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**ON THE RANK OF TORSION-FREE MODULES OVER
TAME HEREDITARY ALGEBRAS**

by

István Ágoston

A thesis submitted to
the Faculty of Graduate Studies and Reserach
in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

November, 1990

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István Ágoston

in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

Chair, Department of Mathematics and Statistics

Thesis Supervisor

External Examiner

Carleton University

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Abstract

The methods and machinery of the theory of finite dimensional representations of algebras is, in contrast to the study of infinite dimensional representations, well developed and, as a consequence, the understanding of these representations is quite advanced. The problems related to the infinite dimensional representations are incomparably more complex and the respective theory is rather limited. Nevertheless, the importance of pursuing research in this area was underlined by recent investigations, showing for example how the category of infinite dimensional modules can explain certain problems arising in finite dimensional representation theory.

The goals of this thesis are quite modest. In a paper published in 1979 Ringel developed a general approach to the theory of infinite dimensional representations over hereditary algebras; in particular, his study extended previously known results for modules over Kronecker algebras to the class of tame hereditary algebras ([R2]). There is also a striking similarity between this theory and the theory of infinite abelian groups. On the other hand neither of the correspondences is "perfect". Thus I tried to look at some of these occurring similarities and differences. In particular in Chapter 3 a new approach is offered for the notion of rank of torsion-free modules, based on a suitable linear dependency relation. Torsion-free modules without proper torsion-closed submodules are discussed in Chapter 4. Finally the last chapter deals with the existence of purely simple modules of large rank.

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Introduction

The representation theory of (finite dimensional) algebras — at least in the form that we know it today — is a relatively new subject that grew out mainly from the research efforts concentrated around the so-called *Brauer–Thrall conjectures*. By the time these conjectures were settled in the late sixties and early seventies, the newly developed methods initiated a whole new area of research. Most of these tools were, however restricted to the study of representations satisfying finiteness conditions.

The restriction is not so severe if we know that very important properties of algebras can be obtained just by studying the category of finite dimensional modules over them. Moreover if the algebra is of finite representation type then the category of all representations is completely determined by the category of finite dimensional ones in the sense that every module is a direct sum of finite dimensional indecomposable modules.

In the same time while the new — homological, geometrical, combinatorial etc. — methods on finite dimensional representations were developed, the research on infinite dimensional representations was focused mainly on some specific classes of algebras. Thus considerable results were obtained in the study of modules over the so-called *Kronecker algebras* (cf. [A], [AF], [Fi] etc.). Most of these results were later generalized for tame hereditary algebras ([R2]). Unfortunately, when trying

to generalize these results to wild hereditary algebras one has to abandon a lot of the features characterizing the tame situation (cf. for example [Lu]). Nevertheless the importance of the infinite dimensional representations was emphasized again by some recent investigations pointing out important connections between infinite dimensional and finite dimensional modules, and resulting in better understanding of some purely finite dimensional phenomena (cf. [DR4], [CB]).

Unfortunately the tools of the infinite dimensional representation theory, which were developed so far are scarce, especially if we compare the situation to the finite dimensional machinery. Thus it seems to be desirable to try to adapt some already existing methods from other fields.

One of the main features of the theory of infinite dimensional representations over tame hereditary algebras is the strong similarity of the theory with the theory of (infinite) abelian groups. Thus a lot of concepts known and used for abelian groups has now its counterpart in the theory of algebras. For example in [R2] Ringel carried over such concepts as *torsion* and *torsion-free*, *divisible*, *reduced modules*, *torsion-free rank* etc. (Some of these were already used for modules over Kronecker algebras.) This "transfusion" turned out to be quite useful. Among other things this leads to a general classification of all the indecomposable modules into finite dimensional modules, torsion-free regular or Prüfer modules. Of course, this classification is still not satisfactory, since there is no complete set of numerical invariants given for torsion-free regular modules, although the concept of *rank* provides a good subdivision of the class. Another concept that can be applied from the theory of abelian groups (or as a matter of fact, from general module theory) is the notion of

purity. Ringel also showed the existence of a unique torsion-free divisible module, denoted by Q , which plays a key role in the theory, just like the set of rationals, $\mathbb{Q}$ does in the case of abelian groups.

However, it turns out that there is also a considerable difference between the two situations. For example while for torsion-free abelian groups the notion of torsion closed subgroups coincides with the notion of pure subgroups, in the general situation purity is a more restrictive concept. Consequently purely simple modules (i.e. modules with no non-trivial pure submodules) are in abundance (cf. for example [O3]), while purely simple torsion-free abelian groups are rather scarce (only the rank ones are such). This also means that the study of purely simple modules might lead to non-trivial results, while in the meantime we also have a stronger control over the class than with the less restrictive notion of indecomposability. Similarly, while the torsion-free rank of abelian groups seems to be a natural generalization of various dimension-like concepts, the behaviour of the rank of torsion-free modules over tame hereditary algebras is less “desirable”.

In my thesis I discuss some of these issues in more detail. Throughout the thesis, if not stated otherwise, all the algebras will be tame hereditary, finite dimensional over a field k . In general we will not assume that k is algebraically closed. The symbol $\text{Mod } A$ will stand for the category of all right A -modules, while $\text{mod } A$ will denote the subcategory of modules of finite length. Chapters 1 and 2 introduce some preliminary concepts about finite dimensional algebras and their finite and infinite dimensional modules. In presenting these preliminaries I was guided with the belief that the infinite dimensional representation theory is perhaps less widely known

than the standard finite dimensional techniques. Thus in Chapter 1 I confined myself to a fairly short discussion of the tools of finite dimensional representation theory, occasionally even giving up on the generality of the notions and motivated mainly by their use in the later chapters. On the other hand I tried to make the exposition of Chapter 2 as readable as possible. In doing this I mainly followed the line of [R2].

Chapter 3 deals with the definition of rank for torsion-free modules. In [R2] it is shown that every torsion-free module can be embedded into a direct sum of copies of the module Q in such a way that the quotient is torsion regular. Moreover the number of copies of Q in such an embedding depends only on the module itself. This number is called the rank of the module. Obviously the definition leaves much to desire. First it is not given by means of internal properties of the modules, but by means of the existence of a certain type of homomorphisms. Secondly the connection between the rank and the torsion closed submodule-structure is far less obvious than it is the case for instance with abelian groups. — Based on a result of Ringel we will show that for torsion-free regular modules it is possible to give an internal characterization of the rank. Moreover this characterization will be based on an appropriate dependency relation on a certain set of submodules. Thus the rank will be just the cardinality of a maximal independent set (cf. Theorem 3.11). We will call these sets P -bases where P denotes an arbitrary but fixed projective module of rank 1. Besides the obvious resemblance to the abelian groups, the advantage of this approach is the more explicit information that we get about the submodule structure of the module.

In Chapter 4 we turn to the question of the torsion substructure of torsion-free modules. When dealing with this question, *torsion-closed submodules* are of primary interest. It is easy to see that they form a lattice (which is actually a meet-sub-semilattice of the lattice of all submodules). The building blocks of this structure, i.e. those modules which have no non-trivial torsion-closed submodules will be called *superpurely simple modules*. (Obviously every superpurely simple torsion-free module is purely simple since a pure submodule of a torsion-free module is torsion-closed.) It is easy to give examples of superpurely simple modules with rank larger than 1 (one might say, contrary to our expectations, based on the experience from abelian group theory). But we will also see that all superpurely simple modules of rank larger than 1 are always finite dimensional, and there are only finitely many isomorphism classes of them (cf. Theorem 4.8). Actually we can find these classes very quickly by looking at the Auslander–Reiten graph of the algebra.

Finally in Chapter 5 we make a few comments about the existence of purely-simple modules of large rank. Here we show that the result of Okoh which states that there are no purely simple Kronecker modules of uncountable rank also holds for tame hereditary algebras. Unfortunately the question of existence of purely simple modules of countable (but infinite) rank still remains unsettled.

While I tried to give due credit to the sources of the presented results, very often it would have been difficult to trace down the origin of certain ideas. On many occasions I quoted only one possible reference, leaving to the reader to “dig” further in the past. For example many results were known for Kronecker

algebras but even in these situations I usually referred to Ringel's work [R2]. Those preliminary statements of Chapter 1 and 2 which were left without proper reference are usually either standard knowledge or can be also found in one of the listed sources.

Chapter 1.

Preliminaries about finite dimensional algebras

In the first chapter we shall introduce some of the basic concepts and tools used in the theory of finite dimensional representations over finite dimensional algebras. In a rather informal way we shall concentrate mainly on those notions that are used in the later chapters.

One of the most important developments some twenty years ago in the representation theory of algebras was, without doubt, the appearance of the Auslander-Reiten theory of almost split sequences. Let A be a finite dimensional algebra over a (commutative) field k . Then a short exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} N \longrightarrow 0$$

of finite dimensional modules in $\text{mod } A$ is called an **almost split sequence** or **Auslander-Reiten sequence** (AR-sequence, for short) if: (i) the sequence is not split; (ii) M and N are indecomposable; (iii) for every map $M \xrightarrow{\varphi} M'$ which is not a split monomorphism there is a map $E \xrightarrow{\varphi'} M'$ making the diagram commutative, i.e. $\varphi = \varphi' \alpha$; (iv) for every map $N' \xrightarrow{\psi} N$ which is not a split epimorphism, there exists a map $N' \xrightarrow{\psi'} E$ such that $\beta \psi' = \psi$. Actually these conditions are not independent (cf. for example [R3]). The almost split sequences are unique in the sense that for every indecomposable module M in $\text{mod } A$ which is not injective there is a unique module N and a unique almost split sequence starting with M and ending with N

(each uniqueness is up to isomorphism, of course). And similarly, if $N \in \text{mod } A$ is indecomposable and not projective then we can construct a unique almost split sequence containing N as its end term. Thus if M and N are the starting term and the end term of an almost split sequence, we can associate to the indecomposable module N the indecomposable module M and we shall denote this correspondence by τ_A , i.e. $M = \tau_A N$ and $N = \tau_A^{-1} M$. We say that M is the **AR-translate** of N .

There is an explicit construction for $\tau_A M$. Let $M \in \text{mod } A$ be an indecomposable module which is not projective and let us take the first two terms of the minimal projective resolution of M :

$$P_1 \xrightarrow{\alpha_1} P_0 \xrightarrow{\alpha_0} M \longrightarrow 0.$$

We can now construct the **transpose** of M , denoted by $\text{Tr } M$, by applying the functor $\text{Hom}_A(-, A)$ to the above sequence and then taking the cokernel of $\text{Hom}_A(\alpha_1, A)$:

$$\text{Hom}_A(P_0, A) \xrightarrow{\text{Hom}_A(\alpha_1, A)} \text{Hom}_A(P_1, A) \longrightarrow \text{Tr } M \longrightarrow 0.$$

The k -dual of $\text{Tr } M$ gives $\tau_A M$:

$$0 \longrightarrow \tau_A M \cong D\text{Tr } M \longrightarrow D\text{Hom}_A(P_1, A) \xrightarrow{D\text{Hom}_A(\alpha_1, A)} D\text{Hom}_A(P_0, A).$$

We should notice here that the last sequence gives a minimal injective coresolution to $\tau_A M$. Similarly, we have a construction for τ_A^{-1} : starting with a non-injective module M we obtain $\tau_A^{-1} M \cong \text{Tr } D M$. It is easy to see from the construction above that for an indecomposable module M the dual transpose $D\text{Tr } M = 0$ if and only if M is projective.

In general the correspondences τ_A and τ_A^{-1} are not functorial on the level of $\text{mod } A$; however if A is hereditary, then we have functors in both cases. Recall that a ring R is called **right hereditary** if every right ideal of R is projective. Moreover this is equivalent to the following conditions: (i) submodules of projective modules are projective; (ii) homomorphic images of injective modules are injective; etc. For a finite dimensional k -algebra A the presence of the k -duality implies that being left hereditary is equivalent to being right hereditary. Thus we shall speak about **hereditary** algebras. In homological terms hereditary algebras are exactly those, for which the functors Ext_A^2 (and then also the functors Ext_A^i , $i > 2$) vanish. If A is a hereditary algebra, then the functor $\text{DTr} = \tau_A$ is naturally equivalent to $\tau_A \cong \text{DExt}_A^1(-, A)$, while for the inverse functor we have $\tau_A^{-1} \cong \text{Ext}_A^1(\text{D}-, A)$. In particular we get that the functors τ_A and τ_A^{-1} define bijections between the sets $\text{Hom}_A(M, M')$ and $\text{Hom}_A(\tau_A M, \tau_A M')$ for $M, M' \in \text{mod } A$, provided that M and M' have no projective summands; and similarly we get a bijection between $\text{Hom}_A(M, M')$ and $\text{Hom}_A(\tau_A^{-1} M, \tau_A^{-1} M')$, if the modules M and M' have no injective summands.

Actually when the algebra A is hereditary, the Auslander–Reiten construction coincides with the previously introduced **Coxeter functors**. The idea of these functors originated in the work of Gelfand and Ponomarev (cf. [GP]). The general definition of the Coxeter functors as a composition of the so-called **reflection functors** for algebras over an algebraically closed field can be found in [BGP], and in full generality in [DR2]. The explicit construction makes it possible to find the as-

sociated **Coxeter-transformations**, a linear map computing numerical invariants of the functor.

In order to define the Auslander-Reiten quiver of the algebra A , we will need the concept of irreducible maps. A homomorphism $f : M \rightarrow N$ is called **irreducible** if: (i) it is neither a split epimorphism nor a split monomorphism; (ii) if $f = hg$ for some homomorphisms $g : M \rightarrow L$ and $h : L \rightarrow N$ then either g is a split monomorphism or h is a split epimorphism. We should notice that irreducible maps are either monomorphisms or epimorphisms. It turns out that there is a strong connection between irreducible maps and almost split sequences. Namely, if $M \in \text{mod } A$ is indecomposable and not injective, then a homomorphism $f : M \rightarrow N$ is irreducible if and only if there is a module N' and a map $f' : M \rightarrow N'$ such that the sequence

$$0 \rightarrow M \xrightarrow{(f, f')} N \oplus N' \rightarrow \text{Cok} \begin{pmatrix} f \\ f' \end{pmatrix} \rightarrow 0$$

is an almost split sequence. Similarly if N is indecomposable and non-projective then $f : M \rightarrow N$ is irreducible if and only if there exists a module M' and a map $f' : M' \rightarrow N$ such that the sequence

$$0 \rightarrow \text{Ker} (f, f') \rightarrow M \oplus M' \xrightarrow{(f, f')} N \rightarrow 0$$

is an almost split sequence. Moreover we also have characterizations of irreducible maps going from an indecomposable injective module I or going into an indecomposable projective module P too: they are just the "components" of the maps $I \rightarrow I/\text{Soc } I \rightarrow 0$ and $0 \rightarrow \text{Rad } P \rightarrow P$. It is not difficult to show that the set of irreducible maps $\text{Irr}(M, N)$ between the modules M and N admits a natural $\text{End}(N)$ - $\text{End}(M)$ bimodule structure.

