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**TEther-LABoratory Demonstration System (TE-LAB);
Payload Orientation Determination,
Rotation Rate Control, and
Validation of Stability Criteria**

by

John L. Bergmans, B.Eng.

**A thesis submitted to the Faculty of Graduate Studies and Research in partial
fulfilment of the requirements for the degree of Master of Engineering**

**Department of Mechanical and Aerospace Engineering
Ottawa-Carleton Institute
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Payload Orientation Determination,
Rotation Rate Control, and
Validation of Stability Criteria"**

submitted by

John L. Bergmans, B.Eng.

in partial fulfilment of the requirements for the degree of
Master of Engineering

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**TEther-LABoratory Demonstration System (TE-LAB): Payload Orientation
Determination, Rotation Rate Control, and Validation of Stability Criteria**

Master of Engineering Thesis by John L. Bergmans

Abstract

The TEther-LABoratory Demonstration System (TE-LAB) is a physical simulation of the dynamics of a spinning, tethered rocket payload. The system was developed to obtain a more confident understanding of the dynamics of OEDIPUS-class payloads.

This thesis describes the Payload Orientation Determination System (PODS) and Rotation Rate Control System (RRCS) designed for TE-LAB. PODS permits quantitative observation of the payload dynamics by measuring the Euler angles of the payload using photogrammetry. The RRCS controls the spin rate of the payload, tether and upper rotating table, while minimizing tether twist.

A method of presenting experimental data and theoretical stability criteria using contour plots in the Tension-Spin Rate plane is described. The payload-associated theoretical stability criterion and the tether-associated theoretical stability criteria are found to reliably predict payload stability and tether mode behaviour, respectively.

Acknowledgments

The development of the TE-LAB apparatus and subsequent mathematical modeling validation was made possible by the contributions of a number of individuals.

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Chapter 1

1. INTRODUCTION

A unique use of tethers is in studying the region of space beyond the earth's atmosphere called the ionosphere. It is in this region that the Aurora Borealis (Northern or Southern Lights) occurs. To perform ionospheric plasma experiments, a tethered sounding rocket mission called OEDIPUS-A (Observation of Electrical-field Distribution in Ionospheric Plasma, a Unique Strategy - flight A) was flown in January 1989. The payload (Figure 1.1) consisted of two cylindrical subpayloads, spinning about their minor axes, connected to each other by a tether. The maximum tether deployment distance reached 958 m during the flight, a record at the time. An overview of the flight trajectory and sequence of events is shown in Figure 1.2.

The mission was a scientific success. However, during the flight, the aft subpayload experienced unexpected dynamics. Pre-flight calculations predicted that the cone angle of the subpayloads would not exceed a few degrees. Post-flight analysis showed that the cone angle of the aft subpayload increased throughout the flight and reached a maximum of nearly 40 degrees. In contrast, the cone angle of the forward subpayload remained approximately constant at 15 degrees. Further analysis in the years following the flight was not successful in explaining the anomaly, but did reveal that the dynamics of this configuration are more complex than originally appreciated.

Figure 1.1 OEDIPUS-A payload configuration.

Figure 1.2 OEDIPUS-A flight trajectory and sequence of events.

A follow-on mission of this configuration, OEDIPUS-C, is planned for the winter of 1995/1996. Analytical models are under development to understand the dynamics of OEDIPUS-type configurations, to investigate stabilization approaches for OEDIPUS-C and to modify the payload for this mission. To validate the analytical models, two physical laboratory models of the OEDIPUS configuration have been developed.

This chapter reviews the analyses which occurred before and after the OEDIPUS-A mission, describes activities associated with the OEDIPUS-C Tether Dynamics Experiment and concludes with the scope of this thesis.

1.1. Pre-Flight Analyses of OEDIPUS-A

For the OEDIPUS subpayload pre-flight stability analysis, it was assumed that the effect of the tether on the subpayload dynamics would be negligible and that cone angle growth could be estimated by assuming the subpayloads were free-spinning bodies.

Minor axis free-spinning bodies are known to be unstable by the major axis rule. The major axis rule states that a freely spinning body with damping will always tend to spin about its largest moment of inertia (ie. tend to a flat spin). The rate at which a freely spinning body approaches a flat spin is related to the amount of damping in the system. Typically, for minor axis spinning sounding rockets, the system damping is low and the flights are short enough that coning angle growth is not significant. The analysis of boom damping and estimate of cone angle growth of the OEDIPUS-A aft-subpayload was performed by the NASA Sounding Rocket Division at Wallops Island. It was estimated that the aft-subpayload cone angle would remain below 10 degrees during the 15 minute flight. It was not considered necessary to measure the damping in the forward subpayload booms since these were shorter and stiffer (less damping) than the aft-booms.

The only other analysis of stability in the OEDIPUS system was performed by Berry [1]. In a final report to the payload contractor, Berry examined the stability of the tether dynamics in both the deployment and free-fall phases of the flight. The emphasis of Berry's report was on the stability of the tether dynamics and not the stability of the subpayload dynamics. This report mentions that the *forward* subpayload would experience coning motion due to the angular momentum of the spool and that this coning motion would affect the tether dynamics. The report concludes that the tether deployment

scheme would be well controlled and stable and that rotational amplitudes would be at most 5 feet.

1.2. Post-Flight Analysis of OEDIPUS-A

The Flight Performance Report by McNabb [2] indicates that the performance of the Black Brant 10 rocket vehicle and the OEDIPUS-A payload were generally successful. The total amount of tether deployed was 958 m, a record at the time. Almost all scientific experiments were able to collect ionospheric data. The report indicates, however, that the subpayloads experienced larger-than-predicted coning angles. After initial separation by spring force, the initial coning angles of the forward and aft payloads were 7° and 3.5° respectively. The coning angle of the forward subpayload remained constant at approximately 8° while the coning angle of the aft subpayload grew to 38° towards the end of the flight (Figure 1.3).

The Flight Performance Report attributes these unexpected dynamics to the initial non-zero coning angles of the subpayloads after separation. The report suggests two causes for these initial coning angles. One cause is lateral forces at the subpayload separation plane, ie non-symmetrical spring force. The other cause is off-axis tether tension when the payloads are in close proximity. The report recommends that the rearrangement of spool and tether geometry be considered prior to future flights.

A post-flight investigation by Tyc and Han [3] was the first to study the effect of the tether on the payload dynamics. The investigation models the payload as a spinning rigid body with flexible booms and models the effects of the tether as a constant force on the payload at an angle to the nominal tether axis. A parametric study was performed which compared numerical simulation results as a function of various parameters with the actual flight data.

The flight data and post-flight analysis revealed that the dynamics of the OEDIPUS type configuration is much more complex than originally believed. For this reason, the Tether Dynamics Experiment, to be described in the following section, was initiated.

Figure 1.3 OEDIPUS-A flight data for coning angle of forward and aft subpayloads.

1.3. The OEDIPUS-C Tether Dynamics Experiment

The Tether Dynamics Experiment (TDE) was initiated in 1992 to understand the dynamic phenomena observed in the OEDIPUS-A mission and to prepare for the upcoming OEDIPUS-C mission. This activity involves analytical modeling of the OEDIPUS configuration, ground testing and the recommendation of payload stabilization approaches.

Two analytical models of OEDIPUS-class spinning tethered bodies have been developed; one by Tyc [4] and the other Vigneron et al. [5]. The model by Tyc is intended to directly support OEDIPUS-C payload development by examining system stability for different payload configurations. It provides a simplified model of the OEDIPUS in-flight configuration. The model is supplemented by a companion report recommending stabilization approaches to the payload contractor.

The model by Vigneron et al., hereafter referred to as the "Linear Model", is a model of the TETHER-LABoratory Demonstration System (TE-LAB) apparatus in the ground state. It results in a similar formulation to Tyc's model but is fully linearized in a rigorous manner and includes the effect of gravity experienced by the ground-based tether. The Linear Model is used as the theoretical model for TE-LAB.

Two ground-based testing facilities have been developed in support of the TDE. The first is Modi and Pradhan's "Hanging Spin Test" at the University of British Columbia [6]. This facility performs qualitative tests of a payload mockup hanging on a spinning tether. It experimentally demonstrated the stability criteria derived from Tyc's model and limit-cycle behaviour not predicted by the model.

The second ground-based testing facility is the TETHER-LABoratory demonstration system (TE-LAB) and is the basis of this thesis. TE-LAB is designed to perform *quantitative* tests of a spinning tethered payload. It features a gimbal-mounted payload mockup with easily adjustable inertia ratios, an adjustable tether tension system capable of producing low tensions, a photogrammetry-based Payload Orientation Determination System (PODS), and a Rotation Rate Control System (RRCS) to control the spin rate of the system. TE-LAB was developed by Carleton University and Canadian Space Agency (CSA) personnel. The system was originally installed at the David Florida Laboratory (DFL) in Ottawa. The experimental results presented in this thesis are based on testing at

the DFL. Upon completion of testing, TE-LAB was moved to the CSA laboratory in St-Hubert, Quebec.

1.4. Scope of the Present Work

The objectives of this thesis are to:

1. describe the development and performance of the TE-LAB apparatus, with particular emphasis on the Payload Orientation Determination System (PODS) and the Rotation Rate Control System (RRCS).
2. generate theoretical stability boundaries based on the stability criteria of the Linear Model.
3. describe the results of experiments performed using the TE-LAB apparatus for the purpose of validating the Linear Model.
4. compare the experimental stability boundary with the theoretical stability boundary and document dynamics which suggesting non-linear behaviour.

A companion thesis by McClure, ref.[7], describes the mechanical components of the TE-LAB apparatus in detail, presents a comparison between experimental system frequencies and natural modal frequencies, and examines system mode shapes.

Chapter 2

2. THE TETHER LABORATORY DEMONSTRATION SYSTEM (TE-LAB)

The TETHER-LABoratory Demonstration System (TE-LAB) is a ground-based physical model of the OEDIPUS configuration. Its purpose is to provide a more confident understanding of the OEDIPUS-type flight configuration. This is accomplished by demonstrating OEDIPUS configuration dynamics, analyzing the extent to which the Linear Model is valid, and identifying non-linear dynamic behaviour. The system provides a controlled, ground-based facility in which to simulate in-flight tether and payload dynamics. The system features variable payload inertias, spin rate and tension parameters and quantitative measurements of the payload dynamics. The design of TE-LAB is based on the design requirements specified by the Canadian Space Agency [8].

This chapter describes the design and operation of TE-LAB. The mechanical design section provides an overview of the mechanical systems. Details of the mechanical systems can be found in the thesis by McClure [7]. The section on scaling laws summarizes the approach taken in modeling the flight configuration in the laboratory. The primary responsibility of the author was the design of the Payload Orientation Determination System (PODS) and the Rotation Rate Control System (RRCS). These items are covered in detail in this chapter.

The development of TE-LAB was a team effort. General guidance was provided by Dr. F. Vigneron and Dr. A. Jablonski of the Canadian Space Agency and Prof. D. Staley of Carleton University. Detailed development was performed by the author and B. McClure.

2.1. TE-LAB Mechanical Design

The mechanical system is comprised of the following major components:

- Payload mockup
- Payload gimbals
- Tether
- Drive mechanism
- Spool and tether tension mechanism

A schematic overview of the TE-LAB apparatus is shown in Figure 2.1. A photograph of the apparatus as seen from below is shown in Figure 2.2. Physical properties of the TE-LAB apparatus are summarized in Table 2.1.

A photograph of the TE-LAB payload is shown in Figure 2.3. The payload mockup is an axisymmetric shell made of Delrin. This material is lightweight and well-suited to machining. The lower portion of the payload is flared to increase the range of motion of the payload about the lateral axes. Rings and weights are added to the shell to change its inertia properties. Two photogrammetric targets, mounted on the top plane of the payload, are used by PODS to determine the Euler angles (§2.3).

The payload is mounted on a two-axis gimbal system which permits the payload to rotate about two perpendicular lateral axes relative to the shaft. The position of the payload center of mass is adjusted to be coincident with the origin of the gimbal axes to eliminate moments due to gravity. When combined with the motion of the shaft, the gimbal system allows three-degree of freedom rotation of the payload.

The tether is rigidly fixed to the payload at its lower end. The tether was obtained from the OEDIPUS-A flight backup spool and is thus identical to the tether used during

that mission. The tether passes through an eyelet on the spin axis mounted on the upper rotating table to minimize the driving force applied to the upper end of the tether.

The drive mechanism consists of two vertically mounted geared DC motors. The upper rotating table is driven by the upper motor. The payload drive shaft, gimbals and payload are driven by the lower drive motor. The hysteresis clutch in the lower drive train was installed to provide a future constant-torque mode of operation. This clutch was locked for all tests in the present work by supplying maximum current. Both motors are equipped with shaft encoders to provide shaft position information to the Rotation Rate Control System (§2.4).

A photograph of the upper rotating table and tether tension mechanism is shown in Figure 2.4. The motor of the tether tension mechanism drives a hysteresis clutch which in turn provides a constant torque to the spool. The torque in the spool induces a tension in the tether. Transmitted torque is a function of current drawn by the clutch. Prior to testing, a calibration chart of spool load vs. current was developed. Thus, tether tension could be set simply by adjusting the clutch current. Power for the spool motor and clutch passes from the stationary power supply to the rotating platform via slip rings.

Table 2.1 Physical properties of the TE-LAB apparatus.

Parameter	Configuration	Value	Units
Lateral Inertia (A_0)	Minor-Axis Spinner	0.0061	$\text{kg} \cdot \text{m}^2$
	Major-Axis Spinner	0.0054	
Longitudinal Inertia (C_0)	Minor-Axis Spinner	0.0025	$\text{kg} \cdot \text{m}^2$
	Major-Axis Spinner	0.0063	
Tether Attachment Distance (b)	Minor-Axis Spinner	0.129	m
	Major-Axis Spinner	0.094	
Tether Density (ρ)	Light Tether	0.0030	kg/m
	Heavy Tether	0.030	
Tether Length (l)	All	6.4	m
Tether Tension (T)	All	0 to 5	N
Spin Rate (S)	All	0 to 3	Hz

Figure 2.1 Schematic diagram of TE-LAB apparatus. Length of tether is 6.4 m.

Figure 2.2 Photograph of TE-LAB apparatus as seen from below. The payload, gimbals, tether, and upper rotating table are visible. This is the "heavy" configuration of the tether with small weights attached.

Figure 2.3 Photograph of TE-LAB payload and associated hardware, including geared DC drive motor and photogrammetric targets.

Figure 2.4 Photograph of TE-LAB upper rotating table.

2.2. Scaling Laws of TE-LAB Apparatus

Before the TE-LAB design could be initiated, studies were performed to determine appropriate design and operating parameters for the apparatus. The result of these studies are included in Jablonski et al.[9]. The paper uses a simplified version of Tyc's equations of motion (no booms, no momentum wheel) for a spinning tethered satellite.

The scaling laws indicate that to scale the OEDIPUS configuration in laboratory conditions, very small payload inertias are required. In addition, impossibly high tether tensions and spin rates are required. Therefore, it is not possible to scale the in-flight configuration exactly in a ground-based laboratory model.

The philosophy regarding the inexact scaling is that if TE-LAB can validate a mathematical model which does not completely correspond in scale to the in-flight parameters, then the mathematical model will still likely be valid for in-flight configurations (Vigneron and Jablonski[10]). Therefore, it was decided that payload inertias, tether tensions and spin rates would be chosen which are physically realistic and which follow the trends suggested by the scaling laws. McClure presents details of the selection of payload inertias used in TE-LAB.

2.3. Payload Orientation Determination System (PODS)

2.3.1. Introduction

One of the goals of TE-LAB is to provide quantitative measurements of the system behaviour for comparison with the analytical models. To this end, the Payload Orientation Determination System (PODS) measures the Euler angles of the payload. PODS is a photogrammetry-based sensor which uses the camera coordinates of two payload mounted targets to calculate payload Euler angles.

A Charge Coupled Device (CCD) video camera views the movement of the two payload mounted targets. The image of the targets is transferred to a video frame grabber board inside the PC-compatible computer. The target tracking software, the Digital Electro-Optic Position Sensor for TE-LAB (DEOPST), calculates the coordinates of each target and saves the coordinates to a file. The offline postprocessing software, POST321B, uses inverse kinematics to calculate the payload Euler angles, based on the cartesian coordinates of the targets. A MATLAB program file, FFTPOST2, permits further analysis of the Euler angle time histories by producing Fast Fourier Transforms and exponential curve fits of the data.

This section describes the hardware, software, optics, accuracy and data postprocessing tools of PODS.

2.3.2. PODS Hardware

A COHU 4110 CCD camera is mounted above and slightly offset from the payload as shown in top and side views of the camera and payload geometry (Figure 2.5). A view of the payload and targets as seen from the camera location is shown in Figure 2.6. Passive targets are used on the TE-LAB payload because the wires and batteries associated with providing power to active targets, such as LEDs, interferes with the delicate balance of the payload. The physical arrangement using passive targets is more complex than active targets; however, this is offset by the resulting ease of target mounting and payload balancing. The targets are lightweight aluminum hemispheres, 5/8" in diameter, polished to a mirror surface. The targets are mounted on the top surface of the payload on opposite sides of the spin axis.

Figure 2.5 a) Top and b) side views of PODS hardware and payload. Coordinate frame $x_1y_1z_1$ is the body-centered vertically-aligned inertial coordinate frame.

Figure 2.6 Payload and targets as seen from camera position.

A white reflective sheet is mounted in front of the camera, approximately parallel to the surface of the CCD array to illuminate the passive payload targets. The sheet is positioned such that the camera views the payload through a rectangular cutout in the center of the sheet. A halogen light source, mounted opposite the camera, illuminates the reflective sheet. The target images viewed by the camera are the images of the illuminated sheet reflected by the targets (Figure 2.7). All light illuminating the targets must originate at the reflecting sheet to ensure that the centroid of the image of the sheet appears to originate at the center of the hemisphere. Therefore, the direct light blocking sheet is placed such that the light source does not illuminate the targets directly. All other light sources such as ceiling lights are shaded or turned off.

The image of the targets from the camera is transferred to the target tracking software via an PC-Vision Plus frame grabber board made by Imaging Technologies Inc. mounted in the PC-compatible computer. The gain and offset of the frame grabber are adjusted using DEOPST until the field of view is black except for the images of the two targets. The image size of the targets is further adjusted by varying the location of the light source and the frame grabber gain and offset until the targets are continuously visible, including when directly behind the tether. The resulting image size is approximately

8x8 pixels². This is slightly larger than the 5x5 pixels² target size recommended for DEOPS.

Figure 2.7 Enlarged image of hemispherical payload target as seen by camera under typical lighting conditions.

2.3.3. The Digital Electro-Optic Position Sensor for TE-LAB (DEOPST)

The Digital Electro-Optic Position Sensor for TE-LAB (DEOPST) is a real-time computer program which tracks and records the cartesian coordinates of both payload targets in the camera coordinate frame. The program also includes an interrupt-driven Rotation Rate Control System (RRCS), to be described separately in §2.4, which controls the spin rates of the upper table and payload drive motors.

DEOPST is based on the Digital Electro-Optic Position Sensor (DEOPS) developed by Laurin [11,12] as a Doctoral Thesis at the University of Toronto Institute for Aerospace Studies. Laurin produced two versions of the program: DEOPS for Daisy (DEOPSD) which is used to track the positions of 10 LED targets on the end of flexible arms in a flexible spacecraft simulator; and DEOPS for RADIUS (DEOPSR) which is

used to track the positions of two LED targets on a flexible robotic arm. The DEOPST software is written in Microsoft C and runs on an 486 33MHz PC-compatible computer.

DEOPS tracks a cluster of bright pixels as it moves in the field of view. DEOPST searches a window of 15x15 pixels for a pixel brighter than a certain threshold value. The brightest pixel in a 6x7 pixel area surrounding the threshold pixel is found. A weighted mean centroid calculation is then performed on the 5x5 pixel area surrounding the threshold pixel. The weighted mean equations for the x and y coordinate used in DEOPST are, from Laurin [12]:

$$\bar{x} = \frac{\sum_i \sum_j I'_{ij} i}{\sum_i \sum_j I'_{ij}}, \quad \bar{y} = \frac{\sum_i \sum_j I'_{ij} j}{\sum_i \sum_j I'_{ij}}, \quad I'_{ij} = I_{ij} - I_{ij}^{fpn} \quad (2.1)$$

where I_{ij} is the pixel intensity and I_{ij}^{fpn} is the fixed pattern noise at pixel (i,j) , n is the search area size ($n=5$)

The DEOPSD and DEOPSR documentation states that the target cartesian coordinates are measured to within $\pm 0.1\%$ of the field of view. This value is used in an analytical accuracy analysis (§2.3.8) to estimate the nutation angle accuracy of PODS.

Software Modifications

DEOPST is a modified version of the DEOPSR software. The following modifications were made for DEOPST application:

- increase of allowable target tracking speed.
- implementation of Rotation Rate Control System (§2.4):
 - changes to real-time menu to allow user to operate RRCS.
 - addition of interrupt-driven motor control routines.
 - addition of menu to allow user to specify motor controller parameters.
- target-coordinate save routine copied from DEOPSD software.
- improved resolution of data time values. Output of 200Hz interrupt-driven counter used rather than 18Hz "_bios_timeofday" DOS routine.

Increased Target Speed

Three changes were made to enable DEOPST to track faster moving targets. The first change was to reduce the number of search windows from 5 in DEOPSR to 2, since the TE-LAB application requires only two targets to be tracked. This permits more pixels to be scanned per target. The second change was to expand the size of each search window to 15×15 pixels².

