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CERTAIN INFINITE SUMS INVOLVING
ULTRASPHERICAL POLYNOMIALS AND ULTRASPHERICAL
FUNCTIONS OF THE SECOND KIND

by

Mihr Jahanian Shah, M.Sc.

A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment
of the requirements for the degree of
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1 ABSTRACT

Let $C_n^\lambda(x)$ and $D_n^\lambda(x)$ be the ultraspherical polynomial and ultraspherical function of the second kind, respectively. Askey, Koornwinder and Rahman showed that the integral $\int_{-1}^1 D_n^\lambda(x) C_m^\lambda(x) C_l^\lambda(x) (1-x^2)^{2\lambda-1} dx$ vanishes when $l+m+n$ is odd and $|l-m| < n < l+m$. We find a dual of this result, namely, that $\sum_{n=0}^{\infty} (n+\lambda) \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} C_n^\lambda(\cos \alpha) C_n^\lambda(\cos \beta) D_n^\lambda(\cos \gamma)$ vanishes when $|\alpha-\beta| < \gamma < \alpha+\beta$. We evaluate this sum for $\gamma < |\alpha-\beta|$ and $\gamma > \alpha+\beta$, and apply Hsu's limiting technique to give a derivation of the Bessel function integral $\int_0^\infty x^{1+\nu} Y_\nu(ax) J_\nu(bx) J_\nu(cx) dx$ that has recently been evaluated by Askey, Koornwinder and Rahman by different methods. As a

generalization to above results we will evaluate

$$\int_{-1}^1 D_n^\lambda(x) C_k^\lambda(x) C_l^\lambda(x) C_m^\lambda(x) (1-x^2)^{2\lambda-1} dx \text{ and}$$

$$\sum_{n=0}^{\infty} (n+\lambda) \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} C_n^\lambda(\cos \alpha) C_n^\lambda(\cos \beta) C_n^\lambda(\cos \gamma) D_n^\lambda(\cos \tau). \text{ We will}$$

also compute the Bessel function integrals $\int_0^\infty t^{1-\rho} J_\mu(at) J_\nu(bt) J_\lambda(ct) dt$ and $\int_0^\infty t^{1-\rho} J_\mu(at) J_\nu(bt) Y_\lambda(ct) dt$. It turns out the Sonine's integral formula [57] and the Askey, Koornwinder, and Rahman evaluation

$$\text{of the integrals } \int_{-1}^1 D_n^\lambda(x) C_m^\lambda(x) C_l^\lambda(x) (1-x^2)^{2\lambda-1} dx \text{ and}$$

$$\int_0^\infty x^{1+\nu} Y_\nu(ax) J_\nu(bx) J_\nu(cx) dx \text{ are particular cases of our results.}$$

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Miss Diane Dodds did the typing of this thesis, and her skill will be evident to the reader.

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CHAPTER I

INTRODUCTION1. Hypergeometric Series:

The series

$$(1.1) \quad 1 + \frac{ab}{1!} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!} z^3 + \dots$$

is called the Euler-Gauss function or the ordinary hypergeometric series.

It is usually denoted by the symbol:

$$(1.2) \quad {}_2F_1 [a, b; c; z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

$$(1.3) \quad \text{where } (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2) \dots (a+n-1), & n=1,2,\dots, \\ 1, & n=0. \end{cases}$$

The series (1.1) and (1.2) are defined provided $c \neq 0, -1, -2, -3, \dots$, unless a or b is of the form $-n$, $n=0, 1, 2, \dots$, with $n \leq -c$. If either of the parameters a or b is a negative integer $-n$, the series has only finite number of terms and becomes a polynomial in z .

The theory of hypergeometric function is very old and many great mathematicians have contributed to it. In 1303 Chü [29] evaluated the following sum:

$$(1.4) \quad {}_2F_1 [-n; b; c; 1] = \frac{(c-b)_n}{(c)_n}.$$

In the current literature this sum is known as Chü-Vandermonde's [82] formula. For historical discussion see Askey [9, p.59-60],

Carlitz [27], Needham [62, p.138] and Takács [79]. The term hypergeometric was first used by John Wallis (1616-1703) in his work [85] *Arithmetica Infinitorum* (1655). In 1812, C. F. Gauss (1777-1855) evaluated the following series [51]:

$$(1.5) \quad {}_2F_1(a, b; c; 1) = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)},$$

provided $\operatorname{Re}(c-a-b) > 0$.

The next major advancement was made by E. E. Kummer (1810-1893) in 1836. He showed that the differential equation [59]:

$$(1.6) \quad z(1-z) \frac{d^2 y}{dz^2} + \{c-(1+a+b)z\} \frac{dy}{dz} - aby = 0$$

is satisfied by the function (1.1) and has in all twenty-four solutions expressible in terms of similar Gauss functions [59]. In 1857

G.F.B. Riemann [73] (1826-66) extended this theory by the introduction of his P-symbol. The first integral representations was given by Euler [43] who showed that

$$(1.7) \quad {}_2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt,$$

$\operatorname{Re} b > 0, \operatorname{Re}(c-b) > 0$.

In 1907 E. W. Barnes [23] gave contour integral representations of Kummer's twenty-four functions and later, in 1910 [24], he proved the integral analogue of Gauss theorem (1.5) namely

$$(1.8) \quad \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Gamma(a+t) \Gamma(b+t) \Gamma(c-t) \Gamma(d-t) dt$$

$$= \frac{\Gamma(a+c) \Gamma(a+d) \Gamma(b+c) \Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

From (1.7) and a simple change of variables, $1-t=s$, we obtain:

$$(1.9) \quad {}_2F_1[a, b; c; z] = (1-z)^{-a} {}_2F_1[a, c-b; c; \frac{z}{z-1}],$$

$$|z| < 1, \quad \left| \frac{z}{z-1} \right| < 1.$$

If we use ${}_2F_1[a, b; c; z] = {}_2F_1[b, a; c; z]$, we get

$$(1.10) \quad {}_2F_1[a, b; c; z] = (1-z)^{-b} {}_2F_1[c-a, b; c; \frac{z}{z-1}],$$

$$|z| < 1, \quad \left| \frac{z}{z-1} \right| < 1.$$

From (1.9) and (1.10) it follows that

$$(1.11) \quad {}_2F_1[a, b; c; z] = (1-z)^{c-a-b} {}_2F_1[c-a, c-b; c; z].$$

The above proofs are due to Kummer [59] and the transformations (1.9) and (1.11) are called Kummer's transformations.

Gauss hypergeometric series ${}_2F_1[a, b; c; z]$ can be generalized by introducing p parameters in the numerator and q parameters in the denominator. It seems to have occurred for the first time in the work of Clausen 1828 [30]. He introduced a series with three numerator parameters and two denominator parameters. The generalized hypergeometric series is defined as

$$(1.12) \quad {}_pF_q \left[\begin{matrix} a_1, a_2, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \frac{z^n}{n!}.$$

In general ${}_pF_q$ converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q+1$ and diverges for all $z \neq 0$ if $p > q+1$.

Usually we shall be concerned with the case $p = q+1$, i.e.,

$$(1.13) \quad {}_{p+1}F_p \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{p+1} \\ \beta_1, \dots, \beta_p \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \dots (\alpha_{p+1})_n}{(\beta_1)_n \dots (\beta_p)_n} \frac{z^n}{n!}.$$

The series in (1.13) is called m -balanced if

$$(1.14) \quad m + \sum_{i=1}^{p+1} \alpha_i = \sum_{i=1}^p \beta_i. \quad \left(\text{For } m=1 \text{ it is simply called balanced or Saalschutzhian.} \right)$$

If the parameters satisfy the relations

$$(1.15) \quad 1 + \alpha_1 = \beta_1 + \alpha_2 = \dots = \beta_p + \alpha_{p+1}.$$

the series is said to be well-poised and very well-poised if it is well-poised and

$$\alpha_2 = 1 + \frac{\alpha_1}{2}, \quad \beta_1 = \frac{\alpha_1}{2}.$$

The names 'Saalschutzhian' and well-poised are due to Whipple [87], and the term very well-poised is due to Askey [5].

2. Ultraspherical and Jacobi Polynomials:

Let $\lambda > -1/2$. Ultraspherical polynomials $C_n^\lambda(x)$, $n=0,1,2,\dots$, are orthogonal with respect to the weight function $w(x) = (1-x^2)^{\lambda-1/2}$ on the interval $[-1, 1]$. We define the ultraspherical polynomial $C_n^\lambda(x)$ of

degree n and order λ by the three-term recurrence relation [78]

$$(2.1) \quad n C_n^\lambda(x) = 2(n+\lambda-1) x C_{n-1}^\lambda(x) - (n+2\lambda-2) C_{n-2}^\lambda(x),$$

$$n = 2, 3, 4, \dots,$$

$$C_0^\lambda(x) = 1, C_1^\lambda(x) = 2\lambda x.$$

If we compute $C_n^\lambda(x)$ recursively then it can be shown that

$$(2.2) \quad C_n^\lambda(x) = \frac{(2\lambda)_n}{n!} {}_2F_1 \left[-n, n+2\lambda; \lambda+1/2; \frac{1-x}{2} \right].$$

Another way of defining the ultraspherical polynomials is the generating function namely

$$\sum_{n=0}^{\infty} C_n^\lambda(x) y^n = (1-2xy+y^2)^{-\lambda}, \quad \lambda \neq 0.$$

Other commonly used expressions for $C_n^\lambda(x)$ are (see [78, p.84] and [9])

$$(2.3) \quad C_n^\lambda(x) = \sum_{m=0}^{[n/2]} \frac{(-1)^m \Gamma(n-m+\lambda)}{\Gamma(\lambda) \Gamma(m+1) \Gamma(n-2m+1)} (2x)^{n-2m},$$

$$(2.4) \quad C_n^\lambda(\cos \theta) = \sum_{k=0}^n \frac{(\lambda)_k (\lambda)_{n-k}}{k! (n-k)!} \cos(n-2k)\theta,$$

$$(2.5) \quad \sin \theta C_n^\lambda(\cos \theta) = \frac{2\Gamma(\lambda+1/2) (2\lambda)_n}{n! \Gamma(\lambda+1) \Gamma(1/2)} \sum_{k=0}^{\infty} \frac{(1-\lambda)_k (1)_{n+k}}{k! (\lambda+1)_{n+k}} \sin(n+2k+1)\theta.$$

The series in (2.5) converges for $0 < \theta < \pi$ when $\lambda > 0$.

The ultraspherical polynomial satisfies the following differential equation:

$$(2.6) \quad (1-x^2)y'' - (2\lambda+1)x y' + n(n+2\lambda)y = 0$$

$$y = C_n^\lambda(x).$$

Next we define Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$, $n=0,1,2,\dots$.

Let $\alpha, \beta > -1$, the Jacobi polynomial of degree n is defined with help of three-term recurrence relation [78]

$$(2.7) \quad \begin{aligned} & 2n(n+\alpha+\beta) (2n+\alpha+\beta-2) P_n^{(\alpha, \beta)}(x) \\ &= (2n+\alpha+\beta-1) \{ (2n+\alpha+\beta) (2n+\alpha+\beta-2)x + \alpha^2 - \beta^2 \} P_{n-1}^{(\alpha, \beta)}(x) \\ &- 2(n+\alpha-1) (n+\beta-1) (2n+\alpha+\beta) P_{n-2}^{(\alpha, \beta)}(x), \quad n=2,3,4,\dots, \\ &P_0^{(\alpha, \beta)}(x) = 1, \quad P_1^{(\alpha, \beta)}(x) = \frac{1}{2} (\alpha+\beta+2)x + \frac{1}{2} (\alpha-\beta). \end{aligned}$$

It can be shown that

$$(2.8) \quad P_n^{(\alpha, \beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1[-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}].$$

$P_n^{(\alpha, \beta)}(x)$ satisfies the following differential equation

$$(2.9) \quad (1-x^2)y'' + [\beta-\alpha-(\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0,$$

$$y = P_n^{(\alpha, \beta)}(x).$$

If $\alpha=\beta$, Jacobi's polynomial $P_n^{(\alpha, \beta)}(x)$ becomes an ultraspherical polynomial and we have

$$(2.10) \quad C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda-1/2, \lambda-1/2)}(x), \quad -1 \leq x \leq 1.$$

From the above definition the following limiting formulas hold

$$(2.11) \quad \frac{H_n(x)}{n!} = \lim_{\lambda \rightarrow \infty} \lambda^{-n/2} C_n^\lambda (\lambda^{-1/2} x),$$

$$(2.12) \quad \left(\frac{x}{2}\right)^{-\alpha} J_\alpha(x) = \lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)}\left(\cos \frac{x}{n}\right),$$

where $H_n(x)$ is a Hermite polynomial and $J_\alpha(x)$ is the Bessel function of the first kind.

3. Ultraspherical and Jacobi Functions of the Second Kind:

Let x be arbitrary complex number in the complex plane cut along the segment $[-1, 1]$. Let $\alpha, \beta > -1, n \geq 0$. Excluding the case $n=0, \alpha+\beta+1=0$, a solution $y = Q_n^{(\alpha, \beta)}(x)$ of the differential equation (2.9), which is linearly independent of $P_n^{(\alpha, \beta)}(x)$, can be obtained in the form [78]

$$\begin{aligned} Q_n^{(\alpha, \beta)}(x) &= \frac{1}{2} (x-1)^{-\alpha} (x+1)^{-\beta} \int_{-1}^1 (1+t)^\alpha (1+t)^\beta \frac{P_n^{(\alpha, \beta)}(t)}{x-t} dt \\ &= 2^{n+\alpha+\beta} \frac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} (x-1)^{-n-\alpha-1} (x+1)^{-\beta} \\ (3.1) \quad &\cdot {}_2F_1 \left[\begin{matrix} n+\alpha+1, & n+1 \\ 2n+\alpha+\beta+2 \end{matrix} ; \frac{2}{1-x} \right] \\ &= 2^{n+\alpha+\beta} \frac{\Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)} (x-1)^{-\alpha} (x+1)^{-n-\beta-1} \\ &\cdot {}_2F_1 \left[\begin{matrix} n+\beta+1, & n+1 \\ 2n+\alpha+\beta+2 \end{matrix} ; \frac{2}{1+x} \right]. \end{aligned}$$

In the exceptional case: $n=0, \alpha+\beta+1=0$ we have the following integral representations:

$$\begin{aligned}
 Q_0^{(\alpha, \beta)}(x) &= -2^{\alpha+\beta} \frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \int_{-\infty}^x (t-1)^{-\alpha-1} (t+1)^{-\beta-1} dt \\
 &\quad \text{if } \alpha + \beta + 1 > 0, \\
 (3.2) \quad Q_0^{(\alpha, \beta)}(x) &= -2^{\alpha+\beta} \frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \left\{ \int_{-\infty}^x [(t-1)^{-\alpha-1} (t+1)^{-\beta-1} \right. \\
 &\quad \left. - t^{-\alpha-\beta-2}] dt - \frac{x^{-\alpha-\beta-1}}{\alpha+\beta+1} \right\} \quad \text{if } \alpha + \beta + 1 < 0,
 \end{aligned}$$

$$\begin{aligned}
 Q^{(\alpha)}(x) &= \int_{-\infty}^x [(t-1)^{-\alpha-1} (t+1)^{-\beta-1} - \frac{1}{t+1}] dt + \log(x+1) \\
 &\quad \text{if } \alpha + \beta + 1 = 0
 \end{aligned}$$

Equations (3.1) can be used to define $Q_n^{(\alpha, \beta)}(x)$ on the cut $(-1, 1)$ and to compute $P_n^{(\alpha, \beta)}(x)$ there by using Durand's formulas [37]

$$(3.3) \quad Q_n^{(\alpha, \beta)}(x) = \frac{1}{2} [e^{i\pi\alpha} Q_n^{(\alpha, \beta)}(x+10) + e^{-i\pi\alpha} Q_n^{(\alpha, \beta)}(x-10)],$$

$$(3.4) \quad P_n^{(\alpha, \beta)}(x) = \frac{1}{\pi} [e^{i\pi\alpha} Q_n^{(\alpha, \beta)}(x+10) - e^{-i\pi\alpha} Q_n^{(\alpha, \beta)}(x-10)].$$

The ultraspherical function of the second kind is defined as follows:

[16]

$$(3.5) \quad D_n^\lambda(x) = \frac{2}{\pi} \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} Q_n^{(\lambda-1/2, \lambda-1/2)}(x), \quad -1 < x < 1.$$

Another representation is [78]

$$\begin{aligned}
 (3.6) \quad (1-x)^2 \lambda^{-1/2} D_n^\lambda(x) &= \frac{2^\lambda (\lambda+1/2) (2\lambda)_k}{\Gamma(1/2) \Gamma(\lambda+1) k!} \sum_{\ell=0}^{\infty} \frac{(1-\lambda)_\ell (1)_{\ell+k}}{\ell! (\lambda+1)_{\ell+k}} \cos(k+2\ell+1)\theta, \\
 x &= \cos \theta.
 \end{aligned}$$

This definition of $D_n^\lambda(x)$ leads to another limit

$$(3.7) \quad \lim_{\lambda \rightarrow \infty} \lambda^{-n/2} D_n^\lambda (\lambda^{-1/2} x) = \frac{\pi^{1/2}}{2} \frac{G_n(x)}{n!}$$

where $G_n(x)$ is Hermite function of the second kind. Another important limit which we will prove in Chapter III is the following

$$(3.8) \quad \lim_{n \rightarrow \infty} n^{-\alpha} Q_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} \right) = \frac{\pi}{2} \left(\frac{x}{2} \right)^{-\alpha} Y_\alpha(x)$$

where $Y_\alpha(x)$ is Bessel function of the second kind [72]. Using (2.12) and (3.8) we obtain

$$(3.9) \quad -\frac{1}{2} x^\alpha Y_\alpha(x) = \int_0^\infty \frac{t^{\alpha+1} J_\alpha(t)}{x^2 - t^2} dt, \quad \text{Re } \alpha > -1, x > 0.$$

4. Integrals and Infinite Sums Containing Special Functions:

Integral and infinite sums of products of classical orthogonal polynomials have been a matter of curiosity for a long time (see [42] for an impressive list of integral formulas). Many of these formulas have also proved very useful. During the last fifteen years or so, Askey and some of his co-workers, [3-7, 9-11, 13] have pointed out some interesting applications of such integral and sum formulas. To mention one of the many such applications, Gasper [47, 48] proved that the linearization coefficient in

$$(4.1) \quad \frac{P_m^{(\alpha, \beta)}(x)}{P_m^{(\alpha, \beta)}(1)} \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)} = \sum_{k=|n-m|}^{n+m} g(k, m, n; \alpha, \beta) P_k^{(\alpha, \beta)}(x)$$

are non-negative for all non-negative (k, m, n) if and only if (α, β) belongs to a certain set. Gasper showed that the non-negativity of these coefficients

is all that is required to obtain a convolution structure, a Wiener-Lévy theorem and the positivity of a generalized translation operator for Jacobi polynomials. Orthogonality of the Jacobi polynomials implies that $g(k,m,n; \alpha, \beta)$ in (4.1) is a positive multiple of the integral

$$(4.2) \quad \int_{-1}^1 (1-x)^{\alpha} (1+x)^{\beta} P_k^{(\alpha, \beta)}(x) P_m^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(x) dx.$$

Recently Rahman [67] was able to evaluate this integral as a single series with non-negative terms in the case $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$. As a dual to the linearization problem, Gasper showed that the kernel $K(x, y, z; \alpha, \beta)$ in

$$(4.3) \quad \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)} \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(1)} = \int_{-1}^1 K(x, y, z; \alpha, \beta) \frac{P_n^{(\alpha, \beta)}(z)}{P_n^{(\alpha, \beta)}(1)} \cdot (1-z)^{\alpha} (1+z)^{\beta} dz$$

is non-negative on the set

$$(4.4) \quad U = \{(\alpha, \beta) \mid \alpha \geq \beta \geq -1/2, \alpha > -1/2\}.$$

Orthogonality implies

$$(4.5) \quad K(x, y, z; \alpha, \beta) = \sum_{n=0}^{\infty} h_n^{(\alpha, \beta)} P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) P_n^{(\alpha, \beta)}(z) \frac{n!}{(\alpha+1)_n},$$

where the normalization constant $h_n^{(\alpha, \beta)}$ is given by

$$(4.6) \quad h_n^{(\alpha, \beta)} = \left\{ \int_{-1}^1 (1-x)^{\alpha} (1+x)^{\beta} [P_n^{(\alpha, \beta)}(x)]^2 dx \right\}^{-1} \\ = \frac{(2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1) n!}{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}, \quad \operatorname{Re} \alpha, \beta > -1.$$

Gasper [49, 50] used the non-negativity of this kernel to construct a convolution structure of the Jacobi series and to prove the positivity of a generalized translation operator.