The **Auslander-Reiten quiver** of the algebra A is defined as follows. It is an oriented graph with vertices being the isomorphism classes of indecomposable modules in $\text{mod } A$, and we have an edge going from $[M]$ to $[N]$ if and only if there is an irreducible map $f : M \longrightarrow N$. (In general the graph might attain a valuation, if the dimensions of the bimodules $\text{Irr}(M, N)$ are not 0 or 1, but we will not need this in our examples.)

The characterization of the irreducible maps shows that we may attach an extra structure to the Auslander-Reiten quiver. Let us define the action of τ on vertices simply by denoting $\tau[M] = [\tau_A M]$ for any vertex $[M]$ with M not projective. Then we get a **translation quiver**, i.e. a locally finite oriented graph with an injective map τ defined on a subset of vertices such that if $\tau[N]$ is defined, then there is an edge going from $[M]$ to $[N]$ if and only if there is an edge going from $\tau[N]$ to $[M]$. Notice that the set of vertices on which τ is defined corresponds to the isomorphism classes of non-projective modules, while the image of τ corresponds to the classes of non-injective ones.

There is a general method for constructing finite dimensional hereditary algebras from a given oriented graph $\vec{\Gamma} = (V, E)$. Let us assume that $\vec{\Gamma}$ is an oriented finite graph without oriented cycles. We can define the **path algebra** $k\vec{\Gamma}$ as follows. Let a basis for the underlying k -vector space be given by the set of all oriented paths of $\vec{\Gamma}$ together with all vertices (thought of as paths of zero length). The multiplication structure will be given by defining the product of two basis elements as the natural composition of paths if it is again a path or 0 if the endpoint of the first

path and the starting point of the second do not match. It is not too difficult to check that $k\vec{\Gamma}$ is a finite dimensional hereditary k -algebra. It is also an easy exercise to show that a module $M \in \text{Mod } A$ over the path algebra $A = k\vec{\Gamma}$ can be given as a set of k -vector-spaces, corresponding to the vertices of the graph, together with linear maps indexed by the arrows: $M \sim (M_i, \varphi_\alpha \mid i \in V, \alpha \in E)$, where $\varphi_\alpha \in \text{Hom}_k(M_i, M_j)$ if α is an arrow in E from i to j . The **dimension vector** of such a representation, denoted by $\underline{\dim} M$ is defined to be a vector with the i -th entry equal to $\dim_k M_i$ for $i \in V$. (In general, if $\{P_i \mid i = 1, \dots, n\}$ is a set of representatives of the isomorphism types of indecomposable projective modules, we define the dimension vector by $(\underline{\dim} M)_i = \dim_{\text{End } P_i} \text{Hom}(P_i, M)$.)

Naturally, the path algebra construction can be done for arbitrary graphs, not only for the ones we mentioned. In the general case we do not necessarily get a finite dimensional algebra but we may get one if we allow to have certain **relations** on the graph. These are just linear combinations of paths of length at least two, and we factor $k\vec{\Gamma}$ out with these. The modules over this algebra will be given by those representations, for which the corresponding combination of the linear maps is 0. It is known that if k is algebraically closed, then every finite dimensional algebra has equivalent module category with the representations of such a graph with relations. In particular, if the algebra A is **basic**, i.e. $A/\text{Rad } A$ is the product of division rings, then we can get it as a "path algebra modulo some relations".

In the general case, let us assume now that A is a basic finite dimensional hereditary algebra over k , but k is not necessarily algebraically closed. If $\{P_i \mid i = 1, \dots, n\}$ is the set of the indecomposable projective modules in $\text{Mod } A$,

then $A/\text{Rad } A \cong \prod_{i=1}^n F_i$, where $F_i = \text{End } P_i$ is a division ring for $i = 1, \dots, n$. Let $\text{Rad } A/\text{Rad}^2 A = \bigoplus_{i,j} {}_i M_j$, where ${}_i M_j$ is an F_i - F_j -bimodule. Then the **species** of A consists of the set of the division rings F_i together with the bimodules ${}_i M_j$. Moreover we can attach to the species an oriented graph with vertices $V = \{1, \dots, n\}$ and arrows from i to j if the bimodule ${}_i M_j$ is not zero. We shall also give a valuation on the edges by assigning to the edge $i \rightarrow j$ the pair $(\dim_{F_i}({}_i M_j), \dim({}_i M_j)_{F_j})$.

We may also go in the opposite direction. Let us be given a k -species $\mathcal{S} = (F_i, {}_i M_j)_{1 \leq i, j \leq n}$, i.e. a set of division rings F_i , finite dimensional over k and containing it in their center, together with a set of F_i - F_j -bimodules ${}_i M_j$, also finite dimensional over k , with k acting centrally on them. Then we may construct the **tensor algebra** $\mathcal{T}(\mathcal{S}) = \bigoplus_{\ell=0}^{\infty} M^{(\ell)}$ of the species. Here we define the semisimple algebra $\Lambda = \prod_{i=1}^n F_i$, together with $M^{(0)} = \Lambda$, $M^{(1)} = \bigoplus_{i,j=1}^n {}_i M_j$, and inductively $M^{(\ell)} = M^{(\ell-1)} \otimes_{\Lambda} M^{(1)}$. The multiplication is given by the canonical maps $M^{(\ell)} \otimes_{\Lambda} M^{(t)} \rightarrow M^{(\ell+t)}$. It should be clear that the concept of a tensor algebra generalizes the notion of path algebras.

Although not every finite dimensional hereditary algebra is isomorphic to the tensor algebra of a suitable species, still, it can be shown for example that hereditary algebras of **finite representation type** (i.e. those ones with only finitely many indecomposable modules) are all tensor algebras of some species (cf. [DR1]).

Let us assume again that we have a finite dimensional hereditary algebra A , together with its species $\mathcal{S} = (F_i, {}_i M_j)_{1 \leq i, j \leq n}$ and its oriented diagram $\vec{\Gamma}$. Let us have $f_i = \dim_k F_i$ and $m_{ij} = \dim_k {}_i M_j$ for $i, j = 1, \dots, n$. Then we can attach to A a quadratic form q defined by

$$q(x_1, \dots, x_n) = \sum_i f_i x_i^2 - \sum_{i,j} m_{ij} x_i x_j.$$

It is known that this quadratic form determines the representation type of the algebra. The algebra A is of finite representation type if and only if the corresponding quadratic form q is positive definite which in turn is equivalent to having a finite disjoint union of Dynkin graphs for our diagram. As we mentioned earlier, in this case A can be obtained as a tensor algebra over a k -species. There is a general concept of an algebra of **tame representation type** (see for example [CB]), which is equivalent in the case of hereditary algebras to the fact that the quadratic form q is positive semidefinite. This in turn is equivalent to having a Euclidean graph for our diagram. It is also known that in this case A is either a tensor algebra or it is of type $\tilde{A}(\varepsilon, \delta)$, described in [DR3]. It is easy to check now, that if q is positive semidefinite, then its restriction to the subspace defined by $x_i = 0$ for any $1 \leq i \leq n$ is positive definite.

Let A now be a tame hereditary algebra. An indecomposable module $M \in \text{mod } A$ is called **preprojective** if there is an $n \in \mathbb{N}$ such that $\tau_A^n M$ is projective; it is called **preinjective** if there is an $n \in \mathbb{N}$ such that $\tau_A^{-n} M$ is injective. Finally it is called **regular** if both $\tau_A^n M$ and $\tau_A^{-n} M$ are nonzero for every $n \in \mathbb{N}$. An arbitrary module $M \in \text{mod } A$ is preprojective, regular or preinjective if each of its indecomposable summands is preprojective, regular or preinjective, respectively. It is known that if A is of infinite representation type, then these classes are disjoint. The indecomposable preprojective and preinjective vectors are uniquely described by their dimension vectors. Thus for any fixed $n \in \mathbb{N}$ there are only finitely many indecomposable preprojective or preinjective modules M with $\dim_k M = n$.

There is a linear form δ , called the **defect** which tells us which class the indecomposable module M belongs to: it is preprojective if $\delta(\underline{\dim} M) < 0$; it is regular if $\delta(\underline{\dim} M) = 0$ and it is preinjective if $\delta(\underline{\dim} M) > 0$. We will assume, that δ is normalized, i.e. $\delta(P_i)$ is an integer for each indecomposable projective module, moreover $\delta(P_i) = -1$ for at least one indecomposable projective module. Note that the defect, being a linear form, is an additive function on the dimension vectors of modules in a short exact sequence. An important property of δ is that it is invariant under τ_A , i.e. $\delta(\underline{\dim} M) = \delta(\underline{\dim} \tau_A M)$, moreover it is uniquely determined by the oriented diagram of A . Actually the values $\delta(P_i)$ with P_i being the indecomposable projective corresponding to the vertex i do not even depend on the orientation. Thus for example it is easy to check that for an indecomposable module M we always have $|\delta(M)| \leq 6$. Except for the diagrams $\tilde{A}_{12}$, $\tilde{A}_n$ and $\tilde{C}_n$, every Euclidean diagram has a vertex i with $\delta(P_i) < -1$.

From the fact that $\text{Hom}(M, N) \cong \text{Hom}(\tau_A M, \tau_A N)$ for M and N indecomposable non-projective modules, we can derive the following properties of the Auslander-Reiten quiver of A . If there is a non-zero homomorphism between the indecomposable modules M and N then N being preprojective implies that M is also preprojective. And similarly, if M is preinjective, then N also must be preinjective. Moreover if a connected component of the Auslander-Reiten quiver contains a preprojective (preinjective, regular) element then all of the vertices are preprojective (preinjective, regular).

When A is a tame hereditary algebra, it is possible to give a fairly complete description of the full subcategory of regular modules in $\text{mod } A$. First of all, they

form an abelian exact subcategory, closed under extensions. The simple objects in this category are called **simple regular** modules. If M is a simple regular module, then so is $\tau_A M$. The τ_A -orbits of these are always finite, and in most cases they are kept fixed under the action of τ_A : depending on the unoriented diagram of A , there are at most three non-trivial τ_A -orbits of simple regular modules. If S and S' are simple regular modules, then $\text{Ext}(S, S') \neq 0$ if and only if $S' = \tau_A S$. Thus the subcategory of regular modules in $\text{mod } A$ is the direct sum of subcategories, indexed by the τ_A -orbits of simple regular modules, and each of the indecomposable objects has a unique regular composition series. To every simple regular module S and every number $n \in \mathbb{N}$ corresponds a unique (indecomposable) regular module with regular socle equal to S and regular length equal to n . This module will be denoted by S^n . The set of τ_A -orbits of simple regular modules is always infinite, but in general difficult to describe. If k is algebraically closed, then it can be parametrized in a natural way by the projective line $\mathbb{P}_1(k)$ over k .

Chapter 2.

...about infinite dimensional representations

First we define a torsion theory on $\text{Mod } A$. For an arbitrary module $M \in \text{Mod } A$ denote by $\mathcal{T}(M)$ the trace of the finite dimensional regular and preinjective modules:

$$\mathcal{T}(M) = \sum \{N \subseteq M \mid \dim_k N < \infty, N \text{ regular or preinjective}\}.$$

We call a module **torsion** if $\mathcal{T}(M) = M$ and **torsion-free** if $\mathcal{T}(M) = 0$. Thus a module M is torsion if and only if it is spanned by its finite dimensional regular and preinjective submodules, while M is torsion-free if and only if every finite dimensional submodule of M is preprojective. A finite dimensional module is torsion if and only if its indecomposable direct summands are all regular or preinjective. Let us remark the following. Since the set of finite dimensional torsion submodules of a fixed module M is directed, every finite dimensional submodule of a torsion module M is contained in a finite dimensional torsion submodule of M . The class of torsion modules will be denoted by $\mathcal{T}$, while the class of torsion-free modules by $\mathcal{F}$.

The following shows that we indeed have a torsion theory. To indicate the flavour of the subject, we include a short proof.

Proposition 2.1. *(i) The class of torsion-free modules is closed under taking submodules, extensions and direct products.*

(ii) *The class of torsion modules is closed under taking homomorphic images, extensions and direct sums.*

(iii) *The pair $(\mathcal{T}, \mathcal{F})$ defines a torsion-theory.*

Proof. (i) It is clear from the definition that $N \subseteq M$ implies $\mathcal{T}(N) \subseteq \mathcal{T}(M)$. Thus a submodule of a torsion-free module is torsion-free again. Let us now take a short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ with N and M/N torsion-free. Assume that U is a finite dimensional torsion submodule of M . Then $U/(U \cap N)$ embeds into M/N and as a homomorphic image of U , it is also torsion. By assumption M/N is torsion-free, hence $U/U \cap N = 0$, i.e. $U \subseteq N$. But N is also torsion-free, thus $U = 0$. Hence M is torsion-free, showing that torsion-free modules are closed under extensions. Finally let us consider $M = \prod_{\lambda} M_{\lambda}$ with M_{λ} torsion-free for every λ . If U is a finite dimensional torsion submodule of M then the modules $U_{\lambda} = \pi_{\lambda}(U)$, where π_{λ} denotes the projection of M onto M_{λ} , are also torsion. Thus by assumption they all must be zero. Hence $U = 0$ showing that M is torsion-free. (ii) follows from a similar argument.

