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1.1 Parity Violating Effect in Nuclei

The influence of the known weak interaction on the parity impurity of nuclear states has been investigated quite intensely. We first consider the older models based on the Cabbibo theory. In this theory, the weak interaction can be written as a current-current form with the Hamiltonian density given as⁴:

$$H_w(x) = \frac{G}{\sqrt{2}} (\bar{\psi}_n \gamma_\mu (1+i\gamma_5) \psi_p) (\bar{\psi}_p \gamma_\mu (1+i\gamma_5) \psi_n)$$

Parity is not a good quantum number with such a Hamiltonian or in other words, the states of nuclei are not eigen states of parity with this type of interaction. Hence, we can write the nuclear state $|\psi^J\rangle$ as a dominant state of spin J and parity π $|\phi^{J\pi}\rangle$ mixed with a state $|\phi^{J-\pi}\rangle$ of opposite parity.

$$|\psi^J\rangle = |\phi^{J\pi}\rangle + F|\phi^{J-\pi}\rangle$$

Here F is the mixing amplitude which has the value $\sim 10^{-6}$ in the current-current hypothesis. The theoretical deduction of parity violating potential was first carried out by Blin Stoye⁵. Later, a variety of current models have been considered by Tadic⁶, Michel⁷ and others. Henley⁸ and others⁹ have reviewed the works on parity violating nuclear interactions from an experimental point of view.

But with this weak interaction Hamiltonian one faces difficulty in explaining the experimental facts involving non-leptonic decays. The

* The notation from Sakurai³⁰ has been used.

PARITY VIOLATION EFFECTS FROM ELECTRON
SCATTERING ON NUCLEI AND ATOMS

by

Sikha Roy

A thesis submitted to the Faculty of Graduate Studies
in partial fulfilment of the requirements for the
degree of Master of Science.

Carleton University

Ottawa, Ontario

August 1979

The undersigned hereby recommend to the Faculty of Graduate Studies acceptance of the thesis,

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ABSTRACT

Parity violation effects due to weak interaction in nuclei and atoms, giving rise to the asymmetries in the scattering of left and right polarized electrons have been reviewed. It has been found that the asymmetry is large at low momentum transfer, if nuclei contain admixture of different parity states. A general formula for the asymmetry of scattering of polarized electrons from mixed parity state has been derived. This formula has been applied in the case of scattering from $2S_{1/2} + F 2P_{1/2}$ state of hydrogen atom and the value of the asymmetry has been found to be very small.

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INTRODUCTION

Mirror symmetry, viz, the invariance of a system under space reflection is observed in almost all physical phenomena known to us. It is also known from the theoretical and experimental investigations that the non-invariance of a system under space reflection may be observed in some cases. Lee and Yang¹ first pointed out that the parity invariance does not hold in the weak interactions responsible for β decay and the decay of strange particles. Experiments carried out indeed showed this to be true. Among the basic physical interactions namely strong, electromagnetic, weak and gravitational, it is the weak one which has, so far, been known to be responsible for the breaking of mirror symmetry. In other words, although the parity operator P commutes with the Hamiltonian in strong and electromagnetic interactions, i.e.

$$[H_{\text{strong \& e.m.}}, P] = 0$$

it is not so in the case of weak interactions, i.e.

$$[H_{\text{weak}}, P] \neq 0$$

The parity violating effect due to the weak force between hadrons and leptons in nuclei and atoms comes into play in either of the following two ways:

- 1) In the lowest order of weak interaction, it would arise from the interference of interaction processes involving

the exchange of a photon and those mediated by a neutral heavy boson. This parity violating effect is appreciable (of the order of a few times 10^{-5}) only in high energy regions where q^2 is typically of the order of a few Gev^2/c^2 .

- 2) The admixture of two different parity states due to weak interaction results in the states which are not eigen states of parity and hence leads to parity non-conservation effect. This is sizeable only at low momentum transfer. We are mainly interested in finding the strength of the effect arising due to the second case.

The scattering of low energy polarized electron beam from mixed parity states of nuclei and atoms is usually employed to study the parity non-conservation effect. The axis of spin quantization is chosen along the direction of the incoming electron beam. The electron with positive helicity has its spin direction parallel to the direction of the beam ($\vec{\sigma} \cdot \vec{k} = +1$) and the one with negative helicity has its spin direction anti-parallel to $\vec{k}$. The electron-nucleus (atom) scattering cross-section depending on the helicity of the incoming electron differs in two cases and hence gives rise to left-right asymmetry:

$$\frac{\frac{d\sigma}{d\Omega}|R - \frac{d\sigma}{d\Omega}|L}{\frac{d\sigma}{d\Omega}|R + \frac{d\sigma}{d\Omega}|L}$$

We know that there are two types of weak currents - the charged current and the neutral one. In the case of weak coupling between nucleons, both of them contribute in the admixture of two different

parity states. The contribution due to weak charged current can be calculated by Cabbibo theory². As far as the effect of weak neutral currents is concerned, they produce an enhancement of the values expected from Cabbibo model calculation. Thus the asymmetry due to weak neutral current can be obtained by subtracting the value obtained in Cabbibo theory from the value given by Weinberg Salam³ model which takes into account both of the currents.

On the other hand, although atoms are dominated by electromagnetic interactions, it follows from the theory due to Weinberg and Salam that observable parity violating effect due to weak interaction should also occur. The nucleon-nucleon and electron-electron weak couplings are assumed to have a negligible effect on atomic energy levels and as the parity-conserving part of the electron-nucleon weak interaction is masked by e.m. interaction, the experiments measure only the parity non-conserving part of the electron-nucleon weak neutral current. Hence, in finding the contribution due to weak neutral currents, the atomic case is much simpler to deal with than the nuclear one.

In this thesis, we have first reviewed the work done in the case of nuclei. Then, applying the general technique as is used for nuclei, we have calculated the left-right asymmetry in low energy polarized electron scattering from mixed parity state of atom and thus studied the effect of weak neutral current alone. Thus, the thesis is basically one of an expository nature. But, in the case of atoms, the amplitude of mixing of two states with opposite parity is smaller than that in nuclei.

As a result, compared to nuclear asymmetry, we expect the value of the asymmetry in the electron-atom scattering to be much smaller.

Existence of two neighbouring levels of $2S_{1/2}$ and $2P_{1/2}$ with opposite parity in hydrogen atom affords an opportunity for an investigation of the parity non-conserving weak interaction. Such an interaction leads to the mixing of $2P_{1/2}$ state with $2S_{1/2}$ state. As a result, there is an E1 radiation (due to $2P_{1/2} \rightarrow 1S_{1/2}$ transition) admixture in the M1 radiation emitted in $2S_{1/2} \rightarrow 1S_{1/2}$ transition. Here we have calculated the values of the cross-sections of scattering of left and right polarized electrons from H atom. The choice of hydrogen atom is mainly because of the existence of well-defined states which makes the calculations relatively simple. Thus, it enables us to see the feasibility of low energy experiment to study the effect of weak neutral current without going into more complicated calculations required for heavy atoms.

In Chapter 1, we have reviewed what has been done on the non-conservation of parity in nuclei and atoms on both theoretical and experimental sides. In Chapter 2, we have derived a general expression for the cross-section of inelastic scattering of left and right circularly polarized electrons from a mixed parity state in terms of Coulomb electric and magnetic multipole operators. From the expressions of the cross-sections for left and right polarized electrons, we have calculated the asymmetry. In Chapter 3, based upon the general discussion in Chapter 2, we have investigated the values of the asymmetry in electron-nucleus

scattering. In Chapter 4, taking the initial state as $2S_{1/2} + F2P_{1/2}$ and final state as $1S_{1/2}$ for hydrogen atom we have evaluated the three multipole operators. With these values, the values of cross-sections and hence the asymmetry are numerically calculated for various incident energies and scattering angles. Chapter 5 gives the concluding remarks along with different limitations.

CHAPTER 1

REVIEW ON PARITY NON-CONSERVATION IN NUCLEI AND ATOMS

There has been a great deal of work done both from experimental and theoretical points of view, to study the effect of parity non-conserving weak force in nuclei and atoms. The parity violating effect in electron scattering from atoms and nuclei originates in two ways depending on the magnitude of the momentum transfer. For very high momentum transfer, it originates from the interference of interaction processes involving the exchange of a photon and those mediated by a neutral heavy boson Z^0 . Fig. 1 shows the Feynman diagram of the two processes in lowest order.

Fig. 1

The parity admixture in the initial or final state contributes to the parity violating effect in electron scattering for low momentum transfer. This is depicted in Fig. 2.

Fig. 2

$$\begin{aligned}
\left. \frac{d\sigma}{d\Omega} \right|_L &= \frac{8\pi\alpha^2}{q_\mu} \left(\frac{k_2}{k_1} \right) [V_T(\theta)] \sum_{J=1}^{\infty} \frac{||T^{e1}||^2 + ||T^m||^2}{2J_i+1} \\
&+ V_L(\theta) \sum_{J=0}^{\infty} \frac{||M^{Coul}||^2}{2J_i+1} - 2 V_P(\theta) \sum_{J=1}^{\infty} \frac{||T^{e1}|| ||T^m||^*}{2J_i+1} \left] \frac{1}{1 + \frac{k_2 - k_1 \cos \theta}{E'}}
\end{aligned}
\quad (2.24)$$

2. Asymmetry

Parity admixtures of states in nucleus or atom gives rise to an asymmetry in the inelastic scattering cross-sections for right and left polarized electrons from the mixed state. The asymmetry is defined as:

$$A(q^2) = \frac{\left. \frac{d\sigma}{d\Omega} \right|_R - \left. \frac{d\sigma}{d\Omega} \right|_L}{\left. \frac{d\sigma}{d\Omega} \right|_R + \left. \frac{d\sigma}{d\Omega} \right|_L} \quad (2.25)$$

Substituting the values of $\left. \frac{d\sigma}{d\Omega} \right|_R$ and $\left. \frac{d\sigma}{d\Omega} \right|_L$ from Eqns. (2.22) and (2.24) in the expression (2.25) for asymmetry, we obtain:

$$A(q^2) = \frac{2V_P(\theta) \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{[V_L(\theta) \sum_{J=0}^{\infty} ||M^{Coul}||^2 + V_T(\theta) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}]}$$

where the values of $V_T(\theta)$, $V_L(\theta)$ and $V_P(\theta)$ in the laboratory system are given by the Eqns. (2.16a), (2.16b) and (2.20a) respectively.

In the non-relativistic limit, which is the case of scattering of low energy electrons, the values of $V_T(\theta)$, $V_L(\theta)$ and $V_P(\theta)$ are given by Eqns. (2.17a), (2.18a) and (2.20a) respectively. Using these values, we can write the asymmetry as:

decays of kaon and hyperons demonstrate that the current-current model must be extended to give $\Delta I = \frac{1}{2}$ rule and must suppress the $\Delta I = \frac{3}{2}$ part of the strangeness changing processes. It is for these difficulties the weak neutral current is introduced. A neutral current is one (like the electromagnetic current) which does not change the total charge of any state on which it acts. In Weinberg-Salam's theory³, the weak current has both charged (V-A) and neutral parts.

The experimental evidence¹⁰ for the existence of weak neutral currents in neutrino-hadron scattering has renewed interest in the question of whether such currents can produce measurable effects in charged lepton-hadron reactions. The deep inelastic electron-nucleus scattering is usually employed. The interference of the interaction processes involving a neutral heavy boson Z^0 and the electromagnetic interaction involving photon gives rise to an asymmetry in the inclusive cross-sections for beams¹¹ (or targets¹²) polarized parallel and anti-parallel to the lepton beam momentum. Polarized electron-nucleon and electron-nucleus scattering have been considered by Reya and Schilcher¹³ and by Feinberg¹⁴, respectively. For electron induced processes, the effects of the neutral boson are, in general, of the order of $G^2 q^2 / e^2$ where q^2 is the momentum transfer and G is the weak Fermi coupling constant. Most of the theoretical estimates concerning the contribution due to weak neutral current make use of Weinberg's $SU(2) \times U(1)$ model. Again, using the quark parton model and some general features of weak currents, Wilson¹⁵ has derived a general

formula for the size of the parity violating effect in nuclei. Theoretical calculations show that the size of the asymmetry is very sensitive to the parameters of the model used.

Porrmann and Gari¹⁶ have considered the low energy electron-nucleus scattering as a test for parity-violating effect arising at low momentum transfer. They investigated the excitation of nuclei by longitudinally polarized electrons in view of parity mixture of nuclear states. Gari and Porrmann found that the asymmetry in the cross-section for left and right polarized electrons is significantly large at low momentum transfer and also in some cases they found that the asymmetries caused by parity mixture are comparable to the contribution expected from weak Z^0 boson exchange. We shall discuss this in detail in Chapter 3.

1.2 Parity Violating Effect in Atoms

The parity violating potential between an electron and a nucleus having Z protons and N neutrons associated with the exchange of a heavy vector boson Z^0 has the following form¹⁷:

$$V_{pv} = \frac{G}{4\sqrt{2} m_e} (\bar{\sigma} \cdot \bar{p} \delta^3(r) + \delta^3(r) \bar{\sigma} \cdot \bar{p}) Q$$

where m_e , $\bar{\sigma}$, $\bar{p}$ and r are respectively the mass, spin, momentum and position of the electron, G is the weak coupling constant. The contribution due to nuclear spin is assumed to be negligible. The effective weak charge Q depends on the choice of particular model of weak interaction. For Weinberg-Salam theory,

$$Q = Q_w = -N + (1 - 4 \sin^2 \theta_w) Z$$

Because of the short range nature of the weak force, the parity mixing occurs only between S and P states. Hence, $S_{1/2}$ state becomes:

$$|\bar{S}_{1/2}\rangle = |S_{1/2}\rangle + F |P'_{1/2}\rangle$$

The impurity of opposite parity can be calculated by perturbation theory:

$$F |P'_{1/2}\rangle = \sum_n |nP_{1/2}\rangle \frac{\langle n P_{1/2} | V_{pv} | S_{1/2} \rangle}{E_s - E_{np}}$$

Bouchiat¹⁸ have made an estimate of F in the quantum defect approximation. The expression for the matrix element of V_{pv} between two states is:

$$\langle nP_{1/2} | V_{pv} | S_{1/2} \rangle = i \frac{G Z^2 Q_w k}{\sqrt{2} 4\pi m_e a_0^4 (\nu_{np} \nu_s)^{3/2}}$$

where a_0 is the Bohr radius, ν is an effective radial quantum number given in terms of the quantum defect and k is a relativistic correction. The presence in the matrix element of V_{pv} of the factor $Z^2 Q_w$ favours very strongly heavy atoms.

Since the parity violating mixing in atomic states is tiny, it is necessary to find experimental consequences in which the effects are enhanced and appear to lowest order. One standard procedure to investigate the parity mixing in atomic wave functions is to look in the radiative transition between unpolarized atomic states for a dependence of the

transition probability on the state of circular polarization of the absorbed photon. One observes the resonance absorption of circularly polarized photons by the atom. A transition which would normally be magnetic dipole with amplitude $M1$, gets through parity mixing with neighbouring states, a small electric dipole $E1$. This interference of the $E1$ and $M1$ matrix elements gives rise to a different rate of absorption for left and right circularly polarized photons. The circular polarization is given as:

$$P_c = 2 \operatorname{Im}\left(\frac{E1}{M1}\right) = 2 \frac{|E1|}{|M1|}$$

∴ time reversal invariance holds.

The parity violating effect can be enhanced by considering transitions in which the normal rate is strongly retarded. With tunable lasers one can contemplate the possibility of exciting twice forbidden magnetic transitions between S states and hence get enhanced effect due to weak interaction. Specific laser induced transitions in a variety of atoms have been suggested. Bouchiat and Bouchiat have considered the excitation of the transition photon $+6S_{1/2} \rightarrow 7S_{1/2}$ in atomic cesium. The neutral current effects in atoms are strongly enhanced with increasing atomic number. $6P_{1/2} \rightarrow 6P_{3/2}$ transition in Tl, $(6P^2)^3P_0 \rightarrow (6P^2)^1S_0$ transition in lead have been investigated experimentally.

Moreover the weak interaction leads to a rotation of the plane of polarization of visible light by any substance not containing optically active molecules. The situation concerning the rotation of the plane of polarization in Bi is somewhat confused. First, the Oxford¹⁹ and

Seattle²⁰ groups reported experimental results which were considerably smaller than the value predicted on the basis of Weinberg-Salam model of weak interactions combined with relativistic central-field atomic theory. Subsequently, the Novosibirsk²¹ group have reported results which are in agreement with the expectation from Weinberg-Salam model. The Oxford²² group also has recently reported observing optical rotation of right magnitude in Bi.