Special functions and many of their integral and sum formulas have perhaps been more widely used by physicists than mathematicians. To mention only one example that is relevant to this work, the Clebsch-Gordon coefficient with zero magnetic quantum numbers have the following representation [33, 38].

$$(4.7) \quad [C_{komo}^{no}]^2 = \frac{2n+1}{2} \int_{-1}^1 P_k(x) P_m(x) P_n(x) dx,$$

where $P_n(x) = P_n^{(0,0)}(x)$ is the Legendre polynomial of degree n .

Obviously the integral in (4.7) is as in (4.2) for $\alpha=\beta=0$. From the symmetry and orthogonality properties of Legendre polynomials it is clear that the integral in (4.7) vanishes if $k+m+n$ is odd and the triangle inequality

$$(4.8) \quad |n-m| \leq k \leq n+m$$

is not satisfied. These inequalities must also satisfy for the non-vanishing of the integral (4.2).

In trying to classify the various fluctuation modes in an investigation of the stability properties of some special solutions of the $O(n)$ non-linear 2-dimensional, σ -model Din and Zahrzewski [34] found that the classification depended crucially on the vanishing of the sum

$$\sum_{k=|n-m|}^{n+m} \frac{2k+1}{k(k+1)-m(m+1)} [C_{k0m0}^{no}]^2.$$

Din showed that this vanishing follows from

$$(4.9) \quad \int_{-1}^1 Q_k(x) P_m(x) P_n(x) dx = 0,$$

where $k+m+n$ is odd, $|n-m| < k < n+m$, and $Q_k(x)$ is an appropriately defined Legendre function [40,78] of second kind on the cut $-1 < x < 1$. Then he showed in [33] that (4.9) holds. This is very interesting and somewhat surprising result because the integral (4.9) vanishes on the set complementary to the set where (4.7) vanishes. Din did not compute its value when k, m, n do not satisfy (4.8). The integral (4.9) obviously vanishes when $k+m+n$ is even. Askey [10] showed that

$$(4.10) \quad \int_{-1}^1 Q_{\ell+m+1+2j}(x) P_m(x) P_\ell(x) dx \\ = - \frac{\Gamma(j+1/2) \Gamma(j+\ell+1) \Gamma(j+m+1) \Gamma(j+\ell+m+3/2)}{2\Gamma(j+1) \Gamma(j+\ell+3/2) \Gamma(j+m+3/2) \Gamma(j+\ell+m+2)}$$

for $j = 0, 1, 2, \dots$ and,

$$(4.11) \quad \int_{-1}^1 Q_{\ell+m+1-2j}(x) P_m(x) P_\ell(x) dx \\ = \frac{\Gamma(j-\ell-1/2) \Gamma(\ell+m-j+3/2) \Gamma(j) \Gamma(m-j+1)}{2\Gamma(j-\ell) \Gamma(\ell+m-j+2) \Gamma(j+1/2) \Gamma(m-j+3/2)}, \\ \text{when } \ell < j \leq \left[\frac{\ell+m+1}{2} \right].$$

Dougall's [57] generalization of the integral in (4.7) to ultraspherical polynomials is well-known

$$\int_{-1}^1 C_k^\lambda(x) C_m^\lambda(x) C_n^\lambda(x) (1-x^2)^{\lambda-1/2} dx$$

(4.12)

$$= \frac{\Gamma(1/2) \Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \frac{(\lambda)_{s-k} (\lambda)_{s-m} (\lambda)_{s-n} (2\lambda)_s}{(s-k)! (s-m)! (s-n)! (\lambda+1)_s},$$

where $k+m+n = 2s$ is even and $|n-m| \leq k \leq n+m$ and zero otherwise.

Recently Askey, Koornwinder and Rahman [16] found a generalization of Din's integral. They showed that

$$(4.13) \quad I(k, m, n) = \int_{-1}^1 D_k^\lambda(x) C_m^\lambda(x) C_n^\lambda(x) (1-x^2)^{2\lambda-1} dx$$

vanishes when

- (i) $k+m+n$ is even,
- (ii) $k+m+n$ is odd but $|n-m| < k < n+m$,

They showed that

$$(4.14) \quad I(m+n+1+2k, m, n) = - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_m (2\lambda)_n (2\lambda)_{m+n+1+2k} (1-\lambda)_k (\lambda)_{m+n+1+k}}{m! n! (m+n+1+2k)! k! (2\lambda)_{m+n+1+k}} \cdot \frac{(\lambda+1)_{m+k} (\lambda+1)_{n+k}}{(\lambda+1)_{m+k} (\lambda+1)_{n+k}},$$

$k = 0, 1, 2, \dots$, while

$$(4.15) \quad I(m+n+1-2k, m, n) = - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_m (2\lambda)_n (2\lambda)_{m+n+1-2k} (-\lambda)_{k-m} (\lambda)_{m+n+1-k} (n-k)! \Gamma(k)}{m! n! (m+n+1-2k)! (\lambda)_k (2\lambda)_{m+n+1-k} (\lambda+1)_{n-k} \Gamma(k-m)},$$

where $0 \leq m \leq n$ and $k = 1, 2, \dots, m, m+1, \dots, \left[\frac{m+n+1}{2} \right]$.

In the next chapter we will find duals of (4.13), (4.14) and (4.15) in the same sense as the dual to Dougall's integral (4.12) is Dougall's infinite sum [57]

$$\begin{aligned}
 (4.16) \quad & \sum_{n=0}^{\infty} (n+\lambda) \left\{ \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} \right\}^2 C_n^{\lambda}(\cos \alpha) C_n^{\lambda}(\cos \beta) C_n^{\lambda}(\cos \gamma) \\
 &= 2^{-2\lambda} \pi \{\Gamma(\lambda)\}^{-4} (\sin \alpha \sin \beta \sin \gamma)^{1-2\lambda} \left\{ \sin \frac{\alpha+\beta+\gamma}{2} \sin \frac{\beta+\gamma-\alpha}{2} \right. \\
 &\quad \left. \cdot \sin \frac{\gamma+\alpha-\beta}{2} \sin \frac{\alpha+\beta-\gamma}{2} \right\}^{\lambda-1}
 \end{aligned}$$

if $0 < \alpha, \beta, \gamma < \pi$, $0 < \operatorname{Re} \lambda$, and a spherical triangle can be drawn with sides α, β, γ (assuming that the sum of any two of them is $\leq \pi$) and 0 otherwise.

From Dougall's result (4.12) we have

$$(4.17) \quad \int_{-\infty}^{\infty} e^{-x^2} H_{\ell}(x) H_m(x) H_n(x) dx = \frac{2^s \ell! m! n!}{(s-\ell)! (s-m)! (s-n)!}$$

provided that $\ell + m + n = 2s$ is an even integer and $s \geq \alpha$, $s \geq \beta$, $s \geq \gamma$. In all other cases the integral is zero.

From Askey, Koornwinder and Rahman formulas (4.13), (4.14) and (4.15) we have

$$(4.18) \quad J(\ell, m; n) = \int_{-\infty}^{\infty} e^{-2x^2} H_{\ell}(x) H_m(x) G_n(x) dx$$

vanishes if

(i) $\ell + m + n$ is even,

(ii) $\ell + m + n$ is odd and $|\ell - m| < n < \ell + m$.

Also

$$(4.19) \quad J(m, n; m+n+1+2k) = \frac{(-1)^{k+1} 2^{m+n+k} (m+k)! (n+k)!}{k!}$$

$k = 0, 1, 2, \dots$, while

$$(4.20) \quad J(m, n; m+n+1-2k) = \frac{(-1)^{k-m} 2^{m+n-k} \Gamma(k)}{\Gamma(k-m)}$$

where $0 \leq m \leq n$ and $k=1, 2, \dots, m, m+1, \dots, [\frac{m+n+1}{2}]$.

In the next chapter we will show the dual to (4.13), (4.14) and (4.15) is

$$(4.21) \quad \sum_{n=0}^{\infty} (n+\lambda) \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} C_n^{\lambda}(\cos \beta) C_n^{\lambda}(\cos \gamma) D_n^{\lambda}(\cos \alpha) \\ = \frac{2^{2-2\lambda}}{\Gamma^2(\lambda)} \begin{cases} 0 & \text{if } |\beta-\gamma| < \alpha < \beta+\gamma \leq \pi, \\ 2^{-2\lambda-1} \sin \pi \lambda (-16D)^{-\lambda}, & \text{if } \alpha < |\beta-\gamma| \text{ or } \pi < \beta+\gamma < 2\pi, \\ -2^{-2\lambda-1} \sin \pi \lambda (-16D)^{-\lambda}, & \text{if } \alpha > \beta+\gamma \text{ and } \alpha+\beta+\gamma < 2\pi, \end{cases}$$

where $0 < \operatorname{Re} \lambda < 1/2, 0 < \alpha, \beta, \gamma < \pi$ and

$$(4.22) \quad -16D = \sin \frac{\alpha+\beta+\gamma}{2} \sin \frac{\alpha+\beta-\gamma}{2} \sin \frac{\beta+\gamma-\alpha}{2} \sin \frac{\beta-\gamma-\alpha}{2}.$$

Hsü [57] showed that Sonine's integral formula [86, p.411]

$$(4.23) \quad \int_0^{\infty} J_{\nu}(ax) J_{\nu}(bx) J_{\nu}(cx) x^{1-\nu} dx \\ = \begin{cases} 0 & \text{if } a, b, c \text{ are not sides of a triangle,} \\ \frac{2^{\nu-1} \Delta^{\nu-1/2}}{\sqrt{\pi} (abc)^{\nu} \Gamma(\nu+1/2)}, & \text{if } a, b, c, \text{ are sides of triangle of area } \Delta^{1/2}, \end{cases}$$

follows as limiting cases of both (4.12) and (4.16), provided $\text{Re } \nu > -1/2$.

$J_\nu(z)$ is related to the ultraspherical polynomials through Mehler's formula

$$(4.24) \quad \lim_{n \rightarrow \infty} n^{-2\nu} C_n^{\nu+1/2} \left(\cos \frac{z}{n} \right) = \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)} (2z)^{-\nu} J_\nu(z).$$

This can also be seen from (2.12). One can show using (3.8) that a similar limiting formula holds for $D_n^\lambda(z)$:

$$(4.25) \quad \lim_{n \rightarrow \infty} n^{-2\nu} D_n^{\nu+1/2} \left(\cos \frac{z}{n} \right) = -\frac{\sqrt{\pi}}{\Gamma(\nu+1/2)} (2z)^{-\nu} Y_\nu(z).$$

We will use (4.24) and (4.25) in (4.21) and (4.13) and obtain a formula complementary to (4.23) in the same sense as (4.12) is complementary to (4.13). However, because of the simplicity of Bessel functions, Askey, Koornwinder and Rahman [16] preferred a direct computation to show that

$$(4.26) \quad \int_0^\infty Y_\nu(ax) J_\nu(bx) J_\nu(cx) x^{1+\nu} dx = \begin{cases} 0 & , \text{ if } |b-c| < a < b+c \\ -\frac{2^{-\nu-1} (-\Delta)^{-\nu-1/2}}{\sqrt{\pi} \Gamma(1/2-\nu) (abc)^\nu} & , \text{ if } a < |b-c|, \\ \frac{2^{-\nu-1} (-\Delta)^{-\nu-1/2}}{\sqrt{\pi} \Gamma(1/2-\nu) (abc)^\nu} & , \text{ if } a > b+c. \end{cases}$$

In (4.23) and (4.26) Δ is given by the same algebraic expression, namely,

$$(4.27) \quad \Delta = \frac{1}{16} [(b+c)^2 - a^2] [a^2 - (b-c)^2].$$

However, in (4.23) it is positive and represents the square of the area of the plane triangle of sides a, b, c , whereas in (4.26) Δ is negative because a, b, c do not satisfy the triangle inequality. The complementary nature of the two formulas is vividly displayed by the striking similarity of the two expressions on the right hand sides of (4.23) and (4.26).

In Chapter II we first introduce a dual to the integral in (4.13) and carry out the computations that lead to (4.21). In Chapter III we prove (2.12), (3.8) and give two alternative derivations of (4.26) by applying Hsü's [57] limiting procedure to (4.21) and (4.13). Also, we evaluate an extension of Henrici's [54] integral formula. He considered the following integral

$$(4.28) \quad h(a,b,c) = \int_0^\infty J_\mu(ax) J_\nu(bx) J_\lambda(cx) dx.$$

This integral is convergent for $\text{Re}(\mu+\nu+\lambda) > 1$ and for arbitrary values of a, b , and c , has for $c > a+b$ been represented by Bailey [20] as a product of two hypergeometric series. Bailey's result, along with some special cases, is listed in [42]. Henrici [54] evaluated the integral in (4.28) when a, b , and c satisfy the triangle inequality. The combined result is

$$(4.29) \quad h(a,b,c) = \begin{cases} D(a,b,c) \sin(\mu+\nu-\lambda+1)\frac{\pi}{2} F_1(X) F_2(Y) & \text{if } c > a+b, \\ D(a,b,c) \text{Im} \{ e^{i(\mu+\nu-\lambda+1)\pi/2} F_1\left(\frac{a}{c} e^{-i\phi}\right) F_2\left(\frac{b}{c} e^{-i\psi}\right) \}, & \text{if } |a-b| < c < a+b. \end{cases}$$

where

$$(4.30) \quad F_1(X) = {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu+\lambda+1}{2}, & \frac{\mu+\nu-\lambda+1}{2} \\ \nu+1 \end{matrix} ; x \right],$$

$$F_2(Y) = {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu+\lambda+1}{2}, & \frac{\mu+\nu-\lambda+1}{2} \\ \mu+1 \end{matrix} ; x \right],$$

$$(4.31) \quad X = \frac{c^2 + a^2 - b^2 - 4\sqrt{-\Delta}}{2}, \quad Y = \frac{c^2 - a^2 + b^2 - 4\sqrt{-\Delta}}{2},$$

$$\phi = \tan^{-1} \frac{4\sqrt{\Delta}}{c^2 + a^2 - b^2}, \quad \psi = \tan^{-1} \frac{4\sqrt{\Delta}}{c^2 - a^2 + b^2},$$

$$(4.32) \quad D(a,b,c) = \frac{1}{\pi} \frac{\Gamma(\frac{\mu+\nu+\lambda+1}{2}) \Gamma(\frac{\mu+\nu-\lambda+1}{2})}{\Gamma(\mu+1) \Gamma(\nu+1)} \frac{a^\mu b^\nu}{c^{\mu+\nu+1}}$$

and Δ is same as in (4.27).

A useful extension of Bailey and Henrici's result (4.29) is to evaluate the following integral

$$(4.33) \quad f(a,b,c) = \int_0^\infty t^{1-\rho} J_\mu(at) J_\nu(bt) J_\lambda(ct) dt.$$

We will prove that.

$$\begin{aligned}
 (4.34) \quad f(a,b,c) = & \begin{cases} (c(a,b,c) \sin (\mu+\nu-\lambda-\rho+2) \frac{\pi}{2}) \sum_{n=0}^{\infty} \frac{\frac{(\mu+\nu+\lambda-\rho+2)}{2}_n \frac{(\mu+\nu-\lambda-\rho+2)}{2}_n (1-\rho)_n}{n! (\mu+1)_n (\nu+1)_n} \\ \quad \cdot F_n(X) G_n(Y) \text{ if } c > a+b, \\ (c(a,b,c) \operatorname{Im} \{ e^{\frac{i\pi}{2} (\mu+\nu-\lambda-\rho+2)} \sum_{n=0}^{\infty} \frac{\frac{(\mu+\nu+\lambda-\rho+2)}{2}_n \frac{(\mu+\nu-\lambda-\rho+2)}{2}_n (1-\rho)_n}{n! (\mu+1)_n (\nu+1)_n} \\ \quad \cdot F_n(\frac{a}{c} e^{-i\phi}) G_n(\frac{b}{c} e^{-i\psi}) \}) \\ \quad \text{if } |a-b| < c < a+b, \end{cases}
 \end{aligned}$$

where

$$(4.35) \quad C(a,b,c) = \frac{1}{\pi} \frac{\Gamma(\frac{\mu+\nu+\lambda-\rho+2}{2}) \Gamma(\frac{\mu+\nu-\lambda-\rho+2}{2})}{2^\rho \Gamma(\mu+1) \Gamma(\nu+1)} \frac{a^\mu b^\nu}{c^{\mu+\nu-\rho+2}},$$

$$(4.36) \quad F_n(X) = {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu-\rho+\lambda+2}{2} + n, & \frac{\mu+\nu-\rho-\lambda+2}{2} + n \\ \mu+1+n \end{matrix} ; x \right],$$

$$G_n(Y) = {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu-\rho+\lambda+2}{2} + n, & \frac{\mu+\nu-\rho-\lambda+2}{2} + n \\ \mu+1+n \end{matrix} ; y \right].$$

Setting $\rho=1$ immediately leads to Henrici-Bailey formula for the integral in (4.28). For $\rho=\mu=\lambda=\nu$, the integral on the r.h.s. of (4.33) becomes the integral on the l.h.s. of (4.23). So, we will obtain Sonine's formula from (4.34) in this special case. Gasper showed that [49]:

$$\begin{aligned}
 & K(\cos 2\phi, \cos 2\psi, \cos 2\theta; \lambda, \nu) \\
 (4.37) \quad & = 2^{-\nu-2} \Gamma(\lambda+1) (\sin \phi \sin \psi \sin^2 \theta)^{-\lambda} (\cos \phi \cos \psi \cos \theta)^{-\nu} \\
 & \cdot \int_0^\infty J_\lambda(t \sin \phi \sin \psi) J_\nu(t \cos \phi \cos \psi) J_\nu(t \cos \theta) t^{1-\lambda} dt
 \end{aligned}$$

when

$$\lambda > -1/2, \quad \nu > -1,$$

where $K(x, y, z; \alpha, \beta)$ is defined in (4.5). Clearly, the integral in (4.37) is a special case of the integral in (4.33) i.e., when $\rho=\lambda$, $\mu=\nu$. We will evaluate the integral in (4.37) for $\cos \theta > \cos(\phi+\psi)$ and $|\cos(\phi+\psi)| < \cos \theta < \cos(\phi-\psi)$ in terms of Legendre functions of the first and second kind.

