{\lambda(m^n)} \log E_{x(m^{n-1})} \left\{ \exp \left\{ 2 \left(\frac{\lambda(m^n)}{m^n} \right)^{1/2} \int_0^{m^{n-1}(m-1)} < f_{m^n}(u+m^{n-1}), dx_u > \right\} \right).$$

Let us remark that the fact that $P_{x(m^{n-1})}$ is random does not really matter for establishing the last inequality since

$$\lim_{t\to\infty} \inf_{y\in\mathbb{R}^d} P_y \left(\sup_{0\leq s\leq t} \frac{|x(s)-x(0)|}{(t\lambda(t))^{1/2}} \leq \epsilon \right) = 1 \quad \forall \epsilon > 0.$$

Next for any $M > 0, \delta > 0$

$$P\left(P_{x(m^{n-1})} \left(\sup_{\frac{1}{m}\leq s\leq 1} \left| \frac{1}{m^n} \int_0^{m^n s - m^{n-1}} a(x(u))du - A\left(s - \frac{1}{m}\right) \right| \geq \delta \right) \geq e^{-M\lambda(m^n)} \right) \leq$$

$$e^{M\lambda(m^n)} P\left(\sup_{\frac{1}{m}\leq s\leq 1} \left| \frac{1}{m^n} \int_{m^{n-1}}^{m^n s} a(x(u))du - A\left(s - \frac{1}{m}\right) \right| \geq \delta \right).$$

By Proposition 2.2.7 a) the last term is bounded by $e^{-2\lambda(m^n)}$ if n is large enough. It follows that P.a.s.,

$$\limsup_{n\to\infty} \frac{1}{\lambda(m^n)} \log P_{x(m^{n-1})} \left(\left| \frac{1}{m^n} \int_0^{m^{n-1}(m-1)} < a(x(u)) f_{m^n}(u+m^{n-1}), f_{m^n}(u+m^{n-1}) > du \right.\right.$$

$$\left.\left. -\ell_m \right| \geq \frac{1}{k} \right) = -\infty, \quad \forall m \in \mathbb{N}, m > 1, \quad \forall k \in \mathbb{N}, \text{ where } \ell_m = 2 \int_{[1/m,1]^2} (s \wedge t - \frac{1}{m}) < A\nu(ds), \nu(dt)$$

From the proof of (2.15) and Lemma 2.2.3, we can conclude that

$$P\left(\liminf_{n\to\infty} \frac{1}{\lambda(m^n)} \log P(A_n | \mathcal{F}_{m^{n-1}}) \geq -\ell_m \right) = 1.$$

Since $\sum_n e^{-\theta\lambda(m^n)} = +\infty$ if $\theta < 1$, we see that

$$P\left(\sum_{n=2}^{\infty} P(A_n | \mathcal{F}_{m^{n-1}}) = +\infty \right) = 1$$

if $\ell_m < 1$. Now $\lim_{m\to\infty} \ell_m = |i^*\nu|_A^2$. Therefore if $|i^*\nu|_A^2 < 1$, one can find $m_2 \in \mathbb{N}$ such that $\sup_{m>m_2} \ell_m < 1$.

By Proposition 2.2.9, (2.22) holds if $|i^*\nu|_A^2 < 1$, and we may conclude that (1.4) holds true. □

Proof of Corollary 2.1.3: Suppose that $a(x) = a_1(x)A_1$, where $A_1 \in S_d^+$, $a_1 : \mathbb{R}^d \to \mathbb{R}$ is measurable and $0 < c_1 \leq a(x) \leq c_2 < \infty \quad \forall x \in \mathbb{R}$; assume that (H_1) holds and let $\lambda \in R_1$ be given. Let P_1 be the solution of the martingale problem for A_1 such that $P_1(x(0) = 0) = 1$. Clearly (H_2) is satisfied for λ (in fact for any $\lambda \in R$), so the conclusions of Theorem 2.1.2 are valid. Since $A_1\frac{1}{t}\int_0^t a_1(x(u))du \to A$ P.a.s., we have $A = A_1 a_2, \quad a_2 \in (0,\infty)$ and $\frac{1}{t}\int_0^t a_1(x(u))du \to a_2$ P.a.s. Without loss of generality we may suppose that $a_2 = 1$. By Theorem 1.2.4, $P_1 = P \circ S^{-1}$ where $Sx(t) = x(T_t(x)), t \geq 0$ and $T_t(x) = \inf\{s > 0; \int_0^s a_1(x(u))du > t\}$. If we succeed in proving

$$(2.23) \qquad P\left(\sup_{0\leq t\leq 1}\left|\frac{x(nt) - x(T_{nt})}{(n\lambda(n))^{1/2}}\right| > \epsilon \ i.o.\right) = 0, \quad \forall \epsilon > 0,$$

then the proof will be completed since $P_1 = P \circ S^{-1}$ and the conclusions of Theorem 2.1.2 are valid for P_1. So let us prove (2.23). Since $|T_t - T_s| \leq c_3|t-s|, \quad \forall s,t \geq 0$ for some constant c_3, and T_t is the continuous inverse of $\int_0^t a_1(x(u))du$, it follows from the proof of Proposition 2.2.7b) that

$$P\left(\lim_{n\to\infty}\sup_{0\leq t\leq 1}\left|\frac{T_{nt}}{n} - t\right| = 0\right) = 1.$$

Set

$$B(n,\delta) = \left\{\sup_{0\leq t\leq 1}\left|\frac{T_{nt}}{n} - t\right| \leq \delta\right\}, n \in \mathbb{N}, \delta > 0.$$

Then

$$P(B(n,\delta)^c i.o) = 0 \quad \forall \delta > 0.$$

Thus if

$$A(n,\epsilon) = \left\{\sup_{0\leq t\leq 1}\left|\frac{x(nt) - x(T_{nt})}{(n\lambda(n))^{1/2}}\right| > \epsilon\right\},$$

we have

$$P(A(n,\epsilon)\ i.o.) = P(A(n,\epsilon) \cap B(n,\delta) \quad i.o).$$

Now

$$A(n,\epsilon)\cap B(n,\delta)\subset\left\{\sup_{\substack{0\le s\le t\le 2\\ t-s\le\delta}}\left|\frac{x(nt)-x(ns)}{(n\lambda(n))^{1/2}}\right|>\epsilon\right\}$$

if $0<\delta<2$, and the last set is included in

$$\{w(x_{2n},\delta/2)>\epsilon(n\lambda(n))^{1/2}\}.$$

Since $\left(\frac{2n\lambda(2n)}{n\lambda(n)}\right)^{1/2}\to\sqrt{2}$ and $\{x_n/\sqrt{n\lambda(n)}\}$ is relatively compact in B_d, we see that $P(A(n,\epsilon)\cap B(n,\delta)\ i.o.)=0$ if δ is small. Hence $P(A(n,\epsilon)\ \ i.o.)=0\quad\forall\epsilon>0.$ □

Proof of Corollary 2.1.4: Suppose that (H_1) holds and let $\lambda\in R_1$ be given. Since (1.3) holds, (1.5) follows readily. Since K_A is separable, we just have to prove

(2.24) $\qquad P\left(\left|\frac{x(n)}{\alpha(n)}-y\right|<\epsilon\quad i.o.\right)=1\qquad\forall\epsilon>0,\qquad\forall\quad y\in D,$ where

D is a dense subset of K_A, and $\alpha(t)=(t\lambda(t))^{1/2}, t\ge 1$. Let $\theta\in\mathbb{R}^d\backslash\{0\}$ be given. Since (H_1) holds $P\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t<a(x(u))\theta,\theta>du=<A\theta,\theta>\right)=1$. Suppose that $<A\theta,\theta>=1$, and set $T_t=\inf\{s>0;\int_0^s<a(x(u))\theta,\theta>du>t\}, t\ge 0$. As before, it is easy to see that $P\left(\lim_{t\to\infty}\frac{T_t}{t}=1\right)=1$, and $P\left(\sup_{0\le t\le 1}\left|\frac{x(nt)-x(T_{nt})}{\alpha(n)}\right|\ge\epsilon\ i.o.\right)=0\quad\forall\epsilon>0$. Let P_1 be the solution of the martingale problem for $a_1(x)\equiv 1, x\in\mathbb{R}$ such that $P_1(x(0)=0)=1$. It follows from Theorem 1.2.4. that $P_1=P\circ S^{-1}$, where $Sx(t)=<\theta,x(T_t(x))>, x(\cdot)\in\mathcal{X}$. We already know that the conclusions of Theorem 2.1.2 are valid for P_1. Therefore $P_1\left(\lim_{n\to\infty}\sup<\frac{x_n}{\alpha(n)},\nu>=|i^*\nu|\right)=1\quad\forall\nu\in B_1^*$. Hence

$$P\left(\lim_{n\to\infty}\sup<\frac{x_n}{\alpha(n)},\theta\nu>=|i_1^*\nu|<A\theta,\theta>^{1/2}\right)=1\quad\forall\nu\in B_1^*,\quad\forall\theta\in\mathbb{R}^d,$$

where $i_1^*\nu(t)=\int s\wedge t\nu(ds), t\in[0,1]$. In particular

$$P\left(\lim_{n\to\infty}\sup<\theta,\frac{x(nt)-x(ns)}{\alpha(n)}>=(2|t-s|<A\theta,\theta>)^{1/2}\right)=1\quad\forall\theta\in\mathbb{R}^d, s,t\in[0,1].$$

Note that $|i^*(\theta\nu)|_A=|i_1^*\nu|<A\theta,\theta>^{1/2},\ \forall\theta\in\mathbb{R}^d,\nu\in B_1^*$.

Next define $\bar{a} : \mathbb{R}^{d+1} \to S^{+}_{d+1}$ (resp. $\bar{A} : \mathbb{R}^{d+1} \to S^{+}_{d+1}$) by $(\bar{a}(x,y))_{ij} = (a(x))_{ij}$, $1 \le i,j \le d$ (resp. $\bar{A}_{ij} = A_{ij}, 1 \le i,j \le d$), $(\bar{a}(x,s))_{d+1,d+1} \equiv 1$, $(\bar{a}(x,s))_{ij} = 0$ otherwise (resp. $\bar{A}_{d+1,d+1} = 1$ and $\bar{A}_{ij} = 0$ otherwise), where $x \in \mathbb{R}^d, y \in \mathbb{R}$.

Clearly the martingale problem for $\bar{a}$ is well-posed and if P_2 is the solution such that

$$P_2((x(0), y(0)) = 0) = 1,$$

then $P_2 = P \otimes P_1$, the product measure; moreover (H_1) holds and (1.3) holds with $K_{\bar{A}}$. Therefore

$$P_2\left(\lim_{n\to\infty} \sup < \frac{(x_n, y_n)}{\alpha(n)}, (\theta\nu, \mu) >= |j^*(\theta\nu,\mu)|_{\bar{A}}\right) = 1$$

for every $\nu, \mu \in B_1^*, \theta \in \mathbb{R}^d$, where $j^*(\theta\nu,\mu) = (i^*(\theta\nu), i_1^*\mu)$ and $|j^*(\theta\nu,\mu)|^2_{\bar{A}} = |i^*(\theta\nu)|^2_A + |i_1^*\mu|^2$. The hypotheses of Lemma 2.2.2 are verified so we have

$$P_2\left(\lim_{n\to\infty} \inf \|\frac{(x_n, y_n)}{\alpha(n)} - j^*(\theta\nu,\mu)/|j^*(\theta\nu,\mu)_{\bar{A}}\| = 0\right) = 1.$$

Hence

$$P\left(\lim_{n\to\infty} \inf \|\frac{x_n}{\alpha(n)} - \frac{i^*(\theta\nu)}{(|i_1^*\mu|^2 + |i^*(\theta\nu)|^2_A)^{1/2}}\| = 0\right) = 1 \quad \forall\theta \in \mathbb{R}^d,\ \mu,\nu \in B_1^*.$$

In particular,

$$P\left(\lim_{n\to\infty} \inf \left|\frac{x(n)}{\alpha(n)} - \frac{\lambda\sqrt{2}A\theta}{< A\theta,\theta >^{1/2}}\right| = 0\right) \quad \forall\lambda \in [0,1], \theta \in \mathbb{R}^d,\ \theta \ne 0,$$

proving (2.2.4). □

Remark: Using the same technique as above, Duncan [6] proved in an elementary way some laws of the iterated logarithm for Wiener processes in function spaces.

CHAPTER THREE

Applications to diffusion processes with random coefficients

3.0 Introduction

Let $(\mathcal{E},\rho)$ be a Polish space, $\{\tau_x\}_{x\in\mathbb{R}^d}$ a group of mappings from $\mathcal{E}$ onto $\mathcal{E}$ such that $(e,x)\to\tau_x e$ is measurable and $\tau_{x+y}=\tau_x\tau_y$, μ a probability measure on $\mathcal{E}$ which is invariant and ergodic with respect to $\{\tau_x\}_{x\in\mathbb{R}^d}$ i.e. $\mu\circ\tau_x^{-1}=\mu \quad \forall x\in\mathbb{R}^d$ and if $\tau_x A = A \quad \forall x\in\mathbb{R}^d$, $A\in\mathcal{B}(\mathcal{E})$, then $\mu(A)=0$ or 1. Then we call $S=\big((\mathcal{E},\rho),\mathcal{B}(\mathcal{E}),\{\tau_x : x\in\mathbb{R}^d\},\mu\big)$ a dynamical system.

In what follows we consider three different types of dynamical systems S and three classes of functions from $\mathbb{R}^d\times\mathcal{E}$ into S_d^+ for which the martingale problem is well posed, obtaining a family of probability measures $\{P^e\}_{e\in\mathcal{E}}$.

We shall prove that (H_0) and (H_1) are satisfied μ a.s.

3.1 Diffusion with random coefficients

Let $S=\big((\mathcal{E},\rho),\mathcal{B}(\mathcal{E}),\{\tau_x; x\in\mathbb{R}^d\},\mu\big)$ be a dynamical system. We will say that S is of type I if

i) $(x,e)\to\tau_x e$ is continuous

ii) $\forall$ compact $K \subset \mathcal{E}, \lim_{\delta \downarrow 0} \sup_{e \in K} \sup_{\substack{x,y \in B(0,R) \\ |x-y|<\delta}} \rho(\tau_x e, \tau_y e) = 0, \quad \forall R > 0.$

iii) $\mathcal{E}$ = support of μ.

Remark: (iii) is not a restriction since τ_x is a homeomorphism, so support (μ) is a closed invariant set of μ-measure one.

We will say that S is of type II if i) and iii) holds. Finally S is of type III if $\mathcal{E} = \Omega \times T_d$, where T_d is the d-dimensional torus, Ω is a Polish space, $\{\tau_k\}_{k \in Z^d}$ is a group of homeomorphismes of Ω onto $\Omega, \tau_{k+j} = \tau_k \cdot \tau_j$, where we define $\tau_x(\omega, u) = (\tau_{[x+u]}\omega, u + x - [u+x]), \omega \in \Omega, u \in T_d$, where $[x] = ([x_1], \ldots, [x_d])$ and $[\,\cdot\,]$ is the integer part, and $\mu = \mu_1 \otimes \mathcal{U}$, where μ_1 is invariant and ergodic with respect to $\{\tau_k; k \in Z^d\}$ and $\mathcal{U}$ is the uniform probability measure on T_d. Next let $M_I = \{V : \mathcal{E} \to S_d^+; V$ is continuous and $\lambda|\theta|^2 \leq \; < V(e)\theta, \theta > \; \leq \Lambda|\theta|^2, (e, \theta) \in \mathcal{E} \times \mathbb{R}^d$ for some $0 < \lambda \leq \Lambda < \infty\}$, $M_{II} = \{V = vV_1, v : \mathcal{E} \to \mathbb{R}^+$ is continuous, $0 < \inf_{e \in \mathcal{E}} v(e) \leq \sup_{e \in \mathcal{E}} v(e) < \infty$, and $V_1 \in S_d^+\}$, $M_{III} = M_{II}$.

Let $S = ((\mathcal{E}, \rho), \mathcal{B}(\mathcal{E}), \{\tau_x, x \in \mathbb{R}^d\}, \mu)$ be a dynamical system of type i and let $V \in M_i$ be given, i=I, II or III. Set $V(x, e) = V(\tau_x e)$. If $i = I$ or $II, V(\cdot, e)$ satisfies the hypotheses of Theorem 1.2.2 so the martingale problem is well-posed; if $i = III$, then it follows from Theorem 1.2.4 that the martingale problem is well-posed.

Let P_x^e be the unique solution of the martingale problem for $V(\cdot, e)$ such that $P_x^e(x(0) = x) = 1$, and $P^e = P_0^e, e \in \mathcal{E}$.