Since atoms interact rather strongly with their environment, there would be a possible enhancement of parity non-conserving effect by suitable external perturbation. Azimov and others²³ have studied the electromagnetic radiation by hydrogen like atoms undergoing $2S_{1/2} \rightarrow 1S_{1/2}$ transition in the presence of external magnetic field. When hydrogen atom is located in a magnetic field of about 1 kilo Gauss the shift of $2S_{1/2}$ and $2P_{1/2}$ levels increase the degree of circular polarization of the photons by a factor of five as compared to the value of zero field. Furthermore, one can observe the anisotropy in the photon emission with respect to the direction of the magnetic field and hence can judge the form of the weak interaction. Lewis and Williams²⁴ have carried out experiments with metastable hydrogen atom in electric and magnetic field and found that the parity non-conserving effect though small can be measured in hydrogen atom.

$$A(q^2) = \frac{\frac{2}{|\bar{q}|} \left[\left(\frac{k_1^2 + k_2^2}{2} \right) (k_2 \cos \theta - k_1) + k_1 k_2 (k_1 \cos \theta - k_2) \right] \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \left[\frac{k_1^2 + k_2^2}{2} + k_1 k_2 \cos \theta + 2m^2 \right] \sum_{J=0}^{\infty} ||M^{coul}||^2 + \left(\frac{k_1^2 k_2^2 \sin^2 \theta}{q^2} + \frac{q^2}{2} \right) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}} \quad (2.27)$$

At $\theta=0$, the expression for asymmetry reduces to:

$$A(q^2) \Big|_{\theta=0} = \frac{\frac{2}{|\bar{q}|} \left(\frac{k_2 - k_1}{2} \right)^3 \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \left[\frac{(k_1 + k_2)^2}{2} + 2m^2 \right] \sum_{J=0}^{\infty} ||M^{coul}||^2 + \frac{q^2}{2} \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}}$$

With the increase in incident energy, $(k_2 - k_1)$ decreases and so does the asymmetry. At backscattering angle, i.e. $\theta=\pi$, we have:

$$A(q^2) \Big|_{\theta=\pi} = \frac{-\frac{2}{|\bar{q}|} \left(\frac{k_2 + k_1}{2} \right)^3 \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \left[\frac{(k_2 - k_1)^2}{2} + 2m^2 \right] \sum_{J=0}^{\infty} ||M^{coul}||^2 + \frac{q^2}{2} \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}}$$

In the relativistic limit, neglecting the mass of electron, the asymmetry is given as:

$$A(q^2) = \frac{-2 \left(\frac{\epsilon_1 + \epsilon_2}{2} \right) \tan^2 \theta / 2 \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \sum_{J=0}^{\infty} ||M^{coul}||^2 + \left(\frac{q_\mu^2}{2q^2} + \tan^2 \theta / 2 \right) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}}$$

Here we have used the values $V_T(\theta)$, $V_L(\theta)$ and $V_p(\theta)$ from Eqns. (2.17b),

The final wave function is:

$$\bar{\psi}_f(x) = \sqrt{\frac{m}{\epsilon_2 \Omega}} e^{-ik_2 \cdot x} \bar{u}(k_2)$$

where $k_2(\bar{k}_2, i\epsilon_2)$ is the four momentum of the scattered electron with energy ϵ_2 .

The Feynman diagram for the interaction is given as:

We shall carry out our discussion in the framework of one photon exchange, i.e. to the lowest order in α , the fine structure constant.

The transition matrix is:

$$S_{fi} = -ie \int d^4x \bar{\psi}_f(x) \gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) \psi_i(x) A_\mu(x) \quad (2.1)$$

The external field A comes from the target nucleus or atom.

$$\square A_\mu(x) = -e J_\mu(x) \quad (2.2)$$

where

$$J_\mu(x) = \langle f | \hat{J}_\mu(x) | i \rangle$$

Here, $\hat{J}_\mu(x)(\bar{J}(x), ip(x))$ is the current operator for the target at the point $x(\bar{x}, it)$. $|i\rangle$ and $|f\rangle$ are the initial and final states of the target respectively one of which has the admixture of two different parity states and hence is not a parity eigen state. As a result, $J_\mu(x)$ is a parity non-conserving operator.

In order to integrate Eqn. (2.2) for $A_\mu(x)$, we introduce a propagator $D_F(x-y)$ defined by:

$$\square D_F(x-y) = \delta^4(x-y)$$

The propagator has the Fourier representation:

$$D_F(x-y) = \int \frac{d^4 q}{(2\pi)^4} \frac{e^{iq \cdot (x-y)}}{q_\mu^2}$$

Hence the solution for the potential according to Eqn. (2.2) is:

$$A_\mu(x) = e \int d^4 y D_F(x-y) J_\mu(y) \quad (2.3)$$

Substituting Eqn. (2.3) in Eqn. (2.1), we have:

$$\begin{aligned} S_{fi} &= -i \int d^4 x \int d^4 y (e \bar{\psi}_f(x) \gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) \psi_i(x)) e D_F(x-y) J_\mu(y) \\ &= \frac{-ie^2}{\Omega} \sqrt{\frac{m^2}{\epsilon_1 \epsilon_2}} \bar{u}(k_2) \gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) u(k_1) \frac{1}{2} \int d^4 x e^{i(k_1-k_2) \cdot x} \\ &\quad \int \frac{d^4 q}{(2\pi)^4} e^{iq \cdot (x-y)} \int d^4 y J_\mu(y) \end{aligned}$$

where we have substituted $\psi_i(x)$, $\bar{\psi}_f(x)$ and $D_F(x-y)$.

Now, $q \cdot y = \bar{q} \cdot \bar{y} + \omega t$

Therefore,

$$S_{fi} = \frac{-ie^2}{\Omega} \sqrt{\frac{m^2}{\epsilon_1 \epsilon_2}} \frac{1}{2} \bar{u}(k_2) \gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) u(k_1) \int e^{-i\omega t} dt \int d\bar{y} e^{-i\bar{q} \cdot \bar{y}} J_\mu(y) \quad (2.4)$$

Using Heisenberg equation of motion for the current operator:

$$\hat{J}_\mu(y) = \exp(i \hat{H} t) \hat{J}_\mu(y) \exp(-i \hat{H} t)$$

we have

$$J_\mu(y) = e^{i(E'-E)t} \langle f | \hat{J}_\mu(y) | i \rangle \quad (2.5)$$

where E and E' are the energy of the target in initial state i and final state f respectively.

Using Eqn. (2.5) in Eqn. (2.4), we have:

$$S_{fi} = \frac{-ie^2}{\Omega} \frac{2\pi\delta(E'-E-\omega)}{q_\mu^2} \sqrt{\frac{m^2}{\epsilon_1 \epsilon_2}} \bar{u}(k_2) \gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) u(k_1) J_\mu(\bar{q})$$

where

$$J_\mu(\bar{q}) = \int e^{-i\bar{q} \cdot \bar{y}} \langle f | \hat{J}_\mu(y) | i \rangle d\bar{y} \quad (2.6)$$

Since one of the states $|i\rangle$ or $|f\rangle$ is not a parity eigen state $J_\mu(\bar{q})$ is parity non-conserving.

The square of the S matrix gives the probability of transition.

Substituting $|S_{fi}|^2$ from Eqn. (2.6), the differential cross-section for scattering of right-handed electron from the target is:

$$d\sigma|_R = \frac{e^4}{k_1/\epsilon_1} \frac{m^2}{\epsilon_1 \epsilon_2} 2\pi\delta(E'+\epsilon_2-E-\epsilon_1) \frac{1}{q_\mu^4} |m_{fi}|^2 \frac{d\bar{k}_2}{(2\pi)^3} \quad (2.7)$$

where

$$|m_{fi}|^2 = |\bar{u}(k_2) \gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) u(k_1) J_\mu(\bar{q})|^2$$

Calculation of $|m_{fi}|^2$:

$$|m_{fi}|^2 = |\bar{u}(k_2) \gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) u(k_1) J_\mu(\bar{q})|^2$$

$$= -\text{Tr} \left[\gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2} \right) \left(\frac{-i\gamma k_1 + m}{2m} \right) \gamma_\nu \left(\frac{-i\gamma k_2 + m}{2m} \right) \right] J_\mu J_\nu^*$$

The negative sign is due to the fact that γ_ν anticommutes with γ_4 :

$$\bar{\gamma}_\nu = \gamma_4 \gamma_\nu^* \gamma_4 = -\gamma_\nu$$

The calculation of traces (given in Appendix A) gives:

$$|m_{fi}|^2 = \frac{1}{2m^2} [2(J_\mu Q_\mu)(J_\nu^* Q_\nu) + \frac{1}{2} q_\mu^2 J_\mu J_\mu^* + m\epsilon_{\mu\nu\alpha\beta} (J_\mu J_\nu^* S_\alpha k_{1\beta} - J_\mu J_\nu^* S_\alpha k_{2\beta})] \quad (2.8)$$

We now proceed to make a multipole analysis of the electromagnetic current $J_\mu = (\bar{J}, i\rho)$. As a result of scattering, the target nucleus or the atom makes a transition from the initial state i of angular momentum J_i (magnetic quantum number M_i) to another state f of angular momentum J_f (magnetic quantum number M_f).

We define the Coulomb multipole operator as:

$$M_{JM}^{\text{Coul}} = \int j_J(qx) Y_{JM}(\Omega_x) \rho(\bar{x}) d\bar{x} \quad (2.9)$$

where $\rho(\bar{x})$ denotes the charge density.

$\rho(\bar{q})$ can be expressed as:

$$\begin{aligned}
\frac{1}{2J_i+1} \sum_{M_i, M_f} |\rho(\vec{q})|^2 &= \frac{4\pi}{2J_i+1} \sum_J |\langle J_f || M_J^{\text{Coul}} || J_i \rangle|^2 \\
&= \frac{4\pi}{2J_i+1} \sum_J ||M^{\text{Coul}}||^2
\end{aligned} \tag{A}$$

where

$$\langle J_f || M_J^{\text{Coul}} || J_i \rangle = ||M^{\text{Coul}}||$$

Here, $\langle J_f || M_J^{\text{Coul}} || J_i \rangle$ represents the reduced matrix element of the tensor M_{JM}^{Coul} . It is related to $\langle J_f M_f | M_{JM}^{\text{Coul}} | J_i M_i \rangle$ by Wigner-Eckart theorem,

$$\langle J_f M_f | M_{JM}^{\text{Coul}} | J_i M_i \rangle = (-1)^{J_f - M_f} \begin{pmatrix} J_f & J & J_i \\ -M_f & M & M_i \end{pmatrix} \langle J_f || M_J^{\text{Coul}} || J_i \rangle$$

Here $\begin{pmatrix} J_f & J & J_i \\ -M_f & M & M_i \end{pmatrix}$ represents the Clebsch Gordan coefficient.

We now define the electric and magnetic multipole operators as:

$$T_{JM}^{\text{el}} = \frac{1}{q} \int d\vec{x} \nabla \times [j_J(qx) \vec{Y}_{JJ1}^M(\Omega_x)] \cdot \vec{J}(\vec{x}) \tag{2.10}$$

and

$$T_{JM}^{\text{mag}} = \int d\vec{x} j_J(qx) Y_{JJ1}^M(\Omega_x) \cdot \vec{J}(\vec{x}) \tag{2.11}$$

where T_{JM}^{el} and T_{JM}^{mag} are irreducible tensor operators of rank J.

Let $J_\lambda(\vec{q})$ ($\lambda=0, \pm 1$) be the spherical components of the vector $\vec{J}(\vec{q})$. Then, for the state quantized along any arbitrary direction, we can write:

$$J_{\lambda}(\bar{q}) = -\sum_{JM} (-i)^J \sqrt{2\pi(2J+1)} (-1)^{J_f-M_f} \begin{pmatrix} J_f & J & J_i \\ -M_f & M & M_i \end{pmatrix} \cdot [\langle J_f || T_J^{el} + \lambda T_J^{mag} || J_i \rangle D_{M\lambda}(-\phi_q - \theta_q \phi_q)] \quad (2.12)$$

where $D_{M\lambda}(-\phi_q - \theta_q \phi_q)$ is the rotation operator.

Now, in the initial state, there is an admixture of odd parity with an even parity state and hence both T_J^{el} and T_J^{mag} which have opposite parity, contribute at the same time. But one of them will have its contribution reduced by a factor F , the admixing amplitude.

Using $\sum_M D_{M\lambda}(\omega) D_{M\lambda}^*(\omega) = \delta_{\lambda\lambda}$, we get

$$\begin{aligned} \frac{1}{2J_i+1} \sum_{M_i M_f} J_{\lambda}(\bar{q}) J_{\lambda}(\bar{q})^* &= \frac{2\pi}{2J_i+1} \delta_{\lambda\lambda} \sum_{J=1}^{\infty} \{ |\langle J_f || T_J^{el} || J_i \rangle|^2 \\ &+ \lambda \langle J_f || T_J^{mag} || J_i \rangle \langle J_f || T_J^{el} || J_i^* \rangle + \lambda' \langle J_f || T_J^{el} || J_i \rangle \langle J_f || T_J^{mag} || J_i^* \rangle \\ &+ |\langle J_f || T_J^{mag} || J_i \rangle|^2 \} \\ &= \frac{2\pi}{2J_i+1} \delta_{\lambda\lambda} \sum_{J=1}^{\infty} \{ ||T_J^{el}||^2 + \lambda ||T_J^{mag}|| ||T_J^{el}||^* + \lambda' ||T_J^{el}|| ||T_J^{mag}||^* \\ &+ ||T_J^{mag}||^2 \} \end{aligned} \quad (B)$$

where, for convenience, we have written

$$\langle J_f || T_J^{el} || J_i \rangle = ||T_J^{el}||$$

$$\langle J_f || T_J^{mag} || J_i \rangle = ||T_J^{mag}||$$

Let us now evaluate the terms within the square bracket in Eqn. (2.8)

for $|m_{fi}|^2$.

$$2(J_\mu Q_\mu)(J_\nu^* Q_\nu) = 2(\bar{J} \cdot \bar{Q})(\bar{J}^* \cdot \bar{Q}) + 2\rho\rho^* \epsilon^2 - 2\bar{J} \cdot \bar{Q} \rho^* \epsilon - 2\bar{J}^* \cdot \bar{Q} \rho \epsilon$$

where

$$\epsilon = \frac{1}{2} (\epsilon_1 + \epsilon_2).$$

Let us define a basis of unit vectors with regard to $\bar{q}$. The unit vectors

$$\hat{e}_{q\pm 1} = \mp \frac{1}{\sqrt{2}} (e_x \pm i e_y)$$

$$\hat{e}_{q0} = \frac{\bar{q}}{|\bar{q}|}$$

form an orthogonal basis.

Therefore,

$$\hat{e}_{q\lambda}^+ \cdot \hat{e}_{q\lambda'} = \delta_{\lambda\lambda'}$$

Therefore, we can expand

$$\bar{J}(\bar{q}) = \sum_{\lambda=0,\pm 1} J_\lambda(\bar{q}) \hat{e}_{q\lambda}^+ \quad (2.13)$$

Hence,

$$\begin{aligned} 2(J_\mu Q_\mu)(J_\nu^* Q_\nu) &= 2(J_1 \hat{e}_{q1}^+ \cdot \bar{Q} + J_{-1} \hat{e}_{q-1}^+ \cdot \bar{Q} + J_0 \hat{e}_{q0}^+ \cdot \bar{Q})(J_1^* \hat{e}_{q1} \cdot \bar{Q} \\ &\quad + J_{-1}^* \hat{e}_{q-1} \cdot \bar{Q} + J_0^* \hat{e}_{q0} \cdot \bar{Q}) + 2\rho\rho^* \epsilon^2 - 4\bar{J} \cdot \bar{Q} \rho^* \epsilon \\ &= \{\bar{Q}^2 - (\bar{Q} \cdot \bar{q})^2\} (J_{+1} J_{+1}^* + J_{-1} J_{-1}^*) + \frac{q_\mu^4}{2q^4} (\epsilon_1 + \epsilon_2)^2 \rho\rho^* \\ &= \{\bar{Q}^2 - (\bar{Q} \cdot \bar{q})^2\} \frac{4\pi}{2J_i + 1} \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \} + \frac{q_\mu^4}{2q^4} (\epsilon_1 + \epsilon_2)^2 \frac{4\pi}{2J_i + 1} \\ &\quad \sum_J ||M^{Coul}||^2 \quad (2.14) \end{aligned}$$

using Eqns. A and B.