In trying to find the unitary relation for the Coulomb Potential in Quantum Mechanics and some related series of products of Legendre functions van Haeringen [84] evaluated the following infinite sum:

$$\begin{aligned}
 & \sum_{n=0}^{\infty} (2n+1) P_n(x) P_n(y) P_n(z) Q_n(u) \\
 (4.38) \quad & = [(f+h)^2 - 4g^2]^{-1/2} P_{-1/2}(x)
 \end{aligned}$$

where f , g , and h are functions of x , y , z and u given by:

$$\begin{aligned}
 f &= x^2 y^2 + z^2 + u^2 - 2xyzu - 1, \\
 g &= (1-x^2)^{1/2} (1-y^2)^{1/2} (xy-zu), \\
 (4.39) \quad h &= (1-x^2) (1-y^2), \\
 \chi &= (f-h) [(f+h)^2 - 4g^2]^{-1/2},
 \end{aligned}$$

where x, y, z are arbitrary complex numbers and u is in the complex plane cut along the segment $(-1, 1)$. Van Haeringen's formula (4.38) is easier to find because he was interested only in the cut plane, not because they are physically more relevant but because they are mathematically simpler to compute. His work therefore remained incomplete as far as evaluation of the series (4.38) is concerned on the cut $(-1, 1)$. An extension of (4.38) is the following infinite sum

$$\begin{aligned}
 & \sum_{n=0}^{\infty} (n+\lambda) \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} C_n^{\lambda}(\cos \alpha) C_n^{\lambda}(\cos \beta) C_n^{\lambda}(\cos \gamma) D_n^{\lambda}(\cos \tau) \\
 (4.40) \quad & \left\{ \begin{array}{ll} -\kappa I & \text{if } \cos(\gamma+\tau) < \cos(\alpha+\beta) \text{ and} \\ & \cos(\gamma-\tau) < \cos(\alpha+\beta), \\ -\kappa I & \text{if } \cos(\gamma+\tau) < \cos(\alpha+\beta) \text{ and} \\ & \cos(\alpha+\beta) < \cos(\gamma-\tau) < \cos(\alpha-\beta), \\ 0 & \text{if } \cos(\alpha+\beta) < \cos(\gamma+\tau) < \cos(\alpha-\beta) \text{ and} \\ & \cos(\alpha+\beta) < \cos(\gamma-\tau) < \cos(\alpha-\beta), \\ 0 & \text{if } \cos(\gamma-\tau) < \cos(\alpha-\beta) \text{ and} \\ & \cos(\gamma+\tau) < \cos(\alpha+\beta), \\ \kappa I & \text{if } \cos(\alpha+\beta) < \cos(\alpha+\tau) < \cos(\alpha-\beta) \text{ and} \\ & \cos(\gamma-\tau) > \cos(\alpha-\beta), \\ \kappa I & \text{if } \cos(\gamma+\tau) > \cos(\alpha-\beta) \text{ and} \\ & \cos(\gamma-\tau) > \cos(\alpha-\beta), \end{array} \right.
 \end{aligned}$$

where $0 \leq \alpha, \beta, \gamma \leq \pi$, $0 < \tau < \pi$, $0 < \operatorname{Re} \lambda < 1/2$,

$$\begin{aligned}
 \kappa &= \frac{\sin \lambda \pi}{\pi} \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)} \right]^2 \Gamma(2\lambda), \\
 I &= 2^{2\lambda-1} \pi \frac{[(f+g)^2 - 4g^2]^{-\lambda/2}}{\sin \lambda \pi} C_{-\lambda}^{\lambda}(x), \\
 (4.41) \quad I' &= 2^{2\lambda-1} [g(1+x_1)(1-x_0)]^{-\lambda} \Gamma(\lambda) \Gamma(1-\lambda) {}_2F_1 \left[\begin{matrix} \lambda, \lambda \\ 1 \end{matrix}; \frac{(1+x_0)(1-x_1)}{(1+x_1)(1-x_0)} \right], \\
 x_0 &= \frac{\cos(\gamma+\tau) - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}, \quad x_1 = \frac{\cos(\gamma-\tau) - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}.
 \end{aligned}$$

The infinite sum on the left of (4.38) is dual to the following integral

$$(4.42) \quad I(k; 1, m, n) = \int_{-1}^1 D_k^{\lambda}(x) C_1^{\lambda}(x) C_m^{\lambda}(x) C_n^{\lambda}(x) (1-x^2)^{2\lambda-1} dx.$$

We will evaluate this integral as a multiple of a very well-poised, 2-balanced and terminating ${}_9F_8$ series.

CHAPTER II

AN INFINITE SERIES WITH PRODUCTS OF JACOBI POLYNOMIALS
AND JACOBI FUNCTIONS OF THE SECOND KIND

5. Introduction:

In both pure research and applied sciences, one often obtains a finite or infinite series and needs to know if there exists an equivalent closed form. Series can serve essential purposes. First, it can provide the means to express some given mathematical function as a sum of more elementary functions. Thus for example, functions like $f(x) = \sin x$ as Taylor series in powers of x . One might want this information for many reasons. The second use for computation of a series is a reverse of this process.

Some applications make use of both these processes. For example, one may wish to evaluate an integral by expanding the integrand in series and integrating term by term. One then wants a closed form for the series so obtained. Similarly, a given series can often be summed by expanding the summand in series, interchanging order of the summation, and then summing the new inner series.

Infinite series containing products of special functions are very useful to the mathematicians and physicists. In reference [83] van Haeringen and Kok have listed some of the infinite series of the product of Legendre functions. Let

$$\begin{aligned}
 & F_{mn}(x_1, \dots, x_m; y_1, y_2, \dots, y_n) \\
 (5.1) \quad & = \sum_{\ell=0}^{\infty} (2\ell+1) P_{\ell}(x_1) P_{\ell}(x_2) \dots P_{\ell}(x_m) Q_{\ell}(y_1) Q_{\ell}(y_2) \dots Q_{\ell}(y_n)
 \end{aligned}$$

for $m, n = 0, 1, 2, 3, \dots$, x_i are arbitrary complex numbers and y_i are in the cut plane $C-[-1, 1]$.

$$\begin{aligned}
 (5.2) \quad F_{12}(x; y, z) &= \sum_{\ell=0}^{\infty} (2\ell+1) P_{\ell}(x) Q_{\ell}(y) Q_{\ell}(z) \\
 &= 1/2 W^{-1/2} \ln \left[\frac{yz-x+w^{1/2}}{yz-x-w^{1/2}} \right],
 \end{aligned}$$

$$\begin{aligned}
 (5.3) \quad F_{21}(x, y; z) &= \sum_{\ell=0}^{\infty} (2\ell+1) P_{\ell}(x) P_{\ell}(y) Q_{\ell}(z) \\
 &= W^{-1/2},
 \end{aligned}$$

$$(5.4) \quad F_{11}(y; z) = (z-y)^{-1}$$

where $W = W(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$.

Equation (5.4) is called Heine's formula.

These formulas (5.2)-(5.4) are not valid for y_i on the cut $(-1, 1)$. In most of the problems of the physical interest we need to evaluate the sums of the form (5.1) when $y_i \in (-1, 1)$. The remainder of this chapter is devoted to evaluating the series on the l.h.s. of (4.21) on the cut $(-1, 1)$ which is a generalization of the sum on the l.h.s. of (5.3) because we use $C_n^{\lambda}(x)$ and $D_n^{\lambda}(x)$ instead of $P_n(x)$ and $Q_n(x)$ respectively and

$$(5.5) \quad C_n^{1/2}(x) = P_n(x), \quad D_n^{1/2}(x) = \frac{2}{\pi} Q_n(x).$$

6. The Dual Kernel:

The essential ingredients of this work are contained in Rahman [66] where he gave a more accessible proof and extensions of the following formula of Feldheim [44]:

$$(6.1) \quad F_4(\alpha_1, \alpha_2; \alpha+1, \beta+1; \frac{\rho(1-x)(1-y)}{4}, \frac{\rho(1+x)(1+y)}{4}) \\ = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n (\alpha+\beta+1)_n}{n! (\beta+1)_n (\alpha+1)_{2n}} (\alpha_1)_n (\alpha_2)_n \rho^n {}_2F_1 \left[\begin{matrix} \alpha_1+n, \alpha_2+n \\ \alpha+\beta+2+2n \end{matrix}; \rho \right] \\ \cdot \frac{P_n^{(\alpha, \beta)}(x)}{P_n^{(\alpha, \beta)}(1)} \cdot \frac{P_n^{(\alpha, \beta)}(y)}{P_n^{(\alpha, \beta)}(1)}$$

where the parameters $\alpha, \beta, \alpha_1, \alpha_2, \rho$ are restricted by the requirement of convergence of the infinite series involved, and F_4 is an Appell function [17, 39, 76] defined by

$$(6.2) \quad F_4(\alpha_1, \alpha_2; \beta_1, \beta_2; u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha_1)_{m+n} (\alpha_2)_{m+n}}{m! n! (\beta_1)_m (\beta_2)_m} u^m v^n.$$

In [66] Rahman put $\alpha_1=1, \alpha_2=\beta+1, \rho = \frac{2}{1+z}$ in (6.1) and used (2.12) to give an alternative proof of Gasper's result [49]:

$$K(x, y, z; \alpha, \beta)$$

$$(6.3) \quad \left\{ \begin{array}{l} 0 \quad \text{if } f < |g-h| \\ \frac{\Gamma(\alpha+1) (1-h^2)^{-\alpha} f^{-2\alpha} (f^2 - g^2 - h^2)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta) \Gamma(\beta+1) 2^{\alpha+\beta+1}} {}_2F_1 \left[\begin{matrix} \frac{\beta-\alpha+1}{2}, \frac{\beta-\alpha+2}{2} \\ \beta+1 \end{matrix}; G^{-2} \right], \\ h < f-g, \\ \text{Re } (\alpha-\beta) > 0, \\ \frac{\Gamma(\alpha+1) (1-h^2)^{-\alpha} f^{-2\alpha} (gh)^{\alpha-\beta-1} (1-G^2)^{\alpha-1/2}}{\Gamma(\alpha+1/2) \Gamma(1/2) 2^{\alpha+\beta+2}} {}_2F_1 \left[\begin{matrix} \alpha-\beta, \alpha+\beta \\ \alpha+1/2 \end{matrix}; \frac{1}{2} (1-G) \right], \\ |f-g| < h < f+g, \text{ Re } \alpha > -1/2, \end{array} \right.$$

where $K(x, y, z; \alpha, \beta)$ is defined in (4.5), and

$$(6.4) \quad \begin{aligned} f &= \sin \phi \sin \psi, \quad g = \cos \phi \cos \psi, \quad h = \cos \theta \\ x &= \cos 2\phi, \quad y = \cos 2\psi, \quad z = \cos 2\theta \end{aligned} \quad 0 < \theta, \phi, \psi < \frac{\pi}{2},$$

$$(6.5) \quad G = \frac{g^2 + h^2 - f^2}{2gh}.$$

One could have used (3.8) in (6.1) with the above choices α_1 and α_2 to obtain a different kernel, but it is not the right kernel because it does not vanish in the triangular region $|f-g| < g < f+g$, analogous to the vanishing of the integral (4.13), unless $\alpha=\beta=0$. The reason is similar to the fact that the appropriate weight factor in (4.13) is $(1-x^2)^{2\lambda-1}$ and not $(1-x^2)^{\lambda-1/2}$, as one might expect. As it turns out, the appropriate dual to (4.13) is obtained by making the following identification of parameters in (6.1):

$$(6.6) \quad \alpha_1 = \alpha + \beta + 1, \quad \alpha_2 = \alpha + 1, \quad \rho = \frac{2}{1+z};$$

thus we have, on using the obvious symmetry of F_4

$$(6.7) \quad F_4(\alpha+\beta+1, \alpha+1; \beta+1, \alpha+1, \frac{(1+x)(1+y)}{2(1+z)}, \frac{(1-x)(1-y)}{2(1+z)}) \\ = \sum_{n=0}^{\infty} \frac{n! (\alpha+\beta+1)_n}{(\beta+1)_n (\alpha+\beta+1)_{2n}} (\alpha+\beta+1)_n \left(\frac{2}{1+z}\right)^n \\ \cdot {}_2F_1 \left[\begin{matrix} \alpha+\beta+1+n, \alpha+1+n \\ \alpha+\beta+2+2n \end{matrix}; \frac{2}{1+z} \right] P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y).$$

Using [39], 2.1.4 (23), p.64], the second formula in (3.1) and (4.6), we get.

$$(6.8) \quad F_4(\alpha+\beta+1, \alpha+1; \beta+1, \alpha+1; \frac{(1+x)(1+y)}{2(1+z)}, \frac{(1-x)(1-y)}{2(1+z)}) \\ = 2(z+1)^{\alpha+\beta+1} \sum_{n=0}^{\infty} h_n^{(\alpha, \beta)} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) Q_n^{(\alpha, \beta)}(z),$$

where $-1 \leq x, y \leq 1$ and z is in the cut-plane. Using (3.3) we have, for $-1 \leq x, y \leq 1$ and $-1 < z < 1$,

$$(6.9) \quad L(x, y, z; \alpha, \beta) = \sum_{n=0}^{\infty} h_n^{(\alpha, \beta)} \frac{(\alpha+\beta+1)_n}{(\beta+1)_n} P_n^{(\alpha, \beta)}(x) P_n^{(\alpha, \beta)}(y) Q_n^{(\alpha, \beta)}(z) \\ = \frac{1}{4} (z+1)^{-\alpha-\beta-1} \lim_{\epsilon \rightarrow 0} [e^{i\pi\alpha} A(x, y, z+i\epsilon) \\ + e^{-i\pi\alpha} A(x, y, z-i\epsilon)]$$

where, for abbreviation, we write

$$(6.10) \quad A(x, y, z) = F_4(\alpha+\beta+1, \alpha+1; \beta+1, \alpha+1; \frac{(1+x)(1+y)}{2(1+z)}, \frac{(1-x)(1-y)}{2(1+z)}).$$

There are some convergence difficulties in both (6.8) and (6.9), the series on the l.h.s. of (6.8) may diverge if z , but neither x nor y is close to -1 . One can use an analytic continuation [39, 5.11 (9), p.240] to remove the difficulty with the F_4 function. However, series in the r.h.s. of (6.9) the limit diverges for $\text{Re } \alpha \geq 1/2$. So our results are necessarily restricted to $-1 < \text{Re } \alpha < 1/2$.

7. Computation of $L(x, y, z; \alpha, \beta)$:

The Appell function in (6.10) is absolutely convergent if

$$(7.1) \quad \left| \frac{(1+x)(1+y)}{2(1+z)} \right|^{1/2} + \left| \frac{(1-x)(1-y)}{2(1+z)} \right|^{1/2} < 1.$$

With $\theta, \phi, \psi, f, g, h$ defined as in (6.4) this condition implies

$$(7.2) \quad f + g < h.$$

When this inequality is satisfied, $A(x, y, z)$ is single-valued on the real axis of z -plane, so we simply have

$$(7.3) \quad L(x, y, z; \alpha, \beta) = \frac{1}{2} \cos \pi \alpha (2h^2)^{-\alpha-\beta-1} F_4(\alpha+\beta+1, \alpha+1; \beta+1, \alpha+1; g^2/h^2, f^2/h^2).$$

When (7.1) is not satisfied, $A(x, y, z)$ may not converge, but by [17, 20(v), p.102] it can be expressed in terms of a ${}_2F_1$ which admits many analytic continuations. Let

$$(7.4) \quad g^2/h^2 = -\frac{s}{(1-s)(1-t)}, \quad f^2/h^2 = \frac{t}{(1-s)(1-t)}.$$

Solving for s and t we get

$$(7.5) \quad s = \frac{f^2 + g^2 - h^2 \pm 4\sqrt{-D}}{2f^2}, \quad t = \frac{f^2 + g^2 - h^2 \pm 4\sqrt{-D}}{2g^2},$$

where

$$(7.6) \quad D = ((f+g)^2 - h^2)(h^2 - (f-g)^2)/16$$

is the same as Δ in (4.27) with a, b, c replaced by h, f and g , respectively, and equals the square of the area of a plane triangle of sides f, g, h whenever $D > 0$. However, if $h < |f-g|$ or $h > f+g$, D is negative and, by (7.5), s, t are both real. If f, g, h are real and positive, as they are whenever $0 < \theta, \phi, \psi < \pi/2$ and satisfy the triangle inequality $|f-g| < h < f+g$, it follows from (7.5) that $\text{Im } s \neq 0$, $\text{Im } t \neq 0$.

We could express D in terms of θ, ϕ, ψ as follows:

$$(7.7) \quad -64D = (z-h_+)(z-h_-),$$

where $h_{\pm} = \cos 2(\phi \pm \psi)$.

As a function of z , $\sqrt{-D}$ has a branch-point at h_+ and another at h_- . In the z -plane cut along the real axis from $\min(h_{\pm})$ to $\max(h_{\pm})$, $\sqrt{-D}$ is single-valued and so, for $|\text{Re } z| > h_{\pm}$ we may choose either of the two signs in (7.5). We choose the following branch:

$$(7.8) \quad s = \frac{f^2 + g^2 - h^2 - 4\sqrt{-D}}{2f^2}, \quad t = \frac{f^2 + g^2 - h^2 - 4\sqrt{-D}}{2g^2}.$$

Let us introduce two auxiliary parameters F and G :

$$(7.9) \quad F = \frac{f^2 + g^2 - h^2}{2fg}, \quad G = \frac{g^2 + h^2 - f^2}{2gh}.$$

Of course, G is the same as in (7.5). One can easily check the following identities:

$$(7.10) \quad f^2 g^2 (F^2 - 1) = -4D = g^2 h^2 (G^2 - 1),$$

$$(7.11) \quad s = \frac{g}{f} (F - \sqrt{F^2 - 1}) = \frac{g^2}{f^2} (1 - h/g (G + \sqrt{G^2 - 1})),$$

$$(7.12) \quad t = \frac{f}{g} (F - \sqrt{F^2 - 1}) = 1 - h/g (G + \sqrt{G^2 - 1}).$$

Whenever s, t are real it follows from (7.4) that either

$$(i) \quad s < 0, t < 0,$$

$$(7.13) \quad (ii) \quad 0 < s < 1, t > 1,$$

$$(iii) \quad 0 < t < 1, s > 1.$$

Let us now consider two separate cases.

Case I $D < 0$

This is the case when f, g, h do not satisfy the triangle inequality, that is either $h < |f-g|$ or $f+g < h$. By Bailey [17, 20(v), p.102]

$$(7.14) \quad F_4(\alpha+\beta+1, \alpha+1; \beta+1, \alpha+1; \frac{-s}{(1-s)(1-t)}, \frac{-t}{(1-s)(1-t)}) \\ = (1-t)^{\alpha+\beta+1} {}_2F_1 \left[\begin{matrix} \alpha+\beta+1, \alpha+1 \\ \beta+1 \end{matrix} ; \frac{s(t-1)}{1-s} \right].$$

Denoting $v = \frac{s(t-1)}{1-s}$ and using the quadratic transformation

[39, 2.11(34), p.113]:

$$(7.15) \quad {}_2F_1 \left[\begin{matrix} \alpha_1, \alpha_2 \\ \alpha_1 - \alpha_2 + 1 \end{matrix} ; v \right] = (1+v)^{-\alpha_1} {}_2F_1 \left[\begin{matrix} \alpha_1/2, \frac{\alpha_1+1}{2} \\ \alpha_1 - \alpha_2 + 1 \end{matrix} ; \frac{4v}{(1+v)^2} \right]$$

we have

$$(7.16) \quad F_4[\alpha+\beta+1, \alpha+1; \beta+1, \alpha+1; \frac{-s}{(1-s)(1-t)}, \frac{-t}{(1-s)(1-t)}] \\ = \left(\frac{1-t}{1+v}\right)^{\alpha+\beta+1} {}_2F_1 \left[\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2} \\ \beta+1 \end{matrix} ; \frac{4v}{(1+v)^2} \right]$$

Let us first assume $f+g < h$ for which kernel in terms of the Appell function is given in (7.3). Using (7.16) we show now express it as a ${}_2F_1$ function. From the definition of F and G it follows that $F < -1$, $G > 1$ and hence $s < 0$, $t > 0$, $v > 0$. Using (7.9), (7.11) and (7.12) one can easily show that

$$(7.17) \quad \frac{1-t}{1+v} = \frac{h}{2g} G^{-1} \text{ and } \frac{4v}{(1+v)^2} = G^{-2}.$$

Thus it follows from (7.3), (7.16) and (7.17) that

$$(7.18) \quad L(x, y, z; \alpha, \beta) = 2^{-\alpha-\beta-2} \cos \pi \alpha (g^2 + h^2 - f^2)^{-\alpha-\beta-1} {}_2F_1 \left[\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2} \\ \beta+1 \end{matrix} ; G^{-2} \right], \\ -1 < \operatorname{Re} \alpha < 1/2.$$

For $\alpha = \beta$, this leads to

$$(7.19) \quad L(x, y, z; \beta, \beta) = 2^{-6\beta-4} (-D)^{-(\beta+1/2)} \cos \pi \beta, f+g < h, -1 < \beta < 1/2.$$

Let us now consider the case $h < |f-g|$. First, assume that $f < g$ so that $h < g-f$. In this case $F > 1$, $G > 1$ and consequently $0 < t < 1$ and $s > 1$ while $v > 0$. Hence, the r.h.s. of (7.16) is single-valued with the result that the kernel is given by the same (7.18).