The third change is an improved search window predict routine. When DEOPS is in predict mode, the center of the search window is centered on a predicted location rather than the previous target location. The original DEOPS predict routine predicted the next target coordinate using only the current velocity of the x and y coordinates. DEOPST incorporates the velocity and acceleration of the target x and y coordinates.

The accuracy of the measured target position is not affected by the current prediction routine because the target always remains within the search window. To reduce noise in the predicted location of the search window, a digital filter could be incorporated in the prediction algorithm.

The derivation of the acceleration-based target predict algorithm used in DEOPST is presented here. The predict routine treats the x and y coordinates as one-dimensional positions of a particle undergoing constant acceleration. The x coordinate as a function of time under constant acceleration can be expressed as:

$$x(t) = x_0 + v_{x0}t + 0.5a_{x0}t^2 \quad (2.2)$$

where x_0 is the initial position, v_{x0} is the initial velocity, a_{x0} is the initial acceleration (constant).

The predicted x -coordinate of the target can then be found from:

$$x_p = x_n + v_{xn}(t_p - t_n) + 0.5a_{xn}(t_p - t_n)^2 \quad (2.3)$$

where x_n is the current x -coordinate, v_{xn} is the current velocity in the x direction, a_{xn} is the current acceleration in the x direction, t_n is the current time step value, t_p is the time step value of the next target position

Since the target tracking is synchronized to the video field update rate, each time step size is equal. Therefore, $(t_p - t_n)$ is the period of 1 video field which is 1/60 second. The current particle velocity in the x -direction is:

$$v_{xn} = \frac{\Delta x}{\Delta t} = \frac{(x_n - x_{n-1})}{(t_n - t_{n-1})} \quad (2.4)$$

where x_n is the current target x -coordinate, x_{n-1} is the previous target x -coordinate, $(t_n - t_{n-1})$ is the step size which is the period of 1 video field.

The current particle acceleration is:

$$a_{xn} = \frac{\Delta v_{xn}}{\Delta t} = \frac{(v_{x,n} - v_{x,n-1})}{(t_n - t_{n-1})} = \frac{1}{(t_n - t_{n-1})} \left(\frac{(x_n - x_{n-1})}{(t_n - t_{n-1})} - \frac{(x_{n-1} - x_{n-2})}{(t_{n-1} - t_{n-2})} \right) \quad (2.5)$$

Each time step is equal. ie $(t_{n-1} - t_{n-2}) = (t_n - t_{n-1})$. Therefore:

$$a_{xn} = \frac{(x_n - 2x_{n-1} + x_{n-2})}{(t_n - t_{n-1})^2} \quad (2.6)$$

Substituting into Eq. (2.3):

$$x_p = x_n + \frac{(x_n - x_{n-1})}{(t_n - t_{n-1})}(t_p - t_n) + 0.5 \frac{(x_n - 2x_{n-1} + x_{n-2})}{(t_n - t_{n-1})^2}(t_p - t_n)^2 \quad (2.7)$$

Again each time step is equal. ie: $(t_p - t_{n-1}) = (t_n - t_{n-1})$. Eq. (2.4) reduces to:

$$x_p = 2.5x_n - 2x_{n-1} + 0.5x_{n-2} \quad (2.8)$$

Eq. (2.8) is used to predict the x -coordinate of the next target position. An equation of identical form is used to predict the y coordinate:

$$y_p = 2.5y_n - 2y_{n-1} + 0.5y_{n-2} \quad (2.9)$$

The next search window is centered on the predicted target position (x_p, y_p) .

It was found experimentally that with these modifications, DEOPST can track two targets moving at up to approximately 30 pixels/field as shown in Figure 2.8. This speed permits recording of target position at payload spin rates of up to 3 Hz even during the most unstable behaviour. The 3 Hz spin rate is the maximum rate specified by the design requirements [8].

In an attempt to permit target tracking at spin rates beyond 3 Hz, the change in acceleration ("jerk") was temporarily incorporated into the predict algorithm. It was found that DEOPST could not track the targets as reliably with the jerk term included. This is probably due to the noise in the acceleration signal.

Overview of Operation

A brief overview of the procedure used to perform a typical test using DEOPST is presented here. More detailed instructions can be found in the PODS user's manual.

1. Start DEOPST from the DOS prompt
2. Initialize motor controller parameters
3. Enter data output file name
4. Identify targets using crosshairs on monitor
5. Start motors
6. Start recording target positions to file
7. TE-LAB dynamic test performed
8. End recording of target positions
9. Stop motors
10. Exit DEOPST
11. Postprocess data

Figure 2.8 Magnitude of a) velocity and b) acceleration of target P during a typical unstable test. Spin rate: 3 Hz.

2.3.4. Measurement of TE-LAB Optical Parameters

The CCD to lens distance and effective focal length of the optical system are calculated in this section to document the settings used during testing.

Thin lens

As an approximation of the optical properties of the camera, the camera can be assumed to exhibit the same behaviour as a thin lens (or pinhole camera). The geometry of a thin lens is shown in Figure 2.9.

By similar triangles:

$$\frac{y_{cam}}{l_i} = \frac{y_o}{z - l_i} \quad (2.10)$$

where all lengths are made in real-world units, typically millimeters. Solving for camera coordinates in real-world units:

$$y_{cam} = \frac{l_i}{(z - l_i)} y_o \quad [\text{mm}] \quad (2.11)$$

Figure 2.9. Thin lens geometry. y_o is the object coordinate, y_{cam} is the image coordinate on the CCD plane, z is the CCD to object distance, l_i is the CCD to lens distance, and l_o is the lens to object distance.

The dimensions of the CCD array in the COHU 4110 camera are 6.4 mm by 4.8 mm. The image from the CCD is transferred to frame grabber memory. The dimensions of the image in the frame grabber memory is 640 pixels by 480 pixels. Thus, the relationship between image size in frame grabber memory and on the CCD is 100 frame grabber pixels per mm on the CCD. Therefore, the image coordinates can be expressed in pixels as:

$$y_{cam} = 100 \frac{l_i}{(z - l_i)} y_o \quad [\text{pixels}] \quad (2.12)$$

Measurement of Image Size

To solve for the CCD to lens distance using Eq. (2.12), the image size and object size must be known. Therefore, the following experiment was performed to determine the ratio between these two dimensions.

A set of orthogonal axes with hatch marks at one inch increments along each axis was placed a distance $z=1710$ mm in front of the CCD plane. The cartesian coordinates of the hatch marks were measured in pixel units in the camera frame (x_{cam}, y_{cam} in Figure 2.10). The x -coordinate of the horizontal hatch marks in pixels as a function of the horizontal hatch mark position in inches is shown in Figure 2.11. The relationship for the y -coordinate is shown in Figure 2.12.

Figure 2.10 Test grid pattern used to calculate the ratio of object size to image size as seen in the camera field of view. Test pattern placed 1710 mm from CCD plane.

Figure 2.11 Plot of the x-coordinate of the test pattern hatch marks in camera frame as a function of hatch mark position in real-world coordinates.

Figure 2.12 Plot of the y-coordinate of the test pattern hatch marks in camera frame as a function of hatch mark position in real-world coordinates.

Lines fit to the data demonstrate that at this CCD to object distance, the ratio of image size to object size is 22.75 frame grabber pixels per object inch.

CCD to Lens Distance

Assuming a 1 inch object size, the variables required to calculate the CCD to lens distance are:

object size:	$y_o = 1 \text{ inch} = 25.4 \text{ mm}$
image size:	$y_{cam} = 22.75 \text{ pixels}$
CCD to object distance:	$z = 1710 \text{ mm}$

Rearranging (2.12) the CCD to lens distance is:

$$l_i = \frac{zy_{cam}}{100y_o + y_{cam}} \quad (2.13)$$

Substituting into (2.13):

$$l_i = \frac{(1710)(22.75)}{(100 * 25.4 + 22.75)} = 15.2 \text{ mm} \quad (2.14)$$

Therefore, the CCD to lens distance is 15.2 mm. The CCD to lens distance is controlled by the zoom setting. This zoom setting was chosen since it maximizes the amount of target movement in the camera frame while ensuring that the targets are always in the field of view and DEOPST can track the targets. Maximization of target movement in the camera frame is accomplished by increasing the camera focal length. It will be shown in §2.3.8 that errors due to DEOPS accuracy can be minimized by increasing the camera focal length.

Effective Focal Length

The effective focal length of the optical components can be found by using the focal length equation for a thin lens:

$$\frac{1}{f} = \frac{1}{l_i} + \frac{1}{l_o} \quad (2.15)$$

Eq. (2.15) can be written as:

$$f = \frac{l_o l_i}{l_i + l_o} \quad (2.16)$$

But, from Figure 2.9:

$$l_o = z - l_i \quad (2.17)$$

Therefore:

$$f = \frac{(z - l_i) l_i}{l_i + (z - l_i)} \quad (2.18)$$

Simplifying:

$$f = \frac{(z - l_i) l_i}{z} \quad (2.19)$$

Substituting the parameters of the optical test into Eq (2.19):

$$f = \frac{(1710 - 15.2) \text{mm} 15.2 \text{mm}}{1710 \text{mm}} = 15.1 \text{mm} \quad (2.20)$$

This value is close to the CCD to lens distance. It can be seen by taking the limit of Eq. (2.19) that if the test object is infinitely far from the camera, the focal length will equal the CCD to lens distance:

$$\lim_{z \rightarrow \infty} f = \lim_{z \rightarrow \infty} \left(\frac{(z - l_i) l_i}{z} \right) = l_i \quad (2.21)$$

2.3.5. The Camera-Aligned Payload Centered Coordinate Frame

A three-dimensional view of the camera-aligned and vertically aligned payload centered coordinate systems (systems $x_0y_0z_0$ and $x_1y_1z_1$ respectively) are shown in Figure 2.13. The term "camera-aligned" refers to the fact that the $x_0 - y_0$ plane is parallel to the CCD plane. For this reason, the angle between the y_0 and y_1 axes and the z_0 and z_1 axes is the camera offset angle, γ . The importance of the camera-aligned payload centered coordinate frame is that the calculation of the payload Euler angles is performed based on the coordinates of the targets in this frame. This section will describe the relationship between the camera coordinates and the camera-aligned payload centered coordinate frame.

Figure 2.13 Camera-aligned payload centered ($x_0y_0z_0$) and vertically aligned payload centered ($x_1y_1z_1$) coordinate frames. Angle γ is the camera offset angle. Payload center of gravity is at the origin of both frames.

Figure 2.14 Coordinates of targets in payload fixed frame.

Direction Cosines in Real-World and Camera Coordinates

The coordinates of the targets in the camera-aligned payload centered coordinate frame and the payload-fixed frame are related by:

$$V_4 = CV_0 \quad (2.22)$$

where C is a 3×3 rotation matrix of the form $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$, V is a 3×2 matrix of target coordinates of the form $V_0 = [\bar{p}_0 | \bar{q}_0]$ and $V_4 = [\bar{p}_4 | \bar{q}_4]$ and $\bar{p}$ and $\bar{q}$ are the vector coordinates of targets P and Q , respectively. In the payload-centered camera-aligned coordinate frame, the vector coordinates of the targets are $\bar{p}_0 = [p_{0x} \ p_{0y} \ p_{0z}]^T$ and $\bar{q}_0 = [q_{0x} \ q_{0y} \ q_{0z}]^T$. In the payload-fixed coordinate frame, the vector coordinates of the targets are $\bar{p}_4 = [-r \ 0 \ R]^T$ and $\bar{q}_4 = [r \ 0 \ R]^T$ where r and R are the payload radius and half-height respectively. The payload target coordinates in the payload-fixed

frame are shown in Figure 2.14. As shown in Appendix A.1, the direction cosines of matrix C are:

$$c_{11} = \frac{p_{0x} - q_{0x}}{-2r} \quad (2.23a)$$

$$c_{31} = \frac{p_{0x} + q_{0x}}{2R} \quad (2.23b)$$

$$c_{12} = \frac{p_{0y} - q_{0y}}{-2r} \quad (2.23c)$$

$$c_{32} = \frac{p_{0y} + q_{0y}}{2R} \quad (2.23d)$$

It is shown in §2.3.6 that the Euler angles defining the orientation of the payload are a function of the above direction cosines.

The relationship between the y -coordinates of the camera-aligned payload centered coordinate frame and the CCD frame are shown in Figure 2.15.

Figure 2.15 y - z plane of camera-aligned payload centered frame and camera coordinate frame. L is the distance from the CCD plane to the center of the payload. z is the distance from the CCD plane to the target.

Using the thin lens relationship between image size and object size, Eq. (2.12), the image coordinate of target P is:

$$p_{cam,y} = 100 \left(\frac{l_i}{z - l_i} \right) p_{0,y} \quad [\text{pixels}] \quad (2.24)$$

The CCD to object distance for target P, z , can be found from:

$$z = L - p_{0,z} \quad (2.25)$$

where L is the distance between the CCD plane and the center of the payload and $p_{0,z}$ is the z coordinate of P in the camera-aligned payload centered frame. Substituting Eq. (2.25) into Eq. (2.24) and rearranging, the y -coordinate of target P can be calculated as a function of camera coordinate:

$$p_{0,y} = \frac{1}{100} \left(\frac{L - p_{0,z} - l_i}{l_i} \right) p_{cam,y} \quad (2.26)$$

where all variables are in physical units, typically millimeters, except $p_{cam,y}$ which is in pixels units. Similarly, the x -coordinate of target P is:

$$p_{0,x} = \frac{1}{100} \left(\frac{L - p_{0,z} - l_i}{l_i} \right) p_{cam,x} \quad (2.27)$$

Eqs. (2.26) and (2.27) can be rewritten in a more compact form as:

$$p_{0,x} = s_p p_{cam,x} \quad (2.28a)$$

$$p_{0,y} = s_p p_{cam,y} \quad (2.28b)$$

where $s_p = \frac{1}{100} \left(\frac{L - p_{0,z} - l_i}{l_i} \right)$ is the scaling factor between camera coordinates and real-world coordinates for target P. A similar compact form of the relationship between camera coordinates and real-world coordinates can be derived for target Q:

$$q_{0z} = s_q q_{cam,z} \quad (2.29a)$$

$$q_{0y} = s_q q_{cam,y} \quad (2.29b)$$

where $s_q = \frac{1}{100} \left(\frac{L - q_{0z} - l_i}{l_i} \right)$. In addition, the real-world size of the payload radius and half-height can respectively be expressed as a function of the image size using:

$$r = s_r r_{cam} \quad (2.30a)$$

$$R = s_R R_{cam} \quad (2.30b)$$

where $s_r = \frac{1}{100} \left(\frac{L - z_{vert} - l_i}{l_i} \right)$, $s_R = \frac{1}{100} \left(\frac{L - z_{nu} - l_i}{l_i} \right)$. z_{vert} and z_{nu} are the z -coordinates of the payload radius and half-height during the calibration tests required to measure these quantities (§2.3.7). Substituting into Eqs. (2.20) the direction cosines can be expressed in terms of the camera frame coordinates as:

$$c_{11} = \frac{s_p p_{cam,x} - s_q q_{cam,x}}{-2s_r r_{cam}} \quad (2.31a)$$

$$c_{31} = \frac{s_p p_{cam,x} + s_q q_{cam,x}}{2s_R R_{cam}} \quad (2.31b)$$

$$c_{12} = \frac{s_p p_{cam,y} - s_q q_{cam,y}}{-2s_r r_{cam}} \quad (2.31c)$$

$$c_{32} = \frac{s_p p_{cam,y} + s_q q_{cam,y}}{2s_R R_{cam}} \quad (2.31d)$$

A view of the y - z plane of the camera-aligned and vertically aligned payload centered coordinated frames relative to the CCD plane is shown in Figure 2.16. The minimum value of s_p occurs when the payload is in line with the camera boresight axis. At this location $p_{0z} = R = 120.4$ mm. From the geometry of the apparatus and the CCD to lens distance, the minimum value of s_p is:

$$s_{p,\min} = \frac{L - p_{0z} - l_i}{l} = \frac{1440 - 120.4 - 15.1}{15.1} = 86.4 \quad (2.32)$$

As an estimate of the maximum value of s_p , assume that the payload is inclined along the x_0 axis. In this case $p_{0z} = 0$ and s_p becomes:

$$s_{p,\max} = \frac{L - p_{0z} - l_i}{l} = \frac{1440 - 0 - 15.1}{15.1} = 94.4 \quad (2.33)$$

Figure 2.16 y-z plane of camera-aligned and vertically aligned payload centered frames relative to CCD plane.

Therefore, the largest difference in scaling factors which could be encountered is approximately 9%. Each of the scaling factors (s_p, s_q, s_r , and s_R) is approximately equal since $L \gg \partial z$ over the range of payload motion. Therefore, unless otherwise specified, for the remainder of the present work, it will be assumed that $s_p = s_q = s_r = s_R = \text{constant} = s$. This assumption is equivalent to assuming that out-of-plane displacements of the targets will be neglected, where out-of-plane displacements are defined as displacements in the direction of the camera boresight axis.

Substituting s into Eqs. (2.31) and cancelling, the direction cosines becomes:

$$c_{11} = \frac{p_{cam,x} - q_{cam,x}}{-2r_{cam}} \quad (2.34a)$$

$$c_{31} = \frac{p_{cam,x} + q_{cam,x}}{2R_{cam}} \quad (2.34b)$$

$$c_{12} = \frac{p_{cam,y} - q_{cam,y}}{-2r_{cam}} \quad (2.34c)$$

$$c_{32} = \frac{p_{cam,y} + q_{cam,y}}{2R_{cam}} \quad (2.34d)$$

Comparing Eqs. (2.23) to Eqs. (2.34), it can be seen that the effect of assuming no out-of-plane displacements is that each of the direction cosines (and thus Euler angles) can be expressed entirely in either the camera coordinates frame in pixel units or the camera-aligned payload centered in real-world units. Further, the form of each direction cosine is identical, implying that, when applied consistently, the x - y coordinates in the camera frame are interchangeable with the x - y coordinates in the camera-aligned real world coordinate frame.

Hereafter, unless otherwise specified, all camera coordinates of the targets will be interpreted as projections of the three-dimensional real-world coordinates onto the $x_0 - y_0$ plane and measured in pixel units as shown in Figure 2.17.

Positioning the Origin of the Camera Coordinate Frame at the Center of the Payload

A view of the payload as seen from the location of the camera is shown in Figure 2.18. The dashed box represents the field of view of the camera. The target tracking software, DEOPST, returns coordinates of the targets in the x_0''' , y_0''' coordinate system in the upper left corner of the field of view. The method used to convert from the frame grabber coordinates to the camera-aligned payload centered camera frame is presented here.

Figure 2.17 Camera-aligned and vertically aligned payload centered frames. Using assumption of no out-of-plane displacements, camera coordinates of targets can be interpreted as projection of target onto $x_0 - y_0$ plane, measured in pixel units

Figure 2.18 View of payload in camera frame, and coordinate frames required to place origin of payload centered frame.

The target coordinates are transformed from the top-left centered coordinate system to the bottom-left centered coordinate system ($x_0'' - y_0''$) when saved to file using the equation:

$$x_0'' = x_0''' \quad (2.35a)$$

$$y_0'' = V - y_0''' \quad (2.35b)$$

where V is the vertical dimension of the frame grabber field of view in pixels (480 pixels).

The Vertical Spin Test (§2.3.7) is performed to measure the payload radius and camera offset angle. The test consists of locking the payload gimbals and spinning the payload about the vertical axis at a constant rate. The path of the targets forms an ellipse in the camera frame. The center of the top-of-payload centered camera coordinate frame is found by calculating the average x_0'' and y_0'' coordinates of the targets:

$$x_{0c}'' = \frac{\sum_{i=1}^n (p_{0xi}'' + q_{0xi}'')}{2n} \quad (2.36a)$$

$$y_{0c}'' = \frac{\sum_{i=1}^n (p_{0yi}'' + q_{0yi}'')}{2n} \quad (2.36b)$$

where (p_{0xi}'', p_{0yi}'') are the bottom-left centered camera coordinates of targets P and Q respectively at the i -th time step and n is the number of data points in the Vertical Spin Test. Point (x_{0c}'', y_{0c}'') represents the intersection of the payload spin axis with the top plane of the payload in the bottom-left centered camera frame. The transformation between the top-of-payload centered frame and the bottom-left centered frame is:

$$x_0' = x_0'' - x_{0c}'' \quad (2.37a)$$

$$y_0' = y_0'' - y_{0c}'' \quad (2.37b)$$

A view of the y - z plane of the top-of-payload centered, payload centered and vertically aligned inertial coordinate frame is shown in Figure 2.19. This view is used to

illustrate the relationship between the y'_0 axis of the top-of-payload centered frame (x'_0, y'_0, z'_0) and the y_0 axis of the payload-centered frame (x_0, y_0, z_0) . The x'_0 axis of top-of-payload centered frame and the x_0 axis of the payload-centered frame (x_0, y_0, z_0) are equal.

The following transformation applies between the top-of-payload centered frame and the payload-centered frame:

$$x_0 = x'_0 \quad (2.38a)$$

$$y_0 = y'_0 + R \sin \hat{\gamma} \quad (2.38b)$$

$$z_0 = z'_0 - R \cos \hat{\gamma} \quad (2.38c)$$

Figure 2.19 y-z plane of camera-aligned frames and vertically aligned frame.

