On Some Contributions to Measurement Error Models

by

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in partial fulfillment of
the requirements for the degree of

Doctor of Philosophy

School of Mathematics and Statistics
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**On Some Contributions to Measurement Error
Models**

Submitted by

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in partial fulfillment of the requirements
for the degree of Doctor of Philosophy.

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Abstract

The theory of preliminary test and Stein-type estimation is quite well known for the last five decades or so, but its application remains still limited. This thesis consists of some contributions to the area of measurement error model (ME model) based on the preliminary test approach to estimation which improves on the traditional estimators in ME models.

The main objective of this thesis is to propose and study the properties of the improved estimators of the parameters of interest in various standard statistical models with measurement error.

Chapter two contains the study of the properties of preliminary test and shrinkage estimators of the parameters in a simple linear model with measurement errors when it is suspected that the slope parameter may assume pre specified values with some degree of uncertainty.

In Chapter three, we consider the multiple regression model with measurement errors. Here, we propose the preliminary test estimator and the Stein-type estimators. The expressions of MSE matrix and risk of estimator based on LSE and an unbiased estimator are obtained and studied in terms of the MSE matrix and weighted quadratic risk functions in finite samples assuming reliability matrix to be known.

Chapter four contains the study of preliminary test and Stein-type estimators when the reliability matrix is unknown, but measurement error matrix is known. Asymptotic distributional analysis of these estimators are given under a class of

local alternatives. This method of analysis is new in the area of ME models.

Chapter five contains the study of several simple linear models with measurement error and provide asymptotic properties of the improved estimators of the slope parameters when it is suspected but not sure that parallelism hypothesis may hold. Asymptotic distributional comparisons of the estimators are made under the Pitman local alternative.

Finally, in Chapter six , the results of Chapter 2 and 3 are applied to the prediction of population totals in survey sampling.

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Chapter 1

Introduction

1.0 Introduction

This thesis deals with the estimation of parameters in various standard statistical models with measurement errors when it is a priori suspected that the parameters of interest may assume certain values with some degree of uncertainty. Although the study of measurement error models has a long history since Adcock (1877,1878) and Kummel (1879), no such study has yet been made when uncertain prior information on the parameter of interest existed in given measurement error models although the Bayesian method exists. The object of this thesis is the study of the properties of estimators of the parameters of interest in measurement error models with the uncertain prior information, beginning with already existing properties without the uncertain prior information.

There is a large number of scholarly papers in the literature, as summarized in Kendall and Stuart (1979) with more details given in the monograph by Fuller (1987) and Cheng and Van Ness (1999) containing update progresses in this area.

This thesis contains the study of estimation of parameters in linear models with measurement errors when uncertain prior information on the parameter of interest is available. In data analysis, it is usual to explore the relation between covariates and the response variables via regression analysis. In many situation, the direct observations of the covariates are not available. Instead, due to measuring mechanism or the nature of environment, the covariates are measured with error and then one desires to assess the relationship between the response variables and the covariates based on the observed responses and covariates. These problems arise in many medical, epidemiology, environmental studies where risk factors are partially observed. Details of many examples are given in the monographs by Carroll, Ruppert and Stefanski (1995).

In problems of statistical inference, use of prior information on the parameters of interest in a statistical models usually leads to improved inference procedure. However, when the prior information on the parameters of interests is **uncertain**, then question arises as to how one can incorporate the uncertain prior information into the inference procedure. However, Bancroft (1944) came up with the idea of the testing the uncertain prior information then inference procedure is developed on the result of this preliminary test. For example, if we have two samples $\{(X_{i1}, \dots, X_{in_i}) | i = 1, 2\}$ from two normal distributions $\mathcal{N}(\theta_1, \sigma^2)$ and $\mathcal{N}(\theta_2, \sigma^2)$ respectively, then in order to estimate θ_1 (or θ_2) when it is suspected that θ_1 may be equal to θ_2 , one may use the first sample mean, $\bar{X}_1$ if the hypothesis $H_0 : \theta_1 = \theta_2$ is rejected or use $\bar{\bar{X}} = \frac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1 + n_2}$ (pooled mean) if H_0 is accepted. This procedure is known in the literature as the **preliminary test estimation** (PTE). Mosteller (1948) at once came up with the proposition of using Bayes and a step towards Empirical Bayes method to produce pooled estimate where the weights in the pooled

estimate is appropriately chosen and the prior distribution in the Bayes/Empirical Bayes method is chosen judiciously. Here again the question arises as to how one may choose the prior distribution to apply the Bayes principle. Finally, Stein (1956) brought in so called the ‘paradoxical’ result that the sample mean-vector $\bar{\mathbf{X}}$ as an estimator of the mean-vector $\boldsymbol{\theta}$ in p - variate normal distribution is inadmissible for $p \geq 3$ with respect to quadratic loss function.

The thesis contains the application of the preliminary test approach of Saleh and Sen (1978-86) which includes the methodologies of Bancroft (1944), Stein (1956) and James-Stein (1961) in models with measurement errors. This is a deviation from Bayesian/Empirical Bayes method due to the assumptions at various stages of the ME model and is difficult. We have taken the simplest path which is easy to apply. We consider the following models in the thesis.

1.1 Simple Linear Model

Consider the linear model

$$\left. \begin{aligned} Y_{0t} &= \beta_0 + \beta_1 x_t + e_t \\ X_{0t} &= x_t + u_t \end{aligned} \right\}, \quad t = 1, \dots, n \quad (1.1.1)$$

where $(x_t, e_t, u_t)' \sim \mathcal{N}_3\{(\mu_x, 0, 0)'; \text{Diag}(\sigma_{xx}, \sigma_{ee}, \sigma_{uu})\}$.

Then,

$$\begin{pmatrix} X_{0t} \\ Y_{0t} \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \mu_x \\ \beta_0 + \beta_1 \mu_x \end{pmatrix}, \begin{pmatrix} \sigma_{xx} + \sigma_{uu} & \beta_1 \sigma_{xx} \\ \beta_1 \sigma_{xx} & \beta_1^2 \sigma_{xx} + \sigma_{ee} \end{pmatrix} \right\} \quad (1.1.2)$$

We are interested in the estimation of $(\beta_0, \beta_1)'$ when a priori there exists uncertain information that β_1 may be equal to β_{10} ($= 0$, without loss of generality).

Clearly,

$$(Y_{0t}|X_{0t}) \sim \mathcal{N}\{\nu_0 + \nu_1 X_{0t}, \sigma_{zz}\} \quad (1.1.3)$$

where

$$\nu_0 = \beta_0 + \beta_1 \mu_x(1 - \kappa_{xx}), \quad \nu_1 = \kappa_{xx} \beta_1$$

$$\kappa_{xx} = \sigma_{xx}(\sigma_{xx} + \sigma_{uu})^{-1}, \quad \text{where } (0 < \kappa_{xx} < 1) \text{ and}$$

$$\sigma_{zz} = \sigma_{ee} + \beta_1^2 \sigma_{xx}(1 - \kappa_{xx}).$$

Notice that there are 6 parameters of the linear models and there are many different configurations that lead to the same distributions of the observation models and is not identifiable. Therefore, we need to impose conditions on the parameters to overcome the identifiability problem and obtain consistent estimators. Here we assume that the **reliability ratio** (κ_{xx}) is known.

Using the LSE principle in the conditional model (1.1.3) we may obtain the estimate of β_1 and β_0 as

$$\tilde{\beta}_{1n} = \kappa_{xx}^{-1} \tilde{\nu}_{1n} = \kappa_{xx}^{-1} S_{XX}^{-1} S_{XY} \quad \text{and} \quad \tilde{\beta}_{0n} = \bar{Y}_0 - \tilde{\beta}_{1n} \bar{X}_0, \quad (1.1.4)$$

where

$$S_{X_0 X_0} = \sum_{i=1}^n (X_{0i} - \bar{X}_0)^2,$$

$$S_{X_0 Y_0} = \sum_{i=1}^n (X_{0i} - \bar{X}_0)(Y_{0i} - \bar{Y}_0) \quad \text{and}$$

$$S_{Y_0 Y_0} = \sum_{i=1}^n (Y_{0i} - \bar{Y}_0)^2.$$

The corresponding test of the null-hypothesis $H_0 : \beta_1 = 0$ vs $H_A : \beta_1 \neq 0$, is given by the test-statistic

$$\begin{aligned} \mathcal{L}_n &= \frac{\kappa_{xx}^2 \tilde{\beta}_{1n}^2 S_{X_0 X_0}}{\tilde{\sigma}_{zz}}, \\ \text{and } \tilde{\sigma}_{zz} &= \frac{1}{n-2} \sum_{i=1}^n \left[(Y_{0i} - \bar{Y}_0) - \tilde{\nu}_{1n} (X_{0i} - \bar{X}_0) \right]^2 \end{aligned} \quad (1.1.5)$$

which follows the central F -distribution with $(1, n-2)$ degree of freedom (d.f.). Let $\mathcal{L}_{n,\alpha}$ be the upper $\alpha\%$ critical value from the null-distribution of $\mathcal{L}_n$. Then, we define the preliminary test estimator (PTE) of β_1 as

$$\hat{\beta}_{1n}^{PT} = \tilde{\beta}_{1n} I(\mathcal{L}_n \geq \mathcal{L}_{n,\alpha}) = \tilde{\beta}_{1n} - \tilde{\beta}_{1n} I(\mathcal{L}_n < \mathcal{L}_{n,\alpha}) \quad (1.1.6)$$

where $I(A)$ is the indicator function of the set A and

$$\hat{\beta}_{0n}^{PT} = \bar{Y} - \hat{\beta}_{1n}^{PT} \bar{X}_0 \quad (1.1.7)$$

Further, we present the properties when the reliability ratio (κ_{xx}) is not known instead the measurement variance (σ_{uu}) is known. The estimators of β_1 and β_0 are given

$$\begin{aligned} \tilde{\beta}_{1n} &= (S_{X_0 X_0} - n\sigma_{uu})^{-1} S_{X_0 Y_0}, \text{ if } S_{X_0 X_0} - n\sigma_{uu} > 0 \\ \tilde{\beta}_{0n} &= \bar{Y}_0 - \tilde{\beta}_{1n} \bar{X}_0 \end{aligned} \quad (1.1.8)$$

with the asymptotic distribution of $\sqrt{n} \{ (\tilde{\beta}_{0n} - \beta_0), (\tilde{\beta}_{1n} - \beta_1) \}'$ converges in distribution to a bivariate normal distribution with mean vector $(0, 0)'$ and covariance matrix

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \quad (1.1.9)$$

with

$$\begin{aligned}\Gamma_{11} &= \mu_x^2 \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} + \sigma_{Xv}^2) + \sigma_{vv}, \\ \Gamma_{12} &= \Gamma_{21} = -\mu_x \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} + \sigma_{Xv}^2), \\ \Gamma_{22} &= \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} + \sigma_{Xv}^2).\end{aligned}\tag{1.1.10}$$

where

$$v_t = Y_{0t} - \beta_0 - \beta_1 X_{0t} = e_t - \beta_1 u_t, \quad \sigma_{Xv} = \sigma_{uv} = -\beta_1 \sigma_{uu}.\tag{1.1.11}$$

The test statistic for $H_0 : \beta_1 = 0$ is given by

$$\mathcal{L}_n^* = \frac{n \tilde{\beta}_{1n}^2}{\tilde{\Gamma}_{22}}.\tag{1.1.12}$$

with the consistent estimator of Γ_{22} as defined in Fuller (1987) by

$$\tilde{\Gamma}_{22} = \tilde{\sigma}_{xx}^{-2} (\tilde{\sigma}_{XX} \tilde{\sigma}_{vv} + \tilde{\sigma}_{Xv}^2).\tag{1.1.13}$$

Side-by-Side we define the continuous estimator of β_1 apposite to PTE as

$$\begin{aligned}\hat{\beta}_{1n}^S &= \tilde{\beta}_{1n} - c |\mathcal{L}_n^{-\frac{1}{2}}| \tilde{\beta}_{1n} \\ &= \tilde{\beta}_{1n} - c \left(\frac{\sqrt{\tilde{\sigma}_{zz}} \sqrt{S_{X_0 X_0}}}{\kappa_{xx} |\tilde{\beta}_{1n}|} \right) \tilde{\beta}_{1n}, \quad (c > 0), \\ \text{and } \hat{\beta}_{0n}^S &= \bar{Y} - \hat{\beta}_{1n}^S \bar{X}_0.\end{aligned}\tag{1.1.14}$$

Note that as $\mathcal{L}_n \rightarrow \infty$, $\hat{\beta}_{1n}^S \rightarrow \tilde{\beta}_{1n}$, but $\hat{\beta}_{1n}^S$ changes sign as $\mathcal{L}_n^{-\frac{1}{2}}$ go past c . We shall study the finite sample and asymptotic properties of $\tilde{\beta}_{1n}$, $\hat{\beta}_{1n}^{PT}$ and $\hat{\beta}_{1n}^S$ in Chapter 2.

1.2 Multiple Regression Model

We consider the multiple regression model with the measurement error model described as

$$\left. \begin{aligned} Y_{0t} &= \beta_0 + \beta' \mathbf{x}_t + e_t \\ \mathbf{X}_{0t} &= \mathbf{x}_t + \mathbf{u}_t \end{aligned} \right\}, \quad t = 1, \dots, n \quad (1.2.1)$$

where β_0 is the intercept and $\beta = (\beta_1, \dots, \beta_p)'$ is the regression parameter with $\mathbf{x}_t = (x_{1t}, \dots, x_{pt})'$, $\mathbf{u}_t = (u_{1t}, \dots, u_{pt})'$ and $\mathbf{X}_{0t} = (X_{01t}, \dots, X_{0pt})'$. e_t is the response error and u_{it} is the measurement error in the i^{th} regression variable x_{it} . Note that x_{it} is not observable while X_{0it} is the corresponding observed value. Similarly, Y_{0t} is the observed response. We assume that

$$(\mathbf{x}_t, e_t, \mathbf{u}_t)' \sim \mathcal{N}\{(\boldsymbol{\mu}'_x, 0, \mathbf{0})'; \text{Blockdiag}(\Sigma_{xx}, \sigma_{ee}, \Sigma_{uu})\} \quad (1.2.2)$$

where $\boldsymbol{\mu}'_x = (\mu_{x_1}, \dots, \mu_{x_p})'$.

Clearly, $(Y_{0t}, \mathbf{X}'_{0t})'$ follows a $(p+1)$ -variate normal distribution with mean-vector $(\beta_0 + \beta' \boldsymbol{\mu}_x, \boldsymbol{\mu}'_x)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} \beta' \Sigma_{xx} \beta + \sigma_{ee} & \beta' \Sigma_{xx} \\ \Sigma_{xx} \beta & \Sigma_{xx} + \Sigma_{uu} \end{pmatrix}, \quad (1.2.3)$$

Thus,

$$\begin{aligned} E[Y_{0t} | \mathbf{X}_{0t}] &= \nu_0 + \boldsymbol{\nu}' \mathbf{X}_{0t} \\ V[Y_{0t} | \mathbf{X}_{0t}] &= \beta' \Sigma_{xx} (\mathbf{I} - \mathbf{K}_{xx}) \beta + \sigma_{ee} = \sigma_{zz}, \text{ (say)} \end{aligned} \quad (1.2.4)$$

where

$$\mathbf{K}_{xx} = (\Sigma_{xx} + \Sigma_{uu})^{-1} \Sigma_{xx} = \Sigma_{X_0 X_0}^{-1} \Sigma_{xx},$$

$$\Sigma_{X_0X_0} = \Sigma_{xx} + \Sigma_{uu},$$

$$\nu_0 = \beta_0 + \beta'(\mathbf{I} - \mathbf{K}'_{xx})\mu_x,$$

$$\nu = \mathbf{K}_{xx}\beta \text{ and } \beta = \mathbf{K}_{xx}^{-1}\nu.$$

If Reliability Matrix ($\mathbf{K}_{xx}$) is known, then conditionally, the estimators of β_0 and β are given by

$$\begin{aligned}\tilde{\beta}_n &= \mathbf{K}_{xx}^{-1}\mathbf{S}_{X_0X_0}^{-1}\mathbf{S}_{X_0Y_0} \\ \tilde{\beta}_{0n} &= \bar{Y}_0 - \tilde{\beta}'_n\bar{\mathbf{X}}_0\end{aligned}\tag{1.2.5}$$

where

$$\mathbf{S}_{X_0X_0} = \left((S_{X_{0i}X_{0j}}) \right),$$

$$\mathbf{S}_{X_0Y_0} = (S_{X_{01}Y_1}, \dots, S_{X_{0p}Y_p}),$$

$$\bar{\mathbf{X}}_0 = (\bar{X}_{01}, \dots, \bar{X}_{0p})', \text{ and}$$

$\bar{Y}_0$ is the mean-vector of the responses.

We define a class of estimators of β when β is suspected to be $\mathbf{0}$ as

$$\beta_n^* = \left(1 - g(\mathcal{L}_n)\right)\tilde{\beta}_n\tag{1.2.6}$$

where $g(\cdot)$ is a measurable function of the test-statistic $\mathcal{L}_n$ defined by

$$\mathcal{L}_n = \frac{\tilde{\nu}'_n \mathbf{S}_{X_0X_0} \tilde{\nu}_n}{p\tilde{\sigma}_{zz}} = \frac{\tilde{\beta}'_n \mathbf{K}'_{xx} \mathbf{S}_{X_0X_0} \mathbf{K}_{xx} \tilde{\beta}_n}{p\tilde{\sigma}_{zz}}\tag{1.2.7}$$

and

$$\tilde{\sigma}_{zz} = \frac{1}{m} \sum_{t=1}^n \left\{ (Y_{0t} - \bar{Y}_0) - (\mathbf{X}_{0t} - \bar{\mathbf{X}}_0)' \tilde{\nu} \right\}^2\tag{1.2.8}$$

where $m = n - p - 1$. We study the properties of the class of estimators β_n^* for chosen g -function. We present the results in Chapter 3.

Further, we present the results and properties of the estimators when the reliability matrix $(\mathbf{K}_{xx})$ is not known instead the measurement variance (Σ_{uu}) is known. The basic estimator of β and β_0 are given by

$$\begin{aligned}\tilde{\beta}_n &= \begin{cases} (\mathbf{S}_{X_0X_0} - n\Sigma_{uu})^{-1}\mathbf{S}_{X_0Y_0}, & \text{if } \mathbf{S}_{X_0X_0} - n\Sigma_{uu} \text{ is positive definite} \\ \mathbf{0}, & \text{otherwise,} \end{cases} \\ \tilde{\beta}_{0n} &= \bar{Y}_0 - \tilde{\beta}_n' \bar{X}_0\end{aligned}\tag{1.2.9}$$

and the class of estimators of β is defined by

$$\beta_n^* = \left(1 - g(\mathcal{L}_n^*)\right) \tilde{\beta}_n\tag{1.2.10}$$

where $g(\mathcal{L}_n^*)$ is a measurable function of $\mathcal{L}_n^*$ and

$$\mathcal{L}_n^* = n\tilde{\beta}_n' \hat{\mathbf{G}}_n^{-1} \tilde{\beta}_n.\tag{1.2.11}$$

We obtain

$$\sqrt{n}(\tilde{\beta}_n - \beta) \sim \mathcal{N}_p(\mathbf{0}, \mathbf{G})\tag{1.2.12}$$

where

$$\mathbf{G} = \Sigma_{xx}^{-1} \left[(\beta' \Sigma_{uu} \beta + \sigma_{ee}) \Sigma_{X_0X_0} + \Sigma_{uu} \beta \beta' \Sigma_{uu} \right] \Sigma_{xx}^{-1}\tag{1.2.13}$$

with the consistent estimator of $\mathbf{G}$ defined by

$$\begin{aligned}\hat{\mathbf{G}}_n &= \left(\frac{1}{n} \mathbf{S}_{X_0X_0} - \Sigma_{uu} \right)^{-1} \left[\left(\tilde{\beta}_n' \Sigma_{uu} \tilde{\beta}_n + \tilde{\sigma}_{ee} \right) \frac{1}{n} \mathbf{S}_{X_0X_0} \right. \\ &\quad \left. + \Sigma_{uu} \tilde{\beta}_n \tilde{\beta}_n' \Sigma_{uu} \right] \left(\frac{1}{n} \mathbf{S}_{X_0X_0} - \Sigma_{uu} \right)^{-1}.\end{aligned}\tag{1.2.14}$$

In order to obtain the reasonable asymptotic distributions of the estimators, we introduce the Pitman local alternatives and study the properties of the class of estimator β_n^* for chosen g -function in Chapter 4.

1.3 Several Simple Linear Models: Parallelism Hypothesis

Consider the simple linear models

$$\left. \begin{aligned} Y_{\alpha_j}^0 &= \theta_\alpha + \beta_\alpha x_{\alpha_j} + e_{\alpha_j} \\ X_{\alpha_j}^0 &= x_{\alpha_j} + u_{\alpha_j} \end{aligned} \right\} \quad j = 1, \dots, n_\alpha, \alpha = 1, \dots, p \quad (1.3.1)$$

where $e_{\alpha_j} \sim \mathcal{N}(0, \sigma_{ee})$ and

$$(x_{\alpha_j}, e_{\alpha_j}, u_{\alpha_j})' \sim \mathcal{N}_3((\mu_{x_j}, 0, 0)' : \text{Diag}(\sigma_{x_\alpha x_\alpha}, \sigma_{ee}, \sigma_{u_\alpha u_\alpha})) \quad (1.3.2)$$

Then, $(Y_{\alpha_j}^0, X_{\alpha_j}^0)'$ follows the bivariate normal distribution

$$\begin{pmatrix} Y_{\alpha_j}^0 \\ X_{\alpha_j}^0 \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_\alpha + \beta_\alpha \mu_{x_\alpha} \\ \mu_{x_\alpha} \end{pmatrix}; \Sigma_{\alpha\alpha} \right\} \quad (1.3.3)$$

where

$$\Sigma_{\alpha\alpha} = \begin{pmatrix} \beta_\alpha^2 \mu_{x_\alpha} + \sigma_{ee} & \beta_\alpha \sigma_{x_\alpha x_\alpha} \\ \beta_\alpha \sigma_{x_\alpha x_\alpha} & \sigma_{x_\alpha x_\alpha} + \sigma_{u_\alpha u_\alpha} \end{pmatrix} \quad (1.3.4)$$

Hence,

$$E[Y_{\alpha_j}^0 | X_{\alpha_j}^0] = \nu_{\alpha_0} + \nu_{\alpha_1} X_{\alpha_j}^0 \quad (1.3.5)$$

where

$$\nu_{\alpha_0} = \theta_\alpha + \beta_\alpha \mu_{x_\alpha} (1 - \kappa_{x_\alpha x_\alpha})$$

$$\nu_{\alpha_1} = \kappa_{x_\alpha x_\alpha} \beta_\alpha$$

$$\kappa_{x_\alpha x_\alpha} = \sigma_{x_\alpha x_\alpha} (\sigma_{x_\alpha x_\alpha} + \sigma_{u_\alpha u_\alpha})^{-1} \quad \text{and}$$

$$\text{Var}[Y_{\alpha_j}^0 | X_{\alpha_j}^0] = \sigma_{ee} + \beta_{\alpha}^2 \sigma_{x_{\alpha} x_{\alpha}} (1 - \kappa_{x_{\alpha} x_{\alpha}}) .$$

Clearly,

$$\begin{pmatrix} \bar{Y}_{\alpha}^0 \\ \bar{X}_{\alpha}^0 \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_{\alpha} + \beta_{\alpha} \mu_{x_{\alpha}} \\ \mu_{x_{\alpha}} \end{pmatrix} ; \frac{1}{n_{\alpha}} \Sigma_{\alpha\alpha} \right\} \quad (1.3.6)$$

We need to impose additional conditions to overcome the identifiability problem.

For this reason we consider the following conditions:

- (i) $\mu_{x_{\alpha}} = (\mu_{x_1}, \dots, \mu_{x_p})'$, for all $\alpha = 1, \dots, p$,
 - (ii) $\sigma_{x_{\alpha} x_{\alpha}} = \sigma_{xx}$ for all $\alpha = 1, \dots, p$,
 - (iii) $\sigma_{u_{\alpha} u_{\alpha}} = \sigma_{uu}$ for all $\alpha = 1, \dots, p$, and
 - (ii) the reliability ratio, κ_{xx} is known
- (1.3.7)

Define

$$\begin{aligned} \bar{Y}_{\alpha}^0 &= \frac{1}{n_{\alpha}} \sum_{j=1}^{n_{\alpha}} Y_{\alpha_j}^0, \\ \bar{X}_{\alpha}^0 &= \frac{1}{n_{\alpha}} \sum_{j=1}^{n_{\alpha}} X_{\alpha_j}^0, \\ S_{X_{\alpha}^0 X_{\alpha}^0} &= \sum_{j=1}^{n_{\alpha}} (X_{\alpha_j}^0 - \bar{X}_{\alpha}^0)^2, \\ S_{Y_{\alpha}^0 Y_{\alpha}^0} &= \sum_{j=1}^{n_{\alpha}} (Y_{\alpha_j}^0 - \bar{Y}_{\alpha}^0)^2 \text{ and} \\ S_{X_{\alpha}^0 Y_{\alpha}^0} &= \sum_{j=1}^{n_{\alpha}} (X_{\alpha_j}^0 - \bar{X}_{\alpha}^0)(Y_{\alpha_j}^0 - \bar{Y}_{\alpha}^0). \end{aligned}$$

Then,

$$\mathbf{S}_{\alpha\alpha} = \begin{pmatrix} S_{Y_{\alpha}^0 Y_{\alpha}^0} & S_{X_{\alpha}^0 Y_{\alpha}^0} \\ S_{X_{\alpha}^0 Y_{\alpha}^0} & S_{X_{\alpha}^0 X_{\alpha}^0} \end{pmatrix} \sim W_p(\Sigma_{\alpha}, n_{\alpha} - 1), \quad (1.3.8)$$

where

$$\Sigma_\alpha = \begin{pmatrix} \beta_\alpha^2 \sigma_{xx} + \sigma_{ee} & \beta_\alpha \sigma_{xx} \\ \beta_\alpha \sigma_{xx} & \sigma_{xx} + \sigma_{uu} \end{pmatrix}. \quad (1.3.9)$$

This allows us to estimate $\sigma_{X^0 X^0} = \sigma_{xx} + \sigma_{uu}$ by pooling the $S_{X_\alpha^0 X_\alpha^0}$'s, i.e. by $S_{X^0 X^0}^0 = \sum_{\alpha=1}^p S_{X_\alpha^0 X_\alpha^0}$ since

$$E\left(\frac{S_{X^0 X^0}^0}{n-p}\right) = \sigma_{X^0 X^0}. \quad (1.3.10)$$

Clearly, $\frac{S_{X^0 X^0}^0}{\sigma_{X^0 X^0}}$ follows the central chi-square distribution with $n-p$ degree of freedom (d.f.).

We introduce Pitman local alternatives

$$K_{(n)} : \beta_{(n)} = \beta_0 \mathbf{1}_p + n^{-\frac{1}{2}} \boldsymbol{\delta}, \quad \boldsymbol{\delta} \text{ fixed finite vector.} \quad (1.3.11)$$

and define the statistic

$$\mathcal{L}_n = \frac{\kappa_{xx}^2 \tilde{\boldsymbol{\beta}}_n' \mathbf{H}_n' \mathbf{S}_{X^0 X^0} \mathbf{H}_n \tilde{\boldsymbol{\beta}}_n}{(p-1) \tilde{\sigma}_{zz}^{(0)}} \quad (1.3.12)$$

where $\mathbf{H}_n = \mathbf{I} - p - \mathbf{A}_n$ and $\mathbf{A}_n = \frac{\mathbf{1}_p \mathbf{1}_p' \mathbf{S}_{X^0 X^0}}{\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p}$, to test the hypothesis $H_0 : \boldsymbol{\beta} = \beta_0 \mathbf{1}_p$, which has asymptotically a noncentral chi-square distribution with $(p-1)$ d.f. and non-centrality parameter $\frac{1}{2} \Delta^2$ with $\Delta^2 = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \left(\boldsymbol{\delta}' \mathbf{J}' \boldsymbol{\Lambda}_0 \mathbf{J} \boldsymbol{\delta} \right)$ under $K_{(n)}$, where $\mathbf{J} = \mathbf{I}_p - \mathbf{1}_p \mathbf{1}_p' \boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}_0 = \text{diag}(\lambda_1, \dots, \lambda_p)$.

We study the properties of the class of estimators of $\boldsymbol{\beta}$ defined by

$$\boldsymbol{\beta}_n^* = \mathbf{A}_n \tilde{\boldsymbol{\beta}}_n + (\mathbf{I}_p - \mathbf{A}_n) \tilde{\boldsymbol{\beta}}_n g(\mathcal{L}_n) \quad (1.3.13)$$

in Chapter 5.

1.4 Application in Sampling Survey

We consider the prediction of the population total, which is a function of the estimators of the regression coefficient parameter β under regression superpopulation model when the explanatory variables are measured with errors. Let $\mathcal{P} = \{1, \dots, N\}$ denote a finite population of N units, where N is known. Associated with the t^{th} unit of $\mathcal{P}$, there is a $(p+1)$ -vector $(Y_t, \mathbf{x}_t)'$, $t = 1, \dots, N$, assumed to be a random sample from a bivariate normal distribution. To gain information about the population total $T = \sum_{t=1}^N Y_t$, a sample s of size n is selected from $\mathcal{P}$ according to some specified sampling plan. We use the superpopulation approach (Cassel et al., 1977) to survey sampling. Under this perspective according to the conditionality principle (Basu, 1975), the sampling plan is not relevant for inference. Let $\mathbf{r} = \mathcal{P} - s$ be the unobserved part of $\mathcal{P}$. We note that a function of $\bar{\mathbf{X}}_r$ or $\bar{\mathbf{X}}_s$ is independent of $\mathbf{S}_{XX}$.

We consider the following distributions:

$$\begin{pmatrix} \bar{Y}_s \\ \bar{\mathbf{X}}_s \end{pmatrix} \sim N_{p+1} \left(\begin{pmatrix} \mu_Y \\ \mu_x \end{pmatrix}; \frac{1}{n} \begin{pmatrix} \sigma_{Y_0 Y_0} & \Sigma_{X_0 Y_0} \\ \Sigma_{X_0 Y_0} & \Sigma_{X_0 X_0} \end{pmatrix} \right), \quad (1.4.1)$$

and

$$\begin{pmatrix} \bar{\mathbf{X}}_s \\ \bar{\mathbf{X}}_r \end{pmatrix} \sim N_{2p} \left(\begin{pmatrix} \mu_x \\ \mu_x \end{pmatrix}; \begin{pmatrix} \frac{1}{n} \Sigma_{X_0 X_0} & 0 \\ 0 & \frac{1}{N-n} \Sigma_{X_0 X_0} \end{pmatrix} \right), \quad (1.4.2)$$

where

$$\bar{\mathbf{X}}_r = (\bar{X}_{r1}, \dots, \bar{X}_{rp})', \text{ where } \bar{X}_{rt} = \frac{1}{N-n} \sum_{t \in r} X_{rt} \text{ and}$$

$$\bar{\mathbf{X}}_s = (\bar{X}_{s1}, \dots, \bar{X}_{sp})', \text{ where } \bar{X}_{st} = \frac{1}{n} \sum_{t \in s} X_{st}.$$

Also we have

$$\left(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s\right) \sim \mathcal{N}_p\left(\mathbf{0}, \frac{N}{n(N-n)} \boldsymbol{\Sigma}_{XX}\right). \quad (1.4.3)$$

Let $T = n\bar{Y}_s + (N - n)\bar{Y}_r$. Then we consider the (generic) predictors of T of the form

$$T^* = N\bar{Y}_s + (N - n)(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)\boldsymbol{\nu}^* \quad (1.4.4)$$

where $\boldsymbol{\beta}^*$ is an estimator of $\boldsymbol{\beta}$ defined by $(1 - g(\mathcal{L}_n))\tilde{\boldsymbol{\nu}}_n$ when $\boldsymbol{\beta}$ is suspected to be $\mathbf{0}$, where $g(\cdot)$ is a measurable function of the test-statistic $\mathcal{L}_n$. The test statistic is defined by

$$\mathcal{L}_n = \frac{\tilde{\boldsymbol{\nu}}_n' \mathbf{S}_{X_0 X_0} \tilde{\boldsymbol{\nu}}_n}{p \tilde{\sigma}_{zz}} \quad (1.4.5)$$

and

$$\tilde{\sigma}_{zz} = \frac{1}{m} \sum_{t=1}^n \left\{ (Y_{0t} - \bar{Y}_0) - (\mathbf{X}_{0t} - \bar{\mathbf{X}}_0)' \tilde{\boldsymbol{\nu}}_n \right\}^2, \quad (1.4.6)$$

where $m = n - p$.

For the prediction of population total, we substitute $\boldsymbol{\nu}^*$ to the predictor of T^* in (1.4.4) and study the properties of the predictors based on the estimator $\boldsymbol{\nu}^*$ for chosen g -function. We present the results of the finite sample in Chapter 6 for $p \geq 1$.

Chapter 2

Simple Linear Model

2.0 Introduction

In this chapter, we consider the simple linear model and introduce the preliminary test estimation (PTE) of the parameters with measurement error (ME model) when the slope of the linear model is suspected, but not sure that it may be equal to 0 (zero) without loss of generality.

Expressions of the bias, MSE and efficiencies of estimators of parameters with measurement error for the conditional model are introduced when reliability coefficient is known as well as the expressions for the model without measurement error. Conditional model results are compared to the standard model without measurement error and unconditional expressions and analysis is provided for finite samples.

Asymptotic theory under local alternatives is developed when the variance of the measurement error or the ratio of the variance of the model error relative to the variance of the measurement error is known. Asymptotic expressions of bias

and MSE of the estimators along with their efficiencies are obtained and graphs and tables are provided to see these results and to determine optimum level of significance for minimum guaranteed efficiency. Finally, an alternative method of estimation is provided and discussed its merit compared to the Preliminary Test (PT) method estimation.

In section 1, we consider the estimation without measurement error. With measurement error, we have the conditional expressions of biases and MSEs of estimators of the parameters as well as unconditional expressions and comparisons when the reliability ratio is known in section 2. In section 3, we introduce Pitman-type local alternative hypothesis to obtain reasonable asymptotic distribution of $\mathcal{L}_n$. The asymptotic biases and MSE's expressions are obtained and compared when the variance of measurement is known. In section 4, we consider an alternative estimator to PTE. The expression of bias and MSE and comparison are presented.

2.1 Model Without Measurement Error

2.1.1 Simple Linear Model

Consider the standard simple linear model given by

$$Y_{0t} = \beta_0 + \beta_1 X_{0t} + e_t, \quad (2.1.1)$$

where $e_1, \dots, e_n$ are independently identically distributed as $N(0, \sigma_{ee})$.

2.1.2 Estimation of β_0 and β_1

We obtain the MLE/LSE of $(\beta_0, \beta_1)'$ are given by $(\tilde{\beta}_{0n}, \tilde{\beta}_{1n})'$ where

$$(i) \tilde{\beta}_{1n} = \frac{S_{X_0 Y_0}}{S_{X_0 X_0}} \quad \text{and} \quad (ii) \tilde{\beta}_{0n} = \bar{Y}_0 - \tilde{\beta}_{1n} \bar{X}_0. \quad (2.1.2)$$

The exact distribution of $(\tilde{\beta}_{0n}, \tilde{\beta}_{1n})'$ follows the bivariate normal distribution

$$\begin{pmatrix} \tilde{\beta}_{0n} \\ \tilde{\beta}_{1n} \end{pmatrix} \sim N_2 \left\{ \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \sigma_{ee} \begin{pmatrix} \frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0 X_0}} & -\frac{\bar{X}_0}{S_{X_0 X_0}} \\ -\frac{\bar{X}_0}{S_{X_0 X_0}} & \frac{1}{S_{X_0 X_0}} \end{pmatrix} \right\}. \quad (2.1.3)$$

An unbiased estimators of σ_{ee} is given by

$$\tilde{\sigma}_{ee} = \frac{1}{m} \sum_{t=1}^n \left\{ (Y_{0t} - \bar{Y}_0) - \tilde{\beta}_{1n} (X_{0t} - \bar{X}_0) \right\}^2, \quad (2.1.4)$$

where $m = n - 2$. The exact distribution of $\frac{m\tilde{\sigma}_{ee}}{\sigma_{ee}}$ follows a central chi-square distribution with m degree of freedom (d.f.) independent of the distribution of $(\tilde{\beta}_{0n}, \tilde{\beta}_{1n})'$.

2.1.3 The Test Statistic

Suppose we want to test the null hypothesis $H_0 : \beta_1 = 0$ against the alternative $H_A : \beta_1 \neq 0$, we consider the likelihood ratio statistic, $\mathcal{L}_n$ given by

$$\mathcal{L}_n = \frac{\tilde{\beta}_{1n}^2 S_{X_0 X_0}}{\tilde{\sigma}_{ee}}. \quad (2.1.5)$$

The exact distribution of $\mathcal{L}_n$ follows a non central F -distribution with $(1, m)$ d.f. with noncentrality parameter $\frac{1}{2}\Delta^2$ where $\Delta^2 = \frac{\beta_1^2 S_{X_0 X_0}}{\sigma_{ee}}$.

Under H_0 , it follows a central F -distribution. At the α -level of significance we obtain the critical value F_α from this distribution and reject H_0 if $\mathcal{L}_n \geq F_\alpha$ otherwise accept H_0 . From now on we shall write F_α for $F_{1,m}(\alpha)$.

2.1.4 Preliminary Test Estimation (PTE) of β_0 and β_1 with Uncertain Prior Information

Suppose that β_1 is suspected to have a prior value zero (without loss of generality). The prior information that $\beta_1 = 0$ is **uncertain** and the doubt on this uncertainty may be removed by testing the hypothesis $H_0 : \beta_1 = 0$ against the alternative $H_A : \beta_1 \neq 0$. As a result of this test one may choose $\tilde{\beta}_{1n}$ from the sample or 0 from the uncertain prior information. Thus, the preliminary test estimator (PTE) of β_1 may be written as

$$\hat{\beta}_{1n}^{PT} = \tilde{\beta}_{1n}I(\mathcal{L}_n \geq F_\alpha) = \tilde{\beta}_{1n} - \tilde{\beta}_{1n}I(\mathcal{L}_n < F_\alpha). \quad (2.1.6)$$

where $I(A)$ is the indicator function of the set A and F_α is the upper α -level critical value from central F with $(1, m)$ d.f.. The PTE procedure leads to the choice of one of the two values, namely, $\tilde{\beta}_1$ or 0 and depends on the level of significance, α .

For the estimation of β_0 we have three possible estimators, namely,

(iii) the restricted estimator (RE)

$$\hat{\beta}_{0n} = \bar{Y} \quad (\text{under } H_0), \quad (2.1.7)$$

(iv) the unrestricted estimator(UE),

$$\tilde{\beta}_{0n} = \bar{Y} - \tilde{\beta}_{1n}\bar{X}_0, \quad (2.1.8)$$

and (v) the preliminary test estimator(PTE),

$$\hat{\beta}_{0n}^{PT} = \tilde{\beta}_{0n} + \tilde{\beta}_{1n}\bar{X}_0I(\mathcal{L}_n < F_\alpha) \quad (2.1.9)$$

respectively.

2.1.5 Bias and MSE of the Estimators of β_0 and β_1

Since we are interested in comparing the UE, RE and PTE of the estimators with respect to bias and MSE, we obtain the bias (b) and the MSE (M) expressions of the estimators of β_0 and β_1 .

Theorem 2.1.1

$$(i) \ b_1(\tilde{\beta}_{1n}) = 0, \ M_1(\tilde{\beta}_{1n}) = \frac{\sigma_{ee}}{S_{X_0X_0}}; \quad (2.1.10)$$

$$(ii) \ b_2(\hat{\beta}_{1n}^{PT}) = -\beta_1 G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right), \quad (2.1.11)$$

$$\begin{aligned} M_2(\hat{\beta}_{1n}^{PT}) &= M_1(\tilde{\beta}_{1n}) - \frac{\sigma_{ee}}{S_{X_0X_0}} \left[G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) \right. \\ &\quad \left. - \Delta^2 \left\{ 2G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) - G_{5,m}\left(\frac{1}{5}F_\alpha; \Delta^2\right) \right\} \right]; \end{aligned} \quad (2.1.12)$$

$$(iii) \ b_1(\hat{\beta}_{0n}) = \beta_1 \bar{X}_0, \ M_1(\hat{\beta}_{0n}) = \sigma_{ee} \left(\frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0X_0}} \Delta^2 \right); \quad (2.1.13)$$

$$(iv) \ b_2(\tilde{\beta}_{0n}) = 0, \ M_2(\tilde{\beta}_{0n}) = \sigma_{ee} \left(\frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0X_0}} \right); \quad (2.1.14)$$

$$(v) \ b_3(\hat{\beta}_{0n}^{PT}) = \beta_1 \bar{X}_0 G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right), \quad (2.1.15)$$

$$\begin{aligned} M_3(\hat{\beta}_{0n}^{PT}) &= M_2(\tilde{\beta}_{0n}) - \sigma_{ee} \frac{\bar{X}_0^2}{S_{X_0X_0}} \left[G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) \right. \\ &\quad \left. - \Delta^2 \left\{ 2G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) - G_{5,m}\left(\frac{1}{5}F_\alpha; \Delta^2\right) \right\} \right]. \end{aligned} \quad (2.1.16)$$

where $G_{\nu_1, \nu_2}(x; \Delta^2)$ stands for the cdf of the noncentral F -distribution with (ν_1, ν_2) d.f. and noncentrality parameter $\frac{1}{2}\Delta^2$ given by

$$G_{\nu_1, \nu_2}(x; \Delta^2) = \sum_{r=0}^{\infty} \frac{1}{\Gamma(r+1)} e^{-\frac{\Delta^2}{2}} \left(\frac{1}{2} \Delta^2 \right)^r I_y \left(\frac{\nu_1}{2} + r; \frac{\nu_2}{2} \right) \quad (2.1.17)$$

where $y = \frac{\nu_1 x}{\nu_2 + \nu_1 x}$ and

$$I_y\left(\frac{\nu_1}{2} + r; \frac{\nu_2}{2}\right) = \frac{\Gamma(\frac{\nu_1}{2} + r + \frac{\nu_2}{2})}{\Gamma(\frac{\nu_1}{2} + r)\Gamma(\frac{\nu_2}{2})} \int_0^y x^{\frac{\nu_1}{2} + r - 1} (1 - x)^{\frac{\nu_2}{2} - 1} dx, \quad (2.1.18)$$

is the Incomplete Beta function. The power of the test is given by (2.1.17) by letting $y = \frac{F_\alpha}{m + F_\alpha}$.

For (2.1.11) and (2.1.12) in (ii) and for (2.1.15) and (2.1.16) in (v) see Judge and Bock (1978) and Saleh (2001) and for (2.1.17) see Johnson and Kotz (1970).

2.1.6 Comparison of the MSE

Following Ahmed and Saleh(1989), we summarize the results as follows: The relative efficiency based on the mean-square-errors of $\hat{\beta}_{1n}^{PT}$ compared to $\tilde{\beta}_{1n}$ is given by

$$\begin{aligned} MRE(\hat{\beta}_{1n}^{PT} : \tilde{\beta}_{1n}) &= \frac{M_1(\tilde{\beta}_{1n})}{M_3(\hat{\beta}_{1n}^{PT})} = MRE(\alpha, \Delta^2) \quad (\text{say}) \\ &= \left\{1 + g(\Delta^2)\right\}^{-1} \end{aligned} \quad (2.1.19)$$

where

$$g(\Delta^2) = -G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) + \Delta^2 \left\{2G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) - G_{5,m}\left(\frac{1}{5}F_\alpha; \Delta^2\right)\right\}.$$

Note that $MRE(\hat{\beta}_{1n}^{PT}; \tilde{\beta}_{1n})$ is a function of $(\alpha : \Delta^2)$ and for fixed α ($0 < \alpha < 1$) it decreases crossing the 1-line to a minimum, then increases towards the 1-line as Δ^2 tends to infinity. Further,

(i) $MRE(\hat{\beta}_{1n}^{PT}; \tilde{\beta}_{1n}) \gtrless 1$ according as

$$\Delta^2 \gtrless \frac{G_{3,m}(\frac{1}{3}F_\alpha : \Delta^2)}{2G_{3,m}(\frac{1}{3}F_\alpha : \Delta^2) - G_{5,m}(\frac{1}{5}F_\alpha : \Delta^2)} < 1. \quad (2.1.20)$$

(ii) Maximum attainable efficiency for fixed α is attained for $\Delta^2 = 0$ and equals

$$\left\{1 - G_{3,m}\left(\frac{1}{3}F_\alpha; 0\right)\right\}^{-1} \quad (2.1.21)$$

and as $\Delta^2 \rightarrow \infty$, the efficiency goes to unity.

(iii) PTE can not be uniformly better than $\tilde{\beta}_{1n}$ but one may determine a PTE with minimum guaranteed efficiency, say, E_0 by choosing α^* by the maximin rule

$$\max_{\alpha} \min_{\Delta^2} MRE(\alpha : \Delta^2) \geq E_0. \quad (2.1.22)$$

(iv) As $\alpha \rightarrow 1$, $\hat{\beta}_{1n}^{PT} \rightarrow \tilde{\beta}_{1n}$.

Figure 2.1 displays the graph of the efficiency function of PTE relative to UE of β_1 ($n = 6$, $\kappa_{xx}^2 = 0.8$ and $\alpha = 0.15$) against Δ^2 .

The relative efficiency expression for $\hat{\beta}_{0n}$ and $\hat{\beta}_{0n}^{PT}$ compared to $\tilde{\beta}_{0n}$ are given by

$$MRE_1(\hat{\beta}_{0n} : \tilde{\beta}_{0n}) = \left\{\frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0X_0}}\right\} \left\{\frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0X_0}} \Delta^2\right\}^{-1}, \quad (2.1.23)$$

$$MRE_2(\hat{\beta}_{0n}^{PT} : \tilde{\beta}_{0n}) = \left\{1 + g_2(\Delta^2)\right\}^{-1} \quad (2.1.24)$$

respectively, where

$$\begin{aligned} g_2(\Delta^2) &= -\frac{\bar{X}_0^2}{S_{X_0X_0}} \left\{\frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0X_0}}\right\}^{-1} \left[G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) \right. \\ &\quad \left. - \Delta^2 \left\{ 2G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) - G_{5,m}\left(\frac{1}{5}F_\alpha; \Delta^2\right) \right\} \right]. \end{aligned} \quad (2.1.25)$$

$MRE_1(\hat{\beta}_0; \tilde{\beta}_{0n})$ is a decreasing function of Δ^2 and has its maximum value $(1 + \frac{n\bar{X}_0^2}{S_{X_0X_0}})$ at $\Delta^2 = 0$ and decreases to 0 as $\Delta^2 \rightarrow \infty$. The efficiency equals one at $\Delta^2 = 1$. Thus, if $0 < \Delta^2 < 1$, then $\hat{\beta}_{0n}$ is better than $\tilde{\beta}_{0n}$ otherwise opposite conclusion holds. As for the PTE $\hat{\beta}_{0n}^{PT}$, the relative efficiency is a function of (α, Δ^2) and for fixed α , it has the maximum value

$$\left\{1 + g_2(0)\right\}^{-1} = \left[1 - \left\{\frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0X_0}}\right\}^{-1} \frac{\bar{X}_0^2}{S_{X_0X_0}} G_{3,m}\left(\frac{1}{3}F_\alpha; 0\right)\right]^{-1} \quad (2.1.26)$$

at $\Delta^2 = 0$. It decreases crossing the 1-line to a minimum, then increases towards the 1-line as $\Delta^2 \rightarrow \infty$. Further

(i) $MRE(\hat{\beta}_{0n}^{PT} : \tilde{\beta}_{0n}) \gtrless 1$ according as

$$\Delta^2 \gtrless \frac{G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right)}{2G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) - G_{5,m}\left(\frac{1}{5}F_\alpha; \Delta^2\right)}. \quad (2.1.27)$$

As $\Delta^2 \rightarrow \infty$, the efficiency tends to one.

(ii) For the minimum guaranteed efficiency E_0 , optimum α^* is determined by the equation

$$\max_{\alpha} \min_{\Delta^2} MRE_2(\alpha : \Delta^2) \geq E_0. \quad (2.1.28)$$

(iii) As $\alpha \rightarrow 1$, $\hat{\beta}_{0n}^{PT} \rightarrow \tilde{\beta}_{0n}$ and as $\alpha \rightarrow 0$, $\hat{\beta}_{0n}^{PT} \rightarrow \hat{\beta}_{0n}$.

2.2 Model with Measurement Error

2.2.1 Simple Linear Model

Consider the simple linear model with measurement errors.

$$\begin{aligned} Y_{0t} &= \beta_0 + \beta_1 x_t + e_t \\ X_{0t} &= x_t + u_t, \quad t = 1, 2, \dots, n \end{aligned} \quad (2.2.1)$$

where e_t is the measurement error in the study variable and u_t is the measurement error in the regressor variable. x_t is unobservable and X_{0t} is the corresponding observed value. Similarly, Y_{0t} is the observed response. Furthermore,

$$(x_t, e_t, u_t)' \sim N_3\left\{(\mu_x, 0, 0)'; \text{Diag}(\sigma_{xx}, \sigma_{ee}, \sigma_{uu})\right\}. \quad (2.2.2)$$

Clearly, $(Y_{0t}, X_{0t})'$ follows a bivariate normal distribution with mean $(\beta_0 + \beta_1\mu_x, \mu_x)'$ and covariance matrix

$$\begin{pmatrix} \sigma_{YY} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{XX} \end{pmatrix} = \begin{pmatrix} \beta_1^2 \sigma_{xx} + \sigma_{ee} & \beta_1 \sigma_{xx} \\ \beta_1 \sigma_{xx} & \sigma_{xx} + \sigma_{uu} \end{pmatrix} = \Sigma \quad (\text{say}). \quad (2.2.3)$$

Also the conditional distribution of Y_{0t} given X_{0t} is again normal with mean

$$E(Y_{0t}|X_{0t}) = \nu_0 + \nu_1 X_{0t} \quad (2.2.4)$$

and variance

$$\sigma_{zz} = \text{Var}(Y_{0t}|X_{0t}) = \sigma_{ee} + \beta_1^2 \sigma_{xx} (1 - \kappa_{xx}), \quad (2.2.5)$$

where

$$\nu_0 = \beta_0 + \beta_1 \mu_x (1 - \kappa_{xx}), \quad \nu_1 = \kappa_{xx} \beta_1, \quad (2.2.6)$$

and

$$\kappa_{xx} = \sigma_{xx}(\sigma_{xx} + \sigma_{uu})^{-1} \quad (\text{Reliability Ratio}). \quad (2.2.7)$$

Further,

$$\begin{pmatrix} \bar{Y}_0 \\ \bar{X}_0 \end{pmatrix} \sim N_2 \left\{ \begin{pmatrix} \beta_0 + \beta_1 \mu_x \\ \mu_x \end{pmatrix}, \frac{1}{n} \Sigma \right\} \quad (2.2.8)$$

and

$$\mathbf{S} = \begin{pmatrix} S_{Y_0 Y_0} & S_{X_0 Y_0} \\ S_{X_0 Y_0} & S_{X_0 X_0} \end{pmatrix} \sim W_2(\Sigma; n-1) \quad (2.2.9)$$

where

$$\begin{aligned} \bar{X}_0 &= \frac{1}{n} \sum_{t=1}^n X_{0t}, \quad \bar{Y}_0 = \frac{1}{n} \sum_{t=1}^n Y_{0t}, \\ S_{X_0 X_0} &= \sum_{t=1}^n (X_{0t} - \bar{X}_0)^2, \quad S_{Y_0 Y_0} = \sum_{t=1}^n (Y_{0t} - \bar{Y}_0)^2 \\ \text{and} \quad S_{X_0 Y_0} &= \sum_{t=1}^n (X_{0t} - \bar{X}_0)(Y_{0t} - \bar{Y}_0). \end{aligned} \quad (2.2.10)$$

Here $W_2(\Sigma; n-1)$ stands for the Wishart distribution which is independent of the distribution of $(\bar{Y}_0, \bar{X}_0)'$. Clearly,

$$E\left(\frac{\mathbf{S}}{n-1}\right) = \Sigma \quad (2.2.11)$$

which means that the unbiased estimators of the element of Σ are the corresponding elements of $\frac{\mathbf{S}}{n-1}$. Hence $E\left(\frac{S_{XX}}{n-1}\right) = \sigma_{xx} + \sigma_{uu} = \sigma_{XX}$ and $\frac{S_{X_0X_0}}{\sigma_{xx} + \sigma_{uu}}$ follows a central chi-square distribution with $n-1$ degrees of freedom independent of $(\bar{Y}_0, \bar{X}_0)'$.

2.2.2 Conditional Estimation and Test

In this section, we consider the finite sample estimation of the intercept and slope parameters in a measurement error model when κ_{xx} is known. Here we consider first the conditional model only to compare the results with the standard model to ascertain the influence of measurement errors on the estimators. The conditional model is given by

$$Y_{0t} = \nu_0 + \nu_1 X_{0t} + Z_t; \quad t = 1, 2, \dots, n \quad (2.2.12)$$

where $Z_t \sim N(0, \sigma_{zz})$. This model is exactly similar to the standard model except that (i) ν_0 is a translation of β_0 , (ii) ν_1 is a scaled version of β_1 and (iii) the variance, σ_{zz} is inflated version of σ_{ee} . The conditional estimation of $(\beta_0, \beta_1)'$ is given by

$$\begin{pmatrix} \beta_{0n}^* \\ \beta_{1n}^* \end{pmatrix} = \begin{pmatrix} \bar{Y}_0 - \beta_{1n}^* \bar{X}_0 \\ \kappa_{xx}^{-1} \tilde{\nu}_{1n} \end{pmatrix} \quad (2.2.13)$$

Then,

$$\begin{pmatrix} \beta_{0n}^* - \beta_0 \\ \beta_{1n}^* - \beta_1 \end{pmatrix} \sim N_2 \left\{ \begin{pmatrix} \beta_1 \tau \\ 0 \end{pmatrix}, \sigma_{zz} \begin{pmatrix} \frac{1}{n} + \frac{\bar{X}_0^2}{\kappa_{xx}^2 S_{X_0X_0}} & -\frac{\bar{X}_0}{\kappa_{xx}^2 S_{X_0X_0}} \\ -\frac{\bar{X}_0}{\kappa_{xx}^2 S_{X_0X_0}} & \frac{1}{\kappa_{xx}^2 S_{X_0X_0}} \end{pmatrix} \right\} \quad (2.2.14)$$

where $\tau = -(1 - \kappa_{xx})(\bar{X}_0 - \mu_x)$. Thus, β_{1n}^* is an unbiased estimator of β_1 and β_{0n}^* is an unbiased estimator of $\beta_0 - \beta_1(1 - \kappa_{xx})(\bar{X}_0 - \mu_x)$ and the marginal distribution of β_{1n}^* is $N\left(\beta_1, \frac{\sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}}\right)$ while the marginal distribution of β_{0n}^* is $N\left(\beta_0 + \beta_1 \tau, \sigma_{zz}\left(\frac{1}{n} + \frac{\bar{X}_0^2}{\kappa_{xx}^2 S_{X_0 X_0}}\right)\right)$.

At this point we indicate that the standard model is obtained by taking $\kappa_{xx} = 1$ in (2.2.13) and (2.2.14).

Test of β_1

In order to test the null hypothesis $H_0 : \beta_1 = 0$ against the alternative $H_A : \beta_1 \neq 0$, we consider the likelihood ratio test statistic

$$\mathcal{L}_n^* = \frac{\kappa_{xx}^2 \beta_{1n}^{*2} S_{X_0 X_0}}{\tilde{\sigma}_{zz}}. \quad (2.2.15)$$

The exact distribution of $\mathcal{L}_n^*$ follows a noncentral F -distribution with $(1, m)$ d.f. with noncentrality parameter $\frac{1}{2} \Delta^{*2}$ where

$$\begin{aligned} \Delta^{*2} &= \frac{\kappa_{xx}^2 \beta_1^2 S_{X_0 X_0}}{\sigma_{zz}} = \kappa_{xx}^2 \Delta^2 \left\{ 1 + \beta_1^2 (1 - \kappa_{xx}) \frac{\sigma_{xx}}{\sigma_{ee}} \right\}^{-1} \\ &< \Delta^2 = \frac{\beta_1^2 S_{X_0 X_0}}{\sigma_{ee}} \end{aligned} \quad (2.2.16)$$

The PTE of β_1 is given by

$$\beta_{1n}^{*PT} = \beta_{1n}^* - \beta_{1n}^* I(\mathcal{L}_n^* < F_\alpha). \quad (2.2.17)$$

The bias and MSE expressions of β_{1n}^* and β_{1n}^{*PT} are given by

$$(i) \ b_1(\beta_{1n}^*) = 0, \ M_1(\beta_{1n}^*) = \frac{\sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} \left(> M_1(\tilde{\beta}_{1n}) \right) \quad (2.2.18)$$

and

$$\begin{aligned}
(ii) \quad b_2(\beta_{1n}^{*PT}) &= -\beta_1 G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^{*2}\right), \\
M_2(\beta_{1n}^{*PT}) &= M_1(\beta_{1n}^*) - \frac{\sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} \left[G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^{*2}\right) \right. \\
&\quad \left. - \Delta^{*2} \left\{ 2G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^{*2}\right) - G_{5,m}\left(\frac{1}{5}F_\alpha; \Delta^{*2}\right) \right\} \right].
\end{aligned} \tag{2.2.19}$$

Figure 2.1 displays the graph of the efficiency function of the PTE's relative to UE of β_1 with measurements error ($n = 6, \kappa_{xx}^2 = 0.8$ and $\alpha = 0.15$) against Δ^2 .

Comparison of β_{1n}^* and β_{1n}^{*PT} (measurement) relative to $\tilde{\beta}_{1n}$ and $\hat{\beta}_{1n}^{PT}$ (standard) respectively.

We observe that the relative efficiency of β_{1n}^* compared to $\tilde{\beta}_{1n}$ (standard model) is

$$MRE(\beta_{1n}^* : \tilde{\beta}_{1n}) = \kappa_{xx}^2 \left\{ 1 + \beta_1^2 (1 - \kappa_{xx}) \frac{\sigma_{xx}}{\sigma_{ee}} \right\}^{-1} \leq 1 \tag{2.2.20}$$

which shows that measurement errors has increased the variability in the unrestricted estimator of β_1 . The amount of increase depends on the quantity κ_{xx}^2 and $\beta_1^2 (1 - \kappa_{xx}) \frac{\sigma_{xx}}{\sigma_{ee}}$.

Also we find that $\Delta^{*2} \leq \Delta^2$ which implies that

$$G_{\nu_1, \nu_2}(x; \Delta^{*2}) \geq G_{\nu_1, \nu_2}(x; \Delta^2) \tag{2.2.21}$$

Consequently $M_2(\beta_{1n}^{*PT}) \geq M_2(\hat{\beta}_{1n}^{PT})$ for all (Δ^{*2}, Δ^2) . Thus the variability in β_{1n}^{*PT} is more than the variability in $\hat{\beta}_{1n}^{PT}$ for the standard model.

Figure 2.1: Graph of $MRE(\hat{\beta}_{1n}^{PT} : \tilde{\beta}_{1n})$ and $MRE(\beta_{1n}^{*PT} : \beta_{1n}^*)$

Figure 2.2: Graph of $MRE(\beta_{1n}^{*PT} : \hat{\beta}_{1n}^{PT})$

The relative efficiency of β_{1n}^{*PT} compared to $\hat{\beta}_{1n}^{PT}$ is given by the expression

$$\begin{aligned} MRE(\beta_{1n}^{*PT} : \hat{\beta}_{1n}^{PT}) &= \kappa_{xx}^2 \left\{ 1 + \beta_1^2 (1 - \kappa_{xx}) \frac{\sigma_{xx}}{\sigma_{ee}} \right\}^{-1} \\ &\times \left[1 - G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^2 \right) + \Delta^2 \left\{ 2G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^2 \right) \right. \right. \\ &\quad \left. \left. - G_{5,m} \left(\frac{1}{5} F_\alpha; \Delta^2 \right) \right\} \right] \left[1 - G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) \right. \\ &\quad \left. + \Delta^{*2} \left\{ 2G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) - G_{5,m} \left(\frac{1}{5} F_\alpha; \Delta^{*2} \right) \right\} \right]^{-1}. \end{aligned} \quad (2.2.22)$$

Note that as $\alpha \rightarrow 1$, $MRE(\beta_{1n}^{*PT} : \hat{\beta}_{1n}^{PT}) \rightarrow MRE(\beta_{1n}^* : \tilde{\beta}_{1n})$. Figure 2.2 displays the graph of the efficiency function of the PTE(with measurement errors) relative to PTE (without measurement errors) of β_1 ($n = 6$, $\kappa_{xx}^2 = 0.8$ and $\alpha = 0.15$). Now we proceed to the estimation of β_0 in the next section.

Estimation of β_0

As in section 2, we have three estimators, namely,

- (i) the restricted estimator,

$$\bar{\beta}_{0n} = \bar{Y}, \quad (2.2.23)$$

- (ii) the unrestricted estimator,

$$\beta_{0n}^* = \bar{Y}_0 - \beta_{1n}^* \bar{X}_0 \quad (2.2.24)$$

- and (iii) the PT estimator,

$$\beta_{0n}^{*PT} = \beta_{0n}^* + \beta_{1n}^* I(\mathcal{L}_n^* < F_\alpha). \quad (2.2.25)$$

The bias and MSE expressions of $\bar{\beta}_{0n}$ are given by

$$b_1(\bar{\beta}_{0n}) = \beta_1 \{ \mu_x + \kappa_{xx} (\bar{X}_0 - \mu_x) \} \quad (2.2.26)$$

$$\text{and } M_1(\bar{\beta}_{0n}) = \sigma_{zz} \left\{ \frac{1}{n} + \frac{\Delta^{*2}}{\kappa_{xx}^2 S_{X_0 X_0}} \left[\mu_x (1 - \kappa_{xx}) + \bar{X}_0 \kappa_{xx} \right]^2 \right\} \quad (2.2.27)$$

The bias and MSE of β_{0n}^* are then given by

$$\begin{aligned} b_2(\beta_{0n}^*) &= -\beta_1(1 - \kappa_{xx})(\bar{X}_0 - \mu_x), \\ M_2(\beta_{0n}^*) &= \sigma_{zz} \left[\frac{1}{n} + \frac{\bar{X}_0^2}{\kappa_{xx}^2 S_{X_0 X_0}} \right. \\ &\quad \left. + \frac{\Delta^{*2}}{\kappa_{xx}^2 S_{X_0 X_0}} (1 - \kappa_{xx})^2 (\bar{X}_0 - \mu_x)^2 \right] \left(\geq M_1(\bar{\beta}_{0n}) \right). \end{aligned} \quad (2.2.28)$$

The bias and MSE expression of β_{0n}^{*PT} are given by

$$\begin{aligned} b_3(\beta_{0n}^{*PT}) &= -\beta_1(1 - \kappa_{xx})(\bar{X}_0 - \mu_x) + \bar{X}_0 \beta_1 G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right), \\ \text{and } M_3(\beta_{0n}^{*PT}) &= M_2(\beta_{0n}^*) - \frac{\bar{X}_0^2 \sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) \\ &\quad + \frac{\sigma_{zz} \Delta^{*2}}{\kappa_{xx}^2 S_{X_0 X_0}} \left[2\bar{X}_0 \left\{ \kappa_{xx} \bar{X}_0 + (1 - \kappa_{xx}) \mu_x \right\} \right. \\ &\quad \left. \times G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) - \bar{X}_0^2 G_{5,m} \left(\frac{1}{5} F_\alpha; \Delta^{*2} \right) \right]. \end{aligned} \quad (2.2.29)$$

The relative efficiency of $\bar{Y}_0$ compared to $\hat{\beta}_{0n}$ in the case of standard model is given by

$$\begin{aligned} MRE(\bar{\beta}_{0n}; \hat{\beta}_{0n}) &= \frac{\sigma_{ee}}{\sigma_{zz}} \left(\frac{1}{n} + \frac{\bar{X}_0^2}{S_{X_0 X_0}} \right) \\ &\quad \times \left(\frac{1}{n} + \frac{\Delta^{*2}}{\kappa_{xx}^2 S_{X_0 X_0}} [\mu_x(1 - \kappa_{xx}) + \bar{X}_0 \kappa_{xx}]^2 \right)^{-1} \end{aligned} \quad (2.2.30)$$

As $\Delta^{*2} \rightarrow \infty$,

$$MRE(\bar{\beta}_{0n}; \hat{\beta}_{0n}) \rightarrow 0,$$

and as $\Delta^{*2} \rightarrow 0$,

$$MRE(\bar{\beta}_{0n}; \hat{\beta}_{0n}) \rightarrow \left(1 + \frac{n\bar{X}_0^2}{S_{X_0 X_0}} \right) \left\{ 1 + \beta_1^2 (1 - \kappa_{xx}) \frac{\sigma_{xx}}{\sigma_{ee}} \right\}^{-1}.$$

Figure 2.3: Graph of $MRE(\bar{\beta}_0 : \hat{\beta}_0)$ Figure 2.4: Graph of $MRE(\beta_{0n}^{*PT} : \hat{\beta}_{0n}^{PT})$

The relative efficiency of β_{0n}^* compared to $\tilde{\beta}_{0n}$ is given by

$$\begin{aligned} MRE(\beta_{0n}^* : \tilde{\beta}_{0n}) &= \frac{\sigma_{ee}}{\sigma_{zz}} \left[1 + \frac{n\bar{X}_0^2}{S_{X_0X_0}} \right] \left[1 + \frac{n\bar{X}_0^2}{\kappa_{xx}^2 S_{X_0X_0}} \right. \\ &\quad \left. + \frac{n\Delta^{*2}}{\kappa_{xx}^2 S_{X_0X_0}} (1 - \kappa_{xx})^2 (\bar{X}_0 - \mu_x)^2 \right]^{-1} \end{aligned} \quad (2.2.31)$$

and the relative efficiency of β_{0n}^{*PT} compared to $\hat{\beta}_{0n}^{PT}$ is given by

$$\begin{aligned} MRE(\beta_{0n}^{*PT} : \hat{\beta}_{0n}^{PT}) &= \frac{\sigma_{ee}}{\sigma_{zz}} \left[1 + \frac{n\bar{X}_0^2}{S_{X_0X_0}} - \frac{n\bar{X}_0^2}{S_{X_0X_0}} \left\{ G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^2 \right) \right. \right. \\ &\quad \left. \left. - \Delta^2 \left(2G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^2 \right) - G_{5,m} \left(\frac{1}{5} F_\alpha; \Delta^2 \right) \right) \right\} \right] \\ &\quad \times \left[1 + \frac{n\Delta^{*2}}{\kappa_{xx}^2 S_{X_0X_0}} \left\{ (1 - \kappa_{xx})^2 (\bar{X}_0 - \mu_x)^2 \right. \right. \\ &\quad \left. \left. + \frac{n\bar{X}_0^2}{\kappa_{xx}^2 S_{X_0X_0}} \left(1 - G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) \right) \right. \right. \\ &\quad \left. \left. + 2\bar{X}_0 \left\{ \kappa_{xx} \bar{X}_0 + (1 - \kappa_{xx}) \mu_x \right\} G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) \right. \right. \\ &\quad \left. \left. - \bar{X}_0^2 G_{5,m} \left(\frac{1}{5} F_\alpha; \Delta^{*2} \right) \right\} \right]^{-1} \end{aligned} \quad (2.2.32)$$

When κ_{xx} varies from zero to 1, $MRE(\beta_{0n}^{*PT} : \hat{\beta}_{0n}^{PT})$ varies from 0 to 1. Figure 2.3 and Figure 2.4 display the graphs of the efficiency functions of restricted estimator (with measurement errors) relative to restricted estimator (without measurement errors) of β_0 and PTE (with measurement errors) relative to PTE (without measurement errors) of β_0 , respectively ($n = 6$, $\kappa_{xx}^2 = 0.8$, $\alpha = 0.15$, $\frac{\bar{X}_0^2}{S_{X_0X_0}} = 2$ and $\frac{\mu_x}{\bar{X}_0} = 2$).