We finally look for the case $g < f$ with $h < f-g$. Here $F > 1$ and $G < -1$ so that $0 < s < 1$, $t > 1$ and $v > 0$. The r.h.s. of (7.16) is double-valued in this case, so we take $\arg(1-t) = -\pi$ in the upper half-plane and $\arg(1-t) = \pi$ in the lower half-plane. Then from (6.9), (7.16) and (7.16) we get

$$(7.20) \quad L(x, y, z; \alpha, \beta) = 2^{-\alpha-\beta-2} \cos \pi(\beta+1) (f^2 - g^2 - h^2)^{-\alpha-\beta-1} \cdot {}_2F_1 \left[\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2}; G^{-2} \right].$$

In the ultraspherical case $\alpha = \beta$ this reduces to:

$$(7.21) \quad L(x, y, z; \beta, \beta) = -2^{-6\beta-4} (-D)^{-(\beta+1/2)} \cos \pi\beta, \quad h < f-g, \quad -1 < \operatorname{Re} \beta < 1/2,$$

Case II $D > 0$

In this case f, g, h do form the sides of a plane triangle so that

$$(7.22) \quad |f-g| < h < f+g.$$

As a function of z , $\sqrt{-D}$ is purely imaginary and double-valued across the cut $\min(h_{\pm}) < z < \max(h_{\pm})$ and we take the branch $i\sqrt{D}$ in the upper half-plane and $-i\sqrt{D}$ in the lower half-plane. So in the upper half-plane

$$(7.23) \quad s = \frac{f^2 + g^2 - h^2 - 4i\sqrt{D}}{2f^2}, \quad t = \frac{f^2 + g^2 - h^2 - 4i\sqrt{D}}{2g^2},$$

while in the lower half-plane

$$(7.24) \quad s = \frac{f^2 + g^2 - h^2 + 4i\sqrt{D}}{2f^2}, \quad t = \frac{f^2 + g^2 - h^2 + 4i\sqrt{D}}{2g^2}.$$

In this case $-1 < F, G < 1$. Also, from (7.12)

$$t = 1 - \frac{hG}{g} - \frac{hG}{g} \sqrt{1-G^{-1}}.$$

In order that this agrees with the choice of branches in (7.23) and (7.24) we need to choose $\arg(1-G^{-2}) = \pi$ in the upper half and $-\pi$ in the lower half-plane. So

$$(7.25) \quad \pm \pi = \arg(1-G^{-2}) = \arg(-G^{-2})$$

in the upper and lower half planes, respectively. From (6.10), (3.16) and (3.17) we have by [39, 2.9(34), p.107]

$$\begin{aligned} A(x,y,z) &= \left(\frac{h}{2gG}\right)^{\alpha+\beta+1} {}_2F_1 \left[\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta+2}{2} \\ \beta+1 \end{matrix} ; G^{-2} \right] \\ (7.26) \quad &= \left(\frac{h}{2gG}\right)^{\alpha+\beta+1} \frac{\Gamma(\beta+1) \Gamma(1/2) (-G^{-2})^{\frac{\alpha+\beta+1}{2}}}{\Gamma(\frac{\alpha+\beta+2}{2}) \Gamma(\frac{\beta-\alpha+1}{2})} \\ &\quad \cdot {}_2F_1 \left[\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2} \\ \frac{\alpha+\beta+2}{2} \end{matrix} ; G^2 \right] \\ &\quad + \frac{\Gamma(\beta+1) \Gamma(-1/2) (-G^{-2})^{\frac{\alpha+\beta+2}{2}}}{\Gamma(\frac{\alpha+\beta+1}{2}) \Gamma(\frac{\beta-\alpha}{2})} {}_2F_1 \left[\begin{matrix} \frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+2}{2} \\ 3/2 \end{matrix} ; G^2 \right]. \end{aligned}$$

Hence:

$$\begin{aligned}
& \lim_{\epsilon \rightarrow 0} [e^{i\pi\alpha} A(x, y, z+i\epsilon) + e^{-i\pi\alpha} A(x, y, z-i\epsilon)] \\
(7.27) \quad &= 2 \left(\frac{h}{2g}\right)^{\alpha+\beta+1} \frac{\Gamma(\beta+1) \Gamma(1/2)}{\Gamma(\frac{\alpha+\beta+1}{2}) \Gamma(\frac{\beta-\alpha+1}{2})} \cos \frac{\alpha-\beta-1}{2} \pi {}_2F_1 \left[\frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2}; G^2 \right]_{1/2} \\
&\quad - 2 \left(\frac{h}{2g}\right)^{\alpha+\beta+1} \frac{\Gamma(\beta+1) \Gamma(-1/2)}{\Gamma(\frac{\alpha+\beta+1}{2}) \Gamma(\frac{\beta-\alpha}{2})} \cos \frac{\alpha-\beta}{2} \pi {}_2F_1 \left[\frac{\alpha+\beta+2}{2}, \frac{\alpha-\beta+2}{2}; G^2 \right]_{3/2}.
\end{aligned}$$

Thus, in the region $|f-g| < h < f+g$,

$$\begin{aligned}
(7.28) \quad L(x, y, z; \alpha, \beta) &= \frac{1}{2} (4gh)^{-\alpha-\beta-1} \left\{ \frac{\Gamma(\beta+1) \Gamma(1/2)}{\Gamma(\frac{\alpha+\beta+2}{2}) \Gamma(\frac{\beta-\alpha+1}{2})} \right. \\
&\quad \cdot \sin \frac{\alpha-\beta}{2} \pi {}_2F_1 \left[\frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2}; G^2 \right]_{1/2} \\
&\quad \left. + \frac{\Gamma(\beta+1) \Gamma(-1/2)}{\Gamma(\frac{\alpha+\beta+1}{2}) \Gamma(\frac{\beta-\alpha}{2})} \cos \frac{\alpha-\beta}{2} \pi {}_2F_1 \left[\frac{\alpha+\beta+2}{2}, \frac{\alpha-\beta+1}{2}; G^2 \right]_{3/2} \right\}.
\end{aligned}$$

Using the reflection formula for the gamma function:

$$(7.29) \quad \Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad 0 < \operatorname{Re} z < 1,$$

and [39, 2.11(3), p.111], this can be further simplified to

$$\begin{aligned}
(7.30) \quad L(x, y, z; \alpha, \beta) &= -\frac{1}{2} (4gh)^{-\alpha-\beta-1} \frac{\pi \Gamma(\beta+1)}{\Gamma(\alpha+\frac{3}{2}) \Gamma(\frac{\beta-\alpha}{2}) \Gamma(\frac{\beta-\alpha+1}{2})} \\
&\quad \cdot {}_2F_1 \left[\frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2}; \frac{1}{2} (1-G) \right] \\
&= -2^{-2\alpha-3} (2gh)^{-\alpha-\beta-1} \frac{\sqrt{\pi} \Gamma(\beta+1)}{\Gamma(\alpha+\frac{3}{2}) \Gamma(\beta-\alpha)} {}_2F_1 \left[\frac{\alpha+\beta+1}{2}, \frac{\alpha-\beta+1}{2}; \frac{1}{2} (1-G) \right], \\
&\quad -1 < \operatorname{Re} \alpha < 1/2.
\end{aligned}$$

$$(7.31) \quad L(x, y, z; \beta, \beta) = 0 \text{ when } |f-g| < g < f+g, -1 < \operatorname{Re} \beta < 1/2.$$

Noting that

$$(7.32) \quad h_n^{(\lambda-1/2, \lambda-1/2)} = 2^{1-2\lambda} (n+\lambda) \Gamma(n+2\lambda) n! / \Gamma^2(n+\lambda+1/2),$$

formulas (7.19), (7.21) and (7.31) can be written as:

$$(7.33) \quad \sum_{n=0}^{\infty} (n+\lambda) \frac{\Gamma(n+1)}{\Gamma(2\lambda+n)} C_n^{\lambda}(\cos 2\lambda) C_n^{\lambda}(\cos 2\lambda) D_n^{\lambda}(\cos 2\lambda) \\ = \frac{2^{2\lambda}}{\pi} \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \right]^2 \begin{cases} 0 & \text{if } |\cos(\phi+\psi)| < \cos \theta < \cos(\phi-\psi), \\ 2^{-6\lambda-1} \sin \pi \lambda (-D)^{-\lambda} & \text{if } \cos(\phi-\psi) < \cos \theta \text{ or } 0 < \cos \theta < \cos(\phi+\psi), \\ -2^{-6\lambda-1} \sin \pi \lambda (-D)^{-\lambda} & \text{if } 0 < \cos \theta < -\cos(\phi+\psi), \end{cases}$$

where D is defined in (7.6), but can be given explicitly in terms of

θ, ϕ, ψ as

$$(7.34) \quad 16D = \sin(\theta+\phi+\psi) \sin(\theta+\phi-\psi) \sin(\theta+\psi-\phi) \sin(\phi+\psi-\theta).$$

Corollary

$$(7.35) \quad \sum_{n=0}^{\infty} (n+\lambda) \frac{\Gamma(n+1)}{\Gamma(2\lambda+n)} C_n^{\lambda}(\cos 2\phi) [C_n^{\lambda}(\cos 2\psi) D_n^{\lambda}(\cos 2\theta) + D_n^{\lambda}(\cos 2\psi) C_n^{\lambda}(\cos 2\theta)]$$

$$= \begin{cases} 0 & , \text{ if } |\cos(\phi+\psi)| < \cos \theta < \cos(\phi-\psi) \\ & \text{and } |\cos(\theta+\psi)| < \cos \psi < \cos(\theta-\phi), \text{ and } \cos \phi > \max(\cos \theta, \cos \psi), \\ -\frac{2^{-4\lambda}}{\pi} \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \right]^2 \sin \lambda \pi (-D)^{-\lambda} & , \text{ if } \cos \phi < \cos(\theta-\psi), \end{cases}$$

Convergence of the series on the l.h.s. of (6.33) and (6.35) requires

$-1/2 < \operatorname{Re} \lambda < 1$ except for some exceptional values of θ, ϕ and ψ .

CHAPTER III

CERTAIN INTEGRALS INVOLVING BESSEL FUNCTIONS OF
THE FIRST KIND AND SECOND KIND

8. Introduction:

The object of this chapter is to evaluate certain integrals which contain Bessel functions. The integrals involving Bessel functions are not only of great interest to the Pure Mathematicians, but they are of extreme importance in many branches of Mathematical Physics. Let us quote the random-walk example from Watson [86]. A man starts from point 0 and walks a distance a_1 in a straight line; he then turns through an arbitrary angle and walks a distance a_2 in a second straight line. He repeats this process n times. Now let $P_n(\gamma; a_1, a_2, \dots, a_n)$ denote the probability that after n stretches the distance from the starting point shall be less than γ . A straightforward calculation leads to [86]:

$$(8.1) \quad P_n(\gamma; a_1, a_2, \dots, a_n) = \gamma \int_0^{\infty} J_1(\gamma t) \prod_{m=1}^n J_0(a_m t) dt.$$

For $n=2$ the above integral can be evaluated using Henrici's integral (4.28) with $\mu=1$, $\nu=\lambda=0$. In this chapter we will give two different methods of evaluating the integral (4.26) using Hsü's [57] limiting procedure. We will give an extension of Henrici's result (4.29) and discuss some of its useful special cases.

9. Important Limits:

In this section we prove (2.12) and (3.8). Consider the following transformation [39, 2.9 (34), p.107]:

$$(9.1) \quad {}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} ; z \right] = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} (-z)^{-a} {}_2F_1 \left[\begin{matrix} a, 1-c+a \\ 1-b+a \end{matrix} ; z^{-1} \right] \\ + \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} (-z)^{-b} {}_2F_1 \left[\begin{matrix} b, 1-c+b \\ 1-a+b \end{matrix} ; z^{-1} \right].$$

Using (3.1) and (9.1) we get

$$(9.2) \quad Q_n^{(\alpha, \beta)}(x) = \frac{2^{\alpha+\beta+1} \Gamma(\alpha) \Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1) (x-1)^\alpha (x+1)^\beta} {}_2F_1 \left[\begin{matrix} n+1, -n-\alpha-\beta \\ 1-\alpha \end{matrix} ; \frac{1-x}{2} \right] \\ + \frac{2^{\beta-1} \Gamma(-\alpha) \Gamma(n+\alpha+1)}{\Gamma(n+1) (x+1)^\beta} {}_2F_1 \left[\begin{matrix} \alpha+n+1, -n-\beta \\ 1+\alpha \end{matrix} ; \frac{1-x}{2} \right].$$

In the upper half-plane $x-1=e^{i\pi}(1-x)$ and in the lower half-plane $x-1=e^{-i\pi}(1-x)$. Therefore, in the upper half-plane

$$Q_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} + i0 \right) = \frac{2^{\alpha+\beta-1} \Gamma(\alpha) \Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1) e^{i\pi\alpha} (1-\cos x/n)^\alpha (1+\cos x/n)^\beta} \\ \cdot {}_2F_1 \left[\begin{matrix} n+1, -n-\alpha-\beta \\ 1-\alpha \end{matrix} ; \frac{1-\cos \frac{x}{n}}{2} \right] \\ + \frac{2^{\beta-1} \Gamma(-\alpha) \Gamma(n+\alpha+1)}{\Gamma(n+1) \Gamma(1+\cos x/n)^\beta} {}_2F_1 \left[\begin{matrix} n+\alpha+1, -n-\beta \\ 1+\alpha \end{matrix} ; \frac{1-\cos \frac{x}{n}}{2} \right]$$

for large n , with the aid of Sterling's formula we get

$$\begin{aligned}
 (9.3) \quad Q_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} + i0 \right) &= \frac{2^{2\alpha-1} \Gamma(\alpha) n^\alpha}{x^{2\alpha} e^{i\alpha\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(1-\alpha)_k!} \left(\frac{x}{2} \right)^{2k} \\
 &+ 2^{-1} \Gamma(-\alpha) n^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{(1+\alpha)_k!} \left(\frac{x}{2} \right)^{2k}.
 \end{aligned}$$

The definition of the Bessel function of the first kind in [72] is

$$(9.4) \quad J_\alpha(z) = \frac{(z/2)^\alpha}{\Gamma(1+\alpha)} {}_0F_1 \left(-; 1+\alpha; -\frac{z^2}{4} \right).$$

Using (7.29) and (9.4) equation (9.3) becomes

$$(9.5) \quad \lim_{n \rightarrow \infty} n^{-\alpha} Q_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} + i0 \right) = -\frac{2^{\alpha-1}}{x^\alpha} \frac{\pi}{\sin \pi\alpha} \left[J_\alpha(x) - \frac{J_{-\alpha}(x)}{e^{i\pi\alpha}} \right].$$

Likewise, in the lower half-plane

$$(9.6) \quad \lim_{n \rightarrow \infty} n^{-\alpha} Q_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} - i0 \right) = -\frac{2^{\alpha-1}}{x^\alpha} \frac{\pi}{\sin \pi\alpha} \left[J_\alpha(x) - \frac{J_{-\alpha}(x)}{e^{-i\pi\alpha}} \right].$$

Using Durand's formula (3.3) and (3.4) we obtain the following two important limits

$$\lim_{n \rightarrow \infty} n^{-\alpha} P_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} \right) = \left(\frac{x}{2} \right)^{-\alpha} J_\alpha(x),$$

$$\lim_{n \rightarrow \infty} n^{-\alpha} Q_n^{(\alpha, \beta)} \left(\cos \frac{x}{n} \right) = -\frac{\pi}{2} \left(\frac{x}{2} \right)^{-\alpha} Y_\alpha(x).$$

10. Derivation of (4.26) using (4.21):

In order that we may apply Hsu's [57] limiting procedure directly to our formulas we now replace θ, ϕ, ψ by $\frac{\alpha}{2}, \frac{\pi-\beta}{2}, \frac{\pi-\gamma}{2}$, respectively, in (7.33) use the duplication formula for the gamma function, and the symmetry property of the ultraspherical polynomials, namely,

$$C_n^\lambda(-x) = (-1)^n C_n^\lambda(x).$$

Formula (7.33) then leads to (4.21) while (7.34) becomes (4.22).

Following Hsu we now set

$$(10.1) \quad \alpha = ah, \beta = bh, \gamma = ch$$

in (4.21), where a, b, c, h are arbitrary positive numbers. If $\max(a, b, c) > \frac{1}{2}(a+b+c)$, that is, a, b, c do not form the sides of a plane triangle, and h is sufficiently small, then α, β, γ will not form the sides of a spherical triangle. Obviously, this is the case that applies to (4.21). For small h

$$\begin{aligned} & 2^{2-2\lambda} \{\Gamma(\lambda)\}^{-2} \cdot 2^{-2\lambda-1} \sin \pi \lambda (-16D)^{-\lambda} \\ & \approx 2^{-4\nu-1} \{\Gamma(\nu+1/2)\}^{-2} \cos \pi \nu (-\Delta)^{-\nu-1/2} h^{-4\nu-2} \end{aligned}$$

where $\lambda = \nu + 1/2$ and Δ is the same as defined in (4.27).

Let us split the sum on the l.h.s. of (4.21) into two ranges:

$0 \leq n \leq [\omega/h]$ and $1 + [\omega/h] \leq n$, where ω is an arbitrary positive number independent of h . From (4.24) and (4.25) we have, for

$$nz = O(1),$$

$$(10.3) \quad n^{-2\nu} C_n^{\nu+1/2}(\cos z) = \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)} (2nz)^{-\nu} J_\nu(nz) + \epsilon_n,$$

$$n^{-2\nu} D_n^{\nu+1/2}(\cos z) = -\frac{\sqrt{\pi}}{\Gamma(\nu+1/2)} (2nz)^{-\nu} Y_\nu(nz) + \epsilon'_n,$$

where $\epsilon_n, \epsilon'_n \rightarrow 0$ as $n \rightarrow \infty$. Hence

$$(10.4) \quad \begin{aligned} & n^{-6\nu} C_n^{\nu+1/2}(\cos \beta) C_n^{\nu+1/2}(\cos \gamma) D_n^{\nu+1/2}(\cos \alpha) \\ &= -\pi^{3/2} \{\Gamma(\nu+1/2)\}^{-3} (2nh)^{-3\nu} (abc)^{-\nu} J_\nu(bnh) J_\nu(cnh) Y_\nu(anh) \\ & \quad + \eta_n \end{aligned}$$

where $\eta_n \rightarrow 0$ as $n \rightarrow \infty$. Also

$$(10.5) \quad (n+\nu+1/2) \frac{\Gamma(n+1)}{\Gamma(n+2\nu+1)} = n^{1-2\nu} (1+\eta'_n)$$

where $\eta'_n = O(n^{-1})$ as $n \rightarrow \infty$.

