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**NON-SELFADJOINT VIBRATION, CONTROL AND
STABILITY OF THIN ELASTIC
RECTANGULAR PLATES**

by

BO LIN, B.Sc.

*A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements for the degree of*

Master of Engineering

**Department of Mechanical and Aerospace Engineering
Ottawa-Carleton Institute for Mechanical and Aerospace Engineering
Carleton University
Ottawa, Ontario K1S 5B6**

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**NON-SELFADJOINT VIBRATION, CONTROL AND STABILITY
OF THIN ELASTIC RECTANGULAR PLATES**

submitted by

Bo Lin

B.Sc.

in partial fulfillment of the requirements for
the degree of Master of Engineering

Chairman,

Department of Mechanical and Aerospace Engineering

Thesis Supervisor

Carleton University

August, 1993

To my parents and my grandmother

Abstract

Green's function approach for two dimensional selfadjoint problems is reviewed in detail and assumptions on which such an approach is based are highlighted. Subsequently, by introducing corresponding assumptions, Green's function approach is extended to be applicable to two dimensional non-selfadjoint systems. The analytical solutions in integral form with unsymmetric Green's function as kernel, have been obtained for non-selfadjoint systems. Advantages of this approach have been shown to be its versatility, ease of application and generality of the form of solution for a given set of boundary conditions under any loading.

Specifically, dynamic response of thin plates under follower forces have been considered. Equations of motion and corresponding boundary conditions have been derived in non-dimensional forms. The analytical solutions for two different types of boundary conditions are obtained through Green's function approach. The system's dynamic characteristics, e.g. non-dimensional eigenvalues and eigenfunctions are either plotted or tabulated. The rapid convergence of the solution was shown through a numerical example.

Non-selfadjoint plate problems introduced by application of non-selfadjoint control law are also considered. The effectiveness of the control law is evaluated through Green's function approach. Stability characteristics of the control system are investigated using Liapunov's first and second method, which yield the upper and lower bounds for the actual stability boundary of the system, respectively.

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Principal Symbols

B	—	spatial boundary
b, l	—	plate lateral dimension
c	—	plate control parameter
c^*	—	non-dimensional control parameter
D	—	flexural rigidity of the plate $D = Eh^3(1 - \nu^2)/12$
$\mathcal{D}$	—	partial differential operator
$\hat{\mathcal{D}}$	—	adjoint operator of $\mathcal{D}$
$\mathcal{D}_s$	—	selfadjoint operator
E	—	Young's modulus of the plate material
g	—	uniformly distributed monogenetic force
g^*	—	non-dimensional monogenetic force parameter
G	—	Green's function
G_s	—	symmetric Green's function
H	—	Hamiltonian of the plate
$\mathcal{L}$	—	control operator
M	—	Liapunov functional
M_s	—	upper bound of $\sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2}$
M_v	—	upper bound of eigenfunction values
M_ψ	—	upper bound of adjoint eigenfunction values
M_{mi}	—	fundamental solutions

N_{m_i}	—	fundamental adjoint solution
$\mathcal{N}$	—	non-selfadjoint operator
$U[\]_B$	—	vector of linear differential operators representing boundary conditions
K	—	transverse driving force
Q	—	load parameter space
q	—	uniformly distributed follower force
q_{crit1}	—	critical q
q^*	—	non-dimensional follower force parameter
$\vec{q}$	—	load parameter vector
t	—	time
V	—	eigenfunction
W	—	plate transverse displacement
x, y	—	plate spatial coordinates
$\vec{x}$	—	spatial coordinate vector
γ, σ	—	non-dimensional spatial dummy variables
$\vec{\xi}$	—	dummy spatial coordinate vector
δ	—	Dirac delta function
μ	—	mass density per unit area
ν	—	Poisson ratio of the plate material
ξ, η	—	non-dimensional spatial coordinates
ω	—	natural frequency
λ	—	non-dimensional frequency parameter
ϕ	—	aspect ratio of the plate
τ	—	dummy variable of t
ψ	—	adjoint eigenfunction of V
Ω	—	integral area of plate
Ω_0	—	frequency of the harmonic driving force

Chapter 1

Introduction

In general, initial-boundary value problems are almost unavoidable when constructing mathematical models of problems in engineering mechanics. Specifically, such problems play a major part in the investigation of vibrational response and structural control of elastic systems [1]. In dealing with various engineering problems encountered in vibration of elastic continua and the associated initial-boundary value problems, two fundamental approaches have been recognized, i.e., Numerical Methods, e.g., [2, 3, 4, 5], and Analytical Methods. Undoubtedly, within recent years, considerable advances have been made in obtaining numerical solutions to challenging engineering problems, in general, and complicated problems, in particular, in structural vibration and control. However, the fact remains that numerical methods will result in numerical solutions only, and therefore, any important qualitative conclusions regarding the character of such solutions can be arrived at only by means of logical *induction*. Whereas besides its challenge and elegance, an analytical solution will always allow the exploitation of the much superior logical process of *deduction* in arriving at any conclusions. It is in view of such an advantage that an analytical approach is adapted for the present work.

Within the realm of analytical methods, one of the most powerful approaches is Green's function approach which is based on the concept of equivalence of integral and differential equations. In this approach, the analytical solution is presented in the

form of an integral involving a kernel, the Green's function, which was first introduced by G. Green as early as 1828.

Moreover, it is quite interesting to note that Green's function approach is considered to be a powerful method not only by those who are interested in analytical solutions but also by those who are primarily interested in numerical methods. The main reason for this is the fact that using Green's function approach an initial-boundary value problem can be reduced to an integral equation representation, which is often much more amenable to numerical analysis than its differential equation representation with associated initial-boundary conditions.

Thus for many initial-boundary value problems, construction of the corresponding Green's function becomes an essential task. There are several ways of obtaining this function for a given problem, for example, by explicitly seeking an inverse operator to the original differential operator [6], by conformal mapping [7] or by eigenfunction method [7], etc. Among these methods, eigenfunction method seems to be a more general and more powerful method which could be applied to rather complicated problems.

For a brief review of this approach, a simple selfadjoint equilibrium problem, representing mathematical formulation of a conservative static system will be considered in the following. The concept of Green's function and the essential features of the eigenfunction method to obtain this function for the equilibrium problem is illustrated.

Consider a domain Ω of the independent space variable $\vec{x}$, i.e., an interval on the x-axis or a domain either in x, y-plane or in x, y, z-space which has a piecewise smooth boundary B . Suppose the state of a continuum occupying Ω is characterized by a function $W(\vec{x})$. Let $\mathcal{D}[W]$ be a selfadjoint linear differential operator in the independent variables $\vec{x}$, defined in Ω , which arises from the variation of the system's potential energy. Let μ represent the mass density at any point of Ω and let $K(\vec{x})$ represent a given external force. Then the governing differential equation of the equilibrium problem reads:

$$\mathcal{D}[W(\vec{x})] = K(\vec{x}) , \quad (1.1)$$

with associated boundary conditions:

$$U[W(\vec{x})]_B = \{0\} , \quad (1.2)$$

where the subscript B indicates that various expressions are to be taken at the boundary of domain Ω .

If we consider a free vibration of the same system with the same boundary conditions, the governing differential equation becomes:

$$\mathcal{D}[u(\vec{x}, t)] + \mu u_{tt}(\vec{x}, t) = 0 \quad (1.3)$$

where $u(\vec{x}, t)$ is time varying response and subscript t representing derivative with respect to time. Boundary conditions will be the same as (1.2). The eigenvibration of the system is distinguished by requiring u in (1.3) to be separable as the product of a spatial function $v(\vec{x})$ and time function $T(t)$, i.e.,

$$u(\vec{x}, t) = v(\vec{x})T(t) . \quad (1.4)$$

Each such eigenvibration is associated with a constant value ω , an eigenvalue, with the property

$$\ddot{T} + \omega^2 T = 0 , \quad (1.5)$$

hence,

$$T(t) = a \cos \omega t + b \sin \omega t , \quad (1.6)$$

and,

$$\mathcal{D}[v(\vec{x})] = \mu \omega^2 v(\vec{x}) , \quad (1.7)$$

where $v(\vec{x})$ must satisfy the boundary conditions (1.2). The eigenvalue problem is to determine values of the parameter ω_i^2 (eigenvalues) for which the homogeneous differential equation (1.3) has nontrivial solutions $v_i(\vec{x})$ (eigenfunctions) under the prescribed boundary conditions.

In case of a finite domain Ω the following statements are in general true. The eigenvalues ω_i^2 form a denumerable infinite sequence $\omega_1^2, \omega_2^2, \dots$. There exists a system of associate eigenfunctions $v_1(\vec{x}), v_2(\vec{x}), \dots$ which is complete and which satisfies the orthogonality relations:

$$\int_{\Omega} v_i v_j d\vec{x} = \delta_{ij} . \quad (1.8)$$

Moreover the expansion theorem holds: Every function $H(\vec{x})$ with continuous $\mathcal{D}[H(\vec{x})]$ which satisfies the prescribed boundary conditions (1.2) may be expanded in an absolutely uniformly ¹ convergent series in the eigenfunctions:

$$H(\vec{x}) = \sum_{i=1}^{\infty} c_i v_i(\vec{x})$$

$$c_i = \int_{\Omega} H(\vec{\xi}) v_i(\vec{\xi}) d\vec{\xi} . \quad (1.9)$$

¹the definition of absolutely and uniformly convergence of a series can be found in [26]

where c_i are the corresponding generalized Fourier series expansion coefficients.

On the basis of these properties, the equilibrium problem (1.1-1.2) can be solved in the following way:

Applying the above expansion theorem, both $W(\vec{x})$ and $K(\vec{x})$ can be expanded in terms of eigenfunctions.

$$W(\vec{x}) = \sum_{i=1}^{\infty} a_i v_i(\vec{x}) , \quad (1.10)$$

$$K(\vec{x}) = \sum_{i=1}^{\infty} k_i v_i(\vec{x}) , \quad (1.11)$$

$$k_i = \int_{\Omega} K(\vec{\xi}) v_i(\vec{\xi}) d\vec{\xi} . \quad (1.12)$$

where coefficients a_i are to be determined in terms of k_i .

Substituting (1.10-1.11) into (1.1) and using (1.7) one has:

$$\mathcal{D} \left[\sum_{i=1}^{\infty} a_i v_i(\vec{x}) \right] = \sum_{i=1}^{\infty} k_i v_i(\vec{x}) ,$$

i.e.,

$$\sum_{i=1}^{\infty} a_i \mu \omega_i^2 v_i(\vec{x}) = \sum_{i=1}^{\infty} k_i v_i(\vec{x}) . \quad (1.13)$$

Comparing the coefficient of $v_i(\vec{x})$ yields

$$\begin{aligned} a_i &= \frac{1}{\mu \omega_i^2} k_i \\ &= \frac{1}{\mu \omega_i^2} \int_{\Omega} K(\vec{\xi}) v_i(\vec{\xi}) d\vec{\xi} , \end{aligned} \quad (1.14)$$

so that substituting (1.14) into (1.10) yields

$$W(\vec{x}) = \sum_{i=1}^{\infty} \frac{v_i(\vec{x})}{\mu\omega_i^2} \int_{\Omega} K(\vec{\xi})v_i(\vec{\xi})d\vec{\xi}. \quad (1.15)$$

If the summation and integration can prove to be interchangeable, one arrives at the Green's function for the equilibrium problem (1.1-1.2)

$$G(\vec{x}, \vec{\xi}) = \sum_{i=1}^{\infty} \frac{v_i(\vec{x})v_i(\vec{\xi})}{\mu\omega_i^2}, \quad (1.16)$$

such that the solution of the equilibrium problem (1.1-1.2) could be written in the form

$$W(\vec{x}) = \int_{\Omega} G(\vec{x}, \vec{\xi})K(\vec{\xi})d\vec{\xi}. \quad (1.17)$$

If the external force $K(\vec{\xi})$ is set to be $\delta(\vec{\xi}-\vec{\xi}_0)$, the Dirac Delta function, then equation (1.17) yields

$$W(\vec{x}) = G(\vec{x}, \vec{\xi}_0). \quad (1.18)$$

It seems that the Green's function can be interpreted as an influence function which represents the response of the domain Ω as a function of $\vec{x}$, due to a unit load at $\vec{\xi} = \vec{\xi}_0$. Then in view of linear elasticity of the system, $G(\vec{x}, \vec{\xi})K(\vec{\xi})d\vec{\xi}$ is the deflection due to an incremental load $K(\vec{\xi})d\vec{\xi}$ at $\vec{\xi}$ so that equation (1.17) represents the superposition of the resulting incremental deflections.

Also from the Green's function expression in (1.16) it is easily found that G is a symmetric function, i.e.,

$$G(\vec{x}, \vec{\xi}) = G(\vec{\xi}, \vec{x}) . \quad (1.19)$$

The symmetry of the Green's function expresses a reciprocal theorem in the linear elastic system: If the force 1, applied at the point $\vec{\xi}$, produces the deflection $G(\vec{x}, \vec{\xi})$ at the point $\vec{x}$, then the force 1 acting at $\vec{x}$ produces the same deflection $G(\vec{\xi}, \vec{x})$ at $\vec{\xi}$.

The underlying mathematical theory for Green's function approach when applied to selfadjoint and separable problems is well established in the literature, and there exists a good deal of information on the application of this method to such "traditional" problem in engineering mechanics, in general, and to vibration problems in particular. However, at least in the realm of structural dynamics and vibrations, the investigation of linearly elastic non-selfadjoint and even possibly non-separable boundary value problems are of much recent interest, e.g., [8, 9, 10, 11, 12]. In fact, the possibility of expanding Green's function approach to include separable non-selfadjoint systems was not accomplished until 1983 [8, 9]. The main interest in these problems stems from the fact that elastic systems involving mechanically non-conservative forces result in mathematically non-selfadjoint problems. Such forces occur in the context of certain mechanical systems such as rockets, engines, pipes conveying fluid, parts of turbines and electric machines, etc. [5, 13, 14, 15, 16, 17]. Early investigations of homogeneous form of such problems are due to Pfluger [13], Beck [19], Ziegler [20] and Bolotin [21], with subsequent contributions by Leipholz to include inhomogeneous problems e.g. [8, 9].

Moreover, in modern structural design, more and more flexible and slender continuous structural systems require the application of some active feedback control in order to control excessive deflections and/or accelerations caused by their vibration. The application of such control laws may introduce non-selfadjointness to the system. Such more recent problems in "*Control of Structures*" have made the investigation of non-selfadjoint systems even more necessary.

The present work is concerned with using Green's function approach to determine dynamic characteristics and vibration response of the elastic plates as two-dimensional non-selfadjoint systems. In this context, subsequent to presentation and development of necessary theoretical considerations, firstly, elastic analysis of thin elastic plates under uniformly distributed tangential follower type forces is considered. Specifically, the Green's function for two separate types of boundary conditions, i.e., all edges simply supported and two opposite edges free with two opposite edges simply supported, are determined. The corresponding eigenvalues and mode shapes have been obtained and a numerical example is provided to show the response of a typical plate.

Secondly, the non-selfadjoint control of a thin plate in the presence of unidirectional (conservative) forces is considered. The plate is modelled as a distributed parameter system which is controlled also by distributed control law such that the governing differential equation of the controlled plate is non-selfadjoint. This problem turns out to be a multiple parameter system. In analyzing this problem, specific practical problems encountered in "*Control of Structures*" such as spillover effect and stability of the system are also fully investigated and dealt with.

Chapter 2

Green's Function Approach to Two-Dimensional Selfadjoint and Non-Selfadjoint Multiple Parameter Problems

2.1 Introduction

It is well established in the literature that Green's function approach is a powerful method that can be used to solve initial-boundary value problems which describe behavior of various mechanical, thermal and electrical systems. In the context of problems in mechanics, there is a good deal of information on the application of this method to separable, selfadjoint and linearly elastic systems. Therefore, let a two-dimensional mathematically selfadjoint system be considered first, so that the details and assumptions on which the Green's function approach is based can be reviewed briefly and highlighted. Subsequently, one may try to find corresponding assumptions for non-selfadjoint systems. Using these assumptions one may then make non-selfadjoint systems of a certain kind amenable to the Green's function approach as well.

2.2 Two-Dimensional Selfadjoint System with Separable Variables

Consider a spatially two-dimensional conservative elastic system (thin plate). The motion of such an elastic system when subjected to an external driving force $K(\vec{x}, t)$ is described by the following set of equations:

$$\mu \ddot{W}(\vec{x}, t) + \mathcal{D}_s[\vec{q}, W(\vec{x}, t)] = K(\vec{x}, t), \quad (2.1)$$

$$U_s[W(\vec{x}, t)]_B = \{0\}, \quad (2.2)$$

$$W(\vec{x}, 0) = f(\vec{x}), \quad \dot{W}(\vec{x}, 0) = h(\vec{x}). \quad (2.3)$$

In (2.1), μ is the mass density per unit area, $W(\vec{x}, t)$ is the transverse deflection of the mid-plane of elastic plate, $\vec{x}$ is the vector of spatial coordinates, t is the time, $(\dot{})$ denotes differentiation with respect to time, $\mathcal{D}_s$ is a selfadjoint operator with respect to $\vec{x}$ as the whole system is assumed to be conservative. In the most general case, considering a multiple parameter system, $\vec{q}$ is a load parameter vector with components $q_1 \dots q_p$ in the parameter space Q . For selfadjoint system all q_i represent conservative type of loading. For the sake of simplicity one may assume two components q_1 and q_2 in the following deliberations. The selfadjoint operator $\mathcal{D}_s[\vec{q}, W(\vec{x}, t)]$ is composed of two parts, such that,

$$\mathcal{D}_s[\vec{q}, W(\vec{x}, t)] = E[W(\vec{x}, t)] + S[\vec{q}, W(\vec{x}, t)], \quad (2.4)$$

with

$$S[\vec{q}, W(\vec{x}, t)] = q_1 S_1[W(\vec{x}, t)] + q_2 S_2[W(\vec{x}, t)], \quad (2.5)$$

where, E is the elastic and therefore selfadjoint partial differential operator with the highest order of n and involving continuous coefficients and only spatial derivatives and, S_1 and S_2 are linear selfadjoint operators of an order $m \leq n - 2$ and involving continuous coefficients and only spatial derivatives. In (2.2), the boundary conditions, the U_s is a vector of linear operators with respect to $\vec{x}$, and the subscript B indicates that various expressions are to be taken at the boundary of the elastic system. Moreover, in (2.3), the initial conditions, f and g are bounded, sufficiently smooth, prescribed functions that satisfy boundary conditions (2.2).

2.2.1 Domain of Stability

The homogeneous form of the original problem given by (2.1) and (2.2), i.e.,

$$\mu \ddot{W}(\vec{x}, t) + \mathcal{D}_s[\vec{q}, W(\vec{x}, t)] = 0, \quad (2.6)$$

$$U_s[W(\vec{x}, t)]_B = \{0\}, \quad (2.7)$$

define the free vibration of the system in the presence of a certain load $\vec{q}$.

Assuming separability of variables, i.e.,

$$W(\vec{x}, t) = V(\vec{x})e^{i\omega t}, \quad (2.8)$$

the coordinate generating system of the problem is obtained as:

$$-\mu\omega_{ij}^2 V_{ij}(\vec{x}) + \mathcal{D}_s[\vec{q}, V_{ij}(\vec{x})] = 0, \quad (2.9)$$

$$U_s[V_{ij}(\vec{x})]_B = \{0\}, \quad (2.10)$$

where V_{ij} and ω_{ij} are eigenfunctions and eigenvalues of the system respectively. Equations (2.9) and (2.10) lead in a well known way to the frequency equation,

$$F(\mu\omega^2, \vec{q}) \equiv F(\mu\omega^2, q_1, q_2) = 0 . \quad (2.11)$$

which yields the eigenvalues ω_{ij}^2 as functions of the load parameter vector $\vec{q}$,

$$\omega_{ij}^2 = \omega_{ij}^2(\vec{q}) , \quad (2.12)$$

and corresponding infinite-many eigenfunctions,

$$V_{ij}(\vec{x}) \equiv V_{ij}(\omega_{ij}^2, \vec{q}, \vec{x}) , \quad (2.13)$$

It is noteworthy that problem (2.6-2.7) is a perturbation of a more fundamental problem, namely,

$$\mu \ddot{W}(\vec{x}, t) + E[W(\vec{x}, t)] = 0 . \quad (2.14)$$

$$U_s[W(\vec{x}, t)]_B = \{0\} . \quad (2.15)$$

Problem (2.6-2.7) bifurcates from (2.14-2.15) by virtue of the parameter vector $\vec{q}$. Setting $\vec{q} = \{0\}$, operator $\mathcal{D}_s$ collapses to E and makes problem (2.6-2.7) degenerate into problem (2.14-2.15). On the other hand problem (2.14-2.15) describes the free vibration of the unloaded thin elastic plate and is therefore selfadjoint and positive definite. The eigenvalues $(\omega_{ij}^0)^2$ of (2.14-2.15) which follow from (2.11) for $\vec{q} = \{0\}$,

i.e., from the characteristic equation

$$F(\mu(\omega^0)^2, \{0\}) = 0 , \quad (2.16)$$

are all positive and represent a denumerably infinite set. The corresponding eigenfunctions $V_{ij}^0(\vec{x})$ are orthogonal uniformly bounded real functions which form a complete set. Now, let $q_2 = 0$, while q_1 increases such that $q_1 > 0$ (Fig 2.1). Then (2.11) reduces to

$$F_1(\mu\omega^2, q_1, 0) = 0 ,$$

from which, the various branches of the eigenvalue curve in $(\mu\omega^2, q_1, 0)$ -plane follow. For $\{q_1, q_2\}^T = \{0\}$ the branches start at the points $\mu(\omega_{ij}^0)^2$ on the $\mu\omega_{ij}^2$ -axis. For the selfadjoint problem there is a certain bound $\{q_{1crit}, 0\}^T$ of $\{q_1, 0\}^T$ below which due to continuous dependence of F_1 on the parameter q_1 , the eigenvalue branches will remain in the positive quadrant, as q_1 increases, so that properties holding for $\mu(\omega_{ij}^0)^2$ and V_{ij}^0 at $\{q_1, q_2\}^T = \{0\}$ will continue to hold for the corresponding eigenvalues and eigenfunctions for all

$$\{0\} \leq \begin{Bmatrix} q_1 \\ 0 \end{Bmatrix} < \begin{Bmatrix} q_{1crit} \\ 0 \end{Bmatrix} .$$

Next for a given $q_1 = \alpha < q_{1crit}$, let q_2 increase from zero. Then (2.11) becomes

$$F_2(\mu\omega^2, \alpha, q_2) = 0 ,$$

from which, the various branches of the eigenvalue curve in $(\mu\omega^2, \alpha, q_2)$ – plane follow. These branches start at the points $\mu(\omega_{ij}^\alpha)^2$ on the $\mu(\omega_{ij})^2$ -axis. Once again, for the selfadjoint problem there is a certain bound $\{\alpha, q_{2crit}\}^T$ of $\{\alpha, q_2\}^T$ below which due to continuous dependence of F_2 on the parameter q_2 , the eigenvalue branches will remain in the positive quadrant as q_2 increases so that properties holding for $\mu(\omega_{ij}^\alpha)^2$ and V_{ij}^α at $\{q_1, q_2\}^T = \{\alpha, 0\}^T$ will continue to hold for the corresponding eigenvalues and eigenfunctions for all

$$\{0\} \leq \begin{Bmatrix} \alpha \\ q_2 \end{Bmatrix} \leq \begin{Bmatrix} \alpha \\ q_{2crit} \end{Bmatrix}.$$

Thus one recognizes a domain $\mathcal{S}$ bounded by a surface defined by

$$0 \leq \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \leq \begin{Bmatrix} q_{1crit}; q_2 = 0 \\ q_{2crit}(\alpha) : \forall 0 \leq \alpha < q_{1crit} \end{Bmatrix}.$$

Beyond the domain $\mathcal{S}$, the ω_{ij}^2 become negative. As a consequence, ω_{ij} become imaginary and $V_{ij}(\vec{x})$ become complex. Moreover, beyond domain $\mathcal{S}$, the natural modes of the system cease to be stable functions of time due to (2.8). Hence the domain $\mathcal{S}$ determines the stability domain of the natural modes. By virtue of the definition, in the stability domain $\mathcal{S}$ of problem (2.6–2.7), the eigenfunctions $V_{ij}(\vec{x})$ are all uniformly bounded real functions and eigenvalues ω_{ij}^2 are all real positive quantities so that problem (2.9–2.10) is positive definite.

