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**EVALUATION OF DISTRIBUTIONS OF WIENER
FUNCTIONALS**

by

Joseph Denes

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

July, 1991

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C. W. L. Garner

Thesis Supervisor

External Examiner

Carleton University
July 1991

Abstract

The effectiveness of an iterative method, referred to as the Q - method, is illustrated by using it to evaluate the Laplace transforms of Brownian motion occupation times for various domains. Through inversion of these Laplace transforms the distributions of the occupation times themselves and related functionals are evaluated.

In Chapter 2 the classical results of Lévy and Kac are rederived and the latter extended to tied down Brownian motion.

In Chapter 3 the domain $\{x \geq a, a = \text{constant}\}$ is considered. Occupation times for both unrestricted and tied down Brownian motion are evaluated. A number of formulas for the distribution of maxima of Brownian motion are derived and a new proof of the Kolmogorov - Petrowsky upper class theorem, not requiring monotonicity conditions is given.

In Chapter 4 the Laplace transform of the occupation time in the domain $x \geq b$ or $-a \geq x$, $a, b \geq 0$ is computed, and this used in Chapter 5 to evaluate the distribution of local time for both tied down and unrestricted Brownian motion. In addition to this the distribution of the oblique local time, i.e. the limiting behaviour of occupation time between two parallel lines that enclose an angle with the time axis is evaluated also.

In Chapter 6 the finite dimensional distributions of local time with respect to the space variable is computed and based upon this a new proof of the Ray - Knight theorem is given.

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Chapter 1

Introduction

This work is concerned with the study of the distribution of functionals of Brownian motion and tied down Brownian motion processes and in particular occupation times and their relation to the Brownian local time.

Let X_t denote the standard Brownian motion process, that is the real valued stochastic process uniquely determined by the following properties:

- 1.) Homogeneity in time i.e. for all constant $t_0 \geq 0$ the processes $X_{t_0+t} - X_{t_0}$ have the same joint distribution functions.
- 2.) The increments of the process for disjoint time intervals are mutually independent i.e. for every $n \geq 1$ and $0 \leq t_1 < t_2 \dots < t_n$ the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are mutually independent.
- 3.) $P(X_0 = 0 \text{ and } X_t \text{ is a continuous function of } t \geq 0) = 1$

For the definition, existence and uniqueness of Brownian motion see [Knight] Chapter 1. By Brownian motion tied down at (t, x) is meant the Brownian motion $\{X_\tau : 0 \leq \tau \leq t\}$ conditioned on $X_t = x$. The special case $x = 0$ and $t = 1$ is called the Brownian bridge process which we shall denote by B_τ . Let $O_0(t)$ be the occupation time defined by $O_0(t) = O_0(X, t) = \lambda(\tau : X_\tau \geq 0, 0 \leq \tau \leq t)$ where λ is the Lebesgue measure. The arcsine law discovered by P. Lévy [Lévy 1] states

that for the Brownian motion process X_t the probability distribution of $O_0(t)$ is

$$P(O_0(t) \leq \xi) = \frac{2}{\pi} \arcsin \sqrt{\frac{\xi}{t}}. \quad (1.1)$$

We seek the following generalisation of this: let $x = f(t)$ be a curve in the (x, t) plane dividing it into two regions, one that is above the curve and the other that is below it. Let $O_f(t)$ be the occupation time $O_f(t) = \lambda(\tau : X_\tau \geq f(\tau), 0 \leq \tau \leq t)$. The probability distribution $P(O_f(t) \leq \xi)$ is a natural analogue of P. Lévy's arcsine law, which corresponds to the case $f(t) = 0$. We can even consider the more general case: the probability distribution of the time spent by the Brownian motion in the domain $\{g(\tau) \leq x \leq f(\tau), \text{ for } 0 \leq \tau \leq t\}$ or outside of it. We shall study these problems not only for the Brownian motion but for the Brownian bridge and the tied down Brownian motion process as well. The "arcsine law" of the Brownian bridge is due to M. Kac [Kac 1] who found that

$$P(\lambda(B_\tau \geq 0) \leq \xi) = \frac{\xi}{t} \quad (1.2)$$

$(0 \leq \tau \leq t)$

that is the "arcsine law" of the Brownian bridge process is the uniform distribution.

The occupation time distributions are related to the Brownian local time, discovered by Lévy [Lévy 1]. Consider the domain $D_b = \{(x, \tau) : 0 \leq \tau \leq t, -b \leq x \leq b\}$ where b is a positive constant and denote the occupation time in this domain by $O_{D_b}(t)$. Lévy has proved that $\frac{O_{D_b}(t)}{2b}$ converges with probability 1 to a random variable $L(t)$, that is equal in distribution to $M_t = \sup_{0 \leq \tau \leq t} X_\tau$. In fact Lévy has shown more, he proved that the joint distribution of $Y_t = M_t - X_t$ and M_t is the same as the joint distribution of $|X_t|$ and $L(t)$. For the proof see [Knight] Chapter 5. Lévy's result was extended by H. F. Trotter. The occupation time defined by $O_A(t) = \lambda(X_\tau \in A, 0 \leq \tau \leq t)$ for A Borel is a random measure, almost surely absolutely continuous with respect to λ . The Radon - Nikodym derivative of $O_A(t)$ with respect to Lebesgue measure, denoted by $L(a, t)$ is called the local time or occupation density at a . Lévy's local time corresponds to $a = 0$, that is $L(t) = L(0, t)$.

Trotter has shown that $L(a, t)$ is continuous with probability 1 in both variables a and t . See [Trotter].

Let $\Gamma = \{X : X \in C[0, t], \lim_{b \downarrow 0} \frac{O_{D_b}(X, t)}{2b} \text{ does not exist} \}$. By Trotter's result Γ is a set of Wiener measure zero. If $X_0 \in C[0, t]$ such that $X_0(0) = 0$ and X_0 has a derivative of bounded variation on $(0, t)$ then let $T\Gamma$ be the translation of Γ by $-X_0$ that is $T\Gamma = \{X : X = \tilde{X} - X_0, \tilde{X} \in \Gamma\}$. It follows from the Cameron - Martin theorem for the transformations of Wiener integrals under translation (see [Cameron - Martin] Theorem 2.) that

$$P(\Gamma) = e^{-\frac{1}{2} \int_0^t X_0'^2 d\tau} E \left(e^{-\int_0^t X_0' dX} : T\Gamma \right).$$

Since the left hand side is zero the measure of $T\Gamma$ is zero as well. This means that for almost all $X \in C[0, t]$ $\lim_{b \rightarrow 0} \frac{O_{D_b}(X - X_0, t)}{2b}$ exists. Let us take for X_0 the special case $X_0(t) = mt$, $m \geq 0$. Now the occupation time of $X - mt$ in D_b is;

$$\lambda(-b \leq X_\tau - m\tau \leq b) = \lambda(-b + m\tau \leq X_\tau \leq b + m\tau).$$

Clearly the last expression is the occupation time of X in the domain $D_{m,b} = \{(x, \tau) : 0 \leq \tau \leq t, -b + m\tau \leq x \leq b + m\tau\}$. Hence the limit $\lim_{b \rightarrow 0} \frac{O_{D_{m,b}}}{2b}$ exists almost surely. We call this limit the oblique local time of the Brownian motion.

The method of investigation used in this thesis is related to that of Kac [Kac 2] and Rosenblatt [Rosenblatt]. Let the function $V(x, t)$ be defined on $R \times [0, \infty)$ by the following expression: $V(x, t) = \begin{cases} 1 & \text{if } (x, t) \in D \\ 0 & \text{if } (x, t) \notin D \end{cases}$ where D is a measurable subset of $R \times [0, \infty)$. Then $O_D(t) = \int_0^t V(X_\tau, \tau) d\tau$ generalises O_f and $O_f = O_D$ if $D = \{(x, \tau) : x \geq f(\tau)\}$. Consider now the conditional expectation $E(e^{-u \int_0^t V(X_\tau, \tau) d\tau} | X_t = x)$ where u is a positive constant. Clearly this can be written in the Laplace transform form:

$$E(e^{-u \int_0^t V(X_\tau, \tau) d\tau} | X_t = x) = \int_0^\infty e^{-u\xi} dP_\xi(O_D(t) \leq \xi | X_t = x)$$

Thus the inversion of this conditional expectation will yield the the distribution of the occupation time $O_D(t)$ for the tied down Brownian process $\{X_\tau \mid X_t = x\}_{0 \leq \tau \leq t}$. For the non tied down Brownian motion we obtain

$$E(e^{-u \int_0^t V(X_\tau, \tau) d\tau}) = \int_{-\infty}^{\infty} dx \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E(e^{-u \int_0^t V(X_\tau, \tau) d\tau} | X_t = x)$$

and again an inverse Laplace transform will yield the distribution of the occupation time. Let us denote by $Q(x, t)$ the function $\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E(e^{-u \int_0^t V(X_\tau, \tau) d\tau} | X_t = x)$. Kac [Kac 2] has proved that in the case when $V(x, t) = V(x)$ is a piecewise continuous non negative function, the Laplace transform of $Q(x, t)$

$$\Psi(x, s) = \int_0^{\infty} e^{-st} Q(x, t) dt$$

is the fundamental solution of the differential equation:

$$\frac{1}{2} \Psi''(x, s) - (s + uV) \Psi(x, s) = 0.$$

Following Kac, Rosenblatt [Rosenblatt] has proved that $Q(x, t)$ satisfies the partial differential equation

$$\frac{\partial Q}{\partial t} = \frac{1}{2} \frac{\partial^2 Q}{\partial x^2} - uVQ.$$

The difficulty in applying these differential equations is that $V(x, t)$ must be a function such that the ensuing differential equation can be reduced to some known equation whose solution is expressible in terms of known functions. Also in the case of Kac's equation one frequently encounters a Laplace transform inversion that is difficult to carry out.

For this reason we have developed an alternative approach. The basic method to be used in this thesis is to directly evaluate the expression

$$Q(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E(e^{-u \int_0^t V(X_\tau, \tau) d\tau} | X_t = x)$$

by expanding the exponential

$$e^{-u \int_0^t V d\tau} = 1 - \frac{u \int_0^t V d\tau}{1!} + \frac{u^2 (\int_0^t V d\tau)^2}{2!} - \frac{u^3 (\int_0^t V d\tau)^3}{3!} \dots,$$

and integrating term by term:

$$\begin{aligned}
 Q(x, t) = & \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - u \int_0^t d\tau \int_{f(\tau)}^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} \\
 & + u^2 \int_0^t d\tau \int_{f(\tau)}^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \int_0^\tau d\rho \int_{f(\rho)}^\infty d\eta \frac{e^{-\frac{(\xi-\eta)^2}{2(\tau-\rho)}}}{\sqrt{2\pi(\tau-\rho)}} \frac{e^{-\frac{\eta^2}{2\rho}}}{\sqrt{2\pi\rho}} - \dots
 \end{aligned} \tag{1.3}$$

In (1.3) let the coefficients of 1, $-u$, u^2 , $-u^3$, $u^4 \dots$ respectively be denoted by $G_0(x, t)$, $G_1(x, t)$, $G_2 \dots$. Then

$$Q(x, t) = G_0(x, t) - uG_1(x, t) + u^2G_2(x, t) - u^3G_3(x, t) \dots \tag{1.4}$$

Let T denote the transform

$$TF = TF(x, t) = \int_0^t d\tau \int_{f(\tau)}^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} F(\xi, \tau). \tag{1.5}$$

Then it follows that $G_0 = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$, $G_1 = TG_0$, $G_2 = TG_1 \dots$ etc .

In the following chapters $Q(x, t)$ will be evaluated for some classes of $D = \{(x, t) : x \geq f(t)\}$ by this iteration and the distribution of O_f will be determined.

In Chapter 2 we shall evaluate the function Q for the case $f(t) = 0$. This is the Paul Lévy's arcsine law case. This will prove to be of fundamental importance for the other Q functions that will be considered, since they will be expressed in terms of this. From Q we shall derive the occupation time distribution for R_+ for the Brownian motion and the tied down Brownian motion.

In Chapter 3 the Q functions will be computed for the case $f(t) = a = \text{constant}$. We shall develop expressions for the probability that the Brownian paths stay in a domain whose boundary is composed of straight lines and as an application of this we shall give a new proof of the Kolmogorov - Petrowsky test for upper class

functions. Then we extend Lévy's arcsine law by computing the distribution of the occupation time of $[a, \infty)$.

In Chapter 4 the Q function for the domain $\{(x, \tau) : -a \leq x \leq b, 0 \leq \tau \leq t\}$ and an expression for the probability that the Brownian motion paths stay in this domain is derived.

The rest of the work is taken up by the study of Brownian local times using the results of Chapter 4. We evaluate the distributions of Brownian local time for unrestricted and tied down Brownian motion and for the oblique local time as well.

In Chapter 6 formulas for the finite dimensional distributions for the local time with respect to space variables are derived and based upon these a new proof of the Ray - Knight theorem is given.

Similar results to the formulas on R_+ occupation times (2.21), (2.22) and to the formula (5.7) for the local time for the tied down Brownian motion have recently been obtained by Karatzas and Shreve. They have derived an expression for joint densities of $\{O_0(t), L(t), X_t\}$ (for the details see their book [Karatzas - Shreve] on page 423). The proofs given in this thesis are however more elementary than theirs.

We shall end with a brief discussion of some directions for future research.

Chapter 2

The occupation time of R_+ for Brownian motion.

2.1 Introduction.

In this chapter we will give a proof of the classical results (1.1) and (1.2) and generalize the latter to the case of the tied down Brownian motion. First we determine the Q function by using our method and then invert the Laplace transform.

2.2 Evaluation of Q .

This is the case $f(t) = 0$. The fundamental formulas (1.3), (1.4) now become:

$$Q = G_0 - uG_1 + u^2G_2 - \dots$$
$$G_0 = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}, \quad G_1 = \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}}, \quad (2.1)$$
$$G_{n+1} = \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} G_n(\xi, \tau)$$

$G_1, G_2, G_3 \dots$ will be called the 1st integral, the 2nd integral, the 3rd integral etc... It will be useful to distinguish between the values of $Q, G_1, G_2 \dots$ for $x \geq 0$ and for $x < 0$. In the first case they will be called the upper function and the upper integrals, in the second case they will be called the lower function and the lower integrals.

It is clear that the upper integrals play the more important role, for the $n+1$ -th integral $G_{n+1} = \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} G_n(\xi, \tau)$ is the transform of the n -th upper integral.

Examining the expression

$$G_{n+1} = \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} G_n(\xi)$$

it can be seen that it is

$$\int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * G_n(\xi, t)$$

where $*$ denotes the convolution of the two functions with respect to t . In general:

$$f * g = f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau.$$

For the sake of brevity some more notation will now be introduced:

$$\varphi(x) = \varphi(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}, \quad \psi(x) = \psi(x, t) = \frac{x e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}}.$$

It is not hard to prove that

$$\varphi(x) = \psi(x) * \frac{1}{\sqrt{2\pi t}} \quad (2.2)$$

or the more general expression:

$$\varphi(x+a) = \psi(x) * \varphi(a) \quad (2.3)$$

provided that $x \geq 0$, and $a \geq 0$

It also holds that

$$\psi(a+b) = \psi(a) * \psi(b) \text{ provided } a \geq 0 \text{ and } b \geq 0 \quad (2.4)$$

Furthermore for all continuous functions $f(t)$

$$\psi(x) * f(t) = \int_0^t \frac{x e^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} f(\tau) d\tau \longrightarrow f(t) \quad (2.5)$$

as $x \rightarrow 0$ ($x > 0$)

These fundamental formulas are known from the theory of the heat equation. The last formula known as Duhamel's formula and expresses the solution of the heat equation $\frac{\partial Q}{\partial t} = \frac{1}{2} \frac{\partial^2 Q}{\partial x^2}$ with initial condition $Q(x, t) |_{t=0} = 0$ and boundary condition $Q(x, t) |_{x=0} = f(t)$ [Widder].

Theorem 2.1 For $x \geq 0$ (upper function)

$$\begin{aligned} Q(x, t) &= e^{-ut} \int_0^t d\tau \frac{x e^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} \frac{e^{u\tau} - 1}{u\sqrt{2\pi\tau^3}} \\ &= \frac{e^{-ut}}{2\pi} \int_0^t d\tau \frac{x e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{e^{u\tau} - 1}{u\tau^{\frac{3}{2}}}. \end{aligned} \quad (2.6)$$

For $x \leq 0$ (lower function)

$$\begin{aligned} Q(x, t) &= \int_0^t d\tau \frac{x e^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} \frac{1 - e^{-u\tau}}{u\sqrt{2\pi\tau^3}} \\ &= \frac{|x|}{2\pi} \int_0^t d\tau \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{1 - e^{-u\tau}}{u\tau^{\frac{3}{2}}}. \end{aligned} \quad (2.7)$$

The proof will be carried out in several steps.

Lemma 2.1 The first integral G_1 for $x \geq 0$ can be expressed as:

$$G_1 = t \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{1}{2} \psi(x) * \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} = \frac{t e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{1}{2} \int_0^t \frac{x e^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} \frac{\tau^{\frac{1}{2}}}{\sqrt{2\pi}} d\tau.$$

Proof:

$$\begin{aligned}
G_1 &= \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} = \\
&= \int_0^t d\tau \int_{-\infty}^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} - \int_0^t d\tau \int_{-\infty}^0 d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} = \\
&\quad \int_0^t d\tau \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x+\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} = \\
&= t\varphi(x) - \int_0^\infty d\xi \varphi(x+\xi) * \varphi(\xi) = t\varphi(x) - \psi(x) * \int_0^\infty d\xi \varphi(\xi) * \varphi(\xi) .
\end{aligned}$$

It remains now to evaluate the integral

$$\begin{aligned}
\int_0^\infty d\xi \varphi(\xi) * \varphi(\xi) &= \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{\xi^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} = \\
&= \int_0^t \frac{d\tau}{2\pi(t-\tau)^{\frac{1}{2}}\tau^{\frac{1}{2}}} \int_0^\infty d\xi e^{-\frac{\xi^2}{2\tau(t-\tau)}} = \\
&= \int_0^t \frac{d\tau}{2\pi\tau^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}} \sqrt{\frac{2\tau(t-\tau)}{t}} \int_0^\infty e^{-\xi^2} d\xi = \\
&= \int_0^t \frac{d\tau}{2\pi\tau^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}} \sqrt{\frac{2\tau(t-\tau)}{t}} \frac{\sqrt{\pi}}{2} = \\
&= \frac{1}{2} \frac{1}{\sqrt{2\pi t}} \int_0^t d\tau = \frac{1}{2} \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} .
\end{aligned}$$

Hence the first integral:

$$G_1 = t \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{1}{2} \psi(x) * \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} = \frac{te^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{1}{2} \int_0^t \frac{xe^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} \frac{\tau^{\frac{1}{2}}}{\sqrt{2\pi}} d\tau . \blacksquare$$

Let us introduce the following notation:

$$F_0(x) = F_0(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \quad F_1(x, t) = \psi(x) * \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} \quad F_2 = \psi(x) * \frac{t^{\frac{3}{2}}}{\sqrt{2\pi}}$$

and in general

$$F_n = \psi(x) * \frac{t^{n-\frac{1}{2}}}{\sqrt{2\pi}}.$$

Clearly $F_0 = \psi(x) * \frac{t^{-\frac{1}{2}}}{\sqrt{2\pi}} = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$ and $G_1 = tF_0 - \frac{1}{2}F_1$

Before proceeding to the computation of the second, third ... upper integrals we shall prove:

Lemma 2.2 *Let T be as defined by (1.3) with $f \equiv 0$.*

$$Tt^k F_n = \frac{t^{k+1}}{k+1} F_n - F_{n+k+1} \frac{1}{2(k+1)} \left(1 + \frac{n(n+1) \dots (n+k)}{(n+\frac{1}{2}) \dots (n+k+\frac{1}{2})} \right) \quad (2.8)$$

for $n, k = 0, 1, 2, \dots$

Proof:

First take the $n = 0$ case. The lefthand side of (2.8) becomes

$$\begin{aligned} & \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \tau^k \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} = \\ & \int_0^t d\tau \tau^k \int_{-\infty}^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} - \int_0^t d\tau \tau^k \int_{-\infty}^0 d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} = \\ & \int_0^t d\tau \tau^k \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \int_0^\infty d\xi \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} * t^k \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} = \\ & \frac{t^{k+1}}{k+1} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \psi(x) * \int_0^\infty d\xi \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} * t^k \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}}. \end{aligned}$$

It remains now to evaluate the last integral on the right hand side.

$$\begin{aligned} \int_0^\infty d\xi \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} * t^k \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} &= \int_0^t \frac{d\tau \tau^k}{2\pi \sqrt{(t-\tau)\tau}} \int_0^\infty d\xi e^{-\frac{\xi^2}{2(t-\tau)}} e^{-\frac{\xi^2}{2\tau}} \\ &= \int_0^t \frac{d\tau \tau^k}{2\pi \sqrt{(t-\tau)\tau}} \sqrt{\frac{2(t-\tau)\tau}{t}} \frac{\sqrt{\pi}}{2} = \frac{1}{2} \frac{1}{\sqrt{2\pi t}} \int_0^t \tau^k d\tau = \end{aligned}$$

$$\frac{1}{2(k+1)} \frac{t^{k+1-\frac{1}{2}}}{\sqrt{2\pi}}$$

substituting this into the equation above gives:

$$T t^k F_0 = \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \tau^k F_0(\xi, \tau) = \frac{t^{k+1}}{k+1} F_0 - \frac{1}{2(k+1)} F_{k+1} \quad (2.9)$$

and this completes the proof of the case $n = 0$.

Next the transform of $t^k F_n$ will be computed for $n > 0$.

Recall $F_n = \psi(x) * \frac{t^{n-\frac{1}{2}}}{\sqrt{2\pi}}$. First F_n will be transformed into a more manageable form. It is clear that for any non negative integer $m = 0, 1, 2, \dots$

$$\frac{1}{\sqrt{t}} * t^m = \frac{m!}{\frac{1}{2} \frac{3}{2} \dots m + \frac{1}{2}} t^{m+1-\frac{1}{2}}.$$

The constant factor is $B(m+1, \frac{1}{2})$ where $B(p, q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$ the beta function or Eulerian integral of the first kind.

Hence $t^{n-\frac{1}{2}} = \frac{1}{B(n, \frac{1}{2})} \frac{1}{\sqrt{t}} * t^{n-1}$ and substituting this into the expression for F_n gives:

$$\begin{aligned} F_n &= \frac{1}{B(n, \frac{1}{2})} \psi(x) * \frac{1}{\sqrt{2\pi t}} * t^{n-1} = \frac{1}{B(n, \frac{1}{2})} \varphi(x) * t^{n-1} \\ &= \frac{1}{B(n, \frac{1}{2})} \int_0^t \frac{e^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \tau^{n-1} d\tau. \end{aligned}$$

Now we shall compute the transform of $t^k F_n$

$$\begin{aligned} T t^k F_n &= \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \tau^k F_n(\xi) = \\ &= \int_0^t d\tau \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \tau^k \int_0^\tau d\rho \frac{e^{-\frac{\xi^2}{2(\tau-\rho)}}}{\sqrt{2\pi(\tau-\rho)}} \frac{\rho^{n-1}}{B(n, \frac{1}{2})}. \end{aligned}$$

Integrating first with respect to ξ

$$\int_0^\infty d\xi () = \int_{-\infty}^\infty d\xi () - \int_{-\infty}^0 d\xi ()$$

and changing the sign of ξ in the second integral gives:

$$\int_0^t d\tau \tau^k \int_0^\tau d\rho \frac{e^{-\frac{\tau^2}{2(t-\rho)}}}{\sqrt{2\pi(t-\rho)}} \frac{\rho^{n-1}}{B(n, \frac{1}{2})} - \int_0^t d\tau \frac{e^{-\frac{(x+\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \tau^k F_n(\xi) .$$

Consider the first term. By changing the order of integration this can be written as:

$$\begin{aligned} & \int_0^t d\rho \frac{e^{-\frac{\tau^2}{2(t-\rho)}}}{\sqrt{2\pi(t-\rho)}} \frac{\rho^{n-1}}{B(n, \frac{1}{2})} \int_\rho^t \tau^k d\tau = \\ & \int_0^t d\rho \frac{e^{-\frac{\tau^2}{2(t-\rho)}}}{\sqrt{2\pi(t-\rho)}} \frac{\rho^{n-1}}{B(n, \frac{1}{2})} \frac{t^{k+1} - \rho^{k+1}}{k+1} = \\ & \frac{t^{k+1}}{k+1} \int_0^t d\rho \frac{e^{-\frac{\tau^2}{2(t-\rho)}}}{\sqrt{2\pi(t-\rho)}} \frac{\rho^{n-1}}{B(n, \frac{1}{2})} - \frac{1}{(k+1) B(n, \frac{1}{2})} \int_0^t d\rho \frac{e^{-\frac{\tau^2}{2(t-\rho)}}}{\sqrt{2\pi(t-\rho)}} \rho^{n+k} = \\ & \frac{t^{k+1}}{k+1} F_n - F_{n+k+1} \frac{B(n+k+1, \frac{1}{2})}{(k+1) B(n, \frac{1}{2})} \end{aligned}$$

and this is the simplified form of the first term. Note that the second term can be written as:

$$\int_0^\infty \varphi(x+\xi) * t^k F_n(\xi) d\xi = \psi(x) * \int_0^\infty \varphi(\xi) * t^k F_n(\xi) d\xi .$$

Now the integral can be written as:

$$\begin{aligned} & \int_0^t d\tau \int_0^\tau d\rho \frac{\tau^k \rho^{n-1}}{B(n, \frac{1}{2})} \int_0^\infty d\xi \frac{e^{-\frac{\tau^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2(\tau-\rho)}}}{\sqrt{2\pi(\tau-\rho)}} = \\ & \int_0^t d\tau \int_0^\tau d\rho \frac{\tau^k \rho^{n-1}}{B(n, \frac{1}{2})} \frac{1}{2} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{(t-\rho)}} = \\ & \int_0^t \frac{d\rho \rho^{n-1} (t-\rho)^{-\frac{1}{2}}}{2 \sqrt{2\pi} B(n, \frac{1}{2})} \int_\rho^t \tau^k d\tau = \\ & \int_0^t \frac{d\rho \rho^{n-1}}{2 \sqrt{2\pi} B(n, \frac{1}{2})} \frac{t^{k+1} - \rho^{k+1}}{k+1} = \\ & \frac{1}{2 \sqrt{2\pi} B(n, \frac{1}{2}) (k+1)} \left\{ t^{k+1} \int_0^t \frac{d\rho \rho^{n-1}}{(t-\rho)^{\frac{1}{2}}} - \int_0^t \frac{d\rho \rho^{n+k}}{(t-\rho)^{\frac{1}{2}}} \right\} . \end{aligned}$$

The last two integrals are beta function integrals: $\int_0^t \frac{d\rho \rho^{n-1}}{(t-\rho)^{\frac{1}{2}}} = t^{n-\frac{1}{2}} B(n, \frac{1}{2})$

$\int_0^t \frac{d\rho \rho^{n+k}}{(t-\rho)^{\frac{1}{2}}} = t^{n+k+1-\frac{1}{2}} B(n+k+1, \frac{1}{2})$. Substituting these gives for the second term:

$$\psi(x) * \frac{t^{n+k+1-\frac{1}{2}}}{\sqrt{2\pi}} \frac{1}{2(k+1)} \{1 - B(n+k+1, \frac{1}{2})\}$$

and adding up the first and second terms finally gives:

$$Tt^k F_n = \frac{t^{k+1}}{k+1} F_n - F_{n+k+1} \frac{1}{2(k+1)} (1 + \frac{B(n+k+1, \frac{1}{2})}{B(n, \frac{1}{2})}) =$$

$$\frac{t^{k+1}}{k+1} F_n - F_{n+k+1} \frac{1}{2(k+1)} (1 + \frac{n(n+1) \dots (n+k)}{(n+\frac{1}{2}) \dots (n+k+\frac{1}{2})})$$

Putting $n = 0$ gives the formula for $t^k F_0$ so it is correct for all $n, k = 0, 1, 2, \dots$

■

The evaluation of the second and third integrals.

Recall that:

$$G_1 = TF_0 = tF_0 - \frac{1}{2}F_1.$$

The second integral for $x \geq 0$:

$$G_2 = TG_1 = T(tF_0) - \frac{1}{2}TF_1 =$$

$$\frac{t^2}{2}F_0 - F_2 \frac{1}{2 \cdot 2} - \frac{1}{2}(tF_1 - F_2 \frac{1}{2}(1 + \frac{1}{1+\frac{1}{2}})).$$

Hence:

$$G_2 = \frac{t^2}{2}F_0 - \frac{tF_1}{2} + \frac{F_2}{3!}.$$

Now

$$TG_2 = T(\frac{t^2 F_0}{2} - \frac{tF_1}{2}) =$$

$$(\frac{t^3}{2 \cdot 3}F_0 - F_3 \frac{1}{2 \cdot 2 \cdot 3}) - (\frac{t^2 F_1}{2 \cdot 2} - F_3 \frac{1}{2 \cdot 2} (1 + \frac{1 \cdot 2}{\frac{3}{2} \cdot \frac{5}{2}})) + \frac{1}{6}(tF_2 - F_3 \frac{1}{2}(1 + \frac{2}{2+\frac{1}{2}}))$$

that is

$$G_3 = \frac{t^3}{3!1!}F_0 - \frac{t^2}{2!2!}F_1 + \frac{t}{1!3!}F_2 - \frac{1}{4!}F_3.$$

The expressions for the first, second and third integrals suggest:

Lemma 2.3

$$G_n = \frac{t^n}{n!1!}F_0 - \frac{t^{n-1}}{(n-1)!2!}F_1 + \frac{t^{n-2}}{(n-2)!3!}F_2 - \dots + (-1)^n \frac{F_n}{n!}. \quad (2.10)$$

Proof: We prove this by induction as follows:

$$\begin{aligned} G_n &= \sum_{k=0}^n (-1)^k \frac{F_k}{(k+1)!} \frac{t^{n-k}}{(n-k)!} \\ TG_n &= \sum_{k=0}^n \frac{(-1)^k}{(k+1)!(n-k)!} T(t^{n-k} F_k) = \\ &= \sum_{k=0}^n (-1)^k \frac{t^{n+1-k} F_k}{(n+1-k)!(k+1)!} - \frac{F_{n+1}}{2} \sum_{k=0}^n \frac{(-1)^k}{(k+1)!(n+1-k)!} \left(1 + \frac{k(k+1)\dots n}{(k+\frac{1}{2})\dots(n+\frac{1}{2})}\right) \end{aligned} \quad (2.11)$$

What remains now is to simplify the second term:

$$\begin{aligned} &\frac{F_{n+1}}{2} \left\{ \sum_{k=0}^n \frac{(-1)^{k+1}}{(k+1)!(n+1-k)!} + \sum_{k=0}^n \frac{(-1)^{k+1} k(k+1)\dots n}{(k+1)!(n+1-k)!(k+\frac{1}{2})\dots(n+\frac{1}{2})} \right\} = \\ &= \frac{F_{n+1}}{2} \{A + B\} \end{aligned}$$

$$\begin{aligned} A &= \sum_{k=0}^n \frac{(-1)^{k+1}}{(k+1)!(n+1-k)!} = \sum_{k=1}^{n+1} \frac{(-1)^k}{k!(n+2-k)!} = \sum_{k=1}^{n+1} \frac{(-1)^k}{(n+2)!} \binom{n+2}{k} \\ &= \frac{1}{(n+2)!} \left(0 - \binom{n+2}{0} - (-1)^{n+2} \binom{n+2}{n+2} \right) = \frac{-1 - (-1)^n}{(n+2)!} \end{aligned}$$

The second summation is somewhat more involved.

$$B = \sum_{k=0}^n \frac{(-1)^{k+1} k(k+1)\dots n}{(k+1)!(n+1-k)!(k+\frac{1}{2})\dots(n+\frac{1}{2})} =$$

$$\begin{aligned} &(-1)^{n+1} \left(\frac{n}{n+\frac{1}{2}} \frac{1}{(n+1)!1!} - \frac{n(n-1)}{(n+\frac{1}{2})(n+\frac{1}{2}-1)} \frac{1}{n!2!} + \right. \\ &\quad \left. \frac{n(n-1)(n-2)}{(n+\frac{1}{2})(n+\frac{1}{2}-1)(n+\frac{1}{2}-2)} \frac{1}{(n-1)!3!} - \dots \right) = \end{aligned}$$

$$\begin{aligned}
& \frac{(-1)^{n+1}}{(n+2)!} \left(\frac{n \cdot (n+2)}{(n+\frac{1}{2})!} - \frac{n(n-1) \cdot (n+2)(n-1)}{(n+\frac{1}{2})(n+\frac{1}{2}-1) \cdot 2!} \dots \right) = \\
& \frac{(-1)^n}{(n+2)!} \left(\frac{-n \cdot (-n-2)}{(-n-\frac{1}{2}) \cdot 1!} + \frac{(-n)(-n+1) \cdot (-n-2)(-n-2+1)}{(-n-\frac{1}{2})(-n-\frac{1}{2}+1) \cdot 2!} \dots \right) = \\
& \frac{(-1)^n}{(n+2)!} \{ {}_2F_1(-n, -n-2; -n-\frac{1}{2}; 1) - 1 \}
\end{aligned}$$

where ${}_2F_1(a, b; c; x)$ is the Gaussian hypergeometric series:

$$1 + \frac{a \cdot b \cdot x}{c \cdot 1!} + \frac{a(a+1) \cdot b(b+1) \cdot x^2}{c(c+1) \cdot 2!} \dots$$

Now it is known from the theory of the hypergeometric functions see [Erdely] that ${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$ whenever ${}_2F_1(a, b; c; 1)$ is convergent. This is clearly the case for ${}_2F_1(-n, -n-2; -n-\frac{1}{2}; 1)$ since it has only finitely many nonvanishing terms.

Now

$$\begin{aligned}
& \frac{(-1)^n}{(n+2)!} \left(\frac{\Gamma(-n-\frac{1}{2})\Gamma(n+\frac{3}{2})}{\Gamma(-\frac{1}{2})\Gamma(\frac{3}{2})} - 1 \right) \\
& = \frac{(-1)^n}{(n+2)!} \left(\frac{\sin(-\frac{\pi}{2})}{\sin \pi(-n-\frac{1}{2})} - 1 \right) = \frac{(-1)^n}{(n+2)!} \left(\frac{1}{\cos \pi n} - 1 \right) \\
& = \frac{(-1)^n}{(n+2)!} ((-1)^n - 1) = \frac{1 + (-1)^{n+1}}{(n+2)!} .
\end{aligned}$$

Hence:

$$A + B = \frac{-1 - (-1)^n}{(n+2)!} + \frac{1 + (-1)^{n+1}}{(n+2)!} = \frac{2(-1)^{n+1}}{(n+2)!} .$$

Substituting this into equation (2.11) gives

$$TG_n = \sum_{k=0}^n \frac{(-1)^k t^{n+1-k} F_k}{(n+1-k)!(k+1)!} + \frac{F_{n+1}(-1)^{n+1}}{(n+2)!} = \sum_{k=0}^{n+1} \frac{(-1)^k t^{n+1-k} F_k}{(n+1-k)!(k+1)!}$$

and this completes the proof. ■

Proof of the Theorem for $x \geq 0$.

We shall prove that the upper function has the form:

$$\begin{aligned}
 Q(x, t) &= e^{-ut} \cdot \left(\psi(x) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right) \\
 &= e^{-ut} \int_0^t d\tau \frac{x e^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} \frac{e^{u\tau} - 1}{u\sqrt{2\pi\tau^3}} \\
 Q &= T_0 - uT_1 + u^2T_2 - u^3T_3 \dots \\
 &= F_0 - u\left(\frac{tF_0}{1!1!} - \frac{F_1}{2!}\right) + u^2\left(\frac{t^2F_0}{2!1!} - \frac{tF_1}{1!2!} + \frac{F_2}{3!}\right) \\
 &\quad - u^3\left(\frac{t^3F_0}{3!1!} - \frac{t^2F_1}{2!2!} + \frac{tF_2}{1!3!} - \frac{F_3}{4!}\right) \dots \\
 &= \left(1 - \frac{ut}{1!} + \frac{u^2t^2}{2!} - \frac{u^3t^3}{3!} \dots\right) \left(\frac{F_0}{1!} + \frac{uF_1}{2!} + \frac{u^2F_2}{3!} \dots\right) \\
 &= e^{-ut} \cdot \left(\psi(x) * \left(\frac{1}{1! \sqrt{2\pi t}} + \frac{u}{2!} \frac{t^{\frac{1}{2}}}{\sqrt{2\pi}} + \frac{u^2}{3!} \frac{t^{\frac{3}{2}}}{\sqrt{2\pi}} \dots \right) \right) \\
 &= e^{-ut} \cdot \left(\psi(x) * \frac{\frac{ut}{1!} + \frac{u^2t^2}{2!} + \frac{u^3t^3}{3!} \dots}{u\sqrt{2\pi t^3}} \right) = e^{-ut} \cdot \left(\psi(x) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right) \\
 &= e^{-ut} \int_0^t d\tau \frac{x e^{-\frac{x^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} \frac{e^{u\tau} - 1}{u\sqrt{2\pi\tau^3}}
 \end{aligned}$$

and this is the final result for the upper function. ■

Proof of the Theorem for $x \leq 0$.

We shall prove that:

$$\begin{aligned}
 Q(x, t) &= \psi(|x|) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \\
 &= \frac{|x|}{2\pi} \int_0^t d\tau \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{1 - e^{-u\tau}}{u\tau^{\frac{3}{2}}}.
 \end{aligned}$$

Let $x < 0$

$$\begin{aligned}
 G_{n+1}(x) &= \int_0^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * G_n(\xi) = \int_0^\infty d\xi \frac{e^{-\frac{(|x|+\xi)^2}{2t}}}{\sqrt{2\pi t}} * G_n(\xi) \\
 &= \psi(|x|) * \int_0^\infty d\xi \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} * G_n(\xi) = \psi(|x|) * G_{n+1}(0)
 \end{aligned}$$

or

$$G_n(x) = \psi(|x|) * G_n(0) .$$

All that is left now is to determine $G_n(0)$

$$G_n = \frac{t^n F_0}{n!1!} - \frac{t^{n-1} F_1}{(n-1)!2!} + \frac{t^{n-2} F_2}{(n-2)!3!} - \dots + (-1)^n \frac{F_n}{(n+1)!} .$$

But $F_k(x, t) = \psi(x) * \frac{t^{k-\frac{1}{2}}}{\sqrt{2\pi}}$ so $F_k(0, t) = \frac{t^{k-\frac{1}{2}}}{\sqrt{2\pi}}$. Hence:

$$\begin{aligned} G_n(0) &= \frac{t^{n-\frac{1}{2}}}{\sqrt{2\pi}} \left(\frac{1}{n!1!} - \frac{1}{(n-1)!2!} + \frac{1}{(n-2)!3!} - \dots \right) = \\ &= \frac{t^{n-\frac{1}{2}}}{\sqrt{2\pi}} \frac{1}{(n+1)!} \left(\binom{n+1}{1} - \binom{n+1}{2} + \binom{n+1}{3} - \dots \right) = \\ &= \frac{1}{(n+1)!} \frac{t^{n-\frac{1}{2}}}{\sqrt{2\pi}} \end{aligned} \quad (2.12)$$

$$\text{Thus } G_n(x) = \frac{1}{(n+1)!} \psi(|x|) * \frac{t^{n-\frac{1}{2}}}{\sqrt{2\pi}} = \frac{F_n}{(n+1)!} .$$

We conclude that the lower function is given by

$$\begin{aligned} Q &= \psi(|x|) * \frac{1}{\sqrt{2\pi}} \left(\frac{1}{1!t^{\frac{1}{2}}} - \frac{ut^{\frac{1}{2}}}{2!} + \frac{u^2 t^{\frac{3}{2}}}{3!} - \frac{u^3 t^{\frac{5}{2}}}{4!} \dots \right) = \\ &= \psi(|x|) * \frac{\frac{ut}{1!} - \frac{u^2 t^2}{2!} + \frac{u^3 t^3}{3!} - \frac{u^4 t^4}{4!} \dots}{u\sqrt{2\pi}t^{\frac{3}{2}}} = \\ &= \psi(|x|) * \frac{1 - e^{-ut}}{u\sqrt{2\pi}t^{\frac{3}{2}}} \end{aligned}$$

that is:

$$\begin{aligned} Q(x, t) &= \psi(|x|) * \frac{1 - e^{-ut}}{u\sqrt{2\pi}t^{\frac{3}{2}}} \\ &= \frac{|x|}{2\pi} \int_0^t d\tau \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{1 - e^{-u\tau}}{u\tau^{\frac{3}{2}}} \end{aligned}$$

when $x \leq 0$. ■

2.3 The formulas of Lévy and Kac.

In this section we use the expressions for $Q(x, t)$ to derive the arcsine law of Lévy and Kac's formula for the distribution of the R_+ occupation time of the Brownian bridge process. First we shall prove:

Theorem 2.2

$$E(e^{-uO_0(t)}, X_t \leq 0) = \frac{1}{2\pi u} \int_0^t \frac{1 - e^{-u\tau}}{(t-\tau)^{\frac{1}{2}} \tau^{\frac{3}{2}}} d\tau \quad (2.13)$$

$$E(e^{-uO_0(t)}, X_t \geq 0) = \frac{e^{-ut}}{2\pi} \int_0^t \frac{e^{u\tau} - 1}{u(t-\tau)^{\frac{1}{2}} \tau^{\frac{3}{2}}} d\tau. \quad (2.14)$$

Proof:

Consider

$$\int_{-\infty}^0 Q(x, t) dx$$

this is

$$E(e^{-uO_0(t)}, X_t \leq 0).$$

Now we have shown (2.7)

$$Q(x, t) = \frac{|x|}{2\pi} \int_0^t \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \cdot \frac{1 - e^{-u\tau}}{u\tau^{\frac{3}{2}}} d\tau$$

when $X_t \leq 0$.

Hence

$$\int_{-\infty}^0 Q(x, t) dx = \frac{1}{2\pi u} \int_0^t \frac{1 - e^{-u\tau}}{(t-\tau)^{\frac{1}{2}} \tau^{\frac{3}{2}}} d\tau$$

and this proves the first part of the theorem.