Substituting Eqs. (2.37a) and (2.37b) into Eqs. (2.38a) and (2.38b), respectively, results in the following transformation between the camera coordinates output by DEOPST and the camera-aligned payload centered coordinates:

$$x_0 = x_0'' - x_{0c}'' \quad (2.39a)$$

$$y_0 = y_0'' - y_{0c}'' + R \sin \hat{\gamma} \quad (2.39b)$$

where (x_0'', y_0'') are the target coordinates output by DEOPST, (x_{0c}'', y_{0c}'') are the coordinates of the intersection of the payload spin axis and top plane of the payload in the left-bottom centered coordinate frame, R is the payload half-height in pixels, $\hat{\gamma}$ is the camera offset angle.

The theory and procedure used to calculate the payload and camera geometry (r , R and γ) based on the Vertical Spin Test and the Nutation Angle Test data are presented in §2.3.7.

2.3.6. PODS Kinematic Equations

The relationship between the camera-aligned payload centered coordinate frame and the payload-fixed coordinate frame, Eq. (2.22), is repeated here:

$$V_4 = CV_0 \quad (2.40)$$

where, as defined in detail in §2.3.5, V_0 is the matrix of target coordinates in the camera-aligned frame, V_4 is the matrix target coordinates in the payload-fixed frame and C is a rotation matrix of the form:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \quad (2.41)$$

As shown briefly in §2.3.5 and in full detail in Appendix A.1, the direction cosines can be solved as a function of payload geometry and camera-aligned payload centered coordinates.

PODS calculates the payload Euler angles based on the camera coordinates of the targets in a three-step process. First, the direction cosines of the rotation matrix between

camera coordinates and payload-fixed coordinates are calculated. Then, Euler angles in a 1-2-3 rotation sequence are calculated from the direction cosines. Finally the Euler angles in a 3-2-1 rotation sequence are calculated based on the Euler angles from the 1-2-3 rotation sequence. Computation of the Euler angles is performed offline by the target coordinate postprocessing software POST321B, to be described in §2.3.10.

This indirect method of calculating the Euler angles in the 3-2-1 sequence is a reflection of the history of PODS development. The original rotation sequence chosen for PODS was a 3-1-3 sequence which output the nutation angle as its second rotation. This sequence suffers from a singularity when the nutation angle is 0 - the region where TE-LAB operates when the payload is stable. To eliminate the singularity, a 1-2-3 rotation sequence was chosen for PODS development. Comparison of Euler angle frequencies to damped natural frequencies from the Linear Model eigenvalue analysis required that PODS be modified to the 3-2-1 rotation sequence of the Linear Model.

The derivation of the indirect method of solving for the 3-2-1 Euler angles is presented in Appendix A. A brief overview of the process is shown here. A method for calculating the 3-2-1 Euler angles directly from the direction cosines is presented in Appendix A.4. This method is computationally-efficient and should prove useful if an attempt is made to calculate the Euler angles in real-time in DEOPST.

To avoid confusion between the two rotation sequences, all angles and axes in the 1-2-3 sequence are noted by caret marks (^). Angles without caret marks correspond to the 3-2-1 sequence. Since the camera offset is constant and is always the first rotation from frame 0 to 4, $\gamma = \hat{\gamma}$ and $C_1(-\gamma) = C_1(-\hat{\gamma})$.

Solving for Euler Angles in a 1-2-3 Rotation Sequence

The rotation matrix in a 1-2-3 Euler angle sequence between frames 0 and 4 is:

$$C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma}) = C_3(\hat{\psi})C_2(\hat{\eta})C_1(\hat{\phi})C_1(-\hat{\gamma}) \quad (2.42)$$

where $C_1(-\hat{\gamma})$ is the rotation matrix due to the camera offset angle.

$$C_1(-\hat{\gamma}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{\hat{\gamma}} & -S_{\hat{\gamma}} \\ 0 & S_{\hat{\gamma}} & C_{\hat{\gamma}} \end{bmatrix}; \quad (2.43)$$

and the payload Euler angle rotation matrices corresponding to the 1-2-3 rotation sequence in Figure 2.20 are:

$C_1(\hat{\phi})$, the rotation about $\hat{x}_1$,

$$C_1(\hat{\phi}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{\hat{\phi}} & S_{\hat{\phi}} \\ 0 & -S_{\hat{\phi}} & C_{\hat{\phi}} \end{bmatrix}; \quad (2.44)$$

$C_2(\hat{\eta})$, the rotation about $\hat{y}_2$,

$$C_2(\hat{\eta}) = \begin{bmatrix} C_{\hat{\eta}} & 0 & -S_{\hat{\eta}} \\ 0 & 1 & 0 \\ S_{\hat{\eta}} & 0 & C_{\hat{\eta}} \end{bmatrix}; \quad (2.45)$$

$C_3(\hat{\psi})$, the rotation about $\hat{z}_3$,

$$C_3(\hat{\psi}) = \begin{bmatrix} C_{\hat{\psi}} & S_{\hat{\psi}} & 0 \\ -S_{\hat{\psi}} & C_{\hat{\psi}} & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad (2.46)$$

Setting $C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma})$ equal to C from Eq. (2.40) and performing algebraic manipulation on the nine resulting equations (Appendix A.2), the Euler angles, expressed as functions of the direction cosines are:

Angle $\hat{\eta}$, the rotation about the $\hat{y}_2$ axis, is:

$$\hat{\eta} = \sin^{-1}(c_{31}) \quad (2.47)$$

Figure 2.20 1-2-3 Euler angle rotation sequence between vertically aligned frame (1) and payload fixed frame (4).

Figure 2.21 3-2-1 Euler angle rotation sequence between vertically aligned frame (1) and payload fixed frame (4).

Angle $\hat{\phi}$, the rotation about the $\hat{x}_1$ axis, is:

$$\hat{\phi} = \sin^{-1} \left(\frac{(\pm\sqrt{1-c_{32}^2-c_{31}^2})S_{\hat{\gamma}} - c_{32}C_{\hat{\gamma}}}{\pm\sqrt{1-c_{31}^2}} \right) \quad (2.48)$$

The positive solution for each of the square root terms is used. Angle $\hat{\psi}$, the rotation about the $\hat{z}_1$ axis, is:

$$\hat{\psi} = \tan^{-1} \left(\frac{S_{\hat{\psi}}}{C_{\hat{\psi}}} \right), \quad (2.49)$$

$$\text{where } C_{\hat{\psi}} = \frac{c_{11}}{C_{\hat{\eta}}} \text{ and } S_{\hat{\psi}} = \frac{C_{\hat{\eta}}c_{12} - (C_{\hat{\gamma}}S_{\hat{\phi}}S_{\hat{\eta}} - S_{\hat{\gamma}}C_{\hat{\phi}}S_{\hat{\eta}})c_{11}}{(C_{\hat{\gamma}}C_{\hat{\phi}} + S_{\hat{\gamma}}S_{\hat{\phi}})C_{\hat{\eta}}}.$$

Solving for Euler Angles in a 3-2-1 Rotation Sequence

The rotation matrix C for a 3-2-1 rotation scheme is:

$$C(\psi, \theta, \phi, -\gamma) = C_1(\psi)C_2(\theta)C_3(\phi)C_1(-\gamma) \quad (2.50)$$

where $C_1(-\gamma)$ is the rotation about x_0 due to camera offset angle as shown in Figure 2.13,

$$C_1(-\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{\gamma} & -S_{\gamma} \\ 0 & S_{\gamma} & C_{\gamma} \end{bmatrix}; \quad (2.51)$$

and the rotation matrices corresponding to the 3-2-1 Euler angle sequence in Figure 2.21 are:

$C_3(\phi)$, the rotation about z_1 ,

$$C_3(\phi) = \begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad (2.52)$$

$C_2(\theta)$, the rotation about y_2 ,

$$C_2(\theta) = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}; \quad (2.53)$$

$C_1(\psi)$, the rotation about x_3 ,

$$C_1(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\psi & S_\psi \\ 0 & -S_\psi & C_\psi \end{bmatrix}; \quad (2.54)$$

The PODS postprocessing program, POST321B, currently uses the indirect method to solve for the Euler angles in the 3-2-1 sequence. Equating the rotation matrix of the 1-2-3 sequence with the rotation matrix of the 3-2-1 sequence:

$$C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma}) = C(\psi, \theta, \phi, -\gamma) \quad (2.55)$$

The indirect method then solves the 3-2-1 Euler angles based on the nine equations resulting from the above equality. Appendix A.3 presents the details of the solution. The solution is simplified by the fact that the camera offset angle γ is equal in both the 3-2-1 and 1-2-3 rotation sequences; ie $\gamma = \hat{\gamma}$. The resulting Euler angles in the 3-2-1 system follow:

The angle ϕ , the rotation about z_1 axis, is:

$$\phi = \tan^{-1} \left(\frac{(S_\theta C_\psi S_{\hat{\eta}} + C_\theta S_\psi)}{C_\psi C_{\hat{\eta}}} \right) \quad (2.56)$$

The angle ψ , the rotation about x_3 axis, is:

$$\psi = \sin^{-1} \left(\frac{(S_{\hat{\phi}} C_{\hat{\psi}} + C_{\hat{\phi}} S_{\hat{\psi}} S_{\hat{\eta}})}{C_{\hat{\theta}}} \right) \quad (2.57)$$

The angle θ , the rotation about y_2 axis, is:

$$\theta = \sin^{-1} (C_{\hat{\phi}} C_{\hat{\psi}} S_{\hat{\eta}} - S_{\hat{\phi}} S_{\hat{\psi}}) \quad (2.58)$$

2.3.7. PODS Geometry Measurement Method

To calculate the geometric properties required in the above kinematics and to properly position the camera coordinate frame, two calibration tests are performed. The tests involve measuring the target coordinates in the camera frame when the payload is in known positions. The first, the "Vertical Spin Test", is used to determine the payload radius (r) and camera offset angle (γ). The second, the "Nutation Angle Test" is used to determine the payload half-height (R). The geometric properties are calculated before processing of Linear Model validation tests by the postprocessing program POST321B (§2.3.10).

The angles presented in the following equations refer to a 1-2-3 rotation sequence, again as a result of the history of the development of the PODS calibration system. As above, the relation between target coordinates in the camera frame and payload-fixed frame is:

$$V_4 = CV_0 \quad (2.59)$$

Using the 1-2-3 rotation scheme and including the camera offset, rotation matrix C is:

$$C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma}) = C_3(\hat{\psi})C_2(\hat{\eta})C_1(\hat{\phi})C_1(-\hat{\gamma}) \quad (2.60)$$

Vertical Spin Test

The Vertical Spin Test is so named because it involves recording the payload target positions as the payload spins about the inertial vertical axis with the gimbals locked; angles $\hat{\phi}$ and $\hat{\eta}$ are 0° for all spin angles. Eq. (2.59) is:

$$\begin{bmatrix} -r & r \\ 0 & 0 \\ R & R \end{bmatrix} = \begin{bmatrix} C_{\hat{\psi}} & S_{\hat{\psi}}C_{\hat{\gamma}} & -S_{\hat{\psi}}C_{\hat{\gamma}} \\ -S_{\hat{\psi}} & C_{\hat{\psi}}C_{\hat{\gamma}} & -C_{\hat{\psi}}S_{\hat{\gamma}} \\ 0 & S_{\hat{\gamma}} & C_{\hat{\gamma}} \end{bmatrix} \begin{bmatrix} p_{0x} & q_{0x} \\ p_{0y} & q_{0y} \\ p_{0z} & q_{0z} \end{bmatrix} \quad (2.61)$$

When the payload spin angle $\hat{\psi}$ reaches 0° , (Figure 2.22), Eq. (2.59) becomes:

$$\begin{bmatrix} -r & r \\ 0 & 0 \\ R & R \end{bmatrix} = \begin{bmatrix} p_{0x} & q_{0x} \\ C_{\hat{\gamma}}p_{0y} - S_{\hat{\gamma}}p_{0z} & C_{\hat{\gamma}}q_{0y} - S_{\hat{\gamma}}q_{0z} \\ S_{\hat{\gamma}}p_{0x} + C_{\hat{\gamma}}p_{0z} & S_{\hat{\gamma}}q_{0x} + C_{\hat{\gamma}}q_{0z} \end{bmatrix} \quad (2.62)$$

Solving for the payload radius:

$$r = -p_{0x} \quad (2.63a)$$

and $r = q_{0x} \quad (2.63b)$

Using Eqs. (2.39) to transform Eqs. (2.63) to bottom-left centered coordinates:

$$r = -p_{ox}'' + x_{0c}'' \quad (2.64a)$$

$$r = q_{ox}'' - x_{0c}'' \quad (2.64b)$$

A similar solution exists at $\hat{\psi}=180^\circ$:

$$r = p_{ox}'' - x_{0c}'' \quad (2.65a)$$

$$r = -q_{ox}'' + x_{0c}'' \quad (2.65b)$$

Figure 2.22 Payload as seen in camera frame during vertical spin test at $\hat{\psi} = 0^\circ$.

Figure 2.23 Payload as seen in camera frame during vertical spin test at $\hat{\psi} = 90^\circ$.

Since the geometric properties are not known before these tests are performed, the spin angle cannot be measured and Eqs. (2.64) and (2.65) cannot be used directly. The postprocessing software calculates the payload radius based on the maximum x -coordinate of the time histories of both targets. The following algorithm is used:

$$r = \max([\max(q''_{0xi}), \max(p''_{0xi})]) - x''_{0c} \quad (2.66)$$

where $\max(\vec{a})$ is a function which returns the maximum value in vector $\vec{a}$, p''_{0x} and q''_{0x} are the x -coordinate time histories of targets P and Q respectively in the bottom-left centered coordinate frame, x''_{0c} is the x -coordinate of the ellipse center relative to the bottom-left centered camera coordinate frame.

The camera offset angle $\hat{\gamma}$ is found by examining the coordinates of the targets when the spin angle is 90° (Figure 2.23). Solving for the camera coordinates in Eq. (2.61) at this spin angle:

$$\begin{bmatrix} p_{0x} & q_{0x} \\ p_{0y} & q_{0y} \\ p_{0z} & q_{0z} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -C_{\hat{\gamma}}r + S_{\hat{\gamma}}R & C_{\hat{\gamma}}r + S_{\hat{\gamma}}R \\ S_{\hat{\gamma}}r + C_{\hat{\gamma}}R & -S_{\hat{\gamma}}R + C_{\hat{\gamma}}R \end{bmatrix} \quad (2.67)$$

The y -coordinates of the targets are:

$$p_{0y} = -C_{\hat{\gamma}}r + S_{\hat{\gamma}}R \quad (2.68a)$$

$$q_{0y} = C_{\hat{\gamma}}r + S_{\hat{\gamma}}R \quad (2.68b)$$

Transforming Eqs. (2.68) to bottom-left centered coordinates and simplifying:

$$p''_{0y} - y''_{0c} = -C_{\hat{\gamma}}r \quad (2.69a)$$

$$q''_{0y} - y''_{0c} = C_{\hat{\gamma}}r \quad (2.69b)$$

Solving for the camera offset angle in Eqs. (2.69):

$$\hat{\gamma} = \cos^{-1} \left(\frac{-\ddot{p}_{0y} + \ddot{y}_{0c}}{r} \right) \Big|_{\hat{\phi}=0, \hat{\eta}=0, \hat{\psi}=90^\circ} \quad (2.70a)$$

$$\hat{\gamma} = \cos^{-1} \left(\frac{\ddot{q}_{0y} - \ddot{y}_{0c}}{r} \right) \Big|_{\hat{\phi}=0, \hat{\eta}=0, \hat{\psi}=90^\circ} \quad (2.70b)$$

Similarly, when the spin angle is 270° , the camera offset angle can be found from:

$$\hat{\gamma} = \cos^{-1} \left(\frac{\ddot{p}_{0y} - \ddot{y}_{0c}}{r} \right) \Big|_{\hat{\phi}=0, \hat{\eta}=0, \hat{\psi}=270^\circ} \quad (2.71a)$$

$$\hat{\gamma} = \cos^{-1} \left(\frac{-\ddot{q}_{0y} + \ddot{y}_{0c}}{r} \right) \Big|_{\hat{\phi}=0, \hat{\eta}=0, \hat{\psi}=270^\circ} \quad (2.71b)$$

As with the calculation of the payload radius above, the spin angle cannot be measured directly at this stage using inverse kinematics. Therefore, the spin angle information which can be implied from the target movement in the camera frame must be used.

The spin angle reaches 90° when $\ddot{p}_{0y}$ is at a minimum. For this condition, Eq. (2.70) can be used to solve for $\hat{\gamma}$. The spin angle reaches 270° when $\ddot{q}_{0y}$ is at a minimum. For this condition Eq. (2.71) can be used to solve for $\hat{\gamma}$. The postprocessor calculates the camera offset angle using either Eq. (2.70b) or Eq. (2.71a), depending on which time-history has the smallest y-coordinate by using the following algorithm:

$$\hat{\gamma} = \cos^{-1} \left(\frac{\min([\min(\ddot{q}_{0y_i}), \min(\ddot{p}_{0y_i})]) - \ddot{y}_{0c}}{r} \right) \quad (2.72)$$

where $\min(\vec{a})$ is a function which returns the minimum value in vector $\vec{a}$, $\ddot{p}_{0y}$ and $\ddot{q}_{0y}$ are the y-coordinate time history vectors of targets P and Q respectively in the bottom-left

centered coordinate frame, y''_{oc} is the y-coordinate of the ellipse center relative to the bottom-left centered camera coordinate frame.

Nutation Angle Test

The Nutation Angle Test is so-named because it involves measuring the coordinates of payload targets while the payload is at a known nutation angle. The payload is rotated about the $\hat{y}_2$ axis by an angle $\hat{\eta}$, while $\hat{\phi} = 0$ (Figure 2.24).

Figure 2.24 Payload as seen in camera frame during Nutation Angle Test. Spin angle $\hat{\psi}$ is set arbitrarily.

Angle $\hat{\eta}$ is measured using a digital level. The payload spin angle $\hat{\psi}$ can be set arbitrarily for this test. Rotation matrix $C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma})$ becomes:

$$C|_{\hat{\phi}=0} = \begin{bmatrix} C_{\hat{\psi}}C_{\hat{\eta}} & C_{\hat{\gamma}}S_{\hat{\psi}} - S_{\hat{\gamma}}C_{\hat{\psi}}S_{\hat{\eta}} & -S_{\hat{\gamma}}S_{\hat{\psi}} - C_{\hat{\gamma}}C_{\hat{\psi}}S_{\hat{\eta}} \\ -S_{\hat{\psi}}C_{\hat{\eta}} & C_{\hat{\gamma}}C_{\hat{\psi}} + S_{\hat{\gamma}}S_{\hat{\psi}}S_{\hat{\eta}} & -S_{\hat{\gamma}}C_{\hat{\psi}} + C_{\hat{\gamma}}S_{\hat{\psi}}S_{\hat{\eta}} \\ S_{\hat{\eta}} & S_{\hat{\gamma}}C_{\hat{\eta}} & C_{\hat{\gamma}}C_{\hat{\eta}} \end{bmatrix} \quad (2.73)$$

Substituting into (2.59)

$$\begin{bmatrix} -r & r \\ 0 & 0 \\ R & R \end{bmatrix} = \begin{bmatrix} C_{\psi}C_{\hat{\eta}} & C_{\psi}S_{\psi} - S_{\psi}C_{\psi}S_{\hat{\eta}} & -S_{\psi}S_{\psi} - C_{\psi}C_{\psi}S_{\hat{\eta}} \\ -S_{\psi}C_{\hat{\eta}} & C_{\psi}C_{\psi} + S_{\psi}S_{\psi}S_{\hat{\eta}} & -S_{\psi}C_{\psi} + C_{\psi}S_{\psi}S_{\hat{\eta}} \\ S_{\hat{\eta}} & S_{\psi}C_{\hat{\eta}} & C_{\psi}C_{\hat{\eta}} \end{bmatrix} \begin{bmatrix} p_{0x} & q_{0x} \\ p_{0y} & q_{0y} \\ p_{0z} & q_{0z} \end{bmatrix} \quad (2.74)$$

Solving for the camera coordinates:

$$\begin{bmatrix} p_{0x} & q_{0x} \\ p_{0y} & q_{0y} \\ p_{0z} & q_{0z} \end{bmatrix} = \begin{bmatrix} (-C_{\psi}C_{\hat{\eta}}r + S_{\hat{\eta}}R) & (C_{\psi}C_{\hat{\eta}}r + S_{\hat{\eta}}R) \\ ((-C_{\psi}S_{\psi} + S_{\psi}C_{\psi}S_{\hat{\eta}})r + S_{\psi}C_{\hat{\eta}}R) & ((C_{\psi}S_{\psi} - S_{\psi}C_{\psi}S_{\hat{\eta}})r + S_{\psi}C_{\hat{\eta}}R) \\ ((S_{\psi}S_{\psi} + C_{\psi}C_{\psi}S_{\hat{\eta}})r + C_{\psi}C_{\hat{\eta}}R) & ((-S_{\psi}S_{\psi} - C_{\psi}C_{\psi}S_{\hat{\eta}})r + C_{\psi}C_{\hat{\eta}}R) \end{bmatrix} \quad (2.75)$$

Adding the x -coordinates of both targets:

$$p_{0x} + q_{0x} = 2S_{\hat{\eta}}R \quad (2.76)$$

The angle $\hat{\eta}$ is measured using the digital level. The payload half-height, R can be found from:

$$R = \frac{p_{0x} + q_{0x}}{2 \sin \hat{\eta}} \bigg|_{\hat{\eta}=0} \quad (2.77)$$

The postprocessor uses the average of all the left-bottom centered x -coordinates recorded during a Nutation Angle Test:

$$R = \frac{1}{\sin \hat{\eta}} \left(\frac{\sum_{i=1}^n (p_{0xi}'' + q_{0xi}'')}{2n} - x_{0c}'' \right) \quad (2.78)$$

where p_{0ni}'' and q_{0ni}'' are the i th x -coordinates in the bottom-left centered frame of targets P and Q respectively recorded during the Nutation Angle Test and n is number of readings recorded during the test.