Equation (2.9) can be rewritten as,

$$\mathcal{D}_s[\vec{q}, V_{ij}(\vec{x})] = \mu\omega_{ij}^2 V_{ij}(\vec{x}), \quad (2.17)$$

Due to selfadjointness, the following condition holds,

$$\int_{\Omega} V_{kl}(\vec{x}) \mathcal{D}_s[\vec{q}, V_{ij}(\vec{x})] d\vec{x} = \int_{\Omega} V_{ij}(\vec{x}) \mathcal{D}_s[\vec{q}, V_{kl}(\vec{x})] d\vec{x} , \quad (2.18)$$

where Ω is the domain of integration with respect to $\vec{x}$.

Equation (2.9) yields

$$\int_{\Omega} V_{kl}(\vec{x}) \mathcal{D}_s[\vec{q}, V_{ij}(\vec{x})] d\vec{x} = \int_{\Omega} \mu \omega_{ij}^2 V_{kl}(\vec{x}) V_{ij}(\vec{x}) d\vec{x} , \quad (2.19)$$

$$\int_{\Omega} V_{ij}(\vec{x}) \mathcal{D}_s[\vec{q}, V_{kl}(\vec{x})] d\vec{x} = \int_{\Omega} \mu \omega_{kl}^2 V_{ij}(\vec{x}) V_{kl}(\vec{x}) d\vec{x} , \quad (2.20)$$

such that from (2.18), one has,

$$\mu(\omega_{ij}^2 - \omega_{kl}^2) \int_{\Omega} V_{kl}(\vec{x}) V_{ij}(\vec{x}) d\vec{x} = 0 , \quad (2.21)$$

since $\omega_{ij}^2 \neq \omega_{kl}^2$ for $i \neq k$ or $j \neq l$, then after proper normalization, the orthonormality of $V_{ij}(\vec{x})$ is obtained, i.e.,

$$\int_{\Omega} V_{ij}(\vec{x}) V_{kl}(\vec{x}) d\vec{x} = \delta_{ik} \delta_{jl} , \quad (2.22)$$

Based on the theorem stated by Collatz [22] and the selfadjointness and positive definiteness of problem (2.9–2.10), the infinitely many eigenfunctions $V_{ij}(\vec{x})$ generated by (2.9–2.10) form a complete set.

Also based on the selfadjointness and positive definiteness of problem (2.9–2.10) Mercer's theorem applies and Green's Function of problem (2.9–2.10) is a symmetric function which can be expanded in terms of the eigenfunctions of the system, i.e.,

$$G_s(\vec{x}, \vec{\xi}) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x}) V_{ij}(\vec{\xi})}{\mu \omega_{ij}^2} , \quad (2.23)$$

as an absolutely and uniformly convergent expansion for $\vec{x}, \vec{\xi} \in \Omega$. Setting $\vec{\xi} = \vec{x}$ in (2.23) yields,

$$G_s(\vec{x}, \vec{x}) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})V_{ij}(\vec{x})}{\mu\omega_{ij}^2}, \quad (2.24)$$

integrating both sides of this relation yields,

$$\int_{\Omega} G_s(\vec{x}, \vec{x})d\vec{x} = \int_{\Omega} \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})V_{ij}(\vec{x})}{\mu\omega_{ij}^2}d\vec{x}. \quad (2.25)$$

Due to absolute and uniform convergence of the series on the right-hand side of the above equation, one may interchange the summation and integration to obtain

$$\begin{aligned} \int_{\Omega} G_s(\vec{x}, \vec{x})d\vec{x} &= \int_{\Omega} \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})V_{ij}(\vec{x})}{\mu\omega_{ij}^2}d\vec{x} \\ &= \sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2} \int_{\Omega} V_{ij}(\vec{x})V_{ij}(\vec{x})d\vec{x} \\ &= \sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2}, \end{aligned} \quad (2.26)$$

where the last equality in (2.26) is justified due to orthonormality of eigenfunctions. Moreover since G_s is bounded, one finally has,

$$\sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2} = \int_{\Omega} G_s(\vec{x}, \vec{x})d\vec{x} = M_s < \infty. \quad (2.27)$$

In summary, in the range of stability domain $\mathcal{S}$ of problem (2.6–2.7), a set of infinitely many eigenvalues ω_{ij}^2 and the corresponding eigenfunctions $V_{ij}(\vec{x})$ can be obtained, which have the following properties:

- I. (i) the ω_{ij}^2 are real, positive quantities;
 (ii) the ω_{ij}^2 satisfy the condition:

$$\sum_{i,j=1}^{\infty} \frac{1}{\mu \omega_{ij}^2} = M_s < \infty ; \quad (2.28)$$

- (iii) the eigenfunctions $V_{ij}(\vec{x})$ are uniformly bounded real functions so that

$$|V_{ij}(\vec{x})| < M_v \quad \text{for all } i, j \text{ and } \vec{x} \in \Omega ; \quad (2.29)$$

- (iv) the $V_{ij}(\vec{x})$ are orthonormal so that

$$\int_{\Omega} V_{ij}(\vec{x}) V_{kl}(\vec{x}) d\vec{x} = \delta_{ik} \delta_{jl} ; \quad (2.30)$$

- (v) the $V_{ij}(\vec{x})$ form a complete set of functions.

As will be shown later the above properties warrant the applicability of Green's function approach to the selfadjoint problem (2.1-2.3).

2.2.2 Solution to the Selfadjoint Problem

Apply Laplace transformation such that

$$W(\vec{x}, s) = \int_0^{\infty} e^{-st} W(\vec{x}, t) dt , \quad (2.31)$$

$$K(\vec{x}, s) = \int_0^{\infty} e^{-st} K(\vec{x}, t) dt . \quad (2.32)$$

are the Laplace transformations of $W(\vec{x}, t)$ and $K(\vec{x}, t)$. Then the Laplace transformed problem (2.1- 2.3) becomes,

$$\mu s^2 W(\vec{x}, s) - \mu s f(\vec{x}) - \mu h(\vec{x}) + \mathcal{D}_s[\vec{q}, W(\vec{x}, s)] = \bar{K}(\vec{x}, s), \quad (2.33)$$

where the initial conditions (2.3) have also been duly considered.

Since the system of eigenfunctions $V_{ij}(\vec{x})$ is an infinite complete set, the series,

$$W(\vec{x}, s) = \sum_{i,j=1}^{\infty} A_{ij}(s) V_{ij}(\vec{x}), \quad (2.34)$$

$$\bar{K}(\vec{x}, s) = \sum_{i,j=1}^{\infty} B_{ij}(s) V_{ij}(\vec{x}), \quad (2.35)$$

$$f(\vec{x}) = \sum_{i,j=1}^{\infty} F_{ij} V_{ij}(\vec{x}), \quad (2.36)$$

$$h(\vec{x}) = \sum_{i,j=1}^{\infty} H_{ij} V_{ij}(\vec{x}), \quad (2.37)$$

can be set up at least formally, where A_{ij} , B_{ij} , F_{ij} and H_{ij} are the generalized Fourier coefficients such that,

$$B_{ij}(s) = \int_{\Omega} \bar{K}(\vec{\xi}, s) V_{ij}(\vec{\xi}) d\vec{\xi}, \quad (2.38)$$

$$F_{ij} = \int_{\Omega} f(\vec{\xi}) V_{ij}(\vec{\xi}) d\vec{\xi}, \quad (2.39)$$

$$H_{ij} = \int_{\Omega} h(\vec{\xi}) V_{ij}(\vec{\xi}) d\vec{\xi}, \quad (2.40)$$

and the coefficients $A_{ij}(s)$ remain to be determined.

Substituting (2.17) and (2.38-2.40) into (2.33) and collecting terms of V_{ij} appropriately yields,

$$\sum_{i,j=1}^{\infty} \mu A_{ij}(s)(s^2 + \omega_{ij}^2)V_{ij}(\vec{x}) = \sum_{i,j=1}^{\infty} (B_{ij} + \mu s F_{ij} + \mu H_{ij})V_{ij}(\vec{x}) . \quad (2.41)$$

Comparing coefficients of the V_{ij} leads to

$$A_{ij}(s) = \frac{B_{ij}(s)/\mu + sF_{ij} + H_{ij}}{s^2 + \omega_{ij}^2} , \quad (2.42)$$

which determines the coefficients in the series for $W(\vec{x}, s)$, i.e.,

$$W(\vec{x}, s) = \sum_{i,j=1}^{\infty} \frac{B_{ij}(s)/\mu + sF_{ij} + H_{ij}}{s^2 + \omega_{ij}^2} V_{ij}(\vec{x}) , \quad (2.43)$$

so that using (2.38-2.40) in (2.43), one gets

$$W(\vec{x}, s) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})}{s^2 + \omega_{ij}^2} \int_{\Omega} \left[\frac{K(\vec{\xi}, s)}{\mu} + s f(\vec{\xi}) + h(\vec{\xi}) \right] V_{ij}(\vec{\xi}) d\vec{\xi} . \quad (2.44)$$

Applying the inverse Laplace transformation yields,

$$\begin{aligned} W(\vec{x}, t) = & \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}} \sin[\omega_{ij}(t - \tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \\ & + \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij}t) h(\vec{\xi}) d\vec{\xi} \\ & + \sum_{i,j=1}^{\infty} \int_{\Omega} V_{ij}(\vec{x})V_{ij}(\vec{\xi}) \cos(\omega_{ij}t) f(\vec{\xi}) d\vec{\xi} . \end{aligned} \quad (2.45)$$

This is the solution to the initial-boundary value problem (2.1-2.3) provided the convergence of the series in (2.45) can be proved.

Using integration by parts with respect to τ , the first term on the right-hand side of (2.45) becomes

$$\begin{aligned}
 & \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \sin[\omega_{ij}(t-\tau)] K(\vec{\xi},\tau) d\tau d\vec{\xi} \\
 = & - \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \cos[\omega_{ij}(t-\tau)] \frac{\partial K(\vec{\xi},\tau)}{\partial \tau} d\tau d\vec{\xi} \\
 & + \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} K(\vec{\xi},t) d\vec{\xi} \\
 & - \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} K(\vec{\xi},0) \cos(\omega_{ij}t) d\vec{\xi}.
 \end{aligned} \tag{2.46}$$

Then using properties I (ii) and I (iii), i.e.,

$$\sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2} = M_s < \infty ;$$

and

$$|V_{ij}(\vec{x})| < M_v \quad \text{for all } i, j \text{ and } \vec{x} \in \Omega,$$

and based on the following assumptions:

i) The integral of $|\int_0^{\infty} \frac{\partial K}{\partial \tau} d\tau|$ is bounded ¹, i.e.,

$$\left| \int_0^{\infty} \frac{\partial K}{\partial \tau} d\tau \right| = K^* < \infty ; \tag{2.47}$$

¹Leipholz in [9] imposed a stronger condition that required $\int_0^{\infty} |\frac{\partial K}{\partial \tau}| d\tau$ to be bounded. But it can be shown that a weaker condition presented here is enough

ii) $K(\vec{x}, t)$ is uniformly bounded, i.e.,

$$|K(\vec{x}, t)| < K^{**} < \infty ; \quad (2.48)$$

it can be shown that all three series on the right-hand side of (2.46) converge absolutely and uniformly [9], so that summation and integration in these series can be interchanged. Hence a new quantity can be introduced:

$$G_s(\vec{x}, \vec{\xi}, t, \tau) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \cos[\omega_{ij}(t - \tau)] , \quad (2.49)$$

which can be interpreted as the Green's function to problem (2.1-2.3). Obviously,

$$G_s(\vec{x}, \vec{\xi}, t, t) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} , \quad (2.50)$$

$$G_s(\vec{x}, \vec{\xi}, t, 0) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \cos(\omega_{ij}t) , \quad (2.51)$$

so that (2.46) becomes,

$$\begin{aligned} & \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \sin[\omega_{ij}(t - \tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \\ &= - \int_{\Omega} \int_0^t G_s(\vec{x}, \vec{\xi}, t, \tau) \frac{\partial K(\vec{\xi}, \tau)}{\partial \tau} d\tau d\vec{\xi} \\ & \quad + \int_{\Omega} G_s(\vec{x}, \vec{\xi}, t, t) K(\vec{\xi}, t) d\vec{\xi} \\ & \quad - \int_{\Omega} G_s(\vec{x}, \vec{\xi}, t, 0) K(\vec{\xi}, 0) d\vec{\xi} . \end{aligned} \quad (2.52)$$

Moreover, using the Hilbert-Schmidt expansion theorem, it can be shown that because all ω_{ij} are real, the series in the second and third terms on the right-hand side of equation (2.45) are also absolutely and uniformly convergent [9]. Thus summations and integrations can also be interchanged to yield:

$$\begin{aligned} \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij}t) h(\vec{\xi}) d\vec{\xi} &= \int_{\Omega} \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})V_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij}t) h(\vec{\xi}) d\vec{\xi} \\ &= -\mu \int_{\Omega} \frac{\partial G_s(\vec{x}, \vec{\xi}, t, 0)}{\partial t} h(\vec{\xi}) d\vec{\xi} \end{aligned} \quad (2.53)$$

and

$$\begin{aligned} \sum_{i,j=1}^{\infty} \int_{\Omega} V_{ij}(\vec{x})V_{ij}(\vec{\xi}) \cos(\omega_{ij}t) f(\vec{\xi}) d\vec{\xi} &= \int_{\Omega} \sum_{i,j=1}^{\infty} V_{ij}(\vec{x})V_{ij}(\vec{\xi}) \cos(\omega_{ij}t) f(\vec{\xi}) d\vec{\xi} \\ &= -\mu \int_{\Omega} \frac{\partial^2 G_s(\vec{x}, \vec{\xi}, t, 0)}{\partial t^2} f(\vec{\xi}) d\vec{\xi} \end{aligned} \quad (2.54)$$

Using equations (2.52-2.54) in (2.45) leads to,

$$\begin{aligned} W(\vec{x}, t) &= - \int_{\Omega} \int_0^t G_s(\vec{x}, \vec{\xi}, t, \tau) \frac{\partial K(\vec{\xi}, \tau)}{\partial \tau} d\tau d\vec{\xi} \\ &\quad + \int_{\Omega} G_s(\vec{x}, \vec{\xi}, t, t) K(\vec{\xi}, t) d\vec{\xi} \\ &\quad - \int_{\Omega} G_s(\vec{x}, \vec{\xi}, t, 0) K(\vec{\xi}, 0) d\vec{\xi} \\ &\quad - \mu \int_{\Omega} \frac{\partial G_s(\vec{x}, \vec{\xi}, t, 0)}{\partial t} h(\vec{\xi}) d\vec{\xi} \\ &\quad - \mu \int_{\Omega} \frac{\partial^2 G_s(\vec{x}, \vec{\xi}, t, 0)}{\partial t^2} f(\vec{\xi}) d\vec{\xi}, \end{aligned} \quad (2.55)$$

which is the solution to the initial-boundary value problem (2.1-2.3).

The solution is obtained in terms of the so-called Green's function $G_*(\vec{x}, \vec{\xi}, t, \tau)$, which is symmetric and which itself may be represented as an infinite series expansion in terms of the eigenfunctions $V_{ij}(\vec{x})$ and eigenvalues ω_{ij} of the coordinate generating system (2.9) and (2.10). One may notice that the convergence of solution (2.55) relies heavily on the properties I , which come not only from the selfadjointness but also from the positive definiteness of the system. The selfadjoint system is positive definite only in the domain S — the stability domain of the system. In summary, Green's function approach is applicable for a selfadjoint problem in the domain of stability S .

In the following section, it is shown that for non-selfadjoint problems with a so-called "domain of regularity", Green's function approach is still applicable.

2.3 Two-Dimensional Non-Selfadjoint System with Separable Variables

Mathematically selfadjointness has been proven to be a very important concept and has found a lot of applications in physics and engineering practice. However an important class of mechanical problems such as problems in fluid-flow-induced vibrating systems, fast moving vehicles surrounded by viscous media, some structural control systems etc., which turn out to involve mechanically non-conservative forces result in mathematically non-selfadjoint problems. Following the pattern established for selfadjoint separable problems, it can be shown that the solution to such non-selfadjoint problems can also be obtained by a suitable extension of Green's function concept.

Consider a two-dimensional non-conservative elastic system. The motion of such a system is described by the following set of conditions:

$$\mu \ddot{W}(\vec{x}, t) + \mathcal{D}[\vec{q}, W(\vec{x}, t)] = K(\vec{x}, t), \quad (2.56)$$

$$U[W(\vec{x}, t)]_B = \{0\}, \quad (2.57)$$

$$W(\vec{x}, 0) = f(\vec{x}), \quad \dot{W}(\vec{x}, 0) = h(\vec{x}), \quad (2.58)$$

where all terms represent the same variables as in equation (2.1-2.3) except that the differential operator $\mathcal{D}$ is now a non-selfadjoint operator which is composed of two parts, such that,

$$\mathcal{D}[\vec{q}, W(\vec{x}, t)] = E[W(\vec{x}, t)] + N[\vec{q}, W(\vec{x}, t)] \quad (2.59)$$

with

$$N[\vec{q}, W(\vec{x}, t)] = q_1 S_1[W(\vec{x}, t)] + q_2 N_1[W(\vec{x}, t)], \quad (2.60)$$

where, E and S_1 are the same as that in $\mathcal{D}_s$. N_1 is a linear non-selfadjoint operator of an order $m \leq n - 2$ and involving continuous coefficients and only spatial derivatives. Then the load parameters q_1 and q_2 represent conservative and non-conservative types of loading, respectively. Moreover, as in the selfadjoint system, assumptions (2.47-2.48) regarding the perturbation $K(\vec{x}, t)$ hold true.

2.3.1 Domain of Regularity

First consider the homogeneous form of (2.56-2.57), i.e.,

$$\mu \ddot{W}(\vec{x}, t) + \mathcal{D}[\vec{q}, W(\vec{x}, t)] = 0, \quad (2.61)$$

$$U[W(\vec{x}, t)]_B = \{0\}, \quad (2.62)$$

and assume the separability of variables,

$$W(\vec{x}, t) = V(\vec{x})e^{i\omega t}. \quad (2.63)$$

Inserting (2.63) into (2.61-2.62), the coordinate generating system of the problem is obtained as:

$$-\mu\omega_{ij}^2 V_{ij}(\vec{x}) + \mathcal{D}[\vec{q}, V_{ij}(\vec{x})] = 0, \quad (2.64)$$

$$U[V_{ij}(\vec{x})]_B = \{0\}, \quad (2.65)$$

which will lead to the eigenvalue equation,

$$F(\mu\omega^2, \vec{q}) = 0. \quad (2.66)$$

It is obvious that when $\vec{q}$ is set to zero, i.e., $\vec{q} = \{0\}$, the non-selfadjoint operator $\mathcal{D}$ degenerates into elastic selfadjoint operator E , which indicates that problem (2.61-2.62) can also be considered as the perturbation of the fundamental selfadjoint problem (2.14-2.15). As has been shown before, the eigenvalues $(\omega_{ij}^0)^2$ and the eigenfunctions V_{ij}^0 of this fundamental problem have the well known properties, i.e., $(\omega_{ij}^0)^2$ form a real positive denumerably infinite set, so that $\sum_{i,j=1}^{\infty} \frac{1}{\mu(\omega_{ij}^0)^2}$ is bounded, i.e.,

$$\sum_{i,j=1}^{\infty} \frac{1}{\mu(\omega_{ij}^0)^2} = M_0 < \infty;$$

and V_{ij}^0 are uniformly bounded real functions which form a complete set.

Now starting with problem (2.14-2.15) and properties of $(\omega_{ij}^0)^2$ and V_{ij}^0 , and following a set of deliberations similar to those presented in Section (2.2.1) for the non-selfadjoint problem (2.61-2.62), one can identify a “regularity” domain $\mathcal{PR}$ defined by

$$0 \leq \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \leq \begin{Bmatrix} q_{1crit}; \quad q_2 = 0 \\ q_{2crit}(\alpha) : \forall \quad 0 \leq \alpha < q_{1crit} \end{Bmatrix}. \quad (2.67)$$

For $q_2 = 0$, q_{1crit} corresponds to the largest value of q_1 for which all the eigenvalue branches remain in the positive quadrant of $(\mu\omega^2 - q_1)$ plane, Fig.(2.2). Whereas, for any given α , $q_{2crit}(\alpha)$ corresponds to the largest value of q_2 for which the coordinates of all points on branches of the eigenvalues curve on $(\mu\omega^2 - q_2)$ plane are Positive and Regular. Here, regularity is generalized to the extent that extrema of the eigenvalue curve are not considered as regular points. Also assume that every branch of the eigenvalue curve crosses the $\mu\omega^2$ -axis with non-zero slope, which implies,

$$\frac{\partial q_1}{\partial \mu\omega^2} \neq 0, \quad \text{and} \quad \frac{\partial q_2}{\partial \mu\omega^2} \neq 0 \quad (2.68)$$

It will be shown in the following that in the domain $\mathcal{PR}$, a non-selfadjoint system has properties similar to that of a selfadjoint system within its stability domain $\mathcal{S}$.