LEMMA 3.1.1 There exists a unique probability measure $\hat{\mu}$ on $\mathcal{E}$ such that $\hat{\mu}$ is equivalent to μ (i.e $\mu \ll \hat{\mu}$ and $\hat{\mu} \ll \mu$) and

$$\mu \left\{ e; P^e \left(\lim_{t \to \infty} \frac{1}{t} \int_0^t V(\tau_{x(u)} e) du = E_{\hat{\mu}}(f) \right) = 1 \right\} = 1$$

$\forall f \in C_B(\mathcal{E})$. In particular, for almost every $e \in \mathcal{E}, (H_0)$ and (H_1) holds for $P = P^e$ and $A = E_{\hat{\mu}}(V)$. Moreover if $V \in M_I$ (and S is of type I) then the Markov process $\tau_{x(t)} e$ has

$\hat{\mu}$ as an ergodic invariant measure and the corresponding probability transition $q(t,e,de')$ given by $q(t,e,A) = P^e(\tau_{x(t)}e \in A), A \in \mathcal{B}(\mathcal{E}), t \geq 0, e \in \mathcal{E}$, is Feller continuous.

Proof: Suppose first that S is of type I and $V \in M_I$. It follows from Theorem 1.2.5 that $P^e \circ x(t)^{-1}$ is equivalent to the Lebesque measure if $t > 0$; let $p(e,t,\cdot)$ be its density. We will prove that $e \to P^e$ from $\mathcal{E}$ into $M(X)$ is continuous and $(e,t) \to p(e,t,\cdot)$ from $\mathcal{E} \times (0,\infty)$ into $L^1(\mathbb{R}^d)$ is continuous. Let $e_n \to e$ and $t_n \to t, e_n, e \in \mathcal{E}$ and $t_n, t \in (0,\infty), n \geq 1$. By Theorem 1.2.5 we only need to prove

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)V(\tau_x e_n)dx = \int_{\mathbb{R}^d} f(x)V(\tau_x e)dx, \quad \forall f \in C_0^\infty(\mathbb{R}^d) \tag{1.1}$$

$$V(\cdot,e_n) \text{ and } V(\cdot,e) \in \mathcal{H}_d^{loc}(\lambda_R, \Lambda_R, \delta_R), \quad \forall R > 0 \tag{1.2}$$

Clearly (1.1) holds. By the definition of M_i, there exists $0 \leq \lambda \leq \Lambda < \infty$ so that $\lambda|\theta|^2 \leq V(e)\theta, \theta >\leq \Lambda|\theta|^2 \quad \forall (e,\theta) \in \mathcal{E} \times \mathbb{R}^d$. In particular it is valid for $\tau_x e_n, \tau_x e, x \in \mathbb{R}^d$. Next let $\delta_R(r) = \sup_n \sup_{\substack{|x-y|\leq r \\ x,y \in B(0,R)}} |V(\tau_x e_n) - |V(\tau_y e_n)|, r > 0, R > 0$. Since $e_n \to e$, $|V(\tau_x e) - V(\tau_y e)| \leq \delta_R(|x-y|)$ whenever $x, y \in \overline{B(0,R)}$. Hence (1.2) holds if $\lim_{r\to 0} \delta_R(r) = 0 \quad \forall R > 0$. Let $K = \{e_n\}_{n\geq 1} \cup \{e\}$. Then K is compact in $\mathcal{E}$. It follows from the definition of S_I ((ii)), that $\delta_R(r) \to 0$ as $r \downarrow 0$. Hence (1.2) holds.

Let $f \in C_b(E)$. Suppose that $(e_n, t_n) \in \mathcal{E} \times [0,\infty) \to (e,t) \in \mathcal{E} \times [0,\infty)$.

Then $$\left|\int_{\mathcal{E}} f(e')q(e_n,t_n,de') - \int_{\mathcal{E}} f(e')q(e,t,de')\right| =$$

$$\left|E^{e_n}\left(f(\tau_{x(t_n)}e_n) - E^e\left(f(\tau_{x(t)}e)\right)\right| \leq$$

$$E^{e_n}\left(|f(\tau_x(tn)e_n) - f(\tau_x(t)e)|\right) + \left|E^{e_n}\left(\tilde{f}(x(t)\right) - E^e\left(\tilde{f}(x(t))\right)\right|, \tag{1.3}$$

where $\tilde{f}(x) = f(\tau_x e) \in C_b(\mathbb{R}^d)$. Since $P^{e_n} \Longrightarrow P^e$, the last term of (1.3) goes to 0 as $n \to \infty$. Next set

$$A_n(r,R) = \{|x(t_n)| \leq R\} \cap \{|x(t)| \leq R\} \cap \{|x(t_n) - x(t)|\} \leq r, r > 0, R > 0 \quad n \geq 1.$$

Then $\lim_{r\downarrow 0}\limsup_{n\to\infty} E^{e_n}\left(|f(\tau_{x(t_n)}e_n) - f\left(\tau_{x(t)}e\right)| 1_{A_n(r,R)}\right) = 0$ for every $R > 0$ since f is continuous and S_t satisfies (ii). Next

$$p(n,r,R) \doteq E^{e_n}\left(|f(\tau_{x(t_n)}e_n) - f\left(\tau_{x(t)}e\right)| 1_{A_n}(r,R)^c\right) \le$$

$$2\sup_{e'}|f(e')|P^{e_n}(A_n(r,R)^c) \le$$

$$4d\sup_{e'}|f(e')|\left(\exp\left\{-\frac{r^2}{2\Lambda d|t_n - t|}\right\} + \exp\left\{-\frac{R^2}{2\Lambda d t_n}\right\} + \exp\left\{-\frac{R^2}{2\Lambda d t}\right\}\right)$$

where $\exp\left\{-\frac{c^2}{0}\right\} = 0, c > 0$, by Theorem 1.2.1. Hence

$$\lim_{R\to\infty}\lim_{r\downarrow 0}\limsup_{n\to\infty} p(n,r,R) = 0$$

proving that $q(t,e,de')$ is Feller continuous. Next let $P(A) = P^{\tau_y e}(A - y), A \in \mathcal{B}(\mathcal{X}), y \in \mathbb{R}^d$ fixed. Then $P(x(0) = y) = P^{\tau_y e}(x(0) = 0) = 1$; if $\theta \in \mathbb{R}^d, A \in \mathcal{F}_s$, and $t \ge s \ge 0$, we have

$$E^P\left\{e^{<\theta,x(t)-y> - \frac{1}{2}\int_0^t <V(x(u),e)\theta,\theta> du} 1_A\right\} =$$

$$E^{\tau_y e}\left\{e^{<\theta,x(t)> - \frac{1}{2}\int_0^t <V(y+x(u),e)\theta,\theta> du} 1_{A-y}\right\}$$

Since $V(y + x(u)e) = V(x(u),\tau_y e)$, we see that

$$\exp\left\{< \theta, x(t) - y > -\frac{1}{2}\int_0^t < V(x(u),e)\theta,\theta > du\right\}$$

is P-martingale, and it follows that $P^{\tau_y e} = P_y^e \quad \forall(y,e) \in \mathbb{R}^d \times \mathcal{E}$. Next if $f \in C_b(\mathcal{E})$, we have

$$S_{t+s}f(e) = \int_{\mathcal{E}} f(e;)q(t+s,e,de') = E^e\left(f(\tau_{x(t+s)}e\right) = E^e\left(E^e_{x(s)}\left(f(\tau_{x(t)}e)\right)\right)$$

From the above calculations,

$$E_y^e\left(f(\tau_{x(t)}e)\right) = E^{\tau_y e}\left(f(\tau_{x(t)}(\tau_y e)\right) = S_t f(\tau_y e).$$

Hence

$$S_{t+s}f(e) = E^e\left(S_t f\left(\tau_{x(s)}e\right)\right) = S_s(S_t f)(e) \qquad \text{i.e. } S_{t+s} = S_t \circ S_s$$

for every $s, t \geq 0$. Therefore $\tau_{x(t)}e$ is a Markov process with state space $\mathcal{E}$. The existence and uniqueness of $\hat{\mu}$ with the prescribed properties now follow from results of G. Papanicolaou and S.R.S. Varadhan [10].

Let us now suppose that S is of type II (III) and $V \in M_{II}(M_{III})$. Then $V(e) = v(e)V_1$, where $V_1 \in S_d^+, v \in C_b(\mathcal{E})$ and $0 < c_1 \leq v(e) \leq c_2 < \infty \quad \forall e \in \mathcal{E}$ for some constants c_1, c_2. It follows from Theorem 1.2.4 that $P^e = P_1 \circ R_e^{-1}$, where $P_1 = \mathcal{W}_{V_1}$ and $R_e x(t) = x(T_{t,e}(x(\cdot)))$, where

$$T_{t,e}(x(\cdot)) = \inf\left\{ s > 0; \int_0^s 1/v(x(u), e)du > t \right\}, t \geq 0, e \in \mathcal{E}.$$

Thus $e \to P^e$ is measurable. Now define $\hat{\mu}(A) = \int_A c/v(e)\mu(de)$, where $\frac{1}{c} = \int_{\mathcal{E}} 1/v(e)\mu(de)$, and $A \subset \mathcal{B}(\mathcal{E})$. Let $f \in C_b(\mathcal{E})$ be given. Then

$$P^e\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\tau_{x(u)}e)du = E_{\hat{\mu}}(f)\right) = 1 \quad \text{iff}$$

$$(1.4) \qquad P_1\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\tau_{x(T_{u,e})}e)du = E_{\hat{\mu}}(f)\right) = 1 \qquad \text{iff}$$

$$P_1\left(\lim_{t\to\infty} \frac{1/t\int_0^t \left(\frac{f}{v}\right)(\tau_{x(u)}e)du}{\frac{1}{t}\int_0^t \frac{1}{v}(\tau_{x(u)}e)du} = \frac{\int \left(\frac{f}{v}\right)(e)\mu(de)}{\int \frac{1}{v}(e)\mu(de)}\right) = 1.$$

Since $0 < c_1 \leq v(e) \leq c_2 < \infty \quad \forall e \in \mathcal{E}$, (1.3) holds if

$$P_1\left(\lim_{t\to\infty} \frac{1}{t}\int_0^t g_i(\tau_{x(u)}e)du = E_\mu(g_i)\right) = 1,$$

where $g_1 = f/v$ and $g_2 = 1/v$. Clearly $\tau_{x(t)}e$ is a Markov process (under P_1) and μ is an invariant measure. It follows from the definition of S that for any $f \in L^2(d\mu)$,

$$\int_{\mathcal{E}} f(\tau_x e)f(e)\mu(de) = \int_{\mathbf{R}^d} e^{i<x,\lambda>}\nu_f(d\lambda)$$

for some finite (positive) measure ν_f and $\nu_f\{0\} = (E_\mu(f))^2$. Therefore

$$E_\mu\left(E_1\left(f(\tau_{x(t)}e)f(e)\right)\right) = E_1\left(\int_{\mathbf{R}^d} e^{i<x(t),\lambda>}\nu_f(d\lambda)\right) =$$

$$\int_{\mathbf{R}^d} e^{-\frac{t}{2}<V_1\lambda,\lambda>}\nu_f(d\lambda) \to \nu_f\{0\} = (E_\mu(f))^2,$$

proving that μ is also an ergodic probability measure for $\tau_{x(t)}e$. Thus

$$\mu\left\{e; P_1\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t f(\tau_{x(u)}e)du = E_\mu(f)\right) = 1\right\} = 1$$

for every $f \in C_b(\mathcal{E})$, which in turn proves that (1.3) holds for almost every e with respect to μ. □

COROLLARY 3.1.2 Suppose that $S = ((\mathcal{E},\rho), \mathcal{B}(\mathcal{E}), \{\tau_x; x \in \mathbb{R}^d\}, \mu)$ is a dynamical system of type I, II or III. If $A \in S_d^+$ and $P = \mathcal{W}_A$ then $\tau_{x(t)}e$ is a reversible Markov process with transition probability $q(e,t,B) = P(\tau_{x(t)}e \in B), B \in \mathcal{B}(\mathcal{E}), e \in \mathcal{E}, t \geq 0$, and μ is an invariant ergodic measure for the process.

Having proved that (H_1) holds for $P^e\mu$ a.s., the next question is: does (H_2) hold for some $\lambda \in R, \mu$ a.s.? We will try to answer this difficult question in Chapter Four.

For the moment we just state:

PROPOSITION 3.1.3: Let $\hat{\mu}$ be as in Lemma 3.1.1 and set $A = E_{\hat{\mu}}(V)$. For a fixed $\lambda \in R$, let $\theta_0 = \inf\left\{\theta > 0;\ \int_1^\infty \frac{e^{-\theta\lambda(t)}}{t}dt < \infty\right\}$. If $\theta_0 < \infty$, then for any $\delta > 0$

$$\mu\left(\limsup_{t\to\infty}\frac{1}{\lambda(t)}\log P^e\left(|\frac{1}{t}\int_0^t V(x(u),e)du - A| \geq \delta\right) \leq \theta_0 + p(\delta)\right) = 1,$$

where

$$p(\delta) = \lim_{\substack{\delta'\to\delta \\ 0<\delta'<\delta}}\ \limsup_{t\to\infty}\frac{1}{\lambda(t)}\left(\int_{\mathcal{E}} P^e\left(|\frac{1}{t}\int_0^t V(x(u),e)du - A| \geq \delta'\right)\hat{\mu}(de)\right).$$

In particular if $\lambda \in R_1$ and

$$\limsup_{t\to\infty}\frac{1}{\lambda(t)}\log\left(\int_{\mathcal{E}} P^e\left(|\frac{1}{t}\int_0^t V(x(u),e)du - A| \geq \delta\right)\hat{\mu}(de)\right) = -\infty$$

for every $\delta > 0$, then

$$\mu\left(\limsup_{t\to\infty}\frac{1}{\lambda(t)}\log P^e\left(|\frac{1}{t}\int_0^t V(x(u),e)du - A| \geq \delta\right) = -\infty, \quad \forall\delta > 0\right) = 1.$$

Proof The proof is similar to the proof of Lemma 4.2.1. □

CHAPTER FOUR

The Hypothesis H_2 when $P = P^e$

4.0 Introduction

We have established condition for the validity of L.I.L. and L.D.P. for a class of diffusion processes. What we want to do in this chapter is to find conditions which guarantee that the results we have proved in Chapters Two and Three hold "almost surely" for dynamical systems of the three types we introduced earlier.

Let $S = (\mathcal{E}, \mathcal{B}(\mathcal{E}), \{\tau_x; x \in \mathbb{R}^d\},\ \mu)$ be a dynamical system of type I, II or III, and suppose that $V : \mathcal{E} \to S_d^+ \in M_I, M_{II}$ or M_{III}. Then the L.D.P. holds for P^e whenever

$$e \in G_{\lambda,\mu} = \{e \in \mathcal{E}; \limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P^e\left(\left|\frac{1}{t}\int_0^t V(x(u),e)du - E_{\hat{\mu}}(V)\right| \geq \delta\right) = -\infty, \forall \delta > 0\ \lambda \in R,$$

where P^e is the solution of the martingale problem for $V(\cdot, e)$ and $P^e(x(0) = 0) = 1$. As we said before, we want to find conditions which guarantee that $\hat{\mu}(G_{\lambda,\mu}) = 1$.

Clearly we just have to find those α's $\in R$ for which

$$(0.1) \qquad \limsup_{t\to\infty} \frac{1}{\alpha(t)} \log P^e\left(\left|\frac{1}{t}\int_0^t V(x(u),e)du - E_{\hat{\mu}}(V)\right| \geq \delta\right) < 0 \quad \hat{\mu} \quad a.s.$$

for every $\delta > 0$. For if (0.1) holds for some $\alpha \in R$, then $\hat{\mu}(G_{\lambda,\mu}) = 1$ whenever $\lim_{t\to\infty} \alpha(t)/\lambda(t) = \infty, \lambda \in R$.

When S is of type I and $V \in M_I$, we will only study the case $\alpha(t) = t$ and $\mathcal{E}$ compact. This will be done in Section 1.

When S is of type II or III, and $V = (1/v)I_d, v \in C_b(\mathcal{E}), \inf_{e\in\mathcal{E}} v(e) > 0$, (0.1) is equivalent to

$$(0.2) \qquad \limsup_{t\to\infty} \frac{1}{\alpha(t)} \log P\left(\left|\frac{1}{t}\int_0^t v(x(u),e)du - E_\mu(v)\right| \geq \delta\right) < 0 \quad \mu \quad a.s$$

for every $\delta > 0$, where P is the standard Wiener measure starting from 0 at time 0.

REMARK: For sake of simplicity we will not consider the case $V = (1/v)A, A \in S_d^+$, since it is similar to the case $A = I_d$.

In Section 2, we study large deviations for $\frac{1}{\lambda(t)}\int_0^t V(x(u),e)du$ where $x(\cdot)$ is a d-dimensional Wiener process and $V \in C_b(\mathcal{E})$. We obtain sufficient conditions on λ, V, μ in order that

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P\otimes\mu\left(\left|\frac{1}{t}\int_0^t V(x(u),e)du - E_\mu(V)\right| \geq \delta\right) < 0 \quad \forall\delta > 0.$$

We then apply our results to function of the form $V(x,\omega,u) = v(\omega([x+u])), (\omega,u) \in \Omega\times T_d$ and $\{\omega(k)\}_{k\in Z^d}$ are i.i.d.

