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*Dynamic Soil-structure Analysis by
boundary element method.*

University — Université

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Degree for which thesis was presented — Grade pour lequel cette thèse fut présentée

ph.D.

Year this degree conferred — Année d'obtention de ce grade

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**DYNAMIC SOIL-STRUCTURE ANALYSIS
BY
BOUNDARY ELEMENT METHOD**

by

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BSc, CAIRO UNIVERSITY, 1975

MSc, CAIRO UNIVERSITY, 1981

A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfilment of the
requirements for the degree of
DOCTOR OF PHILOSOPHY
IN
CIVIL ENGINEERING

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ABSTRACT

A method is developed to analyze the earthquake response of a two-dimensional soil-structure system. The semi-infinite soil domain is modelled by boundary elements, while the structure is modelled as an assemblage of finite elements. A substructure technique of analysis is used and the soil material as well as the superstructure material are considered homogeneous, isotropic and linearly elastic.

The analysis is carried out by first resolving the earthquake excitation into a series of harmonic components. The steady state response of the system to each harmonic component is obtained next. The component response is then Fourier synthesized to obtain the resultant response in time domain. The Fourier analysis is accomplished by using the Fast Fourier Transform method.

The substructure method used in the analysis requires that a dynamic stiffness matrix corresponding to the connection degrees of freedom at the base of the superstructure be obtained for the soil for each component frequency of the excitation. This is best accomplished by first obtaining the flexibility matrix of the soil and then inverting it.

Results are presented in the thesis for the steady state vibration of a rigid strip footing and compared with those available in the literature. This is followed by an investigation of the coupling effects of two footings. The frequency domain analysis of a system consisting of superstructure and substructure is described next and the results of the analysis are presented.

It is concluded that the proposed method of analysis is quite effective in the solution of the complex interaction problem of soil-structure vibration.

ACKNOWLEDGEMENTS

The author wishes to express his sincere thanks to his supervisor, professor J.L. Humar, for his guidance, encouragement and support throughout the course of this study.

Special thanks are due to professor C.L. Tan for many fruitful discussions.

The author is also grateful to professor A.P.S. Selvadurai for his valuable assistance and suggestions.

This work would not have been possible without the scholarship awarded to the author by Carleton University.

Special thanks are due to the author's wife for her care and encouragement throughout the course of this study.

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Chapter 1

INTRODUCTION

1.1 General

It is well recognized that the response of a structure to a dynamic loading, such as due to an earthquake, can be significantly affected by dynamic interaction with the foundation. For important structures, e. g., nuclear reactors and dams, it is therefore imperative that a rational earthquake analysis include foundation/interaction effects.

Much of the past research on analysis of earthquake response of structures including the effects of structure-soil interaction may be classified into two categories, depending on whether the analysis is carried out by a direct method or by a substructure method.

The direct method involves the analysis of a finite element idealization of the combined structure-soil system (1-8). The method can handle non-homogeneous soil properties, non-linear soil as well as complex geometric shapes of the soil domain. However, a serious limitation of the method is that the soil model must have finite boundaries. When the soil consists of a broad, deep layer or has an infinite extent, artificial boundaries must be introduced. Such boundaries will, however, reflect the propagating waves and may invalidate the results of analysis. To avoid such spurious reflections, the boundaries must be placed far away from zone of interest. Thus a very large finite element mesh will be required to model the infinite extent of the soil. As will be discussed later, special type of boundaries which absorb the outgoing waves and suppress their reflection have been used by some researchers (5,6). However, even such special boundaries are not very successful in modelling the radiation damping effect or the energy loss in the wave propagating to infinity.

The direct method requires finite element modelling of the combined soil-structure system. Because of the large finite element mesh required to model infinite soil domains, the problem size may become very large. Modal decomposition may be used in such cases to reduce the size of the

problem, but the first few modes of the entire system may not adequately describe the structural deflection response (1).

It should be noted that a recent development in the finite element technique which is based on the introduction of infinite elements at the boundaries has made it possible to reduce the size of the finite element mesh required to model the half plane (9). Where such element are effective, the direct method of analysis may be quite useful.