2.2.3 Unconditonal Expressions and Comparisons

In this section, we consider the unconditional bias and MSE expressions of β_{1n}^* and β_{1n}^{*PT} . These are obtained by integrating the conditional bias and MSE expressions

Figure 2.5: Graph of Power functions

with respect to the marginal distribution of $S_{X_0X_0}$, namely

$$\frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} e^{-\frac{S_{X_0X_0}}{2\sigma_{XX}}} \left(\frac{S_{X_0X_0}}{\sigma_{XX}} \right)^{\frac{n-1}{2}-1} d\left(\frac{S_{X_0X_0}}{\sigma_{XX}} \right), \quad (2.2.33)$$

together with the marginal distribution of $\bar{X}_0$. Thus, letting

$$\Delta_u^{*2} = \frac{\beta_1^2 \kappa_{xx}^2 \sigma_{XX}}{\sigma_{zz}} \quad (2.2.34)$$

and

$$\begin{aligned} & G_{1+2i,m}^{(j)} \left(\frac{1}{1+2i} F_\alpha : \Delta_u^{*2} \right) \\ &= \sum_{r \geq 0} \frac{2^{j-2} \Gamma(\frac{n-1}{2} + r + j - 2) (\Delta_u^{*2})^r}{\Gamma(r+1) \Gamma(\frac{n-1}{2}) (1 + \Delta_u^{*2})^{\frac{n-1}{2} + r + j - 2}} I_y \left(\frac{1+2i}{2} + r : \frac{m}{2} \right) \end{aligned} \quad (2.2.35)$$

Table 2.1: Power for $\alpha = 0.15, \alpha = 0.2$ and $\alpha = 0.25$

$\alpha \backslash \Delta^2$	0	1	2	3	4	5
0.15	0.15001	0.78181	0.93188	0.97303	0.98738	0.99335
0.20	0.20001	0.83498	0.95177	0.98153	0.99153	0.99560
0.25	0.25001	0.87156	0.96447	0.98675	0.99402	0.99693

for $j = 1, 2$ and $i = 1, 2$, where $y = \frac{F_\alpha}{n-1+F_\alpha}$. For $j = 2$ we get the unconditional cdf of $\mathcal{L}_n^*$ under H_A and the power function of the test at α -level. Figure 2.5 displays the graph of the power functions ($\alpha = 0.15, 0.2$ and 0.25) and Table 2.1 gives the power values for $\Delta^2 = 0$ to 5 ($\alpha = 0.15, 0.2$ and 0.25).

Estimation of β_1

The unconditional expressions of bias and MSE of β_{1n}^* are then obtained as

$$(i) \ b_1^*(\beta_{1n}^*) = 0, \ M_1^*(\beta_{1n}^*) = \frac{\sigma_{zz}}{(n-3)\kappa_{xx}^2\sigma_{XX}}; \quad (2.2.36)$$

$$\text{and } (ii) \ b_2^*(\beta_{1n}^{*PT}) = -\beta_1 G_{3,m}^{(2)}\left(\frac{1}{3}F_\alpha; \Delta_u^{*2}\right),$$

$$\begin{aligned} M_2^*(\beta_{1n}^{*PT}) &= M_1^*(\beta_{1n}^*) - \frac{\sigma_{zz}}{\kappa_{xx}^2\sigma_{XX}} \left[G_{3,m}^{(1)}\left(\frac{1}{3}F_\alpha; \Delta_u^{*2}\right) \right. \\ &\quad \left. - \Delta_u^{*2} \left\{ 2G_{3,m}^{(2)}\left(\frac{1}{3}F_\alpha; \Delta_u^{*2}\right) - G_{5,m}^{(2)}\left(\frac{1}{5}F_\alpha; \Delta_u^{*2}\right) \right\} \right], \end{aligned} \quad (2.2.37)$$

respectively.

The efficiency of β_{1n}^{*PT} with respect to β_{1n}^* is

$$\begin{aligned} MRE^*(\beta_{1n}^{*PT} : \beta_{1n}^*) &= \frac{M_1^*(\beta_{1n}^*)}{M_2^*(\beta_{1n}^{*PT})} = \left[1 + (n-3)g_u(\Delta_u^{*2}) \right]^{-1} \\ \text{where } g_u(\Delta_u^{*2}) &= -G_{3,m}^{(1)}\left(\frac{1}{3}F_\alpha; \Delta_u^{*2}\right) + \Delta_u^{*2} \left\{ 2G_{3,m}^{(2)}\left(\frac{1}{3}F_\alpha; \Delta_u^{*2}\right) \right. \\ &\quad \left. - G_{5,m}^{(2)}\left(\frac{1}{5}F_\alpha; \Delta_u^{*2}\right) \right\} \end{aligned} \quad (2.2.38)$$

Figure 2.6 presents the graph of MSE of PTE of β_1 and Figure 2.7 displays the graph of the efficiency function of the PTE relative to UE of β_1 ($n = 6$, $\kappa_{xx}^2 = 0.8$ and $\alpha = 0.15$).

Now the conclusions which are similar to section 2 follow:

(i) $MRE^*(\beta_{1n}^{*PT} : \beta_{1n}^*) \gtrless 1$ according as

$$\Delta_u^{*2} \lessgtr \frac{G_{3,m}^{(1)}(\frac{1}{3}F_\alpha : \Delta_u^{*2})}{2G_{3,m}^{(2)}(\frac{1}{3}F_\alpha : \Delta_u^{*2}) - G_{5,m}^{(2)}(\frac{1}{5}F_\alpha : \Delta_u^{*2})} < 1. \quad (2.2.39)$$

(ii) Maximum attainable efficiency occurs at $\Delta_u^{*2} = 0$ with the value

$$\begin{aligned} \max_{0 \leq \Delta_u^{*2} \leq \infty} MRE^*(\beta_{1n}^{*PT} : \beta_{1n}^*) = \\ \left\{ 1 - (n-3)G_{3,m}^{(1)}\left(\frac{1}{3}F_\alpha : 0\right) \right\}^{-1} = \left\{ 1 - G_{3,m}\left(\frac{1}{3}F_\alpha; 0\right) \right\}^{-1} \end{aligned} \quad (2.2.40)$$

The efficiency goes to unity as $\Delta_u^{*2} \rightarrow \infty$.

(iii) PTE can not be uniformly better than β_{1n}^* but one may determine a PTE with minimum guaranteed efficiency, say, E_0 by choosing α^* by the maximin rule

$$\max_{\alpha} \min_{\Delta_u^{*2}} MRE^*(\alpha : \Delta_u^{*2}) \geq E_0. \quad (2.2.41)$$

Table 2.3 provides the minimum(E_{min}) and maximum(E_{max}) relative efficiency of the PTE relative to UE of β_1 for varying values of α and Δ_u^{*2} -value at which the minimum relative efficiency occur. To choose an optimum level of significance α^* for given fixed efficiency $MRE^* = E^*$ (say), one goes through the tabular values of efficiency and α , then choose α^* corresponding the closest efficiency to E^* . This value α^* guarantees the minimum value of efficiency E^* which may increase if H_0 is true. For example, if the experimenter wants to have an estimator which has relative efficiency no less than 0.7426, then using Table 2.3, one would use $\alpha^* = 0.25$

Figure 2.6: Graph of $M_2(\beta_{1n}^{*PT})$ Figure 2.7: Graph of $MRE(\beta_{1n}^{*PT} : \beta_{1n}^*)$

Figure 2.8: Graph of $M_2(\beta_{0n}^*)$ and $M_3(\beta_{0n}^{*PT})$

because it maximizes $MRE^*(\alpha, 0)$ and the maximum attainable relative efficiency is 1.5108.

(iv) As $\alpha \rightarrow 1$, $\beta_{1n}^{*PT} \rightarrow \beta_{1n}^*$.

Estimation of β_0

Now we consider the unconditional expressions of bias and MSE of $\bar{\beta}_{0n}$, β_{0n}^* and β_{0n}^{*PT} . They are

$$\begin{aligned}
 (i) \quad b_1^*(\bar{\beta}_{0n}) &= \beta_1 \mu_x, \\
 M_1^*(\bar{\beta}_{0n}) &= \sigma_{zz} \left[\frac{1}{n} + \Delta_u^{*2} \left\{ \frac{1}{n} + \frac{\mu_x^2}{\kappa_{xx}^2 \sigma_{XX}} \right\} \right]; \\
 (ii) \quad b_2^*(\beta_{0n}^*) &= 0, \\
 M_2^*(\beta_{0n}^*) &= \sigma_{zz} \left[\frac{1}{n} + \frac{1}{n(n-3)\kappa_{xx}^2} + \frac{1}{(n-3)\kappa_{xx}^2} \frac{\mu_x^2}{\sigma_{XX}} \right. \\
 &\quad \left. + \frac{\Delta_u^{*2}}{n\kappa_{xx}^2} (1 - \kappa_{xx})^2 \right];
 \end{aligned} \tag{2.2.42}$$

$$\begin{aligned}
 (iii) \quad b_3^*(\beta_{0n}^{*PT}) &= \beta_1 \mu_x G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right), \\
 M_3^*(\beta_{0n}^{*PT}) &= M_2^*(\beta_{0n}^*) - \frac{\sigma_{zz} \mu_x^2}{\kappa_{xx}^2 \sigma_{XX}} \left(\frac{\sigma_{XX}}{n\mu_x^2} + 1 \right) G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right) \\
 &\quad + \frac{\sigma_{zz} \mu_x^2 \Delta_u^{*2}}{\kappa_{xx}^2 \sigma_{XX}} \left[2 \left(\frac{\kappa_{xx} \sigma_{XX}}{n\mu_x^2} + 1 \right) G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right) \right. \\
 &\quad \left. - \left(\frac{\sigma_{XX}}{n\mu_x^2} + 1 \right) G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha; \Delta_u^{*2} \right) \right]
 \end{aligned} \tag{2.2.43}$$

The relative efficiency of $\bar{\beta}_{0n}$ and β_{0n}^{*PT} compared to β_{0n}^* are given by

$$\begin{aligned}
 MRE_1^*(\bar{\beta}_{0n} : \beta_{0n}^*) &= \left[\frac{1}{n} + \frac{1}{n(n-3)\kappa_{xx}^2} \right. \\
 &\quad \left. + \frac{1}{(n-3)\kappa_{xx}^2} \frac{\mu_x^2}{\sigma_{XX}} + \frac{\Delta_u^{*2}}{n\kappa_{xx}^2} (1 - \kappa_{xx})^2 \right] \\
 &\quad \times \left[\frac{1}{n} + \Delta_u^{*2} \left\{ \frac{1}{n} + \frac{\mu_x^2}{\kappa_{xx}^2 \sigma_{XX}} \right\} \right]^{-1}
 \end{aligned} \tag{2.2.44}$$

and

$$\begin{aligned}
MRE_2^*(\beta_{0n}^{*PT} : \beta_{0n}^*) &= \left[1 - \frac{\mu_x^2}{\kappa_{xx}^2 \sigma_{XX}} \left\{ \frac{1}{n} + \frac{1}{n(n-3)\kappa_{xx}^2} \right. \right. \\
&\quad \left. \left. + \frac{\mu_x^2}{(n-3)\kappa_{xx}^2 \sigma_{XX}} + \frac{\Delta_u^{*2}}{n\kappa_{xx}^2} (1 - \kappa_{xx})^2 \right\} \right]^{-1} \\
&\times \left[\left(\frac{\sigma_{XX}}{n\mu_x^2} + 1 \right) G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right) \right. \\
&\quad - \Delta_u^{*2} \left\{ 2 \left(\frac{\kappa_{xx} \sigma_{XX}}{n\mu_x^2} + 1 \right) G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right) \right. \\
&\quad \left. \left. - \left(\frac{\sigma_{XX}}{n\mu_x^2} + 1 \right) G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha; \Delta_u^{*2} \right) \right\} \right]^{-1} \quad (2.2.45)
\end{aligned}$$

respectively.

Figure 2.8 presents the graphs of MSE of UE of β_0 and PTE of β_0 . Figure 2.9 and Figure 2.10 display the graphs of the efficiency functions of restricted estimator relative to UE of β_0 and PTE relative to UE of β_0 , respectively ($n = 6$, $\kappa_{xx}^2 = 0.8, \alpha = 0.15$ and $\frac{\mu_x^2}{\sigma_{XX}} = 3$). $MRE_1^*(\bar{\beta}_{0n} : \beta_{0n}^*)$ is a decreasing function of Δ_u^{*2} and has its maximum value $1 + \frac{1}{(n-3)\kappa_{xx}^2} \left(1 + \frac{n\mu_x^2}{\sigma_{XX}} \right)$ at $\Delta_u^{*2} = 0$ and decreases to 0 as $\Delta_u^{*2} \rightarrow \infty$. The efficiency is one if $\Delta_u^{*2} = (n-3)^{-1} \left\{ 1 - 2(1 - \kappa_{xx})(1 + \frac{n\mu_x^2}{\sigma_{XX}})^{-1} \right\}^{-1} = \Delta_{00}^{*2}$ (say). Thus, if $0 < \Delta_0^{*2} < \Delta_{00}^{*2}$, then $\bar{\beta}_{0n}$ is better than β_{0n}^* otherwise opposite conclusion holds.

Now, consider the efficiency $MRE_2^*(\beta_{0n}^{*PT} : \beta_{0n}^*)$. The graph of $MRE_2^*(\beta_{0n}^{*PT} : \beta_{0n}^*)$ for fixed α has a maximum at $\Delta_u^{*2} = 0$ with maximum value

$$\begin{aligned}
&\left[1 - \left\{ \frac{1}{n} + \frac{1}{n(n-3)\kappa_{xx}^2} + \frac{\mu_x^2}{(n-3)\kappa_{xx}^2 \sigma_{XX}} \right\}^{-1} \right. \\
&\quad \left. \times \left\{ \left(\frac{1}{n\kappa_{xx}^2} + \frac{\mu_x^2}{\kappa_{xx}^2 \sigma_{XX}} \right) G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : 0 \right) \right\} \right]^{-1}
\end{aligned}$$

then monotonically decreases crossing the one-line to a minimum, then increases as Δ_u^{*2} increases towards the 1 as $\Delta_u^{*2} \rightarrow \infty$. Further,

Figure 2.9: Graph of $MRE_1(\bar{\beta}_{0n} : \beta_{0n}^*)$

Figure 2.10: Graphs of $MRE_2(\beta_{0n}^{*PT} : \beta_{0n}^*)$

(i) $MRE_2^*(\beta_{0n}^{*PT} : \beta_{0n}^*) \gtrless 1$ according as

$$\begin{aligned} \Delta_u^{*2} &\gtrless \left[\left(1 + \frac{\sigma_{XX}}{n\mu_x^2} \right) G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right) \right] \\ &\times \left[2(1 - \kappa_{xx}) G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right) \right. \\ &+ \left(1 + \frac{\sigma_{XX}}{n\mu_x^2} \right) \left(2\kappa_{xx} G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha; \Delta_u^{*2} \right) \right. \\ &\left. \left. - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha; \Delta_u^{*2} \right) \right) \right]^{-1}. \end{aligned} \quad (2.2.46)$$

(ii) For the minimum guaranteed efficiency E_0 , optimum α^* is determined by the equation

$$\max_{\alpha} \min_{\Delta^2} MRE_2^*(\alpha : \Delta^2) \geq E_0. \quad (2.2.47)$$

Table 2.2 provides the minimum(E_{min}) and maximum(E_{max}) relative efficiency of the PTE relative to UE of β_0 for the sample size $n=6$ and for selected $\kappa_{xx}^2 = \frac{\sigma_{xx}}{\sigma_{XX}} = 0.8$, varying values of α and Δ_u^{*2} -value at which the minimum relative efficiency occur. To choose an optimum level of significance α^* for given fixed efficiency $MRE_2^* = E^*$ (say), one goes through the tabular values of efficiency and α , then choose α^* corresponding the closest efficiency to E^* . This value α^* guarantees the minimum value of efficiency E^* which may increase if H_0 is true. For example, with $\frac{\mu_x^2}{\sigma_{XX}} = 1$ and $\kappa_{xx}^2 = 0.8$, if the experimenter wants to have an estimator which has relative efficiency no less than 0.6753, then using Table 2.2, one would use $\alpha^* = 0.15$ because it maximizes $MRE_2^*(\alpha, 0)$ and the maximum attainable relative efficiency is 1.6347.

(iii) As $\alpha \rightarrow 1$, $\beta_{0n}^{*PT} \rightarrow \beta_{0n}^*$ and as $\alpha \rightarrow 0$, $\beta_{0n}^{*PT} \rightarrow \bar{\beta}_{0n}$.

α	κ_{xx}^2	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.05	$E_{\max}$	1.4654	1.6123	1.7469	1.8708	1.9852	2.0911	2.1895	2.2812	2.3667	2.4468	2.7804	3.0329	3.2306	3.3896	3.5202	3.6295	3.7222	3.8019
	$E_{\min}$	0.6281	0.5757	0.5407	0.5158	0.4971	0.4826	0.4710	0.4615	0.4536	0.4469	0.4246	0.4121	0.4040	0.3983	0.3942	0.3910	0.3885	0.3864
	$\Delta_{\min}^2$	2.6840	2.6755	2.6730	2.6726	2.6730	2.6738	2.6747	2.6757	2.6766	2.6775	2.6812	2.6837	2.6856	2.6870	2.6881	2.6889	2.6896	2.6902
0.10	$E_{\max}$	1.3462	1.4440	1.5295	1.6048	1.6717	1.7315	1.7853	1.8340	1.8781	1.9185	2.0767	2.1868	2.2678	2.3299	2.3790	2.4189	2.4518	2.4795
	$E_{\min}$	0.7436	0.7001	0.6698	0.6476	0.6305	0.6170	0.6060	0.5970	0.5894	0.5829	0.5609	0.5482	0.5400	0.5342	0.5300	0.5267	0.5240	0.5219
	$\Delta_{\min}^2$	2.0583	2.0484	2.0442	2.0422	2.0413	2.0408	2.0407	2.0407	2.0408	2.0409	2.0419	2.0427	2.0434	2.0439	2.0443	2.0447	2.0450	2.0452
0.15	$E_{\max}$	1.2635	1.3322	1.3903	1.4402	1.4834	1.5212	1.5546	1.5842	1.6108	1.6347	1.7255	1.7861	1.8294	1.8619	1.8872	1.9074	1.9239	1.9377
	$E_{\min}$	0.8112	0.7760	0.7508	0.7319	0.7173	0.7055	0.6959	0.6879	0.6811	0.6753	0.6554	0.6438	0.6362	0.6309	0.6269	0.6238	0.6213	0.6193
	$\Delta_{\min}^2$	1.7633	1.7535	1.7490	1.7466	1.7453	1.7445	1.7440	1.7438	1.7436	1.7435	1.7436	1.7438	1.7441	1.7443	1.7445	1.7447	1.7448	1.7450
0.20	$E_{\max}$	1.2027	1.2523	1.2934	1.3280	1.3575	1.3829	1.4050	1.4245	1.4418	1.4572	1.5146	1.5520	1.5782	1.5976	1.6126	1.6245	1.6341	1.6422
	$E_{\min}$	0.8575	0.8292	0.8087	0.7930	0.7807	0.7708	0.7626	0.7558	0.7500	0.7450	0.7277	0.7175	0.7108	0.7061	0.7025	0.6998	0.6976	0.6958
	$\Delta_{\min}^2$	1.5825	1.5730	1.5685	1.5660	1.5645	1.5636	1.5630	1.5626	1.5623	1.5621	1.5618	1.5618	1.5618	1.5619	1.5620	1.5621	1.5622	1.5622
0.25	$E_{\max}$	1.1564	1.1929	1.2226	1.2472	1.2679	1.2857	1.3010	1.3143	1.3261	1.3365	1.3749	1.3994	1.4165	1.4290	1.4386	1.4461	1.4523	1.4573
	$E_{\min}$	0.8914	0.8689	0.8523	0.8396	0.8295	0.8213	0.8146	0.8089	0.8040	0.7998	0.7852	0.7766	0.7708	0.7667	0.7637	0.7613	0.7594	0.7579
	$\Delta_{\min}^2$	1.4578	1.4486	1.4442	1.4417	1.4402	1.4392	1.4385	1.4380	1.4377	1.4374	1.4368	1.4367	1.4367	1.4367	1.4367	1.4367	1.4368	1.4368
0.30	$E_{\max}$	1.1204	1.1475	1.1692	1.1870	1.2019	1.2145	1.2253	1.2347	1.2429	1.2502	1.2767	1.2934	1.3049	1.3134	1.3198	1.3249	1.3289	1.3323
	$E_{\min}$	0.9171	0.8995	0.8863	0.8761	0.8680	0.8613	0.8558	0.8512	0.8472	0.8437	0.8317	0.8245	0.8198	0.8164	0.8138	0.8118	0.8102	0.8089
	$\Delta_{\min}^2$	1.3658	1.3570	1.3526	1.3501	1.3486	1.3475	1.3468	1.3463	1.3459	1.3457	1.3450	1.3447	1.3446	1.3446	1.3446	1.3446	1.3446	1.3446
0.35	$E_{\max}$	1.0922	1.1122	1.1282	1.1411	1.1519	1.1610	1.1687	1.1754	1.1812	1.1864	1.2050	1.2167	1.2247	1.2305	1.2349	1.2384	1.2412	1.2435
	$E_{\min}$	0.9371	0.9234	0.9131	0.9050	0.8986	0.8933	0.8889	0.8852	0.8820	0.8793	0.8696	0.8637	0.8599	0.8571	0.8550	0.8534	0.8521	0.8510
	$\Delta_{\min}^2$	1.2949	1.2864	1.2822	1.2797	1.2782	1.2771	1.2764	1.2759	1.2755	1.2752	1.2744	1.2741	1.2739	1.2739	1.2738	1.2738	1.2738	1.2738
0.40	$E_{\max}$	1.0698	1.0846	1.0963	1.1057	1.1135	1.1201	1.1256	1.1304	1.1346	1.1383	1.1515	1.1597	1.1653	1.1693	1.1724	1.1748	1.1768	1.1784
	$E_{\min}$	0.9527	0.9422	0.9343	0.9281	0.9231	0.9190	0.9156	0.9126	0.9101	0.9080	0.9003	0.8957	0.8927	0.8904	0.8888	0.8875	0.8865	0.8856
	$\Delta_{\min}^2$	1.2389	1.2306	1.2264	1.2240	1.2225	1.2214	1.2207	1.2202	1.2198	1.2194	1.2186	1.2183	1.2181	1.2180	1.2179	1.2179	1.2179	1.2179
0.45	$E_{\max}$	1.0521	1.0629	1.0714	1.0782	1.0838	1.0885	1.0925	1.0959	1.0989	1.1015	1.1109	1.1167	1.1206	1.1235	1.1256	1.1273	1.1287	1.1298
	$E_{\min}$	0.9650	0.9571	0.9511	0.9464	0.9426	0.9395	0.9369	0.9347	0.9327	0.9311	0.9252	0.9217	0.9193	0.9176	0.9163	0.9153	0.9145	0.9138
	$\Delta_{\min}^2$	1.1936	1.1855	1.1814	1.1791	1.1775	1.1765	1.1758	1.1752	1.1748	1.1745	1.1736	1.1732	1.1730	1.1729	1.1729	1.1728	1.1728	1.1728
0.50	$E_{\max}$	1.0381	1.0459	1.0520	1.0569	1.0608	1.0642	1.0670	1.0694	1.0715	1.0734	1.0799	1.0840	1.0867	1.0887	1.0902	1.0914	1.0923	1.0931
	$E_{\min}$	0.9746	0.9688	0.9644	0.9609	0.9581	0.9558	0.9538	0.9522	0.9508	0.9495	0.9451	0.9425	0.9407	0.9394	0.9384	0.9377	0.9371	0.9366
	$\Delta_{\min}^2$	1.1565	1.1487	1.1447	1.1423	1.1408	1.1398	1.1390	1.1385	1.1381	1.1378	1.1368	1.1365	1.1363	1.1361	1.1361	1.1360	1.1360	1.1359

Table 2.2: Minimum and Maximum Efficiency of PTE relative to UE of β_0 for various values of α when $\kappa_{xx}^2 = 0.8$ (Unconditional case)

2.3 Asymptotic Properties of Estimators when

σ_{uu} or $r = \frac{\sigma_{ee}}{\sigma_{uu}}$ is Known

2.3.1 Estimation when σ_{uu} is Known

Consider model (2.2.1) together with (2.2.2) and assume that σ_{uu} is known. Then, the moment estimators of the slope and intercept parameters are

$$\beta_{1n}^* = (m_{XX} - \sigma_{uu})^{-1} m_{XY}, \text{ if } m_{XX} - \sigma_{uu} > 0 \quad (2.3.1)$$

$$\beta_{0n}^* = \bar{Y}_0 - \beta_{1n}^* \bar{X}_0. \quad (2.3.2)$$

with

$$\tilde{\mu}_x = \bar{X}_0, \tilde{\sigma}_{xx} = m_{XX} - \sigma_{uu}, \tilde{\sigma}_{ee} = m_{YY} - \beta_{1n}^* m_{XY}. \quad (2.3.3)$$

These estimators may also be obtained by substituting a consistent estimator for κ_{xx} which is $(m_{XX} - \sigma_{uu})^{-1} m_{XX}$. A little algebra leads to (2.3.1) noting that m_{XX} is an unbiased moment estimator of $\sigma_{XX} = \sigma_{xx} + \sigma_{uu}$.

The asymptotic distribution of $\sqrt{n} \{(\beta_{0n}^* - \beta_0), (\beta_{1n}^* - \beta_1)\}'$ converges in distribution to a bivariate normal distribution with mean vector $(0, 0)'$ and covariance matrix

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \quad (2.3.4)$$

with

$$\begin{aligned} \Gamma_{11} &= \mu_x^2 \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} + \sigma_{Xv}^2) + \sigma_{vv}, \\ \Gamma_{12} &= \Gamma_{21} = -\mu_x \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} + \sigma_{Xv}^2), \\ \Gamma_{22} &= \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} + \sigma_{Xv}^2). \end{aligned} \quad (2.3.5)$$

where

$$\begin{aligned}
v_t &= Y_{0t} - \beta_0 - \beta_1 X_{0t} = e_t - \beta_1 u_t, \\
\sigma_{vv} &= \sigma_{ee} + \beta_1^2 \sigma_{uu} \\
\sigma_{Xv} &= \sigma_{uv} = -\beta_1 \sigma_{uu}.
\end{aligned} \tag{2.3.6}$$

See Fuller (1987) for the proof.

Consistent estimators of Γ_{11} , Γ_{22} and Γ_{12} are obtained by substitution principle as follow:

$$\begin{aligned}
\tilde{\Gamma}_{11} &= \bar{X}_0^2 \tilde{\sigma}_{xx}^{-2} (\tilde{\sigma}_{XX} \tilde{\sigma}_{vv} + \tilde{\sigma}_{Xv}^2) + \tilde{\sigma}_{vv}, \\
\tilde{\Gamma}_{12} &= \tilde{\Gamma}_{21} = -\bar{X}_0^2 \tilde{\sigma}_{xx}^{-2} (\tilde{\sigma}_{XX} \tilde{\sigma}_{vv} + \tilde{\sigma}_{Xv}^2), \\
\tilde{\Gamma}_{22} &= \tilde{\sigma}_{xx}^{-2} (\tilde{\sigma}_{XX} \tilde{\sigma}_{vv} + \tilde{\sigma}_{Xv}^2).
\end{aligned} \tag{2.3.7}$$

where

$$\tilde{\sigma}_{XX} = \tilde{\sigma}_{xx} + \sigma_{uu}, \quad \tilde{\sigma}_{vv} = \tilde{\sigma}_{ee} + \beta_{1n}^{*2} \sigma_{uu}, \quad \tilde{\sigma}_{Xv} = -\beta_{1n}^* \sigma_{uu}.$$

Now for the test of the hypothesis $H_0 : \beta_1 = 0$, consider the statistic

$$\mathcal{L}_n^* = \frac{n\beta_{1n}^{*2}}{\tilde{\Gamma}_{22}} \tag{2.3.8}$$

As $n \rightarrow \infty$ $\mathcal{L}_n$ follows a chi-square distribution with one d.f. under H_0 . Then the preliminary test estimator of β_1 (when β_1 may be equal to 0) is defined by

$$\beta_{1n}^{*PT} = \beta_{1n}^* - \beta_{1n}^* I(\mathcal{L}_n^* < \mathcal{L}_{n,\alpha}), \tag{2.3.9}$$

where $\mathcal{L}_{n,\alpha}$ is the α -level critical value of the distribution of $\mathcal{L}_n^*$ and $\mathcal{L}_{n,\alpha} \rightarrow \chi_{1,\alpha}^2$ as $n \rightarrow \infty$.

2.3.2 Estimation when $r = \frac{\sigma_{ee}}{\sigma_{uu}}$ is Known.

Now, consider the case when the ratio of variance is known $\left(r = \frac{\sigma_{ee}}{\sigma_{uu}}\right)$. Then, retaining the same notation of estimators the moment estimators of the slope and intercept parameters are given by

$$\beta_{1n}^* = \frac{m_{YY} - r m_{XX} + \sqrt{(m_{YY} - r m_{XX})^2 + 4r(m_{XY})^2}}{2m_{XY}} \quad (2.3.10)$$

and

$$\beta_{0n}^* = \bar{Y}_0 - \beta_{1n}^* \bar{X}_0, \quad (2.3.11)$$

where

$$\begin{aligned} \tilde{\mu}_x &= \bar{X}_0, \quad \tilde{\sigma}_{ee} = r\tilde{\sigma}_{uu}, \\ m_{XX} &= \tilde{\sigma}_{xx} + \tilde{\sigma}_{uu}, \quad m_{XY} = \beta_{1n}^* \tilde{\sigma}_{xx}, \\ m_{YY} &= \beta_{1n}^{*2} \tilde{\sigma}_{xx} + r\tilde{\sigma}_{uu}. \end{aligned} \quad (2.3.12)$$

Then, the asymptotic distribution of $\left\{ \sqrt{n}(\beta_{0n}^* - \beta_0), \sqrt{n}(\beta_{1n}^* - \beta_1) \right\}'$ converges in distribution to a bivariate normal distribution with mean vector $(0, 0)'$ and covariance matrix

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{bmatrix} \quad (2.3.13)$$

with

$$\begin{aligned} \Gamma_{11} &= \mu_x^2 \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} - \sigma_{Xv}^2) + \sigma_{vv}, \\ \Gamma_{12} &= \Gamma_{21} = -\mu_x \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} - \sigma_{Xv}^2), \\ \Gamma_{22} &= \sigma_{xx}^{-2} (\sigma_{XX} \sigma_{vv} - \sigma_{Xv}^2) \end{aligned} \quad (2.3.14)$$

where

$$v_t = Y_{0t} - \beta_0 - \beta_1 X_{0t} = e_t - \beta_1 u_t,$$

$$\sigma_{Xv} = \sigma_{uv} = -\beta_1 \sigma_{uu},$$

$$\text{and } \sigma_{vv} = \sigma_{ee} + \beta_1^2 \sigma_{uu}.$$

For the proof see the theorem 1.3.1 in Fuller (1987). Consistent estimators of Γ_{11} , Γ_{22} and Γ_{12} are obtained by substitution principle. In this too, we define the test statistics for the null hypothesis $H_0 : \beta_1 = 0$ case

$$\mathcal{L}_n^* = \frac{n\beta_{1n}^{*2}}{\tilde{\Gamma}_{22}} \quad (2.3.15)$$

which follows the central chi-square distribution with one d.f. under H_0 and define the PTE of β_1 as the same as (2.3.9).

2.3.3 Pitman-Type Local Alternative Hypothesis

In order to compute the asymptotic bias and MSE of estimators we first consider the asymptotic distribution of β_{1n}^* and β_{1n}^{*PT} in the following manner: First note that under the fixed alternative $H_\delta : \beta_1 = \delta (\neq 0)$, we have

$$\sqrt{n}\beta_{1n}^* = \sqrt{n}(\beta_{1n}^* - \beta_1) + \sqrt{n}\delta \quad (2.3.16)$$

and

$$\mathcal{L}_n^* = \frac{[\sqrt{n}(\beta_{1n}^* - \beta_1) + \sqrt{n}\delta]^2}{\tilde{\Gamma}_{22}} \quad (2.3.17)$$

By standard central limit theorem together with Slutsky's theorem

$$\frac{\sqrt{n}(\beta_{1n}^* - \beta_1)}{\tilde{\Gamma}_{22}^{1/2}} \sim N(0, 1) \quad (2.3.18)$$

and $\sqrt{n}\delta \rightarrow \infty$ as $n \rightarrow \infty$. It follows that $\mathcal{L}_n^* \rightarrow \infty$ as $n \rightarrow \infty$. Consequently, for all fixed alternatives H_δ , $P\{\mathcal{L}_n^* > x\}$ tends to 1 for all x .

Now consider the quadratic difference between $\sqrt{n}\beta_{1n}^*$ and $\sqrt{n}\beta_{1n}^{*PT}$ which is

$$\frac{n(\beta_{1n}^* - \beta_{1n}^{*PT})^2}{\tilde{\Gamma}_{22}} = \mathcal{L}_n^* I(\mathcal{L}_n^* < \mathcal{L}_{n,\alpha}) \leq \mathcal{L}_{n,\alpha} I(\mathcal{L}_n^* < \mathcal{L}_{n,\alpha}) \quad (2.3.19)$$

Also, $E(I(\mathcal{L}_n^* < \mathcal{L}_{n,\alpha})) = P_\delta(\mathcal{L}_n^* < \mathcal{L}_{n,\alpha})$. Therefore, for all $\delta \neq 0$,

$$\lim_{n \rightarrow \infty} E \left\{ \frac{n(\beta_{1n}^* - \beta_{1n}^{*PT})^2}{\tilde{\Gamma}_{22}} \right\} \leq \chi_{1,\alpha}^2 \lim_{n \rightarrow \infty} P_\delta(\mathcal{L}_n^* < \chi_{1,\alpha}^2) = 0 \quad (2.3.20)$$

which implies that

$$\frac{\sqrt{n}(\beta_{1n}^{*PT} - \beta_1)}{\tilde{\Gamma}_{22}^{1/2}} = \frac{\sqrt{n}(\beta_{1n}^* - \beta_1)}{\tilde{\Gamma}_{22}^{1/2}} + o_p(1) \quad (2.3.21)$$

Hence the asymptotic bias and MSE of $\sqrt{n}\beta_{1n}^{*PT}$ is the same as that of $\sqrt{n}\beta_{1n}^*$ under fixed alternative H_δ . To obtain reasonable asymptotic distribution of $\mathcal{L}_n$ as well as $\sqrt{n}\beta_{1n}^*$ we consider the sequence of local Pitman type alternatives $K_{(n)}$ which converges to H_0 as $n \rightarrow \infty$ defined by

$$K_{(n)} : \beta_{1(n)} = n^{-1/2}\delta, \quad \text{where } \delta \text{ is a fixed number.} \quad (2.3.22)$$

Thus we obtain following theorem

Theorem 2.3.1 Under $\{K_{(n)}\}$,

(i) $\mathcal{L}_n$ asymptotically follows a noncentral chi-square distribution with one d.f. and noncentrality parameter $\frac{1}{2}\Delta_0^2$ where

$$\Delta_0^2 = \frac{\delta^2 \sigma_{xx}^2}{\sigma_{XX} \sigma_{ee}} \quad (2.3.23)$$

and

(ii)

$$\frac{\sqrt{n}(\beta_{1n}^{*PT} - \beta_{1(n)})}{\tilde{\Gamma}_{22}^{1/2}} = Z - (Z + \Delta_0)I((Z + \Delta_0)^2 < \chi_{1,\alpha}^2) \quad (2.3.24)$$

where $Z \sim N(0, 1)$.

Proof: Consider first

$$\frac{\sqrt{n}\beta_{1n}^*}{\tilde{\Gamma}_{22}^{1/2}} = \frac{\sqrt{n}(\beta_{1n}^* - \beta_{1(n)})}{\tilde{\Gamma}_{22}^{1/2}} + \frac{\delta}{\tilde{\Gamma}_{22}^{1/2}} \quad (2.3.25)$$

Then we observe that

$$\frac{\sqrt{n}(\beta_{1n}^* - \beta_{1(n)})}{\tilde{\Gamma}_{22}^{1/2}} \sim N(0, 1) \quad \text{and} \quad \frac{\sqrt{n}\beta_{1n}^*}{\tilde{\Gamma}_{22}^{1/2}} \sim N(\Delta_0, 1) \quad (2.3.26)$$

Consequently, $\mathcal{L}_n^* \xrightarrow{\mathcal{D}} \chi_1^2(\Delta_0^2)$ under $K_{(n)}$.

Next we observe that

$$\begin{aligned} \frac{\sqrt{n}(\beta_{1n}^{*PT} - \beta_{1(n)})}{\tilde{\Gamma}_{22}^{1/2}} &= \frac{\sqrt{n}(\beta_{1n}^* - \beta_{1(n)})}{\tilde{\Gamma}_{22}^{1/2}} - \frac{\sqrt{n}\beta_{1n}^*}{\tilde{\Gamma}_{22}^{1/2}}I(\mathcal{L}_n^* < \mathcal{L}_{n,\alpha}) \\ &\xrightarrow{\mathcal{D}} Z - (Z + \Delta_0)I((Z + \Delta_0)^2 < \chi_{1,\alpha}^2) \end{aligned} \quad (2.3.27)$$

where $Z \sim N(0, 1)$

2.3.4 Asymptotic Bias and MSE

Based on theorem 1 one easily obtains the asymptotic distributional bias and MSE expression for the estimators of $(\beta_0, \beta_1)'$ given in the following two theorems.

Theorem 2.3.2 *Under $\{K_{(n)}\}$, the asymptotic distributional bias and MSE expressions of β_{1n}^* and β_{1n}^{*PT} are given by*

$$\begin{aligned}
 (i) \quad b_1(\beta_{1n}^*) &= 0, \quad M_1(\beta_{1n}^*) = \sigma_{ee} \left(\frac{\sigma_{XX}}{\sigma_{xx}^2} \right) \\
 (ii) \quad b_2(\beta_{1n}^{*PT}) &= -\delta H_3(\chi_\alpha^2 : \Delta_0^2) \\
 M_2(\beta_{1n}^{*PT}) &= M(\beta_{1n}^*) - \sigma_{ee} \left(\frac{\sigma_{XX}}{\sigma_{xx}^2} \right) \left[H_3(\chi_\alpha^2 : \Delta_0^2) \right. \\
 &\quad \left. - \Delta_0^2 \left\{ 2H_3(\chi_\alpha^2 : \Delta_0^2) - H_5(\chi_\alpha^2 : \Delta_0^2) \right\} \right]. \quad (2.3.28)
 \end{aligned}$$

The asymptotic efficiency of β_{1n}^{*PT} relative to β_{1n}^* is then given by

$$ARE(\beta_{1n}^{*PT} : \beta_{1n}^*) = \left[1 + g_1^*(\Delta_0^2) \right]^{-1} \quad (2.3.29)$$

where

$$\begin{aligned}
 g_1^*(\Delta_0^2) &= -H_3(\chi_\alpha^2 : \Delta_0^2) \\
 &\quad + \Delta_0^2 \left\{ 2H_3(\chi_\alpha^2 : \Delta_0^2) - H_5(\chi_\alpha^2 : \Delta_0^2) \right\}. \quad (2.3.30)
 \end{aligned}$$

Figure 2.11 displays the behavior of the efficiency function on PTE relative to UE of β_1 ($\kappa_{xx}^2 = 0.8$ and $\alpha = 0.15$). The next theorem pertains to the estimators of β_0 .

Theorem 2.3.3 *Under $\{K_{(n)}\}$, the asymptotic distributional bias and MSE of the three estimators of β_0 are*

$$(i) \quad b_1(\bar{\beta}_0^*) = \delta\mu_x, \quad M_1(\bar{\beta}_0^*) = \sigma_{ee} \left(1 + \frac{\mu_x^2 \sigma_{XX} \Delta_0^2}{\sigma_{xx}^2} \right), \quad (2.3.31)$$

$$(ii) \quad b_2(\beta_{0n}^*) = 0, \quad M_2(\beta_{0n}^*) = \sigma_{ee} \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2} \right), \quad (2.3.32)$$

$$\begin{aligned}
 (iii) \quad b_3(\beta_{0n}^{*PT}) &= \delta\mu_x H_3(\chi_\alpha^2 : \Delta_0^2), \\
 M_3(\beta_{0n}^{*PT}) &= \sigma_{ee} + \frac{\sigma_{ee} \mu_x^2 \sigma_{XX}}{\sigma_{xx}^2} \left[1 - H_3(\chi_\alpha^2 : \Delta_0^2) \right. \\
 &\quad \left. + \Delta_0^2 \left\{ 2H_3(\chi_\alpha^2 : \Delta_0^2) - H_5(\chi_\alpha^2 : \Delta_0^2) \right\} \right]. \quad (2.3.33)
 \end{aligned}$$

Consequently, the asymptotic relative efficiencies of the estimates of β_0 compared to β_0^* are given by

$$\begin{aligned} ARE_1(\bar{\beta}_0^* : \beta_{0n}^*) &= \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2}\right) \left(1 + \frac{\mu_x^2 \sigma_{XX} \Delta_0^2}{\sigma_{xx}^2}\right)^{-1}, \\ ARE_2(\beta_{0n}^{*PT} : \beta_{0n}^*) &= \left\{1 + g_2^*(\Delta_0^2)\right\}^{-1} \end{aligned} \quad (2.3.34)$$

respectively, where

$$\begin{aligned} g_2^*(\Delta_0^2) &= -\frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2} \left\{1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2}\right\}^{-1} \left[H_3(\chi_\alpha^2 : \Delta_0^2) \right. \\ &\quad \left. - \Delta_0^2 \left\{ 2H_3(\chi_\alpha^2 : \Delta_0^2) - H_5(\chi_\alpha^2 : \Delta_0^2) \right\} \right]. \end{aligned} \quad (2.3.35)$$

Figure 2.12 displays the behavior of the efficiency function of PTE relative to UE of β_0 ($\kappa_{xx}^2 = 0.8$, $\alpha = 0.15$ and $\frac{\mu_x^2}{\sigma_{XX}} = 3$).

2.3.5 Analysis of the Asymptotic Properties

$ARE^*(\beta_{1n}^{*PT} : \beta_{1n}^*)$ is a function of (α, Δ_0^2) and for fixed α ($0 < \alpha < 1$) it decreases crossing the 1-line to a minimum, then increases towards the 1-line as Δ_0^2 tends to infinity. Further, $ARE^*(\beta_{1n}^{*PT} : \beta_{1n}^*) \gtrless 1$ according as

$$\Delta_0^2 \gtrless \frac{H_3(\chi_\alpha^2 : \Delta_0^2)}{2H_3(\chi_\alpha^2 : \Delta_0^2) - H_5(\chi_\alpha^2 : \Delta_0^2)} < 1. \quad (2.3.36)$$

Maximum attainable efficiency occurs at $\Delta_0^2 = 0$ with the value

$$\max_{0 \leq \Delta_0^2 \leq \infty} ARE^*(\beta_{1n}^{*PT} : \beta_{1n}^*) = \{1 - H_3(\chi_\alpha^2 : 0)\}^{-1} \quad (2.3.37)$$

The efficiency goes to unity as $\Delta_0^2 \rightarrow \infty$.

PTE can not be uniformly better than β_{1n}^* but one may determine a PTE with minimum guaranteed efficiency, say, E_0 by choosing α^* by the maximum rule

$$\max_{\alpha} \min_{\Delta_0^2} ARE(\alpha, \Delta_0^2) \geq E_0 \quad (2.3.38)$$

Figure 2.11: Graph of $ARE(\beta_{1n}^{*PT} : \beta_{1n}^*)$

Figure 2.12: Graph of $ARE(\beta_{0n}^{*PT} : \beta_{0n}^*)$

As $\alpha \rightarrow 1$, $\beta_{1n}^{*PT} \rightarrow \beta_{1n}^*$.

Table 2.3 provides the minimum (E_{min}) and maximum (E_{max}) asymptotic relative efficiency of the PTE relative to UE of β_1 for varying values of α and Δ_0^2 -value at which the minimum relative efficiency occur. See page 41 for an example of choosing α^* .

Under the null-hypothesis $H_0 : \beta_1 = 0$ i.e. $\Delta_0^2 = 0$, $ARE_1 = (1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2}) \geq 1$. Thus, $\hat{\beta}_0 = \bar{Y}_0$ performs better than $\tilde{\beta}_0$ in the range $0 \leq \Delta_0^2 \leq 1$. As Δ_0^2 increase $\hat{\beta}_0$ loses efficiency monotonically, ultimately becomes 0. Performance of $\hat{\beta}_0^{PT}$ depends on the size of the test α and Δ_0^2 . Under the H_0 , the ARE_2 becomes

$$\begin{aligned} & \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2}\right) \left\{1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2} \left[1 - H_3(\chi_\alpha^2; 0)\right]\right\}^{-1} \\ &= \left[1 - \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2}\right)^{-1} \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2} H_3(\chi_\alpha^2; 0)\right]^{-1} \geq 1. \end{aligned} \quad (2.3.39)$$

Consider the non-null situation. Then, as $\alpha \rightarrow 0$,

$$\begin{aligned} ARE_2 &= \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2}\right) \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2} \Delta_0^2\right)^{-1} \\ &\rightarrow \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2}\right) \left(1 + \frac{\mu_x^2 \sigma_{XX}}{\sigma_{xx}^2} \Delta_0^2\right)^{-1} = ARE_1 \end{aligned} \quad (2.3.40)$$

If $\alpha \rightarrow 1$, $ARE_2 \rightarrow 1$. However, as $\Delta_0^2 \rightarrow \infty$, $ARE_2 \rightarrow 1$ and $ARE_1 = 0$. As a function of Δ_0^2 , $ARE_2(\alpha, \Delta_0^2)$ has its maximum value at $\Delta_0^2 = 0$ for $\alpha \neq 0$. But as Δ_0^2 moves away from the origin, ARE_2 decreases crossing the 1-line to a minimum value $\frac{1}{1+g_2^*(\Delta_{min}^2)}$ at $\Delta_0^2 = \Delta_{min}^2$, then increase monotonically towards 1. The minimum value attended depends on α , the level of significance. For each α , one can compute the minimum value and the corresponding Δ_0^2 -value.

In order to choose an estimator with a minimum guaranteed efficiency, E_0 , we adopt the following procedure: If $0 \leq \Delta_0^2 \leq 1$, then we choose $\hat{\beta}_0$ since $R.E. \geq 1$

α		MRE	ARE
0.05	$E_{\max}$	4.85504	3.58294
	$E_{\min}$	0.36679	0.40580
	$\Delta^2_{\min}$	2.69644	4.68205
0.10	$E_{\max}$	2.80028	2.27642
	$E_{\min}$	0.50123	0.51223
	$\Delta^2_{\min}$	2.04785	3.96923
0.15	$E_{\max}$	2.08937	1.79357
	$E_{\min}$	0.59975	0.59499
	$\Delta^2_{\min}$	1.74639	3.58902
0.20	$E_{\max}$	1.72807	1.53889
	$E_{\min}$	0.67819	0.66466
	$\Delta^2_{\min}$	1.56307	3.34095
0.25	$E_{\max}$	1.51084	1.38197
	$E_{\min}$	0.74258	0.72478
	$\Delta^2_{\min}$	1.43728	3.16322
0.30	$E_{\max}$	1.36750	1.27664
	$E_{\min}$	0.79601	0.77694
	$\Delta^2_{\min}$	1.34485	3.02878
0.35	$E_{\max}$	1.26739	1.20217
	$E_{\min}$	0.84041	0.82201
	$\Delta^2_{\min}$	1.27392	2.92344
0.40	$E_{\max}$	1.19491	1.14778
	$E_{\min}$	0.87713	0.86058
	$\Delta^2_{\min}$	1.21788	2.83892
0.45	$E_{\max}$	1.14125	1.10728
	$E_{\min}$	0.90723	0.89316
	$\Delta^2_{\min}$	1.17271	2.76998
0.50	$E_{\max}$	1.10105	1.07680
	$E_{\min}$	0.93160	0.92022
	$\Delta^2_{\min}$	1.13581	2.71315

Table 2.3: Maximum and minimum guaranteed efficiencies

in this interval. However, if Δ_0^2 is unknown and we have to select an estimator with minimum guaranteed efficiency E_0 (since uniformly high efficiency estimator is impossible), we solve the equation for α given below

$$\max_{\alpha} \min_{\Delta_0^2} E(\alpha, \Delta_0^2) = E_0. \quad (2.3.41)$$

Table 2.4 provides the minimum (E_{min}) and maximum (E_{max}) asymptotic relative efficiency of the PTE relative to UE of β_0 for selected $\kappa_{xx}^2 = \frac{\sigma_{xx}}{\sigma_{xx}} = 0.8$, varying values of α and Δ_0^2 -value at which the minimum relative efficiency occur. See page 41 for an example of choosing α^* .

2.4 An Alternative Approach to Preliminary Test Estimation of the Slope Parameter

2.4.1 Introduction

Our basic problem is the estimation of parameters $(\beta_0, \beta_1)'$ of simple linear model with measurement error when the slope of the linear model is suspected to be zero. Since x_t is unobservable, we estimate the parameters based on $\{(Y_{0t}, X_{0t})', t = 1, 2, \dots, n\}$, the corresponding observable values. Towards this goal, we propose preliminary test estimation (PTE) of β_0 and β_1 and study the finite sample case in the previous section 2.2.

In the study of the properties of the preliminary test estimators, we notice that the optimum preliminary test estimators (i) heavily depend on the level of significance (generally $\alpha \geq .20$) which is a non-standard choice of level of significance and (ii) provide only two choices for the estimator, namely, the restricted estimator or the unrestricted estimator on the result of the preliminary test.

α	κ^2/β_{xx}	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
0.05	$E_{\max}$	1.0871	1.1685	1.2447	1.3163	1.3836	1.4471	1.5070	1.5636	1.6172	1.6681	1.8873	2.0616	2.2033	2.3209	2.4200	2.5047	2.5779	2.6418
	$E_{\min}$	0.8601	0.7735	0.7146	0.6720	0.6397	0.6144	0.5941	0.5773	0.5633	0.5514	0.5115	0.4888	0.4741	0.4638	0.4562	0.4504	0.4458	0.4420
	$\Delta^2_{\min}$	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820	4.6820
	$\Delta^2_{\max}$	1.0664	1.1263	1.1805	1.2299	1.2750	1.3163	1.3544	1.3896	1.4222	1.4524	1.5765	1.6681	1.7385	1.7943	1.8396	1.8771	1.9087	1.9356
0.10	$E_{\max}$	0.9043	0.8400	0.7938	0.7591	0.7319	0.7102	0.6923	0.6775	0.6648	0.6540	0.6169	0.5952	0.5809	0.5709	0.5634	0.5576	0.5529	0.5492
	$E_{\min}$	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692	3.9692
	$\Delta^2_{\min}$	1.0517	1.0971	1.1372	1.1730	1.2051	1.2340	1.2602	1.2841	1.3059	1.3259	1.4056	1.4621	1.5042	1.5368	1.5628	1.5841	1.6017	1.6166
	$\Delta^2_{\max}$	0.9297	0.8802	0.8434	0.8151	0.7925	0.7742	0.7589	0.7461	0.7351	0.7256	0.6926	0.6728	0.6598	0.6504	0.6435	0.6381	0.6337	0.6302
0.15	$E_{\max}$	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890	3.5890
	$E_{\min}$	1.0405	1.0753	1.1056	1.1322	1.1556	1.1766	1.1953	1.2123	1.2276	1.2415	1.2960	1.3336	1.3611	1.3821	1.3987	1.4121	1.4231	1.4324
	$\Delta^2_{\min}$	0.9469	0.9083	0.8790	0.8560	0.8375	0.8222	0.8094	0.7986	0.7892	0.7811	0.7524	0.7351	0.7235	0.7151	0.7089	0.7040	0.7001	0.6969
	$\Delta^2_{\max}$	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409	3.3409
0.20	$E_{\max}$	1.0317	1.0585	1.0815	1.1015	1.1190	1.1344	1.1481	1.1604	1.1714	1.1814	1.2199	1.2460	1.2648	1.2791	1.2903	1.2993	1.3066	1.3128
	$E_{\min}$	0.9595	0.9294	0.9062	0.8876	0.8726	0.8600	0.8495	0.8404	0.8326	0.8258	0.8015	0.7866	0.7766	0.7694	0.7639	0.7596	0.7562	0.7534
	$\Delta^2_{\min}$	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632	3.1632
	$\Delta^2_{\max}$	1.0247	1.0453	1.0628	1.0779	1.0909	1.1024	1.1125	1.1215	1.1296	1.1369	1.1646	1.1831	1.1964	1.2064	1.2141	1.2204	1.2255	1.2297
0.30	$E_{\max}$	0.9691	0.9457	0.9274	0.9127	0.9006	0.8904	0.8818	0.8745	0.8681	0.8624	0.8423	0.8298	0.8214	0.8152	0.8106	0.8069	0.8040	0.8016
	$E_{\min}$	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288	3.0288
	$\Delta^2_{\min}$	1.0190	1.0348	1.0481	1.0594	1.0692	1.0777	1.0852	1.0918	1.0977	1.1031	1.1232	1.1365	1.1460	1.1531	1.1586	1.1630	1.1666	1.1696
	$\Delta^2_{\max}$	0.9765	0.9585	0.9442	0.9327	0.9231	0.9151	0.9082	0.9023	0.8972	0.8926	0.8763	0.8660	0.8591	0.8540	0.8502	0.8471	0.8447	0.8427
0.35	$E_{\max}$	1.0145	1.0264	1.0364	1.0448	1.0521	1.0584	1.0639	1.0688	1.0732	1.0770	1.0917	1.1013	1.1081	1.1132	1.1171	1.1202	1.1227	1.1249
	$E_{\min}$	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389
	$\Delta^2_{\min}$	1.0109	1.0198	1.0271	1.0334	1.0387	1.0433	1.0474	1.0509	1.0541	1.0569	1.0674	1.0743	1.0792	1.0828	1.0856	1.0878	1.0896	1.0911
	$\Delta^2_{\max}$	0.9823	0.9686	0.9577	0.9488	0.9413	0.9351	0.9297	0.9251	0.9210	0.9174	0.9044	0.8963	0.8907	0.8866	0.8835	0.8811	0.8791	0.8775
0.40	$E_{\max}$	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389
	$E_{\min}$	1.0145	1.0264	1.0364	1.0448	1.0521	1.0584	1.0639	1.0688	1.0732	1.0770	1.0917	1.1013	1.1081	1.1132	1.1171	1.1202	1.1227	1.1249
	$\Delta^2_{\min}$	0.9823	0.9686	0.9577	0.9488	0.9413	0.9351	0.9297	0.9251	0.9210	0.9174	0.9044	0.8963	0.8907	0.8866	0.8835	0.8811	0.8791	0.8775
	$\Delta^2_{\max}$	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389	2.8389
0.45	$E_{\max}$	1.0109	1.0198	1.0271	1.0334	1.0387	1.0433	1.0474	1.0509	1.0541	1.0569	1.0674	1.0743	1.0792	1.0828	1.0856	1.0878	1.0896	1.0911
	$E_{\min}$	0.9869	0.9766	0.9684	0.9617	0.9560	0.9512	0.9471	0.9436	0.9404	0.9377	0.9276	0.9213	0.9169	0.9137	0.9113	0.9094	0.9078	0.9065
	$\Delta^2_{\min}$	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700	2.7700
	$\Delta^2_{\max}$	1.0080	1.0145	1.0198	1.0244	1.0282	1.0315	1.0344	1.0370	1.0392	1.0413	1.0488	1.0537	1.0571	1.0597	1.0616	1.0632	1.0645	1.0655
0.50	$E_{\max}$	0.9905	0.9830	0.9769	0.9719	0.9677	0.9642	0.9611	0.9584	0.9561	0.9540	0.9465	0.9417	0.9384	0.9359	0.9341	0.9326	0.9314	0.9305
	$E_{\min}$	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131	2.7131
	$\Delta^2_{\min}$	1.0080	1.0145	1.0198	1.0244	1.0282	1.0315	1.0344	1.0370	1.0392	1.0413	1.0488	1.0537	1.0571	1.0597	1.0616	1.0632	1.0645	1.0655
	$\Delta^2_{\max}$	0.9905	0.9830	0.9769	0.9719	0.9677	0.9642	0.9611	0.9584	0.9561	0.9540	0.9465	0.9417	0.9384	0.9359	0.9341	0.9326	0.9314	0.9305

Table 2.4: Minimum and Maximum Efficiency of PTE relative to UE of β_0 for various values of α when $\kappa_{xx}^2 = 0.8$ (Asymptotic case)

To eliminate the dependency on the level of significance and the extreme choices for the estimator, we devise an **alternative continuous** estimator which will **shrink** towards a targeted prior value of the parameter under consideration and provide interpolated values of the estimator between the extreme choices. The interpolated values depend on the value (not the test result) of the test-statistic.

2.4.2 An Alternative Estimation

Here, we consider the estimators $\tilde{\beta}_{1n}$, $\hat{\beta}_{1n}^{PT}$ of β_1 and $\tilde{\beta}_{0n}$, $\hat{\beta}_{0n}$ and $\hat{\beta}_{0n}^{PT}$ of β_0 given at (2.1.2) and (2.1.6) and (2.1.7), (2.1.9) respectively. Thus bias and mse of the estimators are given in Theorem 2.1.1..

Now, we consider the new estimator

$$\hat{\beta}_{1n}^S = \tilde{\beta}_{1n} - \frac{c\sqrt{\tilde{\sigma}_{ee}}\tilde{\beta}_{1n}}{\sqrt{S_{X_0X_0}}|\tilde{\beta}_{1n}|} \quad (2.4.1)$$

of β_1 in addition to $\tilde{\beta}_{1n}$ and $\hat{\beta}_{1n}^{PT}$ and the estimator

$$\hat{\beta}_{0n}^S = \bar{Y}_0 - \hat{\beta}_{1n}^S \bar{X}_0 \quad (2.4.2)$$

of β_0 in addition to $\tilde{\beta}_{0n}$, $\hat{\beta}_{0n}$ and $\hat{\beta}_{0n}^{PT}$.

Similarly, for the model (2.2.4) we consider the estimators β_{1n}^* and β_{1n}^{*PT} of β_1 together with the new estimator

$$\beta_{1n}^{*S} = \beta_{1n}^* - \frac{c\sqrt{\tilde{\sigma}_{zz}}\beta_{1n}^*}{\kappa_{xx}\sqrt{S_{X_0X_0}}|\beta_{1n}^*|} \quad (2.4.3)$$

and the new estimator of β_0 as

$$\beta_{0n}^{*S} = \bar{Y}_0 - \beta_{1n}^{*S} \bar{X}_0 \quad (2.4.4)$$

where c is the non-negative shrinkage constant. Now we consider the following theorems.

Theorem 2.4.1 If $Z \sim \mathcal{N}(\Delta, 1)$, then

$$E\left[\frac{Z}{|Z|}\right] = 1 - 2\Phi(-\Delta) \quad (2.4.5)$$

where Φ is the cdf of the standard normal distribution.

Proof.

$$E\left[\frac{Z}{|Z|}\right] = \int_{\Delta}^{\infty} \Phi(z)dz - \int_{-\Delta}^{\infty} \Phi(z)dz = 1 - 2\Phi(-\Delta) \quad (2.4.6)$$

Theorem 2.4.2 If $Z \sim \mathcal{N}(\Delta, 1)$, then

$$E(|Z|) = \sqrt{\frac{2}{\pi}} e^{-\Delta^2/2} + \Delta \{2\Phi(\Delta) - 1\} \quad (2.4.7)$$

Proof. The pdf of $|Z|$ is

$$\phi_{|Z|}(z) = \phi(z - \Delta) + \phi(z + \Delta) \quad (2.4.8)$$

and we have

$$\begin{aligned} E(|Z|) &= \int_0^{\infty} z\phi(z - \Delta)dz + \int_0^{\infty} z\phi(z + \Delta)dz \\ &= \int_{\Delta}^{\infty} z\phi(z)dz + \int_{-\Delta}^{\infty} z\phi(z)dz \\ &\quad + \Delta \left\{ \int_{-\Delta}^{\infty} \phi(z)dz - \int_{\Delta}^{\infty} \phi(z)dz \right\} \\ &= \int_{\Delta}^{\infty} z\phi(z)dz + \int_{-\Delta}^{\infty} z\phi(z)dz + \Delta \{2\Phi(\Delta) - 1\} \\ &= \sqrt{\frac{2}{\pi}} e^{-\Delta^2/2} + \Delta \{2\Phi(\Delta) - 1\} \end{aligned} \quad (2.4.9)$$

where $Z \sim \mathcal{N}(\Delta, 1)$ with $\Delta = \frac{\beta_1 \sqrt{S_{X_0 X_0}}}{\sigma_z}$.

The bias and mse expressions for the S-estimator $\hat{\beta}_{1n}^S$ of β_1 may be obtained as

$$b(\beta_{1n}^S) = -\frac{c}{\sqrt{S_{X_0X_0}}} E(\sqrt{\tilde{\sigma}_{ee}}) E\left[\frac{\tilde{\beta}_{1n}}{|\tilde{\beta}_{1n}|}\right] \quad (2.4.10)$$

$$\begin{aligned} &= -\frac{c^* \sqrt{\sigma_{ee}}}{\sqrt{S_{X_0X_0}}} K_n (2\Phi(\Delta) - 1) \\ M(\beta_{1n}^{*S}) &= \frac{\sigma_{ee}}{S_{X_0X_0}} \left\{ 1 - \frac{2}{\pi} K_n \left[2e^{-\frac{\Delta^2}{2}} - 1 \right] \right\} \end{aligned} \quad (2.4.11)$$

where $c = \sqrt{\frac{2}{\pi}} K_n$ and $K_n = \sqrt{\frac{2}{n-2} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}}$

2.4.3 Bias and MSE Expression and Comparison of the Slope Parameter

The bias and mse expressions for the S-estimator $\hat{\beta}_{1n}^{*S}$ of β_1 may be obtained as

$$\begin{aligned} b(\beta_{1n}^{*S}) &= -\frac{c}{\kappa_{xx} \sqrt{S_{X_0X_0}}} E(\sqrt{\tilde{\sigma}_{zz}}) E\left[\frac{\beta_{1n}^*}{|\beta_{1n}^*|}\right] \\ &= -\frac{c}{\kappa_{xx} \sqrt{S_{X_0X_0}}} E(\sqrt{\tilde{\sigma}_{zz}}) E\left[\frac{Z}{|Z|}\right] \\ &= -\frac{c\sigma_z}{\kappa_{xx} \sqrt{S_{X_0X_0}}} K_n (2\Phi(\Delta) - 1) \end{aligned} \quad (2.4.12)$$

where $K_n = \sqrt{\frac{2}{n-2} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}}$.

$$\begin{aligned}
M(\beta_{1n}^{*S}) &= E[(\beta_{1n}^* - \beta_1)^2] + \frac{c^2}{\kappa_{xx}^2 S_{X_0 X_0}} E(s_z^2) \\
&\quad - 2 \frac{c}{\kappa_{xx} \sqrt{S_{X_0 X_0}}} E(s_z) E\left[\frac{\beta_{1n}^* (\beta_{1n}^* - \beta_1)}{|\beta_{1n}^*|}\right] \\
&= \frac{\sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} + \frac{c^2 \sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} \\
&\quad - \frac{2c\sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} K_n \left\{ E(|Z|) - \Delta E\left[\frac{Z}{|Z|}\right] \right\} \\
&= \frac{\sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} \left\{ 1 + c^2 - 2cK_n \sqrt{\frac{2}{\pi}} e^{-\Delta^2/2} \right\}, \\
&= \frac{\sigma_{zz}}{\kappa_{xx}^2 S_{X_0 X_0}} \left\{ 1 - \frac{2}{\pi} K_n^2 [2 e^{-\Delta^2/2} - 1] \right\} \\
&\quad \text{using } c = \sqrt{\frac{2}{\pi}} K_n
\end{aligned} \tag{2.4.13}$$

Note that the value of c which minimizes (2.4.13) depends on Δ^2 and is given by $c^* = \sqrt{\frac{2}{\pi}} K_n e^{-\Delta^2/2}$. To make c^* independent of Δ^2 , we choose $c^0 = \sqrt{\frac{2}{\pi}} K_n$.