Thus the summation over the first range contributes

$$(10.6) \quad \begin{aligned} & h^{4\nu+2} \sum_{n=0}^{[\omega/h]} (n+\nu+1/2) \frac{\Gamma(n+1)}{\Gamma(n+2\nu+1)} C_n^{\nu+1/2}(\cos \beta) C_n^{\nu+1/2}(\cos \gamma) \\ & \quad \cdot D_n^{\nu+1/2}(\cos \alpha) \\ &= -\pi^{3/2} \{\Gamma(\nu+1/2)\}^{-3} (8abc)^{-\nu} h \sum_{n=0}^{[\omega/h]} (nh)^{1+\nu} J_\nu(bnh) J_\nu(cnh) Y_\nu(anh) \\ & \quad + h \sum_{n=0}^{[\omega/h]} (nh)^{4\nu+1} \eta'_n (1+\eta'_n). \end{aligned}$$

The principal part of (10.6), namely,

$$(10.7) \quad h \sum_{n=0}^{[\omega/h]} (nh)^{1+\nu} J_{\nu}(bnh) J_{\nu}(cnh) Y_{\nu}(anh)$$

$$\rightarrow \int_0^{\omega} x^{1+\nu} J_{\nu}(bx) J_{\nu}(cx) Y_{\nu}(ax) dx$$

as $h \rightarrow 0$, provided the integral converges which, for fixed ω , requires that $\operatorname{Re} \nu > -1/2$. Since $(nh)^{1+\nu} J_{\nu}(bnh) J_{\nu}(cnh) Y_{\nu}(anh) = (nh)^{1+2\nu} O(1)$, the remainder part clearly approaches 0 as $h \rightarrow 0$ if $\operatorname{Re} \nu > -1/2$.

In order to deal with the second range of n for the sum in (4.21) we need the following asymptotic formulas

$$(10.8) \quad C_n^{\nu+1/2}(\cos \theta) = \frac{2}{\Gamma(\nu+1/2)} (2 \sin \theta)^{-\nu-1/2} n^{\nu-1/2}$$

$$\cdot [\cos \{(n+\nu+1/2)\theta - 1/2\nu\pi - 1/4\pi\} + R],$$

$$|R| < C (n \sin \theta)^{-1}$$

and

$$(10.9) \quad D_n^{\nu+1/2}(\cos \theta) = \frac{2}{\Gamma(\nu+1/2)} (2 \sin \theta)^{-\nu-1/2} n^{\nu-1/2}$$

$$\cdot [\cos \{(n+\nu+1/2)\theta - 1/2\nu\pi + 1/4\pi\} + R'],$$

$$|R'| < C' (n \sin \theta)^{-1},$$

where $0 < \theta < \pi$, $-1/2 < \nu < 1/2$ and C, C' are independent of n and θ .

The first of these formulas is due to Stieltjes [57, eq.(4.2) ; see also ref. 78, eq. (8.21.18), p.198]. The second formula (10.9) does not seem to be explicitly given in the literature, but can be easily deduced by using the known asymptotic formula for Legendre function of the second kind [39, 3.9.1 (1), p.163] and its relation with the

ultraspherical function. Now,

$$\begin{aligned}
 (10.10) \quad & h^{4v+2} n^{1-2v} (1+\eta'_n) C_n^{v+1/2}(\cos \beta) C_n^{v+1/2}(\cos \gamma) D_n^{v+1/2}(\cos \alpha) \\
 &= \left\{ \frac{2^{1/2-v}}{\Gamma(v+1/2)} \right\}^3 h^{4v+2} (\sin \alpha \sin \beta \sin \gamma)^{-v-1/2} n^{v-1/2} \\
 &\quad \cdot \left[\cos \left(N\beta - \frac{v\pi}{2} - \frac{\pi}{4} \right) + \rho_1 \right] \\
 &\quad \cdot \left[\cos \left(N\gamma - \frac{v\pi}{2} - \frac{\pi}{4} \right) + \rho_2 \right] \cdot \left[\cos \left(N\alpha - \frac{v\pi}{2} + \frac{\pi}{4} \right) + \rho_3 \right] \\
 &= \left\{ \frac{2^{1/2-v}}{\Gamma(v+1/2)} \right\}^3 h^{4v+2} (\sin \alpha \sin \beta \sin \gamma)^{-v-1/2} n^{v-1/2} \\
 &\quad \cdot \left[\cos \left(N\beta - \frac{v\pi}{2} - \frac{\pi}{4} \right) \right. \\
 &\quad \cdot \left. \cos \left(N\gamma - \frac{v\pi}{2} - \frac{\pi}{4} \right) \cos \left(N\alpha - \frac{v\pi}{2} + \frac{\pi}{4} \right) + \rho \right],
 \end{aligned}$$

$$N = n+v+1/2,$$

where ρ_1, ρ_2, ρ_3 and consequently ρ have the same type of bounds as R and R' have in (10.8) and (10.9). Using $\eta'_n = O(n^{-1})$ we find that $\rho = (nh)^{-1} O(1)$.

Since

$$\begin{aligned}
 (10.11) \quad & 4 \cos \left(N\beta - \frac{v\pi}{2} - \frac{\pi}{4} \right) \cos \left(N\gamma - \frac{v\pi}{2} - \frac{\pi}{4} \right) \cos \left(N\alpha - \frac{v\pi}{2} + \frac{\pi}{4} \right) \\
 &= \cos \left\{ N(\alpha+\beta+\gamma) - \frac{3v\pi}{2} - \frac{\pi}{4} \right\} + \cos \left\{ N(\beta+\gamma-\alpha) - \frac{v\pi}{2} - \frac{3\pi}{4} \right\} \\
 &+ \cos \left\{ N(\beta-\gamma+\alpha) - \frac{v\pi}{2} + \frac{\pi}{4} \right\} + \cos \left\{ N(\gamma-\beta+\alpha) - \frac{v\pi}{2} + \frac{\pi}{4} \right\}
 \end{aligned}$$

the principal term in the summation over (10.10) on the second range reduces to an expression of type

$$(10.12) \quad p^{\nu+1/2} \sum_{n=1+\lfloor \omega/h \rfloor}^{\infty} n^{\nu-1/2} \cos(nq+\tau),$$

where p and q are $\sim h$ and τ is a fixed constant. The bound for (10.11) is

$$(10.13) \quad p^{\nu+1/2} (\omega h^{-1})^{\nu-1/2} q^{-1} = \omega^{\nu-1/2} O(1).$$

Therefore, the contribution of the remainder of the term is

$$(10.14) \quad p^{\nu+1/2} \sum_{n=1+\lfloor \omega/h \rfloor}^{\infty} n^{\nu-1/2} (nh)^{-1} = O(1) h^{\nu-1/2} \sum_{n=1+\lfloor \omega/h \rfloor}^{\infty} n^{\nu-3/2} \\ = \omega^{\nu-1/2} O(1).$$

Therefore, as $h \rightarrow 0$,

$$(10.15) \quad h^{4\nu+2} \sum_{n=0}^{\infty} (n+\nu+1/2) \frac{\Gamma(n+1)}{\Gamma(n+2\nu+1)} C_n^{\nu+1/2}(\cos \beta) C_n^{\nu+1/2}(\cos \gamma) \\ \cdot D_n^{\nu+1/2}(\cos \alpha) \\ = -\pi^{3/2} \{\Gamma(\nu+1/2)\}^{-3} (8abc)^{-\nu} \int_0^{\omega} x^{1+\nu} \Psi_{\nu}(ax) J_{\nu}(bx) J_{\nu}(cx) dx \\ + \omega^{\nu-1/2} O(1) \quad \text{provided} \quad -1/2 < \nu < 1/2.$$

Using (4.21), (10.1) (10.2) and (10.15) we have for $\omega \rightarrow \infty$,

$$(10.16) \quad \int_0^{\infty} x^{1+\nu} Y_{\nu}(ax) J_{\nu}(bx) J_{\nu}(cx) dx$$

$$= \begin{cases} 0, & \text{if } |b-c| < a < b+c, \\ \frac{2^{-\nu-1} \Gamma(\nu+1/2) \cos \pi \nu}{\pi^{3/2} (abc)^{-\nu}} (-\Delta)^{-\nu-1/2}, & \text{if } a > b+c, \\ \frac{-2^{-\nu-1} \Gamma(\nu+1/2) \cos \pi \nu}{\pi^{3/2} (abc)^{-\nu}} (-\Delta)^{-\nu-1/2}, & \text{if } a < |b-c|. \end{cases}$$

This immediately leads to (4.26) when we use $\Gamma(1/2-\nu) \Gamma(1/2+\nu) = \pi \sec \pi \nu$.

Although the asymptotic formulas (10.8) and (10.9) that we had to use to derive (10.13) and (10.14) are valid for real ν and $-1/2 < \nu < 1/2$, by analytic continuation, equation (10.16) obviously holds true also for complex ν with $-1/2 < \operatorname{Re} \nu < 1/2$.

11. Derivation of (4.26) Using (4.13), (4.14) and (4.15):

We consider the equations (4.13), (4.14) and (4.15). Let a, b, c be arbitrary positive numbers and N a positive integer such that

$$(11.1) \quad m = 2 [1/2 b N], \quad n = 2 [1/2 c N], \quad k = \pm ([1/2 a N] - [1/2 b N] - [1/2 c N]),$$

where we take positive sign in (4.14) and negative sign in (4.15) in the value of k in (11.1).

If a, b, c do not form the sides of a triangle then ℓ, m, n will not form the sides of a triangle provided N is large and $\ell = m+n+1+2k$ for '+' sign $k = 0, 1, 2, \dots$, and for '-' sign $k = 1, 2, \dots, m, m+1, \dots, [\frac{m+n+1}{2}]$, $0 \leq m \leq n$.

For large N , the r.h.s. of (4.15) can be written as

$$(11.2) \quad R_1 = - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_m (2\lambda)_n (2\lambda)_{m+n+1+2k}}{m! n! (m+n+1+2k)!} \frac{(1-\lambda)_k (\lambda)_{m+n+1+k}}{k! (2\lambda)_{m+n+1+k}} \\ \cdot \frac{(m+k)! (n+k)!}{(\lambda+1)_{m+k} (\lambda+1)_{n+k}}, \quad k = 0, 1, 2, \dots, \text{ while}$$

$$R_1 = \frac{\sin \pi \lambda}{\pi} \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \right]^2 (abc)^{2\lambda-1} \left(\frac{a-b-c}{2} \right)^{-\lambda} \left(\frac{a+b+c}{2} \right)^{-\lambda} \left(\frac{a-b+c}{2} \right)^{-\lambda} \\ \cdot \left(\frac{a+b-c}{2} \right)^{-\lambda} N^{2\lambda-3}.$$

Setting $\lambda = \nu + 1/2$ and using the Legendre duplication formula

$$\Gamma(2z) = \Gamma(z) \Gamma(z+1/2) 2^{2z-1} \pi^{-1/2}.$$

we get

$$(11.3) \quad R_1 = - \frac{2^{-\nu-1} (-\Delta)^{-\nu-1/2}}{\pi^{1/2} \Gamma(1/2-\nu) (abc)^\nu} N^{2\nu-2}.$$

Similarly for the r.h.s. of (4.15) (call it R_2)

$$(11.4) \quad R_2 = \frac{2^{-\nu-1} (-\Delta)^{-\nu-1/2}}{\pi^{1/2} \Gamma(1/2-\nu) (abc)^{-\nu}} N^{2\nu-2}.$$

For abbreviation, we write

$$(11.5) \quad K = - (abc)^{-\nu} 2^{3\nu-1} \pi^{-3/2} \Gamma^3(\nu+1/2) \quad \text{and}$$

$$\begin{aligned}
 (11.6) \quad I &= K N^{-2\nu+2} \int_{-1}^1 D_l^{\nu+1/2}(x) C_m^{\nu+1/2}(x) C_n^{\nu+1/2}(x) (1-x^2)^{2\nu} dx \\
 &= 2 K N^{-2\nu+2} \int_0^{\pi/2} D_l^{\nu+1/2}(\cos \theta) C_m^{\nu+1/2}(\cos \theta) \\
 &\quad \cdot C_n^{\nu+1/2}(\cos \theta) \sin \theta^{4\nu+1} d\theta.
 \end{aligned}$$

Let ω be an arbitrary chosen positive number independent of N ; we can break up the interval $[0, 1/2\pi]$ of integration into two intervals from 0 to ω/N and from ω/N to $1/2\pi$. The integral corresponding to the first of these intervals gives, according to (4.24) and (4.25),

$$\begin{aligned}
 (11.7) \quad & 2 K N^{-2\nu+2} \int_0^{\omega} D_l^{\nu+1/2}(\cos \frac{\phi}{N}) C_m^{\nu+1/2}(\cos \frac{\phi}{N}) C_n^{\nu+1/2}(\cos \frac{\phi}{N}) \\
 & \quad \sin \left(\frac{\phi}{N}\right)^{4\nu+1} \frac{d\phi}{N} \\
 & \rightarrow \int_0^{\omega} x^{1+\nu} Y_\nu(ax) J_\nu(bx) J_\nu(cx) dx \quad \text{as } N \rightarrow \infty
 \end{aligned}$$

In the integral corresponding to the other interval we use (10.8) and (10.9) to obtain

$$\begin{aligned}
 (11.8) \quad & K N^{-2\nu+2} D_l^{\nu+1/2}(\cos \theta) C_m^{\nu+1/2}(\cos \theta) C_n^{\nu+1/2}(\cos \theta) \sin \theta^{4\nu+1} \\
 & = 2^{-3\nu+3/2} N^{-2\nu+2} \{\Gamma^{-3}(\nu+1/2) (\sin \theta)^{3\nu+1/2} (\ell_{mn})^{\nu-1/2}\} \\
 & \quad \cdot \{\cos(\ell'\theta - 1/2\nu\pi + 1/4\pi) + R_1\} \{\cos(m'\theta - 1/2\nu\pi - 1/4\pi) + R_2\} \\
 & \quad \cdot \{\cos(n'\theta - 1/2\nu\pi - 1/4\pi) + R_3\} \\
 & = \left(\frac{2}{\pi}\right)^{3/2} (abc)^{-\nu} N^{-2\nu+2} (\sin \theta)^{3\nu+1/2} \{\cos(\ell'\theta - 1/2\nu\pi + 1/4\pi)
 \end{aligned}$$

$$\cos (m'\theta - 1/2v\pi - 1/4\pi) \cos (n'\theta - 1/4v\pi - 1/4\pi) + R\}$$

where R_1, R_2, R_3 and consequently R are bounds of the given type as mentioned in (10.8) and (10.9). The symbols l', m', n' are

$$l' = l+v+1/2, \quad m' = m+v+1/2, \quad n' = n+v+1/2.$$

Using (10.11), the part $(\omega/N, 1/2\pi)$ of the integral corresponding to the principal term of (11.8) can be reduced to the expression of the type

$$(11.9) \quad p^{v+1/2} \int_{\omega/N}^{1/2} \sin \theta^{v-1/2} \cos (Q\theta + \tau) d\theta$$

where p and Q are $\sim N$, and τ is a fixed constant. By the second mean-value theorem, (11.9) becomes

$$(11.10) \quad p^{v+1/2} (\sin \omega/N)^{v-1/2} O(Q^{-1}) = \omega^{v-1/2} O(1).$$

This holds uniformly with respect to ω . The contribution of the remaining term is

$$(11.11) \quad p^{1/2+v} \int_{\omega/N}^{1/2\pi} (\sin \theta)^{v-1/2} (N \sin \theta)^{-1} d\theta \\ = N^{v-1/2} (\omega/N)^{v-1/2} = \omega^{v-1/2} O(1);$$

thus, for $\omega \rightarrow \infty$ the statement in (4.26) is true provided $-1/2 < v < 1/2$.

By analytic continuation equation (4.26) obviously holds true for complex v with $-1/2 < \operatorname{Re} v < 1/2$.

Corollary:

$$(11.12) \quad \int_0^{\infty} x^{1+\nu} [Y_{\nu}(ax) J_{\nu}(bx) + Y_{\nu}(bx) J_{\nu}(ax)] J_{\nu}(cx) dx$$

$$= \begin{cases} 0, & \text{if } c < \min(a, b) \text{ and a triangle can be drawn} \\ & \text{with sides } a, b, c \text{ with area } \Delta^{1/2}, \\ \frac{2^{-\nu} (-\Delta)^{-\nu-1/2}}{\sqrt{\pi} \Gamma(1/2-\nu) (abc)^{-\nu}}, & \text{if } c > a+b, \\ -1/2 < \operatorname{Re} \nu < 1/2. \end{cases}$$

12. An Extension of Henrici's Result:

In this section we will evaluate the following integral.

$$(12.1) \quad f(a, b, c) = \int_0^{\infty} t^{1-\rho} J_{\mu}(at) J_{\nu}(bt) J_{\lambda}(ct) dt.$$

This integral is convergent for $\operatorname{Re}(\mu+\nu+\lambda-\rho) > -2$ and for arbitrary positive a, b, c when $a+b < c$ and $|a-b| < c < a+b$. During the calculation we shall consistently put

$$\alpha = a^2, \beta = b^2, \gamma = c^2.$$

Without loss of generality we may assume that $c \geq b \geq a$ since this can always be achieved by permutation of the indices. We start out by considering the function

$$(12.2) \quad g(z) = \int_0^{\infty} t^{1-\rho} J_{\mu}(at) J_{\nu}(bt) K_{\lambda}(zt) dt$$

where a and b are considered as fixed parameters. To ensure convergence, we must now assume temporarily that $\operatorname{Re}(\mu+\nu+\lambda-\rho) > -2$. We first

evaluate (12.2) for large real values of z ; later z will also take certain complex values. Putting the expansion [55, p.736]

$$(12.3) \quad J_\mu(at) J_\nu(bt) = \frac{2^{-\mu-\nu} (at)^\mu (bt)^\nu}{\Gamma(\mu+1) \Gamma(\nu+1)} \sum_{n=0}^{\infty} \frac{(\beta-\alpha)^n t^{2n}}{4^n (\mu+1)_n (\nu+1)_n} P_n^{(\mu, \nu)} \left(\frac{\beta+\alpha}{\beta-\alpha} \right)$$

into (11.2) and integrating term by term by using the formula

$$(12.4) \quad \int_0^\infty K_\lambda(zt) t^{k-1} dt = \frac{1}{4} \left(\frac{z}{2} \right)^{-k} \Gamma\left(\frac{k+\lambda}{2}\right) \Gamma\left(\frac{k-\lambda}{2}\right),$$

$\operatorname{Re}(k-\lambda) > 0,$

we find, writing $\xi = z^2$

$$(12.5) \quad g(z) = \frac{\Gamma\left(\frac{\mu+\nu+\lambda-p+2}{2}\right) \Gamma\left(\frac{\mu+\nu-p-\lambda+2}{2}\right)}{2^p \Gamma(\mu+1) \Gamma(\nu+1)} \left(\frac{\alpha^\mu \beta^\nu}{\xi^{\mu+\nu+2-p}} \right)^{1/2} \\ \sum_{n=0}^{\infty} \frac{\left(\frac{\mu+\nu+\lambda-p+2}{2}\right)_n \left(\frac{\mu+\nu-p-\lambda+2}{2}\right)_n}{(\mu+1)_n (\nu+1)_n} \left(\frac{\beta-\alpha}{\xi} \right)^n P_n^{(\mu, \nu)} \left(\frac{\beta+\alpha}{\beta-\alpha} \right).$$

Using $P_n^{(\mu, \nu)}(y) = \frac{(1+\nu)_n}{n!} \left(\frac{y-1}{2} \right)^n {}_2F_1 \left[\begin{matrix} -n, -n-\mu \\ \nu+1 \end{matrix}; \frac{y+1}{y-1} \right]$, the summation on the r.h.s. of (12.5) equals

$$(12.6) \quad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{\mu+\nu+\lambda-p+2}{2}\right)_n \left(\frac{\mu+\nu-p-\lambda+2}{2}\right)_n}{(-n)_m (-n-\mu)_m} \frac{(1+\nu)_n}{n!} \left(\frac{\beta-\alpha}{\xi} \right)^n \left(\frac{\alpha}{\beta-\alpha} \right)^n \\ \frac{(1+\nu)_m}{m!} (\beta/\alpha)^m \\ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(\frac{\mu+\nu+\lambda-p+2}{2}\right)_{m+n} \left(\frac{\mu+\nu+2-\lambda-p}{2}\right)_{m+n}}{m! n! (\mu+1)_n (\nu+1)_m} \left(\frac{\alpha}{\xi} \right)^n \left(\frac{\beta}{\xi} \right)^m \\ = F_4 \left(\frac{\mu+\nu+\lambda-p+2}{2}, \frac{\mu+\nu-\lambda-p+2}{2}; \mu+1, \nu+1; \frac{\alpha}{\xi}, \frac{\beta}{\xi} \right)$$

where the Appel function is as defined in (6.2) and is absolutely convergent if

$$\left| \frac{\alpha}{\xi} \right|^{1/2} + \left| \frac{\beta}{\xi} \right|^{1/2} < 1. \text{ Using the relation}$$

$$(12.7) \quad J_{\lambda}(tc) = \frac{1}{\pi} \{ e^{i\lambda\pi/2} K(itc+i0) - e^{-i\lambda\pi/2} K(-itc-i0) \}.$$