Consider now the case $X_t \geq 0$.

$$E(e^{-uO_0(t)}, X_t \geq 0) = \int_0^\infty Q(x, t) dx.$$

We have shown (2.6) that $Q(x, t)$ in this case is given by

$$\frac{x e^{-ut}}{2\pi} \int_0^t \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{e^{u\tau} - 1}{u\tau^{\frac{3}{2}}} d\tau .$$

Hence

$$\int_0^\infty Q(x, t) dx = \frac{e^{-ut}}{2\pi} \int_0^t \frac{e^{u\tau} - 1}{u(t-\tau)^{\frac{1}{2}} \tau^{\frac{3}{2}}} d\tau$$

and this completes the proof. ■

Theorem 2.3

$$P(O_0(t) \leq \xi, X_t \leq 0) = \frac{1}{\pi} \left(\arcsin \sqrt{\frac{\xi}{t}} + \frac{\sqrt{\xi(t-\xi)}}{t} \right) . \quad (2.15)$$

Proof:

Substituting $\tau = t\bar{\tau}$ into (2.13) and retaining τ after the substitution gives:

$$\frac{1}{2\pi t} \int_0^1 \frac{1 - e^{-u\tau}}{u(t-\tau)^{\frac{1}{2}} \tau^{\frac{3}{2}}} d\tau .$$

Substituting $\tau = \sin^2 \vartheta$ gives

$$\frac{1}{\pi u t} \int_0^{\frac{\pi}{2}} \frac{1 - e^{-u t \sin^2 \vartheta}}{\sin^2 \vartheta} d\vartheta$$

and integration by parts gives:

$$= \frac{-\cot \vartheta (1 - e^{-u t \sin^2 \vartheta}) \Big|_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^{-u t \sin^2 \vartheta} \cdot \frac{2u t \sin \vartheta \cos^2 \vartheta}{\sin^3 \vartheta} d\vartheta}{\pi u t}$$

and finally it becomes:

$$\frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-u t \sin^2 \vartheta} \cos^2 \vartheta d\vartheta .$$

Thus:

$$E(e^{-u O_0(t)}, X_t \leq 0) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-u t \sin^2 \vartheta} \cos^2 \vartheta d\vartheta .$$

Put $t \sin^2 \vartheta = \xi$ that is $\vartheta = \arcsin \sqrt{\frac{\xi}{t}}$ and it becomes:

$$\begin{aligned} & \frac{2}{\pi} \int_0^1 e^{-u\xi} (1 - \sin^2 \arcsin \sqrt{\frac{\xi}{t}}) \cdot d\xi \arcsin \sqrt{\frac{\xi}{t}} \\ &= \frac{2}{\pi} \int_0^1 e^{-u\xi} d\xi \arcsin \sqrt{\frac{\xi}{t}} - \frac{2}{\pi} \int_0^1 e^{-u\xi} \frac{\xi}{t} d\xi \arcsin \sqrt{\frac{\xi}{t}} . \end{aligned}$$

But

$$\begin{aligned} \frac{\xi}{t} \frac{d}{d\xi} \arcsin \sqrt{\frac{\xi}{t}} &= \frac{d}{d\xi} \int \frac{\xi}{t} \frac{d}{d\xi} \arcsin \sqrt{\frac{\xi}{t}} \\ \int \frac{\xi}{t} \frac{d}{d\xi} \arcsin \sqrt{\frac{\xi}{t}} d\xi &= \frac{\xi}{t} \arcsin \sqrt{\frac{\xi}{t}} - \frac{1}{t} \int \arcsin \sqrt{\frac{\xi}{t}} d\xi . \end{aligned}$$

It is easy to see that:

$$\int \arcsin \sqrt{\frac{\xi}{t}} d\xi = \xi \arcsin \sqrt{\frac{\xi}{t}} - \frac{t}{2} \arcsin \sqrt{\frac{\xi}{t}} + \frac{\sqrt{\xi(t-\xi)}}{2}$$

and therefore

$$\int \frac{\xi}{t} \frac{d}{d\xi} \arcsin \sqrt{\frac{\xi}{t}} d\xi = \frac{1}{2} \arcsin \sqrt{\frac{\xi}{t}} - \frac{\sqrt{\xi(t-\xi)}}{2t} .$$

Thus

$$\begin{aligned} \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-ut \sin^2 \vartheta} \cos^2 \vartheta d\vartheta &= \frac{2}{\pi} \int_0^1 e^{-u\xi} d\xi \left(\arcsin \sqrt{\frac{\xi}{t}} - \frac{1}{2} \arcsin \sqrt{\frac{\xi}{t}} + \frac{\sqrt{\xi(t-\xi)}}{2t} \right) = \\ &= \frac{1}{\pi} \int_0^1 e^{-u\xi} d\xi \left(\arcsin \sqrt{\frac{\xi}{t}} + \frac{\sqrt{\xi(t-\xi)}}{t} \right) . \end{aligned}$$

Hence it follows:

$$P(O_0(t) \leq \xi, X_t \leq 0) = \frac{1}{\pi} \left(\arcsin \sqrt{\frac{\xi}{t}} + \frac{\sqrt{\xi(t-\xi)}}{t} \right)$$

and this completes the proof. ■

For the case $X_t \geq 0$ we prove:

Theorem 2.4

$$P(O_0(t) \leq \xi, X_t \geq 0) = \frac{1}{\pi} \left(\arcsin \sqrt{\frac{\xi}{t}} - \frac{\sqrt{\xi(t-\xi)}}{t} \right) \quad (2.16)$$

Proof:

We have seen (2.14)

$$\int_0^\infty Q(x, t) dx = \frac{e^{-ut}}{2\pi} \int_0^t \frac{e^{u\tau} - 1}{u\tau^{\frac{3}{2}}} d\tau$$

and this (through the substitutions $\tau = t\bar{\tau}$ and $\bar{\tau} = \sin^2 \vartheta$ as in the previous case) becomes :

$$\begin{aligned} & \frac{e^{-ut}}{\pi ut} \left\{ \cot \vartheta (e^{ut \sin^2 \vartheta} - 1) \right\} \bigg|_{\vartheta=0}^{\vartheta=\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cot \vartheta \cdot ut \cdot 2 \sin \vartheta \cos \vartheta \cdot e^{ut \sin^2 \vartheta} d\vartheta = \\ & = \frac{2}{\pi} e^{-ut} \int_0^{\frac{\pi}{2}} \cos^2 \vartheta e^{ut \sin^2 \vartheta} d\vartheta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-ut \cos^2 \vartheta} \cdot \cos^2 \vartheta d\vartheta. \end{aligned}$$

Thus:

$$\begin{aligned} E(e^{-uO_0(t)}, X_t \geq 0) &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^2 \vartheta e^{-ut \cos^2 \vartheta} d\vartheta = \\ &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin^2 \vartheta e^{-ut \sin^2 \vartheta} d\vartheta \end{aligned}$$

again using the substitution $\vartheta = \arcsin \sqrt{\frac{\xi}{t}}$ yields:

$$\frac{2}{\pi} \int_0^1 \frac{\xi}{t} e^{-u\xi} d\xi \arcsin \sqrt{\frac{\xi}{t}}.$$

But

$$\frac{\xi}{t} \frac{d}{d\xi} \arcsin \sqrt{\frac{\xi}{t}} = \frac{d}{d\xi} \int \frac{\xi}{t} \frac{d}{d\xi} \arcsin \sqrt{\frac{\xi}{t}}$$

and it has been shown that

$$\int \frac{\xi}{t} \frac{d}{d\xi} \arcsin \sqrt{\frac{\xi}{t}} = \frac{1}{2} \arcsin \sqrt{\frac{\xi}{t}} - \frac{\sqrt{\xi(t-\xi)}}{2t}.$$

Therefore

$$\frac{2}{\pi} \int_0^1 \frac{\xi}{t} e^{-u\xi} d\xi \arcsin \sqrt{\frac{\xi}{t}} = \frac{1}{\pi} \int_0^1 e^{-u\xi} d\xi \left(\arcsin \sqrt{\frac{\xi}{t}} - \frac{\sqrt{\xi(t-\xi)}}{t} \right).$$

Thus

$$P(O_0(t) \leq \xi, X_t \geq 0) = \frac{1}{\pi} \left(\arcsin \sqrt{\frac{\xi}{t}} - \frac{\sqrt{\xi(t-\xi)}}{t} \right) \text{ for } x_t \geq 0. \quad \blacksquare$$

Adding equations (2.15) and (2.16) finally gives:

Theorem 2.5 (*Lévy's arcsine law of the Brownian motion*)

$$P(O_0(t) \leq \xi) = \frac{2}{\pi} \arcsin \sqrt{\frac{\xi}{t}}. \quad (2.17)$$

We now deal with the evaluation of $P(O_0(t) \leq \xi \mid X_t = x)$. For now we consider only the case $x = 0$ and prove:

Theorem 2.6 [*Kac 1*]

$$P(O_0(t) \leq \xi \mid X_t = 0) = \frac{\xi}{t}$$

Proof:

Starting with (2.7)

$$Q(x, t) = \frac{|x|}{2\pi} \int_0^t \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{1 - e^{-u\tau}}{u\tau^{\frac{3}{2}}} d\tau \text{ when } x \leq 0.$$

Clearly because of (2.5) when $x = 0$ this is $\frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}}$. So

$$Q(0, t) = \frac{1}{\sqrt{2\pi t}} E(e^{-uO_0(t)} \mid X_t = 0) = \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}}$$

or

$$E(e^{-uO_0(t)} \mid X_t = 0) = \frac{1 - e^{-ut}}{ut} \quad (2.18)$$

that is

$$\frac{1 - e^{-ut}}{ut} = \int_0^t e^{-u\xi} d_\xi P(O_0(t) \leq \xi \mid x_t = 0).$$

But

$$\frac{1 - e^{-ut}}{ut} = \int_0^t e^{-u\xi} \frac{d\xi}{t} = \int_0^t e^{-u\xi} d\left(\frac{\xi}{t}\right),$$

and therefore

$$P(O_0(t) \leq \xi \mid X_t = x) = \frac{\xi}{t} \quad (\text{uniform distribution}).$$

This is the analogue of the arcsine law for the Brownian bridge process. ■

2.4 The second arcsine law of Brownian motion.

Lévy discovered another law of the Brownian motion that is expressed by the $\frac{2}{\pi} \arcsin \sqrt{\frac{\xi}{t}}$ formula. If $t > 0$, then $X_t \neq 0$ with probability 1. Since the Brownian motion paths are continuous and $X_0 = 0$ there must be a last zero before t . Let this be denoted by LZ

$$LZ = LZ(t) = \sup\{\tau : X_\tau = 0, \tau \leq t\}.$$

Now consider the process Y_τ defined on the time interval $0 \leq \tau \leq T \leq t$ where its value $Y_\tau = X_\tau$ and conditioned on $T = LZ(t)$. The character of Y_τ is expressed in the following:

Theorem 2.7 *The process Y_τ , $0 \leq \tau \leq LZ(t)$ is a Brownian motion process tied down at $LZ(t)$.*

Proof: We shall prove this by computing the finite dimensional distributions of Y_τ . The finite dimensional distributions of the Brownian motion process $\{X_\tau, 0 \leq \tau \leq T\}$ conditioned on $X_T = 0$ are:

$$\begin{aligned} P(X_{\tau_1} \leq a_1, X_{\tau_2} \leq a_2, \dots, X_{\tau_n} \leq a_n \mid X_T = 0) = \\ = \int_{-\infty}^{a_1} d\xi_1 \int_{-\infty}^{a_2} d\xi_2 \dots \int_{-\infty}^{a_n} d\xi_n \sqrt{\frac{T}{\tau_1}} \frac{e^{-\frac{\xi_1^2}{2\tau_1} - \frac{(\xi_2 - \xi_1)^2}{2(\tau_2 - \tau_1)} - \dots - \frac{(\xi_n - \xi_{n-1})^2}{2(\tau_n - \tau_{n-1})} - \frac{\xi_n^2}{2(T - \tau_n)}}}{\sqrt{2\pi(\tau_2 - \tau_1) \dots 2\pi(T - \tau_n)}} \end{aligned} \quad (2.19)$$

where $0 \leq \tau_1 < \tau_2 < \dots < \tau_n < T$.

It is enough to prove that the distribution $P(X_{\tau_1} \leq a_1, X_{\tau_2} \leq a_2, \dots, X_{\tau_n} \leq a_n | T = LZ(t))$ is expressed by (2.19) when $0 \leq \tau_1 < \tau_2 < \dots < \tau_n < LZ(t)$. Using the well known identity (see [Knight] page 19 Theorem 2.1)

$$\tilde{X}_t \stackrel{D}{=} tX_{\frac{1}{t}}$$

we can write:

$$\begin{aligned} & P(X_{\tau_1} \leq a_1, X_{\tau_2} \leq a_2, \dots, X_{\tau_n} \leq a_n | T = LZ(t)) \\ &= P(X_{\frac{1}{\tau_1}} \leq \frac{a_1}{\tau_1}, X_{\frac{1}{\tau_2}} \leq \frac{a_2}{\tau_2}, \dots, X_{\frac{1}{\tau_n}} \leq \frac{a_n}{\tau_n} | T = LZ(t)) \\ &= P(\frac{1}{\tau_1} X_{\frac{1}{\tau_1}} \leq \frac{a_1}{\tau_1}, \frac{1}{\tau_2} X_{\frac{1}{\tau_2}} \leq \frac{a_2}{\tau_2}, \dots, \frac{1}{\tau_n} X_{\frac{1}{\tau_n}} \leq \frac{a_n}{\tau_n} | T = LZ(t)) \\ &= P(\tilde{X}_{\frac{1}{\tau_n}} \leq \frac{a_n}{\tau_n}, \dots, \tilde{X}_{\frac{1}{\tau_1}} \leq \frac{a_1}{\tau_1} | \frac{1}{T} = FZ(\frac{1}{t})) . \end{aligned}$$

For the time reversed Brownian motion process $\frac{1}{LZ(t)}$ is the first zero that is not less than $\frac{1}{t}$. It is a stopping time. Hence by the strong Markov property of the Brownian motion the processes $X_\tau, 0 \leq \tau \leq \frac{1}{LZ(t)}$ and $X_\tau, \frac{1}{LZ(t)} \leq \tau < \infty$ are independent of each other. Hence:

$$\begin{aligned} & P(\tilde{X}_{\frac{1}{\tau_n}} \leq \frac{a_n}{\tau_n}, \dots, \tilde{X}_{\frac{1}{\tau_1}} \leq \frac{a_1}{\tau_1} | \tilde{X}_{\frac{1}{t}} = 0) = \\ &= \int_{-\infty}^{\frac{a_n}{\tau_n}} d\xi_n \int_{-\infty}^{\frac{a_{n-1}}{\tau_{n-1}}} d\xi_{n-1} \dots \int_{-\infty}^{\frac{a_1}{\tau_1}} d\xi_1 \frac{\exp(-\frac{\xi_n^2}{2(\frac{1}{\tau_n} - \frac{1}{T})} - \frac{(\xi_{n-1} - \xi_n)^2}{2(\frac{1}{\tau_{n-1}} - \frac{1}{\tau_n})} \dots - \frac{(\xi_1 - \xi_2)^2}{2(\frac{1}{\tau_1} - \frac{1}{\tau_2})})}{\sqrt{2\pi(\frac{1}{\tau_n} - \frac{1}{T})} 2\pi(\frac{1}{\tau_{n-1}} - \frac{1}{\tau_n}) \dots 2\pi(\frac{1}{\tau_1} - \frac{1}{\tau_2})} . \end{aligned}$$

Substituting $\xi_n = \frac{\bar{\xi}_n}{\tau_n}, \dots, \xi_1 = \frac{\bar{\xi}_1}{\tau_1}$ the last integral after some simplification becomes:

$$\int_{-\infty}^{\frac{a_n}{\tau_n}} d\xi_n \dots \int_{-\infty}^{\frac{a_1}{\tau_1}} d\xi_1 \sqrt{\frac{LZ(t)}{\tau_1}} \frac{\exp(-A)}{\sqrt{2\pi(LZ(t) - \tau_n) 2\pi(\tau_n - \tau_{n-1}) \dots 2\pi(\tau_2 - \tau_1)}}$$

where:

$$A = -\frac{\xi_n^2 LZ(t)}{2\tau_n(LZ(t) - \tau_n)} - \frac{(\tau_{n-1}\xi_n - \tau_n\xi_{n-1})^2}{2\tau_n\tau_{n-1}(\tau_n - \tau_{n-1})} \dots - \frac{(\tau_1\xi_2 - \tau_2\xi_1)^2}{2\tau_1\tau_2(\tau_2 - \tau_1)} .$$

The last integral is the same as (2.19) except for the exponent A . It remains to be proved that:

$$A = -\frac{\xi_n^2}{2(LZ(t) - \tau_n)} - \frac{(\xi_n - \xi_{n-1})^2}{2(\tau_n - \tau_{n-1})} \dots - \frac{\xi^2}{2\tau_1}.$$

This clearly true for $n = 1$ and in general truth can be established by induction and this completes the proof. ■

What is the distribution of LZ ? The answer is given by the following result of Lévy

Theorem 2.8 *The second arcsine law of the Brownian motion. [Lévy 1]*

$$P(LZ(t) \leq \xi) = \frac{2}{\pi} \arcsin \sqrt{\frac{\xi}{t}}.$$

Proof:

It has been proved that: (2.15)

$$P(O_0(t) \leq \xi, X_t \leq 0) = \frac{1}{\pi} \left(\arcsin \sqrt{\frac{\xi}{t}} + \frac{\sqrt{\xi(t-\xi)}}{t} \right).$$

But

$$P(O_0(t) \leq \xi, X_t \leq 0) = P(LZ(t) \leq \xi) + \int_{\xi}^t P(LZ(t) = \tau) P(O_0(LZ(t)) \leq \xi) d\tau$$

because of Theorem 2.7 $P(O_0(LZ(t)) \leq \xi) = \frac{\xi}{LZ(t)} = \frac{\xi}{\tau}$ so we can write:

$$\begin{aligned} P(O_0(t) \leq \xi, X_t \leq 0) &= P(LZ(t) \leq \xi) + \int_{\xi}^t P(LZ(t) = \tau) \frac{\xi}{\tau} d\tau \\ &= \frac{1}{\pi} \left(\arcsin \sqrt{\frac{\xi}{t}} + \frac{\sqrt{\xi(t-\xi)}}{t} \right). \end{aligned}$$

Differentiating both sides with respect to ξ gives:

$$\begin{aligned} P(LZ(t) = \xi) + \int_{\xi}^t \frac{P(LZ(t) = \tau)}{\tau} d\tau - P(LZ = \xi) &= \\ = \frac{1}{2\pi} \left\{ \frac{1}{\sqrt{\xi(t-\xi)}} + \frac{1}{t} \sqrt{\frac{t-\xi}{\xi}} - \frac{1}{t} \sqrt{\frac{\xi}{t-\xi}} \right\}. \end{aligned}$$

Differentiating once more via ξ gives:

$$-\frac{P(LZ = \xi)}{\xi} = -\frac{1}{2\pi} \frac{1}{\xi^{\frac{3}{2}}(t - \xi)^{\frac{1}{2}}}$$

so:

$$P(LZ = \xi) = \frac{1}{2\pi} \frac{1}{\xi^{\frac{1}{2}}(t - \xi)^{\frac{1}{2}}} .$$

Hence:

$$P(LZ \leq \xi) = \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} .$$

More precisely:

$$P(LZ \leq \xi, X_t \leq 0) = \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} .$$

Thus by symmetry

$$P(LZ \leq \xi) = \frac{2}{\pi} \arcsin \sqrt{\frac{\xi}{t}}$$

and this is the so called second arcsine law of the Brownian motion. ■

Next we shall derive the analog of the above theorem for the tied down Brownian motion, that is we answer the question: What is the distribution of the last zero of a tied down Brownian process ?

Theorem 2.9

$$P(LZ \leq \tau \mid X_t = x) = \frac{2}{\sqrt{\pi}} \int_0^{|x| \sqrt{\frac{\tau}{2t(t-\tau)}}} e^{-\xi^2} d\xi . \quad (2.20)$$

Proof:

Let $x < 0$

$$E(e^{-uO_0(t)} \mid X_t = x) = \int_0^t d\tau P(LZ = \tau \mid X_t = x) E(e^{-uO_0(\tau)} \mid X_\tau = 0) .$$

Now multiplying both sides by $\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$ and using (2.18) and Theorem 2.7 we get:

$$Q(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \int_0^t d\tau P(LZ = \tau \mid X_t = x) \frac{1 - e^{-u\tau}}{u\tau} .$$

Hence for all $u > 0$

$$\frac{|x|}{2\pi} \int_0^t \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{1-e^{-u\tau}}{u\tau^{\frac{3}{2}}} d\tau = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \int_0^t d\tau P(LZ = \tau | x_t = x) \frac{1-e^{-u\tau}}{u\tau}.$$

Thus

$$\begin{aligned} P(LZ = \tau | X_t = x) &= \frac{|x|}{2\pi} \sqrt{2\pi t} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \\ &= \frac{|x| t^{\frac{1}{2}}}{\sqrt{2\pi(t-\tau)^3 \tau}} e^{-x^2 \frac{\tau}{2t(t-\tau)}} \end{aligned}$$

and by integration we get:

$$P(LZ \leq \tau | x_t = x) = \frac{2}{\sqrt{\pi}} \int_0^{|x| \sqrt{\frac{\tau}{2t(t-\tau)}}} e^{-\xi^2} d\xi.$$

As a consequence of symmetry this holds for all x of course. ■

Let us remark that from this the second arcsine law of the Brownian motion can be derived by integration. Namely we have:

$$\begin{aligned} P(LZ \leq \tau) &= \int_{-\infty}^{\infty} dx \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(LZ \leq \tau | X_t = x) = \\ &= \frac{2}{\sqrt{2\pi t}} \int_0^{\infty} dx e^{-\frac{x^2}{2t}} \frac{2}{\sqrt{\pi}} \int_0^{x \sqrt{\frac{\tau}{2t(t-\tau)}}} e^{-\xi^2} d\xi = \frac{4}{\pi} \int_0^{\infty} dx \cdot e^{-x^2} \int_0^{x \sqrt{\frac{\tau}{t(t-\tau)}}} e^{-\xi^2} d\xi = \\ &= \frac{4}{\pi} \int_0^{\infty} dx \cdot e^{-x^2} x \int_0^{\sqrt{\frac{\tau}{t-\tau}}} e^{-x^2 \xi^2} d\xi = \frac{2}{\pi} \int_0^{\sqrt{\frac{\tau}{t-\tau}}} \frac{d\xi}{1+\xi^2} \int_0^{\infty} 2x(1+\xi^2) e^{-x^2(1+\xi^2)} dx = \\ &= \frac{2}{\pi} \int_0^{\sqrt{\frac{\tau}{t-\tau}}} \frac{d\xi}{1+\xi^2} = \frac{2}{\pi} \arctan \sqrt{\frac{\tau}{t-\tau}} = \frac{2}{\pi} \arcsin \sqrt{\frac{\tau}{t}} \end{aligned}$$

and this is a second proof of Lévy's formula.

2.5 The occupation time of R_+ for the tied down Brownian motion.

This is the distribution $P(O_0(t) \leq \xi | X_t = x)$. For $x = 0$ this has been shown to be the $\frac{\xi}{t}$ uniform distribution. We shall prove:

Theorem 2.10

$$\begin{aligned}
 (a) \quad P(O_0(t) \leq \xi \mid X_t = x) &= \\
 &= \frac{\xi}{t} + |x| \sqrt{\frac{2\xi(t-\xi)}{\pi t^3}} e^{-\frac{x^2 \xi}{2t(t-\xi)}} + \operatorname{erf}\left(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}\right) \cdot \left(1 - \frac{\xi}{t}\right) \\
 &\quad - \frac{x^2 \xi}{t^2} \cdot \left(1 - \operatorname{erf}\left(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}\right)\right)
 \end{aligned} \tag{2.21}$$

when $x \leq 0$

and

$$\begin{aligned}
 (b) \quad P(O_0(t) \leq \xi \mid X_t = x) &= \\
 &= \frac{\xi}{t} - x \sqrt{\frac{2\xi(t-\xi)}{\pi t^3}} e^{-\frac{x^2(t-\xi)}{2t\xi}} - \frac{\xi}{t} \operatorname{erf}\left(x \sqrt{\frac{t-\xi}{2t\xi}}\right) \\
 &\quad + \frac{x^2}{t} \left(1 - \frac{\xi}{t}\right) \left(1 - \operatorname{erf}\left(x \sqrt{\frac{t-\xi}{2t\xi}}\right)\right)
 \end{aligned} \tag{2.22}$$

when $x \geq 0$,

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\xi^2} d\xi$.

Proof:

Two methods of computation will be given.

Method 1. (The inversion of $Q(x, t)$.)

If $x < 0$ then

$$Q(x, t) = \frac{|x|}{2\pi} \int_0^t \frac{e^{-\frac{x^2}{2(t-\tau)}}}{(t-\tau)^{\frac{3}{2}}} \frac{1 - e^{-u\tau}}{u\tau^{\frac{3}{2}}} d\tau.$$

Integrating by part this becomes:

$$Q(x, t) = \frac{|x|}{2\pi} \int_0^t e^{-u\tau} d\tau \int_{\tau}^t \frac{e^{-\frac{x^2}{2(t-\xi)}}}{\xi^{\frac{3}{2}}(t-\xi)^{\frac{3}{2}}} d\xi.$$

Thus $\frac{|x|}{2\pi} \int_{\tau}^t \frac{e^{-\frac{x^2}{2(t-\xi)}}}{\xi^{\frac{1}{2}}(t-\xi)^{\frac{1}{2}}} d\xi$ is the density function multiplied by $\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$.

So

$$P(O_0(t) \leq \xi | X_t = x) = \sqrt{2\pi t} e^{+\frac{x^2}{2t}} \frac{|x|}{2\pi} \int_0^{\xi} d\eta \int_{\eta}^t \frac{e^{-\frac{x^2}{2(t-\sigma)}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma.$$

Now

$$\int_{\eta}^t \frac{e^{-\frac{x^2}{2(t-\sigma)}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma = \int_0^{t-\eta} \frac{e^{-\frac{x^2}{2(t-\sigma)}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma.$$

Integrating by parts

$$\begin{aligned} \frac{|x|}{2\pi} \int_0^{\xi} d\eta \int_0^{t-\eta} \frac{e^{-\frac{x^2}{2(t-\sigma)}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma &= \frac{|x|}{2\pi} \xi \int_0^{t-\xi} \frac{e^{-\frac{x^2}{2(t-\sigma)}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma + \frac{|x|}{2\pi} \int_0^{\xi} \sigma \frac{e^{-\frac{x^2}{2(t-\sigma)}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma \\ &= \frac{|x|}{2\pi} \left\{ \xi \int_0^{t-\xi} \frac{e^{-\frac{x^2}{2\sigma}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma + \int_{t-\xi}^t \frac{e^{-\frac{x^2}{2\sigma}}}{\sigma^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} d\sigma \right\}. \end{aligned}$$

Putting $\sigma = \frac{1}{\sigma^2}$ gives:

$$\frac{|x|}{\pi} \left\{ \xi \int_{\frac{1}{t-\xi}}^{\infty} \frac{\sigma}{(\sigma^2 t - 1)^{\frac{3}{2}}} e^{-\frac{x^2 \sigma^2}{2}} d\sigma + \int_{\frac{1}{t}}^{\frac{1}{t-\xi}} \frac{\sigma}{(\sigma^2 t - 1)^{\frac{1}{2}}} e^{-\frac{x^2 \sigma^2}{2}} d\sigma \right\}.$$

Let now $\sigma^2 t - 1 = \bar{\sigma}^2$ or $\sigma = \sqrt{\frac{1+\bar{\sigma}^2}{t}}$ then after the substitution we get:

$$\begin{aligned} \frac{|x|}{\pi} \left\{ \frac{\xi}{t^2} \int_{\sqrt{\frac{t}{t-\xi}}}^{\infty} \frac{1+\sigma^2}{\sigma^2} e^{-\frac{x^2}{2t}(1+\sigma^2)} d\sigma + \frac{1}{t} \int_0^{\sqrt{\frac{t}{t-\xi}}} e^{-\frac{x^2}{2t}(1+\sigma^2)} d\sigma \right\} &= \\ = \frac{|x|}{\pi t^2} e^{-\frac{x^2}{2t}} \int_{\sqrt{\frac{t}{t-\xi}}}^{\infty} e^{-\frac{\sigma^2 x^2}{2t}} d\sigma + \frac{|x|}{\pi t^2} e^{-\frac{x^2}{2t}} \int_{\sqrt{\frac{t}{t-\xi}}}^{\infty} \frac{e^{-\frac{\sigma^2 x^2}{2t}}}{\sigma^2} d\sigma \\ + \frac{|x|}{\pi t} e^{-\frac{x^2}{2t}} \int_0^{\sqrt{\frac{t}{t-\xi}}} e^{-\frac{\sigma^2 x^2}{2t}} d\sigma. \end{aligned}$$

This can be simplified further:

$$\begin{aligned} \frac{\xi \sqrt{2t}}{\pi t^2} e^{-\frac{x^2}{2t}} \int_{|x| \sqrt{\frac{t}{2t(t-\xi)}}}^{\infty} e^{-\sigma^2} d\sigma + \frac{x^2 \xi}{t^2 \pi \sqrt{2t}} e^{-\frac{x^2}{2t}} \int_{|x| \sqrt{\frac{t}{2t(t-\xi)}}}^{\infty} \frac{e^{-\sigma^2}}{\sigma^2} d\sigma \\ + \frac{e^{-\frac{x^2}{2t}}}{\pi} \sqrt{\frac{2}{t}} \int_0^{|x| \sqrt{\frac{t}{2t(t-\xi)}}} e^{-\sigma^2} d\sigma. \end{aligned}$$

Using the identity

$$\int \frac{e^{-\sigma^2}}{\sigma^2} d\sigma = -\frac{e^{-\sigma^2}}{\sigma} - 2 \int e^{-\sigma^2} d\sigma$$

and writing $F(x)$ for $\int_0^x e^{-\sigma^2} d\sigma$ this becomes:

$$\begin{aligned} & \frac{\xi\sqrt{2t}}{\pi t^2} e^{-\frac{x^2}{2t}} \left(\frac{\sqrt{\pi}}{2} - F\left(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}\right) \right) \\ & + \frac{x^2\xi}{t^2\pi} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2t}} \left(\frac{e^{-\frac{x^2\xi}{2t(t-\xi)}}}{|x| \sqrt{\frac{\xi}{2t(t-\xi)}}} - 2\left(\frac{\sqrt{\pi}}{2} - F\left(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}\right)\right) \right) \\ & + \frac{e^{-\frac{x^2}{2t}}}{\pi} \sqrt{\frac{2}{t}} F\left(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}\right). \end{aligned} \quad (2.23)$$

Multiplying by $e^{-\frac{x^2}{2t}} \sqrt{2\pi t}$ and rearranging gives now:

$$\begin{aligned} P(O_0(t) \leq \xi \mid X_t = x) = \\ \frac{\xi}{t} + |x| \sqrt{\frac{2\xi(t-\xi)}{\pi t^3}} e^{-\frac{x^2\xi}{2t(t-\xi)}} + \frac{2}{\sqrt{\pi}} F\left(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}\right) \cdot \left(1 - \frac{\xi}{t}\right) \\ - \frac{x^2\xi}{t^2} \cdot \left(1 - \frac{2}{\sqrt{\pi}} F\left(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}\right)\right) \end{aligned}$$

when $x \leq 0$.

Replacing $\frac{2}{\sqrt{\pi}} F(\cdot)$ by $\text{erf}(\cdot)$ we see that this is the same as (2.21). Putting $x = 0$ on the right hand side all terms except the first vanish, so $P(O_0(t) \leq \xi \mid X_t = 0) = \frac{\xi}{t}$ as it was shown before. This completes the first proof of part (a).

Method 2. With the knowledge of the distributions of the last zeros of the tied down Brownian motion we can also compute $P(O_0(t) \leq \xi \mid X_t = x)$ $x < 0$ as follows:

$$\begin{aligned} P(O_0(t) \leq \xi \mid X_t = x) &= \\ &= P(LZ \leq \xi \mid X_t = x) + \int_{\xi}^t d\tau P(LZ = \tau \mid X_t = x) P(O_0(\tau) \leq \xi \mid X_{\tau} = 0) \\ &\quad \text{because of Theorem (2.7) in the last integral } P(O_0(\tau) \leq \xi \mid X_{\tau} = 0) = \frac{\xi}{\tau} \\ &= P(LZ \leq \xi \mid X_t = x) + \int_{\xi}^t d\tau P(LZ = \tau \mid X_t = x) \frac{\xi}{\tau}. \end{aligned}$$

We have shown that the distribution of the last zero of a tied down Brownian process:
(see 2.20)

$$P(LZ \leq \tau \mid X_t = x) = \frac{2}{\sqrt{\pi}} \int_0^{|x| \sqrt{\frac{\tau}{2t(t-\tau)}}} e^{-\xi^2} d\xi.$$

By substituting this into the previous equation we get:

$$P(O_c'(t) \leq \xi \mid X_t = x) = P(LZ \leq \xi \mid X_t = x) + \int_{\xi}^t P(LZ = \tau \mid X_t = x) \frac{\xi}{\tau} d\tau.$$

Then

$$\begin{aligned} P(O_0(t) \leq \xi \mid X_t = x) &= \\ &= \frac{2}{\sqrt{\pi}} \int_0^{|x| \sqrt{\frac{\xi}{2t(t-\xi)}}} e^{-\eta^2} d\eta + \xi \int_{\xi}^t \frac{1}{\sqrt{2\pi}} \frac{|x| t^{\frac{1}{2}}}{\tau^{\frac{1}{2}}(t-\tau)^{\frac{3}{2}}} e^{-\frac{\tau^2}{2t(t-\tau)}} \frac{d\tau}{\tau} = \\ &= \frac{2}{\sqrt{\pi}} \int_0^{|x| \sqrt{\frac{\xi}{2t(t-\xi)}}} e^{-\eta^2} d\eta + \frac{|x| \xi \sqrt{t}}{\sqrt{2\pi}} \int_{\xi}^t \frac{e^{-\frac{\tau^2}{2t(t-\tau)}}}{\tau^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}} d\tau = \\ &= \frac{2}{\sqrt{\pi}} F(|x| \sqrt{\frac{\xi}{2t(t-\xi)}}) + \xi \sqrt{\frac{t}{2\pi}} |x| e^{\frac{x^2}{2t}} \int_{\xi}^t \frac{e^{-\frac{\tau^2}{2t(t-\tau)}}}{\tau^{\frac{3}{2}}(t-\tau)^{\frac{3}{2}}} d\tau = \\ &= \frac{2}{\sqrt{\pi}} F + \frac{|x| \xi e^{\frac{x^2}{2t}}}{\sqrt{2\pi t^3}} \int_{\frac{t}{t-\xi}}^{\infty} \frac{e^{-\frac{\tau^2}{2t}\tau}}{(\tau-1)^{\frac{3}{2}}} \tau^3 \frac{d\tau}{\tau^2} = \\ &= \frac{2}{\sqrt{\pi}} F + \frac{|x| \xi e^{\frac{x^2}{2t}}}{\sqrt{2\pi t^3}} \int_{\frac{t}{t-\xi}-1}^{\infty} \frac{e^{-\frac{\tau^2}{2t}(\tau+1)}}{\tau^{\frac{3}{2}}} (\tau+1) d\tau = \\ &= \frac{2}{\sqrt{\pi}} F + \frac{\xi |x|}{\sqrt{2\pi t^3}} \int_{\frac{t}{t-\xi}}^{\infty} e^{-\frac{\tau^2}{2t}\tau} \left(\frac{1}{\tau^{\frac{1}{2}}} + \frac{1}{\tau^{\frac{3}{2}}} \right) d\tau = \\ &= \frac{2}{\sqrt{\pi}} F + \frac{2\xi}{\sqrt{\pi t}} \int_{|x| \sqrt{\frac{\xi}{2t(t-\xi)}}}^{\infty} e^{-\tau^2} d\tau + \frac{2\xi x^2}{2\sqrt{\pi t^2}} \int_{|x| \sqrt{\frac{\xi}{2t(t-\xi)}}}^{\infty} \frac{e^{-\tau^2}}{\tau^2} d\tau. \end{aligned}$$

Using

$$\int \frac{e^{-\tau^2}}{\tau^2} d\tau = -\frac{e^{-\tau^2}}{\tau} - 2 \int e^{-\tau^2} d\tau$$

gives

$$\frac{\xi}{t} + |x| \sqrt{\frac{2\xi(t-\xi)}{\pi t^3}} e^{-\frac{x^2}{2t(t-\xi)}} + \left(1 - \frac{\xi}{t}\right) \frac{2}{\sqrt{\pi}} F - \frac{\xi x^2}{t^2} \left(1 - \frac{2}{\sqrt{\pi}} F\right)$$

and this is the desired result. This completes the second proof of (a).

It remains yet to evaluate $P(O_0(t) \leq \xi \mid X_t = x)$ for the case $x > 0$. Although this could be done by either of the methods used in the case $x < 0$, there is a much simpler way:

$$\begin{aligned}
 P(O_0(t) \leq \xi \mid X_t = x, x > 0) &= \\
 &= P(\lambda(X_r \leq 0) \leq \xi \mid X_t = -x) = P(t - \xi \leq \lambda(X_r \geq 0) \mid X_t = -x) = \\
 &= 1 - \left\{ \frac{t - \xi}{t} + x \sqrt{\frac{2\xi(t - \xi)}{\pi t^3}} e^{-\frac{x^2(t - \xi)}{2t\xi}} + (1 - \frac{t - \xi}{t}) \operatorname{erf}\left(x \sqrt{\frac{t - \xi}{2t\xi}}\right) \right. \\
 &\quad \left. - \frac{x^2(t - \xi)}{t^2} \left[1 - \operatorname{erf}\left(x \sqrt{\frac{t - \xi}{2t\xi}}\right) \right] \right\}.
 \end{aligned}$$

Thus

$$\begin{aligned}
 P(O_0(t) \leq \xi \mid X_t = x, x > 0) &= \\
 \frac{\xi}{t} - x \sqrt{\frac{2\xi(t - \xi)}{\pi t^3}} e^{-\frac{x^2(t - \xi)}{2t\xi}} - \frac{\xi}{t} \operatorname{erf}\left(x \sqrt{\frac{t - \xi}{2t\xi}}\right) + \frac{x^2}{t} \left(1 - \frac{\xi}{t}\right) \left[1 - \operatorname{erf}\left(x \sqrt{\frac{t - \xi}{2t\xi}}\right) \right]. \blacksquare
 \end{aligned}$$

Chapter 3

Boundary crossing probabilities and occupation times above a fixed level.

3.1 Introduction

In this chapter we derive the Q function for the case $f(t) = a$ where a is a positive constant. Using this Q we shall evaluate the distribution of maxima of the Brownian motion for various domains with boundaries composed of straight lines. As an application of these we shall give a new proof of the Kolmogorov - Petrowsky law of upper class functions. Then we shall give formulas for the distribution of the occupation time $O_a(t)$.

3.2 The evaluation of Q_a .

Let a be a positive constant and let the function Q be denoted in this case by $Q_a(x, t)$. We shall see that it can be expressed in terms of $Q(x, t)$.

Theorem 3.1 *The function Q_a has the form:*

$$\begin{aligned} (a) \quad Q_a(x, t) &= \psi(a) * Q(x - a, t) \\ &= \psi(a) * (e^{-ut}(\psi(x - a) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}})) \quad \text{when } x \geq a \end{aligned} \quad (3.1)$$

and

$$(b) \quad Q_a(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \psi(2a-x) * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \quad \text{when } x \leq a. \quad (3.2)$$

Proof:

The fundamental formulas (1.3) and (1.4) now become:

$$Q_a = \varphi(x) - u \int_0^t d\tau \int_a^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} + u^2 \int_0^t d\tau \int_a^\infty \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \int_0^\tau d\rho \int_a^\infty d\eta \frac{e^{-\frac{(\xi-\eta)^2}{2(\tau-\rho)}}}{\sqrt{2\pi(\tau-\rho)}} \frac{e^{-\frac{\eta^2}{2\rho}}}{\sqrt{2\pi\rho}} \dots$$

Consider the first integral :

$$G_1 = \int_a^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} = \int_0^\infty d\xi \frac{e^{-\frac{(x-a-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{(\xi+a)^2}{2t}}}{\sqrt{2\pi t}} = \psi(a) * \int_0^\infty d\xi \frac{e^{-\frac{(x-a-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}}$$

but the integral in the last expression is the first integral for the case $f(t) = 0$.

Hence:

$$G_{1(a)}(x) = G_1(x-a) * \psi(a).$$

Now the second integral:

$$G_{2(a)} = \int_a^\infty d\xi \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * G_2(\xi-a) * \psi(a) = \int_0^\infty d\xi \frac{e^{-\frac{(x-a-\xi)^2}{2t}}}{\sqrt{2\pi t}} * G_1(\xi) * \psi(a) = G_2(x-a) * \psi(a)$$

and in general:

$$G_{n(a)}(x) = \psi(a) * G_n(x-a).$$

Hence:

$$Q_a(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - u\psi(a) * G_1(x-a) + u^2\psi(a) * G_2(x-a) - \dots \quad (3.3)$$

Notice that when $x \geq a$

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} = \psi(a) * \frac{e^{-\frac{(x-a)^2}{2t}}}{\sqrt{2\pi t}}$$

and substituting this into equation (3.3) gives for $x \geq a$

$$Q_a(x, t) = \psi(a) * \left(\frac{e^{-\frac{(x-a)^2}{2t}}}{\sqrt{2\pi t}} - uG_1(x-a) + u^2G_2(x-a) - \dots \right) =$$

$$\psi(a) * Q(x-a, t) = \psi(a) * \left(e^{-ut} \left(\psi(x-a) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right) \right)$$

and this completes the proof of part (a).

Now consider the lower function, that is the case $x \leq a$:

$$\begin{aligned} Q_a(x, t) &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \psi(a) * \frac{e^{-\frac{(x-a)^2}{2t}}}{\sqrt{2\pi t}} \\ &\quad + \psi(a) * \left(\frac{e^{-\frac{(x-a)^2}{2t}}}{\sqrt{2\pi t}} - uG_1(x-a) + u^2G_2(x-a) - \dots \right) \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \psi(a) * Q(x-a, t) \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \psi(a) * \psi(a-x) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \psi(2a-x) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \end{aligned}$$

and this is the same as (b). ■

3.3 Distribution of maxima of the Brownian motion.

We have shown in the last section that:

$$Q_a(x, t) = \psi(a) * \left(e^{-ut} \cdot \left(\psi(x-a) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right) \right) \text{ when } x \geq a$$

and

$$Q_a(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \psi(2a-x) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \text{ when } x \leq a.$$

Let $u \rightarrow \infty$, clearly

$$\lim_{u \rightarrow \infty} Q_a(x, t) = \begin{cases} 0 & \text{when } x \geq a \\ \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} & \text{when } x \leq a. \end{cases}$$

But

$$\begin{aligned} \lim_{u \rightarrow \infty} e^{\frac{x^2}{2t}} \sqrt{2\pi t} Q_a(x, t) &= \lim_{u \rightarrow \infty} E(e^{-uO_a(t)} | X_t = x) \\ &= P(X_\tau \leq a | X_t = x)_{0 \leq \tau \leq t} = \begin{cases} 0 & \text{when } x \geq a \\ 1 - e^{-\frac{2a(a-x)}{t}} & \text{when } x \leq a. \end{cases} \end{aligned}$$

so we have proved:

Theorem 3.2

$$P(X_\tau \leq a | X_t = x)_{0 \leq \tau \leq t} = \begin{cases} 0 & \text{when } x \geq a \\ 1 - e^{-\frac{2a(a-x)}{t}} & \text{when } x \leq a. \end{cases} \quad (3.4)$$

We shall now demonstrate the usefulness of formula (3.4). It will be used to evaluate the probability of the Brownian motion paths staying in a one sided domain limited by straight lines e.g. $P(X_\tau \leq a)_{0 \leq \tau \leq t}$, $P(X_\tau \leq a + b\tau | X_t = x)_{0 \leq \tau \leq t}$,

$P(X_\tau \leq a + b\tau)_{0 \leq \tau \leq t}$. We shall also describe a method to compute the probability of the Brownian paths staying in a one sided domain whose boundary is made of arbitrary many straight line segments.