The unconditional expressions of bias and MSE of $\hat{\beta}_{1n}^{*S}$ may be obtained as

$$\begin{aligned}
b^*(\beta_{1n}^{*S}) &= -\frac{c\sqrt{\sigma_{zz}}K_n}{\kappa_{xx}} E_{S_{X_0 X_0}} \left[\frac{1}{\sqrt{S_{X_0 X_0}}} (2\Phi(\Delta) - 1) \right], \\
M^*(\beta_{1n}^{*S}) &= \frac{\sigma_{zz}}{\kappa_{xx}^2 \sigma_{XX}} \frac{1}{n-3} \left[1 + \frac{2}{\pi} K_n^2 \left(1 - \frac{2}{(1 + \Delta_u^{*2})^{\frac{n-1}{2}-1}} \right) \right].
\end{aligned} \tag{2.4.14}$$

Comparison

The relative efficiency of β_{1n}^{*PT} with respect to β_{1n}^* is

$$RE(\beta_{1n}^{*PT} : \beta_{1n}^*) = \left[1 + (n-3)g_u(\Delta_u^{*2}) \right]^{-1},$$

that of β_{1n}^{*S} with respect to β_{1n}^* is

$$RE(\beta_{1n}^{*S} : \beta_{1n}^*) = \left[1 + \frac{2}{\pi} K_n^2 \left(1 - \frac{2}{(1 + \Delta_u^{*2})^{\frac{n-1}{2}-1}} \right) \right]^{-1}$$

and the relative efficiency of β_{1n}^{*S} and β_{1n}^{*PT} is given by

$$RE(\beta_{1n}^{*S} : \beta_{1n}^{*PT}) = \frac{[1 - (n-3)g_u(\Delta_u^{*2})]^{-1}}{1 + \frac{2}{\pi} K_n^2 \left(1 - \frac{2}{(1 + \Delta_u^{*2})^{\frac{n-1}{2} - 1}} \right)}$$

where

$$g_u(\Delta_0^2) = -G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) + \Delta_u^{*2} \left\{ 2G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) - G_{5,m} \left(\frac{1}{5} F_\alpha; \Delta^{*2} \right) \right\}.$$

If $\Delta_u^{*2} = 0$,

$$RE(\beta_{1n}^{*S} : \beta_{1n}^*) = \left[1 - \frac{2}{\pi} K_n^2 \right]^{-1} (\geq 1)$$

while

$$RE(\beta_{1n}^{*PT} : \beta_{1n}^*) = \left[1 - (n-3)G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha; 0 \right) \right]^{-1} \geq 1$$

depending on the size of α . As $\Delta_u^{*2} \rightarrow \infty$,

$$RE(\beta_{1n}^{*S} : \beta_{1n}^*) \rightarrow \left[1 + \frac{2}{\pi} K_n^2 \right]^{-1}$$

while

$$RE(\beta_{1n}^{*PT} : \beta_{1n}^*) \rightarrow 1.$$

In general, $RE(\beta_{1n}^{*S} : \beta_{1n}^*)$ decreases from $\left[1 - \frac{2}{\pi} K_n^2 \right]^{-1}$ at $\Delta_u^{*2} = 0$ and crosses the 1-line at $\Delta_u^{*2} = \frac{n-3}{2}\sqrt{2} - 1$ then drops to the minimum value $\left[1 + \frac{2}{\pi} K_n^2 \right]^{-1} (\leq 1)$ as $\Delta_u^{*2} \rightarrow \infty$. The loss of efficiency is $1 - \left[1 + \frac{2}{\pi} K_n^2 \right]^{-1}$ while the gain in efficiency is $\left[1 - \frac{2}{\pi} K_n^2 \right]^{-1}$. Thus, for $0 \leq \Delta_u^{*2} \leq \left(\frac{n-3}{2}\sqrt{2} - 1 \right)$, β_{1n}^{*S} performs better than β_{1n}^* , otherwise β_{1n}^* performs better outside this interval.

In the case of β_{1n}^{*PT} , the graph of $RE(\beta_{1n}^{*PT}; \beta_{1n}^*)$ decreases to a minimum at $\Delta_u^{*2} = \Delta_0^2$ then increases to 1 as $\Delta_u^{*2} \rightarrow \infty$. It performs better than β_{1n}^* in the range $0 \leq \Delta_u^{*2} \leq K(F_{1,m}(\alpha; \Delta_u^{*2}))$, where $K(F_{1,m}(\alpha; \Delta_u^{*2})) = \frac{G_{3,m}^{(2)}(\frac{1}{3} F_\alpha; \Delta_u^{*2})}{2G_{3,m}^{(2)}(\frac{1}{3} F_\alpha; \Delta_u^{*2}) - G_{5,m}^{(2)}(\frac{1}{5} F_\alpha; \Delta_u^{*2})}$. Thus the range of better performance of β_{1n}^{*S} compare to β_{1n}^{*PT} is increased by an amount $\left(\frac{n-3}{2}\sqrt{2} - 1 - K(F_{1,m}(\alpha; \Delta_u^{*2})) \right)$.

The following Table 2.5 gives the performance of β_{1n}^{*S} and β_{1n}^{*PT} (for each selected level of significance and sample sizes, n)

Figure 2.13: MRE of S-estimator and PTE relative to UE

The first two rows of the Table gives the maximum and minimum relative efficiency of S-estimator for selected sample sizes. The maximum relative efficiency of S-estimator increases as the sample size n increases and as $n \rightarrow \infty$ it tends to $\frac{\pi}{\pi-2}$ while the minimum efficiency decreases as n increases. Finally as $n \rightarrow \infty$, it tends to $\frac{\pi}{\pi+2}$. The maximum relative efficiency at $\Delta^2 = 0$ is $\left[1 - \frac{2}{\pi} K_n^2\right]^{-1}$ which varies depending on the sample sizes.

The remaining rows of the Table contains the minimum relative efficiency, E_0 of the PTE at $\Delta_u^{*2} = \Delta_0$ which has been recorded for each $\alpha = 0.05(0.10)0.45$ with the corresponding efficiency (E_{Δ_0}) of S-estimator for $\Delta = \Delta_0$. The graph of the relative efficiency of S-estimator against Δ_0 is plotted in figure 2.13 along with the graph of relative efficiency of PTE showing the relative positions of the efficiency

over the Δ_0 -values.

From the figure, with $n = 10$ it is clear that the relative efficiency of PTE ($\alpha = 0.05$) compared to UE is smaller for $\Delta > 0.1136$ and that of S-estimator relative to UE is smaller for $\Delta > \frac{10-3}{2}\sqrt{2} - 1 = 0.2199$. Thus, S-estimator dominates UE over the wider interval $[0, 0.2199]$ relative to the PTE ($\alpha = 0.05$) in the interval $[0, 0.1136]$. Also, from figure, S-estimator has higher relative efficiency than PTE over all Δ_u^{*2} at which efficiency of PTE is smallest.

In general, there is no uniform domination of S-estimator over the PTE for all Δ and every α . When, Δ is near the origin, S-estimator over performs PTE for $\alpha < 0.20$. Also, at $\Delta = 0$, the relative efficiency of S-estimator is much higher than that of the PTE except for small α . Further, S-estimator is independent of α . Thus, considering the overall performance of S-estimator relative to PTE, S-estimator is preferable to PTE for practical situation since it produces interpolated estimators which is free from the level of significance, α .

$\alpha \backslash n$		10	15	20	25	30	35	40
	$E^{\max}$	2.6261	2.6692	2.6903	2.7029	2.7112	2.7171	2.7215
	$E_{\min}$	0.6176	0.6152	0.6141	0.6135	0.6131	0.6128	0.6125
0.05	E_{Δ_0}	0.6408	0.6466	0.6498	0.6518	0.6532	0.6542	0.6550
	E_0	0.3350	0.3600	0.3720	0.3790	0.3836	0.3868	0.3893
	Δ_0	6.1009	5.5303	5.2857	5.1514	5.0656	5.0063	4.9629
0.15	E_{Δ_0}	0.6827	0.6892	0.6924	0.6943	0.6955	0.6964	0.6971
	E_0	0.5404	0.5601	0.5693	0.5747	0.5782	0.5807	0.5825
	Δ_0	4.1626	3.9403	3.8429	3.7875	3.7520	3.7272	3.7090
0.25	E_{Δ_0}	0.7133	0.7182	0.7206	0.7220	0.7229	0.7236	0.7241
	E_0	0.6844	0.6990	0.7059	0.7098	0.7124	0.7143	0.7156
	Δ_0	3.4811	3.3612	3.3066	3.2757	3.2557	3.2417	3.2314
0.35	E_{Δ_0}	0.7361	0.7395	0.7410	0.7420	0.7426	0.7420	0.7433
	E_0	0.7941	0.8043	0.8090	0.8117	0.8135	0.8148	0.8157
	Δ_0	3.1163	3.0429	3.0110	2.9922	2.9801	2.9716	2.9653
0.45	E_{Δ_0}	0.7536	0.7555	0.7564	0.7569	0.7572	0.7576	0.7578
	E_0	0.8756	0.8820	0.8850	0.8867	0.8878	0.8887	0.8892
	Δ_0	2.8885	2.8442	2.8240	2.8125	2.8050	2.7979	2.7941

$E^{\max}$ is the maximum efficiency of S-estimator

$E_{\min}$ is the minimum efficiency of S-estimator

E_0 is the minimum efficiency of PTE

E_{Δ_0} is the minimum efficiency of S-estimator at Δ_0

Δ_0 is the value of Δ at which the minimum efficiency of PTE occurs

Table 2.5: Maximum and Minimum Guaranteed Efficiencies

Chapter 3

Multiple Regression Analysis: Finite Sample Case

3.0 Introduction

Consider the multiple regression model with measurement errors, namely,

$$\left. \begin{aligned} Y_{0t} &= \beta_0 + \boldsymbol{\beta}' \mathbf{x}_t + e_t \\ \mathbf{X}_{0t} &= \mathbf{x}_t + \mathbf{u}_t \end{aligned} \right\}; \quad t = 1, \dots, n \quad (3.0.1)$$

where β_0 is the intercept and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ is the vector of regression parameters with $\mathbf{x}_t = (x_{1t}, \dots, x_{pt})'$, $\mathbf{u}_t = (u_{1t}, \dots, u_{pt})'$ and $\mathbf{X}_{0t} = (X_{01t}, \dots, X_{0pt})'$. e_t is the response error in the study variable and u_{it} is the measurement error in the i^{th} regression variable x_{it} . Note that x_{it} is unobservable and X_{0it} is the corresponding observed value. Similarly, Y_{0t} is the observed response. We assume that

$$(\mathbf{x}'_t, e_t, \mathbf{u}'_t)' \sim \mathcal{N}_{2p+1} \left\{ (\boldsymbol{\mu}'_x, 0, \mathbf{0}')'; \text{Blockdiag}(\boldsymbol{\Sigma}_{xx}, \sigma_{ee}, \boldsymbol{\Sigma}_{uu}) \right\} \quad (3.0.2)$$

where $\boldsymbol{\mu}_x = (\mu_{x_1}, \dots, \mu_{x_p})'$.

Clearly, $(Y_{0t}, \mathbf{X}'_{0t})'$ follows a $(p+1)$ -variate normal distribution with mean-vector $(\beta_0 + \beta' \mu_x, \mu'_x)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{Y_{0t}Y_{0t}} & \Sigma_{Y_{0t}X_{0t}} \\ \Sigma_{X_{0t}Y_{0t}} & \Sigma_{X_{0t}X_{0t}} \end{pmatrix} = \begin{pmatrix} \beta' \Sigma_{xx} \beta + \sigma_{ee} & \beta' \Sigma_{xx} \\ \Sigma_{xx} \beta & \Sigma_{xx} + \Sigma_{uu} \end{pmatrix} \quad (3.0.4)$$

Thus, the distribution of $(\bar{Y}_0, \bar{\mathbf{X}}'_0)'$ is a $(p+1)$ -variate normal with mean-vector $(\beta_0 + \beta' \mu_x, \mu'_x)'$ and covariance matrix $\frac{1}{n} \Sigma$. Now the conditional distribution of Y_{0t} given $\mathbf{X}_{0t} = (X_{01t}, \dots, X_{0pt})'$ is normal with conditional mean and variance given by

$$E[Y_{0t} | \mathbf{X}_{0t}] = \nu_0 + \nu' \mathbf{X}_{0t} \quad (3.0.5)$$

$$Var[Y_{0t} | \mathbf{X}_{0t}] = \beta' \Sigma_{xx} (\mathbf{I} - \mathbf{K}_{xx}) \beta + \sigma_{ee} = \sigma_{zz} (\text{say}) \quad (3.0.6)$$

where $\mathbf{K}_{xx}$, the **reliability matrix**, is given by

$$\begin{aligned} \mathbf{K}_{xx} &= (\Sigma_{xx} + \Sigma_{uu})^{-1} \Sigma_{xx} \\ &= \Sigma_{X_0 X_0}^{-1} \Sigma_{xx} \text{ with } \Sigma_{X_0 X_0} = \Sigma_{xx} + \Sigma_{uu} \end{aligned} \quad (3.0.7)$$

and

$$\nu_0 = \beta_0 + \beta' (\mathbf{I}_p - \mathbf{K}'_{xx}) \mu_x, \quad \nu = \mathbf{K}_{xx} \beta \text{ and } \beta = \mathbf{K}_{xx}^{-1} \nu. \quad (3.0.8)$$

Our basic problem is the estimation of the regression vector (slope), $\beta = (\beta_1, \dots, \beta_p)'$ when it is suspected but not sure that β may be null-vector, i.e. $\beta = 0$. Towards this goal we propose four estimators of β , namely, (i) the unrestricted estimators (UE) of β (ii) the preliminary test estimator (PTE), (iii) the Stein-type estimator (SE) and (iv) the Positive-rule Stein-type estimator (PRSE). The finite sample properties of these estimators in the conditional model as well as unconditional case are studied based on the MSE matrices and the weighted risks under quadratic loss functions.

Consider the matrix of sum of squares

$$\mathbf{S} = \begin{pmatrix} S_{Y_0 Y_0} & \mathbf{S}_{Y_0 X_0} \\ \mathbf{S}_{X_0 Y_0} & \mathbf{S}_{X_0 X_0} \end{pmatrix} \quad (3.0.9)$$

where

$$(i) \ S_{Y_0 Y_0} = \sum_{t=1}^n (Y_{0t} - \bar{Y}_0)^2, \ \mathbf{S}_{X_0 X_0} = ((S_{X_{0i} X_{0j}}))_{p \times p},$$

$$(iii) \ S_{X_{0i} X_{0j}} = \sum_{t=1}^n (X_{0it} - \bar{X}_{0i})(X_{0jt} - \bar{X}_{0j}), \ i, j = 1, \dots, p,$$

$$(iv) \ S_{X_{0i} Y_0} = \sum_{t=1}^n (X_{0it} - \bar{X}_{0i})(Y_{0t} - \bar{Y}_0),$$

$$(v) \ \bar{X}_{0i} = \frac{1}{n} \sum_{t=1}^n X_{0it} \text{ and } \bar{Y}_0 = \frac{1}{n} \sum_{t=1}^n Y_{0t}.$$

Thus $\mathbf{S} \stackrel{\mathcal{D}}{\simeq} W_{p+1}(\boldsymbol{\Sigma}; n-1)$ where $W_{p+1}(\cdot; \cdot)$ stands for the Wishart distribution with $n-1$ degrees of freedom(d.f.). Clearly, $\frac{1}{n-1}\mathbf{S}$ is the unbiased estimator of $\boldsymbol{\Sigma}$. Consequently, $\frac{1}{n-1}\mathbf{S}_{X_0 X_0}$ is the unbiased estimator of $\boldsymbol{\Sigma}_{X_0 X_0}$ and $\mathbf{S}_{X_0 X_0}$ is independent of $(\bar{Y}_0, \bar{\mathbf{X}}_0)'$.

Notice that there are so many parameters in the model and there are so many different configurations that lead to the same distributions. As such the model is not identifiable and we need to impose additional conditions to overcome the difficulty of estimation. Generally it is assumed that (1) reliability matrix $\mathbf{K}_{xx}$ is known or (2) the measurement variance $\boldsymbol{\Sigma}_{uu}$ or $\sigma_{ee}\boldsymbol{\Sigma}_{uu}^{-1}$ is known. We shall only consider the case where $\mathbf{K}_{xx}$ is known. Information on the matrix $\mathbf{K}_{xx}$ can be obtained from reliability studies of the predictors, X_{0t} (Dunn, 1989 and Gleser, 1992) without evaluating the measurement errors in Y_{0t} 's. Gleser and Hwang (1987) and Gleser (1992) show that $\mathbf{K}_{xx}$ plays a crucial role in determining accurate values for $\boldsymbol{\nu}$ or $\boldsymbol{\beta}$ in the estimation procedure of these parameters.

3.1 Estimation of the Slope Parameters when Reliability Matrix ($\mathbf{K}_{xx}$) is Known.

First we consider the conditional model based on (3.0.5) and (3.0.6), namely,

$$Y_{0t} = \nu_0 + \boldsymbol{\nu}'\mathbf{X}_{0t} + Z_t \quad (3.1.1)$$

where $Z_t \sim \mathcal{N}(0, \sigma_{zz})$ and σ_{zz} is defined by (3.0.6). Then the least-square estimator(LSE) of $\boldsymbol{\nu}$ as

$$\tilde{\boldsymbol{\nu}}_n = \mathbf{S}_{X_0X_0}^{-1} \mathbf{S}_{X_0Y_0},$$

where $\mathbf{S}_{X_0X_0} = ((S_{X_{0i}X_{0j}}))$ and $\mathbf{S}_{X_0Y_0} = (S_{X_{01}Y_0}, \dots, S_{X_{0p}Y_0})'$

Clearly,

$$E(\tilde{\boldsymbol{\nu}}_n) = \boldsymbol{\nu}, \quad \text{cov}(\tilde{\boldsymbol{\nu}}_n) = \sigma_{zz} \mathbf{S}_{X_0X_0}^{-1}. \quad (3.1.2)$$

Thus, the LSE of $\boldsymbol{\beta}$ when $\mathbf{K}_{xx}$ is known is given by

$$\tilde{\boldsymbol{\beta}}_n = \mathbf{K}_{xx}^{-1} \tilde{\boldsymbol{\nu}}_n. \quad (3.1.3)$$

This means that conditionally,

$$E(\tilde{\boldsymbol{\beta}}_n) = \boldsymbol{\beta} \quad (3.1.4)$$

$$\text{Cov}(\tilde{\boldsymbol{\beta}}_n) = (\mathbf{K}_{xx}' \mathbf{S}_{X_0X_0} \mathbf{K}_{xx})^{-1} \sigma_{zz}. \quad (3.1.5)$$

We define the estimator of β_0 as

$$\tilde{\beta}_{0n} = \bar{Y}_0 - \bar{\mathbf{X}}_0' \tilde{\boldsymbol{\beta}}_n, \quad \text{with } \bar{\mathbf{X}}_0 = (\bar{X}_{01}, \dots, \bar{X}_{0p})' \quad (3.1.6)$$

and the estimator of σ_{zz} as

$$\tilde{\sigma}_{zz} = \frac{1}{m} \sum_{t=1}^n \left\{ (Y_{0t} - \bar{Y}_0) - (\mathbf{X}_{0t} - \bar{\mathbf{X}}_0)' \tilde{\boldsymbol{\nu}}_n \right\}^2, \quad (3.1.7)$$

where $m = n - p - 1$.

Now, the conditional distribution of $\tilde{\nu}_n$ given $\mathbf{X}_0$ is $\mathcal{N}_p(\boldsymbol{\nu}, \mathbf{S}_{X_0 X_0}^{-1} \sigma_{zz})$ and that of $\tilde{\boldsymbol{\beta}}_n$ given $\mathbf{X}_0$ is

$$\mathcal{N}_p\left\{\boldsymbol{\beta}, \left(\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}\right)^{-1} \sigma_{zz}\right\}. \quad (3.1.8)$$

We define the conditional test statistic

$$\mathcal{L}_n = \frac{\tilde{\boldsymbol{\beta}}'_n (\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}) \tilde{\boldsymbol{\beta}}_n}{p \tilde{\sigma}_{zz}} = \frac{\tilde{\nu}'_n \mathbf{S}_{X_0 X_0} \tilde{\nu}_n}{p \tilde{\sigma}_{zz}} \quad (3.1.9)$$

for testing $H_0 : \boldsymbol{\nu} = \mathbf{0}$, or $\boldsymbol{\beta} = \mathbf{0}$ against $H_A : \boldsymbol{\nu} \neq \mathbf{0}$ or $\boldsymbol{\beta} \neq \mathbf{0}$. The exact distribution of $\mathcal{L}_n$ follows a noncentral F -distribution with (p, m) d.f., where $m = n - p - 1$ and noncentrality parameter $\frac{1}{2} \Delta^2$ where

$$\Delta^2 = \frac{\boldsymbol{\beta}' (\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}) \boldsymbol{\beta}}{\sigma_{zz}} = \frac{\boldsymbol{\nu}' \mathbf{S}_{X_0 X_0} \boldsymbol{\nu}}{\sigma_{zz}}. \quad (3.1.10)$$

Empirical Bayes Estimators of $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$

For the full model (3.0.1) we again consider the associated conditional model (3.1.1) given by

$$Y_{0t} = \nu_0 + \boldsymbol{\nu}' \mathbf{X}_{0t} + Z_t, \quad t = 1, \dots, n \quad (3.1.11)$$

where $\boldsymbol{\nu} = \mathbf{K}_{xx} \boldsymbol{\beta}$. For known $\mathbf{K}_{xx}$ the LSE of $\boldsymbol{\beta}$ is $\tilde{\boldsymbol{\beta}}_n = \mathbf{K}_{xx}^{-1} \tilde{\nu}_n$ as given by (3.1.3). Also,

$$\tilde{\nu}_n \sim \mathcal{N}_p\left\{\boldsymbol{\nu}, \sigma_{zz} \mathbf{S}_{X_0 X_0}^{-1}\right\}$$

and

$$\tilde{\boldsymbol{\beta}}_n \sim \mathcal{N}_p\left\{\boldsymbol{\beta}, \sigma_{zz} \left(\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}\right)^{-1}\right\} \quad (3.1.12)$$

For this conditional case, let us assume that the prior of β is

$$\nu \sim \mathcal{N}_p\left\{\mathbf{0}, \tau^2 \mathbf{S}_{X_0 X_0}^{-1}\right\}$$

and

$$\beta \sim \mathcal{N}_p\left\{\mathbf{0}, \tau^2 \left(\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}\right)^{-1}\right\} \quad \text{for some } \tau^2 > 0, \quad (3.1.13)$$

incorporating the suspected value of ν and β to be the null vector, $\mathbf{0}$. Hence, the conditional distribution of ν given $\tilde{\nu}_n$ and β given $\tilde{\beta}_n$ are

$$\mathcal{N}_p\left\{\nu^*, \left(\frac{\tau^2}{\tau^2 + \sigma_{zz}}\right) \mathbf{S}_{X_0 X_0}^{-1}\right\}$$

and

$$\mathcal{N}_p\left\{\beta^*, \left(\frac{\tau^2}{\tau^2 + \sigma_{zz}}\right) \left(\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}\right)^{-1}\right\} \quad (3.1.14)$$

respectively, where

$$\nu^* = \left(\frac{\tau^2 \sigma_{zz}}{\tau^2 + \sigma_{zz}}\right) \tilde{\nu}_n = \left(1 - \frac{\sigma_{zz}}{\tau^2 + \sigma_{zz}}\right) \tilde{\nu}_n$$

and

$$\beta^* = \left(\frac{\tau^2 \sigma_{zz}}{\tau^2 + \sigma_{zz}}\right) \tilde{\beta}_n = \left(1 - \frac{\sigma_{zz}}{\tau^2 + \sigma_{zz}}\right) \tilde{\beta}_n. \quad (3.1.15)$$

Here ν^* and β^* are the Bayes estimators of ν and β .

Thus an Empirical Bayes Estimator (EBE) of ν or β can be obtained by using an estimate of $\frac{\sigma_{zz}}{\tau^2 + \sigma_{zz}}$. For this we consider the marginal distribution of $\tilde{\nu}_n$ or $\tilde{\beta}_n$ and of $\tilde{\sigma}_{zz}$ given by (3.1.7). The distribution of these random variables are independent and are given by

$$\begin{aligned} (i) \quad & \tilde{\nu}_n \sim \mathcal{N}_p\left\{\mathbf{0}, (\tau^2 + \sigma_{zz}) \mathbf{S}_{X_0 X_0}^{-1}\right\} \\ (ii) \quad & \tilde{\beta}_n \sim \mathcal{N}_p\left\{\mathbf{0}, (\tau^2 + \sigma_{zz}) \left(\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}\right)^{-1}\right\} \end{aligned} \quad (3.1.16)$$

and

$$(iii) \quad \frac{m\tilde{\sigma}_{zz}}{\sigma_{zz}} \sim \chi_m^2, \quad m = n - p - 1. \quad (3.1.17)$$

Hence,

$$(\tau^2 + \sigma_{zz})^{-1} \tilde{\nu}'_n \mathbf{S}_{X_0 X_0} \tilde{\nu}_n \stackrel{\mathcal{D}}{=} \chi_p^2$$

and

$$(\tau^2 + \sigma_{zz})^{-1} \tilde{\beta}'_n \left(\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx} \right) \tilde{\beta}_n \stackrel{\mathcal{D}}{=} \chi_p^2. \quad (3.1.18)$$

Consequently,

$$\frac{\sigma_{zz}}{\tau^2 + \sigma_{zz}} \mathcal{L}_n \stackrel{\mathcal{D}}{=} F_{p,m} \quad (3.1.19)$$

where

$$\mathcal{L}_n = \frac{\tilde{\nu}'_n \mathbf{S}_{X_0 X_0} \tilde{\nu}_n}{p\tilde{\sigma}_{zz}} = \frac{\tilde{\beta}'_n \left(\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx} \right) \tilde{\beta}_n}{p\tilde{\sigma}_{zz}} \quad (3.1.20)$$

Now,

$$\frac{\tau^2 + \sigma_{zz}}{\sigma_{zz}} E(\mathcal{L}_n^{-1}) = \frac{p}{p-2}, \quad p > 2 \quad (3.1.21)$$

Hence $\frac{\tau^2 + \sigma_{zz}}{\sigma_{zz}}$ can be estimated by a scalar multiple of $\mathcal{L}_n^{-1}$. More generally, let $g(\mathcal{L}_n)$, a real-valued function of $\mathcal{L}_n$, be an estimator of $\frac{\tau^2 + \sigma_{zz}}{\sigma_{zz}}$. Then, the estimator (3.1.15) becomes

$$\nu^*(\hat{\tau}) = \left\{ 1 - g(\mathcal{L}_n) \right\} \tilde{\nu}_n, \text{ and } \beta^*(\hat{\tau}) = \left\{ 1 - g(\mathcal{L}_n) \right\} \tilde{\beta}_n, \quad (3.1.22)$$

respectively. We assert that these estimators are the conditional EBE of ν and β given $\mathbf{X}_{0t}$, respectively .

3.2 Preliminary Test and Stein-type Estimators of ν and β when K_{xx} is Known

Let us consider two class of estimators of β defined by

$$\nu_n^* = \left\{ 1 - g(\mathcal{L}_n) \right\} \tilde{\nu}_n \quad (3.2.1)$$

and

$$\beta_n^* = \left\{ 1 - g(\mathcal{L}_n) \right\} \tilde{\beta}_n, \quad (3.2.2)$$

respectively, where $g(\mathcal{L}_n)$ is a monotone decreasing function of $\mathcal{L}_n$.

This class includes the usual estimators by setting $g(\mathcal{L}_n)$ as follows:

- (i) If $g(\mathcal{L}_n) = 0$, then the unrestricted estimator (UE) of β are defined by $\nu_n^* = \tilde{\nu}_n$ or by $\beta_n^* = \tilde{\beta}_n$,
- (ii) If $g(\mathcal{L}_n) = 1$, then the restricted estimator (RE) of β are defined by $\nu_n^* = 0$ or by $\beta_n^* = 0$,
- (iii) If $g(\mathcal{L}_n) = I(\mathcal{L}_n < F_{p,m}(\alpha))$, then the preliminary test estimator (PTE) of ν and β are defined by $\nu_n^* = \hat{\nu}_n^{PT} = \tilde{\nu}_n I(\mathcal{L}_n > F_{p,m}(\alpha)) = \tilde{\nu}_n - \tilde{\nu}_n I(\mathcal{L}_n < F_{p,m}(\alpha))$ and by $\beta_n^* = \hat{\beta}_n^{PT} = \tilde{\beta}_n I(\mathcal{L}_n > F_{p,m}(\alpha)) = \tilde{\beta}_n - \tilde{\beta}_n I(\mathcal{L}_n < F_{p,m}(\alpha))$, respectively, where $F_{p,m}(\alpha)$ is the upper α -level critical value of the finite sample distribution of $\mathcal{L}_n$ under H_0 ,
- (iv) If $g(\mathcal{L}_n) = c\mathcal{L}_n^{-1}$, then the shrinkage estimator (SE) of ν and β are defined by $\nu_n^* = \hat{\nu}_n^S = (1 - c\mathcal{L}_n^{-1})\tilde{\nu}_n$, and by $\beta_n^* = \hat{\beta}_n^S = (1 - c\mathcal{L}_n^{-1})\tilde{\beta}_n$, respectively, where $m = n - p - 1$ and $c = \frac{(p-2)m}{p(m+2)}$. Note that c is obtained by minimizing the unweighted risk of the estimators (See the appendix of this chapter).

Finally,

- (v) If $g(\mathcal{L}_n) = 1 - (1 - c\mathcal{L}_n^{-1})I(\mathcal{L}_n > c)$, then the positive-rule shrinkage estimator (PRSE) of $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ are defined by $\boldsymbol{\nu}_n^* = \hat{\boldsymbol{\nu}}_n^{S+} = (1 - c\mathcal{L}_n^{-1})I(\mathcal{L}_n > c)\tilde{\boldsymbol{\nu}}_n$, and by $\boldsymbol{\beta}_n^* = \hat{\boldsymbol{\beta}}_n^{S+} = (1 - c\mathcal{L}_n^{-1})I(\mathcal{L}_n > c)\tilde{\boldsymbol{\beta}}_n$, respectively.

3.3 Unconditional Power Function, Bias-vector, MSE matrix and Risk of the Estimators of $\boldsymbol{\beta}$ when $\mathbf{K}_{xx}$ is Known

We are basically interested in the unconditional MSE matrices and risks of $\boldsymbol{\beta}$. Unconditional expressions may be obtained by integrating the conditional expressions with respect to the Wishart distribution $W_p(n-1, \boldsymbol{\Sigma}_{X_0X_0})$ of $\mathbf{S}_{X_0X_0}$. It is known from Rao (1973) that

$$V_n = \frac{\boldsymbol{\beta}'\mathbf{C}\boldsymbol{\beta}}{\boldsymbol{\beta}'\boldsymbol{\psi}\boldsymbol{\beta}} \sim \chi_{n-1}^2$$

where $\mathbf{C} = \mathbf{K}_{xx}'\mathbf{S}_{X_0X_0}\mathbf{K}_{xx}$ and $\boldsymbol{\psi} = \mathbf{K}_{xx}'\boldsymbol{\Sigma}_{X_0X_0}\mathbf{K}_{xx}$ and the pdf of V_n is given by

$$\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n-1}{2})}e^{-\frac{1}{2}V_n}V_n^{\frac{n-1}{2}-1}d(V_n). \quad (3.3.1)$$

Defining $\Delta^2 = \frac{\boldsymbol{\beta}'\mathbf{C}\boldsymbol{\beta}}{\sigma_{zz}} = \Delta_0^2 V_n$, where $\Delta_0^2 = \frac{\boldsymbol{\beta}'\boldsymbol{\psi}\boldsymbol{\beta}}{\sigma_{zz}}$ together with the marginal distribution of $\bar{X}_0$. We obtain the unconditional power function of the test-statistics given by (3.1.9), bias, quadratic bias, MSE and risk expressions of the four estimators by

$$\mathbf{M}^{*(i)} = E_{V_n}(\mathbf{M}^{(i)}) \quad \text{and} \quad R^{*(i)} = \text{tr}(\mathbf{Q}_0\mathbf{M}^{*(i)}), \quad i = 1, 2, \quad (3.3.2)$$

where $\mathbf{M}^{(i)}$ stands for any conditional MSE matrices of the estimators $\tilde{\nu}_n$ and $\tilde{\beta}_n$ and any positive semi-definite matrix $\mathbf{Q}_0$, used in the loss function. The following theorem will be used in calculating of the expressions for $M^{*(i)}$ and $R^{*(i)}$.

Theorem 3.3.1 *Let $\mathbf{C} \sim W_p(n-1, \Sigma)$.*

Then for any measurable function g and fixed vector β ,

$$E[g(\beta' \mathbf{C} \beta) \mathbf{C}^{-1}] = \left(\frac{n-3}{m-1} \right) E \left[\frac{g(\beta' \Sigma \beta \chi_{n-1}^2)}{\chi_{n-1}^2} \right] \Sigma^{-1} \quad (3.3.3)$$

where $m = n - p - 1$.

Proof. If $\mathbf{T}$ is nonsingular matrix such that $\Sigma = \mathbf{T} \mathbf{T}'$, then $\mathbf{C}$ has the distribution of $\mathbf{T} \mathbf{W} \mathbf{T}'$, where $\mathbf{W} \sim W_p(n-1, \mathbf{I})$ (Cholesky Decomposition). Then

$$\begin{aligned} E[g(\beta' \mathbf{C} \beta) \mathbf{C}^{-1}] &= E[g(\beta' \mathbf{T} \mathbf{W} \mathbf{T}' \beta) (\mathbf{T} \mathbf{W} \mathbf{T}')^{-1}] \\ &= E[g(\beta' \mathbf{T} \mathbf{W} \mathbf{T}' \beta) (\mathbf{T}')^{-1} \mathbf{W}^{-1} \mathbf{T}^{-1}] \\ &= (\mathbf{T}')^{-1} E[g(\beta' \mathbf{T} \mathbf{W} \mathbf{T}' \beta) \mathbf{W}^{-1}] \mathbf{T}^{-1} \end{aligned}$$

Consider $\mathbf{P}$ be an orthogonal matrix such that

$$\mathbf{P} \alpha = \left(\sqrt{\alpha' \alpha}, 0, \dots, 0 \right)' \quad \text{for any } p \times 1 \text{ vector } \alpha \quad (3.3.4)$$

Let $\mathbf{V} = \mathbf{P} \mathbf{W} \mathbf{P}'$, which has a $\mathbf{W} \sim W_p(n-1, \mathbf{I})$ distribution. Then

$$\begin{aligned} E[g(\alpha' \mathbf{W} \alpha) \mathbf{W}^{-1}] &= E[g(\alpha' \mathbf{P}' \mathbf{V} \mathbf{P} \alpha) (\mathbf{P}' \mathbf{V} \mathbf{P})^{-1}] \\ &= E[g(\alpha' \alpha V_{11}) \mathbf{P}' \mathbf{V}^{-1} \mathbf{P}], \quad \text{where } V_{11} = (\mathbf{V})_{11}. \end{aligned}$$

Now we consider $\alpha' E[g(\alpha' \alpha V_{11}) \mathbf{P}' \mathbf{V}^{-1} \mathbf{P}] \alpha$.

Since $\alpha' \mathbf{P}' \mathbf{V} \mathbf{P} \alpha = (\sqrt{\alpha' \alpha}, 0, \dots, 0) \mathbf{V} (\sqrt{\alpha' \alpha}, 0, \dots, 0)' = \alpha' \alpha V_{11}$,

$$\begin{aligned}
 & \alpha' E \left[g(\alpha' \alpha V_{11}) \mathbf{P}' \mathbf{V}^{-1} \mathbf{P} \right] \alpha \\
 = & E \left[g(\alpha' \alpha V_{11}) \alpha' \alpha V^{11} \right] \quad \text{where } V^{11} = (\mathbf{V}^{-1})_{11} \\
 = & \alpha' E \left[\frac{g(\alpha' \alpha V_{11})}{V_{11}} \frac{1}{\mathbf{G}^{-1}} \right] \alpha \\
 = & \alpha' \left(\frac{n-3}{m-1} \right) E \left[\frac{g(\alpha' \alpha V_{11})}{V_{11}} \right] \alpha
 \end{aligned} \tag{3.3.5}$$

Thus

$$E \left[g(\alpha' \alpha V_{11}) \mathbf{P}' \mathbf{V}^{-1} \mathbf{P} \right] = \left(\frac{n-3}{m-1} \right) E \left[\frac{g(\alpha' \alpha \chi_{n-1}^2)}{\chi_{n-1}^2} \right] \mathbf{I} \tag{3.3.6}$$

By letting $\alpha = \mathbf{T}' \beta$,

$$\begin{aligned}
 & E[g(\beta' \mathbf{T} \mathbf{W} \mathbf{T}' \beta) \mathbf{W}^{-1}] \\
 = & E[g(\beta' \mathbf{T} \mathbf{T}' \beta V_{11}) \mathbf{P}' \mathbf{V}^{-1} \mathbf{P}] \\
 = & \left(\frac{n-3}{m-1} \right) E \left[\frac{g(\beta' \Sigma \beta \chi_{n-1}^2)}{\chi_{n-1}^2} \right] \mathbf{I}
 \end{aligned} \tag{3.3.7}$$

Therefore,

$$\begin{aligned}
 E[g(\beta' \mathbf{C} \beta) \mathbf{C}^{-1}] &= (\mathbf{T}')^{-1} E[g(\beta' \mathbf{T} \mathbf{W} \mathbf{T}' \beta) \mathbf{W}^{-1}] \mathbf{T}^{-1} \\
 &= (\mathbf{T}')^{-1} \left(\frac{n-3}{m-1} \right) E \left[\frac{g(\beta' \Sigma \beta \chi_{n-1}^2)}{\chi_{n-1}^2} \right] \mathbf{T}^{-1} \\
 &= \left(\frac{n-3}{m-1} \right) E \left[\frac{g(\beta' \Sigma \beta \chi_{n-1}^2)}{\chi_{n-1}^2} \right] (\mathbf{T}')^{-1} \mathbf{T}^{-1} \\
 &= \left(\frac{n-3}{m-1} \right) E \left[\frac{g(\beta' \Sigma \beta \chi_{n-1}^2)}{\chi_{n-1}^2} \right] \Sigma^{-1}.
 \end{aligned} \tag{3.3.8}$$

The theorem is proved.

We have used the following facts for the above theorem. Let $\mathbf{V} \sim W_p(n-1, \mathbf{I})$,

where $\mathbf{V} = \begin{bmatrix} V_{11} & \mathbf{Z} \\ \mathbf{Z} & \mathbf{A} \end{bmatrix}$. $\frac{1}{V^{11}} = V_{11} - \mathbf{Z}' \mathbf{A}^{-1} \mathbf{Z}$ i.e. $V^{11} = \frac{1}{V_{11} - \mathbf{Z}' \mathbf{A}^{-1} \mathbf{Z}} = (V_{11})^{-1} \mathbf{G}$,

where $\mathbf{G}$ is independent of V_{11} because $\mathbf{C} = V_{11} - \mathbf{Z}'\mathbf{A}^{-1}\mathbf{Z}$ and $\mathbf{D} = \mathbf{Z}'\mathbf{A}^{-1}\mathbf{Z}$ are independently distributed as $\chi^2_{(n-p)}$ and $\chi^2_{(p-1)}$, respectively [see 9.2 p. 596 of Rao (1973)], and thus $\mathbf{G}^{-1} = \frac{\mathbf{D}}{\mathbf{C}+\mathbf{D}} = \frac{V_{11}-\mathbf{Z}'\mathbf{A}^{-1}\mathbf{Z}}{V_{11}}$ has a Beta $(\frac{n-p}{2}, \frac{p-1}{2})$ distribution independent of V_{11} . $(V^{-1})_{ii}$ is independent of V_{11} for $i \geq 2$ [see (viii)(a), p538 of Rao(1973)] and $((V^{-1})_{ii})^{-1} \sim \chi^2_{(n-p)}$.

3.3.1 Power Function of the Test

Since our estimators involve the test-statistics for testing $\boldsymbol{\nu} = \boldsymbol{\beta} = \mathbf{0}$ against $\boldsymbol{\nu} \neq \mathbf{0}$ ($\boldsymbol{\beta} \neq \mathbf{0}$), we first display the power function of the test which is obtained by integrating the power function of the F test to be

$$\begin{aligned} & 1 - G_{p,m}(F_\alpha : \Delta_0^2) \\ &= 1 - \sum_{r \geq 0} \frac{\Delta_0^{2r} \Gamma(\frac{n-1}{2} + r)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r}} I_y\left(\frac{p}{2} + r : \frac{m}{2}\right) \end{aligned} \quad (3.3.9)$$

where $\Delta_0^2 = \frac{\boldsymbol{\beta}' \mathbf{K}_{xx} \sum X_0 X_0 \mathbf{K}_{xx} \boldsymbol{\beta}}{\sigma_{zz}}$, $y = \frac{p F_\alpha}{m + p F_\alpha}$ and $m = n - p - 1$.

Associated with the power function of the test we shall need the following expressions to derive the bias, MSE matrices and risk of the estimators.

$$\begin{aligned} & G_{p+2i,m}^{(j)}\left(\frac{p}{p+2i} F_\alpha : \Delta_0^2\right) \\ &= \sum_{r \geq 0} \frac{2^{j-2} \Delta_0^{2r} \Gamma(\frac{n-1}{2} + r + j - 2)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r + j - 2}} I_y\left(\frac{p+2i}{2} + r : \frac{m}{2}\right) \end{aligned} \quad (3.3.10)$$

$$\begin{aligned}
(1) \quad & E^{(j)}[\chi_{p+2i}^{-2}(\Delta_0^2)] \\
&= \sum_{r \geq 0} \frac{2^{j-2} \Delta_0^{2r} \Gamma(\frac{n-1}{2} + r + j - 2)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r + j - 2}} \\
&\quad \times \frac{1}{(p+2i-2+2r)} \tag{3.3.11}
\end{aligned}$$

$$\begin{aligned}
(2) \quad & E^{(j)}[\chi_{p+2i}^{-4}(\Delta_0^2)] \\
&= \sum_{r \geq 0} \frac{2^{j-2} \Delta_0^{2r} \Gamma(\frac{n-1}{2} + r + j - 2)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r + j - 2}} \\
&\quad \times \frac{1}{(p+2i-2+2r)(p+2i-4+2r)} \tag{3.3.12}
\end{aligned}$$

$$\begin{aligned}
(3) \quad & E^{(j)}[\chi_{p+2i}^{-2}(\Delta_0^2) I(\chi_{p+2i}^2(\Delta_0^2) < p-2)] \\
&= \sum_{r \geq 0} \frac{2^{j-2} \Delta_0^{2r} \Gamma(\frac{n-1}{2} + r + j - 2)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r + j - 2}} \\
&\quad \times \frac{1}{(p+2i-2+2r)} H_{p+2i-2+2r}(p-2:0) \tag{3.3.13}
\end{aligned}$$

$$\begin{aligned}
(4) \quad & E^{(j)}[\chi_{p+2i}^{-4}(\Delta_0^2) I(\chi_{p+2i}^2(\Delta_0^2) < p-2)] \\
&= \sum_{r \geq 0} \frac{2^{j-2} \Delta_0^{2r} \Gamma(\frac{n-1}{2} + r + j - 2)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r + j - 2}} \\
&\quad \times \frac{1}{(p+2i-2+2r)(p+2i-4+2r)} \\
&\quad \times H_{p+2i-4+2r}(p-2:0) \tag{3.3.14}
\end{aligned}$$

We note some properties of the expectations.

- (1). $E^{(i)}[\chi_{p+2}^{-2}(\Delta_0^2)] - (p-2)E^{(i)}[\chi_{p+2}^{-4}(\Delta_0^2)] = \Delta_0^2 E^{(i+1)}[\chi_{p+4}^{-4}(\Delta_0^2)]$
- (2). $2E^{(i)}[\chi_{p+4}^{-4}(\Delta_0^2)] = E^{(i)}[\chi_{p+2}^{-2}(\Delta_0^2)] - E^{(i)}[\chi_{p+4}^{-2}(\Delta_0^2)]$ for $i = 1, 2$

3.3.2 Bias Expressions of β_n^* and Comparisons

The bias and quadratic bias expressions are following.

Bias and Quadratic Bias for $\tilde{\beta}_n$

$$\mathbf{b}_1^{(2)}(\tilde{\beta}_n) = 0, \quad B_1^{(2)}(\tilde{\beta}_n) = 0$$

Bias and Quadratic Bias for $\hat{\beta}_n^{PT}$

$$\begin{aligned} \mathbf{b}_2^{(2)}(\hat{\beta}_n^{PT}) &= -\beta G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right) \\ B_2^*(\hat{\beta}_n^{PT}) &= \Delta_0^2 \left\{ G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right) \right\}^2 \end{aligned} \quad (3.3.15)$$

Bias and Quadratic Bias for $\hat{\beta}_n^S$

$$\begin{aligned} \mathbf{b}_3^{(2)}(\hat{\beta}_n^S) &= -\frac{(p-2)m}{m+2} E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \\ B_3^{(2)}(\hat{\beta}_n^S) &= \frac{(p-2)^2 m^2}{(m+2)^2} \Delta_0^2 \left\{ E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right\}^2 \end{aligned} \quad (3.3.16)$$

Bias and Quadratic Bias for $\hat{\beta}_n^{S+}$

$$\begin{aligned} \mathbf{b}_4^{(2)}(\hat{\beta}_n^{S+}) &= -c\beta \left[E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2)) \right. \\ &\quad - E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < pc)) \\ &\quad \left. + \frac{1}{c} G_{p+2,m}^{(2)}(pc; \Delta_0^2) \right] \\ B_4^{(2)}(\hat{\beta}_n^{S+}) &= c^2 \Delta_0^2 \left[E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2)) \right. \\ &\quad - E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < pc)) \\ &\quad \left. + \frac{1}{c} G_{p+2,m}^{(2)}(pc; \Delta_0^2) \right]^2 \end{aligned} \quad (3.3.17)$$

where $c = \frac{(p-2)m}{(p+2)(m+2)}$.

Comparison of $B_3^{(2)}$ with $B_2^{(2)}$

$$\begin{aligned} B_2^{(2)} - B_3^{(2)} &= \Delta_0^2 \left[\left\{ G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \right\}^2 \right. \\ &\quad \left. - \frac{(p-2)^2 m^2}{(m+2)^2} \left\{ E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right\}^2 \right] \stackrel{>}{<} 0 \end{aligned} \quad (3.3.18)$$

whenever

$$G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \stackrel{>}{<} \frac{(p-2)m}{(m+2)} E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \quad (3.3.19)$$

so that $B_2^{(2)} \stackrel{\geq}{<} B_3^{(2)}$ whenever (3.3.19) holds.

Comparison of $B_4^{(2)}$ with $B_2^{(2)}$

$$\begin{aligned} B_2^{(2)} - B_4^{(2)} &= \Delta_0^2 \left[\left\{ G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \right\}^2 - c^2 \left\{ E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2)) \right. \right. \\ &\quad \left. \left. - E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2)) I(F_{p+2,m}(\Delta_0^2) < pc) \right\} \right. \\ &\quad \left. + \frac{1}{c} G_{p+2,m}^{(2)}(pc; \Delta_0^2) \right\}^2 \Big] \end{aligned} \quad (3.3.20)$$

which may be positive or negative for some Δ^2 and for fixed α .

Comparison of $B_3^{(2)}$ with $B_4^{(2)}$

$$\begin{aligned} B_3^{(2)} - B_4^{(2)} &= \frac{(p-2)^2 m^2}{(m+2)^2} \Delta_0^2 \left[\left\{ E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right\}^2 \right. \\ &\quad \left. - \frac{1}{(p+2)^2} \left\{ E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2)) \right. \right. \\ &\quad \left. \left. - E^{(2)}(F_{p+2,m}^{-2}(\Delta_0^2)) I(F_{p+2,m}(\Delta_0^2) < pc) \right\} \right. \\ &\quad \left. + \frac{1}{c} G_{p+2,m}^{(2)}(pc; \Delta_0^2) \right\}^2 \Big] \geq 0 \end{aligned} \quad (3.3.21)$$

Clearly, $\tilde{\beta}_n$ is unbiased and $\hat{\beta}_n^{PT}$, $\hat{\beta}_n^S$ and $\hat{\beta}_n^{S+}$ are biased. Under H_0 , they are all unbiased. Also, as $\Delta^2 \rightarrow \infty$, the bias reduces to zero for all estimators. In general, as Δ^2 moves away from the origin, the quadratic bias function increases to a maximum then decreases zero as Δ^2 tends to infinity.

3.3.3 MSE and Risk Expressions of β_n^* and Comparisons

MSE and Risk for the Estimator $\tilde{\beta}_n$

MSE and risk expressions of $\tilde{\beta}_n$ are given by

$$\begin{aligned} M_1^{(2)}(\tilde{\beta}_n) &= \frac{\sigma_{zz}}{m-1} \psi^{-1} \\ R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q}_0) &= \frac{1}{m-1} \sigma_{zz} \text{tr} \left\{ \mathbf{Q}_0 \psi^{-1} \right\}, \end{aligned} \quad (3.3.22)$$

MSE and Risk for the Estimator $\hat{\beta}_n^{PT}$

The MSE and risk expressions are given by

$$\begin{aligned} M_2^{(2)}(\hat{\beta}_n^{PT}) &= M_1^{(2)}(\tilde{\beta}_n) \\ &\quad - \sigma_{zz} \left(\frac{n-3}{m-1} \right) G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \psi^{-1} \\ &\quad + \beta \beta' \left\{ 2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \right. \\ &\quad \left. - G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right) \right\} \\ R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0) &= R_1^{(2)}(\tilde{\beta}_n) \\ &\quad - \sigma_{zz} \text{tr} \left\{ \mathbf{Q}_0 \psi^{-1} \right\} \left[\left(\frac{n-3}{m-1} \right) G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \right] \\ &\quad + \beta' \mathbf{Q}_0 \beta \left\{ 2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \right. \\ &\quad \left. - G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right) \right\}, \end{aligned} \quad (3.3.23)$$

MSE comparison:

Comparison of $\hat{\beta}_n^{PT}$ with $\tilde{\beta}_n$

$$\begin{aligned} \mathbf{M}_1^{(2)}(\tilde{\beta}_n) - \mathbf{M}_2^{(2)}(\hat{\beta}_n^{PT}) &= \sigma_{zz} \left(\frac{n-3}{m-1} \right) G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \psi^{-1} \\ &\quad - \beta \beta' \left\{ 2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) - G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right) \right\} \end{aligned} \quad (3.3.24)$$

This difference is positive semi-definite whenever for any nonzero vector $\mathbf{b}$,

$$\begin{aligned} &\sigma_{zz} \left(\frac{n-3}{m-1} \right) \mathbf{b}' \psi^{-1} \mathbf{b} G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \\ &- \mathbf{b}' \beta \beta' \mathbf{b} \left\{ 2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) - G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right) \right\} \\ &\geq 0 \end{aligned} \quad (3.3.25)$$

Alternatively, (3.3.24) is positive-semi definite if, for all $\mathbf{b}$, we have

$$\frac{\mathbf{b}' \beta \beta' \mathbf{b}}{\sigma_{zz} \mathbf{b}' \psi^{-1} \mathbf{b}} \leq \frac{\frac{(n-3)}{(m-1)} G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right)}{2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) - G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right)}, \quad \forall \mathbf{b}$$

Now,

$$\max_{\mathbf{b}} \frac{\mathbf{b}' \beta \beta' \mathbf{b}}{\sigma_{zz} \mathbf{b}' \psi^{-1} \mathbf{b}} = \frac{1}{\sigma_{zz}} \beta' \psi \beta = \Delta_0^2$$

Therefore, it reduces to

$$\Delta_0^2 \leq \frac{\frac{(n-3)}{(m-1)} G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right)}{2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) - G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right)}, \quad \forall \mathbf{b} \quad (3.3.26)$$

Hence, $\hat{\beta}_n^{PT}$ is superior to $\tilde{\beta}_n$ in this range of Δ_0^2 , otherwise $\tilde{\beta}_n$ is superior. Further note that under H_0 ,

$$\begin{aligned} \mathbf{M}_1^{(2)}(\tilde{\beta}_n) - \mathbf{M}_2^{(2)}(\hat{\beta}_n^{PT}) &= \sigma_{zz} \left(\frac{n-3}{m-1} \right) G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \psi^{-1} \\ &\geq 0 \end{aligned} \quad (3.3.27)$$

Risk comparison:

Comparison of $\hat{\beta}_n^{PT}$ with $\tilde{\beta}_n$

In this case the risk-difference $R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q}_0) - R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0) \geq 0$ according as

$$\Delta_0^2 \leq \frac{\frac{(n-3)}{(m-1)} \text{tr}\{\mathbf{Q}_0 \psi^{-1}\} G_{p+2,m}^{(1)}(\frac{p}{p+2} F_\alpha; \Delta_0^2)}{ch_{max}\{\mathbf{Q}_0 \psi^{-1}\} 2G_{p+2,m}^{(2)}(\frac{p}{p+2} F_\alpha; \Delta_0^2) - G_{p+4,m}^{(2)}(\frac{p}{p+4} F_\alpha; \Delta_0^2)} \quad (3.3.28)$$

Then, $\hat{\beta}_n^{PT}$ performs better than $\tilde{\beta}_n$ in this range of Δ_0^2 . On the other hand $\tilde{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$.

Some tabular values of efficiencies based on risk are given for $n = 9$ and $p = 4$ when $\mathbf{Q}_0 = \psi$.

Table 3.1: Maximum and Minimum Guaranteed Efficiencies

$(n, p) \backslash \alpha$		.05	.10	.15	.20	.25
(9,4)	E_{max}	10.9783	5.7345	3.9634	3.0704	2.5317
	E_{min}	0.6562	0.7835	0.8571	0.9049	0.9347
	Δ_0	4.3457	3.8844	3.6190	3.3379	3.1795
$(n, p) \backslash \alpha$		.3	.35	.4	.45	.5
(9,4)	E_{max}	2.1716	1.9143	1.7217	1.5727	1.4546
	E_{min}	0.9546	0.9682	0.9777	0.9843	0.9891
	Δ_0	3.0649	2.9804	2.9171	2.8694	2.8336

MSE and Risk for the Estimator $\hat{\beta}_n^S$

The MSE matrix and the risk of $\hat{\beta}_n^S$ are given by

$$\begin{aligned}
 \mathbf{M}_3^{(2)}(\hat{\beta}_n^S) &= \mathbf{M}_1^{(2)}(\tilde{\beta}_n) \\
 &- \sigma_{zz} \frac{(p-2)(n-3)m}{(m+2)(m-1)} \left\{ 2E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
 &- (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \left. \right\} \psi^{-1} \\
 &+ \frac{(p-2)m}{m+2} \beta \beta' \left\{ 2E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
 &- 2E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) + (p-2)E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \left. \right\}
 \end{aligned}$$

$$\begin{aligned}
 R_3^{(2)}(\hat{\beta}_n^S : \mathbf{Q}_0) &= R_1^{(2)}(\tilde{\beta}_n) \\
 &- \sigma_{zz} \text{tr} \left\{ \mathbf{Q}_0 \psi^{-1} \right\} \left[\frac{(p-2)(n-3)m}{(m+2)(m-1)} \left\{ 2E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \right. \\
 &- (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \left. \left. \right\} \right] \\
 &+ \frac{(p-2)m}{m+2} \beta' \mathbf{Q}_0 \beta \left\{ 2E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
 &- 2E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) + (p-2)E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \left. \right\} \quad (3.3.29)
 \end{aligned}$$

MSE comparison:

Comparison of $\hat{\beta}_n^S$ with $\tilde{\beta}_n$

$$\begin{aligned}
 \mathbf{M}_1^{(2)}(\tilde{\beta}_n) - \mathbf{M}_3^{(2)}(\hat{\beta}_n^S) &= \sigma_{zz} \frac{(p-2)(n-3)m}{(m+2)(m-1)} \\
 &\times \left\{ 2E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) - (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \right\} \psi^{-1} \\
 &- \frac{(p+2)(p-2)m}{m+2} \beta \beta' E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \quad (3.3.30)
 \end{aligned}$$

This difference is positive semi-definite whenever for any nonzero vector $\mathbf{b}$,

$$\begin{aligned}
& \frac{(p-2)m(n-3)}{(m+2)(m-1)} \mathbf{b}' \psi^{-1} \mathbf{b} \left\{ E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) + \Delta_0^2 E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \right\} \\
& - \frac{(p-2)m(p+2)}{m-1} \mathbf{b}' \beta \beta' \mathbf{b} E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \geq 0 \\
& \text{i.e.} \\
& \frac{n-3}{m-1} \left\{ E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) + \Delta_0^2 E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \right\} \\
& - (p+2) \Delta_0^2 E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \geq 0
\end{aligned} \tag{3.3.31}$$

which is a contradiction. Hence $\hat{\beta}_n^S$ doesn't dominate $\tilde{\beta}_n$ uniformly with respect to the MSE criterion.

Comparison of $\hat{\beta}_n^S$ with $\hat{\beta}_n^{PT}$

Under $H_0 : \beta = 0$,

$$\begin{aligned}
\mathbf{M}_3^{(2)}(\hat{\beta}_n^S) &= \mathbf{M}_2^{(2)}(\hat{\beta}_n^{PT}) + \sigma_{zz} \frac{(n-3)}{(m-1)} \left[G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) \right. \\
&\quad \left. - \frac{(p-2)m}{m+2} E^{(1)}(\chi_{p+2}^{-2}(0)) \right] \psi^{-1} \\
&= \mathbf{M}_2^{(2)}(\hat{\beta}_n^{PT}) + \sigma_{zz} \frac{(n-3)}{(m-1)} \left[G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) \right. \\
&\quad \left. - \frac{p+2}{p} \frac{c}{n-3} \right] \psi^{-1} > \mathbf{M}_2^{(2)}(\hat{\beta}_n^{PT}),
\end{aligned} \tag{3.3.32}$$

where $c = \frac{(p-2)m}{(m+2)(p+2)}$ and

$$\begin{aligned}
G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) &= \frac{1}{n-3} I_y \left(\frac{p+2}{2} : \frac{m}{2} \right) \\
&= \frac{1}{n-3} F_{p+2,m} \left(\frac{p}{p+2} F_\alpha : 0 \right)
\end{aligned} \tag{3.3.33}$$

where $y = \frac{pF_\alpha}{m+pF_\alpha}$, such that $\frac{p}{p+2} F_\alpha$ satisfies the inequality

$$\left\{ \alpha : \frac{p}{p+2} F_\alpha > F_{p+2,m}^{-1} \left(\frac{p+2}{p} c : 0 \right) \right\} \tag{3.3.34}$$

The MSE of $\hat{\beta}_n^S$ is smaller than that of the MSE $\hat{\beta}_n^{PT}$ when the critical value $\frac{p}{p+2}F_\alpha$ satisfies the opposite inequality in (3.3.45). This means that the estimator $\hat{\beta}_n^S$ does not always dominate $\hat{\beta}_n^{PT}$ under H_0 for all α and we can order the MSE under H_0 as follows

$$\hat{\beta}_n^{PT} \succ \hat{\beta}_n^S \succ \tilde{\beta}_n \quad (3.3.35)$$

when α satisfies (3.3.34). Otherwise we have

$$\hat{\beta}_n^S \succ \hat{\beta}_n^{PT} \succ \tilde{\beta}_n. \quad (3.3.36)$$

Risk comparison:

Comparison of $\hat{\beta}_n^S$ with $\tilde{\beta}_n$

The risk-difference may be expressed by

$$\begin{aligned} & R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q}_0) - R_3^{(2)}(\hat{\beta}_n^S : \mathbf{Q}_0) \\ &= \sigma_{zz} \text{tr} \left\{ \mathbf{Q}_0 \psi^{-1} \right\} \frac{(p-2)m(n-3)}{(m+2)(m-1)} \\ &\times \left\{ (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) + 2\Delta_0^2 E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \right\} \\ &- \frac{(p-2)m(p+2)}{(m+2)} \beta' \mathbf{Q}_0 \beta E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \\ &= \sigma_{zz} \text{tr} \left\{ \mathbf{Q}_0 \psi^{-1} \right\} \frac{(p-2)m(n-3)}{(m+2)(m-1)} \left[(p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \right. \\ &+ \left. 2\Delta_0^2 E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \left\{ 1 - \frac{(m-1)(p+2)}{2(n-3)\text{tr}\{\mathbf{Q}_0 \psi^{-1}\}} \frac{\beta' \mathbf{Q}_0 \beta}{\beta' \psi \beta} \right\} \right]. \end{aligned} \quad (3.3.37)$$

This risk difference is positive for all $\mathbf{Q}_0$ satisfying the condition

$$\left\{ \mathbf{Q}_0 : \frac{\text{tr}\{\mathbf{Q}_0 \psi^{-1}\}}{\text{ch}_{\max}\{\mathbf{Q}_0 \psi^{-1}\}} > \frac{(p+2)(m-1)}{2(n-3)} \right\} \quad (3.3.38)$$

Comparison of $\hat{\beta}_n^S$ with $\hat{\beta}_n^{PT}$

Under $H_0 : \beta = 0$,

$$\begin{aligned}
R_3^{(2)}(\hat{\beta}_n^S : \mathbf{Q}_0) &= R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0) + \sigma_{zz} \frac{(n-3)}{(m-1)} \text{tr}\{\mathbf{Q}_0 \psi^{-1}\} \\
&\times \left[G_{p+2,m}^{(1)}\left(\frac{p}{p+2} F_\alpha : 0\right) - \frac{(p-2)^2 m}{m+2} E^{(1)}(\chi_{p+2}^{-4}(0)) \right] \\
&= R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0) + \sigma_{zz} \frac{(n-3)}{(m-1)} \text{tr}\{\mathbf{Q}_0 \psi^{-1}\} \\
&\times \left[G_{p+2,m}^{(1)}\left(\frac{p}{p+2} F_\alpha : 0\right) - \frac{p+2}{p} \frac{c}{n-3} \right] \\
&> R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0)
\end{aligned} \tag{3.3.39}$$

where $c = \frac{(p-2)m}{(m+2)(p+2)}$ and

$$\begin{aligned}
G_{p+2,m}^{(1)}\left(\frac{p}{p+2} F_\alpha : 0\right) &= \frac{1}{n-3} I_y\left(\frac{p+2}{2} : \frac{m}{2}\right) \\
&= \frac{1}{n-3} F_{p+2,m}\left(\frac{p}{p+2} F_\alpha : 0\right)
\end{aligned} \tag{3.3.40}$$

where $y = \frac{pF_\alpha}{m+pF_\alpha}$, such that $\frac{p}{p+2} F_\alpha$ satisfies the inequality

$$\left\{ \alpha : \frac{p}{p+2} F_\alpha > F_{p+2,m}^{-1}\left(\frac{(p+2)c}{p} : 0\right) \right\}. \tag{3.3.41}$$

The risk of $\hat{\beta}_n^S$ is smaller than that of $\hat{\beta}_n^{PT}$ when the critical value $\frac{p}{p+2} F_\alpha$ satisfies the opposite inequality in (3.3.41). This means the estimator $\hat{\beta}_n^S$ does not always dominate $\hat{\beta}_n^{PT}$ under H_0 for all α .

MSE and Risk for the Estimator $\hat{\beta}_n^{S+}$

The MSE and risk are given by

$$\begin{aligned}
M_4^{(2)}(\hat{\beta}_n^{S+}) &= M_3^{(2)}(\hat{\beta}_n^S) - \sigma_{zz} \frac{(n-3)}{m-1} \left[G_{p+2,m}^{(1)}(c; \Delta_0^2) \psi^{-1} \right. \\
&\quad - c \left\{ 2E^{(1)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad \left. - cE^{(1)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right\} \psi^{-1} \Big] \\
&\quad - \beta \beta' \left[G_{p+4,m}^{(2)}(c_1; \Delta_0^2) - 2G_{p+2,m}^{(2)}(c; \Delta_0^2) \right. \\
&\quad + c \left\{ 2E^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad - 2E^{(2)}(F_{p+4,m}^{-1}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \\
&\quad \left. \left. + c_1 E^{(2)}(F_{p+4,m}^{-2}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right\} \right]. \\
R_4^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) &= R_3^{(2)}(\hat{\beta}_n^S) - \sigma_{zz} \text{tr} \left\{ \mathbf{Q}_0 \psi^{-1} \right\} \frac{(n-3)}{m-1} \left[G_{p+2,m}^{(1)}(c; \Delta_0^2) \right. \\
&\quad - c \left\{ 2E^{(1)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad \left. - cE^{(1)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right\} \Big] \\
&\quad - \beta' \mathbf{Q}_0 \beta \left[G_{p+4,m}^{(2)}(c_1; \Delta_0^2) - 2G_{p+2,m}^{(2)}(c; \Delta_0^2) \right. \\
&\quad + c \left\{ 2E^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad - 2E^{(2)}(F_{p+4,m}^{-1}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \\
&\quad \left. \left. + c_1 E^{(2)}(F_{p+4,m}^{-2}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right\} \right]. \tag{3.3.42}
\end{aligned}$$

where $m = n - p - 1$, $c = \frac{(p-2)m}{(p+2)(m+2)}$, $c_1 = \frac{(p-2)m}{(p+4)(m+2)}$ and $\psi = \mathbf{K}_{xx}' \Sigma_{XX} \mathbf{K}_{xx}$.

The notations in the MSE and risk are given by

$$\begin{aligned}
(1) \quad & E^{(j)} \left[F_{p+2i,m}^{-1}(\Delta_0^2) I \left(F_{p+2i,m}(\Delta_0^2) < \frac{(p-2)m}{(p+2i)(m+2)} \right) \right] \\
&= \sum_{r \geq 0} \frac{2^{j-2} \Delta_0^{2r} \Gamma(\frac{n-1}{2} + r + j - 2)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r + j - 2}} \\
&\quad \times \frac{p+2i}{p+2i-2+2r} I_{y^*} \left(\frac{p+2i-2}{2} + r; \frac{m+2}{2} \right) \tag{3.3.43}
\end{aligned}$$

$$\begin{aligned}
(2) \quad & E^{(j)} \left[F_{p+2i,m}^{-2}(\Delta_0^2) I \left(F_{p+2i,m}(\Delta_0^2) < \frac{(p-2)m}{(p+2i)(m+2)} \right) \right] \\
&= \sum_{r \geq 0} \frac{2^{j-2} \Delta_0^{2r} \Gamma(\frac{n-1}{2} + r + j - 2)}{\Gamma(\frac{n-1}{2}) \Gamma(r+1) (\Delta_0^2 + 1)^{\frac{n-1}{2} + r + j - 2}} \\
&\quad \times \frac{(p+2i)^2(m+2)}{m(p+2i-2+2r)(p+2i-4+2r)} \\
&\quad \times I_{y^*} \left(\frac{p+2i-4}{2} + r; \frac{m+4}{2} \right), \tag{3.3.44}
\end{aligned}$$

where $y^* = \frac{p-2}{m+p}$.

