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BOUNDS ON CLASSICAL RAMSEY NUMBERS

by

Faiz Bannani B.Sc., M.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics
Carleton University
Ottawa, Canada
November, 1988

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ISBN 0-315-51136-2

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Bounds on Classical Ramsey Numbers

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ABSTRACT

In this thesis we describe the different methods used to find Bounds on the classical Ramsey numbers. We introduce a sample space for random "cyclic graphs" and use this space to derive an implicit lower bound for the Ramsey numbers $R(s,t)$. We also generalize Spencer's results for finding lower bounds on $R(s,t)$, to obtain lower bounds on $R_k(s_1, s_2, \dots, s_r)$. We give a constructive lower bound for $R(s)$ of order s^3 and for $R(2 \log s, s)$ of order s^2 . Some new lower bounds for small Ramsey numbers are also established by random generation of cyclic graphs. We conclude our thesis by giving a summary for the known upper bounds for the Ramsey numbers.

ACKNOWLEDGEMENTS

I would like to express my sincerest gratitude to Professor John Dixon for his guidance, suggestions and continued encouragement throughout the course of this thesis.

I thank the Institution of Energy, Tripoli, (S.P.L.A.J.) for its financial sponsorship of my studies in Canada.

I am most grateful to Mrs. Diane Berezowski for her great patience, and for an excellent job in typing this thesis.

Lastly, I cannot find within me words strong enough to express the gratitude I feel towards my parents, whose love, support and constant encouragement have been a guiding light in my life, but especially so during these years of study so far from home. May God bless you both!

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CHAPTER 1

INTRODUCTION

1.1 Some notations and definitions

In this section we give some notations which will be used in the next chapters. We first start with some graph notations and definitions.

Let $G = (V, E)$ be a graph with vertex set V and edge set E , then the number n of vertices of G is called the *order* of G , and the number q of edges in G is called the *size* of G . For $S \subseteq V$ we denote by $G(S)$ the graph with vertex set S and edge set all $(v, u) \in E$ such that v and u belong to S . We call $G(S)$, an *induced subgraph* of G . The *complement* of a graph G , denoted by $\overline{G}$, is the graph having the same vertex set as G , but two vertices of $\overline{G}$ are adjacent in $\overline{G}$ if and only if they are not adjacent in G . A *complete graph* is a graph in which every two vertices are joined by an edge. We denote the complete graph of order n by K_n . A maximal complete subgraph of a graph G is called a *clique* in G . The *clique number* of a graph G is the order of the largest clique in G , and is denoted by $\beta(G)$ or simply β when there is no danger of confusion. A subset of the vertex set of a graph G is called an *independent set* of vertices if the subgraph induced by it has no edges. The *independence number* of G is denoted by $\alpha(G)$, or simply α , and is defined to be the cardinality of a largest independent set in G . Note that $\alpha(G) = \beta(\overline{G})$. Following Chung and Grinstead (1983), we define an (s, t, n, ϵ) -graph to be a graph of order n , size ϵ which has no complete subgraph of order s and no independent set of vertices of cardinality t . If ϵ is unspecified the graph

will be called an (s, t, n) -graph, and if n is also unspecified, then the graph will be called an (s, t) -graph.

Let S be a set. We denote the cardinality of S by $|S|$.

Following the notations in Graham, Rothschild and Spencer (1980), we denote the set $\{1, 2, \dots, n\}$ by $[n]$. We sometimes use $[n]$ to refer to an arbitrary set of cardinality n . We denote the set $\{Y : Y \subset S, |Y| = k\}$ by $[S]^k$, and when $S = [n]$ we simply write $[n]^k$ for $[[n]]^k$. An r -coloring of a set S is a map

$$X : S \rightarrow [r].$$

For $s \in S$, $X(s)$ is called the color of s . A set $T \subseteq S$ is called monochromatic (under X) if X is constant on T .

Definition. We write $n \rightarrow (s_1, \dots, s_r)^k$ if, for every r -coloring of $[n]^k$, there exists $i, 1 \leq i \leq r$, and a subset T of $[n], |T| = s_i$, so that $[T]^k$ is colored i .

In case $s_1 = s_2 = \dots = s_r = s$ we use the shortened

$$n \rightarrow (s)_r^k.$$

and in this case we say that every r -coloring of $[n]^k$ yields a monochromatic $[s]^k$. If r is not indicated it is assumed to be 2. Thus $n \rightarrow (s)_2^k, n \rightarrow (s)^k$ and $n \rightarrow (s, s)^k$ are identical relations. Also if k is not indicated it is assumed to be 2.

Definition. The Ramsey function $R_k(s_1, \dots, s_r)$ denotes the minimal n such that

$$n \rightarrow (s_1, \dots, s_r)^k.$$

If $s_1 = s_2 = \dots = s_r = s$ we denote the Ramsey function by $R_k(s; r)$, and if k or r are not indicated they are assumed to be 2, and we write $R(s_1, \dots, s_r)$ for $R_2(s_1, \dots, s_r)$, $R_k(s)$ for $R_k(s; 2)$ and $R(s)$ for $R_2(s)$.

We distinguish the case $r = k = 2$. $R(s, t)$ denotes the smallest integer n , such that for every 2-coloring of $[n]^2$, there exists $T \subseteq [n]$, $|T| = s$ and $[T]^2$ is colored blue or $T' \subseteq [n]$, $|T'| = t$ and $[T']^2$ is colored red.

In graph theory terms, $R(s, t)$ has a special interpretation: Let G be a graph of order n . Associate with G a two coloring of $[n]^2$ in which $\{i, j\}$ in $[n]^2$ is colored red if and only if $\{i, j\}$ is an edge in G , and blue otherwise. This gives a bijective correspondence between the set of all graphs of order n and all 2-colorings of $[n]^2$. Thus $R(s, t)$ can be defined as the smallest integer n such that every graph of order n contains a complete subgraph of order s or an independent set of cardinality t .

1.2 Ramsey's Theorem.

In the previous section we gave some notations and definitions without actually bothering about whether the terms are well defined. In this section we shall give the theorem that guarantees that these terms are well defined. This theorem is known as Ramsey's Theorem after Frank Plumpton Ramsey.

We first start with a simple example in which we evaluate $R(3, 3)$. We first prove the following: Any graph of order six must contain a triangle or an independent set of size three.

To prove this, let G be a graph of order six with vertices a, b, c, d, e, f . The vertex a is either joined to three vertices or not joined to three vertices. If a is joined at least to three vertices, say b, c, d , then if any two of these vertices are

joined then together with a they form a triangle, and hence G contains a triangle. Otherwise b, c and d form an independent set of size three. The case a is not joined to three vertices is similar (consider the dual graph $\overline{G}$).

The above shows that $R(3, 3) \leq 6$. To show that $R(3, 3) \geq 6$ we need to find a graph G of order 5 with $\beta(G) = \alpha(G) = 2$. Consider the graph with vertex set $\{0, 1, 2, 3, 4\}$ and such that two vertices i and j are adjacent if and only if $(i - j)$ modulo 5 $\in \{1, 4\}$. This graph is called the 5-circuit, and belongs to a family of graphs called cyclic graphs which will be defined in later chapters. It can be easily shown that $\alpha(G) = \beta(G) = 2$. Consequently we have $R(3, 3) = 6$.

The following theorem shows that $R(s, t)$ is well defined and gives an upper bound on $R(s, t)$.

Theorem 1.2.1. The function $R(s, t)$ is well defined and

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1). \quad (1.2.1)$$

Proof. We use a double induction on s and t . It is easily seen that for any $k \geq 1, R(2, k) = R(k, 2) = k$. Assume by induction that $R(s, t - 1)$ and $R(s - 1, t)$ exist. Let G be an (s, t) -graph of order n , and v be a vertex of G . Let $N(v)$ and $NN(v)$ be the sets of neighbors and non-neighbors of the vertex v in G , respectively. It can be easily shown that $G(N(v))$ and $G(NN(v))$ are $(s - 1, t)$ and $(s, t - 1)$ -graphs, respectively. Thus we have

$$|N(v)| \leq R(s - 1, t) - 1$$

$$|NN(v)| \leq R(s, t - 1) - 1,$$

and hence

$$n \leq R(s - 1, t) + R(s, t - 1) - 1.$$

Since G is an arbitrary (s, t) -graph it follows that

$$R(s, t) \leq R(s - 1, t) + R(s, t - 1). \quad (1.2.1)$$

This proves the theorem.

Note that in the above theorem, if there is an (s, t) -graph G of order $R(s - 1, t) + R(s, t - 1) - 1$, then each vertex in G has exactly $R(s - 1, t) - 1$ neighbors. Thus the number of edges in G is $(R(s - 1, t) + R(s, t - 1) - 1)(R(s - 1, t) - 1)/2$, which must be integer. This is possible only if $R(s - 1, t)$ or $R(s, t - 1)$ is odd. Thus if both $R(s - 1, t)$ and $R(s, t - 1)$ are even then (1.2.1) holds with strict inequality.

The next theorem shows that $R_k(s_1, s_2, \dots, s_r)$ is well defined. We state the theorem without proof. The theorem was first proved in Ramsey (1930). One may refer to Graham, Rothschild and Spencer (1980) for a proof for the theorem.

Theorem 1.2.2. (Ramsey's Theorem). For all $k, s_1, \dots, s_r$ there exists n_0 so that for all $n \geq n_0$

$$n \rightarrow (s_1, \dots, s_r)^k,$$

and hence $R_k(s_1, \dots, s_r)$ is well defined.

The problem of finding the Ramsey numbers is a very difficult problem. Thus we are usually satisfied with partial results such as finding lower and upper bounds for these numbers. This will be the subject of the coming chapters. The main results established in this thesis are given in sections 2.6, 3.5, 4.2 and 4.3.

CHAPTER 2

NONCONSTRUCTIVE LOWER BOUNDS FOR THE RAMSEY NUMBERS

2.1 Introduction

In this chapter we discuss the use of probabilistic arguments and results to prove some interesting results concerning Ramsey numbers. Probabilistic methods prove the existence of certain structures with some property without constructing them. The problem of constructing such structures is usually harder than proving their existence.

P. Erdős played a major role in the development of this area. In 1947 he was able to prove, using some counting arguments, that $R(s) > \frac{s2^{s/2}}{e\sqrt{2}}$. His method is discussed in section 2.2.

Spencer (1975) used a theorem due to Lovász known as the Lovász Local Theorem, to improve the lower bound obtained by Erdős by a factor of 2. He also obtained lower bounds on $R(s, t)$ for fixed s as t approaches infinity. We give his main results in section 2.3.

In sections 2.4 and 2.5 we give other methods, which are also based on probabilistic arguments, to obtain some lower bounds on $R(s)$. However these lower bounds are not better than the ones obtained by Spencer 1979. The first method is called The Deletion Method and was discovered in 1982 by Jim Shearer. The second method is called The Recoloring Method. Both methods are described in Spencer (1985).

In section 2.6 we use “Random Cyclic graphs” to derive an implicit bound on $R(s)$. We end the section with an open problem.

Section 2.7 generalizes Spencer’s results given in section 2.3, to obtain lower bounds for $R_k(s_1, \dots, s_r)$, $R_k(s; r)$ and $R_k(s, s, \dots, s, s_r)$.

We conclude this Chapter with tables for lower bounds on $R(s, t)$ obtained by nonconstructive methods. These tables are given in section 2.8.

We now give some preliminary results that we shall use throughout this chapter.

Let A be an event in a probability space. By $P(A)$ we mean the probability that the event A occurs. We shall be using the following simple result, which is due to Markov.

Proposition 2.1.1 (Markov’s inequality). Let X be a non-negative integer valued random variable. Then for any $a > 0$, a real

$$P(X \geq a) \leq \frac{E(X)}{a}$$

where $E(X) = \sum_{i=0}^{\infty} iP(X = i)$ is the expected value of X .

Following the notations in Bollobas (1985), by $\mathcal{G}(n, p)$ we mean the set of all graphs of order n , in which an edge is inserted with probability p . $0 \leq p \leq 1$. independently of any other edge. If $G \in \mathcal{G}(n, p)$ and G has m edges then

$$P(G) = p^m(1 - p)^{\binom{n}{2} - m}.$$

By the binomial theorem it is clear that $\sum P(G) = 1$ where the sum is taken over all G in $\mathcal{G}(n, p)$.

By the map $\psi : \text{edge} \rightarrow \text{red}, \text{nonedge} \rightarrow \text{blue}$ one may think of $\mathcal{G}(n, p)$ as the set of all 2-colored complete graphs of order n in which an edge is colored red with probability p and blue with probability $1 - p$ independently of the color of any other edge.

We shall also be using the following inequality,

$$\binom{n}{s} \leq \frac{n^s}{s!} \leq \frac{1}{\sqrt{2\pi s}} \left(\frac{ne}{s}\right)^s < \left(\frac{ne}{s}\right)^s$$

which is a consequence of the following formula discovered by James Stirling in the eighteenth century, and usually referred to as Stirling's formula.

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}}$$

where $0 < \theta < 1$ and θ depends on n .

2.2 The Erdős Lower Bound on $R(s)$.

To show that $R(s) > n$, we need to prove the existence of an (s, s, n) -graph. Erdős (1947) used a very simple counting argument to obtain a lower bound on the Ramsey number $R(s)$. There is a modern version for the argument Erdős used, however, we shall give the original proof given by Erdős.

Theorem 2.2.1 Let $s \geq 3$. Then

$$R(s) > 2^{s/2}.$$

Proof. Let $n \leq 2^{s/2}$. Clearly the number of labeled graphs of order n is $2^{n(n-1)/2}$. The number of labeled graphs containing a fixed s -complete subgraph

is $2^{n(n-1)/2} / 2^{s(s-1)/2}$. Thus the number of graphs with n vertices containing any complete subgraph of order s is less than

$$\binom{n}{s} 2^{n(n-1)/2} / 2^{s(s-1)/2} < \frac{n^s}{s!} \frac{2^{n(n-1)/2}}{2^{s(s-1)/2}} < \frac{2^{n(n-1)/2}}{2} \quad (2.2.1)$$

for $n \leq 2^{s/2}$ and $s \geq 3$.

Similarly the number of graphs with n vertices containing an independent set of order s is less than $\frac{2^{n(n-1)/2}}{2}$. Hence the number of non- (s, s, n) -graphs is strictly less than $2^{n(n-1)/2}$. Thus there exists at least one (s, s, n) -graph. Hence $R(s) > n$.

The above theorem is still valid if we taken $n \leq \frac{s2^{s/2}}{e}$, and in (2.2.1) in the second inequality we replace $\binom{n}{s}$ by $\frac{1}{2} \left(\frac{en}{s}\right)^s$. This gives the lower bound

$$R(s) > \frac{s2^{s/2}}{e\sqrt{2}} \quad s \geq 3 \quad (2.2.2)$$

which is known as the Erdős lower bound on $R(s)$.

2.3 Spencer's Lower Bounds on $R(s, t)$.

In 1974 Joel Spencer used the probability space $\mathcal{G}(n, p)$ to derive lower bounds on $R(s, t)$. By using the Lovász Local Theorem, he was able to improve the Erdős lower bound on $R(s)$ by a factor of 2. He also proved that for fixed s , and as t tends to infinity $R(s, t) > t^{\frac{s+1}{2} + o(1)}$. We shall state the main results of his paper but postpone the proofs to section 2.7 when each result becomes a special case of its generalization.

Theorem 2.3.1. If

$$4 \binom{s}{2} \binom{n}{s-2} 2^{1-\binom{s}{2}} < 1$$

then $R(s) > n$.

Using Stirling's formula the following is an immediate result.

Corollary 2.3.1.

$$R(s) \geq \left\{ \frac{1}{e} + o(1) \right\} s 2^{(s+1)/2}. \quad (2.3.1)$$

Theorem 2.3.2. If there exists p , $0 \leq p \leq 1$, such that

$$\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$$

then $R(s, t) > n$.

Corollary 2.3.2. For s fixed, $t \rightarrow \infty$

$$R(s, t) > t^{\frac{(s-1)}{2} - o(1)}$$

Let $A \subseteq \{1, \dots, n\}$, $|A| = a$. Denote by $f(a, b, n)$ the number of $B \subseteq \{1, \dots, n\}$, $|B| = b$, $B \neq A$, such that $|A \cap B| \geq 2$. Then we have

Theorem 2.3.3. Let $s \leq t \leq n$. If there exists p , $0 < p < 1$, so that

$$f(t, s, n) p^{\binom{s}{2}} + f(t, t, n) (1-p)^{\binom{t}{2}} < \frac{1}{4}$$

then $R(s, t) > n$.