In the substructure method of analysis, the system is analyzed in two stages. In the first stage the frequency dependent stiffness (or impedance) coefficients of the soil medium with respect to soil-structure interface degrees of freedom are obtained separately from the structure. These resulting impedance coefficients are then used in conjunction with the structural stiffness coefficients to set up the equations of motion in the frequency domain. The solution to these equations followed by a Fourier synthesis gives the structural response (1,11-14).

The efficiency of the substructure method depends, among other considerations, on the procedure used to obtain the dynamic soil impedance matrix. Of the various procedures available the following are the most effective: (i) mathematical formulation of an idealized viscoelastic half space, (ii) finite element modelling of the soil medium, and (iii) boundary element formulation of the soil medium.

The mathematical formulation of a viscoelastic half space (14-16) though quite effective is limited to those cases where the soil surface is plane and soil consists either of an infinite homogeneous layer or is an homogeneous half space.

The finite element procedure is equally effective and can also handle complex geometry and non-homogeneous soils. The method is however beset with the difficulties that arise in the use of finite elements to solve problems of wave propagation in an infinite medium.

In the boundary element method, the soil is modelled by boundary elements. The method is very effective in modelling infinite soil domains particularly when the soil can be treated as homogeneous and linearly elastic.

The main objective of this study is to develop an accurate and efficient substructure technique to solve the soil-structure interaction problem in which the infinite soil medium is modelled by boundary elements and the superstructure is modelled as an assemblage of finite elements as shown in Fig. (1.1). It is shown that the technique can be very effective provided the analysis is carried out in the frequency domain.

Figure (1.1) Modelling of Soil-Structure System

The substructure technique presented in this work provides a simpler alternative to the existing techniques. The boundary element formulation of the soil domain has the ability to model both infinite and finite domains with complex geometries using a small number of elements which results in a smaller system of equations compared to that obtained in a finite element formulation.

The method relies on a Green's function solution for a point harmonic load in an infinite space. In this respect it is similar to the mathematical formulation of the viscoelastic half space presented by Chopra and Dasgupta (15,16). However, the half space solution seeks to satisfy the displacement and traction conditions on the boundary and is therefore very difficult to obtain. In fact, solutions exist only for simple geometries and a horizontal surface. On the other hand, Green's solution of the boundary element method is a simple singular solution for the infinite space and does not have to satisfy a restricting set of boundary conditions. In other words, the method is simpler and can be used to analyze a soil domain with a complex geometry.

Although the boundary element method has been used in some recent research to solve the problem of rigid foundations resting on an infinite soil, the method has not been applied to the problem of soil-structure interaction. To the knowledge of the author, this is the first comprehensive attempt to use this method to solve the problems of this class.

The work presented here deals with the earthquake analysis of structure foundation systems, such as a gravity dam resting on a soil medium. The treatment of the problem in this work is restricted to two dimensional case and several assumptions are made to simplify the problem. These assumptions include:

- (1) A plane strain situation exists.
- (2) Soil medium consists of a homogeneous, linearly elastic half plane.
- (3) Material damping for soil is not considered but radiation damping is taken into account.
- (4) The superstructure material is linearly elastic.
- (5) The damping coefficient corresponding to each mode of the superstructure is independently selected.

1.2 Review of Past Work

1.2.1 General

In the earlier research seeking to understand the dynamic behaviour of ground-borne structures, it was generally accepted that as a first step the foundation should be regarded as a rigid body resting on the soil modelled as a homogeneous isotropic elastic half plane.

Attention was first brought to this problem by the work of Lamb (17) on the dynamics of the earth due to application of an idealized point force. The amount of work involved in this relatively straight forward problem of Lamb gave a foretaste of the tremendous mathematical difficulties that awaited any attempt at analysing the dynamic response of foundations on the earth surface. Ever since then researchers have sought various ways of solving the problem. The major contribution made so far may be grouped under three categories depending on the method of approach, namely:

- (a) Finite model of semi-infinite half space;
- (b) Analytical methods of dual integral equations, and
- (c) numerical methods such as finite element and boundary element method.