Again, using Eqns. (2.13), (A) and (B), we can write:

$$\begin{aligned} \frac{1}{2} q_{\mu}^2 J_{\mu} J_{\mu}^* &= \frac{1}{2} q_{\mu}^2 \frac{4\pi}{2J_1+1} \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \} \\ &\quad - \frac{1}{2} \frac{q_{\mu}^4}{q^2} \frac{4\pi}{2J_1+1} \sum_{J=0}^{\infty} ||M^{Coul}||^2 \end{aligned} \quad (2.15)$$

From Eqns. (2.14) and (2.15), we have:

$$\begin{aligned} 2(J_{\mu} Q_{\mu})(J_{\nu}^* Q_{\nu}^*) + \frac{1}{2} q_{\mu}^2 J_{\mu} J_{\mu}^* &= \frac{4\pi}{2J_1+1} [V_T(\theta) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \} \\ &\quad + V_L(\theta) \sum_J ||M^{Coul}||^2] \end{aligned} \quad (2.16)$$

where

$$V_T(\theta) = \bar{Q}^2 - (\bar{Q} \cdot \frac{\bar{q}}{q})^2 + \frac{1}{2} q_{\mu}^2 \quad (2.16a)$$

and

$$V_L(\theta) = \frac{q_{\mu}^4}{2q^4} [(\epsilon_1 + \epsilon_2)^2 - q^2] \quad (2.16b)$$

$$\begin{aligned} V_T(\theta) &= \frac{1}{4} (\bar{k}_1 + \bar{k}_2)^2 - \frac{1}{4q^2} \{ (\bar{k}_2 + \bar{k}_1)(\bar{k}_2 - \bar{k}_1) \}^2 + \frac{1}{2} \{ q^2 - (\epsilon_2 - \epsilon_1)^2 \} \\ &= \frac{1}{4q^2} \{ (\bar{k}_2 - \bar{k}_1)^2 (\bar{k}_1 + \bar{k}_2)^2 - (k_2^2 - k_1^2)^2 \} + (\epsilon_1 \epsilon_2 - k_1 k_2 \cos \theta - m^2) \\ &\quad + \frac{k_1^2 k_2^2 \sin^2 \theta}{q^2} \\ &= \frac{k_1^2 k_2^2 \sin^2 \theta}{q^2} + \epsilon_1 \epsilon_2 - k_1 k_2 \cos \theta - m^2 \\ &= \frac{k_1^2 k_2^2 \sin^2 \theta}{q^2} + \frac{q^2}{2} \end{aligned} \quad (2.17a)$$

$$\epsilon_1 \epsilon_2 \approx \frac{k_1^2 + k_2^2}{2} + m^2$$

Neglecting mass of electron, we can write:

$$\begin{aligned} V_T(\theta) &= \frac{4k_1^2 k_2^2 \sin^2 \theta/2 \cos^2 \theta/2}{q^2} + k_1 k_2 (1 - \cos \theta) \quad \because \epsilon_1 \epsilon_2 \approx k_1 k_2 \\ &= 2k_1 k_2 \cos^2 \theta/2 \left(\frac{q_\mu^2}{2q^2} + \tan^2 \theta/2 \right) \end{aligned} \quad (2.17b)$$

$$\begin{aligned} V_L(\theta) &= \frac{q_\mu^4}{2q^4} [(\epsilon_1 + \epsilon_2)^2 - q^2] \\ &= \frac{q_\mu^4}{q^4} (\epsilon_1 \epsilon_2 + m^2 + k_1 k_2 \cos \theta) \\ &= \frac{q_\mu^4}{q^4} \left(\frac{k_1^2 + k_2^2}{2} + 2m^2 + k_1 k_2 \cos \theta \right) \end{aligned} \quad (2.18a)$$

Neglecting mass of electron, we have:

$$\begin{aligned} V_L(\theta) &= \frac{q_\mu^4}{q^4} (k_1 k_2 + k_1 k_2 \cos \theta) \\ &= \frac{q_\mu^4}{q^4} 2k_1 k_2 \cos^2 \theta/2 \end{aligned} \quad (2.18b)$$

Let us now evaluate the third term in $|m_{fi}|^2$.

$$\begin{aligned} -m \epsilon_{\mu\nu\alpha\beta} (J_\mu J_\nu^* S_\alpha k_{2\beta} - J_\mu J_\nu^* S_\alpha k_{1\beta}) &= -m (J_1 J_2^* - J_2 J_1^*) (S_3 k_{24} - S_4 k_{23} - S_3 k_{14} + S_4 k_{13}) \\ &= -i (J_1 J_2^* - J_2 J_1^*) (\epsilon_1 \epsilon_2 \hat{k}_1 \cdot \hat{q} - |\hat{k}_1| \hat{k}_2 \cdot \hat{q} - \epsilon_1^2 \hat{k}_1 \cdot \hat{q} + |\hat{k}_1| \hat{k}_1 \cdot \hat{q}) \end{aligned}$$

since

$$k_1 = (\bar{k}_1 \cdot \hat{e}_x, \bar{k}_1 \cdot \hat{e}_y, \bar{k}_1 \cdot \hat{q}, i\epsilon_1)$$

$$k_2 = (\bar{k}_2 \cdot \hat{e}_x, \bar{k}_2 \cdot \hat{e}_y, \bar{k}_2 \cdot \hat{q}, i\epsilon_2)$$

$$S = \left(\frac{\epsilon_1}{m} \bar{k}_1 \cdot \hat{e}_x, \frac{\epsilon_1}{m} \bar{k}_1 \cdot \hat{e}_y, \frac{\epsilon_1}{m} \bar{k}_1 \cdot \hat{q}, \frac{i}{m} |\bar{k}| \right)$$

$$\begin{aligned} \text{3rd term} &= -1(J_1 J_2^* - J_2 J_1^*) \frac{1}{|\bar{q}|} [k_2 \cos \theta (\epsilon_1 \epsilon_2 + k_1^2 - m^2) - k_1 (\epsilon_1 \epsilon_2 + k_2^2 - m^2)] \\ &= 2V_p(\theta) \frac{4\pi}{2J_i + 1} \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^* \end{aligned} \quad (2.19)$$

using Eqn. (B),

where

$$\begin{aligned} V_p(\theta) &= \frac{1}{|\bar{q}|} [k_2 \cos \theta (\epsilon_1 \epsilon_2 + k_2^2 - m^2) - k_1 (\epsilon_1 \epsilon_2 + k_1^2 - m^2)] \\ &= \frac{1}{|\bar{q}|} \left[\left(\frac{k_1^2 + k_2^2}{2} \right) (k_2 \cos \theta - k_1) + k_1 k_2 (k_1 \cos \theta - k_2) \right] \end{aligned} \quad (2.20a)$$

Again,

$$\begin{aligned} V_p(\theta) &= \frac{1}{|\bar{q}|} [k_1 k_2 \cos \theta (\epsilon_1 + \epsilon_2) - k_1 k_2 (\epsilon_1 + \epsilon_2)] \text{ neglecting mass of electron} \\ &= - \frac{(\epsilon_1 + \epsilon_2)}{|\bar{q}|} 2k_1 k_2 \sin^2 \theta / 2 \end{aligned} \quad (2.20b)$$

From Eqns. (2.8), (2.16) and (2.19), we have:

$$\begin{aligned} |m_{fi}|^2 &= \frac{1}{2m^2} \frac{4\pi}{2J_i + 1} [V_T(\theta) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \} + V_L(\theta) \\ &\quad \sum_{J=0}^{\infty} ||M^{\text{Coul}}||^2 + 2V_p(\theta) \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*] \end{aligned} \quad (2.21)$$

Substituting $|m_{fi}|^2$ from Eqn. (2.21) in Eqn. (2.7), we obtain the differential cross-section for scattering of a right polarized electron.

$$\begin{aligned} d\sigma|_R = & \frac{2\pi\delta(E'+\epsilon_2-E-\epsilon_1)}{q_\mu^4} \frac{e^4}{k_1\epsilon_2} \frac{1}{2} \frac{4\pi}{2J_i+1} V_T(\theta) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 \\ & + ||T^m||^2 \} + V_L(\theta) \sum_J ||M^{Coul}||^2 + 2V_p(\theta) \\ & \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^* \frac{d\bar{k}_2}{(2\pi)^3} \end{aligned}$$

Writing $d\bar{k}_2 = k_2 \epsilon_2 d\epsilon_2 d\Omega$ and integrating over $d\epsilon_2$, we obtain:

$$\begin{aligned} \frac{d\sigma}{d\Omega}|_R = & \frac{8\pi\alpha^2}{q_\mu^4} \left(\frac{k_2}{k_1}\right) [V_T(\theta) \sum_{J=1}^{\infty} \{ \frac{||T^{e1}||^2 + ||T^m||^2}{2J_i+1} \} \\ & + V_L(\theta) \sum_{J=0}^{\infty} \frac{||M^{Coul}||^2}{2J_i+1} + 2V_p(\theta) \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*] \\ & \cdot \frac{1}{1 + \frac{k_2 - k_1 \cos\theta}{E'}} \end{aligned} \quad (2.22)$$

For left polarized incident electrons, the transition matrix element is:

$$S_{fi} = -ie \int d^4x \bar{\psi}_f(x) \gamma_\mu \frac{(1-i\gamma_5 \gamma \cdot S)}{2} \psi_i(x) A_\mu(x) \quad (2.23)$$

In this case, we have used the spin projection operator = $\frac{1-i\gamma_5 \gamma \cdot S}{2}$.

Proceeding exactly in the same manner as for right polarized electrons with S_{fi} given by Eqn. (2.23), we obtain the cross-section:

$$\begin{aligned}
\left. \frac{d\sigma}{d\Omega} \right|_L &= \frac{8\pi\alpha^2}{4} \left(\frac{k_2}{k_1} \right) [V_T(\theta) \sum_{J=1}^{\infty} \frac{||T^{e1}||^2 + ||T^m||^2}{2J+1} \\
&+ V_L(\theta) \sum_{J=0}^{\infty} \frac{||M^{Coul}||^2}{2J+1} - 2V_P(\theta) \sum_{J=1}^{\infty} \frac{||T^{e1}|| ||T^m||^*}{2J+1}] \frac{1}{1 + \frac{k_2 - k_1 \cos \theta}{E'}}
\end{aligned}
\quad (2.24)$$

2. Asymmetry

Parity admixtures of states in nucleus or atom gives rise to an asymmetry in the inelastic scattering cross-sections for right and left polarized electrons from the mixed state. The asymmetry is defined as:

$$A(q^2) = \frac{\left. \frac{d\sigma}{d\Omega} \right|_R - \left. \frac{d\sigma}{d\Omega} \right|_L}{\left. \frac{d\sigma}{d\Omega} \right|_R + \left. \frac{d\sigma}{d\Omega} \right|_L} \quad (2.25)$$

Substituting the values of $\left. \frac{d\sigma}{d\Omega} \right|_R$ and $\left. \frac{d\sigma}{d\Omega} \right|_L$ from Eqns. (2.22) and (2.24) in the expression (2.25) for asymmetry, we obtain:

$$A(q^2) = \frac{2V_P(\theta) \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{[V_L(\theta) \sum_{J=0}^{\infty} ||M^{Coul}||^2 + V_T(\theta) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}]}$$

where the values of $V_T(\theta)$, $V_L(\theta)$ and $V_P(\theta)$ in the laboratory system are given by the Eqns. (2.16a), (2.16b) and (2.20a) respectively.

In the non-relativistic limit, which is the case of scattering of low energy electrons, the values of $V_T(\theta)$, $V_L(\theta)$ and $V_P(\theta)$ are given by Eqns. (2.17a), (2.18a) and (2.20a) respectively. Using these values, we can write the asymmetry as:

$$A(q^2) = \frac{\frac{2}{|\bar{q}|} \left[\left(\frac{k_1^2 + k_2^2}{2} \right) (k_2 \cos \theta - k_1) + k_1 k_2 (k_1 \cos \theta - k_2) \right] \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \left[\frac{k_1^2 + k_2^2}{2} + k_1 k_2 \cos \theta + 2m^2 \right] \sum_{J=0}^{\infty} ||M^{coul}||^2 + \left(\frac{k_1^2 k_2^2 \sin^2 \theta}{\bar{q}^2} + \frac{\bar{q}^2}{2} \right) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}} \quad (2.27)$$

At $\theta=0$, the expression for asymmetry reduces to:

$$A(q^2) \Big|_{\theta=0} = \frac{\frac{2}{|\bar{q}|} \left(\frac{k_2 - k_1}{2} \right)^3 \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \left[\frac{(k_1 + k_2)^2}{2} + 2m^2 \right] \sum_{J=0}^{\infty} ||M^{coul}||^2 + \frac{\bar{q}^2}{2} \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}}$$

With the increase in incident energy, $(k_2 - k_1)$ decreases and so does the asymmetry. At backscattering angle, i.e. $\theta=\pi$, we have:

$$A(q^2) \Big|_{\theta=\pi} = \frac{-\frac{2}{|\bar{q}|} \left(\frac{k_2 + k_1}{2} \right)^3 \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \left[\frac{(k_2 - k_1)^2}{2} + 2m^2 \right] \sum_{J=0}^{\infty} ||M^{coul}||^2 + \frac{\bar{q}^2}{2} \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}}$$

In the relativistic limit, neglecting the mass of electron, the asymmetry is given as:

$$A(q^2) = \frac{-2 \left(\frac{\epsilon_1 + \epsilon_2}{2} \right) \tan^2 \theta/2 \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\frac{q_\mu}{4} \sum_{J=0}^{\infty} ||M^{coul}||^2 + \left(\frac{q_\mu^2}{2q} + \tan^2 \theta/2 \right) \sum_{J=1}^{\infty} \{ ||T^{e1}||^2 + ||T^m||^2 \}}$$

Here we have used the values $V_T(\theta)$, $V_L(\theta)$ and $V_p(\theta)$ from Eqns. (2.17b),

(2.18b) and (2.20b) respectively. From Eqn. (2.28), we find that the asymmetry $A(q^2)$ vanishes for $\theta=0$ and obtains its maximum value for $\theta=\pi$ for fixed momentum transfer.

$$A(q^2) \Big|_{\theta=\pi} = \frac{-2 \sum_{J=1}^{\infty} ||T^{e1}|| ||T^m||^*}{\sum_{J=1}^{\infty} ||T^{e1}||^2 + ||T^m||^2} \quad (2.29)$$

CHAPTER 3

ASYMMETRY IN POLARIZED ELECTRON SCATTERING FROM NUCLEONS

AND NUCLEI

Some of the unified gauge theories predict the existence of neutral current interaction between charged leptons and hadrons. An aspect of these neutral current interactions that has been used in various proposals to detect such interaction is the occurrence of parity non-conserving terms which can lead to asymmetries such as dependence of cross-section on longitudinal polarization.

Deep inelastic scattering of polarized electrons from nuclei and nucleons has been investigated theoretically for possible test of weak neutral currents. The axis of spin quantization chosen along the direction of the incoming electron beam and the left-right asymmetry is defined as (given in Eqn. (2.25)):

$$A(q^2) = \frac{\left. \frac{d\sigma}{d\Omega} \right|_R - \left. \frac{d\sigma}{d\Omega} \right|_L}{\left. \frac{d\sigma}{d\Omega} \right|_R + \left. \frac{d\sigma}{d\Omega} \right|_L} \quad (3.1)$$

The labels refer to right-handed (R) electrons when the electron spin is parallel to the beam direction $\vec{k}$ (positive helicity) and to left-handed (L) electrons when the spin is antiparallel to $\vec{k}$ (negative helicity).

The asymmetry as defined in Eqn. (3.1) in polarized electron scattering can be obtained in two ways depending on the magnitude of the momentum transfer in the scattering process.

At very high momentum transfer, the left-right asymmetry arises due to the interference of the interaction processes involving the exchange of photons and those mediated by neutral heavy Z^0 boson. The effects of Z^0 boson are, in general, of the order of Gq^2/e^2 where q^2 is the momentum transfer squared. Hence, as q^2 gets larger, the value of asymmetry becomes sizeable.

In the general context of gauge theories with spontaneously broken symmetry, the electromagnetic currents interact via the exchange of virtual photons and the weak neutral currents by exchanging virtual Z^0 boson. Hence, in the lowest order, Feynman's diagram for the two processes can be given as:

Reya and Schilcher¹⁴ have considered the elastic electron-nucleon scattering.

$$e^-(p,s) + N(P,S) \rightarrow e^-(p',s') + N(P',S')$$

where the symbols in the parentheses denote the momenta and spins of

the corresponding particles. Assuming time-reversal invariance and only first class currents, the form factor for the two processes may be defined as:

$$\begin{aligned} \langle P' | J_Z^\mu | P \rangle &= \bar{u}(P'S') F_Z^\mu u(P,S) \\ &= \bar{u}(P'S') [g_V^0 \gamma^\mu - g_A^0 \gamma^\mu \gamma_5 - f_V^0 (P+P')^\mu - h_A^0 (P-P')^\mu \gamma_5] u(PS) \end{aligned} \quad (3.2a)$$

and

$$\begin{aligned} \langle P' | J^\mu | P \rangle &= \bar{u}(P'S') [F_1 \gamma^\mu + i F_2 \sigma^{\mu\nu} (P'-P)_\nu] u(P,S) \\ &= \bar{u}(P'S') [G_M \gamma^\mu - F_2 (P+P')^\mu] u(PS) \end{aligned} \quad (3.2b)$$

with $G_M = F_1 + 2M F_2$.

Here, the form factors g_V^0 , g_A^0 , f_V^0 , h_A^0 and the electromagnetic nucleon form factors are real functions of $q^2 = (P'-P)^2 = (p-p')^2$.

In the Weinberg's $SU(2) \times U(1)$ model,

$$J_Z^\mu = J_3^\mu - 2 \sin^2 \theta_w J^\mu$$

where $\sin^2 \theta_w$ is the only free parameter of the theory and is given by $g'^2/g^2 + g'^2$, g and g' are the triplet and singlet coupling constants.

The amplitude for the right-handed electron for photon exchange can be written as:

$$m_R^\gamma = - \frac{ie^2}{q} \bar{u}(p',s') \frac{\gamma_\mu (1 + \gamma_5 \not{R})}{2} u(p,s) \bar{u}(P'S') [G_M \gamma^\mu - F_2 (P+P')^\mu] u(PS) \quad (3.3)$$

and the amplitude for Z^0 exchange is:

$$m_R^{Z^0} = -\frac{iG}{\sqrt{2}} \bar{u}(p', s') \gamma_\mu \frac{1}{2} (4\sin^2\theta_w - 1 + \gamma_5) (1 + \gamma_5 s_R) u(p, s) \bar{u}(P', S') \Gamma_Z^\mu u(P, S) \quad (3.4)$$

where Γ_Z^μ is defined in Eqn. (3.2a).

s_R is the spin four vector of positive helicity electrons.

Substituting $s_L = -s_R$, we can find m_L . The total amplitude with definite helicity is:

$$m_i^{Y+Z^0} = m_i^Y + m_i^{Z^0} \quad (3.5)$$

where $i = R, L$.

Hence, the asymmetry is:

$$A = \frac{|m_R^{Y+Z^0}|^2 - |m_L^{Y+Z^0}|^2}{|m_R^{Y+Z^0}|^2 + |m_L^{Y+Z^0}|^2}$$

which can be calculated using Eqns. (3.3), (3.4) and (3.5).

Reya and Schilcher found the values of asymmetry for different energy limits. For high energy where $-q^2/M^2 \gg 1$ and $(\frac{E}{M})(1-\cos\theta) \gg 1$, the asymmetry takes the form:

$$A = -\sqrt{2} \frac{G q^2}{e^2} \frac{\frac{1}{2} G_M [g_V^0 - (4\sin^2\theta_w - 1) g_A^0] + F_2 f_V^0 M^2 \cos^2\theta/2}{\frac{1}{2} G_M^2 + F_2^2 M^2 \cot^2\theta/2}$$

For a proton target and for $\sin^2\theta_w = 1/3$,

$$\begin{aligned} A^p &= -1.5 \times 10^{-4} \frac{q^2}{M_p^2} \frac{0.41 + 0.29 \cot^2\theta/2}{3.89 + 0.81 \cot^2\theta/2} \\ &\approx -2.2 \times 10^{-5} \frac{q^2}{M_p^2} \text{ for } \theta = 90^\circ. \end{aligned} \quad (3.6)$$

Thus we find that the parity violating effects grow with q^2 . Since high energy elastic electron-nucleon scattering yields also very interesting information on electromagnetic form factors, the experiments should be carried out with as large q^2 as possible.