First consider

$$P(X_\tau \leq a)_{0 \leq \tau \leq t} = \int_{-\infty}^{\infty} dx \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq a | X_t = x)_{0 \leq \tau \leq t} =$$

$$\begin{aligned}
& \int_{-\infty}^a dx \left(\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} \right) = \\
& \int_{-a}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx - \int_{-a}^{\infty} \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} dx = \int_{-a}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx - \int_a^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = \\
& \int_{-a}^a \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = \frac{2}{\sqrt{\pi}} \int_0^a \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2t}} dx = \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx
\end{aligned}$$

and we have proved:

Theorem 3.3

$$P(X_\tau \leq a) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx.$$

This formula is usually derived through the application of the so called "reflection principle". See [Billingsley]

Let us consider again the case:

$$P(X_\tau \leq a \mid X_t = x) = \begin{cases} 0 & \text{when } x \geq a \\ 1 - e^{-\frac{2a(a-x)}{t}} & \text{when } x \leq a. \end{cases}$$

Putting $x = 0$, gives for the Brownian bridge process

$$P(X_\tau \leq a \mid X_t = 0) = 1 - e^{-\frac{2a^2}{t}}$$

or for $t = 1$

$$P(B_\tau \leq a) = 1 - e^{-2a^2},$$

and this is the ordinary Brownian bridge process $B_t = X_t - tX_1$. These formulas can be generalized further:

$$P(X_\tau \leq a \mid X_t = x) = P(X_\tau + x\frac{\tau}{t} \leq a \mid X_t = 0) =$$

$$P(X_\tau \leq a - x\frac{\tau}{t} \mid X_t = 0) = P(X_\tau + k\tau \leq a + (k - \frac{\tau}{t})\tau \mid X_t = 0) =$$

$$P(X_\tau \leq a + (k - \frac{x}{t})\tau \mid X_t = kt),$$

$$0 \leq \tau \leq t$$

Putting x for kt and b for $k - \frac{x}{t}$ gives

$$P(X_\tau \leq a + b\tau \mid X_t = x) = P(X_\tau \leq a \mid X_t = x - bt)$$

$$0 \leq \tau \leq t$$

and using the formula for the tied down Brownian motion yields:

$$P(X_\tau \leq a + b\tau \mid X_t = x) = 1 - e^{-\frac{2a(a+b\tau-x)}{t}} \text{ if } x \leq a + bt \text{ and zero otherwise}$$

$$0 \leq \tau \leq t$$

and for the ordinary Brownian bridge process $B_t = X_t - tX_1$

$$P(B_\tau \leq a + b\tau) = 1 - e^{-2a(a+b)}.$$

$$0 \leq \tau \leq 1$$

For the non tied down Brownian motion:

$$P(X_\tau \leq a + b\tau) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{a+bt} e^{-\frac{x^2}{2t}} P(X_\tau \leq a + b\tau \mid X_t = x) dx$$

$$0 \leq \tau \leq t$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{a+bt} e^{-\frac{x^2}{2t}} (1 - e^{-\frac{2a(a+b\tau-x)}{t}}) dx = I_1 - I_2.$$

Now

$$I_1 = \int_{-\infty}^{a+bt} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = \frac{1}{2\pi} \int_{-\frac{a}{\sqrt{t}}+b\sqrt{t}}^{\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{x^2}{2}} dx = \Phi(\frac{a}{\sqrt{t}} + b\sqrt{t})$$

and

$$I_2 = \int_{-\infty}^{a+bt} \frac{e^{-\frac{x^2}{2t} - \frac{2a(a+b\tau-x)}{t} + \frac{4ax}{2t}}}{\sqrt{2\pi t}} dx = \frac{e^{-2ab}}{\sqrt{2\pi t}} \int_{-\infty}^{a+bt} e^{-\frac{(x-2a)^2}{2t}} dx =$$

$$\frac{e^{-2ab}}{\sqrt{2\pi t}} \int_{-\infty}^{-a+bt} e^{-\frac{x^2}{2t}} dx = \frac{e^{-2ab}}{\sqrt{2\pi}} \int_{-\frac{a}{\sqrt{t}}+b\sqrt{t}}^{-\frac{a}{\sqrt{t}}+b\sqrt{t}} e^{-\frac{x^2}{2}} dx = e^{-2ab} \Phi(-\frac{a}{\sqrt{t}} + b\sqrt{t}).$$

Hence we have:

Theorem 3.4

$$P(X_\tau \leq a + b\tau) = \Phi(\frac{a}{\sqrt{t}} + b\sqrt{t}) - e^{-2ab} \Phi(-\frac{a}{\sqrt{t}} + b\sqrt{t}). \quad (3.5)$$

$$0 \leq \tau \leq t$$

Next consider $P(X_{\tau} \leq c)$. Clearly

$$\begin{aligned}
 P(X_{\tau} \leq c) &= \int P(X_{\tau} \leq c \mid X_a = y) dP_y(X_a \leq y \mid X_b = x) = \\
 &= \sqrt{\frac{b}{2\pi a(b-a)}} \int_{-\infty}^c (1 - e^{-\frac{2c(c-y)}{a}}) e^{-\frac{(y-\frac{ax}{b})^2}{2a(b-a)}} dy = I_1 - I_2 \\
 I_1 &= \sqrt{\frac{b}{2\pi a(b-a)}} \int_{-\infty}^c e^{-\frac{(y-\frac{ax}{b})^2}{2a(b-a)}} dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{c-\frac{ax}{b}}{\sqrt{\frac{a(b-a)}{b}}}} e^{-\frac{y^2}{2}} dy = \Phi\left(\frac{bc - ax}{\sqrt{ab(b-a)}}\right) \\
 I_2 &= \sqrt{\frac{b}{2\pi a(b-a)}} \int_{-\infty}^c e^{-\frac{2c(c-y)}{a} - \frac{(y-\frac{ax}{b})^2}{2a(b-a)}} dy .
 \end{aligned}$$

The exponent can be written as:

$$\begin{aligned}
 &-\frac{1}{2} \left\{ \frac{4c(c-y)}{a} + \frac{b(y - \frac{ax}{b})^2}{a(b-a)} \right\} = \\
 &-\frac{1}{2} \frac{b}{b-a} \left[y - \frac{ax + 2c(b-a)}{b} \right]^2 - \frac{2c^2 - 2cx}{b} .
 \end{aligned}$$

Thus

$$I_2 = \sqrt{\frac{b}{2\pi a(b-a)}} e^{-\frac{2c^2 - 2cx}{b}} \int_{-\infty}^c e^{-\frac{b}{2a(b-a)} \left[y - \frac{ax + 2c(b-a)}{b} \right]^2} dy = e^{-\frac{2c(c-x)}{b}} \Phi\left(\frac{c - \frac{ax + 2c(b-a)}{b}}{\sqrt{\frac{a(b-a)}{b}}}\right) .$$

Combining this with the result for I_1 gives:

Theorem 3.5

$$P(X_{\tau} \leq c \mid X_b = x) = \Phi\left(\frac{bc - ax}{\sqrt{ab(b-a)}}\right) - e^{-\frac{2c(c-x)}{b}} \Phi\left(\frac{2ac - bc - ax}{\sqrt{ab(b-a)}}\right)$$

where x can be any value between $-\infty$ and ∞ .

If $a \rightarrow b$, then

$$\text{if } x < c \quad \frac{bc - ax}{\sqrt{ab(b-a)}} \rightarrow \infty$$

$$\text{if } x > c \quad \frac{bc - ax}{\sqrt{ab(b-a)}} \rightarrow -\infty$$

and

$$\text{if } x < c \quad \frac{2ac - bc - ax}{\sqrt{ab(b-a)}} \rightarrow \infty$$

$$\text{if } x > c \quad \frac{2ac - bc - ax}{\sqrt{ab(b-a)}} \rightarrow -\infty.$$

Thus when $a \rightarrow b$

$$P(X_\tau \leq c \mid X_b = x) \rightarrow 1 - e^{-\frac{2a(c-x)}{b}} \text{ if } x < c \text{ and to zero otherwise.}$$

So this formula is consistent with the previous result as it should be.

Consider once more

$$P(X_\tau \leq c \mid X_b = x) = P(X_\tau + \frac{x\tau}{b} \leq c \mid X_b = 0) =$$

$$P(X_\tau \leq c - \frac{x\tau}{b} \mid X_b = 0) = P(X_\tau + k\tau \leq c + (k - \frac{x}{b})\tau \mid X_b = 0) =$$

$$P(X_\tau \leq c + (k - \frac{x}{b})\tau \mid X_b = kt).$$

From this through a simple transformation we obtain:

$$P(X_\tau \leq c + d\tau \mid X_b = x) = P(X_\tau \leq c \mid X_b = x - bd).$$

Using the previous result for $P(X_\tau \leq c \mid X_b = x)$ then yields:

Theorem 3.6

$$\begin{aligned}
 P(X_\tau \leq c + d\tau \mid X_b = x) &= \\
 &= \Phi\left(\frac{bc - ax + abd}{\sqrt{ab(b-a)}}\right) - e^{-\frac{2c(c-x+bd)}{b}} \Phi\left(\frac{2ac - bc - ax + abd}{\sqrt{ab(b-a)}}\right).
 \end{aligned}$$

Consider now the more general case:

$$P(X_\tau \leq a, \quad X_\tau \leq b, \quad X_\tau \leq c \mid X_t = x)$$

$\begin{matrix} 0 \leq \tau \leq \alpha & \alpha < \tau \leq \beta & \beta < \tau \leq t \end{matrix}$

Clearly $0 < a \leq \infty$ and $x < c$ or otherwise the probability is zero. Now:

$$\begin{aligned}
 &P(X_\tau \leq a, \quad X_\tau \leq b, \quad X_\tau \leq c \mid X_t = x) = \\
 &= \int_{-\infty}^a \int_{-\infty}^b \left\{ P(X_\alpha = \xi, X_\beta = \eta \mid X_t = x) P(X_\tau \leq a \mid X_\alpha = \xi) \right. \\
 &\quad \left. P(X_\tau \leq b - \xi \mid x_{\beta-\alpha} = \eta - \xi) P(X_\tau \leq c - \eta \mid X_{t-\beta} = x - \eta) \right\} d\eta d\xi
 \end{aligned}$$

since the conditioned density function

$$P(X_\alpha = \xi, X_\beta = \eta \mid X_t = x) = \frac{\frac{e^{-\frac{\xi^2}{2\alpha} - \frac{(\eta-\xi)^2}{2(\beta-\alpha)} - \frac{(x-\eta)^2}{2(t-\beta)}}}{\sqrt{2\pi\alpha 2\pi(\beta-\alpha) 2\pi(t-\beta)}}}{\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}},$$

we obtain

$$\begin{aligned} & \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq a, \quad X_\tau \leq b, \quad X_\tau \leq c \mid X_t = x) = \\ &= \int_{-\infty}^a \int_{-\infty}^b \left\{ \frac{e^{-\frac{\xi^2}{2\alpha} - \frac{(\eta-\xi)^2}{2(\beta-\alpha)} - \frac{(x-\eta)^2}{2(t-\beta)}}}{\sqrt{2\pi\alpha 2\pi(\beta-\alpha) 2\pi(t-\beta)}} \right. \\ & \quad \left. \left(1 - e^{-\frac{2\alpha(a-\xi)}{\alpha}}\right) \left(1 - e^{-\frac{2(b-\xi)((b-\xi)-(\eta-\xi))}{(\beta-\alpha)}}\right) \left(1 - e^{-\frac{2(c-\eta)((c-\eta)-(x-\eta))}{(t-\beta)}}\right) \right\} d\eta d\xi = \\ &= \int_{-\infty}^a d\xi \int_{-\infty}^b d\eta \left(\frac{e^{-\frac{\xi^2}{2\alpha}} - e^{-\frac{(2a-\xi)^2}{2t}}}{\sqrt{2\pi\alpha}} \right) \left(\frac{e^{-\frac{(\eta-\xi)^2}{2(\beta-\alpha)}} - e^{-\frac{(2b-\eta-\xi)^2}{2(\beta-\alpha)}}}{\sqrt{2\pi(\beta-\alpha)}} \right) \\ & \quad \cdot \left(\frac{e^{-\frac{(x-\eta)^2}{2(t-\beta)}} - e^{-\frac{(2c-x-\eta)^2}{2(t-\beta)}}}{\sqrt{2\pi(t-\beta)}} \right) \end{aligned}$$

It is also clear that:

$$\begin{aligned} & \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq a, \quad X_\tau \leq b, \quad \mid X_\beta = x) = \\ &= \int_{-\infty}^a d\xi \left(\frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} - \frac{e^{-\frac{(2a-\xi)^2}{2t}}}{\sqrt{2\pi\alpha}} \right) \left(\frac{e^{-\frac{(x-\xi)^2}{2(\beta-\alpha)}}}{\sqrt{2\pi(\beta-\alpha)}} - \frac{e^{-\frac{(2b-x-\xi)^2}{2(\beta-\alpha)}}}{\sqrt{2\pi(\beta-\alpha)}} \right) \end{aligned}$$

and

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq a, \quad \mid X_\alpha = x) = \frac{e^{-\frac{x^2}{2\alpha}}}{\sqrt{2\pi\alpha}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi\alpha}}.$$

Now let $f(t)$ $f(0) > 0$ be a piecewise continuous function made of horizontal line segments:

If k is the constant value taken by $f(t)$ on the interval $[0, t_1]$ and $0 < t \leq t_1$ then:

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq f(\tau) \mid X_t = x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2k-x)^2}{2t}}}{\sqrt{2\pi t}}.$$

This function is generated by taking the heat source function $\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}$ and reflecting it around the $x = k$ straight line and subtracting it from the original source. Let this function be called $\Phi_0(x, t)$.

Now let $t_1 < t \leq t_2$

$$\begin{aligned} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq f(\tau) \mid X_t = x) = \\ = \int_{-\infty}^{\min(k, l)} \frac{e^{-\frac{(x-\xi)^2}{2(t-t_1)}}}{\sqrt{2\pi(t-t_1)}} \Phi_0(\xi, t_1) d\xi - \int_{-\infty}^{\min(k, l)} \frac{e^{-\frac{(2l-x-\xi)^2}{2(t-t_1)}}}{\sqrt{2\pi(t-t_1)}} \Phi_0(\xi, t_1) d\xi \end{aligned}$$

where l is the constant value $f(t)$ takes between t_1 and t_2 .

Again this function was arrived at by integrating $\frac{e^{-\frac{(x-\xi)^2}{2(t-t_1)}}}{\sqrt{2\pi(t-t_1)}} \Phi_0(\xi, t_1)$ from $-\infty$ to the boundary of the domain below $f(\tau)$ reflect it around the straight line $x = l$ and subtract the reflected function from the first integral.

The same procedure is to be followed for every point of discontinuity. Let $\Phi_n(x, t)$ be $\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq f(\tau) \mid X_t = x)$ for the interval $t_n \leq t \leq t_{n+1}$.

Now if t is between the discontinuities t_{n+1} and t_{n+2} then

$$\begin{aligned} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_\tau \leq f(\tau) \mid X_t = x) &= \\ &= \int_{-\infty}^{\min(k,l)} \frac{e^{-\frac{(x-\xi)^2}{2(t-t_{n+1})}}}{\sqrt{2\pi(t-t_{n+1})}} \Phi_n(\xi, t_{n+1}) d\xi - \int_{-\infty}^{\min(k,l)} \frac{e^{-\frac{(2t-x-\xi)^2}{2(t-t_{n+1})}}}{\sqrt{2\pi(t-t_{n+1})}} \Phi_n(\xi, t_{n+1}) d\xi \end{aligned} \quad (3.6)$$

where k is the constant value taken by $f(t)$ in the interval $(t_n, t_{n+1}]$ and l is the value of $f(t)$ in the interval $(t_{n+1}, t_{n+2}]$. The proof of this is the same as in the case $f(t)$ is made of 2 or 3 line segments and it will not be repeated again.

3.4 The Petrowsky - Kolmogorov test.

Consider the following special case for $f(t)$

$$f(t) = \begin{cases} \infty & 0 \leq t \leq \alpha \\ a = \text{constant} & \alpha < t \leq t. \end{cases}$$

That is between 0 and α all possible paths are allowed, but between α and t only those that pass below $f(\tau) = a \geq 0$. We shall prove:

Theorem 3.7

$$P(X_\tau \geq a \mid \alpha \leq \tau \leq t) = \frac{1}{2\pi} \int_0^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau + \frac{1}{2\pi} \int_\alpha^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau \quad (3.7)$$

$$= \frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^{\infty} e^{-x^2} dx + \frac{1}{2\pi} \int_{\alpha}^t \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau \quad (3.8)$$

$$= \frac{1}{2} \int_0^t \frac{ae^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi\tau^3}} d\tau + \frac{1}{2\pi} \int_{\alpha}^t \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau. \quad (3.9)$$

Proof:

According to formula 3.6:

$$\begin{aligned} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(X_{\tau} \leq a \mid X_t = x) &= \\ &= \int_{-\infty}^a \frac{e^{-\frac{(x-\xi)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi - \int_{-\infty}^a \frac{e^{-\frac{(2a-x-\xi)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-t_1)}} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi \\ &= I_1 - I_2. \end{aligned}$$

$$\begin{aligned} I_1 &= \int_{-\infty}^a \frac{e^{-\frac{(x-\xi)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \int_a^{\infty} \frac{e^{-\frac{(x-\xi)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \int_0^{\infty} \frac{e^{-\frac{(\xi+a-x)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{(\xi+a)^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi. \end{aligned}$$

But the first factor in the integrand is the convolution product: $\varphi(\xi+a-x, t-\alpha) = \psi(a-x, t-\alpha) * \varphi(\xi, t-\alpha)$. So

$$I_1 = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \int_0^{\infty} d\xi \int_0^{t-\alpha} d\tau \frac{a-x}{\sqrt{2\pi\tau^3}} e^{-\frac{(a-x)^2}{2\tau}} \cdot \frac{e^{-\frac{\xi^2}{2(t-\alpha-\tau)}}}{\sqrt{2\pi(t-\alpha-\tau)}} \cdot \frac{e^{-\frac{(\xi+a)^2}{2\alpha}}}{\sqrt{2\pi\alpha}}.$$

Now

$$\begin{aligned} \int_{-\infty}^a \frac{a-x}{\sqrt{2\pi\tau^3}} e^{-\frac{(a-x)^2}{2\tau}} dx &= \int_{-a}^{\infty} \frac{a+x}{\sqrt{2\pi\tau^3}} e^{-\frac{(a+x)^2}{2\tau}} dx = \\ \int_0^{\infty} \frac{x}{\sqrt{2\pi\tau^3}} e^{-\frac{x^2}{2\tau}} dx &= \frac{1}{\sqrt{2\pi\tau}}. \end{aligned}$$

Hence

$$\begin{aligned} \int_{-\infty}^a I_1 dx &= \\ &= \int_{-\infty}^a \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx - \int_0^\infty d\xi \frac{e^{-\frac{(\xi+a)^2}{2\alpha}}}{\sqrt{2\pi\alpha}} \int_0^{t-\alpha} d\tau \frac{e^{-\frac{\xi^2}{2(t-\alpha-\tau)}}}{\sqrt{2\pi\tau 2\pi(t-\alpha-\tau)}}. \end{aligned}$$

Let us now simplify I_2

$$\begin{aligned} \int_{-\infty}^a \frac{e^{-\frac{(2a-x-\xi)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi &= \\ &= \int_{-a}^\infty \frac{e^{-\frac{(2a-x+\xi)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi = \\ &= \int_{-a}^\infty \frac{e^{-\frac{((a-x)+(\alpha+\xi))^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{\xi^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi = \int_0^\infty \frac{e^{-\frac{(a-x+\xi)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} \frac{e^{-\frac{(\xi-a)^2}{2\alpha}}}{\sqrt{2\pi\alpha}} d\xi. \end{aligned}$$

Again, using convolution product for the first factor of the integrand this becomes:

$$= \int_0^\infty d\xi \int_0^{t-\alpha} d\tau \frac{a-x}{\sqrt{2\pi\tau^3}} e^{-\frac{(a-x)^2}{2\tau}} \cdot \frac{e^{-\frac{\xi^2}{2(t-\alpha-\tau)}}}{\sqrt{2\pi(t-\alpha-\tau)}} \cdot \frac{e^{-\frac{(\xi-a)^2}{2\alpha}}}{\sqrt{2\pi\alpha}}.$$

The integration by x from $-\infty$ to a gives just as before:

$$\begin{aligned} \int_{-\infty}^a I_2 dx &= \int_0^\infty d\xi \frac{e^{-\frac{(\xi-a)^2}{2\alpha}}}{\sqrt{2\pi\alpha}} \int_0^{t-\alpha} d\tau \frac{e^{-\frac{\xi^2}{2(t-\alpha-\tau)}}}{\sqrt{2\pi\tau 2\pi(t-\alpha-\tau)}} \\ &= \int_0^{t-\alpha} d\tau \frac{e^{-\frac{a^2}{2(t-\tau)}}}{\sqrt{2\pi\tau 2\pi(t-\tau)}} - \int_0^{t-\alpha} \frac{d\tau}{\sqrt{2\pi\tau}} \int_{-\infty}^0 d\xi \frac{e^{-\frac{(\alpha-\xi)^2}{2\alpha}}}{\sqrt{2\pi\alpha}} \frac{e^{-\frac{\xi^2}{2(t-\alpha-\tau)}}}{\sqrt{2\pi(t-\alpha-\tau)}} \\ &= \int_0^{t-\alpha} d\tau \frac{e^{-\frac{a^2}{2(t-\tau)}}}{\sqrt{2\pi\tau 2\pi(t-\tau)}} - \int_0^{t-\alpha} \frac{d\tau}{\sqrt{2\pi\tau}} \int_0^\infty d\xi \frac{e^{-\frac{(\alpha+\xi)^2}{2\alpha}}}{\sqrt{2\pi\alpha}} \frac{e^{-\frac{\xi^2}{2(t-\alpha-\tau)}}}{\sqrt{2\pi(t-\alpha-\tau)}}. \end{aligned}$$

Now:

$$P(X_\tau \leq a) = \int_{-\infty}^a \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(x_\tau \leq a \mid X_t = x) dx$$

$$\begin{aligned}
&= \int_{-\infty}^a (I_1 - I_2) dx = \\
&= \int_{-\infty}^a \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx - \frac{1}{2\pi} \int_0^{t-\alpha} d\tau \frac{e^{-\frac{a^2}{2(t-\tau)}}}{\sqrt{\tau(t-\tau)}} = \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx - \frac{1}{2\pi} \int_{\alpha}^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau .
\end{aligned}$$

So finally we get:

$$\begin{aligned}
P(X_{\tau} \leq a) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx - \frac{1}{2\pi} \int_{\alpha}^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau \\
&= 1 - \frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^{\infty} e^{-x^2} dx - \frac{1}{2\pi} \int_{\alpha}^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau . \quad (3.10)
\end{aligned}$$

Therefore:

$$P(X_{\tau} \geq a) = \frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^{\infty} e^{-x^2} dx + \frac{1}{2\pi} \int_{\alpha}^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau .$$

This proves (3.8) the other two differ from it only in their first terms. Their equivalence can be proved as follows:

$$\frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^{\infty} e^{-x^2} dx = \frac{1}{2} \int_0^t \frac{ae^{-\frac{a^2}{2\tau}}}{\sqrt{2\pi\tau^3}} d\tau = \frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}} * \frac{1}{2} \quad (3.11)$$

$$\begin{aligned}
&= \frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}} * \left(\frac{1}{\sqrt{2\pi t}} * \frac{1}{\sqrt{2\pi t}} \right) \\
&= \frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t}} * \frac{1}{\sqrt{2\pi t}} = \frac{1}{2\pi} \int_0^t \frac{e^{-\frac{a^2}{2(t-\tau)}}}{\sqrt{\tau(t-\tau)}} d\tau . \quad (3.12)
\end{aligned}$$

The first of the above equalities is easily established by differentiating $\frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^{\infty} e^{-x^2} dx$ with respect to t . ■

We make two remarks with respect to this formula:

(I) When $\alpha = 0$ (3.10) should become

$$P(X_{\tau} \leq a) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx .$$

It does indeed for the second integral becomes: $\frac{1}{2\pi} \int_0^t \frac{e^{-\frac{x^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau$. Because of relations (3.11) and (3.12) this is just $\frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^{\infty} e^{-x^2} dx$. So equation (3.10) becomes:

$$P(X_{\tau} \leq a)_{\alpha \leq \tau \leq t} = 1 - \frac{2}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^{\infty} e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx . \blacksquare$$

(II) Let now $a = 0$ so $P(X_{\tau} \leq a)_{\alpha \leq \tau \leq t} = \int_{-\infty}^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx - \frac{1}{2\pi} \int_{\alpha}^t \frac{d\tau}{\sqrt{\tau(t-\tau)}}$ becomes

$$P(X_{\tau} \leq 0)_{\alpha \leq \tau \leq t} = \frac{1}{2} - \frac{1}{2\pi} \int_{\alpha}^t \frac{d\tau}{\sqrt{\tau(t-\tau)}} \quad (3.13)$$

but

$$\begin{aligned} P(X_{\tau} \leq 0)_{\alpha \leq \tau \leq t} &= P(X_{\tau} \leq 0)_{\alpha \leq \tau \leq t} \\ &= P((\text{Last zero} \leq t) \leq \alpha, X_t < 0) = \\ &= \frac{1}{2} P((\text{Last zero} \leq t) \leq \alpha) \end{aligned}$$

therefore the distribution of the Last zero can be computed through 3.13.

$$\begin{aligned} P(X_{\tau} \leq a)_{\alpha \leq \tau \leq t} &= \frac{1}{2} - \frac{1}{2\pi} \int_{\alpha}^t \frac{d\tau}{\sqrt{\tau(t-\tau)}} = \\ &= \frac{1}{2} - \frac{1}{2\pi} \int_{\frac{\alpha}{t}}^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}} . \end{aligned}$$

But $\int_0^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}} = B(\frac{1}{2}, \frac{1}{2}) = \pi$ where B stands for the beta function. So

$$\begin{aligned} \frac{1}{2} - \frac{1}{2\pi} \int_{\frac{\alpha}{t}}^1 \frac{d\tau}{\sqrt{\tau(1-\tau)}} &= \frac{1}{2} - \frac{1}{2\pi} \left(\pi - \int_0^{\frac{\alpha}{t}} \frac{d\tau}{\sqrt{\tau(1-\tau)}} \right) = \\ &= \frac{1}{2\pi} \int_0^{\frac{\alpha}{t}} \frac{d\tau}{\sqrt{\tau(1-\tau)}} = \\ \text{putting } \tau = \sin^2 \varphi &= \frac{1}{2\pi} \int_0^{\arcsin \sqrt{\frac{\alpha}{t}}} \frac{2 \sin \varphi \cos \varphi}{\sin \varphi \cos \varphi} d\varphi = \\ &= \frac{1}{\pi} \arcsin \sqrt{\frac{\alpha}{t}} . \end{aligned}$$

So

$$\begin{aligned}
 P(X_\tau < 0) &= \\
 &= \frac{1}{\pi} \arcsin \sqrt{\frac{\alpha}{t}} = \frac{1}{2} P((\text{Last zero} \leq t) \leq \alpha, X_t < 0)
 \end{aligned}$$

and this is another proof of the second arcsine law of the Brownian motion.

Next using the theorem just proved we shall give a new proof of the Kolmogorov - Petrowsky law of upper class functions. [Petrowsky] and [Ito - McKean] page 33.

Theorem 3.8 *Let $f(t)$ be a positive continuous function on the interval $[0, \infty)$, such that $\lim_{t \rightarrow \infty} \frac{f(t)}{\sqrt{t}} = \infty$. If $\int_0^\infty \frac{f(\tau)}{\tau^{\frac{3}{2}}} e^{-\frac{f(\tau)^2}{2\tau}} d\tau < \infty$, then $f(\tau)$ is an upper class function, that is, for almost all ω there exists a value $T = T(\omega)$ such that if $t > T(\omega)$ then $X_t(\omega) < f(t)$.*

Proof:

Consider an arbitrary function $f(t)$, let it be positive and continuous on the interval $[0, \infty)$.

Let the interval $[0, \infty)$ be subdivided into intervals of equal length say h $0 = t_0 < t_1 < t_2 \dots$

and let $m = \inf_{t_{n-1} < t < t_n} f(t)$ $M = \sup_{t_{n-1} < t < t_n} f(t)$. Clearly $f(t)$ is upper class if and only if that for almost all ω there are only finite many intervals such that $\sup_{t \in (t_{n-1}, t_n)} X_t > f(t)$. Now $\lim_{h \rightarrow 0} M(t) = \lim m(t) = f(t)$ so $f(t)$ is upper class

if $M(t)$ is upper class for sufficiently small h . By the Borel - Cantelli lemma $M(t)$ is upper class if

$$\sum_1^\infty P\left(\sup_{t \in (t_{n-1}, t_n)} X_t > M(t)\right) < \infty.$$

We have seen (3.8) that

$$P(X_\tau \geq a) = \int_{-\infty}^{\frac{a}{\sqrt{2t}}} e^{-x^2} dx + \frac{1}{2\pi} \int_a^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau.$$

But :

$$\frac{1}{2\pi} \int_0^t \frac{e^{-\frac{a^2}{2\tau}}}{\sqrt{\tau(t-\tau)}} d\tau = \frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^\infty e^{-\xi^2} d\xi.$$

Hence:

$$\frac{1}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^\infty e^{-\xi^2} d\xi \leq P(X_\tau \geq a) \leq \frac{2}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^\infty e^{-\xi^2} d\xi$$

Let M_n be the value of $M(t)$ between t_{n-1} and t_n then $\sum_1^\infty P(\sup_{t \in (t_{n-1}, t_n)} X_t > M(t)) < \infty$ if and only if

$$\sum_{n=1}^\infty \int_{\frac{M_n}{\sqrt{2t_n}}}^\infty e^{-\xi^2} d\xi < \infty.$$

The finiteness of the last sum implies that $\lim_{n \rightarrow \infty} \frac{M_n}{\sqrt{2t_n}} = \infty$, for if there were infinite many of the terms below a common finite bound the the sum could not be finite. But:

$$\int_{\frac{M_n}{\sqrt{2t_n}}}^\infty e^{-\xi^2} d\xi = \int_0^{t_n} \frac{M_n}{2\sqrt{2\tau^3}} e^{-\frac{M_n^2}{2\tau}} d\tau.$$

So it means that:

$$\sum_n \int_0^{t_n} \frac{M_n}{2\sqrt{2\tau^3}} e^{-\frac{M_n^2}{2\tau}} d\tau < \infty.$$

Next we shall prove that this happens if and only if:

$$\sum_n \int_{t_{n-1}}^{t_n} \frac{M_n}{2\sqrt{2\tau^3}} e^{-\frac{M_n^2}{2\tau}} d\tau < \infty.$$

That is:

$$\int_0^\infty \frac{M_\tau}{2\sqrt{2\tau^3}} e^{-\frac{M_\tau^2}{2\tau}} d\tau < \infty.$$

To this end consider the quotient :

$$\frac{\int_{t_{n-1}}^{t_n} \frac{M_n}{2\sqrt{2\tau^3}} e^{-\frac{M_n^2}{2\tau}} d\tau}{\int_0^{t_n} \frac{M_n}{2\sqrt{2\tau^3}} e^{-\frac{M_n^2}{2\tau}} d\tau} = \frac{\int_{\frac{M_n}{\sqrt{2t_n}}}^{\infty} e^{-\xi^2} d\xi - \int_{\frac{M_n}{\sqrt{2t_{n-1}}}}^{\infty} e^{-\xi^2} d\xi}{\int_{\frac{M_n}{\sqrt{2t_n}}}^{\infty} e^{-\xi^2} d\xi}.$$

Since $\frac{M_n}{\sqrt{2t_n}} \rightarrow \infty$ as $n \rightarrow \infty$. Now $\int_x^{\infty} e^{-\xi^2} d\xi \sim \frac{1}{2x} e^{-x^2}$. Therefore taking the asymptotic values of the integrals in the last expression we get:

$$\begin{aligned} & \frac{\frac{1}{2} \frac{\sqrt{2t_n}}{M_n} e^{-\frac{M_n^2}{2t_n}} - \frac{1}{2} \frac{\sqrt{2t_{n-1}}}{M_n} e^{-\frac{M_n^2}{2t_{n-1}}}}{\frac{1}{2} \frac{\sqrt{2t_n}}{M_n} e^{-\frac{M_n^2}{2t_n}}} = \\ &= 1 - \sqrt{\frac{t_{n-1}}{t_n}} e^{-\frac{M_n^2}{2} \left(\frac{1}{t_{n-1}} - \frac{1}{t_n} \right)} \quad \text{putting } t_{n-1} = t_n - h \\ &= 1 - \sqrt{\frac{t_{n-1}}{t_n}} e^{-\frac{M_n^2 h}{2t_n t_{n-1}}} \end{aligned}$$

this converges to 1 as $n \rightarrow \infty$. Thus it has been proved that

$$\sum_n P\left(\sup_{\tau \in (t_{n-1}, t_n)} X_\tau > M(\tau)\right) < \infty \quad \text{if and only if}$$

$$\int_0^{\infty} \frac{M_\tau}{\sqrt{\tau^3}} e^{-\frac{M_\tau^2}{2\tau}} d\tau < \infty \quad \text{or} \quad \int_0^{\infty} \frac{M_\tau}{2\tau^{\frac{3}{2}}} e^{-\frac{M_\tau^2}{2\tau}} d\tau < \infty$$

where the lower limit of the integral can be any finite value. This means as well that:

$$\sum_n P\left(\sup_{\tau \in (t_{n-1}, t_n)} X_\tau > f(\tau)\right) < \infty \quad \text{if and only if} \quad \int_0^{\infty} \frac{f(\tau)}{\tau^{\frac{3}{2}}} e^{-\frac{f(\tau)^2}{2\tau}} d\tau < \infty.$$

So by the Borel - Cantelli lemma it follows that if $\int_0^{\infty} \frac{f(\tau)}{\tau^{\frac{3}{2}}} e^{-\frac{f(\tau)^2}{2\tau}} d\tau < \infty$ $f(\tau)$ is an upper class function, that is for almost all ω there exists a value $T = T(\omega)$ such that if $t > T$ $X_t < f(t)$. ■

3.5 Occupation time distributions for the case

$$f(t) = a .$$

In this section we derive formulas for the distribution of the $[a, \infty)$ occupation time of the unrestricted Brownian motion. These are given in the next:

Theorem 3.9

$$\begin{aligned} P(O_a(t) \leq \xi, X_t \geq a) &= \\ &= \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{2}{\pi} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\kappa^2} d\kappa \int_0^{\kappa \sqrt{\frac{t-\xi}{t}}} e^{-\rho^2} d\rho - \frac{\sqrt{\xi(t-\xi)}}{\pi t} e^{-\frac{a^2}{2(t-\xi)}} \quad (3.14) \\ &+ \frac{2a\xi e^{-\frac{a^2}{2t}}}{\pi \sqrt{2t^3}} \int_a^{\infty} \frac{e^{-\rho^2} d\rho}{\sqrt{2t(t-\xi)}} \end{aligned}$$

$$\begin{aligned} P(O_a(t) \leq \xi, X_t \leq a) &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi \\ &+ \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{2}{\pi} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\kappa^2} d\kappa \int_0^{\kappa \sqrt{\frac{t-\xi}{t}}} e^{-\rho^2} d\rho + \frac{\sqrt{\xi(t-\xi)}}{\pi t} e^{-\frac{a^2}{2(t-\xi)}} \quad (3.15) \\ &- \frac{2a\xi e^{-\frac{a^2}{2t}}}{\pi \sqrt{2t^3}} \int_a^{\infty} \frac{e^{-\rho^2} d\rho}{\sqrt{2t(t-\xi)}} \end{aligned}$$

and adding up these gives:

$$\begin{aligned} P(O_a(t) \leq \xi) &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi \quad (3.16) \\ &+ \frac{2}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{4}{\pi} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\kappa^2} d\kappa \int_0^{\kappa \sqrt{\frac{t-\xi}{t}}} e^{-\rho^2} d\rho. \end{aligned}$$

Proof:

We have seen that (3.2), (3.1)

$$\begin{aligned} Q_a(x, t) &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E \left(e^{-uO_a(t)} \mid X_t = x \right) \\ &= \psi(a, t) * \left(e^{-ut} (\psi(x-a) * \frac{e^{ut}-1}{u\sqrt{2\pi t^3}}) \right) \quad \text{when } x \geq a \end{aligned}$$

and

$$\begin{aligned} Q_a(x, t) &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \left(e^{-uO_a(t)} \mid X_t = x \right) \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \psi(a) * \psi(a-x) * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \quad \text{when } x \leq a. \end{aligned}$$

Now

$$\begin{aligned} E \left(e^{-uO_a(t)}, x_t \leq a \right) &= \int_{-\infty}^a dx \left(\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} \right) + \int_{-\infty}^a dx \psi(a) * \psi(a-x) * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} = \\ &= I_1 - I_2. \end{aligned}$$

We have seen that $I_1 = \frac{2}{\pi} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi$. In order to evaluate I_2 we notice that

$$\int_{-\infty}^a \psi(a-x) dx = \int_0^{\infty} \psi(x) dx = \frac{1}{\sqrt{2\pi t}}.$$

Thus

$$\begin{aligned} I_1 + I_2 &= E \left(e^{uO_a(t)}, X_t \leq x \right) = \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \psi(a) * \frac{1}{\sqrt{2\pi t}} * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \\ &= P(X_{\tau} \leq a)_{0 \leq \tau \leq t} + \psi(a) * \frac{1}{\sqrt{2\pi t}} * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}}. \end{aligned}$$

But

$$\psi(a) * \frac{1}{\sqrt{2\pi t}} = \varphi(a, t) = \frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t}}.$$

Hence

$$\psi(a) * \frac{1}{\sqrt{2\pi t}} * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} = \frac{1}{2\pi} \int_0^t \frac{e^{-\frac{a^2}{2(t-\tau)}}}{(t-\tau)^{\frac{1}{2}}} \cdot \frac{1-e^{-u\tau}}{u\tau^{\frac{3}{2}}} d\tau =$$

Now through integrating by parts we have:

$$= -\frac{1}{2\pi} \int_{\tau}^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}} \cdot \frac{1-e^{-u\tau}}{u} \Bigg|_{\tau=0}^{\tau=t} + \frac{1}{2\pi} \int_0^t d\tau e^{-u\tau} \int_{\tau}^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}}$$

the first term clearly vanishes at both endpoints and the second term can be written as the Stieltjes integral:

$$= \frac{1}{2\pi} \int_0^t e^{-u\tau} d\tau \int_0^\tau d\sigma \int_\sigma^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}}.$$

So

$$\begin{aligned} E(e^{-uO_a(t)}, X_t \leq a) &= \\ &= P(x_\tau \leq a)_{0 \leq \tau \leq t} + \frac{1}{2\pi} \int_0^t e^{-u\xi} d\xi \left(\int_0^\xi d\sigma \int_\sigma^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}} \right). \end{aligned}$$

But we have that $P(X_\tau \leq a)_{0 \leq \tau \leq t} = P(O_a(t) = 0)$ so

$$\begin{aligned} P(O_a(t) \leq \xi, X_t \leq a) &= \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \frac{1}{2\pi} \int_0^\xi d\sigma \int_\sigma^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}}. \end{aligned}$$

The second integral can be simplified through integrating by parts

$$\begin{aligned} &\frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \frac{1}{2\pi} \int_0^\xi d\sigma \int_\sigma^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}} = \\ &= \frac{\sigma}{2\pi} \int_\sigma^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}} \Bigg|_{\sigma=0}^{\sigma=\xi} + \frac{1}{2\pi} \int_0^\xi d\sigma \frac{e^{-\frac{a^2}{2(t-\sigma)}}}{(t-\sigma)^{\frac{1}{2}} \sigma^{\frac{1}{2}}} = \\ &\text{the first term vanishes at } \sigma = 0 \\ &= \frac{\xi}{2\pi} \int_\xi^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}} + \frac{1}{2\pi} \int_0^\xi d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{1}{2}}}. \end{aligned}$$

Hence:

$$\begin{aligned} P(O_a(t) \leq \xi, X_t \leq a) &= \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \frac{\xi}{2\pi} \int_\xi^t d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{3}{2}}} + \frac{1}{2\pi} \int_0^\xi d\rho \frac{e^{-\frac{a^2}{2(t-\rho)}}}{(t-\rho)^{\frac{1}{2}} \rho^{\frac{1}{2}}}. \end{aligned} \tag{3.17}$$

Next we shall proceed to evaluate $P(O_a(t) \leq \xi)$. To this end consider:

$$\int_a^\infty Q_a(x, t) dx = \int_a^\infty dx \psi(a) * (e^{-ut} (\psi(x-a) * \frac{e^{ut}-1}{u\sqrt{2\pi t^3}})).$$

Clearly after changing the order of integration and substituting $\int_a^\infty dx \psi(x-a) = \int_0^\infty dx \psi(x) = \frac{1}{\sqrt{2\pi t}}$ gives:

$$\int_a^\infty Q_a(x, t) dx = \psi(a) * (e^{-ut} (\frac{1}{\sqrt{2\pi t}} * \frac{e^{ut}-1}{u\sqrt{2\pi t^3}})).$$

Adding this to the already computed $\int_{-\infty}^a Q_a(x, t) dx$ gives:

$$\begin{aligned} \int_{-\infty}^\infty Q_a(x, t) dx &= E(e^{-uO_a(t)}) = \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \psi(a) * \frac{1}{\sqrt{2\pi t}} * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \\ &+ \psi(a) * (e^{-ut} (\frac{1}{\sqrt{2\pi t}} * \frac{e^{ut}-1}{u\sqrt{2\pi t^3}})) \end{aligned}$$

or

$$\begin{aligned} E(e^{-uO_a(t)}) &= \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \psi(a) * \left(\frac{1}{\sqrt{2\pi t}} * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} + e^{-ut} (\frac{1}{\sqrt{2\pi t}} * \frac{e^{ut}-1}{u\sqrt{2\pi t^3}}) \right) \end{aligned}$$

but the expression in brackets following $\psi(a)$ is just $E(e^{-uO_0(t)})$ (see equations 2.13 and 2.14) and this has been shown to be:

$$\frac{2}{\pi} \int_0^t e^{-u\xi} d(\arcsin \sqrt{\frac{\xi}{t}}) = \frac{1}{\pi} \int_0^t \frac{e^{-u\xi}}{\xi^{\frac{1}{2}}(t-\xi)^{\frac{1}{2}}} d\xi = 2 \cdot \frac{e^{-ut}}{\sqrt{2\pi t}} * \frac{1}{\sqrt{2\pi t}}.$$

But $\psi(a) * \frac{1}{\sqrt{2\pi t}} = \varphi(a) = \frac{e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t}}$ Hence the triple convolution in the last expression becomes:

$$\frac{2}{2\pi} \int_0^t d\xi \frac{e^{-u\xi}}{\xi^{\frac{1}{2}}} \frac{e^{-\frac{a^2}{2(t-\xi)}}}{(t-\xi)^{\frac{1}{2}}} = \frac{1}{\pi} \int_0^t e^{-u\xi} d\xi \left(\int_0^\xi \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho \right).$$

Thus

$$\begin{aligned} E(e^{-uO_a(t)}) &= \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \frac{1}{\pi} \int_0^t e^{-u\xi} d\xi \left(\int_0^\xi \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho \right) \end{aligned}$$

and now we infer from here that:

$$\begin{aligned} P(O_a(t) \leq \xi) &= \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \frac{1}{\pi} \int_0^\xi \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho. \end{aligned} \quad (3.18)$$

The second integral in the last equation when $\xi = t$ becomes

$$\frac{1}{\pi} \int_0^t \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho = \frac{2}{\sqrt{\pi}} \int_{\frac{a}{\sqrt{2t}}}^\infty e^{-\xi^2} d\xi$$

giving $P(O_a(t) \leq t) = 1$ as it should be.