4.1 The Case $\mathcal{E}$ is compact.

Throughtout this section, $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \{\tau_x; x \in \mathbb{R}^d\}, \mu)$ is of type I and $\mathcal{E}$ is compact. Let $V \in M_I$ be given, and let P_x^e be the solution of the martingale problem for $V(\cdot,e)$ starting from x at time 0.

The mapping $x(t) \to \tau_{x(t)}e = e_t$ induces a Markov family of p.m.'s $\{Q_e\}_{e\in\mathcal{E}}$ on $C([0,\infty);\mathcal{E})$ with transition probability $q(t,e,de')$ given by $q(t,e,A) = P^e\left(\tau_{x(t)}e \in A\right), A \in \mathcal{B}(\mathcal{E}), t \geq 0$.

We know that the Markov process e_t is Feller continuous and that there exists a unique $\hat{\mu} \in M(\mathcal{E})$ which is equivalent to μ and is ergodic and invariant for the process e_t. For every $A \in \mathcal{B}(\mathcal{E}), e(\cdot) \in C([0,\infty),\mathcal{E})$, and $t > 0$ let

$$(1.1) \qquad L_{t,e(\cdot)}(A) = \frac{1}{t}\int_0^t 1_A(e(u))du.$$

Then (1.1) defines a measurable mapping from $C([0,\infty);\mathcal{E})$ into $M(\mathcal{E})$. L_t is called the occupation time measure. Set $R_{t,e} = Q_e \circ L_{t,\tau e}^{-1}, t > 0, e \in \mathcal{E}$.

In a series of papers, M.D. Donsker and S.R.S. Varadhan studied L.D.P. for occupation time measures of Markov processes with the Feller property. In particular, we can apply their results of [4] to $R_{t,e}$.

THEOREM 4.1.1: Let L be the generator for the Markov process e_t and let $\mathcal{D}$ be its domain; further let $\mathcal{D}^+$ be the subset of $\mathcal{D}$ consisting of functions u with $\inf_{e\in\mathcal{E}} u(e) > 0$. Then for every closed set $K \subset \mathcal{E}$

$$(1.2) \qquad \limsup_{t\to\infty} \frac{1}{t} \log \sup_{e\in\mathcal{E}} R_{t,e}(K) \leq -\inf_{\nu\in K} I(\nu), \quad \text{where}$$

$I(\nu) = -\inf_{u\in\mathcal{D}^+} \int \frac{Lu}{u}(e)\nu(de)$. In particular (1.2) holds for $K = \{\nu \in M(\mathcal{E}); |E_\nu(V) - E_{\hat{\mu}}(V)| \geq \delta\}$.

Proof: Let $A \in \mathcal{B}(M(\mathcal{E}))$ be given. If $u \in \mathcal{D}^+$, then $Lu/u \in C_b(\mathcal{E})$, so

$$(1.3) \qquad \sup_e R_{t,e}(A) \leq e^{-t\inf_{\nu\in A} E_\nu(-Lu/u)} \cdot \sup_e E^{Q_e}\left\{e^{-\int_0^t Lu/u(e_s)ds}\right\}$$

Using Feynman-Kac formula, we get

$$(1.4) \qquad u(e) = E^{Q_e}\left\{u(e_t)e^{-\int_0^t Lu/u(e_s)ds}\right\} \geq \inf_e u(e) E^{Q^e}\left\{e^{-\int_0^t Lu/u(e_s)ds}\right\}.$$

Therefore (1.3) together with (1.4) yield

$$(1.5) \qquad \limsup_{t\to\infty} \frac{1}{t} \log \sup_{e\in\mathcal{E}} R_{t,e}(A) \leq -\sup_{u\in\mathcal{D}^+} \inf_{\nu\in A} E_\nu(-Lu/u).$$

Since $M(\mathcal{E})$ is compact and $\nu \to E_\nu(Lu/u)$ is continuous for $u \in \mathcal{D}^+$, (1.2) follows Lemma 2,2.3.b). □

REMARK: If $\mathcal{E}$ is not compact, (1.2) still holds for compact subset of $M(\mathcal{E})$; but to have (1.2) hold for closed sets in $M(\mathcal{E})$ requires additional conditions on $q(t,e,de')$ and we don't believe that these conditions are satisfied in general. This is why we have restricted our attention to compact spaces.

Since $I : M(\mathcal{E}) \to [0,\infty]$ is lower semicontinuous, the infimum of I over every closed set ($\equiv$ compact set) is attained. Since we want that $\inf_{\nu:|E_\nu(V)-E_{\hat\mu}(V)|\geq\delta} I(\nu) > 0 \quad \forall\delta > 0$, we just have to prove that $I(\nu) = 0$ iff $\nu = \hat\mu$.

Let $S_t f(e) = \int f(e')q(t,e,de'), t \geq 0, f \in C_b(\mathcal{E})$ and let $S_t^*\nu$ be the unique p.m. on $\mathcal{E}$ satisfying $\int f dS_t^*\nu = E_\nu(S_t f), f \in C_b(\mathcal{E})$. We will say that $\nu \in M(\mathcal{E})$ is invariant iff $S_t^*\nu = \nu, \quad \forall t \geq 0$. Recall that $\hat\mu$ is invariant.

LEMMA 4.1.2: $I(\nu) = 0$ iff ν is invariant.

Proof: Suppose first that μ is invariant and let $u \in \mathcal{D}^+$ be given. By Jensen's inequality

$$(1.6) \qquad \frac{1}{t}\int \log\left(\frac{S_t u}{u}\right)(e)\nu(de) \geq \frac{1}{t}\int (S_t - I)(\log u)(e)\nu(de) = 0, \quad \forall t > 0.$$

Since $u \in \mathcal{D}^+$, $\frac{1}{t}\log\left(\frac{S_t u}{u}\right)(e)$ converges pointwise and boundedly to $Lu/u(e)$ as $t \to 0$. Using (1.6) we get $E_\nu(Lu/u) \geq 0 \quad \forall u \in \mathcal{D}^+$. Hence $I(\nu) = 0$. Next suppose that $I(\nu) = 0$. Since $S_t\mathcal{D}^+ \subset \mathcal{D}^+$, we have by bounded convergence

$$(1.7) \qquad \frac{d}{dt}E_\nu\left(\log\left(\frac{S_t u}{u}\right)\right) = E_\nu\left(L(S_t u)/S_t u\right) \geq 0, \quad \forall t \geq 0, u \in \mathcal{D}^+.$$

It follows from (1.7) and Jensen's inequality that

$$(1.8) \qquad \log\left(\int u(e)S_t^*\nu(de)\right) \geq \int \log u(e)\nu(de), \quad \forall u \in \mathcal{D}^+, t \geq 0.$$

Since D is dense in $C_b(\mathcal{E})$ under pointwise convergence, (1.8) holds for every $u \in C_b(\mathcal{E})$ such that $\inf_{e\in\mathcal{E}} u(e) > 0$. Hence

$$(1.9) \qquad \log\left(\int \exp(u(e))S_t^*\nu(de)\right) \geq E_\nu(u), \quad \forall u \in C_b(\mathcal{E}), t \geq 0.$$

In term of entropy of p.m.'s, (1.9) means that $h(\nu; S_t^*\nu) = 0 \quad \forall t \geq 0$ (Lemma 2.2.5). But $h(\nu_1;\nu_2) = 0$ iff $\nu_1 = \nu_2$. Therefore $S_t^*\nu = \nu \quad \forall t \geq 0$, that is ν is invariant. □

COROLLARY 4.1.3: Suppose that $\lambda \in R, \lim_{t\to\infty}\lambda(t)/t = o$. If $\hat\mu$ is the only invariant p.m. for S_t, then $G_{\lambda,\mu} = \mathcal{E}$. In particular if for every $e \in \mathcal{E}$, there exists $t = t(e) > 0$ such that $q(t,e,de') \ll \hat\mu$ (or μ), then $\hat\mu$ is the only invariant p.m. for S_t.

Proof: All we have to prove is the last statement. So let $e \in \mathcal{E}$ be given and let $t > 0$ be such that $q(t,e,de') \ll \hat{\mu}$. Then

$$\lim_{s\to\infty} \frac{1}{s} \int_t^{s+t} S_u f(e) du = \lim_{s\to\infty} \int \left(\frac{1}{s} \int_0^s S_u f(e') du \right) q(t,e,de') = E_{\hat{\mu}}(f)$$

by the ergodic theorem, for every $f \in C_b(\mathcal{E})$. Clearly this implies that $\hat{\mu}$ is the only invariant p.m. for S_t. □

Example: Let $f \in C_b(\mathbb{R}^d)$ be periodic with period $u = (u_1, \cdots, u_d), u_i \geq 0$. Further let μ_U be the uniform p.m. on $U = [0,u_1] \times \cdots [0,u_d]$.

If $\mathcal{E} = \{\tau_x f = f(x+\cdot); x \in U\}$ with the topology of uniform convergence, and μ is the p.m. on $\mathcal{E}$ induced by μ_U , then $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \{\tau_x; x \in \mathbb{R}^d\}, \mu)$ is a dynamical system of type I, $\mathcal{E}$ is compact and for every $V \in M_I, q(t,e,de') \ll \hat{\mu} \quad \forall t > 0, e \in \mathcal{E}$. Let us prove the last assertion. Suppose that $A \in \mathcal{B}(\mathcal{E})$ is such that $\hat{\mu}(A) = 0$. Then $\hat{\mu}(A) = 0 = \int q(t,e,A)\hat{\mu}(de), \quad \forall t \geq 0$. Hence for every $t > 0, \exists x \in U$ so that $q(t, \tau_x f, A) = 0$. But for every $y \in U$ and $t > 0$,

$$q(t,\tau_y f, A) = P_0^{\tau_y f}\left(\tau_{x(t)+y} f \in A\right) = P_y^f\left(\tau_{x(t)} f \in A\right). \tag{1.10}$$

Next, we know that for every $y \in U, P_y^f \circ (x(t))^{-1}$ is equivalent to Lebesgue measure (Theorem 1.2.5). Since $q(t,\tau_x f, A) = 0$, (1.10) and the last remark imply that $q(t,\tau_y f, A) = 0 \quad \forall y \in U$, i.e. $q(t,e,A) = 0 \quad \forall e \in \mathcal{E}$. Therefore $q(t,e,de')$ is absolutely continuous with respect to $\hat{\mu}$, and in fact, $q(t,e,de')$ is equivalent to $\hat{\mu}$. Before closing this section, let us give another example where $\hat{\mu}$ is the only invariant p.m. for S_t.

A function $f \in C_b(\mathbb{R}^d)$ is called almost periodic (a.p. for short) if $\{\tau_x f = f(x+\cdot); x \in \mathbb{R}^d\}$ has compact closure K_f with respect to the topology of uniform convergence. It is easy to see that K_f has a structure of compact Abelian group compatible with its topology. Therefore there exists a unique invariant (with respect to $\{\tau_x : x \in \mathbb{R}^d\}$) p.m. μ_f on K_f, the so-called Haar measure. Since $\{\tau_x f; x \in \mathbb{R}^d\}$ is dense in K_f, μ_f is ergodic. If we set $\mathcal{E} = K_f, \mu = \mu_f$, then $(\mathcal{E}, \mathcal{B}(\mathcal{E}), \{\tau_x; x \in \mathbb{R}^d\}, \mu)$ is of type I.

The next Propositon is a well-known property of a.p. functions.

PROPOSITION 4.1.4 Any a.p. function f can be uniformly approximated by trigonometric polynomials, i.e. functions of the form $p(x) = a_0 + \sum_{k=1}^{n}(a_k \cos < x, c_k > + b_k \sin < x, c_k >)$ where $a_k, b_k \in \mathbb{R}^d, c_k \in \mathbb{R}^d \backslash \{0\}$, $1 \leq k \leq n, n \in \mathbb{N}$.

Moreover p is a.p. and $\int_{K_p} e(0)\mu_p(de) = a_0$.

LEMMA 4.1.5: Suppose f is a.p., and let $V \in C_b(\mathcal{E}), V > 0$, and $A \in S_d^+$ be given. If S_t is the semigroup corresponding to $(1/V)A$, then $\hat{\mu}$ is the only invariant p.m. for S_t, where $d\hat{\mu}/d_\mu(e) = V(e)/E_\mu(V), e \in \mathcal{E}$.

Proof: Set $T_t^e = \inf\{s > 0; \int_0^s V(x(u), e)du > t\}, t \geq 0, e \in \mathcal{E}$. For every $H \in C_b(\mathcal{E})$,

$$\frac{1}{t}\int_0^t S_s H(e)ds = E^{W_A}\left(\frac{1}{t}\int_0^{T_t^e} H \cdot V\left(\tau_{x(u)}e\right) du\right).$$

Hence $\lim_{t\to\infty} \frac{1}{t}\int_0^t S_s H(e)ds = E_{\hat{\mu}}(H) \quad \forall e \in \mathcal{E}, \quad \forall H \in C_b(\mathcal{E})$ if

$$\lim_{t\to\infty} \sup_{e\in\mathcal{E}} E^{W_A}\left(\left(\frac{1}{t}\int_0^t H(\tau_{x(u)}e)du - E_\mu(H)\right)^2\right) = 0, H \in C_b(\mathcal{E}). \tag{1.11}$$

Since $H(\tau_x e)$ is a.p. and $\{\tau_x f; x \in \mathbb{R}^d\}$ is dense in $\mathcal{E}$, (1.11) holds true if for every a.p. g

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} E^{W_A}\left(\left(\frac{1}{t}\int_0^t g(x(u)+x)du - \int_{K_g} e(0)\mu_g(de)\right)^2\right) = 0. \tag{1.12}$$

By Proposition 4.1.4, we only need to prove (1.12) for trigonometric polynomicals. So let $p(x) = a_0 + \sum_{k=1}^{m} a_k \cos < x, c_k > + \sum_{k=1}^{m} b_k \sin < x, c_k >$, where $a_k, b_k \in \mathbb{R}, c_k \in \mathbb{R}^d\backslash\{0\}, 1 \leq$ $k \leq m$.

Then

$$\sup_{x\in\mathbb{R}^d} E^{W_A}\left(\left(\frac{1}{t}\int_0^t p(x(u)+x)du\right) - a_0\right)^2 \leq 4/t \sum_{k=1}^{m} \frac{(|a_k| + |b_k|)}{< Ac_k, c_k >} \sup_{x\in\mathbb{R}^d} |p(x) - a_0|, t > 0.$$

Since $A \in S_d^+$ and $c_k \neq 0, \quad 1 \leq k \leq n$, we see that (1.12) holds for every trigonometric polynomial. Hence

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t S_u H(e)du = E_{\hat{\mu}}(f) \quad \forall e \in \mathcal{E}, \quad H \in C_b(\mathcal{E}),$$

which implies that $\hat{\mu}$ is unique. □

4.2 Large deviations for $\frac{1}{t}\int_0^t V(x(u),e)du$

Throughout this section, $S = (\mathcal{E}, \mathcal{B}(\mathcal{E}), \{\tau_x; x \in \mathbb{R}^d\}, \mu)$ is a dynamical system of type II or III, $V \in C_b(\mathcal{E})$ and P_x stands for the standard Wiener measure starting from $x \in \mathbb{R}^d$ at time 0, $P = P_0$.

We begin with a lemma which is fundamental in what follows.

LEMMA 4.2.1: Suppose that $\alpha \in R$ and $\int_1^\infty \frac{e^{-\theta\alpha(t)}}{t}dt < \infty, \forall\theta > \theta_0, \theta_0 \in [0,\infty)$. Then for every closed set $F \subset \mathbb{R}^1$,

$$(2.1)\qquad \begin{aligned} &\limsup_{t\to\infty} \frac{1}{\alpha(t)} \log P\left(\frac{1}{t}\int_0^t V(x(u),e)du \in F\right) \leq \\ &\theta_0 + \lim_{\delta\downarrow 0}\limsup_{t\to\infty} \frac{1}{\alpha(t)} \log P\otimes\mu\left(\frac{1}{t}\int_0^t V(x(u),e)du \in F^\delta\right) \quad \mu\text{ . a.s.} \end{aligned}$$

where $F^\delta = \{x \in \mathbb{R}; |x-y| < \delta \text{ for some } y \in F\}, \delta > 0$.

Proof: Let $\delta > 0$ and a be given and choose $1 < c < 1 + \frac{\delta}{2\|V\|}$. Then

$$\begin{aligned} &\mu\left(\limsup_{t\to\infty}\frac{1}{\alpha(t)}\log P\left(\frac{1}{t}\int_0^t V(x(u),e)du \in F\right) > a\right) \leq \\ &\mu\left(\sup_{c^n\leq t\leq c^{n+1}}\frac{1}{\alpha(t)}\log P\left(\frac{1}{t}\int_0^t V(x(u),e)du \in F\right) > a \quad i.o.\right) \leq \\ &\mu\left(P\left(\frac{1}{c^n}\int_0^{c^n} V(x(u),e)du \in F^\delta\right) > e^{\alpha(c^n)a} \quad i.o.\right) \text{ since} \\ &\sup_{c^n\leq t\leq c^{n+1}}\left|\frac{1}{t}\int_0^t V(x(u),e)du - \frac{1}{c^n}\int_0^{c^n} V(x(u),e)du\right| \leq 2(c-1)\|V\|, \end{aligned}$$

for every $e \in \mathcal{E}$ and $V \in C_b(\mathcal{E})$.