Feinberg¹⁵ on the other hand considered the polarization asymmetry coming from the interference between electromagnetic and weak neutral current interactions in the scattering of electrons from nuclei. He pointed out that in contrast to nucleons, in nuclei, one can choose the spin and isospin of the target at will. Moreover, by observing the excitation of definite final states, one can isolate terms with specific isospin or parity properties in the weak and electromagnetic Hamiltonians.

The parity violating Hamiltonian can be written as:

$$H_{pv} = \frac{G_F}{2\pi\alpha\sqrt{2}} \sum_{i=1}^2 (\epsilon_{ei}^{VA} \bar{\psi}_e \gamma^\rho \psi_e A_\rho^i + \epsilon_{ie}^{VA} \bar{\psi}_e \gamma^\rho \gamma_5 \psi_e V_\rho^i) \quad (3.7)$$

G_F is the weak coupling constant. ϵ_{ei}^{VA} and ϵ_{ie}^{VA} are parameters measuring the strength of the interaction. Their magnitudes in the Weinberg model are:

$$\begin{aligned} \epsilon_{e1}^{VA} &= \pi\alpha(1-4\sin^2\theta_w) & \epsilon_{e2}^{VA} &= \pi\alpha(1-4\sin^2\theta_w) \\ \epsilon_{1e}^{VA} &= \pi\alpha(1-\frac{8}{3}\sin^2\theta_w) & \epsilon_{2e}^{VA} &= \pi\alpha(1-\frac{4}{3}\sin^2\theta_w) \end{aligned} \quad (3.8)$$

A_ρ^i and V_ρ^i are hadronic axial vector and vector currents:

$$A_\rho^i = \bar{\psi}_i \gamma_\rho \gamma_5 \psi_i$$

and

$$V_{\rho}^i = \bar{\psi}_i \gamma_{\rho} \psi_i$$

The electromagnetic Hamiltonian is:

$$H_{l\gamma} = \frac{4\pi\alpha}{q} \bar{\psi}_e \gamma_{\rho} \psi_e J_e^{\text{e.m.}} \quad (3.9)$$

The matrix element for elastic scattering of electrons from a nucleus can be written as:

$$M = \bar{u} \gamma_{\rho} u \frac{(P+P')_{\rho}}{2M_N} \frac{4\pi\alpha}{q} \left(\frac{2}{3} F^{(1)} - \frac{1}{3} F^{(2)} \right) + \bar{u} \gamma_{\rho} \gamma_5 u \frac{(P+P')_{\rho}}{2M_N} \\ \cdot \frac{G_F}{2\pi\alpha\sqrt{2}} (\epsilon_{1e}^{VA} F^{(1)} + \epsilon_{2e}^{VA} F^{(2)}) \quad (3.10)$$

Here u is the electron wave function, P and P' are the initial and final four-momentum of the nuclear centre of mass, $F^{(1)}$ and $F^{(2)}$ are the form factors which depend on the nucleus. For a nucleus with ground-state isospin zero, the matrix element reduces to:

$$M = Z F_{ch} \frac{4\pi\alpha}{q} (P+P')_{\rho} \left[\bar{u} \gamma_{\rho} u + \frac{3G_F q^2}{8\pi\alpha\sqrt{2}} (\epsilon_{1e}^{VA} + \epsilon_{2e}^{VA}) \bar{u} \gamma_{\rho} \gamma_5 u \right]$$

where

$$F^{(1)} = F^{(2)} = 3 Z F_{ch}$$

F_{ch} being the nuclear charge form factor.

Hence, we get the elastic cross-sections for left-handed and right-handed electrons

$$\frac{d\sigma_{LR}}{d\Omega} = Z^2 \sigma_{\text{Mott}} |F_{\text{ch}}|^2 \left[1 + \frac{3G_F q^2}{4\pi \alpha^2 \sqrt{2}} (\epsilon_{1e}^{\text{VA}} + \epsilon_{2e}^{\text{VA}}) \right] \quad (3.11)$$

Therefore, the polarization asymmetry is:

$$\begin{aligned} A(q^2) &= \frac{d\sigma_L - d\sigma_R}{d\sigma_L + d\sigma_R} = \frac{3G_F |q^2|}{4\pi \alpha^2 \sqrt{2}} (\epsilon_{1e}^{\text{VA}} + \epsilon_{2e}^{\text{VA}}) \\ &= 10^{-2} \frac{|q^2|}{M_p^2} (\epsilon_{1e}^{\text{VA}} + \epsilon_{2e}^{\text{VA}}) \quad (M_p = \text{proton mass}) \\ &= 4 \times 10^{-4} \frac{|q^2|}{M_p^2} \sin^2 \theta_w \end{aligned} \quad (3.12)$$

in the Weinberg model.

In light nuclei at the values $|q^2| \sim \frac{1}{4} M_p^2$, the asymmetry is $\sim 3 \times 10^{-5}$.

Feinberg also calculated the inelastic electron nucleus scattering. In that case, the asymmetry has the values:

$$\begin{aligned} A(q^2) &= 3 \times 10^{-3} \frac{|q^2|}{M_p^2} (\epsilon_{1e}^{\text{VA}} - \epsilon_{2e}^{\text{VA}}) \\ &= 1.2 \times 10^{-4} (1 - 2\sin^2 \theta_w) \frac{|q^2|}{M_p^2} \end{aligned} \quad (3.13)$$

in the Weinberg model. Using polarized e^- scattering from deuterium at $Q^2 = 1.6(\text{Gev}/c)^2$ Prescott et al^{26,27} obtained the value of asymmetry as $-9.5 \times 10^{-5} Q^2$. These results are consistent with the expectation of Weinberg-Salam model for $\sin^2 \theta_w = 0.224$ and are in good agreement with the value of θ_w from neutrino experiments.

Hence, in both cases (elastic and inelastic electron nucleus scattering) in order to obtain sizeable values for the asymmetry, one

has to have very high momentum transfer q . At low momentum transfer, the asymmetry is likely not to be measurable.

On the other hand, asymmetry in polarized electron-nucleus scattering can also be obtained at low momentum transfer. If the initial or the final state in nuclei contains admixture of two different parity states then there is a weak neutral current interaction. The low momentum transfer corresponds to large wavelength which can probe the nuclear states containing mixed parity states. In such a case, the incident particle interacts coherently with the whole nucleus. This gives rise to different values of cross-sections of scattering for left-handed and right-handed electrons and hence the asymmetry. Gari and Porrmann considered this.

The Feynman diagram to the lowest order is:

The nuclear state $|\psi^J\rangle$ can be written as:

$$|\psi^J\rangle = |\phi^{J\pi}\rangle + F|\phi^{J-\pi}\rangle \quad (3.14)$$

where F is the amplitude of admixture.

Proceeding exactly in the same manner as in Chapter 2, in the approximation of exchange of one photon, we can write the scattering cross-section for right polarized electrons from the nucleus with a state $|\psi^J\rangle$ as given in Eqn. (2.22).

$$\begin{aligned} \frac{d\sigma}{d\Omega}\bigg|_R = & \frac{8\pi\alpha^2}{q_\mu} \left(\frac{k_2}{k_1}\right) \frac{1}{1 + \frac{k_2 - k_1 \cos\theta}{E'}} [V_L(\theta) \sum_{J=0}^{\infty} \frac{||M^{\text{coul}}||^2}{2J_i+1} \\ & + V_T(\theta) \sum_{J=1}^{\infty} \frac{(||T^{\text{el}}||^2 + ||T^{\text{m}}||^2)}{2J_i+1} \\ & + 2V_P(\theta) \sum_{J=1}^{\infty} \frac{||T^{\text{el}}|| ||T^{\text{m}}||^*}{2J_i+1}] \end{aligned} \quad (3.15)$$

Here, $||T^{\text{el}}||$ ($= \langle J_f || T_J^{\text{el}} || J_i \rangle$), $||T^{\text{m}}||$ ($= \langle J_f || T_J^{\text{mag}} || J_i \rangle$) are the operators of the electric and magnetic transverse multipole and $||M^{\text{coul}}||$ ($= \langle J_f || M_J^{\text{coul}} || J_i \rangle$) is the Coulomb multipole operator of order J.

The values of $V_T(\theta)$, $V_L(\theta)$ and $V_P(\theta)$ are given by Eqns. (2.17b), (2.18b) and (2.20b) respectively.

$$V_T(\theta) = \left(\frac{q_\mu^2}{2q} + \tan^2\theta/2\right) 2k_1 k_2 \cos^2\theta/2 \quad (3.16)$$

$$V_L(\theta) = \frac{q_\mu^4}{4q} 2k_1 k_2 \cos^2\theta/2 \quad (3.17)$$

$$V_P(\theta) = - \frac{2(\epsilon_1 + \epsilon_2)}{|q|} 2k_1 k_2 \sin^2\theta/2 \quad (3.18)$$

Similarly, for left-handed incident electrons, the cross-section of inelastic scattering is as given in Eqn. (2.24).

$$\begin{aligned} \frac{d\sigma}{d\Omega}\bigg|_L = & \frac{8\pi\alpha^2}{q_\mu^4} \left(\frac{k_2}{k_1}\right) \frac{1}{1 + \frac{k_2 - k_1 \cos\theta}{E'}} [V_L(\theta) \sum_{J=0}^{\infty} \frac{||M^{\text{coul}}||^2}{2J_i+1} \\ & + V_T(\theta) \sum_{J=1}^{\infty} \frac{\{||T^{\text{el}}||^2 + ||T^{\text{m}}||^2\}}{2J_i+1} \\ & - 2V_p(\theta) \sum_{J=1}^{\infty} \frac{||T^{\text{el}}|| ||T^{\text{m}}||^*}{2J_i+1}] \end{aligned} \quad (3.19)$$

Substituting Eqns. (3.15) and (3.19) and the values of $V_T(\theta)$, $V_L(\theta)$ and $V_p(\theta)$ from Eqns. (3.16) to (3.18) in Eqn. (3.1), we obtain the asymmetry:

$$A(q^2) = - \frac{2(\epsilon_1 + \epsilon_2)}{\bar{q}} \tan^2 \theta/2 \frac{\sum_{J=1}^{\infty} ||T^{\text{el}}|| ||T^{\text{m}}||^*}{\left[\frac{q_\mu}{4} \sum_{J=0}^{\infty} ||M^{\text{coul}}||^2 + \left(\frac{q_\mu^2}{2q} + \tan^2 \theta/2 \right) \sum_{J=1}^{\infty} \{ ||T^{\text{el}}||^2 + ||T^{\text{m}}||^2 \} \right]} \quad (3.20)$$

At the backscattering angle ($\theta=\pi$), we obtain the maximum value for the asymmetry:

$$A(q^2)\big|_{\theta=\pi} = \frac{-2 \sum_{J=1}^{\infty} \langle J_f ||T^{\text{el}}|| J_i \rangle \langle J_f ||T^{\text{mag}}|| J_i \rangle}{\sum_{J=1}^{\infty} \{ |\langle J_f ||T^{\text{el}}|| J_i \rangle|^2 + |\langle J_f ||T^{\text{mag}}|| J_i \rangle|^2 \}} \quad (3.21)$$

from Eqn. (3.20).

For transitions with $J_i, J_f = \frac{1}{2}, 1$ only the asymmetry for backscattering takes the form:

$$A(q^2) = \frac{2\langle J_f || T^{e1} || J_i \rangle}{\langle J_f || T^{mag} || J_i \rangle} \text{ or } \frac{2\langle J_f || T^{mag} || J_i \rangle}{\langle J_f || T^{e1} || J_i \rangle}$$

In this case, because of the parity admixture, both transverse electric and magnetic multipole radiations contribute and it is their interference which gives rise to the asymmetry. Hence, the asymmetry increases with the parity admixture amplitude F and vanishes in the case of nuclear states with definite parity.

In the above discussion, we have not mentioned the weak charged current which is always present together with the weak neutral current. We find that the weak neutral current considered in addition to the charged current increases the value of the parity mixing amplitude F . F is given as:

$$F = \frac{M(\Delta I=1)}{\Delta E}$$

In Cabbibo's model, $M(\Delta I=1) \approx 10^{-7}$ whereas in Weinberg-Salam's model $M(\Delta I=1) \approx 10^{-6}$. As a result of increase of F , the asymmetry also increases. The asymmetries for backscattering at low momentum transfer have been calculated for ${}^2\text{H}$, ${}^{18}\text{F}$ in both Cabbibo and Weinberg-Salam models.

Adelberger et al²³ have measured the parity mixing in the ground ($J^\pi = \frac{1}{2}^+$) and 110 kev ($J^\pi = \frac{1}{2}^-$) excited state of ${}^{19}\text{F}$ and found the asymmetry to be $\approx -1.8 \pm 0.9 \times 10^{-4}$.

We now summarize the main features of the two processes viz the first process in which asymmetry is caused by Z^0 boson exchange (Fig. 1), and the second process in which asymmetry arises due to mixing of different parity states in nucleus (Fig. 2). In both cases, larger values for asymmetry are expected for ~~forbidden~~ allowed transitions. But, in the following aspects, they differ.

- 1) The first process produces sizeable asymmetry at large momentum transfer, and it is difficult to measure the asymmetry in electron scattering experiments because of the small value of the cross-section. On the other hand, the second process involving parity mixture in nuclear states produces sizeable asymmetry at low momentum transfer and, in some cases, the asymmetries are larger by orders of magnitude compared to the contribution from the first process. Thus, it provides an opportunity to test the weak interaction theories at low momentum transfer.
- 2) In the calculation of the asymmetry due to Z^0 boson exchange as done by Feinberg et al, the effects of neutral currents alone were obtained. But in the second process, the effect of neutral current enters as an enhancement over the values expected from the Cabbibo model calculation only. Hence, in this case, effect of neutral current may be obtained by subtracting the contribution of charged current given by Cabbibo's model from the total contribution.

- 3) In the second case, the asymmetry vanishes for transition between two spinless states (i.e. $J_i = J_f = 0$) as may be seen from Eqn. (3.21). But the first process due to Z^0 boson exchange produces a non-vanishing contribution to the asymmetry for such transitions.

CHAPTER 4

INELASTIC SCATTERING OF POLARIZED ELECTRON FROM HYDROGEN ATOM

SECTION A

In the previous chapter, we have seen that in the scattering of polarized electron from a mixed parity state in nucleus, at low momentum transfer, the weak neutral current has an effective contribution. But there is always present the contribution due to weak charged currents together with the neutral currents. On the other hand, parity impurity of the atomic states arise from neutral currents alone. Hence, in order to study the effect due to weak neutral current, we investigate the atomic case. Here, we consider the scattering from hydrogen atom in which $2S_{1/2}$ state is nearly degenerate with $2P_{1/2}$ state with opposite parity. Because of the small energy difference between these two states and of the high degree of forbiddenness of $2S_{1/2} \rightarrow 1S_{1/2}$ single photon transition, the effect of parity mixing seems to be large. But, the small value of the admixture amplitude imposes the need for very great sensitivity in any new parity test. Using the same technique as for nuclei, we have calculated the order of magnitude of the asymmetry in the inelastic scattering of longitudinally polarized electrons from hydrogen atom and thus have tried to find whether or not it is experimentally feasible to measure such an asymmetry. We assume that the incident electron has low energy

(of the order of several hundred e.v.s) so that the scattering takes place mainly from the orbital electron bound in $2S_{1/2}$ state with a small admixture of $2P_{1/2}$ state. The nuclear effect is taken to be negligible.

F being the admixture amplitude, the initial state is:

$$|i\rangle = |2S_{1/2} + F 2P_{1/2}\rangle$$

The final state is:

$$|f\rangle = |1S_{1/2}\rangle$$

The reaction we are interested in is:

$$e^-(k_1, s) + e^-(2S_{1/2} + F 2P_{1/2}) \rightarrow e^-(k_2) + e^-(1S_{1/2})$$

The wave functions for initial and final states are given by the exact solution to Dirac's differential equation for an electron in a Coulomb field. Following Bethe Salpeter²⁹, we can write the solution as:

$$\psi = \begin{pmatrix} g(r) \sqrt{\frac{\ell+m+1/2}{2\ell+1}} & Y_{\ell, m-1/2} \\ -g(r) \sqrt{\frac{\ell-m+1/2}{2\ell+1}} & Y_{\ell, m+1/2} \\ -if(r) \sqrt{\frac{\ell-m+3/2}{2\ell+3}} & Y_{\ell+1, m-1/2} \\ -if(r) \sqrt{\frac{\ell+m+3/2}{2\ell+3}} & Y_{\ell+1, m+1/2} \end{pmatrix} \quad (4.1)$$

for $j=\ell+1/2$. Here, $g(r)$ and $f(r)$ are the radial wave functions.

For $j=l-\frac{1}{2}$, we can write:

$$\psi = \begin{pmatrix} g(r) \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}} & Y_{l,m-\frac{1}{2}} \\ g(r) \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}} & Y_{l,m+\frac{1}{2}} \\ -if(r) \sqrt{\frac{l+m-\frac{1}{2}}{2l-1}} & Y_{l-1,m-\frac{1}{2}} \\ if(r) \sqrt{\frac{l-m-\frac{1}{2}}{2l-1}} & Y_{l-1,m+\frac{1}{2}} \end{pmatrix} \quad (4.2)$$

From Eqn. (4.1), we can write the wave function $\psi_{1S_{\frac{1}{2}}}$ for the final state ($j=l+\frac{1}{2}=\frac{1}{2}$, $l=0$, $m=\frac{1}{2}$).

$$\psi_{1S_{\frac{1}{2}}} = \begin{pmatrix} g_{1S} Y_{00} \\ 0 \\ -if_{1S} \frac{1}{\sqrt{3}} Y_{10} \\ -if_{1S} \sqrt{\frac{2}{3}} Y_{11} \end{pmatrix}$$

or

$$\psi_{1S_{\frac{1}{2}}} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} g_{1S} \\ 0 \\ -if_{1S} \cos\theta \\ -if_{1S} \sin\theta e^{i\theta} \end{pmatrix} \quad (4.3)$$

since $Y_{00} = 1/\sqrt{4\pi}$, $Y_{10} = \sqrt{3/4\pi} \cos\theta$, $Y_{11} = \sqrt{3/8\pi} \sin\theta$.

g_{1S} and f_{1S} are the radial wave functions for hydrogen atom ($Z=1$).