Now

$$P(O_a(t) \leq \xi, X_t \geq a) = P(O_a(t) \leq \xi) - P(O_a(t) \leq \xi, X_t \leq a).$$

Substituting (3.17) and (3.18) this becomes

$$\begin{aligned} P(O_a(t) \leq \xi, X_t \geq a) &= \\ &= \frac{1}{2\pi} \int_0^\xi \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho - \frac{\xi}{2\pi} \int_\xi^t \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{3}{2}}(t-\rho)^{\frac{1}{2}}} d\rho. \end{aligned} \quad (3.19)$$

The formulas (3.17), (3.18) and (3.19) will now be transformed so that when $a = 0$ they go into formulas (2.15), (2.16) and (2.17).

$$\begin{aligned} \frac{1}{2\pi} \int_0^\xi \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho &= \frac{1}{2\pi} \int_{t-\xi}^t \frac{e^{-\frac{a^2}{2\rho}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho = \\ &= \frac{1}{2\pi} \int_{\frac{t-\xi}{t}}^1 \frac{e^{-\frac{a^2}{2t\rho}}}{\rho^{\frac{1}{2}}(1-\rho)^{\frac{1}{2}}} d\rho \text{ putting } \rho = \frac{1}{t} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \int_1^{\frac{t}{1-\xi}} \frac{e^{-\frac{a^2}{2t}\rho}}{(\rho-1)^{\frac{1}{2}}\rho} d\rho = \frac{1}{2\pi} \int_0^{\frac{\xi}{1-\xi}} \frac{e^{-\frac{a^2}{2t}(\rho+1)}}{\rho^{\frac{1}{2}}(\rho+1)} d\rho \\
&= \frac{1}{\pi} \int_0^{\sqrt{\frac{\xi}{1-\xi}}} \frac{e^{-\frac{a^2}{2t}(\rho^2+1)}}{(\rho^2+1)} d\rho \\
&\quad \text{substituting } \frac{e^{-\frac{a^2}{2t}(\rho^2+1)}}{(\rho^2+1)} = \frac{1}{\rho^2+1} - \int_0^{\frac{a^2}{2t}} e^{-\kappa(\rho^2+1)} d\kappa \text{ gives} \\
&= \frac{1}{\pi} \int_0^{\sqrt{\frac{\xi}{1-\xi}}} \frac{d\rho}{\rho^2+1} - \frac{1}{\pi} \int_0^{\sqrt{\frac{\xi}{1-\xi}}} d\rho \int_0^{\frac{a^2}{2t}} e^{-\kappa(\rho^2+1)} d\kappa = \\
&= \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{1}{\pi} \int_0^{\frac{a^2}{2t}} e^{-\kappa} d\kappa \int_0^{\sqrt{\frac{\xi}{1-\xi}}} d\rho e^{-\kappa\rho^2} = \\
&= \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{1}{\pi} \int_0^{\frac{a^2}{2t}} \frac{e^{-\kappa}}{\sqrt{\kappa}} d\kappa \int_0^{\sqrt{\kappa\frac{\xi}{1-\xi}}} e^{-\rho^2} d\rho = \\
&= \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{2}{\pi} \int_0^{\frac{a^2}{2t}} e^{-\kappa^2} d\kappa \int_0^{\sqrt{\kappa\frac{\xi}{1-\xi}}} e^{-\rho^2} d\rho.
\end{aligned}$$

Thus

$$\frac{1}{2\pi} \int_0^{\xi} \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{1}{2}}} d\rho = \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{2}{\pi} \int_0^{\frac{a^2}{2t}} e^{-\kappa^2} d\kappa \int_0^{\sqrt{\kappa\frac{\xi}{1-\xi}}} e^{-\rho^2} d\rho.$$

The second integral in equation (3.19) becomes:

$$\begin{aligned}
&\frac{\xi}{2\pi} \int_{\xi}^t \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{3}{2}}(t-\rho)^{\frac{1}{2}}} d\rho = \frac{\xi}{2\pi} \int_0^{t-\xi} \frac{e^{-\frac{a^2}{2\rho}}}{\rho^{\frac{1}{2}}(t-\rho)^{\frac{3}{2}}} d\rho = \\
&= \frac{\xi}{2\pi t} \int_0^{\frac{t-\xi}{t}} \frac{e^{-\frac{a^2}{2t\rho}}}{\rho^{\frac{1}{2}}(1-\rho)^{\frac{3}{2}}} d\rho = \frac{\xi}{2\pi t} \int_{\frac{t}{1-\xi}}^{\infty} \frac{e^{-\frac{a^2}{2t}\rho}}{(\rho-1)^{\frac{3}{2}}} d\rho = \\
&= \frac{\xi e^{-\frac{a^2}{2t}}}{2\pi t} \int_{\frac{\xi}{1-\xi}}^{\infty} \frac{e^{-\frac{a^2}{2t}\rho}}{\rho^{\frac{3}{2}}} d\rho \quad \text{putting } \rho = \bar{\rho}^2 \\
&= \frac{\xi e^{-\frac{a^2}{2t}}}{\pi t} \int_{\sqrt{\frac{\xi}{1-\xi}}}^{\infty} \frac{e^{-\frac{a^2}{2t}\rho^2}}{\rho^2} d\rho = \\
&= \frac{a\xi e^{-\frac{a^2}{2t}}}{\pi\sqrt{2t^3}} \int_{a\sqrt{\frac{\xi}{2t(1-\xi)}}}^{\infty} \frac{e^{-\rho^2}}{\rho^2} d\rho
\end{aligned}$$

using

$$\int \frac{e^{-\rho^2}}{\rho^2} = -\frac{e^{-\rho^2}}{\rho} - 2 \int e^{-\rho^2} d\rho \text{ gives}$$

$$\frac{\xi}{2\pi} \int_{\xi}^t \frac{e^{-\frac{a^2}{2(t-\rho)}}}{\rho^{\frac{3}{2}}(t-\rho)^{\frac{1}{2}}} d\rho = \frac{\sqrt{\xi(t-\xi)}}{\pi t} e^{-\frac{a^2}{2(t-\xi)}} - \frac{2a\xi e^{-\frac{a^2}{2t}}}{\pi\sqrt{2t^3}} \int_a^{\infty} \frac{e^{-\rho^2}}{\sqrt{2t(t-\xi)}} d\rho.$$

Hence

$$\begin{aligned} P(O_a(t) \leq \xi, X_t \geq a) &= \\ &= \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{2}{\pi} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\kappa^2} d\kappa \int_0^{\sqrt{\frac{\xi}{t-\xi}}} e^{-\rho^2} d\rho - \frac{\sqrt{\xi(t-\xi)}}{\pi t} e^{-\frac{a^2}{2(t-\xi)}} + \\ &+ \frac{2a\xi e^{-\frac{a^2}{2t}}}{\pi\sqrt{2t^3}} \int_a^{\infty} \frac{e^{-\rho^2}}{\sqrt{2t(t-\xi)}} d\rho \\ P(O_a(t) \leq \xi, X_t \leq a) &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \\ &+ \frac{1}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{2}{\pi} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\kappa^2} d\kappa \int_0^{\sqrt{\frac{\xi}{t-\xi}}} e^{-\rho^2} d\rho + \frac{\sqrt{\xi(t-\xi)}}{\pi t} e^{-\frac{a^2}{2(t-\xi)}} - \\ &- \frac{2a\xi e^{-\frac{a^2}{2t}}}{\pi\sqrt{2t^3}} \int_a^{\infty} \frac{e^{-\rho^2}}{\sqrt{2t(t-\xi)}} d\rho \end{aligned}$$

and adding up these gives:

$$\begin{aligned} P(O_a(t) \leq \xi) &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\xi^2} d\xi + \\ &+ \frac{2}{\pi} \arcsin \sqrt{\frac{\xi}{t}} - \frac{4}{\pi} \int_0^{\frac{a}{\sqrt{2t}}} e^{-\kappa^2} d\kappa \int_0^{\sqrt{\frac{\xi}{t-\xi}}} e^{-\rho^2} d\rho. \quad \blacksquare \end{aligned}$$

3.6 The distribution of $O_a(t)$ for the tied down Brownian motion with $X_t \leq a$.

Consider again the formula:

$$\begin{aligned} Q_a(x, t) &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} (e^{-uO_a(t)} | X_t = x) = \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \psi(a) * \psi(a-x) * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \quad \text{when } x \leq a. \end{aligned}$$

Compare this formula with that of the lower function for the case $a = 0$

$$Q(x, t) = \psi(|x|) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}}.$$

If $|x|$ is replaced by $2a - x$ we get the second term in the expression for $Q_a(x, t)$. Thus $P(O_a(t) \leq \xi | X_t = x)$ is the sum of $P(X_r \leq a)_{0 \leq r \leq t}$ arising from the first term and the inversion of the second term. But that inversion has already been done in section 2.5. In the formula (2.23) we just have to write $2a - x$ instead of $|x|$ and again we shall have to multiply by $e^{-\frac{x^2}{2t}}\sqrt{2\pi t}$! Then we get:

$$\begin{aligned} P(O_a(t) \leq \xi | x_t = x) &= 1 - e^{-\frac{2a(a-x)}{t}} + \\ &+ e^{-\frac{2a(a-x)}{t}} \left[\frac{\xi}{t} + (2a - x) \sqrt{\frac{\xi(t - \xi)}{\pi t^3}} e^{-\frac{(2a-x)^2}{2t(t-\xi)}} + (1 - \frac{\xi}{t}) \operatorname{erf}((2a - x) \sqrt{\frac{\xi}{2t(t - \xi)}}) \right. \\ &\left. - \frac{(2a - x)^2}{t^2} \xi (1 - \operatorname{erf}((2a - x) \sqrt{\frac{\xi}{2t(t - \xi)}})) \right]. \end{aligned}$$

Putting $x = a$ gives the special case:

$$P(O_a(t) \leq \xi | X_t = a) = P(O_0(t) \leq \xi | X_t = -a).$$

The case $X_t = x > a$ cannot be done as simply as the case $X_t \leq a$, as we shall see in the next section. It will require much more effort.

3.7 The distribution of $O_a(t)$ for the tied down Brownian motion with $X_t \geq a$.

We have seen (3.1) that when $x \geq a$

$$\begin{aligned} Q_a(x, t) &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E \left(e^{-uQ_a(t)} | X_t = x \right) \\ &= \psi(a) * \left(e^{-ut} \left(\psi(x - a) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right) \right) \\ &= \psi(a) * e^{-ut} \psi(x - a) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}}. \end{aligned}$$

To invert this, first we shall prove the following identity.

Lemma 3.1 Let $F(t)$ and $G(t)$ two differentiable functions such that $F(0) = G(0) = 0$, then

$$\begin{aligned} \left[e^{-ut} \left(G * \frac{1}{\sqrt{2\pi t}} \right)' \right] * F - e^{-ut} G * \left(F * \frac{1}{\sqrt{2\pi t}} \right)' \\ = F * e^{-ut} G * \frac{1 - e^{-ut}}{2\sqrt{2\pi t^3}}. \end{aligned} \quad (3.20)$$

Proof:

If A and B are two functions such that $A(0) = B(0) = 0$ then clearly

$$\frac{d}{dt} A * B = A' * B = A * B'.$$

Now:

$$\begin{aligned} e^{-ut} \left(G * \frac{1}{\sqrt{2\pi t}} \right)' &= \left(e^{-ut} \left(G * \frac{1}{\sqrt{2\pi t}} \right) \right)' + ue^{-ut} \left(G * \frac{1}{\sqrt{2\pi t}} \right) \\ &= \left(e^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} \right)' + ue^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}}. \end{aligned}$$

Substituting this into the lefthand side of (3.20) gives

$$\begin{aligned} &ue^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} * F + \left(e^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} \right)' * F - e^{-ut} G * \left(\frac{1}{\sqrt{2\pi t}} * F \right)' \\ &= ue^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} * F \\ &+ \frac{d}{dt} \left\{ e^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} * F - e^{-ut} G * \frac{1}{\sqrt{2\pi t}} * F \right\} \\ &= ue^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} * F + \frac{d}{dt} \left\{ F * e^{-ut} G * \frac{e^{-ut} - 1}{\sqrt{2\pi t}} \right\} \\ &= ue^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} * F + F * e^{-ut} G * \frac{d}{dt} \frac{e^{-ut} - 1}{\sqrt{2\pi t}} \\ &= ue^{-ut} G * \frac{e^{-ut}}{\sqrt{2\pi t}} * F - F * e^{-ut} G * \frac{ue^{-ut}}{\sqrt{2\pi t}} - F * e^{-ut} G * \frac{e^{-ut} - 1}{2\sqrt{2\pi t^3}} \\ &= F * e^{-ut} G * \frac{1 - e^{-ut}}{2\sqrt{2\pi t^3}} \end{aligned}$$

and this is the end of the proof of identity (3.20). ■

Applying this to our problem when neither a nor $x - a$ are zero, we get:

$$\psi(a) * e^{-ut} \psi(x-a) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} =$$

(Putting $\psi(a)$ for F and $\psi(x-a)$ for G in 3.20)

$$\begin{aligned} &= \left(\left[e^{-ut} \left(\psi(x-a) * \frac{1}{\sqrt{2\pi t}} \right) \right] * \psi(a) - e^{-ut} \psi(x-a) * \left(\psi(a) * \frac{1}{\sqrt{2\pi t}} \right) \right) \frac{2}{u} \\ &= \left[e^{-ut} \varphi'(x-a) * \psi(a) - e^{-ut} \psi(x-a) \varphi'(a) \right] \frac{2}{u}. \end{aligned}$$

Writing this now in integral form instead of the covolution product:

$$\frac{2}{u} \int_0^t d\xi e^{-u\xi} \varphi'(x-a, \xi) \psi(a, t-\xi) - \frac{2}{u} \int_0^t d\xi e^{-u\xi} \psi(x-a, \xi) \varphi'(a, t-\xi). \quad (3.21)$$

Let $x-a = A$ and $a = B$ so

$$\varphi(A) = \frac{e^{-\frac{A^2}{2t}}}{\sqrt{2\pi t}}, \quad \varphi'(A) = \frac{A^2}{2} \frac{e^{-\frac{A^2}{2t}}}{\sqrt{2\pi t^3}} - \frac{e^{-\frac{A^2}{2t}}}{2\sqrt{2\pi t^3}}.$$

Substituting these into (3.21) yields:

$$\begin{aligned} &= \frac{1}{u} \int_0^t d\xi e^{-u\xi} \frac{e^{-\frac{A^2}{2\xi}}}{\sqrt{2\pi}} \left(\frac{A^2}{\xi^{5/2}} - \frac{1}{\xi^{3/2}} \right) \frac{e^{-\frac{B^2}{2(t-\xi)}}}{\sqrt{2\pi}} \frac{B}{(t-\xi)^{3/2}} \\ &- \frac{1}{u} \int_0^t d\xi e^{-u\xi} \frac{e^{-\frac{A^2}{2\xi}}}{\sqrt{2\pi}} \frac{A}{\xi^{3/2}} \frac{e^{-\frac{B^2}{2(t-\xi)}}}{\sqrt{2\pi}} \left(\frac{B^2}{(t-\xi)^{5/2}} - \frac{1}{(t-\xi)^{3/2}} \right) \end{aligned}$$

from this we get through integration by parts:

$$\begin{aligned} &\int_0^t d\xi e^{-u\xi} \int_0^\xi \frac{e^{-\frac{A^2}{2\xi}}}{\sqrt{2\pi}} \left(\frac{A^2}{\rho^{5/2}} - \frac{1}{\rho^{3/2}} \right) e^{-\frac{B^2}{2(t-\rho)}} \frac{B}{\sqrt{2\pi}} \frac{1}{(t-\rho)^{3/2}} d\rho \\ &- \int_0^t d\xi e^{-u\xi} \int_0^\xi \frac{e^{-\frac{A^2}{2\xi}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \left(\frac{B^2}{(t-\rho)^{5/2}} - \frac{1}{(t-\rho)^{3/2}} \right) d\rho. \end{aligned}$$

So

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(O_a(t) \leq \xi | X_t = x, x > a) =$$

$$\begin{aligned} & \int_0^\xi d\eta \int_0^\eta d\eta \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \left(\frac{A^2}{\rho^{5/2}} - \frac{1}{\rho^{3/2}} \right) \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B}{(t-\rho)^{3/2}} \\ & - \int_0^\xi d\eta \int_0^\eta d\rho \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \left(\frac{B^2}{(t-\rho)^{5/2}} - \frac{1}{(t-\rho)^{3/2}} \right). \end{aligned}$$

Integrating by parts again this becomes:

$$\begin{aligned} & \xi \int_0^\xi d\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \left(\frac{A^2}{\rho^{5/2}} - \frac{1}{\rho^{3/2}} \right) \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B}{(t-\rho)^{3/2}} \\ & - \int_0^\xi d\rho \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \left(\frac{A^2}{\rho^{3/2}} - \frac{1}{\rho^{1/2}} \right) \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B}{(t-\rho)^{3/2}} \\ & + (t-\xi) \int_0^\xi d\rho \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \left(\frac{B^2}{(t-\rho)^{5/2}} - \frac{1}{(t-\rho)^{3/2}} \right) \\ & - \int_0^\xi d\rho \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \left(\frac{B^2}{(t-\rho)^{3/2}} - \frac{1}{(t-\rho)^{1/2}} \right). \end{aligned}$$

These can be broken up into the sum of eight integrals:

$$I - J - K + L + M - N - P + Q$$

Where:

$$\begin{aligned} I &= \xi \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A^2}{\rho^{5/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B}{(t-\rho)^{3/2}} d\rho \\ J &= \xi \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{1}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B}{(t-\rho)^{3/2}} d\rho \\ K &= \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A^2}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B}{(t-\rho)^{3/2}} d\rho \\ L &= \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{1}{\rho^{1/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B}{(t-\rho)^{3/2}} d\rho \\ M &= (t-\xi) \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B^2}{(t-\rho)^{5/2}} d\rho \\ N &= (t-\xi) \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{1}{(t-\rho)^{3/2}} d\rho \end{aligned}$$

$$\begin{aligned}
P &= \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{B^2}{(t-\rho)^{3/2}} d\rho \\
O &= \int_0^\xi \frac{e^{-\frac{A^2}{2\rho}}}{\sqrt{2\pi}} \frac{A}{\rho^{3/2}} \frac{e^{-\frac{B^2}{2(t-\rho)}}}{\sqrt{2\pi}} \frac{1}{(t-\rho)^{1/2}} d\rho.
\end{aligned}$$

Before proceeding with the simplification of these integrals some formulas will be given that will be of use later.

Let ϕ be, as usual $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\xi^2/2} d\xi$. Let a and b be two real numbers and

$$E = e^{-\frac{a^2 t}{2} - \frac{b^2}{2t}}.$$

It is easy to establish that

$$\frac{E}{\sqrt{2\pi}} \frac{1}{t^{1/2}}$$

is the derivative of

$$\frac{1}{a} \left[e^{ab} \phi \left(a\sqrt{t} + \frac{b}{\sqrt{t}} \right) - e^{-ab} \phi \left(-a\sqrt{t} + \frac{b}{\sqrt{t}} \right) \right] \quad (3.22)$$

$$\frac{E}{\sqrt{2\pi}} \frac{1}{t^{3/2}}$$

is the derivative of

$$-\frac{1}{b} \left[e^{ab} \phi \left(a\sqrt{t} + \frac{b}{\sqrt{t}} \right) + e^{-ab} \phi \left(-a\sqrt{t} + \frac{b}{\sqrt{t}} \right) \right] \quad (3.23)$$

$$\frac{E}{\sqrt{2\pi}} \frac{1}{t^{5/2}}$$

is the derivative of:

$$\frac{2}{b^2} \frac{E}{\sqrt{2\pi}} + \frac{1}{b^2} \left(a - \frac{1}{b} \right) e^{ab} \phi \left(a\sqrt{t} + \frac{b}{\sqrt{t}} \right) - \frac{1}{b^2} \left(a + \frac{1}{b} \right) e^{-ab} \phi \left(-a\sqrt{t} + \frac{b}{\sqrt{t}} \right). \quad (3.24)$$

We shall return now to the evaluation of the integrals I, J, K, L, M, N, P and Q

Evaluation of I:

$$\begin{aligned}
I &= \frac{\xi A^2 B}{2\pi} \int_0^\xi \frac{e^{-\frac{A^2}{2\rho} - \frac{B^2}{2(t-\rho)}}}{\rho^{5/2} (t-\rho)^{3/2}} d\rho \\
&= \frac{\xi A^2 B}{2\pi t^3} \int_0^\xi \frac{e^{-\frac{A^2}{2\rho} - \frac{B^2}{2t(1-\rho)}}}{\rho^{5/2} (1-\rho)^{3/2}} d\rho.
\end{aligned}$$

Taking the reciprocal of ρ gives:

$$\begin{aligned}
&= \frac{\xi A^2 B}{2\pi t^3} \int_{\frac{t}{\xi}}^{\infty} e^{-\frac{A^2 \rho}{2t} - \frac{B^2 \rho}{2t(\rho-1)}} \cdot \frac{\rho^2}{(\rho-1)^{3/2}} d\rho \text{ putting } \rho+1 \rightarrow \rho \\
&= \frac{\xi A^2 B}{2\pi t^3} e^{-\frac{A^2}{2t} - \frac{B^2}{2t}} \int_{\frac{t-\xi}{\xi}}^{\infty} e^{-\frac{A^2 \rho}{2t} - \frac{B^2}{2t\rho}} \frac{(\rho+1)^2}{\rho^{3/2}} d\rho \\
&= \frac{\xi A^2 B}{2\pi t^3} e^{-\frac{A^2}{2t} - \frac{B^2}{2t}} \int_{\frac{t-\xi}{\xi}}^{\infty} e^{-\frac{A^2 \rho}{2t} - \frac{B^2}{2t\rho}} \rho^{1/2} d\rho \\
&\quad + \frac{\xi A^2 B}{2\pi t^3} e^{-\frac{A^2}{2t} - \frac{B^2}{2t}} \int_{\frac{t-\xi}{\xi}}^{\infty} e^{-\frac{A^2 \rho}{2t} - \frac{B^2}{2t\rho}} \left(\frac{2}{\rho^{1/2}} + \frac{1}{\rho^{3/2}} \right) d\rho.
\end{aligned}$$

Taking the reciprocal of ρ in the first integral we get

$$\begin{aligned}
&= \frac{\xi A^2 B}{2\pi t^3} e^{-\frac{A^2}{2t} - \frac{B^2}{2t}} \int_0^{\frac{t}{t-\xi}} e^{-\frac{B^2 \rho}{2t} - \frac{A^2}{2t\rho}} \frac{d\rho}{\rho^{5/2}} \\
&\quad + \frac{\xi A^2 B}{2\pi t^3} e^{-\frac{A^2}{2t} - \frac{B^2}{2t}} \int_{\frac{t-\xi}{\xi}}^{\infty} e^{-\frac{A^2 \rho}{2t} - \frac{B^2}{2t\rho}} \left(\frac{2}{\rho^{1/2}} + \frac{1}{\rho^{3/2}} \right) d\rho.
\end{aligned}$$

Let

$$\begin{aligned}
\phi(--) &= \phi\left(-\frac{A}{\sqrt{t}}\sqrt{\frac{t-\xi}{\xi}} - \frac{B}{\sqrt{t}}\sqrt{\frac{\xi}{t-\xi}}\right) \\
\phi(-+) &= \phi\left(-\frac{A}{\sqrt{t}}\sqrt{\frac{t-\xi}{\xi}} + \frac{B}{\sqrt{t}}\sqrt{\frac{\xi}{t-\xi}}\right).
\end{aligned}$$

Using formulas (3.22), (3.23), (3.24) and rearranging we get

$$\begin{aligned}
I &= \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \left[\frac{2B}{t^{3/2}} e^{\frac{AB}{t}} e^{-\frac{A^2}{2t} \frac{t-\xi}{\xi} - \frac{B^2}{2t} \frac{\xi}{t-\xi}} \cdot \sqrt{\frac{t-\xi}{\xi}} \right. \\
&\quad + e^{\frac{2AB}{t}} \phi(--) \left[\frac{1}{t} \frac{B}{A} - \frac{(A-B)^2}{t^2} \right] \\
&\quad + \phi(-+) \left[\frac{1}{t} \frac{B}{A} + \frac{(A+B)^2}{t^2} \right] \Big].
\end{aligned}$$

The rest of the integrals can be evaluated in the same way as I. We produce only the final results:

$$J = \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \frac{\xi}{t} \left[e^{\frac{2AB}{t}} \phi(--) \left(\frac{B}{A} - 1 \right) + \phi(-+) \left(\frac{B}{A} + 1 \right) \right]$$

$$\begin{aligned}
K &= \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \left[e^{\frac{2AB}{t}} \phi(-) \left(\frac{AB}{t} - \frac{A^2}{t} \right) + \phi(-) \left(\frac{AB}{t} + \frac{A^2}{t} \right) \right] \\
N &= \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \frac{t - \xi}{t} \left[e^{\frac{2AB}{t}} \phi(-) \left(1 - \frac{A}{B} \right) + \phi(-) \left(1 + \frac{A}{B} \right) \right] \\
P &= \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \left[e^{\frac{2AB}{t}} \phi(-) \left(\frac{B^2 - AB}{t} \right) + \phi(-) \left(\frac{B^2 + AB}{t} \right) \right] \\
M &= \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \frac{t - \xi}{\xi} \left[e^{\frac{2AB}{t}} \phi(-) \left(\frac{(B - A)^2}{t} - \frac{A}{B} \right) + \phi(-) \left(\frac{(B + A)^2}{t} + \frac{A}{B} \right) \right] \\
L &= \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \left(\phi(-) + e^{\frac{2AB}{t}} \phi(-) \right) \\
Q &= \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} \left(\phi(-) + e^{\frac{2AB}{t}} \phi(-) \right).
\end{aligned}$$

Adding these up and multiplying by $e^{+\frac{\xi^2}{2t}} \sqrt{2\pi t}$ gives:

$$\begin{aligned}
&P(O_a(t) \leq \xi | X_t = x, x \geq a) \\
&= e^{\frac{2AB}{t}} \phi(-) \left[\frac{2\xi(B - A)^2}{t^2} + \frac{(A - B)^2 - B^2 + A^2}{t} + \frac{2\xi}{t} - 1 \right] + \phi(-) \\
&+ \frac{2(B - A)}{\sqrt{2\pi t^3}} e^{\frac{AB}{t} - \frac{A^2}{2t} \frac{t - \xi}{\xi} - \frac{B^2}{2t} \frac{\xi}{t - \xi}} \sqrt{\xi(t - \xi)}.
\end{aligned}$$

Replacing $\phi(x)$ by:

$$\phi(x) = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right) \right)$$

and putting

$$A = x - a$$

$$B = a$$

finally gives:

Theorem 3.10 For $x \geq a$

$$P(O_a(t) \leq \xi | X_t = x) = e^{+\frac{2(x-a)a}{t}} \left(1 - \operatorname{erf} \left((x - a) \sqrt{\frac{t - \xi}{2t\xi}} + a \sqrt{\frac{\xi}{2t(t - \xi)}} \right) \right)$$

$$\begin{aligned}
& \cdot \left(\frac{\xi}{t} - \frac{(2a-x)^2}{t^2} \xi + \frac{(x-a)(x-2a)}{t} \right) \\
& - \frac{e^{+\frac{2(x-a)a}{t}}}{2} \left(1 - \operatorname{erf} \left((x-a) \sqrt{\frac{t-\xi}{2t\xi}} + a \sqrt{\frac{\xi}{2t(t-\xi)}} \right) \right) \\
& + \frac{1}{2} \left(1 + \operatorname{erf} \left(-(x-a) \sqrt{\frac{t-\xi}{2t\xi}} + a \sqrt{\frac{\xi}{2t(t-\xi)}} \right) \right) \\
& + (2a-x) \sqrt{\frac{2\xi(t-\xi)}{2\pi t^3}} e^{+\frac{(x-a)a}{t}} e^{-\frac{(x-a)^2(t-\xi)}{2t\xi} - \frac{a^2\xi}{2t(t-\xi)}}.
\end{aligned}$$

It is easy to see that when $a = 0$ this goes into

$$\begin{aligned}
& P(O_o(t) \leq \xi | X_t = x, x \geq 0) \\
& = \frac{\xi}{t} - x \sqrt{\frac{2\xi(t-\xi)}{\pi t^3}} e^{-\frac{x^2(t-\xi)}{2t\xi}} - \frac{\xi}{t} \operatorname{erf} \left(x \sqrt{\frac{t-\xi}{2t\xi}} \right) \\
& + \frac{x^2}{t} \left(1 - \frac{\xi}{t} \right) \left(1 - \operatorname{erf} \left(x \sqrt{\frac{t-\xi}{2t\xi}} \right) \right)
\end{aligned}$$

when $x = a$ then it becomes

$$\begin{aligned}
P(O_a(t) | X_t = a) &= P(O_o(t) | X_t = -a) \\
&= \frac{\xi}{t} + a \sqrt{\frac{2\xi(t-\xi)}{\pi t^3}} e^{-\frac{a^2\xi}{2t(t-\xi)}} \\
&+ \operatorname{erf} \left(a \sqrt{\frac{\xi}{2t(t-\xi)}} \right) \left(1 - \frac{\xi}{t} \right) - \frac{x^2\xi}{t^2} \left(1 - \operatorname{erf} \left(a \sqrt{\frac{\xi}{2t(t-\xi)}} \right) \right).
\end{aligned}$$

Chapter 4

Brownian motion between two parallel lines.

4.1 Introduction.

In this chapter we shall study the Brownian motion's behaviour with respect to the two sided domain: $\{(x, \tau) : -a \leq x \leq b, 0 \leq \tau \leq t\}$ where $a, b \geq 0$. The Q function pertaining to this domain will be determined first, then we shall derive the probability that the Brownian motion and the tied down Brownian motion stay within the barriers $-a$ and b that is, $-a \leq X_\tau \leq b$ for $0 \leq \tau \leq t$. We shall prove that:

$$\begin{aligned} P \left(-a \leq X_\tau \leq b \mid X_t = x \right) &= \\ &= 1 - \left(e^{-\frac{2b(b-x)}{t}} + e^{-\frac{2a(a+x)}{t}} \right) + \left(e^{-\frac{(2b+2a)(b+a-x)}{t}} \right. \\ &\quad \left. e^{-\frac{(2b+2a)(b+a+x)}{t}} \right) - \left(e^{-\frac{(4b+2a)(2b+a-x)}{t}} + e^{-\frac{(2b+4a)(b+2a+x)}{t}} \right) \\ &+ \dots \end{aligned}$$

4.2 Determination of the Q function.

In the present case for $Q(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E \left(e^{-u\lambda \left(X_{\tau} \mathbb{I}_{(a,b)} \right)} \middle| X_t = x \right)$ we can distinguish three different cases depending in which of the three regions x lies.

$Q(x, t)$ is the upper function when $x \geq b$, we speak of middle function when $b \geq x \geq -a$ and of lower function when $x \leq -a$. The formulas (1.3) and (1.4) now become:

$$Q(x, t) = G_0(x, t) - u G_1(x, t) + u^2 G_2(x, t) - \dots$$

$\begin{matrix} (-a, b) & (-a, b) & (-a, b) & (-a, b) \end{matrix}$

where

$$G_0 = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}, \quad G_1 = \int_0^t d\tau \int_{\xi \in (-a, b)} \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} d\xi$$

$$G_{n+1} = \int_0^t d\tau \int_{\xi \in (-a, b)} \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} G_n(\xi, \tau) d\xi$$

$\begin{matrix} (-a, b) & (-a, b) \end{matrix}$

$G_1, G_2, \dots$ will be called the first integral, the second integral..., and again we have to distinguish upper, middle and lower integrals depending upon the values of x . When $Q(x, t)$, G_0 , G_1 , $G_2, \dots$ are not indexed by $(-a, b)$ then they have the same meaning as in Chapter II.

First we shall compute the first few G functions and then deduce their general form.

Lemma.

The first upper integral, that is when $x \geq b$, is

$$\psi(b) * G_1(x - b) + \frac{\psi(2a + b)}{2B_1} * G_0(x - b) * 1.$$

The first lower integral, that is when $x \leq -a$, is

$$\psi(a) * G_0(-x - a) + \psi(a + 2b) * G_0(-x - a) * \frac{1}{2!B_1}.$$

Proof:

Let $x \geq b$. The first integral then:

$$\begin{aligned}
 & \int_0^t d\tau \int_b^\infty \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} d\xi + \int_0^t d\tau \int_{-\infty}^{-a} \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} d\xi \\
 &= \int_b^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} d\xi + \int_{-\infty}^{-a} \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} d\xi \\
 &= \int_0^\infty \frac{e^{-\frac{(x-b-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{(\xi+b)^2}{2t}}}{\sqrt{2\pi t}} d\xi + \int_a^\infty \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} d\xi \\
 &= \int_0^\infty \frac{e^{-\frac{(x-b-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{(\xi+b)^2}{2t}}}{\sqrt{2\pi t}} d\xi + \int_0^\infty \frac{e^{-\frac{(x+a+\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{(\xi+a)^2}{2t}}}{\sqrt{2\pi t}} d\xi \\
 &= \psi(b) * \int_0^\infty \frac{e^{-\frac{(x-b-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} d\xi + \psi(x+a) * \psi(a) * \int_0^\infty \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} d\xi.
 \end{aligned}$$

The integral in the first term is $G_1(x-b)$; according to 2.1 the integral in the second term is $G_1(0) = \frac{1}{2} \frac{t^{1/2}}{\sqrt{2\pi}}$.

Thus the first integral becomes:

$$\psi(b) * G_1(x-b) + \frac{1}{2} \psi(x+a) * \psi(a) * \frac{t^{1/2}}{\sqrt{2\pi}}.$$

For present and later use consider the following integral:

$$t^n * \frac{1}{t^{1/2}} = \int_0^t \frac{(t-\tau)^n}{\tau^{1/2}} d\tau = t^{n+1/2} \int_0^1 \frac{(1-\tau)^n}{\tau^{1/2}} d\tau \quad n = 0, 1, \dots$$

The last integral is just a beta function integral

$$B\left(n+1, \frac{1}{2}\right) = \frac{\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+1+\frac{1}{2})} = \frac{n!}{\frac{1}{2} \cdot \frac{3}{2} \dots (n+\frac{1}{2})}.$$

Let $B(n+1, \frac{1}{2})$ be denoted by B_{n+1} then

$$t^{n+1/2} = \frac{1}{B_{n+1}} t^n * \frac{1}{t^{1/2}} \quad n = 0, 1, 2, \dots$$

Now consider the second term in the first integral

$$\begin{aligned}
 & \frac{1}{2} \psi(x+a) * \psi(a) * \frac{t^{1/2}}{\sqrt{2\pi}} = \frac{1}{2} \frac{1}{B_1} \psi(x+a) * \psi(a) * 1 * \frac{1}{\sqrt{2\pi t^{1/2}}} \\
 &= \frac{1}{2B_1} \psi(2a+b) * \psi(x-b) * \frac{1}{\sqrt{2\pi t}} * 1 = \frac{1}{2B_1} \psi(2a+b) * G_0(x-b) * 1.
 \end{aligned}$$

Hence the first integral for $x \geq b$

$$\psi(b) * G_1(x - b) + \frac{\psi(2a + b)}{2B_1} * G_0(x - b) * 1$$

and this completes the evaluation of the first upper integral.

Next we evaluate the first lower integral, that is for $x \leq -a$,

$$\begin{aligned} & \int_b^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} d\xi + \int_{-\infty}^{-a} \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} d\xi \\ &= \int_0^\infty \frac{e^{-\frac{(b-x+\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{(\xi+b)^2}{2t}}}{\sqrt{2\pi t}} d\xi + \int_0^\infty \frac{e^{-\frac{(-x-a-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \frac{e^{-\frac{(\xi+a)^2}{2t}}}{\sqrt{2\pi t}} d\xi \\ &= \psi(a) * G_1(-x - a) + \dots \end{aligned}$$

Now $b - x = (b + a) + (-a - x)$ and substituting this into the first term gives

$$\begin{aligned} &= \psi(a) * G_1(-x - a) + \psi(b + a - a - x) * \psi(b) * \frac{1}{2!} \frac{t^{1/2}}{\sqrt{2\pi}} \\ &= \psi(a) * G_0(-x - a) + \psi(a + 2b) * G_0(-x - a) * \frac{1}{2! B_1} \end{aligned}$$

and this completes the evaluation of the first lower integral. ■

Determination of the second integral.

First consider again the case $x \geq b$

$$\begin{aligned} & \int_b^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \left\{ \psi(b) * G_1(\xi - b) + \psi(2a + b) * G_0(\xi - b) * \frac{1}{2! B_1} \right\} d\xi \\ &+ \int_{-\infty}^{-a} \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \left\{ \psi(a) * G_1(-\xi - a) + \psi(a + 2b) * G_0(-\xi - a) * \frac{1}{2! B_1} \right\} d\xi \\ &= \int_0^\infty \frac{e^{-\frac{(x-b-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \left\{ \psi(b) * G_1(\xi) + \psi(2a + b) * G_0(\xi) * \frac{1}{2! B_1} \right\} d\xi \\ &+ \int_0^\infty \frac{e^{-\frac{(x+a+\xi)^2}{2t}}}{\sqrt{2\pi t}} * \left\{ \psi(a) * G_1(\xi) + \psi(a + 2b) * G_0(\xi) * \frac{1}{2! B_1} \right\} d\xi \\ &= \psi(b) * G_2(x - b) + \psi(2a + b) * G_1(x - b) * \frac{1}{2! B_1} \end{aligned}$$

this comes from the first term, and the second term:

$$+ \psi(x + a + a) * G_2(0) + \psi(x + a + a + 2b) * G_1(0) * \frac{1}{2!B_1}.$$

But it was shown (see 2.12) that:

$$G_0(0) = \frac{1}{\sqrt{2\pi t}}, G_1(0) = \frac{1}{2!} \frac{t^{1/2}}{\sqrt{2\pi}}, G_2(0) = \frac{1}{3!} \frac{t^{3/2}}{\sqrt{2\pi}}, G_3(0) = \frac{1}{4!} \frac{t^{5/2}}{\sqrt{2\pi}}$$

and in general:

$$G_n(0) = \frac{1}{(n+1)!} \frac{t^{n-1/2}}{\sqrt{2\pi}}$$

substituting these into the last expression gives:

$$= \dots + \psi(x + a + a) * \frac{1}{3!} \frac{t^{3/2}}{\sqrt{2\pi}} + \psi(x + a + a + 2b) * \frac{1}{2!} \frac{t^{1/2}}{\sqrt{2\pi}} * \frac{1}{2!B_1}.$$

Putting

$$t^{3/2} = \frac{1}{B_2} t * \frac{1}{t^{1/2}}, t^{1/2} = \frac{1}{B_1} * \frac{1}{t^{1/2}}$$

gives

$$= \dots + \psi(x + 2a) * \frac{1}{\sqrt{2\pi t}} * \frac{t}{3!B_2} + \psi(x + 2a + 2b) * \frac{1}{\sqrt{2\pi t}} * \frac{1}{2!B_1} * \frac{1}{2!B_1}.$$

But $x \geq b$ and writing $x - b + b$ for x gives

$$= \dots + \psi(b + 2a) * G_0(x - b) * \frac{t}{3!B_2} + \psi(2a + 3b) * G_0(x - b) * \frac{1}{2!B_1} * \frac{1}{2!B_1}.$$

So, the second upper integral becomes

$$\begin{aligned} \psi(b) * G_2(x - b) + \psi(2a - b) * G_1(x - b) * \frac{1}{2!B_1} \\ + \psi(2a + b) * G_0(x - b) * \frac{t}{3!B_2} + \psi(2a + 3b) * G_0(x - b) * \frac{1}{2!B_1} * \frac{1}{2!B_1}. \end{aligned}$$

The second lower integral, that is when $x \leq -a$, through essentially the same transformation becomes:

$$\begin{aligned} \psi(a) * G_2(-x - a) + \psi(a + 2b) * G_1(-x - a) * \frac{1}{2!B_1} \\ + \psi(a + 2b) * G_0(-x - a) * \frac{t}{3!B_2} \\ + \psi(3a + 2b) * G_0(-x - a) * \frac{1}{2!B_1} * \frac{1}{2!B_1}. \end{aligned}$$

The same way the third integral becomes for $x \geq b$

$$\begin{aligned}
& \psi(b) * G_3(x-b) + \psi(2a+b) * G_2(x-b) * \frac{1}{2!B_1} \\
& + \psi(2a+b) * G_1(x-b) * \frac{t}{3!B_2} + \psi(2a+3b) * G_1(x-b) * \frac{1}{2!B_1} * \frac{1}{2!B_1} \\
& + \psi(2a+b) * G_0(x-b) * \frac{t^2}{4!B_3} + \psi(2a+3b) * G_0(x-b) * \frac{t}{3!B_2} * \frac{1}{2!B_1} \\
& + \psi(2a+3b) * G_0(x-b) * \frac{1}{2!B_1} * \frac{t}{3!B_2} \\
& + \psi(4a+3b) * G_0(x-b) * \frac{1}{2!B_1} * \frac{1}{2B_1} * \frac{1}{2B_1}.
\end{aligned}$$

Now the lower integral, that is the case $x \leq -a$ comes from the upper integral by substituting $-a-x$ for $x-b$ and a for b and b for a .

$$\begin{aligned}
& \psi(a) * G_3(-x-a) + \psi(a+2b) * G_2(-x-a) * \frac{1}{2!B} \\
& + \psi(a+2b) * G_1(-x-a) * \frac{t}{3!B_2} + \psi(3a+2b) * G_1(-x-a) * \frac{1}{2B_1} * \frac{1}{2B_1} \\
& + \psi(a+2b) * G_0(-x-a) * \frac{t^2}{4!B_3} + \psi(3a+2b) * G_0(-x-a) * \frac{t}{3!B_2} * \frac{1}{2!B_1} \\
& + \psi(3a+2b) * G_0(-x-a) * \frac{1}{2!B_1} * \frac{t}{3!B_2} \\
& + (3a+4b) * G_0(-x-a) * \frac{1}{2!B_1} * \frac{1}{2!B_1} * \frac{1}{2!B_1}.
\end{aligned}$$

From these now we can “assemble” the function $Q_{(-a,b)}$. Consider the $x \geq b$ case.