MSE comparison:

Comparison of $\hat{\beta}_n^{S+}$ with $\hat{\beta}_n^{PT}$

Under $H_0 : \beta = 0$,

$$\begin{aligned}
M_4^{(2)}(\hat{\beta}_n^{S+}) &= M_2^{(2)}(\hat{\beta}_n^{PT}) \\
&+ \frac{(n-3)}{(m-1)} \left[G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) - \frac{p+2}{p} \frac{c}{n-3} \right. \\
&\quad \left. - E^{(1)} \{ (1 - cF_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \} \right] \psi^{-1} \\
&> M_2^{(2)}(\hat{\beta}_n^{PT}) \tag{3.3.45}
\end{aligned}$$

for all α satisfying the condition

$$\begin{aligned}
&\left\{ \alpha : F_{p+2,m} \left(\frac{p}{p+2} F_\alpha \right) \geq \right. \\
&\quad \left. \frac{p+2}{p} c + E^{(1)} [(1 - cF_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c)] \right\}. \tag{3.3.46}
\end{aligned}$$

The MSE of $\hat{\beta}_n^{S+}$ is smaller than that of $\hat{\beta}_n^{PT}$ when the critical value $\frac{p}{p+2} F_\alpha$ satisfies the opposite inequality in (3.3.46). This means the estimator $\hat{\beta}_n^{S+}$ does not always dominate $\hat{\beta}_n^{PT}$ under H_0 for all α .

For reasonable Δ_0^2 near 0, the risk of $\hat{\beta}_n^{PT}$ is smaller than the MSE of $\hat{\beta}_n^{S+}$ for α satisfying (3.3.46) and for $p \geq 3$. However, the MSE of $\hat{\beta}_n^{S+}$ is below the MSE of

$\tilde{\beta}_n$ while the MSE of $\hat{\beta}_n^{PT}$ exceeds the MSE of $\tilde{\beta}_n$ at some intermediate values of Δ_0^2 depending on α .

Comparison of $\hat{\beta}_n^{S+}$ with $\hat{\beta}_n^S$

$$\begin{aligned}
& \mathbf{M}_3^{(2)}(\hat{\beta}_n^S) - \mathbf{M}_4^{(2)}(\hat{\beta}_n^{S+}) \\
&= \sigma_{zz} \left(\frac{n-3}{m-1} \right) \left[E^{(1)} \{ (1 - cF_{p+2,m}^{-1}(\Delta_0^2))^2 I(F_{p+2,m}(\Delta_0^2) < c) \} \psi^{-1} \right. \\
&+ \sigma_{zz} \beta \beta' E^{(2)} \{ (1 - c_1 F_{p+4,m}^{-1}(\Delta_0^2))^2 I(F_{p+4,m}(\Delta_0^2) < c_1) \} \\
&+ \left. 2E^{(2)} \{ (F_{p+2,m}^{-1}(\Delta_0^2) - 1) I(F_{p+2,m}(\Delta_0^2) < c) \} \right] \quad (3.3.47)
\end{aligned}$$

and hence $\mathbf{M}_3^{(2)}(\hat{\beta}_n^S) - \mathbf{M}_4^{(2)}(\hat{\beta}_n^{S+}) \geq 0$ since

$$\begin{aligned}
(0 < F_{p+2i,m}(\Delta_0^2) < c_i) &\Leftrightarrow (c_i F_{p+2i,m}(\Delta_0^2) - 1) \geq 0 \\
E^{(j)} \{ (c_i F_{p+2i,m}^{-1}(\Delta_0^2) - 1) I(F_{p+2i,m}(\Delta_0^2) < c_i) \} &\geq 0 \\
E^{(j)} \{ (c_i F_{p+2i,m}^{-1}(\Delta_0^2) - 1)^2 I(F_{p+2i,m}(\Delta_0^2) < c_i) \} &\geq 0. \quad (3.3.48)
\end{aligned}$$

Risk comparison:

Comparison of $\hat{\beta}_n^{S+}$ with $\tilde{\beta}_n$

The risk-difference may be expressed in a quadratic form by

$$\begin{aligned}
& R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q}_0) - R_4^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) \\
&= R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q}_0) - R_3^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) + \sigma_{zz} \frac{(n-3)}{(m-1)} \left[\text{tr} \{ \mathbf{Q}_0 \psi^{-1} \} \right. \\
&\times E^{(1)} \{ (1 - cF_{p+2,m}^{-1}(\Delta_0^2))^2 I(F_{p+2,m}(\Delta_0^2) < c) \} \\
&+ \Delta_0^2 E^{(2)} \{ (1 - c_1 F_{p+4,m}^{-1}(\Delta_0^2))^2 I(F_{p+4,m}(\Delta_0^2) < c_1) \} \\
&+ \left. 2E^{(2)} \{ (F_{p+2,m}^{-1}(\Delta_0^2) - 1) I(F_{p+2,m}(\Delta_0^2) < c) \} \right] \quad (3.3.49)
\end{aligned}$$

and hence $R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q}_0) - R_4^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) \geq 0$ for satisfying (3.3.46) since

$$\begin{aligned} (0 < F_{p+2i,m}(\Delta_0^2) < c_i) &\Leftrightarrow (c_i F_{p+2i,m}(\Delta_0^2) - 1) \geq 0 \\ E^{(j)}\{(c_i F_{p+2i,m}^{-1}(\Delta_0^2) - 1)I(F_{p+2i,m}(\Delta_0^2) < c_i)\} &\geq 0 \\ E^{(j)}\{(c_i F_{p+2i,m}^{-1}(\Delta_0^2) - 1)^2 I(F_{p+2i,m}(\Delta_0^2) < c_i)\} &\geq 0. \end{aligned} \quad (3.3.50)$$

Comparison of $\hat{\beta}_n^{S+}$ with $\hat{\beta}_n^{PT}$

Under $H_0 : \beta = 0$, the risk-difference may be written by

$$\begin{aligned} &R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0) - R_4^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) \\ &= R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q}_0) - R_3^{(2)}(\hat{\beta}_n^S : \mathbf{Q}_0) + \sigma_{zz} \left(\frac{n-3}{m-1} \right) \\ &\times \text{tr}[\mathbf{Q}_0 \psi^{-1}] \left[E^{(1)} \left\{ (1 - c F_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right. \\ &- \left. G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) \right] \\ &= \sigma_{zz} \left(\frac{n-3}{m-1} \right) \\ &\times \text{tr}[\mathbf{Q}_0 \psi^{-1}] \left[E^{(1)} \left\{ (1 - c F_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right. \\ &- \left. \left\{ G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) - \frac{p+2}{p} \frac{c}{n-3} \right\} \right]. \end{aligned} \quad (3.3.51)$$

Thus,

$$\begin{aligned} &R_4^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) = R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0) \\ &+ \left(\frac{n-3}{m-1} \right) \text{tr}[\mathbf{Q}_0 \psi^{-1}] \left[G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) - \frac{p+2}{p} \frac{c}{n-3} \right. \\ &- \left. E^{(1)} \left\{ (1 - c F_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right] \\ &> R_2^{(2)}(\hat{\beta}_n^{PT} : \mathbf{Q}_0) \end{aligned} \quad (3.3.52)$$

for all α satisfying the condition

$$\begin{aligned} &\left[\alpha : F_{p+2,m} \left(\frac{p}{p+2} F_\alpha \right) \geq \right. \\ &\left. \frac{(p+2)c}{p} + E^{(1)} \left\{ (1 - c F_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right]. \end{aligned} \quad (3.3.53)$$

Figure 3.1: Graph of Unconditional Risk of $\tilde{\beta}_n$, $\hat{\beta}_n^{PT}$, $\hat{\beta}_n^S$ and $\hat{\beta}_n^{S+}$

The risk of $\hat{\beta}_n^{S+}$ is smaller than that of $\hat{\beta}_n^{PT}$ when the critical value $\frac{p}{p+2}F_\alpha$ satisfies the opposite inequality in (3.3.53). This means the estimator $\hat{\beta}_n^{S+}$ does not always dominate $\hat{\beta}_n^{PT}$ under H_0 for all α .

Comparison of $\hat{\beta}_n^{S+}$ with $\hat{\beta}_n^S$

The risk-difference may be expressed in a quadratic form by

$$\begin{aligned}
 & R_3^{(2)}(\hat{\beta}_n^S : \mathbf{Q}_0) - R_4^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) \\
 &= \sigma_{zz} \left(\frac{n-3}{m-1} \right) \left[\text{tr}\{\mathbf{Q}_0 \psi^{-1}\} E^{(1)} \left\{ (1 - c F_{p+2,m}^{-1}(\Delta_0^2))^2 I(F_{p+2,m}(\Delta_0^2) < c) \right\} \right. \\
 &+ \Delta_0^2 E^{(2)} \left\{ (1 - c_1 F_{p+4,m}^{-1}(\Delta_0^2))^2 I(F_{p+4,m}(\Delta_0^2) < c_1) \right\} \\
 &+ \left. 2E^{(2)} \left\{ (F_{p+2,m}^{-1}(\Delta_0^2) - 1) I(F_{p+2,m}(\Delta_0^2) < c) \right\} \right] \quad (3.3.54)
 \end{aligned}$$

and hence $R_3^{(2)}(\hat{\beta}_n^S : \mathbf{Q}_0) - R_4^{(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}_0) \geq 0$ for the same reason that

(3.3.48) holds.

Thus under $H_0 : \beta = 0$, we can order the dominance picture under (3.3.53) as follows:

$$\hat{\beta}_n^{PT} \succ \hat{\beta}_n^{S+} \succ \hat{\beta}_n^S \succ \tilde{\beta}_n \quad (3.3.55)$$

For reasonable Δ_0^2 near 0, the risk of $\hat{\beta}_n^{PT}$ is smaller than the risk of $\hat{\beta}_n^{S+}$ for α satisfying (3.3.53) and for $p \geq 3$. Thus neither $\hat{\beta}_n^{PT}$ nor $\hat{\beta}_n^{S+}$ dominates the order except that they share common property that as $\Delta_0^2 \rightarrow \infty$, the risk of both becomes p . However, the risk of $\hat{\beta}_n^{S+}$ is below the risk of $\tilde{\beta}_n$ while the risk of $\hat{\beta}_n^{PT}$ exceeds the risk of $\tilde{\beta}_n$ at some intermediate values of Δ_0^2 depending on α . Figure 3.1 displays the graph of unconditional risk of the four estimator with $\alpha = 0.15, n = 9$, and $p = 4$.

3.3.4 Bias, Qudratic Bias, MSE Matrix and Risk Expressions for ν_n^*

The quadratic bias of the four estimator may be obtained by using the covariance matrix $\frac{\sigma_{zz}}{m-1} \Sigma_{X_0 X_0}^{-1}$. The bias and quadratic bias of the four estimators evolving from ν_n^* are given as follows:

Bias and Qudratic Bias for $\tilde{\nu}_n$

$$\begin{aligned} \mathbf{b}_1^{(1)}(\tilde{\nu}_n) &= -(\mathbf{I} - \mathbf{K}_{xx})\beta \\ (m-1)B_1^{(1)}(\tilde{\nu}_n) &= \sigma_{zz}^{-1} \beta' (\mathbf{I} - \mathbf{K}_{xx})' \Sigma_{X_0 X_0} (\mathbf{I} - \mathbf{K}_{xx}) \beta \end{aligned} \quad (3.3.56)$$

Bias and Qudratic Bias for $\hat{\nu}_n^{PT}$

$$\begin{aligned}
\mathbf{b}_2^{(1)}(\hat{\nu}_n^{PT}) &= -(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\beta} - \mathbf{K}_{xx}\boldsymbol{\beta} \\
&\quad \times G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_{p,m}(\alpha); \Delta_0^2\right) \\
(m-1)B_2^{(1)}(\hat{\nu}_n^{PT}) &= \sigma_{zz}^{-1}\boldsymbol{\beta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{X_0X_0}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\beta} \\
&\quad + \Delta_0^2\left\{G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_{p,m}(\alpha); \Delta_0^2\right)\right\}^2 \\
&\quad + 2\sigma_{zz}^{-1}\boldsymbol{\beta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{X_0X_0}\mathbf{K}_{xx} \\
&\quad \times G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_{p,m}(\alpha); \Delta_0^2\right). \tag{3.3.57}
\end{aligned}$$

Bias and Qudratic Bias for $\hat{\nu}_n^S$

$$\begin{aligned}
\mathbf{b}_3^{(1)}(\hat{\nu}_n^S) &= -(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\beta} - \frac{(p-2)m}{m+2}\mathbf{K}_{xx}\boldsymbol{\beta}E^{(2)}[\chi_{p+2}^{-2}(\Delta_0^2)] \\
(m-1)B_3^{(1)}(\hat{\nu}_n^S) &= \sigma_{zz}^{-1}\boldsymbol{\beta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{X_0X_0}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\beta} \\
&\quad + \frac{(p-2)^2m^2}{(m+2)^2}\Delta_0^2\left\{E^{(2)}[\chi_{p+2}^{-2}(\Delta_0^2)]\right\}^2 \\
&\quad + \sigma_{zz}^{-1}\boldsymbol{\beta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{X_0X_0}\mathbf{K}_{xx}\boldsymbol{\beta}E^{(2)}[\chi_{p+2}^{-2}(\Delta_0^2)] \tag{3.3.58}
\end{aligned}$$

Bias and Qudratic Bias for $\hat{\nu}_n^{S+}$

$$\begin{aligned}
\mathbf{b}_4^{(1)}(\hat{\nu}_n^{S+}) &= -(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\beta} - \frac{(p-2)m}{m+2}\mathbf{K}_{xx}\boldsymbol{\beta}\left\{E^{(2)}[F_{p+2,m}^{-2}(\Delta_0^2)]\right. \\
&\quad \left.- E^{(2)}\left[F_{p+2,m}^{-2}(\Delta_0^2)I\left(F_{p+2,m}(\Delta_0^2) < \frac{(p+2)m}{m+2}\right)\right]\right\} \\
&\quad + \frac{m+2}{m(p-2)}G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_{p,m}(\alpha); \Delta_0^2\right)\}
\end{aligned}$$

$$\begin{aligned}
(m-1)B_4^{(1)}(\hat{\nu}_n^{S+}) &= \sigma_{zz}^{-1}\beta'(\mathbf{I} - \mathbf{K}_{xx})'\Sigma_{X_0X_0}(\mathbf{I} - \mathbf{K}_{xx})\beta \\
&+ \frac{(p-2)^2m^2}{(m+2)^2}\Delta_0^2\left\{E^{(2)}[F_{p+2,m}^{-2}(\Delta_0^2)]\right. \\
&- E^{(2)}\left[F_{p+2,m}^{-2}(\Delta_0^2)I\left(F_{p+2,m}(\Delta_0^2) < \frac{(p+2)m}{m+2}\right)\right] \\
&+ \frac{m+2}{m(p-2)}G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_{p,m}(\alpha); \Delta_0^2\right)\left.\right\}^2 \\
&+ \frac{2(p-2)m}{(m+2)}\sigma_{zz}^{-1}\beta'(\mathbf{I} - \mathbf{K}_{xx})\Sigma_{xx}\mathbf{K}_{xx}\beta\left\{E^{(2)}[F_{p+2,m}^{-2}(\Delta_0^2)]\right. \\
&- E^{(2)}\left[F_{p+2,m}^{-2}(\Delta_0^2)I\left(F_{p+2,m}(\Delta_0^2) < \frac{(p+2)m}{m+2}\right)\right] \\
&+ \frac{m+2}{m(p-2)}G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_{p,m}(\alpha); \Delta_0^2\right)\left.\right\} \tag{3.3.59}
\end{aligned}$$

MSE matrices and risk expressions for the class of ν_n^* are given as follows.

MSE matrix and the risk expressions of $\tilde{\nu}_{1n}$ are given by

$$\begin{aligned}
M_1^{(1)}(\tilde{\nu}_n) &= \frac{\sigma_{zz}}{m-1}\Sigma_{X_0X_0}^{-1} + (\mathbf{I} - \mathbf{K}_{xx})\beta\beta'(\mathbf{I} - \mathbf{K}_{xx})' \\
R_1^{(1)}(\tilde{\nu}_n : \mathbf{Q}_0) &= \frac{\sigma_{zz}}{m-1}\text{tr}[\mathbf{Q}_0\Sigma_{X_0X_0}^{-1}] + \beta'(\mathbf{I} - \mathbf{K}_{xx})'\mathbf{Q}_0(\mathbf{I} - \mathbf{K}_{xx})\beta \tag{3.3.60}
\end{aligned}$$

MSE matrix and risk expressions of $\hat{\nu}_{1n}^{PT}$ are given by

$$\begin{aligned}
\mathbf{M}_2^{(1)}(\hat{\nu}_n^{PT}) &= \mathbf{M}_1^{(1)}(\tilde{\nu}_n) \\
&- \frac{(n-3)\sigma_{zz}}{m-1}G_{p+2,m}^{(1)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right)\Sigma_{X_0X_0}^{-1} \\
&+ \mathbf{K}_{xx}\beta\beta'\mathbf{K}_{xx}'\left\{2G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right)\right. \\
&- \left.G_{p+4,m}^{(2)}\left(\frac{p}{p+4}F_\alpha; \Delta_0^2\right)\right\} \\
&+ \left[(\mathbf{I} - \mathbf{K}_{xx})\beta\beta'\mathbf{K}_{xx}' + \mathbf{K}_{xx}\beta\beta'(\mathbf{I} - \mathbf{K}_{xx})'\right] \\
&\times G_{p+2,m}^{(1)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right) \tag{3.3.61}
\end{aligned}$$

$$\begin{aligned}
R_2^{(1)}(\hat{\nu}_n^{PT} : \mathbf{Q}_0) &= R_1^{(1)}(\tilde{\nu}_n : \mathbf{Q}_0) \\
&- \frac{(n-3)\sigma_{zz}}{m-1} G_{p+2,m}^{(1)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right) \text{tr}[\Sigma_{X_0X_0}^{-1} \mathbf{Q}_0] \\
&+ \boldsymbol{\beta}' \mathbf{K}'_{xx} \mathbf{Q}_0 \mathbf{K}_{xx} \boldsymbol{\beta} \left\{ 2G_{p+2,m}^{(2)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right) \right. \\
&- \left. G_{p+4,m}^{(2)}\left(\frac{p}{p+4}F_\alpha; \Delta_0^2\right) \right\} \\
&+ \text{tr} \left[\left\{ (\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{K}'_{xx} + \mathbf{K}_{xx} \boldsymbol{\beta} \boldsymbol{\beta}' (\mathbf{I} - \mathbf{K}_{xx})' \right\} \mathbf{Q}_0 \right] \\
&\times G_{p+2,m}^{(1)}\left(\frac{p}{p+2}F_\alpha; \Delta_0^2\right)
\end{aligned} \tag{3.3.62}$$

The MSE matrix and risk of $\hat{\nu}_n^S$ are given by

$$\begin{aligned}
\mathbf{M}_3^{(1)}(\hat{\nu}_n^S) &= \mathbf{M}_1^{(1)}(\tilde{\nu}_n) \\
&- \sigma_{zz} \frac{(p-2)(n-3)m}{(m+2)(m-1)} \left\{ 2E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- \left. (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \right\} \Sigma_{X_0X_0}^{-1} \\
&+ \frac{(p-2)m}{m+2} \mathbf{K}_{xx} \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{K}'_{xx} \left\{ 2E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- \left. 2E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) + (p-2)E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \right\} \\
&+ \frac{(p-2)m}{m+2} \left[(\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{K}'_{xx} + \mathbf{K}_{xx} \boldsymbol{\beta} \boldsymbol{\beta}' (\mathbf{I} - \mathbf{K}_{xx})' \right] \\
&\times E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2))
\end{aligned} \tag{3.3.63}$$

$$\begin{aligned}
R_3^{(1)}(\hat{\nu}_n^S : \mathbf{Q}_0) &= R_1^{(1)}(\tilde{\nu}_n : \mathbf{Q}_0) \\
&- \sigma_{zz} \frac{(p-2)(n-3)m}{(m+2)(m-1)} \left\{ 2E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \left. \right\} \text{tr}[\Sigma_{X_0 X_0}^{-1} \mathbf{Q}_0] \\
&+ \frac{(p-2)m}{m+2} \boldsymbol{\beta}' \mathbf{K}'_{xx} \mathbf{Q}_0 \mathbf{K}_{xx} \boldsymbol{\beta} \left\{ 2E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- 2E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) + (p-2)E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \left. \right\} \\
&+ \frac{(p-2)m}{m+2} \left[\text{tr} \left\{ (\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{K}'_{xx} \right. \right. \\
&+ \left. \left. \mathbf{K}_{xx} \boldsymbol{\beta} \boldsymbol{\beta}' (\mathbf{I} - \mathbf{K}_{xx})' \right\} \mathbf{Q}_0 \right] E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) \quad (3.3.64)
\end{aligned}$$

The MSE matrix and risk expressions of $\hat{\nu}_{1n}^{S+}$ are given by

$$\begin{aligned}
\mathbf{M}_4^{(1)}(\hat{\nu}_n^{S+}) &= \mathbf{M}_3^{(1)}(\hat{\nu}_n^S) - \sigma_{zz} \frac{(n-3)}{m-1} \left[G_{p+2,m}^{(1)}(c; \Delta_0^2) \right. \\
&- c \left\{ 2E^{(1)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&- \left. cE^{(1)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right\} \left. \right] \Sigma_{X_0 X_0}^{-1} \\
&- \mathbf{K}_{xx} \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{K}'_{xx} \left[G_{p+4,m}^{(2)}(c_1; \Delta_0^2) - 2G_{p+2,m}^{(2)}(c; \Delta_0^2) \right. \\
&+ c \left\{ 2E^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&- \left. 2E^{(2)}(F_{p+4,m}^{-1}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right. \\
&+ \left. \left. c_1 E^{(2)}(F_{p+4,m}^{-2}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right\} \right] \\
&+ \left[(\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{K}'_{xx} + \mathbf{K}_{xx} \boldsymbol{\beta} \boldsymbol{\beta}' (\mathbf{I} - \mathbf{K}_{xx})' \right] \\
&\times \left\{ G_{p+2,m}^{(1)}(c; \Delta_0^2) + cE^{(2)}(\chi_{p+2}^2(\Delta_0^2)) \right. \\
&- \left. cE^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right\} \quad (3.3.65)
\end{aligned}$$

$$\begin{aligned}
R_4^{(1)}(\hat{\nu}_n^{S+} : \mathbf{Q}_0) &= R_3^{(1)}(\hat{\nu}_n^S : \mathbf{Q}_0) - \sigma_{zz} \frac{(n-3)}{m-1} \left[G_{p+2,m}^{(1)}(c; \Delta_0^2) \right. \\
&\quad - c \left\{ 2E^{(1)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad \left. - cE^{(1)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right\} \left. \right] \text{tr}[\Sigma_{X_0 X_0}^{-1} \mathbf{Q}_0] \\
&\quad - \beta' \mathbf{K}'_{xx} \mathbf{Q}_0 \mathbf{K}_{xx} \beta \left[G_{p+4,m}^{(2)}(c_1; \Delta_0^2) - 2G_{p+2,m}^{(2)}(c; \Delta_0^2) \right. \\
&\quad + c \left\{ 2E^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad \left. - 2E^{(2)}(F_{p+4,m}^{-1}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right. \\
&\quad \left. + c_1 E^{(2)}(F_{p+4,m}^{-2}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right\} \left. \right] \\
&\quad + \left[\text{tr} \left\{ (\mathbf{I} - \mathbf{K}_{xx}) \beta \beta' \mathbf{K}'_{xx} + \mathbf{K}_{xx} \beta \beta' (\mathbf{I} - \mathbf{K}_{xx})' \right\} \mathbf{Q}_0 \right] \\
&\quad \times \left\{ G_{p+2,m}^{(1)}(c; \Delta_0^2) + cE^{(2)}(\chi_{p+2}^2(\Delta_0^2)) \right. \\
&\quad \left. - cE^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right\} \quad (3.3.66)
\end{aligned}$$

where $m = n - p - 1$, $c = \frac{(p-2)m}{(p+2)(m+2)}$ and $c_1 = \frac{(p-2)m}{(p+4)(m+2)}$.

As regards the estimators ν_n^* similar results hold and is not repeated.

3.3.5 Canonical Risk Forms of ν_n^* with β_n^*

There exists a nonsingular $p \times p$ matrix $\mathbf{T}$ that simultaneously diagonalizes Σ_{XX} and Σ_{xx} . Thus,

$$\mathbf{T}' \Sigma_{XX} \mathbf{T} = \mathbf{I}_p, \quad \mathbf{T}' \Sigma_{xx} \mathbf{T} = \mathbf{D} = \text{diag}(\delta_1, \dots, \delta_p) \quad (3.3.67)$$

where $\delta_1 \leq \dots \leq \delta_p$ are the ordered eigenvalues of $\Sigma_{XX}^{-1} \Sigma_{xx} = \mathbf{K}_{xx}$ and $0 \leq \delta_i \leq 1$, $i = 1, \dots, p$, since the $\mathbf{T}$ standardizes the observed predictor vectors $\mathbf{X}_t$, i.e. $\text{cov}(\mathbf{T}' \mathbf{X}_t) = \mathbf{I}$, so that the δ_i are to be the scalar reliabilities of the elements of $\mathbf{X}_t \mathbf{T}$.

It is easily seen that $\mathbf{K}_{xx} = \mathbf{T} \mathbf{D} \mathbf{T}^{-1}$ so that the i th column of $\mathbf{T}$ is the right eigenvector of $\mathbf{K}_{xx}$ corresponding the eigenvalue δ_i , $i = 1, \dots, p$.

Let $\boldsymbol{\theta} = \mathbf{T}^{-1}\boldsymbol{\beta}$. Then the corresponding estimator of $\boldsymbol{\theta}$ are $\tilde{\boldsymbol{\theta}}_n = \mathbf{T}^{-1}\tilde{\boldsymbol{\beta}}_n$ and $\hat{\boldsymbol{\theta}}_n = \mathbf{T}^{-1}\tilde{\boldsymbol{\nu}}_n$, respectively.

Now consider the estimators $\mathbf{T}^{-1}\boldsymbol{\nu}_n^*$ and $\mathbf{T}^{-1}\boldsymbol{\beta}_n^*$ of $\mathbf{T}^{-1}\boldsymbol{\beta} = \boldsymbol{\theta}$, where $\boldsymbol{\nu}_n^* = (1 - g(\mathcal{L}_n))\tilde{\boldsymbol{\nu}}_n$ and $\boldsymbol{\beta}_n^* = (1 - g(\mathcal{L}_n))\tilde{\boldsymbol{\beta}}_n$, respectively.

Canonical risk form of $\tilde{\boldsymbol{\nu}}_n$ and $\tilde{\boldsymbol{\beta}}_n$

$$\begin{aligned} R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) &= \frac{\sigma_{zz}}{m-1} \text{tr}[\mathbf{I}] + \boldsymbol{\theta}'(\mathbf{I} - \mathbf{D})^2 \boldsymbol{\theta} \\ R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) &= \frac{\sigma_{zz}}{m-1} \text{tr}[\mathbf{D}^{-2}] \end{aligned} \quad (3.3.68)$$

The difference of risks is given by

$$R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) - R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) = \frac{\sigma_{zz}}{m-1} \text{tr}[\mathbf{I} - \mathbf{D}^{-2}] + \boldsymbol{\theta}'(\mathbf{I} - \mathbf{D})^2 \boldsymbol{\theta} \quad (3.3.69)$$

The risk of $\hat{\boldsymbol{\theta}}_n$ is smaller than the risk of $\tilde{\boldsymbol{\theta}}_n$ if

$$\Delta_0^2 \leq \frac{\sigma_{zz} \text{tr}[\mathbf{D}^{-2} - \mathbf{I}]}{(m-1)Ch_{\min}[\mathbf{A}_1 \mathbf{D}^{-2}]} \quad (3.3.70)$$

where $\mathbf{A}_1 = (\mathbf{I} - \mathbf{D})^2$ and the risk of $\tilde{\boldsymbol{\theta}}_n$ is smaller than the risk of $\hat{\boldsymbol{\theta}}_n$ if

$$\Delta_0^2 > \frac{\sigma_{zz} \text{tr}[\mathbf{D}^{-2} - \mathbf{I}]}{(m-1)Ch_{\max}[\mathbf{A}_1 \mathbf{D}^{-2}]} \quad (3.3.71)$$

Clearly, we find that $\hat{\boldsymbol{\theta}}_n$ dominates $\tilde{\boldsymbol{\theta}}_n$ under the null-hypotheses. Figure 3.2 shows the entire dominance picture of $\hat{\boldsymbol{\theta}}_n$ and $\tilde{\boldsymbol{\theta}}_n$ thereby of $\tilde{\boldsymbol{\nu}}_n$ and $\tilde{\boldsymbol{\beta}}_n$.

Canonical risk form of $\hat{\nu}_n^{PT}$ and $\hat{\beta}_n^{PT}$

$$\begin{aligned}
R_2^{(1)}(\hat{\theta}_n^{PT}) &= R_1^{(1)}(\hat{\theta}_n) - \sigma_{zz} \frac{n-3}{m-1} G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \text{tr}[\mathbf{I}] \\
&+ \boldsymbol{\theta}' \mathbf{D}^2 \boldsymbol{\theta} \left\{ 2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \right. \\
&- \left. G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right) \right\} \\
&+ 2\boldsymbol{\theta}' (\mathbf{I} - \mathbf{D}) \mathbf{D} \boldsymbol{\theta} G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \\
R_2^{(2)}(\tilde{\theta}_n^{PT}) &= R_1^{(2)}(\tilde{\theta}_n) - \sigma_{zz} \frac{n-3}{m-1} G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \text{tr}[\mathbf{D}^{-2}] \\
&+ \boldsymbol{\theta}' \boldsymbol{\theta} \left\{ 2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \right. \\
&- \left. G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta_0^2 \right) \right\} \tag{3.3.72}
\end{aligned}$$

Clearly, we find that $\hat{\theta}_n^{PT}$ dominates $\tilde{\theta}_n^{PT}$ under the null-hypotheses. Figure 3.3 shows the entire dominance picture of $\hat{\theta}_n^{PT}$ and $\tilde{\theta}_n^{PT}$ thereby of $\hat{\nu}_n^{PT}$ and $\tilde{\beta}_n^{PT}$.

Canonical risk form of $\hat{\nu}_n^S$ and $\hat{\beta}_n^S$

$$\begin{aligned}
R_3^{(1)}(\hat{\theta}_n^S) &= R_1^{(1)}(\hat{\theta}_n) \\
&- \sigma_{zz} \frac{(p-2)(n-3)m}{(m+2)(m-1)} \left\{ 2E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- \left. (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \right\} \text{tr}[\mathbf{I}] \\
&+ \frac{(p-2)m}{m+2} \boldsymbol{\theta}' \mathbf{D}^2 \boldsymbol{\theta} \left\{ 2E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- \left. 2E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) + (p-2)E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \right\} \\
&+ \frac{2(p-2)m}{m+2} E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) \boldsymbol{\theta}' \mathbf{D} (\mathbf{I} - \mathbf{D}) \boldsymbol{\theta} \tag{3.3.73}
\end{aligned}$$

$$\begin{aligned}
R_3^{(2)}(\tilde{\boldsymbol{\theta}}_n^S) &= R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) \\
&- \sigma_{zz} \frac{(p-2)(n-3)m}{(m+2)(m-1)} \left\{ 2E^{(1)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \left. \right\} \text{tr}[\mathbf{D}^{-2}] \\
&+ \frac{(p-2)m}{m+2} \boldsymbol{\theta}' \boldsymbol{\theta} \left\{ 2E^{(2)}(\chi_{p+2}^{-2}(\Delta_0^2)) \right. \\
&- 2E^{(2)}(\chi_{p+4}^{-2}(\Delta_0^2)) + (p-2)E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \left. \right\} \quad (3.3.74)
\end{aligned}$$

Clearly, we find that $\hat{\boldsymbol{\theta}}_n^S$ dominates $\tilde{\boldsymbol{\theta}}_n^S$ under the null-hypotheses. Figure 3.4 shows the entire dominance picture of $\hat{\boldsymbol{\theta}}_n^S$ and $\tilde{\boldsymbol{\theta}}_n^S$ thereby of $\hat{\nu}_n^S$ and $\tilde{\beta}_n^S$.

Canonical risk form of $\hat{\nu}_n^{S+}$ and $\hat{\beta}_n^{S+}$

$$\begin{aligned}
R_4^{(1)}(\hat{\boldsymbol{\theta}}_n^{S+}) &= R_3^{(1)}(\hat{\boldsymbol{\theta}}_n^S) - \sigma_{zz} \frac{(n-3)}{m-1} \left[G_{p+2,m}^{(1)}(c; \Delta_0^2) \right. \\
&- c \left\{ 2E^{(1)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&- cE^{(1)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \left. \right\} \left. \right] \text{tr}[\mathbf{I}] \\
&- \boldsymbol{\theta}' \mathbf{D}^2 \boldsymbol{\theta} \left[G_{p+4,m}^{(2)}(c_1; \Delta_0^2) - 2G_{p+2,m}^{(2)}(c; \Delta_0^2) \right. \\
&+ c \left\{ 2E^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&- 2E^{(2)}(F_{p+4,m}^{-1}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \\
&+ c_1 E^{(2)}(F_{p+4,m}^{-2}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \left. \right\} \left. \right] \\
&+ 2\boldsymbol{\theta}' \mathbf{D}(\mathbf{I} - \mathbf{D}) \boldsymbol{\theta} \left\{ G_{p+2,m}^{(1)}(c; \Delta_0^2) \right. \\
&+ cE^{(2)}(\chi_{p+2}^2(\Delta_0^2)) - cE^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \left. \right\} \quad (3.3.75)
\end{aligned}$$

Figure 3.2: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n$ and the MLE $\tilde{\theta}_n$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

Figure 3.3: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n^{PT}$ and the MLE $\tilde{\theta}_n^{PT}$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

Figure 3.4: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n^S$ and the MLE $\tilde{\theta}_n^S$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

Figure 3.5: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n^{S+}$ and the MLE $\tilde{\theta}_n^{S+}$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

$$\begin{aligned}
R_4^{(2)}(\tilde{\theta}_n^{S+}) &= R_3^{(2)}(\tilde{\theta}_n^S) - \sigma_{zz} \frac{(n-3)}{m-1} \left[G_{p+2,m}^{(1)}(c; \Delta_0^2) \right. \\
&\quad - c \left\{ 2E^{(1)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad \left. - cE^{(1)}(F_{p+2,m}^{-2}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right\} \left. \right] \text{tr}[\mathbf{D}^{-2}] \\
&\quad - \boldsymbol{\theta}' \boldsymbol{\theta} \left[G_{p+4,m}^{(2)}(c_1; \Delta_0^2) - 2G_{p+2,m}^{(2)}(c; \Delta_0^2) \right. \\
&\quad + c \left\{ 2E^{(2)}(F_{p+2,m}^{-1}(\Delta_0^2) I(F_{p+2,m}(\Delta_0^2) < c)) \right. \\
&\quad \left. - 2E^{(2)}(F_{p+4,m}^{-1}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right. \\
&\quad \left. + c_1 E^{(2)}(F_{p+4,m}^{-2}(\Delta_0^2) I(F_{p+4,m}(\Delta_0^2) < c_1)) \right\} \left. \right] \quad (3.3.76)
\end{aligned}$$

where $m = n - p - 1$, $c = \frac{(p-2)m}{(p+2)(m+2)}$ and $c_1 = \frac{(p-2)m}{(p+4)(m+2)}$.

Clearly, we find that $\hat{\theta}_n^{S+}$ dominates $\tilde{\theta}_n^{S+}$ under the null-hypotheses. Figure 3.5 shows the entire dominance picture of $\hat{\theta}_n^{S+}$ and $\tilde{\theta}_n^{S+}$ thereby of $\tilde{\nu}_n^{S+}$ and $\tilde{\beta}_n^{S+}$.

Appendix

Conditional Bias, Mean-Square-Error and Quadratic Risk of the Estimators of Slope Parameter β ($\mathbf{K}_{xx}$ known).

In this section, we consider the bias, quadratic bias, mean-square-error matrix and quadratic risks of the four estimators defined in section 2. Thus, we may write the bias-vectors of $\tilde{\nu}^*$ and $\tilde{\beta}^*$ as

$$\mathbf{b}^{(1)} = E(\nu_n^* - \beta) = -(\mathbf{I} - \mathbf{K}_{xx})\beta - \mathbf{K}_{xx}\beta E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]$$

and

$$\mathbf{b}^{(2)} = E(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}) = -\boldsymbol{\beta}E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]$$

by Appendix B of Judge and Bock (1978) where $F_{\nu,m}(\Delta^2)$ stands for the non-central F-variable with (ν, m) d.f. and non-centrality parameter $\frac{1}{2}\Delta^2$. To evaluate the properties of the bias we consider the quadratic bias (QB) defined by

$$\begin{aligned} B^{(1)} &= \sigma_{zz}^{-1}\mathbf{b}^{(1)'}\mathbf{S}_{X_0X_0}\mathbf{b}^{(1)} = \boldsymbol{\beta}'(\mathbf{I} - \mathbf{K}_{xx})'\mathbf{S}_{X_0X_0}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\beta} \\ &+ \boldsymbol{\beta}'\mathbf{K}_{xx}'\mathbf{S}_{X_0X_0}\mathbf{K}_{xx}\boldsymbol{\beta}\left\{E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\right\}^2 \\ &- 2\boldsymbol{\beta}'(\mathbf{I} - \mathbf{K}_{xx})\mathbf{S}_{X_0X_0}\mathbf{K}_{xx}\boldsymbol{\beta}E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right] \end{aligned}$$

and

$$\begin{aligned} B^{(2)} &= \sigma_{zz}^{-1}\mathbf{b}^{(2)'}\mathbf{K}_{xx}'\mathbf{S}_{X_0X_0}\mathbf{K}_{xx}\mathbf{b}^{(2)} \\ &= \Delta^2\left\{E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\right\}^2 \end{aligned}$$

Thus, QB of $\boldsymbol{\nu}_n^*$ and $\boldsymbol{\beta}_n^*$ may be studied as a function of Δ^2 , which is the non-centrality parameter of the F -distribution. To obtain the mean-square-error (MSE) matrices of the estimators as

$$\begin{aligned} \mathbf{M}^{(1)} &= E\left[(\boldsymbol{\nu}_n^* - \boldsymbol{\beta})(\boldsymbol{\nu}_n^* - \boldsymbol{\beta})'\right] \\ &= \sigma_{zz}\mathbf{S}_{X_0X_0}^{-1} + (\boldsymbol{\nu} - \boldsymbol{\beta})(\boldsymbol{\nu} - \boldsymbol{\beta})' \\ &- \sigma_{zz}\mathbf{S}_{X_0X_0}^{-1}\left\{2E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\right. \\ &- E\left[g^2\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\left.\right\} + \sigma_{zz}\boldsymbol{\nu}\boldsymbol{\nu}'\left\{2E\left[g\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right]\right. \\ &- 2E\left[g\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right] + E\left[g^2\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right]\left.\right\} \\ &+ (\boldsymbol{\beta}(\boldsymbol{\beta} - \boldsymbol{\nu})' + (\boldsymbol{\beta} - \boldsymbol{\nu})\boldsymbol{\beta}')E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right] \end{aligned}$$

and

$$\begin{aligned}
\mathbf{M}^{(2)} &= E\left[(\boldsymbol{\beta}_n^* - \boldsymbol{\beta})(\boldsymbol{\beta}_n^* - \boldsymbol{\beta})'\right] \\
&= \sigma_{zz}\mathbf{C}^{-1} - \sigma_{zz}\mathbf{C}^{-1}\left\{2E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\right. \\
&\quad \left.- E\left[g^2\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\right\} \\
&\quad + \boldsymbol{\beta}\boldsymbol{\beta}'\left\{2E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right] - 2E\left[g\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right]\right. \\
&\quad \left.+ E\left[g^2\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right]\right\}
\end{aligned}$$

where $\mathbf{C} = \mathbf{K}'_{xx}\mathbf{S}_{X_0X_0}\mathbf{K}_{xx}$. The quadratic risk under the loss function

$$\mathcal{L}(\boldsymbol{\nu}_n^*, \boldsymbol{\beta}) = (\boldsymbol{\nu}_n^* - \boldsymbol{\beta})'\mathbf{Q}(\boldsymbol{\nu}_n^* - \boldsymbol{\beta})$$

and

$$\mathcal{L}(\boldsymbol{\beta}_n^*, \boldsymbol{\beta}) = (\boldsymbol{\beta}_n^* - \boldsymbol{\beta})'\mathbf{Q}(\boldsymbol{\beta}_n^* - \boldsymbol{\beta})$$

with the positive semi-definite matrix $\mathbf{Q}$ is given by

$$\begin{aligned}
R^{(1)} &= E\left[(\boldsymbol{\nu}_n^* - \boldsymbol{\beta})(\boldsymbol{\nu}_n^* - \boldsymbol{\beta})'\right] \\
&= \sigma_{zz}\text{tr}[\mathbf{S}_{X_0X_0}^{-1}\mathbf{Q}] + (\boldsymbol{\nu} - \boldsymbol{\beta})'\mathbf{Q}(\boldsymbol{\nu} - \boldsymbol{\beta}) \\
&\quad - \sigma_{zz}\text{tr}[\mathbf{S}_{X_0X_0}^{-1}\mathbf{Q}]\left\{2E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\right. \\
&\quad \left.- E\left[g^2\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]\right\} + \sigma_{zz}\boldsymbol{\nu}'\mathbf{Q}\boldsymbol{\nu}\left\{2E\left[g\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right]\right. \\
&\quad \left.- 2E\left[g\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right] + E\left[g^2\left(\frac{p+4}{p}F_{p+4,m}(\Delta^2)\right)\right]\right\} \\
&\quad + \text{tr}[(\boldsymbol{\beta}(\boldsymbol{\beta} - \boldsymbol{\nu})' + (\boldsymbol{\beta} - \boldsymbol{\nu})\boldsymbol{\beta}')\mathbf{Q}]E\left[g\left(\frac{p+2}{p}F_{p+2,m}(\Delta^2)\right)\right]
\end{aligned}$$

$$\begin{aligned}
R^{(2)} &= \sigma_{zz} \text{tr}[\mathbf{C}^{-1} \mathbf{Q}] \left[1 - \left\{ 2E \left[g \left(\frac{p+2}{p} F_{p+2,m}(\Delta^2) \right) \right] \right. \right. \\
&\quad \left. \left. - E \left[g^2 \left(\frac{p+2}{p} F_{p+2,m}(\Delta^2) \right) \right] \right\} \right] \\
&\quad + \boldsymbol{\beta}' \mathbf{Q} \boldsymbol{\beta} \left\{ 2E \left[g \left(\frac{p+2}{p} F_{p+2,m}(\Delta^2) \right) \right] \right. \\
&\quad \left. - 2E \left[g \left(\frac{p+4}{p} F_{p+4,m}(\Delta^2) \right) \right] + E \left[g^2 \left(\frac{p+4}{p} F_{p+4,m}(\Delta^2) \right) \right] \right\}
\end{aligned}$$

Optimal c-Value for Stein-type Estimator

The Stein-type estimator $\hat{\boldsymbol{\beta}}_n^S$ is given by

$$\hat{\boldsymbol{\beta}}_n^S = (1 - c\mathcal{L}_n^{-1})\tilde{\boldsymbol{\beta}}_n$$

where $\mathcal{L}_n = \frac{\tilde{\boldsymbol{\beta}}_n' \mathbf{C} \tilde{\boldsymbol{\beta}}_n}{p\hat{\sigma}_{zz}}$, $\mathbf{C} = \mathbf{K}'_{xx} \mathbf{S}_{X^0 X^0} \mathbf{K}_{xx}$ and $\tilde{\boldsymbol{\beta}}_n \sim \mathcal{N}(\boldsymbol{\beta}, \mathbf{C}^{-1} \sigma_{zz})$.

Then the unweighted risk for $\hat{\boldsymbol{\beta}}_n^S$ is given by

$$\begin{aligned}
R(\hat{\boldsymbol{\beta}}_n^S) &= \sigma_{zz} \text{tr}(\mathbf{C}^{-1}) \left[1 - \left\{ 2E \left[\frac{cp}{p+2} F_{p+2,m}^{-1} \right] - E \left[\left(\frac{cp}{p+2} \right)^2 F_{p+2,m}^{-2} \right] \right\} \right] \\
&\quad + \boldsymbol{\beta}' \boldsymbol{\beta} \left\{ 2E \left[\frac{cp}{p+2} F_{p+2,m}^{-1} \right] - 2E \left[\frac{cp}{p+4} F_{p+4,m}^{-1} \right] \right. \\
&\quad \left. + E \left[\left(\frac{cp}{p+4} \right)^2 F_{p+4,m}^{-2} \right] \right\} \\
&= \sigma_{zz} \text{tr}(\mathbf{C}^{-1}) \left[1 - \left\{ 2cpE[\chi_{p+2}^{-2}(\Delta^2)] - \frac{c^2 p^2 m(m+2)}{m^2} E[\chi_{p+2}^{-4}(\Delta^2)] \right\} \right] \\
&\quad + \boldsymbol{\beta}' \boldsymbol{\beta} \left\{ 2cpE[\chi_{p+2}^{-2}(\Delta^2)] - 2cpE[\chi_{p+4}^{-2}(\Delta^2)] \right. \\
&\quad \left. + \frac{c^2 p^2 m(m+2)}{m^2} E[\chi_{p+4}^{-4}(\Delta^2)] \right\}
\end{aligned}$$

Note that

1. $E \left[F_{p+2,m}^{-1}(\Delta^2) \right] = (p+2)E \left[\chi_{p+2}^{-2}(\Delta^2) \right]$
2. $E \left[F_{p+2,m}^{-2}(\Delta^2) \right] = (p+2)^2 E \left[\frac{\chi_m^2}{m} \right]^2 E \left[\chi_{p+2}^{-4}(\Delta^2) \right]$

$$\begin{aligned}
&= \left(\frac{p+2}{m}\right)^2 (2m + m^2) E[\chi_{p+2}^{-4}(\Delta^2)] \\
&= \frac{m(m+2)(p+2)^2}{m^2} E[\chi_{p+2}^{-4}(\Delta^2)]
\end{aligned}$$

To obtain c value which minimize the unweighted risk, we have

$$\begin{aligned}
\frac{\partial R(\hat{\beta}_n^S)}{\partial c} &= \sigma_{zz} \Delta^2 (\mathbf{C}^{-1}) \left\{ -2E[\chi_{p+2}^{-2}(\Delta^2)] \right. \\
&\quad + \frac{2cp^2m(m+2)}{m^2} E[\chi_{p+4}^{-2}(\Delta^2)] \left. \right\} \\
&\quad + \boldsymbol{\beta}' \boldsymbol{\beta} \left\{ 2pE[\chi_{p+2}^{-2}(\Delta^2)] - 2pE[\chi_{p+4}^{-2}(\Delta^2)] \right. \\
&\quad + \left. \frac{cp^2m(m+2)}{m^2} E[\chi_{p+4}^{-4}(\Delta^2)] \right\} = 0
\end{aligned}$$

Therefore,

$$c = \frac{m}{(m+2)p} \frac{E[\chi_{p+2}^{-2}(\Delta^2)] - 2\boldsymbol{\beta}' \boldsymbol{\beta} E[\chi_{p+4}^{-4}(\Delta^2)]}{E[\chi_{p+2}^{-4}(\Delta^2)] + \boldsymbol{\beta}' \boldsymbol{\beta} E[\chi_{p+4}^{-4}(\Delta^2)]}$$

Under $H_0 : \boldsymbol{\beta} = 0$,

$$c = \frac{m}{(m+2)p} \frac{E[\chi_{p+2}^{-2}(0)]}{E[\chi_{p+2}^{-4}(0)]} = \frac{m(p-2)}{(m+2)p}$$

since $E[\chi_{p+2}^{-2}(0)] = \frac{1}{p}$ and $E[\chi_{p+2}^{-4}(0)] = \frac{1}{p(p-2)}$.

Chapter 4

Multiple Regression Analysis: Asymptotic Case

4.0 Introduction

Consider the multiple regression model with measurement errors again, namely,

$$\left. \begin{array}{l} Y_{0t} = \beta_0 + \boldsymbol{\beta}'\mathbf{x}_t + e_t \\ \mathbf{X}_{0t} = \mathbf{x}_t + \mathbf{u}_t \end{array} \right\}; \quad t = 1, \dots, n \quad (4.0.1)$$

where β_0 is the intercept and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ is the regression parameters with $\mathbf{x}_t = (x_{1t}, \dots, x_{pt})'$, $\mathbf{u}_t = (u_{1t}, \dots, u_{pt})'$ and $\mathbf{X}_{0t} = (X_{01t}, \dots, X_{0pt})'$. e_t is the response error in the study variable and u_{it} is the measurement error in the i^{th} regression variable x_{it} . Note that x_{it} is unobservable and X_{0it} is the corresponding observed value. Similarly, Y_{0t} is the observed response. We assume that

$$(\mathbf{x}'_t, e_t, \mathbf{u}'_t)' \sim \mathcal{N}_{2p+1}\left\{(\boldsymbol{\mu}'_x, 0, \mathbf{0}')'; \text{Blockdiag}(\boldsymbol{\Sigma}_{xx}, \sigma_{ee}, \boldsymbol{\Sigma}_{uu})\right\} \quad (4.0.2)$$

where $\boldsymbol{\mu}'_x = (\mu_{x_1}, \dots, \mu_{x_p})'$. Further, we assume that $\boldsymbol{\Sigma}_{uu}$ is known but not $\boldsymbol{\Sigma}_{xx}$.

Clearly, $(Y_{0t}, \mathbf{X}'_{0t})'$ follows a $(p+1)$ -variate normal distribution with mean-vector $(\beta_0 + \beta' \mu_x, \mu'_x)'$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{Y_{0t}Y_{0t}} & \Sigma_{Y_{0t}X_{0t}} \\ \Sigma_{X_{0t}Y_{0t}} & \Sigma_{X_{0t}X_{0t}} \end{pmatrix} = \begin{pmatrix} \beta' \Sigma_{xx} \beta + \sigma_{ee} & \beta' \Sigma_{xx} \\ \Sigma_{xx} \beta & \Sigma_{xx} + \Sigma_{uu} \end{pmatrix} \quad (4.0.4)$$

Thus, the distribution of $(\bar{Y}_0, \bar{\mathbf{X}}'_0)'$ is a $(p+1)$ -variate normal with mean-vector $(\beta_0 + \beta' \mu_x, \mu'_x)'$ and covariance matrix $\frac{1}{n} \Sigma$.

Now the conditional distribution of Y_{0t} given $\mathbf{X}_{0t} = (X_{01t}, \dots, X_{0pt})'$ is normal with conditional mean and variance given by

$$E[Y_{0t} | \mathbf{X}_{0t}] = \nu_0 + \nu' \mathbf{X}_{0t} \quad (4.0.5)$$

$$Var[Y_{0t} | \mathbf{X}_{0t}] = \beta' \Sigma_{xx} (\mathbf{I} - \mathbf{K}_{xx}) \beta + \sigma_{ee} = \sigma_{zz} \text{ (say)} \quad (4.0.6)$$

where the matrix of reliability given by

$$\mathbf{K}_{xx} = (\Sigma_{xx} + \Sigma_{uu})^{-1} \Sigma_{xx} = \Sigma_{X_0 X_0}^{-1} \Sigma_{xx},$$

$$\Sigma_{X_0 X_0} = \Sigma_{xx} + \Sigma_{uu} \text{ and} \quad (4.0.7)$$

$$\nu_0 = \beta_0 + \beta' (\mathbf{I}_p - \mathbf{K}'_{xx}) \mu_{xx}, \nu = \mathbf{K}_{xx} \beta \text{ and } \beta = \mathbf{K}_{xx}^{-1} \nu \quad (4.0.8)$$

Our basic problem is the estimation of the regression vector $\beta = (\beta_1, \dots, \beta_p)'$ when $\mathbf{K}_{xx}$ is unknown and it is suspected but not sure that β may be null-vector, i.e. $\beta = 0$. Towards this goal we propose four estimators of β , namely, (i) the unrestricted estimators (UE) of β , (ii) the preliminary test estimator (PTE), (iii) the Stein-type estimator (SE) and (iv) the Positive-rule Stein-type estimator (PRSE).

The finite sample properties of these estimators in the conditional model as well as unconditional case have been studied based on the MSE matrices and the weighted risks under quadratic loss function in Chapter 3. In this Chapter, we study the asymptotic properties which is forced on the topic since $\mathbf{K}_{xx}$ is unknown.

4.1 Estimation of the Slope Parameters when $\mathbf{K}_{xx}$ is Unknown

Gleser (1992) showed that the MLE of ν_0 , $\boldsymbol{\nu}$ and σ_{zz} are just the ordinary least square estimators (OLS), namely,

$$\tilde{\nu}_{0n} = \bar{Y}_0 - \bar{\mathbf{X}}_0' \tilde{\boldsymbol{\nu}}_n, \quad \tilde{\boldsymbol{\nu}}_n = \mathbf{S}_{X_0 X_0}^{-1} \mathbf{S}_{X_0 Y_0} \quad (4.1.1)$$

and

$$\tilde{\sigma}_{zz} = \frac{1}{n} \sum_{t=1}^n (Y_{0t} - \tilde{\nu}_{0n} - \tilde{\boldsymbol{\nu}}_n' \bar{\mathbf{X}}_0)^2 \quad (4.1.2)$$

provided

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\boldsymbol{\nu}}' \mathbf{K}_{xx}^{-1} \boldsymbol{\Sigma}_{uu} \tilde{\boldsymbol{\nu}} \geq 0 \quad (4.1.3)$$

where

- (i). $S_{Y_0 Y_0} = \sum_{t=1}^n (Y_{0t} - \bar{Y}_0)^2$
- (ii). $\mathbf{S}_{X_0 X_0} = ((S_{X_{0i} X_{0j}}))$
- (iii). $S_{X_{0i} X_{0j}} = \sum_{t=1}^n (X_{0it} - \bar{X}_{0i})(X_{0jt} - \bar{X}_{0j}), \quad i, j = 1, \dots, p$
- (iv). $S_{X_{0i} Y_0} = \sum_{t=1}^n (X_{0it} - \bar{X}_{0i})(Y_{0t} - \bar{Y}_0)$
- (v). $\bar{X}_{0i} = \frac{1}{n} \sum_{t=1}^n X_{0it}$ and $\bar{Y}_0 = \frac{1}{n} \sum_{t=1}^n Y_{0t}$ (4.0.10)

Further, when $\boldsymbol{\Sigma}_{uu}$ is known and $\mathbf{K}_{xx}$ unknown then the reliability matrix is estimated by

$$\hat{\mathbf{K}}_{xx} = \mathbf{S}_{X_0 X_0}^{-1} \hat{\boldsymbol{\Sigma}}_{xx} = \mathbf{S}_{X_0 X_0}^{-1} (\mathbf{S}_{X_0 X_0} - n \boldsymbol{\Sigma}_{uu}). \quad (4.1.4)$$

Thus, the MLE of β_0 , β and σ_{ee} are given by

$$\tilde{\beta}_{0n} = \tilde{\nu}_{0n} - \tilde{\beta}'_n(\mathbf{I}_p - \hat{\mathbf{K}}_{xx})\bar{\mathbf{X}}_0, \quad \tilde{\beta}_n = \hat{\mathbf{K}}_{xx}^{-1}\tilde{\nu}_n \quad (4.1.5)$$

and

$$\tilde{\sigma}_{ee} = \tilde{\sigma}_{zz} - \tilde{\beta}'_n \Sigma_{uu} \hat{\mathbf{K}}_{xx} \tilde{\beta}_n. \quad (4.1.6)$$

Finally, the explicit form of the estimators of β_0 and β are given by

$$\tilde{\beta}_{0n} = \bar{Y}_0 - \tilde{\beta}'_n \bar{\mathbf{X}}_0, \quad \tilde{\beta}_n = (\mathbf{S}_{X_0X_0} - n\Sigma_{uu})^{-1}\mathbf{S}_{X_0Y_0} \quad (4.1.7)$$

provided $\tilde{\sigma}_{ee} = S_{Y_0Y_0} - \mathbf{S}'_{X_0Y_0}(\mathbf{S}_{X_0X_0} - n\Sigma_{uu})^{-1}\mathbf{S}_{X_0Y} \geq 0$.

It may be noted that the estimators described above are consistent for the respective parameter. Hence, we obtain the unconditional asymptotic distribution of $\sqrt{n}(\tilde{\beta}_n - \beta)$ given below.

Theorem 4.1.1 *Consider the model from (4.0.1) to (4.0.4) assuming Σ_{xx} and Σ_{uu} are positive definite matrices and $\sigma_{ee} > 0$. Then,*

$$\sqrt{n}(\tilde{\beta}_n - \beta) \rightarrow \mathcal{N}_p(\mathbf{0}, \mathbf{G}), \quad \text{as } n \rightarrow \infty, \quad (4.1.8)$$

where

$$\mathbf{G} = \Sigma_{xx}^{-1} \left[\left(\beta' \Sigma_{uu} \beta + \sigma_{ee} \right) \Sigma_{X_0X_0} + \Sigma_{uu} \beta \beta' \Sigma_{uu} \right] \Sigma_{xx}^{-1} \quad (4.1.9)$$

Proof. The proof of the theorem parallel that of a more general result given in Fuller (1987, Theorem 2.2.1).

A consistent estimator of $\mathbf{G}$ may be obtained by substituting in $\mathbf{G}$, the consistent estimators given by (4.1.5) and (4.1.6). Let us denote the consistent estimator

of $\mathbf{G}$ by $\hat{\mathbf{G}}_n$ as follows.

$$\begin{aligned} \hat{\mathbf{G}}_n = & \left(\frac{1}{n} \mathbf{S}_{X_0 X_0} - \Sigma_{uu} \right)^{-1} \left[\left(\tilde{\boldsymbol{\beta}}_n' \Sigma_{uu} \tilde{\boldsymbol{\beta}}_n + \tilde{\sigma}_{ee} \right) \frac{1}{n} \mathbf{S}_{X_0 X_0} \right. \\ & \left. + \Sigma_{uu} \tilde{\boldsymbol{\beta}}_n \tilde{\boldsymbol{\beta}}_n' \Sigma_{uu} \right] \left(\frac{1}{n} \mathbf{S}_{X_0 X_0} - \Sigma_{uu} \right)^{-1}. \end{aligned} \quad (4.1.10)$$

We may write

$$\hat{\mathbf{G}}_n = \mathbf{G} + o_p(1). \quad (4.1.11)$$

Since we suspect that $\boldsymbol{\beta}$ may be equal to $\mathbf{0}$, we consider the Wald-type test-statistic $\mathcal{L}_n^*$ to test the null-hypothesis $H_0 : \boldsymbol{\beta} = \mathbf{0}$ against $H_A : \boldsymbol{\beta} \neq \mathbf{0}$, defined by

$$\mathcal{L}_n^* = n \left(\tilde{\boldsymbol{\beta}}_n' \hat{\mathbf{G}}_n^{-1} \tilde{\boldsymbol{\beta}}_n \right) \quad (4.1.12)$$

where the asymptotic distribution of $\mathcal{L}_n^*$ follows a central chi-square distribution with p d.f. under H_0 .

4.2 Preliminary Test and Stein-type Estimators of $\boldsymbol{\beta}$ when Σ_{uu} is Known

Now we consider a class of estimators defined by $\boldsymbol{\beta}_n^* = (1 - g(\mathcal{L}_n^*)) \tilde{\boldsymbol{\beta}}_n$, where $g(\mathcal{L}_n^*)$ is a monotone function of $\mathcal{L}_n^*$. This class includes the usual estimators by setting $g(\mathcal{L}_n^*)$ as follows:

- (i) If $g(\mathcal{L}_n^*) = 0$, then the unrestricted estimator (UE) of $\boldsymbol{\beta}$ is defined by $\boldsymbol{\beta}_n^* = \tilde{\boldsymbol{\beta}}_n$,
- (ii) If $g(\mathcal{L}_n^*) = 1$, then the restricted estimator (RE) of $\boldsymbol{\beta}$ is defined by $\boldsymbol{\beta}_n^* = \mathbf{0}$,
- (iii) If $g(\mathcal{L}_n^*) = I(\mathcal{L}_n^* < \chi_\alpha^2)$, then the preliminary test estimator (PTE) of $\boldsymbol{\beta}$ is defined by $\boldsymbol{\beta}_n^* = \hat{\boldsymbol{\beta}}_n^{PT} = \tilde{\boldsymbol{\beta}}_n I(\mathcal{L}_n^* > \chi_\alpha^2) = \tilde{\boldsymbol{\beta}}_n - \tilde{\boldsymbol{\beta}}_n I(\mathcal{L}_n^* < \chi_\alpha^2)$, where χ_α^2

is the upper α -level critical value of the asymptotic distribution of $\mathcal{L}_n^*$ under H_0

- (iv) If $g(\mathcal{L}_n^*) = (p-2)\mathcal{L}_n^{*-1}$, then the shrinkage estimator (SE) of β is defined by $\beta_n^* = \hat{\beta}_n^S = \left(1 - (p-2)\mathcal{L}_n^{*-1}\right)\tilde{\beta}_n$, where $(p-2)$ is the mode of the asymptotic distribution of $\mathcal{L}_n^*$ under H_0 .

Note that $(p-2)$ is the optimum constant value of c by minimizing the unweighted risk of Stein-type estimator, $\tilde{\beta}_n^S(c) = (1 - c\mathcal{L}_n^{*-1})\tilde{\beta}_n$.

Finally,

- (v) If $g(\mathcal{L}_n^*) = 1 - \left(1 - (p-2)\mathcal{L}_n^{*-1}\right)I(\mathcal{L}_n^* > p-2)$, then the positive-rule shrinkage estimator (PRSE) of β is defined by $\beta_n^* = \hat{\beta}_n^{S+} = \left(1 - (p-2)\mathcal{L}_n^{*-1}\right)I(\mathcal{L}_n^* > p-2)\tilde{\beta}_n$.

Note that PTE is a discontinuous estimator and takes only two values, namely, $\mathbf{0}$ or $\tilde{\beta}_n$ depending on the result of the test which heavily depends on the size α of the test. Now replacing $I(\mathcal{L}_n^* < \chi_\alpha^2)$ by its smooth version $(p-2)\mathcal{L}_n^{*-1}$, we obtain the smooth estimator $\hat{\beta}_n^S$. Next we combine $\mathbf{0}$ with $\hat{\beta}_n^S$ via the test using $(p-2)$ as the critical value to obtain $\hat{\beta}_n^{S+}$, i.e.

$$\begin{aligned}\hat{\beta}_n^{S+} &= \mathbf{0} \times I(\mathcal{L}_n^* < p-2) + \tilde{\beta}_n \left(1 - (p-2)\mathcal{L}_n^{*-1}\right)I(\mathcal{L}_n^* > p-2) \\ &= \left(1 - (p-2)\mathcal{L}_n^{*-1}\right)I(\mathcal{L}_n^* > p-2)\tilde{\beta}_n\end{aligned}\quad (4.2.1)$$

Thus, PRSE is a preliminary test estimator and it improves on $\hat{\beta}_n^S$. Here, the critical value is fixed at $p-2$.

Similary, consider the class of estimator of β based on $\tilde{\nu}_n$ defined by

$$\nu_n^* = (1 - g(\mathcal{L}_n^*))\tilde{\nu}_n. \quad (4.2.2)$$

Accordingly, we know the four estimators $\tilde{\nu}$, $\hat{\nu}^{PT}$, $\hat{\nu}^S$ and $\hat{\nu}^{S+}$ defined by choosing $g(\mathcal{L}_n^*) = 0, 1, I(\mathcal{L}_n^* > \chi_\alpha^2), (p-2)\mathcal{L}_n^{*-1}$ and $1 - [1 - (p-2)\mathcal{L}_n^{*-1}]I(\mathcal{L}_n^* > p-2)$, respectively.

4.3 Asymptotic Properties of the Estimators

In this section we consider the asymptotic distribution of the estimators, β_n^* . First we consider the asymptotic distribution of the four estimators under **fixed alternatives** $H_\delta : \beta = \delta (\delta \neq 0)$ through the Theorem stated below:

Theorem 4.3.1 *Under fixed alternatives $H_\delta : \beta = \delta, \delta \neq 0$,*

$$(1). \mathcal{L}_n^* \rightarrow \infty \text{ as } n \rightarrow \infty$$

$$(2). \sqrt{n}(\beta_n^* - \beta) = \sqrt{n}(\tilde{\beta}_n - \beta) + o_p(1)$$

Proof. To prove (1) consider

$$\sqrt{n}\tilde{\beta}_n = \sqrt{n}(\tilde{\beta}_n - \beta) + \sqrt{n}\delta. \quad (4.3.1)$$

Then

$$\mathcal{L}_n^* = n(\tilde{\beta}_n - \beta)' \hat{\mathbf{G}}_n^{-1} (\tilde{\beta}_n - \beta) + n\delta' \hat{\mathbf{G}}_n^{-1} \delta + 2n(\tilde{\beta}_n - \beta)' \hat{\mathbf{G}}_n^{-1} \delta. \quad (4.3.2)$$

Now, $\sqrt{n}\hat{\mathbf{G}}_n^{-1/2}(\tilde{\beta}_n - \beta) \sim \mathcal{Z}$, $\mathcal{Z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$ and $n\delta' \hat{\mathbf{G}}_n^{-1} \delta \rightarrow \infty$ as $n \rightarrow \infty$, it follows that $\mathcal{L}_n^* \rightarrow \infty$, as $n \rightarrow \infty$.

Consequently,

$$\lim_{n \rightarrow \infty} P_{\beta}(\mathcal{L}_n^* > k) = 1 \quad \text{for all } k \in \mathcal{R}_1. \quad (4.3.3)$$

To prove (2), consider the quadratic differences

$$n \|\tilde{\beta}_n - \hat{\beta}_n^*\|_{\hat{\mathbf{G}}_n^{-1}}^2 \quad (4.3.4)$$

where $\beta_n^* = (1 - g(\mathcal{L}_n))\tilde{\beta}_n$.