Experimental Verification of PODS-measured Geometry

To verify the results of the above calibration tests, physical measurements of the above parameters are compared with calculated values. The payload and camera geometry for these tests is shown in Figure 2.25.

Figure 2.25 Payload geometry and relative position of CCD plane.

The payload radius in physical units is calculated as follows. From the Vertical Spin Test, the payload radius was measured to be $r=57.3$ pixels. The size of this image on the CCD is $r=0.573$ mm. The distance from the CCD to the center of the payload is used as the CCD to object distance. From Figure 2.25, $z=1296$ mm.

From §2.3.4, the CCD to lens distance is $l_i=15.2$ mm. Rearranging Eq. (2.10) to solve for the physical size of the payload radius, based on its image size:

$$r_o = \frac{r_{cam}(z-l_i)}{l_i} \quad (2.79)$$

where r_{cam} is the image size of the payload radius, and r_o is the object size of the payload radius. Substituting into Eq. (2.79), the payload radius is:

$$r_o = \frac{0.573\text{mm}(1296-15.2)\text{mm}}{15.2\text{mm}} = 48.3\text{mm} \quad (2.80)$$

A similar procedure is used to calculate the physical size of the payload half-height. The PODS-measured payload dimensions and the camera offset angle are summarized in Table 2.2. These values validate the calibration tests as well as the optical system parameters presented in §2.3.4.

Table 2.2. Physical measurements and PODS-based measurements of payload geometry and camera offset angle.

Variable	Physical Measurement	Calculated by Calibration Tests	Experimental Error
Payload Radius, r	47.1 mm	48.3mm	+2.5%
Payload Half-Height, R	120.4 mm	125.3mm	+4.1%
Camera Offset-angle, γ	30.6°	31.6°	+3.3%

2.3.8. Analytical Error Analysis

Two analytical error analyses are presented here to assess the effect of the target position measurement error of DEOPST and out-of-plane displacement on the overall error of PODS. The analysis of the simplified system yields valuable insight into factors effecting the accuracy of PODS.

Nutation Angle Error for Simplified System

The nutation angle (or cone angle) is an appropriate subject for assesment of PODS accuracy behaviour since it is a function of the two lateral Euler angles used in a 3-2-1 Euler angle sequence. The nutation angle can be found from the equation:

$$\nu = \cos^{-1}(\cos \theta \cos \psi) \quad (2.81)$$

where ψ and θ are the rotation about the x and y axes of the payload in a 3-2-1 Euler angles sequence respectively. The nutation angle in three-dimensional coordinates is shown in Figure 2.26.

A simplified arrangement of the TE-LAB payload is shown in Figure 2.27. The payload has only one target at the intersection of its top plane and the payload spin axis. and is restricted to rotate about the y axis. The nutation angle for this configuration is:

$$\nu = \sin^{-1}\left(\frac{p_{0x}}{R}\right) \quad (2.82)$$

where p_{0x} is the x coordinate of target P and R is the payload half-height.

The error of a function of several variables can be found using the linearized error equation of a general function $z = f(x, y, \dots)$, (ref. [13]):

$$S(z) = \left[f_x^2 S_x^2(x) + f_y^2 S_y^2(y) + \dots \right]^{1/2} \quad (2.83)$$

Figure 2.26 Nutation angle (or cone angle) is the angle between the payload spin axis and the vertical axis.

Figure 2.27 Payload geometry for simplified system. Point P is located at the intersection of the payload top plane and spin axis.

where f_i is the partial derivative of f with respect to variable i , $S(i)$ is the error of variable i . Applying the error equation to Eq. (2.82) and assuming no error in the measurement of R :

$$S(v) = \left[v_{p_{0x}}^2 S(p_{0x})^2 \right]^{1/2} = \frac{1}{\sqrt{R^2 - p_{0x}^2}} S(p_{0x}) \quad (2.84)$$

From Eq.(2.82), $p_{0x} = R \sin v$. Substituting into (2.84):

$$S(v) = \frac{1}{\sqrt{R^2 - R^2 \sin^2 v}} S(p_{0x}) \quad (2.85)$$

Eq. (2.85) can be reduced to the following error equation for the nutation angle of the simplified system:

$$S(v) = \frac{1}{R \cos v} S(p_{0x}) \quad (2.86)$$

The three variables each suggest factors affecting PODS accuracy. The error of the nutation angle measurement can be decreased by increasing the payload half-height, decreasing the nutation angle or decreasing the error of the target position measurement. A closer investigation of each factor follows.

Increasing the payload half-height can be accomplished by using a physically larger payload or by increasing the apparent size of the payload in the camera frame. The latter option involves increasing the camera focal length (zooming in).

The $\cos v$ term in the numerator of Eq. (2.86) indicates that the error of the measured nutation angle decreases with decreasing nutation angle. This behaviour is due to the nature of the accuracy of the inverse sine function. The $\cos v$ term predicts the same decrease in error as expected due to the assumption of no out-of-plane displacements, to be presented shortly.

There are two factors contributing to the target position error, $S(p_{0z})$. The first factor is the accuracy with which DEOPST can measure the position of a target in the camera frame. The second factor contributing to target position error is the assumption of no out-of-plane displacement to be discussed below.

Estimated Error due to DEOPST

According to Laurin [11] the position measurement accuracy of the DEOPSD and DEOPSR systems is $\pm 0.1\%$ of the field of view. For a field-of-view of 640×480 pixels, the maximum position error will be ± 0.64 pixels. Assuming a nutation angle of 30° , a typical payload half-height image size of 149 pixels and substituting into Eq.(2.86):

$$S(v) = \frac{1}{(149 \text{ pixels})(\cos 30^\circ)} (\pm 0.64 \text{ pixels}) = \pm 4.96 \times 10^{-3} \text{ rad} \quad (2.87)$$

The estimated error of the nutation angle due to DEOPST accuracy, in degrees, is:

$$S(v) = \pm 0.28^\circ \quad (2.88)$$

Estimated Error due to Out-of-Plane Displacement

This analysis will estimate the nutation angle error of the simplified system due to the assumption of no out-of-plane displacement. The payload and camera of the simplified system are shown in Figure 2.28. All dimensions, including camera-frame dimensions, are in real-world units. Recall from §2.3.5 that the no out-of-plane displacement assumption (ie. the CCD to object distance is constant) permits the cancellation of all real-world to CCD plane scaling factors. This assumption also enables the camera coordinates to be used in the kinematic equations in place of the real-world coordinates when applied consistently. The analysis presented here will calculate the nutation angle error using the difference between the actual and apparent real-world coordinates of the target, based on the camera coordinate of the target.

Since the payload target in the simplified system is located on the payload spin axis, the out-of-plane displacement is only due to change in nutation angle. In the TE-LAB apparatus, the out-of-plane displacements of the targets are more complex. The camera is offset slightly from the vertical and there are two targets mounted at a distance

$$p_{0x}|_{z=z_1} = \frac{(z_1 - l_i)}{l_i} p_{cam,x} \quad (2.89)$$

Accounting for the out-of-plane displacement, the actual real-world coordinates of the target are:

$$p_{0x}|_{z=z_2} = \frac{(z_2 - l_i)}{l_i} p_{cam,x} = \frac{(z_1 + R(1 - \cos v) - l_i)}{l_i} p_{cam,x} \quad (2.90)$$

The real-world coordinate measurement error can be expressed as the difference between the measured coordinate and the actual coordinate:

$$S(p_{0x}) = p_{0x}|_{z=z_1} - p_{0x}|_{z=z_2} = -\frac{R(1 - \cos v)}{l_i} p_{cam,x} \quad (2.91)$$

Rearranging Eq. (2.90), the camera coordinate of the target, using the actual CCD to object distance z_2 , is:

$$p_{cam,x} = \frac{l_i}{(z_2 - l_i)} p_{0x} = \frac{l_i}{(z_1 + R(1 - \cos v) - l_i)} p_{0x} \quad (2.92)$$

Rearranging Eq. (2.82), the real-world coordinate of the target can be expressed as a function of nutation angle as:

$$p_{0x} = R \sin v \quad (2.93)$$

Substituting Eq. (2.92) and (2.93) into (2.91) and solving for the error in the measured real-world coordinate as a function of nutation angle:

$$S(p_{0x}) = -\frac{R^2(1 - \cos v) \sin v}{(z_1 + R(1 - \cos v) - l_i)} \quad (2.94)$$

Substituting the error in measured real-world coordinates into the error equation of the nutation angle, Eq. (2.86), and converting to degrees:

$$S(v) = -\tan v \frac{R(1 - \cos v)}{(z_1 + R(1 - \cos v) - l_i)} \frac{180}{\pi} \quad [\text{degrees}] \quad (2.95)$$

Eq. (2.95) is plotted as a function of nutation angle in Figure 2.29 for the lengths used in TE-LAB ($z_1 = 1296 \text{ mm}$, $R = 120.4 \text{ mm}$, $l_i = 15.2 \text{ mm}$). The magnitude of the estimated error is small for small nutation angles but becomes significant for larger angles. The assumption of no out-of-plane displacement creates a bias error in the nutation angle measurement.

Figure 2.29 Estimated error in nutation angle measurement as a function of nutation angle for simplified system.

The nutation angle error due to the assumption of no out-of-plane displacements for the TE-LAB nutation angle range is estimated to be:

$$S(v) = 0^\circ / -0.41^\circ \quad (2.96)$$

This error could be minimized by incorporating the out-of-plane displacements of the targets in the payload kinematics. Laurin [11] uses this approach in measuring the

angular positions of the Daisy ribs. The three-dimensional real-world coordinates of the rib-mounted targets are calculated using a recursive routine. The angular positions of the Daisy ribs are then calculated based on the real-world coordinates of the targets.

Analysis of No Out-of-plane Displacement Assumption

An examination of the properties of the thin lens equation reveals conditions for which the out-of-plane displacement does not affect the image size. The thin lens equation for image size Eq. (2.10) is repeated here:

$$y_{cam} = \frac{l_i}{(z - l_i)} y_o \quad (2.97)$$

The change in image size with respect to object position is:

$$\frac{\partial y_{cam}}{\partial z} = -\frac{l_i}{(z - l_i)^2} y_o \quad (2.98)$$

There are two conditions when the object size becomes independent of the out-of-plane displacement. The first is the condition that the CCD to object distance is infinitely large relative to both the size of the object and the CCD to lens distance:

$$\lim_{z \rightarrow \infty} \frac{\partial y_{cam}}{\partial z} = 0 \quad (2.99)$$

The second condition is that the CCD to lens distance is made as small as possible (zooming out):

$$\lim_{l_i \rightarrow 0} \frac{\partial y_{cam}}{\partial z} = 0 \quad (2.100)$$

Neither of these conditions is practical in the limit because both result in an image size of zero. However, the conditions do suggest useful means of selecting parameters based on the optics of the thin lens.

Selection of Camera Focal Length

From Eq. (2.86), it can be seen that the error of the nutation angle can be reduced by increasing the apparent size of the payload half-height, R , in the camera frame, ie zooming in. Eq. (2.97) suggests that the error due to out-of-plane displacement can be

neglected as long as the CCD to lens distance is small (zooming out). These two properties dictate opposing focal length settings. The former suggests maximizing the focal length while the latter suggests minimizing the focal length. A conceptual diagram of these error and the resulting total error is shown in Figure 2.30.

Figure 2.30 Conceptual diagram of PODS errors due to camera focal length.

There are two strategies which could be used to improve the accuracy of PODS:

- 1) Incorporate out-of-plane displacements in kinematics and increase focal length to maximum possible. This corresponds to Point 1 on graph.
- 2) Find focal length which minimizes the total error. Point 2 on graph.

During PODS development, it was assumed that the effect of out-of-plane displacements on the system accuracy would be negligible. The focal length was increased to the maximum possible in an attempt to reduce errors due to DEOPST accuracy only; Point 3 in Figure 2.30. Despite the fact that this is solution does not result in optimum accuracy, the experimental accuracy of the system proved to be acceptable as presented below.

2.3.9. Experimental Error Analysis

Tests were performed to assess the experimental error of PODS. The tests consisted of measuring the nutation angle as a function of payload spin angle using PODS. The nutation angle measured by PODS is then compared to the nominal nutation angle measured using a digital level. The experimental nutation angle data contains trends which identify possible error sources in the system.

Accuracy Test Procedure

The payload spin angle (Angle $\hat{\psi}$ in a 1-2-3 rotation sequence) was fixed at approximately zero degrees. The payload nutation angle was then measured as a function of vertical spin angle (Angle ϕ in a 3-2-1 rotation sequence). The tests involved spinning the payload drive shaft by hand for approximately two revolutions at a rate of approximately 1/3 rev/sec. Tests were performed for fixed nutation angles of 12.0° and 30.1° as measured by a digital level. The tests were repeated for payload spin angle of 90° . Both payload spin angle configurations are shown in Figure 2.31.

Figure 2.31 Two payload spin angle configurations ($\hat{\psi}=0^\circ$ and $\hat{\psi}=90^\circ$) used in experimental accuracy tests. Nutation angles of $\nu=12.0^\circ$ and $\nu=30.1^\circ$ were tested.

Accuracy Test Results

The nutation angle measured by PODS as a function of vertical spin angle, ϕ , is plotted in Figure 2.32 for $\hat{\psi}=0^\circ$. A summary of results is presented in Table 2.3. This data will be the subject of the current analysis. Similar results and conclusions can be obtained from the tests conducted at $\hat{\psi}=90^\circ$. The scatter in the data can be classified into two groups; large scale scatter and small scale scatter.

Figure 2.32 Nutation angle as a function of spin angle measured by PODS for nominal nutation angles of a) 30.1° and b) 12.0° .

Table 2.3 Results of PODS accuracy tests.

Nutation Angle (degrees)	Error of PODS-Measured Nutation Angle (degrees)	Magnitude of Maximum Small-Scale Data Scatter (degrees)
30.1	+1.5/-2	0.3
12.0	+/-0.5	0.3

Analysis of Large Scale Data Scatter

The minimum nutation angle occurs at a vertical spin angle of approximately 90° for both tests. This spin angle corresponds to the maximum out-of-plane displacement of the targets away from the camera. This behaviour is identical to that predicted by the out-of-plane displacement error analysis in §2.3.8.

The vertical spin angle resulting in a zero bias is approximately the point where the payload is in the x_1z_1 plane ($\phi=0^\circ, 180^\circ$). This suggests that the bias is relative to the vertical axis and not the camera boresight axis as predicted by the simplified analytical model.

At a vertical spin angle of 270°, the targets are closer to the camera than when the payload is in the vertical position. The simplified analysis revealed that a negative nutation angle bias results when the payload moves away from the camera. By observation of the experimental data, there appears to be a similar positive nutation angle bias when the targets move towards the camera.

The magnitude of the maximum experimental errors (1.5° and 2°) is larger than the 0.4° predicted. This is likely due to oversimplifications made in the analytical model. In addition, the experimental nutation angle, computed using Eq. (2.81), is based on the two lateral Euler angles which each have a certain experimental error. The linearized error equation, Eq. (2.83), indicates that this functional form will compound the errors contributed by the individual terms in the function, increasing the nutation angle error.

Analysis of Small Scale Data Scatter

The size of the small scatter band is 0.3° for both nutation angles. This value is smaller than the 0.56° band ($\pm 0.28^\circ$) predicted by the error analysis due to DEOPST accuracy of §2.3.8. The size of the scatter band appears to be independent of nutation

angle. This suggests that the small scale scatter is not entirely caused by position measurement error as predicted by Eq. (2.86). The remaining small scale data scatter may be due to surface imperfections on the targets, vibrations in the payload, the use of oversized target images or other as yet unidentified causes.

Summary of PODS Accuracy

If the large scale scatter proves to be entirely due to the out-of-plane displacement, it should be possible to eliminate it by modifying the kinematics to include this displacement, as discussed above. Only the small scale scatter due to DEOPST accuracy and other effects would remain. The resulting nutation measurement would have a high degree of accuracy. For this reason, the implementation of the out-of-plane displacements in the kinematic equations is an important item for future work.

This experimental error analysis has revealed that the current accuracy of the PODS-measured nutation angle is acceptable. Due to the fact that the nutation angle is a function of the two lateral Euler angles, one of which (θ) is used almost exclusively for TE-LAB data analysis, it will be assumed that the accuracy of all PODS-measured Euler angles is acceptable for the present work.

2.3.10. Experimental Data Postprocessing Tools

The data postprocessing tools convert the cartesian coordinates of the targets into payload Euler angle time-histories and permit analysis of the time-histories. Two programs were developed to process and analyze the experimental data. The first program, POST321B, uses inverse kinematics to produce the payload Euler angle time histories. The Euler angle time histories are then used by FFTPOST2, under user guidance, to produce Fast Fourier Transforms of the data and to fit exponential curves to the data for stability analyses.

Target Position Postprocessor, POST321B

The POST321B program performs two functions to compute the Euler angle time histories. The first is the calculation of payload geometry and camera offset angle using input from the Vertical Spin Test and the Nutation Angle Test and implementing the calibration equations presented in §2.3.7. The second function of POST321B is the Euler angle computation. POST321B uses the inverse kinematic equations of §2.3.6 to convert the target cartesian coordinates to Euler angles in a 3-2-1 rotation sequence.

POST321B is written in Microsoft C and runs under DOS. The letter "B" is used to indicate that this version of the program can be run in batch mode. The program enters batch mode if a filename is included on the command line when the program is called. Prior to running in batch mode, an input file is prepared which includes the files to be processed, corresponding calibration tests and output filenames. The program then processes all the named files unattended. The batch mode speeds up postprocessing and reduces user input errors.

Euler Angle Postprocessor, FFTPOST2

FFTPOST2 is a MATLAB program file used to analyze the Euler angle time histories generated by POST321B. The program has two analysis features. The first is the Fast Fourier Transform feature. It allows the user to select a window of the time history data to be analyzed. This window of data is then processed by the *fft* routine in MATLAB. The *fft* routine returns the magnitude and phase response of the time history over the frequency spectrum. The magnitude and phase of the response, from 0 Hz to 30 Hz (half the sampling frequency of 60Hz), are plotted. The user can then zoom in on the part of the frequency spectrum of interest. Finally, the user can identify the frequency peaks of interest. The frequencies are labeled on the spectrum plot and are stored in a file for each spin rate. The frequencies can then be plotted as a function of spin rate for comparison with damped natural frequencies of the system predicted by the Linear Model. McClure, ref.[7], presents details of this procedure.

The second analysis feature of the program is the exponential curve fit. The user identifies two points on the envelope of Euler angle time history. An exponential curve is then fitted to the two points as follows:

The general form of an exponential equation is:

$$y = ae^{bt} \quad (2.101)$$

where a is a scaling factor and b is the exponential time coefficient. The user chooses two points from the time history; (t_1, y_1) and (t_2, y_2) . Substituting into Eq. (2.101) and solving for a in both equations:

$$a = y_1 e^{-bt_1} \quad (2.102)$$

$$a = y_2 e^{-bt_2} \quad (2.103)$$

Equating the right sides of Eqs. (2.102) and (2.103):

$$y_1 e^{-bt_1} = y_2 e^{-bt_2} \quad (2.104)$$

Solving for b :

$$b = \frac{1}{(t_2 - t_1)} \ln \frac{y_2}{y_1} \quad (2.105)$$

The exponential time coefficient b requires only the change in time for the envelope to change from y_1 to y_2 . The exponential time coefficient b is therefore independent of the absolute time of the test. Details of the data analysis using the exponential time coefficient are presented in Chapter 4.

Substituting into Eq. (2.103) and solving a :

$$a = y_2 \left(\frac{y_2}{y_1} \right)^{\frac{-t_2}{(t_2 - t_1)}} \quad (2.106)$$

The scaling factor a is presented here for completeness only. It is not used to analyze the experimental data in the present work.

2.4. Rotation Rate Control System (RRCS)

2.4.1. Introduction

The Rotation Rate Control System (RRCS) controls the angular velocity of both the payload and upper rotating table drive motors. By explicitly controlling motor position as a function of time, the angular velocity of the motors can be implicitly controlled. In addition, the angular difference between the motors (and tether twist) is implicitly minimized by commanding both motors to simultaneously follow the same position profile.

An overview of the RRCS signal flow is shown in Figure 2.33. A reference shaft position profile is generated corresponding to the desired velocity profile. The software controller compares this reference position with the current shaft position of each motor and calculates a digital control signal for each motor. A digital to analog converter converts the digital signal to an analog voltage signal v_{out} . The analog signal drives an amplifier located near each of the DC motors driving the upper rotating table and payload. The amplifiers in turn power the motors. A shaft encoder attached to each motor generates pulses as the motor turns. These pulse signals are amplified by the line drivers and transmitted back to the pulse counter circuit in the computer. The pulses are accumulated by a counter circuit to be read by the controller.

The following sections present details of the RRCS hardware, software, reference algorithms, control algorithms and an evaluation of the system performance.

Figure 2.33 Overview of signal flow in the Rotation Rate Control System.

2.4.2. RRCS Hardware

Motor Amplifier and Line Driver Circuit

The motor amplifier and line driver circuit (Figure 2.34) performs two functions. The first is to provide a current source to the drive motor associated with the circuit. The second function is to amplify the shaft encoder signal for transmission to the pulse counter circuit. Two copies of this circuit exist in the TE-LAB apparatus. One is associated with the upper-rotating table and the other is associated with the payload drive motor. Each circuit is mounted in a small aluminum box located near its respective motor.

the threshold input voltage. The threshold voltage is the input voltage level at which the FET starts to produce a nominal output current.