Let us denote $G_n(x-b)$ for brevity by G_n then, $Q_{(-a,b)}(x,t) = n = 0, 1, 2, \dots$.
The 0th integral $-u$ (1st integral) $+ u^2$ (2nd integral) $- u^3$ (3rd integral) $\dots =$

$$\begin{aligned}
& \psi(b) * G_0 \\
& - u \left\{ \psi(b) * G_1 + \psi(2a+b) * G_0 * \frac{1}{2!B_1} \right\} \\
& + u^2 \left\{ \psi(b) * G_2 + \psi(2a+b) * G_1 * \frac{1}{2!B_1} + \psi(2a+b) * G_0 * \frac{t}{3!B_2} \right. \\
& \left. + \psi(2a+3b) * G_0 * \frac{1}{2!B_1} * \frac{1}{2!B_1} \right\}
\end{aligned}$$

$$\begin{aligned}
& - u^3 \left\{ \psi(b) * G_3 + \psi(2a+b) * G_2 * \frac{1}{2!B_1} + \psi(2a+b) * G_1 * \frac{t}{3!B_2} \right. \\
& + \psi(2a+3b) * G_1 * \frac{1}{2!B_1} * \frac{1}{2!B_1} + \psi(2a+b) * G_0 * \frac{t^2}{4!B_3} \\
& + \psi(2a+3b) * G_0 * \frac{1}{2!B_1} * \frac{t}{3!B_2} + \psi(2a+3b) * G_0 * \frac{t}{3!B_2} * \frac{1}{2!B_1} \\
& \left. + \psi(4a+3b) * G_0 * \frac{1}{2!B_1} * \frac{1}{2!B_1} * \frac{1}{2!B_1} \right\} + \dots \\
& = \psi(b) * (G_0 - uG_1 + u^2G_2 - u^3G_3 \dots) \\
& - u\psi(2a+b) * \left(G_0 * \frac{1}{2!B_1} - uG_1 * \frac{1}{2!B_1} - G_0 * \frac{ut}{3!B_2} \right. \\
& \left. + u^2G_2 * \frac{1}{2!B_1} + uG_1 * \frac{ut}{3!B_2} + G_0 * \frac{u^2t^2}{4!B_3} \dots \right) \\
& + u^2\psi(2a+3b) * \left(G_0 * \frac{1}{2!B_1} * \frac{1}{2!B_1} - uG_1 * \frac{1}{2!B_1} * \frac{1}{2!B_1} \right. \\
& \left. - G_0 * \frac{tu}{3!B_2} * \frac{1}{2!B_1} - G_0 * \frac{1}{2!B_1} * \frac{t}{3!B_2} \dots \right) \\
& - u^3\psi(4a+3b) \left(G_0 * \frac{1}{2!B_1} * \frac{1}{2!B_1} * \frac{1}{2!B_1} \dots \right).
\end{aligned}$$

Let

$$G_0 - uG_1 + u^2G_2 \dots = Q(x-b, t) = Q(x-b) = Q$$

and

$$B = B(ut) = \frac{1}{2!B_1} - \frac{ut}{3!B_2} + \frac{u^2t^2}{4!B_3} - \frac{u^3t^3}{5!B_4} + \dots$$

So the upper function becomes

$$\begin{aligned}
& Q * \psi(b) - uQ\psi(2a+b) * B + u^2Q * \psi(2a+3b) * B * B \\
& - u^3Q * \psi(4a+3b) * B * B * B + u^4Q * \psi(4a+5b) * B * B * B * B \\
& - \dots
\end{aligned}$$

The lower function, that is the case $x \leq -a$, is obtained if $Q(x-b)$ is replaced by $Q = Q(-x-a)$ and in the ψ 's a and b are exchanged.

Hence the lower function is:

$$Q * \{ \psi(a) - u\psi(a+2b) * B + u^2\psi(3a+2b) * B * B - u^3\psi(3a+4b) * B * B * B \dots \}.$$

It remains now to compute the middle function, that is the case $b \geq x \geq a$. If we had calculated the middle integrals this could be done easily, but we did not.

$Q(x, t)$ (let us denote it by) $\tilde{Q}(x, t)$ satisfies the integral equation

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - u \int_{\xi \notin (a, b)} \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * \tilde{Q}(\xi, t) d\xi = \tilde{Q}(x, t).$$

Now this can be used to compute the middle function since on the left hand side only the upper and the lower functions appear.

Let the upper function be written in the form

$$Q(b-x) * K$$

and the lower function

$$Q(-x-a) * L$$

where

$$K = \psi(b) - u\psi(a+b) * B + u^2\psi(2a+3b) * B * B \dots$$

$$L = \psi(a) - u\psi(a+2b) * B + u^2\psi(3a+2b) * B * B \dots$$

Obviously K and L do not depend on x .

So

$$\begin{aligned} \tilde{Q}(x, t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} &- K * u \int_b^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * Q(\xi - b) d\xi \\ &- L * u \int_{-\infty}^{-a} \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * Q(-\xi - a) d\xi. \end{aligned}$$

After a change of variables in the integrals this becomes

$$\begin{aligned} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} &- K * u \int_0^\infty \frac{e^{-\frac{(b-x+\xi)^2}{2t}}}{\sqrt{2\pi t}} * Q(\xi) d\xi - L * u \int_0^\infty \frac{e^{-\frac{(x+a+\xi)^2}{2t}}}{\sqrt{2\pi t}} * Q(\xi) d\xi \\ &= \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - (K * \psi(b-x) + L * \psi(a+x)) * u \int_0^\infty \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} * Q(\xi) d\xi. \end{aligned}$$

Since Q is the upper function for $f(t) = 0$ that is for the case $V(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$,
so it satisfies the integral equation:

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - u \int_0^\infty \frac{e^{-\frac{(x-\xi)^2}{2t}}}{\sqrt{2\pi t}} * Q(\xi) d\xi = Q(x, t).$$

Putting $x = 0$ and $Q(0, t) = \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}}$ gives

$$u \int_0^\infty \frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} * Q(\xi) d\xi = \frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}}.$$

So finally we get for the middle function:

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - (K * \psi(b-x) + L * \psi(a+x)) * \left(\frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \right).$$

Here we give all three functions:

Upper function ($x \geq b$)

$$Q(x-b) * (\psi(b) - u\psi(2a+b) * B + u^2\psi(2a+3b) * B * B - u^3\psi(4a+3b) * B * B * B + \dots). \quad (4.1)$$

The middle function: $b \geq x \geq a$

$$\begin{aligned} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} &= \left(\frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2b-x) \\ &- \left(\frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a+x) \\ &+ u \left(\frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a+2b-x) * B \\ &+ u \left(\frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a+2b+x) * B \\ &- u^2 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a+4b-x) * B * B \\ &- u^2 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(4a+2b+x) * B * B + \dots \end{aligned} \quad (4.2)$$

The lower function:

$$Q(-a-x) * (\psi(a) - u\psi(a+2b) * B + u^2\psi(3a+2b) * B * B - u^3\psi(3a+4b) * B * B * B + \dots) \quad (4.3)$$

These however have to be simplified further. In the case of $f(t) = 0$. The upper function had the form

$$e^{-ut} \left(\psi(x) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right)$$

it was relatively easy to invert this integral, with one integration by parts, but now the u 's are much more scattered, their powers $1, u, u^2, \dots$ appear as coefficients of the ψ -function, and in addition the B function

$$B = \frac{1}{2!B_1} - \frac{ut}{3!B_2} + \frac{u^2t^2}{4!B_3} \dots$$

is a power series in u 's.

Furthermore, it was easy to obtain the limit $u \rightarrow \infty$ in case of $f(t) = a$ but at this stage it does not appear easy for the present case. To this end, we shall first examine the B function and especially the convolutions $B * B, B * B * B, \dots$

4.3 Properties of the "B" function.

In this section we shall derive several identities satisfied by the "B" function.

Lemma 4.1 *The Laplace transform of the "B" function:*

$$\mathcal{L}(B) = \left(\frac{\sqrt{s+u} - \sqrt{s}}{u} \right)^2$$

Lemma 4.2

$$B = \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \quad (4.4)$$

Lemma 4.3

$$B^{*n} = \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2n} = \frac{t^{n-1}}{2^{2n}\Gamma(n)} {}_1F_1 \left(\frac{2n+1}{2} \mid 2n+1, -ut \right) \quad (4.5)$$

$$B^{*n} = \frac{\Gamma(2n+1)t^{n-1}}{2^{2n}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} \int_0^1 e^{-ut\xi} \xi^{n-\frac{1}{2}}(1-\xi)^{n-\frac{1}{2}} d\xi. \quad (4.6)$$

First we shall show that B is a confluent hypergeometric function or Kummer function.

$$B = \frac{1}{2!B_1} - \frac{ut}{3!B_2} + \frac{u^2t^2}{4!B_3} - \frac{u^3t^3}{5!B_4} + \dots$$

Let ut be denoted by x (temporarily) using

$$\begin{aligned} B_n &= B\left(n+1, \frac{1}{2}\right) = \frac{(n-1)!}{\frac{1}{2} \frac{3}{2} \dots n - \frac{1}{2}} \\ B &= \frac{1}{1.2.\frac{1}{2}} - \frac{x}{1.2.3.\frac{1}{2}} + \frac{x^2}{1.2.3.4.\frac{1}{2}\frac{3}{2}} - \frac{x^3}{1.2.3.4.5.\frac{1}{2}\frac{3}{2}\frac{5}{2}} \\ &= \frac{\frac{1}{2}}{1.2} - \frac{\frac{1}{2} \frac{3}{2} x}{1.2.3.1} + \frac{\frac{1}{2} \frac{3}{2} \frac{5}{2} x^2}{1.2.3.4.2!} - \frac{\frac{1}{2} \frac{3}{2} \frac{5}{2} \frac{7}{2} x^3}{1.2.3.4.5.3!} \\ &= \frac{1}{4} \left(1 - \frac{\frac{3}{2} x}{3.1!} + \frac{\frac{3}{2} \frac{5}{2} x^2}{3.4.2!} - \frac{\frac{3}{2} \frac{5}{2} \frac{7}{2} x^3}{3.4.5.3!} + \dots \right) \\ &= \frac{1}{4} {}_1F_1\left(\frac{3}{2}, 3; -x\right). \end{aligned}$$

Where ${}_1F_1(a, b; x)$ is the confluent hypergeometric function or Kummer function

$$F(a, b, x) = 1 + \frac{a}{b} \frac{x}{1!} + \frac{a(a+1)}{b(b+1)} \frac{x^2}{2!} + \frac{a(a+1)(a+2)}{u(b+1)(b+2)} \frac{x^3}{3!} \dots$$

Next we compute the Laplace transform of B .

Let $F(a, c; t)$ be an arbitrary Kummer function. The Laplace transform for $\alpha \geq 0$

$$\int_0^\infty e^{-st} t^{\alpha-1} F(a, c; kt) dt$$

can be evaluated by taking the expansion of $F(a, c)$ and integrating term by term.

Let $(a)_n = a(a+1)\dots(a+n-1)$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} k^n \int_0^\infty e^{-st} t^{\alpha+n-1} dt = \sum_n \frac{(a)_n}{(c)_n n!} k^n \frac{\Gamma(\alpha+n)}{s^{\alpha+n}} \\ &= \frac{\Gamma(\alpha)}{s^\alpha} \sum \frac{(a)_n}{(c)_n n!} \frac{k^n}{s^n} = \frac{\Gamma(\alpha)}{s^\alpha} {}_2F_1\left(a, \alpha; c; \frac{k}{s}\right) \end{aligned}$$

where ${}_2F_1$ is the ordinary Gaussian hypergeometric series

$${}_2F_1(a, b, c, x) = 1 + \frac{ab}{c} \frac{x}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{x^2}{2!} + \dots$$

Thus

$$\mathcal{L}(t^{\alpha-1} F(a, c; kt)) = \frac{\Gamma(\alpha)}{s^\alpha} {}_2F_1\left(a, \alpha, c; \frac{k}{s}\right). \quad (1.7)$$

The right hand side is convergent only when $\left|\frac{k}{s}\right| < 1$ (or at most if $\left|\frac{k}{s}\right| \leq 1$ but since both sides are analytic functions of s the Laplace transform can be computed through analytic continuation of the expression on the right hand side for all $\operatorname{Re} s > 0$. Applying this to the case of the B function

$$\begin{aligned} \mathcal{L}(B) &= \frac{1}{4} \int_0^\infty e^{-st} t^{1/2} {}_1F_1\left(\frac{3}{2}, 3, -ut\right) dt \\ &= \frac{1}{4s} {}_2F_1\left(\frac{3}{2}, 1; 3; -\frac{u}{s}\right). \end{aligned}$$

Consider now the function

$$\begin{aligned} {}_2F_1\left(\frac{3}{2}, 1; 3; -x\right) &= 1 - \frac{3}{2}x + \frac{3 \cdot 5}{3 \cdot 4}x^2 - \frac{3 \cdot 5 \cdot 7}{3 \cdot 4 \cdot 5}x^3 + \frac{3 \cdot 5 \cdot 7 \cdot 9}{3 \cdot 4 \cdot 5 \cdot 6}x^4 \dots \\ &= \frac{8}{x^2} \left(1 + \frac{x}{2} - \sqrt{1+x}\right). \end{aligned}$$

The last equality can be seen by expanding $\sqrt{1+x}$. Substituting $\frac{u}{s}$ for x gives

$$\begin{aligned} \mathcal{L}(B) &= \frac{1}{4s} \frac{8}{\frac{u^2}{s^2}} \left(1 + \frac{u}{2s} - \sqrt{1 + \frac{u}{s}}\right) \\ &= \frac{2s}{u^2} \left(1 + \frac{u}{2s} - \sqrt{1 + \frac{u}{s}}\right) = \frac{1}{u^2} (2s + u - 2\sqrt{s(s+u)}) \\ &= \frac{1}{u^2} ((s+u) + s - 2\sqrt{s(s+u)}) \\ &= \left(\frac{\sqrt{s+u} - \sqrt{s}}{u}\right)^2 \end{aligned}$$

and this completes the proof of lemma 1. ■

The proof of Lemma 2. Notice that $\sqrt{s+u} - \sqrt{s}$ is the Laplace transform of $\frac{1-e^{-ut}}{2\sqrt{\pi t^3}}$! This can be verified by expanding the exponential and taking the Laplace

transform term by term. Thus:

$$B = \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}}. \quad \blacksquare$$

Proof of lemma 3. First we prove the identity:

$$2^n \left(\frac{\sqrt{1+x}-1}{x} \right)^n = {}_2F_1 \left(\frac{n}{2}, \frac{n+1}{2}, n+1; -x \right) \quad \text{if } |x| < 1$$

or putting $-x$ for x

$$2^n \left(\frac{1 - \sqrt{1-x}}{x} \right)^n = {}_2F_1 \left(\frac{n}{2}, \frac{n+1}{2}, n+1; x \right).$$

The function ${}_2F_1(a, b; c, x)$ satisfies the hypergeometric differential equation of Gauss

$$y'' + \left(\frac{c}{x} + \frac{1+a+b-c}{x-1} \right) y' + \frac{ab}{x(x-1)} y = 0.$$

This differential equation at zero has a regular singularity with exponents 0 and $1-c$ at 1 with exponents 0, $c-a-b$ and at ∞ with exponents a, b .

Now we concern ourselves only with the singularity at zero. It is enough to prove that

$$2^n \left(\frac{1 - \sqrt{1-x}}{x} \right)^n$$

satisfies the hypergeometric differential equation with

$$a = \frac{n}{2}, \quad b = \frac{n+1}{2}, \quad c = n+1 \quad (n \geq 1)$$

and has exponent 0 at $x=0$, and finally that its value is 1 at zero.

The differential equation in the present case:

$$y'' + \left(\frac{n+1}{x} + \frac{\frac{1}{2}}{x-1} \right) y' + \frac{n(n+1)}{x(x-1)} y = 0$$

and it is to be proved that

$$y = \left(\frac{1 - \sqrt{1-x}}{x} \right)^n \quad \text{satisfies it. The derivatives of } y \text{ can be written:}$$

$$y' = y \frac{n}{2x} \left(\frac{1}{\sqrt{1-x}} - 1 \right)$$

$$y'' = y \frac{n^2}{4x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right)^2 + y \frac{n}{2} \left(-\frac{1}{x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right) + \frac{1}{2x(1-x)^{3/2}} \right).$$

Substituting these into the differential equation we get:

$$\left\{ \frac{n^2}{4x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right)^2 + \frac{n}{2} \left(-\frac{1}{x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right) + \frac{1}{2x(1-x)^{3/2}} \right) \right. \\ \left. + \frac{n+1}{x} \frac{n}{2x} \left(\frac{1}{\sqrt{1-x}} - 1 \right) + \frac{\frac{1}{2}}{x-1} \frac{n}{2x} \left(\frac{1}{\sqrt{1-x}} - 1 \right) \right. \\ \left. + \frac{n(n+1)}{4x(x-1)} \right\} y .$$

The coefficients of n^2 :

$$\frac{1}{4x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right)^2 + \frac{1}{2x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right) + \frac{1}{4x(x-1)} \\ = \frac{1}{4x^2} \left[\left(\frac{1}{\sqrt{1-x}} - 1 \right)^2 - 2 \left(\frac{1}{\sqrt{1-x}} - 1 \right) + 1 \right] - \frac{1}{4x^2} + \frac{1}{4x(x-1)} \\ = \frac{1}{4x^2} \left(\frac{1}{\sqrt{1-x}} - 1 + 1 \right)^2 + \frac{1}{4x^2} \left(\frac{x}{x-1} - 1 \right) \\ = \frac{1}{4x^2} \frac{1}{(1-x)} + \frac{1}{4x^2} \frac{1}{x-1} = 0 .$$

Next remains only to collect the coefficients of n

$$\frac{1}{2} \left(-\frac{1}{x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right) + \frac{1}{4x(1-x)^{3/2}} \right) \\ + \frac{1}{2x^2} \left(\frac{1}{\sqrt{1-x}} - 1 \right) + \frac{\frac{1}{2}}{2(x-1)x} \left(\frac{1}{\sqrt{1-x}} - 1 \right) \\ + \frac{1}{4x(x-1)} \\ = \frac{1}{4x(1-x)^{3/2}} + \frac{1}{4x(x-1)\sqrt{1-x}} - \frac{1}{4x(x-1)} + \frac{1}{4x(x-1)} = 0 .$$

Hence $2^n \left(\frac{1-\sqrt{1-x}}{x} \right)^n$ satisfies the differential equation. Now taking the limit at zero

$$\lim_{x \rightarrow 0} 2 \frac{1 - \sqrt{1-x}}{x} = 2 \frac{d\sqrt{x}}{dx} \Big|_{x=1} = 1$$

shows that $2^n \left(\frac{1-\sqrt{1-x}}{x} \right)^n$ is a solution of the above hypergeometric differential equation with exponent zero.

The other solution that does not concern us is

$$x^{-n} F\left(-\frac{n}{2}, -\frac{n}{2} + \frac{1}{2}, 1 - n; x\right) \text{ for } n > 1$$

or in the general case:

$$x^{1-c} F(a + 1 - c, b + 1 - c, 2 - c; x) .$$

Thus it has been proved that:

$$2^n \left(\frac{\sqrt{1+x} - 1}{x} \right)^n = {}_2F_1 \left(\frac{n}{2}, \frac{n+1}{2}, n+1; -x \right) .$$

Put now for x , $x = \frac{u}{s}$ so the above identity becomes

$$\frac{2^n}{u^n} s^{\frac{n}{2}} (\sqrt{s+u} - \sqrt{s})^n = {}_2F_1 \left(\frac{n+1}{2}, \frac{n}{2}, n+1; -\frac{u}{s} \right)$$

or

$$(\sqrt{s+u} - \sqrt{s})^n = \frac{u^n}{2^n s^{\frac{n}{2}}} {}_2F_1 \left(\frac{n+1}{2}, \frac{n}{2}, n+1; -\frac{u}{s} \right) .$$

We have shown (4.7) that

$$\frac{1}{s^\alpha} {}_2F_1 \left(a, \alpha, c, \frac{k}{s} \right)$$

is the Laplace transform of

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)} {}_1F_1 (a, c; kt) .$$

So $(\sqrt{s+u} - \sqrt{s})^n$ is the Laplace transform of

$$\frac{u^n}{2^n} \frac{t^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)} {}_1F_1 \left(\frac{n+1}{2}, n+1; -ut \right) .$$

Hence

$$\left(\frac{1 - e^{-ut}}{2\sqrt{\pi t^3}} \right)^{*n} = \frac{u^n t^{\frac{n}{2}-1}}{2^n \Gamma\left(\frac{n}{2}\right)} {}_1F_1 \left(\frac{n+1}{2}, n+1, -ut \right)$$

or

$$\left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*n} = \frac{t^{\frac{n}{2}-1}}{2^n \Gamma\left(\frac{n}{2}\right)} {}_1F_1 \left(\frac{n+1}{2}, n+1, -ut \right) \quad (4.8)$$

where $*n$ stands for n times convolution. This formula and expression (4.1)

$$B = \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}}$$

allow a considerable simplification of the convolutions B^{*n} appearing in the upper, middle and lower functions. For

$$B^{*n} = \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2n} = \frac{t^{n-1}}{2^{2n}\Gamma(n)} {}_1F_1 \left(\frac{2n+1}{2}, 2n+1, -ut \right).$$

We shall make use of the Euler-Pochhammer-integral representation of the Kummer function (see [Erdely])

$$F(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{x\xi} \xi^{a-1} (1-\xi)^{c-a-1} d\xi \quad (4.9)$$

and we have also

$$B^{*n} = \frac{\Gamma(2n+1)t^{n-1}}{2^{2n}\Gamma(n)\Gamma(n+\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_0^1 e^{-u\xi} \xi^{n-\frac{1}{2}} (1-\xi)^{n-\frac{1}{2}} d\xi$$

and this is the end of the proof of lemma 3. ■

4.4 The evaluation of $P \left(-a \leq X_\tau \leq b \mid X_t = x \right)_{0 \leq \tau \leq t}$

Clearly we have to concern ourselves only with the case $-a \leq x \leq b$ for all other x 's the probability is zero, that is we consider only the middle function.

$$P \left(-a \leq X_\tau \leq b \mid X_t = x \right)_{0 \leq \tau \leq t} = \lim_{u \rightarrow \infty} E \left(e^{-u\lambda(X_\tau \notin (a,b))} \mid X_t = x \right).$$

But the middle function $\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E \left(e^{-u\lambda \left(X_\tau \notin (a,b) \right)} \mid X_t = x \right)_{0 \leq \tau \leq t}$ can be written as: (see 4.2)

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi}} = \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2b-x)B$$

$$\begin{aligned}
& - \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a - x)B \\
& + u \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a + 2b - x) * B \\
& + u \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a + 2b + x) * B \\
& - u^2 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(2a + 4b - x) * B^{*2} \\
& - u^2 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(4a + 2b + x) * B^{*2} \\
& + u^3 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(4a + 4b - x) * B^{*3} \\
& + u^3 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(4a + 4b + x) * B^{*3} \\
& - u^4 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(4a + 6b - x) * B^{*4} \\
& - u^4 \left(\frac{1}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \right) * \psi(6a + 4b + x) * B^{*4} \\
& + \dots \\
& + \dots \\
& = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \left(\frac{e^{-\frac{(2b-x)^2}{2t}}}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} * \psi(2b - x) \right) \\
& - \left(\frac{e^{-\frac{(2a-x)^2}{2t}}}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} * \psi(a - x) \right) \\
& + \left(\frac{e^{-\frac{(2a+2b-x)^2}{2t}}}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} * \psi(2b + 2a - x) \right) * uB \\
& + \left(\frac{e^{-\frac{(2a+2b+x)^2}{2t}}}{\sqrt{2\pi t}} - \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} * \psi(2b + 2a + x) \right) * uB \\
& + \dots
\end{aligned}$$

Taking the limit term by term, we see that the first and second terms become as

$u \rightarrow \infty$

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi}} - \frac{e^{-\frac{(2b-x)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2a+x)^2}{2t}}}{\sqrt{2\pi t}}.$$

The rest of the terms have the general form of

$$\varphi(k) * u^n B^{*n} = \psi(k) * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} * u^n B^{*n}.$$

Let p be an arbitrary positive number, consider the integral

$$u^n \int_0^P B^{*n} dt \text{ as } u \rightarrow \infty$$

substituting (4.5) gives

$$\frac{u^n}{2^{2n}\Gamma(n)} \int_0^P t^{n-1} {}_1F_1 \left(\frac{2n+1}{2}, 2n+1; -ut \right) dt =$$

by changing the independent variable we get

$$= \frac{1}{2^{2n}\Gamma(n)} \int_0^{uP} t^{n-1} {}_1F_1 \left(\frac{2n+1}{2}, 2n+1, -t \right) dt$$

and this becomes as $u \rightarrow \infty$

$$\frac{1}{2^{2n}\Gamma(n)} \int_0^\infty t^{n-1} {}_1F_1 \left(\frac{2n+1}{2}, 2n+1, -t \right) dt.$$

Using the integral representation (4.9) of ${}_1F_1$ this becomes

$$\frac{\Gamma(2n+1)}{2^{2n}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} \int_0^\infty t^{n-1} dt \int_0^1 e^{-t\xi} \xi^{n-\frac{1}{2}} (1-\xi)^{n-\frac{1}{2}} d\xi$$

$$= \frac{\Gamma(2n+1)}{2^{2n}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) + \left(n+\frac{1}{2}\right)} \int_0^1 d\xi \xi^{n-\frac{1}{2}} \int_0^\infty e^{-t\xi} t^{n-1} dt$$

$$\text{substituting } t = \frac{\bar{t}}{\xi}$$

$$= \frac{\Gamma(2n+1)}{2^{2n}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} \int_0^1 d\xi \xi^{-\frac{1}{2}} (1-\xi)^{n-\frac{1}{2}} \underbrace{\int_0^\infty e^{-\bar{t}} \bar{t}^{n-1} d\bar{t}}_{\Gamma(n)}$$

$$= \frac{\Gamma(2n+1)}{2^{2n}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} \underbrace{\int_0^1 \xi^{-\frac{1}{2}} (1-\xi)^{n-\frac{1}{2}} d\xi}_{B\left(\frac{1}{2}, n+\frac{1}{2}\right)}$$

$$\begin{aligned}
& \frac{\Gamma(2n+1)}{2^{2n}\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \\
&= \frac{\Gamma(2n+1)\Gamma\left(\frac{1}{2}\right)}{2^{2n}\Gamma\left(n+\frac{1}{2}\right)\Gamma(n+1)} = \frac{(2n)!}{2^{2n}n!\frac{1}{2}\frac{3}{2}\dots n-\frac{1}{2}} \\
&= \frac{1.2.3\dots 2n}{2.4\dots 2n.1.3\dots(2n-1)} = 1.
\end{aligned}$$

On the other hand, for the very same reason

$$\int_p^q u^n B^{*n} dt \rightarrow 0 \text{ as } u \rightarrow \infty.$$

Hence if $f(t)$ is a continuous function for nonnegative t then

$$\int_0^t f(t-\tau) u^n B^{*n}(\tau) d\tau \rightarrow f(t) \text{ as } u \rightarrow \infty.$$

So

$$\varphi(k) * u^n B^{*n} \rightarrow \varphi(k) \text{ as } u \rightarrow \infty.$$

Now it remains to examine

$$\begin{aligned}
& \psi(k) * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} u^n B^{*n} \\
&= \psi(k) * \sqrt{2} \frac{1-e^{-ut}}{2u\sqrt{\pi t^3}} * \left(\frac{1-e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2n} u^n \\
&= \psi(k) * \sqrt{2} u^n \left(\frac{1-e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2n+1} \\
&\text{using the formula 4.8 this becomes} \\
&= \psi(k) * \sqrt{2} u^n \frac{t^{n-\frac{1}{2}}}{2^{2n+1}\Gamma\left(\frac{2n+1}{2}\right)} {}_1F_1(n+1, 2n+2, -ut).
\end{aligned}$$

Now

$$\int_0^P \frac{u^n t^{n-\frac{1}{2}}}{2^{2n+1}\Gamma\left(\frac{2n+1}{2}\right)} {}_1F_1(\cdot, \cdot, -ut) dt = \frac{1}{\sqrt{u}} \int_0^\infty \frac{t^{n-\frac{1}{2}}}{2^{2n+1}\Gamma\left(\frac{2n+1}{2}\right)} {}_1F_1(\cdot, \cdot, -t) dt$$

and this vanishes as $u \rightarrow \infty$. Hence.

$$\psi(k) * \frac{1-e^{-ut}}{u\sqrt{2\pi t^3}} * u^n B^{*n} \rightarrow 0 \text{ as } u \rightarrow \infty.$$

Thus it has been proved that:

$$\begin{aligned} & \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P \left(-a \leq X_r \leq b | X_t = x \right) = \\ & \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \left(\frac{e^{-\frac{(2b-x)^2}{2t}}}{\sqrt{2\pi t}} + \frac{e^{-\frac{(2a+x)^2}{2t}}}{\sqrt{2\pi t}} \right) + \left(\frac{e^{-\frac{(2b+2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \frac{e^{-\frac{(2b+2a+x)^2}{2t}}}{\sqrt{2\pi t}} \right) \\ & - \left(\frac{e^{-\frac{(4b+2a-x)^2}{2t}}}{\sqrt{2\pi t}} + \frac{e^{-\frac{(2b+4a+x)^2}{2t}}}{\sqrt{2\pi t}} \right) + \dots \end{aligned} \quad (1.10)$$

and thus

$$\begin{aligned} & P \left(-a \leq X_r \leq b | X_t = x \right) = \\ & = 1 - \left(e^{-\frac{2b(b-x)}{t}} + e^{-\frac{2a(a+x)}{t}} \right) + \left(e^{-\frac{(2b+2a)(b+a-x)}{t}} \right. \\ & \quad \left. e^{-\frac{(2b+2a)(b+a+x)}{t}} \right) - \left(e^{-\frac{(4b+2a)(2b+a-x)}{t}} + e^{-\frac{(2b+4a)(b+2a+x)}{t}} \right) \\ & + \dots \end{aligned}$$

Putting $x = 0$ gives

$$\begin{aligned} & P(-a \leq X_r \leq b | X_t = 0) = \\ & = 1 - \left(e^{-\frac{2b^2}{t}} + e^{-\frac{2a^2}{t}} \right) + \left(e^{-\frac{2(a+b)^2}{t}} + e^{-\frac{2(a-b)^2}{t}} \right) \\ & - \left(e^{-\frac{2(2b+a)^2}{t}} + e^{-\frac{2(2a+b)^2}{t}} \right) + \left(e^{-\frac{2(2a+2b)^2}{t}} + e^{-\frac{2(2a-2b)^2}{t}} \right) \\ & - \dots \\ & = 1 - \left(e^{-\frac{2b^2}{t}} + e^{-\frac{2a^2}{t}} \right) + 2e^{-\frac{2(a+b)^2}{t}} - \left(e^{-\frac{2(2b+a)^2}{t}} + e^{-\frac{2(2a+b)^2}{t}} \right) \\ & + 2e^{-\frac{2(2a+2b)^2}{t}} - \dots \end{aligned}$$

When $a = b$ this is the limiting distribution of the two-sided Kolmogorov-Smirnov statistic:

$$\begin{aligned} & P \left(|X_r| \leq b | X_t = 0 \right) = \\ & = 1 - 2e^{-\frac{2b^2}{t}} + 2e^{-\frac{2(2b)^2}{t}} - 2e^{-\frac{2(3b)^2}{t}} + 2e^{-\frac{2(4b)^2}{t}} - \dots \end{aligned}$$

Next $P\left(-a \leq X_r \leq b \mid 0 \leq r \leq t\right)$ will be evaluated where X_t is the Brownian motion process

$$P\left(-a \leq X_r \leq b \mid 0 \leq r \leq t\right) = \int_{-a}^b \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P\left(-a \leq X_r \leq b \mid X_t = x\right) dx .$$

Integrating expression 4.10 we get

$$\begin{aligned} P\left(-a \leq X_r \leq b \mid 0 \leq r \leq t\right) &= \\ &= -1 + 2 \left[\phi\left(\frac{b}{\sqrt{t}}\right) + \phi\left(\frac{a}{\sqrt{t}}\right) \right] - 2 \left[\phi\left(\frac{a+2b}{\sqrt{t}}\right) + \phi\left(\frac{2a+b}{\sqrt{t}}\right) \right] \\ &\quad + 2 \left[\phi\left(\frac{3a+2b}{\sqrt{t}}\right) + \phi\left(\frac{2a+3b}{\sqrt{t}}\right) \right] - \dots \\ &= \left[\operatorname{erf}\left(\frac{b}{\sqrt{2t}}\right) + \operatorname{erf}\left(\frac{a}{\sqrt{2t}}\right) \right] - \left[\operatorname{erf}\left(\frac{a+2b}{\sqrt{2t}}\right) + \operatorname{erf}\left(\frac{2a+b}{\sqrt{2t}}\right) \right] \\ &\quad + \left[\operatorname{erf}\left(\frac{3a+2b}{\sqrt{2t}}\right) + \operatorname{erf}\left(\frac{2a+3b}{\sqrt{2t}}\right) \right] - \dots \end{aligned}$$

Chapter 5

The local time of Brownian motion.

5.1 Introduction.

In this chapter first we shall evaluate the distribution of Brownian local time. This will be done through computing the limiting distribution of $L_{-b,b}(t) = \frac{\lambda\left(X_r \in (-b,b) \atop 0 \leq r \leq t\right)}{2b}$ as $b \rightarrow 0$. Paul Lévy discovered that this limiting distribution is the same as the distribution of $\sup_{0 \leq r \leq t} X_r$ [Lévy 1]. After rederiving Lévy's result we consider the analogous question for the case of the tied down Brownian motion and the Brownian bridge process. Then we evaluate the distribution of the oblique local time defined in chapter 1. This will be done by computing the limiting distribution of $L_{-b,b}^m(t) = \frac{\lambda\left(X_r \in T \atop 0 \leq r \leq t\right)}{2b}$ as $b \rightarrow 0$ where T is the domain: $T = \{(x, \tau) : 0 \leq \tau \leq t), -b + m\tau \leq x \leq b + m\tau, m \geq 0\}$.

The following characterisations of the local-time $L(t)$ are well known: (see pages 43 and 48 of Ito-McKean)

1. If N_b is the number of times that X_r crosses down from b to 0 up to the time t then

$$\lim_{b \rightarrow 0} bN_b = 2L(t) \text{ a.s.} \quad (5.1)$$

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NBS 1010a ANSI/ISO #2 EQUIVALENT

2. If η_b is the number of excursion intervals in $(0, t)$ with length $\geq b$ then

$$\lim_{b \rightarrow 0} \sqrt{\frac{\pi b}{2}} \eta_b = 2L(t) \text{ a.s.} \quad (5.2)$$

3. If ξ_b is the total length of excursion intervals $< b$ then

$$\lim_{b \rightarrow 0} \sqrt{\frac{\pi}{2b}} \xi_b = 2L(t) \text{ a.s.} \quad (5.3)$$

We shall examine how property 1 can be extended to the oblique local time case, by computing the limiting distribution of bN_b as $b \rightarrow 0$.

5.2 The local time of the unrestricted Brownian motion.

In this section we shall be concerned with $\lambda \left(X_\tau \in (-a, b) \right)_{0 \leq \tau \leq t}$ rather than as before with $\lambda \left(X_\tau \notin (-a, b) \right)_{0 \leq \tau \leq t}$, that is we shall examine the behaviour of the time spent in the interval $(-a, b)$ by the Brownian motion. To be more precise we shall study the limiting behaviour of $\frac{\lambda \left(X_\tau \in (-a, b) \right)_{0 \leq \tau \leq t}}{a+b}$ as the width of the zone $(-a \leq x \leq b, 0 \leq \tau \leq t)$ $a+b \rightarrow 0$.

First we shall examine the case of the unrestricted Brownian motion. Consider the expectations

$$\begin{aligned} E \left(e^{-u\lambda(X_\tau \notin (a,b))} | X_t = x \right) &= \\ \text{using: } \lambda(X_\tau \notin (-a,b)) &= t - \lambda(X_\tau \in (-a,b)) \\ &= e^{-ut} E \left(e^{u\lambda(X_\tau \in (-a,b))} | X_t = x \right) \end{aligned}$$

Hence

$$E \left(e^{u\lambda(X_\tau \in (-a,b))} | X_t = x \right) = e^{ut} E \left(e^{-u\lambda(X_\tau \notin (a,b))} | X_t = x \right)$$

Clearly this equation is true for the unrestricted Brownian motion as well since

$$E \left(e^{-\lambda(X_T \mathbb{1}_{(-a,b)})} \right) = \int_{-\infty}^{\infty} dx Q_{(-a,b)}(x, t) \quad (5.4)$$

To simplify the investigation let the strip be equally spaced on both sides of $x = 0$, that is let $a = b$. Thus the (5.4) integral will be the sum of the integral of the upper function from b to ∞ , the integral of the lower function from $-\infty$ to $-b$, and the integral of the middle function from $-b$ to b .

We can however, dispense with the integration of the middle function since we are interested only on the limiting behaviour of the expectation as $b \rightarrow 0$, and as $b \rightarrow 0$ all x will ultimately fall outside of the interval $(-b, b)$.

We have shown (4.1) that the upper function:

$$Q * (\psi(b) - \psi(2a + b) * uB + \psi(2a + 3b) * u^2 B^{*2} \dots)$$

with

$$Q = Q(x - b) = e^{-ut} \left(\psi(x - b) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right)$$

and the lower function (4.3):

$$Q * (\psi(a) - \psi(a + 2b) * uB + \psi(3a + 2b) * u^2 B^{*2} \dots)$$

with

$$Q = e^{-ut} \left(\psi(-x - a) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right)$$

and

$$B = \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2}.$$

Putting $a = b$ for both functions we have

$$Q * (\psi(b) - \psi(3b) * uB + \psi(5b) * u^2 B^{*2} \dots)$$

the only difference being in Q . For the upper function it is $Q(x - b)$ and for the lower function it is $Q(-x - b)$. But

$$\int_b^{\infty} Q(x - b) dx = \int_0^{\infty} Q(x) dx$$

and

$$\int_{-\infty}^{-b} Q(-x-b)dx = \int_0^{\infty} Q(x)dx$$

and

$$\begin{aligned} \int_0^{\infty} Q(x)dx &= e^{-ut} \int_0^{\infty} dx \psi(x) \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \\ &= e^{-ut} \left(\frac{1}{\sqrt{2\pi t}} * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} \right) = \frac{e^{-ut}}{\sqrt{2\pi t}} * \frac{1 - e^{-ut}}{u\sqrt{2\pi t^3}} \\ &= \frac{e^{-ut}}{\sqrt{\pi t}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}}. \end{aligned}$$

Hence the sum of the integrals of the upper and lower functions is

$$2 \frac{e^{-ut}}{\sqrt{\pi t}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * (\psi(b) - \psi(3b) * uB + \dots).$$

Using $B = \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2}$ this becomes:

$$2 \frac{e^{-ut}}{\sqrt{\pi t}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * \psi(b) * \sum_{n=1}^{\infty} \psi(2nb) (-1)^n * u^n \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2n}.$$

We are going to compute the limiting distribution of $L_{-b,b}(t) = \frac{\lambda \left(X_r \in (-b,b) \right)}{2b}$ as $b \rightarrow 0$.

By multiplying the integral of the upper and lower function by e^{ut} we have

$$\begin{aligned} &e^{ut} \left(\frac{2e^{-ut}}{\sqrt{\pi t}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * \psi(b) * \sum_{n=0}^{\infty} \psi(2nb) * (-1)^n u^n \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2n} \right) \\ &= \frac{2}{\sqrt{\pi t}} * \frac{e^{ut} - 1}{2u\sqrt{\pi t^3}} * e^{ut} \psi(b) * \sum_{n=0}^{\infty} e^{ut} \psi(2nb) * (-1)^n u^n \left(\frac{e^{ut} - 1}{2u\sqrt{\pi t^3}} \right)^{*2n}. \end{aligned}$$

This gives $E \left(e^{u\lambda(X_r \in (-b,b))} \right)$ except that the value contributed by the middle integral is missing.