Therefore

$$f(a,b,c) = \frac{iC(\alpha,\beta,\gamma)}{\pi} [e^{i\pi/2(\mu+\nu-\lambda-\rho+2)} A(\alpha,\beta,\gamma+i0) - e^{-i\pi/2(\mu+\nu-\lambda-\rho+2)} A(\alpha,\beta,\gamma-i0)]$$

where we are denoting

$$(12.8) \quad C(\alpha,\beta,\gamma) = \frac{\Gamma(\frac{\mu+\nu+\lambda-\rho+2}{2}) \Gamma(\frac{\mu+\nu-\lambda-\rho+2}{2})}{2^{\rho} \Gamma(\mu+1) \Gamma(\nu+1)} \left(\frac{\alpha^{\mu} \beta^{\nu}}{\gamma^{\mu+\nu-\rho+2}} \right)^{1/2}$$

and

$$(12.9) \quad A(\alpha,\beta,\gamma) = F_4 \left(\frac{\mu+\nu+\lambda-\rho+2}{2}, \frac{\mu+\nu-\lambda-\rho+2}{2}; \mu+1, \nu+1; \frac{\alpha}{\gamma}, \frac{\beta}{\gamma} \right),$$

where the Appell function on the r.h.s. of (12.9) is absolutely convergent if

$$\left| \frac{\alpha}{\gamma} \right|^{1/2} + \left| \frac{\beta}{\gamma} \right|^{1/2} < 1.$$

In terms of a, b, c this condition means

$$(12.10) \quad a + b < c.$$

But, if this inequality is satisfied then $A(\alpha,\beta,\gamma)$ is single-valued on the real axis and so

$$(12.11) \quad f(a,b,c) = \frac{2}{\pi} C(\alpha,\beta,\gamma) A(\alpha,\beta,\gamma) \sin(\mu+\nu-\lambda-\rho+2) \frac{\pi}{2}.$$

When (12.10) is not satisfied, $A(\alpha, \beta, \gamma)$ may or may not be single-valued on the real axis but it can still be formally expressed as in (12.9). Let

$$(12.12) \quad \frac{\alpha}{\gamma} = x(1-y), \quad \frac{\beta}{\gamma} = y(1-x).$$

Solving for x and y we obtain

$$(12.13) \quad x = \frac{\gamma + \beta - \alpha - 4\sqrt{-\Delta}}{2\gamma}, \quad y = \frac{\gamma + \alpha - \beta - 4\sqrt{-\Delta}}{2\gamma}$$

where Δ is defined in (4.27). Here the signs of the square roots are chosen such that the roots are asymptotically equal to $\pm \gamma$ for large values of γ , so that

$$x = -\frac{\alpha}{\gamma} + O(\gamma^{-2}), \quad y = -\frac{\beta}{\gamma} + O(\gamma^{-2}).$$

We now observe that Δ is positive and represents the square of the area of the plane triangle with sides a, b, c and so x, y are complex numbers when

$$(12.14) \quad |a-b| < c < a+b.$$

Also, x, y are real when the inequality (12.14) is violated. So, when

(12.14) is satisfied we can write

$$(12.15) \quad x = \frac{\gamma + \beta - \alpha - 4i\sqrt{\Delta}}{2\gamma}, \quad y = \frac{\gamma + \alpha - \beta - 4i\sqrt{\Delta}}{2\gamma}.$$

Introduce the angle variables ϕ and ψ as

$$(12.16) \quad \tan \phi = \frac{4\sqrt{\Delta}}{\gamma + \beta - \alpha}, \quad \tan \psi = \frac{4\sqrt{\Delta}}{\gamma + \alpha - \beta}$$

so that

$$(12.17) \quad x = \sqrt{\frac{a}{b}} e^{i\phi}, \quad y = \sqrt{\frac{b}{c}} e^{i\psi}.$$

By the assumption that $c \geq b \geq a$ we have $|x| \leq 1$ and $|y| \leq 1$.

Let us now use Burchnall and Chaundy's expansion formula for an F_4 [76, eq.(8.4.14), p.223] to get

$$(12.18) \quad F_4 \left(\frac{\mu+\nu+\lambda-\rho+2}{2}, \frac{\mu+\nu-\lambda-\rho+2}{2}; \mu+1, \nu+1; x(1-y), y(1-x) \right) \\ = \sum_{n=0}^{\infty} \frac{\left(\frac{\mu+\nu+\lambda-\rho+2}{2} \right)_n \left(\frac{\mu+\nu-\lambda-\rho+2}{2} \right)_n (1-\rho)_n}{n! (\mu+1)_n (\nu+1)_n} x^n y^n \\ \cdot {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu+\lambda-\rho+2}{2} + n, \frac{\mu+\nu-\lambda-\rho+2}{2} + n \\ \mu+1+n \end{matrix}; x \right] \\ \cdot {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu+\lambda-\rho+2}{2} + n, \frac{\mu+\nu-\lambda-\rho+2}{2} + n \\ \nu+1+n \end{matrix}; y \right].$$

Hence

$$(12.19) \quad f(a,b,c) = \frac{2}{\pi} C(\alpha, \beta, \gamma) \sin(\mu+\nu-\lambda-\rho+2)\frac{\pi}{2} \\ \sum_{n=0}^{\infty} \frac{\left(\frac{\mu+\nu+\lambda-\rho+2}{2} \right)_n \left(\frac{\mu+\nu-\lambda-\rho+2}{2} \right)_n}{n! (\mu+1)_n (\nu+1)_n} \\ \cdot x^n y^n \cdot (1-\rho)_n {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu+\lambda-\rho+2}{2} + n, \frac{\mu+\nu-\lambda-\rho+2}{2} + n \\ \mu+1+n \end{matrix}; x \right] \\ \cdot {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu+\lambda-\rho+2}{2} + n, \frac{\mu+\nu-\lambda-\rho+2}{2} + n \\ \nu+1+n \end{matrix}; y \right].$$

where $c > a + b$, and x, y are real defined by (12.13)

$$(12.20) \quad f(a, b, c) = \frac{2}{\pi} C(\alpha, \beta, \gamma) \operatorname{Im} e^{1/\pi 2 (\mu + \nu - \lambda - \rho + 2)}$$

$$= \sum_{n=0}^{\infty} \frac{\left(\frac{\mu + \nu + \lambda - \rho + 2}{2}\right)_n \left(\frac{\mu + \nu - \lambda - \rho + 2}{2}\right)_n}{n! (\mu + 1)_n (\nu + 1)_n}$$

$$x^n y^n (1-\rho)_n \cdot {}_2F_1 \left[\begin{matrix} \frac{\mu + \nu + \lambda - \rho + 2}{2} + n, \frac{\mu + \nu - \lambda - \rho + 2}{2} + n \\ \mu + 1 + n \end{matrix} ; x \right]$$

$$\cdot {}_2F_1 \left[\begin{matrix} \frac{\mu + \nu + \lambda - \rho + 2}{2} + n, \frac{\mu + \nu - \lambda - \rho + 2}{2} + n \\ \nu + 1 + n \end{matrix} ; y \right],$$

where $|a-b| < c < a+b$ and x, y are complex numbers defined by (2.17).

Special Cases

I. The substitution $\rho=1$ in (12.19) and (12.20) immediately leads to Henrici's result (4.28).

II. If we let $\lambda = \rho = \mu = \lambda$, the l.h.s. of (12.1) becomes

$$\int_0^1 t^{1-\nu} J_\nu(at) J_\nu(bt) J_\nu(ct) dt = g(a, b, c), \quad (\text{say})$$

and the r.h.s. of (12.19) gives

$$g(a, b, c) = 0 \text{ when } c > a+b.$$

The r.h.s. of (12.18) gives

$$(12.21) \quad \sum_{n=0}^{\infty} \frac{(1)_n (1-\nu)_n}{n! (\nu+1)_n} x^n y^n {}_1F_0 \left[\begin{matrix} n+1 \\ - \end{matrix} ; x \right] {}_1F_0 \left[\begin{matrix} n+1 \\ - \end{matrix} ; y \right]$$

$$= [(1-x)(1-y)]^{-1} {}_2F_1 \left[\begin{matrix} 1, 1-\nu \\ 1+\nu \end{matrix} ; \frac{xy}{(1-x)(1-y)} \right].$$

Introducing two auxiliary parameters

$$s = \frac{-x}{1-x}, \quad t = \frac{-y}{1-y}.$$

It can be easily checked that in the upper half-plane

$$(1-x)^{-1} = 1-s = \frac{c}{b} e^{i\psi},$$

$$(1-y)^{-1} = 1-t = \frac{c}{a} e^{i\phi}.$$

Therefore in the upper half-plane

$$\begin{aligned} (12.22) \quad & [(1-x)(1-y)]^{-1} {}_2F_1 \left[\begin{matrix} 1, 1-\nu \\ 1+\nu \end{matrix}; \frac{xy}{(1-x)(1-y)} \right] \\ &= -\frac{\nu c^2}{a b} e^{i(\phi+\psi)} \int_0^1 \{ (1-u)(1-u e^{2i(\phi+\psi)}) \}^{\nu-1} du \\ &= -\frac{\nu c^2}{a b} \int_0^1 e^{i(\phi+\psi)} [(1-e^{-i(\phi+\psi)}u)(1-e^{i(\phi+\psi)}u)]^{\nu-1} du. \end{aligned}$$

The contribution from the lower half-plane is just the complex conjugate of this. Hence

$$\begin{aligned} (12.23) \quad g(a,b,c) &= \frac{-1}{2^\nu \pi c^2} \frac{1}{\Gamma(\nu+1)} \left(\frac{ab}{c}\right)^\nu \frac{\nu c^2}{ab} \\ &\quad \int_0^1 e^{i(\phi+\psi)} [(1-e^{-i(\phi+\psi)}u)(1-e^{i(\phi+\psi)}u)]^{\nu-1} du \\ &\quad - \int_0^1 e^{-i(\phi+\psi)} [(1-e^{-i(\phi+\psi)}u)(1-e^{i(\phi+\psi)}u)]^{\nu-1} du \\ &= \frac{(ab)^{\nu-1}}{\pi 2^\nu \Gamma(\nu)} \frac{2^{2\nu-1}}{c^\nu} \sin 2\nu-1(\phi+\psi) \frac{\Gamma(\nu) \Gamma(\nu)}{\Gamma(2\nu)}. \end{aligned}$$

If we compute $\sin(\phi+\psi)$ from (12.16)

$$\sin(\phi+\psi) = \frac{2\Delta^{1/2}}{ab}$$

and use the relation

$$\Gamma(2v) = 2^{2v-1} \pi^{-1/2} \Gamma(v) \Gamma(v+1/2)$$

we obtain the Sonine's formula (4.23) for $|a-b| < c < |a+b|$.

III. Let us put $\mu=v$, $\rho=\lambda$, (12.19) becomes

$$\begin{aligned} (12.24) \quad & \int_0^{\infty} t^{1-v} J_v(at) J_v(bt) J_v(ct) dt \\ &= \frac{(ab)}{c^{2v-\lambda+2}} \frac{2^{1-\lambda}}{\pi} \frac{\Gamma(v-\lambda+1)}{\Gamma(v+1)} \sin(v-\lambda+1)\pi \sum_{n=0}^{\infty} \frac{(v-\lambda+1)_n (1-\lambda)_n}{n! (v+1)_n} x^n y^n \\ & \quad \cdot {}_1F_0 \left[\begin{matrix} v-\lambda+1+n \\ - \end{matrix} ; x \right] \cdot {}_1F_0 \left[\begin{matrix} v-\lambda+1+n \\ - \end{matrix} ; y \right] \\ &= \frac{2^{1-\lambda}}{\pi} \frac{\Gamma(v-\lambda+1)}{\Gamma(v+1)} \frac{\pi (ab)^v c^{\lambda-2v-2}}{\Gamma(v-\lambda+1) \Gamma(\lambda-v)} [(1-x)(1-y)]^{\lambda-v-1} \\ & \quad \cdot {}_2F_1 \left[\begin{matrix} v-\lambda+1, 1-\lambda \\ 1+v \end{matrix} ; \frac{xy}{(1-x)(1-y)} \right]. \end{aligned}$$

Using the quadratic transformation [39, 2.11(1), p.110]

$$= \frac{2^{1-\lambda} (ab)^v c^{\lambda-2v-2}}{\Gamma(1+v) \Gamma(\lambda-v)} (1-x-y)^{\lambda-v-1} {}_2F_1 \left[\begin{matrix} v-\lambda+1/2, \lambda+v/2 \\ 1+v \end{matrix} ; \frac{-4xy(1-x)(1-y)}{(1-x-y)^2} \right]$$

Using [39, 3.2 (39), p.135] the definition of Legendre function $Q_v^\mu(x)$ of the second kind and the substitution

$$(12.25) \quad \frac{c^2 - a^2 - b^2}{2ab} = \cosh A$$

we get

$$(12.26) \quad \int_0^\infty t^{1-\lambda} J_\nu(at) J_\nu(bt) J_\nu(ct) dt \\ = \frac{(ab)^{\lambda-1} \cos \nu\pi \sinh^{\mu-1/2} A}{\sqrt{\frac{1}{2}} \pi^{3/2} c^\lambda} Q_{\nu-1/2}^{1/2-\lambda} (\cosh A)$$

which is [86, 13.46 (5), p.412].

If, however, $|a-b| < c < a+b$, and we write

$$(12.27) \quad \cos B = \frac{a^2 + b^2 - c^2}{2ab}.$$

It can be easily checked that

$$B = \phi + \psi.$$

Therefore,

$$(12.28) \quad \int_0^\infty t^{1-\lambda} J_\nu(at) J_\nu(bt) J_\nu(ct) dt \\ = \frac{-i 2^{-\lambda}}{\Gamma(\lambda) \pi} \frac{(ab)^\nu}{c^{2\nu+2-\lambda}} \left(\frac{c}{ab}\right)^{2-\lambda+\nu+1} \frac{e^{1(\phi+\psi)}}{e^{-1(\phi+\psi)}} u^{\nu-\lambda} [(1-e^{-1(\phi+\psi)}) u] \\ \cdot [(1-e^{1(\phi+\psi)}) u]^{\lambda-1} \\ = \frac{2^{1+\lambda} \sin B^{2\lambda-1}}{i(\nu-\lambda)B} \frac{\Gamma(\lambda)}{\Gamma(2\lambda)} \frac{(ab)^{1-\lambda}}{c^\lambda} {}_2F_1 \left[\begin{matrix} \lambda-\nu, \lambda \\ 2\lambda \end{matrix} ; -2i \sin B e^{iB} \right].$$

Using the transformation formulas [39],

$$(12.29) \quad {}_2F_1 \left[\alpha_1, \alpha_2; 2\alpha_2; z \right] = [1/2 + 1/2 (1-z)^{1/2}]^{-2\alpha_1} \\ \cdot {}_2F_1 \left[\alpha_1, \alpha_1 - \alpha_2 + 1/2; \alpha_2 + 1/2; \left[\frac{1-(1-z)^{1/2}}{1+(1-z)^{1/2}} \right]^2 \right]$$

and (1.9), we obtain after some simplifications,

$$(12.30) \quad \int_0^\infty t^{1-\lambda} J_\nu(at) J_\nu(bt) J_\nu(ct) dt \\ = \frac{2^{-\lambda} \sin B^{2\lambda-1}}{\Gamma(\nu+1/2) \Gamma(1/2)} \frac{(ab)^{1-\lambda}}{c^\lambda} (\cos B/2)^{1-2\lambda} {}_2F_1 \left[\begin{matrix} 1/2+\nu, 1/2-\nu \\ \lambda+1/2 \end{matrix}; \sin^2 B/2 \right] \\ = \frac{(ab)^{\lambda-1}}{(2\pi)^{1/2}} \frac{\sin B^{\lambda-1/2}}{c^\lambda} P_{\nu-1/2}^{1/2-\lambda}(\cos B), \quad \operatorname{Re}(\nu), \operatorname{Re}(\lambda) > -1/2.$$

In the last step we have used the definition of Legendre function of the first kind [39, 3.2 (14), p.124]. The results of this case have been used by Gasper [49] in order to prove non-negativity of the kernel $K(x, y, z; \alpha, \beta)$ defined in (4.5).

IV. $\rho = \mu + \lambda - \nu$.

In this case for $c > a+b$, we obtain from (12.19)

$$(12.19) \quad \int_0^\infty t^{1+\nu-\mu-\lambda} J_\nu(at) J_\nu(bt) J_\nu(ct) dt \\ = \frac{2^{1+\nu-\mu-\lambda}}{\pi} \frac{\Gamma(\nu-\lambda+1)}{\Gamma(1+\mu)} \frac{a^\mu b^\nu}{c^{2\nu-\lambda+2}} \sin(\nu-\lambda+1)\pi [(1-x)(1-y)]^{\lambda-\nu-1}$$

$$\cdot F_1 \left(\nu - \lambda + 1; \mu - \nu, 1 + \nu - \lambda - \mu; \mu + 1; \frac{-x}{1-x}, \frac{xy}{(1-x)(1-y)} \right)$$

$$= \frac{2^{1+\nu-\mu-\lambda}}{\Gamma(1+\mu) \Gamma(\lambda-\nu)} \frac{a^\mu b^\nu}{c^{2\nu-\lambda+2}} \frac{(1-x)^{-\nu} (1-y)^{-\mu}}{(x+y-1)^{1+\nu-\lambda-\mu}}$$

$$\cdot F_1 \left(\mu + \lambda - \nu; \mu - \nu, 1 + \nu - \lambda - \mu; \mu + 1; x, \frac{xy}{x+y-1} \right)$$

for $\text{Re}(\lambda - \nu) > 0$,

(using the formula [39, 5.11 (1), p.239]).