It is obvious that in the domain $\mathcal{PR}$, all the eigenvalues ω_{ij}^2 are real and positive owing to the definition of this domain.

Using equation (2.66), one can derive the function,

$$\mu\omega_{ij}^2 = \mu\omega_{ij}^2(\vec{q}), \quad (2.69)$$

for which

$$\mu\omega_{ij}^2 = \mu(\omega_{ij}^0)^2 + \left[\frac{\partial \mu\omega_{ij}^2}{\partial q_1} \Big|_{\vec{q}=\{0\}} \right] dq_1 + \left[\frac{\partial \mu\omega_{ij}^2}{\partial q_2} \Big|_{\vec{q}=\{0\}} \right] dq_2. \quad (2.70)$$

Considering (2.68), equation (2.70) yields,

$$\mu\omega_{ij}^2 = \mu(\omega_{ij}^0)^2 + \beta_{1ij} + \beta_{2ij} , \quad (2.71)$$

where β_{1ij} and β_{2ij} are bounded. Let $B^2 = \max(|\beta_{1ij}| + |\beta_{2ij}|)$, $i, j = 1, 2, 3, \dots$, and $d\vec{q}$ so small that $B^2 < \mu(\omega_{ij}^0)^2$. Then,

$$\mu\omega_{ij}^2 > \mu(\omega_{ij}^0)^2 - B^2 ,$$

and,

$$1 > 1 - \frac{B^2}{\mu(\omega_{ij}^0)^2} > 1 - \frac{B^2}{\mu(\omega_{11}^0)^2} = \varepsilon > 0$$

hold. Hence, by virtue of (2.66)

$$\begin{aligned} \sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2} &\leq \sum_{i,j=1}^{\infty} \frac{1}{\mu(\omega_{ij}^0)^2 - B^2} \\ &= \sum_{i,j=1}^{\infty} \frac{1}{\mu(\omega_{ij}^0)^2 [1 - \frac{B^2}{\mu(\omega_{ij}^0)^2}]} \\ &< \sum_{i,j=1}^{\infty} \frac{1}{\mu(\omega_{ij}^0)^2 \varepsilon} \\ &= \frac{1}{\varepsilon} \sum_{i,j=1}^{\infty} \frac{1}{\mu(\omega_{ij}^0)^2} \\ &= \frac{M_0}{\varepsilon} = M_n < \infty. \end{aligned} \quad (2.72)$$

Which proves convergence of the series $\sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2}$ for $\vec{q}^1 = \{0, 0\}^T + \{dq_1, dq_2\}^T$. This can then be used to prove the convergence of the series at $\vec{q}^2 = \vec{q}^1 + d\vec{q}$ and so

on. In this way, the convergence of the series $\sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2}$ is established successively for any value of $\vec{q}$ in the domain $\mathcal{PR}$

Moreover in the domain $\mathcal{PR}$, the eigenfunctions $V_{ij}(\vec{x})$ are real functions which satisfy boundary conditions (2.65) as do the eigenfunctions V_{ij}^0 of the fundamental problem. Hence V_{ij} are admissible functions with respect to V_{ij}^0 so that V_{ij} can be expanded in terms of V_{ij}^0 . According to Collatz [22] this series converges absolutely and uniformly, i.e.,

$$|V_{ij}(\vec{x})| \leq \sum_{i,j=1}^{\infty} |\alpha_{ij}| |V_{ij}^0(\vec{x})|. \quad (2.73)$$

where α_{ij} are corresponding generalized Fourier series expansion coefficients. By virtue of convergence, the partial sums of (2.73) form a convergent sequence. But a convergent sequence is bounded. Hence $V_{ij}(\vec{x})$ are uniformly bounded functions.

Because of the non-selfadjointness of the problem, eigenfunctions $V_{ij}(\vec{x})$ generated by (2.64-2.65) are no longer orthogonal. These functions will have to be supplemented by an adjoint system of functions $\psi_{ij}(\vec{x})$ with respect to which $V_{ij}(\vec{x})$ are bi-orthonormal.

Equation (2.64) can be rewritten as,

$$\mathcal{D}[\vec{q}, V_{ij}(\vec{x})] = \mu\omega_{ij}^2 V_{ij}(\vec{x}) \quad (2.74)$$

Let the adjoint problem

$$\tilde{\mathcal{D}}[\vec{q}, \psi_{ij}(\vec{x})] = \mu\omega_{ij}^2 \psi_{ij}(\vec{x}), \quad (2.75)$$

$$\tilde{U}[\psi_{ij}(\vec{x})]_{\mathcal{B}} = \{0\}, \quad (2.76)$$

be introduced, which satisfies the condition

$$\int_{\Omega} V_{ij}(\vec{x}) \tilde{\mathcal{D}}[\vec{q}, \psi_{kl}(\vec{x})] d\vec{x} = \int_{\Omega} \psi_{kl}(\vec{x}) \mathcal{D}[\vec{q}, V_{ij}(\vec{x})] d\vec{x}. \quad (2.77)$$

Then the adjoint problem generates the same eigenvalues $\mu\omega_{ij}^2$ as the original problem (2.63 - 2.64). By virtue of (2.74) and (2.75),

$$\int_{\Omega} V_{ij}(\vec{x}) \tilde{\mathcal{D}}[\vec{q}, \psi_{kl}(\vec{x})] d\vec{x} = \int_{\Omega} \mu\omega_{kl}^2 V_{ij}(\vec{x}) \psi_{kl}(\vec{x}) d\vec{x}, \quad (2.78)$$

$$\int_{\Omega} \psi_{kl}(\vec{x}) \mathcal{D}[\vec{q}, V_{ij}(\vec{x})] d\vec{x} = \int_{\Omega} \mu\omega_{ij}^2 V_{ij}(\vec{x}) \psi_{kl}(\vec{x}) d\vec{x}, \quad (2.79)$$

substituting (2.78-2.79) into (2.77) yields,

$$\mu(\omega_{ij}^2 - \omega_{kl}^2) \int_{\Omega} V_{ij}(\vec{x}) \psi_{kl}(\vec{x}) d\vec{x} = 0. \quad (2.80)$$

Since $\omega_{ij}^2 \neq \omega_{kl}^2$ for $i \neq k$ and $j \neq l$ it necessarily follows that

$$\int_{\Omega} V_{ij}(\vec{x}) \psi_{kl}(\vec{x}) d\vec{x} = 0 \quad \text{for } i \neq k \text{ or } j \neq l. \quad (2.81)$$

After proper normalization, one has,

$$\int_{\Omega} V_{ij}(\vec{x}) \psi_{kl}(\vec{x}) d\vec{x} = \delta_{ik} \delta_{jl}, \quad (2.82)$$

which is the bi-orthonormality of $V_{ij}(\vec{x})$ and $\psi_{kl}(\vec{x})$.

It should be mentioned that by the same reasoning as for the $V_{ij}(\vec{x})$, one can conclude that the adjoint eigenfunctions $\psi_{ij}(\vec{x})$ are also uniformly bounded.

Finally, the completeness of the set of eigenfunctions $V_{ij}(\vec{x})$ and the set of adjoint eigenfunction $\psi_{ij}(\vec{x})$ are to be proved.

Choose an arbitrary real-valued function $H(\vec{x})$ which satisfies the boundary conditions (2.64) and is sufficiently-often differentiable so that $\mathcal{D}[H(\vec{x})]$ is a bounded quantity. Expanding $H(\vec{x})$ in terms of $V_{ij}(\vec{x})$, i.e.,

$$H(\vec{x}) = \sum_{i,j=1}^{\infty} a_{ij} V_{ij}(\vec{x}) , \quad (2.83)$$

by virtue of (2.82) and (2.75), one has

$$a_{ij} = \int_{\Omega} H(\vec{\xi}) \psi_{ij}(\vec{\xi}) d\vec{\xi} = \int_{\Omega} \frac{H \tilde{\mathcal{D}}[\psi_{ij}(\vec{\xi})]}{\mu \omega_{ij}^2} d\vec{\xi} . \quad (2.84)$$

Hence using (2.77),

$$H(\vec{x}) = \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{\psi_{ij}(\vec{\xi}) \mathcal{D}[H]}{\mu \omega_{ij}^2} d\vec{\xi} V_{ij}(\vec{x}) . \quad (2.85)$$

Assuming the upper bound for $V_{ij}(\vec{x})$ and $\psi_{ij}(\vec{x})$ is M and the area of the integration domain is S_{Ω} , then,

$$|H(\vec{x})| \leq S_{\Omega} M^2 |\mathcal{D}[H]| \sum_{i,j=1}^{\infty} \frac{1}{\mu \omega_{ij}^2} . \quad (2.86)$$

But (2.72) holds so that

$$|H(\vec{x})| \leq S_{\Omega} M^2 |\mathcal{D}[H]| M_n < \infty . \quad (2.87)$$

Consequently, the series (2.83) is absolutely and uniformly convergent, which implies that the $V_{ij}(\vec{x})$ form a complete set. The completeness of the set $\psi_{ij}(\vec{x})$ can be shown accordingly.

Summarizing, one can state that in the domain $\mathcal{PR}$, the non-selfadjoint system possesses the following properties,

- II.* (i) the eigenvalues ω_{ij}^2 are real, positive quantities;
(ii) the ω_{ij}^2 satisfy the conditions:

$$\sum_{i,j=1}^{\infty} \frac{1}{\mu \omega_{ij}^2} = M_n < \infty ; \quad (2.88)$$

(iii) the eigenfunctions $V_{ij}(\vec{x})$ and the adjoint eigenfunctions $\psi_{ij}(\vec{x})$ are uniformly bounded real functions so that,

$$|V_{ij}(\vec{x})| < M_V \quad \text{and} \quad |\psi_{ij}(\vec{x})| < M_\psi \quad \text{for all } i, j \text{ and } \vec{x} \in \Omega \quad (2.89)$$

(iv) the $V_{ij}(\vec{x})$ are bi-orthonormal to the adjoint eigenfunctions $\psi_{ij}(\vec{x})$ so that equation (2.82) holds;

(v) $V_{ij}(\vec{x})$ and $\psi_{ij}(\vec{x})$ represent two complete sets of functions.

Properties *II*, which correspond completely with properties *I* of the selfadjoint problem can justify Green's function approach for the case of non-selfadjoint problem.

2.3.2 Solution to the Non-Selfadjoint Problem

Following the approach suggested by Leipholz [9] for the simpler unidimensional case one may proceed as follows.

Applying Laplace transformation to equation (2.56 - 2.58) yields,

$$\mu s^2 \bar{W}(\vec{x}, s) - \mu s f(\vec{x}) - \mu h(\vec{x}) + \mathcal{D}[\vec{q}, \bar{W}(\vec{x}, s)] = \bar{K}(\vec{x}, s) , \quad (2.90)$$

where $\bar{W}(\vec{x}, s)$ and $\bar{K}(\vec{x}, s)$ are Laplace transformations of $W(\vec{x}, t)$ and $K(\vec{x}, t)$, respectively.

Since $V_{ij}(\vec{x})$ is a complete set, one can consider the following expansions:

$$\overline{W}(\vec{x}, s) = \sum_{i,j=1}^{\infty} A_{ij}(s) V_{ij}(\vec{x}) , \quad (2.91)$$

$$\overline{K}(\vec{x}, s) = \sum_{i,j=1}^{\infty} B_{ij}(s) V_{ij}(\vec{x}) , \quad (2.92)$$

$$f(\vec{x}) = \sum_{i,j=1}^{\infty} F_{ij} V_{ij}(\vec{x}) , \quad (2.93)$$

$$h(\vec{x}) = \sum_{i,j=1}^{\infty} H_{ij} V_{ij}(\vec{x}) , \quad (2.94)$$

where in contrast to (2.38–2.40), by virtue of (2.82), the coefficients of expansions are

$$B_{ij}(s) = \int_{\Omega} \overline{K}(\vec{\xi}, s) \psi_{ij}(\vec{\xi}) d\vec{\xi} , \quad (2.95)$$

$$F_{ij} = \int_{\Omega} f(\vec{\xi}) \psi_{ij}(\vec{\xi}) d\vec{\xi} , \quad (2.96)$$

$$H_{ij} = \int_{\Omega} h(\vec{\xi}) \psi_{ij}(\vec{\xi}) d\vec{\xi} , \quad (2.97)$$

Employing (2.91–2.94) in (2.90) and using (2.74) yields,

$$\sum_{i,j=1}^{\infty} \mu A_{ij}(s) (s^2 + \omega_{ij}^2) V_{ij}(\vec{x}) = \sum_{i,j=1}^{\infty} (B_{ij} + \mu s F_{ij} + \mu H_{ij}) V_{ij}(\vec{x}) , \quad (2.98)$$

from which one obtains,

$$A_{ij}(s) = \frac{B_{ij}(s)/\mu + s F_{ij} + H_{ij}}{s^2 + \omega_{ij}^2} . \quad (2.99)$$

Hence.

$$W(\vec{x}, s) = \sum_{i,j=1}^{\infty} \frac{B_{ij}(s)/\mu + sF_{ij} + H_{ij}}{s^2 + \omega_{ij}^2} V_{ij}(\vec{x}) . \quad (2.100)$$

Using (2.94–2.96), this expression can be given as

$$W(\vec{x}, s) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})}{s^2 + \omega_{ij}^2} \int_{\Omega} \left[\frac{K(\vec{\xi}, s)}{\mu} + s f(\vec{\xi}) + h(\vec{\xi}) \right] \psi_{ij}(\vec{\xi}) d\vec{\xi} . \quad (2.101)$$

The Inverse Laplace transformation yields,

$$\begin{aligned} W(\vec{x}, t) = & \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi})}{\mu \omega_{ij}} \sin[\omega_{ij}(t - \tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \\ & + \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij} t) h(\vec{\xi}) d\vec{\xi} \\ & + \sum_{i,j=1}^{\infty} \int_{\Omega} V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi}) \cos(\omega_{ij} t) f(\vec{\xi}) d\vec{\xi} . \end{aligned} \quad (2.102)$$

This is a formal solution, which becomes meaningful if the convergence of the various series in (2.102) can be established.

For the following investigation assume that, as in Section (2.2.2), conditions (2.47) and (2.48) for the perturbation K hold. Also the operations are limited in domain $\mathcal{PR}$ so that properties II are valid. Moreover, assume that functions h and f are bounded and sufficiently-often differentiable so that

$$\mathcal{D}[\vec{q}, h] = \phi_h(\vec{q}, \vec{x}) , \quad \mathcal{D}[\vec{q}, f] = \phi_f(\vec{q}, \vec{x})$$

exist and that

$$|\phi_h(\vec{q}, \vec{x})| < M_{\phi} , \quad |\phi_f(\vec{q}, \vec{x})| < M_{\phi} . \quad (2.103)$$

Initially consider the first integral on the right-hand side in (2.102). Integration by parts with respect to τ yields

$$\begin{aligned}
& \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \sin[\omega_{ij}(t-\tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \\
&= - \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \cos[\omega_{ij}(t-\tau)] \frac{\partial K(\vec{\xi}, \tau)}{\partial \tau} d\tau d\vec{\xi} \\
&+ \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} K(\vec{\xi}, t) d\vec{\xi} \\
&- \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} K(\vec{\xi}, 0) \cos(\omega_{ij}t) d\vec{\xi}. \tag{2.104}
\end{aligned}$$

By virtue of (2.89), one has

$$\begin{aligned}
& \left| \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \sin[\omega_{ij}(t-\tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \right| \\
&\leq \sum_{i,j=1}^{\infty} \frac{M_V M_{\psi} S_{\Omega}}{\mu\omega_{ij}^2} \left| \int_0^t \frac{\partial K}{\partial \tau} d\tau \right| + \frac{2M_V M_{\psi} S_{\Omega}}{\mu\omega_{ij}^2} |K|_{\max}. \tag{2.105}
\end{aligned}$$

Using the assumptions for perturbation K i.e. (2.47) and (2.48) and property II (ii) in (2.105) yields

$$\begin{aligned}
& \left| \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \sin[\omega_{ij}(t-\tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \right| \\
&\leq M_V M_{\psi} S_{\Omega} (K^* + 2K^{**}) \sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2} \\
&\leq M_V M_{\psi} S_{\Omega} (K^* + 2K^{**}) M_n < \infty. \tag{2.106}
\end{aligned}$$

which shows that the first integral in (2.102) is absolutely and uniformly convergent so that in (2.104) the summation and integration can be interchanged leading to

$$\begin{aligned}
& \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \sin[\omega_{ij}(t-\tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \\
&= - \int_{\Omega} \int_0^t \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \cos[\omega_{ij}(t-\tau)] \frac{\partial K(\vec{\xi}, \tau)}{\partial \tau} d\tau d\vec{\xi} \\
&+ \int_{\Omega} \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} K(\vec{\xi}, t) d\vec{\xi} \\
&- \int_{\Omega} \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} K(\vec{\xi}, 0) \cos(\omega_{ij}t) d\vec{\xi} .
\end{aligned} \tag{2.107}$$

Introducing as the unsymmetric Green's function the expression

$$G(\vec{x}, \vec{\xi}, t, \tau) = \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \cos[\omega_{ij}(t-\tau)] , \tag{2.108}$$

equation (2.107) becomes

$$\begin{aligned}
& \sum_{i,j=1}^{\infty} \int_{\Omega} \int_0^t \frac{V_{ij}(\vec{x})\psi_{ij}(\vec{\xi})}{\mu\omega_{ij}^2} \sin[\omega_{ij}(t-\tau)] K(\vec{\xi}, \tau) d\tau d\vec{\xi} \\
&= - \int_{\Omega} \int_0^t G(\vec{x}, \vec{\xi}, t, \tau) \frac{\partial K(\vec{\xi}, \tau)}{\partial \tau} d\tau d\vec{\xi} \\
&+ \int_{\Omega} G(\vec{x}, \vec{\xi}, t, t) K(\vec{\xi}, t) d\vec{\xi} \\
&- \int_{\Omega} G(\vec{x}, \vec{\xi}, t, 0) K(\vec{\xi}, 0) d\vec{\xi} .
\end{aligned} \tag{2.109}$$

Next consider the second integral in equation (2.102). Since $\int_{\Omega} \psi_{ij}(\vec{\xi}) h(\vec{\xi}) d\vec{\xi} = h_{ij}^*$ are the generalized Fourier coefficients of $h(\vec{x})$ with respect to complete eigenfunction set $V_{ij}(\vec{x})$, the second integral of (2.102) can be rewritten to yield,

$$\sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij} t) h(\vec{\xi}) d\vec{\xi} = \sum_{i,j=1}^{\infty} V_{ij}(\vec{x}) \frac{\sin(\omega_{ij} t)}{\omega_{ij}} h_{ij}^* . \quad (2.110)$$

Yet $|V_{ij}(\vec{x})| < M$ and $|\sin(\omega_{ij} t)/\omega_{ij}| \leq 1/\omega_{11}$. Therefore,

$$\left| \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij} t) h(\vec{\xi}) d\vec{\xi} \right| < \frac{M}{\omega_{11}} \sum_{i,j=1}^{\infty} |h_{ij}^*| . \quad (2.111)$$

Applying operator $\mathcal{D}$ to function h yields

$$\mathcal{D}[\vec{q}, h] = \phi_h(\vec{q}, \vec{x}) . \quad (2.112)$$

Expanding $h(\vec{x})$ and $\phi_h(\vec{q}, \vec{x})$ with respect to $V_{ij}(\vec{x})$ yields,

$$h(\vec{x}) = \sum_{i,j=1}^{\infty} h_{ij}^* V_{ij}(\vec{x}) , \quad \text{where} \quad h_{ij}^* = \int_{\Omega} \psi_{ij}(\vec{\xi}) h(\vec{\xi}) d\vec{\xi} , \quad (2.113)$$

$$\phi_h(\vec{q}, \vec{x}) = \sum_{i,j=1}^{\infty} \phi_{ij}^* V_{ij}(\vec{x}) , \quad \text{where} \quad \phi_{ij}^* = \int_{\Omega} \psi_{ij}(\vec{\xi}) \phi_h(\vec{q}, \vec{\xi}) d\vec{\xi} . \quad (2.114)$$

Using (2.113), (2.114) in (2.112) leads to

$$\sum_{i,j=1}^{\infty} h_{ij}^* \mathcal{D}[\vec{q}, V_{ij}(\vec{x})] = \sum_{i,j=1}^{\infty} \phi_{ij}^* V_{ij}(\vec{x}) . \quad (2.115)$$

Substituting relation $\mathcal{D}[\vec{q}, V_{ij}(\vec{x})] = \mu\omega_{ij}^2 V_{ij}(\vec{x})$ into (2.115) and comparing the coefficients of $V_{ij}(\vec{x})$ yields

$$h_{ij}^* = \phi_{ij}^* / \mu\omega_{ij}^2 = \frac{\int_{\Omega} \psi_{ij}(\vec{\xi}) \phi_h(\vec{q}, \vec{\xi}) d\vec{\xi}}{\mu\omega_{ij}^2}. \quad (2.116)$$

Substituting (2.116) into (2.111) and using (2.88), (2.89) and (2.103) leads to

$$\begin{aligned} \left| \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij}t) h(\vec{\xi}) d\vec{\xi} \right| &< \frac{M_V}{\omega_{11}} \sum_{i,j=1}^{\infty} \int_{\Omega} |\psi_{ij}(\vec{\xi})| |\phi_h(\vec{q}, \vec{\xi})| d\vec{\xi} \frac{1}{\mu\omega_{ij}^2} \\ &< \frac{S_{\Omega} M_V M_{\psi} M_{\phi}}{\omega_{11}} \sum_{i,j=1}^{\infty} \frac{1}{\mu\omega_{ij}^2} \\ &< \frac{S_{\Omega} M_V M_{\psi} M_{\phi} M_n}{\omega_{11}} < \infty. \end{aligned} \quad (2.117)$$

This prove that the second integral in formal solution (2.102) converges absolutely and uniformly. It should be noted that demonstrating the boundness of this integral, actually, constitutes an extension of Hilbert-Schmidt's expansion theorem to the realm of multi-dimensional non-selfadjoint systems. It can be proved accordingly that the third term in (2.102) is also absolutely and uniformly convergent. Thus the summation and integration in these two terms can be interchanged, which yield

$$\begin{aligned} \sum_{i,j=1}^{\infty} \int_{\Omega} \frac{V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij}t) h(\vec{\xi}) d\vec{\xi} &= \int_{\Omega} \sum_{i,j=1}^{\infty} \frac{V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi})}{\omega_{ij}} \sin(\omega_{ij}t) h(\vec{\xi}) d\vec{\xi} \\ &= -\mu \int_{\Omega} \frac{\partial G(\vec{x}, \vec{\xi}, t, 0)}{\partial t} h(\vec{\xi}) d\vec{\xi}, \end{aligned} \quad (2.118)$$

and

$$\begin{aligned}
\sum_{i,j=1}^{\infty} \int_{\Omega} V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi}) \cos(\omega_{ij}t) f(\vec{\xi}) d\vec{\xi} &= \int_{\Omega} \sum_{i,j=1}^{\infty} V_{ij}(\vec{x}) \psi_{ij}(\vec{\xi}) \cos(\omega_{ij}t) f(\vec{\xi}) d\vec{\xi} \\
&= -\mu \int_{\Omega} \frac{\partial^2 G(\vec{x}, \vec{\xi}, t, 0)}{\partial t^2} f(\vec{\xi}) d\vec{\xi}. \quad (2.119)
\end{aligned}$$

Finally substituting (2.109), (2.118) and (2.119) into (2.102) results in the actual solution to non-selfadjoint problem (2.56 - 2.58), i.e.,

$$\begin{aligned}
W(\vec{x}, t) &= - \int_{\Omega} \int_0^t G(\vec{x}, \vec{\xi}, t, \tau) \frac{\partial K(\vec{\xi}, \tau)}{\partial \tau} d\tau d\vec{\xi} \\
&\quad + \int_{\Omega} G(\vec{x}, \vec{\xi}, t, t) K(\vec{\xi}, t) d\vec{\xi} \\
&\quad - \int_{\Omega} G(\vec{x}, \vec{\xi}, t, 0) K(\vec{\xi}, 0) d\vec{\xi} \\
&\quad - \mu \int_{\Omega} \frac{\partial G(\vec{x}, \vec{\xi}, t, 0)}{\partial t} h(\vec{\xi}) d\vec{\xi} \\
&\quad - \mu \int_{\Omega} \frac{\partial^2 G(\vec{x}, \vec{\xi}, t, 0)}{\partial t^2} f(\vec{\xi}) d\vec{\xi}. \quad (2.120)
\end{aligned}$$

2.4 Conclusions

It has been shown in Section 2.2 and Section 2.3 that Green's function approach is applicable to both selfadjoint and non-selfadjoint two-dimensional problems, under proper conditions such that,

- all operations are limited to the stability domain S for selfadjoint system and to the domain $\mathcal{PR}$ for non-selfadjoint system. In the stability domain S , a selfadjoint system possesses properties I . In the domain $\mathcal{PR}$, a non-selfadjoint system has properties II which correspond to properties I .