Put $-u$ for u . Then

$$\begin{aligned} &E \left(e^{-u\lambda(X_r \in (-b,b))} \right) \\ &= \frac{2}{\sqrt{\pi t}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * e^{-ut} \psi(b) \sum_{n=0}^{\infty} e^{-ut} \psi(2nb) * u^n \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2n} \quad (5.5) \\ &\quad + \text{term due to the middle integral.} \end{aligned}$$

Now

$$\begin{aligned}
& \int e^{-u\xi} dP_\xi(L(t) \leq \xi) \\
&= \lim_{b \rightarrow 0} \int e^{-u\xi} dP_\xi(L_{-b,b}(t) \leq \xi) \\
&= \lim_{b \rightarrow 0} \int e^{-u\xi} dP_\xi\left(\lambda(X_\tau \in (-b, b)) \leq 2b\xi\right) \\
&= \lim_{b \rightarrow 0} \int e^{-\frac{u}{2b}\xi} dP_\xi(\lambda(X_\tau \in (-b, b)) \leq \xi) \\
&= \lim_{b \rightarrow 0} E\left(e^{-\frac{u}{2b}\lambda(X_\tau \in (-b, b))}\right).
\end{aligned}$$

Thus $\int e^{-u\xi} dP_\xi(L(t) \leq \xi)$ is computed by putting $\frac{u}{2b}$ instead of u in $E(e^{-u\lambda(X_\tau \in (-b, b))})$, taking the limit $b \rightarrow 0$, and then inverting the limit to get the distribution of $L(t)$. Rather than working directly with $E(e^{-\lambda(X_\tau \in (-b, b))})$ we shall use its Laplace transform, and then we will have to perform two inversions in order to get the distribution of $L(t)$.

The Laplace transform of

$$\frac{2}{\sqrt{\pi t}} * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * e^{-ut}\psi(b) * \sum_{n=0}^{\infty} e^{-ut}\psi(2nb) * u^n \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}}\right)^{2n}$$

is

$$\frac{2}{\sqrt{s}} \cdot \frac{\sqrt{s+u} - \sqrt{s}}{u} \cdot e^{-\sqrt{2(s+u)}b} \cdot \sum_{n=0}^{\infty} e^{-\sqrt{2(s+u)}2nb} u^n \left(\frac{\sqrt{s+u} - \sqrt{s}}{u}\right)^{2n}.$$

For $\frac{1}{\sqrt{s}}$ is the Laplace transform of $\frac{1}{\sqrt{\pi t}}$ and $\frac{\sqrt{s+u} - \sqrt{s}}{u}$ is the Laplace transform of $\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}}$ and $e^{-\sqrt{2s} \cdot 2nb}$ is the Laplace transform of $\psi(2nb)$. The series

$$\sum_{n=0}^{\infty} e^{-\sqrt{2(s+u)} \cdot 2nb} u^n \left(\frac{\sqrt{s+u} - \sqrt{s}}{u}\right)^{2n} = \sum_{n=0}^{\infty} e^{-\sqrt{2(s+u)} \cdot 2nb} \left(\frac{\sqrt{s+u} - \sqrt{s}}{\sqrt{u}}\right)^{2n}$$

is convergent if b is large enough, and being a simple geometric series its sum is:

$$\frac{1}{1 - e^{-\sqrt{2(s+u)} \cdot 2b} \left(\frac{\sqrt{s+u} - \sqrt{s}}{\sqrt{u}}\right)^2}.$$

Hence the Laplace transform can be written as

$$\frac{2}{\sqrt{s}} \cdot \frac{\sqrt{s+u} - \sqrt{s}}{u} \frac{e^{-\sqrt{2(s+u)} \cdot b}}{1 - e^{-\sqrt{2(s+u)} 2b} \left(\frac{\sqrt{s+u} - \sqrt{s}}{\sqrt{u}} \right)^2}.$$

Let us put $\frac{u}{2b}$ for u . Then we get:

$$\begin{aligned} & \frac{2}{\sqrt{s}} \frac{\sqrt{s + \frac{u}{2b}} - \sqrt{s}}{\frac{u}{2b}} \cdot \frac{e^{-\sqrt{(bs+u)} \sqrt{b}}}{1 - e^{-\sqrt{bs+u} 2\sqrt{b}} \left(\frac{\sqrt{2bs+u} - \sqrt{2bs}}{\sqrt{u}} \right)^2} \\ &= \frac{2}{\sqrt{s}} \sqrt{2b} \frac{\sqrt{2bs+u} - \sqrt{2bs}}{u} \frac{e^{-\sqrt{(bs+u)} \sqrt{b}}}{1 - e^{-\sqrt{bs+u} 2\sqrt{b}} + \left(-\frac{4bs}{u} + 2\sqrt{\frac{2bs}{u}} \sqrt{1 + \frac{2bs}{u}} \right) e^{-\sqrt{bs+u} 2\sqrt{b}}} \end{aligned}$$

Taking the limit $b \rightarrow 0$ this becomes

$$\begin{aligned} & \frac{2}{\sqrt{s}} \sqrt{2} \frac{\sqrt{u}}{u} \frac{1}{\sqrt{u} \cdot 2 + 2\sqrt{\frac{2s}{u}}} \\ &= \frac{2\sqrt{2}}{\sqrt{s}} \cdot \frac{1}{\sqrt{u}} \frac{1}{2\sqrt{u} + 2\sqrt{\frac{2s}{u}}} \\ &= \sqrt{\frac{2}{s}} \frac{1}{u + \sqrt{2s}}. \end{aligned}$$

Now what remains is to invert this function first with respect to s and then with respect to u .

$$\begin{aligned} \sqrt{\frac{2}{s}} \frac{1}{u + \sqrt{2s}} &= \sqrt{\frac{2}{s}} \frac{\sqrt{2s} - u}{2s - u^2} \\ &= \frac{2}{\sqrt{2s}} \frac{\sqrt{2s} - u}{2s - u^2} = \frac{2}{2s - u^2} - \frac{2u}{\sqrt{2s}(2s - u^2)} \\ &= \frac{1}{s - \frac{u^2}{2}} - \frac{u}{\sqrt{2s} \left(s - \frac{u^2}{2} \right)}. \end{aligned}$$

This can now be easily inverted since $\frac{1}{s - \frac{u^2}{2}}$ is the Laplace transform of $e^{\frac{u^2}{2}t}$ and $\frac{1}{\sqrt{2s}}$ is the Laplace transform of $\frac{1}{\sqrt{2\pi t}}$. So the result of the inversion is:

$$e^{+\frac{u^2}{2}t} - \frac{u}{\sqrt{2\pi}} \int_0^t d\xi e^{\frac{u^2}{2}(\iota - \xi)} \frac{d\xi}{\xi^{1/2}}.$$

In order to bring this into the form $\int e^{-u\xi} f(\xi) d\xi$ we continue with

$$\begin{aligned}
&= e^{\frac{u^2}{2}t} - \frac{u}{\sqrt{2\pi}} e^{\frac{u^2}{2}t} \int_0^t e^{-\frac{u^2}{2}\xi} \frac{d\xi}{\xi^{1/2}} \quad \text{Putting: } \xi = \bar{\xi}^2 \\
&= e^{\frac{u^2}{2}t} - \frac{2u}{\sqrt{2\pi}} e^{\frac{u^2}{2}t} \int_0^{\sqrt{t}} e^{-\frac{u^2}{2}\xi^2} d\xi \quad \text{substituting } \frac{u\xi}{\sqrt{2}} = \bar{\xi} \\
&= e^{\frac{u^2}{2}t} - \frac{2u}{u} e^{\frac{u^2}{2}t} \frac{\sqrt{2}}{\sqrt{2\pi}} \int_0^{\sqrt{t}} e^{-\bar{\xi}^2} d\bar{\xi} \\
&= e^{\frac{u^2}{2}t} - \frac{2}{\sqrt{\pi}} e^{\frac{u^2}{2}t} \int_0^{u\sqrt{\frac{t}{2}}} e^{-\bar{\xi}^2} d\bar{\xi} \\
&= \frac{2}{\sqrt{\pi}} e^{\frac{u^2}{2}t} \int_{u\sqrt{\frac{t}{2}}}^{\infty} e^{-\bar{\xi}^2} d\bar{\xi} = \frac{2}{\sqrt{\pi}} e^{\frac{u^2}{2}t} \int_0^{\infty} e^{-(\bar{\xi} + u\sqrt{\frac{t}{2}})^2} d\bar{\xi} \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-(\bar{\xi}^2 + \frac{2u\bar{\xi}\sqrt{t}}{\sqrt{2}})} d\bar{\xi} \quad \text{Let } \bar{\xi} = \frac{\xi}{\sqrt{2t}} \\
&= \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{2t}} \int_0^{\infty} e^{-u\xi} e^{-\frac{\xi^2}{2t}} d\xi = \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{2t}} \int_0^{\infty} e^{-u\xi} d\xi \int_0^{\xi} e^{-\frac{\eta^2}{2t}} d\eta \\
&= \int_0^{\infty} e^{-u\xi} d\xi \frac{2}{\sqrt{\pi}} \int_0^{\frac{\xi}{\sqrt{2t}}} e^{-\eta^2} d\eta.
\end{aligned}$$

Thus we proved:

Theorem 5.1 *The distribution of Brownian local time:*

$$P(L(t) \leq \xi) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{\xi}{\sqrt{2t}}} e^{-\eta^2} d\eta.$$

5.3 The local time of the tied down Brownian motion.

We shall examine now the behaviour of the tied-down Brownian motion as $b \rightarrow 0$. Consider only the $x \geq 0$ case. (The $x < 0$ case being similar.)

The upper function

$$\begin{aligned}
\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E \left(e^{-u\lambda(X_{\tau} \notin (-b, b))} | X_t = x \right) &= Q * \left(\psi(b) - \psi(3b) * uB + \psi(5b)u^2B^{*2} \dots \right) \\
Q &= e^{-ut} \left(\psi(x-b) * \frac{e^{+ut} - 1}{u\sqrt{\pi t^3}} \right).
\end{aligned}$$

So:

$$\begin{aligned} & \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E \left(e^{u\lambda(X_r \in (b,b))} \mid X_t = x \right) \\ &= \psi(x-b) * \frac{e^{ut} - 1}{u\sqrt{2\pi t^3}} * \left(e^{ut}\psi(b) - e^{ut}\psi(3b) * u \left(\frac{e^{ut} - 1}{2u\sqrt{\pi t^3}} \right)^{*2} \dots \right). \end{aligned}$$

Putting now $-u$ for u gives

$$\begin{aligned} & \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} E \left(e^{-u\lambda(X_r \in (b,b))} \mid X_t = x \right) \\ &= \psi(x-b) * \frac{1 - e^{-ut}}{u\sqrt{\pi t^3}} * \left(e^{-ut}\psi(b) + e^{-ut}\psi(3b) * u \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2} \dots \right) \\ &= \sqrt{2}\psi(x-b) * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * \left(e^{-ut}\psi(b) + e^{-ut}\psi(3b) * u \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2} \right. \\ & \quad \left. + e^{-ut}\psi(5b) * u^2 \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*4} \dots \right) \\ &= \sqrt{2}\psi(x-b) * \frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} * e^{-ut}\psi(b) * \sum_{n=0}^{\infty} e^{-ut}\psi(2nb) * u^n \left(\frac{1 - e^{-ut}}{2u\sqrt{\pi t^3}} \right)^{*2}. \end{aligned}$$

Comparing this with formula (5.5) we see the only difference is that here we have $\sqrt{2}\psi(x-b)$ instead of $\frac{2}{\sqrt{\pi t}}$. Neither of them depends on u so they play no role in the limiting process. Taking the Laplace transformation again we have:

$$\begin{aligned} & \sqrt{2}e^{-\sqrt{2s}(x-b)} \frac{\sqrt{s+u} - \sqrt{s}}{u} \cdot e^{-\sqrt{2(s+u)}b} \sum_{n=0}^{\infty} e^{-\sqrt{2(s+u)}2b} \left(\frac{\sqrt{s+u} - \sqrt{s}}{\sqrt{u}} \right)^{2n} \\ &= \sqrt{2}e^{-\sqrt{2s}(x-b)} \frac{\sqrt{s+u} - \sqrt{s}}{u} \frac{e^{\sqrt{2(s+u)}b}}{1 - e^{-\sqrt{2(s+u)}2b}} \left(\frac{\sqrt{s+u} - \sqrt{s}}{\sqrt{u}} \right)^2. \end{aligned}$$

Putting $\frac{u}{2b}$ instead of u and taking the limit $b \rightarrow 0$ gives

$$e^{-\sqrt{2sx}} \frac{1}{u + \sqrt{2s}}.$$

It remains to execute the two required inversions on

$$\frac{e^{-\sqrt{2sx}}}{u + \sqrt{2s}} = e^{-\sqrt{2sx}} \frac{\sqrt{2s} - u}{2s - u^2} = \frac{e^{-\sqrt{2sx}}}{2} \frac{\sqrt{2s} - u}{s - \frac{u^2}{2}}.$$

$$\begin{aligned}
&= \frac{1}{2} e^{-\sqrt{2sx}} \frac{\sqrt{2s}}{s - \frac{u^2}{2}} - \frac{1}{2} e^{-\sqrt{2sx}} \frac{u}{s - \frac{u^2}{2}} \\
&= \frac{e^{-\sqrt{2sx}}}{\sqrt{2s}} \frac{s}{s - \frac{u^2}{2}} - \frac{1}{2} e^{-\sqrt{2sx}} \frac{u}{s - \frac{u^2}{2}}
\end{aligned}$$

This last function can now be easily inverted with respect to s .

For if x is an arbitrary function of t then

$$\mathcal{L}(x') = -x(0) + s\mathcal{L}(x)$$

Now $\frac{e^{-\sqrt{2sx}}}{\sqrt{2s}}$ is the Laplace transform of $\frac{e^{-\frac{x^2}{2t}}}{\sqrt{\pi t}}$ which is zero at zero. So $s \frac{e^{-\sqrt{2sx}}}{\sqrt{2s}}$ is the Laplace transform of

$$\frac{d}{dt} \left(\frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \right) = \frac{1}{2} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi}} \left(\frac{x^2}{\sqrt{t^{5/2}}} - \frac{1}{t^{3/2}} \right)$$

Thus the inversion with respect to s gives:

$$\begin{aligned}
&\int_0^t \frac{e^{-\frac{x^2}{2\tau}}}{2\sqrt{2\pi}} \left(\frac{x^2}{\tau^{5/2}} - \frac{1}{\tau^{3/2}} \right) e^{\frac{u^2}{2}(t-\tau)} d\tau - \frac{u}{2} \int_0^t \frac{x e^{-\frac{x^2}{2\tau}}}{\sqrt{2\pi}} \frac{1}{\tau^{3/2}} e^{\frac{u^2}{2}(t-\tau)} d\tau \\
&= \frac{e^{\frac{u^2}{2}t}}{2} \int_0^t \left(x^2 \frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{5/2}} - \frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{3/2}} - ux \frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{3/2}} \right) d\tau
\end{aligned}$$

where $E = e^{-\frac{u^2}{2}\tau - \frac{x^2}{2\tau}}$.

Let us observe that $\frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{5/2}}$ is the derivative of:

$$\frac{2}{x^2} \frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{1/2}} + \frac{1}{x^2} \left(u - \frac{1}{x} \right) e^{ux} \phi(++) - \frac{1}{x^2} \left(u + \frac{1}{x} \right) e^{-ux} \phi(-+)$$

and $\frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{3/2}}$ is the derivative of

$$-\frac{1}{x} \left(e^{ux} \Phi(++) + e^{-ux} \Phi(-+) \right)$$

where

$$\Phi(++) = \Phi \left(u\sqrt{t} + \frac{x}{\sqrt{t}} \right), \quad \Phi(-+) = \Phi \left(-u\sqrt{t} + \frac{x}{\sqrt{t}} \right)$$

and

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^k e^{-\xi^2/2} d\xi.$$

Substituting these into the last integral gives:

$$\begin{aligned} & \frac{e^{\frac{u^2}{2t}}}{2} \left\{ x^2 \left[\frac{2}{x^2} \frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{1/2}} + \frac{1}{x^2} \left(u - \frac{1}{x} \right) e^{ux} \Phi(++) - \frac{1}{x^2} \left(u + \frac{1}{x} \right) e^{-ux} \Phi(-+) \right] \right\}_{\tau=0}^{\tau=t} \\ & + \frac{1}{x} \left[e^{ux} \Phi(++) + e^{-ux} \Phi(-+) \right]_{\tau=0}^{\tau=t} + \frac{ux}{x} \left[e^{ux} \Phi(++) + e^{-ux} \Phi(-+) \right]_{\tau=0}^{\tau=t} \Big\} \\ & = \frac{e^{\frac{u^2}{2}t}}{2} \left\{ \frac{2E}{\sqrt{2\pi}} \frac{1}{\tau^{1/2}} + 2ue^{ux} \Phi(++) \right\}_{\tau=0}^{\tau=t} = e^{\frac{u^2}{2}t} \left[\frac{E}{\sqrt{2\pi}} \frac{1}{\tau^{1/2}} + ue^{ux} \Phi(++) \right]_{\tau=0}^{\tau=t} \\ & = e^{\frac{u^2}{2}t} \left[\frac{e^{-\frac{u^2}{2}\tau - \frac{\tau^2}{2\tau}}}{\sqrt{2\pi}} \frac{1}{\tau^{1/2}} + ue^{ux} \Phi \left(u\sqrt{\tau} + \frac{x}{\sqrt{\tau}} \right) \right]_{\tau=0}^{\tau=t} \\ & = e^{\frac{u^2}{2}t} \left(\frac{e^{-\frac{u^2}{2}t - \frac{\tau^2}{2t}}}{\sqrt{2\pi t}} + ue^{ux} \left(\Phi \left(u\sqrt{t} + \frac{x}{\sqrt{t}} \right) - 1 \right) \right) \\ & = e^{\frac{u^2}{2}t} \left(\frac{e^{-\frac{u^2}{2}t - \frac{\tau^2}{2t}}}{\sqrt{2\pi t}} - \frac{ue^{ux}}{\sqrt{2\pi}} \int_{u\sqrt{t} + \frac{x}{\sqrt{t}}}^{\infty} e^{-\xi^2/2} d\xi \right) \\ & = \frac{e^{-\frac{\tau^2}{2t}}}{\sqrt{2\pi t}} - \frac{ue^{\frac{u^2}{2}t + ux}}{\sqrt{\pi}} \int_{u\sqrt{\frac{t}{2}} + \frac{x}{\sqrt{2t}}}^{\infty} e^{-\xi^2} d\xi \\ & = \frac{e^{-\frac{\tau^2}{2t}}}{\sqrt{2\pi t}} - \frac{ue^{\frac{u^2}{2}t + ux}}{\sqrt{\pi}} \int_0^{\infty} e^{-\left(\xi + u\sqrt{\frac{t}{2}} + \frac{x}{\sqrt{2t}}\right)^2} d\xi \\ & = \frac{e^{-\frac{\tau^2}{2t}}}{\sqrt{2\pi t}} - \frac{ue^{\frac{u^2}{2}t + ux}}{\sqrt{\pi t}} \int_0^{\infty} d\xi e^{-\left(\xi^2 + \frac{u^2}{2}t + \frac{\tau^2}{2t} + 2\xi \left(u\sqrt{\frac{t}{2}} + \frac{x}{\sqrt{2t}} \right) + ux \right)} \\ & = \frac{e^{-\frac{\tau^2}{2t}}}{\sqrt{2\pi t}} - \frac{u}{\sqrt{\pi}} \int_0^{\infty} d\xi e^{-\left(\xi^2 + \frac{\tau^2}{2t} + 2\xi u\sqrt{\frac{t}{2}} + \frac{2\xi x}{\sqrt{2t}}\right)}. \end{aligned}$$

If we put $\xi = \frac{\bar{\xi}}{\sqrt{2t}}$, this becomes:

$$= \frac{e^{-\frac{\tau^2}{2t}}}{\sqrt{2\pi t}} - \frac{u}{\sqrt{2\pi t}} \int_0^{\infty} d\bar{\xi} e^{-\left(\frac{\bar{\xi}^2}{2t} + \frac{\tau^2}{2t} + u\bar{\xi} + \frac{2\bar{\xi}x}{2t}\right)} = \frac{e^{-\frac{\tau^2}{2t}}}{\sqrt{2\pi t}} - \frac{u}{\sqrt{2\pi t}} \int_0^{\infty} d\bar{\xi} e^{-u\bar{\xi}} \cdot e^{-\frac{(\bar{\xi}+x)^2}{2t}}.$$

Integrating by parts gives

$$\begin{aligned}
& \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} + \left[\frac{e^{-u\xi} e^{-\frac{(\xi+x)^2}{2t}}}{\sqrt{2\pi t}} \right]_{\xi=0}^{\xi=\infty} + \frac{1}{\sqrt{2\pi t}} \int_0^{\infty} d\xi e^{-u\xi} e^{-\frac{(x+\xi)^2}{2t}} \cdot \frac{\xi+x}{t} \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^{\infty} d\xi e^{-u\xi} e^{-\frac{(\xi+x)^2}{2t}} \cdot \frac{\xi+x}{t} \\
&= \frac{1}{\sqrt{2\pi t}} \int_0^{\infty} e^{-u\xi} d\xi \int_0^{\xi} e^{-\frac{(\eta+x)^2}{2t}} \frac{\eta+x}{t} d\eta \\
&= \int_0^{\infty} e^{-u\xi} d\xi \left(\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} \right).
\end{aligned}$$

Then

$$\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(L(t) \leq \xi | X_t = x) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} \quad (5.6)$$

or $P(L(t) \leq \xi) = 1 - e^{-\frac{\xi(\xi+2x)}{2t}}$. It was assumed that $x \geq 0$; for arbitrary x

$$P(L(t) \leq \xi | X_t = x) = 1 - e^{-\frac{\xi(\xi+2|x|)}{2t}} \quad (5.7)$$

that is, x is to be replaced by $|x|$.

A simple integration shows that

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} P(L(t) \leq \xi | X_t = x) dx = P(L(t) \leq \xi).$$

That is,

$$\begin{aligned}
& \int_{-\infty}^{\infty} \left(\frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(|x|+\xi)^2}{2t}}}{\sqrt{2\pi t}} \right) dx = 1 - \int_0^{\infty} \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} - \int_{-\infty}^0 \frac{e^{-\frac{(\xi-x)^2}{2t}}}{\sqrt{2\pi t}} dx \\
&= 1 - 2 \int_0^{\infty} \frac{e^{-\frac{(x+\xi)^2}{2t}}}{\sqrt{2\pi t}} dx = 1 - 2 \int_{\xi}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} dx = 1 - \frac{2}{\sqrt{\pi}} \int_{\frac{\xi}{\sqrt{2t}}}^{\infty} e^{-x^2} dx \\
&= \frac{2}{\sqrt{\pi}} \int_0^{\frac{\xi}{\sqrt{2t}}} e^{-x^2} dx = P(L(t) \leq \xi).
\end{aligned}$$

Consider now the more general problem: let

$$L_{-b,b}(t_1) = \lambda(X_{\tau} \in (-b,b)) / 2b \leq \xi | X_t = x, \quad 0 \leq \tau \leq t_1 \leq t$$

Now:

$$\begin{aligned} \lim_{b \rightarrow 0} P(L_{-b,b}(t_1) \leq \xi) &= P(L(t_1) \leq \xi | X_t = x) \\ &= \sqrt{\frac{t}{2\pi t_1(t-t_1)}} \int_{-\infty}^{\infty} \left(1 - e^{-\frac{\xi(\xi+2|z|)}{2t_1}}\right) e^{-\frac{z^2}{2t_1} - \frac{(x-z)^2}{2(t-t_1)} + \frac{x^2}{2t}} dz \end{aligned}$$

where $\sqrt{\frac{t}{2\pi t_1(t-t_1)}} e^{-\frac{z^2}{2t_1} - \frac{(x-z)^2}{2(t-t_1)} + \frac{x^2}{2t}}$ is the density function of the tied-down Brownian motion $X_t = x$ and

$$1 - e^{-\frac{\xi(\xi+2|z|)}{2t_1}} = P(L(t_1) \leq \xi | X_{t_1} = x).$$

Now we shall evaluate the above integral. It can be written as $I_1 - I_2 - I_3$.

Substituting:

$$-\frac{z^2}{2t_1} - \frac{(x-z)^2}{2(t-t_1)} + \frac{x^2}{2t} = -\frac{t}{2t_1(t-t_1)} \left(z - \frac{xt_1}{t}\right)^2$$

we obtain

$$\begin{aligned} I_1 &= \sqrt{\frac{t}{2\pi t_1(t-t_1)}} \int_{-\infty}^{\infty} dz e^{-\frac{t}{2t_1(t-t_1)} \left(z - \frac{xt_1}{t}\right)^2} = 1 \\ I_2 &= \sqrt{\frac{t}{2\pi t_1(t-t_1)}} \int_0^{\infty} dz e^{-\frac{\xi(\xi+2z)}{2t_1} - \frac{t}{2t_1(t-t_1)} \left(z - \frac{xt_1}{t}\right)^2} \\ I_3 &= \sqrt{\frac{t}{2\pi t_1(t-t_1)}} \int_{-\infty}^0 dz e^{-\frac{\xi(\xi-2z)}{2t_1} - \frac{t}{2t_1(t-t_1)} \left(z - \frac{xt_1}{t}\right)^2} \\ &= \sqrt{\frac{t}{2\pi t_1(t-t_1)}} \int_0^{\infty} dz e^{-\frac{\xi(\xi+2z)}{2t_1} - \frac{t}{2t_1(t-t_1)} \left(z + \frac{xt_1}{t}\right)^2}. \end{aligned}$$

First we shall compute I_2 :

$$\begin{aligned} \frac{\xi(\xi+2z)}{t_1} + \frac{t}{t_1(t-t_1)} \left(z - \frac{xt_1}{t}\right)^2 &= \\ \frac{\xi^2 + 2\xi z}{t_1} + \frac{t}{t_1(t-t_1)} \left(z^2 - \frac{2zxt_1}{t} + \frac{x^2 t_1^2}{t^2}\right) &= \\ = \frac{t}{t_1(t-t_1)} \left(z + \frac{\xi(t-t_1) - xt_1}{t}\right)^2 + \frac{\xi(\xi+2x)}{t}. \end{aligned}$$

Substituting this for the exponent, we get:

$$\begin{aligned}
I_2 &= \sqrt{\frac{t}{2\pi t_1(t-t_1)}} \int_0^\infty dz e^{-\frac{t}{2t_1(t-t_1)} \left(z + \frac{\xi(t-t_1) - xt_1}{t}\right)^2 - \frac{\xi(\xi+2x)}{2t}} \\
&= \sqrt{\frac{t}{2\pi t_1(t-t_1)}} e^{-\frac{\xi(\xi+2x)}{2t}} \int_{\frac{\xi(t-t_1) - xt_1}{t}}^\infty e^{-\frac{t}{2t_1(t-t_1)} z^2} dz \\
&= \frac{e^{-\frac{\xi(\xi+2x)}{2t}}}{\sqrt{2\pi}} \int_{\frac{\xi(t-t_1) - xt_1}{\sqrt{t t_1(t-t_1)}}}^\infty e^{-z^2/2} dz \\
&= e^{-\frac{\xi(\xi+2x)}{2t}} \Phi\left(\frac{xt_1 - \xi(t-t_1)}{\sqrt{t t_1(t-t_1)}}\right)
\end{aligned}$$

and this is the value of I_2 .

If in I_2 we put $-x$ for x we get I_3 .

Hence

$$I_3 = e^{-\frac{\xi(\xi-2x)}{2t}} \Phi\left(\frac{-xt_1 - \xi(t-t_1)}{\sqrt{t t_1(t-t_1)}}\right)$$

and this gives:

Theorem 5.2

$$\begin{aligned}
P(L(t_1) \leq \xi | X_t = x) \\
0 \leq t_1 \leq t \\
= 1 - e^{-\frac{\xi^2}{2t}} \left[e^{-\frac{\xi x}{t}} \Phi\left(\frac{xt_1 - \xi(t-t_1)}{\sqrt{t t_1(t-t_1)}}\right) + e^{\frac{\xi x}{t}} \Phi\left(\frac{-xt_1 - \xi(t-t_1)}{\sqrt{t t_1(t-t_1)}}\right) \right].
\end{aligned}$$

For the special case $x = 0$, that is for the Brownian bridge process:

$$P(L(t_1) \leq \xi | X_t = 0) = 1 - 2e^{-\frac{\xi^2}{2t}} \Phi\left(\frac{-\xi(t-t_1)}{\sqrt{t t_1(t-t_1)}}\right).$$

Remark:

When $t_1 \rightarrow t$ this last formula goes into

$$1 - e^{-\frac{\xi^2 + 2\xi x}{2t}} = P(L(t) \leq \xi | X_t = x)$$

as it is expected.

The method used in this section can also be used to compute distribution of Brownian local time when the Brownian motion is conditioned to stay in an interval $[-a, b]$ or compute the distribution of $L(FC(a))$ where $FC(a)$ is the stopping time $t = \{\inf \tau : X_\tau = a\}$.

5.4 The distribution of the oblique local time.

We shall consider now the limiting behaviour of the Brownian motion between two lines that are not parallel to the t axis: $x = mt + b$ and $x = mt - b$. Let the domain between the two parallel lines to be denoted by $T_{m,b}$.

We want to find the limiting distribution of

$$L_{-b,b}^m = \frac{O_{T_{m,b}}}{2b}.$$

as $b \rightarrow 0$.

Since the law of X_τ conditioned on $X_t = x$ and $X_\tau + \frac{x\tau}{t}$ conditioned on $X_t = 0$ agree

$$\begin{aligned} \lambda\left(-a \leq X_\tau \leq b \mid \substack{0 \leq \tau \leq t \\ X_t = x}\right) &= \lambda\left(-a \leq X_\tau + \frac{x\tau}{t} \leq b \mid X_t = 0\right) \\ &= \lambda\left(-a - \frac{x\tau}{t} \leq X_\tau \leq b - \frac{x\tau}{t} \mid X_t = 0\right) \\ &= \lambda\left(-a + k\tau - \frac{x\tau}{t} \leq X_\tau + k\tau \leq b + k\tau - \frac{x\tau}{t} \mid X_t = 0\right) \\ &= \lambda\left(-a + \frac{(kt - x)\tau}{t} \leq X_\tau \leq b + \frac{(kt - x)\tau}{t} \mid X_t = kt\right) \end{aligned}$$

$$\text{Let any } m = \frac{kt - x}{t}$$

we have that

$$\lambda\left(-a + m\tau \leq X_\tau \leq b + m\tau \mid \substack{0 \leq \tau \leq t \\ X_t = x}\right) = \lambda\left(-a \leq X_\tau \leq b \mid \substack{0 \leq \tau \leq t \\ X_t = x - mt}\right).$$

Thus $L_{-b,b}^m(t \mid X_t = x) = \frac{\lambda(-b+m\tau \leq X_\tau \leq b+m\tau)_{X_t=x}}{2b}$ has the same limiting distribution

as $L_{-b,b}(t \mid X_t = x - mt)$ when $b \rightarrow 0$. Hence for the unrestricted Brownian motion:

$$\begin{aligned}
P(L^m(t) \leq \xi) &= \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \left(1 - e^{-\frac{\xi(\xi+2|x-mt|)}{2t}}\right) dx \\
&= 1 - \int_{mt}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} e^{-\frac{\xi(\xi+2(x-mt))}{2t}} dx \\
&\quad - \int_{-\infty}^{mt} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} e^{-\frac{\xi(\xi+2(mt-x))}{2t}} dx \\
&= 1 - I_1 - I_2 \\
I_1 &= \int_{mt}^{\infty} \frac{e^{-\frac{x^2 + \xi^2 + 2\xi x - 2\xi mt}{2t}}}{\sqrt{2\pi t}} dx \\
&= \frac{e^{+m\xi}}{\sqrt{2\pi t}} \int_{mt}^{\infty} e^{-\frac{(x+\xi)^2}{2t}} dx = \frac{e^{+m\xi}}{\sqrt{2\pi t}} \int_{mt+\xi}^{\infty} e^{-\frac{x^2}{2t}} dx \\
&= \frac{e^{+m\xi}}{\sqrt{2\pi}} \int_{\frac{mt+\xi}{\sqrt{t}}}^{\infty} e^{-x^2} dx = e^{+m\xi} \Phi\left(-\frac{\xi}{\sqrt{t}} - m\sqrt{t}\right) \\
I_2 &= \int_{-\infty}^{+mt} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} e^{-\frac{\xi(\xi+2(mt-x))}{2t}} dx = \int_{-mt}^{\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} e^{-\frac{\xi(\xi+2(mt+x))}{2t}} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-mt}^{\infty} e^{-\frac{x^2 + \xi^2 + 2\xi mt + 2\xi x}{2t}} dx \\
&\quad = \frac{e^{-m\xi}}{\sqrt{2\pi t}} \int_{-mt}^{\infty} e^{-\frac{(x+\xi)^2}{2t}} dx = \frac{e^{-m\xi}}{\sqrt{2\pi t}} \int_{\xi-mt}^{\infty} e^{-\frac{x^2}{2t}} dx \\
&= \frac{e^{-m\xi}}{\sqrt{2\pi}} \int_{\frac{-mt+\xi}{\sqrt{t}}}^{\infty} e^{-x^2} dx = e^{-m\xi} \Phi\left(+\frac{m}{\sqrt{t}} - \xi\sqrt{t}\right).
\end{aligned}$$

Thus we have:

Theorem 5.3

$$P(L^m(t) \leq \xi) = 1 - \left[e^{+m\xi} \Phi\left(-\frac{\xi}{\sqrt{t}} - m\sqrt{t}\right) + e^{-m\xi} \Phi\left(-\frac{\xi}{\sqrt{t}} + m\sqrt{t}\right) \right]. \quad (5.8)$$

When $m = 0$ 5.8 becomes:

$$1 - \Phi\left(-\frac{\xi}{\sqrt{t}}\right) - \Phi\left(-\frac{\xi}{\sqrt{t}}\right) = 2\Phi\left(\frac{\xi}{\sqrt{t}}\right) - 1 = \frac{2}{\sqrt{\pi}} \int_0^{\frac{\xi}{\sqrt{t}}} e^{-x^2} dx$$

in agreement with the classical result.

5.5 Properties of the local times, and up and down crossings.

We shall examine how relation between the down crossings from a fixed level of the Brownian motion and the local time mentioned in section 5.1 (formula 5.1) can be extended to the oblique local time by our method. First we shall treat the case of non oblique down crossing, then following the same path evaluate the general case.

Theorem 5.4 *If N_b is the number of times that X_τ crosses down from the fixed level of height b to 0 up to the time t then*

$$\lim_{b \rightarrow 0} bN_b = 2L(t) \text{ in distribution}$$

Proof:

1. What is the probability of no up-crossing at all? It is clearly

$$\begin{aligned} P(X_\tau \leq b)_{0 \leq \tau \leq t} &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{b}{\sqrt{2t}}} e^{-x^2} dx \\ &= 1 - \int_0^t \frac{b}{\sqrt{2\pi\tau^3}} e^{-\frac{b^2}{2\tau}} d\tau = 1 - \Psi(b) * 1. \end{aligned}$$

2. What is the probability of 1 upcrossing but no down crossing? Case U.
It is

$$\begin{aligned} &\int_0^t \frac{d}{d\tau} P(\sup_{0 \leq w \leq \tau} X_w \geq b) \cdot P(X_w \geq -b)_{0 \leq w \leq t-\tau} d\tau \\ &= \int_0^t \frac{d}{d\tau} P(\sup_{0 \leq w \leq \tau} X_w \geq b) \cdot P(X_w \geq -b)_{0 \leq w \leq t-\tau} d\tau \\ &= \Psi(b) * (1 - \psi(b) * 1) = \Psi(b) * 1 - \Psi(b) * \Psi(b) * 1. \end{aligned}$$

3. What is the probability of 1 upcrossing followed by 1 downcrossing? UD
it is

$$\begin{aligned}
& \int_0^t d\tau \frac{d}{d\tau} P \left(\sup_{0 \leq w \leq \tau} X_w \geq b \right) P \left(\begin{array}{c} \text{There is only 1 downcrossing} \\ \text{from zero to } -b \\ \text{on } 0, t - \tau \end{array} \right) \\
&= \int_0^t d\tau \Psi(b, \tau) P \left(\begin{array}{c} \text{There is only 1 upcrossing} \\ \text{from zero to } b \\ \text{on } (0, t - \tau) \end{array} \right) \\
&= \Psi(b) - (\Psi(b) * 1 - \Psi(b) * \Psi(b) * 1) \\
&= (\Psi(b)^{*2} - \Psi(b)^{*3}) * 1
\end{aligned}$$

and so on.

The probability of upcrossing/downcrossing followed by an upcrossing again is (UDU)

$$(\Psi^{*3} - \Psi^{*4}) * 1 .$$

Rather than to count only down crossings we consider the number of consecutive first crossings that is the first upcrossing followed by a first downcrossing which in turn is followed by a first upcrossing again... The probability of $U_1 D_2 U_3 D_4 \dots X_n$ where X is U or D depending where n is odd or even number

$$\begin{aligned}
& (\Psi^{*n} - \Psi^{*(n+1)}) * 1 \\
&= (\Psi(nb) - \Psi((n+1)b)) * 1 \\
&= \int_0^t (\Psi(nb, \tau) - \Psi((n+1)b, \tau)) d\tau .
\end{aligned}$$

The probability of the number of crossings $\leq n$ at

$$\begin{aligned}
& (1 - \Psi(b) * 1) + (\Psi(b) * 1 - \Psi(b)^{*2} * 1) \\
&+ \dots (\Psi(b)^{*n} * 1 - \Psi(b)^{*(n+1)} * 1) \\
&= 1 - \Psi(b)^{*(n+1)} * 1 = 1 - \Psi((n+1)b) * 1 \\
&= 1 - \int_0^t \Psi((n+1)b, \tau) d\tau .
\end{aligned}$$

Now for the limiting distribution that if $\frac{nb}{b \rightarrow 0} \rightarrow \xi$ this becomes

$$1 - \int_0^t \Psi(\xi, \tau) d\tau = 1 - \frac{2}{\sqrt{\pi}} \int_{-\frac{\xi}{\sqrt{2t}}}^{\infty} e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \int_{-\frac{\xi}{\sqrt{2t}}}^{\frac{\xi}{\sqrt{2t}}} e^{-x^2} dx$$

and this is what we wanted to prove. ■

Now consider the oblique crossings case. We shall prove:

Theorem 5.5 *Let N_b be the number of consecutive first crossings of the lines $x = mt + b$ and $x = mt$ then*

$$\lim_{b \rightarrow 0} P(bN_b \leq \xi) = e^{-m\xi} P(L^m(t) \leq \xi) + (1 - e^{-m\xi}).$$

Proof:

We shall evaluate the distribution of the number of first crossings of the lines $x = b + mt$ and $x = mt$.

We have seen (3.5) that

$$P\left(X_\tau \leq \frac{b}{\sqrt{t}} + m\sqrt{t} \mid 0 \leq \tau \leq t\right) = \Phi\left(\frac{b}{\sqrt{t}} + m\sqrt{t}\right) - e^{-2bm} \Phi\left(-\frac{b}{\sqrt{t}} + m\sqrt{t}\right).$$

The probability of no upcrossing is $P\left(X_\tau \leq \frac{b}{\sqrt{t}} + m\sqrt{t} \mid 0 \leq \tau \leq t\right)$. The probability of 1 upcrossing and no downcrossing

$$-\frac{d}{dt} P\left(X_\tau \leq \frac{b}{\sqrt{t}} + m\sqrt{t} \mid 0 \leq \tau \leq t\right) * P\left(X_\tau \leq \frac{b}{\sqrt{t}} - m\sqrt{t} \mid 0 \leq \tau \leq t\right).$$

The second factor in the convolution product should be $P(X_\tau \geq -b + m\tau)$ because of symmetry however it can be written as above.

Now the case UD

$$-\frac{d}{dt} P(X_\tau \leq b + mT) * -\frac{d}{dt} P(X_\tau \leq b - m\tau) * P(X_\tau \leq b + m\tau).$$

Denoting

$$\begin{aligned} & -\frac{d}{dt}P(X_\tau \leq b + m\tau) \text{ by } P'(+) \\ & -\frac{d}{dt}P(X_\tau \leq b + m\tau) \text{ by } P'(-) \\ & P(X_\tau \leq b + m\tau) \text{ by } P(+) \text{ and } P(X_\tau \leq b - m\tau) \text{ by } P(-) \end{aligned}$$

$$\begin{aligned} P(UDU) &= P'(+) * P'(-) * P'(+) * P(-) \\ P(UDUD) &= P'(+) * P'(-) * P'(+) * P'(-) * P(+). \end{aligned}$$

Substituting the actual values into the convolutions

$$\begin{aligned} P(+) &= \Phi\left(\frac{b}{\sqrt{t}} + m\sqrt{t}\right) - e^{-2bm}\Phi\left(-\frac{b}{\sqrt{t}} + m\sqrt{t}\right) \\ P(-) &= \Phi\left(\frac{b}{\sqrt{t}} - m\sqrt{t}\right) - e^{+2bm}\Phi\left(-\frac{b}{\sqrt{t}} - m\sqrt{t}\right) \\ P'(+) &= -\frac{d}{dt}P(+) = e^{-bm}\frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}e^{-\frac{m^2}{2}t} = e^{-bm}\Psi(b)e^{-\frac{m^2}{2}t} \\ P'(-) &= -\frac{d}{dt}P(-) = e^{bm}\frac{b}{\sqrt{2t^3}}e^{-\frac{b^2}{2t}}e^{-\frac{m^2}{2}t} = e^{+bm}\Psi(b)e^{-\frac{m^2}{2}t}. \end{aligned}$$

By integrating these derivatives we find that

$$\begin{aligned} P(\pm) &= \Phi\left(\frac{b}{\sqrt{t}} + m\sqrt{t}\right) - e^{\mp 2bm}\Phi\left(-\frac{b}{\sqrt{t}} + m\sqrt{t}\right) \\ &= 1 - e^{\mp bm} \int_0^t \Psi(b, \tau) e^{-\frac{m^2}{2}\tau} d\tau = 1 - e^{\mp bm} \Psi(b) e^{-\frac{m^2}{2}t} * 1. \end{aligned}$$

Hence

$$\begin{aligned} P(\text{zero crossing}) &= 1 - e^{-bm}\Psi(b)e^{-\frac{m^2}{2}t} * 1 \\ P(U) &= e^{-bm}\Psi(b)e^{-\frac{m^2}{2}t} * \left(1 - e^{bm}\Psi(b)e^{-\frac{m^2}{2}t} * 1\right) \\ P(UD) &= e^{-bm}\Psi(b)e^{-\frac{m^2}{2}t} * e^{+bm}\Psi(b)e^{-\frac{m^2}{2}t} * \left(1 - e^{bm}\Psi(b)e^{-\frac{m^2}{2}t}\right) \\ P(UDU) &= e^{-bm}\Psi(b)e^{-\frac{m^2}{2}t} * e^{+bm}\Psi(b)e^{-\frac{m^2}{2}t} * e^{-bm}\Psi(b)e^{-\frac{m^2}{2}t} \\ &\quad * \left(1 - e^{+bm}\Psi(b)e^{-\frac{m^2}{2}t} * 1\right). \end{aligned}$$

Let us denote $\Psi(b)e^{-\frac{m^2}{2}t}$ temporarily only by Ψ then:

$$\begin{aligned} P(\text{zero crossing}) &= 1 - e^{-bm}\Psi * 1 \\ P(U) &= e^{-bm}\Psi * 1 - \Psi * \Psi * 1 \\ P(UD) &= \Psi * \Psi * 1 - e^{-mb}\Psi * \Psi * \Psi * 1 \\ P(\underbrace{UD \dots}_{n \text{ times}}) &= \text{either } \Psi^{*n} * 1 - e^{-mb}\Psi^{*(n+1)} * 1 \\ &\quad \text{or } e^{-mb}\Psi^{*n} * 1 - \Psi^{*(n+1)} * 1 \end{aligned}$$

depending upon n being even or odd number as $b \rightarrow 0$ $e^{-mb} \rightarrow 1$ so it can ignored.