Although an extensive review of the past work under the first two categories is presented elsewhere (18), it is reviewed in this section too. A description of the third approach is deferred to the next section.

Finite Model of Semi-infinite Half Space

This approach, which involves replacing the entire half space by a finite arrangement of masses, springs and damping elements, dates back to the work of Ang and Rainer (19) on a lumped parameter model for the treatment of plane strain problems. Agbabian, et al(20), among others, introduced finite boundaries into the Ang and Rainer model with damping elements introduced at the boundaries to account for geometric damping. Not surprisingly they found the reliability of the model to be dependent on its size.

Many variations of this modelling exist. They include confining the computation to a restricted near field volume of the half space. Lysmer and Waas(21), achieved this by the introduction of boundaries designed to transmit D'Alembert forces. Ang and Newmark(22) and Gupta et al(23) used a lumped parameter model to analyze coupled sliding and rocking vibration of an infinitely long rigid strip footing and found agreement with analytical results of Karasudhi, Keer and Lee(24)

who estimated the coupled motion by superposition of the pure modes. Representing the half space by a finite number of masses results in a stiff system and therefore to obtain good results the model itself may have to approach the size of the half space.

The Method of Dual Integral Equations

The method of dual integration can be used to solve the problem of massless rigid plate resting on a linearly elastic half space under the action of a steady state harmonic excitation and a considerable amount of work has been done in this area (24-29). The technique involves the solution of a mixed boundary value problem with a stress condition over the free surface and a displacement condition under the footing. Vertical and horizontal translations as well as torsional vibration are each treated separately. For each case, the partial differential equations of motion are reduced to a set of dual integral equations by introducing appropriate dependent variables of Hankel and complex Fourier transforms. Applying the condition that the surface of the half space is represented by a single function of position and taking account of the other boundary conditions, the equations can be solved in the transformed domain. The original variables can be determined by using the inversion theorem and solving the inversed equation by a series of expansion procedures. To overcome the almost formidable task of solving the transformed pair of integral equations, many researchers assume a contact stress distribution under the footing (25).

The dual integral equations method is also a powerful tool in dealing with axisymmetric and asymmetric dynamic responses of a rigid circular disc embedded in bonded contact with an isotropic elastic medium of infinite extent (30-31). Recently, Selvadurai (32) has developed an approximate solution for the low frequency dynamic response of a rigid plate embedded in an infinite space.

Although the dual integral equations provide an exact formulation of the mathematical problem and several techniques are available to deal with the resulting system of integral equations, it nevertheless presents considerable mathematical difficulty except in simple cases and has no use in the analysis of foundations of arbitrary shapes whose area of contact with the half space can not be described by any simple algebraic expression.

1.2.2 Numerical Solution Approach

Several numerical methods have been attempted to simplify the solution of the soil-structure interaction problem. The more important among the numerical solutions available are: 1) the finite element solution and (2) the boundary element solution.

The Finite Element Solution (FEM)

A considerable amount of work has been done on the problem of soil-structure interaction using the finite element method (1-14).

The FEM allows the simultaneous discretization of the structure and the soil and is equally applicable to embedded foundations, and nonhomogeneous soil and non-linear soil properties. As stated earlier, the past work on the analysis of earthquake response of structures including the effects of soil-structure interaction may be classified into two categories, depending on whether the analysis is carried out by a direct method or by a substructure method.

The direct method of analysis was used in the earlier attempts to analyse the problem by finite element technique. Wilson (2) analyzed a two dimensional model of a gravity dam on a layered foundation by modelling the entire system by finite elements and carrying out the analysis in the time domain by a numerical integration of the equations of motion. A similar procedure was used by Finn (3) and by Ghosh and Wilson (4), the latter for the solution of axisymmetric structures, except that instead of a direct integration of the equations of motion those researchers employed a modal superposition technique utilizing the first few modes of the combined soil-structure system. An extensive summary of the contributions is available in reference (5).