The line driver subcircuit amplifies shaft encoder signals for transmission to the pulse counter board. The low current analog pulse signal from the shaft encoder is converted into a high current (approximately 300 mA) for transmission along the 40 foot long cables to the pulse counter circuit in the computer.

Shaft Encoder Pulse Counter Board

The counter board (Figure 2.35) includes two 12-bit counter chips (74HC4040) which count pulses from each of the shaft encoders. The eight least-significant bit lines of each counter chip are connected to a buffer (74AL374). When the buffer is selected, the corresponding encoder count will appear on the buffer output lines. This value is read by the control software (§2.4.3) as an 8-bit word through the digital input port on the RTI-815 board.

RTI-815 Analog / Digital Board

An Analog Devices RTI-815 Analog / Digital board is installed in the PC-compatible computer. This board is the interface between the control system software and electronics. The 16 analog input channels have 12 bit resolution (4096 levels over the full input range) and are selected to have an input range of 0 to +10 volts. The two analog output channels have 12 bit resolution and are selected to have an output range of 0 to +10 volts. The board also includes eight 1-bit digital input lines, eight 1-bit digital output lines and three user-programmable timers. A summary of the channels currently used are summarized in Table 2.4.

Figure 2.35 Schematic of shaft encoder pulse counter board.

Table 2.4 Input and output channels of the RTI-815 Analog / Digital board used by the Rotation Rate Control System.

Channel	Description	Comments
Digital Output, Bit 0	Buffer chip select	0 - upper motor position 1 - lower motor position
Digital Output, Bit 1	Counter reset	Resets both counter chips
Digital In, (Bits 0 - 7)	Counter chip value	Number of accumulated shaft encoder pulses counted by selected counter chip
Analog Out, Channel 0 and 1	Control signal for upper and lower motor respectively	Analog value of $v_{out}(kT)$ for corresponding motor

One of the RTI-815 timers is used to generate the control system timing signal. The timer is programmed to generate a 200 Hz pulse signal. The output line of the timer is physically wired to the interrupt request line, IRQ2, of the PC. At each occurrence of a pulse on the IRQ2 line, the control software is executed. Details of the control software are presented below (§2.4.3).

The fact that the digital input port is limited to an 8-bit word size places a theoretical limit on the maximum shaft speed of the system. The maximum shaft speed due to control system limitations is found as follows:

$$\omega_{\max} = \frac{N_{\max} [\text{pulses/sample}]}{T [\text{seconds/sample}] * E [\text{pulses/motor rev}] * G [\text{motor rev/output shaft rev}]} \quad (2.108)$$

where $N_{\max}$ is the maximum number of pulses which can be counted per sample (8 bits = 256), T is the sampling period (0.005 sec), E is the number of shaft encoder pulses per motor revolution (500), G is the number of gearbox output shaft revolutions per motor revolution (14). Substituting into (2.108) and solving:

$$\omega_{\max} = 7.3 \frac{\text{output shaft revs}}{\text{sec}} \quad (2.109)$$

Other limitations to the maximum motor speed include the manufacturer's maximum recommended speed of 4.8Hz due to gearbox limitations. A further limitation to maximum speed is the photogrammetric target tracking software which can only track targets up to a rotation rate of 3Hz using the current camera focal length. Therefore, it can be seen that the 8-bit word size is not a limit on the simulation capability of TE-LAB.

2.4.3. RRCS Software

The purpose of the RRCS is to control the rates of both drive motors while the positions of the payload targets are recorded by DEOPST. To accomplish this goal using a single computer, the RRCS software was integrated into three areas of the DEOPST software as follows.

Real-Time Control Software

Before starting the real-time target tracking function `rttr()`, the function `install()` modifies the interrupt vector table and initializes the RTI-815 timer. The interrupt table is modified to cause the processor to immediately execute the code in the function `controller()` when an interrupt signal is detected on the IRQ2 line. Upon exiting the controller function, the processor returns to the code it had been executing in `rttr()` prior to the interrupt. The timer generates interrupt signals at 200 Hz on the IRQ2 line, thus causing `controller()` to be executed every 5 milliseconds.

The function `controller()` calculates a reference shaft position for the current time step based on Eqs.(2.126 to 2.129). The upper and lower motor shaft positions are measured via the RTI-815 board by selecting the respective buffer chip and reading the digital input port value. The control algorithm (§2.4.5) computes an appropriate control signal for each motor. The control signal of each motor is sent to the analog output channels of the RTI-815 board.

The controller routine executes quickly enough that it does not interfere with the operation of the photogrammetric target tracking function, `rttr()`. The two functions, `controller()` and `rttr()` can be said to execute concurrently.

User-Defined Gains

A menu area accessed from the `deopst()` main menu allows the operator to set motor controller parameters. The parameters include not only the controller gains, but also motor ramp time and final motor speed.

Operator Control from Real-Time Menu

The real-time menu of `rttr()` has been expanded to allow the operator to start and stop the motors as well as to perform system emergency stops. The start and stop commands initiate the corresponding position profiles described in §2.4.4. The emergency stop command turns off the power to the motor by setting the control signal level to zero. Operator commands are passed to `controller()` by the changing global variables in the software indicating the motor operating status.

2.4.4. Reference Shaft Position Algorithms

The sensor being used in this controller measures shaft position and not shaft velocity. Therefore, the RRCS is designed to control the motor shaft position as a function of time and implicitly control shaft velocity. The position profiles used to control the motors are time integrals of the desired velocity profiles. This section describes the desired velocity profiles and the corresponding continuous and discrete time shaft position profiles.

Angular Velocity Profiles

In ramp-up mode, the angular velocity of the motors is increased linearly with time from standstill to a desired final velocity (Line 1, in Figure 2.36).

Figure 2.36 Reference motor shaft velocity profiles.

Each line is of the form:

$$\omega(t) = At + B \quad (2.110)$$

The slope of line 1 is:

$$A = \frac{\Delta\omega}{\Delta t} = \frac{\omega_{test}}{t_r} \quad (2.111)$$

where ω_{test} is the angular velocity at which the test is to be run, and t_r is the ramp time to reach ω_{test} . Since $\omega_1(0) = 0$, constant B in Eq. (2.110) is zero. The reference velocity for ramp-up mode is:

$$\omega_1(t) = \left(\frac{\omega_{test}}{t_r} \right) t \quad (2.112)$$

In constant spin rate mode, line 2, the angular velocity is constant as a function of time. Substituting into Eq. (2.107), the slope A is zero and the constant B is the test velocity ω_{test} . The reference velocity for constant spin rate mode is:

$$\omega_2(t) = \omega_{test} \quad (2.113)$$

Due to the differences in inertia of the payload and upper-rotating table, the two motors do not slow down at the same rate when the power to each is turned off. The ramp-down modes were implemented in the system to ensure that the twist in the tether is minimized during the motor-stop procedure.

There are two ramp-down modes in the system. The first ramp-down mode occurs while the angular velocity of the motors is being ramped-up (Line 3a). This could occur if the operator chooses to cancel the test before the system reaches the test velocity. The second ramp-down mode occurs after the system has reached the test velocity (Line 3b). This mode would normally be encountered after a test has been successfully completed.

The slope of both ramp-down modes is the negative slope of the ramp-up mode. Therefore, the slope A for line 3a is:

$$A = -\frac{\omega_{test}}{t_r} \quad (2.114)$$

Substituting into (2.110):

$$\omega_{3a}(t) = -\frac{\omega_{test}}{t_r} t + B \quad (2.115)$$

The constant B is found from the intersection of lines 1 and 3a in Figure 2.36:

$$\omega_{3a}(t_{s1}) = \omega_1(t_{s1}) = \left(\frac{\omega_{test}}{t_r} \right) t_{s1} \quad (2.116)$$

where t_{s1} is the time at which the user selects the ramp-down mode. Substituting into (2.115), solving for B and simplifying, the equation for line 3a is::

$$\omega_{3a}(t) = \frac{\omega_{test}}{t_r} (-t + 2t_{s1}) \quad (2.117)$$

The equation for line 3b has the same slope as line 3a. Using the intersection of lines 2 and 3b to find the constant B :

$$\omega_{3b}(t_{s2}) = \omega_2(t_{s2}) = \omega_{test} \quad (2.118)$$

where t_{s2} is the time at which the user selects the ramp-down mode. Substituting into (2.110) and rearranging, the equation for line 3b is:

$$\omega_{3b}(t) = \frac{\omega_{test}}{t_r} (-t + t_r + t_{s2}) \quad (2.119)$$

Angular Position Profiles

Since only shaft position can be measured and controlled, the controller will require reference shaft position profiles rather than reference shaft velocity profiles. Therefore, the velocity profiles must be integrated. Integrating $\omega_1(t)$:

$$\begin{aligned} \theta_1(t) &= \int \omega_1(t) dt = \int \frac{\omega_{test}}{t_r} t dt \\ \theta_1(t) &= \frac{1}{2} \frac{\omega_{test}}{t_r} t^2 + C \end{aligned} \quad (2.120)$$

But $\theta_1(0) = 0$. Therefore, the constant of integration C is zero. The ramp-up position profile is:

$$\theta_1(t) = \frac{1}{2} \frac{\omega_{test}}{t_r} t^2 \quad (2.121)$$

Similarly, integrating the remaining velocity profiles produces the following position profiles. The constant speed position profile is:

$$\theta_2(t) = \omega_{\max} \left(t - \frac{1}{2} t_r \right) \quad (2.122)$$

The ramp-down during ramp-up position profile is:

$$\theta_{3a}(t) = \frac{\omega_{\max}}{t_r} \left(-\frac{1}{2} t^2 + 2t_{s1}t - t_{s1}^2 \right) \quad (2.123)$$

The ramp-down from constant speed position profile is:

$$\theta_{3b}(t) = \frac{\omega_{\max}}{t_r} \left(-\frac{1}{2} t^2 + t_r t + t_{s2}t - t_{s2}^2 - t_r^2 \right) \quad (2.124)$$

Discrete-Time Angular Position Profiles

The motor shaft positions are measured at a rate of 200 Hz. This corresponds to a sampling period T of 0.005 s = 5 ms. The reference motor shaft position profile must be discretized in steps of 5 ms to allow for direct comparison with measured shaft positions. The transformation between continuous and discrete time is:

$$t = kT \quad (2.125)$$

where t is time in seconds, k is time step number, T is the sampling period in seconds. Substituting discrete variables into (2.121) produces the discrete-time ramp-up position profile:

$$\theta_1(kT) = \frac{1}{2} \frac{\omega_{\max}}{t_r} k^2 T^2 \quad (2.126)$$

The remaining discrete-time position profiles are found using a similar procedure. The constant speed profile is:

$$\theta_2(kT) = \omega_{\max} \left(kT - \frac{1}{2} t_r \right) \quad (2.127)$$

The ramp-down during ramp-up profile is:

$$\theta_{3a}(kT) = \frac{\omega_{ref}}{t_r} \left(-\frac{1}{2} k^2 T^2 + 2t_{r1}kT - t_{r1}^2 \right) \quad (2.128)$$

The ramp-down from constant speed profile is:

$$\theta_{3b}(kT) = \frac{\omega_{ref}}{t_r} \left(-\frac{1}{2} k^2 T^2 + t_r kT + t_{r2}kT - t_{r2}^2 - t_r^2 \right) \quad (2.129)$$

2.4.5. Control Algorithm

The control algorithm uses Proportional, Integral and Derivative error terms and corresponding gains to minimize the error between each of the motor output shaft positions and the reference shaft position profiles. The current shaft position error is:

$$\theta_{error}(kT) = \theta_{ref}(kT) - \theta_{shaft}(kT) \quad (2.130)$$

The accumulated error is:

$$\theta_{integrated_error}(kT) = \theta_{integrated_error}((k-1)T) + \theta_{error}(kT) \quad (2.131)$$

The change in shaft position error is:

$$\theta_{derivative_error}(kT) = \theta_{error}(kT) - \theta_{error}((k-1)T) \quad (2.132)$$

The output voltage to the FET driving the corresponding motor is then calculated by the following PID-type equation:

$$v_{out}(kT) = G * SAMPLE_PERIOD * [ge * \theta_{error}(kT) + gd * \theta_{derivative_error}(kT) + gi * \theta_{integrated_error}(kT)] + 1200 \quad (2.133)$$

where G is used as motor on/off switch (0=off, 1=on), $SAMPLE_PERIOD$ is the sampling period of the controller, ge is the error gain, gi is the integral gain, gd is the derivative gain, 1200 is an offset used to raise v_{out} to the threshold voltage of the FET (§2.4.2). The

sampling period term is included in Eq. (2.133) to scale the control algorithm if the sampling period is changed.

The motor controller gains were found using a systematic experimental approach. For each motor controller algorithm, the error gain was adjusted until the motor could follow the reference position profile. Then, the integral and derivative gains were adjusted to fine tune the total error and damping characteristics respectively of the motor response. The controller gains which are currently used in the TE-LAB apparatus are shown in Table 2.5.

Table 2.5 Controller gains used in TE-LAB apparatus.

	Upper-Rotating Table Algorithm	Payload Algorithm
Error Gain: g_e	100	200
Integral Gain: g_i	5000	4000
Derivative Gain: g_d	0.5	0.5

2.4.6. RRCS Performance

Two types of data are used to evaluate the performance of the Rotation Rate Control System. The first is shaft encoder related data and the second is PODS-related data. The PODS data permits an evaluation of RRCS performance using a sensor independent of the control loop.

Performance Based on Shaft Encoder Data

RRCS performance was assessed based on shaft encoder data of tests spanning the TE-LAB spin rate operating range. For each spin rate, the test with the lowest possible tension which resulted in maximum nutation angle growth was chosen. Three parameters of the RRCS were evaluated at each spin rate:

1. The variation of payload shaft speed from nominal shaft speed.
2. The variation of upper rotating table shaft speed from nominal shaft speed.
3. The difference between payload position and upper rotating table position.
This measurement represents the amount of twist experienced by the tether.

Figures 2.37 2.38 and 2.39 respectively present these parameters for nutation angles less than 15° . The poorest performance is found at the lowest spin rate, 0.5 Hz. Payload shaft speed variation is $+30/-10\%$ of nominal, upper table shaft speed variation is $+26/-18\%$ of nominal and tether twist is $+8/-11^\circ$. Performance improves as the spin rate increases. For higher spin rates the performance figures are approximately as follows: payload shaft speed variation, $\pm 5\%$; upper table shaft speed variation, $\pm 5\%$; tether twist, $\pm 2^\circ$.

For nutation angles between 15 and 30 degrees, the payload shaft speed and tether twist show wider scatter and less dependance on spin rate. Payload shaft speed variation is approximately $+30/-12\%$ and tether twist is $+10/-11^\circ$ throughout the spin rate range. Upper table shaft speed variation remains the same as above at these larger nutation angles.

This analysis indicates that for small nutation angles, the control system performance improves with increasing shaft speed. The performance of the control system deteriorates at higher nutation angles because of the rapidly varying loading about the motor spin axis under these conditions.

2 of/de 2

PM-1 3½" x 4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

Figure 2.37 Variation of payload shaft speed from nominal shaft speed as a function of nominal shaft speed for nutation angles less than 15°.

Figure 2.38 Variation of upper rotating table shaft speed from nominal speed as a function of nominal speed for nutation angles less than 15°.

Figure 2.39 Tether twist (difference in payload and upper table shaft position) as a function of nominal shaft speed for nutation angles less than 15° .

Performance Based on PODS Data

The Fast Fourier Transforms of the payload Euler angles, when the gimbals are unlocked, exhibit an anomaly in the spin rate which prompted further evaluation of the RRCS performance. According to the Linear Model, a peak in the frequency domain is expected at exactly the payload spin rate. The FFTs of the experimental data, however, reveal a peak in all tests at approximately +4% of the nominal spin rate.

This behaviour is demonstrated by examining the x-coordinate of payload target P in the camera frame as the payload spins at a nominal speed of 1Hz with the gimbals locked (Vertical Spin Test). It can be seen in Figure 2.40 that the target completes eight cycles in 7.74 seconds. The payload shaft speed is therefore 1.034 Hz, a +3% error.

A close examination of the `controller()` function revealed the probable cause of this error. The command to reset the counter chips is in a loop which is executed twice, once for each motor. Thus, during each call to the `controller()` function, the counter chips are reset twice. The effect of resetting the counter chips twice is that some shaft encoder pulses are not counted by the control system. Therefore, the controller runs the motors slightly faster to compensate for these "missing" pulses.

If this is the cause of the constant motor speed error, it is easily fixed by deleting the reset commands from the above-mentioned loop and placing a single reset command after both motor counts are read.

RRCS Performance Summary

The RRCS provides satisfactory performance throughout the TE-LAB spin rate range, particularly at higher spin rates and nutation angles below 15 degrees. For the remainder of the present work, all spin rates will refer to the nominal value.

$$\Delta t = 29.74 - 22 = 7.74 \text{ sec}$$

Number of cycles: 8

$$\text{Frequency} = \frac{\text{Number of cycles}}{\Delta t} = \frac{8 \text{ cycles}}{7.74 \text{ sec}} = 1.034 \text{ Hz}$$

Figure 2.40 PODS output of the x-coordinate of payload target P during Vertical Spin Test at nominal rotation rate of 1 Hz.

Chapter 3

3. THEORETICAL STABILITY BOUNDARIES

An overview of the derivation of the theoretical stability boundaries is presented in this chapter. The derivation is based on examining the positive definiteness of the stiffness matrix of the equations of motion in the Linear Model. The resulting theoretical stability boundaries for the TE-LAB configurations are then presented.

3.1. Equations of Motion from the Linear Model

The Linear Model, developed by Vigneron et al. [5] is a rigorously linearized analytical model of the dynamics of a spinning payload and tether system in a gravity field (Figure 3.1).

Figure 3.1 Analytical model of spinning payload and tether system in a gravity field in reference equilibrium state.

The linearized equations of motion of the system, without boom deformations are written in matrix form as:

$$[M](\ddot{x}) + ([G] + [D])(\dot{x}) + [K](x) = 0 \quad (3.1)$$

where $[M]$ is a positive-definite symmetric mass matrix which includes the payload inertia, tether mass and boom mass, $[G]$ is a skew-symmetric matrix which describes gyroscopic effects of the payload and tether, $[D]$ is a symmetric damping matrix which describes payload and tether damping and $[K]$ is a symmetric stiffness matrix which includes the material stiffness of the booms, the tension-induced and gravity-induced stiffness of the tether, and negative stiffness terms contributed by centrifugal forces acting on the tether. The state vector of the Linear Model is:

$$\{x\}^T = \{\psi, \theta, q_9^T, q_{10}^T\} \quad (3.2)$$

where ψ and θ are the Euler angles corresponding to a local rotation about the x and y axis respectively in a 3-2-1 Euler angle rotation sequence, and q_9 and q_{10} are modal coordinate column matrices corresponding to lateral (u,v) components of the tether displacement. The dimensions of each modal coordinate column matrix is $(n \times 1)$ where n is the number of assumed tether modes. Using n assumed tether modes, the dimensions of the state vector are $(2n + 2) \times 1$. The dimensions of each matrix in Eq. (3.1) are $(2n + 2) \times (2n + 2)$.

The stiffness matrix $[K]$ can be positive definite, negative definite or indefinite. The sign-definiteness of the stiffness matrix is used below to determine the stability of the system.

3.2. Linear Model Stability Criteria

If damping in a system is pervasive, the stability of a system with damping can be determined solely by examining whether the stiffness matrix is positive definite, Tyc et al. [15]. If damping is not pervasive, the Routh-Hurwitz criterion or an eigenvalue approach could be used. Tyc et al. assume that damping in the tether-payload system is pervasive since all state variables are coupled through the motions of the payload.

Tyc [16] derived the TE-LAB stability criteria by examining the conditions under which the stiffness matrix $[K]$ in Eq. (3.1) remains positive definite. The derivation uses three assumed tether modes and does not include boom deflections. A brief overview of Tyc's derivation of the stability criteria [16], rederived in TE-LAB notation by Jablonski [17], is presented here.