Choose $l_\delta > \limsup_{t\to\infty}\frac{1}{\alpha(t)}\log P\otimes\mu\left(\frac{1}{t}\int_0^t V(x(u),e)du \in F^\delta\right)$. Then for n large enough

$$(2.2)\qquad \mu\left(P\left(\frac{1}{c^n}\int_0^{c^n} V(x(u),e)du \in F^\delta\right) > e^{a\alpha(c^n)}\right) \leq e^{-(a-l_\delta))\alpha(c^n)}.$$

Next $\sum_n e^{-\theta\alpha(c^n)} < \infty$ for some $c > 1$ iff $\int_1^\infty \frac{e^{-\theta\alpha(t)}}{t} dt < \infty, \theta > 0$. Using Borel-Cantelli Lemma and (2.2) we get $\mu\left(\limsup_{t\to\infty} \frac{1}{\alpha(t)} \log P\left(\frac{1}{t}\int_0^t V(x(u),e)du \in F\right) \leq a\right) = 1$, when ever $a > \theta_0 + l_\delta$, for some $\delta > 0$. Letting $\delta \downarrow 0$, we obtain (2.1). □

LEMMA 4.2.2: Let F be the spectral measure corresponding to $V - E(V)$ i.e. $E_\mu(V(x)V(0)) - E_\mu^2(V(0)) = \int_{\mathbb{R}^d} e^{i<\lambda,x>} F(d\lambda), x \in \mathbb{R}^d$. If $\int |\lambda|^{-\beta} F(d\lambda) < \infty$ for some $\beta > 0$, then

$$(2.3) \qquad \limsup_{t\to\infty} \frac{1}{\log t} \log P\left(\left|\frac{1}{t}\int_0^t V(x(u),e)du - E_\mu(v)\right| \geq \delta\right) \leq -\frac{\min(2,\beta)}{2}, \mu \quad a.s.$$

Proof: First of all, $\log t \notin R$, but $\max(1, \log t) \in R$ and since they are equal for $t \geq e$, we will work with $\log t$. Now

$$P \otimes \mu\left(\left|\frac{1}{t}\int_0^t V(x(u),e)du - E_\mu(V)\right| \geq \delta\right) \leq$$

$$\frac{1}{\delta^2} E^{P\otimes\mu}\left(\left(\frac{1}{t}\int_0^t V(x(u),e)du - E_\mu(V)\right)^2\right) = \frac{2}{\delta^2 t^2}\int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-\frac{s}{2}|\lambda|^2} F(d\lambda)duds.$$

If $\beta \geq 2$, then $\frac{2}{\delta^2 t^2}\int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-\frac{s}{2}|\lambda|^2} F(d\lambda)duds \leq \frac{4}{\delta^2 t}\int \frac{1}{|\lambda|^2} F(d\lambda)$. Hence

$$\limsup_{t\to\infty} \frac{1}{\log t} \log P \otimes \mu\left(\left|\frac{1}{t}\int_0^t V(x(u),e)du - E_\mu(V)\right| \geq \delta\right) \leq -1.$$

Since

$$\int_0^\infty \frac{e^{-\theta \log t}}{t} dt = \frac{1}{\theta} < \infty \quad \forall\theta > 0, \text{ (2.3) follows from Lemma 4.2.1.}$$

Next suppose that $\beta \in (0,2)$. Then

$$\sup_{\theta>0} \theta^\beta e^{-\frac{u}{2}\theta^2} = \left(\frac{\beta}{u}\right)^{\beta/2} e^{-\beta/2}, u > 0.$$

Thus

$$\frac{2}{\delta^2 t^2}\int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-\frac{s}{2}|\lambda|^2} F(d\lambda)duds \leq \frac{2}{\delta^2} \frac{\beta^{\beta/2} e^{-\beta/2} t^{-\beta/2}}{(1-\beta/2)(2-\beta/2)} \int |\lambda|^{-\beta} F(d\lambda),$$

and

$$\limsup_{t\to\infty} \frac{1}{\log t} P \otimes \mu\left(\left|\frac{1}{t}\int_0^t V(x(u),e)du - E_\mu(V)\right| \leq \delta\right) \leq -\frac{\beta}{2}, \quad \forall\delta > 0$$

By Lemma 4.2.1 we conclude that (2.3) holds true. □

REMARK: Let $H = L^2(d\mu)$ with scalar product $< f, g >= E_\mu(fg)$, and let,

$$T_t f(e) = \int_{\mathbb{R}^d} \frac{e^{-\frac{|x|^2}{2t}}}{(2\pi t)^{d/2}} f(\tau_x e) dx, \quad f \in H.$$

Clearly $(T_t f, g) = (T_t g, f), f, g \in H$ and

$$(T_t f, f) = \int_{\mathbb{R}^d} e^{-\frac{t}{2}|x|^2} F_f(dx),$$

where F_f is the spectral measure corresponding to $f \in H$. Therefore $e_t = \tau_{x(t)} e$ is a reversible ergodic Markov process with invariant measure μ; its infinitesimal generator L is self-adjoint and

$$\text{domain of } (-L)^{1/2} = \mathcal{D}((-L)^{1/2}) = \{f \in H; \int |x|^{-2} F_f(dx) < \infty\}$$

(see Corollary 3.2.1).

It follows from Theorem 1.1.5 that for any $f \in \mathcal{D}((-L)^{1/2})$ $\frac{1}{\sqrt{n}} \int_0^{tn} f(\tau_{x(u)} e) du$ satisfies a functional central limit theorem relative to $P \otimes \mu$ with limiting variance $\sigma^2 = 4 \int |x|^{-2} F_f(dx)$.

In particular if $V \in C_b(\mathcal{E})$ satisfies the hypothesis of Lemma 4.2.2 with $\beta \geq 2$, then

$$\left(\frac{1}{\sqrt{n}} \int_0^{tn} V(x(u), e) du - t\sqrt{n} E_\mu(V), t \in [0,1] \right)$$

converges weakly as $n \to \infty$ to the Wiener process $\mathcal{W}_{\sigma^2}$ on $C([0,1]; \mathbb{R})$, where

$$\sigma^2 = 4 \int |x|^{-2} F_{V - E_\mu(V)}(dx).$$

We will now develop some tools to find upper bounds for $P \otimes \mu \left(\frac{1}{t} \int_0^t V(x(u), e) du \in F \right)$, where F is a closed subset of $\mathbb{R}$.

Our approach is based on the following observations: Suppose that $D \subset \mathbb{R}^d$ is a bounded domain (a domain is an open and connected nonempty set) and let

$$T_D = \inf\{t > 0, x(t) \in D^c\}.$$

Further let $p(t,F) = P \otimes \mu(< V, L_t > \in F)$, where $< V, L_t > = \frac{1}{t}\int_0^t V(x(u)e)du$, and $|A| = \int_A dx, A \in \mathcal{B}(\mathbb{R}^d)$. Then

$$\begin{aligned} p(t,F) &= E_\mu\left(\frac{1}{|D|}\int_D P_x(< V, L_t > \in F)\,dx\right) \\ &= E_\mu\left(\frac{1}{|D|}\int_D P_x(< V, L_t > \in F, T_D > t)\,dx\right) \\ &+ E_\mu\left(\frac{1}{|D|}\int_D P_x(< V, L_t > \in F, T_D \le t)\,dx\right). \end{aligned}$$

= (1) + (2) say.

If G is open, $G \subsetneq D$ and $|G| > 0$,

$$(2) \le \frac{1}{|D|}\int_G P_x(T_D \le t)\,dx + \left(1 - \frac{|G|}{|D|}\right) p(t,F).$$

Hence

$$\begin{aligned} p(t,F) \le \frac{|D|}{|G|} E_\mu\left(\frac{1}{|D|}\int_D P_x(< V, L_t > \in F, T_D > t)\,dx\right) \\ + \frac{1}{|G|}\int_G P_x(T_D \le t)\,dx. \end{aligned} \tag{2.4}$$

Similarly

$$\begin{aligned} E^{P\otimes\mu}\left(e^{\int_0^t V(x(u),e)du}\right) \le \frac{|D|}{|G|} E_\mu\left(\frac{1}{|D|}\int_D E_x\left(e^{\int_0^t V(x(u),e)du}; T_D > t\right) dx\right) \\ + \frac{e^{t\|V\|}}{|G|}\int_G P_x(T_D \le t)\,dx. \end{aligned} \tag{2.5}$$

For "nice" domains D, we know the asymptotic behaviour of $P_x((T_D \le t)$. We will now study the behaviour of $\int_D P_x(< V, L_t > \in F, T_D > t)\,dx$. To this end, suppose that $V : \mathbb{R}^d \to \mathbb{R}$ is bounded and measurable.

For any $\theta \in \mathbb{R}$,

$$\begin{aligned} \int_D P_x\left(\frac{1}{t}\int_0^t V(x(u))du \ge \theta, T_D > t\right) dx \le \\ \inf_{\lambda > 0}\left(e^{-\theta\lambda t}\int_D E_x\left(e^{\lambda\int_0^t V(x(u))du}; T_D > t\right) dx\right). \end{aligned} \tag{2.6}$$

Let $C_0^\infty(D) = \{f \in C_0^\infty(\mathbb{R}^d);$ support $(f) \subset D\}$. For every $f \in C_0^\infty(D)$, we define $\nabla f(x) = \left(\frac{\partial}{\partial x_1} f(x), \cdots, \frac{\partial}{\partial x_d} f(x)\right)$ and $\Delta_0 f(x) = \frac{1}{2}\sum_1^d \frac{\partial^2}{\partial x_k^2} f(x)$. Further let $H_0^1(D)$ be the completion of $C_0^\infty(D)$ with respect to the norm $\|f\|_1 = \left(\int_D |f^2(x)|dx + \frac{1}{2}\int_D |\nabla f(x)|^2 dx\right)^{1/2}$. As usual, $L^2(D) = L^2(D, dx)$ is the real Hilbert space with scalar product $(f,g) = \int_D f(x)g(x)dx, f,g \in L^2(D)$ (since we will be dealing with self-adjoint operators, we need only to consider real-valued functions).

Next we define the Dirichlet Laplacian $\Delta_D = \Delta$ to be the unique self-adjoint operator on $L^2(D)$ satisfying

(i) $\Delta f = \Delta_0 f, \forall f \in C_0^\infty(D)$;

(ii) domain of $(-\Delta)^{1/2} = H_0^1(D)$.

We will write $\mathcal{D}(D)$ to designate the domain of Δ.

We now state some properties of Δ. For more details, we refer the reader to [12].

PROPOSITION 4.2.3 Let D be a bounded domain and let $V : \mathbb{R}^d \to \mathbb{R}$ be bounded and measurable. Then

a) $\Delta + V$ is self-adjoint with domain $\mathcal{D}(\Delta)$;

b) $\forall f \in L^2(D), e^{t(\Delta+V)} f(x) = E_x\left(f(x(t))e^{\int_0^t V(x(u))du}; T_D > t\right), x \in D$;

c) $\{f \in \mathcal{D}(D); |f| \le 1, I(f) + (Vf,f) \le a\}$ is compact in $L^2(D)$, where $I(f) = (-\Delta f, f), |f|^2 = (f,f)$, and $a \in \mathbb{R}$;

d) $\forall t > 0, e^{t(\Delta+V)}$ maps $L^2(D)$ into $C_b(D)$ and if $f \ge 0$ $a.e., f \not\equiv 0$ on $D, e^{t(\Delta+V)} f(x) > 0 \quad \forall x \in D$;

e) $\Delta + V$ has a discrete spectrum $\{\lambda_k; k \ge 1\}$, where $\lambda_1 > \lambda_2 \ge \cdots \ge \lambda_k$, and there exists a unique $f \in C_b(D) \cap \mathcal{D}(D), f(x) > 0 \quad \forall x \in D$ and $(\Delta + V)f = \lambda_1 f$ on D.

f) $\lambda_1 = \sup_{f \in \mathcal{M}(D)} (Vf,f) - I(f) = \sup_{f \in \mathcal{C}(D)} (Vf,f) - I(f)$, where $\mathcal{M}(D) = \{f \in \mathcal{D}(D); |f| = 1\}$ and $\mathcal{C}(D) = C_0^\infty(D) \cap \mathcal{M}(D)$.

It follows from (2.6) and Proposition 4.2.3 that

$$\frac{1}{|D|}\int_D P_x\left(\frac{1}{t}\int_0^t V(x(u))du \ge \theta, T_D > t\right) \le \exp\left(-t\sup_{\lambda>0}(\lambda\theta - c(\lambda))\right),$$

where $c(\lambda) = c_V(\lambda) = \sup_{f \in \mathcal{M}(D)} \lambda(Vf,f) - I(f)$.

Using Proposition 4.2.3 and perturbation theory, we can see that c is convex, finite and everywhere differentiable with derivative $c'(\lambda) = (Vf_\lambda, f_\lambda)$, where $(\Delta + \lambda V)f_\lambda = c(\lambda)f_\lambda$, and f_λ satisfies e) of Proposition 4.2.3.

If we let $J_0(\theta) = \sup_{\lambda>0}(\lambda\theta - c(\lambda)), \theta \in \mathbb{R}$, then

$$(2.8) \qquad J_0(\theta) = \begin{cases} I(f_0), & \theta \le c'(0) \\ I(f_\lambda), & \theta = c'(\lambda), \lambda \ge 0 \\ \lim_{\lambda\uparrow\infty} I(f_\lambda), & \theta = \lim_{\lambda\uparrow\infty} c'(\lambda) = \theta_\infty < \infty \\ +\infty, & \theta > \theta_\infty \end{cases}$$

Next define $J_1(\theta) = \inf_{\substack{f\in\mathcal{M}(D) \\ (Vf,f)\ge\theta}} I(f)$, where $\inf_{f\in\emptyset} I(f) = +\infty$.

LEMMA 4.2.4 $J_0 = J_1$.

Proof: Clearly $J_1(\theta) \ge J_0(\theta)$ $\forall\theta \in \mathbb{R}$ and (2.8) yields

$$J_1(\theta) \le J_0(\theta), \quad \forall\theta \in \mathbb{R}\backslash\{\theta_\infty\}.$$

If $J_0(\theta_\infty) = +\infty$ we are done. So suppose that $J_0(\theta_\infty) < \infty$. By Proposition 4.2.3 c) one can find $\lambda_n \uparrow \infty$ so that $f_{\lambda_n} \to f, f \in \mathcal{M}(D), (Vf,f) = \theta_\infty$ and $I(f) \le J_0(\theta_\infty)$ Hence $J_1(\theta_\infty) \le I(f) \le J_0(\theta_\infty)$ which completes the proof. As a by-product of our proof we get

$$(2.9) \qquad J(x) := \sup_{\lambda\in\mathbb{R}}(\lambda x - c(\lambda)) = \inf_{\substack{f\in\mathcal{M}(D) \\ (Vf,f)=x}} I(f)$$

REMARK: It follows from Proposition 4.2.3 f) that J is a decreasing function of D.

Next, combining (2.7) - (2.9) we obtain

$$(2.10) \qquad \frac{1}{|D|}\int_D P_x\left(\frac{1}{t}\int_0^t V(x(u))du \in F, T_D > t\right)dx \le 2e^{-t\inf_{x\in F} J(x)}, t > 0,$$

$$\text{and} \quad \inf_{x\in F} J(x) = \inf_{\substack{f\in\mathcal{M}(D) \\ (Vf,f)\in F}} I(f), F \text{ closed}$$

We will now return to $V \in C_B(\mathcal{E})$. J as defined by (2.9) depends on x and e.

PROPOSITION 4.2.5 J is jointly measurable in x and e.

Proof: Since $J(e,x) = \sup_{\lambda\in\mathbb{R}}(\lambda x - c(e,\lambda))$, we only need to show that c is jointly measurable. This will be done if we prove

$$c(e,\lambda) = \lim_{t\to\infty}\frac{1}{t}\log\left(\frac{1}{|D|}\int_D E_x\left(e^{\lambda\int_0^t V(x(u),e)du}; T_D > t\right)dx\right). \tag{2.11}$$

Since $\frac{1}{|D|}\int_D E_x\left(e^{\lambda\int_0^t V(x(u),e)de}; T_D > t\right)dx \le e^{tc(e,\lambda)}$, we have
$\limsup_{t\to\infty}\frac{1}{t}\log\left(\frac{1}{|D|}\int_D E_x\left(e^{\lambda\int_0^t V(x(u,e))du}; T_D > t\right)dx\right) \le c(e,\lambda)$.