One of the major drawbacks of the FEM is that it must deal with a finite model while the soil is usually infinite. To overcome this difficulty, several ingenious techniques have been developed. Lysmer (6) used a finite model to represent an infinite foundation media by introducing special viscous boundaries with vertical and horizontal dashpot dampers at the boundary nodes of the system. In the Lysmer's technique, the damping values are proportional to the soil density and wave velocities while the size of the model depends on the element size and the Rayleigh wave length. Other studies available in this area are those due to Waas (7) and Laing (8). Liang has studied embedded foundation as well as the interaction between adjacent footings. Both researchers have used special semi-infinite elements to model the infinite extent of the soil layer in the horizontal direction only. Recently, special infinite elements were employed successfully by Medina and Taylor for modelling the far field of a half plane soil medium (9).

An alternative technique for calculating the effect of soil structure interaction is the substructure method. In the substructure method, soil and building are treated as two different substructures. The soil can then be analyzed by any method including finite elements (1,10).

The mathematical formulation of the substructure technique was first presented by Sandi (11). Chopra and Perumalswami (12) made use of the substructure technique for the problem of a dam modelled by finite elements while the soil was treated as an elastic half space. Chakrabarti and Chopra utilized the idea for the dynamic response analyses of a dam-reservoir system (13). More recently they employed the same technique for the response analyses of a complete dam-reservoir-soil system (14).

The Boundary Elements Method (BEM)

As a numerical technique, the BEM has many interesting advantages over the FEM. The FEM can not easily model infinite domains and therefore its application to soil structure problem is beset with difficulties. On the other hand, the BEM is very well suited to solve soil structure interaction problems as it is very effective in the analysis of infinite and semi infinite domains such as the soil. The BEM also reduces the size of the problem since only the elements on the boundary need to be defined. By contrast, finite element method requires definition of many interior nodes. This is of particular disadvantage in the soil-structure problem where the only tractions and displacements of interest are on the surface of the soil and not in the interior of the soil domain.

The BEM has been applied to the determination of the dynamic stiffnesses of rigid foundations resting on or embedded in an elastic half space. Solutions have been obtained for both two dimensional and three dimensional problems (33-42) for transient as well as steady state vibration. Several researchers have concluded that the boundary element method seems to provide an accurate and computationally feasible procedure for the solution of two and three dimensional problems in soil-structure interaction and often demonstrates clear advantages over the FEM.

The existing applications of boundary element method have generally been limited to the solution of isolated rigid foundations resting on the half space. The potential of the BEM method in the solution of a complete substructure problem has not been fully investigated.

1.3 Objectives and Scope of Work

This research presents a method for the dynamic analysis of a foundation-structure system in which a substructure approach is used and where the foundation substructure is analyzed by using a boundary element analysis technique while the superstructure is modelled by finite elements.

The work was approached in the following stages:

- (1) Review and investigation of the finite element formulation of the general dynamic analysis of structure-foundation systems in the frequency domain. The formulation uses a substructure approach, The foundation is analyzed separately and its effect incorporated into the equations of the structure.
- (2) Formulation of the boundary element solution for the steady state elastodynamic response of a soil half plane or layer to obtain the frequency dependent compliance coefficients needed to include foundation interaction effects in the analysis, and the development of the necessary computer programs.
- (3) Verification of the formulation mentioned in the preceding paragraph by solving the problem of harmonic response of rigid footing resting on a half plane and comparing the solution with the existing classical solutions available in the literature. The investigation has been extended to include the study of coupling effects between two rigid strip footings under steady state vibration as well as the study of the effect of embedment on the dynamic response of footings.
- (4) Development of the substructure procedure and the necessary computer programs to obtain the earthquake response of the structure foundation system using a frequency domain analysis procedure.

1.4 Format of The Thesis

The thesis is divided into eight chapters as follows:

Chapter 1 introduces the problem, reviews the previous work, and describes the objectives and scope of the present work and the format of the thesis.

Chapter 2 deals with the general formulation of the structure-foundation problem. The problem is formulated in the frequency domain. Advantages of using the substructuring approach over the direct solution approach are discussed.