(iii) To finish the proof we have to show that $T \in \mathcal{T}$ if and only if $\text{Hom}(T, F) = 0$ for every $F \in \mathcal{F}$ and similarly, $F \in \mathcal{F}$ holds if and only if $\text{Hom}(T, F) = 0$ for every $T \in \mathcal{T}$. It is clear from the definition, that $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$. On the other hand it is also an easy consequence of the definition that for a module F the property $\text{Hom}(\mathcal{T}, F) = 0$ implies that $\mathcal{T}(F) = 0$, i.e. F is torsion-free. To show that $\text{Hom}(\mathcal{T}, \mathcal{F}) = 0$ implies $\mathcal{T} \in \mathcal{T}$, it is enough to check that $\mathcal{T}(M/\mathcal{T}(M)) = 0$ for every module M . Assume $\mathcal{T}(M) \subset U \subseteq M$ is a submodule for which $U/\mathcal{T}(M)$ is

finite dimensional indecomposable. Choose a finite dimensional submodule $U' \subseteq M$ such that $U' + \mathcal{T}(M) = U$. The module $U' \cap \mathcal{T}(M)$ is finite dimensional, hence by our previous remark there is a finite dimensional torsion submodule U'' containing $U' \cap \mathcal{T}(M)$ and then $U' \cap \mathcal{T}(M) = U' \cap U''$. Since the finite dimensional submodule $U' + U''$ is not contained in $\mathcal{T}(M)$, it is not torsion, hence it must have a non-zero preprojective direct summand, say P . Thus $U' + U'' = P \oplus V$ for some $V \subseteq M$. Since U'' is torsion, there is no non-zero map from U'' to P , thus $U'' \subseteq V$. But then we have:

$$\begin{aligned} U/\mathcal{T}(M) &= U' + \mathcal{T}(M)/\mathcal{T}(M) \cong U'/U' \cap \mathcal{T}(M) = U'/U' \cap U'' \cong \\ &\cong U' + U''/U'' = (P \oplus V)/U'' \cong P \oplus (V/U''). \end{aligned}$$

Since by assumption $U/\mathcal{T}(M)$ was indecomposable, it must be isomorphic to P , i.e. it is preprojective. Thus $\mathcal{T}(M/\mathcal{T}(M)) = 0$, and this finishes the proof.

Remark. The above statements are not entirely independent. It is easy to show that (iii) in general implies (i) and (ii). The detailed statement and the proof for (i) serve merely illustrative purposes.

A submodule $N \subseteq M$ is called **torsion-closed in M** if the quotient module M/N is torsion-free. Now we easily get the following statement.

Corollary 2.2. (i) If $L \subseteq N \subseteq M$ are such that L is torsion closed in N and N is torsion-closed in M , then L is torsion-closed in M .

(ii) If $\{N_\lambda \subseteq M \mid \lambda \in \Lambda\}$ is a set of torsion-closed submodules in M then their intersection $N = \bigcap_{\lambda \in \Lambda} N_\lambda$ is also torsion-closed in M .

Proof. (i) This follows from Proposition 2.1 since the class $\mathcal{F}$ is closed under extensions.

(ii) The module M/N is a subdirect product of the torsion-free modules M/N_λ . Hence by Proposition 2.1 it is torsion-free.

Now we may define the torsion closure of a submodule. For $N \subseteq M$ the **torsion closure** of N in M is the intersection of all torsion-closed submodules in M which contain N . We will denote this module by $\overline{N}^M$. From Corollary 2.2 it is clear that $\overline{N}^M$ is the smallest torsion-closed submodule of M containing N . It should be equally clear that $\overline{N}^M$ is just the full preimage of $\mathcal{T}(M/N)$ in M . Finally the following technical statement is also easy to verify.

Lemma 2.3. *If $L \subseteq N \subseteq M$ then $\overline{L}^N \subseteq N \cap \overline{L}^M$.*

Proof. The embedding $N/N \cap \overline{L}^M \hookrightarrow M/\overline{L}^M$ shows that $N \cap \overline{L}^M$ is torsion-closed in N , containing L as a submodule. Hence the statement follows. (Notice that $\overline{L}^N$ is not necessarily equal to $N \cap \overline{L}^M$ since a torsion module can easily have torsion-free submodules.)

Another concept we shall be dealing with is the notion of pure submodules. We can define it using the following proposition.

Proposition 2.4. *The following are equivalent for a submodule N of the module M :*

- (i) For every intermediate submodule $N \subseteq W \subseteq M$, if W/N is of finite length, then N is a direct summand of W .
- (ii) For every submodule $W \subseteq N$, if N/W is of finite length, then N/W is a direct summand of M/W .
- (iii) For every finite system of equations $\sum_j c_{ij}x_j = n_i$ with $c_{ij} \in A$ and $n_i \in N$ for $i = 1, \dots, r$, $j = 1, \dots, s$, if it is solvable in M , then it is also solvable in N . In short: $C \cdot M^s \cap N^r = C \cdot N^s$ for every $r \times s$ matrix C with coefficients in the algebra A .

Proof. We refer here to [R2] 1.F. See also [Z], [C].

We say that N is a **pure submodule** of M if it satisfies the equivalent conditions of Proposition 2.4. It is clear that if N is a direct summand of M then it is pure. It is also an easy corollary of (ii) that for a submodule N of finite length being pure in M is equivalent to being a direct summand of M . We should notice the connection between Proposition 2.4 and the similar results for abelian groups (cf. [Fu]).

Unlike the situation with abelian groups, where the intersection of pure subgroups of torsion-free abelian groups is again pure, over tame hereditary algebras the intersection of pure submodules is not necessarily pure. Take for example the path algebra of $\tilde{E}_6$ with the arrows all pointing from the center towards the "branches". Let P_2 and S_2 denote the indecomposable projective and the simple module, respectively, corresponding to a vertex adjacent to the center of the graph. Denote

by S_1 the socle of P_2 . Consider the module $M = P_2 \oplus S_2$. It is torsion-free, as both P_2 and S_2 are preprojective. If we denote by a and b the generators of $P_2 \subseteq M$ and $S_2 \subseteq M$, respectively, then the submodule generated by $a + b$ is also a direct summand (S_2 is its complement). So the submodules generated by a and by $a + b$ are both pure, but their intersection, isomorphic to S_1 , is not.

On the other hand purity behaves somewhat better with respect to unions:

Proposition 2.5. *Let $\{N_\lambda \mid \lambda \in \Lambda\}$ be a directed set of pure submodules in M . Then $N = \bigcup_{\lambda \in \Lambda} N_\lambda$ is again pure in M . In particular the union of a directed set of direct summands of M is pure in M .*

Proof. See, for example, [R2] 1.F.(c).

Unlike for abelian groups, there are finite dimensional modules over a tame hereditary algebra such that their torsion-closed submodules are not necessarily pure. However, the converse statement is true.

Proposition 2.6. *If N is a pure submodule of the torsion-free module M then it is torsion-closed.*

Proof. It is easy, so we shall give it here. We just have to show that if $U \subseteq M$ is a submodule containing N and U/N is of finite length then U is preprojective. But by assumption N is pure, thus U splits as $U = N \oplus U'$ for some $U' \subseteq M$ and here $U/N \cong U'$. Since M was torsion-free, U' and hence U/N must be preprojective. Thus N is torsion-closed.

One of the significant applications of Proposition 2.5 makes it possible that instead of dealing with arbitrary modules, most of the time we can focus our attention to two extreme situations: one is when the module is finite dimensional, and the other is when the module has no finite dimensional direct summands. Namely the following is true. (For the proof we again refer to [R2] Theorem 1.G.)

Theorem 2.7. *Let M be an arbitrary module. Then there is a pure submodule $N \subseteq M$ such that N is the direct sum of finite dimensional modules and the quotient has no finite dimensional direct summands.*

Actually when it comes to finite dimensional direct summands, we can check their existence quite well using the following two functors. For an arbitrary module M define $\mathcal{P}(M)$ to be the reject of the finite dimensional preprojectives on M , i.e.

$$\mathcal{P}(M) = \bigcap \{ \text{Ker } f \mid f \in \text{Hom}(M, P) \text{ with } P \text{ preprojective, } \dim_k P < \infty \}.$$

And similarly to this we can define $\mathcal{J}(M)$ as the trace of the preinjectives:

$$\mathcal{J}(M) = \sum \{ \text{Im } f \mid f \in \text{Hom}(I, M) \text{ with } I \text{ preinjective, } \dim_k I < \infty \}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are any subsets of the set of all indecomposable preprojective or preinjective modules, respectively, then we can also define the "partial functors" $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{J}_{\mathcal{B}}$, by restricting the choice of modules for P and I in the definitions above.

While the functors $\mathcal{P}$ and $\mathcal{J}$ appear to be dual to each other, there is a considerable difference between their behaviour. First of all, for every module M we have $\mathcal{J}(M/\mathcal{J}(M)) = 0$; on the other hand $\mathcal{P}(\mathcal{P}(M)) \neq \mathcal{P}(M)$ in general. (Thus in the case of $\mathcal{P}$ we can iterate the process, and define by transfinite induction

$\mathcal{P}^\lambda$ for an ordinal $\lambda = \mu + 1$ as $\mathcal{P}^\lambda(M) = \mathcal{P}(\mathcal{P}^\mu(M))$, and for λ a limit ordinal as $\mathcal{P}^\lambda = \bigcap_{\mu < \lambda} \mathcal{P}^\mu(M)$. Obviously this process will stop somewhere; we denote the obtained submodule by $\mathcal{P}^\infty(M)$, i.e. $\mathcal{P}^\infty(M) = \bigcap_{\lambda} \mathcal{P}^\lambda(M)$.) Actually Ringel has shown that for every ordinal λ there is a module for which $\mathcal{P}$ is stabilized exactly after λ steps (cf. [R2] Theorem 2.6). Nevertheless we have the following parallel statements.

Theorem 2.8. *Let $\mathcal{A}$ be a finite subset of the set of the isomorphism types of all indecomposable preprojective modules of finite length. Assume $\mathcal{A}$ is closed under taking predecessors (i.e. $P \in \mathcal{A}$ and $\text{Hom}(P', P) \neq 0$ for some indecomposable preprojective module P' implies $P' \in \mathcal{A}$). Then for every module M there is a decomposition*

$$M = \mathcal{P}_{\mathcal{A}}(M) \oplus \left(\bigoplus_{P \in \mathcal{A}} U_P \right),$$

where U_P is a (not necessarily unique) submodule of M , isomorphic to a direct sum of copies of P . In particular this also implies that $\mathcal{P}_{\mathcal{A}}(\mathcal{P}_{\mathcal{A}}(M)) = \mathcal{P}_{\mathcal{A}}(M)$ for any module M and set $\mathcal{A}$ as above.

Theorem 2.9. *Let $\mathcal{B}$ be a finite subset of the set of the isomorphism types of all indecomposable preinjective modules of finite length. Assume $\mathcal{B}$ is closed under taking successors (i.e. $I \in \mathcal{B}$ and $\text{Hom}(I, I') \neq 0$ for some indecomposable preinjective module I' implies $I' \in \mathcal{B}$). Then for every module M the submodule $\mathcal{J}_{\mathcal{B}}(M)$ is a direct summand of M . Moreover $\mathcal{J}_{\mathcal{B}}(M)$ is the direct sum of some (not*

necessarily unique) submodules V_I for $I \in \mathcal{B}$ such that V_I is isomorphic to a direct sum of copies of I .

Proof. A detailed proof for both statements can be found in [R2], cf. Propositions 2.1 and 3.1. Here we include only the idea of the argument. If $\mathcal{A}$ consists of a single projective P , every homomorphism $f \in \text{Hom}(M, P)$ splits, thus the statement becomes clear. Then intuitively we can proceed by realizing that indecomposable preprojective modules can be defined via relative projectivity; etc. — (The theorem can also be deduced from the Brenner–Butler Theorem (cf. [HR]).)

There is a further difference between the behaviour of $\mathcal{P}$ and $\mathcal{J}$. On the one hand the existence of modules with the property that $\mathcal{P}(\mathcal{P}(M)) \neq \mathcal{P}(M)$ shows that $\mathcal{P}(M)$ is not necessarily a direct summand of M . (Notice that in Theorem 2.8 $\mathcal{A}$ must be finite!) On the other hand, $\mathcal{J}(M)$ is always a pure submodule of M , and when the algebra is tame, it is even a direct summand. Namely, we have the following characterization of tameness.

Theorem 2.10. *Let A be a finite dimensional hereditary algebra of infinite representation type. Then the following are equivalent:*

- (i) *A is of tame representation type.*
- (ii) *For every A -module M the submodule $\mathcal{J}(M)$ is a direct summand of M .*
- (iii) *If N is a submodule of finite length of the module M where $\mathcal{J}(M) = 0$ then $\mathcal{J}(M/N)$ is also of finite length.*

Proof. See [R2] 3.6–3.9. For some further characterizations, see also [Le].

The main reason for discussing these functors here is, as we have already mentioned, that they give a nice criterion for the existence of finite dimensional preprojective or preinjective direct summands. It is not difficult to get the following corollaries.

Corollary 2.11. (i) *A module M has no finite dimensional preprojective direct summands if and only if $\mathcal{P}(M) = M$.*

(ii) *A module M has no finite dimensional preinjective direct summands if and only if $\mathcal{J}(M) = 0$.*

Moreover we can derive the purity of certain submodules, too.

Corollary 2.12. (i) *The torsion submodule $T(M)$ is always pure in M .*

(ii) *If N is a torsion-closed submodule of M such that $\mathcal{P}(N) = N$ (for example N has no finite dimensional direct summands), then N is pure in M .*

This brings us to another important concept. We call a module **regular** if it has no finite dimensional preinjective or preprojective direct summands. It is clear that for finite dimensional modules this concept coincides with the usual terminology. Thus Corollary 2.11 gives us that M is regular if and only if $\text{Hom}(M, P) = 0$ for every finite dimensional preprojective module P and $\text{Hom}(I, M) = 0$ for every finite dimensional preinjective module I . So for instance it is easy to see that if a

submodule N is torsion-closed in M and M is regular, then the quotient module M/N is again regular. In general we have the following proposition, the proof of which follows from the criteria mentioned above.