Next let $f \in C(D)$ be given and set $r = \sup_x f^2(x) < \infty$. Then

$$\frac{1}{|D|}\int_D E_x\left(e^{\int_0^t V(x(u),e)du}; T_D > t\right) \ge \frac{1}{|D|r}\left(e^{t(\Delta+V_e)}f, f\right). \tag{2.12}$$

By Jensen's inequality, (2.12) yields

$$\liminf_{t\to\infty}\frac{1}{t}\log\left(\frac{1}{|D|}\int_D E_x\left(e^{\int_0^t V(x(u),e)du}\right)dx\right) \ge (V_e f, f) - I(f) \tag{2.13}$$

Since (2.13) holds for every $f \in C(D)$, and $c(e,\lambda) = \sup_{f\in C(D)} \lambda(V_e f, f) - I(f)$, (2.11) follows by Proposition 4.2.3 f). □

The proof of the next lemma follows closely the proof of the last proposition, so we omit it; L.A. Pastur used it to find the asymptotic behaviour of some Wiener integrals (see [11]).

LEMMA 4.2.6 If $f \in C_0^\infty(\mathbb{R}^d)$ set $k(f) = \int |f(y)|dy \cdot \sup|f(y)|$. For every $V \in C_b(\mathcal{E})$ and $f \in C = \{f \in C_0^\infty(\mathbb{R}^d); \int f^2(x)dx = 1\}$,

$$E^{P\otimes\mu}\left(\exp\left(\int_0^t V(x(u),e)du\right)\right) \ge \frac{1}{k(f)}\exp(-tI(f))\,E_\mu\left(\exp\left(t\int V(x,e)f^2(x)dx\right)\right), t \ge 0.$$

We are now in a position to prove

LEMMA 4.2.7: If $V \in C_b(\mathcal{E})$,

$$\lim_{t\to\infty}\frac{1}{t}\log E^{P\otimes\mu}\left(\exp\left(\int_0^t V(x(u),e)du\right)\right) = \sup_{f\in C}\sup\{a; \mu((V_e f, f) - I(f) > a) > 0\}. \tag{2.14}$$

Proof: From Lemma 4.2.6

$$\liminf_{t\to\infty}\frac{1}{t}\log E^{P\otimes\mu}\left(\exp\left(\int_0^t V(x(u),e)du\right)\right) \geq$$
$$\liminf_{t\to\infty}\frac{1}{t}\log E_\mu\left(\exp\left(t\left((V_e f,f)-I(f)\right)\right)\right) =$$
$$\sup\{a:\mu((V_e f,f)-I(f)>a)>0\} \quad \forall f\in C.$$

Next, using (2.5) we get

$$E^{P\otimes\mu}\left(\exp\left(\int_0^t V(x(u),e)du\right)\right) \leq \frac{|D|}{|G|}E_\mu\left(e^{tc(e,1)}\right)+\frac{e^{t\|V\|}}{|G|}\int_G Px(T_D\leq t)\,dx.$$

If we take, $D=B(0,n),G=B(0,\theta n),\theta\in(0,1)$ then $c_n(e,1)\uparrow\sup_{f\in C}(V_e f,f)-I(f)$ and letting n goes to infinity and then letting $\theta\uparrow 1$, we obtain

$$E^{P\otimes\mu}\left(\exp\left(\int_0^t V(x(u),e)du\right)\right) \leq E_\mu\left(\exp\left(t\left(\sup_{f\in C}(V_e f,f)-I(f)\right)\right)\right) =$$
$$\exp\left\{t\left(\sup_{f\in C}(V_e f,f)-I(f)\right)\right\} \quad \mu \quad a.s.$$

since μ is ergodic and $\sup_{f\in C}(V_e f,f)-I(f)$ is measurable and invariant under $\{\tau_x;x\in\mathbb{R}^d\}$.

Therefore, to prove (2.14) we only need to show that

(2.15) $$\sup_{f\in C}(V_e f,f)-I(f)\leq\sup_{f\in C}\sup(a:\mu((V_e f,f)-I(f)>a)>0),\mu \quad a.s.$$

Set $\ell=\sup_{f\in C}\sup(a;\mu((V_e f,f)-I(f)>a)>0)$. Suppose first that S is of type II. Then for every $f\in C,\mu(e;(V_e f,f)-I(f)\leq\ell)=1$. Since $e\to(V_e f,f)$ is continuous $\{e;(V_e f,f)-I(f)\leq\ell\}$ is closed and has μ measure 1. Hence it is equal to $\mathcal{E}$ for every $f\in C$, i.e.

$$\sup_{f\in C}\sup_{e\in\mathcal{E}}(V_e f,f)-I(f)\leq\sup_{f\in C}\sup(a;\mu((V_e f,f)-I(f)>a)>0).$$

Next suppose that S is of type III. Then $\mathcal{E}=\Omega\times T_d$, $\mu=\mu_1\otimes U$ and $\tau_x(\omega,u)=(\tau_{[x+u]}\omega,x+u-[x+u])$. Since $\int V(\tau_x(\omega,u))f^2(x)dx=\int V(\tau_{[x]}\omega,x-[x])f^2(x-u)dx$, we have $\sup_{f\in C}(V_{(\omega,u)}f,f)-I(f)=\sup_{f\in C}(V_{(\omega,0)}f,f)-I(f)=\sup_{f\in C}\sup_{u\in T_d}(V_{(\omega,u)}f,f)-I(f)$. Moreover

$$\mu((\omega,u);(V_{(\omega,u)}f,f)-I(f)>a)>0 \text{ for some } f\in C \iff$$
$$\mu_1(\omega;(V_{(\omega,u)}f,f)-I(f)>a)>0 \text{ for some } u\in T_d \text{ and some}$$
$$f\in C \iff \mu_1(\omega;(V_{(\omega,0)}f,f)>a)>0 \text{ for some } f\in C.$$

Hence

$$\sup_{f\in\mathcal{C}}\sup(a;\mu((V_e f,f)-I(f)>a)>0)$$

$$\sup_{f\in\mathcal{C}}\sup\left(a;\mu_1\left((V_{(\omega,0)}f,f)-I(f)>a\right)>0\right)=\ell.$$

As before, $\{\omega\in\Omega,(V(\omega,0)f,f)-I(f)\leq\ell\}$ is a closed set of μ_1-measure 1 for every $f\in\mathcal{C}$, which implies that

$$\sup_{f\in\mathcal{C}}\sup_{\omega\in\Omega}\left(V_{(\omega,0)}f,f\right)-I(f)\leq\ell.$$

Therefore we have proved that (2.15) holds and

$$(2.16)\qquad \sup_{f\in\mathcal{C}}\sup_{e\in\mathcal{E}}(V_e f,f)-I(f)=\sup_{f\in\mathcal{C}}\sup(a;\mu((V_e f,f)-I(f)>a)>0)\qquad\square$$

THEOREM 4.2.8

$$(2.17)\qquad \limsup_{t\to\infty}\frac{1}{t}\log p(t,F)\leq-\sup_{\delta>0}\inf_{\substack{f\in\mathcal{C}\\ \mu((V_e f,f)\in F^\delta)>0}}I(f),\quad F\text{ closed.}$$

If S is of type II or if $d=1$ (and S is of type II or III), then

$$(2.18)\quad \liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{1}{t}\int_0^t V(x(u),e)du\in\mathcal{O}\right)\geq-\inf_{\substack{f\in\mathcal{C}\\ \mu((V_e f,f)\in\mathcal{O})>0}}I(f)\quad\mu\quad a.s.,\mathcal{O}\text{ open}$$

Proof: Let $\mathcal{O}$ open be given. $C_0^\infty(D)$ being dense in $H_0^1(D)$,

$$\inf_{\substack{f\in\mathcal{M}(D)\\ (V_e f,f)\in\mathcal{O}}}I(f)=\inf_{\substack{f\in\mathcal{C}(D)\\ (V_e f,f)\in\mathcal{O}}}I(f).$$

Since $\inf_{\substack{f\in\mathcal{C}\\ (V_e f,f)\in\mathcal{O}}}I(f)=\underline{J}(e,\mathcal{O})$ is the limit when $D\uparrow\mathbb{R}^d$ of the measurable functions $\inf_{\substack{f\in\mathcal{M}(D)\\ (V_e f,f)\in\mathcal{O}}}I(f)$, $\underline{J}(e,\mathcal{O})$ is measurable; moreover $\underline{J}(\tau_x e,\mathcal{O})=\underline{J}(e,\mathcal{O})\quad\forall x\in\mathbb{R}^d, e\in\mathcal{E}$. By ergodicity $\underline{J}(e,\mathcal{O})=cte\quad\mu\quad a.s$ and it is not difficult to see that in fact $\underline{J}(e,\mathcal{O})=\inf_{\substack{f\in\mathcal{C}\\ \mu((V_e f,f)\in\mathcal{O})>0}}I(f),\quad\mu\quad a.s.$ With this identification, (2.17) can be proved the same way we proved (2.14).

Under our hypotheses, we have from [3]

$$\liminf_{t\to\infty}\frac{1}{t}\log P\left(\frac{1}{t}\int_0^t V(x(u),e)du\in\mathcal{O}\right)\geq-\underline{J}(e,\mathcal{O}),\quad e\in\mathcal{E},\mathcal{O}\text{ open.}$$

Since $\underline{J}(e, \mathcal{O}) = \inf_{\substack{f \in C \\ \mu((V_e f, f) \in \mathcal{O}) > 0}} I(f) \quad \mu \quad a.s.$, (2.18) follows. □

COROLLARY 4.2.9 Suppose that the dynamical system is of type II or suppose that $d = 1$. Then

$$\mu\left(\limsup_{t\to\infty} \frac{1}{t} \log P\left(\left|\frac{1}{t}\int_0^t V(x(u), e)du - E_\mu(V)\right| \geq \delta\right) < 0 \quad \forall \delta > 0\right) \text{ is}$$

equal to 0 or 1 according as $\lim_{a\downarrow 0} \sup_{f \in C_a} \sup_{e \in \mathcal{E}} |(V_e f, f) - E_\mu(V)|$ is greater than or equal to 0, where $C_a = \{f \in C; I(f) \leq a\}, a > 0$.

Moreover

$$\tilde{c}(\lambda) = \sup_{f \in C} \sup_{e} \lambda(V_e f, f) - I(f)$$

is differentiable at 0 iff

$$\lim_{a\downarrow 0} \sup_{f \in C_a} \sup_{e} |(V_e f, f) - E_\mu(V)| = 0$$

Proof: Without loss of generality we may suppose that $E_\mu(V) = 0$. It follows from Lemma 4.2.1 and Theorem 4.2.8 that

$$\mu\left(\lim_{t\to\infty} \sup \frac{1}{t} \log P\left(\left|\frac{1}{t}\int_0^t V(x(u), e)du\right| \geq \delta\right) < 0 \quad \forall \delta > 0\right) = 0 \text{ or } 1$$

according as

$$\inf_{\substack{f \in C \\ \mu(|(V_e f, f)| > \delta) > 0}} I(f) := J_2(\delta)$$

is equal to 0 for some $\delta > 0$ or $J_2(\delta) > 0 \quad \forall \delta > 0$.

Now if $J_2(\delta) > a > 0$, then $\mu(|(V_e f, f)| \leq \delta) = 1 \quad \forall f \in C_a$. From the proof of Lemma 4.2.7 we conclude that $\sup_{\substack{f \in C \\ I(f) \leq a}} \sup_{e \in \mathcal{E}} |(V_e f, f)| \leq \delta > 0$. Hence $J_2(\delta) > 0 \quad \forall \delta > 0$ implies

(2.19) $$\lim_{a\downarrow 0} \sup_{f \in C_a} \sup_{e} |(V_e f, f)| \leq \delta, \quad \forall \delta > 0$$

Clearly (2.19) implies that $J_2(\delta) > 0 \quad \forall \delta > 0$.

To complete the proof, we just have to show

$$\lim_{\lambda\downarrow 0} \bar{c}(\lambda)/\lambda = \lim_{a\downarrow 0} \sup_{f\in C_a} \sup_{e} (V_e f, f) \tag{2.20}$$

for if (2.20) holds, it also holds for $-V$. Thus

$$\lim_{\lambda\downarrow 0} \bar{c}(-\lambda)/(-\lambda) = \lim_{a\downarrow 0} \inf_{f\in C_a} \inf_{e} (V_e f, f) \tag{2.21}$$

Hence $\bar{c}$ is differentiable at 0 iff (2.20) and (2.21) are equal, which in turn is equivalent to (2.19).

Since $\bar{c}$ is convex and $\bar{c}(0) = 0$, $\frac{\bar{c}(\lambda)}{\lambda} \downarrow \ell$. Therefore if $\epsilon > 0$ is given, one can find $\lambda > 0$ so that $(V_e f, f) \leq I(f)/\lambda + \ell + \epsilon, \quad \forall f \in C, \forall e \in \mathcal{E}$. Thus

$$\lim_{a\downarrow 0} \sup_{f\in C_a} \sup_{e} (V_e f, f) \leq \ell + \epsilon, \quad \forall \epsilon > 0. \tag{2.22}$$

On the other hand, for every $\epsilon > 0, \exists f_n \in C, e_n \in \mathcal{E}$ such that $(V_{e_n} f_n, f_n) - nI(f_n)) \geq \ell - \epsilon \quad \forall n$. Hence $I(f_n) \leq \frac{\|V\|+|\ell-\epsilon|}{n}$, and $(V_{e_n} f_n, f_n) \geq \ell - \epsilon$. Thus

$$\lim_{a\downarrow 0} \sup_{f\in C_a} \sup_{e\in\mathcal{E}} (V_e f, f) \geq \ell - \epsilon \quad \forall \epsilon > 0. \tag{2.23}$$

Therefore (2.20) follows from (2.22) and (2.23). □

REMARK:

1) Corollary 4.2.9 suggests that for "random systems" S (in opposition to deterministic systems), the rate $\lambda(t) = t$ gives rise to phase transition.
2) When $\mathcal{E} = K_f, f \cdot a.p.$, it is easy to see that $\bar{c}$ is differentiable at 0.

We will now find sufficient conditions which guarantee that $\lim_{t\to\infty} \sup \frac{1}{\lambda(t)} \log p(t, F) < 0$, F closed, $E_\mu(V) \in F^c, \lambda \in R$.

To this end we only need to consider $F = [\theta, \infty), \theta > E_\mu(V)$. Since we want to use (2.4) to get upper bounds, we see that D must increase with t; a natural candidate is $\alpha(t)D = \{\alpha(t)x; x \in D\}$.

So let D be a bounded domain, and take $G = B(x,r)$, where $B(x,2r) \subset D, x \in D, r > 0$. Then for any $\alpha > 0$.

$$\frac{1}{|\alpha G|}\int_{\alpha G} P_x(T_{\alpha D} \le t)\,dx = \frac{1}{|G|}\int_G P_x\left(T_D \le t/\alpha^2\right) \le 2de^{-\frac{r^2\alpha^2}{2dt}}.$$

Let

$$(2.24) \qquad -A(D) = \inf_{\substack{G\subset D\\ G \text{ open}, G\neq\emptyset}} \limsup_{t\to\infty} \frac{1}{t}\log\left(\frac{1}{|G|}\int_G P_x(T_D \le \frac{1}{t})dx\right).$$

Then $A(D) > 0$ and if we set $\alpha(t) = (t\lambda(t))^{1/2}$, we get from (2.4):

$$(2.25) \qquad \begin{aligned} &\limsup_{t\to\infty} \frac{1}{\lambda(t)}\log P\otimes\mu\left(\frac{1}{t}\int_0^t V(x(u),e)du \ge \theta\right) \le \\ &\max(-A(D), \quad \limsup_{t\to\infty}\frac{1}{\lambda(t)}\log E_\mu\left(e^{-tJ_t}\right)), \text{ where} \\ &J_t = \inf\{I(f); \quad f\in\mathcal{M}(\alpha(t)D), (V_e f, f) \ge \theta\}. \end{aligned}$$

Next let

$$(2.26) \qquad \mathcal{I}_\lambda(a,\theta,D) = -\limsup_{t\to\infty}\frac{1}{\lambda(t)}\log\mu\left(J_t \le a\frac{\lambda(t)}{t}\right), a > 0$$

Then for every $m,n \in \mathbb{N}, \mathrm{E}_\mu(e^{-tJ_t}) \le$

$$\sum_{k=1}^m E_\mu\left(e^{-tJ_t}, \frac{t}{\lambda(t)}J_t \in \left[\frac{k-1}{n},\frac{k}{n}\right]\right) + E_\mu\left(e^{-tJ_t}; J_t \ge \frac{\lambda(t)m}{tn}\right)$$

$$\le \sum_{k=1}^m e^{-\lambda(t)\frac{(k-1)}{n}}\mu\left(J_t \le \frac{\lambda(t)k}{tn}\right) + e^{-\lambda(t)\frac{m}{n}}.$$

Hence

$$\limsup_{t\to\infty}\frac{1}{\lambda(t)}\log E_\mu\left(e^{-tJ_t}\right) \le \max\left(-\frac{m}{n}, \frac{1}{n} - \min_{1\le k\le n}\left(\mathcal{I}_\lambda(k/n) + \frac{k}{n}\right)\right)$$

$$\le \max\left(-\frac{m}{n}, \frac{1}{n} - \inf_{a>0}\left(\mathcal{I}_\lambda(a,\theta,D) + a\right)\right).$$

Letting m and n go to infinity, it follows from the above and (2.25) that

$$(2.27) \qquad \begin{aligned} &\limsup_{t\to\infty}\frac{1}{\lambda(t)}\log P\otimes\mu\left(\frac{1}{t}\int_0^t V(x(u),e)du \ge \theta\right) \le \\ &-\sup_D \min\left(A(D), \inf_{a>0}\left(a + \mathcal{I}_\lambda(a,\theta,D)\right)\right) \end{aligned}$$

where the sup is taken over bounded domains.