When a, b, c are sides of a triangle i.e., $|a-b| < c < a+b$ we have

$$\int_0^\infty t^{1+\nu-\mu-\lambda} J_\nu(at) J_\nu(bt) J_\nu(ct) dt$$

$$= -\frac{1}{\pi} \frac{a^\mu b^\nu}{c^{2\nu-\lambda+2}} \frac{2^{\nu-\mu-\lambda}}{\Gamma(\lambda+\mu-\lambda)} \left(\frac{c}{ab}\right)^{2\nu-\lambda+1} \int_0^\infty v^{\nu-\lambda} \frac{e^{iB}}{e^{-iB}} [(1-ve^{-iB})(1-ve^{iB})]^{\mu+\lambda-\nu-1} dv$$

$$= \frac{\left(\frac{ab}{2}\right)^{\mu+\lambda-\nu-1} (\sin B)^{2\lambda+2\mu-2\nu-1}}{c^\lambda \sqrt{\pi} \Gamma(\lambda+\mu-\mu+\nu+1/2)} e^{i(\lambda-\nu)B} (b-ae^{-iB})^{\nu-\mu}$$

$$\cdot F_1 \left(\lambda + \mu - \nu; \lambda - \nu, \mu - \nu; 2\lambda + 2\mu - 2\nu; 1 - e^{2iB}, \frac{1 - e^{2iB}}{1 - \frac{b}{a} e^{iB}} \right).$$

By setting $\cos \tau = \frac{b^2 + c^2 - a^2}{2bc}$

so that

$$1 - \frac{c}{b} e^{-i\tau} = \frac{a}{b} e^{iB}$$

we obtain

$$\int_0^{\infty} t^{1+\nu-\mu-\lambda} J_{\mu}(at) J_{\nu}(bt) J_{\lambda}(ct) dt$$

$$= \frac{(a/2)^{\mu+\lambda-\nu-1} b^{\lambda-1} (\sin B)^{2\lambda+2\mu-2\nu-1}}{c^{\lambda} \sqrt{\pi} \Gamma(\lambda+\mu-\nu+1/2)} e^{i(\lambda-\nu)B} e^{i(\nu-\mu)\tau}$$

$$= F_1(\lambda+\mu-\nu; \lambda-\nu, \mu-\nu; 2\lambda+2\mu-2\nu; 1-e^{2iB}, 1-e^{-2i\tau}).$$

V. If $\rho=0$ and $\lambda=\mu-\lambda$ then for $c > a+b$

$$\int_0^{\infty} t J_{\mu}(at) J_{\nu}(bt) J_{\mu-\nu}(ct) dt$$

$$= \frac{2}{\pi} \frac{a^{\mu} b^{\nu}}{c^{\mu+\nu+2}} \sin(\nu+1)\pi \frac{(1-x-y)}{(1-x)^{\nu} (1-y)^{\mu}};$$

for $|a-b| < c < a+b$, we have

$$\int_0^{\infty} t J_{\mu}(at) J_{\nu}(bt) J_{\mu-\nu}(ct) dt$$

$$= -\frac{i}{\pi} \frac{a^{\mu} b^{\nu}}{c^{\mu+\nu+2}} \left[e^{i(\nu+1)\pi} \frac{4i\Delta/c^2}{(b/c e^{-i\psi})^{\nu} (a/c e^{-i\phi})^{\mu}} \right.$$

$$\left. + e^{-i(\nu+1)\pi} \frac{4i\Delta/c^2}{(b/c e^{i\phi})^{\nu} (a/c e^{i\psi})^{\mu}} \right]$$

$$= \frac{8\sqrt{\Delta}}{\pi c^4} \cos(\phi\mu + \psi\nu + \pi\nu).$$

VI. Let

$$(12.31) \quad f_1(a, b, c) = \int_0^{\infty} t^{1-\rho} J_{\mu}(at) J_{\nu}(ct) Y_{\lambda}(ct) dt.$$

Then, using the relation

$$(12.32) \quad Y_{\lambda}(tc) = \frac{1}{\pi} [e^{i\pi\lambda/2} K_{\lambda}(itc+10) + e^{-i\pi\lambda/2} K_{\lambda}(-itc-10)]$$

we obtain

$$(13.33) \quad f_1(a,b,c) = \frac{2}{\pi} C(\alpha, \beta, \gamma) \cos(\mu+\nu-\lambda-\rho+2) \frac{\pi}{2} \\ \cdot \sum_{n=0}^{\infty} \frac{(\frac{\mu+\nu-\rho+\lambda+2}{2})_n (\frac{\mu+\nu-\rho-\lambda+2}{2})_n (1-\rho)_n}{(\mu+1)_n (v+1)_n n!} x^n y^n \\ \cdot {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu-\rho+\lambda+2}{2} + n, \frac{\mu+\nu-\rho-\lambda+2}{2} + n \\ \mu+1+n \end{matrix} ; x \right] \\ \cdot {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu-\rho+\lambda+2}{2} + n, \frac{\mu+\nu-\rho-\lambda+2}{2} + n \\ v+1+n \end{matrix} ; y \right],$$

if $c > a+b$ and

$$(12.34) \quad f_1(a,b,c) = \frac{2}{\pi} C(\alpha, \beta, \gamma) \operatorname{Re} \{ e^{i\pi/2 (\mu+\nu-\lambda-\rho+2)} \sum_{n=0}^{\infty} \frac{(\frac{\mu+\nu+\lambda-\rho+2}{2})_n}{(\mu+1)_n} \}$$

$$\frac{(\frac{\mu+\nu-\lambda-\rho+2}{2})_n (1-\rho)_n}{(v+1)_n n!}$$

$$\cdot {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu-\rho+\lambda+2}{2} + n, \frac{\mu+\nu-\lambda-\rho+2}{2} + n \\ \mu+1+n \end{matrix} ; x \right] \\ \cdot {}_2F_1 \left[\begin{matrix} \frac{\mu+\nu-\rho+\lambda+2}{2} + n, \frac{\mu+\nu-\lambda-\rho+2}{2} + n \\ v+1+n \end{matrix} ; y \right],$$

if $|a-b| < c < a+b$.

If we put $\lambda=\mu=\nu$ and $\rho=-\nu$ in the r.h.s. of (12.33) we obtain

$$\frac{2^{\nu+1}}{\pi} \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)} \frac{(ab)^\nu}{c^{3\nu+1}} \cos(\nu+1)\pi (1-x-y)^{-(1+2\nu)}.$$

Using (12.13) and

$$\sqrt{\pi} \Gamma(2\nu+1) = 2^{2\nu} \Gamma(\nu+1/2) \Gamma(\nu+1),$$

$$\Gamma(1/2-\nu) \Gamma(1/2+\nu) = \frac{\pi}{\cos \pi \nu},$$

we get

$$\int_0^{\infty} t^{1+\nu} J_\nu(at) J_\nu(bt) Y_\nu(ct) dt = \frac{2^{-\nu-1} (-\Delta)^{-\nu-1/2}}{\sqrt{\pi} \Gamma(1/2-\nu) (abc)^{-\nu}},$$

if $c > a+b$.

But, if $|a-b| < c < a+b$ the integral on the l.h.s. of (12.35) is given by

$$\begin{aligned} & \frac{2^{\nu+1}}{\pi} \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)} \frac{(ab)^\nu}{c^{3\nu+1}} \operatorname{Re} \left\{ e^{i\pi(\nu+1)} \left(\frac{4i\sqrt{\Delta}}{c^2} \right)^{-2\nu-1} \right\} \\ &= \frac{2^{\nu+1}}{\pi} \frac{\Gamma(2\nu+1)}{\Gamma(\nu+1)} \frac{(ab)^\nu}{c^{3\nu+1}} \operatorname{Re} \left[i \left(\frac{4\sqrt{\Delta}}{c^2} \right)^{-2\nu-1} \right] \\ &= 0. \end{aligned}$$

This is the same as Askey, Koornwinder and Rahman's formula (4.26).

If we set

$$\rho = -\mu, \lambda = \nu$$

$a = \cos \phi \cos \psi$, $b = \sin \phi \sin \psi$, $c = \cos \theta$, then we obtain that,

$$(12.36) \quad \sum_{n=0}^{\infty} h_n^{(\mu, \nu)} P_n^{(\mu, \nu)}(\cos 2\phi) P_n^{(\mu, \nu)}(\cos 2\psi) Q_n^{(\mu, \nu)}(\cos 2\theta) \cdot \frac{(\mu+\nu+1)_n}{(\nu+1)_n}$$

$$= 2^{-2\mu-2\nu-3} \frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)} (\cos \phi \cos \psi)^{-\mu} (\sin \phi \sin \psi \cos \theta)^{-\nu}$$

$$\cdot \int_0^{\infty} t^{1+\mu} J_{\mu}(\cos \phi \cos \psi t) J_{\nu}(\sin \phi \sin \psi t) Y_{\nu}(\cos \theta t) dt,$$

where $0 \leq \phi, \psi \leq \pi$, $0 < \theta < \pi$ and $h_n^{(\mu, \nu)}$ is the same as $h_n^{(\alpha, \beta)}$ defined in (4.6) with (α, β) replaced by (μ, ν) . The series in (12.36) has been evaluated in section 7.

Remark:

$$\int_0^{\infty} t^{1-\rho} J_{\mu}(at) J_{\nu}(bt) H_{\lambda}^1(ct) dt$$

$$= \begin{cases} \frac{2}{\pi} C(\alpha, \beta, \gamma) A(\alpha, \beta, \gamma) e^{i\pi/2 (\mu+\nu-\lambda-\rho+2)}, & \text{if } c > a+b, \\ \frac{2}{\pi} C(\alpha, \beta, \gamma) F_4\left(\frac{\mu+\nu+\lambda-\rho+2}{2}, \frac{\mu+\nu-\lambda-\rho+2}{2}, \mu+1, \nu+1; x(1-y) y(1-x)\right) \\ \cdot e^{i\pi/2 (\mu+\nu-\lambda-\rho+2)}, & \text{if } |a-b| < c < a+b, \end{cases}$$

where

$$H_{\lambda}^1(ct) = J_{\lambda}(ct) + i Y_{\lambda}(ct),$$

x and y are defined in (12.17).

CHAPTER IV

SOME GENERALIZATIONS13. Introduction:

In this chapter we generalize the summation formula (4.21) and the integral formula (4.13). If we substitute α , β or γ equal zero in (4.34) this reduces to (4.21). To prove (4.34) we will use the product formulas for ultraspherical polynomials; namely [9, 40]

$$(13.1) \quad C_n^\lambda(x) C_n^\lambda(y) = C_n^\lambda(1) C_\lambda \int_0^\pi C_n^\lambda(xy + (1-x^2)^{1/2} (1-y^2)^{1/2} \cos \phi) \sin \phi^{2\lambda-1} d\phi,$$

$$(13.2) \quad C_m^\lambda(x) C_n^\lambda(y) = \sum_{j=0}^{\min(m,n)} \frac{(m+n+\lambda-2j) (\lambda)_j (\lambda)_{m-j} (\lambda)_{n-j} (2\lambda)_{m+n-j}}{(m+n+\lambda-j) j! (m-j)! (n-j)! (\lambda)_{m+n-j}},$$

$$\cdot \frac{(m+n-2j)!}{(2\lambda)_{m+n-2j}} C_{m+n-2j}^\lambda(x)$$

$$\text{where } C_\lambda = \left\{ \int_0^\pi \sin^{2\lambda-1} \phi d\phi \right\}^{-1} = \frac{\Gamma(2\lambda)}{2^{2\lambda-1} \Gamma^2(\lambda)}.$$

The formula (13.2) will be used to evaluate

$$(13.3) \quad I(k; l, m, n) = \int_{-1}^1 D_k^\lambda(x) C_l^\lambda(x) C_m^\lambda(x) C_n^\lambda(x) (1-x^2)^{2\lambda-1} dx.$$

It turns out that the regions where $I(k; l, m, n)$ is non-zero it is a multiple of terminating, very-well poised and 2-balanced ${}_9F_8$. A ${}_9F_8$ generalized hypergeometric series having the said properties can be transformed as follows [17]:

$$\begin{aligned}
 (13.4) \quad & {}_9F_8 \left[\begin{matrix} a, 1 + \frac{1}{2}a, & b, & c, & d, & e, & f, \\ \frac{1}{a}a, & 1+a-b, 1+a-c, 1+a-d, 1+a-e, 1+a-f, \\ & & g, -m \\ & & & ; 1 \\ & & & & 1+a-g, 1+a+m \end{matrix} \right] \\
 & = \frac{(1+a)_m (1+k-l)_m (1+k-f)_m (1+k-g)_m}{(1+k)_m (1+a-l)_m (1+a-f)_m (1+a-g)_m} \\
 & \cdot {}_9F_8 \left[\begin{matrix} k, 1 + \frac{1}{2}k, k+b-a, k+c-a, k+d-a, & e, & f, \\ \frac{1}{2}k, & 1+a-b, 1+a-c, 1+a-d, 1+k-e, 1+k-f, \\ & & g, -m \\ & & & ; 1 \\ & & & & 1+k-g, 1+k+m \end{matrix} \right]
 \end{aligned}$$

where $k = 1+2a - b - c - d$, and the parameters are subject to the 2-balanced restriction

$$b + c + d + e + f + g - m = 2 + 3a.$$

The formula (13.4) is one of the most general known transformations existing between two terminating, 2-balanced and very well-poised series.

14. Evaluation of the Sum (4.37):

The important equation which will be used in this work is (8.33) which can be re-written as:

$$(14.1) \quad \sum_{n=0}^{\infty} (n+\lambda) \frac{\Gamma(n+1)}{\Gamma(n+2\lambda)} C_n^{\lambda}(y) C_n^{\lambda}(z) D_n^{\lambda}(u)$$

$$= \begin{cases} 0 & \text{if } yz - \sqrt{(1-y^2)(1-z^2)} < u < yz + \sqrt{(1-y^2)(1-z^2)}, \\ \pm \frac{2^{2\lambda-1}}{\pi} \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \right]^2 \{ (1-y^2)(1-z^2) - (u-yz)^2 \}^{-\lambda} \sin \lambda\pi & \text{if } 0 < u < yz - \sqrt{(1-y^2)(1-z^2)} \\ & \text{and } u > yz + \sqrt{(1-y^2)(1-z^2)}. \end{cases}$$

Let us replace y by $xy + \sqrt{(1-x^2)(1-y^2)} \cos \phi$ ($x, y \neq \pm 1$) in (14.1) and multiply both sides by $(\sin \phi)^{2\lambda-1}$ and integrate the resultant equation with respect to ϕ over the interval $[0, \pi]$, keeping in mind equation (13.1).

Notice that the integrand on the r.h.s. of new equation obtained after replacing y by $xy + \sqrt{(1-x^2)(1-y^2)} \cos \phi$ is zero when

$$(14.2) \quad |xyz + (1-x^2)^{1/2} (1-y^2)^{1/2} z \cos \phi - u| < \sqrt{\{1 - (xy + (1-x^2)^{1/2} (1-y^2)^{1/2} \cos \phi)^2\} (1-z^2)}.$$

Squaring both sides of the above inequality and simplifying gives

$$(14.3) \quad \left(\cos \phi - \frac{(uz-xy) + \sqrt{(1-u^2)(1-z^2)}}{\sqrt{(1-x^2)(1-y^2)}} \right) \left(\cos \phi - \frac{(uz-xy) - \sqrt{(1-u^2)(1-z^2)}}{\sqrt{(1-x^2)(1-y^2)}} \right) < 0,$$

which implies

$$x_0 < \cos \phi < x_1,$$

where

$$(14.4) \quad x_0 = \frac{(uz-xy) - \sqrt{(1-u^2)(1-z^2)}}{\sqrt{(1-x^2)(1-y^2)}}, \quad x_1 = \frac{(uz-xy) + \sqrt{(1-u^2)(1-z^2)}}{\sqrt{(1-x^2)(1-y^2)}}.$$

Also, we take the position sign in (14.1) when $\cos \phi < x_0$ and the negative sign when $\cos \phi > x_1$.

For $z \neq \pm 1$, we have $x_0 < x$. Putting $x = \cos \alpha$, $y = \cos \beta$, $z = \cos \gamma$, $u = \cos \tau$, we get

$$(14.5) \quad x_0 = \frac{\cos(\gamma+\tau) - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}, \quad x_1 = \frac{\cos(\gamma-\tau) - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}.$$

There are six possibilities:

(a) $x_0, x_1 < -1$, which gives

$$\cos(\gamma+\tau) < \cos(\alpha+\beta),$$

$$\cos(\gamma-\tau) < \cos(\alpha+\beta).$$

(b) $x_0 < -1$, $-1 < x_1 < 1$ which becomes

$$\cos(\gamma+\tau) < \cos(\alpha+\beta),$$

$$\cos(\alpha+\beta) < \cos(\gamma-\tau) < \cos(\alpha-\beta).$$

(c) $-1 < x_0, x_1 < 1$ which implies

$$\cos(\alpha+\beta) < \cos(\gamma+\tau) < \cos(\alpha-\beta),$$

$$\cos(\alpha+\beta) < \cos(\gamma-\tau) < \cos(\alpha-\beta).$$

(d) $x_0 < -1, x_1 > 1$

$$\cos(\gamma+\tau) < \cos(\alpha-\beta),$$

$$\cos(\gamma-\tau) > \cos(\alpha-\beta).$$

(e) $-1 < x_0 < 1, x_1 > 1$

$$\cos(\alpha+\beta) < \cos(\gamma+\tau) < \cos(\alpha-\beta),$$

$$\cos(\gamma-\tau) > \cos(\alpha-\beta).$$

(f) $x_0, x_1 > 1$

$$\cos(\gamma+\tau) > \cos(\alpha-\beta),$$

$$\cos(\gamma-\tau) > \cos(\alpha-\beta).$$

In order to evaluate the series (4.40) in all the six regions mentioned in (a) - (f), we need to consider the following three integrals.

$$(14.6) \quad I = \int_0^\pi (f + 2g \cos \phi + h \cos^2 \phi)^{-\lambda} \sin^{2\lambda-1} \phi \, d\phi,$$

$$(14.7) \quad I_1 = \int_{-1}^{x_0} (f + 2gt + ht^2)^{-\lambda} (1-t^2)^{\lambda-1} dt,$$

$$(14.8) \quad I_2 = \int_{x_1}^1 (f + 2gt + ht^2)^{-\lambda} (1-t^2)^{\lambda-1} dt,$$

where f, g, h are functions of x, y, z and are defined in (4.37).

In order to evaluate the integral I , we need the following integral representation of the ultraspherical function of the first kind [40].

$$(14.9) \quad C_\alpha^\lambda(x) = \frac{\pi \sin \alpha \pi}{\sin \lambda \pi} \int_0^\infty (1+2tx+t^2)^{-\lambda} t^{-\alpha-1} dt,$$

where $-2 < \operatorname{Re} \lambda < \operatorname{Re} \alpha < 0$, $|\arg(x \pm 1)| < \pi$.

If we set

$$(14.10) \quad t = (f+h-2g)^{1/2} (f+h+2g)^{-1/2} \tan^2 \frac{\phi}{2} = A \tan^2 \frac{\phi}{2}$$

in (14.6), where $A = (f+h-2g)^{1/2} (f+h+2g)^{-1/2}$, we get

$$(14.11) \quad I = \frac{A^{-\lambda} 2^{2\lambda-1}}{(f+h-2g)^\lambda} \int_0^\infty (t^2 + 2\chi t + 1)^{-\lambda} t^{\lambda-1} dt$$

where χ is defined in (4.39). Then (14.9) gives,

$$I = \frac{2^{2\lambda-1} \pi [(f+h)^2 - 4g^2]^{-\lambda/2}}{\sin \lambda \pi} C_{-\lambda}^\lambda(\chi).$$

If we make use of (2.2) and the quadratic transformation (7.15) and [39, (2.11) (2), p.111] we get

$$(14.12) \quad I = \frac{2^{2\lambda-1} \pi [(f+h)^2 - 4g^2]^{-\lambda/2}}{\sin \lambda \pi \Gamma(1-\lambda) \Gamma(2\lambda)} \Gamma(\lambda) {}_2F_1 \left[\begin{matrix} \lambda/2, \lambda/2+1/2 \\ \lambda+1/2 \end{matrix} ; 1-x^{-2} \right].$$

We now consider

$$\begin{aligned} I_1 &= g^{-\lambda} \int_{-1}^{x_0} (x_0-t)^{-\lambda} (x_1-t)^{-\lambda} (1-t)^{\lambda-1} (1+t)^{\lambda-1} dt \\ &= 2^{2\lambda-1} g^{-\lambda} \int_0^{1+x_0/2} (1+x_0-2u)^{-\lambda} (1+x_1-2u)^{-\lambda} u^{\lambda-1} (1-u)^{\lambda-1} du \end{aligned}$$

$$(14.13) \quad = 2^{\lambda-1} g^{-\lambda} (1+x_1)^{-\lambda} \int_0^1 t^{\lambda-1} (1-t)^{-\lambda} \left(1 - \frac{1+x_0}{1+x_1} t\right)^{-\lambda} \left(1 - \frac{1+x_0}{2} t\right)^{\lambda-1} dt.$$

Using [39, 9.3 (4), p.77],

$$(14.14) \quad I_1 = 2^{\lambda-1} g^{-\lambda} (1+x_1)^{-\lambda} \Gamma(\lambda) \Gamma(1-\lambda) F_1(\lambda; \lambda, 1-\lambda; 1; \frac{1+x_0}{1+x_1}, \frac{1+x_0}{2}).$$

The Appell's function F_1 in (14.14) is absolutely convergent for

$$(14.15) \quad \left| \frac{1+x_0}{1+x_1} \right| < 1 \text{ and } \left| \frac{1+x_0}{2} \right| < 1.$$

Assuming temporarily the inequalities in (14.15), if we now use

[17, 9.5 (2), p.79]

$$(14.16) \quad F_1(\alpha; \beta, \beta; \beta+\beta'; x, y) = (1-y)^{-\alpha} {}_2F_1 \left[\begin{matrix} \alpha, \beta \\ \beta + \beta \end{matrix} ; \frac{x-y}{1-y} \right],$$

we get

$$(14.17) \quad I_1 = 2^{2\lambda-1} g^{-\lambda} [(1-x_0)(1+x_1)]^{-\lambda} {}_2F_1 \left[\begin{matrix} \lambda, \lambda \\ 1 \end{matrix} ; \frac{(1+x_0)(1-x_1)}{(1+x_1)(1-x_0)} \right].$$

Since ${}_2F_1$ in (14.17) is absolutely convergent so the condition in (14.15) can be removed by analytic continuation.