- for both cases, the perturbation K satisfies conditions (2.47-2.48).
- for both selfadjoint and non-selfadjoint systems, the initial conditions $f(\vec{x})$ and $h(\vec{x})$ must be bounded, sufficiently smooth functions that satisfy the necessary boundary condition.

Under the above conditions, the selfadjoint problem (2.1-2.3) has a closed form solution, (2.55), in terms of a symmetric Green's function which is given by equation (2.49).

Correspondingly, the non-selfadjoint problem (2.56-2.58) also has a closed form solution (2.120) in terms of a Green's function which, in contrast to selfadjoint problem, is unsymmetric as shown in (2.108).

Chapter 3

Dynamic Characteristics and Response of a Spatially Two-Dimensional Elastic System with Follower Forces

3.1 Introduction

As was pointed out earlier, elastic systems involving mechanically non-conservative external loads result in mathematically non-selfadjoint problems. Specifically, in the context of linearly elastic components of structural systems subjected to dynamic loads, two types of non-conservative forces may be present, namely, follower type forces and control forces. While analysis and treatment of problems with control forces will be undertaken in Chapter 4, the present chapter will deal with non-selfadjoint problems arising from follower forces. Such problems are of specific interest in the context of certain mechanical and structural systems such as rockets, jet engines, flow-induced vibrations, etc.. So far, the main emphasis of the existing literature in this area seems to have been on the analysis of spatially one-dimensional rod-type problems, e.g., Beal[23], Glaser[17], Solarz[24], Leipholz[8, 9], Afagh[25], Leipholz and Afagh[11], J. X. Lee[27]. However, in many engineering problems such as in the design of aircraft lifting and stabilizing devices; stationary flat structural components exposed to high velocity gas flow; parts of silo walls subjected to frictional forces caused by the flowing bulk material in a silo, etc., the planar structural component

may be modelled as a two-dimensional plate subjected to tangential follower forces. In this chapter, a typical spatially two-dimensional non-selfadjoint problem, which physically describes a thin elastic rectangular plate subjected to an external time varying, transverse perturbation under constant uniformly distributed tangential follower force is investigated. Two kinds of boundary conditions are considered: (i) four edges simply supported (which is represented as S-S-S-S), (ii) two opposite edges simply supported with the other two opposite edges free (which is represented as S-F-S-F). For the more involved S-F-S-F boundary condition, the solution obtained by using the Green's function method, as outlined in Chapter 2, is given in detail. Similar procedure can be followed in obtaining a solution for the case of S-S-S-S. The dynamic characteristics are obtained for both cases and the response of a S-F-S-F plate is determined demonstrating a numerical example.

3.2 Problem Formulation

Consider the transverse small vibration of a classical rectangular plate under constant uniformly distributed follower force q and subjected to external driving force $K(x, y, t)$, Fig. (3.1). The governing differential equation is obtained as [28, 29],

$$\begin{aligned} \mu \ddot{W}(x, y, t) + D \left[\frac{\partial^4 W(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 W(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y, t)}{\partial y^4} \right] \\ + q(l-x) \frac{\partial^2 W(x, y, t)}{\partial x^2} = K(x, y, t) . \end{aligned} \quad (3.1)$$

Assume that edges $y = 0, b$ are simply supported and edges $x = 0, l$ are free, then the boundary conditions are:

$$W(x, 0, t) = \frac{\partial^2 W(x, 0, t)}{\partial y^2} = W(x, b, t) = \frac{\partial^2 W(x, b, t)}{\partial y^2} = 0 , \quad (3.2)$$

$$\frac{\partial^2 W(0, y, t)}{\partial x^2} + \nu \frac{\partial^2 W(0, y, t)}{\partial y^2} = 0, \quad (3.3)$$

$$\frac{\partial^3 W(0, y, t)}{\partial x^3} + (2 - \nu) \frac{\partial^3 W(0, y, t)}{\partial x \partial y^2} = 0, \quad (3.4)$$

$$\frac{\partial^2 W(l, y, t)}{\partial x^2} + \nu \frac{\partial^2 W(l, y, t)}{\partial y^2} = 0, \quad (3.5)$$

$$\frac{\partial^3 W(l, y, t)}{\partial x^3} + (2 - \nu) \frac{\partial^3 W(l, y, t)}{\partial x \partial y^2} = 0, \quad (3.6)$$

where,

- x, y — spatial coordinates of the plate mid-plane
- W — transverse deflection of the plate
- l, b — dimension of the plate in the x and y directions respectively
- μ — constant mass density per unit area
- ν — Poisson's ratio for plate material
- D — flexural rigidity of the plate $D = Eh^3(1 - \nu^2)/12$
- q — constant uniformly distributed tangential follower force
- K — transverse driving force

One may non-dimensionalize the coordinates x and y by introducing new variables,

$$\begin{aligned} \xi &= \frac{x}{l}, \quad x \in [0, l], \quad \xi \in [0, 1], \\ \eta &= \frac{y}{b}, \quad y \in [0, b], \quad \eta \in [0, 1], \\ \phi &= \frac{t}{l}, \end{aligned} \quad (3.7)$$

substituting (3.7) into (3.1-3.6) yields

$$\begin{aligned} \frac{\mu b^4}{D} \ddot{W}(\xi, \eta, t) + \frac{\partial^4 W(\xi, \eta, t)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 W(\xi, \eta, t)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 W(\xi, \eta, t)}{\partial \xi^4} \\ + \frac{q l^3}{D} \phi^4 (1 - \xi) \frac{\partial^2 W(\xi, \eta, t)}{\partial \xi^2} = \frac{b^4}{D} K(\xi, \eta, t), \end{aligned} \quad (3.8)$$

with the boundary conditions

$$W(\xi, 0, t) = \frac{\partial^2 W(\xi, 0, t)}{\partial \eta^2} = W(\xi, 1, t) = \frac{\partial^2 W(\xi, 1, t)}{\partial \eta^2} = 0, \quad (3.9)$$

$$\frac{\partial^2 W(0, \eta, t)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 W(0, \eta, t)}{\partial \eta^2} = 0, \quad (3.10)$$

$$\frac{\partial^3 W(0, \eta, t)}{\partial \xi^3} + \frac{(2 - \nu)}{\phi^2} \frac{\partial^3 W(0, \eta, t)}{\partial \xi \partial \eta^2} = 0, \quad (3.11)$$

$$\frac{\partial^2 W(1, \eta, t)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 W(1, \eta, t)}{\partial \eta^2} = 0, \quad (3.12)$$

$$\frac{\partial^3 W(1, \eta, t)}{\partial \xi^3} + \frac{(2 - \nu)}{\phi^2} \frac{\partial^3 W(1, \eta, t)}{\partial \xi \partial \eta^2} = 0. \quad (3.13)$$

3.3 Modal Generating System

Consider the homogeneous form of equation (3.8)

$$\begin{aligned} \frac{\mu b^4}{D} \ddot{W}(\xi, \eta, t) + \frac{\partial^4 W(\xi, \eta, t)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 W(\xi, \eta, t)}{\partial \eta^2 \partial \xi^2} \\ + \phi^4 \frac{\partial^4 W(\xi, \eta, t)}{\partial \xi^4} + \frac{q l^3}{D} \phi^4 (1 - \xi) \frac{\partial^2 W(\xi, \eta, t)}{\partial \xi^2} = 0 \end{aligned} \quad (3.14)$$

Assuming separability of variables, such that,

$$W(\xi, \eta, t) = V(\xi, \eta)e^{i\omega t} \quad (3.15)$$

and introducing non-dimensional quantities

$$\lambda^2 = \frac{\mu b^4}{D} \omega^2 \quad \text{and} \quad q^* = \frac{l^3}{D} q. \quad (3.16)$$

Substitution of (3.15-3.16) into (3.14) and boundary conditions (3.9-3.13) will yield the following modal generating system:

$$\begin{aligned} \mathcal{D}[q^*, V(\xi, \eta)] \equiv & \frac{\partial^4 V(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 V(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 V(\xi, \eta)}{\partial \xi^4} \\ & + q^* \phi^4 (1 - \xi) \frac{\partial^2 V(\xi, \eta)}{\partial \xi^2} = \lambda^2 V(\xi, \eta), \end{aligned} \quad (3.17)$$

with boundary conditions

$$V(\xi, 0) = \frac{\partial^2 V(\xi, 0)}{\partial \eta^2} = V(\xi, 1) = \frac{\partial^2 V(\xi, 1)}{\partial \eta^2} = 0, \quad (3.18)$$

$$\frac{\partial^2 V(0, \eta)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 V(0, \eta)}{\partial \eta^2} = 0, \quad (3.19)$$

$$\frac{\partial^3 V(0, \eta)}{\partial \xi^3} + \frac{(2 - \nu)}{\phi^2} \frac{\partial^3 V(0, \eta)}{\partial \xi \partial \eta^2} = 0, \quad (3.20)$$

$$\frac{\partial^2 V(1, \eta)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 V(1, \eta)}{\partial \eta^2} = 0, \quad (3.21)$$

$$\frac{\partial^3 V(1, \eta)}{\partial \xi^3} + \frac{(2 - \nu)}{\phi^2} \frac{\partial^3 V(1, \eta)}{\partial \xi \partial \eta^2} = 0, \quad (3.22)$$

where $\mathcal{D}[q^*, V(\xi, \eta)]$ is the non-selfadjoint operator of the system.

Assume the modal solution has the form [30]:

$$V_m(\xi, \eta) = X_m(\xi) \sin(m\pi\eta), \quad m = 1, 2, 3, \dots, \infty \quad (3.23)$$

which duly satisfies (3.18), the boundary conditions along edges $\eta = 0$ and $\eta = 1$.

Substituting (3.23) into (3.17) yields

$$\left[(m\pi)^4 X_m(\xi) - 2\phi^2(m\pi)^2 \frac{d^2 X_m(\xi)}{d\xi^2} + \phi^4 \frac{d^4 X_m(\xi)}{d\xi^4} + q^* \phi^4 (1 - \xi) \frac{d^2 X_m(\xi)}{d\xi^2} \right] \sin(m\pi\eta) = \lambda^2 X_m(\xi) \sin(m\pi\eta), \quad (3.24)$$

where, comparing the coefficients of $\sin(m\pi\eta)$, one obtains

$$\begin{aligned} (m\pi)^4 X_m(\xi) - 2\phi^2(m\pi)^2 \frac{d^2 X_m(\xi)}{d\xi^2} + \phi^4 \frac{d^4 X_m(\xi)}{d\xi^4} \\ + q^* \phi^4 (1 - \xi) \frac{d^2 X_m(\xi)}{d\xi^2} = \lambda^2 X_m(\xi). \end{aligned} \quad (3.25)$$

Assume a power series solution for $X_m(\xi)$, i.e.,

$$X_m(\xi) = \sum_{n=1}^{\infty} a_{mn} (1 - \xi)^{n-1}, \quad m = 1, 2, 3, \dots, \infty \quad (3.26)$$

where a_{mn} are the unknown coefficients to be determined. Using (3.26) in (3.25) yields,

$$\sum_{n=5}^{\infty} \left\{ \phi^4 (n-1)(n-2)(n-3)(n-4) a_{mn} - 2\phi^2 (m\pi)^2 (n-3)(n-4) a_{m(n-2)} \right.$$

$$+q^*\phi^4(n-4)(n-5)a_{m(n-3)} + [(\pi m)^4 - \lambda^2] a_{m(n-4)} \} (1-\xi)^{n-5} = 0, \quad (3.27)$$

which leads to the recursion formula

$$\begin{aligned} a_{mn} &= \frac{[\lambda^2 - (\pi m)^4] a_{m(n-4)} - q^*\phi^4(n-4)(n-5)a_{m(n-3)}}{\phi^4(n-1)(n-2)(n-3)(n-4)} \\ &+ \frac{2(\pi m)^2(n-3)(n-4)a_{m(n-2)}}{\phi^2(n-1)(n-2)(n-3)(n-4)}, \quad (n \geq 5), \end{aligned} \quad (3.28)$$

$$\text{for } m = 1, 2, 3, \dots, \infty.$$

Using this recursion formula, any a_{mn} with $n \geq 5$ can be represented in terms of a_{m1} , a_{m2} , a_{m3} , a_{m4} , ($m = 1, 2, 3, \dots, \infty$) so that $X_m(\xi)$ can be represented as,

$$X_m(\xi) = \sum_{j=1}^4 a_{mj} M_{mj}(\xi) \quad \text{for } m = 1, 2, 3, \dots, \infty, \quad (3.29)$$

with

$$M_{m1}(\xi) = 1 + \frac{[\lambda^2 - (\pi m)^4]}{4!\phi^4} (1-\xi)^4 + \frac{2[\lambda^2 - (\pi m)^4](\pi m)^2}{6!\phi^6} (1-\xi)^7 + \dots,$$

$$M_{m2}(\xi) = (1-\xi) + \frac{[\lambda^2 - (\pi m)^4]}{5!\phi^4} (1-\xi)^5 + \dots,$$

$$M_{m3}(\xi) = (1-\xi)^2 + \frac{2!2(\pi m)^2}{4!\phi^4} (1-\xi)^4 - \frac{2q^*}{5!} (1-\xi)^5$$

$$+ \frac{2![\lambda^2 + 3(\pi m)^4]}{6!\phi^4} (1-\xi)^6 + \dots,$$

$$M_{m4}(\xi) = (1 - \xi)^3 + \frac{3!2(m\pi)^2}{5!\phi^2}(1 - \xi)^5 - \frac{3!3q^*}{6!}(1 - \xi)^6 + \dots \quad (3.30)$$

In (3.29), coefficients a_{m1} , a_{m2} , a_{m3} , a_{m4} play the role of constants of integration and $M_{mj}(\xi)$, $j = 1, 2, 3, 4$ represent the fundamental solutions to the problem.

The general solution to (3.17) can be given as

$$\begin{aligned} V_m(\xi, \eta) &= X_m(\xi) \sin(m\pi\eta) \\ &= \sum_{j=1}^4 a_{mj} M_{mj}(\xi) \sin(m\pi\eta), \quad m = 1, 2, 3, \dots, \infty. \end{aligned} \quad (3.31)$$

To determine the coefficients of integration a_{m1} through a_{m4} , the remaining boundary conditions (3.19-3.22) are applied to (3.31) to yield,

$$\begin{aligned} &a_{m1} \left[M''_{m1}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m1}(0) \right] + a_{m2} \left[M''_{m2}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m2}(0) \right] \\ &+ a_{m3} \left[M''_{m3}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m3}(0) \right] + a_{m4} \left[M''_{m4}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m4}(0) \right] = 0, \end{aligned} \quad (3.32)$$

$$\begin{aligned} &a_{m1} \left[M'''_{m1}(0) - \frac{2-\nu}{\phi^2}(m\pi)^2 M'_{m1}(0) \right] + a_{m2} \left[M'''_{m2}(0) - \frac{2-\nu}{\phi^2}(m\pi)^2 M'_{m2}(0) \right] \\ &+ a_{m3} \left[M'''_{m3}(0) - \frac{2-\nu}{\phi^2}(m\pi)^2 M'_{m3}(0) \right] \\ &+ a_{m4} \left[M'''_{m4}(0) - \frac{2-\nu}{\phi^2}(m\pi)^2 M'_{m4}(0) \right] = 0, \end{aligned} \quad (3.33)$$

$$a_{m3} = \frac{\nu}{2\phi^2}(m\pi)^2 a_{m1} , \quad (3.34)$$

$$a_{m4} = \frac{2-\nu}{6\phi^2}(m\pi)^2 a_{m2} . \quad (3.35)$$

Substituting (3.34) and (3.35) into (3.32) and (3.33) and collecting the coefficients of a_{m1} and a_{m2} leads to

$$A_m a_{m1} + B_m a_{m2} = 0 , \quad (3.36)$$

$$C_m a_{m1} + D_m a_{m2} = 0 , \quad (3.37)$$

where

$$\begin{aligned} A_m &= M''_{m1}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m1}(0) \\ &+ \frac{\nu}{2\phi^2}(m\pi)^2 \left[M''_{m3}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m3}(0) \right] , \end{aligned} \quad (3.38)$$

$$\begin{aligned} B_m &= M''_{m2}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m2}(0) \\ &+ \frac{2-\nu}{6\phi^2}(m\pi)^2 \left[M''_{m4}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M_{m4}(0) \right] , \end{aligned} \quad (3.39)$$

$$\begin{aligned} C_m &= M'''_{m1}(0) - \frac{2-\nu}{\phi^2}(m\pi)^2 M'_{m1}(0) \\ &+ \frac{\nu}{2\phi^2}(m\pi)^2 \left[M'''_{m3}(0) - \frac{\nu}{\phi^2}(m\pi)^2 M'_{m3}(0) \right] , \end{aligned} \quad (3.40)$$

$$\begin{aligned} D_m &= M'''_{m2}(0) - \frac{2-\nu}{\phi^2}(m\pi)^2 M'_{m2}(0) \\ &+ \frac{2-\nu}{6\phi^2}(m\pi)^2 \left[M'''_{m4}(0) - \frac{2-\nu}{\phi^2}(m\pi)^2 M'_{m4}(0) \right] . \end{aligned} \quad (3.41)$$

Obviously, A_m, B_m, C_m, D_m are polynomials in λ and q^* .

In order to get a non-trivial solution of a_{m1}, a_{m2} , the determinant of (3.36 3.37) should be zero, i.e.,

$$\begin{vmatrix} A_m & B_m \\ C_m & D_m \end{vmatrix} = 0, \quad (3.42)$$

which leads to the characteristic equation of the system

$$F(\lambda, q^*, \phi, m) = A_m D_m - C_m B_m = 0. \quad (3.43)$$

For any given q^*, ϕ and m , eigenvalues λ_{mi} , $m, i = 1, 2, 3, \dots, \infty$ can be obtained by solving this equation.

By virtue of (3.36), one has

$$a_{m2} = -\frac{A_m(\lambda_{mi})}{B_m(\lambda_{mi})} a_{m1}. \quad (3.44)$$

Using (3.44) in (3.35) yields

$$a_{m4} = -\frac{(2-\nu)(m\pi)^2 A_m(\lambda_{mi})}{6\phi^2 B_m(\lambda_{mi})} a_{m1}. \quad (3.45)$$

Substituting (3.34), (3.44) and (3.45) into (3.31) leads to

$$\begin{aligned}
V_{m1}(\xi, \eta) = & a_{m1} \left[M_{m1}(\xi, \lambda_{mi}) - \frac{A_m(\lambda_{mi})}{B_m(\lambda_{mi})} M_{m2}(\xi, \lambda_{mi}) + \frac{\nu}{\phi^2} (m\pi)^2 M_{m3}(\xi, \lambda_{mi}) \right. \\
& \left. - \frac{(2-\nu)(m\pi)^2 A_m(\lambda_{mi})}{6\phi^2 B_m(\lambda_{mi})} M_{m4}(\xi, \lambda_{mi}) \right] \sin(m\pi\eta)
\end{aligned} \quad (3.46)$$

which is the mode shape corresponding to eigenvalue λ_{mi} .

3.4 Adjoint Problem

According to (3.17), the non-selfadjoint operator of the system is

$$\begin{aligned}
\mathcal{D}[q^*, V(\xi, \eta)] = & \frac{\partial^4 V(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 V(\xi, \eta)}{\partial \eta^2 \partial \xi^2} \\
& + \phi^4 \frac{\partial^4 V(\xi, \eta)}{\partial \xi^4} + q^* \phi^4 (1 - \xi) \frac{\partial^2 V(\xi, \eta)}{\partial \xi^2}
\end{aligned} \quad (3.47)$$

with boundary conditions (3.18-3.22). The adjoint problem can be developed by satisfying Green's identity, i.e.,

$$\int_0^1 \int_0^1 V(\xi, \eta) \hat{\mathcal{D}}[q^*, \psi(\xi, \eta)] d\psi d\xi = \int_0^1 \int_0^1 \psi(\xi, \eta) \mathcal{D}[q^*, V(\xi, \eta)] d\psi d\xi \quad (3.48)$$

where

$\hat{\mathcal{D}}$ — adjoint differential operator to $\mathcal{D}$.