Chapters 3 to 6 present the dynamics analysis of the foundation to obtain the frequency dependent stiffness coefficients needed in the formulation of the equations of motion. Chapter 3 presents the formulation of the boundary element equations for the elastodynamics problem of a soil half plane. Chapter 4 presents the problem of the vibration of an infinitely long rectangular rigid footing on an elastic half plane. Chapter 5 investigates the coupling effect between two adjacent rigid strip footings. Chapter 6 deals with the behaviour of soil layer under the action of harmonic

unit load and develops a procedure for assembling the stiffness matrix of the soil medium with respect to a specified set of degrees of freedom.

In chapter 7 the problem of the analysis of the structure using the foundation complex stiffnesses is dealt with. It is noted that a direct method of solving the problem requires very large computational effort and storage. A mode superposition technique is therefore used for the expression of structural response even though in the classical sense mode shapes do not exist. This technique expresses the displacements of the structure in terms of the normal modes of an associated structural system. Because the response of the structure is essentially contained in the first few modes, this results in a drastic reduction in the size of the problem and consequently in a large saving in storage and computing time required. The technique is compared with a direct method of analysis for a harmonic free field input to demonstrate the effectiveness of modal analysis. Finally, the response of a dam resting on an elastic half plane to the horizontal component of El-Centro earthquake is presented.

Chapter 8 summarizes the investigation and presents the more important conclusions.

Chapter 2

FORMULATION OF STRUCTURE-FOUNDATION PROBLEM

2.1 Introduction

As stated earlier, the effect of structure-foundation interaction is often too important to be ignored and must therefore be included in the analysis. The complex nature of the problem however necessitates the use of numerical techniques in tackling it. These techniques include finite element method and boundary element method. A brief review of the previous research in this area was presented in chapter 1.

In the substructure technique proposed in this work, the soil is modelled by boundary elements while the superstructure is modelled by finite elements. The boundary element formulation of the soil medium is presented in chapter 3.

This chapter presents the finite element formulation of the superstructure. This is followed by the presentation of the general formulation of the soil-structure interaction problem. In general, two different techniques have commonly been used to analyze the response of a structure-soil system idealised as an assemblage of finite elements, namely a direct or 'one pass method', and a substructure method. A complete review of the concepts involved as well as the details can be found in the literature (1,43). However, a brief description of the two methods is presented here for the sake of completeness.

2.2 Finite Element Modelling of the Superstructure

The formulation of the plane strain problem using the finite element method is well developed and has been used quite often. Details of the formulation may be found in the literature (43). The basic idea in the method is a discretization of the continuum into a finite number of degrees of freedom by dividing it up into discrete elements interconnected at their nodes. Each element has a small finite number of independent degrees of freedom. The displacement field within an

element is defined in terms of the nodal displacements of the element. The displacement in the entire continuum is approximated by a combination of displacement fields in each element, which are usually chosen to satisfy the compatibility at the element boundaries. The combination which best represents the actual solution to the continuous problem is obtained by solving equations of equilibrium, one for each degree of freedom, formed as described later. The discretization and the imposed displacement fields are chosen such that as the mesh becomes finer and finer the approximate solution 'converges' in an energy norm to the actual solution (43).

A 6-degrees-of freedom triangular element is used in this study, as it is efficient and available in the form of existing routines. Using standard procedures an elemental stiffness matrix can be calculated once the displacement field has been assumed. The stiffness matrix is a function of the geometric and constitutive properties of the element; the detailed derivation is described in Appendix A. The total stiffness matrix, K , for the entire set of degrees of freedom can be routinely assembled from the element stiffness matrices. A lumped mass matrix which leads to a diagonal mass matrix was found to be simpler and adequately accurate (1). Hence, the lumped mass matrix is used in this study and is denoted by M . The derivation of the damping matrix C , is discussed later.

Once M , C , K are known, one can write the equations of equilibrium

$$M\ddot{r} + C\dot{r} + Kr = F(t) \quad (2.1.a)$$

where r is the vector containing the nodal displacements, and $F(t)$ represents the prescribed nodal forces.