The next step consists in studying $\mu\left(J_t \leq \frac{\lambda(t)}{t}a\right), a > 0$.

Suppose that $V : \mathbb{R}^d \to \mathbb{R}$ is measurable and let D be a bounded domain. Then

$$\inf_{\substack{f\in\mathcal{M}(D)\\(Vf,f)\geq\theta}} I(f) \leq a \iff \sup_{f\in K_a}(Vf,f) \geq \theta, \quad \text{where}$$
$$K_a = \{f \in \mathcal{M}(D); I(f) \leq a\}.$$

Let $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ be the eigenvalues of $-\Delta = -\Delta_D$, and let $f_1, \cdots, f_k, \cdots$ be the corresponding eigenvectors. Further let $N(\lambda) = \{\lambda_k; \lambda_k \leq \lambda\}, l > 0$ and $P_\lambda =$ projection on the subspace generated by $\{f_k; \lambda_k \leq \lambda\}$.

Since

$$|(Vf,f) - (Vg,g)| \leq \|V\|(|f| + |g|)(|f-g|), \quad f,g \in L^2(D),$$

and

$$|f - P_\lambda f|^2 \leq I(f)/\lambda, \quad f \in \mathcal{D}(D),$$

we get

$$\sup_{f\in K_a}(Vf,f) \geq \theta \Rightarrow \sup_{f\in P_{a/\epsilon^2}(K_a)}(Vf,f) \geq \theta - 2\epsilon\|V\|$$

Now $P_{a/\epsilon^2}(K_a)$ is isometrically isomorphic to a compact subset of the closed unit ball of $\mathbb{R}^{N(a/\epsilon^2)}$.

For any $m \in \mathbb{N}$, let $n(\epsilon, m)$ be the minimal numbers of closed balls of radius ϵ required to cover the closed unit ball in $\mathbb{R}^m$. Then $\frac{1}{\epsilon^m} \leq n(\epsilon, m) \leq \left(\frac{(1+2\epsilon)}{\epsilon}\right)^m, \epsilon > 0, m \in \mathbb{N}$.

It follows that $P_{a/\epsilon^2}(K_a)$ can be covered by $n(\epsilon, N(a/\epsilon^2))$ closed balls of radius 3ϵ with centers in K_a; since $C_0^\infty(D)$ is dense in $H_0^1(D)$, we see that for any $a' > a$ one can find

$$f_1, \cdots, f_{n(\epsilon,N(a/\epsilon^2))}, f_i \in C(D) \text{ and } I(f_i) < a', \text{ so that}$$

$$\sup_{f\in K_a}(Vf,f) \geq \theta \Rightarrow \max_{1\leq i\leq n(\epsilon,N(a^2/\epsilon^2))}(Vf_i,f_i) \geq \theta - 9\epsilon\|V\|.$$

Note that the choice of $\{f_i\}$ does not depend on V.

Since $\{\lambda_k/\alpha^2\}$ are the eigenvalues of $-\Delta_{(\alpha D)}$, we obtain:

LEMMA 4.2.10: Let D be a domain and let $\alpha(t) = (t\lambda(t))^{1/2}$. Then

$$\mu\left(J_t \leq a\frac{\lambda(t)}{t}\right) \leq \left(\frac{(1+2\epsilon)}{\epsilon}\right)^{N(\lambda^2(t)a/\epsilon^2)} \inf_{a'>a} \sup_{\substack{f\in C(\alpha(t)D) \\ I(f)\leq a'}} \mu\left(e;(V_e f, f) \geq \theta - 9\epsilon\|V\|\right),$$

$a, \epsilon > 0, \theta \in \mathbb{R}, \lambda \in R$.

DEFINITION 4.2.11 We will say that a bounded set $A \subset \mathbb{R}^d$ is contented if for any $\epsilon > 0$, one find two sets of disjoint sets $\{I_k\}_{k=1}^n \{J_k\}_{k=1}^m$, where each I_k, J_k is of the form $(a,b], \cup_1^m J_k \subset A \subset \cup_1^n I_k$, and $\sum_1^n |I_k| \leq \epsilon + \sum_1^m |J_k|$.

LEMMA 4.2.12 (Weyl's Lemma). If D is a contented domain,

$$\lim_{\lambda\to\infty} N(\lambda)/\lambda^{d/2} = \frac{|D|}{(2\pi)^{d/2}}\Gamma\left(\frac{d+2}{2}\right)^{-1}, \text{ where } \Gamma(\cdot) \text{ is the}$$

gamma function. Moreover for any bounded domain D, $N(\lambda) \leq |D|\lambda^{d/2}\left(\frac{e}{\pi d}\right)^{d/2}$.

Proof: Let D be a bounded domain, and let $0 < \lambda_1 < \lambda_2 \leq \cdots$, be the eigenvalues of $-\Delta_D$. By Proposition 4.2.3,

$$e^{t\Delta_D} f(x) = E_x\left(f(x(t)); T_D > t\right), f \in L^2(d), x \in D.$$

Since $P_x(x(t) \in A, T_D > t) \leq P_x(x(t) \in A), x \in D, A \subset D, A \in \mathcal{B}(\mathbb{R}^d)$, we see that $e^{t\Delta_D} f(x) = \int_D f(y)P_D(t,x,y)dy$, and $P_D(t,x,y) \leq \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}}, x, y \in D, t > 0$.

It follows that $\int_0^\infty e^{-\lambda t} dN(\lambda) = \sum_k e^{-\lambda_k t} = \int_D P_D(t,x,x)dx \leq \frac{|D|}{(2\pi t)^{d/2}}, t > 0$. Hence $N(\lambda) \leq |D|\lambda^{d/2}\left(\frac{e}{\pi d}\right)^{d/2}, \lambda > 0$.

Next suppose that $D = \prod_1^d (a_k, b_k), a_k < b_k$. Then it is easy to see that the eigenvalues are $\left\{\frac{\pi^2}{2}\sum_1^d j_k^2/(b_k - a_k), j_k \geq 1\right\}$.

Thus $\lim_{t\downarrow 0} t^{d/2} \int_0^\infty e^{-t\lambda} dN(\lambda) = \prod_1^d \left(\frac{b_k - a_k}{\sqrt{2\pi}}\right) = \frac{|D|}{(2\pi)^{d/2}}$.

If D is also contented, for any $\epsilon > 0$, one can find disjoint open intervals $\{I_k\}_{k=1}^n$ such that $\sum_1^n |I_k| \geq |D| - \epsilon, I_k \subset D, 1 \leq k \leq n$. Now

$$\int_0^\infty e^{-\lambda t} dN(\lambda) = \int_D P_D(t,x,x)dx \geq \sum_{k=1}^n \int_{Ik} P_D(t,x,x)dx \geq \sum_{k=1}^n \int_{Ik} P_{I_k}(t,x,x)dx.$$

Therefore $\liminf_{t\downarrow 0} t^{d/2} \int_0^\infty e^{-\lambda t} dN(\lambda) \geq \frac{|D| - \epsilon}{(2\pi)^{d/2}}$. Since the last inequality holds for every $\epsilon > 0$ and

$$\limsup_{t\downarrow 0} t^{d/2} \int_0^\infty e^{-\lambda t} dN(\lambda) \leq \frac{|D|}{(2\pi)^{d/2}},$$

we get

$$\lim_{t\downarrow 0} t^{d/2} \int_0^\infty e^{-\lambda t} dN(\lambda) = \frac{|D|}{(2\pi)^{d/2}} \quad \text{if } D \text{ is contented} \tag{2.28}$$

Hence

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{d/2}} = \frac{|D|}{(2\pi)^{d/2}} \left(\Gamma\left(\frac{d+2}{2}\right)\right)^{-1}$$

follows from (2.28) and Karamata's Theorem. □

DEFINITION 4.2.13 If D is bounded domain of $\mathbb{R}^d$, $a > 0, \theta > E_\mu(V), \lambda \in R_\circ$ and $\alpha(t) = (t\lambda(t))^{1/2}$ let

$$\gamma_\lambda(a,\theta,D) = -\limsup_{t\to\infty} \frac{1}{\lambda(t)^d} \log \left(\sup_{\substack{f \in C(\alpha(t)D) \\ I(f) \leq a\frac{\lambda(t)}{t}}} \mu(e; (V_e f, f) \geq \theta) \right).$$

and let $\kappa_\lambda(a,\theta,D) =$

$$\max\left(0, \sup_{0<\epsilon<\frac{\theta - E_\mu(V)}{9\|V\|}} \left(\sup_{a'>a} \left(\gamma_\lambda(a', \theta - 9\epsilon\|V\|, D) - \frac{a^{d/2}|D|}{(2\pi\epsilon^2)^{d/2}} \left(\Gamma\left(\frac{d+2}{2}\right)\right)^{-1} \log\left(\frac{2(1+2\epsilon)}{\epsilon}\right)\right)\right)\right).$$

The next Theorem follows directly from Lemma 4.2.10 and Lemma 4.2.12, so we omit the proof.

THEOREM 4.2.14 Let D be contented domain of $\mathbb{R}^d$ and let $V \in C_b(\mathcal{E})$ be given. Further let $\mathcal{I}_\lambda$ be defined as in (2.26).

If $d = 1$, $\mathcal{I}_\lambda(a,\theta,D) \geq \kappa_\lambda(a,\theta,D)$, $a > 0, \theta > E_\mu(V)$.

If $d > 1$, $\mathcal{I}_\lambda(a,\theta,D) = +\infty$ whenever $\kappa_\lambda(a,\theta,D) > 0$, $a > 0, \theta > E_\mu(V)$.

COROLLARY 4.2.15 Suppose that the hypotheses of Theorem 4.2.14 are satisfied. If $d = 1$

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P \otimes \mu \left(\frac{1}{t} \int_0^t V(x(u), e) du \geq \theta \right) \leq$$
$$- \sup_D \min \left(A(D), \inf_{a>0} (a + \kappa_\lambda(a, \theta, D)) \right), \quad \theta > E_\mu(V).$$

If $d > 1$,

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P \otimes \mu \left(\frac{1}{t} \int_0^t V(x(u), e) du \geq \theta \right) \leq$$
$$- \sup_D \min \left(A(D), \sup(a \geq 0; \kappa_\lambda(a, \theta, D) > 0) \right), \quad \theta < E_\mu(V),$$

where we define $\kappa_\lambda(0, \theta, D) = +\infty, \quad \theta > E_\mu(V)$.

Proof: All these results are consequences of (2.27) and Theorem 4.2.14. □

Combining Corollary 4.2.15 and Lemma 4.2.1, we obtain

COROLLARY 4.2.16 Suppose that $\lambda \in R$, $\int_1^\infty \frac{e^{-x\lambda(t)}}{t} dt < \infty$, $\forall x > 0$ and $V \in C_b(\mathcal{E})$. Then

$$\mu \left(e; \limsup_{t\to\infty} \frac{1}{\lambda(t)} log P \left(\frac{1}{t} \int_0^t V(x(u), e) du > \theta \right) < 0 \quad \forall \theta > E_\mu(V) \right) = 1$$

if for every $\theta > E_\mu(V)$, one can find a contented domain D (which may depend on θ) such that $\kappa_\lambda(a, \theta, D) > 0$ for some $a > 0$.

REMARK: If we apply Corollary 4.2.15 to $\lambda(t) = t$, we get (2.17).

At this point, we would like to know if our method is good enough to determine a critical rate, where a critical rate is defined as follows: $\lambda \in R$ is a critical rate if

$$\limsup_{t\to\infty} \frac{1}{\lambda(t)} \log P \otimes \mu \left(\left| \frac{1}{t} \int_0^t V(x(u), e) du - E_\mu(V) \right| \geq \theta \right) < 0 \quad \forall \theta > 0$$

and if $$\lim_{t\to\infty} \frac{1}{\alpha(t)} \log P \otimes \mu \left(\left| \frac{1}{t} \int_0^t V(x(u), e) du - E_\mu(V) \right| \geq \theta \right) = 0$$

for some $\theta > 0$ whenever $\alpha \in R$ and $\lim_{t\to\infty} \lambda(t)/\alpha(t) = 0$.

We believe that when $d = 1$, our method gives a critical rate (when it exists), but when $d > 1$, our method may fail. For the rest of the section we will study the following model of dynamical system of type III: we will take $\Omega = \{\omega : Z^d \to \mathbb{R}\}$ (with the product topology) $\tau_k\omega(\cdot) = \omega(k + \cdot), k \in Z^d$ and μ such that the r.v.'s $\omega(k)$ are i.i.d; moreover we will suppose that $V(\omega, u) = v(\omega(0)), \forall \omega \in \Omega, u \in T_d$, where $v \in C_b(\mathbb{R})$.

Since no topology will be involved in our calculations there is no loss of generality if we consider instead that $(\Omega, \mathcal{F}, \mu)$ is a probability space, $\{\xi_k\}_{k \in Z^d}, U$ are random variables defined on $\Omega, \{\xi_k\}_{k\in Z^d}$ are i.i.d, bounded and independent from U which is uniformly distributed over $[0,1)^d$; we will also assume that $E(\xi_k) = 0, E(\xi_k^2) = 1$, where E will denote expectation with respect to $P \otimes \mu$. Finally set $V(x) = \xi_{[x+y]}, x \in \mathbb{R}^d$, and $\xi_0 = \xi$.

We begin with

PROPOSITION 4.2.17 Suppose that $p = P(\xi > \theta) > 0, \theta > 0$ fixed. Then

$$\liminf_{t\to\infty} \frac{1}{t^{d/d+2}} \log P \otimes \mu\left(\frac{1}{t}\int_0^t V(x(u))du > \theta\right) \geq -\left(\frac{d+2}{2}\right)\pi^{2d/d+2}\left(\log\frac{1}{p}\right)^{2/d+2}.$$

Proof Let $m > 0$ be given. Then for every $x \in D_m = (-m, m)^d$,

$$P_x(T_{D_m} > t) \geq e^{-\frac{\pi^2 dt}{8m^2}} \prod_1^d \sin\frac{\pi(x_k + m)}{2m} = e^{-\frac{\pi^2 dt}{8m^2}} f_m(x).$$

Hence

$$P \otimes \mu\left(\frac{1}{t}\int_0^t V(x(u))du > \theta\right) \geq$$
$$\mu\left(\xi_{[x]} > \theta \quad \forall x \in D_m\right) e^{-\frac{\pi^2 dt}{8m^2}} \int_{[0,1)^d} f_m(x)dx, \quad m > 0.$$

Clearly

$$\mu\left(\xi_{[x]} > \theta \quad \forall x \in D_m\right) \geq \mu(\xi > \theta)^{(2(m+1))^d} = p^{(2(m+1))^d},$$

and $\displaystyle\int_{[0,1)^d} f_m(x)dx = \left(\frac{2m}{\pi}\right)^d \prod_1^d \left(1 - \cos\pi\frac{(m+1)}{2m}\right) \sim \left(\frac{2m}{\pi}\right)^d$, m large.

Taking $m = \frac{a}{2}t^{1/d+2}, a > 0$, we get

$$\liminf_{t\to\infty} \frac{1}{t^{d/d+2}} \log P \otimes \mu\left(\frac{1}{t}\int_0^t V(x(u))du > \theta\right) \geq -\inf_{a>0}\left(\frac{\pi^2 d}{2a^2} + a^d \log\frac{1}{p}\right).$$

Since $\displaystyle\inf_{a>0}\left(\frac{\pi^2\lambda}{2a^2} + a^d\log\frac{1}{p}\right) = \left(\frac{d+2}{2}\right)\pi^{2d/d+2}\left(\log\frac{1}{p}\right)^{2d/d+2}$, the proof is complete. □

REMARK: $-\left(\frac{d+2}{2}\right)\pi^{2d/d+2}\left(\log\frac{1}{p}\right)^{2/d+2}$ is not the best lower bound we can find (see [5]), but it is sufficient for our purpose. After the last Proposition, we conjecture

Conjecture: $t^{d/d+2}$ is a critical rate.

We will show below that our conjecture is true when $d = 1$.