$\psi(\xi, \eta)$ — adjoint eigenfunctions to $V(\xi, \eta)$.

Substituting (3.47) into (3.48) yields,

$$\begin{aligned}
 & \int_0^1 \int_0^1 V(\xi, \eta) \tilde{\mathcal{D}} [q^*, \psi(\xi, \eta)] d\psi d\xi \\
 = & \int_0^1 \int_0^1 \psi(\xi, \eta) \left[\frac{\partial^4 V(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 V(\xi, \eta)}{\partial \eta^2 \partial \xi^2} \right. \\
 & \left. + \phi^4 \frac{\partial^4 V(\xi, \eta)}{\partial \xi^4} + q^* \phi^4 (1 - \xi) \frac{\partial^2 V(\xi, \eta)}{\partial \xi^2} \right] d\psi d\xi . \quad (3.49)
 \end{aligned}$$

Applying Gauss's integral theorem and boundary conditions (3.18-3.22) yields,

$$\begin{aligned}
 & \int_0^1 \int_0^1 V(\xi, \eta) \tilde{\mathcal{D}} [q^*, \psi(\xi, \eta)] d\psi d\xi \\
 = & \int_0^1 \int_0^1 V(\xi, \eta) \left[\frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 \psi(\xi, \eta)}{\partial \xi^4} \right. \\
 & + \frac{ql^3}{D} \phi^4 (1 - \xi) \frac{\partial^2 \psi(\xi, \eta)}{\partial \xi^2} - 2q^* \phi^4 \frac{\partial \psi(\xi, \eta)}{\partial \xi} \left. \right] d\psi d\xi \\
 & - \int_0^1 B_1 d\xi + \int_0^1 B_2 d\eta , \quad (3.50)
 \end{aligned}$$

where B_1, B_2 are boundary terms such that

$$\begin{aligned}
 B_1 = & 2\phi^2 \psi_{\xi\xi}(\xi, 0) V_\eta(\xi, 0) + 2\phi^2 \psi_{\xi\xi}(\xi, 1) V_\eta(\xi, 1) \\
 & + \psi(\xi, 0) V_{\eta\eta\eta}(\xi, 0) + \psi_{\eta\eta}(\xi, 0) V_\eta(\xi, 0) \\
 & + \psi(\xi, 1) V_{\eta\eta\eta}(\xi, 1) + \psi_{\eta\eta}(\xi, 1) V_\eta(\xi, 1) , \quad (3.51)
 \end{aligned}$$

$$\begin{aligned}
B_z = & \left[\nu \phi^2 \psi(1, \eta) V_{\eta\eta\xi}(1, \eta) - (2 - \nu) \phi^2 \psi_\xi(1, \eta) V_{\eta\eta}(1, \eta) \right. \\
& + \left. \phi^4 \psi_{\xi\xi}(1, \eta) V_\xi(1, \eta) - \phi^4 \psi_{\xi\xi\xi}(1, \eta) V(1, \eta) \right] \\
& + \left[\nu \phi^2 \psi(0, \eta) V_{\eta\eta\xi}(0, \eta) - (2 - \nu) \phi^2 \psi_\xi(0, \eta) V_{\eta\eta}(0, \eta) \right. \\
& + \left. \phi^4 \psi_{\xi\xi}(0, \eta) V_\xi(0, \eta) - \phi^4 \psi_{\xi\xi\xi}(0, \eta) V(0, \eta) \right] \\
& + \left[q^* \phi^4 \psi(0, \eta) V_\xi(0, \eta) - q^* \phi^4 \psi_\xi(0, \eta) V(0, \eta) \right. \\
& + \left. q^* \phi^4 \psi(0, \eta) V(0, \eta) + q^* \phi^4 \psi(1, \eta) V(1, \eta) \right] . \tag{3.52}
\end{aligned}$$

The subscripts in (3.51–3.52) indicate differentiation.

To satisfy the Green's identity, both boundary terms $\int_0^1 B_1 d\xi$ and $\int_0^1 B_2 d\eta$ should vanish respectively. After imposing the following minimum number of conditions $\int_0^1 B_1 d\xi$ becomes zero:

$$\psi(\xi, 0) = \frac{\partial^2 \psi(\xi, 0)}{\partial \eta^2} = \psi(\xi, 1) = \frac{\partial^2 \psi(\xi, 1)}{\partial \eta^2} = 0 . \tag{3.53}$$

After further integration by parts with respect to ξ and applying (3.53), $\int_0^1 B_2 d\eta$ becomes,

$$\begin{aligned}
\int_0^1 B_2 d\eta = & \int_0^1 \left[\nu \phi^2 \psi_{\eta\eta}(1, \eta) V_\xi(1, \eta) - (2 - \nu) \phi^2 \psi_{\xi\eta\eta}(1, \eta) V(1, \eta) \right. \\
& + \left. \phi^4 \psi_{\eta\eta}(1, \eta) V_\xi(1, \eta) - \phi^4 \psi_{\xi\xi\xi}(1, \eta) V(1, \eta) \right] d\eta
\end{aligned}$$

$$\begin{aligned}
& + \int_0^1 \left[\nu \phi^2 \psi_{\eta\eta}(0, \eta) V_\xi(0, \eta) - (2 - \nu) \phi^2 \psi_{\xi\eta\eta}(0, \eta) V(0, \eta) \right. \\
& \quad \left. + \phi^4 \psi_{\eta\eta}(0, \eta) V_\xi(0, \eta) - \phi^4 \psi_{\xi\xi\xi}(0, \eta) V(0, \eta) \right] d\eta \\
& + \int_0^1 \left[q^* \phi^4 \psi(0, \eta) V_\xi(0, \eta) - q^* \phi^4 \psi_\xi(0, \eta) V(0, \eta) \right. \\
& \quad \left. + q^* \phi^4 \psi(0, \eta) V(0, \eta) + q^* \phi^4 \psi(1, \eta) V(1, \eta) \right] d\eta \quad (3.54)
\end{aligned}$$

then $\int_0^1 B_2 d\eta$ vanishes after imposing the following minimum number of conditions:

$$\frac{\partial^2 \psi(0, \eta)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 \psi(0, \eta)}{\partial \eta^2} + q^* \psi(0, \eta) = 0, \quad (3.55)$$

$$\frac{\partial^3 \psi(0, \eta)}{\partial \xi^3} + \frac{(2 - \nu)}{\phi^2} \frac{\partial^3 \psi(0, \eta)}{\partial \xi \partial \eta^2} + q^* \frac{\partial \psi(0, \eta)}{\partial \xi} - q^* \psi(0, \eta) = 0, \quad (3.56)$$

$$\frac{\partial^2 \psi(1, \eta)}{\partial \xi^2} + \frac{\nu}{\phi^2} \frac{\partial^2 \psi(1, \eta)}{\partial \eta^2} = 0, \quad (3.57)$$

$$\frac{\partial^3 \psi(1, \eta)}{\partial \xi^3} + \frac{(2 - \nu)}{\phi^2} \frac{\partial^3 \psi(1, \eta)}{\partial \xi \partial \eta^2} - q^* \psi(1, \eta) = 0, \quad (3.58)$$

Thus using boundary conditions (3.53) and (3.55-3.58) reduces (3.50) into

$$\begin{aligned}
& \int_0^1 \int_0^1 V(\xi, \eta) \tilde{\mathcal{D}}[q^*, \psi(\xi, \eta)] d\psi d\xi \\
& = \int_0^1 \int_0^1 V(\xi, \eta) \left[\frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 V(\xi, \eta)}{\partial \xi^4} \right. \\
& \quad \left. + \frac{q l^3}{D} \phi^4 (1 - \xi) \frac{\partial^2 \psi(\xi, \eta)}{\partial \xi^2} - 2q^* \phi^4 \frac{\partial \psi(\xi, \eta)}{\partial \xi} \right] d\psi d\xi. \quad (3.59)
\end{aligned}$$

In summary, the adjoint problem of the system is obtained as,

$$\begin{aligned}\tilde{\mathcal{D}}[q^*, \psi(\xi, \eta)] &= \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 \psi(\xi, \eta)}{\partial \xi^4} \\ &+ \frac{\eta l^3}{D} \phi^4 (1 - \xi) \frac{\partial^2 \psi(\xi, \eta)}{\partial \xi^2} - 2q^* \phi^4 \frac{\partial \psi(\xi, \eta)}{\partial \xi} = \lambda^2 \psi(\xi, \eta)\end{aligned}\quad (3.60)$$

with the related boundary conditions (3.53) and (3.55 - 3.58). The adjoint problem has the same eigenvalues λ_{m_i} as the original problem (3.17-3.22).

For the adjoint eigenfunctions $\psi_{m_i}(\xi, \eta)$, assume they have the form,

$$\psi_{m_i}(\xi, \eta) = S_{m_i}(\xi) \sin(m\pi\eta), \quad m = 1, 2, 3, \dots, \infty. \quad (3.61)$$

Obviously, $\psi_{m_i}(\xi, \eta)$ satisfy boundary conditions (3.53). Substituting (3.61) into (3.60) and comparing the coefficient of $\sin m\pi\eta$ one has:

$$\begin{aligned}(m\pi)^4 S_{m_i}(\xi) - 2\phi^2 (m\pi)^2 \frac{d^2 S_{m_i}(\xi)}{d\xi^2} + \phi^4 \frac{d^4 S_{m_i}(\xi)}{d\xi^4} \\ + q^* \phi^4 (1 - \xi) \frac{d^2 S_{m_i}(\xi)}{d\xi^2} - 2q^* \phi^2 \frac{d S_{m_i}(\xi)}{d\xi} = \lambda^2 S_{m_i}(\xi).\end{aligned}\quad (3.62)$$

Assume a power series solution for $S_{m_i}(\xi)$, i.e.,

$$S_{m_i}(\xi) = \sum_{n=1}^{\infty} b_{m_n} (1 - \xi)^{n-1}. \quad (3.63)$$

Then substituting (3.63) into (3.62) yields

$$\begin{aligned}
& \sum_{n=5}^{\infty} \left\{ \phi^4(n-1)(n-2)(n-3)(n-4)b_{mn} \right. \\
& \quad - 2\phi^2(m\pi)^2(n-3)(n-4)b_{m(n-2)} + q^*\phi^4(n-4)(n-5)a_{m(n-3)} \\
& \quad \left. + 2q^*\phi^4(n-4)b_{m(n-3)} + [(m\pi)^4 - \lambda^2] b_{m(n-4)} \right\} (1-\xi)^{n-5} = 0, \quad (3.64)
\end{aligned}$$

which leads to the recursion formula:

$$\begin{aligned}
b_{mn} = & \frac{[\lambda^2 - (m\pi)^4] b_{m(n-4)} - q^*\phi^4(n-3)(n-4)b_{m(n-3)}}{\phi^4(n-1)(n-2)(n-3)(n-4)} \\
& + \frac{2(m\pi)^2(n-3)(n-4)b_{m(n-2)}}{\phi^2(n-1)(n-2)(n-3)(n-4)}, \quad (n \geq 5), \quad (3.65)
\end{aligned}$$

$$\text{for } m = 1, 2, 3, \dots, \infty$$

From which b_{mn} can be determined for all $n \geq 5$, $m = 1, 2, \dots, \infty$. Similarly, $S_{mi}(\xi)$ can be represented in terms of four fundamental solutions $N_{mj}(\xi)$, $j=1, 2, 3, 4$, i.e.,

$$S_{mi}(\xi) = \sum_{j=1}^4 b_{mj} N_{mj}(\xi), \quad (3.66)$$

where

$$N_{m1}(\xi) = 1 + \frac{[\lambda_{mj}^2 - (m\pi)^4]}{4!\phi^4} (1-\xi)^4 + \frac{2[\lambda_{mj}^2 - (m\pi)^4] (m\pi)^2}{6!\phi^8} (1-\xi)^6 + \dots,$$

$$N_{m2}(\xi) = (1-\xi) - \frac{2!q^*}{4!} (1-\xi)^4$$

$$\begin{aligned}
& + \frac{[\lambda_{m_j}^2 - (m\pi)^4]}{5!\phi^4}(1-\xi)^5 - \frac{2!2q^*(m\pi)^2}{6!\phi^4}(1-\xi)^6 + \dots, \\
N_{m3}(\xi) &= (1-\xi)^2 + \frac{2!2(m\pi)^2}{4!\phi^4}(1-\xi)^4 - \frac{3!q^*}{5!}(1-\xi)^5 + \frac{2![\lambda_{m_j}^2 - (m\pi)^4]}{6!(1-\xi)^6} + \dots, \\
N_{m4}(\xi) &= (1-\xi)^3 + \frac{3!2(m\pi)^2}{5!\phi^4}(1-\xi)^5 - \frac{4!q^*}{6!}(1-\xi)^6 + \dots
\end{aligned} \tag{3.67}$$

and b_{m1} , b_{m2} , b_{m3} and b_{m4} are constants of integration.

Thus the general solution to the adjoint problem is:

$$\begin{aligned}
\psi_{m_j}(\xi, \eta) &= S_{m_j}(\xi) \sin(m\pi\eta) \\
&= \sum_{j=1}^4 b_{m_j} N_{m_j}(\xi) \sin(m\pi\eta) \quad m = 1, 2, 3, \dots, \infty.
\end{aligned} \tag{3.63}$$

The constants of integration b_{m1} , b_{m2} , b_{m3} and b_{m4} can be determined by means of the adjoint boundary conditions (3.55-3.58). Applying (3.57), (3.58) to (3.68) yields:

$$b_{m3} = \frac{\nu}{2\phi^2}(m\pi)^2 b_{m1} \tag{3.69}$$

$$b_{m4} = \frac{(2-\nu)}{6\phi^2}(m\pi)^2 b_{m2} - \frac{q^*}{6} b_{m1}. \tag{3.70}$$

Applying (3.55) to (3.68) and using (3.69-3.70) leads to:

$$b_{m2} = -\frac{E_m(\lambda_{m1})}{F_m(\lambda_{m1})} b_{m1}, \tag{3.71}$$

where

$$\begin{aligned}
 E_m = & \left\{ N''_{m1}(0) + \left[q^* - \frac{\nu}{\phi^2} (m\pi)^2 \right] N_{m1}(0) \right\} \\
 & + \frac{\nu(m\pi)^2}{2\phi^2} \left\{ N''_{m3}(0) + \left[q^* - \frac{\nu}{\phi^2} (m\pi)^2 \right] N_{m3}(0) \right\} \\
 & - \frac{q^*}{6} \left\{ N''_{m4}(0) + \left[q^* - \frac{\nu}{\phi^2} (m\pi)^2 \right] N_{m4}(0) \right\} , \quad (3.72)
 \end{aligned}$$

$$\begin{aligned}
 F_m = & \left\{ N''_{m2}(0) + \left[q^* - \frac{\nu}{\phi^2} (m\pi)^2 \right] N_{m2}(0) \right\} \\
 & + \frac{(2-\nu)}{6\phi^2} \left\{ N''_{m4}(0) + \left[q^* - \frac{\nu}{\phi^2} (m\pi)^2 \right] N_{m4}(0) \right\} . \quad (3.73)
 \end{aligned}$$

Using (3.71) in (3.70) yields

$$b_{m4} = - \left[\frac{(2-\nu)}{6\phi^2} (m\pi)^2 \frac{E_m(\lambda_{m1})}{F_m(\lambda_{m1})} + \frac{q^*}{6} \right] b_{m1} . \quad (3.74)$$

Substituting (3.69), (3.71) and (3.74) into (3.68) leads to:

$$\begin{aligned}
 \psi_{m1}(\xi, \eta) = & b_{m1} \left\{ N_{m1}(\xi, \lambda_{m1}) - \frac{E_m(\lambda_{m1})}{F_m(\lambda_{m1})} N_{m2}(\xi, \lambda_{m1}) + \frac{\nu}{\phi^2} (m\pi)^2 N_{m3}(\xi, \lambda_{m1}) \right. \\
 & \left. - \left[\frac{(2-\nu)(m\pi)^2 E_m(\lambda_{m1})}{6\phi^2 F_m(\lambda_{m1})} + \frac{q^*}{6} \right] N_{m4}(\xi, \lambda_{m1}) \right\} \sin(m\pi\eta) \quad (3.75)
 \end{aligned}$$

which is the adjoint eigenfunction of the system corresponding to eigenvalue λ_{m1} .

3.5 Green's Function of the Problem

In chapter 2, it has been shown that $V_{ij}(\xi, \eta)$ and $\psi_{kl}(\xi, \eta)$ are bi-orthogonal, i.e.,

$$\int_0^1 \int_0^1 V_{ij}(\xi, \eta) \psi_{kl}(\xi, \eta) d\xi d\eta = 0 \text{ when } i \neq k \text{ or } j \neq l, \quad (3.76)$$

and

$$\int_0^1 \int_0^1 V_{ij}(\xi, \eta) \psi_{ij}(\xi, \eta) d\xi d\eta = c_{ij} \neq 0. \quad (3.77)$$

One may normalize $V_{ij}(\xi, \eta)$ and $\psi_{kl}(\xi, \eta)$ by setting

$$V_{ij}^n(\xi, \eta) = \frac{1}{\sqrt{c_{ij}}} V_{ij}(\xi, \eta), \quad (3.78)$$

$$\psi_{ij}^n(\xi, \eta) = \frac{1}{\sqrt{c_{ij}}} \psi_{ij}(\xi, \eta). \quad (3.79)$$

Then $V_{ij}^n(\xi, \eta)$ and $\psi_{kl}^n(\xi, \eta)$ are bi-orthonormal, i.e.,

$$\int_0^1 \int_0^1 V_{ij}^n(\xi, \eta) \psi_{kl}^n(\xi, \eta) d\xi d\eta = \delta_{ik} \delta_{jl}. \quad (3.80)$$

Using equation (3.16), the frequencies ω_{ij} are determined as:

$$\omega_{ij} = \sqrt{\frac{D}{\mu b^4}} \lambda_{ij} \quad (3.81)$$

In view of equation (2.108), define non-dimensional dummy variables:

$$\begin{aligned}\gamma &= \frac{x}{l}, \quad x \in [0, l], \quad \gamma \in [0, 1], \\ \sigma &= \frac{y}{b}, \quad y \in [0, b], \quad \sigma \in [0, 1].\end{aligned}\quad (3.82)$$

The unsymmetric Green's Function of the problem is

$$\begin{aligned}G(\xi, \gamma, \eta, \sigma, t, \tau) &= \sum_{i,j=1}^{\infty} \frac{V_{ij}^n(\xi, \eta) \psi_{ij}^n(\gamma, \sigma)}{\mu \omega_{ij}^2} \cos[\omega_{ij}(t - \tau)] \\ &= \sum_{i,j=1}^{\infty} \frac{V_{ij}(\xi, \eta) \psi_{ij}(\gamma, \sigma)}{\mu c_{ij} \omega_{ij}^2} \cos[\omega_{ij}(t - \tau)],\end{aligned}\quad (3.83)$$

where $V_{ij}(\xi, \eta)$, $\psi_{ij}(\gamma, \sigma)$, c_{ij} and ω_{ij} are expressed in equations (3.46), (3.75), (3.77) and (3.81) respectively. Thus the solution to problem (3.1-3.3), i.e. the response $W(x, y, t)$ may now be calculated using equation (2.120).

3.6 Dynamic Characteristics of the System

3.6.1 Results for Typical Values of ϕ and ν

S-F-S-F Plate

In this section typical results for an S-F-S-F plate with $\phi = 1$ and $\nu = 0.3$ are presented as an example. The influence of the plate aspect ratio ϕ and Poisson ratio ν on the system characteristics will be discussed in the subsequent sections.

Based on the theory presented in Chapter 2, Green's function approach used above is meaningful only in the corresponding domain $\mathcal{PR}$ of the system within which the stability of the system prevails. By using the characteristic equation (3.43), the eigenvalue system can be obtained and for a given aspect ratio ϕ and Poisson ratio ν , one may plot non-dimensional eigenvalues versus load parameter q^* as shown in Fig (3.2) for $\phi = 1$ and $\nu = 0.3$. From this plot one concludes that for $\phi = 1$ and

$\nu = 0.3$, a S-F-S-F plate under follower forces loses stability by flutter ¹ at the first critical load $q_{crit1}^* = 91.55$. This first critical load corresponds to coalescing of the first two eigenvalue branches beyond which the eigenvalues become complex conjugates.

As a practical design aid, the first five non-dimensional eigenvalues with the domain $\mathcal{PR}$ are also tabulated in Table (3.1) at discrete levels of q^*/q_{crit1}^* .

Typical two dimensional mode shapes for $q^*/q_{crit1}^* = 0$ and 0.5 are presented in Fig. (3.3). Furthermore to get a more clear indication of the influence of follower forces on system's mode shapes, also the first five mode shapes, at an appropriate η , are given for $q^*/q_{crit1}^* = 0, 0.25, 0.5$ and 0.75 in Fig. (3.4). It can be seen that the first four modes are quite sensitive to the magnitude of q^* .

S-S-S-S Plate

Similarly for $\phi = 1$ and $\nu = 0.3$, the first five eigenvalues of the S-S-S-S plate, under foilower forces, are plotted as functions of q^* up to the first critical load $q_{crit1}^* = 67.42$, as shown in Fig. (3.5). The loss of stability in this case is by divergence which occurs when the first eigenvalue becomes zero at $q^* = q_{crit1}^*$. This result is consistent with the conclusion obtained by Leipholz in [31] where he showed that a S-S-S-S plate under follower forces is always a divergent system regardless of its aspect ratio and Poisson ratio.

Also for design purpose, the first five non-dimensional eigenvalues within the domain $\mathcal{PR}$, are also tabulated in Table (3.2) at discrete levels of q^*/q_{crit1}^* .

Typical two dimensional mode shapes for $q^*/q_{crit1}^* = 0$ and 0.5 are presented in Fig. (3.6). The first five mode shapes for appropriate η are also given for $q^*/q_{crit1}^* = 0, 0.25, 0.5$ and 0.75 in Fig. (3.7), from which the influence of q^* on the mode shapes is quite apparent. Although such an influence does not seem to be as pronounced as in the case of S-F-S-F boundary conditions.

For both S-F-S-F and S-S-S-S plates, in the special case, $q^* = 0$, the problem is reduced to a selfadjoint one with a symmetric Green's function, for $V_{ij}(\xi, \eta) = \psi_{ij}(\xi, \eta)$. This represents the elastic plate problem when the follower force are absent.

¹more details of divergent and flutter instability can be found in Section 4.7

As expected, the eigenvalues λ_m^0 and the eigenfunctions $V_{ij}^0(\xi, \eta)$, corresponding to various modes of free vibration of the plate, are in perfect agreement with the results obtained using classical approaches [30].