After manipulations, the state vector with three assumed tether modes and no boom deflections is:

$$\{x\}^T = \{q_{10,1}, q_{10,2}, q_{10,3}, \psi, q_{9,1}, q_{9,2}, q_{9,3}, \theta\} \quad (3.3)$$

The corresponding stiffness matrix is:

$$[K] = \begin{bmatrix} A_{11} & X_{12} & 0 & \mathcal{E}_1 & 0 & 0 & 0 & 0 \\ X_{12} & A_{22} & X_{23} & \mathcal{E}_2 & 0 & 0 & 0 & 0 \\ 0 & X_{23} & A_{33} & \mathcal{E}_3 & 0 & 0 & 0 & 0 \\ \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{11} & X_{12} & 0 & \mathcal{E}_1 \\ 0 & 0 & 0 & 0 & X_{12} & A_{22} & X_{23} & \mathcal{E}_2 \\ 0 & 0 & 0 & 0 & 0 & X_{23} & A_{33} & \mathcal{E}_3 \\ 0 & 0 & 0 & 0 & \mathcal{E}_1 & \mathcal{E}_2 & \mathcal{E}_3 & \theta \end{bmatrix} \quad (3.4)$$

The elements of $[K]$ are:

$$A_{nn} = TIX_{T,nn} - \rho gl^2 X_{g,nn} - S^2 \rho l^3 m_{t,nn} \quad (3.5a)$$

$$\theta = (C - A)S^2 + bT \left(1 + \frac{b}{l}\right) - \rho glb \left(1 + \frac{b}{2l}\right) \quad (3.5b)$$

$$\mathcal{E}_n = S^2 \rho l^2 b \mathcal{E}_n - \rho glb \mathcal{E}_{g,n} \quad (3.5c)$$

$$X_{nm} = -\rho gl^2 X_{g,nm} \quad (3.5d)$$

where T is the tension applied to the top of the tether by the spool mechanism, l is the tether length, S is the spin rate, ρ is the linear tether density, g is the acceleration due to gravity and b is the distance between the payload center of mass and the tether attachment point. The moment of inertia terms are:

$$A = A_0 + \frac{1}{3}\rho lb^2 \quad (3.6a)$$

$$C = C_0 \quad (3.6b)$$

where A_0 and C_0 are the transverse and longitudinal moments of inertia of the payload respectively. Numerical values for parameters in the above equations can be found in Table 2.1. Coefficients $x_{T,nm}$, $x_{g,nm}$, $m_{i,nm}$, ρ_n , $g_{g,n}$ and $x_{g,nm}$ are dimensional matrix coefficients based on the assumed tether modes. These coefficients are:

$$x_{T,nm} = \begin{cases} n^2\pi^2 & n = m \\ 0 & n \neq m \end{cases} \quad (3.7a)$$

$$x_{g,nm} = \begin{cases} \frac{n^2\pi^2}{2} & n = m \\ 0 & n + m \text{ even}, n \neq m \\ \frac{4nm(n^2 + m^2)}{(n^2 - m^2)^2} & n + m \text{ odd}, n \neq m \end{cases} \quad (3.7b)$$

$$m_{i,nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad (3.7c)$$

$$\rho_n = \frac{\sqrt{2}}{n\pi} \quad (3.7d)$$

$$g_{g,n} = \begin{cases} \frac{2\sqrt{2}}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \quad (3.7e)$$

Since $[K]$ is block diagonal, the positive definiteness of only one block of $[K]$ needs to be proven to show that $[K]$ is positive definite. The upper left block of $[K]$ is defined to be $[\tilde{K}]$:

$$[\tilde{K}] = \begin{bmatrix} A_{11} & x_{12} & 0 & \varepsilon_1 \\ x_{12} & A_{22} & x_{23} & \varepsilon_2 \\ 0 & x_{23} & A_{33} & \varepsilon_3 \\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & 0 \end{bmatrix} \quad (3.8)$$

For stability:

$$\det[\tilde{K}] > 0 \quad (3.9)$$

Set:

$$\det[\tilde{K}] = \lambda_1 \lambda_2 \lambda_3 \lambda_4 \quad (3.10)$$

It is shown that each of the following conditions must be met for system stability:

$$\begin{aligned} \lambda_1 &> 0 \\ \lambda_2 &> 0 \\ \lambda_3 &> 0 \\ \lambda_4 &> 0 \end{aligned} \quad (3.11a-d)$$

After considerable manipulation of Eq. (3.10), the derivation [16,17] concludes that the system will be stable when each of the following individual criteria are met:

I) Tether-associated stability criteria:

$$\lambda_n > 0 \quad (n=1,2,3 \text{ corresponding to tether modes } 1,2,3) \quad (3.12)$$

where:

$$\lambda_n = T x_{i,nn} - \rho g l^2 x_{g,nn} - S^2 \rho l^3 m_{i,nn} - \rho^2 g^2 l^4 \frac{x_{g,n-1,n}^2}{\lambda_{n-1}} \quad (3.13)$$

$$k_n = \rho g l b \left(\frac{l}{b} \frac{k_{n-1}}{\lambda_{n-1}} x_{s,n-1,n} - z_{s,n} + \frac{l S^2}{g} \alpha_n \right) \quad (3.14)$$

and $k_0 = 0, \lambda_0 = 1$ and $x_{s,0,1} = 0$.

II) Payload-associated stability criterion:

$$\lambda_4 > 0 \quad (3.15)$$

where:

$$\lambda_4 = (C_0 - A_0) - \rho l b^2 \left[\frac{1}{3} + \frac{1}{\rho l b^2 S^2} \sum_{n=1}^{N=3} \frac{k_n^2}{\lambda_n} \right] + \frac{b T}{S^2} \left(1 + \frac{b}{l} \right) - \frac{\rho g l b}{S^2} \left(1 + \frac{b}{2l} \right) \quad (3.16)$$

The system stability criteria can be defined as the condition that each of the individual stability criteria must be true; ie. $\lambda_n > 0, n=1,2,3,4$.

3.3. TE-LAB Theoretical Stability Boundaries

The contour plot was found to be a useful method to present theoretical data. The individual stability criteria ($\lambda_1, \lambda_2, \lambda_3, \lambda_4$) can be plotted on a contour plot as a function of both tether tension and spin rate. The contour plot represents a function of two variables by using contour curves to connect solutions of equal value. A common form of contour plot is the topographic map which represents geographic elevation as a function of two-dimensional position using contours to connect points of equal elevation.

The system stability criteria of the minor-axis payload with light tether configuration of the TE-LAB apparatus will be developed here in detail based on the sign of each of the individual stability criteria. The system stability criteria of the other TE-LAB configurations will also be presented here.

Minor-Axis Spinner with Light Tether

The tether-associated stability criteria for this configuration are shown in Figures 3.2-4. The contour plot of the payload-associated stability criteria is shown in Figure 3.5. The contour lines connect solutions of equal value. The contour labels indicate the value of the solution along the contour. For clarity, very dense contour levels are not labelled.

Figure 3.2 Contour plot of first tether-associated criteria, $\lambda_1(T, S)$, for minor-axis spinner with light tether in tension-spin rate plane.

Figure 3.3 Contour plot of second tether-associated criteria, $\lambda_2(T, S)$, for minor-axis spinner with light tether in tension-spin rate plane.

Figure 3.4 Contour plot of third tether-associated criteria, $\lambda_3(T, S)$, for minor-axis spinner with light tether in tension-spin rate plane.

Figure 3.5 Contour plot of payload-associated criteria, $\lambda_4(T, S)$, for minor-axis spinner with light tether in tension-spin rate plane.

The dense contour regions in the plot of the second tether-associated criteria, λ_2 , (Figure 3.3) correspond to the region near $\lambda_1 = 0$. The λ_{n-1} term in the denominator of tether-associated criteria (Eq. 3.12) causes a singularity in the solution of λ_n when $\lambda_{n-1} = 0$. The value of λ_2 reaches infinity at $\lambda_1 = 0$ and very large positive and negative values in the region of $\lambda_1 = 0$. This effect is also observed in the solution of the third tether-associated criteria, λ_3 (Figure 3.4). The dense contours of the payload-associated criteria are caused by the λ_n term in the denominator of this criteria (Eq. 3.13). As each of the tether-associated criteria approach zero, the payload-associated criteria approaches infinity.

The stability boundary for each individual criteria is defined as the curve at the boundary between positive and negative solutions of the criteria; ie the curve where $\lambda_n(T,S)=0$. Solutions of a criterion which pass through zero due to a singularity (ragged curves surrounding the singularity) are not considered as part of the stability boundary for that criterion.

Each of these contour plots was generated by calculating the stability criteria at 0.04 Hz and 0.04 N intervals in the range 0 to 4 Hz and 0 to 5 N. The resulting data was then passed to the *contour* routine in MATLAB for plotting.

Superimposing the four individual stability boundaries onto one plot results in the system stability criteria for the minor-axis spinner with light tether (Figure 3.6). Curve P represents the payload stability boundary ($\lambda_4 = 0$) and the Tn curves represent the boundaries of the respective tether-associated stability criteria. In the unshaded region, all stability criteria are met ($\lambda_n(T,S) > 0$ for all $n=1,2,3,4$) indicating that, in theory, the system is asymptotically stable. In the shaded region, at least one stability criterion is not met ($\lambda_n(T,S) < 0$) indicating that, in theory, the system is unstable. Thus, only the payload-associated criteria dictates the system stability for this configuration.

Figure 3.6 System stability criteria in the tension-spin rate plane for the minor-axis spinner with light tether. Curve P represents the payload stability boundary, $\lambda_4 = 0$ and curves T_n represent the n th tether stability boundary $\lambda_{1-3} = 0$.

The horizontal line labelled Minimum Tension denotes the minimum tension required to support the weight of the tether. This tension is calculated by the equation $T_{\min} = \rho g l$. Solutions below this line are not physically possible because the tension generated by the spool at levels below $T_{\min}$ cannot support the tether weight.

Minor-Axis Spinner with a Heavy Tether

The system stability criteria of the minor-axis spinner with a heavy tether is shown in Figure 3.7. Each of the individual criteria is shifted up and to the left. This plot indicates that the effect of the heavy tether is to destabilize the system. The system stability is dictated by both the first tether-associated criteria and the payload-associated criteria. The jump in the payload-associated criteria is caused by the line of singularities of the first tether-associated criteria.

Major-Axis Spinner with Light Tether

The system stability criteria of the major-axis spinner with light tether are shown in Figure 3.8. The payload-associated criterion is negative only at tensions below the minimum physically possible tension, as seen in a magnified view of the low tension and spin rate region of the plot (Figure 3.9). Therefore, for all physically possible tensions, the system stability is dictated by the first tether-associated criteria.

Figure 3.7 System stability criteria in the tension-spin rate plane for the minor-axis spinner with heavy tether.

Figure 3.8 System stability criteria in the tension-spin rate plane for the major-axis spinner with light tether.

Figure 3.9 Enlargement of system stability criteria in the tension-spin rate plane for the major-axis spinner with light tether.

Chapter 4

4. EXPERIMENTAL TEST PROCEDURES AND SAMPLE RESULTS

The TE-LAB tests are designed to compare the behaviour of the apparatus with that predicted by the Linear Model. The tests are divided into three types; the Free-Spinning Body test, the Spinning Tether test and the Spinning Tether and Endbody test. The first two tests examine the dynamics of the system in simplified configurations while the third test examines the dynamics of the complete tether and endbody system.

4.1. Free-Spinning Body Test

The Free-Spinning Body test is designed to compare the behaviour of the payload without a tether to the behaviour predicted by the major-axis rule for a free-spinning body. Since the major axis spinner is a stable configuration, a torque-free initial impulse condition ("kick") was applied to the major axis spinner to test the stability of the vertical equilibrium state. The torque-free initial condition consists of two taps in quick succession (<0.5 sec) with two hard objects (plastic blocks or screwdriver handles) to opposite sides of the payload. The taps are applied to opposite sides of the payload to minimize the amount of torque applied. The kick initial condition is also used in the Spinning Tether and Endbody test. The minor axis spinner is unstable, therefore, a typical

test involves stabilizing the spinning payload by hand and releasing it. Three tests were performed at 1, 2 and 3 Hz spin rates for both the major and minor axis spinners.

POST321B was used to generate the 3-2-1 Euler angle time histories of the payload response from the target cartesian coordinates. For each test, an exponential curve is fit to the time history envelope of the rotation about the y axis in a 3-2-1 Euler angle rotation sequence (hereafter referred to as "rotation angle θ ") using FFTPOST2 as described in §2.3.10. The result of a typical run is shown in Figure 4.1. This run is for a major axis spinner with a nominal spin rate of 2 Hz subjected to an initial kick. The negative exponential time coefficient of the curve fit ($b = -0.022$) indicates that the time history converges.

Figure 4.1 Partial time history of rotation angle θ and exponential curve fit for major axis payload spinning at 2 Hz.

4.2. Spinning Tether Test

The objective of the Spinning Tether test is to investigate the tether-associated stability criteria without the dynamics of the payload. The payload gimbals are locked, eliminating movement of the payload relative to the drive shaft. This configuration can then be considered to be a vertical spinning tether.

Tether modes were found by visual observation of the spinning tether as the tension is slowly lowered from a maximum (5 N) to a minimum (approx. 0.15 N) for a particular spin rate. A "tether mode" was declared when the tether is in the form of a steady, spinning sinusoid. The mode number is determined by counting the number of amplitude maxima in the spinning tether. Alternately, the mode number can be determined by the number of tether nodes + one. A schematic of the tether in the third mode is shown in Figure 4.2 to illustrate the assessment of the mode number.

Figure 4.2 Example of spinning tether in third mode. Number of amplitude maxima: 3. Number of tether nodes+1: 3.

Tension values which resulted in a tether mode were recorded. This procedure was performed at the following spin rates: 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5 and 4.0 Hz. This test was performed four times with the light tether. Tether length for all tests was 6.4 m.

Tether modes are plotted as a function of tension and spin rate in Figures 4.3 and 4.4. Figure 4.3 presents the tether modes of interest to this work since these will be compared in §5.2 to theoretical behaviour predicted by the Linear Model. Higher modes are shown in Figure 4.4. Some data points are missing for modes 5 and 6 since it is difficult to experimentally find the proper tether tensions which induce the higher tether modes at low spin rates.

Figure 4.3 Tension and spin rates at which the first tether three modes occurred for a spinning 6.4 m tether.

Figure 4.4 Tension and spin rates at which modes 3 through 6 occur for a spinning 6.4 m tether.

4.3. Spinning Tether and Endbody Test

Procedure

Tests of the complete spinning tether and endbody system are grouped into three categories; Minor axis spinner with light tether, minor axis spinner with heavy tether, and major axis spinner with light tether. Spin rates and tensions for each system configuration are summarized in Table 4.1. To examine repeatability of the tests, the minor axis spinner with light tether configuration was tested twice.

For payloads configurations which appear to be stable, there are two initial conditions which are applied. The first is the torque-free "kick" initial condition described in §4.1. The second is the "tether pluck" initial condition. In the pluck initial condition, the tether is pulled by hand several inches from the spin axis and released. The effect of the pluck initial condition is to subject the tether to an initial displacement. Configurations which appear unstable are subjected to the third initial condition, the "no initial condition". In this case, the payload is stabilized by hand about the vertical and then released.

Table 4.1 Summary of Spinning Tether and Endbody Tests.

Spinner Type	Tether Type	Tether Tension (N)	Nominal Spin Rate (Hz), (for each tension level)	No. of times tested
Minor axis	light	0.5, 0.89, 1.42, 2.76, 4.54	0.0, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75, 2.0, 2.25, 2.5, 2.75, 3.0	2
Minor axis	heavy	2.02, 3.51, 4.54	0.0, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0	1
Major axis	light	0.49, 1.41, 4.54	0.0, 0.5, 0.75, 1.0, 1.5, 2.0, 2.5, 3.0	1

Data Processing and Sample Results

Each of the runs was post-processed using POST321B to generate the payload Euler angle time histories. Exponential time coefficients and Fast Fourier Transforms were generated for each of the time histories of rotation angle θ using FFTPOST2.

Two sample runs of typical payload behaviour are presented here; first a converging time response and then a diverging time response. Sample run A (Figure 4.5) shows the full time history of the response of angle θ due to a kick for first series of tests on the minor axis spinner with light tether at a spin rate of 1 Hz and tension of 2.76 N. The exponential curve fit for this response is shown in Figure 4.6. The exponential time coefficient b is -0.061 indicating convergence of the time-history. The frequency spectrum of this response is shown in Figure 4.7. The result of the "constant" error of the Rotation Rate Control System can be seen in the frequency spectrum. The Linear Model predicts a peak in the frequency response of the θ angle at the payload spin rate while the frequency spectrum displays a peak at 1.02 Hz. This suggests an error of +2% for the spin rate.

Sample run B is from the same series of tests and at the same tension but higher spin rate (1.75 Hz). The initial condition for this run is the "no-initial condition". Figure 4.8 shows the full time history of the response of angle θ starting from the vertical. The exponential time coefficient resulting from the curve fit for this response is 0.020 (Figure 4.9), indicating divergence of the time history. The frequency spectrum of this response is shown in Figure 4.10. As in sample run A, the "constant" error in the payload spin rate can be seen in this figure (+4% error).

Figure 4.5 Sample run A. Full time history response of angle θ for minor axis spinner due to "kick" initial condition. Spin Rate: 1 Hz., Tension: 2.76 N. (First minor axis/light tether test series)

Figure 4.6 Magnification of sample run A and exponential curve fit applied to response. $b=-0.061 < 0$ indicating time history convergence.

Figure 4.7 Frequency response of sample run A. a) Time history data used in FFT analysis; b) Magnitude of frequency response; c) Phase of frequency response.

Figure 4.8 Sample run B. Full time history response of angle θ for minor axis spinner due to "no-initial condition". Spin Rate: 1.75 Hz, Tension: 2.76 N. (First minor axis/light tether test series)

Figure 4.9 Magnification of sample run B and exponential curve fit applied to response. $b=0.020>0$ indicating time history divergence.

Figure 4.10 Frequency response of sample run B. a) Time history data used in FFT analysis; b) Magnitude of frequency response; c) Phase of frequency response.

Stability of Vertical Orientation and Sign of Exponential Time Coefficient of Curve Fits to Experimental Data

Time histories of angle θ which diverge from zero and have a positive exponential time coefficient, indicate that the vertical orientation of the payload is unstable. A distinction must be made, however, between a run which is entirely unstable and a run for which the vertical orientation is unstable. In general, in a run for which the vertical orientation is unstable, angle θ diverges until the payload strikes the drive shaft. In this case, the divergence of angle θ correctly indicates system instability. There are some runs which have an unstable vertical orientation, yet angle θ only diverges to a finite large angle limit cycle without striking the payload shaft. These runs are said to be asymptotically orbitally stable. The divergence of angle θ correctly indicates that the vertical orientation is unstable; however, the system is not strictly unstable.

All runs during which the time history of angle θ converges are stable. There are theoretically two possible types of stable tests. The first is the asymptotically stable run during which the angle θ and the tether displacement converges to zero. This type of stability is not conclusively observable experimentally due to small mechanical imbalances and imperfections in the system. The second type of convergent run is asymptotically orbitally stable. During this type of run, the angle θ converges to a small limit cycle (typically $\cong \pm 2^\circ$) and the tether displacement is small and non-zero.

For the purpose of the present work, the stability of the vertical orientation is of utmost interest. Therefore, the following terminology and the meaning of the experimentally measured exponential time coefficient b will be used in the remainder of the present work regarding the convergence/divergence of angle θ . The term "stable" will refer to asymptotically orbitally stable behaviour of the payload ($b < 0$). The term "unstable" will refer to both the large limit cycle behaviour (asymptotically orbitally stable) and the unstable behaviour of angle θ . For both "unstable" cases, $b > 0$. This terminology is summarized in Table 4.2.

Table 4.2 Summary of relationship between sign of exponential time coefficient, b , of experimental data and stability of vertical payload orientation

Sign of Exponential Time Coefficient	Possible Stability Type	Stability of Vertical Payload Orientation
$b < 0$	asymptotically orbitally stable (θ converges to small magnitude limit cycle $\cong \pm 2^\circ$)	stable
	asymptotically stable (θ converges to 0°)	stable, but full convergence to 0° never observed
$b > 0$	asymptotically orbitally stable (θ diverges from vertical orientation, then converges to large magnitude limit cycle $\gg \pm 2^\circ$)	unstable
	unstable (θ diverges from vertical orientation until payload hits shaft)	unstable

Data Presentation Methods

To provide an overview of the experimental payload stability as a function of tension and spin rate for each TE-LAB configuration, the exponential time coefficients are presented as contour plots in the tension-spin rate plane. Figure 4.11 is an example of such a contour plot for the first series of tests on the minor-axis spinner with light tether. The exponential time coefficients from the two sample runs presented above are included in this figure. Points A and B represent sample runs A and B respectively.

A useful feature of the contour plot is that it provides a systematic method for defining the experimental stability boundary of the vertical payload orientation; hereafter referred to as the experimental payload stability boundary. For the present work, this boundary is defined to be the contour corresponding to $b=0$. The experimental payload stability boundary is shown as the bold curves in the sample contour plot. A comparison of the experimental payload stability boundary and the theoretical system stability is presented in the following chapter.

Figure 4.11 Contour plot of exponential time coefficients in tension-spin rate plane for first series of tests on minor axis spinner with light tether. Points A and B represent the respective sample runs presented above. Dark contours indicate experimental payload stability boundary.

An overview of the system natural frequencies can be obtained by plotting the system frequencies as a function of spin rate for a particular tension. A plot of the system frequencies for the first series of tests on the minor-axis spinner with light tether at a tether tension of 2.76 N is shown in Figure 4.12. Some system frequencies can be seen to form regular patterns. McClure [7] compares the system frequencies to the natural modal frequencies using an eigenvalue analysis of the Linear Model. The system frequencies of sample runs A and B are shown at 1 and 1.75 Hz respectively, as indicated.

Figure 4.12 System frequencies as a function of spin rate for first series of tests on minor axis spinner with light tether at tension of 2.76 N. System frequencies of sample runs A and B are shown at 1 and 1.75 Hz respectively.

Chapter 5

5. COMPARISON OF EXPERIMENTAL RESULTS WITH ANALYTICAL MODELING

In this chapter, the experimental system stability is compared to the theoretical system stability to determine to what extent the stability analysis of the Linear Model is valid. The stability of the payload without tether is examined to ensure that the payload simulates a free-spinning body. The stability of the spinning tether dynamics and occurrence of tether modes with the payload gimbals locked are studied. Finally, the stability of the spinning tether and payload system with gimbals unlocked is examined. In addition, sample runs are presented to illustrate dynamic behaviour of the spinning tether and payload system which is not predicted by the Linear Model.

5.1. Free-Spinning Body

The exponential time coefficients for both major and minor-axis spinners as a function of spin rate are shown in Figure 5.1. All major axis tests result in negative exponential time coefficients (stable) while all minor axis tests result in positive exponential time coefficients (unstable). This verifies that the two configurations of the gimbal-mounted payload follow the major axis rule for stability of a free-spinning body, in the current spin rate range. In addition, this behaviour contributes to the validation of the design and fabrication of the TE-LAB payload and gimbal system.