Let $\sqrt{n}\hat{\mathbf{G}}_n^{-1/2}(\tilde{\beta}_n - \beta) = \mathbf{Z} + o_p(1)$, Then

$$\begin{aligned} n \|\tilde{\beta}_n - \beta_n^*\|_{\hat{\mathbf{G}}_n^{-1}}^2 &= (n\tilde{\beta}_n' \hat{\mathbf{G}}_n^{-1} \beta_n) g^2(\mathcal{L}_n^*) \\ &= \mathcal{L}_n^* g^2(\mathcal{L}_n^*) \end{aligned} \quad (4.3.5)$$

It may be verified that

$$\lim_{n \rightarrow \infty} E[\mathcal{L}_n^* g^2(\mathcal{L}_n^*)] = 0 \quad \text{for} \quad (4.3.6)$$

$$g(\mathcal{L}_n^*) = I(\mathcal{L}_n^* > \chi_{\alpha}^2).$$

$$g(\mathcal{L}_n^*) = (p - 2)\mathcal{L}_n^{*-1} \quad \text{and}$$

$$g(\mathcal{L}_n^*) = 1 - [1 - (p - 2)\mathcal{L}_n^{*-1}]I(\mathcal{L}_n^* > p - 2).$$

Hence

$$\sqrt{n}(\beta_n^* - \beta) = \sqrt{n}(\tilde{\beta}_n - \beta) + o_p(1) \quad (4.3.7)$$

Thus, under fixed alternative, all the estimators are asymptotically MSE/risk equivalent in distribution which is $\mathcal{N}_p(\mathbf{0}, \mathbf{G})$.

Similary,

$$\lim_{n \rightarrow \infty} E \left[n \|\tilde{\nu}_n - \nu_n^*\|_{(\hat{\mathbf{K}}_{xx} \hat{\mathbf{G}}_n \hat{\mathbf{K}}_{xx}')^{-1}}^2 \right] = 0$$

Hence

$$\sqrt{n}(\boldsymbol{\nu}_n^* - \boldsymbol{\nu}) = \sqrt{n}(\tilde{\boldsymbol{\nu}}_n - \boldsymbol{\nu}) + o_p(1)$$

and all the estimators $\boldsymbol{\nu}_n^*$ are asymptotically MSE and risk equivalent.

In order to obtain reasonable comparison, we need the asymptotic distributions of the four estimators which are different. Thus, we consider the sequence of local alternatives $\{K_{(n)}\}$ defined by

$$K_{(n)} : \boldsymbol{\beta}_{(n)} = n^{-\frac{1}{2}}\boldsymbol{\delta}, \quad \boldsymbol{\delta} \text{ fixed finite vector} \quad (4.3.8)$$

Theorem 4.3.2 Under $\{K_{(n)}\}$ and assumptions as in theorem 4.1.1,

$$\begin{aligned} (i) \quad & \hat{\mathbf{G}}_n \rightarrow \mathbf{G}^* \text{ in probability as } n \rightarrow \infty. \\ (ii) \quad & \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0}, \mathbf{G}^*) \\ (iii) \quad & \sqrt{n}(\tilde{\boldsymbol{\nu}}_n - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathcal{N}\left(-(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}, \sigma_{ee}\boldsymbol{\Sigma}_{XX}^{-1}\right) \\ (iv) \quad & \lim_{n \rightarrow \infty} P\left\{\mathcal{L}_n^* \leq x | K_{(n)}\right\} = H_p(x, \Delta^2) \end{aligned} \quad (4.3.9)$$

where $\mathbf{G}^* = \sigma_{ee}\boldsymbol{\Sigma}_{xx}^{-1}\boldsymbol{\Sigma}_{X_0X_0}\boldsymbol{\Sigma}_{xx}^{-1}$ and $H_\nu(\cdot; \Delta^2)$ is the cdf of a noncentral chi-square distribution with ν d.f. and noncentrality parameter $\frac{1}{2}\Delta^2$ with $\Delta^2 = \boldsymbol{\delta}'(\mathbf{G}^*)^{-1}\boldsymbol{\delta}$.

Proof. (i) First note that $\mathbf{G} \rightarrow \mathbf{G}^*$ under $K_{(n)}$, therefore $\hat{\mathbf{G}}_n \rightarrow \mathbf{G}^*$ in probability under $K_{(n)}$.

Further, for (ii) and (iii),

$$\begin{aligned} \sqrt{n}(\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)}) & \xrightarrow{\mathcal{D}} \mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{G}^*) \\ \sqrt{n}(\tilde{\boldsymbol{\nu}}_n - \boldsymbol{\beta}_{(n)}) & \xrightarrow{\mathcal{D}} \mathcal{N}\left(-(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}, \sigma_{ee}\boldsymbol{\Sigma}_{XX}^{-1}\right) \end{aligned} \quad (4.3.10)$$

See Chapter 7 of Sen and Singer (1993).

(iv) Consider the distribution of $\mathcal{L}_n^*$ as $n \rightarrow \infty$. By (i)

$$\mathcal{L}_n^* \xrightarrow{\mathcal{D}} \|\mathbf{Z} + \boldsymbol{\delta}^*\|^2 \xrightarrow{\mathcal{D}} \chi_p^2(\Delta^2) \quad (4.3.11)$$

where $\chi_p^2(\Delta^2)$ stands for a non-central chi-square variable with p d.f. and non-centrality parameter $\frac{1}{2}\Delta^2$, $\Delta^2 = \boldsymbol{\delta}' \mathbf{G}^{*-1} \boldsymbol{\delta} = \|\boldsymbol{\delta}^*\|^2$.

Thus we may write

$$\begin{aligned} (1). \quad & \sqrt{n} \hat{\mathbf{G}}_n^{-\frac{1}{2}} (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathbf{Z} \\ (2). \quad & \sqrt{n} \hat{\mathbf{G}}_n^{-\frac{1}{2}} (\hat{\boldsymbol{\beta}}_n^{PT} - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathbf{Z} - (\mathbf{Z} + \boldsymbol{\delta}^*) I(\|\mathbf{Z} + \boldsymbol{\delta}^*\|^2 < \chi_p^2) \\ (3). \quad & \sqrt{n} \hat{\mathbf{G}}_n^{-\frac{1}{2}} (\hat{\boldsymbol{\beta}}_n^S - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathbf{Z} - (p-2) \frac{\mathbf{Z} + \boldsymbol{\delta}^*}{\|\mathbf{Z} + \boldsymbol{\delta}^*\|^2} \\ (4). \quad & \sqrt{n} \hat{\mathbf{G}}_n^{-\frac{1}{2}} (\hat{\boldsymbol{\beta}}_n^{S+} - \boldsymbol{\beta}_{(n)}) \xrightarrow{\mathcal{D}} \mathbf{Z} - (\mathbf{Z} + \boldsymbol{\delta}^*) I(\|\mathbf{Z} + \boldsymbol{\delta}^*\|^2 < p-2) \\ & - (p-2) \frac{\mathbf{Z} + \boldsymbol{\delta}^*}{\|\mathbf{Z} + \boldsymbol{\delta}^*\|^2} I(\|\mathbf{Z} + \boldsymbol{\delta}^*\|^2 > p-2) \end{aligned}$$

where $\mathbf{Z} \sim \mathcal{N}_p(\mathbf{0}, \mathbf{I}_p)$ and $\boldsymbol{\delta}^* = (\mathbf{G}^*)^{-\frac{1}{2}} \boldsymbol{\delta}$ with $\mathbf{G}^* = \sigma_{ee} \boldsymbol{\Sigma}_{xx}^{-1} \boldsymbol{\Sigma}_{X_0 X_0} \boldsymbol{\Sigma}_{xx}^{-1}$.

4.4 Asymptotic Distributional Bias (ADB), Quadratic Bias (ADQB), Mean-Square Error Matrix (ADMSE) and Quadratic Risk (ADQR) of ν_n^* and β_n^* of Slope Parameter β .

In this section, we consider the asymptotic distributional bias (ADB), the asymptotic distributional quadratic bias (ADQB), the asymptotic distributional mean-

square error matrix (ADMSE) and the asymptotic distributional quadratic risks (ADQR) of the four estimators defined in section 3. Thus, we may write the ADB for the estimator β_n^* as

$$\mathbf{b}^{*(1)}(\tilde{\nu}_n) = -(\mathbf{I} - \mathbf{K}_{xx})\delta - \mathbf{K}_{xx}\delta E\left[g(\chi_{(p+2)}^2(\Delta^2))\right] \quad (4.4.1)$$

and

$$\mathbf{b}^{*(2)}(\tilde{\beta}_n) = -\delta E\left[g(\chi_{(p+2)}^2(\Delta^2))\right] \quad (4.4.2)$$

by Appendix B of Judge and Bock(1978) where $\chi_\nu^2(\Delta^2)$ stands for the non-central χ -variable with ν d.f. and non-centrality parameter $\frac{1}{2}\Delta^2$. To evaluate the properties of the bias we consider the quadratic bias(ADQB) defined by

$$\begin{aligned} B^{*(1)}(\tilde{\nu}_n) &= \mathbf{b}^{*(1)'}\Sigma_{XX}\mathbf{b}^{*(1)} = \Delta^2\left\{E\left[g(\chi_{p+2}^2(\Delta^2))\right]\right\}^2 \\ &+ \delta'(\mathbf{I} - \mathbf{K}_{xx})'\Sigma_{XX}(\mathbf{I} - \mathbf{K}_{xx})\delta \\ &+ \delta'(\mathbf{I} - \mathbf{K}_{xx})'\Sigma_{XX}\mathbf{K}_{xx}\delta E\left[g(\chi_{p+2}^2(\Delta^2))\right] \\ &+ \delta'\mathbf{K}_{xx}'\Sigma_{XX}(\mathbf{I} - \mathbf{K}_{xx})\delta\left[g(\chi_{p+2}^2(\Delta^2))\right] \end{aligned} \quad (4.4.3)$$

and

$$B^{*(2)}(\tilde{\beta}_n) = \mathbf{b}^{*(2)'}\mathbf{G}^{*-1}\mathbf{b}^{*(2)} = \Delta^2\left\{E\left[g(\chi_{p+2}^2(\Delta^2))\right]\right\}^2. \quad (4.4.4)$$

Thus, ADQB of β_n^* may be studied as a function of Δ^2 , the non-centrality parameter of the χ^2 -distribution.

The mean-square-error (ADMSE) matrix of the estimator β_n^* is given by

$$\begin{aligned}
\mathbf{M}^{*(1)}(\boldsymbol{\nu}_n^*) &= \sigma_{ee} \boldsymbol{\Sigma}_{XX}^{-1} + (\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\delta} \boldsymbol{\delta}' (\mathbf{I} - \mathbf{K}_{xx})' \\
&\quad - \sigma_{ee} \boldsymbol{\Sigma}_{XX}^{-1} \left\{ 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] - E \left[g^2(\chi_{p+2}^2(\Delta^2)) \right] \right\} \\
&\quad + \mathbf{K}_{xx} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{K}_{xx}' \left\{ 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] - 2E \left[g(\chi_{p+4}^2(\Delta^2)) \right] \right. \\
&\quad \left. + E \left[g^2(\chi_{p+4}^2(\Delta^2)) \right] \right\} + \left[(\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{K}_{xx}' \right. \\
&\quad \left. + \mathbf{K}_{xx} \boldsymbol{\nu} \boldsymbol{\nu}' (\mathbf{I} - \mathbf{K}_{xx})' \right] E \left[g(\chi_{p+2}^2(\Delta^2)) \right]. \tag{4.4.5}
\end{aligned}$$

$$\begin{aligned}
\mathbf{M}^{*(2)}(\boldsymbol{\beta}_n^*) &= \mathbf{G}^* \left[1 - \left\{ 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] - E \left[g^2(\chi_{p+2}^2(\Delta^2)) \right] \right\} \right] \\
&\quad + \boldsymbol{\delta} \boldsymbol{\delta}' \left\{ 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] - 2E \left[g(\chi_{p+4}^2(\Delta^2)) \right] \right. \\
&\quad \left. + E \left[g^2(\chi_{p+4}^2(\Delta^2)) \right] \right\} \tag{4.4.6}
\end{aligned}$$

The quadratic risk (ADQR) under the loss functions

$$L(\boldsymbol{\beta}_n^*, \boldsymbol{\beta}_{(n)}) = n(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})' \mathbf{Q}(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})$$

and

$$L(\boldsymbol{\nu}_n^*, \boldsymbol{\beta}_{(n)}) = n(\boldsymbol{\nu}_n^* - \boldsymbol{\beta}_{(n)})' \mathbf{Q}(\boldsymbol{\nu}_n^* - \boldsymbol{\beta}_{(n)}) \tag{4.4.7}$$

with the positive semi-definite matrix $\mathbf{Q}$ is given by

$$\begin{aligned}
R^{*(1)}(\boldsymbol{\nu}_n^*) &= \sigma_{ee} \text{tr}[\boldsymbol{\Sigma}_{XX}^{-1} \mathbf{Q}] + \boldsymbol{\delta}' (\mathbf{I} - \mathbf{K}_{xx})' \mathbf{Q} (\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\delta} \\
&\quad - \sigma_{ee} \text{tr}[\boldsymbol{\Sigma}_{XX}^{-1} \mathbf{Q}] \left\{ 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] - E \left[g^2(\chi_{p+2}^2(\Delta^2)) \right] \right\} \\
&\quad + \mathbf{K}_{xx} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{K}_{xx}' \left\{ 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] - 2E \left[g(\chi_{p+4}^2(\Delta^2)) \right] \right. \\
&\quad \left. + E \left[g^2(\chi_{p+4}^2(\Delta^2)) \right] \right\} + \text{tr} \left[(\mathbf{I} - \mathbf{K}_{xx}) \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{K}_{xx}' \mathbf{Q} \right. \\
&\quad \left. + \mathbf{K}_{xx} \boldsymbol{\nu} \boldsymbol{\nu}' (\mathbf{I} - \mathbf{K}_{xx})' \mathbf{Q} \right] E \left[g(\chi_{p+2}^2(\Delta^2)) \right]. \tag{4.4.8}
\end{aligned}$$

$$\begin{aligned}
R^{*(2)}(\beta_n^*) &= \text{tr}[\mathbf{G}^* \mathbf{Q}] \left\{ 1 - 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] + E \left[g^2(\chi_{p+2}^2(\Delta^2)) \right] \right\} \\
&+ \delta' \mathbf{Q} \delta \left\{ 2E \left[g(\chi_{p+2}^2(\Delta^2)) \right] - 2E \left[g(\chi_{p+4}^2(\Delta^2)) \right] \right. \\
&+ \left. E \left[g^2(\chi_{p+4}^2(\Delta^2)) \right] \right\}
\end{aligned} \tag{4.4.9}$$

From the expressions from (4.4.1) to (4.4.9), we obtain the corresponding asymptotic distributional bias, quadratic bias, MSE-matrix and the quadratic risks of UE, PTE, SE and PRSE.

4.4.1 ADB and ADQB Expressions of β_n^* and Comparisons

The asymptotic distributional bias and quadratic bias expressions of $\tilde{\beta}_n$, $\hat{\beta}_n^{PT}$, $\hat{\beta}_n^S$ and $\hat{\beta}_n^{S+}$ are given by

$$\mathbf{b}_1^{(2)}(\tilde{\beta}_n) = \mathbf{0}, \text{ and } B_1^{(2)} = 0, \tag{4.4.10}$$

The asymptotic distributional bias and quadratic bias expressions of $\hat{\beta}_n^{PT}$ are given by

$$\begin{aligned}
\mathbf{b}_2^{(2)}(\hat{\beta}_n^{PT}) &= -\delta H_{p+2}(\chi_\alpha^2; \Delta^2) \\
B_2^{(2)} &= \Delta^2 \left\{ H_{p+2}(\chi_\alpha^2; \Delta^2) \right\}^2.
\end{aligned} \tag{4.4.11}$$

$$(4.4.12)$$

ADQB of $\hat{\beta}_n^{PT}$ is always bigger than that of $\tilde{\beta}_n$ except at the null hypothesis i.e. for $\Delta^2 = 0$.

The asymptotic distributional bias and quadratic bias expressions of $\hat{\beta}_n^S$ are given

by

$$\begin{aligned} \mathbf{b}_3^{(2)}(\hat{\beta}_n^S) &= -(p-2)\delta E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \\ B_3^{(2)} &= (p-2)^2\Delta^2\left\{E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\}^2 \end{aligned} \quad (4.4.13)$$

Comparisons $B_3^{(2)}$ with $B_2^{(2)}$

In the case of PTE and SE, the QB difference shows that

$$\begin{aligned} B_2^{(2)} - B_3^{(2)} &= \Delta^2\left\{H_{p+2}(\chi_\alpha^2; \Delta^2) + E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\} \\ &\times \left\{H_{p+2}(\chi_\alpha^2; \Delta^2) - E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\} \end{aligned}$$

Thus the graph of QB of SE remains below the graph of QB of PTE whenever

$$H_{p+2}(\chi_\alpha^2; \Delta^2) \geq E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \text{ for all } (\alpha, \Delta^2) \quad (4.4.14)$$

Otherwise, the graph of QB of PTE remains below the graph of QB of SE. In general, the graph of QB of PTE and SE intersects at some point Δ^2 for fixed α and the bias of PTE is worse than that of SE.

The asymptotic distributional bias and quadratic bias expression of $\hat{\beta}_n^{S+}$ given by

$$\begin{aligned} \mathbf{b}_4^{(2)}(\hat{\beta}_n^{S+}) &= -(p-2)\delta\left\{E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \right. \\ &\quad \left. - E\left[\chi_{p+2}^{-2}(\Delta^2)I\left(\chi_{p+2}^2(\Delta^2) < p-2\right)\right] \right. \\ &\quad \left. + \frac{1}{p-2}H_{p+2}((p-2); \Delta^2)\right\} \\ B_4^{(2)} &= (p-2)^2\Delta^2\left\{E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \right. \\ &\quad \left. - E\left[\chi_{p+2}^{-2}(\Delta^2)I\left(\chi_{p+2}^2(\Delta^2) < p-2\right)\right] \right. \\ &\quad \left. + \frac{1}{p-2}H_{p+2}((p-2); \Delta^2)\right\}^2 \end{aligned}$$

Comparisons $B_4^{(2)}$ with $B_2^{(2)}$

Similarly, the QB-difference of PTE and PRSE is given by

$$\begin{aligned}
B_2^{(2)} - B_4^{*(2)} &= \Delta^2 \left\{ H_{p+2}(\chi_\alpha^2; \Delta^2) \right. \\
&+ (p-2)E \left[\chi_{p+2}^{-2} I \left(\chi_{p+2}^2(\Delta^2) > p-2 \right) \right] \Big\} \\
&+ H_{p+2}(p-2; \Delta^2) \Big\{ H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+2}(p-2; \Delta^2) \\
&- (p-2)E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) > p-2 \right) \right] \Big\}
\end{aligned} \tag{4.4.15}$$

which may be positive or negative for some Δ^2 and fixed α . Hence, for fixed α ,

$$B_2^{(2)} < B_4^{(2)} \quad \text{or} \quad B_2^{(2)} > B_4^{(2)} \quad \text{for some } \Delta^2. \tag{4.4.16}$$

Comparisons $B_4^{(2)}$ with $B_3^{(2)}$

First note that for all Δ^2 ,

$$\begin{aligned}
B_3^{(2)} - B_4^{(2)} &= (p-2)^2 \Delta^2 \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \\
&+ \frac{1}{p-2} H_{p+2}(p-2; \Delta^2) \\
&- E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \Big\} \\
&\times \left\{ E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&- \frac{1}{p-2} H_{p+2}(p-2; \Delta^2) \Big\} \geq 0
\end{aligned} \tag{4.4.17}$$

Thus the graph of the QB of PRSE remains below the graph of the QB of SE, that is to say, the bias of PRSE is always smaller than that of SE, we may order the estimators according to the bias as follow:

$$\tilde{\beta}_n < \hat{\beta}_n^{S+} < \hat{\beta}_n^S \tag{4.4.18}$$

Clearly, $\tilde{\beta}_n$ is asymptotically unbiased and $\hat{\beta}_n^{PT}$, $\hat{\beta}_n^S$ and $\hat{\beta}_n^{S+}$ are biased. Under H_0 , they are all unbiased. Also, as $\Delta^2 \rightarrow \infty$, the bias reduces to zero for all estimators.

For PTE, as $\alpha \rightarrow 0$, the quadratic bias goes to zero

In general as Δ^2 moves away from the origin, the quadratic bias function increases to a maximum then decreases towards zero as Δ^2 tends to infinity. The graphical representation of QB depicts these findings.

4.4.2 ADMSE and ADQR Expressions of β_n^* and Comparisons

MSE and Risk for the estimator $\tilde{\beta}_n$

The MSE and risk are given by

$$M_1^{(2)} = G^* \text{ and } R_1^{(2)} = \text{tr}[G^*Q] \quad (4.4.19)$$

MSE and Risk for the estimator $\hat{\beta}_n^{PT}$

The MSE and risk are given by

$$\begin{aligned} M_2^{(2)} &= G^* \left\{ 1 - H_{p+2}(\chi_\alpha^2; \Delta^2) \right\} \\ &\quad + \delta\delta' \left\{ 2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2) \right\} \\ R_2^{(2)} &= \text{tr}[G^*Q] \left\{ 1 - H_{p+2}(\chi_\alpha^2; \Delta^2) \right\} \\ &\quad + \delta'Q\delta \left\{ 2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2) \right\} \end{aligned} \quad (4.4.20)$$

Risk Analysis

In this case the risk-difference $R_1^{*(2)}(\tilde{\beta}_n : Q) - R_2^{(2)}(\tilde{\beta}_n^{PT} : Q) \geq 0$ according as

$$\Delta^2 \leq \frac{\text{tr}(G^*Q)}{Ch_{max}(G^*Q)} \frac{H_{p+2}(\chi_\alpha^2; \Delta^2)}{\{2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2)\}} \quad (4.4.21)$$

Then, $\tilde{\beta}_n^{PT}$ performs better than $\tilde{\beta}_n$ in this range of Δ^2 . On the other hand $\tilde{\beta}_n$

Figure 4.1: QB of $\tilde{\beta}$, $\hat{\beta}^{PT}$, $\hat{\beta}^S$ and $\hat{\beta}^{S+}$ with $p = 4$

Figure 4.2: QB of $\tilde{\beta}$, $\hat{\beta}^{PT}$, $\hat{\beta}^S$ and $\hat{\beta}^{S+}$ with $p = 8$

performs better than $\tilde{\beta}_n^{PT}$ if

$$\Delta^2 > \frac{\text{tr}(\mathbf{G}^*\mathbf{Q})}{Ch_{\min}(\mathbf{G}^*\mathbf{Q})} \frac{H_{p+2}(\chi_\alpha^2 : \Delta^2)}{\{2H_{p+2}(\chi_\alpha^2 : \Delta^2) - H_{p+4}(\chi_\alpha^2 : \Delta^2)\}} \quad (4.4.22)$$

where $Ch_{\min}(\mathbf{A})$ and $Ch_{\max}(\mathbf{A})$ are the minimum and maximum characteristic values of the matrix $\mathbf{A}$.

Under H_0 , the risk-difference becomes $\text{tr}(\mathbf{G}^*\mathbf{Q}H_{p+2}(\chi_\alpha^2 : 0))$. Hence, the relative gain in using $\tilde{\beta}_n^{PT}$ against $\tilde{\beta}_n$ is $100H_{p+2}(\chi_\alpha^2 : 0)\%$ at α -level of significance test. An optimum PTE with minimum guaranteed efficiency E_0 may be obtained at α^* -level of significance by finding smallest α for which ,

$$\max_{\alpha} \min_{\Delta^2} E(\alpha, \Delta^2) \geq E_0 \quad (4.4.23)$$

where

$$E(\alpha, \Delta^2) = \frac{R_1^{(2)}(\tilde{\beta}_n : \mathbf{Q})}{R_2^{(2)}(\tilde{\beta}_n^{PT} : \mathbf{Q})}$$

Now setting $\mathbf{Q} = (\mathbf{G}^*)^{-1}$, we have

$$E(\alpha, \Delta^2) = \left[1 - H_{p+2}(\chi_\alpha^2 : \Delta^2) + \frac{1}{p} \Delta^2 \left\{ 2H_{p+2}(\chi_\alpha^2 : \Delta^2) - H_{p+4}(\chi_\alpha^2 : \Delta^2) \right\} \right]^{-1} \quad (4.4.24)$$

Table 4.1 gives the maximum and minimum asymptotic efficiency of $\hat{\beta}_n^{PT}$ relative to $\tilde{\beta}_n$ for chosen α -values and $p = 4(2)14$. For $p = 8$, if PTE of β is to chosen with at least .80 efficiency, we choose $\alpha^* = 0.05$ as the level of the test.

MSE Comparision

First we compare $\tilde{\beta}_n$ and $\hat{\beta}_n^{PT}$. In this case MSE matrix difference $\mathbf{M}_1^{(2)} - \mathbf{M}_2^{(2)}$ is

α	p	4	6	8	10	12	14
0.05	$E_{\max}$	6.75908	7.89292	8.72360	9.37375	9.90415	10.34988
	$E_{\min}$	0.67741	0.76147	0.81707	0.85639	0.88538	0.90738
	$\Delta^2_{\min}$	8.70073	10.82463	12.79251	14.66773	16.48109	18.24932
0.1	$E_{\max}$	3.92601	4.49118	4.89829	5.21311	5.46767	5.67991
	$E_{\min}$	0.75506	0.82544	0.87014	0.90071	0.92260	0.93879
	$\Delta^2_{\min}$	7.73232	9.77526	11.68294	13.51002	15.28335	17.01821
0.15	$E_{\max}$	2.89783	3.26722	3.53047	3.73248	3.89487	4.02965
	$E_{\min}$	0.80429	0.86374	0.90061	0.92530	0.94267	0.95530
	$\Delta^2_{\min}$	7.16494	9.15150	11.01743	12.81104	14.55626	16.26681
0.2	$E_{\max}$	2.35590	2.62511	2.81549	2.96075	3.07701	3.17317
	$E_{\min}$	0.84032	0.89067	0.92141	0.94171	0.95579	0.96591
	$\Delta^2_{\min}$	6.76058	8.70106	10.53304	12.29940	14.02168	15.71231
0.25	$E_{\max}$	2.01845	2.22631	2.37244	2.48345	2.57199	2.64502
	$E_{\min}$	0.86844	0.91105	0.93681	0.95363	0.96518	0.97339
	$\Delta^2_{\min}$	6.44492	8.34493	10.14718	11.88969	13.59187	15.26500
0.3	$E_{\max}$	1.78732	1.95346	2.06973	2.15774	2.22775	2.28535
	$E_{\min}$	0.89120	0.92715	0.94874	0.96272	0.97225	0.97897
	$\Delta^2_{\min}$	6.18490	8.04786	9.82294	11.54366	13.22745	14.88455
0.35	$E_{\max}$	1.61900	1.75471	1.84936	1.92081	1.97751	2.02408
	$E_{\min}$	0.91005	0.94021	0.95827	0.96989	0.97776	0.98326
	$\Delta^2_{\min}$	5.96289	7.79095	9.54048	11.24068	12.90716	14.54916
0.4	$E_{\max}$	1.49112	1.60350	1.68169	1.74059	1.78724	1.82549
	$E_{\min}$	0.92589	0.95099	0.96604	0.97567	0.98215	0.98666
	$\Delta^2_{\min}$	5.76832	7.56281	9.28777	10.96826	12.61807	14.24553
0.45	$E_{\max}$	1.39096	1.48475	1.54992	1.59894	1.63770	1.66944
	$E_{\min}$	0.93933	0.96002	0.97245	0.98039	0.98571	0.98938
	$\Delta^2_{\min}$	5.59431	7.35598	9.05691	10.71809	12.35159	13.96478
0.5	$E_{\max}$	1.31074	1.38927	1.44382	1.48480	1.51718	1.54367
	$E_{\min}$	0.95079	0.96763	0.97781	0.98430	0.98862	0.99160
	$\Delta^2_{\min}$	5.43611	7.16522	8.84227	10.48427	12.10153	13.70053

Table 4.1: Maximum and Minimum of Risk Based Efficiencies

given by

$$\begin{aligned} \mathbf{M}_1^{(2)} - \mathbf{M}_2^{(2)} &= \mathbf{G}^* H_{p+2}(\chi_\alpha^2; \Delta^2) \\ &\quad - \delta\delta' \left\{ 2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2) \right\} \end{aligned} \quad (4.4.25)$$

Thus, for a given non-zero vector ℓ , we have

$$\begin{aligned} \ell'(\mathbf{M}_1^{(2)} - \mathbf{M}_2^{(2)})\ell &= \ell' \mathbf{G}^* \ell H_{p+2}(\chi_\alpha^2; \Delta^2) \\ &\quad - \ell' \delta\delta' \ell \left\{ 2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2) \right\} \end{aligned} \quad (4.4.26)$$

R.H.S is non-negative for all ℓ whenever

$$\begin{aligned} \max_{\ell} \frac{\ell' \delta\delta' \ell}{\ell' \mathbf{G}^* \ell} \left\{ 2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2) \right\} \\ \leq H_{p+2}(\chi_\alpha^2; \Delta^2) \end{aligned} \quad (4.4.27)$$

or

$$\Delta^2 \leq \frac{H_{p+2}(\chi_\alpha^2; \Delta^2)}{2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2)} \quad (4.4.28)$$

since $\max_{\ell} \frac{\ell' \delta\delta' \ell}{\ell' \mathbf{G}^* \ell} = \Delta^2$.

Thus $\hat{\beta}_n^{PT}$ is better than $\tilde{\beta}_n$ whenever (4.4.28) holds otherwise $\tilde{\beta}_n$ is better than $\hat{\beta}_n^{PT}$.

MSE and Risk for the estimator $\hat{\beta}_n^S$

The MSE and risk are given by

$$\begin{aligned}
\mathbf{M}_3^{(2)} &= \mathbf{G}^* \left\{ 1 - (p-2) \left[2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \right. \\
&\quad \left. \left. - (p-2)E \left[\chi_{p+2}^{-4}(\Delta^2) \right] \right] \right\} \\
&\quad + (p-2)\delta\delta' \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] - 2E \left[\chi_{p+4}^{-2}(\Delta^2) \right] \right. \\
&\quad \left. + (p-2)E \left[\chi_{p+4}^{-4}(\Delta^2) \right] \right\} \\
R_3^{(2)} &= \text{tr}[\mathbf{G}^* \mathbf{Q}] \left\{ 1 - (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \right. \\
&\quad \left. \left. - (p-2)E \left[\chi_{p+2}^{-4}(\Delta^2) \right] \right\} \right\} \\
&\quad + (p-2)\delta' \mathbf{Q} \delta \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \\
&\quad \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) \right] + (p-2)E \left[\chi_{p+4}^{-4}(\Delta^2) \right] \right\} \tag{4.4.29}
\end{aligned}$$

Risk Analysis

The risk of $\hat{\beta}_n^S$ may be rewritten as

$$\begin{aligned}
&\text{tr}(\mathbf{G}^* \mathbf{Q}) - p(p-2)\text{tr}(\mathbf{G}^* \mathbf{Q}) \left\{ (p-2)E[\chi_{p+2}^{-4}(\Delta^2)] \right. \\
&\quad \left. + 2 \left(1 - \frac{(p+2)\delta' \mathbf{Q} \delta}{2\Delta^2 \text{tr}(\mathbf{G}^* \mathbf{Q})} \right) \Delta^2 E[\chi_{p+4}^{-4}(\Delta^2)] \right\} \tag{4.4.30}
\end{aligned}$$

The risk-difference $R_1^{*(2)}(\tilde{\beta}_n : \mathbf{Q}) - R_3^{*(2)}(\hat{\beta}_n^S : \mathbf{Q}) \geq 0$ for all Δ^2 if $\mathbf{Q}$ satisfies the condition

$$\left\{ \mathbf{Q} : \frac{\text{tr}(\mathbf{G}^* \mathbf{Q})}{Ch_{max}(\mathbf{G}^* \mathbf{Q})} \geq \frac{p+2}{2} \right\} \tag{4.4.31}$$

Thus, $\hat{\beta}_n^S$ uniformly dominates $\tilde{\beta}_n$. As $\Delta^2 \rightarrow \infty$, the risk-difference goes to zero.

Consider the risk-difference of $R_3^{*(2)}(\hat{\beta}_n^S : \mathbf{Q}) - R_2^{*(2)}(\hat{\beta}_n^{PT} : \mathbf{Q})$ under H_0 given by

$$\text{tr}(\mathbf{G}^* \mathbf{Q}) \left[H_{p+2}(\chi_\alpha^2 : 0) - \frac{p-2}{p} \right] \tag{4.4.32}$$

It is non-negative if $H_{p+2}(\chi_\alpha^2 : 0) > \frac{p-2}{p}$. Thus, $\hat{\beta}_n^{PT}$ dominates $\hat{\beta}_n^S$ for a set of the level of significance α for which

$$H_{p+2}(\chi_\alpha^2 : 0) > \frac{p-2}{p} = 1 - \frac{2}{p} \quad (4.4.33)$$

Thus, neither $\hat{\beta}_n^{PT}$ nor $\hat{\beta}_n^S$ dominate each other uniformly. In general, $\hat{\beta}_n^S$ should be used whenever $p \geq 3$ but for $p \leq 2$, $\hat{\beta}_n^{PT}$ is preferable in case β is close to 0.

MSE Comparision

Next, we consider the MSE matrix differences $\mathbf{M}_1^{(2)} - \mathbf{M}_3^{(2)}$ and $\mathbf{M}_2^{(2)} - \mathbf{M}_3^{(2)}$. Now,

$$\begin{aligned} \mathbf{M}_1^{(2)} - \mathbf{M}_3^{(2)} &= (p-2)\mathbf{G}^* \left\{ 2E[\chi_{p+2}^{-2}(\Delta^2)] - (p-2)E[\chi_{p+2}^{-4}(\Delta^2)] \right\} \\ &\quad - (p^2-4)\delta\delta'E[\chi_{p+4}^{-4}(\Delta^2)] \end{aligned} \quad (4.4.34)$$

The R.H.S is non-negative whenever for a givn non-zero vector ℓ we have

$$(p+2)\frac{\ell'\delta\delta'\ell}{\ell'\mathbf{G}^*\ell}E[\chi_{p+4}^{-4}(\Delta^2)] \leq E[\chi_{p+2}^{-2}(\Delta^2)] + \Delta^2E[\chi_{p+4}^{-4}(\Delta^2)] \quad (4.4.35)$$

We have for all ℓ ,

$$(p+2)\Delta^2E[\chi_{p+4}^{-4}(\Delta^2)] \leq E[\chi_{p+2}^{-2}(\Delta^2)] + \Delta^2E[\chi_{p+4}^{-4}(\Delta^2)] \quad (4.4.36)$$

or

$$p\Delta^2E[\chi_{p+4}^{-4}(\Delta^2)] \leq (p-2)E[\chi_{p+2}^{-4}(\Delta^2)] \quad (4.4.37)$$

which is a contradiction. Hence, $\hat{\beta}_n^S$ does not dominate $\tilde{\beta}_n$ uniformly.

The comparion of $\hat{\beta}_n^{PT}$ and $\hat{\beta}_n^S$ is obtained by considering

$$\begin{aligned} \mathbf{M}_2^{(2)} - \mathbf{M}_3^{(2)} &= \mathbf{G}^* \left\{ H_{p+2}(\chi_{p+2}^2; \Delta^2) \right. \\ &\quad - (p-2) \left[2E[\chi_{p+2}^{-2}(\Delta^2)] - (p-2)E[\chi_{p+2}^{-4}(\Delta^2)] \right] \Big\} \\ &\quad + \delta\delta' \left\{ \left[2H_{p+2}(\chi_{p+2}^2; \Delta^2) - H_{p+4}(\chi_{p+2}^2; \Delta^2) \right] \right. \\ &\quad \left. - (p^2-4)E[\chi_{p+4}^{-4}(\Delta^2)] \right\} \end{aligned} \quad (4.4.38)$$

Thus, the difference $M_2^{(2)} - M_3^{(2)}$ is non-negative semi-definite whenever for some level of significance, α .

$$\begin{aligned} \Delta^2 \leq & \left[H_{p+2}(\chi_{p+2}^2; \Delta^2) - (p-2)\{2E[\chi_{p+2}^{-2}(\Delta^2)] \right. \\ & - (p-2)E[\chi_{p+2}^{-4}(\Delta^2)]\} \left. \right] \left[(p^2-1)E[\chi_{p+4}^{-4}(\Delta^2)] \right. \\ & - \left. \{2H_{p+2}(\chi_{p+2}^2; \Delta^2) - H_{p+4}(\chi_{p+2}^2; \Delta^2)\} \right]^{-1} \end{aligned} \quad (4.4.39)$$

Thus, $\hat{\beta}_n^{PT}$ is better than $\hat{\beta}_n^S$ if Δ^2 satisfies (4.4.39) otherwise $\hat{\beta}_n^S$ is better. Thus, neither $\hat{\beta}_n^{PT}$ nor $\hat{\beta}_n^S$ dominate each other uniformly.

MSE and Risk for the estimator $\hat{\beta}_n^{S+}$

The MSE and risk are given by

$$\begin{aligned} M_4^{(2)} = & M_3^{(2)} - G^* \left[H_{p+2}((p-2); \Delta^2) \right. \\ & - (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\ & - (p-2) E \left[\chi_{p+2}^{-4}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \left. \right\} \left. \right] \\ & - \delta \delta' \left[H_{p+4}((p-2); \Delta^2) - 2H_{p+2}((p-2); \Delta^2) \right. \\ & + (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\ & - 2E \left[\chi_{p+4}^{-2}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \\ & + (p-2) E \left[\chi_{p+4}^{-4}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \left. \right\} \left. \right] \end{aligned}$$

$$\begin{aligned}
R_4^{(2)} &= R_3^{(2)} - \text{tr}[\mathbf{G}^* \mathbf{Q}] \left\{ H_{p+2}((p-2); \Delta^2) \right. \\
&\quad - (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. - (p-2) E \left[\chi_{p+2}^{-4}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \\
&\quad - \boldsymbol{\delta}' \mathbf{Q} \boldsymbol{\delta} \left\{ H_{p+4}((p-2); \Delta^2) - 2H_{p+2}((p-2); \Delta^2) \right. \\
&\quad + (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. \left. + (p-2) E \left[\chi_{p+4}^{-4}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right\} \right\} \quad (4.4.40)
\end{aligned}$$

Risk Analysis

Now consider the risk-difference $R_3^{(2)}(\hat{\boldsymbol{\beta}}_n^S : \mathbf{Q}) - R_4^{(2)}(\hat{\boldsymbol{\beta}}_n^{S+} : \mathbf{Q})$. It is non-negative for all Δ^2 since the risk-difference is

$$\begin{aligned}
&\left\{ \text{tr}(\mathbf{G}^* \mathbf{Q}) E \left[\left(1 - (p-2) \chi_{p+2}^{-2}(\Delta^2) \right)^2 I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&+ \boldsymbol{\delta}' \mathbf{Q} \boldsymbol{\delta} E \left[\left(1 - (p-2) \chi_{p+4}^{-2}(\Delta^2) \right)^2 I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \left. \right\} \\
&+ \boldsymbol{\delta}' \mathbf{Q} \boldsymbol{\delta} E \left[\left((p-2) \chi_{p+2}^{-2}(\Delta^2) - 1 \right) \right] I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \geq 0. \quad (4.4.41)
\end{aligned}$$

Hence, $\hat{\boldsymbol{\beta}}_n^{S+}$ dominates $\hat{\boldsymbol{\beta}}_n^S$ for all $\mathbf{Q}$ satisfying the above condition. This leads to the conclusion that

$$R_1^{(2)}(\tilde{\boldsymbol{\beta}}_n : \mathbf{Q}) \geq R_3^{(2)}(\hat{\boldsymbol{\beta}}_n^S : \mathbf{Q}) \geq R_4^{(2)}(\hat{\boldsymbol{\beta}}_n^{S+} : \mathbf{Q})$$

uniformly in Δ^2 .

Figure 4.3: Graph of Asymptotic Risk of $\tilde{\beta}$, $\hat{\beta}^{PT}$, $\hat{\beta}^S$ and $\hat{\beta}^{S+}$

Now, the risk-difference of $R_4^{*(2)}(\hat{\beta}_n^{S+} : \mathbf{Q}) - R_2^{*(2)}(\hat{\beta}_n^{PT} : \mathbf{Q})$ under H_0 is given by

$$\begin{aligned} & \text{tr}(\mathbf{G}^* \mathbf{Q}) \left[H_{p+2}(\chi_\alpha^2; 0) + (p-2) \left\{ 2E[\chi_{p+2}^{-2} I(\chi_{p+2}^2 < p-2)] \right. \right. \\ & \left. \left. - (p-2)E[\chi_{p+2}^{-4} I(\chi_{p+2}^2 < p-2)] \right\} - \left(\frac{p-2}{p} + H_{p+2}(p-2; 0) \right) \right]. \end{aligned} \quad (4.4.42)$$

Hence, $\hat{\beta}_n^{S+}$ has less risk than $\hat{\beta}_n^{PT}$ when the level of significance satisfies the relation

$$\begin{aligned} H_{p+2}(\chi_\alpha^2; 0) & \geq \left\{ \left(1 - \frac{2}{p} \right) \right. \\ & \left. + H_{p+2}(p-2; 0) - (p-2)E[\chi_{p+2}^{-2} I(\chi_{p+2}^2 < p-2)] \right\}. \end{aligned} \quad (4.4.43)$$

Similar conclusion as before holds for $\hat{\beta}_n^{PT}$ and $\hat{\beta}_n^{S+}$ and neither $\hat{\beta}_n^{PT}$ nor $\hat{\beta}_n^{S+}$

dominate each other uniformly for some level of significance, α .

MSE Comparision

Next we see that

$$\begin{aligned} \mathbf{M}_3^{(2)} - \mathbf{M}_4^{(2)} &= \mathbf{G}^* \left\{ E \left[\left\{ 1 - (p-2)\chi_{p+2}^{-2} \right\}^2 I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\ &\quad + \delta \delta' \left[\left\{ 1 - (1-p)\chi_{p+4}^{-2}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right\}^2 \right] \left. \right\} \\ &\quad + 2\delta \delta' \left[\left\{ (p-2)\chi_{p+2}^{-2} - 1 \right\} \right] \geq \mathbf{0}, \end{aligned} \quad (4.4.44)$$

for all δ . Hence $\hat{\beta}_n^{S+}$ dominates $\hat{\beta}_n^S$ uniformly with respect to the MSE-matrices.

4.4.3 ADB, ADQB, ADMSE Matrix and Risk Expressions of ν_n^*

ADB and ADQB of $\tilde{\nu}_n$

$$\begin{aligned} \mathbf{b}_1^{(1)}(\tilde{\nu}_n) &= -(\mathbf{I} - \mathbf{K}_{xx})\delta, \\ B_1^{(1)}(\tilde{\nu}_n) &= \sigma_{ee}^{-1} \delta' (\mathbf{I} - \mathbf{K}_{xx})' \Sigma_{XX} (\mathbf{I} - \mathbf{K}_{xx}) \delta \end{aligned} \quad (4.4.45)$$

ADB and ADQB of $\hat{\nu}_n^{PT}$

$$\begin{aligned} \mathbf{b}_2^{(1)}(\hat{\nu}_n^{PT}) &= -(\mathbf{I} - \mathbf{K}_{xx})\delta - \mathbf{K}_{xx}\delta H_{p+2}(\chi_\alpha^2; \Delta^2) \\ B_2^{(1)}(\hat{\nu}_n^{PT}) &= \Delta^2 \left\{ H_{p+2}(\chi_\alpha^2; \Delta^2) \right\}^2 \\ &\quad + \sigma_{ee}^{-1} \delta' (\mathbf{I} - \mathbf{K}_{xx})' \Sigma_{XX} (\mathbf{I} - \mathbf{K}_{xx}) \delta \\ &\quad + \sigma_{ee}^{-1} \left\{ \delta' (\mathbf{I} - \mathbf{K}_{xx})' \Sigma_{XX} \mathbf{K}_{xx} \delta \right. \\ &\quad \left. + \delta' \mathbf{K}_{xx}' \Sigma_{XX} (\mathbf{I} - \mathbf{K}_{xx}) \delta \right\} H_{p+2}(\chi_\alpha^2; \Delta^2) \end{aligned} \quad (4.4.46)$$

ADB and ADQB $\hat{\nu}_n^S$

$$\begin{aligned}
\mathbf{b}_3^{(1)}(\hat{\nu}_n^S) &= -(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta} - (p-2)\mathbf{K}_{xx}\boldsymbol{\delta}E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \\
B_3^{(1)}(\hat{\beta}_n^S) &= (p-2)^2\Delta^2\left\{E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right\}^2 \\
&\quad + \sigma_{ee}^{-1}\boldsymbol{\delta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{XX}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta} \\
&\quad + (p-2)\sigma_{ee}^{-1}\left\{\boldsymbol{\delta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{XX}\mathbf{K}_{xx}\boldsymbol{\delta}\right. \\
&\quad \left.+ \boldsymbol{\delta}'\mathbf{K}_{xx}'\boldsymbol{\Sigma}_{XX}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}\right\}H_{p+2}(\chi_\alpha^2; \Delta^2)
\end{aligned}$$

ADB and ADQB of $\hat{\nu}_n^{S+}$

$$\begin{aligned}
\mathbf{b}_4^{(1)}(\hat{\nu}_n^{S+}) &= -(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta} - (p-2)\boldsymbol{\delta}\left\{E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right. \\
&\quad \left.- E\left[\chi_{p+2}^{-2}(\Delta^2)I\left(\chi_{p+2}^2(\Delta^2) < p-2\right)\right]\right. \\
&\quad \left.+ \frac{1}{p-2}H_{p+2}((p-2); \Delta^2)\right\} \\
B_4^{(1)}(\hat{\nu}_n^{S+}) &= (p-2)^2\Delta^2\left\{E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right. \\
&\quad \left.+ \sigma_{ee}^{-1}\boldsymbol{\delta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{XX}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}\right. \\
&\quad \left.- E\left[\chi_{p+2}^{-2}(\Delta^2)I\left(\chi_{p+2}^2(\Delta^2) < p-2\right)\right]\right. \\
&\quad \left.+ \frac{1}{p-2}H_{p+2}((p-2); \Delta^2)\right\}^2 \\
&\quad + \sigma_{ee}^{-1}\left\{\boldsymbol{\delta}'(\mathbf{I} - \mathbf{K}_{xx})'\boldsymbol{\Sigma}_{XX}\mathbf{K}_{xx}\boldsymbol{\delta} + \boldsymbol{\delta}'\mathbf{K}_{xx}'\boldsymbol{\Sigma}_{XX}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}\right\} \\
&\quad \times \left\{H_{p+2}((p-2); \Delta^2) + (p-2)E\left[\chi_{p+2}^{-2}(\Delta^2)\right]\right. \\
&\quad \left.- E\left[\chi_{p+2}^{-2}(\Delta^2)I\left(\chi_{p+2}^2(\Delta^2) < p-2\right)\right]\right\}
\end{aligned}$$

ADMSE and ADQR of $\tilde{\nu}_n$

$$\begin{aligned}
\mathbf{M}_1^{(1)}(\tilde{\nu}_n) &= \sigma_{ee}\boldsymbol{\Sigma}_{XX}^{-1} + (\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta}\boldsymbol{\delta}'(\mathbf{I} - \mathbf{K}_{xx}) \\
R_1^{(1)}(\tilde{\nu}_n) &= \sigma_{ee}\text{tr}[\boldsymbol{\Sigma}_{XX}^{-1}\mathbf{Q}] + \boldsymbol{\delta}'(\mathbf{I} - \mathbf{K}_{xx})\mathbf{Q}(\mathbf{I} - \mathbf{K}_{xx})\boldsymbol{\delta} \quad (4.4.47)
\end{aligned}$$

ADMSE and ADQR of $\hat{\nu}_n^{PT}$

$$\begin{aligned}
\mathbf{M}_2^{(1)}(\hat{\nu}_n^{PT}) &= \sigma_{ee} \Sigma_{XX}^{-1} \left\{ 1 - H_{p+2}(\chi_\alpha^2; \Delta^2) \right\} \\
&\quad + \delta \delta' \left\{ 2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2) \right\} \\
&\quad + (\mathbf{I} - \mathbf{K}_{xx}) \delta \delta' (\mathbf{I} - \mathbf{K}_{xx})' + \left\{ (\mathbf{I} - \mathbf{K}_{xx}) \delta \delta' \mathbf{K}_{xx}' \right. \\
&\quad \left. + \mathbf{K}_{xx} \delta \delta' (\mathbf{I} - \mathbf{K}_{xx})' \right\} H_{p+2}(\chi_\alpha^2; \Delta^2) \\
R_2^{(1)}(\hat{\nu}_n^{PT}) &= \sigma_{ee} \text{tr}[\Sigma_{XX}^{-1} \mathbf{Q}] \left\{ 1 - H_{p+2}(\chi_\alpha^2; \Delta^2) \right\} \\
&\quad + \delta' \mathbf{Q} \delta \left\{ 2H_{p+2}(\chi_\alpha^2; \Delta^2) - H_{p+4}(\chi_\alpha^2; \Delta^2) \right\} \\
&\quad + \delta' (\mathbf{I} - \mathbf{K}_{xx})' \mathbf{Q} (\mathbf{I} - \mathbf{K}_{xx}) \delta + \text{tr} \left\{ (\mathbf{I} - \mathbf{K}_{xx}) \delta \delta' \mathbf{K}_{xx}' \mathbf{Q} \right. \\
&\quad \left. + \mathbf{K}_{xx} \delta \delta' (\mathbf{I} - \mathbf{K}_{xx})' \mathbf{Q} \right\} H_{p+2}(\chi_\alpha^2; \Delta^2) \tag{4.4.48}
\end{aligned}$$

ADMSE and ADQR of $\hat{\nu}_n^S$

$$\begin{aligned}
\mathbf{M}_3^{(1)}(\hat{\nu}_n^S) &= \sigma_{ee} \Sigma_{XX}^{-1} \left\{ 1 - (p-2) \left[2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \right. \\
&\quad \left. \left. - (p-2)E \left[\chi_{p+2}^{-4}(\Delta^2) \right] \right] \right\} + (p-2) \delta \delta' \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \\
&\quad \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) \right] + (p-2)E \left[\chi_{p+4}^{-4}(\Delta^2) \right] \right\} \\
&\quad + (\mathbf{I} - \mathbf{K}_{xx}) \delta \delta' (\mathbf{I} - \mathbf{K}_{xx})' + (p-2) \left\{ (\mathbf{I} - \mathbf{K}_{xx}) \delta \delta' \mathbf{K}_{xx}' \right. \\
&\quad \left. + \mathbf{K}_{xx} \delta \delta' (\mathbf{I} - \mathbf{K}_{xx})' \right\} H_{p+2}(\chi_\alpha^2; \Delta^2) \tag{4.4.49}
\end{aligned}$$

$$\begin{aligned}
R_3^{(1)}(\hat{\nu}_n^S) &= \sigma_{ee} \text{tr}[\Sigma_{XX}^{-1} \mathbf{Q}] \left\{ 1 - (p-2) \left[2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \right. \\
&\quad \left. \left. - (p-2)E \left[\chi_{p+2}^{-4}(\Delta^2) \right] \right] \right\} + (p-2) \delta' \mathbf{Q} \delta \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \\
&\quad \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) \right] + (p-2)E \left[\chi_{p+4}^{-4}(\Delta^2) \right] \right\} \\
&\quad + \delta' (\mathbf{I} - \mathbf{K}_{xx})' \mathbf{Q} (\mathbf{I} - \mathbf{K}_{xx}) \delta + (p-2) \text{tr} \left\{ (\mathbf{I} - \mathbf{K}_{xx}) \delta \delta' \mathbf{K}_{xx}' \mathbf{Q} \right. \\
&\quad \left. + \mathbf{K}_{xx} \delta \delta' (\mathbf{I} - \mathbf{K}_{xx})' \mathbf{Q} \right\} H_{p+2}(\chi_\alpha^2; \Delta^2) \tag{4.4.50}
\end{aligned}$$

ADMSE and ADQR of $\hat{\nu}_n^{S+}$

$$\begin{aligned}
\mathbf{M}_4^{(1)}(\hat{\nu}_n^{S+}) &= \mathbf{M}_3^{*(1)}(\hat{\nu}_n^S) - \sigma_{ee} \Sigma_{XX}^{-1} \left[H_{p+2}((p-2); \Delta^2) \right. \\
&\quad - (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. - (p-2) E \left[\chi_{p+2}^{-4}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \Big] \\
&\quad - \delta \delta' \left[H_{p+4}((p-2); \Delta^2) - 2H_{p+2}((p-2); \Delta^2) \right. \\
&\quad + (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. + (p-2) E \left[\chi_{p+4}^{-4}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right\} \Big] \\
&\quad + (\mathbf{I} - \mathbf{K}_{xx}) \delta \delta' (\mathbf{I} - \mathbf{K}_{xx})' \\
&\quad + \left\{ \delta' (\mathbf{I} - \mathbf{K}_{xx})' \Sigma_{XX} \mathbf{K}_{xx} \delta + \delta' \mathbf{K}_{xx}' \Sigma_{XX} (\mathbf{I} - \mathbf{K}_{xx}) \delta \right\} \\
&\quad \times \left\{ H_{p+2}((p-2); \Delta^2) + (p-2) E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \\
&\quad \left. - E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \tag{4.4.51}
\end{aligned}$$

$$\begin{aligned}
R_4^{(1)}(\hat{\nu}_n^{S+}) &= R_3^{*(1)}(\hat{\nu}_n^S) - \sigma_{ee} \text{tr}[\Sigma_{XX}^{-1} \mathbf{Q}] \left[H_{p+2}((p-2); \Delta^2) \right. \\
&\quad - (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. \left. - (p-2) E \left[\chi_{p+2}^{-4}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \right] \\
&\quad - \delta' \mathbf{Q} \delta \left[H_{p+4}((p-2); \Delta^2) - 2H_{p+2}((p-2); \Delta^2) \right. \\
&\quad \left. + (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \right. \\
&\quad \left. \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right. \right. \\
&\quad \left. \left. + (p-2) E \left[\chi_{p+4}^{-4}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right\} \right] \\
&\quad + \delta' (\mathbf{I} - \mathbf{K}_{xx})' \mathbf{Q} (\mathbf{I} - \mathbf{K}_{xx}) \delta \\
&\quad + \text{tr} \left\{ \delta' (\mathbf{I} - \mathbf{K}_{xx})' \Sigma_{XX} \mathbf{K}_{xx} \delta \mathbf{Q} + \delta' \mathbf{K}_{xx}' \Sigma_{XX} (\mathbf{I} - \mathbf{K}_{xx}) \delta \mathbf{Q} \right\} \\
&\quad \times \left\{ H_{p+2}((p-2); \Delta^2) + (p-2) E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \\
&\quad \left. - E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \tag{4.4.52}
\end{aligned}$$

As regards the estimators ν_n^* similar results hold and is not repeated.

4.4.4 Canonical MSE Matrix and Risk Forms of ν_n^* and β_n^*

There exists a nonsingular $p \times p$ matrix $\mathbf{T}$ that simultaneously diagonalizes Σ_{XX} and Σ_{xx} . Thus,

$$\mathbf{T}' \Sigma_{XX} \mathbf{T} = \mathbf{I}_p, \quad \mathbf{T}' \Sigma_{xx} \mathbf{T} = \mathbf{D} = \text{diag}(\delta_1, \dots, \delta_p) \tag{4.4.53}$$

where $\delta_1 \leq \dots \leq \delta_p$ are the ordered eigenvalues of $\Sigma_{XX}^{-1} \Sigma_{xx} = \mathbf{K}_{xx}$ and $0 \leq \delta_i \leq 1$, $i = 1, \dots, p$, since the $\mathbf{T}$ standardizes the observed predictor vectors $\mathbf{X}_t$, i.e. $\text{cov}(\mathbf{T}' \mathbf{X}_t) = \mathbf{I}$, so that the δ_i are to be the scalar reliabilities of the elements of $\mathbf{T}' \mathbf{X}_t$.

It is easily seen that $\mathbf{K}_{xx} = \mathbf{T} \mathbf{D} \mathbf{T}^{-1}$ so that the i th column of $\mathbf{T}$ is the right eigenvector of $\mathbf{K}_{xx}$ corresponding the eigenvalue δ_i , $i = 1, \dots, p$.

Let $\boldsymbol{\theta} = \mathbf{T}^{-1}\boldsymbol{\beta}$. Then, the corresponding estimator of $\boldsymbol{\theta}$ are $\tilde{\boldsymbol{\theta}}_n = \mathbf{T}^{-1}\tilde{\boldsymbol{\beta}}_n$ and $\hat{\boldsymbol{\theta}}_n = \mathbf{T}^{-1}\tilde{\boldsymbol{\nu}}_n$, respectively.

Now consider the estimators $\mathbf{T}^{-1}\boldsymbol{\nu}_n^*$ and $\mathbf{T}^{-1}\boldsymbol{\beta}_n^*$ of $\mathbf{T}^{-1}\boldsymbol{\beta} = \boldsymbol{\theta}$, where $\boldsymbol{\nu}_n^* = (1 - g(\mathcal{L}_n))\tilde{\boldsymbol{\nu}}_n$ and $\boldsymbol{\beta}_n^* = (1 - g(\mathcal{L}_n))\tilde{\boldsymbol{\beta}}_n$, respectively.

The expression and ADMSE of $\mathbf{T}^{-1}\boldsymbol{\nu}_n^*$ is given by

$$\begin{aligned}
 \mathbf{M}^{*(1)}(\hat{\boldsymbol{\theta}}_n^*) &= \mathbf{T}^{-1}E[(\boldsymbol{\nu}_n^* - \boldsymbol{\beta}_{(n)}) (\boldsymbol{\nu}_n^* - \boldsymbol{\beta}_{(n)})'] \mathbf{T}^{-1'} \\
 &= \sigma_{ee}\mathbf{I} + (\mathbf{I}_p - \mathbf{D})\boldsymbol{\gamma}\boldsymbol{\gamma}'(\mathbf{I} - \mathbf{D}) \\
 &\quad - \sigma_{ee}\mathbf{I} \left\{ 2E\left[g(\chi_{p+2}^2(\Delta^2))\right] - E\left[g^2(\chi_{p+2}^2(\Delta^2))\right] \right\} \\
 &\quad + \mathbf{D}\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{D} \left\{ 2E\left[g(\chi_{p+2}^2(\Delta^2))\right] - 2E\left[g(\chi_{p+4}^2(\Delta^2))\right] \right. \\
 &\quad \left. + E\left[g^2(\chi_{p+4}^2(\Delta^2))\right] \right\} + 2(\mathbf{I} - \mathbf{D})\boldsymbol{\gamma}\boldsymbol{\gamma}'\mathbf{D}E\left[g(\chi_{p+2}^2(\Delta^2))\right].
 \end{aligned} \tag{4.4.54}$$

where $\boldsymbol{\gamma} = \sqrt{n}\mathbf{T}^{-1}\boldsymbol{\beta}$

The expression of ADQR of $\mathbf{T}^{-1}\boldsymbol{\nu}_n^*$ is given by

$$\begin{aligned}
 R^{*(1)}(\hat{\boldsymbol{\theta}}_n^*) &= \sigma_{ee}\text{tr}[\mathbf{I}] + \boldsymbol{\gamma}'(\mathbf{I} - \mathbf{D})^2\boldsymbol{\gamma} \\
 &\quad - \sigma_{ee}\text{tr}[\mathbf{I}] \left\{ 2E\left[g(\chi_{p+2}^2(\Delta^2))\right] - E\left[g^2(\chi_{p+2}^2(\Delta^2))\right] \right\} \\
 &\quad + \boldsymbol{\gamma}'\mathbf{D}^2\boldsymbol{\gamma} \left\{ 2E\left[g(\chi_{p+2}^2(\Delta^2))\right] - 2E\left[g(\chi_{p+4}^2(\Delta^2))\right] \right. \\
 &\quad \left. + E\left[g^2(\chi_{p+4}^2(\Delta^2))\right] \right\} + 2\boldsymbol{\gamma}'\mathbf{D}(\mathbf{I} - \mathbf{D})\boldsymbol{\gamma}E\left[g(\chi_{p+2}^2(\Delta^2))\right].
 \end{aligned} \tag{4.4.55}$$

The expression of ADMSE of $\mathbf{T}^{-1}\boldsymbol{\beta}_n^*$ is given by

$$\begin{aligned} \mathbf{M}^{*(2)}(\tilde{\boldsymbol{\theta}}_n^*) &= \mathbf{T}^{-1}E[(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})']\mathbf{T}^{-1'} \\ &= \sigma_{ee}\mathbf{D}^{-2} - \sigma_{ee}\mathbf{D}^{-2}\left\{2E\left[g(\chi_{p+2}^2(\Delta^2))\right] \right. \\ &\quad \left. - E\left[g^2(\chi_{p+2}^2(\Delta^2))\right]\right\} + \gamma\gamma'\left\{2E\left[g(\chi_{p+2}^2(\Delta^2))\right] \right. \\ &\quad \left. - 2E\left[g(\chi_{p+4}^2(\Delta^2))\right] + E\left[g^2(\chi_{p+4}^2(\Delta^2))\right]\right\} \end{aligned} \quad (4.4.56)$$

The expression of ADQR of $\mathbf{T}^{-1}\boldsymbol{\beta}_n^*$ is given by

$$\begin{aligned} R^{*(2)}(\tilde{\boldsymbol{\theta}}_n^*) &= \sigma_{ee}\text{tr}[\mathbf{D}^{-2}] - \sigma_{ee}\text{tr}[\mathbf{D}^{-2}]\left\{2E\left[g(\chi_{p+2}^2(\Delta^2))\right] \right. \\ &\quad \left. - E\left[g^2(\chi_{p+2}^2(\Delta^2))\right]\right\} + \gamma'\gamma\left\{2E\left[g(\chi_{p+2}^2(\Delta^2))\right] \right. \\ &\quad \left. - 2E\left[g(\chi_{p+4}^2(\Delta^2))\right] + E\left[g^2(\chi_{p+4}^2(\Delta^2))\right]\right\} \end{aligned} \quad (4.4.57)$$

Canonical risk expressions of $\tilde{\nu}_n$ and $\tilde{\beta}_n$

$$\begin{aligned} R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) &= \sigma_{ee}\text{tr}[\mathbf{I}] + \gamma'(\mathbf{I} - \mathbf{D})^2\gamma \\ R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) &= \sigma_{ee}\text{tr}[\mathbf{D}^{-2}] \end{aligned} \quad (4.4.58)$$

The difference of risks is given by

$$R_1^{(1)}(\hat{\boldsymbol{\theta}}_n) - R_1^{(2)}(\tilde{\boldsymbol{\theta}}_n) = \sigma_{ee}\text{tr}[\mathbf{I} - \mathbf{D}^{-2}] + \gamma'(\mathbf{I} - \mathbf{D})^2\gamma \quad (4.4.59)$$

The risk of $\hat{\boldsymbol{\theta}}_n$ is smaller than the risk of $\tilde{\boldsymbol{\beta}}_n$ if

$$\Delta^2 \leq \frac{\sigma_{ee}\text{tr}[\mathbf{D}^{-2} - \mathbf{I}]}{Ch_{\min}[\mathbf{A}_1\mathbf{D}^{-2}]} \quad (4.4.60)$$

where $\mathbf{A}_1 = (\mathbf{I} - \mathbf{D})^2$ and the risk of $\tilde{\boldsymbol{\beta}}_n$ is smaller than $\hat{\boldsymbol{\theta}}_n$ if

$$\Delta^2 > \frac{\sigma_{ee}\text{tr}[\mathbf{D}^{-2} - \mathbf{I}]}{Ch_{\max}[\mathbf{A}_1\mathbf{D}^{-2}]} \quad (4.4.61)$$

Clearly, we find that $\hat{\boldsymbol{\theta}}_n$ dominates $\tilde{\boldsymbol{\theta}}_n$ under the null-hypotheses. Figure 4.4 shows the entire dominance picture of $\hat{\boldsymbol{\theta}}_n$ and $\tilde{\boldsymbol{\theta}}_n$ thereby of $\tilde{\nu}_n$ and $\tilde{\beta}_n$.

Canonical risk expression of $\hat{\nu}_n^{PT}$ and $\hat{\beta}_n^{PT}$

$$\begin{aligned}
R_2^{(1)}(\hat{\theta}_n^{PT}) &= R_1^{(1)}(\hat{\theta}_n) - \sigma_{ee} \text{tr}[\mathbf{I}] H_{p+2}(\chi_\alpha^2 : \Delta^2) \\
&\quad + \gamma' \mathbf{D}^2 \gamma \left\{ 2H_{p+2}(\chi_\alpha^2 : \Delta^2) - H_{p+4}(\chi_\alpha^2 : \Delta^2) \right\} \\
&\quad + 2\gamma' \mathbf{D}(\mathbf{I} - \mathbf{D}) \gamma H_{p+2}(\chi_\alpha^2 : \Delta^2) \\
R_2^{(2)}(\tilde{\theta}_n^{PT}) &= R_1^{(2)}(\tilde{\theta}_n) - \sigma_{ee} \text{tr}[\mathbf{D}^{-2}] H_{p+2}(\chi_\alpha^2 : \Delta^2) \\
&\quad + \gamma' \gamma \left\{ 2H_{p+2}(\chi_\alpha^2 : \Delta^2) - H_{p+4}(\chi_\alpha^2 : \Delta^2) \right\}
\end{aligned} \tag{4.4.62}$$

Clearly, we find that $\hat{\theta}_n^{PT}$ dominates $\tilde{\theta}_n^{PT}$ under the null-hypotheses. Figure 4.5 shows the entire dominance picture of $\hat{\theta}_n^{PT}$ and $\tilde{\theta}_n^{PT}$ thereby of $\hat{\nu}_n^{PT}$ and $\tilde{\beta}_n^{PT}$.