Proposition 2.13. *The class of all regular modules is closed under images of homomorphisms, extensions and direct sums.*

While the structure of torsion-free regular modules is in general unknown, for torsion-regular modules Ringel has given a structure theorem, somewhat similar to the primary decomposition of torsion abelian groups. To give the complete description, we will first define the so called Prüfer modules. As we have seen in Chapter 1, if A is a tame hereditary algebra, the regular modules of finite length form an exact abelian category which is the direct sum of certain subcategories. These subcategories are indexed by the τ_A -orbits of the simple objects. The indecomposable objects are uniserial, and to each simple regular module S (i.e. simple in the category of regular modules) there corresponds a uniquely determined indecomposable object S^n of (regular) length n with (regular) socle isomorphic to S ; etc. Thus for each simple regular module S we have a series of embeddings:

$$0 \subseteq S \subseteq S^2 \subseteq \dots \subseteq S^n \subseteq S^{n+1} \subseteq \dots$$

Taking the direct limit of this system we get a module $S^\omega = \bigcup_{n \in \mathbb{N}} S^n$. Since an automorphism of S^n extends to an automorphism of S^{n+1} , the module S^ω is uniquely defined, up to automorphism. It is easy to see that the defining sequence of submodules is the only sequence of regular submodules and it is just the regular socle

sequence of S^ω . The module S^ω will be called **Prüfer module** with socle S , in reference to the obvious analogy to Prüfer groups. (Note that in the case when the τ_A -orbit of S has more than one elements, S^ω will have as composition factors each of the simple modules in the orbit of S , i.e. the composition factors are not necessarily all isomorphic.) These modules were introduced in [R1].

Now we are ready to state the structure theorem for torsion regular modules.

Theorem 2.14. *Let A be a finite dimensional tame hereditary algebra. Then the full subcategory of torsion regular modules is an exact abelian category. It is a direct product of certain subcategories and these are in one-to-one correspondence with the τ_A -orbits of the finite dimensional simple regular modules. The elements of these subcategories are regular submodules of direct sums of Prüfer modules, the (regular) socle of which belongs to the corresponding orbit.*

Proof. For the proof we refer to [R2] Theorem 4.4.

We should note here that it is possible to give a more explicit description of the subcategories mentioned above. From this explicit characterization we can get a number of statements concerning, for instance the homological properties of the Prüfer modules. In particular we find that they are characterized by the following property.

Proposition 2.15. *Let M be a torsion-regular module. Then we have $\text{Ext}(N, M) = 0$ for every torsion-regular module N if and only if M is a direct sum*

of Prüfer modules.

This leads us to the following definition. A module M will be called **divisible** if and only if $\text{Ext}(S, M) = 0$ for every simple regular module S . Actually the divisible modules satisfy a stronger homological condition. For the proof, we refer to [R2] or [Le], Proposition 4.7.

Proposition 2.16. *Let M be a divisible module. Then $\text{Ext}(N, M) = 0$ for every N such that $\mathcal{J}(N) = 0$.*

Obviously the finite dimensional preinjective modules are all divisible. So are the Prüfer modules, as we have just seen it. Actually there is “not much” left besides them.

Proposition 2.17. *Let A be a tame hereditary algebra. Then any divisible module M is the direct sum of finite dimensional indecomposable preinjectives, Prüfer modules and a torsion-free divisible module.*

Proof. It is clear that the decomposition can only be $M \cong \mathcal{J}(M) \oplus \mathcal{T}(M)/\mathcal{J}(M) \oplus M/\mathcal{T}(M)$. So besides using Theorem 2.10, one just has to show that the class of divisible modules is closed under taking quotients and torsion submodules, moreover the torsion submodule splits off M by Proposition 2.16, since $M/\mathcal{T}(M)$ is regular, while $\mathcal{T}(M)$ is divisible.

We can also derive from here a (partial) classification of indecomposable modules in $\text{Mod } A$ for a tame hereditary algebra A . It shows that for a full picture of the indecomposable modules in $\text{Mod } A$ one would only need a better understanding of the structure of torsion-free regular modules.

Theorem 2.18. *Let A be a tame hereditary algebra and M be an indecomposable module in $\text{Mod } A$. Then one of the following holds:*

- (i) M is of finite length;
- (ii) M is a Prüfer module;
- (iii) M is a torsion-free regular module.

Proof. See [R2] Corollary 4.8.

Before looking at the existence of torsion-free divisible modules it is perhaps appropriate to stop here for a moment and mention one result that is going to be crucial for our further investigations. It also emphasises the importance of the structure theorem for torsion regular modules, as it shows that every regular module can be obtained as the extension of a direct sum of indecomposable projective modules by a torsion-regular module.

Theorem 2.19. *Let A be a connected tame hereditary algebra. Let M be a regular module in $\text{Mod } A$, and P an arbitrary (indecomposable) projective module of defect $\delta(P) = -1$. Then there is a submodule U isomorphic to a direct sum of copies of P such that the quotient module M/U is torsion regular.*

Proof. See [R2] Proposition 4.3.

Returning back to divisibility, we would like to show that there exist torsion-free divisible modules. In fact, for every tame hereditary algebra A there exists a uniquely defined indecomposable torsion-free divisible module Q . The following statement summarizes some of the most important information about torsion-free divisible modules. The proof of the statements can be found in [R2] Chapter 5.

Theorem 2.20. *There exists a unique indecomposable torsion-free divisible module, which we will denote by Q . Its endomorphism ring is a division ring, and $\dim_{\text{End } Q} Q$ is finite. Any torsion-free divisible module is a direct sum of copies of Q . Any torsion-free module M can be embedded into a direct sum Y of copies of Q so that the quotient Y/M is torsion-regular. The number of copies of Q in such an embedding is an invariant of M .*

Example 2.21. Let A be the path algebra of the graph $\tilde{D}_4$ with the orientation where all the arrows point towards the vertex of degree 4. Then Q can be given as the module with $k(x)$ as the underlying vector space for each vertex of degree 1 and $k(x) \oplus k(x)$ the vector space for the vertex of degree 4. The linear maps corresponding to the multiplications with the arrows $\alpha_1, \alpha_2, \alpha_3$ and α_4 are given by:

$$\varphi_{\alpha_1}(p) = (p, 0);$$

$$\varphi_{\alpha_2}(p) = (0, p).$$

$$\varphi_{\alpha_3}(p) = (p, p);$$

$$\varphi_{\alpha_4}(p) = (p, x \cdot p);$$

Here $\text{End } Q = k(x)$, and Q is obviously finite dimensional over $\text{End } Q$.

The theorem above gives rise to the definition of rank. If M is a torsion-free module, then there is an embedding into $\bigoplus_I Q$ for some index set I such that $\bigoplus_I Q/M$ is torsion regular. The **rank** of M is defined as the cardinality of this index set: $\text{rk } M = |I|$.

Here are some basic facts about the rank.

Theorem 2.22. *If P is a finite dimensional preprojective module, then $\text{rk } P = -\delta(P)$, i.e. for finite dimensional modules the rank is just the negative of the defect.*

Theorem 2.23. *Let M be a torsion-free module, N a torsion-closed submodule in M . Then $\text{rk } M = \text{rk } N + \text{rk } M/N$.*

Theorem 2.24. *Let M be torsion-free, and assume that at least one of the numbers $\text{rk } M$ and $\dim_{\text{End } Q} \text{Hom}(M, Q)$ is finite. Then they are both finite and equal.*

Proof. For the proofs see [R2] Proposition 5.6, [DZ1] Proposition 2.2 and [D].

Perhaps a warning notice is in place here. Despite the obvious resemblance to the case of abelian groups, where the group of rationals $\mathbb{Q}$ plays a similar role as the module Q defined above, there are also considerable differences in their behaviour.

For example while for torsion-free abelian groups being of rank 1 is equivalent to being a subgroup of $\mathbb{Q}$, the infinite dimensional module Q has infinite dimensional socle, thus it will contain also submodules of infinite rank. Of course, Theorem 2.23 tells us, that none of these submodules of large rank can be torsion-closed. Thus instead of looking at all the submodules of a torsion-free module, it is appropriate to consider only the torsion-closed ones. Another fact deserving some attention is that large rank doesn't necessarily mean large "complexity" in the (torsion-closed) submodule structure: there are simple preprojective modules having defect smaller than -1 (see Chapter 1), hence by Theorem 2.22 their rank is larger than 1.

As we have seen in Theorem 2.18, the only type of indecomposable modules not fully understood is the class of torsion-free regular indecomposable modules. The notion of rank offers itself as a first natural invariant for any classification. For example it is an easy consequence of Theorem 2.23 that every torsion-free module of rank 1 is indecomposable. Since the question of indecomposability is in general difficult to attack, a suitable subclass arose as the subject of investigations. We already noted that direct summands are obviously pure submodules. So if a module is purely simple, it is necessarily indecomposable.

Contrary to the abelian group case where the purely simple torsion-free groups are just the ones with rank 1, Okoh showed (see, for example [O3]) that there are purely simple Kronecker modules of arbitrarily large (finite) rank. His results were later generalized for all tame hereditary algebras by Dean and Zorzitto (cf. [DZ2]). Thus the notion of purely simple modules eventually might lead to a more fruitful

analysis. Let us just mention here one useful criterion from [DZ1] for determining whether a torsion-free module of finite rank is purely simple.

Proposition 2.25. *Let M be an infinite dimensional torsion-free module of finite rank. Then it is purely simple if and only if every proper torsion-closed submodule of M is of finite length.*

Proof. Assume first that M is not purely simple, and let N be a non-trivial pure submodule. Now either N is infinite dimensional and then by Proposition 2.6 it is an infinite dimensional torsion-closed submodule, or N is finite dimensional in which case Proposition 2.4 (ii) implies that N is a direct summand of M , i.e. $M = N \oplus N'$ for some submodule N' . But then N' is obviously infinite dimensional and also torsion-closed.

Assume now that N is an infinite dimensional torsion-closed submodule of M . Theorem 2.23 allows us to write N as a finite direct sum of indecomposable torsion-free submodules. Obviously all of them must be torsion-closed in M (cf. Corollary 2.2 (i)) and at least one of them will be infinite dimensional. Corollary 2.12 (ii) gives us that this infinite dimensional indecomposable torsion-closed submodule must be pure.

As a final issue let us mention here that in [R2] Ringel has given a full classification for the torsion-free modules of rank 1, at least in the case when the base field k is algebraically closed. The characterization, in terms of equivalence classes

of height functions, is similar to the one for torsion-free abelian groups of rank 1.
(For Kronecker-modules, see also [Fi].)

Some of these topics will be discussed in more detail in the next chapters.

Chapter 3.

P-bases for torsion-free regular modules

As we have seen in Chapter 3, the definition of the rank of torsion-free modules is far from being obvious and it is in some sense rather unnatural, especially if we have in mind the rank of torsion-free abelian groups (or in general some other “dimension-like” concepts). The present approach gives neither an internal characterization of the rank of a module, nor is it linked to any obvious “dependency relation” as it is in the other cases. As a consequence, one can meet difficulties already in proving some elementary statements. In this chapter, we shall define a certain dependence relation on a subset of all submodules of M . Based on an idea of Ringel (cf. Proposition 2.19), we shall see that for torsion-free regular modules it is possible to obtain the rank as the cardinality of a maximal independent system for this dependence relation. This will also reveal some of the properties of the torsion-closed submodules.

The fact that we will be dealing only with regular modules seems to be necessary if we wish to have a fairly complete analogy to the general situation. One should perhaps also realize that — as we have indicated in the previous chapter — the behaviour of the finite dimensional preprojective modules in this respect is rather “irregular”, thus the obstacles for extending fully these ideas are not only of technical nature. It is worth mentioning that the original idea of Ringel does have a

suitable counterpart for general torsion-free modules (in particular we should mention Proposition 2.7 as a useful tool). This result also allows us to restrict ourselves to either the finite dimensional or the regular case. Since for finite dimensional modules the rank can be obtained from the defect, the general situation can be considered fairly satisfactory.

Throughout the chapter we will assume that A is a finite dimensional connected tame hereditary algebra. Let us recall from Chapter 2 that the torsion closure $\overline{N}^M$ of a submodule N in a module M is the (unique) smallest torsion-closed submodule containing N . It can be obtained as the intersection of all torsion-closed submodules of M which contain N , or equivalently, it is just the full preimage of the torsion submodule $\mathcal{T}(M/N)$. Recall also that by Q we denoted the unique indecomposable torsion-free divisible module over A .

The following proposition is of crucial importance.

Proposition 3.1. *Let M be a torsion-free module, $N \leq M$. Then $\text{rk } \overline{N}^M \leq \text{rk } N$.*

Proof. Let $\overline{N} = \overline{N}^M$. Assume first that $\text{rk } N < \infty$, and consider the following exact sequence:

$$0 \longrightarrow N \xrightarrow{\varphi} \overline{N} \xrightarrow{\psi} \overline{N}/N \longrightarrow 0.$$

Applying the functor $\text{Hom}(-, Q)$ to the sequence we get:

$$0 \longrightarrow \text{Hom}(\overline{N}/N, Q) \xrightarrow{\psi^*} \text{Hom}(\overline{N}, Q) \xrightarrow{\varphi^*} \text{Hom}(N, Q).$$

Since $\overline{N}/N$ is torsion, $\text{Hom}(\overline{N}/N, Q) = 0$. Thus φ_* is a monomorphism, so in particular $\dim_E \text{Hom}(\overline{N}, Q) \leq \dim_E \text{Hom}(N, Q)$, where E denotes the endomorphism ring of Q . But from Theorem 2.24 we know that $\dim_E \text{Hom}(N, Q) = \text{rk } N$ and the finiteness of this implies that also $\dim_E \text{Hom}(\overline{N}, Q) = \text{rk } \overline{N}$. Thus $\text{rk } \overline{N} \leq \text{rk } N$, as required.

Let us consider now the case when $\text{rk } N = \kappa \geq \aleph_0$ and let us assume indirectly that $\text{rk } \overline{N} = \lambda > \kappa$. From Theorems 2.7 and 2.19 we get that there exists a submodule $\bigoplus_{i \in I} P_i \leq N$ such that P_i is finite dimensional preprojective for each $i \in I$ and $N / \bigoplus_{i \in I} P_i$ is torsion regular. Now from the definition it is clear that the rank is additive for arbitrary direct sums. Since the class of torsion-regular modules is closed under extensions (see, for example, Theorem 2.14), we also get that $\text{rk } N = \text{rk } \bigoplus_{i \in I} P_i$. Thus, since for every $i \in I$ we have that $\text{rk } P_i < \aleph_0$, we get that $\text{rk } N = |I| = \kappa$.