Corollary 2.3.3. For s fixed, $t \rightarrow \infty$

$$R(s, t) > t^{c(s) + o(1)}$$

where $c(s) = \left(\binom{s}{2} - 2 \right) / (s-2) \sim \frac{s+1}{2} + o(1)$ for large s .

2.4 The Deletion Method:

The Deletion Method was discovered by Jim Shearer in 1982 and was described in Spencer (1985). This method improves the Erdős lower bound by a factor of $\sqrt{2}$. However the lower bound obtained by Spencer (1975) is still better.

Let K_n be a randomly 2-colored complete graph with vertex set V , $|V| = n$. For $S \subset V$ and $|S| = s$ define

$$X_S = \begin{cases} 1 & \text{if } K_n(S) \text{ is monochromatic} \\ 0 & \text{otherwise.} \end{cases}$$

Then $P(X_S = 1) = 2^{1-\binom{s}{2}}$. Let $X = (\sum X_S : S \subset V \text{ and } |S| = s)$. Then X counts the number of monochromatic complete subgraphs of K_n of order s . We have

$$E(X) = \sum E(X_S) = \sum P(X_S = 1) = \binom{n}{s} 2^{1-\binom{s}{2}}.$$

By Markov's inequality we have

$$P(X \geq \epsilon n) \leq \frac{1}{\epsilon n} \binom{n}{s} 2^{1-\binom{s}{2}}.$$

Hence if $\binom{n}{s} 2^{1-\binom{s}{2}} < \epsilon n$, then $P(X < \epsilon n) > 0$. Thus there exists a particular coloring of K_n with less than ϵn monochromatic K_s . From each monochromatic K_s delete one vertex. The resulting graph has $n(1 - \epsilon)$ vertices and contains no monochromatic K_s . Hence we have the following

Theorem 2.4.1. If

$$\binom{n}{s} 2^{1-\binom{s}{2}} < \epsilon n$$

then $R(s) > n(1 - \epsilon)$.

For appropriately chosen ϵ and using Stirling's formula one can show

Corollary 2.4.1. For $s \geq 3$

$$R(s) > \frac{s2^{s/2}}{e}(1 + o(1)).$$

2.5 The Recoloring Method.

Beck (1978) used a method called The Recoloring Method to find bounds on the function $m(n)$ for property B . Spencer (1985) used the same method to asymptotically improve the Erdős lower bound on $R(s)$ by a factor of 2. This gives no improvement on Spencer's lower bound obtained in 1975. In this section we describe this method briefly.

Consider a random 2-coloring of K_n in $\mathcal{G}(n, p)$. Suppose that $\binom{n}{s}2^{1-\binom{s}{2}} < n^{2-\epsilon}$. Then as in section 2.4, there exists a particular coloring of K_n with less than $n^{2-\epsilon}$ monochromatic K_s . We call this coloring the first coloring. Let ℓ be some integer. We call an s -set A near-monochromatic if all but less than ℓ edges have the same color. Note that near-monochromatic includes monochromatic.

Given the first coloring an edge is called critical if it is a red edge in a near-red s -set or a blue-edge in a near blue s -set. Let θ be such that $0 < \theta < 1$. We change the color of each critical edge with probability θ independently of the color of any other critical edge. We call the second coloring The Recoloring. There are three possibilities in which an s -set S can be red after the recoloring,

- (i) S was near-red in the first coloring and none of its red edges changed in the Recoloring.
- (ii) S was near-blue in the first coloring and changed to red.

(iii) S was neither near-red nor near-blue in the first coloring but became Red in the Recoloring.

For $\ell = [100\epsilon^{-2}]^2$ and $\theta = 100/s$, Spencer was able to show that the probability that any of the above three cases occurs is $o(1)$, which tends to zero as n tends to infinity. Similarly for the probability of a blue s -set. We omit the proof in this section and the reader may refer to Spencer (1985).

Thus the following theorem is true.

Theorem 2.5.1. Let $\epsilon > 0$. Then for all sufficiently large n the inequality $\binom{n}{s} 2^{1-\binom{s}{2}} < n^{2-\epsilon}$ implies that $R(s) > n$.

As a consequence of the above theorem we have the following

Corollary 2.5.1 $R(s) > \left(\frac{\sqrt{2}}{\epsilon}\right) s 2^{s/2} (1 + o(1))$

As we shall see in Chapter 5 that the best known upper bound on $R(s)$ is $\frac{1}{6} 2^{2s} s^{-1/2}$, which differs from the lower bounds by the exponent $2s$ vs. $s/2$. It is quite sad that all the attempts to improve the Erdős lower bound on $R(s)$ gained us only a factor of at most 2.

2.6 Random Cyclic graphs and lower Bounds on $R(s)$:

Cyclic graphs have been used to find lower bounds for small Ramsey numbers $R(s, t)$. As we shall see in Chapter 4, in most cases the largest known (s, t, n) -graphs are cyclic, and the lower bounds obtained by these graphs on $R(s)$ seem to be exponential in s . It is hoped that by restricting our sample space $\mathcal{G}(n, p)$ to a sample space whose elements are cyclic graphs one may find better lower bounds on $R(s)$ than those obtained in the previous sections. Unfortunately the lower

bound obtained in this section is given implicitly by a formula. The section is ended by a conjecture that if true would give the lower bound $R(s) \geq c \frac{2^s}{s^2}$, which would be a significant improvement to the known lower bounds on $R(s)$.

We start by describing the idea in more general terms and then apply the results to randomly generated cyclic graphs.

Let $W = \{w_1, w_2, \dots, w_n\}$ be a set of n elements and R an equivalence relation on $[W]^2$. Then R partitions $[W]^2$ into equivalence classes $C_1, C_2, \dots, C_m$ with $m \leq \binom{n}{2}$. Define a graph $G = (V(G), E(G))$ as follows.

$$V(G) = W$$

$$E(G) = \bigcup_{i=1}^k C_{r_i} \text{ for some } r_1, r_2, \dots, r_k \text{ in } \{1, \dots, m\}.$$

We call such graph an equivalence graph. We denote the set of all equivalence graphs which may be induced by W and R by $\mathcal{G}_n(W, R)$. For $S \subseteq W$, set $C(S) = \{C_i : C_i \cap [S]^2 \neq \emptyset\}$.

Theorem 2.6.1. Given a set $W, |W| = n$, and equivalence relation R on $[W]^2$. Let G be a graph in $\mathcal{G}_n(W, R)$. For $i = 1, 2$ let $S_i \subseteq W$ and $|S_i| = s$. If $C(S_1) \supseteq C(S_2)$ and $G(S_1) \cong K_s$, then $G(S_2) \cong K_s$.

Proof. Suppose $G(S_2) \not\cong K_s$. Then there exists e in $[S_2]^2$ which is not an edge in $G(S_2)$. Let C be the equivalence class in which e falls. Then all elements of C are non edges in G . Since $C(S_2) \subseteq C(S_1)$, there is e' in $[S_1]^2$ such that e' is in C . Thus e' is not an edge in G and hence in $G(S_1)$. This is a contradiction to the assumption that $G(S_1) \cong K_s$.

Definition. A set $B(W, R) = \{S_1, S_2, \dots, S_N\}$, where $S_i \in [W]^s$ is called an s -Base if for every S in $[W]^s$ there exists S_i in $B(W, R)$ such that $C(S_i) \subseteq C(S)$

and $B(W, R)$ is minimal with respect to this property. We shall abridge $B(W, R)$ to B whenever there is no danger of confusion.

Corollary 2.6.1. Given a set $W, |W| = n$, and an equivalence relation R on $[W]^2$. Let B be an s -Base and $G \in \mathcal{G}_n(W, R)$. Then G contains a K_s if and only if $G(S) \cong K_s$ for some $S \in B$.

Proof. One direction is trivial. Suppose G contains a K_s . Let S_1 be the vertex set of this complete graph K_s in G . If $S_1 \in B$ we are done. Otherwise by the definition of B there exists S_2 in B such that $C(S_2) \subseteq C(S_1)$. But then by Theorem 2.6.1 $G(S_2) \cong K_s$.

We now define a probability space for equivalence graphs. With the same notations as before, insert the edges of an equivalence class C_i with probability θ independently of any other edges in any other equivalence class $C_j, j \neq i$. Then the probability that an equivalence graph has its set of edges equal to the union of k equivalence classes is given by

$$\binom{m}{k} \theta^k (1 - \theta)^{m-k} \quad 0 \leq \theta \leq 1.$$

where m is the number of all possible equivalence classes. We denote the space consisting of all graphs formed as above by $\mathcal{G}_n(W, R, \theta)$.

Let $G \in \mathcal{G}_n(W, R, \theta)$ and B be an s -Base. Let X_B be the number of s -sets S in B such that $G(S) \cong K_s$, and X_G be the number of s -sets S' in $V(G) = W$ such that $G(S') \cong K_s$. By Corollary 2.6.1 we have $X_B \geq 1$ if and only if $X_G \geq 1$. Thus

$$P(X_G \geq 1) = P(X_B \geq 1) \leq E(X_B) = \sum_{S \in B} \theta^{|C(S)|}$$

Consequently

$$P(X_G \geq 1) \leq \min_{B \in \mathcal{B}} \left\{ \sum_{S \in B} \theta^{|C(S)|} \right\} \quad (2.6.1)$$

where $\mathcal{B}$ is the set of all s -Bases.

We next show that it is irrelevant to take the minimum in (2.6.1). Given a base B , let $C(B) = \{C(S) : S \in B\}$.

Lemma 2.6.1. Let B and B' be two s -Bases then $C(B) = C(B')$.

Proof. Suppose $C(B) \neq C(B')$. We can assume that there is S in B such that $C(S) \not\subset C(B')$. By the definition of B' , there exists S' in B' such that $C(S') \subset C(S)$. By the definition of B there exists S'' in B such that $C(S'') \subset C(S')$. This implies $C(S'') \subset C(S)$. Thus $B - S$ is a base, which contradicts the minimality of B .

By Lemma 2.6.1 we have the following immediate

Corollary 2.6.2. $\min_{B \in \mathcal{B}} \sum_{S \in B} \theta^{|C(S)|} = \sum_{S \in B} \theta^{|C(S)|}$ for any $B \in \mathcal{B}$.

By a similar argument one can also show that if Y_G is the number of t -independent sets in $G \in \mathcal{G}(W, R, \theta)$ then

$$P(Y_G \geq 1) \leq \sum_{S \in B_t} (1 - \theta)^{|C(S)|}$$

for some t -Base B_t .

Consequently we have

$$\begin{aligned} P(G \text{ contains } K_s \text{ or } \bar{G} \text{ contains } K_t) &\leq P(X_G \geq 1) + P(Y_G \geq 1) \\ &\leq \sum_{S \in B_s} \theta^{|C(S)|} + \sum_{S \in B_t} (1 - \theta)^{|C(S)|} \quad (2.6.2) \end{aligned}$$

where in general B_i denotes an i -Base. Hence we have the following

Theorem 2.6.2. Given a set $W = \{w_1, w_2, \dots, w_n\}$ and an equivalence relation R on $[W]^2$, let B_s and B_t be an s -Base and a t -Base respectively. If there exists θ , $0 \leq \theta \leq 1$ such that

$$\sum_{S \in B_s} \theta^{|C(S)|} + \sum_{S \in B_t} (1 - \theta)^{|C(S)|} < 1$$

then $R(s, t) > n$.

Proof. By the condition of the theorem and (2.6.2), there exists a graph with n vertices that contains no K_s and whose complement contains no K_t . Hence $R(s, t) > n$.

Corollary 2.6.3. If

$$\sum_{S \in B_s} 2^{1-|C(S)|} < 1$$

then $R(s, s) > n$.

Proof. Take $t = s$ and $\theta = \frac{1}{2}$ and apply Theorem 2.6.2.

We now apply the previous results to a particular choice of a set W^* and an equivalence relation R^* on $[W^*]^2$.

Let p be an odd prime number. Define the absolute value of an element $a \in \{0, 1, \dots, p-1\}$ by

$$|a|_p = \min\{a, p - a\}.$$

Let $W^* = \{0, 1, \dots, p-1\}$ and R^* be such that

$$\{i, j\} R^* \{k, \ell\} \text{ iff } |i - j|_p = |k - \ell|_p \text{ for any } \{i, j\}, \{k, \ell\} \text{ in } [W^*]^2$$

Clearly R^* is an equivalence relation on $[W^*]^2$ and the equivalence classes are given by

$$C_i = \{\{k, j\} : |k - j|_p = i\} \quad i = 1, 2, \dots, (p-1)/2.$$

Also since p is prime $|C_i| = p$ for all $i = 1, \dots, (p-1)/2$. An equivalence graph induced by W^* and R^* as above is called a cyclic graph. For $S \subseteq W^*$ and $|S| = s$ we have

$$C(S) \doteq \{C_i : |k - j|_p = i \text{ for some } k, j \text{ in } S \text{ and } k \neq j\} \quad (2.6.3)$$

Let $\Delta(S) = S - S = \{(i - j), \text{ mod } p : i, j \in S\}$. Then

$$|C(S)| = \frac{|\Delta(S)| - 1}{2} \quad (2.6.4)$$

The -1 in (2.6.4) counts for the case $i - j = 0$ in $\Delta(S)$.

Let G be an additive group and A, B be subsets of G . Define $A + B = \{a + b : a \in A, b \in B\}$. The following theorem is due to Cauchy (1813) and is proved independently by Davenport H. (1935). It is usually referred to as the Cauchy-Davenport Theorem.

Theorem 2.6.3. If Z_p is the additive group of residues modulo p , p a prime, and A, B are subsets of Z_p such that $A + B \neq Z_p$, then $|A + B| \geq |A| + |B| - 1$.

Corollary 2.6.4. Let p be an odd prime, $S \subseteq \{0, 1, \dots, p-1\}$, $|S| = s \leq \frac{p+1}{2}$. Then

$$\frac{s(s-1)}{2} \geq |C(S)| \geq s-1, \quad (2.6.5)$$

where $C(S)$ is as defined by (2.6.3).

Proof. Since $|S| = s$, the inequality $\frac{s(s-1)}{2} \geq |C(S)|$ is trivial, since there can be at most $\frac{s(s-1)}{2}$ distinct absolute differences in $C(S)$. If $S - S = \{0, 1, \dots, p-1\}$ then $|S - S| = p = 2|C(S)| + 1$ by (2.6.4). Thus $|C(S)| = \frac{p-1}{2} \geq s-1$ by the hypothesis of the corollary. Hence we can assume that $S - S \neq \{0, 1, \dots, p-1\}$. Set $A = -B = S$ in Theorem 2.6.3. Then we have $|S - S| \geq |S| + |S| - 1 = 2s - 1$. By (2.6.4) we have $|S - S| = 2|C(S)| + 1 \geq 2s - 1$. This implies

$$|C(S)| \geq s - 1.$$

Let $B_s = B(W^*, R^*)$ be an s -base. Define

$$N_i(B_s) = |\{S \in B_s : |C(S)| = i\}|. \quad (2.6.6)$$

The following is an immediate consequence of Corollary 2.6.3 and Corollary 2.6.4.

Theorem 2.6.4. Let B_s and $N_i(B_s)$ be as above. If

$$\sum_{i=s-1}^{\frac{s(s-1)}{2}} N_i(B_s) 2^{1-i} < 1 \quad (2.6.7)$$

then $R(s) > p$.

Note that in (2.6.7) if $\frac{N_i(B_s)}{2^{1-i(s-1)}} = 0(p)$ for all i , then Theorem 2.6.4 would imply $R(s) > c \frac{2^s}{s^2} \{1 + 0(1)\}$ where c is a constant. We use a Theorem due to Vosper (1956), to show that $N_{s-1}(B_s) \leq \frac{p-1}{2}$.

Theorem 2.6.5. Let Z_p be the additive group of residues modulo the prime p . Let A and B be subsets of Z_p . then either

$$|A + B| \geq |A| + |B|$$

or one of the following holds.