Canonical risk expressions of $\hat{\nu}_n^S$ and $\hat{\beta}_n^S$

$$\begin{aligned}
R_3^{(1)}(\hat{\theta}_n^S) &= R_1^{(1)}(\hat{\theta}_n) \\
&\quad - (p-2)\sigma_{ee} \text{tr}[\mathbf{I}] \left\{ \left[2E\left[\chi_{p+2}^{-2}(\Delta^2)\right] - (p-2)E\left[\chi_{p+2}^{-4}(\Delta^2)\right] \right] \right\} \\
&\quad + (p-2)\gamma' \mathbf{D}^2 \gamma \left\{ 2E\left[\chi_{p+2}^{-2}(\Delta^2)\right] - 2E\left[\chi_{p+4}^{-2}(\Delta^2)\right] \right. \\
&\quad \left. + (p-2)E\left[\chi_{p+4}^{-4}(\Delta^2)\right] \right\} + 2(p-2)\gamma' \mathbf{D}(\mathbf{I} - \mathbf{D}) \gamma E\left[\chi_{p+2}^{-2}(\Delta^2)\right]
\end{aligned} \tag{4.4.63}$$

$$\begin{aligned}
R_3^{(2)}(\tilde{\theta}_n^S) &= R_1^{(2)}(\tilde{\theta}_n) - (p-2)\sigma_{ee} \text{tr}[\mathbf{D}^{-2}] \left\{ \left[2E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \right. \right. \\
&\quad \left. \left. - (p-2)E\left[\chi_{p+2}^{-4}(\Delta^2)\right] \right] \right\} \\
&\quad + (p-2)\gamma' \gamma \left\{ 2E\left[\chi_{p+2}^{-2}(\Delta^2)\right] \right. \\
&\quad \left. - 2E\left[\chi_{p+4}^{-2}(\Delta^2)\right] + (p-2)E\left[\chi_{p+4}^{-4}(\Delta^2)\right] \right\}
\end{aligned} \tag{4.4.64}$$

Clearly, we find that $\hat{\theta}_n^S$ dominates $\tilde{\theta}_n^S$ under the null-hypotheses. Figure 4.6 shows the entire dominance picture of $\hat{\theta}_n^S$ and $\tilde{\theta}_n^S$ thereby of $\hat{\nu}_n^S$ and $\tilde{\beta}_n^S$.

Figure 4.4: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n$ and the MLE $\tilde{\theta}_n$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

Figure 4.5: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n^{PT}$ and the MLE $\tilde{\theta}_n^{PT}$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

Figure 4.6: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n^S$ and the MLE $\tilde{\theta}_n^S$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

Figure 4.7: Risk bounds as a function of Δ^2 for the LSE $\hat{\theta}_n^{S+}$ and the MLE $\tilde{\theta}_n^{S+}$ for $p = 4$ and $\mathbf{D} = \text{diag}(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

Canonical risk expressions of $\hat{\nu}_n^{S+}$ and $\hat{\beta}_n^{S+}$

$$\begin{aligned}
R_4^{(1)}(\hat{\theta}_n^{S+}) &= R_3^{(1)}(\hat{\theta}_n^S) - \sigma_{ee} \text{tr}[\mathbf{I}] \left[H_{p+2}((p-2); \Delta^2) \right. \\
&\quad - (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. \left. - (p-2) E \left[\chi_{p+2}^{-4}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \right] \\
&\quad - \gamma' \mathbf{D}^2 \gamma \left[H_{p+4}((p-2); \Delta^2) - 2H_{p+2}((p-2); \Delta^2) \right. \\
&\quad \left. + (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \right. \\
&\quad \left. \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right. \right. \\
&\quad \left. \left. + (p-2) E \left[\chi_{p+4}^{-4}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right\} \right] \\
&\quad + 2(p-2) \gamma' \mathbf{D} (\mathbf{I} - \mathbf{D}) \gamma \\
&\quad \times \left\{ H_{p+2}((p-2); \Delta^2) + (p-2) E \left[\chi_{p+2}^{-2}(\Delta^2) \right] \right. \\
&\quad \left. - E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \tag{4.4.65}
\end{aligned}$$

$$\begin{aligned}
R_4^{(2)}(\tilde{\theta}_n^{S+}) &= R_3^{(2)}(\tilde{\theta}_n^S) - \text{tr}[\mathbf{D}^{-2}] \left\{ H_{p+2}((p-2); \Delta^2) \right. \\
&\quad - (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \\
&\quad \left. \left. - (p-2) E \left[\chi_{p+2}^{-4}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right\} \right\} \\
&\quad - \gamma' \gamma \left\{ H_{p+4}((p-2); \Delta^2) - 2H_{p+2}((p-2); \Delta^2) \right. \\
&\quad \left. + (p-2) \left\{ 2E \left[\chi_{p+2}^{-2}(\Delta^2) I \left(\chi_{p+2}^2(\Delta^2) < p-2 \right) \right] \right. \right. \\
&\quad \left. \left. - 2E \left[\chi_{p+4}^{-2}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right. \right. \\
&\quad \left. \left. + (p-2) E \left[\chi_{p+4}^{-4}(\Delta^2) I \left(\chi_{p+4}^2(\Delta^2) < p-2 \right) \right] \right\} \right\} \tag{4.4.66}
\end{aligned}$$

Clearly, we find that $\hat{\theta}_n^{S+}$ dominates $\tilde{\theta}_n^{S+}$ under the null-hypotheses. Figure 4.7 shows the entire dominance picture of $\hat{\theta}_n^{S+}$ and $\tilde{\theta}_n^{S+}$ thereby of $\hat{\nu}_n^{S+}$ and $\tilde{\beta}_n^{S+}$.

Chapter 5

Several Simple Linear Models: Parallelism Hypothesis

5.0 Introduction

For several simple linear models with measurement errors, estimation of parameters is considered when it is suspected with some degree of uncertainty that the lines may be parallel.

We consider p independent bivariate samples $\{(x_{\alpha j}, Y_{\alpha j}^0) | \alpha = 1, \dots, p, j = 1, \dots, n_\alpha\}$ such that $Y_{\alpha j}^0 \sim \mathcal{N}(\theta_\alpha + \beta_\alpha \mu_{x_\alpha}; \sigma_{ee})$ for each pair (α, j) , where $\beta = (\beta_1, \dots, \beta_p)'$ is the slope and $\theta = (\theta_1, \dots, \theta_p)$ is the intercept parameters. It is common to test the parallelism hypothesis $H_0 : \beta = \beta_0 \mathbf{1}_p$ (where β_0 is an unknown scalar) against the alternative H_A : at least one pair of the components of β differ.

Accordingly, we consider five estimators, namely, the unrestricted estimator, the restricted estimator, the preliminary test estimator, Stein-type estimator and the positive-rule Stein-type estimator of β . These estimators are obtained by assuming

that the reliability ratio is known and the asymptotic analysis of the estimators is carried under a sequence of local alternative against parallelism.

The five estimators are compared in an asymptotic set up with respect to bias, MSE matrices and weight risk under a quadratic loss function.

5.1 Model Specification

Consider simple linear models

$$\begin{cases} Y_{\alpha_j}^0 = \theta_\alpha + \beta_\alpha x_{\alpha_j} + e_{\alpha_j} & j = 1, \dots, n_\alpha \\ X_{\alpha_j}^0 = x_{\alpha_j} + u_{\alpha_j} \end{cases} \quad (5.1.1)$$

where $e_{\alpha_j} \sim \mathcal{N}(0, \sigma_{ee})$ is the measurement error in the study variable and u_{α_j} is the measurement error in the repressor variable. Note that x_{α_j} is unobservable and $X_{\alpha_j}^0$ is the corresponding observed value. Furthermore, we assume that

$$(x_{\alpha_j}, e_{\alpha_j}, u_{\alpha_j})' \sim \mathcal{N}_3 \{(\mu_{x_\alpha}, 0, 0) : \text{Diag}(\sigma_{x_\alpha x_\alpha}, \sigma_{ee}, \sigma_{u_\alpha u_\alpha})\} \quad (5.1.2)$$

Then, $(Y_{\alpha_j}^0, X_{\alpha_j}^0)'$ follows a bivariate normal distribution, that is,

$$\begin{pmatrix} Y_{\alpha_j}^0 \\ X_{\alpha_j}^0 \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_\alpha + \beta_\alpha \mu_{x_\alpha} \\ \mu_\alpha \end{pmatrix} : \Sigma_{\alpha\alpha} \right\} \quad (5.1.3)$$

where

$$\Sigma_{\alpha\alpha} = \begin{pmatrix} \sigma_{Y_\alpha^0 Y_\alpha^0} & \sigma_{X_\alpha^0 Y_\alpha^0} \\ \sigma_{X_\alpha^0 Y_\alpha^0} & \sigma_{X_\alpha^0 X_\alpha^0} \end{pmatrix} = \begin{pmatrix} \beta_\alpha^2 \sigma_{x_\alpha x_\alpha} + \sigma_{ee} & \beta_\alpha \sigma_{x_\alpha x_\alpha} \\ \beta_\alpha \sigma_{x_\alpha x_\alpha} & \sigma_{x_\alpha x_\alpha} + \sigma_{u_\alpha u_\alpha} \end{pmatrix} \quad (5.1.4)$$

The conditional distribution of Y_{α_j} given X_{α_j} is again normal with mean

$$E(Y_{\alpha_j}^0 | X_{\alpha_j}^0) = \nu_{\alpha_0} + \nu_{\alpha_1} X_{\alpha_j} \quad (5.1.5)$$

and variance

$$\sigma_{zz}^{(\alpha)} = \sigma_{ee} + \beta_\alpha^2 \sigma_{x_\alpha x_\alpha} (1 - \kappa_{\alpha\alpha}) \quad (5.1.6)$$

where

$$\begin{aligned} \nu_{\alpha_0} &= \theta_\alpha + \beta_\alpha \mu_{x_\alpha} (1 - \kappa_{\alpha\alpha}) \\ \nu_{\alpha_1} &= \kappa_{\alpha\alpha} \beta_\alpha \\ \kappa_{x_\alpha x_\alpha} &= \sigma_{x_\alpha x_\alpha} (\sigma_{x_\alpha x_\alpha} + \sigma_{u_\alpha u_\alpha})^{-1} \quad (\alpha^{th} \text{ reliability ratio}). \end{aligned}$$

Clearly,

$$\begin{pmatrix} \bar{Y}_\alpha^0 \\ \bar{X}_\alpha^0 \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_\alpha + \beta_\alpha \mu_{x_\alpha} \\ \mu_{x_\alpha} \end{pmatrix} : \frac{1}{n_\alpha} \Sigma_{\alpha\alpha} \right\} \quad (5.1.7)$$

and

$$\mathbf{S}_{\alpha\alpha} = \begin{pmatrix} S_{Y_\alpha^0 Y_\alpha^0} & S_{X_\alpha^0 Y_\alpha^0} \\ S_{X_\alpha^0 Y_\alpha^0} & S_{X_\alpha^0 X_\alpha^0} \end{pmatrix} \sim W_2(\Sigma_{\alpha\alpha}, n_\alpha - 1) \quad (5.1.8)$$

where

$$\begin{aligned} \bar{Y}_\alpha^0 &= \frac{1}{n_\alpha} \sum_{j=1}^{n_\alpha} Y_{\alpha_j}^0, \quad \bar{X}_\alpha^0 = \frac{1}{n_\alpha} \sum_{j=1}^{n_\alpha} X_{\alpha_j}^0 \\ S_{X_\alpha^0 X_\alpha^0} &= \sum_{j=1}^{n_\alpha} (X_{\alpha_j}^0 - \bar{X}_\alpha^0)^2, \quad S_{Y_\alpha^0 Y_\alpha^0} = \sum_{j=1}^{n_\alpha} (Y_{\alpha_j}^0 - \bar{Y}_\alpha^0)^2 \quad \text{and} \\ S_{X_\alpha^0 Y_\alpha^0} &= \sum_{j=1}^{n_\alpha} (X_{\alpha_j}^0 - \bar{X}_\alpha^0)(Y_{\alpha_j}^0 - \bar{Y}_\alpha^0) \end{aligned}$$

Here $W_2(\Sigma_{\alpha\alpha}, n_\alpha - 1)$ stands for the Wishart distribution which is independent of the distribution of $(\bar{Y}_\alpha^0, \bar{X}_\alpha^0)'$. Thus we have

$$E\left(\frac{1}{n_\alpha - 1} \mathbf{S}_{\alpha\alpha}\right) = \Sigma_{\alpha\alpha} \quad (5.1.9)$$

which means that the unbiased estimators of the element of $\Sigma_{\alpha\alpha}$ are the corresponding elements of $\frac{1}{n_\alpha-1}\mathbf{S}_{\alpha\alpha}$. Hence, $E(\frac{1}{n_\alpha-1}S_{X_\alpha^0 X_\alpha^0}) = \sigma_{X_\alpha^0 X_\alpha^0}$.

Our basic problem is the estimation of $\beta = (\beta_1, \dots, \beta_p)'$ when it is suspected that the lines may be parallel i.e.

$$H_0 : \beta = \beta_0 \mathbf{1}_p, \quad (5.1.10)$$

where β_0 is a scalar and $\mathbf{1}_p = (1, \dots, 1)'$ — a p -tuple of 1's. Toward this goal, we propose several estimators of β and study their properties.

Notice that there are 6 parameters of the linear models (5.1.1) and there are many different configurations that lead to the same distributions of the observation models and is not identifiable. Therefore, we need to impose conditions on the parameters to overcome the identifiability problem and obtain consistent estimators. For this reason we consider the following conditions:

- (i) $\mu_{x_\alpha} = (\mu_{x_1}, \dots, \mu_{x_p})'$ for all $\alpha = 1, \dots, p$,
 - (ii) $\sigma_{x_\alpha x_\alpha} = \sigma_{xx}$ for all $\alpha = 1, \dots, p$,
 - (iii) $\sigma_{u_\alpha u_\alpha} = \sigma_{uu}$ for all $\alpha = 1, \dots, p$,
 - (iv) the reliability ratio, κ_{xx} known.
- (5.1.11)

5.2 Estimation of Parameters in Parallelism Model

Consider the model (5.1.1) and the assumptions that (i) $\mu_x = (\mu_{x_1}, \dots, \mu_{x_p})'$, (ii) $\sigma_{x_\alpha x_\alpha} = \sigma_{xx}$, (iii) $\sigma_{u_\alpha u_\alpha} = \sigma_{uu}$ for all $\alpha = 1, \dots, p$ and (iv) κ_{xx} known. Then we

notice that

$$\begin{pmatrix} \bar{Y}_\alpha^0 \\ \bar{X}_\alpha^0 \end{pmatrix} \sim \mathcal{N}_2 \left\{ \begin{pmatrix} \theta_\alpha + \beta_\alpha \mu_{x_\alpha} \\ \mu_{x_\alpha} \end{pmatrix} : \frac{1}{n_\alpha} \Sigma_\alpha \right\} \quad (5.2.1)$$

and

$$\mathbf{S}_{\alpha\alpha} = \begin{pmatrix} S_{Y_\alpha^0 Y_\alpha^0} & S_{X_\alpha^0 Y_\alpha^0} \\ S_{X_\alpha^0 Y_\alpha^0} & S_{X_\alpha^0 X_\alpha^0} \end{pmatrix} \sim W_2(\Sigma_\alpha, n_\alpha - 1) \quad (5.2.2)$$

where

$$\Sigma_\alpha = \begin{pmatrix} \beta_\alpha^2 \sigma_{xx} + \sigma_{ee} & \beta_\alpha \sigma_{xx} \\ \beta_\alpha \sigma_{xx} & \sigma_{xx} + \sigma_{uu} \end{pmatrix}. \quad (5.2.3)$$

This allows us to estimate $\sigma_{X^0 X^0} = \sigma_{xx} + \sigma_{uu}$ by pooling the $S_{X_\alpha^0 X_\alpha^0}$'s, i.e. by $S_{X^0 X^0}^0 = \sum_{\alpha=1}^p S_{X_\alpha^0 X_\alpha^0}$ since

$$E\left(\frac{S_{X^0 X^0}^0}{n-p}\right) = \sigma_{X^0 X^0}. \quad (5.2.4)$$

Clearly, $\frac{S_{X^0 X^0}^0}{\sigma_{X^0 X^0}}$ follows the central chi-square distribution with $n-p$ degree of freedom (d.f.).

We obtain the unrestricted estimator of θ and β via the conditional model i.e. $(Y_{\alpha_j}^0 | X_{\alpha_j}^0)$ given by

$$Y_{\alpha_j}^0 = \nu_{\alpha_0} + \nu_{\alpha_1} X_{\alpha_j}^0 + Z_{\alpha_j}; \quad \alpha = 1, \dots, p, \quad (5.2.5)$$

where $Z_{\alpha_j} \sim \mathcal{N}(0, \sigma_{zz}^{(\alpha)})$ where $\sigma_{zz}^{(\alpha)} = \sigma_{ee} + \beta_\alpha \sigma_{xx}(1 - \kappa_{xx})$. This model is exactly similar to the standard model except that (i) ν_{α_0} is a translation of θ_α , (ii) ν_{α_1} is a scaled version of β_α and (iii) $\sigma_{zz}^{(\alpha)}$ is the inflated version of σ_{ee} . The estimator of ν_{α_0} and ν_{α_1} conditionally on $(\bar{X}_\alpha, \mathbf{S}_{\alpha\alpha})$ are given by

$$\begin{aligned} \tilde{\nu}_{\alpha_0} &= \bar{Y}_\alpha - \bar{X}_\alpha \tilde{\nu}_{\alpha_1} \\ \tilde{\nu}_{\alpha_1} &= \frac{S_{X_\alpha^0 Y_\alpha^0}}{S_{X_\alpha^0 X_\alpha^0}}. \end{aligned} \quad (5.2.6)$$

Clearly,

$$\begin{Bmatrix} (\tilde{\nu}_{\alpha 0} - \nu_{\alpha 0}) \\ (\tilde{\nu}_{\alpha 1} - \nu_{\alpha 1}) \end{Bmatrix} \sim \mathcal{N}_2 \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} : \Sigma^{(\alpha)} \right) \quad (5.2.7)$$

where

$$\Sigma^{(\alpha)} = \sigma_{zz}^{(\alpha)} \begin{pmatrix} \frac{1}{n_\alpha} + \frac{\bar{X}_\alpha^{02}}{S_{X_\alpha^0 X_\alpha^0}} & -\frac{\bar{X}_\alpha^0}{S_{X_\alpha^0 X_\alpha^0}} \\ -\frac{\bar{X}_\alpha^0}{S_{X_\alpha^0 X_\alpha^0}} & \frac{1}{S_{X_\alpha^0 X_\alpha^0}} \end{pmatrix}. \quad (5.2.8)$$

Defining $\tilde{\theta}_\alpha = \bar{Y}_\alpha^0 - \bar{X}_\alpha^0 \tilde{\beta}_\alpha$ and $\tilde{\beta}_\alpha = \kappa_{xx}^{-1} \tilde{\nu}_{\alpha 1}$, we obtain

$$\begin{Bmatrix} (\tilde{\theta}_\alpha - \theta_\alpha) \\ (\tilde{\beta}_\alpha - \beta_\alpha) \end{Bmatrix} \sim \mathcal{N}_2 \left(\begin{pmatrix} -(1 - \kappa_{xx})(\bar{X}_\alpha^0 - \mu_{x_\alpha})\beta_\alpha \\ 0 \end{pmatrix} : \Sigma_{\kappa_{xx}}^{(\alpha)} \right) \quad (5.2.9)$$

where

$$\Sigma_{\kappa_{xx}}^{(\alpha)} = \sigma_{zz}^{(\alpha)} \begin{pmatrix} \frac{1}{n_\alpha} + \frac{\bar{X}_\alpha^{02}}{\kappa_{xx}^2 S_{X_\alpha^0 X_\alpha^0}} & -\frac{\bar{X}_\alpha^0}{\kappa_{xx}^2 S_{X_\alpha^0 X_\alpha^0}} \\ -\frac{\bar{X}_\alpha^0}{\kappa_{xx}^2 S_{X_\alpha^0 X_\alpha^0}} & \frac{1}{\kappa_{xx}^2 S_{X_\alpha^0 X_\alpha^0}} \end{pmatrix}. \quad (5.2.10)$$

Further, the unrestricted estimators of $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ are given by the LSE/MLE as follows:

$$\begin{aligned} \tilde{\boldsymbol{\theta}}_n &= (\tilde{\theta}_1, \dots, \tilde{\theta}_p)' = \bar{\mathbf{Y}} - \mathbf{T}_{\bar{\mathbf{X}}} \tilde{\boldsymbol{\beta}}_n \\ \tilde{\boldsymbol{\beta}}_n &= (\tilde{\beta}_1, \dots, \tilde{\beta}_p)' = \kappa_{xx}^{-1} \mathbf{S}_{X^0 X^0}^{-1} \mathbf{S}_{X^0 Y^0} \end{aligned} \quad (5.2.11)$$

where

- (i) $\bar{\mathbf{Y}} = (\bar{Y}_1^0, \dots, \bar{Y}_p^0)'$
- (ii) $\mathbf{T}_{\bar{\mathbf{X}}} = \text{Diag}(\bar{X}_1^0, \dots, \bar{X}_p^0)$
- (iii) $\mathbf{S}_{X^0 X^0} = \text{Diag}(S_{X_1^0 X_1^0}, \dots, S_{X_p^0 X_p^0})$

$$(iv) \mathbf{S}_{X^0 Y^0} = (S_{X_1^0 Y_1^0}, \dots, S_{X_p^0 Y_p^0})'$$

Finally, the unbiased estimator of $\sigma_{zz}^{(\alpha)}$ is given by

$$s_{z\alpha}^2 = (n_\alpha - 2)^{-1} \left\| \mathbf{Y}_\alpha^0 - \tilde{\nu}_{\alpha 0} - \tilde{\nu}_{\alpha 1} \mathbf{X}_\alpha^0 \right\|^2 \quad (5.2.12)$$

The conditional distribution of $\left\{ (\tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta})', (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta})' \right\}'$ is given by

$$\begin{pmatrix} \tilde{\boldsymbol{\theta}}_n - \boldsymbol{\theta} \\ \tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta} \end{pmatrix} \sim \mathcal{N}_{2p} \left\{ \begin{pmatrix} -\beta_0(1 - \kappa_{xx})(\mathbf{T}_{\bar{\mathbf{X}}} - \mathbf{T}_{\boldsymbol{\mu}})\mathbf{1}_p \\ \mathbf{0} \end{pmatrix}; \mathbf{D} \right\} \quad (5.2.13)$$

where

$$\begin{aligned} (i) \quad \mathbf{D} &= \begin{pmatrix} \mathbf{D}_{11} & -\mathbf{D}_{12} \\ -\mathbf{D}_{21} & \mathbf{D}_{22} \end{pmatrix}, \\ (ii) \quad \mathbf{D}_{11} &= \text{Diag} \left\{ \left(\frac{1}{n_1} + \frac{\bar{X}_1^{02}}{\kappa_{xx}^2 S_{X_1^0 X_1^0}} \right) \sigma_{zz}^{(1)}, \dots, \left(\frac{1}{n_p} + \frac{\bar{X}_p^{02}}{\kappa_{xx}^2 S_{X_p^0 X_p^0}} \right) \sigma_{zz}^{(p)} \right\} \\ \mathbf{D}_{22} &= \text{Diag} \left\{ \frac{\sigma_{zz}^{(1)}}{\kappa_{xx}^2 S_{X_1^0 X_1^0}}, \dots, \frac{\sigma_{zz}^{(p)}}{\kappa_{xx}^2 S_{X_p^0 X_p^0}} \right\} \\ (iii) \quad \mathbf{D}_{12} &= \mathbf{D}_{21}' = \text{Diag} \left\{ \frac{\bar{X}_1^0 \sigma_{zz}^{(1)}}{\kappa_{xx}^2 S_{X_1^0 X_1^0}}, \dots, \frac{\bar{X}_p^0 \sigma_{zz}^{(p)}}{\kappa_{xx}^2 S_{X_p^0 X_p^0}} \right\} \end{aligned} \quad (5.2.14)$$

In case, the slopes are equal i.e. $\boldsymbol{\beta} = \beta_0 \mathbf{1}_p$ we estimate the common conditional variance $\sigma_{zz}^{(0)} = \sigma_{ee} + \beta_0^2 \sigma_{xx}(1 - \kappa_{xx})$ by $\hat{\sigma}_{zz}^{(0)}$ defined by

$$\hat{\sigma}_{zz}^{(0)} = (n - 2p)^{-1} \sum_{\alpha=1}^p (n_\alpha - 2) s_{z\alpha}^2 \quad (5.2.15)$$

where $s_{z\alpha}^2$ is given at (5.2.12). We define

$$\mathbf{A}_n = \frac{\mathbf{1}_p \mathbf{1}_p' \mathbf{S}_{X^0 X^0}}{\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p} \quad \text{and} \quad \mathbf{H}_n = \mathbf{I}_p - \mathbf{A}_n. \quad (5.2.16)$$

Then we have the restricted estimators (RE) of $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ as given by (a) and (b) respectively.

$$(a) \quad \hat{\boldsymbol{\theta}}_n = \tilde{\boldsymbol{\theta}}_n + \mathbf{T}_{\bar{\mathbf{X}}} \mathbf{H}_n \tilde{\boldsymbol{\beta}}_n \quad \text{and} \quad (b) \quad \hat{\boldsymbol{\beta}}_n = \mathbf{A}_n \tilde{\boldsymbol{\beta}}_n. \quad (5.2.17)$$

Now we state the following theorem.

Theorem 5.2.1 Under $H_0 : \beta = \beta_0 \mathbf{1}_p$ and assumed conditions,

$$\begin{pmatrix} \hat{\theta}_n - \theta \\ \tilde{\beta}_n - \beta \end{pmatrix} \sim \mathcal{N}_{2p} \left\{ \begin{pmatrix} -\beta_0(1 - \kappa_{xx})(\mathbf{T}_{\bar{\mathbf{X}}} - \mathbf{T}_{\mu})\mathbf{1}_p \\ \mathbf{0} \end{pmatrix}; \sigma_{zz}^{(0)} \begin{pmatrix} \mathbf{D}_{11}^* & \mathbf{D}_{12}^* \\ \mathbf{D}_{21}^* & \mathbf{D}_{22}^* \end{pmatrix} \right\} \quad (5.2.18)$$

where

$$\begin{aligned} \mathbf{D}_{11}^* &= \mathbf{N}^{-1} + \frac{\mathbf{T}_{\bar{\mathbf{X}}} \mathbf{1}_p \mathbf{1}_p' \mathbf{T}_{\bar{\mathbf{X}}}}{\kappa_{xx}^2 (\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p)} \\ \mathbf{D}_{12}^* &= \mathbf{D}_{21}^{*'} = -\frac{\mathbf{T}_{\bar{\mathbf{X}}} \mathbf{1}_p \mathbf{1}_p'}{\kappa_{xx}^2 (\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p)} \\ \mathbf{D}_{22}^* &= \frac{\mathbf{S}_{X^0 X^0}^{-1}}{\kappa_{xx}^2} \end{aligned} \quad (5.2.19)$$

Proof. Under H_0 ,

$$\begin{aligned} E[\hat{\theta}_n - \theta] &= E[\hat{\theta}_n - \theta + \mathbf{T}_{\bar{\mathbf{X}}} \mathbf{H}_n \tilde{\beta}_n] \\ &= -\beta_0(1 - \kappa_{xx})(\mathbf{T}_{\bar{\mathbf{X}}} - \mathbf{T}_{\mu})\mathbf{1}_p + \mathbf{T}_{\bar{\mathbf{X}}} \mathbf{H}_n \beta_0 \mathbf{1}_p \\ &= -\beta_0(1 - \kappa_{xx})(\mathbf{T}_{\bar{\mathbf{X}}} - \mathbf{T}_{\mu})\mathbf{1}_p \end{aligned} \quad (5.2.20)$$

since $\mathbf{H}_n \mathbf{1}_p = \mathbf{0}$.

$$\begin{aligned} \text{cov}[\hat{\theta}_n - \theta, \tilde{\beta}_n - \beta] &= \text{cov}[\tilde{\theta}_n - \theta + \mathbf{T}_{\bar{\mathbf{X}}} \mathbf{H}_n \tilde{\beta}_n, \tilde{\beta}_n - \beta] \\ &= \text{cov}[\tilde{\theta}_n - \theta] + \mathbf{T}_{\bar{\mathbf{X}}} \mathbf{H}_n \text{var } \tilde{\beta}_n \\ &= -\sigma_{zz}^{(0)} \mathbf{D}_{12} + \mathbf{T}_{\bar{\mathbf{X}}} \mathbf{H}_n \mathbf{D}_{22} \\ &= -\sigma_{zz}^{(0)} \mathbf{T}_{\bar{\mathbf{X}}} \mathbf{A}_n \mathbf{D}_{22} = \sigma_{zz}^{(0)} \mathbf{D}_{12}^* \end{aligned} \quad (5.2.21)$$

5.3 Test of Parallelism

For the distribution of $\{(\tilde{\theta}_n - \theta)', (\tilde{\beta}_n - \beta)'\}$ under H_0 , we replace $\sigma_{zz}^{(\alpha)}$ by $\sigma_{zz}^{(0)}$ in (5.2.15) for all $\alpha = 1, \dots, p$. Now consider the test of the null hypothesis $H_0 : \beta = \beta_0 \mathbf{1}_p$. The conditional likelihood ratio test is given by

$$\mathcal{L}_n = \frac{\kappa_{xx}^2 \tilde{\beta}_n' \mathbf{H}_n' \mathbf{S}_{X^0 X^0} \mathbf{H}_n \tilde{\beta}_n}{(p-1) \hat{\sigma}_{zz}^{(0)}}. \quad (5.3.1)$$

Under H_0 , $\mathcal{L}_n$ follows the central F -distribution with $(p-1, n-2p)$ d.f..

To see this, consider the orthogonal matrix $\Gamma = (\Gamma_1, \Gamma_2)$, where Γ_1 is a $p \times (p-1)$ matrix and Γ_2 is a p -vector such that $\Gamma_2' \Gamma_1 = \mathbf{0}$ and $\Gamma_1 \Gamma_1' + \Gamma_2 \Gamma_2' = \mathbf{I}_p$ so that $\Gamma_1 \Gamma_1' = \mathbf{I}_p - \Gamma_2 \Gamma_2'$. Further let us choose $\Gamma_2 = \frac{\mathbf{S}_{X^0 X^0}^{1/2} \mathbf{1}_p}{\sqrt{\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p}}$, then $\Gamma_1 \Gamma_1' = \mathbf{I}_p - \frac{\mathbf{S}_{X^0 X^0}^{1/2} \mathbf{1}_p \mathbf{1}_p' \mathbf{S}_{X^0 X^0}^{1/2}}{\mathbf{1}_p' \mathbf{S}_{X^0 X^0} \mathbf{1}_p}$, which implies that $\mathbf{H}_n = \mathbf{S}_{X^0 X^0}^{-1/2} \Gamma_1 \Gamma_1' \mathbf{S}_{X^0 X^0}^{1/2}$. Now define the random variable $\mathcal{Z} = \frac{\kappa_{xx}}{\sqrt{\sigma_{zz}^{(*)}}} \Gamma_1' \mathbf{S}_{X^0 X^0}^{1/2} \tilde{\beta}_n \sim \mathcal{N}(\frac{\kappa_{xx}}{\sqrt{\sigma_{zz}^{(*)}}} \Gamma_1' \mathbf{S}_{X^0 X^0}^{1/2} \beta, \mathbf{I}_{p-1})$, where $\sigma_{zz}^{(*)} = \frac{1}{p} \sum_{\alpha=1}^p \sigma_{zz}^{(\alpha)}$. Hence $\|\mathcal{Z}\|^2$ follows the noncentral chi-square distribution with $p-1$ d.f. with noncentrality parameter $\frac{1}{2} \Delta^2$, with $\Delta^2 = \frac{\kappa_{xx}^2 \beta' \mathbf{H}_n \mathbf{S}_{X^0 X^0} \mathbf{H}_n \beta}{\sigma_{zz}^{(*)}}$. Therefore, under $H_0 : \beta = \beta_0 \mathbf{1}_p$, $\mathcal{L}_n$ follows the central F -distribution with $(p-1, n-2p)$ d.f. since $\hat{\sigma}_{zz}^{(0)} \sim \sigma_{zz}^{(*)} \chi_{n-2p}^2 = \sigma_{zz}^{(0)} \chi_{n-2p}^2$.

5.4 Estimators of Slopes

We may now define a class of estimators, β_n^* of the form

$$\beta_n^* = \mathbf{A}_n \tilde{\beta}_n + (\mathbf{I}_p - \mathbf{A}_n) \tilde{\beta}_n g(\mathcal{L}_n), \quad (5.4.1)$$

where $\mathbf{A}_n$ is an idempotent matrix of rank 1 and $g(\mathcal{L}_n)$ is a non-decreasing function of the test-statistic $\mathcal{L}_n$ for testing the $H_0 : \beta = \beta_0 \mathbf{1}_p$ against the $H_A : \beta \neq \beta_0 \mathbf{1}_p$. Accordingly, we have the preliminary test estimators of β are as follows:

$$\hat{\beta}_n^{PT} = \tilde{\beta}_n - \mathbf{H}_n \tilde{\beta}_n I(\mathcal{L}_n < \mathcal{L}_{n,\alpha}) \quad (5.4.2)$$

where $\mathcal{L}_{n,\alpha}$ is the α significance level of the distribution of $\mathcal{L}_n$. To overcome the dependence of the estimators of β , namely,

$$\hat{\beta}_n^S = \tilde{\beta}_n - c \mathbf{H}_n \tilde{\beta}_n \mathcal{L}_n^{-1} \quad (5.4.3)$$

where $c = \frac{(p-3)m}{(p-1)(m+2)}$. Notice that we have replaced $I(\mathcal{L}_n < \mathcal{L}_{n,\alpha})$ by $c\mathcal{L}_n^{-1}$ in (5.4.2) to obtain (5.4.3) and impose the condition that $p \geq 4$. The estimator $\hat{\beta}_n^S$ may go past the values $\hat{\beta}_n$. Thus we consider the convex combinations of $\hat{\beta}_n$ and $\hat{\beta}_n^S$ via preliminary test procedure with critical value c . And we obtain the positive-rule Stein-type estimators of β respectively as

$$\begin{aligned} \hat{\beta}_n^{S+} &= \hat{\beta}_n I(\mathcal{L}_n < c) + \hat{\beta}_n^S I(\mathcal{L}_n > c) \\ &= \hat{\beta}_n + (1 - c\mathcal{L}_n^{-1}) I(\mathcal{L}_n > c) \mathbf{H}_n \tilde{\beta}_n \end{aligned} \quad (5.4.4)$$

5.5 Asymptotic Properties of Estimators under Local Alternatives

In order to study the properties of the five estimators, we need the distribution of the statistic, $\mathcal{L}_n$ under the alternative $H_A : \beta \neq \beta_0 \mathbf{1}_p$. Now, under any alternative, the exact distribution of $\mathcal{L}_n$ depends on the exact distribution of $(n - 2p)\hat{\sigma}_{zz}^{(0)}$ which is a weighted sum of central chi-square variables with $(n_\alpha - 2)$ d.f. for given $\alpha (= 1, \dots, p)$.

i.e. $(n - 2p)\hat{\sigma}_{zz}^{(0)} \stackrel{\mathcal{D}}{=} \sum_{\alpha=1}^p \sigma_{zz}^{(\alpha)} \chi_{n_\alpha-2}^2$ by (5.2.15). Even in the conditional situation, the expressions for bias, MSE-matrices and risks are not available due to the lack of the distribution of $\mathcal{L}_n$. Therefore, we take recourse to the asymptotic methods.

Let $n = n_1 + \dots + n_p$ and $\lim_{n \rightarrow \infty} \frac{n_\alpha}{n} = \lambda_\alpha (0 < \lambda_\alpha < 1)$ so that $\sum_{\alpha=1}^p \lambda_\alpha = 1$. Thus, we easily find the distribution by moment method following Fuller (1978) that as $n \rightarrow \infty$

$$\sqrt{n} \begin{pmatrix} \tilde{\theta}_n - \theta \\ \tilde{\beta}_n - \beta \end{pmatrix} \sim \mathcal{N}_{2p} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} : \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{21} & \Gamma_{22} \end{pmatrix} \right\}, \quad (5.5.1)$$

with

$$\begin{aligned} \Gamma_{11} &= \text{Diag} \left\{ \lambda_1^{-1} \left(\sigma_{v_1 v_1} + \frac{\mu_{x_1}^2 \sigma_{zz}^{(1)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \right), \dots, \lambda_p^{-1} \left(\sigma_{v_p v_p} + \frac{\mu_{x_p}^2 \sigma_{zz}^{(p)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \right) \right\} \\ \Gamma_{22} &= \text{Diag} \left\{ \lambda_1^{-1} \frac{\sigma_{zz}^{(1)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}}, \dots, \lambda_p^{-1} \frac{\sigma_{zz}^{(p)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \right\} \\ \Gamma_{12} &= \Gamma'_{21} = -\text{Diag} \left\{ \lambda_1^{-1} \frac{\mu_{x_1}^2 \sigma_{zz}^{(1)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}}, \dots, \lambda_p^{-1} \frac{\mu_{x_p}^2 \sigma_{zz}^{(p)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \right\} \end{aligned} \quad (5.5.2)$$

where $\sigma_{v_\alpha v_\alpha} = \sigma_{ee} + \beta_\alpha^2 \sigma_{uu}$, $\alpha = 1, \dots, p$.

Further, we consider a class $\{K_{(n)}\}$ of Pitman local alternatives

$$K_{(n)} : \beta_{(n)} = \beta_0 \mathbf{1}_p + n^{-\frac{1}{2}} \delta, \quad \delta \text{ fixed finite vector.} \quad (5.5.3)$$

Note that when $\delta = \mathbf{0}$ we deal with the null hypothesis H_0 . Then we obtain the following theorem.

Theorem 5.5.1 Under $\{K_{(n)}\}$ and the assumed conditions, as $n \rightarrow \infty$,

$$(a) \quad \sqrt{n} \begin{pmatrix} \tilde{\theta}_n - \theta \\ \tilde{\beta}_n - \beta \end{pmatrix} \sim \mathcal{N}_{2p} \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} \Gamma_{11}^* & \Gamma_{12}^* \\ \Gamma_{21}^* & \Gamma_{22}^* \end{pmatrix} \right\}$$

with

$$\begin{aligned} \Gamma_{11}^* &= \sigma_{vv}^{(0)} \Lambda_0^{-1} + \sigma_{zz}^{(0)} \frac{\mathbf{T} \boldsymbol{\mu} \Lambda_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \\ \Gamma_{12}^* &= \Gamma_{21}^* = -\sigma_{zz}^{(0)} \frac{\mathbf{T} \boldsymbol{\mu} \Lambda_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \\ \Gamma_{22}^* &= \sigma_{zz}^{(0)} \frac{\Lambda_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \end{aligned} \quad (5.5.4)$$

$$(b) \quad \sqrt{n} \begin{pmatrix} \hat{\theta}_n - \theta \\ \tilde{\beta}_n - \beta \end{pmatrix} \sim \mathcal{N}_{2p} \left\{ \begin{pmatrix} \mathbf{T} \boldsymbol{\mu} \mathbf{J} \delta \\ \delta \end{pmatrix}; \begin{pmatrix} \Gamma'_{11} & \Gamma'_{12} \\ \Gamma'_{21} & \Gamma_{22}^* \end{pmatrix} \right\}$$

with

$$\begin{aligned} \Gamma'_{11} &= \sigma_{vv}^{(0)} \Lambda_0^{-1} + \sigma_{zz}^{(0)} \frac{\mathbf{T} \boldsymbol{\mu} \mathbf{1}_p \mathbf{1}_p' \mathbf{T} \boldsymbol{\mu}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \\ \Gamma'_{12} &= \Gamma'_{21} = -\sigma_{zz}^{(0)} \frac{\mathbf{T} \boldsymbol{\mu} \mathbf{1}_p \mathbf{1}_p'}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \\ \Gamma_{22}^* &= \sigma_{zz}^{(0)} \frac{\Lambda_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \end{aligned}$$

$$(c) \quad \mathcal{L}_n = \frac{\kappa_{xx}^2 \tilde{\beta}_n' \mathbf{H}_n' \mathbf{S}_{X^0 X^0} \mathbf{H}_n \tilde{\beta}_n}{(p-1) \hat{\sigma}_{zz}^{(0)}}$$

has asymptotically a noncentral chis-quared distribution

with $(p-1)$ d.f. and non-centrality parameter $\frac{1}{2} \Delta^2$ with

$$\Delta^2 = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \left(\delta' \mathbf{J}' \Lambda_0 \mathbf{J} \delta \right) \quad (5.5.5)$$

where

$$\sigma_{vv}^{(0)} = \sigma_{ee} + \beta_0^2 \sigma_{uu}, \quad \sigma_{zz}^{(0)} = \sigma_{ee} + \beta_0^2 \sigma_{xx} (1 - \kappa_{xx}),$$

$$\mathbf{T}\boldsymbol{\mu} = \text{Diag}(\mu_{x_1}, \dots, \mu_{x_p}) = \lim_{n \rightarrow \infty} \mathbf{T}_{\tilde{\mathbf{X}}},$$

$$\boldsymbol{\Lambda}_0 = \text{Diag}(\lambda_1, \dots, \lambda_p),$$

$$\lim_{n \rightarrow \infty} \mathbf{H}_n = \mathbf{J} \text{ and } \mathbf{J} = \mathbf{I}_p - \mathbf{1}_p \mathbf{1}_p' \boldsymbol{\Lambda}_0.$$

Proof. For (c)

$$\begin{aligned} \mathcal{L}_n &= \frac{\kappa_{xx}^2 \tilde{\boldsymbol{\beta}}_n' \mathbf{H}_n' \mathbf{S}_{X^0 X^0} \mathbf{H}_n \tilde{\boldsymbol{\beta}}_n}{(p-1) \hat{\sigma}_{zz}^{(0)}} \\ &= \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{(p-1) \sigma_{zz}^{(0)}} \left(n \tilde{\boldsymbol{\beta}}_n' \mathbf{J}' \boldsymbol{\Lambda}_0 \mathbf{J} \tilde{\boldsymbol{\beta}}_n \right) + o_p(1) \end{aligned} \quad (5.5.6)$$

Thus, $\mathcal{L}_n$ follows closely the noncentral chi-square distribution with $p-1$ d.f. as $n \rightarrow \infty$ using Slutsky's theorem since $\sigma_{zz}^{(\alpha)} \rightarrow \sigma_{zz}^{(0)}$, $\mathbf{H}_n \rightarrow \mathbf{J}$, $\frac{1}{n} S_{X_\alpha^0 X_\alpha^0}^0 \rightarrow \sigma_{X^0 X^0}$ and $\frac{1}{n} S_{X_\alpha^0 X_\alpha^0}^0 \rightarrow \lambda_\alpha \sigma_{X^0 X^0}$ in probability as $n \rightarrow \infty$.

Now we consider the asymptotic distributional bias (ADB), quadratic bias (ADQB), MSE matrices (ADMSE) and the quadratic risks (ADQR) of the five estimators. Thus under $\{K_{(n)}\}$ and the regularity conditions of section 1 we may obtain the various related expressions.

5.6 Asymptotic Distributional Bias (ADB), Quadratic Bias (ADQB), Mean-square Error Matrix (ADMSE) and Quadratic Risk (ADQR)

In this section, we consider the bias, quadratic bias, mean-square error matrix and quadratic risks of the four estimators defined in section 4. Thus, we may write the bias-vector for the estimator β_n^* as

$$\begin{aligned} \mathbf{b}^* &= \lim_{n \rightarrow \infty} \sqrt{n} E[\beta_n^* - \beta_{(n)}] = - \lim_{n \rightarrow \infty} \sqrt{n} E[\tilde{\beta}_n g(\mathcal{L}_n)] \\ &= -\delta E[g(\chi_{p+1}^2(\Delta^2))] \end{aligned} \quad (5.6.1)$$

by Appendix B of Judge and Bock (1978) and Saleh (2002) where $\chi_\nu^2(\Delta^2)$ stands for the non-central chi-square variable with ν d.f. and non-centrality parameter $\frac{1}{2}\Delta^2$. To evaluate the properties of the bias we consider the quadratic bias (QB) defined by

$$\mathbf{b}^{*'} \Gamma_{22}^{*-1} \mathbf{b}^* = \Delta^2 \left\{ E[g(\chi_{p+1}^2(\Delta^2))] \right\}^2 \quad (5.6.2)$$

Thus, QB of $\mathbf{b}^*$ may be studied as a function of Δ^2 , the non-centrality parameter of the chi-square distribution.

To obtain the mean-square-error (MSE) matrix of the estimator β_n^* we have to

evaluate

$$\begin{aligned}
\mathbf{M}^* &= \lim_{n \rightarrow \infty} nE \left[(\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)}) (\boldsymbol{\beta}_n^* - \boldsymbol{\beta}_{(n)})' \right] \\
&= \lim_{n \rightarrow \infty} nE \left[\mathbf{A}_n (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)}) (\tilde{\boldsymbol{\beta}}_n - \boldsymbol{\beta}_{(n)})' \mathbf{A}_n' \right] \\
&\quad + \lim_{n \rightarrow \infty} nE \left[\mathbf{H}_n \tilde{\boldsymbol{\beta}}_n \tilde{\boldsymbol{\beta}}_n' \mathbf{H}_n' g^2(\mathcal{L}_n) \right] \\
&\quad - \lim_{n \rightarrow \infty} nE \left[\mathbf{H}_n \tilde{\boldsymbol{\beta}}_n g(\mathcal{L}_n) \boldsymbol{\beta}' \mathbf{H}_n' \right] \\
&\quad - \lim_{n \rightarrow \infty} nE \left[\mathbf{H}_n \boldsymbol{\beta} \tilde{\boldsymbol{\beta}}_n' \mathbf{H}_n' g(\mathcal{L}_n) \right] \\
&\quad + \lim_{n \rightarrow \infty} n \left[\mathbf{H}_n \boldsymbol{\beta} \boldsymbol{\beta}' \mathbf{H}_n' \right]
\end{aligned} \tag{5.6.3}$$

Now, using Appendix B of Judge and Bock(1978) and Chapter 2 of Saleh(2002), we get

$$\lim_{n \rightarrow \infty} nE \left[\mathbf{H}_n \tilde{\boldsymbol{\beta}}_n g(\mathcal{L}_n) \right] = \mathbf{J} \boldsymbol{\delta} E \left[g(\chi_{p+1}^2(\Delta^2)) \right] \tag{5.6.4}$$

and

$$\begin{aligned}
\lim_{n \rightarrow \infty} nE \left[\mathbf{H}_n \tilde{\boldsymbol{\beta}}_n \tilde{\boldsymbol{\beta}}_n' \mathbf{H}_n' g^2(\mathcal{L}_n) \right] &= \mathbf{J} \boldsymbol{\Gamma}_{22}^* \mathbf{J}' E \left[g^2(\chi_{p+1}^2(\Delta^2)) \right] \\
&\quad + \mathbf{J} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{J}' E \left[g^2(\chi_{p+3}^2(\Delta^2)) \right]
\end{aligned} \tag{5.6.5}$$

Similarly,

$$\begin{aligned}
\lim_{n \rightarrow \infty} nE \left[\mathbf{H}_n \tilde{\boldsymbol{\beta}}_n \tilde{\boldsymbol{\beta}}_n' \mathbf{H}_n' g(\mathcal{L}_n) \right] &= \mathbf{J} \boldsymbol{\Gamma}_{22}^* \mathbf{J}' E \left[g(\chi_{p+1}^2(\Delta^2)) \right] \\
&\quad + \mathbf{J} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{J}' E \left[g(\chi_{p+3}^2(\Delta^2)) \right]
\end{aligned} \tag{5.6.6}$$

Finally, we substitute the expressions in (5.6.5) and (5.6.6) to

(5.6.3) to get

$$\begin{aligned}
\mathbf{M}^* &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \mathbf{1}_p \mathbf{1}_p' + \mathbf{J} \boldsymbol{\Gamma}_{22}^* \mathbf{J}' E \left[g^2(\chi_{p+1}^2(\Delta^2)) \right] \\
&\quad + \mathbf{J} \boldsymbol{\delta} \boldsymbol{\delta}' \mathbf{J}' \left[1 - \left\{ 2E \left[g(\chi_{p+1}^2(\Delta^2)) \right] - E \left[g^2(\chi_{p+3}^2(\Delta^2)) \right] \right\} \right]
\end{aligned} \tag{5.6.7}$$

The quadratic risk under the loss function

$$L(\beta_n^*, \beta) = n(\beta_n^* - \beta_{(n)})' \mathbf{Q} (\beta_n^* - \beta_{(n)}) \quad (5.6.8)$$

with the positive semi-definite matrix $\mathbf{Q}$ is given by

$$\begin{aligned} R^* &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr}(\mathbf{1}_p \mathbf{1}_p' \mathbf{Q}) + \text{tr}(\mathbf{J} \Gamma_{22}^* \mathbf{J}' \mathbf{Q}) E \left[g^2(\chi_{p+1}^2(\Delta^2)) \right] \\ &+ \text{tr}(\mathbf{J} \delta \delta' \mathbf{J}' \mathbf{Q}) \left[1 - \left\{ 2E \left[g(\chi_{p+1}^2(\Delta^2)) \right] - E \left[g^2(\chi_{p+3}^2(\Delta^2)) \right] \right\} \right] \end{aligned} \quad (5.6.9)$$

From the expressions (5.6.1), (5.6.2), (5.6.7) and (5.6.9) we obtain the corresponding bias, quadratic bias, MSE-matrix and the quadratic risks of UE, PTE, SE and PRSE's as follows.

5.6.1 ADB and ADQB Expressions and Comparisons

ADB and ADQB of $\tilde{\theta}_n$ and $\tilde{\beta}_n$

$$\begin{aligned} \zeta_1(\tilde{\beta}_n) &= \lim_{n \rightarrow \infty} E \left\{ \sqrt{n}(\tilde{\beta}_n - \beta) | K_{(n)} \right\} = \mathbf{0}, \\ \text{and } B_1(\tilde{\beta}_n) &= 0 \end{aligned} \quad (5.6.10)$$

ADB and ADQB of $\hat{\theta}_n$ and $\hat{\beta}_n$

$$\begin{aligned} \zeta_2(\hat{\beta}_n) &= \lim_{n \rightarrow \infty} E \left\{ \sqrt{n}(\hat{\beta}_n - \beta) | K_{(n)} \right\} = -\mathbf{J} \delta, \\ \text{and } B_2(\hat{\beta}_n) &= \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} (\delta' \mathbf{J}' \Lambda_0 \mathbf{J} \delta) = \Delta^2 \end{aligned} \quad (5.6.11)$$

ADB and ADQB of $\hat{\theta}_n^{PT}$ and $\hat{\beta}_n^{PT}$

$$\begin{aligned}
 \zeta_3(\hat{\beta}_n^{PT}) &= \lim_{n \rightarrow \infty} E \left\{ \sqrt{n}(\hat{\beta}_n^{PT} - \beta) | K_{(n)} \right\} \\
 &= -\mathbf{J} \delta H_{p+1} \left(\chi_{p-1}^2(\alpha) : \Delta^2 \right), \\
 \text{and } B_3(\hat{\beta}_n^{PT}) &= \Delta^2 \left\{ H_{p+1} \left(\chi_{p-1}^2(\alpha) : \Delta^2 \right) \right\}^2
 \end{aligned} \tag{5.6.12}$$

ADB and ADQB of $\hat{\theta}_n^S$ and $\hat{\beta}_n^S$

$$\begin{aligned}
 \zeta_4(\hat{\beta}_n^S) &= \lim_{n \rightarrow \infty} E \left\{ \sqrt{n}(\hat{\beta}_n^S - \beta) | K_{(n)} \right\} \\
 &= -(p-3) \mathbf{J} \delta E \left[\chi_{p+1}^{-2}(\Delta^2) \right], \\
 \text{and } B_4(\hat{\beta}_n^S) &= (p-3)^2 \Delta^2 \left\{ E \left[\chi_{p+1}^{-2}(\Delta^2) \right] \right\}^2
 \end{aligned} \tag{5.6.13}$$

ADB and ADQB of $\hat{\theta}_n^{S+}$ and $\hat{\beta}_n^{S+}$

$$\begin{aligned}
 \zeta_5(\hat{\beta}_n^{S+}) &= \lim_{n \rightarrow \infty} E \left\{ \sqrt{n}(\hat{\beta}_n^{S+} - \beta) | K_{(n)} \right\} \\
 &= -\mathbf{J} \delta \left\{ H_{p+1}(p-3; \Delta^2) \right. \\
 &\quad \left. + (p-3) E \left[\chi_{p+1}^{-2}(\Delta^2) I \left(\chi_{p+1}^2(\Delta^2) > p-3 \right) \right] \right\}, \\
 \text{and } B_5(\hat{\beta}_n^{S+}) &= \Delta^2 \left\{ H_{p+1}(p-3; \Delta^2) \right. \\
 &\quad \left. + (p-3) E \left[\chi_{p+1}^{-2}(\Delta^2) I \left(\chi_{p+1}^2(\Delta^2) > p-3 \right) \right] \right\}^2
 \end{aligned} \tag{5.6.14}$$

Bias Comparison

First we consider the bias comparison of the estimators of the slopes. Clearly, under H_0 , all estimators are unbiased. Also, as $n \rightarrow \infty$, all estimators except $\tilde{\beta}_n$

has unbounded bias. When Δ^2 is moderate, the biases satisfy the relation

$$B_1(\tilde{\beta}_n) = 0 < B_3(\hat{\beta}_n^{PT}) < B_2(\hat{\beta}_n)$$

and

$$B_1(\tilde{\beta}_n) = 0 < B_5(\hat{\beta}_n^{S+}) < B_4(\hat{\beta}_n^S) \quad (5.6.15)$$

5.6.2 ADMSE and ADQR Expressions and Comparisons

We may obtain the asymptotic MSE and the weighted risk expressions as follows.

ADMSE and ADQR of $\tilde{\beta}_n$

$$\begin{aligned} M_1(\tilde{\beta}_n) &= \lim_{n \rightarrow \infty} E \left\{ n(\tilde{\beta}_n - \beta)(\tilde{\beta}_n - \beta)' | K_{(n)} \right\} \\ &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \Lambda_0^{-1} \\ R_1(\tilde{\beta}_n) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr} \left\{ \mathbf{Q} \Lambda_0^{-1} \right\} \end{aligned} \quad (5.6.16)$$

ADMSE and ADQR of $\hat{\beta}_n$

$$\begin{aligned} M_2(\hat{\beta}_n) &= \lim_{n \rightarrow \infty} E \left\{ n(\hat{\beta}_n - \beta)(\hat{\beta}_n - \beta)' | K_{(n)} \right\} \\ &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \mathbf{1}_p \mathbf{1}_p' + \mathbf{J} \delta \delta' \mathbf{J}' \\ R_2(\hat{\beta}_n) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr} \left\{ \mathbf{Q} \mathbf{1}_p \mathbf{1}_p' \right\} + \delta' \mathbf{J}' \mathbf{Q} \mathbf{J} \delta \end{aligned} \quad (5.6.17)$$

Risk Analysis

Now consider the risk analysis of the estimators.

The risk-difference of the two estimators is given by

$$\begin{aligned}
& R_1(\tilde{\beta}_n) - R_2(\hat{\beta}_n) \\
&= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr}(\mathbf{Q} \Lambda_0^{-1}) - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr}(\mathbf{Q} \mathbf{1}_p \mathbf{1}_p') - \delta' \mathbf{J}' \mathbf{Q} \mathbf{J} \delta \\
&= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \{ \text{tr}(\mathbf{Q} [\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p']) \} - \delta' \mathbf{J}' \mathbf{Q} \mathbf{J} \delta
\end{aligned} \tag{5.6.18}$$

Hence, $\hat{\beta}_n$ perform better than $\tilde{\beta}_n$ whenever

$$\Delta^2 \leq \frac{\text{tr}\{\mathbf{Q}[\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p']\}}{Ch_{\min}(\mathbf{Q} \Lambda_0^{-1})} \tag{5.6.19}$$

by Courant theorem in Appendix. If $\mathbf{Q} = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \Lambda_0$, then $\hat{\beta}_n$ performs better than $\tilde{\beta}_n$ whenever $\Delta^2 < p - 1$, otherwise $\tilde{\beta}_n$ performs better. The asymptotic risk difference (RRE) of $\hat{\beta}_n$ compared to $\tilde{\beta}_n$ is given by $ARRE(\hat{\beta}_n : \tilde{\beta}_n) = \frac{p}{1+\Delta^2}$ which is a decreasing function of Δ^2 . At $\Delta^2 = 0$, it attains its maximum efficiency p and drops down to zero as $n \rightarrow \infty$. However, the efficiency belongs to the interval $(1, p)$ when $0 < \Delta^2 < p - 1$ and outside this interval $\tilde{\beta}_n$ performs better than $\hat{\beta}_n$.

MSE Analysis

In this case consider the matrix-difference

$$M_1(\tilde{\beta}_n) - M_2(\hat{\beta}_n) = \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \left[\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p' \right] - \mathbf{J} \delta \delta' \mathbf{J}' \tag{5.6.20}$$

This MSE-difference is positive definite whenever for a given non-zero vector $\ell = (\ell_1, \dots, \ell_p)'$ we have

$$\frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \ell' \left[\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p' \right] \ell - \ell' \mathbf{J} \delta \delta' \mathbf{J}' \ell > 0 \tag{5.6.21}$$

That is,

$$\ell' \left[\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p' \right] \ell \geq \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \ell' \mathbf{J} \delta \delta' \mathbf{J}' \ell. \tag{5.6.22}$$

Since $\ell' \Lambda_0^{-1} \ell > 0$, we consider

$$\max_{\ell} \frac{\ell' \mathbf{J} \Lambda_0^{-1} \ell}{\ell' \Lambda_0^{-1} \ell} \geq \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \max_{\ell} \frac{\ell' \mathbf{J} \delta \delta' \mathbf{J}' \ell}{\ell' \Lambda_0^{-1} \ell}$$

or

$$Ch_{\max}[\mathbf{J}] \geq \Delta^2. \quad (5.6.23)$$

Hence, $\hat{\beta}_n$ performs better than $\tilde{\beta}_n$ if $0 \leq \Delta^2 \leq 1$, otherwise $\tilde{\beta}_n$ performs better than $\hat{\beta}_n$.

Note that the determinant of $M_2(\hat{\beta}_n)$ is zero so that it is meaningless to have AMRE's of $\hat{\beta}_n$ to other estimators.

ADMSE and ADQR of $\hat{\beta}_n^{PT}$

$$\begin{aligned} M_3(\hat{\beta}_n^{PT}) &= \lim_{n \rightarrow \infty} E \left\{ n(\hat{\beta}_n^{PT} - \beta)(\hat{\beta}_n^{PT} - \beta)' | K_{(n)} \right\} \\ &= \frac{\sigma_{zz}^{(0)} \Lambda_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} - \frac{\sigma_{zz}^{(0)} \mathbf{J} \Lambda_0^{-1} \mathbf{J}'}{\kappa_{xx}^2 \sigma_{X^0 X^0}} H_{p+1}(\chi_{p-1}^2(\alpha) : \Delta^2) \\ &\quad + \left(\mathbf{J} \delta \delta' \mathbf{J}' \right) \left\{ 2H_{p+1}(\chi_{p-1}^2(\alpha) : \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha) : \Delta^2) \right\} \\ R_3(\hat{\beta}_n^{PT}) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr} \left\{ \mathbf{Q} \Lambda_0^{-1} \right\} \\ &\quad - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr} \left\{ \mathbf{Q} \mathbf{J} \Lambda_0^{-1} \mathbf{J}' \right\} H_{p+1}(\chi_{p-1}^2(\alpha) : \Delta^2) \\ &\quad + \left(\delta' \mathbf{J}' \mathbf{Q} \mathbf{J} \delta \right) \left\{ 2H_{p+1}(\chi_{p-1}^2(\alpha) : \Delta^2) \right. \\ &\quad \left. - H_{p+3}(\chi_{p-1}^2(\alpha) : \Delta^2) \right\} \end{aligned} \quad (5.6.24)$$

Risk Analysis

The risk -difference shows that $\hat{\beta}_n^{PT}$ performs better than $\tilde{\beta}_n$ whenever

$$\Delta^2 \leq \frac{\text{tr} \{ \mathbf{Q} \mathbf{J} \Lambda_0^{-1} \mathbf{J}' \} H_{p+1}(\chi_{p-1}^2(\alpha) : \Delta^2)}{Ch_{\max}[\mathbf{Q} \Lambda_0^{-1}] \{ 2H_{p+1}(\chi_{p-1}^2(\alpha) : \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha) : \Delta^2) \}} \quad (5.6.25)$$

while $\tilde{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$ if

$$\Delta^2 > \frac{\text{tr}\{\mathbf{QJ}\Lambda_0^{-1}\mathbf{J}'\}H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)}{Ch_{\min}[\mathbf{Q}\Lambda_0^{-1}]\{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}} \quad (5.6.26)$$

If $\mathbf{Q} = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \Lambda_0$, then $\hat{\beta}_n^{PT}$ performs better than $\tilde{\beta}_n$

$$\Delta^2 \leq \frac{(p-1)H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)}{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)} \quad (5.6.27)$$

otherwise $\tilde{\beta}_n$ is better than $\hat{\beta}_n^{PT}$. Now consider the risk-difference of $\hat{\beta}_n$ and $\hat{\beta}_n^{PT}$ under H_0 ,

$$\begin{aligned} & R_2(\hat{\beta}_n) - R_3(\hat{\beta}_n^{PT}) \\ &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx} \sigma_{X^0 X^0}} \text{tr}\{(\mathbf{Q}\mathbf{1}_p \mathbf{1}_p') - \mathbf{Q}\Lambda_0^{-1}\} \\ &+ \frac{\sigma_{zz}^{(0)}}{\kappa_{xx} \sigma_{X^0 X^0}} \text{tr}(\mathbf{QJ}\Lambda_0^{-1}\mathbf{J}')H_{p+1}(\chi_{p-1}^2(\alpha); 0) \stackrel{>}{<} 0 \end{aligned} \quad (5.6.28)$$

accordingly as

$$H_{p+1}(\chi_{p-1}^2(\alpha); 0) \stackrel{>}{<} \frac{\text{tr}[\mathbf{Q}(\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p')]}{\text{tr}[\mathbf{QJ}\Lambda_0^{-1}\mathbf{J}']} \quad (5.6.29)$$

Thus $\hat{\beta}_n^{PT}$ performs better than $\hat{\beta}_n$ when

$$H_{p+1}(\chi_{p-1}^2(\alpha); 0) > \frac{\text{tr}[\mathbf{Q}(\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p')]}{\text{tr}[\mathbf{QJ}\Lambda_0^{-1}\mathbf{J}']} \quad (5.6.30)$$

If $\mathbf{Q} = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \Lambda_0$, then $\hat{\beta}_n^{PT}$ performs better than $\hat{\beta}_n$ when

$$H_{p+1}(\chi_{p-1}^2(\alpha); 0) > 1 \quad (5.6.31)$$

otherwise $\hat{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$. Notice that the asymptotic risk of $\hat{\beta}_n$ is unbounded while that of $\hat{\beta}_n^{PT}$ is bounded. In general, $\hat{\beta}_n^{PT}$ performs better than $\hat{\beta}_n$

whenever

$$\Delta^2 > \frac{\text{tr}\{\mathbf{Q}[\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p' - \mathbf{J} \Lambda_0^{-1} \mathbf{J}' H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)]\}}{Ch_{max}[\mathbf{Q} \Lambda_0^{-1}]\{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}} \quad (5.6.32)$$

and $\hat{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$ whenever

$$\Delta^2 \leq \frac{\text{tr}\{\mathbf{Q}[\Lambda_0^{-1} - \mathbf{1}_p \mathbf{1}_p' - \mathbf{J} \Lambda_0^{-1} \mathbf{J}' H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)]\}}{Ch_{min}[\mathbf{Q} \Lambda_0^{-1}]\{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}} \quad (5.6.33)$$

If $\mathbf{Q} = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \Lambda_0$, then $\hat{\beta}_n^{PT}$ performs better than $\hat{\beta}_n$

$$\Delta^2 > \frac{(p-1)[1 - H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)]}{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)} \quad (5.6.34)$$

otherwise $\hat{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$ for any fixed α .

Now consider the asymptotic risk efficiency of $\hat{\beta}_n^{PT}$ relative to $\tilde{\beta}_n$ when $\mathbf{Q} = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \Lambda_0$

$$\begin{aligned} ARRE(\hat{\beta}_n^{PT} : \tilde{\beta}_n) &= \left[1 - \left(1 - \frac{1}{p} \right) H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) \right. \\ &\quad \left. + \frac{1}{p} \Delta^2 \left\{ 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2) \right\} \right]^{-1} \end{aligned} \quad (5.6.35)$$

We note the following properties of $ARRE(\hat{\beta}_n^{PT} : \tilde{\beta}_n)$.

(i) $ARRE(\hat{\beta}_n^{PT} : \tilde{\beta}_n) \geq 1$ according as

$$\Delta^2 \begin{cases} < \\ > \end{cases} \frac{(p-1)H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)}{\{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}} \quad (\leq p-1) \quad (5.6.36)$$

from (5.6.25) and (5.6.26)

(ii) The maximum efficiency is attained at $\Delta^2 = 0$ with the value

$$\left\{ 1 - \left(1 - \frac{1}{p} \right) H_{p+1}(\chi_{p-1}^2(\alpha); 0) \right\}^{-1} \quad (5.6.37)$$

(iii) $ARRE(\hat{\beta}_n^{PT}; \tilde{\beta}_n)$ decreases as a function of Δ^2 for fixed $\alpha \in (0, 1)$ crossing the 1-line to a minimum, say, at $\Delta^2 = \Delta_{\min}^2$, then monotonically increases towards unity as $\Delta^2 \rightarrow \infty$. Since PTE is not an estimator with uniform dominance over $\tilde{\beta}_n$, one may obtain a PTE with minimum guaranteed efficiency, say, E_0 by solving the inequality

$$\max_{\alpha \in \mathcal{A}} \min_{\Delta^2} ARRE(\hat{\beta}_n^{PT}; \tilde{\beta}_n) \geq E_0. \quad (5.6.38)$$

Tables of maximum and minimum efficiency may be obtained as before.

(iv) As $\alpha \rightarrow 0$, $\hat{\beta}_n^{PT} \rightarrow \beta_n$ while $\hat{\beta}_n^{PT} \rightarrow \tilde{\beta}_n$ as $\alpha \rightarrow 1$.