3.6.2 Influence of the Aspect Ratio on the Critical Load

S-F-S-F Plate

The q_{crit1}^* for an S-F-S-F plate with different aspect ratios were determined. All such q_{crit1}^* were found to correspond to a loss of stability through flutter, indicating that an S-F-S-F plate with q parallel to the supported edge will always become instable through flutter irrespective of the aspect ratio. The corresponding data is presented in Fig. (3.8) which can also be used as an important design aid.

S-S-S-S Plate

As mentioned before, irrespective of the aspect ratio this plate will always lose its stability by means of divergence. Fig. (3.9) shows q_{crit1}^* for various aspect ratios.

The critical divergent load for several typical aspect ratios are given below:

$$\phi = 0.4, \quad q_{crit1}^* = q_{crit1} \frac{l^3}{D} = 331.00,$$

$$\phi = 1.0, \quad q_{crit1}^* = q_{crit1} \frac{l^3}{D} = 67.42,$$

$$\phi = 1.75, \quad q_{crit1}^* = q_{crit1} \frac{l^3}{D} = 32.42,$$

$$\phi = 2.5, \quad q_{crit1}^* = q_{crit1} \frac{l^3}{D} = 25.18.$$

Realizing that the total critical load a plate can carry is

$$P_{crit} = q_{crit1} lb = q_{crit1} l^2 \phi = \left[q_{crit1} \frac{l^3}{D} \right] \phi \frac{D}{l} = q_{crit1}^* \phi \frac{D}{l},$$

it follows that for

$$\phi = 0.4, \quad P_{crit} = q_{crit1}^* \phi \frac{D}{l} = 331.00 \times 0.4 \frac{D}{l} = 132.4 \frac{D}{l},$$

$$\phi = 1.0, \quad P_{crit} = q_{crit1}^* \phi \frac{D}{l} = 67.42 \times 1.0 \frac{D}{l} = 67.42 \frac{D}{l},$$

$$\phi = 1.75, \quad P_{crit} = q_{crit1}^* \phi \frac{D}{l} = 32.42 \times 1.75 \frac{D}{l} = 56.73 \frac{D}{l},$$

$$\phi = 2.5, \quad P_{crit} = q_{crit1}^* \phi \frac{D}{l} = 25.18 \times 2.5 \frac{D}{l} = 62.945 \frac{D}{l}.$$

Setting these values for $\frac{D}{l} = const$ yields the curve in Fig. (3.10). This seems to indicate that around $\phi = 1.75$ the S-S-S-S plate carries smallest total critical load.

3.6.3 Influence of Poisson Ratio

In classical free vibration analysis of rectangular thin plates, i.e., when $q^* = 0$, it is well known that the plate eigenvalues depend on the Poisson ratio ν only when the plate has one or more free edges [32]. This statement is true also in case when non-conservative forces are present. This may be verified by obtaining the characteristic equation for S-S-S-S plate problem which will be independent of ν . It should however be pointed out that due to the relationship between the eigenvalues λ_{ij} and natural frequencies ω_{ij} , the natural frequencies will depend on the Poisson ratio for any boundary conditions. In view of above, the variation of ν will affect the eigenvalues of the S-F-S-F plate but not S-S-S-S plate.

Let the ν change in a S-F-S-F plate under follower forces, it is very interesting to note that at different levels of follower forces within domain $\mathcal{PR}$, the eigenvalues of the system change with ν in very different ways. Fig (3.11) shows a plot of the first five eigenvalues versus ν for a S-F-S-F plate under different levels of follower force $q^* = 0, 10, 20, 50$.

For the classical case when $q^* = 0$, all first five λ decrease with increasing ν . This is a rather surprising result, since an increase in ν will mean an increase in the flexural rigidity of the plate and should raise the natural frequencies. However, as suggested by Bardell [32], for a plate with at least one free edge, anticlastic curvature effects become more pronounced as ν is increased, so that at a particular mode of vibration, the increase in the kinetic energy of the plate may exceed the corresponding increase in flexural strain energy, thus accounting for the reduction in frequency. Bardell's suggestion becomes even more plausible in view of the results obtained in the present work for values of λ in the presence of q^* . Obviously, higher values of $q^* > 0$ make S-F-S-F plates increasingly more rigid. Therefore, as shown in Fig (3.11), at increasingly higher levels of q^* , at least for lower modes, the increase in flexural strain energy may exceed the corresponding increase in the kinetic energy of the plate. Hence as q^* becomes larger, the values of λ for lower frequencies increase as ν increase, i.e., as q^* increases the trend will reverse so that the flexural strain energy overcomes the anticlastic curvature effects.

It is important to know that further investigation shows that variation of ν will not change the flutter nature of a S-F-S-F plate under follower force but will change only the magnitude of the critical flutter load of the system. Figure (3.12) shows how the system's critical flutter load varies with ν .

3.7 A Numerical Example

A numerical example is presented for a S-F-S-F plate under follower force. Consider a S-F-S-F thin plate made of 6061-T6 aluminum with the following properties:

$$\begin{aligned} E &= 69,000 \text{ (MPa) ,} \\ \nu &= 0.32 , \\ \rho &= 2714.4 \text{ (Kg/m}^3\text{) .} \end{aligned}$$

The dimensions of the plate read:

$$\begin{aligned} l &= b = 1 \text{ (m) ,} \\ h &= 0.025 \text{ (m) ,} \end{aligned}$$

which gives

$$\mu = h \times \rho = 67.86 \text{ (Kg/m}^2\text{)} .$$

The plate is subjected to a uniformly distributed, transverse harmonic driving force

$$K(x, y, t) = k \cos(\Omega_0 t) \text{ (N/m}^2\text{)} , \quad (3.84)$$

where

$$k = 10,000 \text{ (N/m}^2\text{)} ,$$

$$\Omega_0 = 6.28 \text{ (rad/s)} .$$

The initial conditions are assumed to be zero, i.e.,

$$W(x, y, 0) = f(x, y) = 0 , \quad (3.85)$$

$$\dot{W}(x, y, 0) = h(x, y) = 0 . \quad (3.86)$$

Obviously transverse driving force $K(x, y, t)$ satisfies condition (2.47-2.48) and conditions (2.103) are also satisfied by the initial conditions (3.85–3.86). Then by virtue of equation (2.120) and using equation (3.83), the response of the plate is

$$\begin{aligned} W(\xi, \eta, t) = & \frac{W\Omega_0^2}{\mu} \sum_{m,n=1}^{\infty} \frac{\cos \omega_{mn} t - \cos \Omega_0 t}{(\Omega_0^2 - \omega_{mn}^2) c_{mn} \omega_{mn}^2} V_{mn}(\xi, \eta) \int_0^1 \int_0^1 \psi_{mn}(\gamma, \sigma) d\gamma d\sigma \\ & + \frac{W \cos \Omega_0 t}{\mu} \sum_{m,n=1}^{\infty} \frac{V_{mn}(\xi, \eta)}{c_{mn} \omega_{mn}^2} \int_0^1 \int_0^1 \psi_{mn}(\gamma, \sigma) d\gamma d\sigma \\ & + \frac{W}{\mu} \sum_{m,n=1}^{\infty} \frac{V_{mn}(\xi, \eta)}{c_{mn} \omega_{mn}^2} \cos \omega_{mn} t \int_0^1 \int_0^1 \psi_{mn}(\gamma, \sigma) d\gamma d\sigma , \end{aligned} \quad (3.87)$$

where the expressions for ω_{mn} , c_{mn} , $V_{mn}(\xi, \eta)$ and $\psi_{mn}(\gamma, \sigma)$ can be found in equations (3.81), (3.77), (3.46) and (3.75) respectively.

Using (3.87), the dynamic response of the plate is determined at two different levels of follower force $q^* = 0$ and $q^* = 0.5q_{crit}^*$. Typical deflections of the plate after

one and two seconds of motion are shown in Fig. (3.13). Furthermore, the motion of the mid point of edge $\xi = 0$, is given in Fig. (3.14).

Finally the rapid convergence of solution (3.87) is illustrated in Table (3.3), where results based on one, three and five modes contribution for two levels of q^* are given. It can be seen that using only the first three modes will result in sufficient accuracy.

Chapter 4

Non-Selfadjoint Control of Thin Elastic Plates in the Presence of Unidirectional Forces

4.1 Introduction

The need for active control of increasingly more flexible and slender, continuous, elastic structures has emerged, in recent years, as an important consideration in design of traditional engineering structures such as bridges and high-rise buildings as well as non-traditional modern structures such as space stations and platforms.

By now, it is well established that employing discretization technique in stability analysis of such controlled systems with distributed parameters could lead to questionable and non-definite results. A previously uncontrolled conservative system may, upon application of a specific control law, become a nonconservative system. Mathematically, this means that provided one is working within the realm of linear system, application of a control may transform an initially selfadjoint system into a non-selfadjoint one. In such a case, if the structure is idealized and designed on the basis of discretized finite dimensional models, the neglected higher modes could have detrimental effect on the stability of the actual infinite dimensional system, i.e., a spillover phenomenon could occur. To avoid this difficulty, Leipholz suggested using closed form solutions when investigating the behavior of non-selfadjoint prob-

lems encountered in the control of linearly elastic continuous systems [33]. Then, using concepts of operator theory and working with functional analysis one can avoid uncertainties due to spillover effect by "determining a lower bound for all possible, infinitely many eigenvalues which the system may possess and which may govern the instability of the system" [33].

In the present work, this approach is extended to include the non-selfadjoint control of the dynamic response of a thin elastic plate in the presence of uniformly distributed in-plane unidirectional forces arising from a conservative force field, for example, gravitational field. The closed form solution and the exact characteristic equation of the controlled system are obtained, which are used to evaluate the effectiveness of the control law and the stability of the controlled system. The stability boundary obtained by using exact characteristic equation of the controlled system are compared with the stability lower bound obtained from the Liapunov's second method. A numerical example is used to show the application of the method.

4.2 Description of the Control Problem

Consider the small vibrations of a thin elastic rectangular plate with four edges simply supported as shown in Fig. (4.1). The plate is exposed to the transverse distributed driving force $K(x, y, t)$ and carries the uniformly distributed in plane unidirectional forces resulting from a conservative force field. Using

$$\nabla^4 \equiv \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

according to the classical small deflection plate theory, the governing differential equation of the motion of such a plate reads,

$$\mu \ddot{W}(x, y, t) + D \nabla^4 W(x, y, t) + [g(l-x)W_x(x, y, t)]_x = K(x, y, t), \quad (4.1)$$

with the boundary conditions,

along edges $x = 0, l$:

$$W(0, y, t) = \frac{\partial^2 W(0, y, t)}{\partial x^2} = W(l, y, t) = \frac{\partial^2 W(l, y, t)}{\partial x^2} = 0, \quad (4.2)$$

along edges $y = 0, b$:

$$W(x, 0, t) = \frac{\partial^2 W(x, 0, t)}{\partial y^2} = W(x, b, t) = \frac{\partial^2 W(x, b, t)}{\partial y^2} = 0, \quad (4.3)$$

where all terms represent the same variables as in equations (3.1–3.6) except that now the g is the uniformly distributed unidirectional, hence conservative, force and the above system is a selfadjoint system.

Let now control of the plate be considered. In general, equation (4.1) will then change into

$$\mu \ddot{W} + D \nabla^4 W + [g(l-x)W_x]_x + C = K, \quad (4.4)$$

where for a control operator $\mathcal{L}$,

$$C = \mathcal{L}W \quad (4.5)$$

is the applied control law. The boundary conditions of the plate are not affected so that (4.4) along with (4.2–4.3) define the new controlled problem.

Depending on the form of the operator $\mathcal{L}$, the controlled problem will turn out to be either a selfadjoint or a non-selfadjoint system. If the former is the case, then the general methodology for selfadjoint system, as reviewed at the beginning of Chapter 2 will apply. However, based on the theoretical developments presented in Section (2.3), the more challenging case of a non-selfadjoint control law will be considered in the following. In order to demonstrate the specifics of the analysis and in view of

deliberations of Chapter 3, let the operator $\mathcal{L}$ be specified as

$$\mathcal{L} \equiv c(l-x) \frac{\partial^2}{\partial x^2} = c \frac{\partial}{\partial x} \left[(l-x) \frac{\partial}{\partial x} \right] + c \frac{\partial}{\partial x},$$

for $c > 0$,

so that according to (4.5) the resulting control law implies a uniformly distributed continuously applied control force of magnitude c which always remains tangential to mid-plane of the plate as shown in Fig. (4.2).

According to such a control law, the system's governing differential equation becomes

$$\begin{aligned} \mu \ddot{W}(x, y, t) + D \left[\frac{\partial^4 W(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 W(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y, t)}{\partial y^4} \right] \\ + (c + g) \frac{\partial}{\partial x} \left[(l - x) \frac{\partial W(x, y, t)}{\partial x} \right] + c \frac{\partial W(x, y, t)}{\partial x} = K(x, y, t), \end{aligned} \quad (4.6)$$

with the boundary conditions (4.2-4.3). Problem (4.6) is a multiple parameter system with c and g as load parameters. It can easily be verified that this is a non-selfadjoint problem with the non-selfadjoint operator:

$$\begin{aligned} \mathcal{N}[g, c, W(x, y, t)] &= E + (c + g) \mathcal{S}_1 + c \mathcal{N}_1 \\ &= D \left[\frac{\partial^4 W(x, y, t)}{\partial x^4} + 2 \frac{\partial^4 W(x, y, t)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y, t)}{\partial y^4} \right] \\ &\quad + (c + g) \frac{\partial}{\partial x} \left[(l - x) \frac{\partial W(x, y, t)}{\partial x} \right] + c \frac{\partial W(x, y, t)}{\partial x}. \end{aligned} \quad (4.7)$$

By introducing the following non-dimensional quantities,

$$\begin{aligned}\xi &= \frac{x}{l}, \quad x \in [0, l], \quad \xi \in [0, 1], \\ \eta &= \frac{y}{b}, \quad y \in [0, b], \quad \eta \in [0, 1], \\ \phi &= \frac{b}{l}, \quad g^* = \frac{l^3}{D}g, \quad c^* = \frac{l^3}{D}c,\end{aligned}\tag{4.8}$$

equation (4.6) changes into,

$$\begin{aligned}&\frac{\mu b^4}{D}\ddot{W}(\xi, \eta, t) + \frac{\partial^4 W(\xi, \eta, t)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 W(\xi, \eta, t)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 W(\xi, \eta, t)}{\partial \xi^4} \\ &+ (c^* + g^*)\phi^4 \frac{\partial}{\partial \xi} \left[(1 - \xi) \frac{\partial W(\xi, \eta, t)}{\partial \xi} \right] + c^* \phi^4 \frac{\partial W(\xi, \eta, t)}{\partial \xi} = \frac{b^4}{D} K(\xi, \eta, t),\end{aligned}\tag{4.9}$$

and the boundary conditions (4.2) and (4.3) become,

along edges $\xi = 0, 1$,

$$W(0, \eta, t) = \frac{\partial^2 W(0, \eta, t)}{\partial \xi^2} = W(1, \eta, t) = \frac{\partial^2 W(1, \eta, t)}{\partial \xi^2} = 0, \tag{4.10}$$

along edges $\eta = 0, 1$,

$$W(\xi, 0, t) = \frac{\partial^2 W(\xi, 0, t)}{\partial \eta^2} = W(\xi, 1, t) = \frac{\partial^2 W(\xi, 1, t)}{\partial \eta^2} = 0. \tag{4.11}$$

Based on Green's function approach discussed in Chapter 2, the controlled plate problem (4.9–4.11) can be solved analytically. The exact characteristic equation and closed form solution can be obtained so that the effects of the control law and the control system may be analyzed completely.

4.3 Modal Generating System

Considering the homogeneous form of (4.9), using separability of variables and introducing the non-dimensional frequency

$$\lambda^2 = \frac{\mu b^4}{D} \omega^2 , \quad (4.12)$$

the modal generating system of the problem can be obtained as

$$\begin{aligned} \mathcal{N}[g^*, c^*, V(\xi, \eta)] \equiv & \frac{\partial^4 V(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 V(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 V(\xi, \eta)}{\partial \xi^4} \\ & + (c^* + g^*)\phi^4 \frac{\partial}{\partial \xi} \left[(1 - \xi) \frac{\partial V(\xi, \eta)}{\partial \xi} \right] + c^* \phi^4 \frac{\partial V(\xi, \eta)}{\partial \xi} = \lambda^2 V(\xi, \eta) , \end{aligned} \quad (4.13)$$

with the associated boundary conditions,

along edges $\xi = 0, 1$,

$$V(0, \eta) = \frac{\partial^2 V(0, \eta)}{\partial \xi^2} = V(1, \eta) = \frac{\partial^2 V(1, \eta)}{\partial \xi^2} = 0 , \quad (4.14)$$

along edges $\eta = 0, 1$,

$$V(\xi, 0) = \frac{\partial^2 V(\xi, 0)}{\partial \eta^2} = V(\xi, 1) = \frac{\partial^2 V(\xi, 1)}{\partial \eta^2} = 0 . \quad (4.15)$$

Modal generating system (4.13–4.15) will generate eigenvalues λ_{ij} and mode shapes $V_{ij}(\xi, \eta)$ of the controlled system.

Assuming the solution for (4.13) has the form,

$$V_m(\xi, \eta) = X_m(\xi) \sin(m\pi\eta), \quad m = 1, 2, 3, \dots, \infty , \quad (4.16)$$

obviously (4.16) satisfies the boundary conditions (4.15) along edges $\eta = 0, 1$.

Substituting (4.16) into (4.13) yields,

$$\begin{aligned} & (m\pi)^4 X_m(\xi) - 2\phi^2(m\pi)^2 \frac{d^2 X_m(\xi)}{d\xi^2} + \phi^4 \frac{d^4 X_m(\xi)}{d\xi^4} \\ & + (c^* + g^*)\phi^4 \frac{d}{d\xi} \left[(1 - \xi) \frac{dX_m(\xi)}{d\xi} \right] + c^* \phi^4 \frac{dX_m(\xi)}{d\xi} = \lambda^2 X_m(\xi) . \end{aligned} \quad (4.17)$$

Assume a power series solution for $X_m(\xi)$, i.e.,

$$X_m(\xi) = \sum_{n=1}^{\infty} a_{mn}(1 - \xi)^{n-1} , \quad m = 1, 2, 3, \dots, \infty \quad (4.18)$$

Substituting (4.18) into (4.17), one obtains the recursion formula for the unknown coefficients a_{mn} :

$$\begin{aligned} a_{mn} = & \frac{[\lambda^2 - (m\pi)^4] a_{m(n-4)} + [(n-5)c^* - (n-4)g^*] \phi^4 (n-4) a_{m(n-3)}}{\phi^4 (n-1)(n-2)(n-3)(n-4)} \\ & + \frac{2(m\pi)^2 (n-3)(n-4) a_{m(n-2)}}{\phi^2 (n-1)(n-2)(n-3)(n-4)} , \quad (n \geq 5), \end{aligned} \quad (4.19)$$

$$\text{for } m = 1, 2, 3, \dots, \infty$$

so that all coefficients a_{mn} with $n \geq 5$ can be determined in terms of a_{m1} , a_{m2} , a_{m3} and a_{m4} , ($m = 1, 2, 3, \dots, \infty$). Thus the general solution $X_m(\xi)$, in terms of the fundamental solutions, is obtained as

$$X_m(\xi) = \sum_{j=1}^4 a_{mj} M_{mj}(\xi) \quad \text{for } m = 1, 2, 3, \dots, \infty , \quad (4.20)$$

with the corresponding fundamental solutions,

$$\begin{aligned}
 M_{m1}(\xi) &= 1 + \frac{[\lambda^2 - (m\pi)^4]}{4!\phi^4} (1 - \xi)^4 + \frac{2[\lambda^2 - (m\pi)^4](m\pi)^2}{6!\phi^6} (1 - \xi)^6 + \dots, \\
 M_{m2}(\xi) &= (1 - \xi) - \frac{g^*}{4!} (1 - \xi)^4 + \frac{[\lambda^2 - (m\pi)^4]}{5!\phi^4} (1 - \xi)^5 + \frac{2(m\pi)^2 g^*}{6!\phi^2} (1 - \xi)^6 + \dots, \\
 M_{m3}(\xi) &= (1 - \xi)^2 + \frac{2!2(m\pi)^2}{4!\phi^2} (1 - \xi)^4 - \frac{2!(c^* + 2g^*)}{5!} (1 - \xi)^5 \\
 &\quad + \frac{2![\lambda^2 + 3(m\pi)^4]}{6!\phi^4} (1 - \xi)^6 + \dots, \\
 M_{m4}(\xi) &= (1 - \xi)^3 + \frac{3!2(m\pi)^2}{5!\phi^2} (1 - \xi)^5 - \frac{3!(2c^* + 3g^*)}{6!} (1 - \xi)^6 + \dots
 \end{aligned} \tag{4.21}$$

By applying boundary conditions (4.14), the constants of integration a_{m1} , a_{m2} , a_{m3} and a_{m4} are determined as,

$$\begin{aligned}
 a_{m1} &= 0, \\
 a_{m3} &= 0, \\
 a_{m2}M_{m2}(0) + a_{m4}M_{m4}(0) &= 0, \\
 a_{m2}M_{m2}''(0) + a_{m4}M_{m4}''(0) &= 0.
 \end{aligned} \tag{4.22}$$

In order to get non-trivial solutions for a_{m2} and a_{m4} , one should have,

$$\begin{vmatrix} M_{m2}(0) & M_{m4}(0) \\ M_{m2}''(0) & M_{m4}''(0) \end{vmatrix} = 0. \tag{4.23}$$

Thus one obtains the characteristic equation of the problem, which is a polynomial

in λ , g^* and c^* .

$$F_m(\lambda, g^*, c^*, \phi) = M_{m2}(0)M''_{m4}(0) - M_{m4}(0)M''_{m2}(0) = 0. \quad (4.24)$$