Following Bethe Salpeter, we have:

$$\begin{aligned}
 g_{1S} &= \left(\frac{2}{a_0}\right)^{3/2} \sqrt{\frac{1+\epsilon_1}{2\Gamma(2\gamma_1+1)}} e^{-\rho_1/2} \rho_1^{\gamma_1-1} \\
 &= \left(\frac{2}{a_0}\right)^{3/2} e^{-r/a_0} \left(\frac{2r}{a_0}\right)^{\gamma_1-1} \sqrt{\frac{1+\epsilon_1}{2\Gamma(2\gamma_1+1)}} \quad \rho_1 = \frac{2r}{a_0} \\
 &= C_1 e^{-r/a_0} r^{\gamma_1-1} \quad (4.3a)
 \end{aligned}$$

C_1 is a constant, its value has been derived in Appendix B.

Here,

a_0 = Bohr radius

$$\gamma_1 = \sqrt{1-\alpha^2} \quad (A)$$

α = fine structure constant

$$\epsilon_1 = \frac{1}{1 + \left(\frac{\alpha}{\gamma_1}\right)^2}$$

$$f_{1S} = -\sqrt{\frac{1-\epsilon_1}{1+\epsilon_1}} g_{1S}$$

$$= -\sqrt{\frac{1-\epsilon_1}{1+\epsilon_1}} C_1 e^{-r/a_0} r^{\gamma_1-1}$$

$$= -C_2 e^{-r/a_0} r^{\gamma_1-1} \quad (4.3b)$$

Here C_2 is a constant. The constants ϵ_1 , γ_1 are given by (A).

The wave function $\psi_{2S_{1_2}}$ for the initial state can be written from Eqn. (4.3) simply by putting g_{2S} and f_{2S} instead of g_{1S} and f_{1S} , the angular part remaining the same.

$$\psi_{2S_{1_2}} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} g_{2S} \\ 0 \\ -if_{2S}\cos\theta \\ -if_{2S}\sin\theta e^{i\phi} \end{pmatrix} \quad (4.4)$$

The explicit form of g_{2S} and f_{2S} is:

$$g_{2S} = \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \sqrt{\frac{1+\epsilon_2}{4N_2(N_2+1)}} e^{-\rho_2/2} \left[N_2 \rho_2^{\gamma_1-1} - \frac{N_2+1}{2\gamma_1+1} \rho_2^{\gamma_1} \right]$$

where

$$\begin{aligned} \rho_2 &= \frac{2r}{N_2 a_0} \\ &= C_3 e^{-r/N_2 a_0} r^{\gamma_1-1} - C_4 e^{-r/N_2 a_0} r^{\gamma_1} \end{aligned} \quad (4.4a)$$

C_3 and C_4 are constants which are evaluated in Appendix. Here,

$$N_2 = \sqrt{2(1+\gamma_1)} \quad (B)$$

$$\epsilon_2 = \frac{1}{\sqrt{1 + \frac{\alpha^2}{(1+\gamma_1)^2}}}$$

γ_1 and a_0 have the same value as given by (A).

$$\begin{aligned}
f_{2S}(r) &= -\sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} [(2\gamma_1+1)(N_2+2) - (N_2+1)\rho_2] \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \\
&\times \sqrt{\frac{1+\epsilon_2}{4N_2(N_2+1)}} e^{-r/N_2 a_0} \left(\frac{2r}{N_2 a_0}\right)^{\gamma_1-1} \frac{1}{2\gamma_1+1} \\
&= -C_5 e^{-r/N_2 a_0} r^{\gamma_1-1} + C_6 e^{-r/N_2 a_0} r^{\gamma_1} \quad (4.4b)
\end{aligned}$$

The constants C_5 and C_6 are evaluated in Appendix B. From Eqn. (3.2), we can write the wave function for $2P_{1/2}$ state ($j=\ell-l_2=1/2$, $\ell=1$, $m=1/2$).

$$\psi_{2P_{1/2}} = \begin{pmatrix} \frac{1}{\sqrt{3}} g_{2P} Y_{10} \\ \sqrt{\frac{2}{3}} g_{2P} Y_{11} \\ -if_{2P} Y_{00} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{4\pi}} \begin{pmatrix} g_{2P} \cos\theta \\ g_{2P} \sin\theta e^{i\phi} \\ -if_{2P} \\ 0 \end{pmatrix} \quad (4.5)$$

where

$$\begin{aligned}
g_{2P} &= \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \sqrt{\frac{1+\epsilon_2}{4N_2(N_2-1)}} \\
&\times e^{-r/N_2 a_0} \left(\frac{2r}{N_2 a_0}\right)^{\gamma_1-1} \frac{1}{(2\gamma_1+1)} [(N_2-2)(2\gamma_1+1) - (N_2-1) \frac{2r}{N_2 a_0}]
\end{aligned}$$

or

$$g_{2P} = C_7 e^{-r/N_2 a_0} r^{\gamma_1-1} - C_8 e^{-r/N_2 a_0} r^{\gamma_1} \quad (4.5a)$$

C_7 and C_8 are constants whose values are given in Appendix B. γ_1 , N_2 , a_0 and ϵ_2 have the same values as given by (A) and (B).

$$\begin{aligned}
 f_{2P} &= -\sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \left[\frac{(2\gamma_1+1)N_2-(N_2-1)\rho_2}{(2\gamma_1+1)(N_2-2)-(N_2-1)\rho_2} \right] g_{2P} \\
 &= -C_9 e^{-r/N_2 a_0} r^{\gamma_1-1} + C_{10} e^{-r/N_2 a_0} r^{\gamma_1}
 \end{aligned} \quad (4.5b)$$

The expressions for these ten constants and their numerical values are given in Appendix B.

To calculate the cross-section of scattering (e^- -H atom) let us first find the values of the three multipole operators.

a) Calculation of the Coulomb multipole operator:

We define the Coulomb multiple operator M_{JM}^{coul} as:

$$\langle f | M_{JM}^{\text{coul}} | i \rangle = \int j_J(qx) Y_{JM}(\Omega) \rho_{fi} d\bar{x}$$

For very low momentum transfer ($q \rightarrow 0$), the small argument limit for spherical Bessel function can be used.

$$j_J(qx) \xrightarrow{q \rightarrow 0} \frac{(qx)^J}{(2J+1)!!}$$

Therefore,

$$\langle f | M_{JM}^{\text{coul}} | i \rangle = \frac{q^J}{(2J+1)!!} \int d\bar{x} x^J Y_{JM}(\Omega) \rho_{fi} \quad (4.6)$$

The charge density ρ_{fi} is given as:

$$\begin{aligned}
 \rho_{fi} &= \psi_{1S_{\frac{1}{2}}}^+ \psi_{2S_{\frac{1}{2}}} \\
 &= \frac{1}{4\pi} (g_{1S} g_{2S} + f_{1S} f_{2S})
 \end{aligned} \quad (4.7)$$

Substituting Eqn. (4.7) in (4.6), we obtain the transition matrix element for the operator M_{JM}^{coul} :

$$\langle f | M_{JM}^{\text{coul}} | i \rangle = \frac{q^J}{(2J+1)!!} \iint r^2 dr r^J Y_{JM}(\Omega) \frac{1}{4\pi} (g_{1S} g_{2S} + f_{1S} f_{2S}) d\Omega$$

where

$$d\bar{x} = r^2 dr d\Omega$$

or

$$\langle f | M_{JM}^{\text{coul}} | i \rangle = \frac{q^J}{(2J+1)!!} \frac{1}{\sqrt{4\pi}} \delta_{JO} \delta_{MO} \int_0^\infty r^{J+2} (g_{1S} g_{2S} + f_{1S} f_{2S}) dr$$

since

$$\int Y_{JM}(\Omega) Y_{00}^* d\Omega = \delta_{JO} \delta_{MO}$$

Using the expressions for g_{1S} , f_{1S} from Eqns. (4.3a), (4.3b) and g_{2S} , f_{2S} from Eqns. (4.4a) and (4.4b) respectively, we have:

$$\langle 1S | M_{00}^{\text{coul}} | 2S \rangle = \frac{1}{\sqrt{4\pi}} \frac{(C_1 C_3 + C_2 C_5) \Gamma(2\gamma_1 + 1)}{(k)^{2\gamma_1 + 1}} - \frac{(C_1 C_4 + C_2 C_6) \Gamma(2\gamma_1 + 2)}{(k)^{2\gamma_1 + 2}} \quad (4.8)$$

$$\text{where } k = \frac{1}{a_0} + \frac{1}{N_2 a_0}$$

Substituting the values of the constants from the Appendix B in (4.8), we have:

$$\langle 1S | M_{00}^{\text{coul}} | 2S_{\frac{1}{2}} \rangle = -2.336 \times 10^{-5}$$

Therefore,

$$\langle 1S_{\frac{1}{2}} | M_{00}^{\text{coul}} | 2S_{\frac{1}{2}} \rangle = \frac{-2.336 \times 10^{-5}}{C(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})}$$

or

$$||M^{\text{coul}}|| = -32.95 \times 10^{-6} \quad (4.9)$$

b) Calculation of the electric multipole operator T_{JM}^{elec} :

For the moving electron, the total current can be considered to be the sum of convection current due to the moving charge and the current associated with the intrinsic magnetization of the electron. Therefore, the total current density is:

$$\vec{J}(\vec{x}) = \vec{J}_e(\vec{x}) + \nabla \times \vec{\mu}_e$$

$\nabla \times \vec{\mu}_e$ denotes the current density due to magnetization. In the limit of very low momentum transfer (i.e. $q \rightarrow 0$), the electric multipole operator can be written as (Appendix C):

$$\langle f | T_{JM}^{\text{el}} | i \rangle = \frac{q^{J-1}}{i(2J+1)!!} \sqrt{\frac{J+1}{J}} \int d\vec{x} x^J Y_{JM}(\Omega) \{ \nabla \cdot \vec{J}_{e_{fi}} + \frac{q^2}{J+1} \nabla \cdot (\vec{r} \times \vec{\mu}_{e_{fi}}) \} \quad (4.10)$$

Now,

$$(\nabla \times \vec{\mu}_{e_{fi}})_\alpha = -\frac{i}{2m} p_\beta (\bar{\psi}_f \sigma_{\alpha\beta} \psi_i) \quad (4.11)$$

where

$$\sigma_{\alpha\beta} = \begin{pmatrix} \sigma_Y & 0 \\ 0 & \sigma_Y \end{pmatrix}$$

From the above expression (4.11), three components of $\vec{\mu}_{e_{fi}}$ can be calculated. Hence, we can write:

$$\vec{r} \times \vec{\mu}_{e_{fi}} = -\hat{\theta} \frac{i}{8m\eta} r (g_{1S} g_{2P} + f_{1S} f_{2P}) \sin\theta \quad (4.12)$$

where $\hat{\theta}$ is the unit vector.

Therefore,

$$\begin{aligned} \nabla(\bar{r} \times \bar{u})_{e_{fi}} &= \frac{1}{r^2 \sin \theta} \frac{r \partial}{\partial \theta} \left[-\frac{i}{8m\pi} r (g_{1S} g_{2P} + f_{1S} f_{2P}) \sin^2 \theta \right] \\ &= -\frac{i}{4m\pi} (g_{1S} g_{2P} + f_{1S} f_{2P}) \cos \theta \end{aligned} \quad (4.13)$$

We use the continuity equation.

$$\begin{aligned} \nabla \cdot \bar{J}_{e_{fi}} &= i q \rho_{fi} \\ &= i q \psi_{1S}^+ \psi_{2P} \\ &= \frac{i q}{4\pi} (g_{1S} g_{2P} + f_{1S} f_{2P}) \cos \theta \end{aligned} \quad (4.14)$$

ψ_{1S}^+ and ψ_{2P} are given in Eqns. (4.3) and (4.5).

Substituting Eqns. (4.13) and (4.14) in (4.10), we have the transition matrix for electric operator given as:

$$\begin{aligned} \langle f | T_{JM}^{el} | i \rangle &= \frac{q^J}{(2J+1)!!} \sqrt{\frac{J+1}{J}} \iint r^{J+2} d\mathbf{r} Y_{JM}(\Omega) \frac{1}{4\pi} (g_{1S} g_{2P} + f_{1S} f_{2P}) \cos \theta d\Omega \\ &= \frac{q^{J+1}}{m \sqrt{J(J+1)}} \frac{1}{(2J+1)!!} \iint r^{J+2} d\mathbf{r} \frac{1}{4\pi} (g_{1S} g_{2P} + f_{1S} f_{2P}) \\ &\quad \cos \theta Y_{JM}(\Omega) d\Omega \\ &= \frac{q^J}{(2J+1)!!} \sqrt{\frac{J+1}{J}} \frac{1}{\sqrt{12\pi}} \delta_{J1} \delta_{M0} \left[\int r^{J+2} (g_{1S} g_{2P} + f_{1S} f_{2P}) dr \right. \\ &\quad \left. - \frac{q}{m(J+1)} \int r^{J+2} (g_{1S} g_{2P} + f_{1S} f_{2P}) dr \right] \end{aligned}$$

since

$$\int Y_{JM}(\Omega) Y_{10}^* d\Omega = \delta_{J1} \delta_{M0}$$

or

$$\langle f | T_{10}^{el} | i \rangle = \left(\frac{q}{3\sqrt{6}\pi} - \frac{q^2}{m6\sqrt{6}\pi} \right) X \quad (4.15)$$

where

$$X = \int_0^\infty r^3 (g_{1S} g_{2P} + f_{1S} f_{2P}) dr \quad (4.16)$$

Substituting g_{1S} , f_{1S} , g_{2P} , f_{2P} from Eqns. (4.3a), (4.3b), (4.5a) and (4.5b) and the values of the constants from Appendix B, we have

$$X = -6.842 \times 10^{-9}$$

Putting $X = -6.842 \times 10^{-9}$ in Eqn. (4.15), we have:

$$\langle f | T_{10}^{el} | i \rangle = -26.62q$$

or

$$\langle 1S || T_1^{el} || 2P \rangle = \frac{-26.62q}{C(\frac{1}{2}, 1, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2})}$$

or

$$|| T^{el} || = -65.21q \quad (4.17)$$

c) Calculation of the magnetic multipole operator T_{JM}^{mag} :

In the limit of very small momentum transfer (i.e. $q \rightarrow 0$), we can write the magnetic multipole operator giving rise to a transition from $2S_{\frac{1}{2}}$ to $1S_{\frac{1}{2}}$ state as:

$$\langle f | T_{JM}^{mag} | i \rangle = \frac{1}{i} \frac{q^J}{(2J+1)!!} \int d\vec{x} x^J Y_{JM}(\Omega) \{ \nabla \cdot \vec{\mu}_{e_{fi}} + \frac{1}{J+1} \nabla(\vec{r} \times \vec{J}_{e_{fi}}) \} \quad (4.18)$$

From the three components of $\bar{\mu}_{e_{fi}}$, we can write:

$$\nabla \cdot \bar{\mu}_{e_{fi}} = - \frac{1}{2m} \frac{1}{4\pi} \frac{\cos\theta}{2} \frac{\partial}{\partial r} (r^2 g_{1S} g_{2S} - r^2 f_{1S} f_{2S}) + \frac{1}{2m} \frac{1}{4\pi} \frac{2\cos\theta}{r} (g_{1S} g_{2S} + f_{1S} f_{2S}) \quad (4.19)$$

The current density $\bar{J}_{e_{fi}}$ is given as:

$$\bar{J}_{e_{fi}} = \psi_{1S}^+ \bar{\alpha} \psi_{2S} \quad (4.20)$$

Using the expressions for ψ_{1S} and ψ_{2S} from Eqns. (4.3) and (4.4) and $\bar{\alpha} = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix}$ we can find the three components of $\bar{J}_{e_{fi}}$ and hence we have:

$$\bar{\nabla} \cdot (\bar{r} \times \bar{J}_{e_{fi}}) = \frac{1}{4\pi} 2(g_{1S} f_{2S} + f_{1S} g_{2S}) \cos\theta \quad (4.21)$$

Substituting Eqns. (4.19) and (4.21) in (4.18), we have:

$$\begin{aligned} \langle f | T_{JM}^{\text{mag}} | i \rangle &= \frac{q^J}{i(2J+1)!!} \sqrt{\frac{J+1}{J}} \int r^{J+2} dr Y_{JM}(\Omega) d\Omega \\ &\quad \left[\left\{ - \frac{1}{2m} \frac{\cos\theta}{4\pi r^2} \frac{\partial}{\partial r} (r^2 g_{1S} g_{2S} - r^2 f_{1S} f_{2S}) \right. \right. \\ &\quad \left. \left. + \frac{1}{2m} \frac{\cos\theta}{2\pi r} (g_{1S} g_{2S} + f_{1S} f_{2S}) \right\} \right. \\ &\quad \left. + \frac{1}{J+1} \frac{1}{4\pi} 2\cos\theta (g_{1S} f_{2S} + f_{1S} g_{2S}) \right] \quad (4.22) \end{aligned}$$

Using the expressions for g_{1S} , g_{2S} , f_{1S} , f_{2S} and substituting the values of the constants from Appendix B, we obtain from Eqn. (4.22):

$$i\langle 1S | T_{10}^{\text{mag}} | 2S \rangle = 2.60 \times 10^{-3} q$$

or

$$\begin{aligned} i\langle 1S | |T_{10}^{\text{mag}}| | 2S \rangle &= \frac{2.60 \times 10^{-3}}{1/\sqrt{6}} q \\ &= 6.369 \times 10^{-3} q \end{aligned}$$

or

$$||T^{\text{mag}}|| = 6.369 \times 10^{-3} q \quad (4.23)$$

Because of the symmetry of the multipole operator under time reversal $||T^{\text{mag}}||$ must be real.