Next, we consider the risk efficiency of $\hat{\beta}_n^{PT}$ relative to $\hat{\beta}_n$ for $\mathbf{Q} = \frac{\kappa_{xx}^2 \sigma_{xx}^0 \chi^0}{\sigma_{zz}^{(0)}} \Lambda_0$ given by

$$ARRE(\hat{\beta}_n^{PT}; \hat{\beta}_n) = p^{-1}(1 + \Delta^2) \left[1 - \left(1 - \frac{1}{p} \right) H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + \Delta^2 p^{-1} \left\{ 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2) \right\} \right]^{-1}. \quad (5.6.39)$$

We note the following properties of $ARRE(\hat{\beta}_n^{PT}; \hat{\beta}_n)$

(i) $ARRE(\hat{\beta}_n^{PT}; \hat{\beta}_n) \geq 1$ according as

$$\Delta^2 \leq \frac{(p-1)\{1 - H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)\}}{\{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}}. \quad (5.6.40)$$

(ii) Under H_0 , the efficiency is given by

$$p^{-1} \left[1 - \left(1 - \frac{1}{p} \right) H_{p+1}(\chi_{p-1}^2(\alpha); 0) \right]^{-1} \quad (\geq p^{-1}) \quad (5.6.41)$$

while that of $\hat{\beta}_n^{PT}$ rel $\tilde{\beta}_n$ is given by

$$\left[1 - \left(1 - \frac{1}{p}\right)H_{p+1}(\chi_{p-1}^2(\alpha); 0)\right]^{-1} \quad (\geq 1) \quad (5.6.42)$$

Thus, under H_0

$$p^{-1} \leq ARRE(\hat{\beta}_n^{PT}; \hat{\beta}_n) \leq ARRE(\hat{\beta}_n^{PT}; \tilde{\beta}_n) \quad (5.6.43)$$

(iii) As $\Delta^2 \rightarrow \infty$, $ARRE(\hat{\beta}_n^{PT}; \hat{\beta}_n) \rightarrow \infty$.

(iv) As $\alpha \rightarrow 0$, $\hat{\beta}_n^{PT} \rightarrow \hat{\beta}_n$ and as $\alpha \rightarrow 1$, $\hat{\beta}_n^{PT} \rightarrow \tilde{\beta}_n$.

MSE Analysis

The MSE matrix difference is given by

$$\begin{aligned} M_1(\tilde{\beta}_n) - M_3(\hat{\beta}_n^{PT}) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \mathbf{J} \Lambda_0^{-1} \mathbf{J}' H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) \\ &\quad - \mathbf{J} \delta \delta' \mathbf{J}' \left\{ 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2) \right\} \end{aligned} \quad (5.6.44)$$

It may be shown by the method in section 5.8.1 that $\hat{\beta}_n^{PT}$ performs better than $\tilde{\beta}_n$ if

$$\Delta^2 < \frac{H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)}{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)} \quad (5.6.45)$$

otherwise, $\tilde{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$. Similarly, we can show that $\hat{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$ if

$$\Delta^2 < \frac{\{1 - H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2)\}}{\{1 - 2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) + H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\}}. \quad (5.6.46)$$

Otherwise, $\hat{\beta}_n^{PT}$ performs better than $\hat{\beta}_n$. However, under H_0 , it is seen that $\hat{\beta}_n$ performs better than $\hat{\beta}_n^{PT}$ whenever $\chi_{p-1}^2(\alpha)$ satisfies the inequality

$$H_{p+1}(\chi_{p-1}^2(\alpha); 0) \leq 1. \quad (5.6.47)$$

Thus, none of the estimators dominates the other.

Now, the MSE based efficiency of $\hat{\beta}_n^{PT}$ relative to $\tilde{\beta}_n$ and β_n are given respectively by

$$\begin{aligned}
 AMRE(\hat{\beta}_n^{PT}; \tilde{\beta}_n) &= \frac{|\mathbf{M}_1(\tilde{\beta}_n)|^{1/p}}{|\mathbf{M}_3(\hat{\beta}_n^{PT})|^{1/p}} \\
 &= \left| \mathbf{I}_p - \mathbf{J}\mathbf{\Lambda}_0^{-1}\mathbf{J}'\mathbf{\Lambda}_0 H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) \right. \\
 &\quad \left. + \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} (\mathbf{J}\delta\delta'\mathbf{J}\mathbf{\Lambda}_0) \{2H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2)\} \right|^{-1/p} \\
 &= \left\{ 1 - H_{p+1}(\chi_{p-1}^2(\alpha); \Delta^2) \right\}^{-(p-1)/p} \\
 &\quad \times \left| 1 + \Delta^2 \left\{ 2H_{p+1,m}(\chi_{p-1}^2(\alpha); \Delta^2) - H_{p+3}(\chi_{p-1}^2(\alpha); \Delta^2) \right\} \right|^{-1/p}, \quad (5.6.48)
 \end{aligned}$$

Note that

(i) Under H_0 ,

$$AMRE(\hat{\beta}_n^{PT}; \tilde{\beta}_n) = \left\{ 1 - H_{p+1}(\chi_{p-1}^2(\alpha); 0) \right\}^{-1}$$

(ii) Under the alternatives, that is, $\Delta^2 > 0$, $AMRE(\hat{\beta}_n^{PT}; \tilde{\beta}_n)$ is decreasing function of Δ^2 for fixed α until $\Delta^2 = \Delta_{\min}^2$, then it increases towards 1 as $\Delta^2 \rightarrow \infty$.

We cannot say the same thing for $MRE(\hat{\beta}_n^{PT}; \hat{\beta}_n)$.

To determine optimum level of significance α to obtain a minimum guaranteed efficiency E_0 we solve the inequality

$$\max_{\alpha} \min_{\Delta^2} AMRE(\hat{\beta}_n^{PT}; \tilde{\beta}_n) \geq E_0. \quad (5.6.49)$$

Some tabular values are given in Table 5.6.1 for various p and n values.

ADMSE and ADQR of $\hat{\beta}_n^S$

$$\begin{aligned}
M_4(\hat{\beta}_n^S) &= \lim_{n \rightarrow \infty} E \left\{ n(\hat{\beta}_n^S - \beta)(\hat{\beta}_n^S - \beta)' | K_{(n)} \right\} \\
&= \frac{\sigma_{zz}^{(0)} \Lambda_0^{-1}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} - \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} (\mathbf{J} \Lambda_0^{-1} \mathbf{J}') \\
&\times \left\{ 2E \left[\chi_{p+1}^{-2}(\Delta^2) \right] - (p-3)E \left[\chi_{p+1}^{-4}(\Delta^2) \right] \right\} \\
&+ (p-3)(p-1) (\mathbf{J} \delta \delta' \mathbf{J}') E \left[\chi_{p+3}^{-4}(\Delta^2) \right] \\
R_4(\hat{\beta}_n^S) &= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr} \left\{ \mathbf{Q} \Lambda_0^{-1} \right\} \\
&- \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \text{tr} \left[\mathbf{Q} (\mathbf{J} \Lambda_0^{-1} \mathbf{J}') \right] \left\{ 2E \left[\chi_{p+1}^{-2}(\Delta^2) \right] \right. \\
&- (p-3)E \left[\chi_{p+1}^{-4}(\Delta^2) \right] \left. \right\} \\
&+ (p-3)(p-1) (\delta' \mathbf{J}' \mathbf{Q} \mathbf{J} \delta) E \left[\chi_{p+3}^{-4}(\Delta^2) \right] \tag{5.6.50}
\end{aligned}$$

Risk Analysis

First note that the asymptotic risk difference

$$\begin{aligned}
&R_1(\tilde{\beta}_n) - R_4(\hat{\beta}_n^S) \\
&= \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx} \sigma_{X^0 X^0}} \left\{ \text{tr} [\mathbf{Q} \mathbf{J} \Lambda_0^{-1} \mathbf{J}'] \left(2E \left[\chi_{p+1}^{-2}(\Delta^2) \right] \right. \right. \\
&- (p-3)E \left[\chi_{p+1}^{-4}(\Delta^2) \right] \left. \right) + (p+1)\Delta^2 E \left[\chi_{p+3}^{-4}(\Delta^2) \right] \left. \right\} \\
&= \frac{(p-3)\sigma_{zz}^{(0)}}{\kappa_{xx} \sigma_{X^0 X^0}} \text{tr} [\mathbf{Q} \mathbf{J} \Lambda_0^{-1} \mathbf{J}'] \left\{ (p-3)E \left[\chi_{p+1}^{-2}(\Delta^2) \right] \right. \\
&+ 2\Delta^2 E \left[\chi_{p+3}^{-4}(\Delta^2) \right] \left(1 - \frac{(p+1)\delta' \mathbf{J}' \mathbf{Q} \mathbf{J} \delta}{2\delta' \mathbf{J}' \Lambda_0 \mathbf{J} \delta} \text{tr} [\mathbf{Q} \mathbf{J} \Lambda_0^{-1} \mathbf{J}'] \right) \left. \right\} \geq 0 \tag{5.6.51}
\end{aligned}$$

for all Δ^2 and $\mathbf{Q}$ matrices which belong to the set

$$\mathcal{Q} = \left\{ \mathbf{Q} : \frac{\text{tr}(\mathbf{Q} \mathbf{J} \Lambda_0 \mathbf{J}')}{Ch_{max}(\mathbf{Q} \Lambda_0^{-1})} \geq \frac{p+1}{2} \right\} \tag{5.6.52}$$

Hence $\hat{\beta}_n^S$ asymptotically dominates $\tilde{\beta}_n$ uniformly for all $\mathbf{Q} \in \mathcal{Q}$.

MSE Analysis

First consider the comparison of $\tilde{\beta}_n$ and $\hat{\beta}_n^S$. In this case, the MSE matrix-difference is given by

$$\begin{aligned} & \mathbf{M}_1(\tilde{\beta}_n) - M_4(\hat{\beta}_n^S) \\ &= \frac{(p-1)\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \mathbf{J} \Lambda_0^{-1} \mathbf{J}' \left\{ 2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)] \right\} \\ & \quad - (p-3)(p-1)(\mathbf{J} \delta \delta' \mathbf{J}') E[\chi_{p+3}^{-4}(\Delta^2)]. \end{aligned} \quad (5.6.53)$$

In order that $\hat{\beta}_n^S$ dominates $\tilde{\beta}_n$ based on MSE matrices, we must show (5.6.54) is non-negative definite. Thus, consider the quadratic form

$$\begin{aligned} & \frac{(p-1)\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \ell' \mathbf{J} \Lambda_0^{-1} \mathbf{J}' \ell \left\{ 2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)] \right\} \\ & \geq (p-3)(p-1) \ell' (\mathbf{J} \delta \delta' \mathbf{J}') \ell E[\chi_{p+3}^{-4}(\Delta^2)]. \end{aligned} \quad (5.6.54)$$

for a given non-zero vector $\ell = (\ell_1, \dots, \ell_p)'$. Thus, dividing by $\ell' \Lambda_0^{-1} \ell (> 0)$, and maximising over all ℓ , we have

$$\begin{aligned} & \max_{\ell} \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \frac{\ell' \mathbf{J} \Lambda_0^{-1} \mathbf{J}' \ell}{\ell' \Lambda_0^{-1} \ell} \left\{ 2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)] \right\} \\ & \geq (p-1) \max_{\ell} \frac{\ell' \mathbf{J} \delta \delta' \mathbf{J}' \ell}{\ell' \Lambda_0^{-1} \ell} E[\chi_{p+3}^{-4}(\Delta^2)] \\ & \Leftrightarrow \left\{ 2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)] \right\} \\ & \geq (p-1) \Delta^2 E[\chi_{p+3}^{-4}(\Delta^2)] \\ & \Leftrightarrow E[\chi_{p+1}^{-2}(\Delta^2)] \geq (p-2) \Delta^2 E[\chi_{p+3}^{-4}(\Delta^2)] \end{aligned} \quad (5.6.55)$$

which is contradiction. $\hat{\beta}_n^S$ doesn't dominate uniformly with respect to the MSE criterion.

The MSE-efficiency of $\hat{\beta}_n^S$ compared to $\tilde{\beta}_n$ is given by

$$\begin{aligned}
AMRE(\hat{\beta}_n^S; \tilde{\beta}_n) &= \frac{|\mathbf{M}_1(\tilde{\beta}_n)|^{1/p}}{|\mathbf{M}_4(\hat{\beta}_n^S)|^{1/p}} \\
&= \left| \mathbf{I}_p - (p-3)\mathbf{J}\mathbf{\Lambda}_0^{-1}\mathbf{J}'\mathbf{\Lambda}_0 \left\{ 2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)] \right\} \right. \\
&\quad \left. + (p-1)(p-3) \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^0} (\mathbf{J}\delta\delta'\mathbf{J}\mathbf{\Lambda}_0) E[\chi_{p+3}^{-4}(\Delta^2)] \right|^{-1/p} \\
&= \left[1 - (p-1)(p-3) \left\{ 2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)] \right\} \Delta^2 \right]^{-1/p} \\
&\times \left[1 - (p-3) \left\{ 2E[\chi_{p+1}^{-2}(\Delta^2)] - (p-3)E[\chi_{p+1}^{-4}(\Delta^2)] \right\} \right]^{-(p-1)/p}
\end{aligned} \tag{5.6.56}$$

Under H_0 , the $AMRE(\hat{\beta}_n^S; \tilde{\beta}_n)$ becomes

$$AMRE(\hat{\beta}_n^S; \tilde{\beta}_n) = \left\{ 1 - \frac{p-3}{p-1} \right\}^{-(p-1)/p} (\geq 1), \tag{5.6.57}$$

and as $\Delta^2 \rightarrow \infty$, $AMRE(\hat{\beta}_n^S; \tilde{\beta}_n) \rightarrow 1$.

ADMSE and ADQR of $\hat{\beta}_n^{S+}$

$$\begin{aligned}
\mathbf{M}_5(\hat{\beta}_n^{S+}) &= \lim_{n \rightarrow \infty} E \left\{ n(\hat{\beta}_n^{S+} - \beta)(\hat{\beta}_n^{S+} - \beta)' | K_{(n)} \right\} \\
&= \mathbf{M}_4(\hat{\theta}_n^S) - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \left\{ \left(\mathbf{J}\mathbf{\Lambda}_0^{-1}\mathbf{J}' \right) \right. \\
&\times E \left[\left(1 - (p-3)\chi_{p+1}^{-2}(\Delta^2) \right)^2 I \left(\chi_{p+1}^2(\Delta^2) > p-3 \right) \right] \\
&+ \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \left(\mathbf{J}\delta\delta'\mathbf{J}' \right) E \left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2) \right)^2 \right. \\
&\times \left. \left. I \left(\chi_{p+3}^2(\Delta^2) > p-3 \right) \right] \right\} - 2 \left(\mathbf{J}\delta\delta'\mathbf{J}' \right) \\
&\times E \left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2) \right) I \left(\chi_{p+3}^2(\Delta^2) > p-3 \right) \right]
\end{aligned}$$

$$\begin{aligned}
R_5(\hat{\beta}_n^{S+}) &= R_4(\hat{\beta}_n^S) - \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \left\{ \text{tr}(\mathbf{QJ}\Lambda_0^{-1}\mathbf{J}') \right. \\
&\quad \times E \left[\left(1 - (p-3)\chi_{p+1}^{-2}(\Delta^2) \right)^2 I(\chi_{p+1}^2(\Delta^2) > p-3) \right] \\
&\quad + \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} (\delta' \mathbf{J}' \mathbf{QJ} \delta) E \left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2) \right)^2 \right. \\
&\quad \times I(\chi_{p+3}^2(\Delta^2) > p-3) \left. \left. \right] \right\} - 2(\delta' \mathbf{J}' \mathbf{QJ} \delta) \\
&\quad \times E \left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2) \right) I(\chi_{p+3}^2(\Delta^2) > p-3) \right]
\end{aligned} \tag{5.6.58}$$

Risk Analysis

Next we compare $\hat{\beta}_n^S$ and $\hat{\beta}_n^{S+}$.

$$\begin{aligned}
&R_4(\hat{\beta}_n^S) - R_5(\hat{\beta}_n^{S+}) \\
&= \frac{\sigma_{zz}^{(0)}}{\kappa_{xx} \sigma_{X^0 X^0}} \left\{ \text{tr}[\mathbf{QJ}\Lambda_0^{-1}\mathbf{J}'] \right. \\
&\quad \times E \left[\left(1 - (p-3)\chi_{p+1}^{-2}(\Delta^2) \right)^2 I(\chi_{p+1}^2(\Delta^2) > p-3) \right] \\
&\quad + \frac{\kappa_{xx} \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} (\delta' \mathbf{J} \mathbf{QJ} \delta) \\
&\quad \times E \left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2) \right)^2 I(\chi_{p+3}^2(\Delta^2) > p-3) \right] \left. \right\} \\
&\quad + 2(\delta' \mathbf{J}' \mathbf{QJ} \delta) E \left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2) \right) \right. \\
&\quad \times I(\chi_{p+3}^2(\Delta^2) > p-3) \left. \right] \geq 0
\end{aligned} \tag{5.6.59}$$

for all δ and $\mathbf{Q}$. Hence $\hat{\beta}_n^{S+}$ asymptotically dominates $\hat{\beta}_n^S$ uniformly. Thus we may order the estimators accordingly to the asymptotic risk as

$$R_5(\hat{\beta}_n^{S+}) < R_4(\hat{\beta}_n^S) < R_1(\tilde{\beta}_n) \tag{5.6.60}$$

for all $(\Delta^2, \mathbf{Q})$ and $\mathbf{Q}$ satisfying (5.6.52). It may be verified that $ARRE(\hat{\beta}_n^S; \tilde{\beta}_n)$ and $ARRE(\hat{\beta}_n^{S+}; \hat{\beta}_n^S)$ are decreasing functions of Δ^2 bounded below by the 1-line.

The risk-efficiency of $\hat{\beta}_n^S$ compared to $\tilde{\beta}_n$ for $\mathbf{Q} = \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} \mathbf{\Lambda}_0$ is given by

$$\begin{aligned} ARRE(\hat{\beta}_n^S; \tilde{\beta}_n) &= \left(1 - \frac{(p-3)(p-1)}{p} \left\{ 2E\left[\chi_{p+1}^{-2}(\Delta^2)\right] \right. \right. \\ &\quad \left. \left. - (p-3)E\left[\chi_{p+1}^{-4}(\Delta^2)\right] - \Delta^2 E\left[\chi_{p+3}^{-4}(\Delta^2)\right] \right\} \right)^{-1} \end{aligned} \quad (5.6.61)$$

Similarly, the risk-efficiency of $\hat{\beta}_n^{S+}$ compared to $\hat{\beta}_n^S$ may be written as

$$\begin{aligned} ARRE(\hat{\beta}_n^{S+}; \hat{\beta}_n^S) &= \left[1 - \{R_3(\hat{\beta}_n^S)\}^{-1} \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \left\{ (p-1) \right. \right. \\ &\quad \times E\left[\left(1 - (p-3)\chi_{p+1}^{-2}(\Delta^2)\right)^2 I\left(\chi_{p+1}^2(\Delta^2) > p-3\right) \right. \\ &\quad \left. \left. + \frac{\kappa_{xx}^2 \sigma_{X^0 X^0}}{\sigma_{zz}^{(0)}} (\delta' \mathbf{J}' \mathbf{\Lambda}_0 \mathbf{J} \delta) E\left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2)\right)^2 \right. \right. \right. \\ &\quad \times I\left(\chi_{p+3}^2(\Delta^2) > p-3\right) - 2\{R_3(\hat{\beta}_n^S)\}^{-1} (\delta' \mathbf{J}' \mathbf{\Lambda}_0 \mathbf{J} \delta) \\ &\quad \left. \left. \left. \times E\left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2)\right) I\left(\chi_{p+3}^2(\Delta^2) > p-3\right) \right] \right\} \right]^{-1} \end{aligned} \quad (5.6.62)$$

MSE Analysis

The MSE-efficiency of $\hat{\beta}_n^{S+}$ compared to $\hat{\beta}_n^S$ may be written as

$$\begin{aligned} AMRE\left(\hat{\beta}_n^{S+}; \hat{\beta}_n^S\right) &= \frac{|\mathbf{M}_4(\hat{\beta}_n^S)|^{1/p}}{|\mathbf{M}_5(\hat{\beta}_n^{S+})|^{1/p}} \\ &= \left| \mathbf{I}_p - \left(\mathbf{M}_4(\hat{\beta}_n^S)\right)^{-1} \times \right. \\ &\quad \left\{ \frac{\sigma_{zz}^{(0)}}{\kappa_{xx}^2 \sigma_{X^0 X^0}} \mathbf{J} \mathbf{\Lambda}_0^{-1} \mathbf{J}' E\left[\left(1 - (p-3)\chi_{p+1}^{-2}(\Delta^2)\right)^2 I\left(\chi_{p+1}^2(\Delta^2) > p-3\right)\right] \right. \\ &\quad \left. - \mathbf{J} \delta \delta' \mathbf{J}' \left\{ E\left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2)\right)^2 I\left(\chi_{p+3}^2(\Delta^2) > p-3\right)\right] \right. \right. \\ &\quad \left. \left. + 2E\left[\left(1 - (p-3)\chi_{p+3}^{-2}(\Delta^2)\right) I\left(\chi_{p+3}^2(\Delta^2) > p-3\right)\right] \right\} \right\} \right|^{-1/p} \end{aligned} \quad (5.6.63)$$

It may be shown that under H_0 , the $AMRE(\hat{\beta}_n^{S+}; \hat{\beta}_n^S) \geq 1$ and as $\Delta^2 \rightarrow \infty$, $AMRE(\hat{\beta}_n^{S+}; \hat{\beta}_n^S)$ tends to 1 from above.

Chapter 6

Application to Survey Sampling: the Prediction of Population Total

6.0 Introduction

In this chapter, we consider the prediction of the population total, which are functions of the estimators of the regression coefficient parameter β under regression superpopulation model when the explanatory variables are measured with error and it is suspected that the regression parameter vector may be the null vector with uncertainty.

Beginning with simple linear model, we introduce the predictors using the PT estimator, the Stein-type estimator, and the positive-rule Stein type estimator in multiple regression model.

Let $\mathcal{P} = \{1, \dots, N\}$ denote a finite population of N units, where N is known. Associated with the t th unit of $\mathcal{P}$, there is a 2-vector $(Y_t, \mathbf{x}_t)'$, $t = 1, \dots, N$, assumed to be a random sample from a bivariate normal distribution. To gain

information about the population total $T = \sum_{t=1}^N Y_t$, a sample s of size n is selected from $\mathcal{P}$ according to some specified sampling plan. We use the superpopulation approach (Cassel et al., 1977) to survey sampling. Under this perspective according to the conditionality principle (Basu, 1975), the sampling plan is not relevant for inference.

Properties of predictors for finite samples are studied and comparisons are made based on MSE criterion. Since the regression models with measurement error are non-identifiable, we consider the reliability ratio known case.

In section 1, we consider the prediction of the population total based on the estimated parameter in the simple linear model without measurement error. Two types of predictors with measurement error are compared in section 2. The predictor based on S-estimator is studied in section 3 and finally, we study the predictions based on Stein-type estimator in multiple linear model in section 4.

6.1 Simple Linear Model: Without measurement errors

In this section, we consider the prediction of T for the standard model. The standard model given by

$$Y_t = \beta_0 + \beta_1 X_t + e_t, \quad t = 1, 2, \dots, n \quad (6.1.1)$$

where X_t is fixed explanatory variable for $t \in s$ and $e_t \sim N(0, \sigma_{ee})$. The Ordinary Least Square (OLS) estimate of β_1 is given by

$$\tilde{\beta}_{1(OLS)} = S_{XX}^{-1} S_{XY} \quad (6.1.2)$$

where

$$\bar{X}_s = \frac{1}{n} \sum_{t \in s} X_t,$$

$$\bar{Y}_s = \frac{1}{n} \sum_{t \in s} Y_t,$$

$$S_{XX} = \sum_{t \in s} (X_t - \bar{X}_s)^2,$$

$$\text{and } S_{XY} = \sum_{t \in s} (X_t - \bar{X}_s)(Y_t - \bar{Y}_s).$$

If β_1 is suspected to be equal to 0 (zero), then we define the preliminary test estimator (PTE) of β_1 as

$$\hat{\beta}_{1(OLS)}^{PT} = \tilde{\beta}_{1(OLS)} - \tilde{\beta}_{1(OLS)} I(F < F_\alpha) \quad (6.1.3)$$

where

$$F = \frac{S_{XX} \tilde{\beta}_{1(OLS)}^2}{\tilde{\sigma}_{ee}}, \quad (6.1.4)$$

is the test statistics for testing $H_0 : \beta_1 = 0$ against the alternative $H_A : \beta_1 \neq 0$ and F_α is the upper α -level critical value of the F -statistic.

These results follow from the fact that

$$\tilde{\beta}_{1(OLS)} \sim N\left(\beta_1, \frac{\sigma_{ee}}{S_{XX}}\right) \quad (6.1.5)$$

and

$$\frac{S_{XX} \tilde{\beta}_{1(OLS)}^2}{\tilde{\sigma}_{ee}} \sim F_{1,m}(\Delta^2), \quad m = n - 2 \quad (6.1.6)$$

where $F_{1,m}(\Delta^2)$ is a non-central F -distribution with $(1, m)$ degree of freedom and non-centrality parameter Δ^2 defined by

$$\Delta^2 = \frac{S_{XX} \beta_1^2}{\sigma_{ee}} \quad (6.1.7)$$

The power function of this test is given by

$$1 - G_{1,m}(F_\alpha; \Delta^2) \geq \alpha = 1 - G_{1,m}(F_\alpha; 0). \quad (6.1.8)$$

This test is unbiased and most powerful for $H_0 : \beta_1 = 0$.

Note that $G_{\nu_1, \nu_2}(x; \Delta^2)$ stands for the cdf of the non-central F -distribution with (ν_1, ν_2) d.f. and non-centrality parameter $\frac{1}{2}\Delta^2$ with

$$G_{\nu_1, \nu_2}(x; \Delta^2) = \sum_{r \geq 0} \frac{e^{-\frac{\Delta^2}{2}} \left(\frac{\Delta^2}{2}\right)^r}{\Gamma(r+1)} I_y\left(\frac{\nu_1}{2} + r : \frac{\nu_2}{2}\right), \quad (6.1.9)$$

where $y = \frac{\nu_1 x}{\nu_2 + \nu_1 x}$ and

$$I_y\left(\frac{\nu_1}{2} + r : \frac{\nu_2}{2}\right) = \frac{1}{B\left(\frac{\nu_1}{2} + r : \frac{\nu_2}{2}\right)} \int_0^y x^{\frac{\nu_1}{2} + r - 1} (1-x)^{\frac{\nu_2}{2} - 1} dx \quad (6.1.10)$$

is the incomplete Beta function.

Let $T = n\bar{Y}_s + (N - n)\bar{Y}_r$, where $\mathbf{r} = \mathcal{P} - \mathbf{s}$ be the unobserved part of $\mathcal{P}$. Then we consider the Generalized Regression Predictors of T of the form

$$\hat{T}_S = N\bar{Y}_s + (N - n)(\bar{X}_r - \bar{X}_s)\tilde{\beta}_{1(OLS)} \quad (6.1.11)$$

where $\tilde{\beta}_{1(OLS)}$ is an estimator of β_1 and $\bar{X}_r = \frac{1}{N-n} \sum_{t \in \mathbf{r}} X_t$, which has been shown to be the best unbiased predictor of T under the simple normal regression model. To obtain the PTE Predictor (PTP) of T we substitute $\hat{\beta}_{1(OLS)}$ by $\hat{\beta}_{1(OLS)}^{PT}$ to obtain the expression given by

$$\hat{T}_S^{PT} = N\bar{Y}_s + (N - n)(\bar{X}_r - \bar{X}_s)\hat{\beta}_{1(OLS)}^{PT}. \quad (6.1.12)$$

To calculate biases and MSE of predictors, we consider

$$\hat{T}_S - T = (N - n)[(\bar{X}_r - \bar{X}_s)(\tilde{\beta}_{1(OLS)} - \beta_1) + (\bar{e}_s - \bar{e}_r)]$$

$$\text{and } \hat{T}_S^{PT} - T = (N - n)[(\bar{X}_r - \bar{X}_s)(\hat{\beta}_{1(OLS)}^{PT} - \beta_1) + (\bar{e}_s - \bar{e}_r)].$$

Bias and MSE of the Predictors

The bias expressions of predictors $\hat{T}_S$ and $\hat{T}_S^{PT}$ in the standard linear model are in the following theorem.

Theorem 6.1.1 *The bias expressions of predictors are given by*

$$b(\hat{T}_S) = E(\hat{T}_S - T) = 0, \quad (6.1.13)$$

$$\begin{aligned} b(\hat{T}_S^{PT}) &= E(\hat{T}_S^{PT} - T) \\ &= -(N - n)\beta_1(\bar{X}_r - \bar{X}_s)G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) \end{aligned} \quad (6.1.14)$$

The mean square error(MSE) of $\hat{T}_{OLS}$ and $\hat{T}_{OLS}^{PT}$ are given by

$$\begin{aligned} M(\hat{T}_S) &= E[(\hat{T}_S - T)^2] \\ &= N\frac{1}{f}(1 - f)\sigma_{ee}[1 + Nf(1 - f)K(X)] \end{aligned} \quad (6.1.15)$$

$$\begin{aligned} M(\hat{T}_S^{PT}) &= E[(\hat{T}_S^{PT} - T)^2] \\ &= M(\hat{T}_S) - \sigma_{ee}N^2(1 - f)^2K(X) \\ &\quad \times \left[G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) - \Delta^2\left\{ 2G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) \right. \right. \\ &\quad \left. \left. - G_{5,m}\left(\frac{1}{5}F_\alpha; \Delta^2\right) \right\} \right]. \end{aligned} \quad (6.1.16)$$

where $f = \frac{n}{N}$ and $K(X) = \frac{(\bar{X}_r - \bar{X}_s)^2}{S_{XX}}$.

Proof. Note that from Chapter 2,

$$\begin{aligned} E[\tilde{\beta}_{1(OLS)} - \beta_1] &= E[\bar{e}_s - \bar{e}_r] = 0 \\ E[\hat{\beta}_{1(OLS)}^{PT} - \beta_1] &= -\beta_1 G_{3,m}\left(\frac{1}{3}F_\alpha; \Delta^2\right) \end{aligned} \quad (6.1.17)$$

Figure 6.1: Graph of $E(\hat{T}_S^{PT} : \hat{T}_S)$

$$\begin{aligned}
 E[(\tilde{\beta}_{1(OLS)} - \beta_1)^2] &= \frac{\sigma_{ee}}{S_{XX}} \\
 E[(\hat{\beta}_{1(OLS)}^{PT} - \beta_1)^2] &= \frac{\sigma_{ee}}{S_{XX}} - \frac{\sigma_{ee}}{S_{XX}} \left[G_{3,m} \left(\frac{1}{3} F_\alpha : \Delta^2 \right) \right. \\
 &\quad \left. - \Delta^2 \left\{ 2G_{3,m} \left(\frac{1}{3} F_\alpha : \Delta^2 \right) - G_{5,m} \left(\frac{1}{5} F_\alpha : \Delta^2 \right) \right\} \right] \\
 E[(\bar{e}_s - \bar{e}_r)^2] &= \frac{N}{n(N-n)} \sigma_{ee}
 \end{aligned} \tag{6.1.18}$$

Finally, the efficiency of $\hat{T}_S^{PT}$ relative to $\hat{T}_S$ is available as

$$\begin{aligned}
 E(\hat{T}_S^{PT} : \hat{T}_S) &= \frac{M(\hat{T}_S)}{M(\hat{T}_S^{PT})} \\
 &= \left[1 + Nf(1-f)K(X)(1 + Nf(1-f)K(X))^{-1} \right. \\
 &\quad \times \left\{ \Delta^2 \left(2G_{3,m} \left(\frac{1}{3} F_\alpha ; \Delta^2 \right) - G_{5,m} \left(\frac{1}{5} F_\alpha ; \Delta^2 \right) \right) \right. \\
 &\quad \left. \left. - G_{3,m} \left(\frac{1}{3} F_\alpha ; \Delta^2 \right) \right\} \right]^{-1}
 \end{aligned} \tag{6.1.19}$$

Note that $E(\hat{T}_S^{PT} : \hat{T}_S)$ is a function of (α, Δ) and for fixed α ($0 < \alpha < 1$) it decreases crossing the 1-line to a minimum, then increases towards the 1-line as Δ tends to infinity. Further,

(i) $E(\hat{T}_S^{PT} : \hat{T}_S) \gtrless 1$ according as

$$\Delta^2 \gtrless \frac{G_{3,m}(\frac{1}{3}F_\alpha; \Delta^2)}{2G_{3,m}(\frac{1}{3}F_\alpha; \Delta^2) - G_{5,m}(\frac{1}{5}F_\alpha; \Delta^2)} = K_S(\alpha, \Delta^2) (\leq 1). \quad (6.1.20)$$

Hence $\hat{T}_S^{PT}$ performs better than $\hat{T}_S$ if $\Delta^2 \leq K_S(\alpha : \Delta^2)$, otherwise $\hat{T}_S$ performs better.

(ii) Maximum efficiency for fixed α is attained for $\Delta^2 = 0$ and equals

$$\left\{ 1 - \frac{Nf(1-f)K(X)}{1 + Nf(1-f)K(X)} G_{3,m}\left(\frac{1}{3}F_\alpha; 0\right) \right\}^{-1} \quad (6.1.21)$$

and as $\Delta^2 \rightarrow \infty$, the efficiency goes to unity.

(iii) PTP can not be uniformly better than $\hat{T}_S$ but one may determine a PTE with minimum guaranteed efficiency, say, E_0 by choosing an optimum level of significance α^* by the maxmin rule

$$\max_{\alpha} \min_{\Delta^2} E(\alpha : \Delta^2) \geq E_0. \quad (6.1.22)$$

The efficiency may go up to (6.1.21) if β_1 is near the origin.

(iv) As $\alpha \rightarrow 1$, $\hat{T}_S^{PT} \rightarrow \hat{T}_S$. Figure 6.1 displays one graph of the efficiency of $\hat{T}_S^{PT}$ relative to $\hat{T}_S$ when $n = 8$, $\alpha = 0.15$, $K(X) = 3$ and $Nf(1-f) = 4$.

6.2 Simple Linear Model: With Measurement Error

6.2.1 Conditional Expressions

In this section, we consider the finite sample estimation of the Total T in a measurement error model when κ_{xx} known. Here we first consider the conditional model to compare the results with standard model to ascertain the influence of measurement errors on the estimators. The conditional model is given by

$$Y_{0t} = \nu_0 + \nu_1 X_{0t} + Z_t; \quad t = 1, 2, \dots, n \quad (6.2.1)$$

where $Z_t \sim N(0, \sigma_{zz})$ and $\sigma_{zz} = \sigma_{ee} + \beta_1^2 \sigma_{xx} (1 - \kappa_{xx})$. This model is exactly similar to the standard model except that (i) ν_0 is a translation of β_0 , (ii) ν_1 is a scaled version of β_1 and (iii) the variance, σ_{zz} is inflated version of σ_{ee} .

For the estimation of the slope parameter under this conditional model, we consider two estimators of ν_1 , namely,

$$\tilde{\nu}_{1(OLS)} = \frac{S_{XY}}{S_{XX}} \quad (6.2.2)$$

which is an unbiased estimator of ν_1 (a biased estimator of β_1) and

$$\tilde{\nu}_{1(B)} = \frac{S_{XY}}{\kappa_{xx} S_{XX}} \quad (6.2.3)$$

which is a biased estimator of ν_1 (an unbiased estimator of β_1). If β_1 is suspected to be equal to 0 (zero), then we define the preliminary test estimator (PTE) of ν_1 as

$$\hat{\nu}_{1(G)}^{PT} = \tilde{\nu}_{1(G)} - \tilde{\nu}_{1(G)} I(F_G < F_\alpha) \quad (6.2.4)$$

where ν_G stands for any estimator of ν_1 and

$$F_G = \frac{S_{XX}\tilde{\nu}_{1(OLS)}^2}{\tilde{\sigma}_{zz}} \text{ or } \frac{S_{XX}\tilde{\nu}_{1(B)}^2\kappa_{xx}^2}{\tilde{\sigma}_{zz}} \quad (6.2.5)$$

depending on the estimators. $I(A)$ is the indicator of the set A and $\tilde{\sigma}_{zz}$ is the estimate of the variance σ_{zz} of the errors (Z) in the conditional model (6.2.1) given by

$$\tilde{\sigma}_{zz} = \frac{1}{n-2} \sum_{t \in s} \left\{ (Y_t - \bar{Y}_s) - \tilde{\nu}_{1(G)}(X_t - \bar{X}_s) \right\}^2, \quad (6.2.6)$$

where F_α is the upper α -level critical value of the F -statistic under the null hypothesis $H_0 : \beta_1 = 0$ against the alternative $H_A : \beta_1 \neq 0$. These results follows from the fact that

$$\tilde{\nu}_{1(OLS)} = \frac{S_{XY}}{S_{XX}} \sim \mathcal{N}\left(\kappa_{xx}\beta_1, \frac{\sigma_{zz}}{S_{XX}}\right) \quad (6.2.7)$$

$$\tilde{\nu}_{1(B)} = \frac{S_{XY}}{\kappa_{xx}S_{XX}} \sim \mathcal{N}\left(\beta_1, \frac{\sigma_{zz}}{\kappa_{xx}^2 S_{XX}}\right) \quad (6.2.8)$$

and

$$\begin{aligned} \frac{S_{XX}\tilde{\nu}_{1(OLS)}^2}{\tilde{\sigma}_{zz}} &\sim F_{1,m}(\Delta^{*2}) \\ \frac{S_{XX}\tilde{\nu}_{1(B)}^2\kappa_{xx}^2}{\tilde{\sigma}_{zz}} &\sim F_{1,m}(\Delta^{*2}), \quad m = n - 2, \end{aligned} \quad (6.2.9)$$

where $F_{1,m}(\Delta^{*2})$ follows a non-central F -distribution with $(1, m)$ degree of freedom and non-centrality parameter $\frac{1}{2}\Delta^{*2}$ and Δ^{*2} defined by

$$\begin{aligned} \Delta^{*2} &= \frac{S_{XX}\beta_1^2\kappa_{xx}^2}{\sigma_{zz}} = \frac{S_{XX}\beta_1^2\kappa_{xx}^2}{\sigma_{ee} + \beta_1^2\sigma_{xx}(1 - \kappa_{xx})} \\ &= \frac{S_{XX}\beta_1^2\kappa_{xx}^2}{\sigma_{ee} + \beta_1^2\sigma_{uu}\kappa_{xx}} = \frac{\Delta^2\kappa_{xx}^2}{1 + \beta_1^2\delta\kappa_{xx}} \leq \Delta^2, \end{aligned} \quad (6.2.10)$$

where $\delta = \frac{\sigma_{uu}}{\sigma_{ee}}$. The predictors $\hat{T}_{OLS}$, $\hat{T}_B$, $\hat{T}_{OLS}^{PT}$ and $\hat{T}_B^{PT}$ are in the expressions of the predictors (6.1.11) and (6.1.12) substituting $\tilde{\nu}_{1(OLS)}$ or $\tilde{\nu}_{1(B)}$ and $\tilde{\nu}_{1(OLS)}^{PT}$ or

$\tilde{\nu}_{1(B)}^{PT}$ instead of $\tilde{\beta}_1$ and $\tilde{\beta}_1^{PT}$. As before, we consider the following expressions to get biases and MSEs.

$$\begin{aligned}\hat{T}_{(*)} - T &= (N - n)[(\bar{X}_r - \bar{X}_s)(\tilde{\nu}_{1(*)} - \nu_1) + (\bar{Z}_s - \bar{Z}_r)] \\ \hat{T}_{(*)}^{PT} - T &= (N - n)[(\bar{X}_r - \bar{X}_s)(\tilde{\nu}_{1(*)}^{PT} - \nu_1) + (\bar{Z}_s - \bar{Z}_r)]\end{aligned}\tag{6.2.11}$$

where $\hat{T}_{(*)}$ and $\hat{T}_{(*)}^{PT}$ are predictors based on $\tilde{\nu}_{1(*)}$ and $\tilde{\nu}_{1(*)}^{PT}$, respectively.

Bias and MSE when κ_{xx} is known

Theorem 6.2.1 *The bias expressions of predictors are*

$$b(\hat{T}_{OLS}) = 0, \tag{6.2.12}$$

$$\begin{aligned}b(\hat{T}_{OLS}^{PT}) &= -(N - n)\beta_1\kappa_{xx}(\bar{X}_r - \bar{X}_s)G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right), \\ b(\hat{T}_B) &= (N - n)(\bar{X}_r - \bar{X}_s)\beta_1(1 - \kappa_{xx}),\end{aligned}\tag{6.2.13}$$

$$\begin{aligned}b(\hat{T}_B^{PT}) &= -(N - n)\beta_1(\bar{X}_r - \bar{X}_s) \\ &\times \left[G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right) - (1 - \kappa_{xx})\right]\end{aligned}\tag{6.2.14}$$

The mean square error(MSE) are given by

$$M(\hat{T}_{OLS}) = N\frac{1}{f}(1 - f)\sigma_{zz}\left[1 + Nf(1 - f)K(X)\right] \tag{6.2.15}$$

$$\begin{aligned}M(\hat{T}_{OLS}^{PT}) &= M(\hat{T}_{OLS}) - \sigma_{zz}N^2(1 - f)^2K(X) \\ &\times \left[G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right) - \Delta^{*2}\left\{2G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right) \right. \right. \\ &\left. \left. - G_{5,m}\left(\frac{1}{5}F_\alpha : \Delta^{*2}\right)\right\}\right]\end{aligned}\tag{6.2.16}$$

$$\begin{aligned}
M(\hat{T}_B) &= N \frac{1}{f} (1-f) \sigma_{zz} \left[1 + Nf(1-f) \frac{K(X)}{\kappa_{xx}^2} \right. \\
&\quad \times \left. \left\{ 1 + (1 - \kappa_{xx})^2 \Delta^{*2} \right\} \right]
\end{aligned} \tag{6.2.17}$$

$$\begin{aligned}
M(\hat{T}_B^{PT}) &= M(\hat{T}_B) - \sigma_{zz} N^2 (1-f)^2 K(X) \\
&\quad \times \left[G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) - \Delta^{*2} \left\{ 2\kappa_{xx} G_{3,m} \left(\frac{1}{3} F_\alpha; \Delta^{*2} \right) \right. \right. \\
&\quad \left. \left. - G_{5,m} \left(\frac{1}{5} F_\alpha; \Delta^{*2} \right) \right\} \right],
\end{aligned} \tag{6.2.18}$$

where $f = \frac{n}{N}$ and $K(X) = \frac{(\bar{X}_r - \bar{X}_s)^2}{S_{XX}}$.

The efficiency of $\hat{T}_{OLS}$ or $\hat{T}_B$ relative to $\hat{T}_S$ are given by

$$\begin{aligned}
E(\hat{T}_{OLS} : \hat{T}_S) &= \frac{\sigma_{ee}}{\sigma_{zz}} = \frac{1}{1 + \beta_1^2 \delta \kappa_{xx}} \left(\leq 1 \right) \\
E(\hat{T}_B : \hat{T}_S) &= \frac{\sigma_{ee}}{\sigma_{zz}} \left[1 + Nf(1-f) K(X) \right] \\
&\quad \times \left[1 + Nf(1-f) \frac{K(X)}{\kappa_{xx}^2} \left\{ 1 + \Delta^{*2} (1 - \kappa_{xx})^2 \right\} \right]^{-1}
\end{aligned} \tag{6.2.19}$$

where $K(X) = \frac{(\bar{X}_r - \bar{X}_s)^2}{S_{XX}}$.

If $\kappa_{xx} \rightarrow 1$, then $E(\hat{T}_{OLS} : \hat{T}_S) = E(\hat{T}_B : \hat{T}_S)$. If $\kappa_{xx} \rightarrow 0$, $E(\hat{T}_{OLS} : \hat{T}_S) = 1$ and $E(\hat{T}_B : \hat{T}_S) \rightarrow 0$. If $\Delta^{*2} = \frac{\nu_1^2 S_{XX}}{\sigma_{zz}} \rightarrow \infty$, $E(\hat{T}_B : \hat{T}_S) \rightarrow 0$. Clearly, the predictor of the total without the measurement errors has less variability than the ones with measurement errors.

For the PTE predictors the expressions of efficiency are given by

$$\begin{aligned}
E(\hat{T}_{OLS}^{PT} : \hat{T}_S^{PT}) &= \frac{\sigma_{ee}}{\sigma_{zz}} \left[1 + Nf(1-f)K(X) \right. \\
&\quad \times \left\{ 1 - G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^2 \right) + \Delta^2 \left(2G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^2 \right) \right. \right. \\
&\quad \left. \left. - G_{5,m} \left(\frac{1}{5}F_\alpha; \Delta^2 \right) \right) \right\} \left[1 + Nf(1-f)K(X) \right. \\
&\quad \times \left\{ 1 - G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^{*2} \right) + \Delta^{*2} \left(2G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^{*2} \right) \right. \right. \\
&\quad \left. \left. - G_{5,m} \left(\frac{1}{5}F_\alpha; \Delta^{*2} \right) \right) \right\} \right]^{-1} \tag{6.2.20}
\end{aligned}$$

where $K(X) = \frac{(\bar{X}_r - \bar{X}_s)^2}{S_{XX}}$.

$$\begin{aligned}
E(\hat{T}_B^{PT} : \hat{T}_S^{PT}) &= \frac{\sigma_{ee}}{\sigma_{zz}} \left[1 + Nf(1-f)K(X) \right. \\
&\quad \times \left\{ 1 - G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^2 \right) + \Delta^2 \left(2G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^2 \right) \right. \right. \\
&\quad \left. \left. - G_{5,m} \left(\frac{1}{5}F_\alpha; \Delta^2 \right) \right) \right\} \left[1 + Nf(1-f) \frac{K(X)}{\kappa_{xx}^2} \right. \\
&\quad \times \left\{ 1 - G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^{*2} \right) + \Delta^{*2} \left(2\kappa_{xx} G_{3,m} \left(\frac{1}{3}F_\alpha; \Delta^{*2} \right) \right. \right. \\
&\quad \left. \left. - G_{5,m} \left(\frac{1}{5}F_\alpha; \Delta^{*2} \right) + (1 - \kappa_{xx})^2 \right) \right\} \right]^{-1} \tag{6.2.21}
\end{aligned}$$

where $K(X) = \frac{(\bar{X}_r - \bar{X}_s)^2}{S_{XX}}$.

Since $\Delta^{*2} \leq \Delta^2$ which implies that

$$G_{\nu_1, \nu_2}(x; \Delta^{*2}) \geq G_{\nu_1, \nu_2}(x; \Delta^2), \tag{6.2.22}$$

clearly we have $M(\hat{T}_{OLS}^{PT}) \geq M(\hat{T}_S^{PT})$ and $M(\hat{T}_B^{PT}) \geq M(\hat{T}_S^{PT})$ for all (Δ^{*2}, Δ^2) . Thus the variability in $\hat{T}_{OLS}^{PT}$ and $\hat{T}_B^{PT}$ is more than the variability in $\hat{T}_S^{PT}$ for the standard model. Note that as $\alpha \rightarrow 1$, $E(\hat{T}_{OLS}^{PT} : \hat{T}_S^{PT}) \rightarrow E(\hat{T}_{OLS} : \hat{T}_S)$ and $E(\hat{T}_B^{PT} : \hat{T}_S^{PT}) \rightarrow E(\hat{T}_B : \hat{T}_S)$.

6.2.2 Unconditional Expressions and Comparisons

Now we consider the unconditional bias and MSE expressions of $\hat{T}_{OLS}$, $\hat{T}_{OLS}^{PT}$, $\hat{T}_B$ and $\hat{T}_B^{PT}$. These are obtained by integrating the conditional bias and MSE expression with respect to the marginal distribution of S_{XX} , which is

$$\frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} e^{-\frac{S_{XX}}{2\sigma_{XX}}} \left(\frac{S_{XX}}{\sigma_{XX}} \right)^{\frac{n-1}{2}-1}, \quad (6.2.23)$$

together with the marginal distribution of the function of $\bar{X}$. Hence defining for any i ,

$$\begin{aligned} & G_{1+2i,m}^{(j)} \left(\frac{1}{1+2i} F_\alpha : \Delta_0^{*2} \right) \\ &= \sum_{r \geq 0} \frac{2^{j-2} \Gamma(\frac{n-1}{2} + r + j - 2) (\Delta_0^{*2})^r}{\Gamma(r+1) \Gamma(\frac{n-1}{2}) (1 + \Delta_0^{*2})^{\frac{n-1}{2} + r + j - 2}} I_y \left(\frac{1+2i}{2} + r : \frac{m}{2} \right), \end{aligned} \quad (6.2.24)$$

where $j = 1, 2$, $y = \frac{F_\alpha}{n-1+F_\alpha}$ and $\Delta_0^{*2} = \frac{\beta_1^2 \sigma_{XX} \kappa_{xx}^2}{\sigma_{zz}}$. $G_{1,m}^{(2)}(x; \Delta_0^{*2})$ is the unconditional cdf of the test statistic under H_A . With this notation we may find the biases and MSEs of the predictors. The following distributions are considered as well.

$$\begin{pmatrix} \bar{Y}_s \\ \bar{X}_s \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \mu_Y \\ \mu_X \end{pmatrix}; \frac{1}{n} \begin{pmatrix} \sigma_{YY} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{XX} \end{pmatrix} \right), \quad (6.2.25)$$

$$\begin{pmatrix} \bar{X}_s \\ \bar{X}_r \end{pmatrix} \sim N_2 \left(\begin{pmatrix} \mu_X \\ \mu_X \end{pmatrix}; \begin{pmatrix} \frac{1}{n} \sigma_{XX} & 0 \\ 0 & \frac{1}{N-n} \sigma_{XX} \end{pmatrix} \right), \quad (6.2.26)$$

and

$$\begin{pmatrix} S_{YY} & S_{XY} \\ S_{XY} & S_{XX} \end{pmatrix} \sim W_2 \left(\Sigma; n-1 \right). \quad (6.2.27)$$

Also we have

$$(\bar{X}_r - \bar{X}_s)^2 \sim \frac{\sigma_{XX}}{Nf(1-f)}\chi_1^2 \quad (6.2.28)$$

The unconditional biases and MSEs are in the following theorem.

Theorem 6.2.2 *The biases are given by*

$$b^*(\hat{T}_{OLS}) = b^*(\hat{T}_B) = b^*(\hat{T}_{OLS}^{PT}) = b^*(\hat{T}_B^{PT}) = 0. \quad (6.2.29)$$

The mean square error(MSE) are given by

$$M^*(\hat{T}_{OLS}) = N \frac{1}{f} (1-f) \frac{n-2}{n-3} \sigma_{zz}, \quad (6.2.30)$$

$$\begin{aligned} M^*(\hat{T}_{OLS}^{PT}) = N \frac{1}{f} (1-f) \sigma_{zz} & \left[\frac{n-2}{n-3} - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \\ & \left. + \Delta_0^{*2} \left\{ 2G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) \right\} \right] \end{aligned} \quad (6.2.31)$$

$$\begin{aligned} M^*(\hat{T}_B) = N \frac{1}{f} (1-f) \sigma_{zz} & \left[1 + \frac{1}{\kappa_{xx}^2} \left\{ \frac{1}{n-3} + \Delta_0^{*2} (1 - \kappa_{xx})^2 \right\} \right] \\ M^*(\hat{T}_B^{PT}) = N \frac{1}{f} (1-f) \sigma_{zz} & \left[1 + \frac{1}{\kappa_{xx}^2} \left\{ \frac{1}{n-3} + \Delta_0^{*2} (1 - \kappa_{xx})^2 \right. \right. \\ & - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) + \Delta_0^{*2} \left(2\kappa_{xx} G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \\ & \left. \left. - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) \right) \right\} \right] \end{aligned} \quad (6.2.32)$$

where $\Delta_0^{*2} = \frac{\beta_1^2 \sigma_{XX} \kappa_{xx}^2}{\sigma_{zz}}$ and $\Delta_0^2 = \frac{\beta_1^2 \sigma_{XX}}{\sigma_{ee}}$.

MSE Comparison

The efficiency of $\hat{T}_B$ relative to $\hat{T}_{OLS}$ is given by

$$\begin{aligned} E^*(\hat{T}_B : \hat{T}_{OLS}) &= \frac{M^*(\hat{T}_{OLS})}{M^*(\hat{T}_B)} \\ &= \frac{n-2}{n-3} \left[1 + \kappa_{xx}^{-2} \left\{ \frac{1}{n-3} + \Delta_0^{*2} (1 - \kappa_{xx})^2 \right\} \right]^{-1} \end{aligned} \quad (6.2.33)$$

As a function of κ_{xx} , the efficiency is an increasing function from 0 to 1. That is, if $\kappa_{xx} \rightarrow 0$, $E(\hat{T}_B : \hat{T}_{OLS}) \rightarrow 0$ and if $\kappa_{xx} \rightarrow 1$, $E(\hat{T}_B : \hat{T}_{OLS}) \rightarrow 1$. And as a function of Δ_0^{*2} , it is a decreasing function with maximum $\frac{n-2}{n-3+\kappa_{xx}} (\leq 1)$ at $\Delta_0^{*2} = 0$. As $\Delta_0^{*2} \rightarrow \infty$, the efficiency goes to 0. Therefore, $\hat{T}_{OLS}$ is a better predictor than $\hat{T}_B$. Figure 6.2 displays the graph of the efficiency of $\hat{T}_B$ relative to $\hat{T}_{OLS}$ when Δ_0^{*2} varies ($n = 8$ and $\kappa_{xx}^2 = 0.8$) and Figure 6.3 ($n = 8$ and $\Delta_0^{*2} = 2$) shows that the efficiency increases as κ_{xx} increases, i.e. $\hat{T}_{OLS}$ becomes more efficient as measurement errors becomes more severe in a finite sample with $n = 8$.

Now we compare $\hat{T}_{OLS}^{PT}$ relative to $\hat{T}_{OLS}$. The efficiency expression is given by

$$E^*(\hat{T}_{OLS}^{PT} : \hat{T}_{OLS}) = \left[1 + \frac{n-3}{n-2} \left\{ \Delta_0^{*2} \left(2G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) \right) - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right\} \right]^{-1} \quad (6.2.34)$$

Note that $E^*(\hat{T}_{OLS}^{PT} : \hat{T}_{OLS})$ is a function of $(\alpha : \Delta_0^{*2})$ and for fixed α ($0 < \alpha < 1$) it decreases crossing the 1-line to a minimum, then increases towards the 1-line as Δ tends to infinity. Further,

(i) $E^*(\hat{T}_{OLS}^{PT} : \hat{T}_{OLS}) \gtrless 1$ according as

$$\Delta_0^{*2} \gtrless \frac{G_{3,m}^{(1)}(\frac{1}{3} F_\alpha; \Delta_u^{*2})}{2G_{3,m}^{(2)}(\frac{1}{3} F_\alpha; \Delta_u^{*2}) - G_{5,m}^{(2)}(\frac{1}{5} F_\alpha; \Delta_u^{*2})} = K_{OLS}^*(\alpha : \Delta_0^{*2}) (\leq 1).$$

Hence $\hat{T}_{OLS}^{PT}$ performs better than $\hat{T}_{OLS}$ if $\Delta_0^{*2} < K_{OLS}^*(\alpha : \Delta_0^{*2})$.

(ii) Maximum attainable efficiency for fixed α is attained for $\Delta_0^{*2} = 0$ and equals

$$\begin{aligned} & \left\{ 1 - \frac{n-3}{n-2} G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha; 0 \right) \right\}^{-1} \\ &= \left\{ 1 - \frac{1}{n-2} G_{3,m} \left(\frac{1}{3} F_\alpha; 0 \right) \right\}^{-1} \end{aligned} \quad (6.2.35)$$

and as $\Delta_0^{*2} \rightarrow \infty$, the efficiency goes to unity.

(iii) PTE can not be uniformly better than $\hat{T}_S$ but one may determine a PTE with minimum guaranteed efficiency, say, E_0 by choosing α^* by the maximin rule

$$\max_{\alpha} \min_{\Delta_0^{*2}} E^*(\alpha : \Delta_0^{*2}) \geq E_0. \quad (6.2.36)$$

The efficiency with this α^* goes to (6.2.35) when β_1 is close to 0.

(iv) As $\alpha \rightarrow 1$, $\hat{T}_{OLS}^{PT} \rightarrow \hat{T}_{OLS}$.

Figure 6.4 displays the graphs of the efficiency of $\hat{T}_{OLS}^{PT}$ relative to $\hat{T}_{OLS}$ and $\hat{T}_B^{PT}$ relative to $\hat{T}_B$ ($n = 8, \alpha = 0.15$ and $\kappa_{xx}^2 = 0.8$). Table 6.1 provides the minimum (E_{min}) and maximum (E_{max}) relative efficiency of using the PTE estimator relative to using UE estimator of the slope parameter for varying values of α and Δ_0^{*2} -value at which the minimum relative efficiency occur. To choose an optimum level of significance α^* for given fixed efficiency $MRE^* = E^*$ (say), one goes through the tabular values of efficiency and α , then choose α^* corresponding the closest efficiency to E^* . This value α^* guarantees the minimum value of efficiency E^* which may increase if H_0 is true. For example, if the experimenter wants to have an OLS predictor which has relative efficiency no less than 0.9231, then using Table 6.1, one would use $\alpha^* = 0.2$ because it maximizes $MRE^*(\alpha, 0)$ and the maximum attainable relative efficiency is 1.0717.

Figure 6.2: Graph of $E^*(\hat{T}_B : \hat{T}_{OLS})$ Figure 6.3: Graph of $E^*(\hat{T}_B : \hat{T}_{OLS})$

Table 6.1: Maximum and Minimum Efficiency of PTP relative to UP of β_1 for various values of α when $n = 8$ and $\kappa_{xx}^2 = 0.8$

α	0.05	0.1	0.15	0.2	0.25
E_{max}	1.1486	1.1155	1.0909	1.0717	1.0565
E_{min}	0.7824	0.8563	0.8966	0.9231	0.9420
Δ_{min}^2	3.2157	2.5261	2.1900	1.9806	1.8348
α	0.3	0.35	0.4	0.45	0.5
E_{max}	1.0442	1.0342	1.0262	1.0197	1.0145
E_{min}	0.9561	0.9668	0.9752	0.9817	0.9867
Δ_{min}^2	1.7266	1.6429	1.5764	1.5227	1.4786

For the estimator $\hat{T}_B$, the efficiency expression is given by

$$\begin{aligned}
 E^*(\hat{T}_B^{PT} : \hat{T}_B) &= \left[1 + \kappa_{xx}^{-2} \left(\frac{1}{n-3} + \Delta_0^{*2} (1 - \kappa_{xx})^2 \right) \right] \\
 &\times \left[1 + \kappa_{xx}^{-2} \left\{ \frac{1}{n-3} - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \right. \\
 &+ \Delta_0^{*2} \left(2\kappa_{xx} G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \\
 &\left. \left. - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) + (1 - \kappa_{xx})^2 \right\} \right]^{-1}. \quad (6.2.37)
 \end{aligned}$$

Note that $E^*(\hat{T}_B^{PT} : \hat{T}_B)$ is a function of $(\alpha : \Delta_0^{*2})$ and for fixed α ($0 < \alpha < 1$) it decreases crossing the 1-line to a minimum, then increases towards the 1-line as Δ tends to infinity. Further,

(i) $E^*(\hat{T}_B^{PT} : \hat{T}_B) \gtrless 1$ according as

$$\Delta_0^{*2} \gtrless \frac{G_{3,m}^{(1)}(\frac{1}{3} F_\alpha; \Delta_u^{*2})}{2\kappa_{xx} G_{3,m}^{(2)}(\frac{1}{3} F_\alpha; \Delta_u^{*2}) - G_{5,m}^{(2)}(\frac{1}{5} F_\alpha; \Delta_u^{*2})} = K_B^*(\alpha : \Delta_0^{*2}) (\leq 1).$$

Hence $\hat{T}_B^{PT}$ performs better than $\hat{T}_B$ if $\Delta_0^{*2} < K_B^*(\alpha : \Delta_0^{*2})$.

(ii) Maximum attainable efficiency for fixed α is attained for $\Delta_0^{*2} = 0$ and equals

$$\begin{aligned}
 & \left[1 - \frac{1}{\kappa_{xx}^2} \left\{ 1 + \frac{1}{(n-3)\kappa_{xx}^2} \right\}^{-1} G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha; 0 \right) \right]^{-1} \\
 = & \left[1 - \frac{1}{(n-3)\kappa_{xx}^2} \left\{ 1 + \frac{1}{(n-3)\kappa_{xx}^2} \right\}^{-1} G_{3,m} \left(\frac{1}{3} F_\alpha; 0 \right) \right]^{-1}
 \end{aligned} \tag{6.2.38}$$

and as $\Delta_0^{*2} \rightarrow \infty$, the efficiency goes to unity.

(iii) PTE can not be uniformly better than $\hat{T}_S$ but one may determine a PTE with minimum guaranteed efficiency, say, E_0 by choosing α^* by the maxmin rule

$$\max_{\alpha} \min_{\Delta_0^{*2}} E^*(\alpha : \Delta_0^{*2}) \geq E_0. \tag{6.2.39}$$

(iv) As $\alpha \rightarrow 1$, $\hat{T}_B^{PT} \rightarrow \hat{T}_B$.

The efficiency of $\hat{T}_B^{PT}$ relative to $\hat{T}_{OLS}^{PT}$ is given by

$$\begin{aligned}
 & E^*(\hat{T}_B^{PT} : \hat{T}_{OLS}^{PT}) \\
 = & \left(\frac{n-2}{n-3} \right) \left[1 + \frac{n-3}{n-2} \left\{ \Delta_0^{*2} \left(2G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \right. \right. \\
 & \left. \left. - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) \right) - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right\} \right] \\
 & \times \left[1 + \kappa_{xx}^{-2} \left\{ \frac{1}{n-3} - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \right. \\
 & \left. \left. + \Delta_0^{*2} \left(2\kappa_{xx} G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \right. \right. \\
 & \left. \left. - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) + (1 - \kappa_{xx})^2 \right\} \right]^{-1}.
 \end{aligned} \tag{6.2.40}$$

where $\Delta_0^{*2} = \frac{\beta_1^2 \sigma_{XX} \kappa_{xx}^2}{\sigma_{zz}}$. Figure 6.5 displays the graph of the efficiency of $\hat{T}_B^{PT}$ relative to $\hat{T}_{OLS}^{PT}$. ($n = 8$, $\alpha = 0.15$ and $\kappa_{xx}^2 = 0.8$). When β_1 is close to 0, we find the following relationship;

$$M^*(\hat{T}_B) \gg M^*(\hat{T}_B^{PT}) > M^*(\hat{T}_{OLS}) > M^*(\hat{T}_{OLS}^{PT}), \tag{6.2.41}$$

otherwise we have

$$M^*(\hat{T}_B^{PT}) > M^*(\hat{T}_B) > M^*(\hat{T}_{OLS}^{PT}) \gg M^*(\hat{T}_{OLS}). \quad (6.2.42)$$

This indicates that $\hat{T}_{OLS}^{PT}$ behave best among $\hat{T}_{OLS}$, $\hat{T}_B^{PT}$ and $\hat{T}_B^{PT}$ in small interval near the origin, otherwise $\hat{T}_{OLS}$ is the best among others.

6.3 S-Predictor: With Measurement Errors

We apply the shrinkage estimator (S) of the previous section to survey sampling for the prediction of the population total based on section 2.4.

6.3.1 Conditional bias and MSE

We consider the conditional model in section 2 and an estimator of ν_1 , namely,

$$\tilde{\nu}_{1n} = \tilde{\nu}_{1(OLS)} = \frac{S_{XY}}{S_{XX}} \sim \mathcal{N}\left(\kappa_{xx}\beta_1, \frac{\sigma_{zz}}{S_{XX}}\right) \quad (6.3.1)$$

which has been shown to be the best predictor of T under the simple normal regression model. than using $\beta_{1n}^* = \frac{S_{XY}}{\kappa_{xx}S_{XX}}$ (See Kim and Saleh(2002)). If β_1 is suspected to be equal to 0 (zero), then we define the preliminary test (PTE) and shrinkage (S) estimator of ν_1 as

$$\hat{\nu}_{1(OLS)}^{PT} = \tilde{\nu}_{1(OLS)} - \tilde{\nu}_{1(OLS)}I(F_{OLS} < F_\alpha) \quad (6.3.2)$$

$$\hat{\nu}_{1(OLS)}^S = \tilde{\nu}_{1(OLS)} - c \frac{\sqrt{\tilde{\sigma}_{zz}}}{\sqrt{S_{XX}}} \frac{\tilde{\nu}_{1(OLS)}}{|\tilde{\nu}_{1(OLS)}|}. \quad (6.3.3)$$

Let $T = n\bar{Y}_s + (N - n)\bar{Y}_r$. Then we consider the (generic) predictors of T of the form

$$\hat{T}_{OLS}^S = N\bar{Y}_s + (N - n)(\bar{X}_r - \bar{X}_s)\hat{\nu}_{1(OLS)}^S \quad (6.3.4)$$

Figure 6.4: Graph of $E^*(\hat{T}_{OLS}^{PT} : \hat{T}_{OLS})$ and $E^*(\hat{T}_B^{PT} : \hat{T}_B)$

Figure 6.5: Graph of $E^*(\hat{T}_B^{PT} : \hat{T}_{OLS}^{PT})$

where $\hat{\nu}_{1(OLS)}^S$ is the shrinkage estimator of $\nu_1 = \kappa_{xx}\beta$.