We finally consider:

$$\begin{aligned}
(14.18) \quad I_2 &= \int_{x_1}^1 (x_0-t)^{-\lambda} (x_1-t)^{-\lambda} (1-t)^{\lambda-1} (1+t)^{\lambda-1} dt \\
&= 2^{2\lambda-1} g^{-\lambda} \int_{\frac{1+x_1}{2}}^1 (1+x_0-2u)^{-\lambda} (1+x_1-2u)^{-\lambda} (1-u)^{\lambda-1} u^{\lambda-1} du \\
&= 2^{2\lambda-2} g^{-\lambda} \left(\frac{1+x_1}{2}\right)^{\lambda-1} \left(\frac{1-x_1}{2}\right)^{\lambda-1} (x_1-x_0)^{-\lambda} (1-x_1)^{1-\lambda} \\
&\quad \cdot \int_0^1 t^{-\lambda} (1-t)^{\lambda-1} \left(1 - \frac{x_1-1}{x_1+1} t\right)^{\lambda-1} \left(1 - \frac{1-x_1}{x_0-x_1} t\right)^{-\lambda} dt \\
&= g^{-\lambda} (1+x_1)^{-\lambda} (x_1-x_0)^{-\lambda} \Gamma(\lambda) \Gamma(1-\lambda) F_1(1-\lambda; 1-\lambda, \lambda; 1; \frac{x_1-1}{x_1+1}, \\
&\quad \frac{x_1-1}{x_1-x_0}).
\end{aligned}$$

Assuming the conditions of convergence in (14.16), we have

$$\begin{aligned}
(14.19) \quad I_2 &= g^{-\lambda} \Gamma(\lambda) \Gamma(1-\lambda) (1+x_1)^{\lambda-1} (x_1-x_0)^{-\lambda} \left(\frac{1-x_0}{x_1-x_0}\right)^{\lambda-1} \\
&\quad \cdot {}_2F_1 \left[\begin{matrix} 1-\lambda, 1-\lambda \\ 1 \end{matrix} ; \frac{(1+x_0)(1-x_1)}{(1+x_1)(1-x_0)} \right].
\end{aligned}$$

By Kummer's transformation (1.11)

$$(14.20) \quad I_2 = I_1 = I' \quad (\text{say})$$

$$(14.21) \quad \text{Let } \kappa = \frac{\sin \lambda \pi}{\pi} \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)} \right]^2 \Gamma(2\lambda).$$

Therefore

$$\begin{aligned}
(14.22) \quad \kappa I &= \frac{\sin \lambda \pi}{\pi} [\Gamma(\lambda+1/2)]^2 2^{2\lambda-1} [(f+h)^2 - 4g^2]^{-\lambda/2} x^{-\lambda} \\
&\quad {}_2F_1 \left[\begin{matrix} \lambda/2, \lambda/2+1/2 \\ \lambda+1/2 \end{matrix} ; 1-x^{-2} \right].
\end{aligned}$$

$${}_K I' = 2^{2\lambda-1} \Gamma(2\lambda) \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)} \right]^2 [g(1+x_1)(1-x_0)]^{-\lambda} \\ \cdot {}_2F_1 \left[\begin{matrix} \lambda, \lambda \\ 1 \end{matrix} ; \frac{(1+x_0)(1-x_1)}{(1+x_1)(1-x_0)} \right].$$

Summarization of the results of this section gives equation (4.40) p.21.

15. Evaluation of the Integral (4.42):

The object of this section is to find a generalization of the Askey, Koornwinder and Rahman integral formula (4.13) in the same sense as (4.40) is a generalization of (4.21). We compute

$$(15.1) \quad I(k; \ell, m, n) = \int_{-1}^1 D_k^\lambda(x) C_\ell^\lambda(x) C_m^\lambda(x) C_n^\lambda(x) (1-x^2)^{2\lambda-1} dx.$$

Using the formulas (4.13), (4.14), (4.15) and the product formula (13.2) for ultraspherical polynomials we obtain

$$(15.2) \quad I(k; \ell, m, n) = \sum_{j=0}^{\min(m,n)} \frac{(m+n+\lambda-2j) (\lambda)_j (\lambda)_{m-j} (\lambda)_{n-j} (2\lambda)_{m+n-j}}{(m+n+\lambda-j) j! (m-j)! (n-j)! (\lambda)_{m+n-j}} \\ \cdot \frac{(m+n-j)!}{(2\lambda)_{m+n-2j}} I(k; \ell, m+n-2j, 0).$$

Clearly, $I(k; \ell, m, n)$ vanishes if $k+\ell+m+n$ is even. So we need only to consider the case when $k+\ell+m+n$ is odd. We know that $I(k; \ell, m+n-2j, 0)$ vanishes when

$$|m+n-2j-\ell| < k < \ell+m+n-2j$$

which becomes

$$\left[\frac{m+n-(\ell+k)}{2} \right] < j < \left[\frac{m+n-|\ell-k|}{2} \right].$$

As we have seen in the previous section, there are six possibilities depending upon the fact that $\lfloor \frac{m+n-(\ell+k)}{2} \rfloor$ and $\lfloor \frac{m+n-|\ell-k|}{2} \rfloor$ are in the set $\{0, 1, 2, 3, \dots, \min(m, n)\}$ or outside it. Since $\min(m, n) = \frac{m+n-|m-n|}{2}$, we have

$$(15.3) \quad \begin{aligned} (i) \quad & \ell+k > m+n \text{ and } |\ell-k| > m+n, \\ (ii) \quad & \ell+k > m+n \text{ and } |m-n| < |\ell-k| < m+n, \\ (iii) \quad & |m-n| < |\ell-k| < m+n \text{ and } |m-n| < \ell+k < m+n, \\ (iv) \quad & \ell+k > m+n \text{ and } |\ell-k| < |m-n|, \\ (v) \quad & |m-n| < \ell+k < m+n \text{ and } |\ell-k| < |m-n|, \\ (vi) \quad & \ell+k < |m-n| \text{ and } |\ell-k| < |m-n|. \end{aligned}$$

In order to compute the integral in (15.1) in all the six regions mentioned above, we first observe that:

$$(15.4) \quad I(\ell+m+n+1+2p; \ell, m, n)$$

$$= \sum_{j=0}^{\min(m,n)} \frac{(m+n+\lambda-2j) (\lambda)_j (\lambda)_{m-j} (\lambda)_{n-j} (2\lambda)_{m+n-j} (m+n-2j)!}{(m+n+\lambda-j)j! (m-j)! (n-j)! (\lambda)_{m+n-j} (2\lambda)_{m+n-2j}}$$

$$\cdot I(\ell+m+n+1+2p; \ell, m+n-2j, 0).$$

Using (4.14) the last equation becomes

$$\begin{aligned} & I(\ell+m+n+1+2p; \ell, m, n) \\ &= - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_\ell (2\lambda)_{\ell+m+n+1+2p}}{\ell! (\ell+m+n+1+2p)!} \sum_{j=0}^{\min(m,n)} \frac{(m+n+\lambda-2j) (\lambda)_j (\lambda)_{m-j}}{(m+n+\lambda-j) j! (m-j)!} \\ &\quad \cdot \frac{(\lambda)_{n-j} (2\lambda)_{m+n-j} (m+n-2j)! (2\lambda)_{m+n-2j} (1-\lambda)_{p+j} (\lambda)_{\ell+m+n+p-j} (\ell+p+j)!}{(n-j)! (\lambda)_{m+n-j} (2\lambda)_{m+n-j} (m+n-2j)! (p+j)! (2\lambda)_{\ell+m+n+p-j} (\lambda+1)_{\ell+p+j}} \\ &\quad \cdot \frac{(m+n+p-j)!}{(\lambda+1)_{m+n+p-j}} \end{aligned}$$

$$\begin{aligned}
 (15.5) \quad &= - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_\ell (2\lambda)_{\ell+m+n+1+2p} (\lambda)_m (\lambda)_n (2\lambda)_{m+n} (1-\lambda)_p (\lambda)_{\ell+m+n+1+p}}{\ell! (\ell+m+n+1+2p)! m! n! (\lambda)_{m+n} p! (2\lambda)_{\ell+m+n+1+p}} \\
 &\quad \frac{(\ell+p)! (m+n+p)!}{(\lambda+1)_{\ell+p} (\lambda+1)_{m+n+p}} \sum_{j=0}^{\min(m,n)} \frac{(\lambda)_j (-m)_j (-n)_j (1-\lambda-m-n)_j (-\ell-m-n-p-2\lambda)_j}{j! (1-\lambda-m)_j (1-\lambda-n)_j (1-2\lambda-m-n)_j (-\ell-m-n-p-\lambda)_j} \\
 &\quad \frac{(-m-n-p-\lambda)_j (1-\lambda+p)_j (1+\ell+p)_j (m+n+\lambda-2j)}{(-m-n-p)_j (1+p)_j (1+\lambda+\ell+p)_j (m+n+\lambda-j)}.
 \end{aligned}$$

Using the fact that

$$(15.6) \quad (-1-\lambda-m-n)_j \left(\frac{-m-n-\lambda+2j}{-m-n-\lambda+j} \right)_j = \left(\frac{(-m-n-\lambda)_j (1-\frac{m+n+\lambda}{2})_j}{(-\frac{m+n+\lambda}{2})_j} \right)_j,$$

we obtain

$$\begin{aligned}
 (15.7) \quad &I(\ell+m+n+1+2p; \ell, m, n) \\
 &= \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_\ell (2\lambda)_{\ell+m+n+1+2p} (\lambda)_m (\lambda)_n (\lambda)_{m+n} (1-\lambda)_p (\lambda)_{\ell+m+n+1+p}}{\ell! (\ell+m+n+1+2p)! m! n! (\lambda)_{m+n} p! (2\lambda)_{\ell+m+n+1+p}} \\
 &\quad \cdot \frac{(\ell+p)! (m+n+p)!}{(\lambda+1)_{\ell+p} (\lambda+1)_{m+n+p}} {}_9F_8 \left[\begin{matrix} -m-n-\lambda, 1-\frac{m+n+\lambda}{2}, -\ell-m-n-p-2\lambda, \\ -\frac{m+n+\lambda}{2}, 1+\lambda+\ell+p, \\ 1-\lambda+p, 1+\ell+p, -m-n-p-\lambda, \lambda, -m, -n \\ -m-n-p, -\ell-m-n-p-\lambda, 1+p, 1-2\lambda-m-n, 1-\lambda-n, 1-\lambda-m \end{matrix} ; 1 \right].
 \end{aligned}$$

Since the ${}_9F_8$ hypergeometric series in (15.7) is terminating, very well-poised and 2-balanced, it can be transformed to another ${}_9F_8$ hypergeometric series which is also terminating, very well-poised, and 2-balanced. So, using (13.4) we have

$$(15.8) \quad I(\ell+m+n+1+2p; \ell, m, n)$$

$$= - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_m (2\lambda)_n}{m! (\ell+m+n+1+2p)! n! \ell! (\lambda)_{\ell+m} p! (2\lambda)_{\ell+m+n+1+p}} \cdot \frac{(m+p)! (\ell+n+p)!}{(\lambda+1)_{m+p} (\lambda+1)_{\ell+n+p}} {}_9F_8 \left[\begin{matrix} -\ell-n-\lambda, 1-\frac{\ell+n+\lambda}{2}, -\ell-m-n-p-2\lambda, 1-\lambda+p, \\ -\frac{\ell+n+\lambda}{2}, 1+\lambda+m+p, -\ell-n-p, \\ 1+m+p, -\ell-n-p-\lambda, \lambda, -\ell, -n \\ -\ell-m-n-p-\lambda, 1+p, 1-2\lambda-\ell-n, 1-\lambda-n, 1-\lambda-\ell \end{matrix} ; 1 \right]$$

$$(15.9) \quad = - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_n (2\lambda)_m}{n! (\ell+m+n+1+2p)! m! \ell! (\lambda)_{1+m} p! (2\lambda)_{\ell+m+n+1+p}} \cdot \frac{(m+p)! (\ell+n+p)!}{(\lambda+1)_{m+p} (\lambda+1)_{\ell+n+p}} {}_9F_8 \left[\begin{matrix} -\ell-n-\lambda, 1-\frac{\ell+n+\lambda}{2}, -\ell-m-n-p-2\lambda, 1-\lambda+p, \\ -\frac{\ell+n+\lambda}{2}, 1+\lambda+n+p, -\ell-n-p, \\ 1+m+p, -\ell-n-p-\lambda, \lambda, -\ell, -n \\ -\ell-m-n-p-\lambda, 1+p, 1-2\lambda-\ell-n, 1-\lambda-n, 1-\lambda-\ell \end{matrix} ; 1 \right],$$

where $p = 0, 1, 2, \dots$

Notice that $I(\ell+m+n+1+2p; \ell, m, n)$ is symmetric with respect to ℓ, m, n ; the symmetry property was not obvious from (15.7) but, it can be seen from (15.8) and (15.9).

Likewise if we use (4.15) and (15.2) we obtain

$$(15.10) \quad I(\ell+m+n+1-2p; \ell, m, n)$$

$$= \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_\ell (2\lambda)_{m+n} (2\lambda)_{\ell+m+n+1-2p}}{\ell! (\lambda)_{m+n} (\ell+m+n+1-2p)! m! n! (\lambda)_p} \frac{(\lambda)_m (\lambda)_n (-\lambda)_{p-\ell}}{p!}$$

$$\frac{(\lambda)_{l+m+n+1-p} (m+n-p)! \Gamma(p)}{(2\lambda)_{l+m+n+1-p} (\lambda+1)_{m+n-p} \Gamma(p-l)} \cdot {}_9F_8 \left[\begin{matrix} -m-n-\lambda, 1-\frac{m+n+\lambda}{2}, -l-m-n+p-2\lambda, \\ -\frac{m+n+\lambda}{2}, 1+\lambda+l-p, \\ -m-n+p-\lambda, 1-\lambda-p, 1+l-p, 1+l-p, \lambda, -m, -n, \\ 1-p, -m-n+p, -l-m-n+p-\lambda, 1-2\lambda-n-m, 1-\lambda-n, 1-\lambda-m \end{matrix} \right]$$

where $m \leq n$, $l \leq k$, $p = 0, 1, 2, \dots, \lfloor \frac{m+n+l+1}{2} \rfloor$.

Equations (15.8) and (15.10) give the values of $I(k; l, m, n)$ in the regions (15.3) (i) and (15.13) (vi). In (15.3) (iii) this is obviously zero. When the inequalities in (15.3) (ii) is satisfied we have

$$(15.11) \quad I(m+n+l+1-p; l-p, m-p, n-p)$$

$$= - \left[\frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda+1)} \right]^2 \frac{(2\lambda)_{l-p} (2\lambda)_{m+n+1-p} (\lambda)_{m-p} (\lambda)_{n-p} (2\lambda)_{m+n-2p} (1-\lambda)_p}{(l-p)! (m+n+1-p)! (m-p)! (n-p)! (\lambda)_{m+n-2p} p!}$$

$$\frac{(\lambda)_{l+m+n+1-2p} l! (m+n-p)!}{(2\lambda)_{l+m+n+1-2p} (\lambda+1)_l (\lambda+1)_{m+n-p}} {}_9F_8 \left[\begin{matrix} -m-n-\lambda+2p, 1-\frac{m+n+\lambda-2p}{2}, \\ -\frac{m+n+\lambda-2p}{2}, \\ p-m, p-n, l-m-n-\lambda+p, -l-m-n-2\lambda+2p, 1-\lambda+p, 1+l, \\ 1-\lambda+p-n, 1-\lambda+p-m, 1+p-l, 1+l+\lambda, -m-n+p, -l-m-n-\lambda+2p, \\ 1-m-n-2\lambda+2p \end{matrix} ; 1 \right],$$

where $p = 0, 1, 2, \dots, \min(l, m, n)$, $m \leq n$

and $l \neq 0, m \neq 0, n \neq 0$.

Next we find $I(k; l, m, n)$ in the region (15.3) (v)

$$(15.12) \quad I(l+m+n+1-5p; l, m, n)$$

$$= \frac{(\lambda+1/2)^2 (2\lambda)_{l-p} (2\lambda)_{l+m+n+1-5p} (\lambda)_{m-p} (\lambda)_{n-p} (2\lambda)_{m+n-2p} (-\lambda)_{2p-l}}{\Gamma(\lambda+1) (l-p)! (l+m+n+1-5p)! (m-p)! (n-p)! (\lambda)_{m+n-2p} (\lambda)_p} \\ \cdot \frac{(\lambda)_{l+m+n-1-4p} (m+n-3p)! \Gamma(p)}{(2\lambda)_{l+m+n+1-4p} (\lambda+1)_{m+n-3p} \Gamma(2p-l)} \\ \cdot {}_9F_8 \left[\begin{matrix} -m-n-\lambda+2p, 1-\frac{m+n+\lambda-2p}{2}, -l-m-n-2\lambda+4p, -m-n+3p-\lambda, \\ -\frac{m+n+\lambda-2p}{2}, 1+\lambda+l-2p, -p+1, \\ 1-\lambda-p, 1+l-2p, \lambda, -m+p, -n+p \\ -m-n+3p, -l-m-n-\lambda+4p, 1-2\lambda-m-n+2p, 1-\lambda-n-p, 1-\lambda-m-p \end{matrix} ; 1 \right],$$

where $p = \lfloor \frac{l}{3} \rfloor + 1, \lfloor \frac{l}{3} \rfloor + 2, \dots, \lfloor \frac{l+m}{3} \rfloor, n \geq m$.

Finally, in the region (15.3) (iii) we have

$$(15.13) \quad I(l+m+n+1-p; l-p, m-p, n-p) + I(l+m+n+1-5p; l-p, m-p, n-p)$$

= l.h.s. of (5.11) + l.h.s. (5.12)

for $\lfloor \frac{l}{3} \rfloor < p < \lfloor \frac{l+m}{3} \rfloor, m \leq n$.

Let us summarize the results of this section.

$$I(k; l, m, n)$$

$$\left\{ \begin{array}{ll} 0 & \text{if } k+l+m+n \text{ is even} \\ (15.7) & \text{if } l+k > m+n \text{ and } |l-k| > m+n, \\ (15.11) & \text{if } l+k > m+n \text{ and } |m-n| < |l-k| < m+n, \\ (15.13) & \text{if } |m-n| < |l-k| < m+n \text{ and } |m-n| < l+k < m+n, \\ 0 & \text{if } l+k > m+n \text{ and } |l-k| < |m-n|, \\ (15.12) & \text{if } |m-n| < l+k < m+n \text{ and } |l-k| < |m-n|, \\ (15.11) & \text{if } l+k < |m-n| \text{ and } |l-k| < |m-n|. \end{array} \right.$$

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$$y = 1 + \frac{\alpha}{1} \cdot \frac{\beta}{\gamma} x + \frac{\alpha \cdot \alpha + 1}{1 \cdot 2} \frac{\beta \cdot \beta + 1}{\gamma \cdot \gamma + 1} x^2 + \text{etc.}$$

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