For given g^* , c^* and ϕ , infinite number of λ_{mi} , ($m, i = 1, 2, 3, \dots, \infty$), which are the eigenvalues of the problem, can be obtained. Then using the third equation in (4.22), i.e.,

$$a_{m4} = -\frac{M_{m2}(0, \lambda_{mi})}{M_{m4}(0, \lambda_{mi})}a_{m2}, \quad (4.25)$$

one has

$$X_{mi}(\xi, \lambda) = a_{m2} \left[M_{m2}(\xi, \lambda_{mi}) - \frac{M_{m2}(0, \lambda_{mi})}{M_{m4}(0, \lambda_{mi})} M_{m4}(\xi, \lambda_{mi}) \right], \quad (4.26)$$

by means of which, the mode shape corresponding to λ_{mi} is

$$V_{mi}(\xi, \eta) = a_{m2} \left[M_{m2}(\xi, \lambda_{mi}) - \frac{M_{m2}(0, \lambda_{mi})}{M_{m4}(0, \lambda_{mi})} M_{m4}(\xi, \lambda_{mi}) \right] \sin m\pi\eta \quad (4.27)$$

$$m, i = 1, 2, 3, \dots, \infty.$$

4.4 Adjoint Eigenfunctions

Following a procedure similar to that outlined in Chapter 3, one can obtain $\tilde{\mathcal{N}}$ the adjoint operator to $\mathcal{N}$ by satisfying Green's identity, i.e.,

$$\int_0^1 \int_0^1 V(\xi, \eta) \tilde{\mathcal{N}} [g^*, c^*, \psi(\xi, \eta)] d\psi d\xi$$

$$\begin{aligned}
&= \int_0^1 \int_0^1 \psi(\xi, \eta) \mathcal{N}[g^*, c^*, V(\xi, \eta)] d\psi d\xi \\
&= \int_0^1 \int_0^1 \psi(\xi, \eta) \left\{ \frac{\partial^4 V(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 V(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 V(\xi, \eta)}{\partial \xi^4} \right. \\
&\quad \left. + (c^* + g^*) \phi^4 \frac{\partial}{\partial \xi} \left[(1 - \xi) \frac{\partial V(\xi, \eta)}{\partial \xi} \right] + c^* \phi^4 \frac{\partial V(\xi, \eta)}{\partial \xi} \right\} d\psi d\xi \quad (4.28)
\end{aligned}$$

After applying Gauss' integral theorem and imposing boundary conditions (4.14) and (4.15), equation (4.28) changes into

$$\begin{aligned}
&\int_0^1 \int_0^1 V(\xi, \eta) \tilde{\mathcal{N}}[g^*, c^*, \psi(\xi, \eta)] d\psi d\xi \\
&= \int_0^1 \int_0^1 V(\xi, \eta) \left\{ \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 \psi(\xi, \eta)}{\partial \xi^4} \right. \\
&\quad \left. + (c^* + g^*) \phi^4 \frac{\partial}{\partial \xi} \left[(1 - \xi) \frac{\partial \psi(\xi, \eta)}{\partial \xi} \right] - c^* \phi^4 \frac{\partial \psi(\xi, \eta)}{\partial \xi} \right\} d\psi d\xi \\
&\quad - \int_0^1 B_1 d\xi + \int_0^1 B_2 d\eta, \quad (4.29)
\end{aligned}$$

where

$$\begin{aligned}
B_1 &= 2\phi^2 \left[\frac{\partial^2 \psi(\xi, 0)}{\partial \xi^2} \frac{\partial V(\xi, 0)}{\partial \eta} + \frac{\partial^2 \psi(\xi, 1)}{\partial \xi^2} \frac{\partial V(\xi, 1)}{\partial \eta} \right] \\
&\quad + \psi(\xi, 0) \frac{\partial^3 V(\xi, 0)}{\partial \eta^3} + \frac{\partial^2 \psi(\xi, 0)}{\partial \eta^2} \frac{\partial V(\xi, 0)}{\partial \eta} \\
&\quad + \psi(\xi, 1) \frac{\partial^3 V(\xi, 1)}{\partial \eta^3} + \frac{\partial^2 \psi(\xi, 1)}{\partial \eta^2} \frac{\partial V(\xi, 1)}{\partial \eta}, \\
B_2 &= \phi^4 \left[\psi(1, \eta) \frac{\partial^3 V(1, \eta)}{\partial \xi^3} + \frac{\partial^2 \psi(1, \eta)}{\partial \xi^2} \frac{\partial V(1, \eta)}{\partial \xi} \right]
\end{aligned}$$

$$\begin{aligned}
& + \phi^4 \left[\psi(1, \eta) \frac{\partial^3 V(1, \eta)}{\partial \xi^3} + \frac{\partial^2 \psi(1, \eta)}{\partial \xi^2} \frac{\partial V(1, \eta)}{\partial \xi} \right] \\
& + 2\phi^2 \left[\psi(1, \eta) \frac{\partial^3 V(1, \eta)}{\partial \xi \partial \eta^2} + \psi(0, \eta) \frac{\partial^3 V(0, \eta)}{\partial \xi \partial \eta^2} \right] \\
& + (c^* + g^*) \phi^4 \psi(0, \eta) \frac{\partial V(0, \eta)}{\partial \xi} , \tag{4.30}
\end{aligned}$$

To satisfy the Green's identity, the boundary terms $\int_0^1 B_1 d\xi$ and $\int_0^1 B_2 d\eta$ should vanish. This will be accomplished by imposing the following minimum boundary conditions for $\psi(\xi, \eta)$, i.e.,

$$\psi(0, \eta) = \frac{\partial^2 \psi(0, \eta)}{\partial \xi^2} = \psi(1, \eta) = \frac{\partial^2 \psi(1, \eta)}{\partial \xi^2} = 0 , \tag{4.31}$$

$$\psi(\xi, 0) = \frac{\partial^2 \psi(\xi, 0)}{\partial \eta^2} = \psi(\xi, 1) = \frac{\partial^2 \psi(\xi, 1)}{\partial \eta^2} = 0 . \tag{4.32}$$

Thus the adjoint problem is:

$$\begin{aligned}
\tilde{\mathcal{N}}[g^*, c^*, \psi(\xi, \eta)] &= \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^4} + 2\phi^2 \frac{\partial^4 \psi(\xi, \eta)}{\partial \eta^2 \partial \xi^2} + \phi^4 \frac{\partial^4 \psi(\xi, \eta)}{\partial \xi^4} \\
&+ (c^* + g^*) \phi^4 \frac{\partial}{\partial \xi} \left[(1 - \xi) \frac{\partial \psi(\xi, \eta)}{\partial \xi} \right] - c^* \phi^4 \frac{\partial \psi(\xi, \eta)}{\partial \xi} = \lambda \phi^4 \psi(\xi, \eta) , \tag{4.33}
\end{aligned}$$

with the associated boundary conditions (4.31–4.32).

Again assume the solution has the form:

$$\psi_m(\xi, \eta) = S_m(\xi) \sin(m\pi\eta), \quad m = 1, 2, 3, \dots, \infty \tag{4.34}$$

which tacitly satisfies the boundary conditions (4.32).

Substituting (4.34) into (4.33) yields,

$$\begin{aligned} & (m\pi)^4 S_m(\xi) - 2\phi^2(m\pi)^2 \frac{d^2 S_m(\xi)}{d\xi^2} + \phi^4 \frac{d^4 S_m(\xi)}{d\xi^4} \\ & + (c^* + g^*)\phi^4 \frac{d}{d\xi} \left[(1 - \xi) \frac{dS_m(\xi)}{d\xi} \right] - c^* \phi^4 \frac{dS_m(\xi)}{d\xi} = \lambda^2 S_m(\xi) . \end{aligned} \quad (4.35)$$

Assuming a power series solution for $S_m(\xi)$, i.e.,

$$S_m(\xi) = \sum_{n=1}^{\infty} b_{mn}(1 - \xi)^{n-1} , \quad m = 1, 2, 3, \dots, \infty \quad (4.36)$$

one can obtain the corresponding recursion formulae for b_{mn} as,

$$\begin{aligned} b_{mn} = & \frac{[\lambda^2 - (m\pi)^4] b_{m(n-4)} + [(n-3)c^* - (n-4)g^*] \phi^4 (n-4) b_{m(n-3)}}{\phi^4 (n-1)(n-2)(n-3)(n-4)} \\ & + \frac{2(m\pi)^2 (n-3)(n-4) b_{m(n-2)}}{\phi^2 (n-1)(n-2)(n-3)(n-4)} , \quad (n \geq 5), \end{aligned} \quad (4.37)$$

for $m = 1, 2, 3, \dots, \infty$

Then, the general solution for $S_m(\xi)$ in terms of fundamental solutions $N_{mj}(\xi)$, $j=1, 2, 3, 4$ is obtained as ,

$$S_m(\xi) = \sum_{j=1}^4 b_{mj} N_{mj}(\xi) \quad \text{for } m = 1, 2, 3, \dots, \infty , \quad (4.38)$$

where

$$\begin{aligned}
N_{m1}(\xi) &= 1 + \frac{[\lambda^2 - (m\pi)^4]}{4!\phi^4} (1 - \xi)^4 + \frac{2[\lambda^2 - (m\pi)^4](m\pi)^2}{6!\phi^6} (1 - \xi)^6 + \dots, \\
N_{m2}(\xi) &= (1 - \xi) - \frac{2c^* + g^*}{4!} (1 - \xi)^4 + \frac{2(m\pi)^2(2c^* + g^*)}{6!} (1 - \xi)^6 + \dots, \\
N_{m3}(\xi) &= (1 - \xi)^2 + \frac{2!2(m\pi)^2}{4!\phi^2} (1 - \xi)^4 - \frac{2!(3c^* + 2g^*)}{5!} (1 - \xi)^5 \\
&\quad + \frac{2![\lambda^2 + 3(m\pi)^4]}{6!\phi^4} (1 - \xi)^6 + \dots, \\
N_{m4}(\xi) &= (1 - \xi)^3 + \frac{3!2(m\pi)^2}{5!\phi^2} (1 - \xi)^5 - \frac{3!(4c^* + 3g^*)}{6!\phi^2} (1 - \xi)^6 + \dots \quad (4.39)
\end{aligned}$$

By applying boundary conditions (4.31), the constants of integration b_{m1} , b_{m2} , b_{m3} and b_{m4} are determined as,

$$b_{m1} = 0 ,$$

$$b_{m3} = 0 ,$$

$$b_{m4} = -\frac{N_{m2}(0, \lambda_{mi})}{N_{m4}(0, \lambda_{mi})} b_{m2} . \quad (4.40)$$

Using (4.40) in (4.38) together with (4.34), one obtains the adjoint eigenfunction corresponding to eigenvalue λ_{mi} as

$$\psi_{mi}(\xi, \eta) = b_{m2} \left[N_{m2}(\xi, \lambda_{mi}) - \frac{N_{m2}(0, \lambda_{mi})}{N_{m4}(0, \lambda_{mi})} N_{m4}(\xi, \lambda_{mi}) \right] \sin m\pi\eta , \quad (4.41)$$

$$m, i = 1, 2, 3, \dots, \infty .$$

4.5 Solution to the Control Problem

Following a procedure similar to that in Section 3.5, $V_{ij}(\xi, \eta)$ and $\psi_{ij}(\xi, \eta)$ can be normalized by setting

$$V_{ij}^n(\xi, \eta) = \frac{1}{\sqrt{c_{ij}}} V_{ij}(\xi, \eta) \quad (4.42)$$

$$\psi_{ij}^n(\xi, \eta) = \frac{1}{\sqrt{c_{ij}}} \psi_{ij}(\xi, \eta) \quad (4.43)$$

where

$$c_{ij} = \int_0^1 \int_0^1 V_{ij}(\xi, \eta) \psi_{ij}(\xi, \eta) d\xi d\eta \neq 0 \quad (4.44)$$

Then the Green's Function of the control problem is the unsymmetric function

$$\begin{aligned} G(\xi, \gamma, \eta, \sigma, t, \tau) &= \sum_{i,j=1}^{\infty} \frac{V_{ij}^n(\xi, \eta) \psi_{ij}^n(\gamma, \sigma)}{\mu \omega_{ij}^2} \cos[\omega_{ij}(t - \tau)] \\ &= \sum_{i,j=1}^{\infty} \frac{V_{ij}(\xi, \eta) \psi_{ij}(\gamma, \sigma)}{\mu c_{ij} \omega_{ij}^2} \cos[\omega_{ij}(t - \tau)], \end{aligned} \quad (4.45)$$

where $V_{ij}(\xi, \eta)$ and $\psi_{ij}(\gamma, \sigma)$ are expressed by equations (4.27) and (4.41), respectively, and ω_{ij} according to (4.12) is

$$\omega_{ij} = \sqrt{\frac{D}{\mu b^4}} \lambda_{ij}.$$

Using (4.45), the solution to the control problem (4.6) is obtained by means of equation (2.120).

4.6 A Numerical Example

A numerical example is presented for a S-S-S-S plate. The following data were used:

$$\begin{aligned} l = b = 1.25 (m), \quad h = 0.0125 (m), \quad \rho = 7870 (kg/m^3), \\ \nu = 0.25, \quad E = 200,000 (Mpa) \end{aligned}$$

The driving force and the unidirectional load are assumed to be

$$K(x, y, t) = 10,000 \cos 300t (N/m^2), \quad (4.46)$$

$$g = g^* \frac{D}{l^3} = 30 \frac{D}{l^3} \quad i.e., \quad g^* = 30. \quad (4.47)$$

In order to assess the efficiency of the control, not the deflection itself but norm, which is defined as

$$J_W(\xi, \eta, t) = \int_0^t W^2(\xi, \eta, \tau) d\tau \quad (4.48)$$

has been determined [33]. This norm reflects the severity of vibrational displacement of any point on the plate throughout time t . For a given point on the plate, after a given time t , the larger the magnitude of J_W is the larger the amplitude of its vibration will be. Since in our case the maximum vibration magnitude always occurs at some point along $\eta = 0.5$, the norm $J_W(\xi, 0.5, t)$ are presented in Fig.(4.3) for $t = 3$ and 5 seconds. Moreover Fig. (4.4) shows a plot of maximum values of the norm along $\eta = 0.5$ as a function of control parameter c^* for $t = 3, 4$ and 5 seconds. These graphs indicate that for a given g^* , a larger value of c^* will translate into more effective control. One may hastily conclude that to improve the effect of control, one will need to keep increasing the value of c^* . However, in doing so, one would make a mistake since larger values of c^* may impair the stability of the control system.

In fact, the stability of a closed loop control system is of vital importance in the design of the system. This aspect of the problem will be thoroughly dealt with in the subsequent section.

4.7 Domain of Regularity and Stability Boundary

The applicability of Green's function approach to non-selfadjoint control of plates in the presence of unidirectional forces is based on the fact that one is working in the domain of regularity of the non-selfadjoint problem within which properties // of the eigenvalues and eigenfunctions of the generating problem hold. Therefore, to determine this domain of regularity is crucial to the analysis of the problem. Considering what has been outlined in Section (2.3.1) about the domain of regularity and the behavior of the eigenvalue curve branches of the generating problem in this domain, it is clear that this domain is the stability domain of the problem and its boundary will be the stability boundary of the system. Therefore, the stable behavior of the system is of vital importance in designing a closed loop control system. Basically there are two forms of structural instability: divergence (static instability) and flutter (oscillatory instability). These two forms of instability differ in the behavior of the disturbance: in the first case the disturbance increase monotonically with time while in the second case the increase in disturbance is of oscillatory nature.

4.7.1 Liapunov's First method

Two methods for investigating stability were proposed by A. M. Liapunov: Liapunov's first method and Liapunov's second method [29]. The first method is also called the indirect method. In applying this method not only the differential equation is used to predict stability, but the solution of the differential equation or at least a characteristic quantity involved in the solution is used also in the stability investigation. Considering a control system with multiple load-control parameters as an example, when all the load-control parameters are zero, the system is usually in the stable

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state. In order to trace the effect of these parameters, one may use the characteristic equation of the system from which the eigenvalues of the system (which are functions of these parameters) can be obtained. Considering an arbitrary path emanating from the origin of the parameter space, the initially stable system may lose stability by divergence or flutter depending on whether one eigenvalue ω_i^2 (ω_i is the natural frequency of the system) vanishes or two eigenvalues ω_i^2 and ω_j^2 coincide at a critical point (with coincident eigenfunctions). Upon passing through such a critical point these two eigenvalues become complex conjugates. Thus if the characteristic equation is assumed to be

$$F(\omega_{ij}^2, \vec{q}) = 0 , \quad (4.49)$$

then in the parameter space, those points corresponding to onset of divergent instability satisfy the condition

$$\omega_{ij}^2(\vec{q}) = 0 , \quad (4.50)$$

and those points corresponding to the onset of flutter instability satisfy the condition

$$\frac{dq_i}{d\omega_{ij}^2(q_i)} = 0 , \quad (4.51)$$

which is equivalent to

$$\frac{\partial F(\omega_{ij}^2, \vec{q})}{\partial \omega_{ij}^2} = 0 , \quad (4.52)$$

because (4.49) yields

$$\frac{\partial F(\omega_{ij}^2, \vec{q})}{\partial \omega_{ij}^2} + \frac{\partial F(\omega_{ij}^2, \vec{q})}{\partial \vec{q}} \frac{\partial \vec{q}}{\partial \omega_{ij}^2(\vec{q})} = 0 , \quad (4.53)$$

using (4.51) in (4.53) results in (4.52).

The locus of points in the parameter space corresponding to zero eigenvalues (onset of divergence) is called the divergence boundary while the points corresponding to coincident eigenvalues (with coincident eigenfunctions) comprise the flutter boundary. The points on these two boundaries which are associated with an initial loss of stability form the stability boundary. Thus if one has fully understood the eigenvalue system of a control problem, using Liapunov's first method of stability investigation mentioned above, theoretically, one may obtain the so-called "exact" stability boundary for the control problem.

continuous dynamic system with infinite degrees of freedom possesses infinite number of eigenvalues. Although one may obtain the exact characteristic equation (for example the characteristic equation (4.24) is an "exact" one in the sense of infinite series) so that one may obtain as accurate and as many eigenvalues as one desires, practically, since most characteristic equations are transcendental equations, one can only obtain the first limited number of eigenvalues by solving the transcendental eigenvalue equation numerically. Thus when applying Liapunov's first method to the problem at hand one inevitably considers the first limited number of eigenvalues and modes to determine the stability of the system. By doing so, one assumes that higher modes other than those obtained will not upset stability. For a selfadjoint system, this assumption holds which is secured by the classical extremum properties of a selfadjoint (conservative) system [34]. But when one is dealing with a non-selfadjoint system, this assumption is not guaranteed any more [34]. The spillover effect, i.e., possible detrimental effect on the stability of the actual infinite dimensional system due to exclusion of higher modes, has been reported in several references [35, 36]. In view of this effect, the above mentioned approach and the resulting stability boundary for a control problem may be questionable.

However, from a practical point of view, Liapunov's first method has been proven to be a very powerful method and has been used extensively not only for selfadjoint systems but also for non-selfadjoint ones. Because, even for non-selfadjoint systems, one may argue that dangerous modes may be of such a high order that they will possibly not be excited thus causing no difficulties. Also such an approach can reveal

some important features of the problem, e.g. type of instability, and the boundary obtained using this method may be considered as an "upper bound" to the actual stability boundary. In contrast to a "lower bound" [29], here "upper bound" is used in the sense of approximating and approaching the actual stability boundary from above, i.e., from unstable region.

Liapunov's first method was used for investigating the stability boundary of the plate control problem (4.2–4.4). Using characteristic equation (4.24) the first six eigenvalues of the system are obtained. By observing the variation of those eigenvalues with the change of parameters g^* and c^* , one may obtain the first critical values of g^* and c^* corresponding to various possible divergence and flutter instabilities. A plot of these critical values is shown in Fig. (4.5) which indicates that the system becomes instable always by divergence and no flutter instability will occur.

4.7.2 Liapunov's Second Method

In using this method one may seek global statements on the properties of any possible solution and therefore on the infinite number of modes of the continuous system by employing appropriate functionals involving the hypothetical solution of the problem and its derivatives. Hence, any danger of spillover possibility can be avoided.

More specifically, Liapunov's second method in stability investigation can be described as follows: a functional M , the Liapunov functional is to be chosen and appropriate conditions are to be imposed on it so that the functional itself is positive definite and the time derivative of M is not positive. If a functional of that kind is found, then it can be concluded that stability of the system is secured. The boundary obtained using Liapunov's second method gives a lower bound to the actual stability boundary.

Let the stability of the control plate problem be investigated using Liapunov's second method.

Consider the homogeneous equation of the control plate problem

$$\begin{aligned} \mu \ddot{W}(x, y, t) + D \nabla^4 W(x, y, t) \\ + (c + g) \frac{\partial}{\partial x} \left[(l - x) \frac{\partial W(x, y, t)}{\partial x} \right] + c \frac{\partial W(x, y, t)}{\partial x} = 0, \end{aligned} \quad (4.54)$$

with the associated boundary conditions (4.2) and (4.3).

The homogeneous equation of the corresponding plate with the same boundary conditions but subjected only to conservative unidirectional forces of density g reads,

$$\mu \ddot{U}(x, y, t) + D \nabla^4 U(x, y, t) + g \frac{\partial}{\partial x} \left[(l - x) \frac{\partial U(x, y, t)}{\partial x} \right] = 0. \quad (4.55)$$

This is a conservative system and therefore the static stability criterion and Rayleigh's inequality may be applied. Let g_{crit} be the buckling load of the plate subjected to conservative load and u a comparison function, then

$$g_{crit} \leq R = D \int_0^l \int_0^b u \nabla^4 u dx dy / \left(- \int_0^l \int_0^b u \frac{\partial}{\partial x} \left[(l - x) \frac{\partial u}{\partial x} \right] dx dy \right). \quad (4.56)$$

Using W (the deflection of the control system) as the comparison function u in (4.56), one gets

$$g_{crit} \leq D \int_0^l \int_0^b W \nabla^4 W dx dy / \left(- \int_0^l \int_0^b W \frac{\partial}{\partial x} \left[(l - x) \frac{\partial W}{\partial x} \right] dx dy \right). \quad (4.57)$$

Applying Gauss's integral theorem

$$\begin{aligned} & - \int_0^l \int_0^b W \frac{\partial}{\partial x} \left[(l - x) \frac{\partial W}{\partial x} \right] dx dy \\ & = \int_0^l \int_0^b (l - x) \left(\frac{\partial W}{\partial x} \right)^2 dx dy - \int_R W (l - x) \frac{\partial W}{\partial x} \cos \alpha dR \end{aligned} \quad (4.58)$$

where R is the contour of the plate and α is the angle made by the normal to the contour with the x direction. Since for simply supported plate $W \equiv 0$ along R , one has,

$$-\int_0^l \int_0^b W \frac{\partial}{\partial x} \left[(l-x) \frac{\partial W}{\partial x} \right] dx dy = \int_0^l \int_0^b (l-x) \left(\frac{\partial W}{\partial x} \right)^2 dx dy . \quad (4.59)$$

Moreover according to Green's formula,

$$\begin{aligned} & \int_0^l \int_0^b W \nabla^4 W dx dy \\ &= \int_0^l \int_0^b (\Delta W)^2 dx dy + \int_R \left(W \frac{\partial \Delta W}{\partial n} - \frac{\partial W}{\partial n} \Delta W \right) dR , \end{aligned} \quad (4.60)$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} . \quad (4.61)$$

Again, since for simply supported plate $W \equiv 0$ and $\Delta W \equiv 0$ along contour R , one has,

$$\int_0^l \int_0^b W \nabla^4 W dx dy = \int_0^l \int_0^b (\Delta W)^2 dx dy . \quad (4.62)$$

Using (4.59) and (4.62) in (4.57) yields

$$D \int_0^l \int_0^b (\Delta W)^2 dx dy \geq g_{crit} \int_0^l \int_0^b (l-x) \left(\frac{\partial W}{\partial x} \right)^2 dx dy . \quad (4.63)$$

This inequality will be used later.