For the moment, we will find upper bounds for $\mu((Vf,f) \geq \theta)$ $\theta > 0, f \in \mathcal{C}$. To this end, let $\Phi(\lambda) = \log E\{e^{\lambda\xi}\}, \lambda \in \mathbb{R}$. Then

$$E\left(e^{\lambda(Vf,f)}\right) = \int_{[0,1)^d} E\left(\exp\left(\sum_{k\in Z^d} \lambda\xi_k \int_{[x]=k} f^2(x-u)dx\right)\right) du$$
$$= \int_{[0,1)^d} \exp\left(\sum_{k\in Z^d} \Phi\left(\lambda \int_{[x]=k} f^2(x-u)dx\right)\right) du$$

By Jensen's inequality $\Phi\left(\lambda\int_{[x]=k} f^2(x-u)dx\right) \leq \int_{[x]=k}\Phi\left(\lambda f^2(x-u)\right)dx$. Therefore

$$E\left(e^{\lambda(Vf,f)}\right) \leq \exp\left(\int \Phi(\lambda f^2(x))dx\right).$$

Note that Jensen's Inequality also gives

$$E\left(e^{\lambda(Vf,f)}\right) \geq$$

(2.29)
$$\exp\left(\int_{[0,1)}^d \sum_k \Phi\left(\lambda \int_{[x]=k} f^2(x-u)dx\right) du\right)$$
$$= \exp\left(\int \Phi\left(\lambda \int_{[0,1)^d} f^2(u-x)du\right) dx\right).$$

LEMMA 4.2.18 Let

$$\psi(\theta) = \inf_{f\in\mathcal{C}} \sup_{\beta>0} \left(\beta\theta - \int \Phi(\beta f^2(x))dx\right), \theta \geq 0.$$

Then

(2.30) $$\psi(\theta) = a^{d/2} \inf_{f\in\mathcal{C}_a} \sup_{\beta>0} \left(\beta\theta - \int \Phi(\beta f^2(x))dx\right), \theta \geq 0, a > 0.$$

$$\psi(\theta) \geq \sup_{\beta>0}\left(\theta\beta - \sup_{f\in C_1}\int \Phi\left(\beta f^2(x)\right)dx\right), \theta \geq 0 \tag{2.31}$$

$$\lim_{\beta\downarrow 0}\frac{1}{\beta}\sup_{f\in C_1}\int \Phi(\beta f^2(x))dx = \lim_{a\downarrow 0}\sup_{f\in C_a}\int \Phi(f^2(x))dx. \tag{2.32}$$

$$\sup_{f\in C_a}\mu\left((Vf,f)\geq\theta\right) \leq e^{-a^{-d/2}\psi(\theta)} \quad \theta\geq 0, a>0. \tag{2.33}$$

Proof: For any $f \in C$, set $f_\theta(x) = f(x/\theta)\theta^{-d/2}, \theta > 0$. Then $f_\theta \in C$ and $I(f_\theta) = 1/\theta^2 I(f)$.

Therefore

$$\psi(\theta) = \inf_{f\in C_a}\sup_{\beta>0}\left(\beta\theta - \int \Phi\left(\beta f^2_{\sqrt{a}}(x)\right)dx\right)$$

$$= \inf_{f\in C_a}\sup_{\beta>0}\left(\beta\theta - a^{d/2}\int \Phi\left(\beta/a^{d/2}f^2(x)\right)dx\right) = a^{d/2}\psi(\theta), \quad \theta\geq 0, a>0.$$

which proves (2.30).

Next let $\phi(\beta) = \sup_{f\in C_1}\int\Phi\left(\beta f^2(x)\right)dx, \beta\geq 0$. Clearly $\phi(a^{d/2})/a^{d/2} = \sup_{f\in C_a}\int\Phi\left(f^2(x)\right)dx$. Since ϕ is convex, and $\phi(0) = 0$ (2.32) follows easily. For every $\beta > 0, \theta \geq 0$ and $f \in C_1$,

$$\theta\beta - \phi(\beta) \leq \theta\beta - \int\Phi\left(\beta f^2(x)\right)dx \leq \sup_{\beta>0}\left(\theta\beta - \int\Phi\left(\beta f^2(x)\right)dx\right). \tag{2.34}$$

Therefore (2.31) follows by taking the sup on the left hand side of (2.34) and by taking the inf on the right hand side of (2.34).

To prove (2.33) note that for every $f \in C_a$, $a > 0, \theta \geq 0$,

$$\mu\left(Vf,f)\geq\theta)\right) \leq \inf_{\beta>0}\left(e^{-\beta\theta}\right)E\left(e^{\beta(Vf,f)}\right)$$

$$\leq \exp\left(-\sup_{\beta>0}\left(\theta\beta - \int\Phi\left(\beta f^2(x)dx\right)\right)\right).$$

Thus (2.33) follows by taking the sup on both sides and by using (2.30). □

LEMMA 4.2.19 $\lim_{a\downarrow 0}\sup_{f\in C_a}\int\Phi\left(f^2(x)\right)dx = 0.$

Proof Set $\theta_1 = \sup(\theta; \mu(\xi > \theta) > 0$. By hypothesis $\theta_1 \in (0, \infty)$. Since $\Phi(0) = 0, \Phi'(0) = 0$ and Φ is convex, $\Phi(\beta)/\beta$ is increasing for $\beta > 0$ and $\lim_{\beta\downarrow 0} \Phi(\beta)/\beta = 0$, $\lim_{\beta\uparrow\infty} \Phi(\beta)/\beta = \theta_1$. Hence

$$\int \Phi(f^2(x))dx \le \Phi(\beta)/\beta + \int_{x; f^2(x)\ge\beta} \Phi\left(f^2(x)\right) dx \le \Phi(\beta)/\beta + \frac{\theta_1}{\beta^\epsilon} \int f^{2+\epsilon}(x)dx,$$

if $\beta, \epsilon > 0$. and $f \in \mathcal{C}$; so it is sufficient to prove

$$\limsup_{a\downarrow 0} \sup_{f\in\mathcal{C}_a} \int f^{2+\epsilon}(x)dx = 0 \text{ for some } \epsilon > 0. \tag{2.35}$$

If $d = 1$ and $f \in \mathcal{C}_a$, $f^2(x) = \int_{-\infty}^{x} 2f(x)\dot{f}(x)dx \le (8a)^{1/2}$. Thus (2.35) holds with $\epsilon = 2$. When $d > 1$, (2.35) is a consequence of the next Lemma. □

LEMMA 4.2.20 Suppose that $f \in C_0^\infty(\mathbb{R}^d)$, d > 1. Then for every $p > 1$,

$$\left(\int_{\mathbf{R}^d} |f(x)|^{pd/d-1}dx\right)^{d-1} \le \left(\frac{2p^2}{d}\right)^{d/2} I(f)^{d/2} \left(\int_{\mathbf{R}^d} |f(x)|^{2(p-1)}dx\right)^{d/2}.$$

Proof Suppose that $f_1, \cdots f_d$ are positive and measurable functions on $\mathbb{R}^d$, f_i does not depend on $x_i, 1 \le i \le d$ and $\int_{\mathbf{R}^{d-1}} f_i(x)dx < \infty$ for every $1 \le i \le d$.

Using the inequality $|\prod_1^m g_k|_{L^1(\mathbf{R})} \le \prod_1^m |g_k|_{L^m(\mathbf{R})}$ it is easy to see that

$$\int_{\mathbf{R}^d} \prod_1^d f_i(x)^{1/d-1}dx \le \prod_1^d \left(\int_{\mathbf{R}^{d-1}} f_i(x)dx\right)^{1/d-1}, d > 1. \tag{2.36}$$

Next for every $f \in C_0^\infty(\mathbb{R}^d)$ and $p > 1$,

$$|f(x)|^p \le p\int_{\mathbf{R}} |\partial_i f(x)||f(x)|^{p-1}dx_i \text{ where } \partial_i = \frac{\partial}{\partial x_i}.$$

Applying (2.36) to $f_i(x) = \int_{\mathbf{R}} |\partial_i f(x)||f(x)|^{p-1}dx_i$, we find

$$\int |f(x)|^{pd/d-1}dx \le p^{d/d-1} \prod_{i=1}^d \left(\int_{\mathbf{R}^d} |\partial_i f(x)|^{p-1}dx\right)^{1/d-1}$$

By Schwarz's inequality

$$\prod_{i=1}^{d}\left(\int_{\mathbb{R}^d}|\partial_i f(x)||f(x)|^{p-1}dx\right)^{1/d-1} \leq$$

$$\left(\int |f(x)|^{2(p-1)}dx\right)^{d/2(d-1)}\prod_{i=1}^{d}\left(\int_{\mathbb{R}^d}|\partial_i f(x)|^2 dx\right)^{1/2(d-1)}.$$

Since

$$\prod_{1}^{m}|x_i| \leq \left(\frac{1}{m}\sum_{1}^{m}|x_i|\right)^m, m \in \mathbb{N}, x_i \in \mathbb{R}$$

we obtain

$$\left(\int |f(x)|^{pd/d-1}dx\right)^{d-1} \leq \left(\frac{2p^2}{d}\right)^{d/2} I(f)^{d/2}\left(\int_{\mathbb{R}^d}|f(x)|^{2(p-1)}dx\right)^{d/2},$$

which is the desired inequality. □

THEOREM 4.2.21

$$\limsup_{t\to\infty}\frac{1}{t^{1/3}}\log P\otimes\mu\left(\left|\frac{1}{t}\int_0^t V(x(u)du\right| \geq \theta\right) < 0 \quad \forall\theta > 0.$$

In particular $t^{1/3}$ is a critical rate if $d = 1$.

Proof: It follows from (2.31) - (2.33) and Lemma 4.2.18-19 that

$$\gamma_\lambda(a,\theta,d) \geq a^{-d/2}\psi(\theta)\left(\liminf_{t\to\infty}\frac{\lambda(t)}{t^{1/3}}\right)^{-3d/2} \quad \forall a > 0, \theta > 0 \quad \text{and } \psi(\theta) = 0 \iff \theta = 0.$$

When $\lambda(t) = t^{1/3}, \kappa_\lambda(a,\theta,D) > 0$ if a is small enough, so the conclusion follows from Corollary 4.2.15.

We have proved in Proposition 4.2.17 that when $d = 1$,

$$\liminf_{t\to\infty}\frac{1}{t^{1/3}}\log P\otimes\mu\left(\frac{1}{t}\int_0^t V(x(u))du > \theta\right) > -c_\theta, c_\theta \in (0,\infty)$$

if θ is small, $\theta > 0$. Hence $\lambda(t) = t^{1/3}$ is a critical rate in dimension 1. □

We conjectured that $t^{d/d+2}$ is a critical rate in dimension d. The next Proposition supports our conjecture.

PROPOSITION 4.2.11 For every $\beta \in \mathbb{R}$,

$$\liminf_{t\to\infty} \frac{1}{t^{d/d+2}} E\left(e^{t^{d/d+2}\left(\frac{\beta}{t}\int_0^t V(x(u))du\right)}\right) \geq c(\beta),$$

$$\text{where } c(\beta) = \sup_{f\in C}\left(\int \Phi\left(\beta f^2(x)\right) dx - I(f)\right).$$

c is convex, $c(0) = 0$, c is differentiable at 0, and $c'(0) = 0$.

Proof: For simplicity set $\lambda(t) = t^{d/d+2}$. From Lemma 4.2.6

$$E\left(e^{\frac{\lambda(t)}{t}\beta\int_0^t V(x(u))du}\right) \geq \frac{1}{k(f)} e^{-tI(f)} E\left(e^{\beta\lambda(t)(V_f,f)}\right), f \in C,$$

where $k(f) = \sup_y |f(y)| \cdot \int |f(y)|dy$. Using (2.29) we find

$$\text{(2.37)}\quad \begin{aligned}&\log E\left(e^{\frac{\lambda(t)}{t}\beta}\int_0^t V(x(u))du\right) \geq \\ &-\log k(f) - tI(f) + \int \Phi\left(\beta\lambda(t)\int_{[0,1)^d} f^2(x-u)dx\right) dx.\end{aligned}$$

If we apply (2.37) to $f_{\sqrt{t/\lambda(t)}}$, we get

$$\text{(2.38)}\quad \begin{aligned}&\log E\left(\exp\left\{\frac{\lambda(t)}{t}\beta\int_0^t V(x(u))du\right\}\right) \geq \\ &-\log k(f) - \lambda(t)I(f) \\ &+\left(\frac{t}{\lambda(t)}\right)^{d/2}\int \Phi\left(\beta\frac{\lambda(t)^{d+2/2}}{t^{d/2}}\int_{[0,1)^d} f^2(x-u\sqrt{\lambda(t)/t})du\right) dx\end{aligned}$$

since $k(f_\theta) = k(f) \quad \forall\theta > 0$.

Now $\lambda(t) = (t/\lambda(t))^{d/2}$ and $\lambda(t)^{d+2/2} = t^{d/2}$. It follows from (2.38) that

$$\liminf_{t\to\infty}\frac{1}{t^{d/d+2}}\log E\left(\exp\left\{\beta t^{-2/d+2}\int_0^t V(x(u))du\right\}\right) \geq$$
$$c(\beta) = \sup_{f\in C}\left(\int \Phi(\beta f^2(x))dx - I(f)\right).$$

Clearly c is convex and $c(0) = 0$. Set $\ell = \lim_{\beta\downarrow 0} c(\beta)/\beta \geq 0$. For any $\epsilon > 0$, one can find $f_n \in C$ so that

$$\int \Phi(f_n^2(x))dx - nI(f_n) \geq n\left(\int \Phi\left(1/nf_n^2(x)\right) dx - I(f_n)\right) \geq \ell - \epsilon.$$

Hence $I(f_n) \to 0$ as $n \to \infty$ and $\ell - \epsilon \le \lim_{a\downarrow 0} \sup_{f\in C_a} \int \Phi(f^2(x))dx = 0$ by Lemma 4.2.19. Therefore $\lim_{\beta\downarrow 0} c(\beta)/\beta = 0$. Since our proof is also valid for $-V$, we find $\lim_{\beta\uparrow 0} c(\beta)/\beta = 0 = \lim_{\beta\downarrow 0} c(\beta)/\beta$.

Before closing this chapter, we will study weak convergence of $1/a_n \int_0^{nt} V(x(u))du$, where a_n is a normalizing sequence.

THEOREM 4.2.23 Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $\{\xi_k\}_{k\in Z^d}$ be i.i.d random variables with mean 0 and variance 1, U a random variable which is uniformly distributed over $[0,1)^d = T_d$ and independent of $\{\xi_k; k \in \mathbb{R}^d\}$. Further let P_x be the Wiener measure over $\mathcal{X} = C([0,\infty); \mathbb{R}^d)$ starting from x at time 0; we write B_t to designate the canonical Wiener process.

Set $V(x) = \xi_{[x+U]}, x \in \mathbb{R}^d, X_n(t) = \int_0^{nt} V(B_u)du$, $n \in \mathbb{N}, t \in [0,1]$. Then under $P \otimes \mu$

a) Case $d = 1$: $X_n/n^{3/4} \Rightarrow Z \in C([0,1]; \mathbb{R})$ where Z has the following representation: $Z(t) = \int_0^\infty \ell_t(x)dZ(x) + \int_0^\infty \ell_t(-x)dZ_2(x)$, where B_1, Z_1, Z_2 are 3 independent 1-dimensional Wiener processes and $\ell_t(\cdot)$ is the local time of B i.e.

$$\int_0^t 1_A(B_u)du = \int_A \ell_t(x)dx, A \in \mathcal{B}(\mathbb{R}), t \ge 0.$$

b) Case $d = 2$: $P \otimes \mu \circ \left(X_n/\sqrt{n \log n}\right)^{-1} \Rightarrow \mathcal{W}_{1/\pi}$ on $C([0,1]; \mathbb{R})$

c) Case $d \ge 3$: $P \otimes \mu \circ (X_n/\sqrt{n})^{-1} \Rightarrow \mathcal{W}_{\sigma_d^2}$ on $C([0,1]; \mathbb{R})$, where $\sigma_d^2 = \int_{\mathbb{R}^d} \frac{4}{|x|^2} f_d(x)dx$ and $f_d(x) = \prod_{i=1}^{d} \left(\frac{1-\cos x_i}{\pi x_i^2}\right), x \in \mathbb{R}^d$.