The bias expressions of $\hat{T}_{OLS}$ and $\hat{T}_{OLS}^{PT}$ are given by

$$\begin{aligned} b(\hat{T}_{OLS}) &= 0, \\ b(\hat{T}_{OLS}^{PT}) &= -(N-n)\beta_1\kappa_{xx}(\bar{X}_r - \bar{X}_s)G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right), \end{aligned} \quad (6.3.5)$$

and the MSE expressions are given by

$$\begin{aligned} M(\hat{T}_{OLS}) &= N\frac{1}{f}(1-f)\sigma_{zz}[1 + Nf(1-f)K(X)] \\ M(\hat{T}_{OLS}^{PT}) &= M(\hat{T}_{OLS}) - \sigma_{zz}N^2(1-f)^2K(X) \\ &\quad \times \left[G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right) - \Delta^{*2}\left\{2G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right) \right. \right. \\ &\quad \left. \left. - G_{5,m}\left(\frac{1}{5}F_\alpha : \Delta^{*2}\right)\right\} \right] \end{aligned} \quad (6.3.6)$$

The bias and MSE expressions of $\hat{T}_{OLS}^S$ are given by

$$\begin{aligned} b(\hat{T}_{OLS}^S) &= -(N-n)c\sigma_zK_n\sqrt{K(X)}\{2\Phi(\Delta^{*2}) - 1\} \\ M(\hat{T}_{OLS}^S) &= M(\hat{T}_{OLS}) \\ &\quad - \sigma_{zz}N^2(1-f)^2K(X)\frac{2}{\pi}K_n^2\{2e^{-\Delta^{*2}/2} - 1\} \end{aligned} \quad (6.3.7)$$

where $f = \frac{n}{N}$, $K(X) = \frac{(\bar{X}_r - \bar{X}_s)^2}{S_{XX}}$ and $K_n = \sqrt{\frac{2}{n-2} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n-2}{2})}}$.

The expression of efficiencies are given by

$$\begin{aligned} E(\hat{T}_{OLS}^{PT} : \hat{T}_{OLS}) &= \left[1 - \sigma_{zz}N^2(1-f)^2K(X)\left\{M(\hat{T}_{OLS})\right\}^{-1} \right. \\ &\quad \times \left\{ G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right) - \Delta^{*2}\left(2G_{3,m}\left(\frac{1}{3}F_\alpha : \Delta^{*2}\right) \right. \right. \\ &\quad \left. \left. - G_{5,m}\left(\frac{1}{5}F_\alpha : \Delta^{*2}\right)\right)\right\} \left. \right]^{-1} \end{aligned} \quad (6.3.8)$$

$$\begin{aligned} E(\hat{T}_{OLS}^S : \hat{T}_{OLS}) &= \left[1 - \sigma_{zz}N^2(1-f)^2K(X)\frac{2}{\pi}K_n^2 \right. \\ &\quad \times \left\{ M(\hat{T}_{OLS}) \right\}^{-1} \left\{ 2e^{-\Delta^{*2}/2} - 1 \right\} \left. \right]^{-1} \end{aligned} \quad (6.3.9)$$

6.3.2 Unconditional Expressions and Comparisons

Now we consider the unconditional bias and MSE expressions of $\hat{T}_{OLS}$, $\hat{T}_{OLS}^{PT}$, and $\hat{T}_{OLS}^S$. These are obtained by integrating the conditional bias and MSE expression with respect to the marginal distribution of S_{XX} , which is

$$\frac{1}{2^{\frac{n-1}{2}} \Gamma(\frac{n-1}{2})} e^{-\frac{1}{2} \chi_{n-1}^2} (\chi_{n-1}^2)^{\frac{n-1}{2}-1}, \quad (6.3.10)$$

together with the marginal distribution of the function of $\bar{X}$. Hence defining for any i ,

$$\begin{aligned} & G_{1+2i,m}^{(j)} \left(\frac{1}{1+2i} F_\alpha : \Delta_0^{*2} \right) \\ &= \sum_{r \geq 0} \frac{2^{j-2} \Gamma(\frac{n-1}{2} + r + j - 2) (\Delta_0^{*2})^r}{\Gamma(r+1) \Gamma(\frac{n-1}{2}) (1 + \Delta_0^{*2})^{\frac{n-1}{2} + r + j - 2}} I_y \left(\frac{1+2i}{2} + r : \frac{m}{2} \right), \end{aligned} \quad (6.3.11)$$

where $j = 1, 2$, $y = \frac{F_\alpha}{n-1+F_\alpha}$ and $\Delta_0^{*2} = \frac{\beta_1^2 \sigma_{XX} \kappa_{xx}^2}{\sigma_{zz}}$. $G_{1,m}^{(2)}(x; \Delta_0^{*2})$ is the unconditional cdf of the test statistic under H_A . With this notation we may find the biases of the predictors

$$b^*(\hat{T}_{OLS}) = b^*(\hat{T}_{OLS}^{PT}) = 0 \quad (6.3.12)$$

and the MSEs of the predictors are given by

$$\begin{aligned} M^*(\hat{T}_{OLS}) &= N \frac{1}{f} (1-f) \frac{n-2}{n-3} \sigma_{zz}, \\ M^*(\hat{T}_{OLS}^{PT}) &= N \frac{1}{f} (1-f) \sigma_{zz} \left[\frac{n-2}{n-3} - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \\ &\quad \left. + \Delta_0^{*2} \left\{ 2G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) \right\} \right] \end{aligned} \quad (6.3.13)$$

The unconditional bias and MSE are given by

$$b^*(\hat{T}_{OLS}^S) = 0 \quad (6.3.14)$$

$$\begin{aligned} M^*(\hat{T}_{OLS}^S) &= N \frac{1}{f} (1-f) \sigma_{zz} \left[\frac{n-2}{n-3} \right. \\ &\quad \left. - \frac{1}{n-3} \frac{2}{\pi} K_n^2 \left\{ \frac{2}{(\Delta_0^{*2} + 1)^{(n-1)/2-1}} - 1 \right\} \right] \end{aligned} \quad (6.3.15)$$

where $\Delta_0^{*2} = \frac{\beta_1^2 \sigma_{XX} \kappa_{xx}^2}{\sigma_{zz}}$.

Now we compare $\hat{T}_{OLS}^{PT}$ relative to $\hat{T}_{OLS}$. The efficiency expression is given by

$$\begin{aligned} E^*(\hat{T}_{OLS}^{PT} : \hat{T}_{OLS}) &= \left[1 + \frac{n-3}{n-2} \left\{ \Delta_0^{*2} \left(2G_{3,m}^{(2)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right. \right. \right. \\ &\quad \left. \left. - G_{5,m}^{(2)} \left(\frac{1}{5} F_\alpha : \Delta_0^{*2} \right) \right) - G_{3,m}^{(1)} \left(\frac{1}{3} F_\alpha : \Delta_0^{*2} \right) \right\} \right]^{-1} \\ E^*(\hat{T}_{OLS}^S : \hat{T}_{OLS}) &= \left[1 - \sigma_{zz} N \frac{1}{f} (1-f) \frac{1}{n-3} \frac{2}{\pi} K_n^2 \left(M^*(\hat{T}_{OLS}) \right)^{-1} \right. \\ &\quad \left. \cdot \left\{ \frac{2}{(\Delta_0^{*2} + 1)^{(n-1)/2-1}} - 1 \right\} \right]^{-1} \end{aligned} \quad (6.3.16)$$

Comparison

In general, $E^*(\hat{T}_{OLS}^S : \hat{T}_{OLS})$ decreases from $\left[1 - \frac{2}{\pi} \frac{1}{n-2} K_n^2 \right]^{-1}$ at $\Delta_u^{*2} = 0$ and crosses the 1-line at $\Delta_u^{*2} = \frac{n-3}{2} \sqrt{2} - 1$ then drops to the minimum value $\left[1 + \frac{2}{\pi} \frac{1}{n-2} K_n^2 \right]^{-1} (\leq 1)$ as $\Delta_u^{*2} \rightarrow \infty$. The loss of efficiency is $1 - \left[1 + \frac{2}{\pi} K_n^2 \right]^{-1}$ while the gain in efficiency is $\left[1 - \frac{2}{\pi} K_n^2 \right]^{-1}$. Thus, for $0 \leq \Delta_u^{*2} \leq \left(\frac{n-3}{2} \sqrt{2} - 1 \right)$, $\hat{T}_{OLS}^S$ performs better than $\hat{T}_{OLS}$, otherwise $\hat{T}_{OLS}$ performs better outside this interval.

In the case of $\hat{T}_{OLS}^{PT}$, the graph of $E^*(\hat{T}_{OLS}^{PT} : \hat{T}_{OLS})$ decreases to a minimum at $\Delta_u^{*2} = \Delta_0^2$ then increases to 1 as $\Delta_u^{*2} \rightarrow \infty$. It performs better than $\hat{T}_{OLS}$ in the range $0 \leq \Delta_u^{*2} \leq K(F_\alpha : \Delta_u^{*2})$. Thus the range of better performance of $\hat{T}_{OLS}^S$ compare to $\hat{T}_{OLS}^{PT}$ is increased by an amount $\left(\frac{n-3}{2} \sqrt{2} - 1 - K(F_{1,m}(\alpha : \Delta_u^{*2})) \right)$, where

$$K(F_{1,m}(\alpha : \Delta_u^{*2})) = \frac{G_{3,m}^{(1)}(\frac{1}{3} F_\alpha : \Delta_u^{*2})}{2G_{3,m}^{(2)}(\frac{1}{3} F_\alpha : \Delta_u^{*2}) - G_{5,m}^{(2)}(\frac{1}{5} F_\alpha : \Delta_u^{*2})}.$$

It is clear that the relative efficiency of PTP compared to UP is smaller for $\Delta_u^{*2} > K(F_{1,m}(\alpha : \Delta_u^{*2}))$ and that of SEP relative to UP is smaller for $\Delta_u^{*2} > \frac{n-3}{2} \sqrt{2} - 1$.

Thus, SEP dominates UP over the wider interval $[0, \sqrt[n-3]{2} - 1]$ relative to the PTP in the interval $[0, K(F_{1,m}(\alpha : \Delta_u^{*2}))]$. Also, SEP has higher relative efficiency than PTP over all Δ_u^{*2} at which efficiency of PTP is smallest.

6.4 Multiple Regression Model: With Measurement Errors

Now we consider the estimators in multiple regression model with measurement errors to have the prediction of the population total, namely,

$$\left. \begin{aligned} Y_{0t} &= \beta_0 + \boldsymbol{\beta}' \mathbf{x}_t + e_t \\ \mathbf{X}_{0t} &= \mathbf{x}_t + \mathbf{u}_t \end{aligned} \right\}; \quad t = 1, \dots, n \quad (6.4.1)$$

where β_0 is the intercept and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)'$ is the regression parameters while $\mathbf{x}_t = (x_{1t}, \dots, x_{pt})'$, $\mathbf{u}_t = (u_{1t}, \dots, u_{pt})'$ and $\mathbf{X}_{0t} = (X_{01t}, \dots, X_{0pt})'$. e_t is the response error in the study variable and u_{it} is the measurement error in the i^{th} regression variable x_{it} . Note that x_{it} is unobservable and X_{0it} is the corresponding observed value. Similarly, Y_{0t} is the observed response. We assume that

$$(\mathbf{x}'_t, e_t, \mathbf{u}'_t)' \sim \mathcal{N}_{2p+1} \left\{ (\boldsymbol{\mu}'_x, 0, \mathbf{0}')'; \text{Blockdiag}(\boldsymbol{\Sigma}_{xx}, \sigma_{ee}, \boldsymbol{\Sigma}_{uu}) \right\} \quad (6.4.2)$$

where $\boldsymbol{\mu}_x = (\mu_{x_1}, \dots, \mu_{x_p})'$.

We consider the following distributions:

$$\begin{pmatrix} \bar{Y}_s \\ \bar{\mathbf{X}}_s \end{pmatrix} \sim N_{p+1} \left(\begin{pmatrix} \mu_Y \\ \boldsymbol{\mu}_X \end{pmatrix}; \frac{1}{n} \begin{pmatrix} \sigma_{YY} & \boldsymbol{\Sigma}_{XY} \\ \boldsymbol{\Sigma}_{XY} & \boldsymbol{\Sigma}_{XX} \end{pmatrix} \right), \quad (6.4.3)$$

where $\bar{\mathbf{X}}_s = (\bar{X}_{s1}, \dots, \bar{X}_{sp})$ and $\bar{\mathbf{X}}_r = (\bar{X}_{r1}, \dots, \bar{X}_{rp})$.

$$\begin{pmatrix} \bar{\mathbf{X}}_s \\ \bar{\mathbf{X}}_r \end{pmatrix} \sim N_{2p} \left(\begin{pmatrix} \boldsymbol{\mu}_X \\ \boldsymbol{\mu}_X \end{pmatrix}; \begin{pmatrix} \frac{1}{n}\boldsymbol{\Sigma}_{XX} & 0 \\ 0 & \frac{1}{N-n}\boldsymbol{\Sigma}_{XX} \end{pmatrix} \right), \quad (6.4.4)$$

Also we have

$$(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \sim \mathcal{N}_p \left(\mathbf{0}, \frac{N}{n(N-n)} \boldsymbol{\Sigma}_{XX} \right) \quad (6.4.5)$$

Further, the distribution of $(\bar{Y}_0, \bar{\mathbf{X}}_0)'$ is a $(p+1)$ -variate normal with mean-vector $(\beta_0 + \boldsymbol{\beta}'\boldsymbol{\mu}_x, \boldsymbol{\mu}_x')$ and covariance matrix $\frac{1}{n}\boldsymbol{\Sigma}$. Now the conditional distribution of Y_{0t} given $\mathbf{X}_{0t} = (X_{01t}, \dots, X_{0pt})'$ is normal with conditional mean and variance given by

$$E[Y_{0t}|\mathbf{X}_{0t}] = \nu_0 + \boldsymbol{\nu}'\mathbf{X}_{0t} \quad (6.4.6)$$

$$Var[Y_{0t}|\mathbf{X}_{0t}] = \boldsymbol{\beta}'\boldsymbol{\Sigma}_{xx}(\mathbf{I} - \mathbf{K}'_{xx})\boldsymbol{\beta} + \sigma_{ee} = \sigma_{zz}(\text{say}) \quad (6.4.7)$$

Consider the matrix of sum of squares

$$\mathbf{S} = \begin{pmatrix} S_{Y_0Y_0} & \mathbf{S}_{Y_0X_0} \\ \mathbf{S}_{X_0Y_0} & \mathbf{S}_{X_0X_0} \end{pmatrix} \quad (6.4.9)$$

where

$$\begin{aligned} \text{(i). } S_{Y_0Y_0} &= \sum_{i=1}^n (Y_{0i} - \bar{Y}_0)^2 \\ \text{(ii). } \mathbf{S}_{X_0X_0} &= ((S_{X_{0i}X_{0j}})) \\ \text{(iii). } S_{X_{0i}X_{0j}} &= \sum_{t=1}^n (X_{0it} - \bar{X}_{0i})(X_{0jt} - \bar{X}_{0j}), \quad i, j = 1, \dots, p \\ \text{(iv). } S_{X_{0i}Y_0} &= \sum_{t=1}^n (X_{0it} - \bar{X}_{0i})(Y_{0t} - \bar{Y}_0) \\ \text{(v). } \bar{X}_{0i} &= \frac{1}{n} \sum_{t=1}^n X_{0it} \text{ and } \bar{Y}_0 = \frac{1}{n} \sum_{t=1}^n Y_{0t} \end{aligned} \quad (6.4.10)$$

Now let $T = n\bar{Y}_s + (N - n)\bar{Y}_r$. Then we consider the (generic) Predictors of T of the form

$$\hat{T}^* = N\bar{Y}_s + (N - n)(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu}^* \quad (6.4.11)$$

where $\boldsymbol{\nu}^*$ is an estimator of $\boldsymbol{\nu}$, $\bar{\mathbf{X}}_r = (\bar{X}_{r1}, \dots, \bar{X}_{rp})'$ and $\bar{X}_{rt} = \frac{1}{N-n} \sum_{t \in r} X_{rt}$.

It is shown to be the best predictor of T based on OLS estimator under regression model with measurement errors. We consider a class of estimators based on OLS estimator defined by

$$\boldsymbol{\nu}^* = \left\{ 1 - g(\mathcal{L}_n) \right\} \tilde{\boldsymbol{\nu}}_{OLS} \quad (6.4.12)$$

where $g(\mathcal{L}_n)$ is a monotone decreasing function of $\mathcal{L}_n$ and the test-statistic $\mathcal{L}_n$ defined by

$$\mathcal{L}_n = \frac{\tilde{\boldsymbol{\beta}}_n' (\mathbf{K}'_{xx} \mathbf{S}_{X_0 X_0} \mathbf{K}_{xx}) \tilde{\boldsymbol{\beta}}_n}{p \tilde{\sigma}_{zz}} = \frac{\tilde{\boldsymbol{\nu}}' (\mathbf{S}_{X_0 X_0}) \tilde{\boldsymbol{\nu}}}{p \tilde{\sigma}_{zz}} \quad (6.4.13)$$

and

$$\tilde{\sigma}_{zz} = \frac{1}{m} \sum_{t=1}^n \left\{ (Y_{0t} - \bar{Y}_0) - (\mathbf{X}_{0t} - \bar{\mathbf{X}}_0)' \tilde{\boldsymbol{\nu}} \right\}^2, \quad m = n - p. \quad (6.4.14)$$

The class of predictors are given by setting $g(\mathcal{L}_n)$.

If $g(\mathcal{L}_n) = 0$, $\boldsymbol{\nu}^* = \tilde{\boldsymbol{\nu}} = \mathbf{S}_{X_0 X_0}^{-1} \mathbf{S}_{X_0 Y}$. Then $\hat{T}^{OLS}$ is called the unrestricted predictor (URP).

If $g(\mathcal{L}_n) = I(\mathcal{L}_n < F_\alpha)$, $\boldsymbol{\nu}^* = \hat{\boldsymbol{\nu}}^{PT} = \tilde{\boldsymbol{\nu}} I(\mathcal{L}_n > F_\alpha)$. Then $\hat{T}^{PT}$ is called the primary test predictor (PTP).

If $g(\mathcal{L}_n) = \frac{(p-2)m}{p(m+2)} \mathcal{L}_n^{-1}$, $\boldsymbol{\nu}^* = \hat{\boldsymbol{\nu}}^S = \left\{ 1 - \frac{(p-2)m}{p(m+2)} \mathcal{L}_n^{-1} \right\} \tilde{\boldsymbol{\nu}}$. Then $\hat{T}^S$ is called the Stein-type predictor (SP).

If $g(\mathcal{L}_n) = 1 - \left\{ \frac{(p-2)m}{p(m+2)} \mathcal{L}_n^{-1} \right\} I(\mathcal{L}_n > F_\alpha)$,

$\boldsymbol{\nu}^* = \hat{\boldsymbol{\nu}}^{S+} = \left\{ 1 - \frac{(p-2)m}{p(m+2)} \mathcal{L}_n^{-1} \right\} I \left\{ \mathcal{L}_n > \frac{(p-2)m}{p(m+2)} \right\} \tilde{\boldsymbol{\nu}}$. Then $\hat{T}^{S+}$ is called the positive-rule Stein-type predictor (PSP).

To obtain the URP, PTP, SP, SPP of T we substitute $\boldsymbol{\nu}^*$ by $\tilde{\boldsymbol{\nu}}$, $\hat{\boldsymbol{\nu}}^{PT}$, $\hat{\boldsymbol{\nu}}^S$ and $\hat{\boldsymbol{\nu}}^{S+}$, that is,

$$\begin{aligned} (1) \quad \hat{T}^{OLS} - T &= (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)(\tilde{\boldsymbol{\nu}} - \boldsymbol{\nu}) + (\bar{z}_s - \bar{z}_r)] \\ (2) \quad \hat{T}^{PT} - T &= (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)(\hat{\boldsymbol{\nu}}^{PT} - \boldsymbol{\nu}) + (\bar{z}_s - \bar{z}_r)] \\ (3) \quad \hat{T}^S - T &= (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)(\hat{\boldsymbol{\nu}}^S - \boldsymbol{\nu}) + (\bar{z}_s - \bar{z}_r)] \\ (4) \quad \hat{T}^{S+} - T &= (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)(\hat{\boldsymbol{\nu}}^{S+} - \boldsymbol{\nu}) + (\bar{z}_s - \bar{z}_r)]. \end{aligned}$$

6.4.1 Conditional Expressions

The bias expressions for the predictors are given by

$$(1) \quad E[\hat{T}^{OLS} - T] = (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' E(\tilde{\boldsymbol{\nu}} - \boldsymbol{\nu})] = 0 \quad (6.4.15)$$

$$\begin{aligned} (2) \quad E[\hat{T}^{PT} - T] &= (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' E(\hat{\boldsymbol{\nu}}^{PT} - \boldsymbol{\nu})] \\ &= -(N - n)(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} G_{p+2,m} \left(\frac{p}{p+2} F_\alpha; \Delta^2 \right) \end{aligned} \quad (6.4.16)$$

$$\begin{aligned}
(3) \quad E[\hat{T}^S - T] &= (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' E(\hat{\boldsymbol{\nu}}^S - \boldsymbol{\nu})] \\
&= -\frac{(N - n)(p - 2)m}{m + 2}(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} E[\chi_{p+2}^{-2}(\Delta^2)] \quad (6.4.17)
\end{aligned}$$

$$\begin{aligned}
(4) \quad E[\hat{T}^{S+} - T] &= (N - n)[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' E(\hat{\boldsymbol{\nu}}^{S+} - \boldsymbol{\nu})] \\
&= -\frac{(N - n)(p - 2)m}{(m + 2)(p + 2)}(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} \left\{ E[F_{p+2,m}^{-1}(\Delta^2)] \right. \\
&\quad \left. - E[F_{p+2,m}^{-1}(\Delta^2) I(F_{p+2,m}(\Delta^2) < pc)] \right. \\
&\quad \left. + \frac{1}{c} G_{p+2,m}(pc; \Delta^2) \right\} \quad (6.4.18)
\end{aligned}$$

The MSE expressions for the predictors are given by

$$\begin{aligned}
(1) \quad E[(\hat{T}^{OLS} - T)^2] &= (N - n)^2 \sigma_{zz} \left\{ (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{XX}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) + \frac{N}{n(N - n)} \right\} \quad (6.4.19)
\end{aligned}$$

$$\begin{aligned}
(2) \quad E[(\hat{T}^{PT} - T)^2] &= (N - n)^2 \sigma_{zz} \left\{ (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{XX}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \right. \\
&\quad - (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{XX}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) G_{p+2,m} \left(\frac{p}{p+2} F_\alpha; \Delta^2 \right) \\
&\quad + (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} \boldsymbol{\nu}' (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \left\{ 2G_{p+2,m} \left(\frac{p}{p+2} F_\alpha; \Delta^2 \right) \right. \\
&\quad \left. \left. - G_{p+4,m} \left(\frac{p}{p+4} F_\alpha; \Delta^2 \right) + \frac{N}{(N - n)n} \right\} \right\} \quad (6.4.20)
\end{aligned}$$

$$\begin{aligned}
(3) \quad E[(\hat{T}^S - T)^2] &= (N - n)^2 \sigma_{zz} \{ (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{XX}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \\
&\quad - \frac{(p-2)m}{m+2} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{XX}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \\
&\quad \times \left[2E(\chi_{p+2}^{-2}(\Delta^2)) - (p-2)E(\chi_{p+2}^{-4}(\Delta^2)) \right. \\
&\quad \left. + \frac{(p-2)m(p+2)}{m+2} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} \boldsymbol{\nu}' (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \right. \\
&\quad \left. \times E(\chi_{p+4}^{-4}(\Delta^2)) + \frac{N}{(N-n)n} \right\} \quad (6.4.21)
\end{aligned}$$

$$\begin{aligned}
(4) \quad E[(\hat{T}^{S+} - T)^2] &= E[(\hat{T}^S - T)^2] \\
&\quad - (N - n)^2 \sigma_{zz} \left\{ (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{XX}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \right. \\
&\quad \left. - (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{XX}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \right. \\
&\quad \times E[(1 - cF_{p+2,m}^{-1}(\Delta^2))^2 I(F_{p+2,m}(\Delta^2) < c)] \\
&\quad \left. - (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} \boldsymbol{\nu}' (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) \right. \\
&\quad \times \left\{ E[(1 - c_1 F_{p+4,m}^{-1}(\Delta^2))^2 I(F_{p+4,m}(\Delta^2) < c_1)] \right. \\
&\quad \left. \left. - 2E[(1 - cF_{p+2,m}^{-1}(\Delta^2)) I(F_{p+2,m}(\Delta^2) < c)] \right\} \right\} \quad (6.4.22)
\end{aligned}$$

where $c = \frac{(p-2)m}{(p+2)(m+2)}$ and $c_1 = \frac{(p-2)m}{(p+4)(m+2)}$.

6.4.2 Unconditional Expressions and Comparisons

Unconditional bias and MSE of $\hat{T}^{OLS}$, $\hat{T}^{PT}$, $\hat{T}_S$ and $\hat{T}^{S+}$ may be obtained by integrating the conditional MSE with respect to the Wishart distribution $W_p(n-1, \Sigma_{X_0 X_0})$ of $\mathbf{S}_{XX}$. As in section 3.5, $V_n = \frac{\boldsymbol{\nu}' \mathbf{S}_{XX} \boldsymbol{\nu}}{\boldsymbol{\nu}' \Sigma_{X_0 X_0} \boldsymbol{\nu}} \sim \chi_{n-1}^2$ and $\Delta^2 = \frac{\boldsymbol{\nu}' \mathbf{S}_{XX} \boldsymbol{\nu}}{\sigma_{zz}} = \Delta_0^2 V_n$, where $\Delta_0^2 = \frac{\boldsymbol{\nu}' \Sigma_{X_0 X_0} \boldsymbol{\nu}}{\sigma_{zz}}$.

The bias of the predictors are given by

$$E^*[\hat{T}^{OLS} - T] = E^*[\hat{T}^{PT} - T] = E^*[\hat{T}^S - T] = E^*[\hat{T}^{S+} - T] = 0 \quad (6.4.23)$$

The Unrestricted Predictor, $\hat{T}^{OLS}$

The MSE expressions for the predictors are given by

$$M_1^*(\hat{T}^{OLS}) = \frac{N(N-n)(n-2)}{n(n-p-2)}\sigma_{zz} \quad (6.4.24)$$

Note that

$$E\left[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{X_0 X_0}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)\right] = \frac{Np}{n(N-n)(n-p-2)}.$$

The Preliminary Test predictor, $\hat{T}^{PT}$

$$\begin{aligned} M_2^*(\hat{T}^{PT}) &= \frac{N(N-n)}{n}\sigma_{zz} \left\{ \frac{n-2}{n-p-2} \right. \\ &\quad - \frac{(n-3)p}{n-p-2} G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha; \Delta_0^2 \right) \\ &\quad + \boldsymbol{\nu}' \Sigma_{XX} \boldsymbol{\nu} \left[2G_{p+2,m}^{(2)} \left(\frac{p}{p+2} F_\alpha; \Delta^2 \right) \right. \\ &\quad \left. \left. - G_{p+4,m}^{(2)} \left(\frac{p}{p+4} F_\alpha; \Delta^2 \right) \right] \right\} \end{aligned} \quad (6.4.25)$$

Note that

$$\begin{aligned} (1) \quad & E\left[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{X_0 X_0}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)\right] = \frac{Np}{n(N-n)(n-p-2)} \\ (2) \quad & E\left[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \mathbf{S}_{X_0 X_0}^{-1} (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) G_{p+2i,m} \left(\frac{p}{p+2i} F_\alpha; \Delta^2 \right)\right] \\ &= \frac{Np(n-3)}{n(N-n)(n-p-2)} G_{p+2i,m}^{(1)} \left(\frac{p}{p+2i} F_\alpha; \Delta_0^2 \right) \\ (3) \quad & E\left[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} \boldsymbol{\nu}' (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)\right] = \frac{N}{n(N-n)} \boldsymbol{\nu}' \Sigma_{X_0 X_0} \boldsymbol{\nu} \\ (4) \quad & E\left[(\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s)' \boldsymbol{\nu} \boldsymbol{\nu}' (\bar{\mathbf{X}}_r - \bar{\mathbf{X}}_s) G_{p+2i,m} \left(\frac{p}{p+2i} F_\alpha; \Delta^2 \right)\right] \\ &= \frac{N}{n(N-n)} \boldsymbol{\nu}' \Sigma_{X_0 X_0} \boldsymbol{\nu} G_{p+2i,m}^{(2)} \left(\frac{p}{p+2i} F_\alpha; \Delta_0^2 \right) \end{aligned}$$

where $\Delta^2 = \frac{\boldsymbol{\nu}' \mathbf{S}_{X_0 X_0} \boldsymbol{\nu}}{\sigma_{zz}}$ and $\Delta_0^2 = \frac{\boldsymbol{\nu}' \Sigma_{X_0 X_0} \boldsymbol{\nu}}{\sigma_{zz}}$.

Comparison of $\hat{T}^{PT}$ with $\hat{T}^{OLS}$

In this case the MSE difference $M_1^* - M_2^* \geq 0$ according as

$$\Delta_0^2 \leq \frac{\frac{(n-3)p}{(m-1)} G_{p+2,m}^{(1)}(\frac{p}{p+2} F_\alpha; \Delta_0^2)}{2G_{p+2,m}^{(2)}(\frac{p}{p+2} F_\alpha; \Delta_0^2) - G_{p+4,m}^{(2)}(\frac{p}{p+4} F_\alpha; \Delta_0^2)} \quad (6.4.26)$$

Then, $\hat{T}^{PT}$ performs better than $\hat{T}^{OLS}$ in this range of Δ_0^2 . On the other hand $\hat{T}^{OLS}$ performs better than $\hat{T}^{PT}$.

The Shrinkage Predictor, $\hat{T}^S$

$$\begin{aligned} M_3^*(\hat{T}^S) &= \frac{N(N-n)}{n} \sigma_{zz} \left\{ \frac{n-2}{n-p-2} - \frac{(n-3)p(p-2)m}{(n-p-2)(m+2)} \right. \\ &\times \left[2E^{(1)}(\chi_{p+2}^{-2}(\Delta^2)) - (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta^2)) \right] \\ &+ \left. \frac{(p-2)m(p+2)}{m+2} E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \boldsymbol{\nu}' \boldsymbol{\Sigma}_{XX} \boldsymbol{\nu} \right\} \end{aligned} \quad (6.4.27)$$

Comparison of $\hat{T}^S$ with $\hat{T}^{OLS}$

The MSE difference may be expressed by

$$\begin{aligned} M_1^* - M_3^* &= \sigma_{zz} \frac{N(N-n)}{n} \left[\frac{(p-2)m(n-3)p}{(m+2)(m-1)} \right. \\ &\times \left\{ (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) + 2\Delta_0^2 E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \right\} \\ &- \left. \frac{(p-2)m(p+2)}{(m+2)} \boldsymbol{\nu}' \boldsymbol{\Sigma}_{XX} \boldsymbol{\nu} E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \right] \\ &= \sigma_{zz} \frac{N(N-n)}{n} \left[\frac{(p-2)m(n-3)}{(m+2)(m-1)} \left\{ (p-2)E^{(1)}(\chi_{p+2}^{-4}(\Delta_0^2)) \right. \right. \\ &+ \left. \left. 2\Delta_0^2 E^{(2)}(\chi_{p+4}^{-4}(\Delta_0^2)) \left\{ 1 - \frac{(m-1)(p+2)}{2(n-3)p} \right\} \right\} \right] > 0 \end{aligned} \quad (6.4.28)$$

The MSE of $\hat{T}^S$ is smaller than that of $\hat{T}^{OLS}$ when $p > 2$.

Comparison of $\hat{T}^S$ with $\hat{T}^{PT}$

Under $H_0 : \beta_1 = 0$,

$$\begin{aligned}
 M_3^* &= M_2^* + \sigma_{zz} \frac{(n-3)p}{(m-1)} \\
 &\times \left[G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) - \frac{(p-2)^2 m}{m+2} E^{(1)}(\chi_{p+2}^{-4}(0)) \right] \\
 &= M_2^* + \sigma_{zz} \frac{(n-3)p}{(m-1)} \\
 &\times \left[G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) - \frac{p+2}{p} \frac{c}{n-3} \right] \\
 &> M_2^*
 \end{aligned} \tag{6.4.29}$$

where $c = \frac{(p-2)m}{(m+2)(p+2)}$ and

$$\begin{aligned}
 G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) &= \frac{1}{n-3} I_y \left(\frac{p+2}{2} : \frac{m}{2} \right) \\
 &= \frac{1}{n-3} F_{p+2,m} \left(\frac{p}{p+2} F_\alpha : 0 \right) \quad \text{where } y = \frac{pF_\alpha}{m+pF_\alpha}
 \end{aligned} \tag{6.4.30}$$

such that $\frac{p}{p+2} F_\alpha$ satisfies the inequality

$$\left\{ \alpha : \frac{p}{p+2} F_\alpha > F_{p+2,m}^{-1} \left(\frac{p+2}{p} c : 0 \right) \right\} \tag{6.4.31}$$

The MSE of $\hat{T}^S$ is smaller than that of the MSE $\hat{T}_{PT}$ when the critical value $\frac{p}{p+2} F_\alpha$ satisfies the opposite inequality in (6.4.31). This means the estimator $\hat{T}^S$ does not always dominate $\hat{T}_{PT}$ under H_0 for all α .

The Positive-Rule Shrinkage Predictor, $\hat{T}^{S+}$

$$\begin{aligned}
M_4^*(\hat{T}^{S+}) &= M^*(\hat{T}^S) - \frac{N(N-n)}{n} \sigma_{zz} \left\{ \frac{(n-3)p}{(n-p-2)} \right. \\
&\times E^{(1)} \left[(1 - cF_{p+2,m}^{-1}(\Delta^2))^2 I(F_{p+2,m}(\Delta^2) < c) \right] \\
&- \boldsymbol{\nu}' \Sigma_{XX} \boldsymbol{\nu} \\
&\times \left\{ E^{(2)} \left[(1 - c_1 F_{p+4,m}^{-1}(\Delta^2))^2 I(F_{p+4,m}(\Delta^2) < c_1) \right] \right. \\
&- \left. \left. 2E^{(2)} \left[(1 - cF_{p+2,m}^{-1}(\Delta^2)) I(F_{p+2,m}(\Delta^2) < c) \right] \right\} \right\}
\end{aligned} \tag{6.4.32}$$

Comparison of $\hat{T}^{S+}$ with $\hat{T}^{OLS}$

The MSE difference may be expressed by

$$\begin{aligned}
M_1^* - M_4^* &= M_1^* - M_3^* \\
&+ \sigma_{zz} \frac{(n-3)}{(m-1)} \left[pE^{(1)} \{ (1 - cF_{p+2,m}^{-1}(\Delta_0^2))^2 I(F_{p+2,m}(\Delta_0^2) < c) \} \right. \\
&+ \Delta_0^2 E^{(2)} \{ (1 - c_1 F_{p+4,m}^{-1}(\Delta_0^2))^2 I(F_{p+4,m}(\Delta_0^2) < c_1) \} \\
&+ \left. 2E^{(2)} \{ (F_{p+2,m}^{-1}(\Delta_0^2) - 1) I(F_{p+2,m}(\Delta_0^2) < c) \} \right]
\end{aligned} \tag{6.4.33}$$

and hence $M_1^* - M_4^* \geq 0$ since

$$\begin{aligned}
(0 < F_{p+2i,m}(\Delta_0^2) < c_i) &\Leftrightarrow (c_i F_{p+2i,m}(\Delta_0^2) - 1) \geq 0 \\
E^{(j)} \{ (c_i F_{p+2i,m}^{-1}(\Delta_0^2) - 1) I(F_{p+2i,m}(\Delta_0^2) < c_i) \} &\geq 0 \\
E^{(j)} \{ (c_i F_{p+2i,m}^{-1}(\Delta_0^2) - 1)^2 I(F_{p+2i,m}(\Delta_0^2) < c_i) \} &\geq 0
\end{aligned} \tag{6.4.34}$$

Comparison of $\hat{T}^{S+}$ with $\hat{T}^{PT}$

Under $H_0 : \beta_1 = 0$, the MSE-difference may be written by

$$\begin{aligned}
 & M_2^* - M_4^* \\
 &= M_1^* - M_3^* + \sigma_{zz} \frac{(n-3)}{(m-1)} \\
 &\times \left[pE^{(1)} \left\{ (1 - cF_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right. \\
 &- \left. pG_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) \right] \\
 &= \sigma_{zz} \frac{(n-3)}{(m-1)} \\
 &\times \left[pE^{(1)} \left\{ (1 - cF_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right. \\
 &- \left. \left\{ G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) - \frac{p+2}{p} \frac{c}{n-3} \right\} \right]. \tag{6.4.35}
 \end{aligned}$$

Thus,

$$\begin{aligned}
 & M_4^* = M_2^* \\
 &+ \frac{(n-3)}{(m-1)} p \left[G_{p+2,m}^{(1)} \left(\frac{p}{p+2} F_\alpha : 0 \right) - \frac{p+2}{p} \frac{c}{n-3} \right. \\
 &- \left. E^{(1)} \left\{ (1 - cF_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right] \\
 &> M_2^* \tag{6.4.36}
 \end{aligned}$$

for all α satisfying the condition

$$\begin{aligned}
 & \left[\alpha : F_{p+2,m} \left(\frac{p}{p+2} F_\alpha \right) \geq \right. \\
 & \left. \frac{p+2}{p} c + E^{(1)} \left\{ (1 - cF_{p+2,m}^{-1}(0))^2 I(F_{p+2,m}(0) < c) \right\} \right] \tag{6.4.37}
 \end{aligned}$$

The risk of $\hat{T}^{S+}$ is smaller than that of the risk $\hat{T}^{PT}$ when the critical value $\frac{p}{p+2} F_\alpha$ satisfies the opposite inequality in (6.4.37). This means the estimator $\hat{T}^{S+}$ does not always dominate $\hat{T}^{PT}$ under H_0 holds for all α .

Figure 6.6: Graph of Unconditional MSE of $\hat{T}_{OLS}$, $\hat{T}^{PT}$, $\hat{T}^S$ and $\hat{T}^{S+}$

Comparison of $\hat{T}^{S+}$ with $\hat{T}^S$

The MSE difference may be expressed by

$$\begin{aligned}
 & M_3^* - M_4^* \\
 &= \sigma_{zz} \frac{(n-3)}{(m-1)} \left[pE^{(1)} \left\{ (1 - cF_{p+2,m}^{-1}(\Delta_0^2))^2 I(F_{p+2,m}(\Delta_0^2) < c) \right\} \right. \\
 &+ \Delta_0^2 E^{(2)} \left\{ (1 - c_1 F_{p+4,m}^{-1}(\Delta_0^2))^2 I(F_{p+4,m}(\Delta_0^2) < c_1) \right\} \\
 &+ \left. 2E^{(2)} \left\{ (F_{p+2,m}^{-1}(\Delta_0^2) - 1) I(F_{p+2,m}(\Delta_0^2) < c) \right\} \right] \quad (6.4.38)
 \end{aligned}$$

and hence $\hat{T}^S - \hat{T}^{S+} \geq 0$ with the same reason that (6.4.34) holds. Thus we can order the dominance picture under (6.4.37) as follow:

$$\hat{T}^{PT} \succ \hat{T}^{S+} \succ \hat{T}^S \succ \tilde{T}^{OLS} \quad (6.4.39)$$

For reasonable Δ_0^2 near 0, the risk of $\hat{T}^{PT}$ is smaller than the risk of $\hat{T}^{S+}$ for α satisfying (6.4.37) and for $p \geq 3$. Thus neither $\hat{T}^{PT}$ and $\hat{T}^{S+}$ dominates the order except that they share common property that as $\Delta_0^2 \rightarrow \infty$, the risk of both becomes p . However, the MSE of $\hat{T}^{S+}$ is below the MSE of $\hat{T}^{OLS}$ while the MSE of $\hat{T}^{PT}$ exceeds the MSE of $\tilde{\beta}_n$ at some intermediate values of Δ_0^2 depending on α . Figure 6.6 displays the graph of unconditional MSE of the four predictors with $\alpha = 0.15, n = 9, N = 12$ and $p = 4$.

Chapter 7

Conclusion

The thesis contains the application of the preliminary test approach to estimators due to Saleh and Sen (1978-1986) which covers the methodologies of Bancroft (1944), Stein (1956) and James-Stein (1961) in models with measurement errors associated with the standard statistical models.

Based on Fuller (1987) and Cheng and Van Ness (1999), we consider the linear model and introduce the preliminary test estimation (PTE) of the parameters with measurement error (ME model) when the slope of the linear model is suspected, but not sure that it may be equal to 0 (zero), without loss of generality. To overcome the non-identifiability of the model, we have imposed two additional conditions in this thesis: the reliability ratio is known or the measurement variance is known (the reliability ratio is unknown). In every case, we have studied the finite sample as well as asymptotic properties of the preliminary test estimators of the slope and intercept parameters under measurement error model. Through the analysis, graphs and tables, methodology is provided to obtain optimum value of the level of significance for the preliminary test estimator to achieve minimum guaranteed

efficiency. It is further evidenced that the MSE expressions of the estimators are inflated due to the presence of measurement errors.

In addition to the preliminary test estimator (PTE), we have studied the finite sample properties of shrinkage estimator (SE) of the slope and intercept parameters under measurement error model. In general, there is no uniform domination of SE over the PTE for all departures parameter Δ_u^{*2} and every α . When, Δ_u^{*2} is near the origin, SE over performs PTE for $\alpha < 0.20$. Also, at $\Delta_u^{*2} = 0$, the relative efficiency of SE is much higher than that of the PTE except for small α . Further, SE is independent of α . Thus, considering the overall performance of SE relative to PTE, SE is preferable to PTE for practical situation since it produces interpolated estimators which is free from the level of significance, α .

The improved estimation of the regression parameters in a multiple regression model with measurement error is considered when it is suspected that the regression parameter vector may be null-vector with some degree of uncertainty. We consider the unrestricted, restricted, preliminary test, Stein-type estimator, positive-rule Stein estimator for the regression parameters. Comparisons of estimators are made based on the MSE matrices and the weighted risks under quadratic loss functions. Finite sample properties of the estimators in the conditional case as well as unconditional case have been studied. When the reliability matrix is unknown, we introduce Pitman local alternatives to study the asymptotic properties of the estimators based on MSE and as well as risk analysis.

We introduced the study of estimation of slope parameters in a parallelism model. We conclude the thesis with the application of the methodology to survey sampling.

In all cases, it is shown that the positive-rule Stein estimator dominates the usual Stein-type estimator which in turn dominate the unrestricted estimator uniformly

for the number of lines more than three. Neither the restricted estimator nor the preliminary test estimator dominate each other uniformly. Similarly, neither the PTE nor SE (PRSE) dominate each other uniformly. However, when $p \geq 3$, PRSE is the preferred choice for application while S-estimator is a preferable choice when β is near the origin (0), otherwise UE preferable when $p \leq 2$.

The problem of polynomial regression and ANOVA model are not covered as it has many apparent difficulties to be resolved which we consider as open problems for future study.

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Appendix

Conditional First and Second Moments

The joint density function of two random variables X and Y will be denoted by $f(x, y)$ and the conditional density $f_{X|Y}(x|Y = y)$ by $f(x|y)$. Let $E(X)$ be the expected value of X , and using E_y to represent expectation over y , we have the two well-known results

1. $E(X) = E_y[E(X|y)]$
2. $\text{Var}(X) = E_y[\text{Var}(X|y)] + \text{Var}_y[E(X|y)]$.

Central Chi-square and Non-central Chi-square Distributions and Properties

If X is $\mathcal{N}(0, 1)$, then X^2 is distributed as a central chi-square variable with one degree of freedom (D.F.) denoted as χ_1^2 . However, if X is $\mathcal{N}(\mu, \sigma^2)$, then X^2 is distributed as a noncentral chi-square variable $\chi_1^2(\Delta^2)$ with one d.f. and noncentrality parameter $\frac{\mu^2}{2\sigma^2} = \frac{1}{2}\Delta^2$. The pdf of a noncentral chi-square variable with ν D.F. and

noncentrality parameter $\frac{1}{2}\Delta^2$ may be written as

$$h_\nu(\chi^2 : \Delta^2) = \sum_{r \geq 0} \frac{e^{-\frac{\Delta^2}{2}}}{\Gamma(r+1)} \left(\frac{\Delta^2}{2}\right)^r h_{\nu+2r}(\chi^2; 0)$$

where $h_{\nu+2r}(\chi^2; 0)$ is the pdf of the central chi-square distribution with $\nu+2r$ D.F..

The pdf of a central chi-square distribution with ν D.F. may be written as

$$h_\nu(\chi^2 : 0) = \frac{\left(\frac{\chi^2}{2}\right)^{\frac{\nu}{2}-1} e^{-\frac{\chi^2}{2}}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})}$$

The mean and variance of this distribution are ν and 2ν respectively.

Some of the results involving noncentral chi-square distributions are given below in theorem form.

Theorem 1 Let χ_ν^2 be a central chi-square variable with ν degrees of freedom and $\phi(\chi_\nu^2)$ be a measurable function, then

1. $E\left[\chi_\nu^2 \phi(\chi_\nu^2)\right] = \nu E\left[\phi(\chi_{\nu+2}^2)\right].$
2. $E\left[\chi_\nu^4 \phi(\chi_\nu^2)\right] = \nu(\nu+2) E\left[\phi(\chi_{\nu+4}^2)\right].$
3. $E\left[\chi_\nu^{-2} \phi(\chi_\nu^2)\right] = \frac{1}{\nu-2} E\left[\phi(\chi_{\nu-2}^2)\right].$
4. $E\left[\chi_\nu^{-4} \phi(\chi_\nu^2)\right] = \frac{1}{(\nu-2)(\nu-4)} E\left[\phi(\chi_{\nu-4}^2)\right].$

Theorem 2 Let $\chi_\nu^2(\Delta^2)$ be a noncentral chi-square variable with ν degrees of freedom and noncentrality parameter $\frac{\Delta^2}{2}$ and let $\phi(\chi_\nu^2(\Delta^2))$ be a measurable function of $\chi_\nu^2(\Delta^2)$ then

$$E\left[\chi_\nu^2(\Delta^2) \phi(\chi_\nu^2(\Delta^2))\right] = \nu E\left[\phi(\chi_{\nu+2}^2(\Delta^2))\right] + \Delta^2 E\left[\phi(\chi_{\nu+4}^2(\Delta^2))\right]$$

Theorem 3 Let $\chi_\nu^2(\Delta^2)$ be a noncentral chi-square variable with ν degrees of freedom and noncentrality parameter $\frac{\Delta^2}{2}$ and let $\phi(\chi_\nu^2(\Delta^2))$ be a measurable function of $\chi_\nu^2(\Delta^2)$ then

$$E\left\{\left[\chi_\nu^2(\Delta^2)\right]^2 \phi(\chi_\nu^2(\Delta^2))\right\} = \nu(\nu+2)E\left[\phi(\chi_{\nu+4}^2(\Delta^2))\right] \\ + 2(\nu+2)\Delta^2 E\left[\phi(\chi_{\nu+6}^2(\Delta^2))\right] + \Delta^4 E\left[\phi(\chi_{\nu+8}^2(\Delta^2))\right]$$

Theorem 4 If $Z \sim \mathcal{N}(\Delta, 1)$ and $\phi(Z^2)$ is a Borel measurable function, then

$$E\left[Z\phi(Z^2)\right] = \Delta E\left[\phi(\chi_3^2(\Delta^2))\right].$$

Theorem 5 If $Z \sim \mathcal{N}(\Delta, 1)$ and $\phi(Z^2)$ is a Borel measurable function, then

$$E\left[Z^2\phi(Z^2)\right] = E\left[\phi(\chi_3^2(\Delta^2))\right] + \Delta^2 E\left[\phi(\chi_5^2(\Delta^2))\right].$$

Theorem 6 Let $\mathbf{Z} = (Z_1, \dots, Z_p)'$ be p -dimensional normal vector distributed as $\mathcal{N}_p(\boldsymbol{\mu}, \mathbf{I}_p)$, then for a measurable function ϕ , we have

$$E\left[\mathbf{Z}\phi(\mathbf{Z}'\mathbf{Z})\right] = \boldsymbol{\mu} E\left[\phi(\chi_{p+2}^2(\Delta^2))\right].$$

Theorem 7 Let $\mathbf{Z} = (Z_1, \dots, Z_p)'$ be p -dimensional normal vector distributed as $\mathcal{N}_p(\boldsymbol{\mu}, \mathbf{I}_p)$, then for a measurable function ϕ , we have

$$E\left[\mathbf{Z}\mathbf{Z}'\phi(\mathbf{Z}'\mathbf{Z})\right] = \mathbf{I}_p E\left[\phi(\chi_{p+2}^2(\Delta^2))\right] + \boldsymbol{\mu}\boldsymbol{\mu}' E\left[\phi(\chi_{p+4}^2(\Delta^2))\right].$$

Theorem 8 Let $\mathbf{Z} = (Z_1, \dots, Z_p)'$ be p -dimensional normal vector distributed as $\mathcal{N}_p(\boldsymbol{\mu}, \mathbf{I}_p)$, then for a measurable function ϕ , we have for a positive definite matrix,

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$$E\left[\mathbf{Z}'\mathbf{A}\mathbf{Z}\phi(\mathbf{Z}'\mathbf{Z})\right] = \text{tr}(\mathbf{A}) E\left[\phi(\chi_{p+2}^2(\Delta^2))\right] + \boldsymbol{\mu}'\mathbf{A}\boldsymbol{\mu} E\left[\phi(\chi_{p+4}^2(\Delta^2))\right].$$

Some Identities:

1. $E[\chi_{p+2}^{-2}(\Delta^2)] = \sum_{r \geq 0} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r (p+2r)^{-1} = E_r(p+2r)^{-1}$, where E_r stands for the expectation with respect to a Poisson variable r with mean $\frac{\Delta^2}{2}$.
2. $E[\chi_{p+2}^{-4}(\Delta^2)] = \sum_{r \geq 0} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r [(p+2r)(p+2r-2)]^{-1}$
 $= E_r[(p+2r)(p+2r-2)]^{-1}$,
3. $\Delta^2 E[\chi_{p+2}^{-2}(\Delta^2)] = 1 - (p-2)E[\chi_p^2(\Delta^2)]$,
4. $\Delta^2 E[\chi_{p+4}^{-4}(\Delta^2)] = E[\chi_{p+2}^{-2}(\Delta^2)] - (p-2)E[\chi_{p+2}^{-4}(\Delta^2)]$
5. $2E[\chi_{p+2}^{-4}(\Delta^2)] = E[\chi_p^{-2}(\Delta^2)] - E[\chi_{p+2}^{-2}(\Delta^2)]$,
6. $E[\chi_p^{-2}(\Delta^2)] \leq E[\chi_p^{-2}] = \frac{1}{p-2}$. ($p > 2$).

Non-central F Distributions and Properties

The noncentral F distribution with ν_1, ν_2 degrees of freedom and noncentrality parameters Δ^2 is defined as the distribution of the ratio, denoted by $F_{\nu_1, \nu_2}(\Delta^2)$, $\frac{\chi_{\nu_1}^2(\Delta^2)/\nu_1}{\chi_{\nu_2}^2/\nu_2}$ while the ratio denoted by $G_{\nu_1, \nu_2}(\Delta^2)$ is defined as $\frac{\chi_{\nu_1}^2(\Delta^2)}{\chi_{\nu_2}^2}$. Note that

$$p_{G_{\nu_1, \nu_2}(\Delta^2)}(g) = \sum_{r=0}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r p_{G_{\nu_1+2r, \nu_2}}(g)$$

and

$$\begin{aligned} P(F_{\nu_1, \nu_2}(\Delta^2) \leq f_0) &= P(G_{\nu_1, \nu_2}(\Delta^2) \leq \frac{\nu_1}{\nu_2} f_0) \\ &= \sum_{r=0}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r I_{\frac{\nu_1 f_0}{\nu_2 + \nu_1 f_0}}\left(\frac{1}{2}\nu_1 + r; \frac{1}{2}\nu_2\right) \end{aligned}$$

where $p_{G_{\nu_1, \nu_2}}(g)$ is the density function of the central G distribution with ν_1, ν_2 degrees of freedom. Then we may have the following expectations.

1. $G_{\nu_1, \nu_2}(g_0 : \Delta^2) = \sum_{r=0}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r I_{\frac{\nu_2 g_0}{\nu_2 + \nu_2 g_0}}\left(\frac{1}{2}\nu_1 + r; \frac{1}{2}\nu_2\right)$
2. $G_{\nu_1, \nu_2}(g_0 : 0) = I_{\frac{\nu_2 g_0}{\nu_2 + \nu_2 g_0}}\left(\frac{1}{2}\nu_1; \frac{1}{2}\nu_2\right)$
3. $E(F_{\nu_1, \nu_2}(\Delta^2)) = \frac{\nu_2(\nu_1 + \Delta^2)}{\nu_1(\nu_2 - 2)}$ where $\nu_2 > 2$
4. $E(F_{\nu_1, \nu_2}^{-1}(\Delta^2)) = \sum_{r=0}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r \frac{\nu_1}{\nu_1 + 2r - 2}$
5. $E[F_{\nu_1, \nu_2}^{-1}(\Delta^2)]I(F_{\nu_1, \nu_2}(\Delta^2) \leq c)$
 $= \sum_{r=0}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r \frac{\nu_1}{\nu_1 + 2r - 2} I_{\frac{\nu_1 c}{\nu_2 + \nu_1 c}}\left(\frac{\nu_1 + 2r - 2}{2}; \frac{\nu_2 + 2}{2}\right)$
6. $E[F_{\nu_1, \nu_2}^{-2}(\Delta^2)]I(F_{\nu_1, \nu_2}(\Delta^2) \leq c) = \sum_{r=0}^{\infty} \frac{e^{-\frac{1}{2}\Delta^2}}{\Gamma(r+1)} \left(\frac{1}{2}\Delta^2\right)^r$
 $\times \frac{\nu_1^2(\nu_2 + 2)}{\nu_2(\nu_1 + 2r - 2)(\nu_1 + 2r - 4)} I_{\frac{\nu_1 c}{\nu_2 + \nu_1 c}}\left(\frac{\nu_1 + 2r - 4}{2}; \frac{\nu_2 + 4}{2}\right)$

Some Multivariate Normal Theory Results

Let $\mathbf{x}$ be a p -variate normal vector with mean vector $\boldsymbol{\mu}$ and covariate Σ be denoted by $\mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$. If $\mathbf{x}_1, \dots, \mathbf{x}_N$ be a random sample of size N from $\mathcal{N}_p(\boldsymbol{\mu}, \Sigma)$.

The joint pdf of $\mathbf{x}_1, \dots, \mathbf{x}_N$ is given by

$$\left\{ (2\pi)^{\frac{p}{2}} |\Sigma| \right\}^{-\frac{N}{2}} \exp \left\{ -\frac{1}{2} \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x}_{\alpha} - \boldsymbol{\mu}) \right\}$$

Let $\bar{\mathbf{x}} = \frac{1}{N} \sum_{\alpha=1}^N \mathbf{x}_{\alpha}$ and let

$$\mathbf{V} = \sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})'$$

Further, we have the decomposition given by

$$\sum_{\alpha=1}^N (\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})' = \mathbf{V} + N(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})(\mathbf{x}_{\alpha} - \bar{\mathbf{x}})'$$

We may note that $\bar{\mathbf{x}}$ and $\frac{1}{N-1}\mathbf{V}$ are unbiased estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ respectively while $\frac{1}{N}\mathbf{V}$ is the MLE of $\boldsymbol{\Sigma}$ and is a biased estimator of $\boldsymbol{\Sigma}$. With respect to $\mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the following properties may be noted.

1. $\bar{\mathbf{X}} \sim \mathcal{N}_p(\boldsymbol{\mu}, \frac{1}{N}\boldsymbol{\Sigma})$
2. $\mathbf{V} \sim W_p(\boldsymbol{\Sigma}, n)$ where $W_p(\boldsymbol{\Sigma}, n)$ stands for the Wishart distribution with $n=N-1$ D.F.. Then the following results may be verified

(a) $E[\mathbf{V}] = n\boldsymbol{\Sigma}$,

(b) $E[\mathbf{VAV}] = n(n+1)\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} + n\text{tr}(\mathbf{A}\boldsymbol{\Sigma})\boldsymbol{\Sigma}$, where $\mathbf{A}$ is symmetric, and

(c) $E[\mathbf{V}^{-1}] = \frac{1}{n-p-1}\boldsymbol{\Sigma}^{-1}$.

- (d) Let (V^{ij}) is the reciprocal of $\mathbf{V} = (V_{ij})$ and (σ^{ij}) the reciprocal of $(\sigma_{ij}) = \boldsymbol{\Sigma}$ exist, then

$$\frac{\sigma^{pp}}{V^{pp}} \sim \chi_{n-p+1}^2 \text{ and is independent of } (V_{ij}),$$

where $i, j = 1, 2, \dots, (p-1)$

- (e) for any $p \times 1$ fixed vector $\mathbf{L}$,

$$\frac{\mathbf{L}'\mathbf{A}\mathbf{L}}{\mathbf{L}'\boldsymbol{\Sigma}\mathbf{L}} \sim \chi_n^2,$$

- (f) for any $p \times 1$ fixed vector $\mathbf{L}$,

$$\frac{\mathbf{L}'\boldsymbol{\Sigma}^{-1}\mathbf{L}}{\mathbf{L}'\mathbf{A}^{-1}\mathbf{L}} \sim \chi_{n-p+1}^2.$$

3. Let $T^2 = Nn\bar{\mathbf{x}}'\mathbf{V}^{-1}\bar{\mathbf{x}}$, then

$$\frac{n-p+1}{p} \frac{T^2}{n} = \frac{N-p}{p} \frac{T^2}{n} \stackrel{\mathcal{D}}{=} F_{p, N-p}(\Delta^2)$$

where $\mathcal{D}$ stands for "equal in distribution" and $F_{p,N-p}(\Delta^2)$ stands for a non-central F -distribution with $(p, N-p)$ D.F. and noncentrality parameter $\frac{\Delta^2}{2}$ where $\Delta^2 = N\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$. Further, we note that

$$\frac{T^2}{n} = \frac{N\bar{\mathbf{x}}'\mathbf{V}^{-1}\bar{\mathbf{x}}}{N\bar{\mathbf{x}}'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}}} N\bar{\mathbf{x}}'\boldsymbol{\Sigma}^{-1}\bar{\mathbf{x}} \stackrel{\mathcal{D}}{=} \frac{\chi_p^2(\Delta^2)}{\chi_{N-p}^2} = \frac{N-p}{p} F_{p,N-p}(\Delta^2)$$

Here $F_{m,n}(\Delta^2)$ stands for the noncentral F -variable with (m, n) D.F. and noncentral parameter $\frac{\Delta^2}{2}$.

Theorem 9 For $\mathbf{X} \sim \mathcal{N}_p(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ then

$$E[\mathbf{X}'\mathbf{A}\mathbf{X}] = \text{tr}[\mathbf{A}\boldsymbol{\Sigma}] + \boldsymbol{\theta}'\mathbf{A}\boldsymbol{\theta}$$

Theorem 10 Let $\mathbf{C} \sim W_p(n, \boldsymbol{\Sigma})$, then for any $p \times 1$ vector z and real-valued differential function h , then we have

$$E[h(z'\mathbf{C}z)\mathbf{C}^{-1}] = \frac{n-2}{n-p-1} E\left[\frac{h(z'\boldsymbol{\Sigma}z\chi_n^2)}{\chi_n^2}\right] \boldsymbol{\Sigma}^{-1}$$

Beta Distribution and Applications

Let $\chi_{\nu_1}^2$ and $\chi_{\nu_2}^2$ be two independent central chi-square variables with ν_1 and ν_2 D.F. respectively. Define

$$X \sim \frac{\chi_{\nu_1}^2}{\chi_{\nu_1}^2 + \chi_{\nu_2}^2},$$

then X distributed as a Beta-variable with the pdf given by

$$\frac{1}{B\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)} x^{\frac{\nu_1}{2}-1} (1-x)^{\frac{\nu_2}{2}-1}, \quad 0 < x < 1.$$

and we have the following expectations.

1. $E(X) = \frac{\nu_1}{\nu_1 + \nu_2}$
2. $E(X^{-1}) = \frac{\nu_1 + \nu_2 - 2}{\nu_1 - 2}$

Matrix Results

Let $\mathbf{A}$ be an $n \times n$ matrix, then the roots of the determinantal (characteristic) equation

$$|\mathbf{A} - \lambda \mathbf{I}| = 0$$

are called the characteristic roots of $\mathbf{A}$ or eigenvalues of $\mathbf{A}$.

Theorem 11 If $\mathbf{A}$ be an $n \times n$ matrix, then there exists an orthogonal matrix $\mathbf{\Gamma}$ such that $\mathbf{\Gamma}'\mathbf{A}\mathbf{\Gamma} = \mathbf{D} = \text{diag}(d_1, \dots, d_n)$. If $\mathbf{A}$ is positive semidefinite then $d_i \geq 0, i = 1, \dots, n$ and if $\mathbf{A}$ is positive definite then $d_i > 0, i = 1, \dots, n$.

Theorem 12 Consider the partition of the nonsingular matrix $\mathbf{A}$ given by

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

where $\mathbf{A}_{22}$ is nonsingular. Let $\mathbf{A}_{11.2} = \mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}$. Then

$$\mathbf{A}^{-1} = \begin{pmatrix} \mathbf{A}_{11.1}^{-1} & -\mathbf{A}_{11.2}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} \\ -\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11.2} & \mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11.2}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} + \mathbf{A}_{22}^{-1} \end{pmatrix}$$

Further,

$$\mathbf{A}_{22.1}^{-1} = \mathbf{A}_{22}^{-1}\mathbf{A}_{21}\mathbf{A}_{11.2}^{-1}\mathbf{A}_{12}\mathbf{A}_{22}^{-1} + \mathbf{A}_{22}^{-1}.$$

Theorem 13 If $\mathbf{A}$ is an $n \times n$ positive definite matrix and $\mathbf{B}$ is an $n \times m$ ($m < n$) matrix with rank m , then $\mathbf{B}'\mathbf{A}\mathbf{B}$ is positive definite.

If $\mathbf{A}$ is an $n \times n$ idempotent matrix, i.e. $\mathbf{A} = \mathbf{A}'$ and $\mathbf{A}\mathbf{A} = \mathbf{A}$ then if the rank of $\mathbf{A}$ is r , $\mathbf{A}$ has r positive roots and $n - r$ roots equal to 0, the rank of $\mathbf{A}$ is equal to $\text{tr}(\mathbf{A})$ and there exists an orthogonal matrix $\mathbf{\Gamma}$ such that

$$\mathbf{\Gamma}'\mathbf{A}\mathbf{\Gamma} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{n-r} \end{pmatrix}.$$

Theorem 14 If $\mathbf{A}$ and $\mathbf{B}$ are two symmetric matrices, a necessary and sufficient condition for an orthogonal matrix $\mathbf{\Gamma}$ to exist such that $\mathbf{\Gamma}'\mathbf{A}\mathbf{\Gamma} = \mathbf{D}_1$ and $\mathbf{\Gamma}'\mathbf{B}\mathbf{\Gamma} = \mathbf{D}_2$ where $\mathbf{D}_1$ and $\mathbf{D}_2$ are both diagonal matrices, is that $\mathbf{A}$ and $\mathbf{B}$ satisfy $\mathbf{AB} = \mathbf{BA}$ i.e. $\mathbf{A}$ and $\mathbf{B}$ commute.

Theorem 15 [Courant Theorem] Let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of an $n \times n$ matrix $\mathbf{A}$ such that $\min \lambda_i = \lambda_1, \max \lambda_i = \lambda_n$ and let $v_1, \dots, v_n$ be the characteristic vectors, then $\mathbf{A} = \lambda_1 v_1 v_1' + \dots + \lambda_n v_n v_n', \mathbf{I} = v_1 v_1' + \dots + v_n v_n' : \sup \left(\frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}\mathbf{x}'} \right) = \lambda_n$ and $\inf \left(\frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}\mathbf{x}'} \right) = \lambda_1$. Hence,

$$Ch_{\min}(\mathbf{A}) \leq \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}\mathbf{x}'} < Ch_{\max}(\mathbf{A})$$

where

$$\min_i \lambda_i = Ch_{\min}(\mathbf{A}) \quad \text{and} \quad \max_i \lambda_i = Ch_{\max}(\mathbf{A})$$

Theorem 16 If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{b}$ is a $n \times 1$ vector, then

$$\max_{\mathbf{x}} \frac{\mathbf{x}'\mathbf{b}\mathbf{b}'\mathbf{x}}{\mathbf{x}'\mathbf{A}\mathbf{x}} = \mathbf{b}'\mathbf{A}^{-1}\mathbf{b}.$$

Theorem 17 If $\mathbf{A}$ is an $n \times n$ positive definite matrix and $\mathbf{b}$ is a $n \times 1$ vector, a necessary and sufficient condition for $(\mathbf{A}^{-1} - \mathbf{b}\mathbf{b}')$ to be positive definite is that $\mathbf{b}'\mathbf{A}\mathbf{b}$ is less than 1. Further, $(\mathbf{b}\mathbf{b}' - \mathbf{A}^{-1})$ cannot be negative definite for n exceeding 1.

Theorem 18 Let A_1 and A_2 be nonsingular matrices of order p_1 and p_2 and B be $p_1 \times p_2$ matrix, then

$$(A + BDB')^{-1} = A^{-1} - A^{-1}B(B'A^{-1}B + D^{-1})^{-1}B'A^{-1}$$

Theorem 19 If A and D are square matrix such that A is non-singular, then

$$\begin{vmatrix} A & C \\ B & D \end{vmatrix} = |A||D - BA^{-1}C|$$

Theorem 20 If A be a $p \times p$ matrix and U a column $p \times 1$ vector, then

$$|A + UU'| = |A||I + U'A^{-1}U| \quad \text{if} \quad |A| \neq 0$$

Theorem 21

$$\text{rank}[A] = \text{rank}[A'] = \text{rank}[A'A] = \text{rank}[A'A]$$

If A be a symmetric matrix and rank 1, then

$$(i) \quad |I + A| = 1 + \text{tr}(A)$$

$$(ii) \quad |A| = \lambda_j, \text{ where } \lambda_j : \text{non-zero eigenvalue.}$$

If A be a symmetric and idempotent matrix with rank 1, then it has only one eigenvalue ($= 1$).

Theorem 22 [Spectral Decomposition] Let A be an $n \times n$ symmetric and positive definite matrix. There exists an orthogonal matrix $T = (t_1, \dots, t_n)$ such that

$$T'AT = \text{diag}(\lambda_1, \dots, \lambda_n) = \Lambda$$

where $\lambda_1 \geq \cdots \geq \lambda_n$ are the ordered eigenvalues of $\mathbf{A}$.

Theorem 23 [Cholesky Decomposition] If $\mathbf{A}$ is positive definite, there exists a unique lower triangular matrix $\mathbf{T}(t_{ij} = 0, i < j)$ with positive diagonal elements such that $\mathbf{A} = \mathbf{T}\mathbf{T}'$.