Now $\Psi^{*n} = \left(\Psi(b)e^{-\frac{m^2}{2}t} \right)^{*n}$. Since the Laplace transform of $x(t)e^{-ut} = (\mathcal{L}(x))(s+u)$ we obtain $\mathcal{L}\left(\Psi(b)e^{-\frac{m^2}{2}t} \right) = e^{-\sqrt{2(s+\frac{m^2}{2})}b}$. Thus $\mathcal{L}(\Psi^{*n}) = e^{-\sqrt{2(s+\frac{m^2}{2})}nb}$ and inverting this again we have that:

$$\Psi^{*n} = \left(\Psi(b)e^{-\frac{m^2}{2}t} \right)^{*n} = \Psi(nb)e^{\frac{m^2}{2}t}.$$

Thus

$$\begin{aligned} P(\underbrace{UD \dots}_{n \text{ times}}) &= e^{-mb}\Psi(nb)e^{-\frac{m^2}{2}t} * 1 - \Psi((n+1)b)e^{-\frac{m^2}{2}t} * 1 \\ &\quad \text{or } \Psi(nb)e^{-\frac{m^2}{2}t} * 1 - e^{-mb}\Psi((n+1)b)e^{-\frac{m^2}{2}t} * 1 \end{aligned}$$

$$\begin{aligned} P(\text{number of crossings} \leq n) &= P(\text{zero crossing}) + P(U) + P(UD) \\ &\quad + \dots + P(\underbrace{UD \dots}_{n \text{ times}}) = 1 - \Psi((n+1)b)e^{-\frac{m^2}{2}t} * 1. \end{aligned}$$

Taking the limit $b \rightarrow 0$ $nb = \xi$ gives

$$1 - \int_0^t \Psi(\xi)e^{-\frac{m^2}{2}\tau} d\tau = \Phi\left(\frac{\xi}{\sqrt{t}} + m\sqrt{t}\right) - e^{-2m\xi}\Phi\left(-\frac{\xi}{\sqrt{t}} + m\sqrt{t}\right).$$

Hence the limiting distribution of $P(bN_b \leq \xi) = P(X_{\tau} \leq \xi + m\tau)$. We have seen (5.8) that the local time in the oblique case has the distribution

$$P(L^m(t) \leq \xi) = 1 - \left[e^{+m\xi}\Phi\left(-\frac{\xi}{\sqrt{t}} - m\sqrt{t}\right) + e^{-m\xi}\Phi\left(-\frac{\xi}{\sqrt{t}} + m\sqrt{t}\right) \right]$$

$$\begin{aligned}
&= 1 - e^{m\xi} \left[1 - \left(\Phi \left(\frac{\xi}{\sqrt{t}} + m\sqrt{t} \right) - e^{-2m\xi} \Phi \left(-\frac{\xi}{\sqrt{t}} + m\sqrt{t} \right) \right) \right] \\
&= 1 - e^{m\xi} \left(1 - P(X_\tau \leq \xi + m\tau)_{0 \leq \tau \leq t} \right) \\
&= e^{m\xi} P \left(X_\tau \leq \xi + m\tau \right)_{0 \leq \tau \leq t} - (e^{m\xi} - 1).
\end{aligned}$$

Hence:

$$\lim_{b \rightarrow 0} P(bN_b \leq \xi) = e^{-m\xi} P(L^m(t) \leq \xi) + (1 - e^{-m\xi})$$

and this completes the proof. ■

Chapter 6

The Ray - Knight theorem.

6.1 Preliminaries.

It has been shown (Theorem 3.3) that

$$P(X_\tau \leq b) = \frac{2}{\sqrt{\pi}} \int_0^{b/\sqrt{2t}} e^{-\xi^2} d\xi$$

hence the density

$$P(FC(b) = t) = \frac{d}{dt} P(X_\tau \leq b) = \psi(b, t) \quad (6.1)$$

where $FC(b)$ (that is first cut level b) is the stopping time $\inf\{t : X_t \geq b\}$.

Likewise we can compute the conditional density

$$P(FC(b) = \tau | X_t = x).$$

For according to Theorem 3.5

$$\begin{aligned} & P(X_\omega \leq b | X_t = x) = \\ & = \Phi\left(\frac{bt - x\tau}{\sqrt{\tau t(t - \tau)}}\right) - e^{\frac{2x(b-x)}{t}} \Phi\left(\frac{2\tau b - bt - \tau x}{\sqrt{t\tau(t - \tau)}}\right). \end{aligned}$$

Differentiating by τ we get

$$\frac{d}{d\tau} P(X_\omega \leq b | X_t = x) = \frac{\psi(b, \tau) \varphi(x - b, t - \tau)}{\varphi(x, t)}.$$

Hence

$$P(FC(b) = \tau | X_t = x) = \frac{\psi(b, \tau) \varphi(x - b, t - \tau)}{\varphi(x, t)}. \quad (6.2)$$

Before proceeding to derive similar expressions but of more complex type, let us remark that we could have derived the expressions (6.1) and (6.2) without the formulas of Theorems 3.3 and 3.5 using only the simple principle expressed in formula (3.6). In case of $P(FC(b) = t)$ we can start with

$$P(X_t \leq b) = \int_{-\infty}^b \left(\frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(2b-\xi)^2}{2t}}}{\sqrt{2\pi t}} \right) d\xi.$$

So

$$P(FC(b) = t) = -\frac{d}{dt} \int_{-\infty}^b \left(\frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(\xi-2b)^2}{2t}}}{\sqrt{2\pi t}} \right) d\xi.$$

Let us remark that both terms of the integrand satisfy the heat equation

$$f_t = \frac{1}{2} f_{\xi\xi}.$$

Thus first differentiating by t under the integral sign then replacing the time derivatives by the second derivatives with respect to ξ , and integrating by ξ gives

$$\begin{aligned} P(FC(b) = t) &= -\frac{1}{2} \frac{d}{dt} \left(\frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(\xi-2b)^2}{2t}}}{\sqrt{2\pi t}} \right) \Big|_{\xi=-\infty}^{\xi=b} \\ &= \frac{1}{2} (\psi(\xi, t) - \psi(\xi - 2b, t)) \Big|_{\xi=b} \\ &= \psi(b, t). \end{aligned}$$

In case of the density expressed in formula (6.2) the procedure has to be modified as follows:

$$-\frac{e^{-\frac{\xi^2}{2t}}}{\sqrt{2\pi t}} P(FC(b) = \tau | X_t = x) =$$

$$\begin{aligned}
&= -\frac{d}{d\tau} \int_{-\infty}^b \frac{e^{-\frac{\xi^2}{2\tau}} - e^{-\frac{(2b-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \frac{e^{-\frac{(x-\xi)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)}} d\xi \\
&= -\frac{d}{d\tau} \int_{-\infty}^b f(\xi, \tau) g(\xi, t-\tau) d\xi.
\end{aligned}$$

Differentiating by τ gives for the integrand

$$\begin{aligned}
(f_\tau g + f g_\tau) &= \frac{1}{2}(f_{\xi\xi} f - f g_{\xi\xi}) \\
&= \frac{1}{2} \frac{d}{d\xi} (f_\xi g - f g_\xi).
\end{aligned}$$

Hence the integral becomes:

$$\begin{aligned}
\frac{1}{2}(f_\xi g - f g_\xi)|_{\xi=-\infty}^{\xi=b} &= \frac{1}{2}(f_\xi g - f g_\xi)|_{\xi=b} \\
&= \frac{1}{2} f_\xi g|_{\xi=b} \\
&= \frac{1}{2} (\psi(b, \tau) - \psi(-b, \tau)) \varphi(x-b, t-\tau) \\
&= \psi(b, \tau) \varphi(x-b, t-\tau).
\end{aligned}$$

Thus:

$$P(FC(b) = \tau | X_t = x) = \frac{\psi(b, \tau) \varphi(x-b, t-\tau)}{\varphi(x, t)}.$$

Next we shall compute, using the method just demonstrated

$$P\left(FC(-a) = \tau, X_\rho \leq b | X_t = x\right) \quad 0 \leq \tau < \alpha < t, \quad a > 0, b > 0.$$

This is clearly,

$$\frac{d}{d\tau} P\left(-a \leq X_\rho \leq b, X_\rho \leq b | X_t = x\right).$$

Combining the method given with formula (3.6) with the result of Section 4.4 we get that

$$\begin{aligned}
&P\left(-a \leq X_\rho \leq b, X_\rho \leq b | X_t = x\right) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} = \\
&\int_{-\infty}^b d\eta \int_a^b d\xi \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} P\left(-a \leq X_\rho \leq b | X_\tau = \xi\right) \frac{e^{-\frac{(\eta-\xi)^2}{2(\alpha-\tau)} - \frac{(2b-\eta-\xi)^2}{2(\alpha-\tau)}}}{\sqrt{2\pi(\alpha-\tau)}} \frac{e^{-\frac{(x-\eta)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}}
\end{aligned}$$

and now according to 4.10

$$f(\xi, \tau) = \frac{e^{-\frac{\xi^2}{2\tau}}}{\sqrt{2\pi\tau}} - \left(\frac{e^{-\frac{(2b-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} + \frac{e^{-\frac{(2a+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \right) + \left(\frac{e^{-\frac{(2b+2a-\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} + \frac{e^{-\frac{(2b+2a+\xi)^2}{2\tau}}}{\sqrt{2\pi\tau}} \right) + \dots$$

and

$$g(\xi, \tau) = \frac{e^{-\frac{(\eta-\xi)^2}{2(\alpha-\tau)}} - e^{-\frac{(2b-\eta-\xi)^2}{2(\alpha-\tau)}}}{\sqrt{2\pi(\alpha-\tau)}}.$$

Differentiating by τ again we get:

$$\begin{aligned} \frac{d}{d\tau}(f(\xi, \tau)g(\xi, \tau)) &= f_{\tau}g + fg_{\tau} = \\ &= \frac{1}{2}(f_{\xi\xi}g - fg_{\xi\xi}) \\ &= \frac{1}{2}\frac{d}{d\xi}(f_{\xi}g - fg_{\xi}) \end{aligned}$$

so the integration with respect to ξ yields

$$\frac{1}{2}(f_{\xi}g - fg_{\xi})|_{\xi=-a}^{\xi=b}.$$

Since

$$f(-a, \cdot) = f(b, \cdot) = g(b, \cdot) = 0$$

this becomes:

$$\begin{aligned} &= -\frac{1}{2}f_{\xi}g|_{\xi=-a} = \\ &= -\frac{1}{2}[\psi(-a, \tau) - (\psi(-a-2b) + \psi(a)) + (\psi(-3a-2b) \\ &\quad + \dots)](\varphi(\eta+a, \alpha-\tau) - \varphi(2b+a-\eta, \alpha-\tau)) \\ &= [\psi(a, \tau) - \psi(a+2b) + \psi(3a+2b) - \psi(3a+4b) \dots \\ &\quad \dots](\varphi(a+\eta, \alpha-\tau) - \varphi(2b+a-\eta, \alpha-\tau)). \end{aligned}$$

Hence we proved that:

$$\begin{aligned} &P\left(FC(-a) = \tau, X_{\rho} \leq b \mid X_t = x\right) \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} = \\ &[\psi(a, \tau) - \psi(a+2b) + \psi(3a+2b) - \psi(3a+4b) \dots] \times \\ &\int_{-\infty}^b (\varphi(a+\eta, \alpha-\tau) - \varphi(2b+a-\eta, \alpha-\tau)) \varphi(x-\eta, t-\alpha) d\eta. \end{aligned}$$

We shall not need this theorem in such general form, consider the case

$$\alpha \rightarrow \tau.$$

Now the integral:

$$\begin{aligned} & \int_{-\infty}^b (\varphi(a + \eta, \alpha - \tau) - \varphi(2b + a - \eta, \alpha - \tau)) \varphi(x - \eta, t - \alpha) d\eta \\ &= \int_{-\infty}^b \frac{e^{-\frac{(\eta+a)^2}{2(\alpha-\tau)}}}{\sqrt{2\pi(\alpha-\tau)}} \frac{e^{-\frac{(x-\eta)^2}{2(t-\alpha)}}}{\sqrt{2\pi(t-\alpha)}} d\eta - \int_{-\infty}^b \frac{e^{-\frac{(\eta-a-2b)^2}{2(\alpha-\tau)}}}{\sqrt{2\pi(\alpha-\tau)}} \frac{e^{-\frac{(x-\eta)^2}{2(t-\alpha)}}}{\sqrt{\pi(t-\alpha)}} d\eta. \end{aligned}$$

The first integral, after substitution of $\bar{\eta} = \eta + a$, becomes

$$\int_{-\infty}^{a+b} \frac{e^{-\frac{\bar{\eta}^2}{2(\alpha-\tau)}}}{\sqrt{\pi(\alpha-\tau)}} \frac{e^{-\frac{(x+a-\bar{\eta})^2}{2(t-\alpha)}}}{\sqrt{\pi(t-\alpha)}} d\bar{\eta}$$

and the limit $\alpha \rightarrow \tau$ is clearly

$$\frac{e^{-\frac{(x+a)^2}{2(t-\tau)}}}{\sqrt{\pi(t-\tau)}} = \varphi(x + a, t - \tau).$$

The second integral, after substituting $\eta - a - 2b = \bar{\eta}$, becomes:

$$\int_{-\infty}^{-a-b} \frac{e^{-\frac{\bar{\eta}^2}{2(\alpha-\tau)}}}{\sqrt{\pi(\alpha-\tau)}} \frac{e^{-\frac{(x-a-2b-\bar{\eta})^2}{2(t-\alpha)}}}{\sqrt{\pi(t-\alpha)}} d\bar{\eta}$$

since $-a - b < 0$ the limit $\alpha \rightarrow \tau$ is clearly zero. Hence we have proved:

Theorem 6.1

$$\begin{aligned} P(FC(-a) = \tau, X_\rho \leq b | X_t = x) = \\ \frac{(\psi(a, \tau) - \psi(a + 2b) + \psi(3a + 2b) - \psi(3a + 4b) \dots) \varphi(x + a, t - \tau)}{\varphi(x, t)} \end{aligned} \quad (6.3)$$

integrating with respect to x from $-\infty$ to ∞ we infer that

$$\begin{aligned} P(FC(-a) = \tau, X_\rho \leq b) = \\ \psi(a, \tau) - \psi(a + 2b) + \psi(3a + 2b) - \psi(3a + 4b) \dots \end{aligned} \quad (6.4)$$

and this gives

$$\begin{aligned}
 P(FC(-a) = \tau, X_\rho \leq b | X_t = x) \\
 &= \frac{P(FC(-a) = \tau, X_\rho \leq b) \cdot \varphi(x+a, t-\tau)}{\varphi(x, t)}.
 \end{aligned} \tag{6.5}$$

Remark. Clearly if we integrate (6.4) with respect to τ from zero to infinity we get the probability that the Brownian motion exits from the interval $[-a, b]$ at level $-a$, that is

$$\begin{aligned}
 P(X_\rho \text{ exits } [-a, b] \text{ at level } -a) \\
 = \int_0^\infty d\tau P(FC(-a) = \tau, X_\rho \leq b).
 \end{aligned} \tag{6.6}$$

To evaluate this integral take the Laplace transform of (6.4) with respect to t .

It is:

$$\begin{aligned}
 & e^{-a\sqrt{2s}} - e^{-(a+2b)\sqrt{2s}} + e^{-(3a+2b)\sqrt{2s}} \dots = \\
 &= \frac{e^{-a\sqrt{2s}} - e^{-(a+2b)\sqrt{2s}}}{1 - e^{-2(a+b)\sqrt{2s}}}.
 \end{aligned}$$

The integral (6.6) is then

$$\lim_{s \rightarrow 0} \frac{e^{-a\sqrt{2s}} - e^{-(a+2b)\sqrt{2s}}}{1 - e^{-2(a+b)\sqrt{2s}}} = \frac{b}{a+b}.$$

Thus we have proved the following:

Theorem 6.2

$$P(X_t \text{ exists from the interval } -a \leq x \leq b \text{ at level } -a) = \frac{b}{a+b}.$$

For other proofs of this theorem using random walks see [Knight] page 29.

6.2 The Ray-Knight Theorem for Brownian Motion.

Let $L_a(t)$ denote the local time of the Brownian motion at level a , where $a \in \mathbb{R}$. Using the results of Chapter 5 we shall prove:

Theorem 6.3 For any $a_0 > 0$ the process $L_a = L_a(FC(a_0))$ $a \leq a_0$ is a Markov process, starting from $L_{a_0} = 0$ at $a = a_0$, with transition density:

$$P(L_b = \eta | L_a = \xi) = \frac{1}{2(a-b)} e^{-\frac{\xi+\eta}{2(a-b)}} I_0\left(\frac{\sqrt{\xi\eta}}{a-b}\right)$$

for $0 \leq b < a$. and

$$P(L_b = \eta | L_a = \xi) = \begin{cases} \frac{1}{2(a-b)} e^{-\frac{\xi+\eta}{2(a-b)}} \sqrt{\frac{\xi}{\eta}} I_1\left(\frac{\sqrt{\xi\eta}}{a-b}\right) & \text{if } \eta > 0 \\ e^{-\frac{\xi}{2(a-b)}} & \text{if } \eta = 0 \end{cases}$$

for $-\infty < b < a \leq 0$.

Here I_0 is the modified Bessel function:

$$I_0(x) = 1 + \frac{(x/2)^2}{1!1!} + \frac{(x/2)^4}{2!2!} + \frac{(x/2)^6}{3!3!} \dots$$

and

$$I_1(x) = \left(\frac{x}{2}\right) \left(1 + \frac{\left(\frac{x}{2}\right)^2}{2!1!} + \frac{\left(\frac{x}{2}\right)^4}{3!2!} + \frac{\left(\frac{x}{2}\right)^6}{4!3!} + \dots\right).$$

This theorem was proved by F.B. Knight for Brownian motion and by D. Ray for general diffusion processes. See [Knight] on page 120 where the references for Ray's works are given also.

We shall prove this theorem by computing the finite dimensional distributions

$$P(L_{a_1}(FC(a_0)) \leq \xi_1, L_{a_2}(FC(a_0)) \leq \xi_2, \dots, L_{a_n}(FC(a_0)) \leq \xi_n) \quad (6.7)$$

$a_0 > a_1 > a_2 > \dots > a_n \geq 0$

This joint distribution of local times will be deduced from the Laplace transform of

$$\psi(a_0, t) P(L_{a_1}(t) \leq \xi_1, \dots, L_{a_n}(t) \leq \xi_n | FC(a_0) = t) \quad (6.8)$$

with respect to t .

If we denote this Laplace transform by $L(s)$ then $\lim_{s \rightarrow 0} L(s)$ will yield the probability distribution (6.7).

The Laplace transform (6.8) will be obtained through successively computing the Laplace transforms of:

1. $\psi(a, t)P(L_0(t) \leq \xi | FC(a) = t)$
2. $\varphi(x, t)P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \leq a | X_t = -|x|) \ x > 0$
3. $\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = x)$
4. $\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, a \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -|x|)$
5. $\psi(b, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta | FC(b) = r)$
6. $\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, a \leq \sup_{0 \leq \rho \leq t} X_\rho \leq b | X_t = -|x|)$
7. $\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, L_b(t) \leq \zeta, b \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = x)$
8. $\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, L_b(t) \leq \zeta | X_t = -|x|)$
9. $\psi(c, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, L_b(t) \leq \zeta | FC(c) = t)$

and so on, that finally the Laplace transform of

$$\psi(k, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta \dots L_h(t) \leq \kappa | FC(k) = t)_{0 \leq a < b \dots < h < k}$$

is obtained, and from this (6.7) can be deduced the way already described.

In the following sections we shall compute the first few of these Laplace transforms and deduce the general formulas for them.

6.3 Evaluation of $P(L_0(t) \leq \xi | FC(a) = t) \psi(a, t)$.

$P(L_0(t) \leq \xi | FC(a) = t) \psi(a, t)$ can be determined from the following integral equation:

$$\begin{aligned} P(L_0(t) \leq \xi | X_t = a) = & \quad (6.9) \\ \int_0^t d\tau P(FC(a) = \tau | X_t = a) P(L_0(\tau) \leq \xi | FC(a) = \tau) \\ & \cdot P(X_\rho \leq a | X_{t-\tau} = 0)_{0 \leq \rho \leq t-\tau} \end{aligned}$$

that is, after having cut for the first time level a the Brownian path may or may not

In the first case the local time $L_0(t)$ is just $L_0(\tau)$ and in the second case it is the sum of $L_0(\tau)$ and the local time of the path between ω and t .

$$\begin{aligned} P(FC(a) = \tau | X_t = a) &= \frac{\psi(a, \tau) \varphi(0, t - \tau)}{\varphi(a, t)} \\ P(FC(a) = \omega | X_{t-\tau} = 0) &= \frac{\psi(a, \omega) \varphi(a, t - \tau - \omega)}{\varphi(0, t - \tau)}. \end{aligned}$$
$$\begin{aligned} \varphi(a, t)P(L_0(t) \leq \xi | X_t = a) = & \quad (6.10) \\ \int_0^t d\tau \psi(a, \tau)P(L_0(\tau) \leq \xi | FC(a) = \tau) \varphi(0, t - \tau)P(X_{\rho} \leq a | X_{t-\tau} = 0) \\ & \quad 0 \leq \rho \leq t - \tau \\ \int_0^\xi dk \int_0^t d\tau \psi(a, \tau)P(L_0(\tau) \leq \xi - k | FC(a) = \tau) \\ \cdot \int_0^{t-\tau} d\omega \psi(a, \omega) \cdot \frac{d}{dk}(\varphi(a, t - \tau - \omega)P(L_0(t - \tau - \omega) \leq k | X_{t-\tau-\omega} = a)). \end{aligned}$$
$$\varphi(a, t)P(L_0(t) \leq \xi | X_t = a) = \varphi(a, t) - \varphi(a + \xi, t) \quad (6.11)$$
$$\varphi(a, t)P\left(X_\rho \leq a | X_t = 0\right) = \frac{1 - e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t}}. \quad (6.12)$$

Let us notice that the integrals with respect to τ, ω are of convolution type. Taking the Laplace transform with respect to t and denoting the Laplace variable by s we get the following integral equation:

$$\frac{e^{-a\sqrt{2s}} - e^{-(a+\xi)\sqrt{2s}}}{\sqrt{2s}} = h(s, \xi) \frac{1 - e^{-2a\sqrt{2s}}}{\sqrt{2s}} + \int_0^\xi dk h(s, \xi - k) e^{-a\sqrt{2s}} \frac{d}{dk} \frac{e^{a\sqrt{2s}} - e^{-(a+k)\sqrt{2s}}}{\sqrt{2s}}$$

or after differentiating with respect to k

$$\frac{e^{-a\sqrt{2s}} - e^{-(a+\xi)\sqrt{2s}}}{\sqrt{2s}} = h(s, \xi) \frac{1 - e^{-2a\sqrt{2s}}}{\sqrt{2s}} + \int_0^\xi dk h(s, \xi - k) e^{-(2a+k)\sqrt{2s}} \quad (6.13)$$

where $h(s, \xi)$ is the Laplace transform of

$$\psi(a, t) P(L_0(t) \leq \xi | FC(a) = t)$$

(6.13) is a Volterra type integral equation for $h(s, \xi)$. The integral with respect to k is a convolution integral again. So it can be solved through an additional Laplace transform with respect to ξ . Let the corresponding Laplace variable be z and the transform of $h(s, \xi)$ be $H(s, z)$, then taking the Laplace transform of (6.13) we get:

$$\frac{e^{-a\sqrt{2s}}}{\sqrt{2s}} \left(\frac{1}{z} - \frac{1}{z + \sqrt{2s}} \right) = H(s, z) \frac{1 - e^{-2a\sqrt{2s}}}{\sqrt{2s}} + H(s, z) \frac{e^{-2a\sqrt{2s}}}{z + \sqrt{2s}}.$$

Solving this for $H(s, x)$ we obtain:

$$H(s, z) = \frac{e^{-a\sqrt{2s}} p}{z(z + p)} \quad (6.14)$$

where $p = p_a = \frac{\sqrt{2s}}{1 - e^{-2a\sqrt{2s}}}$. The inversion of (6.14) with respect to z gives

$$h(s, \xi) = e^{-a\sqrt{2s}} p_a \int_0^\xi dk e^{-pk} = e^{-a\sqrt{2s}} (1 - e^{-p\xi}). \quad (6.15)$$

6.4 Evaluation of $P(L_0 \leq \xi, X_\rho \leq a | X_t = x) \varphi(x, t)$ $x \leq 0, a > 0.$

$P(L_0(t) \leq \xi | X_t = -|x|)$ can be expressed as the sum

$$P(L_0(t) \leq \xi, X_\rho \leq a | X_t = -|x|) + P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \geq a | X_t = -|x|).$$

Thus:

$$\begin{aligned} P(L_0(t) \leq \xi | X_t = -|x|) &= P(L_0(t) \leq \xi, X_\rho \leq a | X_t = -|x|) \quad (6.16) \\ &+ \int_0^t d\tau P(FC(a) = \tau | X_t = -|x|) \int_0^{t-\tau} d\omega P(FC(a) = \omega | X_{t-\tau} = a + |x|) \\ &\cdot \int_0^\xi dk P(L_0(\tau) \leq \xi - k | FC(a) = \tau) \\ &\cdot \frac{d}{dk} P(L_0(t - \tau - \omega) < k | X_{t-\tau-\omega} = a + |x|). \end{aligned}$$

Using (6.2) we get:

$$\begin{aligned} P(FC(a) = \tau | X_t = -|x|) &= \frac{\psi(a, \tau) \varphi(a + |x|, t - \tau)}{\varphi(x, t)} \\ P(FC(a) = \omega | X_{t-\tau} = |x| + a) &= \frac{\psi(a, \omega) \varphi(x, t - \tau - \omega)}{\varphi(|x| + a, t - \tau)}. \end{aligned}$$

Substituting these into (6.16) and multiplying the equation by $\varphi(x, t)$ gives:

$$\varphi(x, t) P(L_0(t) \leq \xi | X_t = -|x|) = \varphi(x, t) P(L_0(t) \leq \xi, X_\rho \leq a | X_t = -|x|)$$

$$+ \int_0^\xi dk \int_0^t d\tau \psi(a, \tau) P(L_0(\tau) \leq \xi - k | FC(a) = \tau) \cdot \\ \cdot \int_0^{t-\tau} d\omega \psi(a, \omega) \frac{d}{dk} (\varphi(x, t - \tau - \omega) P(L_0(t - \tau - \omega) \leq k | X_{t-\tau-\omega} = -|x|)).$$

Taking the Laplace transform of this integral equation with respect to t we get

$$\frac{e^{-|x|\sqrt{2s}}}{\sqrt{2s}} = h(s, \xi) + \\ + \int_0^\xi dk e^{-a\sqrt{2s}} (1 - e^{-p(\xi-k)}) e^{-a\sqrt{2s}} \frac{d}{dk} \frac{(e^{-|x|\sqrt{2s}} - e^{-(|x|+k)\sqrt{2s}})}{\sqrt{2s}}$$

where $h(s, \xi)$ is the Laplace transform of

$$\varphi(x, t) P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \leq a | X_t = -|x|).$$

We made use of (6.11) for the value of

$$\varphi(x, t) P(L_0(t) \leq \xi | X_t = -|x|) = \varphi(x, t) - \varphi(a + |x|, t)$$

and (6.15) for the value of the Laplace transform of

$$\psi(a, t) P(L_0(t) \leq \xi | FC(a) = t).$$

$h(s, \xi)$ could be evaluated by carrying out the integration in the equation above. However, if we take the Laplace transform with respect to ξ , then we get an algebraic expression for $H(s, z)$ (the transform of $h(s, \xi)$):

$$H(s, z) = \frac{e^{-|x|\sqrt{2s}}}{\sqrt{2s}} \left(\frac{1}{z} - \frac{1}{z + \sqrt{2s}} \right) - \frac{e^{-(|x|+2a)\sqrt{2s}}}{z(z + p_a)} \cdot \frac{1}{z + \sqrt{2s}}$$

and from here we have:

$$H(s, z) = \frac{e^{-|x|\sqrt{2s}}}{z(z + p_a)} \quad (6.17)$$

where

$$p = p_a = \frac{\sqrt{2s}}{1 - e^{-2a\sqrt{2s}}}.$$

Inverting this with respect to z we get for the Laplace transform of $\varphi(x, t) P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \leq a | X_t = -|x|)$

$$\frac{e^{-|x|\sqrt{2s}}}{p} (1 - e^{-p\xi}). \quad (6.18)$$

6.5 Evaluation of $P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = x)$ $0 < a < x$

With the help of the above diagram it can be established that

$$P(L_0(t) \leq \xi, L_a(t) \leq \eta, | X_t = x)$$

satisfies the following integral equation

$$\begin{aligned} P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = x) = & \quad (6.19) \\ & \int_0^t d\tau P(FC(a) = \tau | X_t = x) P(L_0(t) \leq \xi | FC(a) = \tau) \\ & \cdot P(L_0(t - \tau) \leq \eta, X_{t-\tau} \leq a | X_{t-\tau} = -(x - a)) + \\ & + \int_0^t d\tau P(FC(a) = \tau | X_t = x) \int_0^{t-\tau} d\omega P(FC(a) = \omega | X_{t-\tau} = -(x - a)) \\ & \cdot \int_0^\xi dk \int_0^\eta d\ell P(L_0(\tau) \leq \xi - k | FC(a) = \tau) P(L_0(\omega) \leq \eta - \ell | FC(a) = \omega) \\ & \cdot \frac{\partial^2}{\partial k \partial \ell} P(L_0(t - \tau - \omega) \leq k, L_a(t - \tau - \omega) \leq \ell | X_{t-\tau-\omega} = x). \end{aligned}$$

Substituting the values for

$$P(FC(a) = \tau | X_t = x) \text{ and } P(FC(a) = \omega | X_{t-\tau} = -(x - a))$$

will result again that all the probability distributions will be multiplied by the appropriate φ or ψ factor.

In all the previous cases we have executed two Laplace transforms: with respect to t and then with respect to ξ . Now we shall execute one more Laplace transform with respect to the new variable η , let w denote the corresponding Laplace variable.

Let $h(s, \xi, \eta)$ be the Laplace transform of

$$\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = x)$$

with respect to t and $H(s, z, w)$ with respect to t, ξ and η , then $H(s, z, w)$ satisfies the following equation:

$$\begin{aligned} H(s, z, w) &= \frac{e^{-a\sqrt{2s}p_a}}{z(z+p_a)} \cdot \frac{e^{-(x-a)\sqrt{2s}}}{w(w+p_a)} + \\ &+ \frac{e^{-a\sqrt{2s}p_a}}{z(z+p_a)} \frac{e^{-a\sqrt{2s}p_a}}{w(z+p_a)} zwH(s, z, w). \end{aligned} \quad (6.20)$$

Here we have used (6.14) for the two transforms of $\psi(a, t)P(L_a(t) \leq \xi | FC(a) = t)$ and (6.17) for the transforms of type $\psi(x, t)P(L_0(t) \leq \xi, X_\rho \leq a | X_t = -|x|)$
 $0 \leq \rho \leq t$

Solving (6.20) for $H(s, z, w)$ gives

$$H(s, z, w) = \frac{e^{-x\sqrt{2s}p_a}}{zw} \frac{1}{z+p_a} \frac{1}{w+p_a - \frac{e^{-2a\sqrt{2s}p_a}}{z+p_a}}. \quad (6.21)$$

Before proceeding further, let us remark that $\frac{e^{\frac{A}{w}}}{w^{n+1}}$ $n = 0, 1, 2, \dots$ is the Laplace transform of

$$\left(\frac{t}{A}\right)^{\frac{n}{2}} I_n(2\sqrt{At}) \quad (6.22)$$

and

$$e^{\frac{A}{w}} - 1 \text{ is the Laplace transform of } \sqrt{\frac{A}{t}} I_1(2\sqrt{At}) \quad (6.23)$$

with respect to t . Here I_n is the modified Bessel function of the first kind

$$I_n(x) = \left(\frac{x}{2}\right)^n \left(\frac{1}{\Gamma(n+1)} + \frac{\left(\frac{x}{2}\right)^2}{1!\Gamma(n+2)} + \frac{\left(\frac{x}{2}\right)^4}{2!\Gamma(n+3)} + \dots \right).$$

These formulas can be verified by using the above series expansion for I_n and taking the Laplace transform term by term.

Returning now to the inversion of

$$H(s, z, w) = \frac{e^{-x\sqrt{2s}} p_a}{zw} \frac{1}{z + p_a} \frac{1}{w + p_a - \frac{e^{-2a\sqrt{2s}} p_a^2}{z + p_a}}.$$

First, inverting with respect to w , we get

$$e^{-x\sqrt{2s}} p_a \int_0^\eta d\ell \frac{1}{z(z + p_a)} e^{-p_a \ell} e^{\frac{p_a^2 e^{-2a\sqrt{2s}} \ell}{z + p_a}}$$

and this in turn can be inverted with respect to z by applying (6.22) with $n = 0$, so we get for the Laplace transform of $P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = x) \varphi(x, t)$

$$e^{-x\sqrt{2s}} p_a \int_0^\xi dk \int_0^\eta d\ell e^{-p_a k - p_a \ell} I_0(2p_a e^{-a\sqrt{2s}} \sqrt{k\ell}).$$

6.6 Evaluation of $P(L_0(t) \leq \xi, L_a(t) \leq \eta, a \leq$

$$\sup_{0 \leq \rho \leq t} X_\rho | X_t = -|x|).$$

Clearly

$$P(L_0(t) \leq \xi, L_a(t) \leq \eta, a \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -|x|) =$$

$$= \int_0^t d\tau P(FC(a) = \tau | X_t = -|x|) \int_0^\xi dk P(L_0(\tau) \leq \xi - k | FC(a) = \tau) \\ \cdot \frac{d}{dk} P(L_0(t - \tau) \leq \eta, L_a(t - \tau) \leq k | X_{t-\tau} = |x| + a).$$

Putting

$$P(FC(a) = \tau | X_t = -|x|) = \frac{\psi(a, \tau) \varphi(|x| + a, t - \tau)}{\varphi(x, t)}$$

and multiplying by $\varphi(x, t)$ gives:

$$\varphi(x, t) P(L_0(t) \leq \xi, L_a(t) \leq \eta, \sup_{0 \leq \rho \leq t} X_\rho \geq a | X_t = -|x|) = \\ = \int_0^\xi dk \int_0^t d\tau \psi(a, \tau) P(L_0(\tau) \leq \xi - k | FC(a) = \tau) \\ \cdot \frac{d}{dk} \varphi(|x| + a, t - \tau) P(L_0(t - \tau) \leq \eta, L_a(t - \tau) \leq k | X_{t-\tau} = |x| + a).$$

Let $H(s, z, w)$ be the Laplace transform of the left hand side with respect to t, ξ, η , then using (6.14) and (6.21) we can establish that

$$H(s, z, w) = \frac{e^{-a\sqrt{2s}} p_a}{z(z + p_a)} \cdot \frac{ze^{-(|x|+a)\sqrt{2s}} p_a}{zw(w + p_a)} \frac{1}{z + p_a - \frac{e^{-2a\sqrt{2s}}}{w + p_a}}.$$

Thus:

$$H(s, z, w) = \frac{e^{-(2a+|x|)\sqrt{2s}} p_a}{zw(z + p_a)^2(w + p_a - \frac{e^{-2a\sqrt{2s}} p_a^2}{z + p_a})} \quad (6.24) \\ p_a = \frac{\sqrt{2s}}{1 - e^{-2a\sqrt{2s}}}.$$

Inverting this with respect to z and w gives the Laplace transform of

$$\varphi(X, t) P(L_0(t) \leq \xi, L_a(t) \leq \eta, a \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -|x|).$$

It is carried out just as in Section 5, using (6.21) and it is:

$$e^{-(a+|x|)\sqrt{2s}} p_a \int_0^\xi dk \int_0^\eta d\ell e^{-p_a k} e^{-p_a \ell} \sqrt{\frac{k}{\ell}} I_1 \left(2p_a e^{-a\sqrt{2s}} \sqrt{k\ell} \right).$$

6.7 Evaluation of $P(L_0(t) \leq \xi, L_a(t) \leq \eta | FC(b) = t) \psi(b, t)$.

This is done much the same way as is in Section 3, but the present case is more complicated, because after having for the first time the Brownian path reached level b , there are now 3 possibilities: 1) it stays above level a ; 2) it goes below level a but does not reach zero; 3) it goes below zero, between τ and t .

Now the integral equation becomes:

$$\begin{aligned}
 & P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = b) \\
 = & \int_0^t d\tau P(FC(b) = \tau | X_t = b) P(L_0(\tau) \leq \xi, L_a(\tau) \leq \eta | FC(b) = \tau) \\
 & \cdot P\left(X_\rho \leq b - a | X_{t-\tau} = 0\right) + \\
 + & \int_0^t d\tau P(FC(b) = \tau | X_t = b) \int_0^\tau d\omega P(L_0(\tau) \leq \xi, L_a(\tau) \leq \eta - \ell | FC(b) = \tau) \\
 & \cdot \int_0^{t-\tau} d\omega P(FC(b-a) = \omega | X_{t-\tau} = 0) \\
 & \cdot \frac{d}{d\ell} P(L_0(t-\tau-\omega) \leq \ell, X_\rho \leq a | X_{t-\tau-\omega} = -(b-a)) \\
 + & \int_0^t d\tau P(FC(b) = \tau | X_t = b)
 \end{aligned}$$

$$\begin{aligned}
& \cdot \int_0^\xi dk \int_0^\eta d\ell P(L_0(\tau) \leq \xi - k, L_a(\tau) \leq \eta - \ell | FC(b) = \tau) \\
& \cdot \int_0^{t-\tau} d\omega P(FC(b-a) = \omega | X_{t-\tau} = 0) \frac{\partial^2}{\partial k \partial \ell} P(L_0(t-\tau-\omega) \leq \ell, \\
& L_a(t-\tau-\omega) \leq k, \sup_{0 \leq \rho \leq t-\tau-\omega} X_\rho \geq a | X_{t-\tau-\omega} = -(b-a)).
\end{aligned}$$

Substituting

$$\begin{aligned}
P(FC(b) = \tau | X_t = b) &= \frac{\psi(b, \tau) \varphi(0, t-\tau)}{\varphi(b, t)} \\
P(FC(b) = \omega | X_{t-\tau} = 0) &= \frac{\psi(b-a, \omega) \varphi(b-a, t-\tau-\omega)}{\varphi(0, t-\tau)}
\end{aligned}$$

gives

$$\begin{aligned}
& \varphi(b, t) P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = b) \\
&= \int_0^t d\tau \psi(b, \tau) P(L_0(\tau) \leq \xi, L_a(\tau) \leq \eta | FC(b) = \tau) \\
& \varphi(0, t-\tau) P(X_\rho \leq b-a | X_{t-\tau} = 0) + \cdot \\
&+ \int_0^\eta d\ell \int_0^t d\tau \psi(b, \tau) P(L_0(\tau) \leq \xi, L_a(\tau) \leq \eta - \ell | FC(b) = \tau) \cdot \\
& \cdot \int_0^{t-\tau} d\omega \psi(b-a, \omega) \frac{d}{d\ell} \varphi(b-a, t-\tau-\omega) \\
& P(L_0(t-\tau-\omega) \leq \ell, \sup_{0 \leq \rho \leq t-\tau-\omega} X_\rho \leq a | X_{t-\tau-\omega} = -(b-a)) \\
&+ \int_0^\xi dk \int_0^\eta d\ell \int_0^t d\tau \psi(b, \tau) P(L_0(\tau) \leq \xi - k, L_a(\tau) \leq \eta - \ell | FC(b) = \tau) \\
& \int_0^{t-\tau} d\omega \psi(b-a, \omega) \frac{\partial^2}{\partial k \partial \ell} \varphi(b-a, t-\tau-\omega) \\
& P(L_0(t-\tau-\omega) \leq k, L_a(t-\tau-\omega) \leq \ell, \\
& a \leq \sup_{0 \leq \rho \leq t-\tau-\omega} X_\rho | X_{t-\tau-\omega} = -(b-a)).
\end{aligned}$$

Let now $H(s, z, w)$ be the Laplace transform of

$$P(L_0(t) \leq \xi, L_a(t) \leq \eta | FC(b) = t) \psi(b, t)$$

with respect to t, ξ, η , then after executing these Laplace transforms we get

$$\frac{e^{-b\sqrt{2s}} p_a}{zw} \frac{1}{(z+p_a)} \frac{1}{w+p_a - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z+p_a}} = \quad (6.25)$$

$$\begin{aligned}
&= H(s, z, w) \left\{ \frac{1 - e^{-2a\sqrt{2}s}}{\sqrt{2}s} + \frac{we^{-2(b-a)\sqrt{2}s}}{w(w+p_a)} + \right. \\
&\quad \left. + \frac{wze^{-2(b-a)\sqrt{2}s}p_a^2}{wz(w+p_a)^2 \left(z + p_a - \frac{p_a^2 e^{-2a\sqrt{2}s}}{w+p_a} \right)} \right\} \quad (6.26)
\end{aligned}$$

here we have used (6.24) to compute the Laplace transform of the left-hand side (6.12), (6.17), (6.24) to compute the Laplace transforms of $\varphi(0, t)P(X_\rho \leq b-a | X_t = 0, \varphi(b-a, t)P(L_0(t) \leq \eta, X_\rho \leq a | X_t = -(b-a))$ and $\varphi(b-a, t)P(L_0(t) \leq \eta, L_a(t) \leq \xi, a \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -(b-a))$.

Solving (6.25) for $H(s, z, w)$ we get

$$H(s, z, w) = \frac{e^{-b\sqrt{2}s}p_ap_{b-a}}{zw} \frac{1}{(z+p_a)} \frac{1}{w+p_a + p_{b-a}e^{-2(b-a)\sqrt{2}s} - \frac{e^{-2a\sqrt{2}s}p_a^2}{z+p_a}}. \quad (6.27)$$

Inverting this with respect to z and w we get for the Laplace transform of

$$\psi(b, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta | FC(b) = t)$$

with respect to t :

$$p_ap_{b-a}e^{-b\sqrt{2}s} \int_0^\xi dk \int_0^\eta d\ell e^{-p_ak} e^{-(p_a+p_{b-a}e^{-2(b-a)\sqrt{2}s})\ell} I_0(2p_ae^{-a\sqrt{2}s}\sqrt{k\ell}).$$

6.8 The general case.

Following the way demonstrated in the preceeding sections 2-7 the Laplace transforms of more general distributions can be obtained and through induction the general cases can be proved.