Proceeding in the same manner as in Chapter 2, we can write the scattering cross-section for right polarized incident electron:

$$\begin{aligned} \frac{d\sigma}{d\Omega} \Big|_R &= \frac{8\pi\alpha^2}{q_\mu} \left(\frac{k_2}{k_1} \right) \frac{1}{1 + \frac{k_2 - k_1 \cos\theta}{E_i}} \left[V_L(\theta) \sum_{J=0}^{\infty} \frac{||M^{\text{coul}}||^2}{2J_i + 1} \right. \\ &\quad + V_T(\theta) \sum_{J=1}^{\infty} F^2 \frac{||T^{e1}||^2 + ||T^m||^2}{2J_i + 1} \\ &\quad \left. + 2V_P(\theta)F \sum_{J=1}^{\infty} \frac{||T^{e1}|| ||T^m||^*}{2J_i + 1} \right] \quad (4.24) \end{aligned}$$

Here, $||M^{\text{coul}}||$, $||T^{e1}||$ and $||T^m||$ are the reduced matrix elements for transition of the orbital electron from $(2S_{1/2} + F 2P_{1/2})$ state to find $1S_{1/2}$ state and their values are given by Eqns. (4.9), (4.17) and (4.23) respectively. For low energy incident electron, the values of $V_L(\theta)$, $V_T(\theta)$ and $V_P(\theta)$ are as follows:

$$V_L(\theta) = \frac{q_\mu^4}{q} \left(\frac{k_1^2 + k_2^2}{2} + k_1 k_2 \cos\theta + 2m^2 \right) \quad (4.25)$$

$$\begin{aligned} V_T(\theta) &= \frac{k_1^2 k_2^2 \sin^2 \theta}{q} + \frac{k_1^2 k_2^2 - 2k_1 k_2 \cos\theta}{2} \\ &= \frac{k_1^2 k_2^2 \sin^2 \theta}{q} + \frac{q}{2} \end{aligned} \quad (4.26)$$

and

$$V_P(\theta) = \frac{1}{|q|} \left[\left(\frac{k_1^2 + k_2^2}{2} \right) (k_2 \cos\theta - k_1) + k_1 k_2 (k_1 \cos\theta - k_2) \right] \quad (4.27)$$

Substituting the values of the multipole operators and putting $J_i = \frac{1}{2}$ we obtain from Eqn. (4.24):

$$\begin{aligned} \frac{d\sigma}{d\Omega} \Big|_R &= \frac{4\pi\alpha^2}{q_\mu} \left(\frac{k_2}{k_1} \right) \frac{1}{1 + \frac{k_2 - k_1 \cos\theta}{E'}} [V_L(\theta) (3.295 \times 10^{-5})^2 \\ &\quad + V_T(\theta) \{ F^2(-65.21q)^2 + (6.369 \times 10^{-3} q)^2 \} \\ &\quad + 2V_P(\theta) F(-65.21q) (6.369 \times 10^{-3} q)] \end{aligned} \quad (4.28)$$

Similarly, the scattering cross-section for left polarized incident electron is:

$$\begin{aligned} \frac{d\sigma}{d\Omega} \Big|_L &= \frac{4\pi\alpha^2}{q_\mu} \left(\frac{k_2}{k_1} \right) \frac{1}{1 + \frac{k_2 - k_1 \cos\theta}{E'}} [V_L(\theta) (3.295 \times 10^{-5})^2 \\ &\quad + V_T(\theta) \{ F^2(65.21q)^2 + (6.369 \times 10^{-3} q)^2 \} \\ &\quad - 2V_P(\theta) F(-65.21) (6.369 \times 10^{-3} q)^2] \end{aligned} \quad (4.29)$$

From Eqns. (4.28) and (4.29), we can write the asymmetry as:

$$A(q^2) = \frac{\frac{2F}{q} \left[\left(\frac{k_1^2 + k_2^2}{2} \right) (k_2 \cos \theta - k_1) + k_1 k_2 (k_1 \cos \theta - k_2) \right] (-65.21q) (6.369 \times 10^{-3} q)}{\frac{q_\mu^4}{q} \left(\frac{k_1^2 + k_2^2}{2} + k_1 k_2 \cos \theta + 2m^2 \right) (32.95 \times 10^{-6})^2 + \left(\frac{k_1^2 k_2^2 \sin^2 \theta}{q^2} + \frac{q^2}{2} \right) [(65.21q)^2 F^2 + (6.369 \times 10^{-3} q)^2]} \quad (4.30)$$

For backscattering ($\theta = \pi$), $\bar{q} = \bar{k}_1 + \bar{k}_2$ is maximum and hence the asymmetry reaches its maximum value for a fixed kinetic energy. Unlike the nuclear case, the asymmetry at $\theta = \pi$ is not constant but has the value given as:

$$A(q^2)_{\theta=\pi} = \frac{-F q^2 ||T^{e1}|| ||T^m||}{\frac{q_\mu^4}{q} \left[\frac{(k_2 - k_1)^2}{2} + 2m^2 \right] ||M^{coul}||^2 + \frac{q^2}{2} [F^2 ||T^{e1}||^2 + ||T^m||^2]} \quad (4.31)$$

where

$$||T^{e1}|| = -65.21q$$

$$||T^m|| = 6.369 \times 10^{-3} q$$

$$||M^{coul}|| = 32.95 \times 10^{-6}$$

Following Forest and Walecka, we can find the transition rate for the hydrogen atom to emit a real photon and make a transition from the state $|2P_{1/2}\rangle$ to a state $|1S_{1/2}\rangle$. It is given as:

$$W_{2P_{1/2} \rightarrow 1S_{1/2}} = 8\pi\alpha k \sum_{J=1}^{\infty} \frac{1}{2J+1} | \langle 1S_{1/2} | T_J^{e1} | 2P_{1/2} \rangle |^2$$

Using the values of $||T^{e1}||$ from Eqn. (4.18), $J_i = \frac{1}{2}$ and $k = 10^{-5} \text{ meV/C}$, we have

$$W_{2P_{1/2} \rightarrow 1S_{1/2}} = 5.93 \times 10^8 \text{ sec}^{-1}$$

This value of $W_{2P_{1/2} \rightarrow 1S_{1/2}}$ is in agreement with that given by Bethe and Salpeter.

Similarly, the transition rate from $2S_{1/2}$ state to $1S_{1/2}$ state by single photon is:

$$W_{2S_{1/2} \rightarrow 1S_{1/2}} = 8\pi\alpha k \frac{1}{2} | \langle 1S_{1/2} || T^{\text{mag}} || 2S_{1/2} \rangle |^2 R$$

where R is the branching ratio

$$\begin{aligned} &= \frac{W_{2S_{1/2} \rightarrow 1S_{1/2}}}{W_{2S_{1/2} \rightarrow 1S_{1/2}}} = 25 \times 10^{-8} \\ &= 14.14 \times 10^{-7} \text{ sec}^{-1} \end{aligned}$$

using

$$||T^m|| = 6.369 \times 10^{-3} k.$$

This agreement of the values of the transition rates with some standard values verifies that the values of $||T^{e1}||$ and $||T^m||$ which we have calculated are correct.

SECTION B. RESULTS

The values of the cross-section are computed for different incident energies and different angles of scattering. From Eqns. (4.28) and (4.29), we can write:

$$\left. \frac{d\sigma}{d\Omega} \right|_{R,L} = a \pm F.C \quad (4.30)$$

where

$$a = d(V_L(\theta) || M^{\text{coul}} ||^2 + V_T(\theta) || T^m ||^2)$$

$$c = d 2V_p(\theta) || T^{e1} || || T^m ||$$

$$d = \frac{4\pi\alpha^2}{q_\mu} \left(\frac{k_2}{k_1} \right) \frac{1}{1 + \frac{k_2 - k_1 \cos\theta}{E}}$$

Here positive and negative signs refer to R and L respectively. We have neglected the F^2 term in $\left. \frac{d\sigma}{d\Omega} \right|_{R,L}$ since it is very small.

Hence, from Eqn. (4.30), we can write the asymmetry as:

$$A(q^2) = F \frac{c}{a} \quad F \text{ being equal to } 10^{-9}.$$

The incident energy is:

$$\epsilon_1 = \frac{k_1^2}{2m} + m$$

or

$$k_1 = \sqrt{2m\epsilon_1 - m^2} \quad (4.32)$$

The final energy is:

$$\epsilon_2 = \frac{k_2^2}{2m} + m$$

and

$$\epsilon_2 - \epsilon_1 = \frac{k_2^2 - k_1^2}{2m}$$

= difference in energies of $2S_{1/2}$ and $1S_{1/2}$ states

$$= 10.2 \times 10^{-6} \text{ mev}$$

Therefore,

$$k_2 = \sqrt{k_1^2 + 2m \times 10.2 \times 10^{-6}} \quad (4.33)$$

The momentum transfer q is calculated from

$$\begin{aligned} \bar{q}^2 &= (\bar{k}_2 - \bar{k}_1)^2 \\ &= k_2^2 + k_1^2 - 2k_1 k_2 \cos \theta \end{aligned} \quad (4.34)$$

and

$$q_\mu^2 = \bar{q}^2 - (\epsilon_2 - \epsilon_1)^2 \quad (4.35)$$

Table 1. Values of asymmetry for different kinetic energies.

k.E in ev	θ in degrees	q in 10^{-3} mev/c	a in 10^{-24} cm ²	c in 10^{-24} cm ²	Asymmetry (c/a)
50	0	.695	.693x10 ³	-.119	-.17x10 ⁻³
	5	.954	.195x10 ³	.830	.43x10 ⁻²
	10	1.479	.338x10 ²	.926	.27x10 ⁻¹
	15	2.075	.873x10	.758	.87x10 ⁻¹
	20	2.692	.308x10	.615	.20
	25	3.316	.134x10	.512	.38
	29	3.814	.764	.450	.59
	30	3.938	.672	.437	.65
100	0	.503	.242x10 ⁴	-.114	-.47x10 ⁻⁴
	5	1.034	.136x10 ³	.173x10	.13x10 ⁻¹
	10	1.875	.126x10 ²	.116x10	.93x10 ⁻¹
	15	2.751	.271x10	.826	.31
	20	3.633	.089x10	.635	.71
	21	3.809	.734	.606	.82
	22	3.985	.614	.580	.95
200	0	.360	.904x10 ⁴	-.111	-.12x10 ⁻⁴
	4	1.073	.114x10 ³	.265x10	.23x10 ⁻¹
	8	2.052	.855x10	.151x10	.18
	12	3.048	.176x10	.104x10	.59
	14	3.547	.957	.895	.94
	15	3.797	.729	.837	.12x10
	16	4.046	.565	.787	.14x10
300	0	.295	.198x10 ⁵	-.110	-.56x10 ⁻⁵
	4	1.268	.582x10 ²	.290x10	.50x10 ⁻¹
	8	2.481	.397x10	.154x10	.39
	11	3.398	.113x10	.113x10	.10x10 ¹
	12	3.704	.799	.104x10	.13x10
	13	4.009	.582	.964	.17x10
400	0	.256	.348x10 ⁵	-.109	-.31x10 ⁻⁵
	2	.755	.461x10 ³	.522x10	.11x10 ⁻¹
	4	1.443	.345x10 ²	.299x10	.87x10 ⁻¹
	6	2.145	.707x10	.205x10	.29
	8	2.851	.227x10	.155x10	.69
	9	3.204	.142x10	.138x10	.97
	10	3.556	.936	.125x10	1.31
	11	3.909	.641	.113x10	1.77

Table 1. (continued)

k.E in ev	θ in degrees	q in 10^{-3} mev/c	a in 10^{-24} cm ²	c in 10^{-24} cm ²	Asymmetry (c/a)
500	0	.229	.540x10 ⁵	-.110	-.20x10 ⁻⁵
	2	.826	.322x10 ³	.555x10	.17x10 ⁻¹
	4	1.603	.227x10 ²	.304x10	.13
	6	2.390	.459x10	.206x10	.45
	8	3.178	.147x10	.155x10	1.06
	9	3.573	.917	.138x10	1.51
	10	3.968	.604	.125x10	2.07

Table 2. Values of asymmetry for different kinetic energies and θ varying from 0° to 180° .

k.E in ev	θ in degrees	q in 10^{-3} mev/c	a in 10^{-24} cm ²	c in 10^{-24} cm ²	Asymmetry (c/a)
50	0	.695	.693x10 ³	-.119	-.17x10 ⁻³
	20	2.692	.308x10	.615	.20
	40	5.170	.226	.337	.15x10
	60	7.521	.503x10 ⁻¹	.233	.46x10 ²
	100	11.494	.091x10 ⁻¹	.152	.17x10 ²
	140	14.091	.402x10 ⁻²	.123	.31x10 ²
	180	14.992	.312x10 ⁻²	.115	.37x10 ²
100	0	.503	.242x10 ⁴	-.114	-.47x10 ⁻⁴
	20	3.633	.089x10	.635	.71
	40	7.104	.061	.328	.54x10
	60	10.371	.133x10 ⁻¹	.224	.17x10 ²
	100	15.879	.239x10 ⁻²	.145	.61x10 ²
	140	19.474	.105x10 ⁻²	.117	.11x10 ³
	180	20.722	.081x10 ⁻²	.109	.13x10 ³
200	0	.360	.904x10 ⁴	-.111	-.12x10 ⁻⁴
	20	5.041	.234	.633	.27x10
	40	9.910	.156x10 ⁻¹	.321	.21x10 ²
	60	14.482	3.41x10 ⁻²	.219	.64x10 ²
	100	22.183	.061x10 ⁻²	.140	.23x10 ³
	140	27.209	.269x10 ⁻³	.113	.42x10 ³
	180	28.954	.209x10 ⁻³	.105	.50x10 ³
300	0	.295	.198x10 ⁵	-.110	-.56x10 ⁻⁵
	20	6.140	.106	.630	.60x10
	40	12.083	.702x10 ⁻²	.319	.45x10 ²
	60	17.663	.153x10 ⁻²	.216	.14x10 ³
	100	27.056	.277x10 ⁻³	.138	.50x10 ³
	140	33.187	.122x10 ⁻³	.110	.90x10 ³
	180	35.315	.095x10 ⁻³	.103	.1x10 ⁴
400	0	.256	.347x10 ⁵	-.109	-.31x10 ⁻⁵
	20	7.071	.060	.628	.11x10 ²
	40	13.922	.398x10 ⁻²	.317	.80x10 ²
	60	20.351	.087x10 ⁻²	.215	.25x10 ³
	100	31.177	.158x10 ⁻³	.137	.86x10 ³
	140	38.242	.071x10 ⁻³	.109	.15x10 ⁴
	180	40.694	.555x10 ⁻⁴	.101	.18x10 ⁴

k.E in ev	θ in degrees	q in 10^{-3} mev/c	a in 10^{-24} cm ²	c in 10^{-24} cm ²	Asymmetry (c/a)
500	0	.229	.540x10 ⁵	-.110	-.20x10 ⁻⁵
	20	7.895	.385x10 ⁻¹	.627	.16x10 ²
	40	15.545	.256x10 ⁻²	.316	.12x10 ³
	60	22.724	.562x10 ⁻³	.214	.38x10 ³
	100	34.813	.104x10 ⁻³	.136	.13x10 ⁴
	140	42.702	.469x10 ⁻⁴	.108	.23x10 ⁴
	180	45.440	.371x10 ⁻⁴	.101	.27x10 ⁴

SECTION C. DISCUSSION OF THE RESULTS

We know that the admixture amplitude of $2P_{1/2}$ state with $2S_{1/2}$ state in hydrogen atom is very small and hence some method of enhancing it can be applied. Using external magnetic field, the two levels can be made closer and thus the maximum value of F (about 10^{-9}) can be attained. Provided F is maximized, we want to find the order of magnitude of the asymmetry for different values of incident energy and scattering angles. From Table 2, we find that for forward scattering angles the cross-section is about 10^{-21} cm^2 but C being very small the asymmetry is of the order of $10^{-4} F$. On the other hand, with the increase of scattering angle, the momentum transfer increases and so does the asymmetry. The asymmetry reaches its maximum value (about $10^3 F$) for backscattering angle, but then the cross-section drops to a very small value (about 10^{-26} cm^2). Again, for a particular value of the scattering angle, the asymmetry increases as the incident energy increases from 50 eV to 500 eV. We notice that for all energies at $\theta=0$, since k_2 is greater than k_1 , $V_p(\theta=0)$ which is proportional to $(k_2 - k_1)^3$ is positive and hence the asymmetry is negative. But as θ increases, $V_p(\theta)$ gets negative which results in positive values for the asymmetry. In Table 1 we have given the values for asymmetry at the angle $(\theta=1/k_1 a_0, a_0 \text{ being the radius of the atom})$ where the first diffraction peak occurs.

CHAPTER 5

CONCLUSIONS

From the results in Chapter 4, we see that the maximum value of the asymmetry in the inelastic scattering of polarized electron from hydrogen atom is about $10^3 F$ where F is the admixture amplitude of $2P_{1/2}$ state with $2S_{1/2}$ state. In an external magnetic field, the maximum value of F can be attained and it is about 10^{-9} and hence the asymmetry is of the order of 10^6 . The large values of asymmetry are obtained at high scattering angles but the cross-sections are very small (about 10^{-26} cm^2) at those angles. Hence it will be very difficult to measure such value of asymmetry experimentally. To conclude, we can say that the parity mixing due to weak neutral coupling of electron with the nucleus in hydrogen atom is very small for low momentum transfer and the expected effects are very difficult to detect experimentally. But, applying the similar technique used for hydrogen atom, we can find the value of asymmetry in the case of scattering of polarized electrons from muonic atoms, the heavier hydrogen-like atoms. In the latter case, the parity mixing amplitude F , which is proportional to $Gm_e^{2\mu}$, is larger because of the mass ratio $(\frac{m_\mu}{m_e})^2 \approx 4 \times 10^4$. Hence, we would expect the value of the asymmetry to be greatly enhanced for muonic atoms.