Next, let

$$M = H + \frac{1}{2} \int_0^l \int_0^b c \frac{\partial W}{\partial x} W \, dx \, dy \quad (4.64)$$

be used as a Liapunov functional, where

$$H = \frac{1}{2} \int_0^l \int_0^b \left[\mu \dot{W}^2 + D(\Delta W)^2 - (c + g)(l - x) \left(\frac{\partial W}{\partial x} \right)^2 \right] dx \, dy \quad (4.65)$$

is the Hamiltonian of the plate.

Differentiating with respect to time, (4.64) yields:

$$\dot{M} = \dot{H} + \frac{1}{2} \int_0^l \int_0^b \left(c \frac{\partial \dot{W}}{\partial x} W + c \frac{\partial W}{\partial x} \dot{W} \right) dx \, dy . \quad (4.66)$$

Using the homogeneous equation (4.54), it can be shown that

$$\dot{H} = - \int_0^l \int_0^b c \frac{\partial W}{\partial x} \dot{W} \, dx \, dy . \quad (4.67)$$

However, recalling that the natural modes W have the form

$$W(x, y, t) = \Phi(x, y) f(t) , \quad (4.68)$$

the second term on the right hand side of equation (4.66) becomes

$$\frac{1}{2} \int_0^l \int_0^b \left(c \frac{\partial \dot{W}}{\partial x} W + c \frac{\partial W}{\partial x} \dot{W} \right) dx \, dy$$

$$\begin{aligned}
&= \frac{1}{2} \int_0^t \int_0^b c \dot{f} f \left(\frac{\partial \Phi}{\partial x} \Phi + \frac{\partial \Phi}{\partial x} \Phi \right) dx dy \\
&= \int_0^t \int_0^b c \frac{\partial(\Phi f)}{\partial x} (\Phi \dot{f}) dx dy \\
&= \int_0^t \int_0^b c \frac{\partial W}{\partial x} \dot{W} dx dy . \tag{4.69}
\end{aligned}$$

Using (4.67) and (4.69) in (4.66) yields

$$\dot{M} = 0 \quad \text{thus} \quad M = \text{const.}$$

Hence, one is justified in using M as a Liapunov functional as long as it's positive definiteness is guaranteed. To this end, let the second term of M on the right-hand side of (4.64) be considered first. Applying Gauss's integral theorem yields that

$$\begin{aligned}
&\frac{1}{2} \int_0^t \int_0^b c \frac{\partial W}{\partial x} W dx dy \\
&= -\frac{1}{2} \int_0^t \int_0^b c W \frac{\partial W}{\partial x} dx dy + \frac{1}{2} \int_R c W^2 \cos \alpha dc . \tag{4.70}
\end{aligned}$$

Once again, since $W \equiv 0$ along contour R , one has

$$\frac{1}{2} \int_R c W^2 \cos \alpha dc = 0 , \tag{4.71}$$

so that from (4.70) it follows that

$$\frac{1}{2} \int_0^t \int_0^b c \frac{\partial W}{\partial x} W dx dy = 0 . \tag{4.72}$$

Using (4.65) and (4.72) in (4.64) yields

$$\begin{aligned}
 M &= \frac{1}{2} \int_0^l \int_0^b \left[\mu \dot{W}^2 + D(\Delta W)^2 - (c + g)(l - x) \left(\frac{\partial W}{\partial x} \right)^2 \right] dx dy \\
 &\geq \frac{1}{2} \int_0^l \int_0^b \left[D(\Delta W)^2 - (c + g)(l - x) \left(\frac{\partial W}{\partial x} \right)^2 \right] dx dy \\
 &\geq \frac{1}{2} \int_0^l \int_0^b \left[g_{crit}(l - x) \left(\frac{\partial W}{\partial x} \right)^2 - (c + g)(l - x) \left(\frac{\partial W}{\partial x} \right)^2 \right] dx dy \\
 &= \frac{1}{2} \int_0^l \int_0^b [g_{crit} - (c + g)] (l - x) \left(\frac{\partial W}{\partial x} \right)^2 dx dy , \tag{4.73}
 \end{aligned}$$

where the last inequality is justified due to (4.63).

The right-hand side of (4.73) is positive definite if one has

$$c + g \leq g_{crit} . \tag{4.74}$$

It is well known in the literature that for a square plate subjected to only load g the buckling load

$$g_{crit} = g_{crit}^* \frac{D}{l^3} = 63.78 \frac{D}{l^3} . \tag{4.75}$$

By using the non-dimensional quantities

$$c^* = \frac{l^3}{D} c \quad \text{and} \quad g^* = \frac{l^3}{D} g . \tag{4.76}$$

one obtains the non-dimensional stability lower bound for the control problem as:

$$c^* + g^* \leq 63.78 \tag{4.77}$$

The stability lower bound (4.77) is plotted in (c^*, g^*) -space, Fig. (4.6). The results obtained using Liapunov's first method are also plotted as dashed line in the same figure.

Leipholtz remarked that Liapunov's second method has two disadvantages. Firstly, Liapunov functional may not be found easily. Secondly, this method is of a very general nature and its results are frequently so general that these results may not be of very great use in practical applications [29].

It seems that the approach used here overcomes these two major disadvantages. One may note that two boundaries in Fig.(4.6) are very close to each other, i.e. both upper bound obtained from Liapunov's first method and lower bound from Liapunov's second method are very close to the actual stability boundary. Since the uncertain region in between the upper bound and the lower bound is so thin, one may conclude that the upper bound obtained using Liapunov's first method (dashed line in Fig. 4.6) is indeed the exact stability boundary.

Chapter 5

Conclusions and Recommendations

5.1 Conclusions

- Green's function approach can be extended to deal with two dimensional selfadjoint as well as non-selfadjoint multiple parameters continuous linear systems. It seems that following similar methodology presented in this work, Green's function approach may be generalized to deal with any kind of continuous selfadjoint and non-selfadjoint dynamic system as long as such a system is a linear elastic one.
- For a two dimensional non-selfadjoint system, it is shown that as long as the parameters of the system are restricted to the domain $\mathcal{PR}$ of the system, then the eigenvalues of the system will all be real and positive and the unsymmetric Green's function of the system can be obtained which is represented as an infinite series of the product of eigenfunctions and adjoint eigenfunctions of the system.
- One of the advantages of using Green's function approach to solve non-selfadjoint problems is the generality of the form of solution such that once the Green's function of the system is obtained, the dynamic response of the system to any form of external perturbation and initial conditions which abide by a certain behavior can easily be obtained in the form of an integral equation solution with the Green's function as the kernel of the equation.

- Under follower forces, plates with four edges simply supported (S-S-S-S) and plates with two opposite edges simply supported and another two edges free (S-F-S-F) represent two different kinds of stability systems. A plate with S-F-S-F boundary always loses stability by flutter at the first critical load regardless of its aspect ratio and Poisson ratio while a S-S-S-S plate always loses stability by divergence at the first critical load for any aspect ratio.
- While the Green's function approach does not necessarily predict the best distributed control law for a continuous system, it does have the capability of evaluating the effectiveness of both selfadjoint and non-selfadjoint controls imposed on a continuous system as an active feedback control law.
- For a continuous system with non-selfadjoint distributed control, applying traditional discretization technique and Liapunov's first method in investigating the system's stability behavior may exhibit spillover effect. The results obtained this way provide a stability upper bound for the system beyond which the system is definitely unstable. On the other hand by applying Liapunov's second method, one may determine the stability condition for a system without solving explicitly the system's governing differential equation. Since no specific form for the solution is assumed when working with Liapunov's second method, one does not know whether the system will fail by divergence or by flutter. However one does avoid spillover effect this way and one obtains a lower bound for the system below which the stability of the system is secure. By choosing a proper Liapunov functional, one may obtain a lower bound for stability which is accurate enough for practical purposes.

5.2 Recommendations

This research has raised a number of new questions for further studies in order to refine the research in this area:

- Plates considered in this work always possess at least two opposite simply supported edges. Thus the eigenfunction solutions to such problems are Levy type

solutions. Plates without two opposite simply supported edges should be considered and the method of superposition [37] may be adopted for solving such problems. Among various possible plate problems, the cantilever plate under follower forces may receive great attention because of its practical importance, particularly for simulating lifting and stabilizing surfaces in the aerospace industry.

- In the context of specific applications, further studies of non-selfadjoint plate systems may include other practical factors such as damping, non-uniformly distributed follower force, non-uniformly distributed mass, non-uniform flexural rigidity and various non-selfadjoint distributed control laws, etc.
- In this work, in order to avoid the spillover effect, Liapunov's second method was used to determine the stability lower bound for a distributed control of a plate. By comparing this lower bound with the upper bound obtained from Liapunov's first method, one can conclude with confidence, that the lower bound obtained is a good approximation to the exact stability boundary. However, one should realize that Liapunov's functional is not necessarily unique. One may have to follow a "trial and error" approach, and based on one's experience to obtain a proper Liapunov functional which would lead to a lower bound that would approximate the exact stability boundary with good accuracy for the control system. In fact, K. Huseyin in [38] has proposed and initiated the research to the estimation of stability boundary for a non-selfadjoint system in the context of matrix analysis by introducing the concept of "corresponding conservative system". An interesting and challenging research would be to try to find the corresponding approach in the context of functional analysis for the stability boundary approximation for the distributed control of continuous systems which could eliminate the spillover effect and offer an easy method for the application engineers.
- By now, the theoretical analysis of a good number of non-selfadjoint problems, especially those involving follower type forces are substantial. However, experimental verification of these results seem to be quite rare. The main difficulty in

such experimental investigations seem to be a monitored implementation and measurement of follower forces. Recently, with the rapid development in the application of piezoelectric material [39, 40], most of these experimental difficulties might be overcome by carefully embedding piezoelectric film or layer in the system. The development of such a technology will also open new possibility in the field of active control of structures.

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Figure 2.1 $(\mu\omega^2, q_1, q_2)$ -Space for a Multiple Parameter Selfadjoint System

Figure 2.2 $(\mu\omega^2, q_1, q_2)$ -Space for a Multiple Parameter Non-selfadjoint System

Figure 3.1 A S-F-S-F Plate with Uniformly Distributed Tangential Follower Force

Figure 3.2 Non-dimensional Eigenvalues versus Non-dimensional Follower Forces for S-F-S-F Plates ($\phi=1$, $\nu=0.3$)

$$q^* = 0$$

$$q^* = 0.5 q_{crk1}^*$$

(a) First Mode $V_{11}(\xi, \eta)$

**Figure 3.3 Two-dimensional Mode Shapes $V_{ij}(\xi, \eta)$ for
S-F-S-F Plates ($\phi=1$, $\nu=0.3$)**

 $q^* = 0$

 $q^* = 0.5 q_{crit1}^*$

(b) Second Mode $V_{12}(\xi, \eta)$

Figure 3.3 (Continued)

 $q^* = 0$

 $q^* = 0.5 q^*_{crit1}$

(c) Third Mode $V_{13}(\xi, \eta)$

Figure 3.3 (Continued)

 $q^* = 0$

 $q^* = 0.5 q_{crit}^*$

(d) Fourth Mode $V_{21}(\xi, \eta)$

Figure 3.3 (Continued)

$$q^* = 0$$

$$q^* = 0.5 q^*_{crit}$$

(e) Fifth Mode $V_{22}(\xi, \eta)$

Figure 3.3 (Continued)

(a) First Mode $V_{11}(\xi, \eta)$ at $\eta=0.5$

**Figure 3.4 First Five Modes at Appropriate η for
S-F-S-F Plates ($\phi=1, \nu=0.3$)**

(b) Second Mode $V_{12}(\xi, \eta)$ at $\eta=0.5$

Figure 3.4 (Continued)

(c) Third Mode $V_{13}(\xi, \eta)$ at $\eta=0.5$

Figure 3.4 (Continued)

(d) Fourth Mode $V_{21}(\xi, \eta)$ at $\eta=0.25$

Figure 3.4 (Continued)

(e) Fifth Mode $V_{22}(\xi, \eta)$ at $\eta=0.25$

Figure 3.4 (Continued)

Figure 3.5 Non-dimensional Eigenvalues versus Non-dimensional Follower Forces for Square S-S-S-S Plates

$$q^* = 0$$

$$q^* = 0.5 q_{crit1}^*$$

(a) First Mode $V_{11}(\xi, \eta)$

Figure 3.6 Two-dimensional Mode Shapes $V_{ij}(\xi, \eta)$ for Square S-S-S-S Plates

$$q^* = 0$$

$$q^* = 0.5 q_{crit1}^*$$

(b) Second Mode $V_{12}(\xi, \eta)$

Figure 3.6 (Continued)

 $q^* = 0$

 $q^* = 0.5 q_{crit}^*$

(c) Third Mode $V_{21}(\xi, \eta)$

Figure 3.6 (Continued)

$$q^* = 0$$

$$q^* = 0.5 q_{crit1}^*$$

(d) Fourth Mode $V_{22}(\xi, \eta)$

Figure 3.6 (Continued)

$$q^* = 0$$

$$q^* = 0.5 q_{crit1}^*$$

(e) Fifth Mode $V_{13}(\xi, \eta)$

Figure 3.6 (Continued)

(a) First Mode $V_{11}(\xi, \eta)$ at $\eta=0.5$

Figure 3.7 First Five Modes at Appropriate η for Square S-S-S-S Plates

(b) Second Mode $V_{12}(\xi, \eta)$ at $\eta=0.5$

Figure 3.7 (Continued)

(c) Third Mode $V_{21}(\xi, \eta)$ at $\eta=0.25$

Figure 3.7 (Continued)

(d) Fourth Mode $V_{22}(\xi, \eta)$ at $\eta=0.25$

Figure 3.7 (Continued)

(e) Fifth Mode $V_{13}(\xi, \eta)$ at $\eta=0.5$

Figure 3.7 (Continued)

Figure 3.8 Critical Flutter Load versus Aspect Ratio of
S-F-S-F Plates with $\nu = 0.3$

Figure 3.9 Critical Divergent Load of S-S-S-S Plates versus Aspect Ratio

**Figure 3.10 Total Critical Divergent Load of
S-S-S-S Plates versus Aspect Ratio**

(a) $q^* = 0$

Figure 3.11 First Five Non-dimensional Eigenvalues versus Poisson Ratio for Square S-F-S-F Plates

(b) $q^* = 10$

Figure 3.11 (Continued)

(c) $q^* = 20$

Figure 3.11 (Continued)

(d) $q^* = 50$

Figure 3.11 (Continued)

**Figure 3.12 Critical Flutter Load of Square S-F-S-F Plates
versus Poisson Ratio**

(a) $q^* = 0$

Figure 3.13 Deflection of a S-F-S-F Plate ($\phi=1$, $\nu=0.32$) Subjected to Harmonic Driving Force

(b) $q^* = 0.5 q_{crit1}^*$

Figure 3.13 (Continued)

(a) $q^* = 0$

Figure 3.14 Response of a S-F-S-F Plate ($\phi=1$, $\nu=0.32$) at Point (0, 0.5) Subjected to Harmonic Driving Force

(b) $q^* = 0.5 q_{crit1}^*$

Figure 3.14 (Continued)

Figure 4.1 A S-S-S-S Plate with Uniformly Distributed Unidirectional Force

Figure 4.2 A S-S-S-S Plate with Unidirectional Force under Control

(a) $t = 3$ seconds

**Figure 4.3 Norms of Response of a Controlled Square S-S-S-S Plate
along $\eta = 0.5$**

(b) $t = 5$ seconds

Figure 4.3 (Continued)

Figure 4.4 Maximum Norms of Response of a Controlled Square S-S-S-S Plate versus c^*

Figure 4.5 Upper Bound of a Controlled Square S-S-S-S Plate Obtained by Liapunov's First Method

**Figure 4.6 Comparison of Lower and Upper Bounds of
a Controlled Square S-S-S-S Plate**

Table 3.1 : Non-dimensional Eigenvalues λ_{ij} for S-F-S-F Plates

q^*/q_{crit1}^* ($q_{crit1}^* = 91.55$)	$\lambda_{ij} = \omega_{ij} b^2 \sqrt{\mu/D} \text{ (} \phi = 1.0, \nu = 0.3 \text{)}$				
	λ_{11}	λ_{12}	λ_{13}	λ_{21}	λ_{22}
0.00	9.631	16.135	36.726	38.945	46.738
0.04	9.780	16.154	36.378	39.031	46.780
0.08	9.922	16.171	36.026	39.115	46.819
0.12	10.058	16.182	35.671	39.196	46.857
0.16	10.189	16.190	35.311	39.274	46.893
0.20	10.314	16.192	34.947	39.351	46.927
0.24	10.434	16.189	34.580	39.425	46.959
0.28	10.550	16.180	34.208	39.496	46.988
0.32	10.661	16.163	33.832	39.566	47.015
0.36	10.768	16.139	33.453	39.633	47.039
0.40	10.871	16.107	33.069	39.698	47.061
0.44	10.970	16.065	32.681	39.761	47.079
0.48	11.066	16.013	32.290	39.823	47.095
0.52	11.159	15.949	31.895	39.882	47.107
0.56	11.249	15.872	31.496	39.939	47.116
0.60	11.337	15.780	31.094	39.995	47.121
0.64	11.423	15.672	30.688	40.049	47.123
0.68	11.508	15.545	30.280	40.101	47.121
0.72	11.593	15.397	29.870	40.152	47.115
0.76	11.678	15.224	29.457	40.201	47.105
0.80	11.766	15.023	29.042	40.248	47.090
0.84	11.860	14.788	28.627	40.294	47.071
0.88	11.965	14.511	28.210	40.338	47.047
0.92	12.092	14.177	27.794	40.381	47.017
0.96	12.270	13.753	27.379	40.423	46.983

Table 3.2 : Non-dimensional Eigenvalues λ_{ij} for S-S-S-S Plates

q^*/q_{crit1} ($q_{crit1} = 67.42$)	$\lambda_{ij} = \omega_{ij} b^2 \sqrt{\mu/D} \text{ (} \phi = 1.0, \nu = 0.3 \text{)}$				
	λ_{11}	λ_{12}	λ_{21}	λ_{22}	λ_{13}
0.00	19.739	49.348	49.348	78.957	98.696
0.04	19.398	48.805	49.213	78.619	98.087
0.08	19.048	48.254	49.077	78.278	97.473
0.12	18.689	47.694	48.940	77.935	96.855
0.16	18.320	47.126	48.802	77.589	96.231
0.20	17.940	46.549	48.663	77.241	95.603
0.24	17.549	45.963	48.523	76.891	94.969
0.28	17.144	45.368	48.382	76.537	94.331
0.32	16.726	44.762	48.240	76.182	93.687
0.36	16.292	44.146	48.097	75.823	93.038
0.40	15.841	43.519	47.952	75.462	92.383
0.44	15.371	42.881	47.807	75.097	91.723
0.48	14.880	42.232	47.661	74.730	91.057
0.52	14.365	41.570	47.513	74.360	90.386
0.56	13.823	40.895	47.364	73.987	89.708
0.60	13.250	40.207	47.213	73.611	89.025
0.64	12.639	39.506	47.062	73.232	88.336
0.68	11.986	38.790	46.909	72.849	87.640
0.72	11.281	38.058	46.754	72.463	86.938
0.76	10.511	37.311	46.598	72.074	86.229
0.80	9.661	36.548	46.440	71.682	85.514
0.84	8.703	35.768	46.281	71.286	84.793
0.88	7.594	34.969	46.120	70.886	84.064
0.92	6.251	34.153	45.957	70.483	83.328
0.96	4.458	33.317	45.792	70.076	82.586

**Table 3.3 : Plate Deflections after One Second of Motion
Using First N modes**

Perturbation : $K(\xi, \eta, t) = 10,000 \cos 6.28t$

(a) $q^*/q_{crit1} = 0.0$

ξ	Deflection W(N) (mm)		
	(N=1)	(N=3)	(N=5)
0.00	2.48422	2.47065	2.47065
0.05	2.39884	2.38842	2.38842
0.10	2.32963	2.32232	2.32232
0.15	2.27379	2.26956	2.26956
0.20	2.22909	2.22786	2.22786
0.25	2.19380	2.19538	2.19538
0.30	2.16657	2.17066	2.17066
0.35	2.14636	2.15255	2.15255
0.40	2.13243	2.14022	2.14022
0.45	2.12428	2.13305	2.13305
0.50	2.12159	2.13069	2.13069
0.55	2.12428	2.13305	2.13305
0.60	2.13243	2.14022	2.14022
0.65	2.14636	2.15255	2.15255
0.70	2.16657	2.17066	2.17066
0.75	2.19380	2.19538	2.19538
0.80	2.22909	2.22786	2.22786
0.85	2.27379	2.26956	2.26956
0.90	2.32963	2.32232	2.32232
0.95	2.39884	2.38842	2.38842
1.00	2.48422	2.47065	2.47065

Table 3.3 (Continued)(b) $q^*/q_{crit}^* = 0.50$

ξ	Deflection $W(N)$ (mm)		
	(N=1)	(N=3)	(N=5)
0.00	1.18842	1.41436	1.41436
0.05	1.19283	1.39788	1.39788
0.10	1.20639	1.39148	1.39148
0.15	1.22857	1.39388	1.39388
0.20	1.25858	1.40368	1.40368
0.25	1.29547	1.41945	1.41945
0.30	1.33820	1.43990	1.43990
0.35	1.38577	1.46385	1.46385
0.40	1.43722	1.49039	1.49039
0.45	1.49177	1.51883	1.51883
0.50	1.54880	1.54877	1.54877
0.55	1.60792	1.58010	1.58010
0.60	1.66900	1.61296	1.61296
0.65	1.73215	1.64776	1.64776
0.70	1.79783	1.68517	1.68517
0.75	1.86677	1.72609	1.72609
0.80	1.94009	1.77171	1.77171
0.85	2.01932	1.82348	1.82348
0.90	2.10650	1.88318	1.88318
0.95	2.20424	1.95298	1.95298
1.00	2.31598	2.03557	2.03557

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