Consider now the following embeddings:

$$\bigoplus_{i \in I} P_i \hookrightarrow N \hookrightarrow \overline{N} \hookrightarrow \bigoplus_J Q = Y$$

with $|I| = \lambda = \text{rk } \overline{N}$ and $Y/\overline{N}$ being torsion regular. Let us notice first that since the modules $N / \bigoplus_{i \in I} P_i$, $\overline{N}/N$ and $Y/\overline{N}$ are all torsion, and torsion modules are closed under extensions (see Proposition 2.1 (ii)), for the torsion closure of $\bigoplus_{i \in I} P_i$ in Y we obtain $\overline{\bigoplus_{i \in I} P_i}^Y = Y$. We will show that this leads to contradiction.

Since $\dim_k P_i < \aleph_0$ for each $i \in I$, we get that $\kappa \leq \dim_k \bigoplus_{i \in I} P_i \leq \kappa \cdot \aleph_0 = \kappa$. This also gives that the set $J' = \{j \in J \mid \pi_j(\bigoplus_{i \in I} P_i) \neq 0\}$, where $\pi_j : Y \rightarrow Q$ is the j th projection, satisfies $|J'| \leq (\dim_k \bigoplus_{i \in I} P_i) \cdot \aleph_0 = \kappa < \lambda = |J|$. Hence $J \setminus J'$

is not empty, i.e. there is an index $j \in J$ such that $\text{Ker } \pi_j \supseteq \bigoplus_{i \in I} P_i$. But then for the torsion-closure of $\bigoplus_{i \in I} P_i$ we get $\overline{\bigoplus_{i \in I} P_i}^Y \subseteq \text{Ker } \pi_j \subset Y$, since $\text{Ker } \pi_j$ is obviously torsion-closed, and this contradicts to what was said above.

The following simple fact will be also useful.

Lemma 3.2. *Let $\mathcal{N} = \{N_i \mid i \in I\}$ be a directed set of torsion-closed submodules of a module M . Then $N = \bigcup_{i \in I} N_i \leq M$ is also torsion-closed in M . In particular the union of a chain of torsion-closed submodules is torsion-closed.*

Proof. Let us take an arbitrary submodule $W \leq M$ such that $N \leq W$ and W/N is finite dimensional. We have to show that W/N is preprojective. Let W' be a finite dimensional preimage of W/N in M , i.e. $W' \leq M$ such that $W' + N = W$. Consider $W' \cap N = W' \cap (\bigcup_{i \in I} N_i) = \bigcup_{i \in I} (N_i \cap W')$. Since W' is finite dimensional and $\mathcal{N}$ is directed, there exists an index $i \in I$ such that $W' \cap N = W' \cap N_i$. Then:

$$W/N = (W' + N)/N \cong W'/W' \cap N = W'/W' \cap N_i \cong (W' + N_i)/N_i \leq M/N_i.$$

Since M/N_i is torsion-free by assumption, W/N is preprojective. Hence N is torsion-closed.

Now we are ready to introduce our dependence relation. As we mentioned earlier, it will not be defined on elements but on certain submodules of M . From now on, throughout the chapter, M will always denote a torsion-free regular module.

Definition 3.3. Let M be a torsion-free regular module and P a fixed

projective module of rank 1. We define $\mathcal{W}_P = \mathcal{W}_P(M) = \{P_i \leq M \mid P_i \cong P\}$.

Then we say:

- (i) $P_j \in \mathcal{W}_P$ **depends on** $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$ if and only if $P_j \subseteq \overline{\langle P_i \mid i \in I \rangle}^M =: \overline{\mathcal{P}}^M$.
- (ii) $\mathcal{P} \subseteq \mathcal{W}_P$ is **independent** if and only if no $P_i \in \mathcal{P}$ depends on $\mathcal{P} \setminus \{P_i\}$.
- (iii) $\mathcal{P} \subseteq \mathcal{W}_P$ **generates** $\mathcal{P}' \subseteq \mathcal{W}_P$ if and only if every $P_i \in \mathcal{P}'$ depends on $\mathcal{P}$.
- (iv) $\mathcal{P} \subseteq \mathcal{W}_P$ is a **generating set for** M if and only if $\mathcal{P}$ generates $\mathcal{W}_P$.
- (v) $\mathcal{P} \subseteq \mathcal{W}_P$ is a **P -basis for** M if and only if $\mathcal{P}$ is an independent generating set for M .

We list now a few elementary properties of this relation.

Proposition 3.4. $P_j \in \mathcal{W}_P$ *depends on* $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$ if and only if $P_j \cap \overline{\mathcal{P}}^M \neq 0$ (i.e. P_j *does not depend on* $\mathcal{P}$ if and only if $P_j \cap \overline{\mathcal{P}}^M = 0$).

Proof. One direction is obvious from the definition. To show the other direction, assume that $P_j \cap \overline{\mathcal{P}}^M \neq 0$. By Proposition 3.1 we have that $\text{rk } P_j = 1$ implies $\text{rk } \overline{P_j}^M = 1$. Since $\overline{P_j}^M \cap \overline{\mathcal{P}}^M$ is torsion closed in $\overline{P_j}^M$ and is non-zero by assumption, we must have $\overline{P_j}^M \subseteq \overline{\mathcal{P}}^M$. Hence P_j depends on $\mathcal{P}$.

We can also show that the torsion closure is an “algebraic closure operation”, i.e. of finitary character.

Proposition 3.5. If $P_j \in \mathcal{W}_P$ depends on $\mathcal{P} \subseteq \mathcal{W}_P$, then P_j depends on a

finite subset $\mathcal{P}'$ of $\mathcal{P}$. In general, if for a submodule $N \subseteq M$ we have $N \cap \overline{\mathcal{P}}^M \neq 0$, then there exists a finite subset $\mathcal{P}' \subseteq \mathcal{P}$ such that $N \cap \overline{\mathcal{P}'}^M \neq 0$.

Proof. In view of Proposition 3.4, it is enough to show the general statement. Let κ be the smallest cardinality for which there exists a subset $\mathcal{P}' \subseteq \mathcal{P}$ of cardinality κ such that $N \cap \overline{\mathcal{P}'}^M \neq 0$. Let n be an element in this intersection. Assume that $\kappa \geq \aleph_0$. Then κ is a limit ordinal. Thus for $\mathcal{P}' = \{P_\mu \mid \mu < \kappa\}$ we get that

$$n \in \overline{\mathcal{P}'}^M = \overline{\{P_\mu \mid \mu < \kappa\}}^M = \overline{\bigcup_{\lambda < \kappa} \{P_\mu \mid \mu < \lambda\}}^M.$$

But by the Proposition 3.2 the union of a chain of torsion closed submodules in M is torsion closed. Hence:

$$n \in \overline{\bigcup_{\lambda < \kappa} \{P_\mu \mid \mu < \lambda\}}^M = \bigcup_{\lambda < \kappa} \overline{\{P_\mu \mid \mu < \lambda\}}^M.$$

This means that there is a $\lambda < \kappa$ such that $N \cap \overline{\{P_\mu \mid \mu < \lambda\}}^M \neq 0$, and since $|\lambda| < \kappa$, this gives a contradiction. Hence $\kappa < \aleph_0$, i.e. there is a finite subset $\mathcal{P}' \subseteq \mathcal{P}$ of the required property.

Remark. Similar argument shows that the statement of Proposition 3.5 is true in a more general setting: if $m \in \overline{N}^M$ for a submodule N , then there is a finite dimensional submodule $N' \subseteq N$ such that $m \in \overline{N'}^M$.

We can now prove the general version of Proposition 3.4.

Proposition 3.6. A set $\mathcal{P} \subseteq \mathcal{W}_P$ is independent if and only if for any subsets $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P}$ if $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ then we have $\overline{\mathcal{P}_1}^M \cap \overline{\mathcal{P}_2}^M = 0$.

Proof. One direction is obvious from the definition. For the other direction, assume that for some disjoint subsets $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P}$ we have $\overline{\mathcal{P}_1}^M \cap \overline{\mathcal{P}_2}^M \neq 0$. We want to show that in this case $\mathcal{P}$ cannot be independent. Applying Proposition 3.5 to the sets $\mathcal{P}_1$ and $\mathcal{P}_2$, we get that there are finite subsets $\mathcal{P}'_1 \subseteq \mathcal{P}_1$ and $\mathcal{P}'_2 \subseteq \mathcal{P}_2$ such that $\overline{\mathcal{P}'_1}^M \cap \overline{\mathcal{P}'_2}^M \neq 0$. Now, it is easy to show that if a finite subset $\mathcal{P}' \subseteq W_P$ is independent then $\text{rk } \overline{\mathcal{P}'}^M = |\mathcal{P}'|$ (cf. also Theorem 3.11, later in this chapter). Namely assuming that the statement is true for sets of cardinality less than n , then for $\mathcal{P}' = \{P_1, P_2, \dots, P_n\}$ we have that $\text{rk } (\overline{\mathcal{P}' \setminus \{P_n\}})^M = n - 1$. But since $\mathcal{P}'$ is independent, $\overline{\mathcal{P}' \setminus \{P_n\}}^M$ is a proper submodule of $\overline{\mathcal{P}'}^M$, hence $\text{rk } \overline{\mathcal{P}'}^M \geq n$. On the other hand from Proposition 3.1 we get that $\text{rk } ((\overline{\mathcal{P}' \setminus \{P_n\}})^M \oplus \overline{P_n})^M \leq n$. Thus the rank must be equal to n .

In this way the independence of $\mathcal{P}'_1 \cup \mathcal{P}'_2$ would give us that $\text{rk } \overline{\mathcal{P}'_1}^M = |\mathcal{P}'_1|$, $\text{rk } \overline{\mathcal{P}'_2}^M = |\mathcal{P}'_2|$ and $\text{rk } (\overline{\mathcal{P}'_1 \cup \mathcal{P}'_2})^M = |\mathcal{P}'_1 \cup \mathcal{P}'_2| = |\mathcal{P}'_1| + |\mathcal{P}'_2|$. On the other hand by assumption we get that $N = \overline{\mathcal{P}'_1}^M \cap \overline{\mathcal{P}'_2}^M$ is a non-zero torsion-closed submodule thus by an argument similar as above we get that $\text{rk } (\overline{\mathcal{P}'_1 \cup \mathcal{P}'_2})^M \leq \text{rk } \overline{\mathcal{P}'_1}^M + \text{rk } \overline{\mathcal{P}'_2}^M - \text{rk } N < |\mathcal{P}'_1| + |\mathcal{P}'_2|$, a contradiction. Hence the set $\mathcal{P}'_1 \cup \mathcal{P}'_2$ is not independent, so $\mathcal{P}$ cannot be independent either.

As one would expect, we have the exchange property, too.

Proposition 3.7. *If P_0 depends on $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$ but it does not depend on $\mathcal{P}' = \{P_1, P_2, \dots, P_{n-1}\}$, then P_n depends on $\mathcal{P}'' = \{P_1, P_2, \dots, P_{n-1}, P_0\}$.*

Proof. Clearly $\overline{\mathcal{P}'}^M \subset \overline{\mathcal{P}}^M$, and since P_n embeds into $\overline{\mathcal{P}}^M / \overline{\mathcal{P}'}^M$ in such

a way that its torsion closure is the whole module, by Proposition 3.1 we get that $\text{rk } \overline{\mathcal{P}}^M / \overline{\mathcal{P}'}^M = 1$. By assumption $\overline{\mathcal{P}'}^M \subset \overline{\mathcal{P}''}^M \subseteq \overline{\mathcal{P}}^M$, hence we have $\text{rk } \overline{\mathcal{P}''}^M / \overline{\mathcal{P}'}^M \geq 1$, and so $\overline{\mathcal{P}''}^M = \overline{\mathcal{P}}^M$. Thus $P_n \subseteq \overline{\mathcal{P}''}^M$, i.e. P_n depends on $\mathcal{P}''$.

We can now separate the different properties, included in Ringel's proof of Theorem 2.19. The next proposition characterizes the independence of a set $\mathcal{P} \subseteq \mathcal{W}_P$, while Proposition 3.9 gives the condition for a set to be a generating set.

Proposition 3.8. *A set $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$ is independent if and only if*

- (i) $P_i \cap \langle P_j \in \mathcal{P} \mid j \neq i \rangle = 0$ for every $i \in I$ (i.e. $\bigoplus_{i \in I} P_i \subseteq M$);
- (ii) $M / \bigoplus_{i \in I} P_i$ is regular.

Proof. Assume first that the conditions (i) and (ii) are satisfied, i.e. we have $M / \bigoplus_{i \in I} P_i$ regular. Then consider the following diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \bigoplus_{i \in I} P_i & \xrightarrow{\iota} & M & \longrightarrow & M / \bigoplus_{i \in I} P_i \longrightarrow 0 \\
 & & \downarrow \pi_i & & & \nearrow \psi & \\
 & & P_i & & & & \\
 & & \downarrow \iota_i & & & & \\
 & & Q & & & &
 \end{array}$$

where ι_i is an embedding of P_i into Q . Since $M / \bigoplus_{i \in I} P_i$ is regular, by Proposition 2.16 we get that there exists a map $\psi : M \rightarrow Q$ such that $\psi \iota = \iota_i \pi_i$. Thus $\text{Ker } \psi \supseteq \overline{\mathcal{P} \setminus \{P_i\}}^M$ and also $\text{Ker } \psi \cap P_i = 0$. Thus $\mathcal{P}$ is independent.

Assume now that $\mathcal{P} \subseteq \mathcal{W}_P$ is independent. Thus by Proposition 3.4 we get that $P_i \cap \langle P_j \in \mathcal{P} \mid j \neq i \rangle \subseteq P_i \cap \overline{(\mathcal{P} \setminus \{P_i\})}^M = 0$. Hence we get (i). To get condition

(ii), assume $|\mathcal{P}| = \kappa$ and $\mathcal{P} = \{P_\mu \mid \mu < \kappa\}$. We will prove by transfinite induction that $M_\lambda = M / \bigoplus_{\mu < \lambda} P_\mu$ is regular for each $\lambda \leq \kappa$. (We will use here Ringel's argument from the proof of [R2], Proposition 4.3.)