Proof: Let $X(t) = \int_0^t V(B_u)du, t \ge 0$, and let $\sigma^2(t) = E\left(X^2(t)\right)$, where E stands for the expectation with respect to $P \otimes \mu$. Then

$$E(X(t)) = 0 \quad \forall t \text{ and } \sigma^2(t) = 2\int_0^t \int_0^s E\left(V(B_u)V(B_s)\right) duds.$$

Now

$$E(V(x)V(y)) = E(V(0)V(y-x)) = E\left(\xi_0 \xi_{[y-x+u]}\right) =$$
$$\int_{\mathbb{R}^d} 1_{[u]=0} 1_{[y-x+u]=0} du = h_d(y-x) = \int_{\mathbb{R}^d} e^{i<\lambda, y-x>} f_d(x)dx,$$

where

$$h_d(x) = \prod^d (1 - |x_i|) 1_{\{|x_i| \le 1\}}, \quad d \geq 1.$$

$$\text{Therefore } \sigma^2(t) = 2\int_0^t \int_0^s \int_{\mathbb{R}^d} E\left(e^{i<\lambda, B_s - B_u>}\right) f_d(\lambda) d\lambda du ds =$$

$$2\int_0^t \int_0^s \int_{\mathbb{R}^d} e^{-\frac{|\lambda|^2}{2}u} f_d(\lambda) d\lambda du ds.$$

Next $\lim_{t\to\infty} \sigma^2(t)/t = +\infty$ if $d = 1,2$ and $\lim_{t\to\infty} \frac{\sigma^2(t)}{t} = \sigma_d^2, \sigma_d^2 = \int_{\mathbb{R}^d} \frac{4}{|\lambda|^2} f_d(\lambda) d\lambda \in (0,\infty)$ when $d \geq 3$. Adapting Corollary 3.1.3 and Theorem 1.1.5 in our setting we obtain c) (see the remark following Lemma 4.2.2).

Next a) follows from an adaptation of the results of H. Kesten and F. Spitzer [7], so we only indicate how to prove it.

Proof of a). By stationarity

$$E\left((X_n(t) - X_n(s))^2 \Big/ n^{3/2}\right) =$$

$$E\left(X_n^2(|t-s|)/n^{3/2}\right) = \sum_{k\in Z} E\left(\frac{1}{n^{3/2}}\left(\int_{k-u}^{k+1-u} \ell_{n|t-s|}(x)dx\right)^2\right) \leq$$

$$\frac{1}{n^{3/2}} \int_{\mathbb{R}} E\left(\ell^2_{n|t-s|}(x)\right) dx,$$

where $\ell_t(\cdot)$ is the local time of B.

From the scaling property of $B, E\left(n^{-3/2}\ell^2_{nt}(x)\right) = \frac{1}{\sqrt{n}} E\left(\ell_t^2(x/\sqrt{n})\right), t \geq 0, x \in \mathbb{R}$. Hence

$$E\left(((X_n(t) - X_n(s))^2 \Big/ n^{3/2}\right) \leq E\left(\int_{\mathbb{R}} \ell^2_{|t-s|}(x)dx\right) = |t-s|^{3/2} E\left(\int_{\mathbb{R}} \ell_1^2(x)dx\right)$$

It follows from Theorem 1.1.4 that $\{P \otimes \mu \circ \left(X_n/n^{3/4}\right)^{-1}\}_{n\geq 1}$ is tight.

Next

$$P \otimes \mu \left(\sup_k \int_k^{k+1} \ell_{nt}(x+u)du > \epsilon n^{3/4}\right) \to 0 \quad \forall t \in [0,1].$$

It follows that for every $0 = t_0 \le t_1 \le \cdots \le t_m \le 1$ and $\alpha_1 \cdots, \alpha_m \in \mathbb{R}$,

$$\lim_{n\to\infty} E\left(\exp\left\{i\sum_{j=1}^{m}\alpha_j(X_n(t_j) - X_n(t_{j-1})\right\}/n^{3/4}\right) =$$

$$E\left(\exp\left\{-\frac{1}{2}\int_{\mathbb{R}}\left(\sum_{1}^{m}\alpha_j(\ell_{t_j}(x) - \ell_{t_{j-1}}(x))\right\}\right)^2 dx\right)$$

which completes the proof.

Proof of b): Set $a(t) = (t\log t)^{1/2}, t \ge 2$, and let $p_t(k) = \int_0^t 1_{[B_s+u]=k}ds$. Further let $\theta(t) = \sum_{k\in Z^2} p_t^2(k), t \ge 0$. From now on we will write P and $\sum_k$ instead of $P \otimes \mu$ and $\sum_{k\in Z^2}$.

Assume

$$A_1 : \lim_{t\to\infty} \sigma^2(t)/a^2(t) = \frac{1}{\pi}$$

and

$$A_2 : \lim_{t\to\infty} E\left(\theta^2(t)/a^4(t)\right) = \frac{1}{\pi^2}, \quad E\left(\theta^2(t)\right) \le t^2\alpha(t) \quad \text{where}$$

α is increasing and $\alpha(t) \sim \frac{1}{\pi^2}(\log t)^2$, which means $\lim_{t\to\infty} \dfrac{\alpha(t)}{(\log t)^2} = \dfrac{1}{\pi^2}$. Then

(2.39) $$\left\{P \circ (X_n/a(n))^{-1}\right\}_{n\ge 2} \text{ is tight (in } C[0,1];\mathbb{R}))$$

and

(2.40) $$\lim_{n\to\infty} E\left(e^{i}\sum_{j=1}^{m}\alpha_j(X_n(t_j) - X_n(t_{j-1}))/a(n)\right) = e^{-\frac{1}{2\pi}\sum_{j=1}^{m}\alpha_j(t_j - t_{j-1})}$$

for any $0 = t_0 \le t_1 \le \cdots \le t_m \le 1$ and $\alpha_j \in \mathbb{R}$.

Clearly (2.39) and (2.40) are equivalent to b). So suppose that A_1 and A_2 hold. To prove (2.39) we just have to prove

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P\left(\sup_{\substack{0\le s\le t\le 1\\ 0\le t-s\le\delta}} |X_n(t) - X_n(s)| > 3\sqrt{2}\epsilon a(n)\right) = 0 \quad \forall\epsilon > 0.$$

Put

$$A(n,\epsilon,\delta) = \left\{ \sup_{\substack{0\le s\le t\le 1\\ 0\le t-s\le\delta}} \sum_k \xi_k^2 \left(p_{nt}(k) - p_{ns}(k)\right)^2 \le \epsilon^2 a^2(n) \right\}, m = [1/\delta].$$

Let $u > 0$ be given and set $\xi_{k,u} = \xi_k 1_{\{|\xi_k|\le u\}}, \zeta_{k,u} = \xi_k - \xi_{k,u}$. Since $p_t(k)$ is increasing in t for fixed k,

$$P(A(n,\epsilon,\delta)^c) \le$$

$$P\left(\sup_{0\le s\le 1} \sum_k \xi_k \left(p_{n(s+\delta)}(k) - p_{ns}(k)\right)^2 > \epsilon^2 a^2(n)\right) \le$$

$$\sum_{j=0}^{m-1} P\left(\sum_k \xi_k^2 \left(p_{n(j+2)\delta}(k) - p_{nj\delta}(k)\right)^2 > \epsilon^2 a^2(n)\right).$$

By definition $\xi_k^2 = \xi_{k,u}^2 + \zeta_{k,u}^2$ and $E\left(\xi_{k,u}^4\right) \le u^2$. Therefore

$$P(A(n,\epsilon,\delta)^c) \le \frac{4u^2}{\delta\epsilon^4 a^4(n)} E\left(\theta^2(2n\delta)\right) + \frac{2}{\delta\epsilon^2 a^2(n)}\sigma^2(2n\delta)E(\zeta_{0,u}^2),$$ where

we have used the stationarity of $\xi_{[U+B_t],u}$ and $\zeta_{[U+B_t],u}$.

It follows from A_1 and A_2 that

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P(A(n,\epsilon,\delta)^c) = 0$$

since

$$\limsup_{n\to\infty} P(A(n,\epsilon,\delta)^c) \le 16u^2\delta/\epsilon^4\pi^2 + \frac{4}{\pi\epsilon^2}E\left(\zeta_{0,u}^2\right)$$

for every $\delta \in [0,1]$ and $u > 0$.

Now suppose that $1 \ge \delta \ge t - s \ge 0$. Then

$$P\left(A(n,\epsilon,\delta) \cap \left\{|X_n(t) - X_n(s)| > \sqrt{2}\epsilon a(n)\right\}\right) \le$$

$$P\left(\sum_{k\ne j} \xi_k\xi_j \left(p_{nt}(k) - p_{ns}(k)\right)\left(p_{nt}(j) - p_{ns}(j)\right) > \epsilon^2 a^2(n)\right) \le$$

$$\frac{1}{\epsilon^4 a^4(n)} E\left(\left(\sum_{k\ne j}\xi_k\xi_j(p_{nt}(k) - p_{ns}(k))(p_{nt}(j) - p_{ns}(j))\right)^2\right) \le$$

$$\frac{2}{\epsilon^4 a^4(n)} E\left(\theta^2(n|t-s|)\right) \le \frac{2|t-s|^2}{\epsilon^4} C_n$$ and $C_n \to 1/\pi^2$, by A_2.

It follows from Theorem 1.1.4 that

$$\limsup_{n\to\infty} P\left(A(n,\epsilon,\delta)\cap\left\{\sup_{j\delta\le s\le(j+1)\delta}|X_n(s)-X_n(j\delta)|>\sqrt{2}\epsilon a(n)\right\}\right)\le\frac{K_1\delta^2}{\epsilon^4}$$

for some finite constant K_1 (ind. from δ and ϵ). Combining the last inequality with

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty} P(A(n,\epsilon,\delta)^c)=0$$

we get (2.39).

We will now prove (2.40). Set $Y_n=\sup_k p_n(k)$ and $q_t(x)=\frac{1}{2\pi t}e^{-\frac{|x|^2}{2t}}$. Clearly

$$P(Y_n>\epsilon)\le(\epsilon a(n))^{-j}\sum_k E(p_n^j(k))\le$$

$$(\epsilon a(n))^{-j}j!\int_{S_n}\int_{(T_2)^j}q_{t_2}(x_2-x_1)\cdots q_{t_j-t_{j-1}}(x_j-x_{j-1})dxdt$$

where

$$\overset{\circ}{S_n}=\{0\le t_1\le\cdots\le t_j\le n\}.$$

Define $h(t,x)=\int_{T_2}q_t(y-x)dy,\quad x\in T_2, t>0$. By elementary calculus $h(t,0)\le h(t,x)\le h(t,x_0)\quad\forall x\in T_2$, where $x_0=\left(\frac{1}{2},\frac{1}{2}\right)$. Therefore $p(Y_n>\epsilon)\le(\epsilon a(n))^{-j}j!n\left(\int_0^n h(t_1x_0)dt\right)^{j-1}$. Now $h(t,x_0)\sim\frac{1}{2\pi t}$ so $\int_0^n h(t,x_0)dt\sim\frac{1}{2\pi}\log n$. Choosing $j=3$, we get $\lim_{n\to\infty}P(Y_n>\epsilon)=0\quad\forall\epsilon>0$. Using this and $p_t(k)\le p_n(k)\quad\forall 0\le t\le n, k\in Z^2$, we can prove easily that

$$\lim_{n\to\infty}\left|E\left(e^{i\tilde{X}_n}\right)-E\left(e^{\frac{1}{2}\sum_k Z_n^2(k)}\right)\right|=0\quad\text{where }\tilde{X}_n=$$

$$\frac{1}{a(n)}\sum_{j=1}^m\alpha_j(X_n(t_j)-X_n(t_{j-1})), Z_n(k)=\frac{1}{a(n)}\sum_{j=1}^m\alpha_j(p_{nt_j}(k)-p_{nt_{j-1}}(k)),$$

and $0=t_0\le t_1\le\cdots\le t_m\le 1,\alpha_j\in\mathbb{R}$ are fixed. Therefore (2.40) will follow if we can prove that

$$\sum_k Z_n^2(k)\xrightarrow{P_r}\frac{1}{\pi}\sum_{j=1}^m\alpha_j^2(t_j-t_{j-1}).$$

Suppose that $t \geq s \geq 0$. By stationarity $\sigma^2(t-s) = E\left((X(t)-X(s))^2\right)$, so $E(\sum_k p_t(k)p_s(k)) = \frac{1}{2}\left(\sigma^2(t)+\sigma^2(s)-\sigma^2(t-s)\right)$ and it follows from A_1 that $E(\sum_k p_{nt}(k)p_{ns}(k)) \sim \frac{s}{\pi}a^2(n)$. Thus

$$0 \leq E\left(\frac{1}{a^2(n)}\sum_k (p_{nt_j}(k)-p_{nt_{j-1}}(k))(p_{nt_\ell}(k)-p_{nt_{\ell-1}}(k))\right) \to 0$$

whenever $j \neq \ell$, and

$$\sum_k Z_n^2(k) \to \frac{1}{\pi}\sum_{j=1}^m \alpha_j^2(t_j - t_{j-1})$$

in probability since $E\left(\left(\frac{1}{a^2(n)}\sum_k (p_{nt}(k)-p_{ns}(k))^2 - \frac{1}{\pi}(t-s)\right)^2\right) \to 0$ by A_2 and stationarity, $0 \leq s \leq t \leq 1$. This completes the proofs of (2.29) and (2.40) assuming that A_1 and A_2 are verified.

Proof of A_1: Since $\sigma^2(t) = 2\int_0^t\int_0^s h_2(x)q_u(x)dx du ds$, h_2 has compact support and $q_u(\cdot) \sim \frac{1}{2\pi u}$ uniformly on compacts, $\sigma^2(t) \sim \frac{t}{\pi}\log t$ follows easily.

Proof of A_2: Since $\theta(t)$ does not depend on $\{\xi_k\}$, let us suppose that ξ_k is Gaussian with mean 0 and variance 1. Then $E(\theta^2(t)) = \frac{1}{3}E(X^4(t)) = 8(C_1(t)+C_2(t)+C_3(t))$, where $S_t = \{0 \leq s_1 \leq \cdots \leq s_4 \leq t\}$ and

$$C_1(t) = \sum_k \int_{S_t}\int_{(T_2)^4} q_{s_2-s_1}(x_2-x_1)q_{s_3-s_2}(x_3+k-x_2)q_{s_4-s_3}(x_4-x_3)dx ds$$

$$C_2(t) = \sum_k \int_{S_t}\int_{(T_2)^4} q_{s_2-s_1}(x_2+k-x_1)q_{s_3-s_2}(x_3-k-x_2)q_{s_4-s_3}(x_4-k-x_3)dx ds$$

$$C_3(t) = \sum_k \int_{S_t}\int_{(T_2)^4} q_{s_2-s_1}(x_2+k-x_1)q_{s_3-s_2}(x_3-x_2)q_{s_4-s_3}(x_4-k-x_3)dx ds$$

Since

$$h(t,0) \leq h(t,x) \leq h(t,x_0) \quad t>0, x \in T_2, x_0 = \left(\frac{1}{2},\frac{1}{2}\right), \text{ we get}$$

$$\int_{S_t} h(s_2-s_1,0)h(s_4-s_3,0)ds \leq C_1(t) \leq \frac{t^2}{2}\left(\int_0^t h(s,x_0)ds\right)^2$$

and we can easily check that $C_1(t) \sim \frac{t^2}{8\pi^2}(\log t)$ and $\int_0^t h(s,x_0)ds \sim \frac{1}{2\pi}\log t$.

Using Cauchy-Schwarz inequality, we get

$$C_2(t) \leq \int_{S_1} \left(\sum_{k} \int_{(T_2)^2} q^2_{s_1-s_2}(x_3 - k - x_2) dx_2 dx_3 \right)^{1/2} \cdot$$

$$\left(\sum_k \int_{(T_2)^2} q^4_{s_2-s_1}(x_2 + k - x_1) dx_1 dx_2 \right)^{1/2} \cdot$$

$$\left(\sum_k \int_{(T_2)^2} q^4_{s_4-s_3}(x_4 + k - x_3) dx_s dx_y \right)^{1/2} ds.$$

Now $\sum_k \int_{(T_2)^2} q_t^j(y + k - x) dx dy = \int_{\mathbb{R}^2} q_t^j(y) dy = \frac{1}{j(2\pi t)^{j-1}}, \quad j \geq 1.$

By elementary calculations we obtain $C_2(t) \leq t^2 C_2'$ for some constant C_2' and similarly $C_3(t) \leq C_3' t^2 d_3(t)$, where $d_3(t) = \int_0^t \left(\int_{(T_2)^2} q_s^2(y - x) dx dy \right)^{1/2} ds \sim \frac{1}{2\pi} \log t$. Hence we can conclude that $E(\theta^2(t)) \sim \frac{1}{\pi^2}(t \log)^2$ and $E(\theta^2(t)) \leq t^2 \alpha(t)$ for some increasing α such that $\alpha(t) \sim \frac{1}{\pi^2}(\log t)^2$ which proves A_2. □

REMARK: 1° Since (2.39) and (2.40) depend only on A_1 and A_2, and not on the properties of Wiener process, and also A_1 and A_2 depend only on the behaviour of $q_t(\cdot)$, we see that we can replace $\{\xi_k\}_{k \in Z^2}, U$ and B_t by $\{\zeta_k\}_{k \in Z}, U'$ = uniform on [0,1) and $x(t)$: symmetric Cauchy process, and we can easily prove that $X'(t) = \int_0^t \zeta_{[x(u)+U']} du$ has the same limiting distribution as $X(t)$ i.e. $X'(nt)/\sqrt{n \log n}$ converges to some Wiener process.

2°: Borodin [2] has proved a more general result than Theroem 4.2.23 where B_t (resp $x(t)$) is replaced by a symmetric random walk on Z^2 (resp).

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