- (i) $Z_p = A + B$.
- (ii) $|A + B| = p - 1$ and $B^c = \bar{c} - A$ for some $\bar{c}$ not in $A + B$ where B^c is the complement of B .
- (iii) $|A| = 1$ or $|B| = 1$
- (iv) A and B are in arithmetic progression with the same difference. That is there is $d \in Z_p$ such that

$$A = \{a, a + d, a + 2d, \dots, a + md\} \quad |A| = m + 1.$$

and

$$B = \{b, b + d, b + 2d, \dots, b + nd\} \quad |B| = n + 1.$$

Corollary 2.6.5. If $1 < s \leq \frac{p-3}{2}$, then

$$N_{s-1}(B_s) \leq \frac{p-1}{2}.$$

Proof. Let $k = N_{s-1}(B_s)$ and $\{S_1, S_2, \dots, S_k\}$ be the s -sets in B_s with $|C(S_i)| = s - 1, i = 1, \dots, k$. Fix i and let $A = -B = S_i$ in the above theorem. Then by (2.6.4) we have

$$|A + B| = |S_i - S_i| = 2|C(S_i)| + 1 = 2(s - 1) + 1 = 2s - 1 < 2s = |S_i| + |S_i| = |A| + |B|.$$

Thus in Theorem 2.6.5 one of the cases (i) - (iv) must hold. We show that (i) - (iii) do not hold under the assumption of the Corollary. Suppose (i) holds. Then we have

$$p = |Z_p| = |A + B| = |S_i - S_i| = 2(s - 1) + 1 = 2s - 1 \leq 2\frac{p-3}{2} + 1 < p$$

a contradiction. Similarly if (ii) holds then we have $p - 1 \leq p - 2$ which is a contradiction, and (iii) does not hold since $s > 1$. Hence for each i , S_i is an arithmetic progression $\{a_i, a_i + d_i, a_i + 2d_i, \dots, a_i + (s - 1)d_i\}$ for some $a_i \in \{0, 1, \dots, p - 1\}$, $d_i \in \{1, \dots, p - 1\}$. Clearly there are at most $p(p - 1)$ such S_i and if $S_k = a \mp S_\ell$ then $C(S_k) = C(S_\ell)$ and since the S_i are elements of an s -base B_s we can have at most $\frac{p(p-1)}{2p} = \frac{(p-1)}{2}$ different S_i in B_s . This proves the theorem.

We conjectured that the lower bound on $R(s)$ that would be obtained from Theorem 2.6.4 is at least $\frac{c}{s^2}2^s$ where c is some constant. This lower bound could be attained if we could show that $N_i(B_s)/2^{i-(s-1)} \leq c_1 p$ for all $s - 1 \leq i \leq \frac{s(s-1)}{2}$, for some constant c_1 : For suppose $N_i(B_s)/2^{i-(s-1)} \leq c_1 p$ for all i , then by Theorem 2.6.4 if

$$\sum_{i=s-1}^{\frac{s(s-1)}{2}} [c_1 p 2^{i-(s-1)}] 2^{1-i} < 1$$

then $R(s) > p$. Choose $p = \frac{c}{s^2}2^s$ where $c = \frac{1}{2c_1}$, then we have

$$\begin{aligned} \sum_{i=s-1}^{\frac{s(s-1)}{2}} [c_1 p 2^{i-(s-1)}] 2^{1-i} &= \sum_{i=s-1}^{\frac{s(s-1)}{2}} \frac{2}{s^2} \\ &= 1 - \frac{3}{s} + \frac{4}{s^2} \\ &< 1 \end{aligned}$$

In Corollary 2.6.5, one can in fact show that $N_{s-1}(B_s) = \frac{p-1}{2}$. Thus in order to have the condition (2.6.7) of Theorem 2.6.4 satisfied, we must at least have the first term $\frac{p-1}{2}2^{1-(s-1)}$ in the sum

$$\sum_{i=s-1}^{\frac{s(s-1)}{2}} N_i(B_s)2^{1-i}$$

less than 1. This implies that $p < 2^{(s-1)} + 1$. The following table gives values of $N_i(B_s)/2^{i-(s-1)}$ for $s = 4, 5, 6, (s-1) \leq i \leq \frac{s(s-1)}{2}$ and p a prime less than $2^{s-1} + 1$.

$N_i(B_s)/2^{i-(s-1)}, s-1 \leq i \leq s(s-1)/2$					
s	p	$i=3$	$i=4$	$i=5$	$i=6$
4	7	1	0	0	0

$N_i(B_s)/2^{i-(s-1)}, s-1 \leq i \leq s(s-1)/2$								
s	p	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$
5	11	5	0	0	0	0	0	0
	13	6	0	0	0	0	0	0
	17	8	0	0	0	0	0	0

$N_i(B_s)/2^{i-(s-1)}, s-1 \leq i \leq s(s-1)/2$												
s	p	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$	$i=10$	$i=11$	$i=12$	$i=13$	$i=14$	$i=15$
6	17	8	0	0	0	0	0	0	0	0	0	0
	19	9	0	0	0	0	0	0	0	0	0	0
	23	11	0	2.75	0	0	0	0	0	0	0	0
	29	14	0	14	7	0	0	0	0	0	0	0
	31	15	0	18.75	11.25	0	$\frac{3}{32}$	0	0	0	0	0

*Note that in the above table $N_i(B_s)$ is a multiple of $\frac{p-1}{2}$ whenever $N_i(B_s) \geq \frac{p-1}{2}$.

We conclude this section with the following open problem.

Problem: Prove or disprove, the following statement

If p is an odd prime such that $p < 2^{s-1} + 1$, then we have

$$N_i(B_s)/2^{i-(s-1)} = O(p)$$

for all i such that $s \leq i \leq \frac{s(s-1)}{2}$.

2.7 Generalization of Spencer's Lower Bounds.

In this section we give non-constructive lower bounds on $R_k(s; r)$ and $R_k(s_1, s_2, \dots, s_r)$. All results of sections 2.2 and 2.5 follow by taking $r = k = 2$. The proofs below follow fairly closely those given by Spencer (1975) for the cases $r = k = 2$.

Consider a random r -coloring of $[n]^k$ in which each set A in $[n]^k$ is colored i with probability p_i , $1 \leq i \leq r$, independently of the color of any other A' in $[n]^k$, where $\sum_{i=1}^r p_i = 1$. We denote by $\mathcal{P}_n^k(p_1, p_2, \dots, p_r)$ the probability space of all randomly r -colored $[n]^k$.

Theorem 2.7.1. If

$$\binom{n}{s} r^{1-\binom{s}{k}} < 1,$$

then $R_k(s; r) > n$.

Proof. Consider a random r -coloring of $[n]^k$ in $\mathcal{P}_n^k(\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r})$. Let $S \subseteq \{1, \dots, n\}$ such that $|S| = s$. Define X_S as follows

$$X_S = \begin{cases} 1 & \text{if } [S]^k \text{ is monochromatic.} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}
 E(X_S) &= P(X_S = 1) \\
 &= \sum_{i=1}^r P([S]^k \text{ is colored } i) \\
 &= \sum_{i=1}^r \left(\frac{1}{r}\right)^{\binom{s}{k}} \\
 &= r^{1-\binom{s}{k}}.
 \end{aligned}$$

Let $X = (\Sigma X_S : S, |S| = s)$. Then X counts the number of monochromatic $[s]^k$ in our r -coloring of $[n]^k$. Now by Markov's inequality

$$\begin{aligned}
 P(X \geq 1) &\leq E(X) = (\Sigma(X_S) : S, |S| = s) \\
 &= \binom{n}{s} r^{1-\binom{s}{k}}.
 \end{aligned}$$

Thus if $\binom{n}{s} r^{1-\binom{s}{k}} < 1$, then $P(X = 0) > 0$. Hence there exists a particular r -coloring of $[n]^k$ which yields no monochromatic $[s]^k$. Hence

$$R_k(s; r) > n.$$

Corollary 2.7.1.

$$R_k(s; r) \geq \frac{s}{e} r^{\frac{1}{s} \binom{s}{k}} \left(\frac{2\pi s}{r^2} \right)^{\frac{1}{2s}}. \quad (2.7.1)$$

Proof. Suppose $n < \frac{s}{e} r^{\frac{1}{s} \binom{s}{k}} \left(\frac{2\pi s}{r^2} \right)^{\frac{1}{2s}}$. Then

$$\binom{n}{s} r^{1-\binom{s}{k}} \leq \frac{1}{\sqrt{2\pi s}} \left(\frac{ne}{s} \right)^s r^{1-\binom{s}{k}} < 1$$

for our choice of n . Thus the condition of Theorem 2.7.1 holds and hence the Corollary follows.

Note that for $r = k = 2$ we have $R(s) := R_2(s; 2) \geq \frac{s2^{s/2}}{e\sqrt{2}}$, which is the Erdős lower bound on $R(s)$.

We now make use of the independence of some events in a collection of events $\{A_S\}$ to improve the lower bound given by 2.7.1. For this we use a theorem due to Erdős and Lovász (1975) which is known as the Lovász Local Theorem.

Lemma 2.7.1. (Lovász Local Theorem). Let G be a finite graph with maximal degree d and vertices $1, \dots, m$. Let $A_i, 1 \leq i \leq m$ be events in a probability space such that A_i is independent of $\{A_j : (i, j) \notin E(G)\}$. Assume $P(A_j) \leq p$ for $1 \leq j \leq m$. If $4dp < 1$ then

$$P(\overline{A}_1, \dots, \overline{A}_m) > 0.$$

Theorem 2.7.2. If

$$4 \binom{s}{k} \binom{n}{s-k} r^{1-\binom{s}{k}} < 1$$

then $R_k(s; r) > n$.

Proof. In $\mathcal{P}_n^k(\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r})$, and for each $S \subseteq \{1, \dots, n\}, |S| = s$, define A_S to be the event that $[S]^k$ is monochromatic. Then $P(A_S) = r^{1-\binom{s}{k}}$. A_S is independent of $\{A_T : |S \cap T| \leq k-1\}$, since S and T share no k -set. By applying the Lovász Local Theorem with

$$d = |\{T : |T| = s, |S \cap T| \geq k\}| \leq \binom{s}{k} \binom{n}{s-k}$$

and $p = r^{1-\binom{s}{k}}$ we have.

$4 \binom{s}{k} \binom{n}{s-k} r^{1-\binom{s}{k}} < 1$ implies $4dp < 1$ which implies

$$P \left(\bigcap_S \overline{A}_S : S \subseteq \{1, \dots, n\}, |S| = s \right) > 0.$$

The last inequality says that there exists a particular r -coloring of $[n]^k$ with no S such that $|S| = s$ and $[S]^k$ is monochromatic. Hence $R_k(s; r) > n$.

The above theorem generalizes Theorem 2.3.1 of Spencer. A simple application of Stirling's Formula gives the following

Corollary 2.7.2. For fixed r and k and as $s \rightarrow \infty$.

$$R_k(s; r) \geq \frac{s}{e} r^{\frac{1}{s-k}} \binom{s}{k} \{1 + o(1)\}.$$

Note that the lower bound obtained by Corollary 2.7.2, improves the lower bound obtained by Corollary 2.7.1 by a factor of $r^{\frac{k}{(s-k)s}} \binom{s}{k}$.

We now give an implicit lower bound on $R_k(s_1, s_2, \dots, s_r)$ and use the result to derive a lower bound in the case when $s_1 = s_2 = \dots = s_{r-1} = s$ where s is fixed and as $s_r \rightarrow \infty$. We always assume $k \leq s_i$ for all i . The following Theorem is a simple generalization of Theorem 2.3.2.

Theorem 2.7.3. For given $n, r, k, s_1, s_2, \dots, s_r$ if there exist $p_1, p_2, \dots, p_r$ where $0 \leq p_i \leq 1$ $i = 1, \dots, r$ and $\sum_{i=1}^r p_i = 1$, such that

$$\binom{n}{s_1} p_1^{\binom{s_1}{k}} + \binom{n}{s_2} p_2^{\binom{s_2}{k}} + \dots + \binom{n}{s_r} p_r^{\binom{s_r}{k}} < 1 \quad (2.7.2)$$

then $R_k(s_1, s_2, \dots, s_r) > n$.

Proof. In $\mathcal{P}_n^k(p_1, p_2, \dots, p_r)$, for each $S_i \subseteq \{1, \dots, n\}$ with $|S_i| = s_i$, define X_{S_i} as follows

$$X_{S_i} = \begin{cases} 1 & \text{if } [S_i]^k \text{ is colored } i. \\ 0 & \text{otherwise.} \end{cases} \quad i = 1, \dots, r.$$

Then $E(X_{S_i}) = p_i \binom{n}{s_i}^{-k}$. Let $X_i = (\sum X_{S_i} : S_i \text{ such that } |S_i| = s_i)$, and $X = \sum_{i=1}^r X_i$. Then X counts the number of sets S with $|S| = s_i$ so that $[S]^k$ is colored i for some i in $\{1, \dots, r\}$. Now we have

$$\begin{aligned} E(X) &= E\left(\sum_{i=1}^r (\sum X_{S_i} : S_i \text{ such that } |S_i| = s_i)\right) \\ &= \sum_{i=1}^r \binom{n}{s_i} p_i \binom{n}{s_i}^{-k}. \end{aligned}$$

Under the hypotheses of the theorem we have

$$P(X \geq 1) \leq E(X) < 1.$$

Hence $P(X = 0) > 0$ which implies the existence of an r -coloring of $[n]^k$ with no i and $S, |S| = s_i$, such that $[S]^k$ is colored i . Hence $R_k(s_1, s_2, \dots, s_r) > n$.

Corollary 2.7.3. For fixed $s_1, s_2, \dots, s_{r-1}$ and sufficiently large s_r , if

$$\log n - \frac{1}{s_r} \binom{s_r}{k} \sum_{i=1}^{r-1} \left(\frac{1}{r}\right)^{\frac{1}{\binom{n}{s_i}^{-k}}} \left(\frac{s_i}{n\epsilon}\right)^{\frac{s_i}{\binom{n}{s_i}^{-k}}} \leq 0$$

then $R_k(s_1, \dots, s_r) > n$.

Proof. For $i = 1, \dots, r-1$ choose $p_i = \left(\frac{1}{r}\right)^{\frac{1}{\binom{n}{s_i}^{-k}}} \left(\frac{s_i}{n\epsilon}\right)^{\frac{s_i}{\binom{n}{s_i}^{-k}}}$. Then we have

$$\binom{n}{s_i} p_i \binom{n}{s_i}^{-k} < \left(\frac{n\epsilon}{s_i}\right)^{s_i} p_i \binom{n}{s_i}^{-k} = \frac{1}{r} \quad \text{for } 1 \leq i \leq r-1.$$

Also since $s_1, s_2, \dots, s_{r-1}$ are fixed and $n \geq s_r$, for sufficiently large s_r , we have $\sum_{i=1}^{r-1} p_i < 1$.

Using the inequality $(1-x) \leq e^{-x}$ we have

$$\begin{aligned} \binom{n}{s_r} p_r^{\binom{s_r}{k}} &= \binom{n}{s_r} \left(1 - \sum_{i=1}^{r-1} p_i \right)^{\binom{s_r}{k}} \\ &\leq \frac{\left\{ n e^{-\frac{1}{s_r} \binom{s_r}{k} \sum_{i=1}^{r-1} p_i} \right\}^{s_r}}{s_r!} \\ &< \frac{1}{r} \end{aligned}$$

if the conditions of the corollary are satisfied. Thus $\sum_{i=1}^r \binom{n}{s_i} p_i^{\binom{s_i}{k}} < 1$ which implies that $R_k(s_1, \dots, s_r) > n$ by Theorem 2.7.3.

Corollary 2.7.4. If $s_1 = s_2 = \dots = s_{r-1} = s$ where s is fixed. Then as $s_r \rightarrow \infty$

$$R_k(s, s, \dots, s, s_r) \geq A(s_r) \frac{1}{B} - \alpha(s_r)$$

where

$$\begin{aligned} A(s_r) &= \frac{1}{s_r} \binom{s_r}{k} (r-1) \left(\frac{1}{r} \right)^{\frac{1}{\binom{s_r}{k}}} \left(\frac{s}{e} \right)^{\frac{1}{\binom{s_r}{k}}} \\ B &= \frac{s}{\binom{s}{k}} \end{aligned}$$

and

$$\alpha(s_r) = \frac{\log \log A(s_r) - \log B}{B \log A(s_r)} \rightarrow 0 \text{ as } s_r \rightarrow \infty.$$

Proof. Under the above assumptions and with the same definitions in the Corollary for $A(s_r)$, B and $\alpha(s_r)$ and by Corollary 2.7.3, for sufficiently large s_r if

$$\log n - A(s_r) n^{-B} \leq 0$$

then $R_k(s, s, \dots, s, s_r) > n$. Let $n = A(s_r)^{\frac{1}{B} - \alpha(s_r)}$, then

$$\begin{aligned} \log n - A(s_r)n^{-B} &= \left(\frac{1}{B} - \alpha(s_r) \right) \log A(s_r) - A(s_r) \{ A(s_r)^{-1+B\alpha(s_r)} \} \\ &\leq \frac{\log A(s_r)}{B} - A(s_r)^{B\alpha(s_r)} \\ &= 0 \end{aligned}$$

for our choice of $\alpha(s_r)$. Hence the corollary follows.