Assume first that $\lambda = \nu + 1$. By assumption $M_\nu = M / \bigoplus_{\mu < \nu} P_\mu$ is regular. Since $\mathcal{P}$ is independent, by Proposition 3.4 we get that P_ν embeds into M_ν as $\hat{P}_\nu$; actually it embeds even into the torsion-free module $\tilde{M}_\nu = M_\nu / T(M_\nu)$ as $\tilde{P}_\nu$. We will show first that $\tilde{M}_\nu / \tilde{P}_\nu$ is regular. Since M was regular, by Corollary 2.11 this is equivalent to showing that $\tilde{M}_\nu / \tilde{P}_\nu$ has no preinjective submodules. Let us assume that $\tilde{W} \subseteq \tilde{M}_\nu$ is a submodule containing $\tilde{P}_\nu$ and $\tilde{W} / \tilde{P}_\nu$ is indecomposable preinjective. But then for the defect of $\tilde{W}$ we get: $\delta(\tilde{W}) = \delta(\tilde{P}_\nu) + \delta(\tilde{W} / \tilde{P}_\nu) = -1 + \delta(\tilde{W} / \tilde{P}_\nu) \geq 0$, and this contradicts the fact that $\tilde{M}_\nu$ is torsion-free. Thus $\tilde{M}_\nu / \tilde{P}_\nu$ is regular. Consider now the following exact sequence:

$$0 \longrightarrow (T(M_\nu) + \hat{P}_\nu) / \hat{P}_\nu \longrightarrow M_\nu / \hat{P}_\nu \longrightarrow (M_\nu / \hat{P}_\nu) / ((T(M_\nu) + \hat{P}_\nu) / \hat{P}_\nu) \longrightarrow 0.$$

Here the first term is isomorphic to $T(M_\nu)$, and since it is the torsion submodule of the regular module M_ν , by Corollary 2.11 it is regular. On the other hand the last term is isomorphic to $\tilde{M}_\nu / \tilde{P}_\nu$, which was just shown to be regular, too. Thus $M_\nu / \hat{P}_\nu \cong M / \bigoplus_{\mu < \lambda} P_\mu = M_\lambda$, as an extension of two regular modules is also regular.

Assume now that λ is a limit ordinal and for every $\mu < \lambda$ the module M_μ is regular. Assume that $M_\lambda = M / \bigoplus_{\mu < \lambda} P_\mu$ has an indecomposable preinjective submodule, i.e. there is a submodule $W \subseteq M$ such that $\bigoplus_{\mu < \lambda} P_\mu \subseteq W$ and $W / \bigoplus_{\mu < \lambda} P_\mu$ is finite dimensional preinjective. Let W' be a finite dimensional submodule of M satisfying $W = W' + \bigoplus_{\mu < \lambda} P_\mu$. Then $W' \cap \bigoplus_{\mu < \lambda} P_\mu = W' \cap \bigoplus_{\mu < \nu} P_\mu$ for some $\nu < \lambda$. Thus:

$$\begin{aligned}
W / \bigoplus_{\mu < \lambda} P_\mu &= (W' + \bigoplus_{\mu < \lambda} P_\mu) / \bigoplus_{\mu < \lambda} P_\mu \cong W' / (W' \cap \bigoplus_{\mu < \lambda} P_\mu) = \\
&= W' / (W' \cap \bigoplus_{\mu < \nu} P_\mu) \cong (W' + \bigoplus_{\mu < \nu} P_\mu) / \bigoplus_{\mu < \nu} P_\mu \subseteq M_\nu,
\end{aligned}$$

and this would contradict the regularity of M_ν . Thus M_λ is also regular. This finishes the proof.

Proposition 3.9. *A subset $\mathcal{P} \subseteq \mathcal{W}_P$ is a generating set for M if and only if $M/\langle \mathcal{P} \rangle$ is torsion, i.e. if and only if $\overline{\mathcal{P}}^M = M$.*

Proof. We will again use the argument of Ringel from the proof of [R2], Proposition 4.3. One direction is obvious from the definitions. For the other direction let us assume that $\mathcal{P}$ is a generating set for M . Assume that $\overline{\mathcal{P}}^M \neq M$. Then the module $\tilde{M} = M/\overline{\mathcal{P}}^M \neq 0$ is a non-zero torsion-free regular module. But then there must exist a non-zero homomorphism from P into $\tilde{M}$, otherwise it would be a module over an algebra of finite representation type, and as such it would be a direct sum of finite dimensional indecomposable submodules, contradicting the fact that $\tilde{M}$ is torsion-free and regular. Thus there is a non-zero homomorphism $\varphi : P \rightarrow \tilde{M}$, and since $\text{rk } P = 1$, we get that φ must be an embedding. But P is projective, so this also gives an embedding of P into M and the image P_0 is disjoint from $\overline{\mathcal{P}}^M$. Thus P_0 does not depend on $\mathcal{P}$, contradicting the assumption that $\mathcal{P}$ was a generating set for M .

Proposition 3.10. *Every independent set $\mathcal{P} \subseteq \mathcal{W}_P$ can be completed to a P -basis of M . Similarly every generating set for M contains a subset which is a P -basis.*

Proof. To apply Zorn's lemma, one just has to show that the set of independent subsets of $\mathcal{W}_P$ is inductive. But this follows easily from Proposition 3.5. Thus we get both statements.

So we can obtain the following description of the rank of a regular module.

Theorem 3.11. *Let $\mathcal{P} = \{P_i \mid i \in I\} \subseteq \mathcal{W}_P$ be an independent set. Then $\text{rk } \overline{\mathcal{P}}^M = |\mathcal{P}|$. In particular if $\mathcal{P}$ is a $\mathcal{P}$ -basis for M then $|\mathcal{P}| = \text{rk } M$.*

Proof. Let us take $N = \overline{\mathcal{P}}^M$. Then the module $N / \bigoplus_{i \in I} P_i$ is just the torsion submodule of the module $M / \bigoplus_{i \in I} P_i$, which is regular by Proposition 3.8. Thus $N / \bigoplus_{i \in I} P_i$ must also be regular. If we now embed the module N into a direct sum Y of copies of the module Q in such a way that Y/N is torsion regular, then $Y / \bigoplus_{i \in I} P_i$ is also torsion regular. Hence $\text{rk } N = \text{rk } \bigoplus_{i \in I} P_i$. But clearly $\text{rk } \bigoplus_{i \in I} P_i = |I| = |\mathcal{P}|$, thus we are done.

Let us close the chapter with a simple application, showing that the connection between independence, rank and regularity can be applied in a wider setting, too. The following proposition generalizes a statement by Ringel (cf. [R2] 6.1 Proposition 2).

Proposition 3.12. *Let M be a torsion-free module of finite rank, and let $N \subseteq M$ be a submodule such that M/N is torsion. Then M/N is regular if and only if $\text{rk } N = \text{rk } M$.*

Proof. Assume first that M/N is regular. Then just as above, an embedding of M into a direct sum $\bigoplus_{i \in I} Q$ with a torsion regular cokernel yields an embedding of N into the same direct sum with a cokernel term which is also torsion regular. Hence $\text{rk } N = \text{rk } M$.

Assume now that $\text{rk } N = \text{rk } M = n$. Then we have the following diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & N & \xrightarrow{\iota_N} & \bigoplus_1^n Q' = Y' & \longrightarrow & Y'/N \longrightarrow 0 \\
 & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' \\
 0 & \longrightarrow & M & \xrightarrow{\iota_M} & \bigoplus_1^n Q'' = Y'' & \longrightarrow & Y''/M \longrightarrow 0
 \end{array}$$

Here φ' is just the embedding of N into M , the modules Y'/N and Y''/M are torsion regular, and $Q' \cong Q'' \cong Q$. (The existence of φ follows from Proposition 2.16.) Since the modules $M/N = M/\varphi'(N)$ and Y''/M are torsion, then so is $Y''/(\iota_M \varphi')(N) = Y''/(\varphi \iota_N)(N)$. Thus $\overline{\text{Im } \varphi}^{Y''} = Y''$. This immediately gives that φ is a monomorphism, otherwise the exact sequence

$$0 \longrightarrow \text{Ker } \varphi \longrightarrow Y' \longrightarrow \text{Im } \varphi \longrightarrow 0$$

would give that $\text{rk Im } \varphi < \text{rk } \bigoplus_1^n Q' = n$, and then by Proposition 3.1 we would have that $\text{rk } \overline{\text{Im } \varphi}^{Y''} \leq \text{rk Im } \varphi < n$, which contradicts to the statement above.

In order to show the regularity of M/N , it is enough to show that φ is an epimorphism, as this would give us by the Snake Lemma that $\text{Ker } \varphi'' \cong \text{Cok } \varphi' = M/N$. Since φ'' is a homomorphism between torsion regular modules, $\text{Ker } \varphi''$ is torsion regular, as required.

Thus assume that $\varphi : Y' \rightarrow Y''$ is any homomorphism satisfying that $\text{Ker } \varphi = 0$ and $\overline{\text{Im } \varphi}^{Y''} = Y''$. We will show that φ must be an epimorphism. Let $\mathcal{P} =$

$\{P_1, P_2, \dots, P_n\}$ be a P -basis for Y' . Thus by Propositions 3.8 and 3.9 we have that $\langle P_1, P_2, \dots, P_n \rangle = \bigoplus_{i=1}^n P_i$ and $Y' / \bigoplus_{i=1}^n P_i$ is torsion regular. Consider now the following diagram:

$$\begin{array}{ccccccc}
 0 & - & \bigoplus_{i=1}^n P_i & \longrightarrow & Y' & \longrightarrow & Y' / \bigoplus_{i=1}^n P_i \longrightarrow 0 \\
 & & \downarrow \varphi' & & \downarrow \varphi & & \downarrow \varphi'' \\
 0 & \longrightarrow & \bigoplus_{i=1}^n \varphi(P_i) & \longrightarrow & Y'' & \longrightarrow & Y'' / \bigoplus_{i=1}^n \varphi(P_i) \longrightarrow 0
 \end{array}$$

Clearly $\varphi' = \varphi|_{\bigoplus_{i=1}^n P_i}$ is an isomorphism. If we can show that $Y'' / \bigoplus_{i=1}^n \varphi(P_i)$ is torsion regular then using again the Snake Lemma we would get that $\text{Cok } \varphi \cong \text{Cok } \varphi''$ is torsion regular. Since Y' is divisible, Proposition 2.16 would imply that φ splits, and Y'' being torsion-free, we would get that $\text{Cok } \varphi = 0$. (For this last argument see also [R2], Lemma 5.1.)

Thus we only need to show that $Y'' / \bigoplus_{i=1}^n \varphi(P_i)$ is torsion regular. But the module $Y' / \bigoplus_{i=1}^n P_i$ is torsion by assumption, and similarly $Y'' / \text{Im } \varphi$, thus $\overline{\bigoplus_{i=1}^n \varphi(P_i)}^{Y''} = Y''$, i.e. we get easily that $Y'' / \bigoplus_{i=1}^n \varphi(P_i)$ is torsion. To show the regularity, notice that $\mathcal{P}' = \{\varphi(P_1), \varphi(P_2), \dots, \varphi(P_n)\} \subseteq \mathcal{W}_P(Y'')$, thus by Proposition 3.8 the quotient in question is regular if and only if $\mathcal{P}'$ is independent. But Proposition 3.9 gives that $\mathcal{P}'$ is a generating set for Y'' and its cardinality is equal to the rank of Y'' , so it must be a P -basis for Y'' . Thus $\bigoplus_{i=1}^n Q'' / \bigoplus_{i=1}^n \varphi(P_i)$ is torsion regular, as required.

Remark 3.13. It is easy to construct examples to show that the above statement is not true in general, i.e. without assuming the finiteness of $\text{rk } N = \text{rk } M$. Take for instance a module Q_1 isomorphic to Q with two disjoint submodules:

$P_0, P_1 \subseteq Q_1$. (Since $\text{Soc } Q_1$ is not simple, one can obviously find such submodules.)

Let us also choose arbitrary non-zero finite dimensional submodules $P_i \subseteq Q_i \cong Q$ for $i = 2, 3, \dots$. Take $M = \bigoplus_{i=1}^{\infty} Q_i$, $N = \bigoplus_{i=0}^{\infty} P_i$. Then obviously $\text{rk } N = \text{rk } M = \aleph_0$ with $\overline{N}^M = M$. But M/N cannot be regular as otherwise one could extend the homomorphism $\bigoplus_{i=0}^{\infty} P_i \xrightarrow{\pi_0} P_0 \xrightarrow{\iota} Q$ to a homomorphism $\bigoplus_{i=1}^{\infty} Q_i \xrightarrow{\psi} Q$. Then $\text{Ker } \psi \supseteq \bigoplus_{i=1}^{\infty} P_i$, hence $\text{Ker } \psi \supseteq \overline{\bigoplus_{i=1}^{\infty} P_i}^M = M$, contradicting the fact that $\psi \neq 0$, since for example $\text{Ker } \psi \cap P_0 = 0$. Thus M/N is not regular.

Chapter 4.

Superpurely simple modules

In Chapter 2 we have noted that there is a considerable difference between the tame hereditary case and the abelian group case when we compare the relation of the rank of a module and the substructure of torsion-closed submodules. We have seen for example that it is possible to have a simple module of large (i.e. larger than 1) rank. In this chapter we are going to investigate this situation. It turns out that the “irregularities” in some sense occur only in the finite dimensional cases and there are only finitely many (necessarily indecomposable) modules which are of rank larger than 1 and without non-trivial torsion-closed submodules. We introduce the following definition.

Definition 4.1. A torsion-free module M is called **superpurely simple** if it has no non-trivial torsion-closed submodules.