Figure 5.1 Exponential time coefficients of major and minor axis free-spinning bodies as a function of spin rate. All major axis configurations are stable and all minor axis configurations are unstable.

5.2. Spinning Tether

The tether stability boundaries predicted by the tether criteria of the Linear Model and corresponding experimental tether modes are shown in Figures 5.2 to 5.4. The tether stability boundaries predict the location of the tether modes quite well. Although the payload dynamics are eliminated in this configuration, it is valid to compare the tether modes with the tether stability boundaries of the Linear Model because the tether-associated criteria, Eq. (3.12), do not contain any payload-related terms.

The stability analysis based on the Linear Model predicts that the entire system will become unstable when the first tether-associated stability criterion is violated. During testing, significant lateral displacements of the spinning tether were only observed when the system was operating at a tether mode. The tether displacements were generally small but non-zero when operating at other conditions, including regions to the right of the first tether stability boundary. The tether may have experienced small limit cycle behaviour in the regions predicted to be unstable by the tether-associated criteria, although this was not explicitly observed.

The fact that the tether displacements remain small in the theoretically unstable region is believed to be due to the elimination of non-linear terms during the linearization process.

Figure 5.2 First theoretical tether stability boundary and locations of first experimental tether modes for 6.4 m light tether.

Figure 5.3 Second theoretical tether stability boundary and locations of second experimental tether modes for 6.4 m light tether.

Figure 5.4 Third theoretical tether stability boundary and locations of third experimental tether modes for 6.4 m light tether.

5.3. Spinning Tether and Endbody

5.3.1. Minor Axis Spinner with Light Tether

Contour plots of the exponential time coefficients for both series of tests on the minor axis spinner with light tether configuration are superimposed on the theoretical stability regions (§3.3) in Figures 5.5 and 5.6. The dashed curves represent stability boundaries predicted by the Linear Model. Curve P is the payload stability boundary and curves T1, T2 and T3 represent the stability boundaries of the first, second and third tether-associated criteria respectively.

The experimental stability boundaries, (bold lines in the figures) follow the theoretical system stability boundary closely. Tests to the right of the theoretical payload stability boundary are typically unstable (except for the unexpected negative exponential time coefficients to be discussed below), and to the left of the theoretical payload stability boundary are stable. Typical stable and unstable behaviours were shown previously in sample runs A and B (Figures 4.5-10).

Figure 5.5 Contour plot of exponential time coefficient for first series of tests on minor axis spinner with light tether superimposed on corresponding theoretical stability boundaries. Bold curves represent experimental stability boundary. Curve P is the theoretical payload stability boundary, curves T_n are n th theoretical tether stability boundaries. Point C indicates sample run C.

Figure 5.6 Contour plot of exponential time coefficient for second series of tests on minor axis spinner with light tether superimposed on corresponding theoretical stability boundaries. Bold curves represent experimental stability boundary. Curve P is the theoretical payload stability boundary, curves T_n are n th theoretical tether stability boundaries. Points D and E indicate sample runs D and E respectively.

Negative Exponential Coefficients Not Predicted by the Linear Model

A phenomenon not predicted by the Linear Model is the occurrence of small regions of negative exponential time coefficients in the theoretically unstable region between the first and second tether modes. Samples of runs in these regions are provided here. Sample run C is from the first series of tests on minor-axis spinner with light tether at 1.75 Hz and 0.89 N. The full time history and exponential curve fit plots of sample run C are shown in Figures 5.7 and 5.8 respectively. Sample run D is from the second series of tests on the minor-axis spinner with light tether at 1.75 Hz and 0.5 N. The corresponding plots for sample run D are presented in Figures 5.9 and 5.10. The run at point D was originally a no-initial condition run. As the run progressed, it became apparent that the nutation angle would not diverge spontaneously. Therefore, the payload was given a "kick" initial condition at $t=920$ sec. It can be seen from the time histories of sample runs C and D that the angle θ converged to a small angle (less than $\pm 2^\circ$), resulting in a negative exponential time coefficient. By the definition of stability defined previously, these runs are considered experimentally stable.

Figure 5.7 Sample run C. Full time history response of angle θ for minor axis spinner with light tether due to "kick" initial condition. Spin Rate: 1.75 Hz, Tension: 0.89 N. (First minor axis/light tether test series)

Figure 5.8 Magnification of sample run C and exponential curve fit applied to response. $b=-0.004<0$ indicating time history convergence and payload stability.

Figure 5.9 Sample run D. Full time history response of angle θ for minor axis spinner with light tether due to "kick" initial condition. Spin Rate: 1.75 Hz, Tension: 0.50 N. (Second minor axis/light tether test series)

Figure 5.10 Magnification of sample run D and exponential curve fit applied to response. $b=-0.011<0$ indicating time history convergence and payload stability.

Large Positive Exponential Coefficients and Large Limit Cycles

Large positive exponential coefficients and large limit cycles occur in runs near the payload stability boundary for both series of tests of this configuration. The run which produced the largest exponential coefficient is examined here as sample run E; 0.5 Hz, 0.5 N for the second series of minor axis spinner with light tether. The coefficient is so large ($b=0.503$) that the contour lines in the region of the test are dense enough to form a black triangle on the plot. The full time history of angle θ , the full time history of the nutation angle and the exponential curve fit of angle θ are shown in Figures 5.11-13. The payload was steadied by hand about the vertical at $t=190$ sec and released at $t=205$ sec. Upon release, the nutation angle grew quickly to 22° . The nutation angle then remained nearly constant at this value until the end of the test 110 seconds later. The exponential curve fit illustrates rapid divergence of the angle θ .

Figure 5.11 Sample run E. Full time history response of angle θ for minor axis spinner with light tether due to "no initial condition". Spin Rate: 0.5 Hz, Tension:0.50 N. (Second minor axis/light tether test series)

Figure 5.12 Full time history response of nutation angle for sample run E.

Figure 5.13 Magnification of angle θ from sample run E and exponential curve fit applied to response. $b=0.503>0$ indicating time history divergence and instability of vertical orientation of payload.

This constant non-zero nutation angle can be considered large limit cycle behaviour for the payload. This response is formally described as asymptotically orbitally stable; however, the vertical orientation is not stable for this run. Therefore, by the earlier definition of payload stability, this run is classified as unstable.

A small number of other runs for this configuration result in large limit cycle behaviour; however, this occurs only for runs near the theoretical payload stability boundary. For example, the run at 1.0 Hz, 1.4 N in the second series of minor axis spinner with light tether tests has an exponential time coefficient of $b=0.13$ and reaches a constant nutation angle of 14.5° . The large value of b at this point in the tension-spin rate plane is responsible for the discrepancy between the theoretical and experimental stability boundaries for this series of tests (Figure 5.6).

5.3.2. Minor Axis Spinner with Heavy Tether

A contour plot of the exponential time coefficients and theoretical stability regions for the minor axis-spinner with heavy tether is shown in Figure 5.14. For most tension and spin rate settings, the stability of the payload is predicted by the stability criteria of the Linear Model. The exception, however, is the region between the first tether stability boundary, T1, and the payload stability boundary, P. The exponential time coefficients of the three tests performed in this region (0.75 Hz, 3.51 N; 1 Hz, 3.51 N and 1 Hz, 4.54 N) have negative values. This phenomenon is similar to that of the Spinning Tether test where the tether stability boundaries predict system instability, but no system instability is detected experimentally.

As with the Spinning Tether test, it is possible that the payload experiences small limit cycle behaviour in this theoretically unstable region. If this behaviour exists, it cannot be distinguished either visually or by examining PODS data. Test data from runs on the stable side (sample run F) and unstable side (sample run G) of the first tether stability boundary are presented here for comparison. Sample run F is the response due to a kick initial condition at 0.5 Hz, 3.5 N, Figure 5.15. Sample run G is the response due to a kick initial condition at 1 Hz, 3.5 N is shown in Figure 5.16. Both tests appear to converge to approximately the same angle.

Figure 5.14 Contour plot of exponential time coefficient for minor axis spinner with heavy tether superimposed on corresponding theoretical stability boundaries. Bold curve represents experimental stability boundary. Curve P is the theoretical payload stability boundary, curves T_n are n th theoretical tether stability boundaries. Point F and G indicates sample runs F and G.

Figure 5.15 Sample run F. Full time history response of angle θ for minor axis spinner with heavy tether due to "kick" initial condition. Spin Rate: 0.5 Hz, Tension: 3.5 N.

Figure 5.16 Sample run G. Full time history response of angle θ for minor axis spinner with heavy tether due to "kick" initial condition. Spin Rate: 1.0 Hz, Tension: 3.5 N.

5.3.3. Major Axis Spinner with Light Tether

A contour plot of the exponential time coefficient and theoretical stability region for the major axis spinner with light tether is shown in Figure 5.17. For this configuration, the theoretical system stability is specified by the first tether stability boundary. The shaded area bounded by T1 represents the region predicted to be unstable by the Linear Model. Recall from Figure 3.8 that the payload stability criteria has a positive value for all physically possible solutions of this configuration. Therefore, the payload stability boundary does not appear in this figure.

All exponential time coefficients are negative for this configuration, indicating that the payload was stable for all tests in this operating range. This is consistent with the behaviour of the Spinning Tether test and the Minor Axis Spinner with Heavy Tether test. Again, the tether stability boundaries do not predict system instability, but possibly only limit cycle behaviour below the resolution of PODS.

Although the experimental payload stability boundary does not match the theoretical stability boundary, there is still a trend in the magnitudes of the exponential time coefficient. The most stable region ($b < -0.07$) occurs at 1Hz , 4.54N which is well into the theoretical stability region. The least stable payload responses ($b > -0.03$) occur in the neighbourhood of the second tether mode, T2.

Figure 5.17 Contour plot of exponential time coefficient for major axis spinner with light tether superimposed on corresponding theoretical stability boundaries. Curves T_n are n th theoretical tether stability boundaries. The theoretical payload stability boundary does not appear in operating range for this configuration.

5.4. Summary

From a practical perspective, there are two important behaviours which are reliably predicted by the theoretical stability criteria:

- The theoretical payload stability criterion closely predicts payload stability (as defined in this work).
- Operation at theoretical tether stability boundaries ($\lambda_{1-3} = 0$) results in clearly observable fundamental tether modes (1st, 2nd, 3rd respectively).

The theoretical stability criteria are not consistent with the following experimental observations, believed to be due to the absence of non-linear terms in the modelling:

Tether-Related

- The appearance of tether modes at the theoretical tether stability boundaries. (Figures 5.2-4).
- Stable behaviour of tether when not operating at a tether mode.
- Stable behaviour of the major axis spinner in the region predicted to be unstable by the first tether criteria (Figure 5.17).

Payload-Related

- Small regions of stable behaviour of the minor-axis payload in the region between the first and second tether modes (Sample run C and D in Figures 5.5-6 respectively).
- Large limit cycle behaviour of payload at payload stability boundary (Sample run E in Figure 5.6).

It can be concluded that the theoretical stability criteria are conservative in predicting stability of the TE-LAB apparatus. No region which was theoretically stable proved to be unstable during TE-LAB testing. However, tests did reveal certain regions, predicted to be unstable, which proved to be experimentally stable.

The experiments performed using TE-LAB and the comparison of experimental to theoretical results have produced a hardware-based insight into the validity of the theoretical stability criteria. In addition, the strengths and weaknesses of using these criteria in the prediction of on-orbit OEDIPUS dynamics have been identified.

Chapter 6

6. CONCLUSIONS AND FUTURE WORK

6.1. Conclusions

On the basis of the experimental testing and theoretical analysis in this work, the following conclusions can be made:

1. TE-LAB testing and comparison to theoretical stability criteria has resulted in a more confident understanding of the dynamics of spinning tethered payload systems.
2. The stability criteria based on the Linear Model predict stability of the TE-LAB system conservatively. That is, no parameter region which was predicted to be stable was experimentally unstable; however, some regions which were predicted to be unstable were found to be stable. Only the theoretical payload stability criteria dictated the experimental stability of the payload during testing.

3. Contour plots of the exponential time coefficient superimposed on the theoretical stability criteria in the Tension-Spin Rate plane are useful for comparing experimental and theoretical stability.
4. The Payload Orientation Determination System was successful in measuring the payload Euler angles throughout the required spin rate range. The experimental accuracy of the nutation angle is; $\pm 0.5^\circ$ for small nutation angles and ± 1.5 - 2° at the maximum nutation angle. Error sources contributing to these accuracy ranges have been identified by analytical accuracy analyses.
5. The performance of the Rotation Rate Control System for nutation angles less than 15° and spin rates above 1Hz is as follows; payload and upper table shaft speed variation: $\pm 5\%$ of nominal, tether twist $\pm 2^\circ$. Performance degrades at low spin rates and large nutation angles. A nearly constant spin rate error of approximately $+4\%$ is present.

6.2. Future Work

The items for future work are designed to expand the understanding of the spinning tethered payload system. This can be achieved through further analysis and testing and through improvements to the TE-LAB apparatus. These items are:

Analysis and Testing

1. Perform tests with a significantly heavier and/or longer tether and major axis spinner to investigate whether there are any conditions at which the tether can destabilize the major axis payload.
2. Compare experimental and theoretical stability in other parameter planes. For example, the tether length-spin rate plane would be useful for studying system stability during tether deployment.
3. Perform a more refined search for stable behaviour in theoretically unstable region for minor axis spinner. This data may be useful for comparison to possible future non-linear modelling efforts.
4. Perform tests of minor axis spinner at more payload stability boundaries to examine which parameters affect the magnitude of the large limit cycle behaviour.

Apparatus

1. Incorporate out-of-plane displacements in PODS kinematics to improve system accuracy.
2. Improve real-time target tracking (using a digital filter, prediction of Euler angles or quaternions etc.) to enable DEOPST to track faster moving targets, allowing higher spin rates and/or larger focal length (higher accuracy).
3. Development of a tether position sensor and tether force sensor to obtain quantitative data on the tether dynamics.
4. Fix constant spin rate error in RRCS.

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Appendix A

APPENDIX A. Derivation of PODS Kinematic Equations

The PODS kinematic equations are derived in detail in this Appendix. A three-step process is used currently in the post-processing software POST321B to calculate the Euler angles. First, direction cosines of the rotation matrix are calculated based on camera coordinates of the two targets. Then, the Euler angles are calculated in a 1-2-3 sequence based on the direction cosines of the rotation matrix. Finally, these angles are converted into a 3-2-1 sequence.

This three step process is due to the fact that during PODS development, a 1-2-3 Euler angle sequence was used. Analysis of test data revealed that PODS should output a 3-2-1 Euler angle sequence used in the Linear Model for direct comparison of the natural frequencies of the system with modal frequencies predicted by an eigenvalues of the Linear Model.

The final section of this Appendix presents a more computationally-efficient method to calculate the Euler angles in a 3-2-1 sequence directly from the direction cosines. The method is not used in the POST321B postprocessor, but is presented here as it is valuable for future work. These algorithms will reduce the amount of computation

required to calculate the Euler angles. This could be useful if an attempt is made to calculate the Euler angles in real-time in the DEOPST program.

Note: To avoid confusion between the two rotation sequences, all angles and axes in the 1-2-3 sequence are noted by carret marks (^). Angles without carret marks correspond to the 3-2-1 sequence.

A.1 Calculation of Direction Cosines from Target Coordinates

The coordinates of the targets in the camera-aligned payload centered coordinate frame (Figure 2.13) and the payload-fixed frame (Figure 2.14) are related by:

$$V_4 = CV_0 \quad (A.1)$$

where C is a 3x3 rotation matrix of the form $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$, V is a 3x2 matrix of target coordinates of the form $V_0 = [\bar{p}_0 | \bar{q}_0]$ and $V_4 = [\bar{p}_4 | \bar{q}_4]$ and $\bar{p}$ and $\bar{q}$ are the vector coordinates of targets P and Q, respectively. In the payload-centered camera-aligned coordinate frame, the vector coordinates of the targets are $\bar{p}_0 = [p_{0x} \ p_{0y} \ p_{0z}]^T$ and $\bar{q}_0 = [q_{0x} \ q_{0y} \ q_{0z}]^T$. In the payload-fixed coordinate frame, the vector coordinates of the targets are $\bar{p}_4 = [-r \ 0 \ R]^T$ and $\bar{q}_4 = [r \ 0 \ R]^T$ where r and R are the payload radius and half-height respectively.

From the orthogonality of rotation matrices, $C^{-1} = C^T$. Pre-multiplying both sides of Eq. (A.1) by C^T :

$$V_0 = C^T V_4 \quad (A.2)$$

$$\begin{bmatrix} p_{0x} & q_{0x} \\ p_{0y} & q_{0y} \\ p_{0z} & q_{0z} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \begin{bmatrix} -r & r \\ 0 & 0 \\ R & R \end{bmatrix} \quad (\text{A.3})$$

Solving for x-y coordinates of targets in camera frame:

$$p_{0x} = -c_{11}r + c_{31}R \quad (\text{A.4})$$

$$q_{0x} = c_{11}r + c_{31}R \quad (\text{A.5})$$

$$p_{0y} = -c_{12}r + c_{32}R \quad (\text{A.6})$$

$$q_{0y} = c_{12}r + c_{32}R \quad (\text{A.7})$$

Solving for the direction cosines;

(A.4)-(A.5) and solving for c_{11} :

$$c_{11} = \frac{p_{0x} - q_{0x}}{-2r} \quad (\text{A.8a})$$

(A.4)+(A.5) and solving for c_{31} :

$$c_{31} = \frac{p_{0x} + q_{0x}}{2R} \quad (\text{A.8b})$$

(A.6)-(A.7) and solving for c_{12} :

$$c_{12} = \frac{p_{0y} - q_{0y}}{-2r} \quad (\text{A.8c})$$

(A.6)+(A.7) and solving for c_{32} :

$$c_{32} = \frac{p_{0y} + q_{0y}}{2R} \quad (\text{A.8d})$$

A.2 Euler Angles in 1-2-3 Sequence

With reference to Figures 2.13 and 2.20, the transformation matrix in a 1-2-3 Euler angle sequence, $C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma})$, between camera coordinate frame (frame 0) and the body-fixed frame (frame 4) is:

$$C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma}) = C_3(\hat{\psi})C_2(\hat{\eta})C_1(\hat{\phi})C_1(-\hat{\gamma}) \quad (\text{A.9})$$

where $C_1(-\hat{\gamma})$ is the rotation matrix due to the camera offset angle.

$$C_1(-\hat{\gamma}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{\hat{\gamma}} & -S_{\hat{\gamma}} \\ 0 & S_{\hat{\gamma}} & C_{\hat{\gamma}} \end{bmatrix}; \quad (\text{A.10})$$

and the payload Euler angle rotation matrices corresponding to the 1-2-3 rotation sequence are;

$C_1(\hat{\phi})$, the rotation about $\hat{x}_1$,

$$C_1(\hat{\phi}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_{\hat{\phi}} & S_{\hat{\phi}} \\ 0 & -S_{\hat{\phi}} & C_{\hat{\phi}} \end{bmatrix}; \quad (\text{A.11})$$

$C_2(\hat{\eta})$, the rotation about $\hat{y}_2$,

$$C_2(\hat{\eta}) = \begin{bmatrix} C_{\hat{\eta}} & 0 & -S_{\hat{\eta}} \\ 0 & 1 & 0 \\ S_{\hat{\eta}} & 0 & C_{\hat{\eta}} \end{bmatrix}; \quad (\text{A.12})$$

$C_3(\hat{\psi})$, the rotation about $\hat{z}_3$,

$$C_3(\hat{\psi}) = \begin{bmatrix} C_{\hat{\psi}} & S_{\hat{\psi}} & 0 \\ -S_{\hat{\psi}} & C_{\hat{\psi}} & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad (\text{A.13})$$

The elements of $C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma})$ are:

$$c_{11} = C_{\hat{\psi}} C_{\hat{\eta}} \quad (\text{A.14})$$

$$c_{12} = C_{\hat{\gamma}} (S_{\hat{\phi}} C_{\hat{\psi}} S_{\hat{\eta}} + C_{\hat{\phi}} S_{\hat{\psi}}) + S_{\hat{\gamma}} (S_{\hat{\phi}} S_{\hat{\psi}} - C_{\hat{\phi}} C_{\hat{\psi}} S_{\hat{\eta}}) \quad (\text{A.15})$$

$$c_{13} = -S_{\hat{\gamma}} (S_{\hat{\phi}} C_{\hat{\psi}} S_{\hat{\eta}} + C_{\hat{\phi}} S_{\hat{\psi}}) + C_{\hat{\gamma}} (S_{\hat{\phi}} S_{\hat{\psi}} - C_{\hat{\phi}} C_{\hat{\psi}} S_{\hat{\eta}}) \quad (\text{A.16})$$

$$c_{21} = -S_{\hat{\psi}} C_{\hat{\eta}} \quad (\text{A.17})$$

$$c_{22} = C_{\hat{\gamma}} (-S_{\hat{\phi}} S_{\hat{\psi}} S_{\hat{\eta}} + C_{\hat{\phi}} C_{\hat{\psi}}) + S_{\hat{\gamma}} (S_{\hat{\phi}} C_{\hat{\psi}} + C_{\hat{\phi}} S_{\hat{\psi}} S_{\hat{\eta}}) \quad (\text{A.18})$$

$$c_{23} = -S_{\hat{\gamma}} (-S_{\hat{\phi}} S_{\hat{\psi}} S_{\hat{\eta}} + C_{\hat{\phi}} C_{\hat{\psi}}) + C_{\hat{\gamma}} (S_{\hat{\phi}} C_{\hat{\psi}} + C_{\hat{\phi}} S_{\hat{\psi}} S_{\hat{\eta}}) \quad (\text{A.19})$$

$$c_{31} = S_{\hat{\eta}} \quad (\text{A.20})$$

$$c_{32} = C_{\hat{\gamma}} (-S_{\hat{\phi}} C_{\hat{\eta}}) + S_{\hat{\gamma}} (C_{\hat{\phi}} C_{\hat{\eta}}) \quad (\text{A.21})$$

$$c_{33} = -S_{\hat{\gamma}} (-S_{\hat{\phi}} C_{\hat{\eta}}) + C_{\hat{\gamma}} (C_{\hat{\phi}} C_{\hat{\eta}}) \quad (\text{A.22})$$