The neutral current interaction in the systems of large spatial dimensions could be investigated more easily in the following manner. The parity violating effects in atoms are strongly enhanced with

increasing atomic number (they go roughly as Z^3). Moreover, with the help of advanced laser technology, we can study the strongly hindered radiative transition where a weak electron-nucleon interaction shows up. This has been applied to study experimentally the parity violation induced by neutral currents in Cs ($Z=55$, $N=78$, $6S_{1/2} \rightarrow 7S_{1/2}$ transition) and thallium ($Z=81$, $N=124$, $6P_{1/2} \rightarrow 7P_{1/2}$ transition) by Henley and others. Though experimentally, it is much easier to find the handedness and hence the weak interaction effect in heavy atoms, theoretical interpretations are much more difficult compared to the case of light atoms.

Gari and Pomrmann calculated the asymmetry due to parity mixture of nuclear states at low momentum transfer. They found the value to be of the order of 10^{-4} in ^{18}F , ^{19}F . Since it is greater than the asymmetry obtained for H atom, it is experimentally feasible to measure such a value in the case of nuclei. The parity violating potential being practically of zero range nature gives matrix elements proportional to the inverse of the volume of the atom or nucleus. Hence, the nucleus appears to be more strongly favoured compared to atoms.

At high momentum transfer, the exchange of heavy neutral boson (Z^0) together with the photon contributes to the asymmetry. This neutral current contribution in the deep inelastic scattering of polarized electrons from nuclei can be calculated using various models. This helps to understand the fundamental nature of weak interaction and hence a great deal of both theoretical analyses and experimental searches are being carried out in this field in recent days.

APPENDIX B

EVALUATION OF THE CONSTANTS OF THE RADIAL WAVE FUNCTIONS
FOR HYDROGEN ATOM

Here, we evaluate the ten constants $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9$ and C_{10} of the radial wave functions for $1S_{1/2}, 2S_{1/2}$ and $2P_{1/2}$ states of hydrogen atom in units of a_0 , the Bohr radius. Following Bethe and Salpeter, we define:

$$\gamma_1 = \sqrt{1-\alpha^2}$$

where

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137} = \text{fine structure constant}$$

or

$$\gamma_1 = .9999733 \quad (\text{B.1})$$

$$N_1 = 1$$

$$N_2 = \sqrt{2(1+\gamma_1)} = 1.9999866 \quad (\text{B.2})$$

$$\epsilon_1 = \left[1 + \left(\frac{\alpha}{\gamma_1} \right)^2 \right]^{-1/2} = .9999733 \quad (\text{B.3})$$

$$\epsilon_2 = \left[1 + \left(\frac{\alpha}{1+\gamma_1} \right)^2 \right]^{-1/2} = .9999933 \quad (\text{B.4})$$

For $1S_{1/2}$ state ($n=1, l=0, j=1/2$), we have:

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APPENDIX A

CALCULATION OF TRACES

In Chapter 2, in the expression for the scattering cross-section, we have:

$$|m_{fi}|^2 = -\text{Tr}[\gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2}\right) \left(\frac{-i\gamma k_1 + m}{2m}\right) \gamma_\nu \left(\frac{-i\gamma k_2 + m}{2m}\right)] J_\mu J_\nu^*$$

Here, we give the details of the calculation of traces in the above expression.

$$\begin{aligned} & -\text{Tr}[\gamma_\mu \left(\frac{1+i\gamma_5 \gamma \cdot S}{2}\right) \left(\frac{-i\gamma k_1 + m}{2m}\right) \gamma_\nu \left(\frac{-i\gamma k_2 + m}{2m}\right)] = X \\ & = -\frac{1}{8m^2} \text{Tr}[-\gamma_\mu \gamma k_1 \gamma_\nu \gamma k_2 - i m \gamma_\mu \gamma_\nu \gamma k_2 - i \gamma_\mu \gamma_5 \gamma S \gamma k_1 \gamma_\nu \gamma k_2 \\ & \quad + m \gamma_\mu \gamma_5 \gamma S \gamma_\nu \gamma k_2 - i m \gamma_\mu \gamma k_1 \gamma_\nu + m^2 \gamma_\mu \gamma_\nu + m \gamma_\mu \gamma_5 \gamma S \gamma k_1 \\ & \quad + i m^2 \gamma_\mu \gamma_5 \gamma S \gamma_\nu] \end{aligned} \quad (A.1)$$

We have

$$\text{Tr}(-i m \gamma_\mu \gamma_\nu \gamma k_2) = \text{Tr}(-i m \gamma_\mu \gamma k_1 \gamma_\nu) = 0 \quad (A.2)$$

and

$$\text{Tr}(i m^2 \gamma_\mu \gamma_5 \gamma S \gamma_\nu) = \text{Tr}(-i \gamma_\mu \gamma_5 \gamma S \gamma k_1 \gamma_\nu \gamma k_2) = 0 \quad (A.3)$$

since the traces of odd number of γ matrices = 0.

Substituting Eqns. (A.2) and (A.3) in (A.1), we have:

$$\begin{aligned}
 X = & \frac{1}{8m^2} \text{Tr}(\gamma_\mu \gamma^k_1 \gamma_\nu \gamma^k_2) - \frac{1}{8m^2} \text{Tr}(m^2 \gamma_\mu \gamma_\nu) - \frac{1}{8m^2} \text{Tr}(m \gamma_\nu \gamma_5 \gamma S \gamma_\nu \gamma^k_2) \\
 & - \frac{1}{8m^2} \text{Tr}(m \gamma_\mu \gamma_5 \gamma S \gamma^k_1 \gamma_\nu) \quad (A.4)
 \end{aligned}$$

Let us now evaluate each trace in Eqn. (A.4).

$$\text{Tr}(\gamma_\mu \gamma^k_1 \gamma_\nu \gamma^k_2) = 4(k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - \delta_{\mu\nu} k_1 k_2) \quad (A.5)$$

$$\text{Tr}(m^2 \gamma_\mu \gamma_\nu) = 4m^2 \delta_{\mu\nu} \quad (A.6)$$

$$\begin{aligned}
 m \text{Tr}(\gamma_\mu \gamma_5 \gamma S \gamma_\nu \gamma^k_2) &= -m \text{Tr}(\gamma_5 \gamma_\nu \gamma S \gamma_\mu \gamma^k_2) \quad \gamma_5 \text{ anticommutes with } \gamma_\mu \\
 &= -4m \epsilon_{\mu\alpha\nu\beta} S_\alpha k_{2\beta} \quad (A.7)
 \end{aligned}$$

$$\begin{aligned}
 m \text{Tr}(\gamma_\mu \gamma_5 \gamma S \gamma^k_1 \gamma_\nu) &= -m \text{Tr}(\gamma_5 \gamma_\mu \gamma S \gamma^k_1 \gamma_\nu) \quad \gamma_5 \text{ anticommutes with } \gamma_\mu \\
 &= -4m \epsilon_{\mu\alpha\beta\nu} S_\alpha k_{1\beta} \quad (A.8)
 \end{aligned}$$

Substituting Eqns. (A.5), (A.6), (A.7) and (A.8) in (A.4), we obtain:

$$\begin{aligned}
 X = & \frac{1}{8m^2} 4(k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - \delta_{\mu\nu} k_1 k_2) - \frac{1}{8m^2} 4m^2 \delta_{\mu\nu} \\
 & + \frac{1}{8m^2} 4m \epsilon_{\mu\alpha\nu\beta} S_\alpha k_{2\beta} + \frac{1}{8m^2} 4m \epsilon_{\mu\alpha\beta\nu} S_\alpha k_{1\beta} \\
 = & \frac{1}{8m^2} 4[k_{1\mu} k_{2\nu} + k_{1\nu} k_{2\mu} - \delta_{\mu\nu} k_1 k_2 - \delta_{\mu\nu} m^2] \\
 & + \frac{1}{8m^2} 4m \epsilon_{\mu\alpha\beta\nu} (S_\alpha k_{1\beta} - S_\alpha k_{2\beta})
 \end{aligned}$$

Hence,

$$|m_{fi}|^2 = \frac{1}{8m^2} [4(k_{1\mu}k_{2\nu} + k_{1\nu}k_{2\mu} - \delta_{\mu\nu}k_1^2 - \delta_{\mu\nu}m^2)J_\mu J_\nu^*] \\ + \frac{1}{8m^2} 4m \epsilon_{\mu\nu\alpha\beta} (J_\mu J_\nu^* S_\alpha k_{1\beta} - J_\mu J_\nu^* S_\alpha k_{2\beta})$$

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$$\gamma_1 = \sqrt{1-\alpha^2}$$

where

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$$N_1 = 1$$

$$N_2 = \sqrt{2(1+\gamma_1)} = 1.9999866 \quad (\text{B.2})$$

$$\epsilon_1 = \left[1 + \left(\frac{\alpha}{\gamma_1} \right)^2 \right]^{-1/2} = .9999733 \quad (\text{B.3})$$

$$\epsilon_2 = \left[1 + \left(\frac{\alpha}{1+\gamma_1} \right)^2 \right]^{-1/2} = .9999933 \quad (\text{B.4})$$

For $1S_{1/2}$ state ($n=1, l=0, j=1/2$), we have:

$$\begin{aligned}
 g_{1S} &= \left(\frac{2}{a_0}\right)^{3/2} \sqrt{\frac{1+\epsilon_1}{2\Gamma(2\gamma_1+1)}} e^{-r/a_0} \left(\frac{2}{a_0}\right)^{\gamma_1-1} \\
 &= C_1 e^{-r/a_0} r^{\gamma_1-1}
 \end{aligned}$$

where

$$\begin{aligned}
 C_1 &= \left(\frac{2}{a_0}\right)^{3/2} \left(\frac{2}{a_0}\right)^{\gamma_1-1} \sqrt{\frac{1+\epsilon_1}{2\Gamma(2\gamma_1+1)}} \\
 &= \left(\frac{1}{a_0}\right)^{\gamma_1+1/2} \times 1.9999492 \text{ using values of } \gamma_1, \epsilon_1 \text{ from (B.1) and (B.3)}
 \end{aligned}$$

or

$$C_1 = 1.9999492 (a_0)^{-\gamma_1-1/2} \quad (\text{B.5})$$

and

$$\begin{aligned}
 f_{1S} &= -\sqrt{\frac{1-\epsilon_1}{1+\epsilon_1}} g_{1S} \\
 &= -\sqrt{\frac{1-\epsilon_1}{1+\epsilon_1}} C_1 e^{-r/a_0} r^{\gamma_1-1} \\
 &= -C_2 e^{-r/a_0} r^{\gamma_1-1}
 \end{aligned}$$

where

$$\begin{aligned}
 C_2 &= \sqrt{\frac{1-\epsilon_1}{1+\epsilon_1}} C_1 \\
 &= \sqrt{\frac{1-\epsilon_1}{1+\epsilon_1}} \left(\frac{1}{a_0}\right)^{\gamma_1+1/2} (2)^{\gamma_1+1/2} \sqrt{\frac{1+\gamma_1}{2\Gamma(2\gamma_1+1)}} \\
 &= \left(\frac{1}{a_0}\right)^{\gamma_1+1/2} \times 7.31 \times 10^{-3}
 \end{aligned}$$

or

$$C_2 = 7.31 \times 10^{-3} (a_0)^{-\gamma_1 - \frac{1}{2}} \quad (B.6)$$

For $2S_{\frac{1}{2}}$ state ($n=2$, $\ell=0$, $j=\frac{1}{2}$),

$$\begin{aligned} g_{2S} &= \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1 + 1}{\Gamma(2\gamma_1 + 1)}} \sqrt{\frac{1 + \epsilon_2}{4N_2(N_2 + 1)}} e^{-\gamma/N_2 a_0} \left(\frac{2r}{N_2 a_0}\right)^{\gamma_1 - 1} \frac{1}{2\gamma_1 + 1} \\ &\quad \cdot [N_2(2\gamma_1 + 1) - (N_2 + 1) \frac{2\gamma}{N_2 a_0}] \\ &= C_3 e^{-r/N_2 a_0} r^{\gamma_1 - 1} - C_4 e^{-r/N_2 a_0} r^{\gamma_1} \end{aligned} \quad (B.7)$$

where

$$C_3 = \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1 + 1}{\Gamma(2\gamma_1 + 1)}} \sqrt{\frac{1 + \epsilon_2}{4N_2(N_2 + 1)}} N_2 \left(\frac{2}{N_2 a_0}\right)^{\gamma_1 - 1}$$

or

$$C_3 = \left(\frac{1}{a_0}\right)^{\gamma_1 + \frac{1}{2}} \times 7.074 \times 10^{-1}$$

using the values of γ_1 , N_2 and ϵ_2 from Eqns. (B.1), (B.2) and (B.4),

or

$$C_3 = 7.074 \times 10^{-1} (a_0)^{-\gamma_1 - \frac{1}{2}} \quad (B.8)$$

And,

$$\begin{aligned}
C_4 &= \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \sqrt{\frac{1+\epsilon_2}{4N_2(N_2+1)}} \frac{(N_2+1)}{(2\gamma_1+1)} \left(\frac{2}{N_2 a_0}\right)^{\gamma_1} \\
&= \left(\frac{1}{a_0}\right)^{\gamma_1+3/2} \left(\frac{2}{N_2}\right)^{\gamma_1+3/2} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{4N_2(N_2+1)\Gamma(2\gamma_1+1)}} \cdot \frac{N_2+1}{2\gamma_1+1} \\
&= \left(\frac{1}{a_0}\right)^{\gamma_1+3/2} \times 1.735 \times 10^{20} \times 2.0389 \times 10^{-21}
\end{aligned}$$

using the values of γ_1 , N_2 and ϵ_2 from Eqns. (B.1), (B.2) and (B.4)

or

$$C_4 = 3.5375 \times 10^{-1} (a_0)^{-\gamma_1-3/2} \quad (B.9)$$

$$\begin{aligned}
f_{2S} &= - \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \left[(2\gamma_1+1)(N_2+2) - (N_2+1) \left(\frac{2r}{N_2 a_0}\right) \right] \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \\
&\quad \cdot \sqrt{\frac{1+\epsilon_2}{4N_2(N_2+1)}} e^{-r/N_2 a_0} \left(\frac{2r}{N_2 a_0}\right)^{\gamma_1-1} \frac{1}{2\gamma_1+1}
\end{aligned}$$

or

$$f_{2S} = -C_5 e^{-r/N_2 a_0} r^{\gamma_1-1} + C_6 e^{-r/N_2 a_0} r^{\gamma_1} \quad (B.10)$$

where

$$C_5 = \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} (N_2+2) \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \sqrt{\frac{1+\epsilon_2}{4N_2(N_2+1)}} \left(\frac{2}{N_2 a_0}\right)^{\gamma_1-1}$$

or

$$\begin{aligned}
C_5 &= \left(\frac{1}{a_0}\right)^{\gamma_1+1/2} \left(\frac{2}{N_2 a_0}\right)^{\gamma_1+1/2} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{4N_2(N_2+1)\Gamma(2\gamma_1+1)}} \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \\
&= \left(\frac{1}{a_0}\right)^{\gamma_1+1/2} \times 6.720 \times 10^9 \times 3.853 \times 10^{-13}
\end{aligned}$$

or

$$C_5 = 2.589 \times 10^{-3} (a_0)^{-\gamma_1 - 1/2} \quad (B.11)$$

and

$$\begin{aligned} C_6 &= \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \frac{(N_2+1)}{2\gamma_1+1} \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \sqrt{\frac{1+\epsilon_2}{4N_2(N_2+1)}} \left(\frac{2}{N_2 a_0}\right)^{\gamma_1} \\ &= \left(\frac{1}{a_0}\right)^{3/2+\gamma_1} \left(\frac{2}{N_2}\right)^{3/2+\gamma_1} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{4N_2(N_2+1)\Gamma(2\gamma_1+1)}} \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \sqrt{\frac{(N_2+1)}{(2\gamma_1+1)}} \\ &= \left(\frac{1}{a_0}\right)^{3/2+\gamma_1} \times 3.177 \times 10^{17} \times 2.039 \times 10^{-21} \end{aligned}$$

or

$$C_6 = 6.477993 \times 10^{-4} (a_0)^{-\gamma_1 - 3/2} \quad (B.12)$$

For $2P_{1/2}$ state ($n=2$, $l=1$, $j=1/2$), we have:

$$\begin{aligned} g_{2P} &= \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \sqrt{\frac{1+\epsilon_2}{4N_2(N_2-1)}} e^{-r/N_2 a_0} \left(\frac{2r}{N_2 a_0}\right)^{\gamma_1-1} (N_2-2) \\ &\quad - \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{\Gamma(2\gamma_1+1)4N_2(N_2-1)}} \frac{(N_2-1)}{2\gamma_1+1} e^{-r/N_2 a_0} \left(\frac{2r}{N_2 a_0}\right)^{\gamma_1} \end{aligned}$$

or

$$g_{2P} = C_7 e^{-r/N_2 a_0} r^{\gamma_1-1} - C_8 e^{-r/N_2 a_0} r^{\gamma_1} \quad (B.13)$$

where

$$\begin{aligned}
 C_7 &= \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{\Gamma(2\gamma_1+1)4N_2(N_2-1)}} (N_2-2) \left(\frac{2}{N_2 a_0}\right)^{\gamma_1-1} \\
 &= \left(\frac{1}{a_0}\right)^{\gamma_1+3/2} \left(\frac{2}{N_2}\right)^{\gamma_1+3/2} (N_2-2) \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{\Gamma(3)4N_2(N_2-1)}} \\
 &= -\left(\frac{1}{a_0}\right)^{\gamma_1+3/2} \times 8.209 \times 10^{-6}
 \end{aligned}$$