The following is a concise formulation of Corollary 2.7.4: For fixed s, r, k there exist a constant $C(s, r, k)$ and $\alpha(s_r)$ where $\alpha(s_r) \rightarrow 0$ as $s_r \rightarrow \infty$ such that

$$R_k(s, s, \dots, s, s_r) \geq C(s, r, k) s_r^{\frac{(k-1)}{r} \binom{s}{k} - \alpha(s_r)}. \quad (2.7.3)$$

Setting $r = k = 2$ in Corollary 2.7.4, it can be easily verified that Corollary 2.3.2 follows.

We now use a generalization of Lovász Local Theorem to improve Theorem 2.7.3 by weakening its condition. This generalization was obtained by Spencer (1975).

Lemma 2.7.2. Let G be a finite graph on vertices $1, \dots, m$. Let $A_i, 1 \leq i \leq m$, be events in a probability space such that A_i is independent of $\{A_j : \{i, j\} \notin E(G)\}$. For $1 \leq i \leq m$ assume

$$\sum_{\{i, j\} \in G} P(A_j) < \frac{1}{4}$$

Assume further, that $P(A_j) < 1$ for all j . Then

$$P(\bar{A}_1 \dots \bar{A}_m) > 0.$$

For $A \subseteq \{1, \dots, n\}$ and $|A| = a$. Let

$$f(a, b, n) = |\{B \subseteq \{1, \dots, n\}, |B| = b, B \neq A \text{ and } |A \cap B| \geq k\}|.$$

Clearly $f(a, b, n) \leq f(a', b, n)$ if $a \leq a'$. The next theorem uses this fact to asymptotically weaken the condition (2.7.2) of Theorem 2.7.3.

Theorem 2.7.4. Let $s_1 \leq s_2 \leq \dots \leq s_{r-1} \leq s_r \leq n$. If there exist $p_i, 0 < p_i < 1$ $i = 1, \dots, r$ and $\sum_{i=1}^r p_i = 1$, such that

$$f(s_r, s_1, n)p_1^{\binom{s_1}{k}} + f(s_r, s_2, n)p_2^{\binom{s_2}{k}} + \dots + f(s_r, s_r, n)p_r^{\binom{s_r}{k}} < \frac{1}{4} \quad (2.7.4)$$

then $R_k(s_1, \dots, s_r) > n$.

Proof. Let $s_1, \dots, s_r, p_1, \dots, p_r, k, n$ satisfy 2.7.4. In $\mathcal{P}_n^k(p_1, \dots, p_r)$ if $S_i \subseteq \{1, \dots, n\}$ and $|S_i| = s_i$, let A_{S_i} be the event $\{|S_i|^k \text{ is colored } i\}$. Then $P(A_{S_i}) = p_i^{\binom{s_i}{k}}$ $1 \leq i \leq r$. Define a graph $G = (V(G), E(G))$ as follows

$$V(G) = \bigcup_{i=1}^r \{S_i : |S_i| = s_i\}$$

$$E(G) = \{(S, T) : S, T \in V(G), S \neq T \text{ and } |S \cap T| \geq k\}.$$

For $S \in \{S_r : |S_r| = s_r\}$ we have

$$\sum_{\{S, S_i\} \in G} P(A_{S_i}) = \sum_{i=1}^r f(s_r, s_i, n)p_i^{\binom{s_i}{k}} < \frac{1}{4} \quad \text{by (2.7.4).}$$

For each $i, 1 \leq i \leq r-1$ and any $S' \in \{S_i : |S_i| = s_i\}$ we have

$$\sum_{\{S', S_j\} \in G} P(A_{S_j}) = \sum_{j=1}^r f(s_i, s_j, n)p_j^{\binom{s_j}{k}} \leq \sum_{j=1}^r f(s_r, s_j, n)p_j^{\binom{s_j}{k}} < \frac{1}{4}$$

since for $s_r \geq s_i$, $f(s_r, s_j, n) \geq f(s_i, s_j, n)$ for all j .

Hence the conditions of Lemma 2.7.2 are satisfied. Consequently

$$P\left(\bigcap_{i=1}^r \left\{ \bigcap_{S_i} \overline{A}_{S_i} : |S_i| = s_i \right\}\right) > 0.$$

This implies $R_k(s_1, s_2, \dots, s_r) > n$.

Corollary 2.7.5. For fixed $s_1, s_2, \dots, s_{r-1}$ and sufficiently large s_r , if

$$\log n - \frac{\binom{s_r}{k}}{s_r} \sum_{i=1}^{r-1} \left\{ \frac{1}{4r \binom{s_r}{k}} \right\}^{\frac{1}{\binom{s_i}{k}}} \left\{ \frac{s_i - k}{ne} \right\}^{\frac{s_i - k}{\binom{s_i}{k}}} \leq 0$$

then $R_k(s_1, \dots, s_r) \geq n$.

Proof. In Theorem 2.7.4, we bound

$$\begin{aligned} f(s_r, s_i, n) &\leq \binom{s_r}{k} \binom{n}{s_i - k} \quad i = 1, \dots, r-1 \\ f(s_r, s_r, n) &\leq \binom{n}{s_r} \end{aligned}$$

Thus if there exist $p_i, 0 < p_i < 1, i = 1, \dots, r$ and $\sum_{i=1}^r p_i = 1$ such that

$$\sum_{i=1}^{r-1} \binom{s_r}{k} \binom{n}{s_i - k} p_i^{\binom{s_i}{k}} + \binom{n}{s_r} p_r^{\binom{s_r}{k}} < \frac{1}{4} \quad (2.7.5)$$

then (2.7.4) holds and hence $R(s_1, \dots, s_r) > n$. For $1 \leq i \leq r-1$ let

$$p_i = \left\{ \frac{1}{4r \binom{s_r}{k}} \right\}^{\frac{1}{\binom{s_i}{k}}} \left\{ \frac{s_i - k}{ne} \right\}^{\frac{s_i - k}{\binom{s_i}{k}}}.$$

Then

$$\sum_{i=1}^{r-1} \binom{s_r}{k} \binom{n}{s_i - k} p_i^{\binom{s_i}{k}} < \sum_{i=1}^{r-1} \binom{s_r}{k} \left(\frac{ne}{s_i - k} \right)^{s_i - k} p_i^{\binom{s_i}{k}} = \frac{r-1}{4r}. \quad (2.7.6)$$

Also since $s_1, s_2, \dots, s_{r-1}$ are fixed for sufficiently large s_r we have $\sum_{i=1}^{r-1} p_i < 1$.

Since $1 - x \leq e^{-x}$ we have,

$$\begin{aligned} \binom{n}{s_r} p_r^{\binom{s_r}{k}} &< \frac{n^{s_r}}{s_r!} \left(1 - \sum_{i=1}^{r-1} p_i \right)^{\binom{s_r}{k}} \\ &= \frac{\left(n e^{-\frac{1}{s_r} \binom{s_r}{k} \sum_{i=1}^{r-1} p_i} \right)^{s_r}}{s_r!} \\ &\leq \frac{1}{4r} \end{aligned} \quad (2.7.7)$$

if the assumptions of the corollary are satisfied. (2.7.6) and (2.7.7) imply (2.7.5) and the corollary is proved.

Corollary 2.7.6. For fixed r, k, s and sufficiently large s_r ,

$$R_k(s, s, \dots, s, s_r) \geq A(s_r)^{\frac{1}{B} - \alpha(s_r)}$$

where

$$A(s_r) = (r-1) \frac{\binom{s_r}{k}}{s_r} \left\{ \frac{1}{4r \binom{s_r}{k}} \right\}^{\frac{1}{\binom{s_r}{k}}} \left\{ \frac{s-k}{e} \right\}^{\frac{s-k}{\binom{s_r}{k}}}$$

$$B = \frac{s-k}{\binom{s}{k}}$$

and

$$\alpha(s_r) = \frac{\log \log A(s_r) - \log B}{B \log A(s_r)} \rightarrow 0 \text{ as } s_r \rightarrow \infty.$$

Proof. Let $s_1 = s_2 = \dots = s_{r-1} = s$. Then for sufficiently large s_r and by Corollary 2.7.5, if

$$\log n - (r-1) \frac{\binom{s_r}{k}}{s_r} \left\{ \frac{1}{4r \binom{s_r}{k}} \right\}^{\frac{1}{\binom{s_r}{k}}} \left\{ \frac{s-k}{ne} \right\}^{\frac{s-k}{\binom{s_r}{k}}} \leq 0 \quad (2.7.8)$$

then $R_k(s, s, \dots, s, s_r) \geq n$. Let $A(s_r), B$ and $\alpha(s_r)$ be as given in the Corollary. Then for $n = A(s_r)^{\frac{1}{B} - \alpha(s_r)}$ one can easily check that (2.7.8) holds and the Corollary follows.

The following is a concise formulation of the above Corollary:

For fixed s, r, k there exist a constant $C'(s, r, k)$ and $\alpha'(s_r)$ where $\lim_{s_r \rightarrow \infty} \alpha'(s_r) = 0$ such that

$$R_k(s, s, \dots, s, s_r) \geq C'(s, r, k) s_r^{\frac{k(k-1)}{s-k} \binom{s}{k} - \frac{k}{s-k} - \alpha'(s_r)} \quad (2.7.9)$$

The lower bound given by (2.7.9) improves that given by (2.7.3) by a factor of $s_r^{\frac{k(k-1)}{s-k} \binom{s}{k} - \frac{k}{s-k} + o(1)}$. Setting $r = k = 2$ in Corollary 2.7.6, one can easily check that Corollary 2.3.3 of Spencer follows.

2.8 Tables for Lower Bounds on $R(s, t)$ Obtained by Non-constructive Methods.

In this section we give tables for lower bounds on $R(s, t), 3 \leq s, t \leq 20$, obtained using the non-constructive methods discussed in previous sections.

Table II (III) gives the lower bounds obtained by finding the largest n that satisfies the conditions of Theorem 2.3.2 (Theorem 2.3.3). Where in Theorem 2.3.3

$$f(t, t, n) = \sum_{i=2}^{t-1} \binom{t}{i} \binom{n-t}{t-i}$$

$$f(t, s, n) = \sum_{i=2}^s \binom{t}{i} \binom{n-t}{s-i}.$$

As we shall see in Chapter 4, these lower bounds are quite poor compared to the best known lower bounds.

Lower Bounds On $R(s, t)$
Obtained by Theorem 2.3.2

Table II

s	t	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
4	4	4	6	7	8	9	11	12	13	15	16	17	18	20	21	22	24	25	26
4			7	9	11	13	15	17	19	21	24	26	28	30	33	35	37	40	42
5				12	14	17	20	24	27	31	34	38	42	46	51	55	60	64	69
6					18	22	27	32	37	43	49	55	62	69	76	84	92	101	109
7						28	34	42	49	58	67	77	87	99	111	124	137	151	166
8							43	53	64	76	89	104	120	137	156	176	197	220	245
9								66	81	98	117	137	161	186	214	244	277	312	350
10									101	123	149	178	210	246	286	330	378	431	488
11										153	187	226	271	321	376	438	507	583	666
12											232	284	343	410	486	572	668	775	893
13												350	428	517	619	735	866	1012	1197
14													528	644	778	931	1106	1305	1528
15														793	966	1166	1397	1660	1959
16															1187	1445	1744	2088	2483
17																1772	2155	2600	3114
18																	2640	3208	3868
19																		3924	4763
20																			5818

Lower Bounds On $R(s, t)$
Obtained by Theorem 2.3.3
Table III

$s \setminus t$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
3	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	20	21	22
4		6	7	9	11	12	14	16	18	20	22	24	26	28	30	32	35	37
5			10	12	15	17	20	23	27	30	34	37	41	45	49	54	58	63
6				15	19	23	28	33	38	44	50	56	63	71	78	86	95	104
7					25	31	37	45	53	62	72	83	94	106	120	134	149	165
8						40	49	60	72	86	101	117	135	155	177	200	225	252
9							63	76	96	115	137	162	189	220	253	290	330	373
10								100	124	151	183	218	258	303	353	408	469	537
11									158	196	239	288	345	409	481	562	652	752
12										248	307	374	452	541	643	757	886	1031
13											388	478	583	704	843	1002	1183	1386
14												602	740	902	1090	1306	1553	1834
15													929	1141	1389	1677	2010	2391
16														1427	1750	2128	2568	3077
17															2189	2671	3245	3913
18																3318	4058	4925
19																	5028	6140
20																		7590

- No lower bound was provided.

CHAPTER 3

CONSTRUCTIVE LOWER BOUNDS FOR THE RAMSEY NUMBERS

3.1 Introduction

As we have seen in Chapter 2, P. Erdős used some counting arguments to show that $R(s) \geq \frac{s2^{\frac{s-1}{2}}}{e} \{1 + o(1)\}$, and that this bound was improved by J. Spencer by a factor of two. Erdős (1969) has asked if one can find, by explicit construction a graph G with $2^{s/2}$ vertices such that neither G nor its complement $\overline{G}$ contains a complete subgraph of order s . He also pointed out that he was not able to give a constructive proof of the much weaker result:

$$R(s) \geq cs^2$$

where c is a constant. As put by F.R.K. Chung (1981) "This problem, however, falls into an interesting category of problems which have the property that for any large n the existence of a "good configuration" is assured by probabilistic methods, (in fact most of the configurations are good), but we cannot explicitly find even one "good configuration". In this chapter we give different constructive lower bounds on $R(s)$. None of these lower bounds attains the order of the Erdős lower bound obtained by non-constructive methods. In fact for any $\epsilon > 0$, the problem of finding a graph with $n > (1 + \epsilon)^s$ vertices such that neither it nor its complement contain a complete subgraph of order s remains open.

H. Abbott (1972) showed by recursive construction that $R(s) > cs^{2.322}$ for some constant c . We use his construction together with a result due to Mathon

(1957), to show that $R(s) > cs^{3.203}$, where c is a constant. These results are given in section 3.2.

In section 3.3, we give a non-recursive construction that shows $R(s) > cs^3$. The same result was obtained by Nagy (1975), but our approach is quite different.

Frankl (1977) showed by construction that $R(s) \geq cs^k$ for any k and some constant c which depends on k . Chung (1981) improved Frankl's lower bound by showing that $R(s) \geq e^{h(s)}$ where $h(s) = c(\log s)^{4/3}/(\log \log s)^{1/3}$. The best known constructive lower bound on $R(s)$ was obtained by Frankl and Wilson (1981). They were able to show that $R(s) \geq e^{f(s)}$, where $f(s) = \frac{(\log s)^2}{(4 \log \log s)}[1 + o(1)]$. All three constructions are given in section 3.4.

In section 3.5 we show that $R(2 \log_2 s, s) \geq cs^2$, where c is a constant.

3.2 Abbott's Constructive Lower Bound.

Abbott (1972) used a recursive construction which shows that $R(s) \geq cs^{c'}$, where c is a constant and $c' = \log 5 / \log 2 \approx 2.322$. We use a result by Mathon R. (1987), together with Abbott's construction to show that $R(s) \geq cs^{c'}$, where c is a constant and $c' = \log 1596 / \log 10 \approx 3.203$.

Let K_p and K_q be two complete graphs with vertex sets $\{u_1, u_2, \dots, u_p\}$ and $\{v_1, v_2, \dots, v_q\}$ respectively. Suppose that the edges of K_p and K_q are colored with r colors $c_1, c_2, \dots, c_r$. Let K_{pq} be a complete graph with vertices w_{ij} , $i = 1, \dots, p$ and $j = 1, \dots, q$. Color the edges of K_{pq} as follows:

Let e be the edge joining w_{ij} and $w_{\ell m}$. If $i = \ell$, color e the same color as the edge joining u_j and u_m in K_q . If $i \neq \ell$ color e the same color as the edge joining v_i and v_ℓ in K_p .