Obviously every torsion-free module of rank 1 is superpurely simple. It is also easy to show that no infinite dimensional torsion-free module M of rank larger than 1 can be superpurely simple. Namely if M is not regular, then it has a finite dimensional direct summand, which is obviously torsion-closed in M . On the other hand if M is torsion-free regular, then by the results in Chapter 3 we have a P -basis

with more than 1 elements, hence the torsion-closure of one basis element will give a nontrivial torsion-closed submodule.

Let us also mention here that the name “superpurely simple” reflects the fact that all superpurely simple modules are purely simple, since by Proposition 2.6 a non-trivial pure submodule would be necessarily torsion-closed.

The following proposition will help us to find all superpurely simple modules.

Proposition 4.2. *Let M be a torsion-free module. Then the following statements are equivalent.*

- (i) *There exists a non-zero homomorphism $\varphi \in \text{Hom}(M, Q)$ with $\text{Ker } \varphi \neq 0$.*
- (ii) *M is not superpurely simple.*
- (iii) *There exists a non-zero submodule $U \subset M$ with $\text{rk } U < \text{rk } M$.*
- (iv) *There exists a non-zero homomorphism $\varphi : V \rightarrow M$ from a torsion-free module V with $\text{rk } V < \text{rk } M$.*

Proof. (i) $\Rightarrow$ (ii) Let $\varphi \in \text{Hom}(M, Q)$ be a non-zero homomorphism with non-zero kernel. Let $U = \text{Ker } \varphi$. Then U is torsion-closed and $U \neq M$.

(ii) $\Rightarrow$ (iii) Let U be a non-trivial torsion-closed submodule of M . Assume first that $\text{rk } M < \infty$. Consider the sequence

$$0 \longrightarrow U \longrightarrow M \longrightarrow M/U \longrightarrow 0.$$

Then Theorem 2.23 implies that $\text{rk } M = \text{rk } U + \text{rk } M/U$. Since $\text{rk } M/U > 0$, we have $\text{rk } U < \text{rk } M$, as required. To show the statement for the case when $\text{rk } M = \infty$, we just have to notice that the rank of a finite dimensional submodule W will be

always finite (cf. for example Theorem 2.22). Thus by Proposition 3.1 the torsion closure $U = \overline{W}^M$ of any finite dimensional submodule W will be torsion closed and of finite rank.

(iii) $\Rightarrow$ (iv) Choose the embedding of U into M .

(iv) $\Rightarrow$ (iii) Take $U = \text{Im } \varphi \leq M$. Then $\text{Im } \varphi \neq 0$ and by Theorem 2.23 $\text{rk Im } \varphi = \text{rk } V - \text{rk Ker } \varphi \leq \text{rk } V < \text{rk } M$.

(iii) $\Rightarrow$ (ii) By Proposition 3.1, if $\text{rk } U < \text{rk } M$, then $\overline{U}^M \neq M$, hence M has a non-trivial proper torsion-closed submodule, as required.

(ii) $\Rightarrow$ (i) For $U \subset M$, where U is torsion-closed in M , the module M/U is torsion-free, hence by Theorem 2.20 we can find a non-zero homomorphism $\varphi : M/U \rightarrow Q$. By combining it with the natural epimorphism $M \rightarrow M/U$, we get a non-zero homomorphism $\tilde{\varphi} : M \rightarrow Q$ with $U \subseteq \text{Ker } \tilde{\varphi}$. Since by assumption $U \neq 0$, we have $\text{Ker } \tilde{\varphi} \neq 0$, as required.

Before going on with the “pursuit” of superpurely simple modules, we should perhaps observe first that superpurely simple modules occur quite frequently: they are the basic building blocks of the torsion structure of torsion-free modules. The following holds.

Proposition 4.3. (i) *Every torsion-free module M contains a nonzero superpurely simple torsion-closed submodule.*

(ii) *If M is a torsion-free module of finite rank, then there exists a chain $0 = U_0 \subset U_1 \subset \dots \subset U_t = M$ of torsion-closed submodules in M such that U_i/U_{i-1} is*

superpurely simple for $i = 1, \dots, t$. If in addition M is also regular, then the chain can be chosen in such a way that the quotients are all of rank 1.

Proof. (i) Choose a non-zero torsion-closed submodule U of minimal rank. The argument of Proposition 4.2 (ii) $\Rightarrow$ (iii) shows that $\text{rk } U < \infty$. Then by Corollary 2.2 (i) any proper torsion-closed submodule of U would be torsion-closed in M and of smaller rank than U , contradicting the minimality of $\text{rk } U$. Hence U is superpurely simple.

(ii) If M is superpurely simple, then we are done. Otherwise choose a torsion-closed non-zero superpurely simple submodule $U_1 \leq M$, given by (i). Then $\text{rk } M/U_1 < \text{rk } M$, hence an induction argument completes the proof of the first part of the statement. For the second part we can take a P -basis for M , for example $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$. If we take $U_j = \overline{\bigoplus_{i=1}^j P_i}^M$, then by Theorem 3.11 we get that $\text{rk } U_j = j$, hence $\text{rk } U_j/U_{j-1} = 1$, as required.

Remark. An example at the end of the chapter (Example 4.11) shows that these chains are by no means unique: neither the isomorphism types of the superpurely quotients, nor the ranks or even the number of them is uniquely determined by the module M .

Another comment we should make here is that one could change the formulation of the statement in (ii) so that it applies to arbitrary torsion-free modules, not only those ones of finite rank. For regular modules with a P -basis $\mathcal{P} = \{P_i \mid i \in I\}$ one can obviously define for every $I' \subseteq I$ the submodules $U_{I'} = \overline{\bigoplus_{i \in I'} P_i}^M$. Choosing a

well-ordering for the index set I and then taking the sequence of initial segments of I we obtain a chain of torsion-closed submodules of the type of the well-ordering, and the consecutive terms will give quotients of rank 1. If M is not regular, i.e. it has finite dimensional preprojective direct summands, we can use Theorem 2.7. First we can obtain a suitable chain for the submodule N , which is a direct sum of finite dimensional submodules, and then we can continue with a chain for the regular quotient module M/N , as described above.

Although, as we mentioned, no uniqueness holds for the chains defined in Proposition 4.3 (ii), one can still read some properties of the module M from them. The following is an easy consequence of the criterion given in Proposition 2.25 for determining whether a module is purely simple or not.

Proposition 4.4. *Let M be an infinite dimensional torsion-free module of finite rank. Then M is purely simple if and only if for every chain $0 = U_0 \subset U_1 \subset \dots \subset U_{t-1} \subset U_t = M$ of torsion-closed submodules with U_i/U_{i-1} superpurely simple for $i = 1, \dots, t$, we have $\dim_k U_{t-1} < \infty$.*

Proof. If M is purely simple, then by Proposition 2.25 every proper torsion-closed submodule must be finite dimensional. Hence $\dim_k U_{t-1} < \infty$. On the other hand if M is not purely simple, then also by Proposition 2.25 there is an infinite dimensional proper torsion-closed submodule $U \subset M$. If we take a chain of torsion-closed submodules of U with superpurely simple quotients and complete it to a chain of M as follows:

$$0 = U_0 \subset U_1 \subset \dots \subset U_i = U \subset U_{i+1} \subset \dots \subset U_{t-1} \subset U_t = M,$$

then $\dim_k U_i$ and hence $\dim_k U_{t-1}$ is infinite.

As a corollary we also get the general version of a result of Okoh (cf. [O3] Proposition 1.2), stated for Kronecker modules.

Corollary 4.5. *Let M be a purely simple infinite dimensional torsion-free module of finite rank and let N be a torsion-closed submodule of M . Then M/N is also purely simple.*

Remark. It would be interesting to know if Corollary 4.5 holds if M is of infinite rank. Actually the existence or non-existence of purely simple modules of infinite rank is still an open question (see more about this in Chapter 5).

We will turn now to examining in more detail the distribution of superpurely simple modules in the Auslander-Reiten quiver of A . Let us recall from Chapter 1 that $\tau_A P = D\text{Tr } P$ denotes the Auslander-Reiten translate of the finite dimensional indecomposable module P . Let us also recall from Theorem 2.22 that for a finite dimensional preprojective module P the rank of P equals the negative of the defect of P which is invariant under τ_A (cf. Chapter 1). Thus $\text{rk } P = \text{rk } \tau_A P$ for any finite dimensional indecomposable preprojective module P which is not projective.

Proposition 4.6. *Let P be a (finite dimensional) indecomposable preprojective module which is not projective. If P is superpurely simple, then so is $\tau_A P$.*

Proof. Assume that $\tau_A P$ is not superpurely simple. Then by Proposition 4.2 (iii) we can find a non-zero homomorphism $\varphi : P' \rightarrow \tau_A P$ from an indecomposable preprojective module P' for which $\text{rk } P' < \text{rk } \tau_A P$. But then $\tau_A^{-1} \varphi : \tau_A^{-1} P' \rightarrow P$ is also non-zero (cf. Chapter 1). Since $\text{rk } \tau_A^{-1} P' = \text{rk } P' < \text{rk } \tau_A P = \text{rk } P$, Proposition 4.2 (iv) implies that P is not superpurely simple.

For a Euclidean diagram Γ the number of indecomposable projective modules of rank 1 over the path algebra $k\vec{\Gamma}$ is independent of the orientation of Γ (see again Chapter 1). We will denote this number by $s(\Gamma)$. The following proposition shows that by choosing an appropriate orientation for Γ , arbitrary number of τ_A -orbits, besides those ones of rank 1 may contain superpurely simple modules.

Proposition 4.7. *Let Γ be a Euclidean diagram on n vertices. Then for every t such that $s(\Gamma) \leq t \leq n$ there exists an orientation $\vec{\Gamma}$ of Γ such that the number of superpurely simple (indecomposable) projective modules over $k\vec{\Gamma}$ equals t .*

Proof. For each vertex i of the graph Γ let $P(i)$ denote the corresponding indecomposable projective module. The algebra $k\vec{\Gamma}$ is hereditary, i.e. submodules of indecomposable projective modules are projective. Hence it follows from Proposition 4.2 that the projective module $P(j)$ is not superpurely simple if and only if there exists a non-zero homomorphism $\varphi : P(i) \rightarrow P(j)$ for some i where $\text{rk } P(i) < \text{rk } P(j)$. It is clear that there is a non-zero homomorphism $\varphi : P(i) \rightarrow P(j)$ if and only if there is an oriented path in $\vec{\Gamma}$ from j to i . Thus $P(j)$ is superpurely simple if

and only if there is no oriented path in $\vec{\Gamma}$ from j to any i with $\text{rk } P(i) < \text{rk } P(j)$. As $\text{rk } P(i)$ is independent of the orientation of Γ , the statement follows by an easy case-by-case analysis, or from Proposition 4.9 below.

Theorem 4.8. *Let Γ be as above. Then the number of superpurely simple modules of rank larger than 1 over $k\vec{\Gamma}$ is finite for any orientation of Γ .*

Proof. As mentioned at the beginning of the chapter, any superpurely simple module of rank larger than 1 must be finite dimensional indecomposable preprojective. Note that for $\Gamma = \tilde{A}_n$ every indecomposable preprojective module is of rank 1. So let us assume that Γ is not $\tilde{A}_n$. Let i be a vertex with $\text{rk } P(i) > 1$ and let t be the distance of the vertex i in the graph Γ from a vertex j with $\text{rk } P(j) < \text{rk } P(i)$. Then regardless of the orientation of Γ , using 4.2 (iv), $\tau_A^{-t}P(i)$ is not superpurely simple. Hence by Proposition 4.6, each τ_A -orbit with rank larger than 1 contains only finitely many superpurely simple modules.

Remark. In Example 4.12 we have listed the maximal number $m(\Gamma)$ of superpurely simple modules of rank larger than 1 over $k\vec{\Gamma}$; we also give the orientations $\vec{\Gamma}$ for which these numbers occur.

The following proposition is purely combinatorial. We include it here, for the sake of completeness, as an alternate proof for the case-by-case checking in Proposition 4.7.

Proposition 4.9. Let $\Gamma = (V, E)$ be a finite connected unoriented graph with a valuation $v : V \rightarrow \{1, 2, \dots, l\}$ on the vertices. A vertex $g \in V$ is called *superpurely simple* for a given orientation of Γ , if there is no oriented path in Γ from g to some vertex $h \in V$ with $v(h) < v(g)$. Then for any m satisfying $|\{g \in V \mid \forall h \in V \ v(g) \leq v(h)\}| \leq m \leq |V|$ there is an orientation of Γ such that the number of superpurely simple vertices of Γ with respect to this orientation is exactly m .

Proof. The proof is due to [L]. We will need the following lemma.

Lemma 4.10. Let $\Gamma = (V, E)$ and v be as above. Let $A = \{g \in V \mid v(g) < t\}$ and $B = \{h \in V \mid v(h) \geq t\}$ for some (fixed) value t . Assume that for every $h \in B$ there is a path leading from h to some vertex $g \in A$. Then there is an orientation of Γ such that the vertices which are superpurely simple with respect to this orientation are exactly the elements of A .

Proof. Choose an orientation according to the following rules:

- (a) $g, h \in A$, $g \rightarrow h$ implies $v(g) \leq v(h)$;
- (b) $g \in A$, $h \in B$ implies that if there is an edge between g and h , then $h \rightarrow g$;
- (c) $g, h \in B$, $g \rightarrow h$ implies that $d_A(g) \geq d_A(h)$, where $d_A(g)$ and $d_A(h)$ denote the distances of g and h from A .

This orientation clearly satisfies the requirements.