In this section the general forms of the Laplace transforms of the the four different types of distributions are given. First for the sake of simplicity we introduce some notations.

Let $z = z_0, z_1, z_2 \dots$ be the Laplace variables corresponding to $\xi = \xi_0, \xi_1, \xi_2 \dots$
Let

$$A = A_0 = z_0 + p_{a_1}, A_1 = z_1 + p_{a_1} + p_{a_2-a_1} e^{-2(a_2-a_1)\sqrt{2s}},$$

$$A_2 = z_2 + p_{a_2-a_1} + p_{a_3-a_2} e^{-2(a_3-a_2)\sqrt{2s}}$$

and in genral

$$A_n = z_n + p_{a_n-a_{n-1}} + p_{a_{n+1}-a_n} e^{-2(a_{n+1}-a_n)\sqrt{2s}}$$

$$\overline{A_1} = z_1 + p_{a_1}, \overline{A_2} = z_2 + p_{a_2-a_1} \dots \overline{A_n} = z_n + p_{a_n-a_{n-1}}$$

$$B_1 = p_{a_1} e^{-2a_1\sqrt{2s}}, B_2 = p_{a_2-a_1} e^{-2(a_2-a_1)\sqrt{2s}} \dots B_n = p_{a_n-a_{n-1}} e^{-2(a_n-a_{n-1})\sqrt{2s}}$$

$$Q_0 = \frac{1}{A_0}, Q_1 = \frac{1}{A_1 + \frac{B_1}{A_0}}, Q_2 = \frac{1}{A_2 + \frac{B_2}{A_1 + \frac{B_1}{A_0}}} \text{ and in general } Q_n = \frac{1}{A_n + B_n Q_{n-1}}$$

and let furthermore

$$\overline{Q_n} = \frac{1}{\overline{A_n} + B_n Q_{n-1}}.$$

1. The Laplace transforms of distributions of type

$$\varphi(x, t) P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n | X_t = x)$$

$$0 < a_1 < a_2 < \dots < a_n \leq x$$

$$\varphi(x, t) P(L_0(t) \leq \xi | X_t = x) :$$

$$\frac{e^{-x\sqrt{2s}}}{z(z + \sqrt{2s})}$$

$$\varphi(x, t) P(L_0(t) \leq \xi, L_a(t) \leq \eta | X_t = x) :$$

$$\frac{e^{-x\sqrt{2s}} p_a}{zw} \frac{1}{(z + p_a)} \frac{1}{\left(w + p_a - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z + p_a}\right)}$$

$$\varphi(x, t) P(L_0(t) \leq \xi, L_a(t) \leq \eta, L_b(t) \leq \zeta | X_t = x) :$$

$$\frac{e^{-x\sqrt{2s}} p_a p_{b-a}}{zw(z+p_a)} \frac{1}{\left(w+p_a+p_{b-a}e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z+p_a}\right)} \cdot$$

$$\frac{1}{v+p_{b-a} - \frac{p_{b-a}^2 e^{-2(b-a)\sqrt{2s}}}{w+p_a+p_{b-a}e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z+p_a}}}.$$

In general the Laplace transform of

$$\varphi(x, t)P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n | X_t = x)$$

$$0 < a_1 < a_2 < \dots < a_n \leq x$$

with respect to $t, \xi_1, \xi_2, \dots, \xi_n$ is

$$\frac{e^{-x\sqrt{2s}}}{z_0 z_1 z_2 \dots z_n} p_{a_1} p_{a_2 - a_1} \dots p_{a_n - a_{n-1}} \cdot Q_0 Q_1 \dots Q_{n-1} \overline{Q_n}.$$

2. The Laplace transforms of distributions of type

$$\varphi(x, t)P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n,$$

$$a_n \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -|x|)$$

$$0 < a_1 < a_2 < \dots < a_n$$

$$\varphi(x, t)P(L_0(t) \leq \xi | X_t = -|x|)$$

$$\frac{e^{-|x|\sqrt{2s}}}{z(z + \sqrt{2s})}$$

$$\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, \sup_{0 \leq \rho \leq t} X_\rho \geq a | X_t = -|x|):$$

$$\frac{e^{-(2a+|x|)\sqrt{2s}}}{zw} \frac{1}{(z + P_a)^2} \frac{1}{\left(w + p_a - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z+p_a}\right)}$$

$$\varphi(x, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, L_b(t) \leq \zeta, b \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -|x|):$$

$$\frac{e^{-(2b+|x|)\sqrt{2s}}}{zw(z + p_a)^2} \frac{1}{\left(w + p_a + p_{b-a}e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z+p_a}\right)^2}$$

$$\frac{1}{v + p_{b-a} - \frac{P_{b-a}^2 e^{2(b-a)\sqrt{2s}}}{w + p_a + p_{b-a} e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z + p_a}}}.$$

In general the Laplace transform of

$$\begin{aligned} & \varphi(x, t) P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n, \\ & a_n \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -|x|) \\ & 0 < a_1 < a_2 < \dots < a_n \end{aligned}$$

with respect to $t, \xi_1, \xi_2, \dots, \xi_n$ is

$$\frac{e^{-(2a_n + |x|)\sqrt{2s}}}{z_0 z_1 z_2 \dots z_n} p_{a_1}^2 p_{a_2 - a_1}^2 \dots p_{a_n - a_{n-1}}^2 \cdot Q_0^2 Q_1^2 \dots Q_{n-1}^2 \overline{Q_n}.$$

3. The Laplace transforms of distributions of type

$$\begin{aligned} & \varphi(x, t) P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n, \\ & a_n \leq \sup_{0 \leq \rho \leq t} X_\rho \leq a_{n+1} | X_t = -|x|) \\ & 0 < a_1 < a_2 < \dots < a_n < a_{n+1} \end{aligned}$$

$$\begin{aligned} & \varphi(x, t) P(L_0(t) \leq \xi, X_\rho \leq a | X_t = -|x|) : \\ & \frac{e^{-x\sqrt{2s}}}{z(z + p_a)} \end{aligned}$$

$$\begin{aligned} & \varphi(x, t) P(L_0(t) \leq \xi, L_a(t) \leq \eta, a \leq \sup_{0 \leq \rho \leq t} X_\rho \leq b | X_t = -|x|) : \\ & \frac{e^{-(2a + |x|)\sqrt{2s}} p_a^2}{z w (z + p_a)^2} \frac{1}{w + p_a + p_{b-a} e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z + p_a}} \end{aligned}$$

$$\begin{aligned} & \varphi(x, t) P(L_0(t) \leq \xi, L_a(t) \leq \eta, L_b(t) \leq \zeta, b \leq \sup_{0 \leq \rho \leq t} X_\rho \leq c | X_t = -|x|) \\ & \frac{e^{-(2b + |x|)\sqrt{2s}} p_a^2 p_{b-a}^2}{z w v (z + p_a)^2} \frac{1}{(w + p_a + p_{b-a} e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z + p_a})^2} \end{aligned}$$

$$\frac{1}{v + p_{b-a} + p_{c-b}e^{-2(c-b)\sqrt{2s}} - \frac{p_{b-a}^2 e^{-2(b-a)\sqrt{2s}}}{w + p_a + p_{b-a}e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z + p_a}}}$$

In general the Laplace transform of

$$\varphi(x, t)P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n,$$

$$a_n \leq \sup_{0 \leq \rho \leq t} X_\rho \leq a_{n+1} | X_t = -|x|)$$

$$0 < a_1 < a_2 < \dots < a_n < a_{n+1}$$

with respect to $t, \xi_1, \xi_2, \dots, \xi_n$ is

$$\frac{e^{-(2a_n+|x|)\sqrt{2s}}}{z_0 z_1 z_2 \dots z_n} p_{a_1}^2 p_{a_2-a_1}^2 \dots p_{a_n-a_{n-1}}^2 \cdot Q_0^2 Q_1^2 \dots Q_{n-1}^2 Q_n$$

4. For proving the Ray-Knight theorem the only important case is:

$$\psi(b, t)P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n | FC(b) = t)$$

$$0 < a_1 < a_2 < \dots < a_n < a_{n+1} = b$$

$$\psi(a, t)P(L_0(t) \leq \xi | FC(a) = t) :$$

$$\frac{e^{-a\sqrt{2s}} p_a}{z(z + p_a)}$$

$$\psi(b, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta | FC(b) = t) :$$

$$\frac{e^{-b\sqrt{2s}} p_a p_{b-a}}{zw(z + p_a)} \frac{1}{w + p_a + p_{b-a}e^{-2(b-a)\sqrt{2s}} - \frac{e^{-2a\sqrt{2s}} p_a^2}{z + p_a}}$$

$$\psi(c, t)P(L_0(t) \leq \xi, L_a(t) \leq \eta, L_b(t) \leq \zeta | FC(c) = t) :$$

$$\frac{e^{-c\sqrt{2s}} p_a p_{b-a} p_{c-b}}{zwv(z + p_a)} \frac{1}{w + p_a + p_{b-a}e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z + p_a}}$$

$$\frac{1}{v + p_{b-a} + p_{c-b}e^{-2(c-b)\sqrt{2s}} - \frac{p_{b-a}^2 e^{-2(b-a)\sqrt{2s}}}{w + p_a + p_{b-a}e^{-2(b-a)\sqrt{2s}} - \frac{p_a^2 e^{-2a\sqrt{2s}}}{z + p_a}}}$$

etc. ...

In general the Laplace transform of

$$\psi(b, t)P(L_0(t) \leq \xi, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2 \dots L_{a_n}(t) \leq \xi_n | FC(b) = t)$$

$$0 < a_1 < a_2 < \dots < a_n < b$$

with respect to $t, \xi_1, \xi_2, \dots, \xi_n$ is

$$\frac{e^{-b\sqrt{2s}}}{z_0 z_1 z_2 \dots z_n} p_{a_1} p_{a_2 - a_1} \dots p_{a_n - a_{n-1}} p_{a_{n+1} - a_n} \cdot Q_0 Q_1 \dots Q_{n-1} Q_n. \quad (6.28)$$

Inverting (6.28) with respect to $z_0, z_1, z_2, \dots, z_n$ by using (6.22) we get for the Laplace transform of

$$\psi(b, t)P(L_0(t) \leq \xi_0, L_{a_1}(t) \leq \xi_1, L_{a_2}(t) \leq \xi_2, \dots, L_{a_n}(t) \leq \xi_n | FC(b) = t)$$

$$0 < a_1 < a_2 < \dots < a_n < b$$

with respect to t

$$p_{a_1} p_{a_2 - a_1} p_{a_3 - a_2} \dots p_{b - a_n} e^{-b\sqrt{2s}} \int_0^{\xi_0} dk_0 \dots \int_0^{\xi_n} dk_n$$

$$e^{-p_{a_1} k_0} e^{-(p_{a_1} + e^{-2(a_2 - a_1)\sqrt{2s}} p_{a_2 - a_1}) k_1} \dots e^{-(p_{a_n - a_{n-1}} + e^{-2(b - a_1)\sqrt{2s}} p_{b - a_n}) k_n}$$

$$I_0 \left(2p_{a_1} e^{-a_1 \sqrt{2s}} \sqrt{k_0 k_1} \right) I_0 \left(2p_{a_2 - a_1} e^{-(a_2 - a_1)\sqrt{2s}} \sqrt{k_1 k_2} \right) \dots$$

$$\dots I_0 \left(2p_{a_n - a_{n-1}} e^{-(a_n - a_{n-1})\sqrt{2s}} \sqrt{k_{n-1} k_n} \right).$$

Taking the limit $s \rightarrow 0$ and using

$$\lim_{s \rightarrow 0} P_x = \frac{\sqrt{2s}}{1 - e^{-2x\sqrt{2s}}} = \frac{1}{2x}$$

gives

$$P(L_0(FC(b)) \leq \xi_0, L_{a_1}(FC(b)) \leq \xi_1, \dots, L_{a_n}(FC(b)) \leq \xi_n) =$$

$$\frac{1}{2a_1} \frac{1}{2(a_2 - a_1)} \dots \frac{1}{2(b - a_n)} \int_0^{\xi_0} dk_0 \dots \int_0^{\xi_n} dk_n \quad (6.29)$$

$$\cdot \exp \left(-\frac{k_0}{2a_1} - \left(\frac{1}{2a_1} + \frac{1}{2(a_2 - a_1)} \right) k_1 - \left(\frac{1}{2(a_2 - a_1)} + \frac{1}{2(a_3 - a_2)} \right) k_2 \right)$$

$$\dots - \left(\frac{1}{2(a_n - a_{n-1})} + \frac{1}{2(b - a_n)} \right) k_n \\ I_0 \left(\frac{\sqrt{k_0 k_1}}{a_1} \right) I_0 \left(\frac{\sqrt{k_1 k_2}}{a_2 - a_1} \right) \dots I_0 \left(\frac{\sqrt{k_{n-1} k_n}}{a_n - a_{n-1}} \right)$$

for $a < a_1 < a_2 < \dots < a_n < b$.

Let now $0 \leq a < a_1 < a_2 < \dots < a_n < b$. It follows from (6.29) that

$$P(L_a(FC(b)) = \xi, L_{a_1}(FC(b)) = \xi_1, \dots, L_{a_n}(FC(b)) = \xi_n) = \\ = \exp \left\{ -\frac{\xi}{2(a_1 - a)} - \left(\frac{1}{2(a_1 - a)} + \frac{1}{2(a_2 - a_1)} \right) \xi_1 \dots \right. \\ \left. \dots - \left(\frac{1}{2(a_n - a_{n-1})} + \frac{1}{2(b - a_n)} \right) \xi_n \right\} \\ \frac{1}{2(a_1 - a)} \frac{1}{2(a_2 - a_1)} \dots \frac{1}{2(b - a_n)} \\ I_0 \left(\frac{\sqrt{\xi \xi_1}}{a_1 - a} \right) I_0 \left(\frac{\sqrt{\xi_1 \xi_2}}{a_2 - a_1} \right) \dots I_0 \left(\frac{\sqrt{\xi_{n-1} \xi_n}}{a_n - a_{n-1}} \right)$$

and

$$P(L_{a_1}(FC(b)) = \xi_1, \dots, L_{a_n}(FC(b)) = \xi_n) = \\ \exp \left\{ -\frac{\xi_1}{2(a_2 - a_1)} \dots - \left(\frac{1}{2(a_n - a_{n-1})} + \frac{1}{2(b - a_n)} \right) \xi_n \right\} \\ \frac{1}{2(a_2 - a_1)} \dots \frac{1}{2(b - a_n)} \\ I_0 \left(\frac{\sqrt{\xi_1 \xi_2}}{a_2 - a_1} \right) \dots I_0 \left(\frac{\sqrt{\xi_{n-1} \xi_n}}{a_n - a_{n-1}} \right).$$

Hence

$$P(L_a = \xi | L_{a_1} = \xi_1, \dots, L_{a_n} = \xi_n) = \\ \frac{e^{-\frac{\xi + \xi_1}{2(a_1 - a)}}}{2(a_1 - a)} I_0 \left(\frac{\sqrt{\xi \xi_1}}{a_1 - a} \right) \\ \dots P(L_a = \xi | L_{a_1} = \xi_1).$$

Thus the process $L_a(FC(b))$ is Markov with transition density

$$P \left(L_a(FC(b)) = \xi \mid L_{a_1}(FC(b)) = \xi_1 \right) = \frac{e^{-\frac{\xi + \xi_1}{2(a_1 - a)}}}{2(a_1 - a)} I_0 \left(\frac{\sqrt{\xi \xi_1}}{a_1 - a} \right)$$

with entrance boundary at b , and this completes the proof of the first part of the Ray-Knight theorem. ■

6.9 The Ray - Knight Theorem (continued).

We shall prove now the second half of the Ray - Knight theorem or rather the equivalent:

Theorem 6.4 *The process $L_b(FC(-a))$ $0 \leq b < \infty, a > 0$ is a Markov process, with entrance boundary at $b = 0$ and with initial value $L_0(FC(-a))$*

Proof: We shall evaluate successively the finite dimensional distributions (or rather their Laplace transforms):

1. $P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \leq b, FC(-a) = t) \quad a, b > 0$
 2. $P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho, FC(-a) = t)$
 3. $P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho \leq c, FC(-a) = t)$
 4. $P(L_0(t) \leq \xi, L_b(t) \leq \eta, L_c(t) \leq \zeta, c \leq \sup_{0 \leq \rho \leq t} X_\rho, FC(-a) = t) \quad 0 < b < c$
- and so on ...

Again, we shall evaluate the first few cases and deduce the general form, which can be proved by induction.

Case 1.

$$P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \leq b, FC(-a) = t) \quad a, b > 0.$$

Consider the Brownian motion paths

$$\{ \sup_{0 \leq \rho \leq t} X_\rho \leq b, X_t = -a \}.$$

For a Brownian motion path after having hit $-a$ for the first time there are two possibilities: it may or may not return to zero before arriving at $-a$ at time t .

This is expressed in the following integral equation:

$$\begin{aligned}
 P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \leq b | X_t = -a) = & \quad (6.30) \\
 \int_0^t d\tau P(FC(-a) = \tau, \sup_{0 \leq \rho \leq t} X_\rho \leq b | X_t = -a) \\
 \cdot P(L_0(\tau) \leq \xi | \sup_{0 \leq \rho \leq \tau} X_\rho \leq b, FC(-a) = \tau) \cdot P(X_\rho \leq a | X_{t-\tau} = 0) \\
 + \int_0^t d\tau P(FC(-a) = \tau, \sup_{0 \leq \rho \leq t} X_\rho \leq b | X_t = -a) \\
 \cdot \int_0^\xi dk P(L_0(\tau) \leq \xi - k | \sup_{0 \leq \rho \leq \tau} X_\rho \leq b, FC(-a) = \tau) \\
 \cdot \int_0^{t-\tau} d\omega P(FC(a) = \omega | X_{t-\tau} = 0) \\
 \cdot \frac{d}{dk} P(L_0(t - \tau - \omega) \leq k, \sup_{0 \leq \rho \leq t-\tau-\omega} X_\rho \leq b | X_{t-\tau-\omega} = -a).
 \end{aligned}$$

According to (6.2)

$$P(FC(a) = \omega | X_{t-\tau} = 0) = \frac{\psi(a, \omega) \varphi(a, t - \tau - \omega)}{\varphi(0, t - \tau)}$$

and by (6.5) we have also:

$$P(FC(-a) = \tau, X_\rho \leq b | X_t = x) = \frac{P(FC(-a) = \tau, X_\rho \leq b) \cdot \varphi(x+a, t-\tau)}{\varphi(x, t)}.$$

Substituting these into the integral equation and multiplying by $\varphi(a, t)$ and rearranging we get:

$$\begin{aligned} & \varphi(a, t) P(L_0(t) \leq \xi, X_\rho \leq b | X_t = -a) = \\ & \int_0^t d\tau P(L_0(\tau) \leq \xi, X_\rho \leq b, FC(-a) = \tau) \varphi(0, t - \tau) P(X_\rho \leq a | X_{t-\tau} = 0) \\ & + \int_0^\xi dk \int_0^t d\tau P(L_0(\tau) \leq \xi - k, X_\rho \leq b, FC(-a) = \tau) \int_0^{t-\tau} d\omega \psi(a, \omega) \\ & \cdot \frac{d}{dk} (\varphi(a, t - \tau - \omega) P(L_0(t - \tau - \omega) \leq k, X_\rho \leq b | X_{t-\tau-\omega} = -a)). \end{aligned}$$

Taking the Laplace transforms with respect to t and ξ and denoting the Laplace variables by s and z we get for $H(s,z)$, the Laplace transform of

$$P(L_0(t) \leq \xi, X_p \leq b, FC(-a) = t)_{0 \leq p \leq t}$$

the following equation:

$$\frac{e^{-a\sqrt{2}s}}{z(z+p_b)} = H(s, z) \left\{ \frac{1 - e^{-2a\sqrt{2}s}}{\sqrt{2}s} + e^{-a\sqrt{2}s} \cdot \frac{ze^{-a\sqrt{2}s}}{z + p_b} \right\}.$$

Solving this for H yields:

$$H(s, z) = \frac{e^{-a\sqrt{2s}} p_a}{z} \frac{1}{z + p_n + p_b e^{-2b\sqrt{2s}}}.$$

Inverting this with respect to z gives:

$$\begin{aligned} h(s, \xi) &= e^{-a\sqrt{2s}} p_a \int_0^\xi dk e^{-(p_a + p_b e^{-2b\sqrt{2s}})k} \\ &= e^{-a\sqrt{2s}} p_a \frac{1 - e^{-(p_a + p_b e^{-2b\sqrt{2s}})\xi}}{p_a + p_b e^{-2b\sqrt{2s}}}. \end{aligned}$$

Let $s \rightarrow 0$ we have that:

$$P(L_0(FC(-a)) \leq \xi, \sup_{0 \leq \rho \leq FC(-a)} X_\rho \leq b) = \frac{b}{a+b} (1 - e^{-\frac{a+b}{2ab}\xi}).$$

Differentiating by ξ we have:

$$P(L_0(FC(-a)) = \xi, L_b(FC(-a)) = 0) = \frac{e^{-(\frac{1}{2a} + \frac{1}{2b})\xi}}{2a}.$$

We have shown that $P(L_0(FC(-a)) = \xi) = \frac{e^{-\frac{\xi}{2a}}}{2a}$ so

$$P(L_b(FC(-a)) = 0 | L_0(FC(-a)) = \xi) = e^{-\frac{\xi}{2b}}.$$

Case 2.

Next we shall evaluate $P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho, FC(-a) = t)$. To this end consider the set of Wiener paths: $\{\sup_{0 \leq \rho \leq t} X_\rho \geq b, X_t = -a\}$. Before hitting for the first time $-a$ a Brownian path may or may not hit b . After having reached $-a$ for the first time there are 3 possibilities: 1) the Brownian path does not return to zero, 2) it returns to zero, but does not reach b , 3) it hits b .

This is expressed in the following integral equation:

$$\begin{aligned}
& P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho | X_t = -a) \\
&= \int_0^\xi dk \int_0^t d\tau P(X_\rho \leq b, FC(-a) = \tau | X_t = -a) \\
&\quad \cdot P(L_0(\tau) \leq \xi - k | X_\rho \leq b, FC(-a) = \tau) \\
&\quad \cdot \int_0^{t-\tau} d\omega P(FC(-a) = \omega | X_{t-\tau} = 0) \\
&\quad \cdot \frac{d}{dk} P(L_0(t - \tau - \omega) \leq k, L_b(t - \tau - \omega) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho | X_{t-\tau-\omega} = -a) \\
&+ \int_0^t d\tau P(b \leq \sup_{0 \leq \rho \leq \tau} X_\rho, FC(-a) = \tau | X_t = -a) \\
&\quad P(L_0(\tau) \leq \xi, L_b(\tau) \leq \eta | b \leq \sup_{0 \leq \rho \leq \tau} X_\rho, FC(-a) = \tau) \\
&\quad \cdot P(X_\rho \leq a | X_{t-\tau} = 0) \\
&+ \int_0^\xi dk \int_0^t d\tau P(\sup_{0 \leq \rho \leq \tau} X_\rho \geq b, FC(-a) = \tau | X_t = -a) \\
&\quad P(L_0(\tau) \leq \xi - k, L_b(\tau) \leq \eta | \sup_{0 \leq \rho \leq \tau} X_\rho \geq b, FC(-a) = \tau) \\
&\quad \cdot \int_0^{t-\tau} d\omega P(FC(-a) = \omega | X_{t-\tau} = 0) \\
&\quad \cdot \frac{d}{dk} P(L_0(t - \tau - \omega) \leq k, \sup_{0 \leq \rho \leq t} X_\rho \leq b | X_{t-\tau-\omega} = -a) \\
&+ \int_0^\xi dk \int_0^\eta d\ell \int_0^t d\tau P(\sup_{0 \leq \rho \leq \tau} X_\rho \geq b, FC(-a) = \tau | X_t = -a) \\
&\quad \cdot P(L_0(\tau) \leq \xi - k, L_b(\tau) \leq \eta - \ell | \sup_{0 \leq \rho \leq \tau} X_\rho \geq b, FC(-a) = \tau) \\
&\quad \cdot \int_0^{t-\tau} d\omega P(FC(-a) = \omega | X_{t-\tau} = 0) \\
&\quad \cdot \frac{\partial^2}{\partial k \partial \ell} P(L_0(t - \tau - \omega) \leq k, L_b(t - \tau - \omega) \leq \ell, \sup_{0 \leq \rho \leq t-\tau-\omega} X_\rho \geq b | X_{t-\tau-\omega} = -a)
\end{aligned}$$

After substituting for the appropriate boundary crossing densities and executing the Laplace transforms with respect to t, ξ, η and denoting the corresponding Laplace variables by s, z, w we get the following equation for $H(s, z, w)$ the Laplace transform of $P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho, FC(-a) = t)$

$$\begin{aligned} & \frac{e^{-(2b+a)\sqrt{2s}}}{zw} \frac{1}{(z+p_b)^2} \frac{1}{w+p_b - \frac{p_b^2 e^{-2b\sqrt{2s}}}{z+p_b}} = \\ & = \frac{e^{-a\sqrt{2s}} p_a}{z(z+p_a+p_b e^{-2b\sqrt{2s}})} \cdot e^{-a\sqrt{2s}} \cdot \frac{z e^{-(2b+a)\sqrt{2s}} p_b^2}{zw(z+p_b)^2(w+p_b - \frac{p_b^2 e^{-2b\sqrt{2s}}}{z+p_b})} \\ & + H(s, z, w) \left\{ \frac{1 - e^{-2a\sqrt{2s}}}{\sqrt{2s}} + e^{-a\sqrt{2s}} z \frac{e^{-a\sqrt{2s}}}{z(z+p_b)} \right. \\ & \left. + e^{-a\sqrt{2s}} zw \frac{e^{-(2b+a)\sqrt{2s}}}{zw(z+p_b)^2(w+p_b - \frac{p_b^2 e^{-2b\sqrt{2s}}}{z+p_b})} \right\}. \end{aligned}$$

Solving this for $H(s, z, w)$ yields:

$$H(s, z, w) = \frac{e^{-(2b+a)\sqrt{2s}} p_b^2 p_a}{zw(z+p_a+p_b e^{-2b\sqrt{2s}})^2(w+p_b - \frac{p_b^2 e^{-2b\sqrt{2s}}}{z+p_a+p_b e^{-2b\sqrt{2s}}})}.$$

Inverting this with respect to z, w we get:

$$e^{-(a+b)\sqrt{2s}} p_a p_b \int_0^\xi dk \int_0^\eta d\ell e^{-(p_a+p_b e^{-2b\sqrt{2s}})k} e^{-p_b \ell} \sqrt{\frac{k}{\ell}} I_1(2p_b e^{-b\sqrt{2s}} \sqrt{k\ell}).$$

Letting $s \rightarrow 0$ gives:

$$\begin{aligned} & P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho, FC(-a) = t) = \\ & = \frac{1}{2a} \frac{1}{2b} \int_0^\xi dk \int_0^\eta d\ell e^{-(\frac{1}{2a} + \frac{1}{2b})k - \frac{\ell}{2b}} \sqrt{\frac{k}{\ell}} I_1\left(\frac{\sqrt{k\ell}}{b}\right). \end{aligned}$$

Case 3.

$$P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho \leq c, FC(-a) = t).$$

Consider the Brownian motion paths

$$\left\{ b \leq \sup_{0 \leq \rho \leq t} X_\rho \leq c, X_t = -a \right\}.$$

Before hitting $-a$ for the first time a Brownian path may stay below level b , in this case however after having hit $-a$ it must hit b before arriving at $-a$ at time t .

On the other hand, if the Brownian path hit b before hitting $-a$ for the first time, say at time $\tau \leq t$, then between τ and t it either stays below zero or hits zero but stay below level b , or hits b .

Based upon this an integral equation can be written up and solved through Laplace transforms again.

Case 4. $P(L_0(t) \leq \xi, L_b(t) \leq \eta, L_c(t) \leq \zeta, c \leq \sup_{0 \leq \rho \leq t} X_\rho, F'(-a) = t)$ Now we shall consider the following set of Brownian paths:

$$\left\{ c \leq \sup_{0 \leq \rho \leq t} X_\rho \right\}.$$

Let the time the Brownian path first hits $-a$ be τ . If between zero and τ it stays below c then it must hit c between τ and t .

If it hits c between zero and τ then between τ and t there are four possibilities:

- 1.) It does not return to zero.
- 2.) Returns to zero, but stays below b .
- 3.) Hits b but stays below c .
- 4.) It hits c .

Again this can be expressed as an integral equation that can be solved through Laplace transforms as before.

Here we list the probability distributions with their Laplace transforms:

$$P(L_0(t) \leq \xi, \sup_{0 \leq \rho \leq t} X_\rho \leq b, FC(-a) = t) : a, b > 0$$

$$\frac{p_a e^{-a\sqrt{2s}}}{z} \cdot \frac{1}{z + p_a + p_b e^{-2b\sqrt{2s}}}$$

$$P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho, FC(-a) = t) :$$

$$\frac{p_a p_b^2 e^{-(2b+a)\sqrt{2s}}}{zw} \cdot \frac{1}{(z + p_a + p_b e^{-2b\sqrt{2s}})^2} \cdot \frac{1}{w + p_b - \frac{p_b^2 e^{-2b\sqrt{2s}}}{z + p_a + p_b e^{-2b\sqrt{2s}}}}$$

$$P(L_0(t) \leq \xi, L_b(t) \leq \eta, b \leq \sup_{0 \leq \rho \leq t} X_\rho \leq c, FC(-a) = t) :$$

$$\frac{p_a p_b^2 e^{-(2b+a)\sqrt{2s}}}{zw} \cdot \frac{1}{(z + p_a + p_b e^{-2b\sqrt{2s}})^2} \cdot \frac{1}{w + p_b + p_{c-b} e^{-2(c-b)\sqrt{2s}} - \frac{p_b^2 e^{-2b\sqrt{2s}}}{z + p_a + p_b e^{-2b\sqrt{2s}}}}$$

In general: let $\xi = \xi_0, \xi_1, \xi_2, \dots, \xi_n$ be positive numbers and $z = z_0, z_1, z_2, \dots, z_n$ be the Laplace variables corresponding to them, furthermore let $0 = b_0 < b_1 < b_2 < \dots < b_n < b_{n+1}$, define $Q_0, Q_1, \dots, Q_n$ as

$$Q_0 = \frac{1}{z + p_a + p_{b_1} e^{-2b_1 \sqrt{2s}}}, \quad Q_1 = \frac{1}{z_1 + p_{b_1} + p_{b_2-b_1} e^{-2(b_2-b_1) \sqrt{2s}} - p_{b_1}^2 e^{-2b_1 \sqrt{2s}} Q_0}$$

$$Q_2 = \frac{1}{z_2 + p_{b_2-b_1} + p_{b_3-b_2} e^{-2(b_3-b_2) \sqrt{2s}} - p_{b_2-b_1}^2 e^{-2(b_2-b_1) \sqrt{2s}} Q_1}$$

and

$$Q_n = \frac{1}{z_n + p_{b_n-b_{n-1}} + p_{b_{n+1}-b_n} e^{-2(b_{n+1}-b_n) \sqrt{2s}} - p_{b_n-b_{n-1}}^2 e^{-2(b_n-b_{n-1}) \sqrt{2s}} Q_{n-1}}$$

$$\overline{Q}_n = \frac{1}{z_n + p_{b_n-b_{n-1}} + -p_{b_n-b_{n-1}}^2 e^{-2(b_n-b_{n-1}) \sqrt{2s}} Q_{n-1}}.$$

The Laplace transform of

$$P(L_0(t) \leq \xi, L_{b_1}(t) \leq \xi_1, \dots, L_{b_n}(t) \leq \xi_n, b_n \leq \sup_{0 \leq \rho \leq t} \leq b_{n+1}, FC(-a) = t)$$

is

$$\frac{e^{-(2b_n+a)\sqrt{2s}} p_a p_{b_1}^2 p_{b_2-b_1}^2 p_{b_n-b_{n-1}}^2 Q_0 Q_1 \dots Q_n}{z_0 z_1 \dots z_n} \quad (6.31)$$

and if $b_{n+1} = \infty$ this gives the Laplace transform of $P(L_0(t) \leq \xi, L_{b_1}(t) \leq \xi_1, \dots, L_{b_n}(t) \leq \xi_n, b_n \leq \sup_{0 \leq \rho \leq t}, FC(-a) = t)$:

$$\frac{e^{-(2b_n+a)\sqrt{2s}} p_a p_{b_1}^2 p_{b_2-b_1}^2 p_{b_n-b_{n-1}}^2 Q_0 Q_1 \dots \overline{Q}_n}{z_0 z_1 \dots z_n} \quad (6.32)$$

The inverse Laplace transform of (6.31) with respect to $z_1, z_2, \dots, z_n$ gives the Laplace transform of $P(L_0(t) \leq \xi, L_{b_1}(t) \leq \xi_1, \dots, L_{b_n}(t) \leq \xi_n, b_n \leq \sup_{0 \leq \rho \leq t} X_\rho \leq b_{n+1}, FC(-a) = t)$:

$$p_a p_{b_1} p_{b_2-b_1} \dots p_{b_n-b_{n-1}} \int_0^\xi dk_0 \int_0^{\xi_1} dk_1 \dots \int_0^{\xi_n} dk_n \quad (6.33)$$

$$\cdot \exp \left\{ -(p_a + p_{b_1} e^{-2b_1 \sqrt{2s}}) k_0 - (p_{b_1} + p_{b_2-b_1} e^{-2(b_2-b_1) \sqrt{2s}}) k_1 \right.$$

$$\begin{aligned}
& -(p_{b_2-b_1} + p_{b_3-b_2} e^{-2(b_3-b_2)\sqrt{2s}})k_2 \dots - (p_{b_n-b_{n-1}} + p_{b_{n+1}-b_n} e^{-2(b_{n+1}-b_n)\sqrt{2s}})k_n \} \\
& \cdot \sqrt{\frac{k_0}{k_1}} I_1(2p_{b_1} e^{-b_1\sqrt{2s}} \sqrt{k_0 k_1}) \sqrt{\frac{k_1}{k_2}} I_1(2p_{b_2-b_1} e^{-(b_2-b_1)\sqrt{2s}} \sqrt{k_1 k_2}) \dots \\
& \dots \sqrt{\frac{k_{n-1}}{k_n}} I_1(2p_{b_n-b_{n-1}} e^{-(b_n-b_{n-1})\sqrt{2s}} \sqrt{k_{n-1} k_n}).
\end{aligned}$$

Putting $s = 0$ gives finally:

$$\begin{aligned}
& P(L_0(FC(-a)) \leq \xi, L_{b_1}(FC(-a)) \leq \xi_1, \dots, L_{b_n}(FC(-a)) \leq \xi_n, \\
& b_n \leq \sup_{0 \leq \rho \leq t} X_\rho \leq b_{n+1}) = \tag{6.34} \\
& \frac{1}{2a} \frac{1}{2b_1} \frac{1}{2(b_2-b_1)} \dots \frac{1}{2(b_n-b_{n-1})} \int_0^\xi dk_0 \int_0^{\xi_1} dk_1 \dots \int_0^{\xi_n} dk_n \\
& \cdot \exp \left\{ -\left(\frac{1}{2a} + \frac{1}{2b_1}\right)k_0 - \left(\frac{1}{2b_1} + \frac{1}{2(b_2-b_1)}\right)k_1 \right. \\
& \left. - \left(\frac{1}{2(b_n-b_{n-1})} + \frac{1}{2(b_{n+1}-b_n)}\right)k_n \right\} \\
& \cdot \sqrt{\frac{k_0}{k_1}} I_1\left(\frac{\sqrt{k_0 k_1}}{b_1}\right) \sqrt{\frac{k_1}{k_2}} I_1\left(\frac{\sqrt{k_1 k_2}}{b_2-b_1}\right) \dots \\
& \dots \sqrt{\frac{k_{n-1}}{k_n}} I_1\left(\frac{\sqrt{k_{n-1} k_n}}{b_n-b_{n-1}}\right).
\end{aligned}$$

Putting $\xi = \infty$ and integrating by k using $\int_0^\infty e^{-st} \sqrt{\frac{t}{A}} I_1(2\sqrt{At}) = \frac{e^{-\frac{A}{s^2}}}{s^2}$ gives:

$$\begin{aligned}
& P(L_{b_1}(FC(-a)) \leq \xi_1, \dots, L_{b_n}(FC(-a)) \leq \xi_n, \\
& b_n \leq \sup_{0 \leq \rho \leq t} X_\rho \leq b_{n+1}) = \tag{6.35} \\
& = \frac{a}{2(a+b_1)^2} \frac{1}{2(b_2-b_1)} \dots \frac{1}{2(b_n-b_{n-1})} \int_0^{\xi_1} dk_1 \dots \int_0^{\xi_n} dk_n \\
& \cdot \exp \left\{ -\left(\frac{1}{2(a+b_1)} + \frac{1}{2(b_2-b_1)}\right)k_1 \right. \\
& \left. - \left(\frac{1}{2(b_n-b_{n-1})} + \frac{1}{2(b_{n+1}-b_n)}\right)k_n \right\}
\end{aligned}$$

$$\begin{aligned} & \cdot \sqrt{\frac{k_1}{k_2}} I_1\left(\frac{\sqrt{k_1 k_2}}{b_2 - b_1}\right) \dots \\ & \dots \sqrt{\frac{k_{n-1}}{k_n}} I_1\left(\frac{\sqrt{k_{n-1} k_n}}{b_n - b_{n-1}}\right). \end{aligned}$$

Now (6.35) is equivalent to:

$$\begin{aligned} P(L_{b_1}(FC(-a)) = \xi_1, \dots, L_{b_n}(FC(-a)) = \xi_n, \\ L_{b_{n+1}}(FC(-a)) = 0) = \\ \frac{a}{2(a+b_1)^2} \frac{1}{2(b_2-b_1)} \dots \frac{1}{2(b_n-b_{n-1})} \\ \cdot \exp \left\{ -\left(\frac{1}{2(a+b_1)} + \frac{1}{2(b_2-b_1)}\right)\xi_1 \right. \\ \left. -\left(\frac{1}{2(b_n-b_{n-1})} + \frac{1}{2(b_{n+1}-b_n)}\right)\xi_n \right\} \\ \cdot \sqrt{\frac{\xi_1}{\xi_2}} I_1\left(\frac{\sqrt{\xi_1 \xi_2}}{b_2 - b_1}\right) \dots \\ \dots \sqrt{\frac{\xi_{n-1}}{\xi_n}} I_1\left(\frac{\sqrt{\xi_{n-1} \xi_n}}{b_n - b_{n-1}}\right). \end{aligned} \quad (6.36)$$

Letting $b_{n+1} \rightarrow \infty$ gives:

$$\begin{aligned} P(L_{b_1}(FC(-a)) = \xi_1, \dots, L_{b_n}(FC(-a)) = \xi_n) = \\ \frac{a}{2(a+b_1)^2} \frac{1}{2(b_2-b_1)} \dots \frac{1}{2(b_n-b_{n-1})} \\ \cdot \exp \left\{ -\left(\frac{1}{2(a+b_1)} + \frac{1}{2(b_2-b_1)}\right)\xi_1 \right. \\ \left. - \frac{1}{2(b_n-b_{n-1})}\xi_n \right\} \\ \cdot \sqrt{\frac{\xi_1}{\xi_2}} I_1\left(\frac{\sqrt{\xi_1 \xi_2}}{b_2 - b_1}\right) \dots \\ \dots \sqrt{\frac{\xi_{n-1}}{\xi_n}} I_1\left(\frac{\sqrt{\xi_{n-1} \xi_n}}{b_n - b_{n-1}}\right) \end{aligned} \quad (6.37)$$

and putting $n + 1$ for n gives:

$$\begin{aligned}
P(L_{b_1}(FC(-a)) = \xi_1, \dots, L_{b_n}(FC(-a)) = \xi_n, \\
L_{b_{n+1}}(FC(-a)) = \xi_{n+1}) = \\
\frac{a}{2(a+b_1)^2} \frac{1}{2(b_2-b_1)} \cdots \frac{1}{2(b_n-b_{n-1})} \frac{1}{2(b_{n+1}-b_n)} \\
\cdot \exp \left\{ -\left(\frac{1}{2(a+b_1)} + \frac{1}{2(b_2-b_1)} \right) \xi_1 \right. \\
\left. - \left(\frac{1}{2(b_n-b_{n-1})} + \frac{1}{2(b_{n+1}-b_n)} \right) \xi_n - \frac{1}{2(b_{n+1}-b_n)} \xi_{n+1} \right\} \\
\cdot \sqrt{\frac{\xi_1}{\xi_2}} I_1\left(\frac{\sqrt{\xi_1 \xi_2}}{b_2-b_1}\right) \cdots \\
\cdots \sqrt{\frac{\xi_{n-1}}{\xi_n}} I_1\left(\frac{\sqrt{\xi_{n-1} \xi_n}}{b_n-b_{n-1}}\right) \sqrt{\frac{\xi_n}{\xi_{n+1}}} I_1\left(\frac{\sqrt{\xi_n \xi_{n+1}}}{b_{n+1}-b_n}\right).
\end{aligned} \tag{6.38}$$

Using (6.36) (6.37) (6.38) we can establish that if $0 \leq b < c$ than:

$$P(L_c(FC(-a)) = \xi | L_b(FC(-a)) = \eta) = \frac{e^{-\frac{\xi+\eta}{2(c-b)}}}{2(c-b)} \sqrt{\frac{\xi}{\eta}} I_1\left(\frac{\sqrt{\xi\eta}}{c-b}\right)$$

and

$$P(L_c(FC(-a)) = 0 | L_b(FC(-a)) = \eta) = e^{-\frac{\eta}{c-b}}$$

and this completes the proof of the Ray - Knight theorem. ■

Chapter 7

Directions for future research.

The object of the thesis has been the systematic development of the Q - function method. We have used it to rederive many known properties as well as a number of new distribution results for Brownian motion. This method can be used to determine the occupation time distribution O_f for arbitrary function $f(t)$. For two sided domains it becomes rather cumbersome as the simple case discussed in Chapter 4 shows. In principle the Q - function method can be extended to the case of the Brownian sheet as well as and higher dimensional time parameter processes.

In Kac's original paper the method developed was used to evaluate distributions of functionals of more general type. The Q method can also be used to evaluate distributions of functionals of the form $\int_0^t X_\tau^{2n} d\tau$ for tied down Brownian motion.

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