Using the above construction Abbott was able to prove that if K_p and K_q do not contain monochromatic K_s and K_t respectively, then K_{pq} does not contain a monochromatic K_ℓ where $\ell = st - s - t + 2$. In other words we have the following

Lemma 3.2.1 If $p \not\rightarrow (s)_r$ and $q \not\rightarrow (t)_r$, then one can show constructively that

$$pq \not\rightarrow (st - s - t + 2)_r.$$

Consider the case $r = 2$. By coloring one hamiltonian cycle in K_5 blue and the other hamiltonian cycle in K_5 red, it is clear that $5 \not\rightarrow (3)_2$. Thus by recursive application of Lemma 3.2.1, one can easily show that for all integers ℓ ,

$$5^\ell \not\rightarrow (2^\ell + 1)_2.$$

Consequently for $n \geq 5$ and ℓ such that $5^{\ell-1} < n \leq 5^\ell$, $n \not\rightarrow (2^\ell + 1)_2$. This implies the following

Theorem 3.2.1. Constructively one can show that

$$R(s) \geq cs^{c'}$$

where c is a constant and $c' = \log 5 / \log 2 \approx 2.322$.

Mathon (1987) constructed a 2-coloring for K_{1596} with no monochromatic K_{11} (see Chapter 4). Thus we have $1596 \not\rightarrow (11)_2$. By recursive application of Lemma 3.2.1, one can easily show that for all integers ℓ

$$(1596)^\ell \not\rightarrow (10^\ell + 1)_2.$$

Consequently for $n \geq 1596$ and ℓ such that $(1596)^{\ell-1} < n \leq (1596)^\ell$, we have $n \not\rightarrow (10^\ell + 1)_2$. Thus we obtained the following

Theorem 3.2.2 Constructively one can show that

$$R(s) \geq cs^{c'}$$

where c is a constant, and $c' = \log 1596 / \log 10 \approx 3.203$.

3.3 A Lower Bound on $R(s)$ of Order s^3 .

In this section we give constructively a lower bound on $R(s)$ of order s^3 . The same result was proved by Nagy (1975), however our approach is quite different and appears to be new.

Let $Z_2^{(s)}$ be the vector space of all s -tuples over Z_2 . For any two vectors

$$v = (v_1, v_2, \dots, v_s) \text{ and } u = (u_1, u_2, \dots, u_s)$$

in $Z_2^{(s)}$ we define

$$v \cdot u = \sum_{i=1}^s v_i u_i$$

in the field Z_2 .

A vector v in $Z_2^{(s)}$ is of length ℓ if it has exactly ℓ nonzero entries. Let V_3 be the set of all vectors in $Z_2^{(s)}$ of length three. Clearly $|V_3| = \binom{s}{3}$. Define a graph $G(V_3)$ on V_3 as follows: The vertex set of $G(V_3)$ is $V(G) = V_3$. For any u, v in $V(G)$, (u, v) is an edge in $G(V_3)$ if and only if $v \cdot u = 0$. Then we have the following

Theorem 3.3.1. Let β be the clique number of $G(V_3)$. Then $s - 2 \leq \beta \leq s$.

Proof. We first show that $s - 2 \leq \beta$. Consider the $s - 2$ vectors

$$\begin{aligned} v_1 &= (1110 \dots 0) \\ v_2 &= (11010 \dots 0) \\ &\vdots \\ v_{s-2} &= (110 \dots 01) \end{aligned}$$

Clearly $(v_i \cdot v_j) = 0$ for all $1 \leq i \neq j \leq s-2$. Hence $\beta \geq s-2$. We now prove that $\beta \leq s$. Let $C = \{u_1, u_2, \dots, u_k\}$ be any clique in $G(V_3)$. Then $(u_i \cdot u_j) = 0$ for all $i \neq j$, and $(u_i \cdot u_i) = 1$, since the length of u_i is three. Consider a linear relation

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k = 0.$$

For any $i, 1 \leq i \leq k$ we have

$$\alpha_1(u_1 \cdot u_i) + \alpha_2(u_2 \cdot u_i) + \dots + \alpha_i(u_i \cdot u_i) + \dots + \alpha_k(u_k \cdot u_i) = 0.$$

This implies that $\alpha_i = 0$. Thus $u_1, u_2, \dots, u_k$ are linearly independent. Hence $k \leq s$. In particular we have $\beta \leq s$.

The next lemma gives a lower bound on the independence number of $G(V_3)$.

Lemma 3.3.1. Let α be the independence number of $G(V_3)$. Then,

$$\alpha \geq \lfloor \frac{s-1}{2} \rfloor.$$

Proof. Let $I = \{v_i\}$ where v_i is the vector in V_3 with one in its first, $(2i)^{th}$ and $(2i+1)^{th}$ components and zero elsewhere $i = 1, 2, \dots, \lfloor \frac{s-1}{2} \rfloor$. Clearly $v_i \cdot v_j = 1$ for all i, j . Thus $\alpha \geq \lfloor \frac{s-1}{2} \rfloor$.

Theorem 3.3.2. Let α be the independence number of $G(V_3)$. If $\lfloor \frac{s-1}{2} \rfloor > 7$, then $\alpha = \lfloor \frac{s-1}{2} \rfloor$.

Proof. Suppose $\lfloor \frac{s-1}{2} \rfloor > 7$. Then by Lemma 3.3.1 we have $\alpha > 7$. Let $I = (v_1, v_2, \dots, v_\alpha)$ be a maximum cardinality independent set. We can assume that $v_1 = (11100\dots 0)$. Since $\alpha > 7$ and $v_i v_j = 1$ for all $j = 2, \dots, \alpha$, there are at least three vectors in I not equal to v_1 with i^{th} component one, for some

$i = 1, 2$, or 3 . We can assume $i = 1$ and $v_2 = (100110 \dots 0)$, $v_3 = (10000110 \dots 0)$, $v_4 = (1000000110 \dots 0)$. If v_5 has zero in its first component then it must share a nonzero component with each of v_1, v_2, v_3, v_4 , which is not possible, since the length of v_5 is three. Thus v_5 can be forced to be taken $v_5 = (100000000110 \dots 0)$. Similarly each $v_i, i \geq 6$ can be forced to be taken as the vector in V_3 with one in its first, $(2i)^{th}$ and $(2i + 1)^{th}$ components and zero elsewhere. Consequently $\alpha \leq \lfloor \frac{s-1}{2} \rfloor$.

Theorem 3.3.3. One can show constructively that for $s > 16$

$$R(s+1) > 2 \binom{s}{3}$$

Proof. Let G be the graph consisting of two components, each component being $G(V_3)$. Then by Theorem 3.3.1, the clique number of G is at most s , and by Theorem 3.3.2, the independence number of G is $2 \lfloor \frac{s-1}{2} \rfloor$. And since $|V(G)| = 2 \binom{s}{3}$, the Theorem follows.

The following table shows that the lower bound on β in Theorem 3.3.1 can be attained, and the condition $\lfloor \frac{s-1}{2} \rfloor > 7$ in Theorem 3.3.2 is necessary.

Table IV. The Clique and Independence Numbers of $G(V_3)$
For Different Values of s .

s	$\binom{s}{3}$	β	α
6	20	4	4
7	35	5	7
8	56	8	7
9	84	8	7
10	120	8	7
11	165	9	7
12	220	12	7

3.4 Best Known Constructive Lower Bounds For $R(s)$.

In this section we outline the ideas behind the best known constructive lower bounds for $R(s)$. The basic idea for these bounds is due to Frankl (1977), who was able to show, by explicit construction, that $R(s) \geq c_k s^k$ for any integer k and constant c_k depending on k . We first give his construction in general:

For any integers p, n, x we construct a graph $G = G(p, n, x)$, such that the vertex set $V(G)$ consists of all p -subsets of $\{1, 2, \dots, n\}$ and for $v_1, v_2 \in V(G)$, v_1 is adjacent to v_2 if and only if $|v_1 \cap v_2| \in \{2ax + b : 0 \leq a, b \leq x - 1\}$.

For any integer k , Frankl (1977) showed that neither $G(2k^2, n, k)$ nor its complement contains a complete subgraph of order $s = d_k n^{2k}$, where d_k is a constant depending only on k . Since $|V(G(2k^2, n, k))| = \binom{n}{2k^2}$, the constructive lower bound $R(s) > c_k s^k$ was obtained.

By tightening up some loose ends in Frankl's results, Chung (1981) was able to show that if we choose

$$\begin{aligned} n &= \lceil \exp[(\log s)^{2/3} / (3 \log \log s)^{1/3}] \rceil \\ x &= \frac{(3 \log s)^{1/3}}{4(\log \log s)^{1/3}} \\ p &= 2x^2, \end{aligned}$$

then neither $G(p, n, x)$ nor its complement contains any complete subgraph on s vertices. Thus the lower bound $R(s) \geq \exp[c(\log s)^{4/3} / (\log \log s)^{1/3}]$ was obtained.

The last constructive lower bound we give in this section was obtained by Frankl and Wilson (1981). Their construction is as follows:

Let $V(G) = \{F \subseteq \{1, 2, \dots, n\} : |F| = q^2 - 1\}$, q is a prime power, and $E(G) = \{(F, F') : |F \cap F'| \not\equiv -1 \pmod{q}\}$. Then neither G nor its complement contains a complete subgraph with more than $\binom{n}{q-1}$ vertices. By setting $n = p^3$, $q = p$, they were able to show that

$$R(s) \geq \exp[(1 + o(1))(\log s)^2 / 4 \log \log s].$$

3.5 Constructive Lower Bound for $R(2 \log_2 s, s)$.

In this section we give a new construction similar to that described in Section 3.3 to obtain a lower bound on $R(2 \log_2 s, s)$, of order s^2 . We construct a graph

G of order 2^{s-1} and show that the clique number of G is at most s and the independence number of G is at most $2^{\lfloor s/2 \rfloor}$.

With the same notations as in Section 3.3, let V_{odd} be the set of all vectors in $Z_2^{(s)}$ of odd length (that is each vector in V_{odd} has an odd number of ones). Clearly $|V_{\text{odd}}| = 2^{s-1}$. Define a graph $G(V_{\text{odd}})$ with vertex set $V(G) = V_{\text{odd}}$ and for any two vertices u, v in $V(G)$, (u, v) is an edge of $G(V_{\text{odd}})$ if and only if $(v, u) = 0$. Then we have the following

Theorem 3.5.1. Let β be the clique number of $G(V_{\text{odd}})$. Then $\beta = s$.

Proof. For $1 \leq i \leq s$, the vectors e_i with one in the i^{th} component and zero elsewhere form a clique in $G(V_{\text{odd}})$. Thus $\beta \geq s$.

Let $C = (u_1, u_2, \dots, u_k)$ be any clique in $G(V_{\text{odd}})$. Suppose we have a linear combination $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k = 0$. Since u_i has odd length, taking the dot product of both sides of this relation with u_i shows that $\alpha_i = 0$ for each i . Thus $u_1, u_2, \dots, u_k$ are linearly independent. Consequently $k \leq s$. In particular we have $\beta \leq s$. Thus we have $\beta = s$.

We next show that the independence number of $G(V_{\text{odd}})$ is at most $2^{\lfloor s/2 \rfloor}$.

Theorem 3.5.2. Let α be the independence number of $G(V_{\text{odd}})$. Then $\alpha \leq 2^{\lfloor s/2 \rfloor}$.

Proof. Suppose that $S = (v_1, v_2, \dots, v_m)$ is a maximum cardinality independent set in $G(V_{\text{odd}})$. Then for any i, j in $\{1, \dots, m\}$ we have $v_i \cdot v_j = 1$. Consider the map $w : v_i \rightarrow u_i = v_i + v_1$. Clearly w is injective on S . Let $S' = \psi(S)$, then we have $u_i \cdot u_j = 0$ for all i, j including $i = j$. Since S is maximal it is easy to show that S' forms a subspace of $Z_2^{(s)}$. Let $\{w_1, w_2, \dots, w_k\}$ be a basis for S' . Form the

matrix $A_{s \times k} = \begin{bmatrix} & \vdots & & \vdots & & \vdots \\ w_1 & & w_2 & & \dots & w_k \\ & \vdots & & \vdots & & \vdots \end{bmatrix}$ whose columns are $w_1, w_2, \dots, w_k$.

Since $w_1, w_2, \dots, w_k$ are linearly independent we have $\text{rank}(A) = \text{rank}(A^t) = k$.

Let $T_A : Z_2^{(k)} \rightarrow Z_2^{(s)}$ and $T_{A^t} : Z_2^{(s)} \rightarrow Z_2^{(k)}$, be two linear transformations defined by $T_A(u) = Au, T_{A^t}(v) = A^t v$ for $u \in Z_2^{(k)}$ and $v \in Z_2^{(s)}$. Since $w_i \in S'$ for all $i = 1, \dots, k$ we have $A^t A = 0$. This implies that $\text{Im} T_A \subseteq \text{Ker } T_{A^t}$, where $\text{Im} T_A = \{Au : u \in Z_2^{(k)}\}$ and $\text{Ker } T_{A^t} = \{v \in Z_2^{(s)} : A^t v = 0\}$. Consequently we have $\dim(\text{Im} T_A) \leq \dim(\text{Ker } T_{A^t})$, which implies $k \leq s - k$. Thus $k \leq \lfloor \frac{s}{2} \rfloor$. Hence $m = |S| = |S'| \leq 2^{\lfloor s/2 \rfloor}$.

The following theorem is an immediate result of the above two theorems.

Theorem 3.5.3. Constructively one can show that for some constant c .

$$R(2 \log_2 s, s) \geq cs^2$$

CHAPTER 4

BEST KNOWN LOWER BOUNDS FOR SMALL RAMSEY NUMBERS

4.1 Introduction

In this chapter we give some constructive methods to obtain good lower bounds for small Ramsey numbers. These methods are based on some symmetry assumptions for the Ramsey graphs. The most frequently used method is due to Greenwood and Gleason (1955).

For given s and t , to show that $R(s, t) > n$ we need to find a graph G with clique and independence numbers $s - 1$ and $t - 1$ respectively. Instead of searching for (s, t) -graphs in the set of all graphs, Greenwood and Gleason restricted their search to the set of cyclic graphs. As we shall see later they were able to find some good lower bounds for $R(s, t)$, and were able to determine some small Ramsey numbers.

In Chapter 2 we defined cyclic graphs of prime order. In this chapter a cyclic graph can be of any order and is defined as follows

Definition. Let $D \subseteq \{1, 2, \dots, n-1\}$ be such that $i \bmod n \in D$ if and only if $-i \bmod n \in D$. Define the graph $G_n[D] = (V, E)$, where $V = \{0, 1, \dots, n-1\}$ and $(i, j) \in E$ if and only if $(i - j) \bmod n \in D$. Then $G_n[D]$ is called cyclic, n is its order, and D is called the determining set.

Note that since $i \bmod n \in D$ if and only if $-i \bmod n \in D$, $G_n[D]$ is well defined. Denote by D^+ the set $\{i \in D : i \leq \lceil \frac{n}{2} \rceil\}$. Clearly $D = D^+ \cup (-D^+)$

where $-D^+ = \{-i : i \in D^+\}$. We shall sometimes identify $G_n[D]$ by n and D^+ . This saves writing half the elements of D .

Following the notations of Chung and Grinstead (1983), we denote the largest n such that a cyclic (s, t, n) -graph exists by $C(s, t)$. Clearly $R(s, t) \geq C(s, t) + 1$.

Let G_i be a graph on n_i vertices, with clique number $s - 1$ and independence number $t_i - 1, i = 1, 2$. Let G be the graph consisting of two components, the components being $G_i, i = 1, 2$. Then G has $n_1 + n_2$ vertices and clique and independence numbers $s - 1$ and $t_1 + t_2 - 2$ respectively. Consequently we have the following

Theorem 4.1.1. For any s and $t \geq 2$

$$R(s, t) \geq \max_{0 \leq i \leq t-1} [R(s, i+1) + R(s, t-i)] - 1$$

Definition. A lower bound on $R(s, t)$ is called trivial if it is obtained directly from table II or III or by the application of Theorem 4.1.1 to the best known lower bounds on $R(s, i+1)$ and $R(s, t-i)$ $0 \leq i \leq t-1$.

Obviously we have $R(1, t) = 1, R(2, t) = t$ and $R(s, t) = R(t, s)$. Thus in the next sections we shall be concerned with finding lower bounds for $R(s, t)$ for s and t greater than or equal to three.

Most of the lower bounds previously obtained on $R(s, t)$, were obtained using exhaustive search in the set of cyclic graphs. Instead of this we used random search in the set of cyclic graphs of given order and given size of the determining set. Using random search we were able to find new lower bounds for larger Ramsey numbers. These bounds will be given in the next sections together with the best known lower bounds for some other Ramsey numbers.