Let us turn now to the proof of the proposition. Fix the value m satisfying

the given conditions. Then we can choose t to be such that $|\{g \in V \mid v(g) < t\}| < m \leq |\{g \in V \mid v(g) \leq t\}|$. Finally, let us define the following subsets of V :

$$A = \{g \in V \mid v(g) < t\};$$

$$B = \{g \in V \mid v(g) = t\};$$

$$B'_i = \{g \in B \mid \text{every path connecting } g \text{ with an element of } A \text{ contains at least } i \text{ elements of } B, \text{ counting } g \text{ too}\};$$

$$B_i = B'_i \setminus B'_{i+1}.$$

Thus B is the disjoint union of the sets $B_1, B_2, \dots, B_s$, and from an element of B_i there is always a path to an element of A not containing any of the elements of $B_{i+1}, \dots, B_s$. Choose now a subset $C \subseteq B$ such that:

$$(i) \quad |A \cup C| = m;$$

$$(ii) \quad \text{if } C \cap B_i \neq \emptyset, \text{ then } C \supseteq \bigcup_{j=i+1}^s B_j.$$

Split the set of vertices V into the following two disjoint sets:

$$V_1 = \{g \in V \setminus C \mid \text{there is a path from } g \text{ to } A \text{ in } \Gamma \setminus C\};$$

$$V_2 = V \setminus V_1.$$

Let us first note that clearly $A \cup (B \setminus C) \subseteq V_1$, and $C \subseteq V_2$. Let G_1 and G_2 denote the subgraphs of G on the vertices V_1 and V_2 , respectively. Apply now Lemma 4.10 for G_1 and G_2 with decompositions $V_1 = A \cup (V_1 \setminus A)$ and $V_2 = C \cup (V_2 \setminus C)$. Both decompositions satisfy the requirements. Complete the so found orientations of G_1 and G_2 to an orientation of G by taking $g \rightarrow h$ if $g \in V_1$, $h \in V_2$ and if there is an edge between g and h . Then the set of superpurely simple vertices is $A \cup C$, since orienting the edges from V_1 to V_2 will preserve this property both for vertices in A and in C .

We shall finish the chapter with some examples.

Example 4.11. The following example shows that for chains of torsion closed submodules with superpurely simple quotients (cf. 4.3 (ii)) neither the isomorphism types of the quotients, nor the number of the elements in these chains is uniquely determined by the module.

Consider the graph $\Gamma = \tilde{E}_8$ with the following orientation:

On Figure 4.1. we show a portion of the preprojective component of the Auslander-Reiten quiver of the corresponding path algebra $k\tilde{\Gamma}$.

The projective module $P(6)$ has three different chains of torsion closed submodules with superpurely simple quotients:

$$(i) \quad 0 < P(4) < P(5) < P(6);$$

here $P(5)/P(4) = \tau_A^{-1}P(1)$ and $P(6)/P(5) = \tau_A^{-2}P(1)$, thus $\text{rk } P(4) = 4$.

$\text{rk } P(5)/P(4) = \text{rk } \tau_A^{-1}P(1) = 1$ and $\text{rk } P(6)/P(5) = \text{rk } \tau_A^{-2}P(1) = 1$.

$$(ii) \quad 0 < P(7) < P(6);$$

here $\text{rk } P(7) = 4$, $\text{rk } P(6)/P(7) = \text{rk } \tau_A^{-1}P(8) = 2$.

$$(iii) \quad 0 < P(9) < P(6);$$

here $\text{rk } P(9) = 3$, $\text{rk } P(6)/P(9) = \text{rk } \tau_A^{-1}P(9) = 3$.

The quotients considered above are all superpurely simple as can be easily checked on the diagram of Figure 4.1.

$\text{rk}P(i)$

Figure 4.1

Example 4.12. In the following we give the maximal number of superpurely simple modules of rank larger than 1 corresponding to some orientation of Euclidean graphs over an algebraically closed field. These numbers, denoted by $m(\Gamma)$ can be easily checked by a simple case-by-case analysis. Note that in the case of the graph

$\tilde{A}_n$ for any orientation without oriented cycles every indecomposable preprojective module is of rank 1, hence $m(\tilde{A}_n) = 0$.

Γ	$m(\Gamma)$
$\tilde{A}_n$	0
$\tilde{D}_n$	$\left[\frac{n-2}{2} \right] \cdot \left[\frac{n-1}{2} \right]$
$\tilde{E}_6$	4
$\tilde{E}_7$	8
$\tilde{E}_8$	13

The following orientations give a maximal number of superpurely simple modules of rank > 1 .

As an illustration we also give the position of the superpurely simple modules in the Auslander-Reiten quiver of $k\tilde{\Gamma}$ for $\Gamma = \tilde{E}_7$ with the orientation given above. The circled vertices correspond to the superpurely simple modules.

Chapter 5.

Purely simple modules of large rank

In the previous chapter we briefly mentioned the problem of the existence of purely simple modules of large rank. In this last chapter we are going to see that there are no torsion-free modules of uncountable rank.

By applying a simple "closure technique" we get the following bound on the size of the smallest pure submodule containing a given subset of the module M .

Proposition 5.1. *Let M be an arbitrary A -module, and let $N \subseteq M$ be a subset of M . Then there is a pure submodule $N' \subseteq M$ such that $N \subseteq N'$ and $|N'| \leq |N| \cdot |k| \cdot \aleph_0$.*

Proof. Clearly we may assume that N is a submodule. Then we can construct a chain of submodules

$$N = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_i \subseteq N_{i+1} \subseteq \cdots$$

so that they satisfy the following condition: $C \cdot M^s \cap (N_i)^r \subseteq C \cdot (N_{i+1})^s$ for every $r \times s$ matrix C with coefficients in the algebra A . Moreover we may choose our modules in such a way that $|N_{i+1}| \leq |N_i| \cdot |k| \cdot \aleph_0$. It is clear that the module $N' = \bigcup_{i=1}^{\infty} N_i$ will contain the subset N , its cardinality will be not greater than $|N| \cdot |k| \cdot \aleph_0$, and by Proposition 2.4 (iii) it will be pure in M .

Naturally, this result gives a bound $|k| \cdot \aleph_0$ on the cardinality of purely simple modules over A . That this bound is sharp can be seen from the existence of large torsion-free modules of rank 1. In particular the module Q will have the given cardinality.

Of course, a bound on the cardinality of a module also gives a bound on its rank. But it turns out that this bound can be improved. The following result was proved by Okoh for Kronecker algebras (cf. [O2]), but the proof can be easily applied to the general situation of tame hereditary algebras.

Theorem 5.2. *Let M be a torsion-free purely simple module of infinite rank. Then M can be written as the union of a chain of finite dimensional torsion closed submodules in M*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_i \subseteq M_{i+1} \subseteq \cdots$$

Proof. Assume M is torsion-free purely simple of infinite rank. Observe first that every torsion-closed submodule of M which is of finite rank must be finite dimensional, otherwise we would get an infinite dimensional indecomposable torsion-closed submodule of finite rank, so by Corollary 2.12 (ii) it would be a proper pure submodule. Thus combining this with Proposition 3.1, we get that the torsion closure of any finite dimensional submodule $W \subseteq M$ is finite dimensional.

Before proceeding with the construction of the submodules M_i let us also recall the well-known fact that for tame hereditary algebras there is a partition of the set of the isomorphism classes of indecomposable preprojective modules $\mathcal{A} = \bigcup_{i=1}^{\infty} \mathcal{A}_i$

so that if $P \in \mathcal{A}_i$ and $P' \in \mathcal{A}_j$ are indecomposable preprojective modules having a homomorphism $0 \neq f \in \text{Hom}(P, P')$ which is not an isomorphism, then $i < j$. Let us fix such a partition and for an indecomposable preprojective module P we define $c(P) = i$ if and only if $P \in \mathcal{A}_i$. It is also well-known that for each positive integer $n \in \mathbb{N}$ there are only finitely many indecomposable preprojective modules with k -dimension equal to n . Thus $\lim_{i \rightarrow \infty} \min \{\dim_k P \mid P \in \mathcal{A}_i\} = \infty$.

We shall now construct the modules M_i by induction. Choose for M_1 an arbitrary finite dimensional torsion-closed submodule of M . (We have seen that such submodules do exist.) Assume we have constructed the finite dimensional torsion-closed submodules $M_1 \subseteq M_2 \subseteq \dots \subseteq M_\ell$ in such a way, that — denoting by $M_{i,j}$ the indecomposable direct summands of M_i — we have $\min \{c(M_{i,j}) \mid j = 1, \dots, n_i\} \geq i$ for $i = 1, \dots, \ell$. We have to construct the submodule $M_{\ell+1}$.

Since none of the indecomposable direct summands of M_ℓ , i.e. $M_{\ell,1}, \dots, M_{\ell,n_\ell}$, are pure submodules, there exist finite dimensional submodules N_j , $j = 1, \dots, n_\ell$ such that $M_{\ell,j} \subseteq N_j$ and the inclusion maps do not split. Denote by N' the torsion closure in M of the submodule generated by the submodules N_j , for $j = 1, \dots, n_\ell$. By our first remark N' is finite dimensional. Write N' as the direct sum of indecomposable direct summands: $N' = \bigoplus_{j=1}^{t_{\ell+1}} N'_j$. Finally define $M_{\ell+1}$ as the sum of those summands of N' for which the restriction of the corresponding projection map from the submodule M_ℓ to N'_j is non-zero. We write $M_{\ell+1} = \bigoplus_{j=1}^{n_{\ell+1}} M_{\ell+1,j}$ with $M_{\ell+1,j}$ indecomposable for $j = 1, \dots, n_{\ell+1}$. Of course, $M_{\ell+1}$ is still torsion closed, finite dimensional, and it contains M_ℓ .

We have to observe first that none of the submodules $M_{\ell j}$ is a direct summand of $M_{\ell+1}$ by our choice of the modules N_j and by the fact that $M_{\ell+1}$ is a direct summand of N' .

Next, it is easy to see, that each indecomposable summand of $M_{\ell+1}$ is a successor of a summand of M_ℓ , since the restrictions of the projection maps are by our choice of $M_{\ell+1}$ non-zero. Moreover none of the projection maps can give an isomorphism between a summand $M_{\ell j}$ of M_ℓ and a summand of $M_{\ell+1}$, since this would imply that $M_{\ell j}$ is also a summand of $M_{\ell+1}$.

Thus using the induction hypothesis, we get by our introductory remarks that $\min \{c(M_{\ell+1,j}) \mid j = 1, \dots, n_{\ell+1}\} \geq \ell + 1$. (This also implies that if we define $d_i = \min \{\dim_k M_{ij} \mid j = 1, \dots, n_i\}$ then $\lim_{i \rightarrow \infty} d_i = \infty$.)

Consider now the union of this chain: $M' = \bigcup_{i=1}^{\infty} M_i$. As a union of a chain of torsion-closed submodules in M , by Lemma 3.2 M' is also torsion-closed in M . The condition on the dimensions of the indecomposable summands of the modules M_i show that M' cannot have any finite dimensional direct summands since otherwise it would be a summand of all but finitely many M_i 's, a clear contradiction. Thus by Corollary 2.12 (ii) we get that M' is a pure submodule of M . Since M was purely simple, we must have $M' = M$, as required.

As an immediate corollary we get the "almost best possible" bound for the rank of torsion-free purely simple modules and a somewhat surprising bound on their dimension. (Recall that by results of Okoh, Dean and Zorzitto, there are purely simple modules of arbitrarily large finite rank.)

Corollary 5.3. *Any module M contains a non-zero pure submodule N so that one of the following is satisfied.*

- (i) $\dim_k N < \infty$.
- (ii) N is a Prüfer module.
- (iii) N is torsion-free module and $\text{rk } N \leq \aleph_0$; moreover if $\text{rk } N = \aleph_0$ then $\dim_k N = \aleph_0$.

Proof. By Corollary 2.12 (i) we get that the torsion submodule $T(M)$ of the module M is always pure. Thus by the transitivity of the purity condition we can restrict ourselves to the cases where M is either torsion or torsion-free.

Assume first that M is torsion. Then by Corollary 2.11 it will either contain a finite dimensional preinjective direct summand or it will be torsion regular. In the latter case it must have a summand which is either finite dimensional regular or a Prüfer module (cf. [R2], Lemma 2 in 4.5).

Assume now that M is torsion-free. Then we may get infinite dimensional torsion-closed submodules of finite rank in which case Corollary 2.12 (ii) gives us a pure submodule of finite rank. Otherwise the construction of Theorem 5.2 will result in a pure submodule of countable rank and of countable dimension.

Unfortunately the question posed by Okoh about the existence of purely simple modules of infinite rank still remains unsolved, despite the fact, that he showed: the construction of the submodules above is “almost” enough to get a purely simple module of countable rank. His original statement was also for Kronecker algebras,

but it easily generalizes for the general tame hereditary case. Namely the following is true.

Proposition 5.4. *M is a torsion-free purely simple module of infinite rank if and only if:*

- (i) $M = \bigcup_{i=1}^{\infty} M_i$ for a chain of finite dimensional preprojective modules $M_1 \subseteq \cdots \subseteq M_i \subseteq \cdots$ such that each M_i is torsion closed in M_{i+1} , moreover denoting by d_i the minimal k -dimension of the indecomposable direct summands of M_i , we have that $\lim_{i \rightarrow \infty} d_i = \infty$.
- (ii) M has no proper infinite dimensional torsion-closed submodules without finite dimensional direct summands.

Proof. If M is a torsion-free purely simple module of infinite rank then in the proof of Theorem 5.2 we have constructed a chain of submodules satisfying condition (i). On the other hand condition (ii) must also be satisfied since otherwise we would get a nontrivial pure submodule (cf. Corollary 2.12 (ii)).

To prove the converse we have to observe first that the existence of a chain of submodules satisfying (i) guarantees that M has no finite dimensional direct summands, hence no finite dimensional pure submodules. But there are no infinite dimensional pure submodules either, because a pure submodule with a finite dimensional summand would give a finite dimensional summand to M , violating condition (i), while without finite dimensional summands it would violate condition (ii). Thus M is purely simple. Finally from 2.1 (i) we get easily that M_i is torsion-

closed in M , hence $\text{rk } M_i \leq \text{rk } M$. Since in the chain we have infinitely many strict inclusions, we have $\text{rk } M_i \longrightarrow \infty$, hence $\text{rk } M = \aleph_0$.

One remark is perhaps in place here. It is not very difficult to construct chains of finite dimensional preprojective modules satisfying condition (i) of the last proposition. If we start with an indecomposable module M_1 , we can take for M_2 the middle term of the Auslander-Reiten sequence starting with M_1 . Then we may continue in this fashion by embedding each indecomposable direct summand of M_2 into its "own" middle term and then taking the direct sum of these (possibly with repetitions) as M_3 etc. Unfortunately, in this construction condition (ii) clearly will be violated. It is the non-constructiveness of this condition which makes the question on the existence of purely simple modules of large rank difficult.

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