or

$$C_7 = -8.209 \times 10^{-6} (a_0)^{-\gamma_1-3/2} \quad (\text{B.14})$$

$$\begin{aligned}
 C_8 &= \left(\frac{1}{a_0}\right)^{3/2+\gamma_1} \left(\frac{2}{N_2}\right)^{3/2+\gamma_1} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{\Gamma(3)4N_2(N_2-1)}} \frac{(N_2-1)}{2\gamma_1+1} \\
 &= \left(\frac{1}{a_0}\right)^{\gamma_1+3/2} \times 2.0436 \times 10^{-3}
 \end{aligned}$$

or

$$C_8 = 2.0436 \times 10^{-1} (a_0)^{-\gamma_1-3/2} \quad (\text{B.15})$$

$$f_{2P} = -\sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \left[\frac{(2\gamma_1+1)N_2-(N_2-1)\left(\frac{2r}{N_2 a_0}\right)}{(2\gamma_1+1)(N_2-2)-(N_2-1)\frac{2r}{N_2 a_0}} \right] g_{2P}$$

or

$$\begin{aligned}
 f_{2P} &= -\sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \left[(2\gamma_1+1)N_2-(N_2-1)\frac{2r}{N_2 a_0} \right] \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \\
 &\quad \sqrt{\frac{1+\epsilon_2}{4N_2(N_2-1)}} e^{-r/N_2 a_0} \left(\frac{2}{N_2 a_0}\right)^{\gamma_1-1} \frac{1}{2\gamma_1+1} \\
 &= -C_9 e^{-r/N_2 a_0} r^{\gamma_1-1} + C_{10} e^{-r/N_2 a_0} r^{\gamma_1}
 \end{aligned} \quad (\text{B.16})$$

where

$$C_9 = \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} N_2 \left(\frac{2}{N_2 a_0}\right)^{3/2} \sqrt{\frac{2\gamma_1+1}{\Gamma(2\gamma_1+1)}} \frac{1+\epsilon_2}{4N_2(N_2-1)} \left(\frac{2}{N_2 a_0}\right)^{\gamma_1-1}$$

$$= \left(\frac{1}{a_0}\right)^{\gamma_1+1/2} \left(\frac{2}{N_2}\right)^{\gamma_1+1/2} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{\Gamma(2\gamma_1+1)4N_2(N_2-1)}} \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} N_2$$

or

$$C_9 = \left(\frac{1}{a_0}\right)^{\gamma_1+1/2} \times 2.2422 \times 10^{-3} \quad (B.17)$$

and

$$C_{10} = \left(\frac{1}{a_0}\right)^{\gamma_1+3/2} \left(\frac{2}{N_2}\right)^{\gamma_1+3/2} \sqrt{\frac{(2\gamma_1+1)(1+\epsilon_2)}{\Gamma(2\gamma_1+1)4N_2(N_2-1)}} \sqrt{\frac{1-\epsilon_2}{1+\epsilon_2}} \frac{(N_2-1)}{(2\gamma_1+1)}$$

$$= \left(\frac{1}{a_0}\right)^{\gamma_1+3/2} \times 3.7393 \times 10^{-4}$$

or

$$C_{10} = 3.7393 \times 10^{-4} (a_0)^{-\gamma_1-3/2} \quad (B.18)$$

APPENDIX C

DERIVATION OF THE EXPRESSIONS FOR THE TRANSVERSE MULTIPOLE OPERATORS WHEN $q \neq 0$

A. ELECTRIC MULTIPOLE OPERATOR

The transverse electric multipole operator is defined as:

$$T_{JM}^{el} = \frac{1}{q} \int d\vec{x} \nabla \times [j_J(qx) \vec{Y}_{JJ1}^M] \cdot \vec{J}(\vec{x}) \quad (C.1)$$

Now, the total current $\vec{J}(\vec{x})$ is a sum of the convection current part $\vec{J}_e(\vec{x})$ and a part $\nabla \times \vec{\mu}_e$ coming from the intrinsic magnetic moment of the electron.

$$\vec{J}(\vec{x}) = \vec{J}_e(\vec{x}) + \nabla \times \vec{\mu}_e(\vec{x}) \quad (C.2)$$

Inserting Eqn. (C.2) in (C.1), we have:

$$\begin{aligned} T_{JM}^{el} &= \frac{1}{q} \int d\vec{x} \{ \nabla \times [j_J(qx) \vec{Y}_{JJ1}^M] \cdot (\vec{J}_e + \nabla \times \vec{\mu}_e) \} \\ &= \frac{1}{q} \int d\vec{x} \{ \nabla \times (j_J(qx) \vec{Y}_{JJ1}^M) \cdot \vec{J}_e + \nabla \times (j_J(qx) \vec{Y}_{JJ1}^M) \cdot \nabla \times \vec{\mu}_e \} \\ &= \frac{1}{q} \int d\vec{x} \{ \nabla \times (j_J(qx) \vec{Y}_{JJ1}^M) \cdot \vec{J}_e + q^2 j_J(qx) \vec{Y}_{JJ1}^M \cdot \vec{\mu}_e \} \end{aligned}$$

since $(\nabla^2 + q^2) j_J(qx) \vec{Y}_{JJ1}^M = 0$

Hence the transition matrix element between the initial state i and the final state f of the electric multipole operator is:

$$\langle f | T_{JM}^{el} | i \rangle = \frac{1}{q} \int d\bar{x} \{ \nabla x (j_J(qx) \bar{Y}_{JJ1}^M) \cdot \bar{J}_{e_{fi}} + q^2 j_J(qx) \bar{Y}_{JJ1}^M \cdot \bar{\mu}_{e_{fi}} \} \quad (C.3)$$

$$\text{Let us put } \bar{Y}_{JJ1}^M = \frac{1}{\sqrt{J(J+1)}} \bar{L} Y_{JM} \quad (C.4)$$

in Eqn. (C.3). Hence, we obtain:

$$\begin{aligned} \langle f | T_{JM}^{el} | i \rangle &= \frac{1}{q \sqrt{J(J+1)}} \int d\bar{x} \{ \nabla x \bar{L} j_J(qx) Y_{JM} \cdot \bar{J}_{e_{fi}} + q^2 \bar{L} j_J(qx) Y_{JM} \cdot \bar{\mu}_{e_{fi}} \} \\ &= \frac{1}{q \sqrt{J(J+1)}} \int d\bar{x} \{ \bar{L} j_J(qx) Y_{JM} (\nabla x \bar{J}_{e_{fi}}) + q^2 \bar{L} j_J(qx) Y_{JM} \cdot \bar{\mu}_{e_{fi}} \} \end{aligned} \quad (C.5)$$

$$\text{since } \nabla \cdot (\bar{L} j_J(qx) Y_{JM} x \bar{J}_{e_{fi}}) = \bar{L} j_J(qx) Y_{JM} \cdot (\nabla x \bar{J}_{e_{fi}}) - (\nabla x \bar{L} j_J(qx) Y_{JM}) \cdot \bar{J}_{e_{fi}},$$

or

$$\begin{aligned} \nabla x \bar{L} j_J(qx) Y_{JM} \cdot \bar{J}_{e_{fi}} &= \bar{L} j_J(qx) Y_{JM} \cdot (\nabla x \bar{J}_{e_{fi}}) - \nabla \cdot (\bar{L} j_J(qx) Y_{JM}) x \bar{J}_{e_{fi}} \\ &= \bar{L} j_J(qx) Y_{JM} \cdot (\nabla x \bar{J}_{e_{fi}}) - \frac{1}{i} \nabla \cdot (\bar{r} x \nabla) j_J(qx) Y_{JM} x \bar{J}_{e_{fi}} \\ &= \bar{L} j_J(qx) Y_{JM} \cdot (\nabla x \bar{J}_{e_{fi}}) - \frac{1}{i} \nabla \cdot (\bar{J}_{e_{fi}} x (\nabla x \bar{r}) j_J(qx) Y_{JM}) \\ &= \bar{L} j_J(qx) Y_{JM} \cdot (\nabla x \bar{J}_{e_{fi}}) \end{aligned}$$

$$\text{since } \nabla x \bar{r} = 0.$$

$\bar{L}$ being a Hermitian operator can be partially integrated over to the transition matrix element with an overall negative sign. Here, we assume

that the transition matrix element of the current density operator are confined in the space of the orbital electron and hence the term at infinity can be neglected.

From Eqn. (C.5), we can write:

$$\begin{aligned} \langle f | T_{JM}^{el} | i \rangle &= \frac{1}{q \sqrt{J(J+1)}} \int d\vec{x} \{ -j_J(qx) Y_{JM} \bar{L} \cdot (\nabla x \vec{J}_{e_{fi}}) - q^2 j_J(qx) Y_{JM} \bar{L} \cdot \vec{\mu}_{e_{fi}} \} \\ &= \frac{q^J}{(2J+1)!! q \sqrt{J(J+1)}} \int d\vec{x} x^J Y_{JM} \{ -\bar{L} \cdot (\nabla x \vec{J}_{e_{fi}}) - q^2 \bar{L} \cdot \vec{\mu}_{e_{fi}} \} \quad (C.6) \end{aligned}$$

since

$$j_J(qx) \xrightarrow{q \rightarrow 0} \frac{q^J x^J}{(2J+1)!!}$$

Now,

$$\bar{L} \cdot (\nabla x \vec{J}_{e_{fi}}) = \frac{1}{i} (\vec{r} \times \nabla) (\nabla x \vec{J}_{e_{fi}})$$

since

$$\bar{L} = \frac{1}{i} (\vec{r} \times \nabla)$$

$$\begin{aligned} \bar{L} \cdot (\nabla x \vec{J}_{e_{fi}}) &= \frac{1}{i} (\nabla x (\nabla x \vec{J}_{e_{fi}})) \cdot \vec{r} \\ &= -\frac{1}{i} \nabla \cdot (\vec{r} x (\nabla x \vec{J}_{e_{fi}})) \quad (C.7) \end{aligned}$$

and

$$\bar{L} \cdot \vec{\mu}_{e_{fi}} = \frac{1}{i} (\vec{r} \times \nabla) \cdot \vec{\mu}_{e_{fi}} = -\frac{1}{i} \nabla \cdot (\vec{r} x \vec{\mu}_{e_{fi}}) \quad (C.8)$$

Substituting Eqn. (C.7) and (C.8) in (C.6), we obtain:

$$\begin{aligned}
\langle f | T_{JM}^{el} | i \rangle &= \frac{q^{J-1}}{i(2J+1)!! \sqrt{J(J+1)}} \int d\bar{x} x^J Y_{JM} \{ \nabla \cdot (\bar{r} x (\nabla x \bar{J}_{e_{fi}})) + q^2 \nabla \cdot (\bar{r} x \bar{\mu}_{e_{fi}}) \} \\
&= \frac{q^{J-1}}{i(2J+1)!!} \frac{(J+1)}{\sqrt{J(J+1)}} \int d\bar{x} x^J Y_{JM} \{ (\nabla \cdot \bar{J}_{e_{fi}}) + \frac{q^2}{(J+1)} \nabla \cdot (\bar{r} x \bar{\mu}_{e_{fi}}) \}
\end{aligned}$$

where we have used the vector identity

$$\int x^J Y_{JM} \nabla \cdot (\bar{r} x (\nabla x \bar{A})) d\bar{x} = (J+1) \int (x^J Y_{JM} (\nabla \cdot \bar{A})) d\bar{x} \quad (C.9)$$

where $\bar{A}$ is any vector.

$$\langle f | T_{JM}^{el} | i \rangle = \frac{q^{J-1}}{i(2J+1)!!} \sqrt{\frac{J+1}{J}} \int d\bar{x} x^J Y_{JM} \{ \nabla \cdot \bar{J}_{e_{fi}} + \frac{q^2}{J+1} \nabla \cdot (\bar{r} x \bar{\mu}_{e_{fi}}) \} \quad (C.10)$$

B. MAGNETIC MULTIPOLE OPERATOR T_{JM}^{mag}

The transverse magnetic multipole operator T_{JM}^{mag} is defined as:

$$T_{JM}^{mag} = \int d\bar{x} j_J(qx) \bar{Y}_{JJ1}^M \cdot \bar{J}(\bar{x}) \quad (C.11)$$

Substituting (C.2) in (C.11), we get:

$$\begin{aligned}
T_{JM}^{mag} &= \int d\bar{x} j_J(qx) \bar{Y}_{JJ1}^M \cdot (\bar{J}_e + \nabla x \bar{\mu}_e) \\
&= \int d\bar{x} \{ j_J(qx) \bar{Y}_{JJ1}^M \cdot \bar{J}_e + j_J(qx) \bar{Y}_{JJ1}^M \cdot (\nabla x \bar{\mu}_e) \} \\
&= \int d\bar{x} \{ j_J(qx) \bar{Y}_{JJ1}^M \cdot \bar{J}_e + (\nabla x j_J(qx) \bar{Y}_{JJ1}^M) \cdot \bar{\mu}_e \}
\end{aligned}$$

Hence the transition matrix element of the magnetic multipole operator between initial and final states is:

$$\langle f | T_{JM}^{\text{mag}} | i \rangle = \int d\vec{x} \{ j_J(qx) \bar{Y}_{JJ1}^M \cdot \vec{J}_{e_{fi}} + (\nabla x j_J(qx) \bar{Y}_{JJ1}^M) \cdot \vec{\mu}_{e_{fi}} \} \quad (C.12)$$

Using Eqn. (C.4) in (C.12), we have:

$$\langle f | T_{JM}^{\text{mag}} | i \rangle = \frac{1}{\sqrt{J(J+1)}} \int d\vec{x} \{ \bar{L} j_J(qx) Y_{JM} \cdot \vec{J}_{e_{fi}} + (\nabla x \bar{L} j_J(qx) Y_{JM}) \cdot \vec{\mu}_{e_{fi}} \}$$

or

$$\begin{aligned} \langle f | T_{JM}^{\text{mag}} | i \rangle &= \frac{1}{\sqrt{J(J+1)}} \int d\vec{x} \{ \bar{L} j_J(qx) Y_{JM} \cdot \vec{J}_{e_{fi}} + \bar{L} j_J(qx) Y_{JM} (\nabla x \vec{\mu}_{e_{fi}}) \} \\ &= \frac{1}{\sqrt{J(J+1)}} \int d\vec{x} \{ -j_J(qx) Y_{JM} \bar{L} \cdot \vec{J}_{e_{fi}} - j_J(qx) Y_{JM} \bar{L} \cdot (\nabla x \vec{\mu}_{e_{fi}}) \} \end{aligned} \quad (C.13)$$

Here, $\bar{L}$ has been partially integrated and assuming that the transition matrix element of the current density operator is confined to the space of the electron in hydrogen atom, the term at infinity is taken to be zero. Putting

$$j_J(qx) \xrightarrow{q \rightarrow 0} \frac{q^J x^J}{(2J+1)!!}$$

in Eqn. (C.13), we have:

$$\langle f | T_{JM}^{\text{mag}} | i \rangle = \frac{q^J}{(2J+1)!!} \frac{1}{\sqrt{J(J+1)}} \int d\vec{x} x^J Y_{JM} \{ -\bar{L} \cdot \vec{J}_{e_{fi}} - \bar{L} \cdot (\nabla x \vec{\mu}_{e_{fi}}) \} \quad (C.14)$$

Now,

$$\bar{L} \cdot \vec{J}_{e_{fi}} = \frac{1}{i} (\vec{r} \times \nabla) \cdot \vec{J}_{e_{fi}} = -\frac{1}{i} \nabla \cdot (\vec{r} \times \vec{J}_{e_{fi}}) \quad (C.15)$$

$$\begin{aligned}
\mathbf{L} \cdot (\nabla \bar{\mu}_{e_{fi}}) &= \frac{1}{i} (\bar{\mathbf{r}} \times \nabla) \cdot (\nabla \bar{\mu}_{e_{fi}}) = \frac{1}{i} (\nabla \times (\nabla \bar{\mu}_{e_{fi}})) \cdot \bar{\mathbf{r}} \\
&= -\frac{1}{i} \nabla \cdot (\bar{\mathbf{r}} \times (\nabla \bar{\mu}_{e_{fi}}))
\end{aligned} \tag{C.16}$$

Substituting Eqn. (C.15) and (C.16) in (C.14), we have:

$$\begin{aligned}
\langle f | T_{JM}^{\text{mag}} | i \rangle &= \frac{1}{i} \frac{q^J}{(2J+1)!!} \frac{1}{\sqrt{J(J+1)}} \int d\bar{x} x^J Y_{JM} \{ \nabla \cdot (\bar{\mathbf{r}} \times \bar{\mathbf{J}}_{e_{fi}}) + \nabla \cdot (\bar{\mathbf{r}} \times (\nabla \bar{\mu}_{e_{fi}})) \} \\
&= \frac{1}{i} \frac{q^J}{(2J+1)!!} \frac{1}{\sqrt{J(J+1)}} \int d\bar{x} x^J Y_{JM} \{ \nabla \cdot (\bar{\mathbf{r}} \times \bar{\mathbf{J}}_{e_{fi}}) + (J+1) \nabla \cdot \bar{\mu}_{e_{fi}} \}
\end{aligned}$$

using the vector identity (C.9).

$$\langle f | T_{JM}^{\text{mag}} | i \rangle = \frac{q^J}{i(2J+1)!!} \sqrt{\frac{J+1}{J}} \int d\bar{x} x^J Y_{JM} \cdot \{ \nabla \cdot \bar{\mu}_{e_{fi}} + \frac{1}{(J+1)} \nabla \cdot (\bar{\mathbf{r}} \times \bar{\mathbf{J}}_{e_{fi}}) \} \tag{C.17}$$

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