Clearly each translation x to $x + b$ is an automorphism of any cyclic graph, and so each cyclic graph has a vertex transitive automorphism group. Thus, in a cyclic graph, each vertex belongs to a clique of maximal cardinality (and similarly to an independent set of maximal cardinality). Denote by $H^0(D)$ the graph induced by the neighbours of 0 in $G_n[D]$, (i.e. the graph induced by D). then $\beta[G_n[D]] = \beta[H^0(D)] + 1$, where in general $\beta(H)$ denotes the clique number of H .

A general description for the random search algorithm is given below, and an implementation for the algorithm in Pascal is given in the Appendix.

Algorithm 4.1.1 Random Search for Cyclic Ramsey Graphs.

Input. n = number of vertices. ℓ = size of the determining set D . s, t = integers greater than or equal to 3.

Output. n and D for which $G_n[D]$ is an (s, t, n) -graph.

METHOD.

```

begin
1.  $b \leftarrow s$ ;
2.  $i \leftarrow 1$ ;
3. Generate at random a determining set  $D \subseteq \{1, \dots, n\}$  of size  $\ell$ ;
4. Generate a new maximal clique  $C$  in  $H^0(D)$ ;
5. if  $|C| + 1 \geq b$  then goto step 1;
6. if all maximal cliques in  $H^0(D)$  are generated then
    begin
7.     if  $i = 2$  then

```

```

      begin
8.       $D \leftarrow \{1, \dots, n\} - D;$ 
9.      return  $s, t, n, D$  and " $R(s, t) \geq n + 1$ ";
10.     stop
      end;
11.    if  $i = 1$  then
      begin
12.       $i \leftarrow 2;$ 
13.       $D \leftarrow \{1, \dots, n - 1\} - D;$ 
14.       $b \leftarrow t;$ 
15.      goto step 4
      end
    end;
16. goto step 4
end.

```

In step 3 in Algorithm 4.1.1, the determining set D is randomly generated as follows:

Case 1: $\ell = |D|$ is even.

When n is even, $\frac{n}{2}$ does not belong to D because $D = -D$ (and $\frac{n}{2} = -\frac{n}{2}$) whilst $|D|$ is even. Thus for any n let $m = \lceil \frac{n}{2} - 1 \rceil$ and $a = \ell/2$. Go through $1, 2, \dots, m$ in order and suppose that $\{\pm i_1, \pm i_2, \dots, \pm i_k\} k < a$ are already in D , and we are at stage $j > i_k$. Then insert j and $-j$ in D with probability $\frac{a - k}{m - j + 1}$. One can easily show that the set D generated in this manner has length ℓ and that each D has probability $\binom{m}{a}^{-1}$ of being selected.

Case 2: $\ell = |D|$ is odd

If ℓ is odd then we must have an element $v \neq 0$ in $\mathbb{Z}_n$ such that $v = -v$. This forces n to be even and $\frac{n}{2}$ to be an element of D . Thus let $m = \frac{n}{2} - 1$, and $a = \frac{\ell - 1}{2}$ and generate D' as in Case 1. Then $D = D' \cup \left\{ \frac{n}{2} \right\}$.

The algorithm used in step 4 to generate the maximal cliques in $H^0(D)$ was taken from Bron and Kerbosch (1973). Their algorithm has the advantage that it tends to produce larger cliques first. Thus detecting that the graph is not an (s, t) -graph tends to occur at early stages. This reduces the overall searching time.

In Section 4.2 we give the best known lower bounds for $R(3, t)$, $3 \leq t \leq 19$. For $15 \leq t \leq 19$ the lower bounds were obtained using Algorithm 4.1.1. The best known lower bounds for $R(4, t)$ and $R(5, t)$ for some values of t are given in Sections 4.3 and 4.4 respectively. The last section of this chapter is section 4.5, in which we give the best known lower bounds for the diagonal Ramsey numbers $R(s)$, $6 \leq s \leq 11$.

We conclude this chapter with a table which summarizes the best known lower bound for $R(s, t)$ for some values of s and t .

4.2 Lower Bounds For $R(3, t)$, $3 \leq t \leq 19$.

As shown in Chapter 1, $R(3, 3) = 6$. Most of the lower bounds established in this section are based on searching in the class of all cyclic graphs. Only three of these bounds were obtained by graphs with no evident structure. In Greenwood and Gleason (1955), the values of $R(3, 4)$ and $R(3, 5)$ were obtained and are given in the following

Theorem 4.2.1 (1) $R(3, 4) = 9$ (2) $R(3, 5) = 14$.

Proof. (1) Consider $G_8[D]$ where $D = \{1, 4, 7\}$. Then $G_8[D]$ is a $(3, 4, 8)$ -graph. This shows that $R(3, 4) \geq 9$. Using the inequality $R(s, t) \leq R(s-1, t) + R(s, t-1)$, given in Chapter 1, which holds with strict inequality if both $R(s-1, t)$ and $R(s, t-1)$ are even we get $R(3, 4) < R(2, 4) + R(3, 3) = 4 + 6 = 10$. Thus we have $R(3, 4) = 9$.

(2) Consider $G_{13}[D]$ where $D = \{1, 5, 8, 12\}$. Then $G_{13}[D]$ is a $(3, 5, 13)$ -graph. This proves that $R(3, 5) \geq 14$. We also have $R(3, 5) \leq R(2, 5) + R(3, 4) = 5 + 9 = 14$. Thus we have $R(3, 5) = 14$.

In Kalbfleisch (1965), it is proved that $C(3, 6) = 16$, $C(3, 7) = 21$, and $C(3, 8) = 26$, and that these are attained by the cyclic graphs $G_{16}[D_1]$, $G_{21}[D_2]$ and $G_{26}[D_3]$ respectively, where $D_1 = \{1, 3, 8, 13, 15\}$, $D_2 = \{1, 3, 8, 13, 18, 20\}$ and $D_3 = \{1, 3, 8, 13, 18, 23, 25\}$. This shows that $R(3, 6) \geq 17$, $R(3, 7) \geq 22$ and $R(3, 8) \geq 27$. In Kalbfleisch (1966) it was shown that $R(3, 6) \geq 18$ and $R(3, 7) \geq 23$. The last two lower bounds were obtained independently by Graver and Yackel (1968) in the following two theorems.

Theorem 4.2.2 $R(3, 6) \geq 18$.

Proof. Let $G_{13}[D]$ be the cyclic $(3, 5)$ -graph on 13 points constructed in the proof of Theorem 4.2.1. Adjoin to $G_{13}[D]$ points $0', 1', 8'$ and $9'$ so that the subgraph G'_{13} spanned by $0', 1', 2, \dots, 7, 8', 9', 10, 11, 12$ is isomorphic to $G_{13}[D]$ under the obvious mapping, and add edges between 0 and $9'$, $0'$ and 9 , 1 and 8 , and $1'$ and 8 . Then the resulting graph is a $(3, 6, 17)$ -graph.

Theorem 4.2.3 $R(3, 7) \geq 23$.

Proof. Let p be a point with edges to the points $p_1, p_2, \dots, p_6$. Let p_{ij} be points for $1 \leq i < j \leq 6$ and let p_{ij} have an edge to p_i and p_j . Finally we joint p_{ij} to p_{kh} with an edge if i, j, k, h are all distinct. The resulting graph has 22 points and contains no independent set of size 7, but it has many triangles. All these triangles are in the subgraph H induced by the points p_{ij} $1 \leq i < j \leq 6$. Now H has a Hamiltonian circuit C (the orders of the points in C is $p_{12}, p_{35}, p_{16}, p_{23}, p_{56}, p_{24}, p_{36}, p_{45}, p_{13}, p_{46}, p_{15}, p_{26}, p_{14}, p_{25}, p_{34}$) passing through each triangle exactly once. Let G denote the graph obtained by deleting the edges of C from the graph described above. Then G is a $(3, 7)$ -graph.

In fact, the lower bounds obtained in Theorem 4.2.2 and Theorem 4.2.3, hold with equality (see Chapter 5).

We noted before that $C(3, 8) \geq 26$. In a lengthy paper Grinstead and Roberts (1982) provided a $(3, 8, 27)$ -graph. This graph has no evident structure. They also were able to show that $R(3, 8) \leq 29$. The reader may refer to Theorem 1 of Grinstead and Roberts (1982), for the proof of the following

Theorem 4.2.4. $28 \leq R(3, 8) \leq 29$.

C. Grinstead and S. Roberts (1982) also noted that they could not find a $(3, 8, 28)$ -graph. At the time of writing this thesis it is not known whether $R(3, 8) = 28$ or $R(3, 8) = 29$.

In Kalbfleisch (1966), the following theorem is proved.

Theorem 4.2.5. $R(3, 9) \geq 36$

Proof. Consider $G_{35}[D]$ with $D = \{1, 7, 11, 16, 19, 24, 28, 34\}$, then $G_{35}[D]$ is a $(3, 9, 35)$ -graph.

Grinstead and Roberts (1982), showed that $R(3, 9) \leq 36$. Together with Theorem 4.2.5 they were able to prove that $R(3, 9) = 36$.

Theorem 4.2.6 gives the best known lower bounds for $R(3, t)$ $10 \leq t \leq 14$. These bounds were established in Kalbfleisch (1966).

Theorem 4.2.6 (1) $R(3, 10) \geq 39$ (2) $R(3, 11) \geq 46$ (3) $R(3, 12) \geq 49$
(4) $R(3, 13) \geq 58$ (5) $R(3, 14) \geq 63$.

It is also known (see Chung and Grinstead (1983)) that $C(3, 10) = 38$, $C(3, 11) = 45$, $C(3, 12) = 48$. We were able to obtain the same lower bounds given in Theorem 4.2.6 using Algorithm 4.1., but we could not improve either of $R(3, 13)$ or $R(3, 14)$. The cyclic graphs obtained by our algorithm are tabulated in Table V in which $N(3, t, \ell, n)$ denotes the number of random cyclic graphs with $|D| = \ell$, and order n generated until a $(3, t, n)$ -graph is found. Note that in a $(3, t, n)$ -graph the neighbors of any vertex must form an independent set: otherwise the graph contains a triangle. Thus we must have $\ell = |D| \leq t - 1$. In fact we choose $\ell = t - 1$, since all previously known largest cyclic graphs have $|D| = t - 1$.

Table V. Cyclic $(3, t)$ -graph for $10 \leq t \leq 14$

t	n	D^+	$N(3, t, t-1, n)$	LOWER BOUNDS FOR $R(3, t)$
10	38	$\{1, 3, 5, 12, 19\}$	151	39
11	45	$\{3, 4, 12, 14, 20\}$	10867	46
12	48	$\{3, 8, 9, 10, 22, 24\}$	1511	49
13	57	$\{2, 7, 8, 17, 18, 27\}$	1764	58
14	62	$\{9, 14, 25, 27, 29, 30, 31\}$	17423	63

Using Algorithm 4.1 we were able to find new lower bounds for $R(3, t)$ for $15 \leq t \leq 19$. These lower bounds are obtained by the cyclic $(3, t, n)$ -graphs $G_n[D]$, whose order n and D^+ are given in the following table.

Table VI. Cyclic $(3, t)$ -graphs for $15 \leq t \leq 19$.

t	n	D^+	$N(3, t, t-1, n)$	LOWER BOUNDS FOR $R(3, t)$
15	68	$\{9, 23, 25, 27, 28, 30, 33\}$	28240	69
16	72	$\{8, 9, 11, 21, 31, 34, 35, 36\}$	16163	73
17	79	$\{1, 3, 7, 16, 20, 22, 28, 33\}$	17362	80
18	86	$\{1, 4, 7, 13, 16, 18, 28, 37, 43\}$	29554	87
19	91	$\{1, 8, 12, 14, 18, 23, 33, 39, 42\}$	3189	92

Consequently we have the following

Theorem 4.2.7 (1) $R(3, 15) \geq 69$ (2) $R(3, 16) \geq 73$ (3) $R(3, 17) \geq 80$
 (4) $R(3, 18) \geq 87$ (5) $R(3, 19) \geq 92$.

It can be easily checked that all the lower bounds discussed in this section are non-trivial.

4.3 Lower Bounds For $R(4, t)$ $4 \leq t \leq 16$.

In this section we give the best known lower bounds for $R(4, t)$ $4 \leq t \leq 9$. All of these bounds were attained by cyclic graphs. Using Algorithm 4.1.1 we were able to obtain some new lower bounds for $R(4, t)$ for some large values of t .

The only known $R(4, t)$ number is $R(4, 4)$. This number was determined by R. Greenwood and A. Gleason (1955). They showed that the cyclic graph $G_{17}[D]$ where $D = \{1, 2, 4, 8, 9, 13, 15, 16\}$ the set of square residues in $\mathbb{Z}_{17}$, is a (4.4.17)-graph. This implies that $R(4, 4) \geq 18$. Also using the inequality $R(s, t) \leq R(s-1, t) + R(s, t-1)$ and since $R(3, 4) = R(4, 3) = 9$, they were able to prove that $R(4, 4) = 18$. Hence we have the following

Theorem 4.3.1. $R(4, 4) = 18$.

In Kalbfleisch (1965), it is shown that $C(4, 5) = 24$, and hence $R(4, 5) \geq 25$. The cyclic (4.5.24)-graph obtained is $G_{24}[D]$, where $D = \{1, 2, 4, 8, 9, 15, 16, 20, 22, 23\}$. This lower bound has not been improved since then. We summarize this result in the following

Theorem 4.3.2 $C(4, 5) = 24$, $R(4, 5) \geq 25$.

The best known lower bound for $R(4, 6)$ is 34, and it was established in Kalbfleisch (1966). It has also been shown that $C(4, 6) = 33$. The cyclic graph

obtained was $G_{33}[D]$, where $D = \{1, 2, 4, 10, 11, 15, 18, 22, 23, 29, 31, 32\}$. Thus we have

Theorem 4.3.3. $C(4, 6) = 33$, $R(4, 6) \geq 34$.

In an unpublished work, Lin Geng (1987) proved that $R(4, 7) \geq 41$ and $R(4, 8) \geq 46$. Using Algorithm 4.1.1 we were able to improve these lower bounds and obtain new lower bounds for $R(4, t)$ $9 \leq t \leq 18$. The cyclic graphs obtained by our algorithm are given in terms of n and D^+ in the following table.

Table VII. Cyclic $(4, t)$ -graphs for $7 \leq t \leq 18$.

t	n	D^*	$\ell = D $	$N(4, t, \ell, n)$	LOWER BOUNDS FOR $R(4, t)$
7	42	$\{2, 3, 6, 7, 12, 13, 14, 17\}$	16	2599	43
8	51	$\{4, 7, 12, 13, 16, 17, 18, 19, 24\}$	18	15783	52
9	59	$\{2, 7, 8, 10, 13, 15, 25, 26, 29\}$	18	17722	60
10	71	$\{1, 2, 6, 12, 13, 17, 21, 22, 26, 27, 29, 35\}$	24	13181	72
11	75	$\{1, 6, 12, 16, 20, 23, 25, 27, 30, 31, 33, 34\}$	24	1690	76
12	81	$\{4, 6, 8, 9, 13, 16, 19, 20, 26, 27, 28, 31\}$	24	13142	82
13	94	$\{3, 6, 8, 10, 23, 24, 28, 29, 36, 40, 41, 43, 45\}$	26	9833	95
14	102	$\{10, 20, 25, 28, 31, 32, 33, 34, 36, 37, 40, 47, 49\}$	26	8658	103
15	110	$\{8, 10, 12, 23, 26, 27, 28, 32, 37, 40, 41, 43, 44, 53\}$	28	3366	111
16	119	$\{10, 11, 15, 16, 19, 24, 29, 32, 35, 36, 38, 41, 43, 55, 56\}$	30	14511	120
17	127	$\{11, 12, 19, 21, 22, 23, 26, 31, 36, 37, 39, 41, 45, 58, 61\}$	30	21890	128
18	134	$\{2, 5, 9, 10, 16, 17, 25, 26, 30, 33, 37, 56, 57, 63, 65, 66\}$	32	3439	135

The above table implies the following

Theorem 4.3.4 (1) $R(4, 7) \geq 43$ (2) $R(4, 8) \geq 51$ (3) $R(4, 9) \geq 60$
 (4) $R(4, 10) \geq 72$ (5) $R(4, 11) \geq 76$ (6) $R(4, 12) \geq 82$
 (7) $R(4, 13) \geq 95$ (8) $R(4, 14) \geq 103$ (9) $R(4, 15) \geq 111$
 (10) $R(4, 16) \geq 120$ (11) $R(4, 17) \geq 128$ (12) $R(4, 18) \geq 135$

In Radziozowski and Kreher (1988) it has been proved that $R(4, 7) \geq 47$, $R(4, 8) \geq 52$, $R(4, 9) \geq 69$. The reader may refer to their paper for a list of all cyclic graphs that attained the above three new lower bounds. These new results made most of the lower bounds obtained in Theorem 4.3.4 trivial, and the theorem may be interesting in the sense that it proves the existence of cyclic $(4, t)$ -graphs of certain orders for given values of t .