Solving for payload Euler angles $\hat{\phi}$, $\hat{\eta}$ and $\hat{\psi}$:

From Eq. (A.20), $\hat{\eta}$, the rotation about the $\hat{y}_2$ axis is:

$$\hat{\eta} = \sin^{-1}(c_{31}) \quad (\text{A.23})$$

$$C_{\hat{\gamma}}(\text{A.21}): C_{\hat{\gamma}} c_{32} = C_{\hat{\gamma}}^2 (-S_{\hat{\phi}} C_{\hat{\eta}}) + C_{\hat{\gamma}} S_{\hat{\gamma}} (C_{\hat{\phi}} C_{\hat{\eta}}) \quad (\text{A.24})$$

$$S_{\hat{\gamma}}(\text{A.22}): S_{\hat{\gamma}} c_{33} = -S_{\hat{\gamma}}^2 (-S_{\hat{\phi}} C_{\hat{\eta}}) + C_{\hat{\gamma}} S_{\hat{\gamma}} (C_{\hat{\phi}} C_{\hat{\eta}}) \quad (\text{A.25})$$

$$(\text{A.25})-(\text{A.24}): c_{33} S_{\hat{\gamma}} - c_{32} C_{\hat{\gamma}} = S_{\hat{\phi}} C_{\hat{\eta}} \quad (\text{A.26})$$

The magnitude of each row and column in a rotation matrix is 1. Therefore:

$$c_{33} = \pm \sqrt{1 - c_{32}^2 - c_{31}^2} \quad (\text{A.27})$$

From trigonometry:

$$\begin{aligned} C_{\eta} &= \pm \sqrt{1 - S_{\eta}^2} \\ &= \pm \sqrt{1 - c_{31}^2} \end{aligned} \quad (\text{A.28})$$

Substituting c_{33} and C_{η} into Eq. (A.26) and solving for ϕ , the rotation about $\hat{x}_1$:

$$\hat{\phi} = \sin^{-1} \left(\frac{(\pm \sqrt{1 - c_{32}^2 - c_{31}^2}) S_{\gamma} - c_{32} C_{\gamma}}{\pm \sqrt{1 - c_{31}^2}} \right) \quad (\text{A.29})$$

From Eq. (A.14):

$$C_{\psi} = \frac{c_{11}}{C_{\eta}} \quad (\text{A.30})$$

Rearranging Eq. (A.15)

$$c_{12} = (C_{\gamma} S_{\phi} S_{\eta} - S_{\gamma} C_{\phi} S_{\eta}) C_{\psi} + (C_{\gamma} C_{\phi} + S_{\gamma} S_{\phi}) S_{\psi} \quad (\text{A.31})$$

Solving for S_{ψ} :

$$S_{\psi} = \frac{c_{12} - (C_{\gamma} S_{\phi} S_{\eta} - S_{\gamma} C_{\phi} S_{\eta}) C_{\psi}}{(C_{\gamma} C_{\phi} + S_{\gamma} S_{\phi})} \quad (\text{A.32})$$

Substituting C_{ψ} into S_{ψ} :

$$S_{\psi} = \frac{C_{\eta} c_{12} - (C_{\gamma} S_{\phi} S_{\eta} - S_{\gamma} C_{\phi} S_{\eta}) c_{11}}{(C_{\gamma} C_{\phi} + S_{\gamma} S_{\phi}) C_{\eta}} \quad (\text{A.33})$$

Angle $\hat{\psi}$ represents the rotation about $\hat{z}_j$, the spin angle. This angle revolves through 360° . To ensure a finite error throughout the range of motion of this angle, the inverse tangent will be used:

$$\hat{\psi} = \tan^{-1} \frac{S_{\hat{\psi}}}{C_{\hat{\psi}}} \quad (\text{A.34})$$

where $C_{\hat{\psi}}$ and $S_{\hat{\psi}}$ are defined in Eqs. (A.30) and (A.33).

A.3 Euler Angles in 3-2-1 Sequence: The Indirect Method

For a given rotation, elements of a rotation matrix will have the same value, regardless of the rotation sequence used. This fact is used in the indirect method of solving for the Euler angles in a 3-2-1 rotation sequence. The indirect method involves equating the direction cosines of the 3-2-1 rotation matrix with the direction cosines of the 1-2-3 rotation matrix and then solving for the Euler angles in the 3-2-1 rotation sequence.

A 3-2-1 Euler angle sequence is shown in Figure 2.21. The 3-2-1 rotation matrix $C(\psi, \theta, \phi, -\gamma)$, between camera coordinate frame (frame 0) and the body-fixed frame (frame 4) is:

$$C(\psi, \theta, \phi, -\gamma) = C_1(\psi)C_2(\theta)C_3(\phi)C_1(-\gamma) \quad (\text{A.35})$$

where $C_1(-\gamma)$ is the rotation about x_0 due to camera offset angle as shown in Figure 2.13,

$$C_1(-\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\gamma & -S_\gamma \\ 0 & S_\gamma & C_\gamma \end{bmatrix}; \quad (\text{A.36})$$

and the rotation matrices corresponding to the 3-2-1 Euler angle sequence are;

$C_3(\phi)$, the rotation about z_1 ,

$$C_3(\phi) = \begin{bmatrix} C_\phi & S_\phi & 0 \\ -S_\phi & C_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad (\text{A.37})$$

$C_2(\theta)$, the rotation about y_2 ,

$$C_2(\theta) = \begin{bmatrix} C_\theta & 0 & -S_\theta \\ 0 & 1 & 0 \\ S_\theta & 0 & C_\theta \end{bmatrix}; \quad (\text{A.38})$$

$C_1(\psi)$, the rotation about x_3 ,

$$C_1(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\psi & S_\psi \\ 0 & -S_\psi & C_\psi \end{bmatrix}; \quad (\text{A.39})$$

The elements of $C(\psi, \theta, \phi, -\gamma)$ are:

$$c_{11} = C_\phi C_\theta \quad (\text{A.40})$$

$$c_{12} = C_\gamma (S_\phi C_\theta) + S_\gamma (-S_\theta) \quad (\text{A.41})$$

$$c_{13} = -S_\gamma (S_\phi C_\theta) + C_\gamma (-S_\theta) \quad (\text{A.42})$$

$$c_{21} = S_\psi C_\phi S_\theta - C_\psi S_\phi \quad (\text{A.43})$$

$$c_{22} = C_\gamma (S_\psi S_\phi S_\theta + C_\psi C_\phi) + S_\gamma (S_\psi C_\theta) \quad (\text{A.44})$$

$$c_{23} = -S_\gamma (S_\psi S_\phi S_\theta + C_\psi C_\phi) + C_\gamma (S_\psi C_\theta) \quad (\text{A.45})$$

$$c_{31} = C_\psi C_\phi S_\theta + S_\psi S_\phi \quad (\text{A.46})$$

$$c_{32} = C_\gamma (C_\psi S_\phi S_\theta - S_\psi S_\phi) + S_\gamma (C_\psi C_\theta) \quad (\text{A.47})$$

$$c_{33} = -S_\gamma (C_\psi S_\phi S_\theta - S_\psi S_\phi) + C_\gamma (C_\psi C_\theta) \quad (\text{A.48})$$

Recall from Eq. (A.1) that the rotation matrix C , transforms coordinates from reference frame 0 to reference frame 4:

$$V_4 = CV_0 \quad (A.49)$$

Using a 1-2-3 rotation sequence, (A.49) is written as:

$$V_4 = C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma})V_0 \quad (A.50)$$

Using a 3-2-1 rotation sequence, (A.49) is written as:

$$V_4 = C(\psi, \theta, \phi, -\gamma)V_0 \quad (A.51)$$

Comparing Eqs. (A.50) and (A.51), it can be seen that the two rotation matrices must be equal, element for element. This can be expressed as:

$$C(\psi, \theta, \phi, -\gamma) = C(\hat{\psi}, \hat{\eta}, \hat{\phi}, -\hat{\gamma}) \quad (A.52)$$

The following algebraic manipulations are performed prior to equating the elements of the two rotation matrices:

$$S_{\hat{\gamma}}(A.15): S_{\hat{\gamma}}c_{12} = S_{\hat{\gamma}}C_{\hat{\gamma}}(S_{\hat{\phi}}C_{\hat{\psi}}S_{\hat{\eta}} + C_{\hat{\phi}}S_{\hat{\psi}}) + S_{\hat{\gamma}}^2(S_{\hat{\phi}}S_{\hat{\psi}} - C_{\hat{\phi}}C_{\hat{\psi}}S_{\hat{\eta}}) \quad (A.53)$$

$$C_{\hat{\gamma}}(A.16): C_{\hat{\gamma}}c_{13} = -C_{\hat{\gamma}}S_{\hat{\gamma}}(S_{\hat{\phi}}C_{\hat{\psi}}S_{\hat{\eta}} + C_{\hat{\phi}}S_{\hat{\psi}}) + C_{\hat{\gamma}}^2(S_{\hat{\phi}}S_{\hat{\psi}} - C_{\hat{\phi}}C_{\hat{\psi}}S_{\hat{\eta}}) \quad (A.54)$$

$$(A.53)+(A.54): S_{\hat{\gamma}}c_{12} + C_{\hat{\gamma}}c_{13} = (S_{\hat{\phi}}S_{\hat{\psi}} - C_{\hat{\phi}}C_{\hat{\psi}}S_{\hat{\eta}}) \quad (A.55)$$

$$S_{\gamma}(A.41): S_{\gamma}c_{12} = S_{\gamma}C_{\gamma}(S_{\phi}C_{\psi}) + S_{\gamma}^2(-S_{\theta}) \quad (A.56)$$

$$C_{\gamma}(A.42): C_{\gamma}c_{13} = -C_{\gamma}S_{\gamma}(S_{\phi}C_{\psi}) + C_{\gamma}^2(-S_{\theta}) \quad (A.57)$$

$$(A.56)+(A.57): S_{\gamma}c_{12} + C_{\gamma}c_{13} = -S_{\theta} \quad (A.58)$$

The camera offset angle is constant regardless of rotation sequence. ie $\hat{\gamma} = \gamma$. Therefore, the right hand sides of Eqs. (A.55) and (A.58) can be equated:

$$-S_{\theta} = (S_{\hat{\phi}}S_{\hat{\psi}} - C_{\hat{\phi}}C_{\hat{\psi}}S_{\hat{\eta}}) \quad (A.59)$$

Solving for θ , the rotation about the y_2 axis:

$$\theta = \sin^{-1}(C_{\phi}C_{\psi}S_{\eta} - S_{\phi}S_{\psi}) \quad (\text{A.60})$$

Performing further algebraic manipulations on the elements of the two rotation matrices:

$$S_{\eta}(\text{A.18}): S_{\eta}c_{22} = S_{\eta}C_{\eta}(-S_{\phi}S_{\psi}S_{\eta} + C_{\phi}C_{\psi}) + S_{\eta}^2(S_{\phi}C_{\psi} + C_{\phi}S_{\psi}S_{\eta}) \quad (\text{A.61})$$

$$C_{\eta}(\text{A.19}): C_{\eta}c_{22} = -C_{\eta}S_{\eta}(-S_{\phi}S_{\psi}S_{\eta} + C_{\phi}C_{\psi}) + C_{\eta}^2(S_{\phi}C_{\psi} + C_{\phi}S_{\psi}S_{\eta}) \quad (\text{A.62})$$

$$(\text{A.61})+(\text{A.62}): C_{\eta}c_{22} + S_{\eta}c_{22} = (S_{\phi}C_{\psi} + C_{\phi}S_{\psi}S_{\eta}) \quad (\text{A.63})$$

$$S_{\gamma}(\text{A.44}): S_{\gamma}c_{22} = S_{\gamma}C_{\gamma}(S_{\psi}S_{\phi}S_{\theta} + C_{\psi}C_{\phi}) + S_{\gamma}^2(S_{\psi}C_{\phi}) \quad (\text{A.64})$$

$$C_{\gamma}(\text{A.45}): C_{\gamma}c_{22} = -C_{\gamma}S_{\gamma}(S_{\psi}S_{\phi}S_{\theta} + C_{\psi}C_{\phi}) + C_{\gamma}^2(S_{\psi}C_{\phi}) \quad (\text{A.65})$$

$$(\text{A.64})+(\text{A.65}): C_{\gamma}c_{22} + S_{\gamma}c_{22} = S_{\psi}C_{\phi} \quad (\text{A.66})$$

Therefore, equating the right hand sides of Eqs. (A.63) and (A.66):

$$S_{\psi}C_{\phi} = (S_{\phi}C_{\psi} + C_{\phi}S_{\psi}S_{\eta}) \quad (\text{A.67})$$

Solving for ψ , the rotation about the x_3 axis:

$$\psi = \sin^{-1}\left(\frac{(S_{\phi}C_{\psi} + C_{\phi}S_{\psi}S_{\eta})}{C_{\theta}}\right) \quad (\text{A.68})$$

Additional algebraic manipulation of the direction cosines:

$$C_{\eta}(\text{A.15}): C_{\eta}c_{12} = C_{\eta}^2(S_{\phi}C_{\psi}S_{\eta} + C_{\phi}S_{\psi}) + C_{\eta}S_{\eta}(S_{\phi}S_{\psi} - C_{\phi}C_{\psi}S_{\eta}) \quad (\text{A.69})$$

$$S_{\eta}(\text{A.16}): S_{\eta}c_{13} = -S_{\eta}^2(S_{\phi}C_{\psi}S_{\eta} + C_{\phi}S_{\psi}) + S_{\eta}C_{\eta}(S_{\phi}S_{\psi} - C_{\phi}C_{\psi}S_{\eta}) \quad (\text{A.70})$$

$$(\text{A.69})-(\text{A.70}): C_{\eta}c_{12} - S_{\eta}c_{13} = (S_{\phi}C_{\psi}S_{\eta} + C_{\phi}S_{\psi}) \quad (\text{A.71})$$

$$C_{\gamma}(\text{A.42}): C_{\gamma}c_{12} = C_{\gamma}^2(S_{\phi}C_{\theta}) + C_{\gamma}S_{\gamma}(-S_{\theta}) \quad (\text{A.72})$$

$$S_{\gamma}(\text{A.26}): S_{\gamma}c_{13} = -S_{\gamma}^2(S_{\phi}C_{\theta}) + S_{\gamma}C_{\gamma}(-S_{\theta}) \quad (\text{A.73})$$

$$(\text{A.72})-(\text{A.73}): C_{\gamma}c_{12} - S_{\gamma}c_{13} = S_{\phi}C_{\theta} \quad (\text{A.74})$$

Equating the right hand sides of (A.71) and (A.74), and again using the fact that $\hat{\gamma} = \gamma$:

$$S_{\phi}C_{\theta} = (S_{\phi}C_{\psi}S_{\eta} + C_{\phi}S_{\psi}) \quad (\text{A.75})$$

Rearranging:

$$S_{\phi} = \frac{(S_{\phi}C_{\psi}S_{\eta} + C_{\phi}S_{\psi})}{C_{\theta}} \quad (\text{A.76})$$

As in the 1-2-3 sequence, the spin angle ϕ rotates through 360° . Therefore, a solution using $\tan^{-1}$ is required. To this end, an equation for C_{ϕ} will be found. Using element c_{11} , the right hand sides of Eqs. (A.14) and (A.40):

$$C_{\phi}C_{\theta} = C_{\psi}C_{\eta} \quad (\text{A.77})$$

Rearranging:

$$C_{\phi} = \frac{C_{\psi}C_{\eta}}{C_{\theta}} \quad (\text{A.78})$$

Dividing Eq. (A.76) by Eq. (A.78) and solving for ϕ , the rotation about the z_1 axis:

$$\phi = \tan^{-1} \left(\frac{(S_{\phi}C_{\psi}S_{\eta} + C_{\phi}S_{\psi})}{C_{\psi}C_{\eta}} \right) \quad (\text{A.79})$$

A.4 Euler Angles in 3-2-1 Sequence: The Direct Method

This section presents a method to calculate the payload Euler angles directly from the direction cosines. The derivation uses the elements of the 3-2-1 rotation matrix, $C(\psi, \theta, \phi, -\gamma)$, presented in §A.3.

$$C_{\gamma}(\text{A.41}): C_{\gamma}c_{12} = C_{\gamma}^2(S_{\phi}C_{\theta}) + C_{\gamma}S_{\gamma}(-S_{\theta}) \quad (\text{A.80})$$

$$S_{\gamma}(\text{A.42}): S_{\gamma}c_{13} = -S_{\gamma}^2(S_{\phi}C_{\theta}) + S_{\gamma}C_{\gamma}(-S_{\theta}) \quad (\text{A.81})$$

$$(\text{A.80})-(\text{A.81}): C_{\gamma}c_{12} - S_{\gamma}c_{13} = S_{\phi}C_{\theta} \quad (\text{A.82})$$

Solving for S_ϕ :

$$S_\phi = \frac{C_\gamma c_{12} - S_\gamma c_{13}}{C_\phi} \quad (\text{A.83})$$

From Eq. (A.40): $C_\phi = \frac{c_{11}}{C_\phi}$ (A.84)

Dividing Eq. (A.83) by Eq. (A.84) and solving for ϕ :

$$\phi = \tan^{-1} \left(\frac{C_\gamma c_{12} - S_\gamma c_{13}}{c_{11}} \right) \quad (\text{A.85})$$

The magnitudes of all rows and columns of rotation matrices are 1. Therefore:

$$c_{11}^2 + c_{12}^2 + c_{13}^2 = 1 \quad (\text{A.86})$$

Solving for c_{13} :

$$c_{13} = \sqrt{1 - c_{11}^2 - c_{12}^2} \quad (\text{A.87})$$

Eq. (A.85) becomes:

$$\phi = \tan^{-1} \left(\frac{C_\gamma c_{12} - S_\gamma (\pm \sqrt{1 - c_{11}^2 - c_{12}^2})}{c_{11}} \right) \quad (\text{A.88})$$

$$C_\gamma (\text{A.47}): C_\gamma c_{32} = C_\gamma^2 (C_\psi S_\phi S_\theta - S_\psi S_\theta) + C_\gamma S_\gamma (C_\psi C_\theta) \quad (\text{A.89})$$

$$S_\gamma (\text{A.48}): S_\gamma c_{33} = -S_\gamma^2 (C_\psi S_\phi S_\theta - S_\psi S_\theta) + S_\gamma C_\gamma (C_\psi C_\theta) \quad (\text{A.90})$$

$$(\text{A.89})-(\text{A.90}): C_\gamma c_{32} - S_\gamma c_{33} = C_\psi S_\phi S_\theta - S_\psi C_\phi \quad (\text{A.91})$$

$$C_\phi (\text{A.91}): C_\phi (C_\gamma c_{32} - S_\gamma c_{33}) = C_\psi C_\phi S_\phi S_\theta - C_\phi^2 S_\psi \quad (\text{A.92})$$

$$S_\phi (\text{A.46}): S_\phi c_{31} = C_\psi C_\phi S_\phi S_\theta + S_\psi S_\phi^2 \quad (\text{A.93})$$

$$(\text{A.92})-(\text{A.93}): S_\psi = S_\phi c_{31} - C_\phi (C_\gamma c_{32} - S_\gamma c_{33}) \quad (\text{A.94})$$

But: $c_{33} = \pm \sqrt{1 - c_{31}^2 - c_{32}^2}$ (A.95)

Substituting c_{33} into Eq. (A.94) and solving for ψ :

$$\psi = \sin^{-1} \left(S_{\phi} c_{31} - C_{\phi} \left(C_{\gamma} c_{32} - S_{\gamma} \left(\pm \sqrt{1 - c_{31}^2 - c_{32}^2} \right) \right) \right) \quad (\text{A.96})$$

$$S_{\gamma} (\text{A.47}): S_{\gamma} c_{32} = S_{\gamma} C_{\gamma} (C_{\psi} S_{\phi} S_{\theta} - S_{\psi} S_{\phi}) + S_{\gamma}^2 (C_{\psi} C_{\theta}) \quad (\text{A.97})$$

$$C_{\gamma} (\text{A.48}): C_{\gamma} c_{33} = -S_{\gamma} C_{\gamma} (C_{\psi} S_{\phi} S_{\theta} - S_{\psi} S_{\phi}) + C_{\gamma}^2 (C_{\psi} C_{\theta}) \quad (\text{A.98})$$

$$(\text{A.97}) + (\text{A.98}): S_{\gamma} c_{32} + C_{\gamma} c_{33} = C_{\psi} C_{\theta} \quad (\text{A.99})$$

Using (A.95) and solving for θ :

$$\theta = \cos^{-1} \left(\frac{S_{\gamma} c_{32} + C_{\gamma} \left(\pm \sqrt{1 - c_{31}^2 - c_{32}^2} \right)}{C_{\psi}} \right) \quad (\text{A.100})$$

END

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FIN