We conclude this section with the following table in which we give the best known lower bound for $R(4, t)$, $4 \leq t \leq 18$. In the table those numbers marked "c" were established by the construction of cyclic graphs, those marked "t" are by our definition of trivial lower bound and those marked "tc" are trivial but there exists a cyclic graph that produces that lower bound.

Table VIII The Best Known Lower Bounds For $R(4, t)$ $4 \leq t \leq 16$.

t	4	5	6	7	8	9	10	11	12	13	14	15	16
LOWER BOUNDS	13 _c	25 _c	34 _c	47 _c	52 _c	69 _c	72 _{tc}	77 _t	86 _t	95 _c	103 _c	115 _t	120 _{tc}

In the next table we give some cyclic $(4, t)$ -graphs obtained during the search for the largest $(4, t)$ -graphs we could possibly get. Note that the non-existence of a cyclic (s, t, n) -graph does not imply the non-existence of a cyclic $(s, t, n+1)$ -graph. For example there does not exist a cyclic $(3, 3, 3)$ -graph, but there exists a $(3, 3, 4)$ -graph and a $(3, 3, 5)$ -graph each being the n -cycle for $n = 4$ and 5 respectively. Also, as we shall see in the next section, there exists a cyclic $(6, 6, 101)$ -graph, but as noted in Radziszowski and Kreher (1988), there does not exist a cyclic $(6, 6, 100)$ -graph.

Table IX. Some cyclic $(4, t, n)$ -graphs.

t	n	D^*	$ D $
8	46	$\{1, 3, 4, 8, 9, 11, 20, 22\}$	16
8	46	$\{5, 10, 12, 13, 16, 19, 20, 21\}$	16
8	49	$\{2, 9, 10, 14, 15, 18, 20, 21\}$	16
8	50	$\{1, 3, 4, 6, 14, 15, 20, 22, 23\}$	18
9	55	$\{3, 4, 6, 10, 15, 18, 19, 20, 24\}$	18
9	57	$\{8, 12, 16, 17, 19, 20, 21, 23, 26\}$	18
10	63	$\{1, 7, 13, 14, 16, 18, 25, 28, 30\}$	16
10	65	$\{9, 12, 15, 17, 18, 20, 23, 24, 25, 31\}$	20
10	67	$\{1, 6, 7, 11, 14, 16, 19, 28, 30, 33\}$	20
11	70	$\{2, 3, 4, 7, 9, 17, 25, 26, 33, 34\}$	20
11	71	$\{3, 10, 11, 14, 15, 20, 21, 22, 27, 34\}$	20
13	87	$\{3, 4, 5, 8, 9, 16, 19, 23, 34, 36, 39, 40\}$	24
13	88	$\{6, 12, 14, 15, 19, 20, 24, 32, 39, 40, 41, 43\}$	24
13	89	$\{1, 4, 5, 10, 13, 18, 25, 26, 28, 32, 37, 44\}$	24
13	90	$\{1, 3, 5, 6, 8, 14, 24, 28, 32, 33, 37, 43\}$	24
13	91	$\{3, 7, 10, 18, 22, 23, 24, 26, 28, 30, 31, 41\}$	24
13	93	$\{4, 12, 14, 16, 17, 18, 19, 23, 24, 31, 34, 40, 44\}$	26
14	100	$\{3, 7, 8, 11, 12, 13, 16, 21, 28, 30, 39, 41, 43, 47\}$	28
15	109	$\{7, 8, 9, 16, 17, 18, 20, 22, 30, 34, 37, 47, 49, 53\}$	28
16	118	$\{1, 6, 10, 15, 18, 20, 22, 23, 33, 35, 38, 40, 42, 46, 47\}$	30

4.4 Lower Bounds For $R(5, t)$ $5 \leq t \leq 8$.

In this section we give the best known lower bounds for $R(5, t)$, $5 \leq t \leq 8$. For $t \geq 5$ no $R(5, t)$ is known. Paul Erdős sometimes makes the following joke in his lectures:

Suppose an evil spirit would tell us, "Unless you tell me the value of $R(5, 5)$ I will exterminate the human race." Our best strategy would perhaps be to get all the computers and computer scientists to work on it. If he would ask for $R(6, 6)$ our best bet would perhaps be to try to destroy him before he destroys us (unfortunately we are very good at destroying [especially ourselves]). If we could be so smart as to be able to compute these numbers without computers, we would not have to pay attention to the evil spirit and could tell him "just try and see what will happen to you."

In Abbott (1965) it has been shown that $R(5, 5) \geq 38$. This lower bound was obtained by the cyclic graph $G_{37}[D]$ where D is the set of squares in the field Z_{37} . Note that since $37 = 4 \times 9 + 1 = 4k + 1$, $-1 \in D$ and the graph $G_{37}[D]$ is well defined.

In Hanson (1976) the number $f(s, t)$, $s, t \geq 3$ is defined to be the largest integer m such that the integers $1, 2, \dots, m$ can be partitioned into two classes C_1 and C_2 in such a way that C_1 does not contain a solution to a system of equations (S_s) and C_2 does not contain a solution to a system of equations (S_t) where for any $k \geq 3$, (S_k) is given by

$$x_{i,j} + x_{j,j+1} = x_{i,j+1} \quad 1 \leq i < j \leq s-1.$$

C_1 and C_2 are called (S_s) -free and (S_t) -free respectively. Given such C_1 and C_2 define a graph G with vertex set $\{0, 1, 2, \dots, f(s, t)\}$, and a vertex i is adjacent to a vertex j if and only if $|i - j| \in C_1$. Suppose that $i_1, i_2, \dots, i_s$, where $i_1 > i_2 > \dots > i_s$, are vertices of a clique in G . Then $i_k - i_\ell \in C_1$ for $1 \leq k < \ell \leq s$. But then

$$(i_k - i_\ell) + (i_\ell - i_{\ell+1}) = (i_k - i_{\ell+1}) \quad 1 \leq k < \ell \leq s-1,$$

is a solution to the system (S_s) in C_1 which is a contradiction to the assumption that C_1 is (S_s) -free. Thus G does not contain a clique of size s . Similarly one can show that G does not contain an independent set of size t . Thus we have

$$R(s, t) \geq f(s, t) + 2.$$

The graph G defined above need not be cyclic. For example the partitioning corresponding to $f(3, 8) = 25$ given by

$$C_1 = \{1, 6, 10, 13, 18, 21, 25\}$$

$$C_2 = \{1 \dots 25\} - C_1.$$

does not give rise to a cyclic graph. In Hanson (1976) it was found that the following partition shows that $f(5, 5) \geq 40$:

$$C_1 = \{1, 3, 7, 8, 12, 13, 16, 17, 18, 19, 22, 23, 24, 25, 28, 29, 33, 36, 38, 40\}$$

$$C_2 = \{1, 2, \dots, 40\} - C_1.$$

In fact it was found that $f(5, 5) = 40$. Note that the graph G induced by the above partitioning is the cyclic graph $G_{41}[D]$ where $D = C_1$. Since $R(s, t) \geq f(s, t) + 2$ we have the following

Theorem 4.4.1. $R(5, 5) \geq 42$.

The lower bound on $R(5, 5)$ given in Theorem 4.4.1 is the best known lower bound for $R(5, 5)$. It was also shown that $f(5, 6) \geq 51$ which implies $R(5, 6) \geq 53$.

In Hanson and Hanson (1977) it was found that $f(5, 6) = 55$, and this was attained by twelve partitionings, each one of which gives rise to a cyclic graph. One such partitioning of the integers $1, 2, \dots, 55$ is

$$C_1 = \{2, 3, 6, 9, 14, 16, 18, 19, 23, 24, 25, 27, 28, 29, 31, 32, 33, 37, 38, 40, 42, 47, 50, 53, 54\}$$

$$C_2 = \{1, 2, \dots, 55\} - C_1.$$

Thus we have the following

Theorem 4.4.2. $R(5, 6) \geq 57$.

The best known lower bounds for $R(5, 7)$ and $R(5, 8)$ are obtained in Radziszowski and Kreher (1988), and we present them in the following theorem

Theorem 4.4.3. (1) $R(5, 7) \geq 76$. (2) $R(5, 8) \geq 94$.

The reader may refer to the above mentioned paper for a list of the cyclic graphs that attained the lower bounds given in Theorem 4.4.3.

4.5 Lower Bounds for $R(s)$ $6 \leq s \leq 11$.

In this section we give the best known lower bounds for the diagonal Ramsey numbers $R(s)$ $6 \leq s \leq 11$. Erdős has conjectured that the largest (s, s) -graph is self complementary and this is true for all known cases. Thus the idea behind finding lower bounds for the diagonal Ramsey numbers is to define a class of self complementary graphs and restrict the search for an (s, s) -graph in that class.

A well-known family of self complementary graphs is "the Paley graphs": Let p be a prime of the form $4k + 1$ where k is an integer. The Paley graph Q_p is defined to be the graph with vertex set $\{0, 1, \dots, p - 1\}$ and two vertices i and j are adjacent if $i - j$ is a square in the field $\mathbb{Z}_p$. The assumption $p = 4k + 1$ implies that -1 is a square and hence Q_p is well defined.

Note that the Paley graphs are cyclic and the graphs obtained in the previous sections to show that $R(3, 3) \geq 6$ and $R(4, 4) \geq 18$ are the Paley graphs Q_5 and Q_{17} respectively. The largest Paley $(5, 5)$ -graph is Q_{37} , and we have shown in the previous section the existence of a $(5, 5, 41)$ -graph. Thus the largest (s, s) -graph need not be a Paley graph.

In Kalbfleisch (1965) it has been shown that Q_{101} is a $(6, 6, 101)$ -graph. Hence we have the following

Theorem 4.5.1. $R(6) \geq 102$.

The lower bound given in the above theorem is the best known lower bound for $R(6)$.

In Burling and Reyner (1972), it has been shown that Q_{109} , Q_{281} and Q_{373} are $(7, 7)$ -graphs, $(8, 8)$ -graphs and $(9, 9)$ -graphs respectively. Thus we have the lower bounds $R(7, 7) \geq 110$, $R(8, 8) \geq 282$ and $R(9, 9) \geq 374$.

In Clapham (1979) a class of self complementary graphs that contains the Paley graphs is defined as follows: "For a prime p with $p = 4n + 1$, let F be the finite field with p elements taken to be the integers $0, 1, \dots, 4n$ modulo p . Let g be a primitive root modulo p , that is, a generator of the cyclic multiplicative group of non-zero elements. Let H be the subgroup generated by g^{2^r} , where r divides n ,

so that H has index $2r$. Partition the non-zero elements of F into two classes C_1 and C_2 each being a union of cosets as follows:

$$C_1 = H \cup gH \cup \dots \cup g^{r-1}H$$

$$C_2 = g^rH \cup g^{r+1}H \cup \dots \cup g^{2r-1}H.$$

Now $g^{2n} = -1$ so $-1 \in H$ and each coset contains with each element also its negative. So the edges of the graph G defined as follows are well defined:

$$V(G) = \{0, 1, 2, \dots, 4n\} \quad \text{and} \quad (i, j) \in E(G) \quad \text{if} \quad i - j \in C_1.$$

Moreover C_1 and C_2 have the property that if i belongs to one class, then $g^r i$ belongs to the other. Therefore $\sigma : i \rightarrow g^r i$ is a permutation of $V(G)$ that maps G onto its complement. Thus G is self-complementary."

Note that for $r = 1$ the graph defined above is the Paley graph. The largest such self complementary graph containing no K_5 was found to occur for $p = 41$. The largest of these graphs containing no K_6 was found to occur for $p = 101$. Thus no improvement for $R(5, 5)$ and $R(6, 6)$ was established. The largest of these graphs containing no K_7 was found to occur for $p = 113, g = 3$ and $r = 2$ giving the lower bound $R(7, 7) \geq 114$ which improves the lower bound obtained by Q_{109} by four.

In Mathon (1987) the lower bound 114 on $R(7)$ was improved dramatically by the following construction.

Let Q_p and Q'_p be two identical copies of the Paley graph with p vertices. Denote by V, V' and E, E' their corresponding vertex and edge sets, respectively.

and let v, v' be two additional vertices. Define a new graph $G(Q_p)$ with vertices W and edges F as follows:

- (i) $W = \{v, v'\} \cup V \cup V'$
- (ii) $F = \{(v, x) | x \in V\} \cup \{(v', x') | x' \in V'\} \cup E \cup E'$
 $\cup \{(x, y') | x \in V, y' \in V' \text{ and } (x, y) \notin E\}.$

The graph $G(Q_p)$ is non-self-complementary. The following theorem is given as a Corollary in Mathon (1987).

Theorem 4.5.2. If the Paley graph Q_p contains no K_s then both $(2p+2)$ -vertex graphs $G(Q_p)$ and the complement of $G(Q_p)$ contain no K_{s+1} .

The following table given in Mathon (1987). gives the clique number α_p of the self complementary Paley graph Q_p for $5 \leq p \leq 1601$.

Table X. Clique numbers for some Paley graphs.

p	a_p	p	a_p	p	a_p	p	a_p	p	a_p	p	a_p
5	2	197	8	433	11	709	10	1009	11	1249	13
13	3	229	9	449	9	733	11	1013	11	1277	11
17	3	233	7	457	11	757	11	1021	11	1289	15
29	4	241	7	461	11	761	11	1033	11	1297	13
37	4	257	7	509	9	769	11	1049	11	1301	13
41	5	269	8	521	9	773	11	1061	12	1321	13
53	5	277	8	541	11	797	9	1069	12	1361	12
61	5	281	7	557	11	809	11	1093	12	1373	14
73	5	293	8	569	9	821	12	1097	12	1381	12
89	5	313	8	577	10	829	11	1109	11	1409	13
97	6	317	9	593	11	853	13	1117	12	1429	13
101	5	337	9	601	11	857	13	1129	13	1433	13
109	6	349	9	613	12	877	13	1153	13	1453	12
113	7	353	9	617	11	881	11	1181	12	1481	15
137	7	373	8	641	11	929	13	1193	11	1489	13
149	7	389	9	653	11	937	11	1201	13	1493	12
157	7	397	9	661	11	941	11	1213	13	1549	13
173	8	401	9	673	11	953	11	1217	13	1553	13
181	7	409	9	677	11	977	13	1229	12	1597	13
193	7	421	9	701	10	997	13	1237	13	1601	13

Since Q_{797} contains no K_{10} it follows that $R(10, 10) \geq 798$. Applying Theorem 4.5.2 to Q_{101} , Q_{281} , and Q_{797} we obtain the lower bounds $R(7, 7) \geq 205$, $R(9, 9) \geq 565$ and $R(11, 11) \geq 1597$, respectively. Note that applying Theorem 4.5.2 to Q_{17} , Q_{37} , Q_{109} and Q_{373} does not improve the lower bounds $R(5) \geq 42$, $R(6) \geq 102$, $R(8) \geq 282$ and $R(10) \geq 798$ obtained before. We summarize these results in the following

Theorem 4.5.3 (1) $R(7,7) \geq 205$ (2) $R(8,8) \geq 281$ (3) $R(9,9) \geq 565$
(4) $R(10,10) \geq 798$ (5) $R(11,11) \geq 1597$.

We summarize the results of this chapter in the following table in which we display the best known lower bounds for $R(s,t)$ for some values of s and t . Those bounds marked "e" hold with equality, those marked "n" are established by the application of Algorithm 4.1.1, those marked "tn" are trivial but for which Algorithm 4.1.1 produced a cyclic graph and those marked "t" are trivial. The unmarked lower bounds are established elsewhere.

Table XI Best known lower bound for some Ramsey numbers $R(s,t)$.

$s \setminus t$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
2	2 _e	3 _e	4 _e	5 _e	6 _e	7 _e	8 _e	9 _e	10 _e	11 _e	12 _e	13 _e	14 _e	15 _e	16 _e	17 _e	18 _e	19 _e
3		6 _e	9 _e	14 _e	18 _e	23 _e	28	36 _e	39	46	49	58	63	69 _n	73 _n	80 _n	87 _n	92 _n
4			18 _e	25	34	47	52 _n	69	72 _{tn}	77 _t	86 _t	95 _n	103 _n	115 _t	120 _{tn}			
5				42	57	76	94											
6					102													
7						205												
8							281											
9								565										
10									798									
11										1597								

CHAPTER 5

UPPER BOUNDS FOR THE RAMSEY NUMBERS

5.1 Introduction

In this chapter we give the best known upper bounds for the Ramsey function $R(s, t)$. We also give a summary for previous results on upper bounds for $R(s, t)$ for small values of s and t . We have shown in Chapter 1 that

$$R(s, t) \leq R(s-1, t) + R(s, t-1), \quad (5.1.1)$$

and that (5.1.1) holds with strict inequality if both $R(s-1, t)$ and $R(s, t-1)$ are both even.

Definition. An upper bound on $R(s, t)$ is trivial if it is obtained from (5.1.1) by replacing each term on the right hand side by its best known upper bound.

In Section 5.2, we give the well-known upper bound on $R(s)$, obtained from (5.1.1) by replacing t by s . We also give an upper bound for $R(s, t)$ for fixed s and large t . This upper bound was obtained by Graver and Yackel (1968),

In Section 5.3, we give a summary for the best known upper bounds for small Ramsey numbers. These upper bounds are based on counting arguments by which, for given s and t , the non-existence of (s, t) -graphs of certain order can be established. We conclude the section with a table in which we summarize the results discussed in the section.

5.2 Upper Bounds for $R(s)$ and $R(s, t)$.

In this section we give the best known upper bounds for the Ramsey functions $R(s)$ and $R(s, t)$ for constant s and sufficiently large t .

From (5.1.1), we immediately obtain

$$R(s, t) \leq \binom{s+t-2}{s-1}.$$

Thus for $t = s$ we have

$$R(s) \leq \binom{2(s-1)}{s-1}.$$

A simple application of Stirling's formula. (see section 2.1), gives

$$R(s) \leq c 4^s s^{-\frac{1}{2}} \quad (5.2.1)$$

We have shown in Chapter 1. that

$$R(s) \geq c' s^{2^{s/2}} \quad (5.2.2)$$

for some constant c' . It is not known if $\lim_{s \rightarrow \infty} R(s)^{\frac{1}{s}}$ exists. By (5.2.1) and (5.2.2) we know that

$$\sqrt{2} \leq \liminf R(s)^{\frac{1}{s}} \leq \limsup R(s)^{\frac{1}{s}} \leq 4.$$

One of the major problems in this area of Ramsey theory is the determination of $\lim_{s \rightarrow \infty} R(s)^{\frac{1}{s}}$ if it exists.

In a lengthy paper Graver and Yackel (1968) have shown that

$$R(3, t) \leq ct^2 \frac{\log \log t}{\log t} \quad (5.2.3)$$

for some constant c .

Also in Erdős (1961) it has been shown, using probabilistic methods, that

$$R(3, t) \geq c' \frac{t^2}{(\log t)^2} \quad (5.2.4)$$

for some constant c . The bounds on $R(3, t)$ given in (5.2.3) and (5.2.4) are the best known bounds for $R(3, t)$. Graver and Yackel (1968) also showed that for fixed s and large t ,

$$R(s, t) \leq ct^{s-1} \frac{\log \log t}{\log t}$$

for some constant c . Together with Corollary 2.3.3 we have

$$c_1 t^{\frac{s+1}{2} + o(1)} \leq R(s, t) \leq c_2 t^{s-1} \frac{\log \log t}{\log t}$$

for some constants c_1 and c_2 . It is conjectured that for fixed s and sufficiently large t , $R(s, t) = ct^{(s-1)+o(1)}$. Note that this is true for $s = 3$.

5.3 Best known upper bounds for small Ramsey numbers.

We conclude this chapter with this section in which we give a summary for the known upper bounds for some small Ramsey numbers. These bounds are based on some computational techniques by mean of which the non-existence of (s, t) graphs of certain order can be established. We shall give some results without proofs. The proofs of most of these results are quite lengthy and the reader may refer to the references from which these results were obtained.

Greenwood and Gleason (1955) used (5.1.1) to prove that $R(3, 4) \leq 9$ and $R(3, 5) \leq 14$, and they also showed that these bounds hold with equality. In Kalbfleisch (1966) it is shown that $R(3, 6) \leq 18$, and that this bound holds with equality. In Kalbfleisch (1967) it is shown that $R(3, 7) \leq 24$, $R(3, 8) \leq 36$,

$R(3, 9) \leq 37$, $R(3, 10) \leq 45$, $R(3, 11) \leq 54$, $R(3, 12) \leq 63$, $R(3, 13) \leq 73$ and $R(3, 14) \leq 84$. Graver and Yackel (1968) improved the upper bound obtained by Kalbfleisch (1967) on $R(3, 7)$ and they were able to prove that $R(3, 7) = 23$. In Grinstead and Roberts (1982), it is proved that $28 \leq R(3, 8) \leq 29$ and that $R(3, 9) = 36$; these two results improve the upper bounds obtained by Kalbfleisch (1967) for $R(3, 8)$ and $R(3, 9)$ each by 1. In Walker (1968) it is shown that $R(3, 11) \leq 44$. For $11 \leq t \leq 14$, the upper bounds obtained by Kalbfleisch (1969) for $R(3, t)$ are still the best known upper bounds for $R(3, t)$.

In Greenwood and Gleason (1955), (5.1.1) was used to show that $R(4, 4) \leq 18$. It is also shown that this bound holds with equality. Walker (1968) proved that $R(4, 5) \leq 29$ and $R(4, 6) \leq 45$. In Walker (1971) it is shown that $R(4, 5) \leq 28$, $R(4, 6) \leq 44$, $R(5, 5) \leq 55$ and $R(5, 6) \leq 94$. For $t \geq 7$, the only known bounds for $R(4, t)$ are the trivial ones.

In Giraud (1973) it is shown that $R(6, 6) \leq 169$ and $R(7, 7) \leq 586$. The known upper bounds for the remaining Ramsey numbers are all trivial.

The following table summarizes the known upper bounds for some Ramsey numbers.

Table XII
Known Upper Bounds For Certain Ramsey Numbers

$s \ t$	3	4	5	6	7	8	9	10	11	12	13	14
3	6_e	9_e	14_e	13_e	23_e	29	36_e	44	54	63	73	84
4		18_e	28	44								
5			55	94								
6				169								
7					580							

In the above table those bounds marked with "e" hold with equality.

APPENDIX A

KNOWN LOWER AND UPPER BOUNDS FOR SMALL RAMSEY NUMBERS

In the following table we display the known exact values and bounds for $R(s, t)$ for some values of s and t . The subscript of each number indicates the reference in which that bound is established. Those numbers subscripted with "n" are established in this thesis using Algorithm 4.1.1. A single number to the left of a " _ " indicates a lower bound on the corresponding Ramsey number with upper bound no better than the trivial one. The undisplayed bounds are trivial.

TABLE XIII KNOWN LOWER AND UPPER BOUNDS FOR THE RAMSEY NUMBERS

r	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
3	9_{22}	14_{22}	18_{27}	23_{21}	$28_{23} - 29_{23}$	36_{23}	$39_{26} - 44_{35}$	$46_{27} - 54_{28}$	$49_{27} - 63_{28}$	$58_{27} - 73_{28}$	$63_{27} - 84_{28}$	69_n	73_n	80_n	87_n	92_n	
4	18_{22}	$25_{26} - 28_{36}$	$34_{27} - 44_{36}$	47_{31}	52_{31}	69_{31}				95_n	103_n						
5		$42_{24} - 55_{36}$	$57_{25} - 94_{36}$	76_{31}	94_{31}												
6			$102_{26} - 169_{19}$														
7					$205_{20} - 586_{19}$												
8						281_6											
9							565_{29}										
10								798_{29}									
11									1597_{29}								

APPENDIX B

AN IMPLEMENTATION OF

ALGORITHM 4.1.1

MAIN VARIABLES IN THE PROGRAM

COUNTER	→	Counts the number of random cyclic graphs generated.
MAXNGRAPH	→	The maximum number of Random cyclic graphs to be generated.
D	→	The determining set of a cyclic graph.
L	→	The cardinality of the determining sets of the random cyclic graphs to be generated.
N	→	Order of the cyclic graphs to be generated.
S.T	→	Integers for which an (S, T, N) -graph is required.
INPUT:		N, L, S, T, MAXNGRAPH
OUTPUT:		N, L, S, T, Clique number, Independence number and D.

PROGRAM

RANCYCLIC (INPUT, OUTPUT);

{THIS PROGRAM GENERATES A RANDOM CYCLIC GRAPH G{D} OF GIVEN }
{ORDER 'N' AND FOR A GIVEN SIZE 'L' FOR THE DETERMINING SET D. }
{IT ALSO TESTS IF THE GENERATED GRAPH IS AN (S,T)-GRAPH. IF IT }
{IS AN (S,T)-GRAPH THEN THE PROGRAM OUTPUTS THE CLIQUE AND IND- }
{PENDENCE NUMBERS AND THE DETERMINING SET D AND STOPS.ELSE IT }
{GENERATES AT RANDOM ANOTHER CYCLIC GRAPH AND TESTS IF IT IS }
{AN (S,T)-GRAPH. THE PROCESS CONTINUES UNTIL AN (S,T)-GRAPH IS }
{FOUND OR THE NUMBER OF GENERATED GRAPHS EXCEEDS "MAXNGRAPH" }
{(THE MAXIMUM NUMBER OF RANDOM CYCLIC GRAPHS TO BE GENERATED)..}

TYPE

TDIARRAY = ARRAY [1 .. 150, 1 .. 150] OF INTEGER;

TYPE

INTARRAY = ARRAY [1 .. 150] OF INTEGER;

LABEL 100, 200, 300;

VAR

COUNTER,
MAXNGRAPH,

I,

J,

L,

M,

LO,

PO,

PI,

C,

S,

T,

N,

NO: INTEGER;

VAR

X: REAL;

VAR

A,

COMPSUB,

ALL: INTARRAY;

VAR

ADJ: TDIARRAY;

VAR

BETA: ARRAY [1 .. 2] OF INTEGER;

VAR

D: SET OF INTEGER;

VAR

NEWNE,

NEWCE,

```

COUNT,
POS,
P,
Z,
SEL,
MINNOD: INTEGER;

```

```

{THE FOLLOWING PROCEDURE GENERATES ALL CLIQUES OF A GIVEN }
{GRAPH G AND TESTS IF THE CARDINALITY OF EACH CLIQUE IS }
{LESS THAN A GIVEN NUMBER. }

```

```

PROCEDURE
  EXTEND (VAR OLD: INTARRAY; NE, CE: INTEGER);
  VAR
    NEW: INTARRAY;
    LABEL 10;
  VAR
    NOD,
    FIXP: INTEGER;
  VAR
    I,
    J: INTEGER;
  VAR
    LOC: INTEGER;
BEGIN
  {EXTEND}

  MINNOD := CE;
  I := 1;
  NOD := 0;
  WHILE (I <= CE) AND (MINNOD <> 0) DO
    BEGIN
      P := OLD [I];
      COUNT := 0;
      J := NE;
      FOR J := J + 1 TO CE DO
        IF (ADJ [P, OLD [J]] = 0) AND (COUNT < MINNOD)
          THEN
            BEGIN
              COUNT := COUNT + 1;
              POS := J
            END;
        IF (COUNT < MINNOD)
          THEN
            BEGIN
              FIXP := P;
              MINNOD := COUNT;
              IF I <= NE
                THEN
                  Z := POS
                ELSE
                  BEGIN
                    Z := I;

```

```

        NOD := 1
    END
    END;
    I := I - 1
END;
FOR NOD := MINNOD + NOD DOWNT0 1 DO
BEGIN
    P := OLD [Z];
    OLD [Z] := OLD [NE + 1];
    SEL := P;
    OLD [NE + 1] := P;
    NEWNE := 0;
    I := 0;
    FOR I := I + 1 TO NE DO
        IF (ADJ [SEL, OLD [I]] = 1)
        THEN
            BEGIN
                NEWNE := NEWNE + 1;
                NEW [NEWNE] := OLD [I]
            END;
        NEWCE := NEWNE;
        I := NE + 1;
        FOR I := I + 1 TO CE DO
            IF (ADJ [SEL, OLD [I]] = 1)
            THEN
                BEGIN
                    NEWCE := NEWCE + 1;
                    NEW [NEWCE] := OLD [I]
                END;
        C := C + 1;
        COMPSUB [C] := SEL;
        IF NEWCE = 0
        THEN
            BEGIN
                IF C > BETA [M]
                THEN
                    BETA [M] := C;
                    IF (BETA [1] + 1 > S - 1)
                    THEN
                        GOTO 100;
                    IF (BETA [2] + 1 > T - 1)
                    THEN
                        GOTO 100;
                END
            ELSE
                IF NEWNE < NEWCE
                THEN
                    EXTEND (NEW, NEWNE, NEWCE);
        C := C - 1;
        NE := NE + 1;
        IF NOD > 1
        THEN
            BEGIN
                Z := NE;

```

10:

```

        Z := Z + 1;

```



```

        IF (ADJ [FIXP, OLD [Z]] = 1)
        THEN
            GOTO 10
        END
    END
END ; 'EXTEND'
END.
RANDOMIZE,

WRITELN ('ENTER THE NUMBER OF VERTICES "N" :');
READ (N);
WRITELN ('ENTER THE SIZE OF THE DETERMINING SET (i.e INTER L)');
READ (L);
WRITELN ('ENTER S AND T :');
READ (S, T);
WRITELN ('ENTER THE MAXIMUM NUMBER OF GRAPHS TO BE GENERATED:');
READ (MAXNGRAPH);
WRITELN ('S =', S, ' T =', T);
WRITELN,
WRITELN ('NUMBER OF VERTICES=', N);
WRITELN ('THE SIZE OF THE DETERMINING SET D=', L);
WRITELN,
WRITELN,
FOR COUNTER := 0 TO MAXNGRAPH DO
    BEGIN
        M := 1;
        P0 := N DIV 2 - L MOD 2;
        L0 := L DIV 2;
        IF ((L MOD 2 = 0) AND (N MOD 2 = 0))
        THEN
            P0 := P0 - 1;
            L1 := P0;
            IF (L MOD 2 = 0)
            THEN
                I := []
            ELSE
                I := [N DIV 2];
            IF THE FOLLOWING A DETERMINING SET D IS RANDOMLY GENERATED]
            FOR I := 1 TO P1 DO
                BEGIN
                    X := RANDOM;
                    IF (X < L0 - P0)
                    THEN
                        BEGIN
                            I := D + [I] - [(-1) MOD N];
                            L0 := L0 - 1;
                        END;
                    P0 := P0 - 1;
                END;
            [END OF GENERATION OF D.]
        NC := 0;
        FOR I := 1 TO N DO

```

```

    IF (I IN D)
    THEN
        BEGIN
            NC := NC - 1;
            A [NO] := 1;
        END;
    FOR I := 1 TO NO DO
    BEGIN
        ADJ [I, I] := 1;
        FOR J := I + 1 TO NO DO
        BEGIN
            IF ((A [I] - A [J]) MOD N IN D)
            THEN
                ADJ [I, J] := 1
            ELSE
                ADJ [I, J] := 0;
                ADJ [J, I] := ADJ [I, J];
            END;
        END;
    END;
    FOR I := 1 TO NO DO
        ALL [I] := I;
    C := 0;
    BETA [1] := 0;
    BETA [2] := 0;
    EXTEND (ALL, 0, NO);
    IF M = 1
    THEN
        BEGIN
            WRITELN;
            WRITE ('COUNTER=', COUNTER);
            WRITE ('      CLIQUE NUMBER=', BETA [1] - 1, ' ');
        END
    ELSE
        BEGIN
            WRITE ('INDEPENDENCE NUMBER =', BETA[2] - 1, ' ');
            WRITELN;
            WRITE ('THE DETERMINING SET D=', [1 .. N - 1] - D);
            GOTO 200;
        END;
    M := 2;
    D := [1 .. N - 1] - D;
    GOTO 300;
100:
    END;
200:
    END. (RANCYCLIC)

```

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