2.3 The Direct Method

The direct method is also known as 'the one pass method'. In this method the structure as well as the soil are represented by finite elements. The finite element idealization of the combined soil-structure system (5) is then analyzed for an excitation applied at the base rock. The results obtained utilizing this approach are highly dependent upon the ability of the finite element model to represent the semi-infinite soil medium. In general, only a finite region of the soil medium is covered by the finite element model, and, consequently, artificial boundaries must be introduced. When the soil medium is semi-infinite as well as in the case when it consists of a broad, deep layer, the artificial boundaries which reflect the propagating waves may invalidate the results of analysis. To avoid these spurious reflections, the boundaries must be placed far away from the zone

of interest. Thus a very large finite element mesh is required to model the infinite extent of the soil. Different types of special boundaries have been used to avoid such large systems. These include: special viscous boundaries with vertical and horizontal dashpot dampers which absorb the outgoing waves and suppress their reflection (6); and special infinite elements at the boundaries (7). Some researchers have attempted to use material damping to account for the radiation of energy that would take place in the semi-infinite medium (5). Recently, Medina and Taylor (9) have developed a special infinite element which appears to be quite effective in modelling the semi-infinite extent of the soil.

The soil-structure interaction problem discussed here is assumed to be a two dimensional plane strain problem in which the structure and the soil are idealised as a finite elements assemblage as shown in Fig. (2.1). The main concern of this study is the analysis of the response of the structure including foundation interaction to an earthquake excitation. Hence, no external forces are presented; $F(t) = 0.0$. The equations of motion of the system become

$$M\ddot{r}^t + C\dot{r}^t + Kr = 0.0 \quad (2.1.b)$$

in which r^t is the total displacements vector comprised of two degrees of freedom, the horizontal and vertical displacements in that order, for each nodal point in the finite element mesh; M , C , and K are the mass, damping and stiffness matrices respectively; and r represents the motion of the system relative to the motion of the base rock.

The total acceleration vector can be represented as the sum of a vector of accelerations relative to the base rock and the vector of nodal quasi-static displacement caused by the motion of the base rock multiplied by the base rock acceleration, thus if the ground motion is assumed to be identical along the entire length of the base rock

$$\ddot{r}^t = \ddot{r} + a_g^u(t)1^u + a_g^v(t)1^v \quad (2.2)$$

in which r is the vector of relative motion, $a_g^u(t)$ and $a_g^v(t)$, are the earthquake accelerations of the base in the horizontal and vertical direction respectively, and

$$\{1^u\}^T = \{1 \ 0 \ 1 \ 0 \dots 1 \ 0\} \quad (2.3.a)$$

$$\{1^v\}^T = \{0 \ 1 \ 0 \ 1 \dots 0 \ 1\} \quad (2.3.b)$$

Substitution of Eqn. (2.2) into Eqns. (2.1) gives

$$M\ddot{r} + C\dot{r} + Kr = -a_g^u(t)E^u - a_g^v(t)E^v \quad (2.4)$$

Figure(2.1) Finite Element Modelling of The Soil-Structure System

where

$$\{E^u\}^T = \{M_1 \ 0 \ M_2 \ 0 \dots M_N \ 0\} \quad (2.5.a)$$

$$\{E^v\}^T = \{0 \ M_1 \ 0 \ M_2 \dots 0 \ M_N\} \quad (2.5.b)$$

and M_N is the lumped mass at nodal point N .

Equations (2.4) can be rewritten as

$$M\ddot{r} + C\dot{r} + Kr = R_t(t) \quad (2.6)$$

where $R_t(t)$ represents the prescribed forces corresponding to the system degrees of freedom;

$$R_t(t) = -a_g^u(t)E^u - a_g^v E^v \quad (2.7)$$

Several solution techniques have been used to deal with the system of equations in the time domain as well as in the frequency domain. In either case, the equations could be solved directly or could be first uncoupled and expressed in terms of generalized displacements by utilizing a modal decomposition for the displacement vector r .

For non-linear problems, modal decomposition or frequency domain analysis are not feasible and analysis must be carried out in the time domain (44). A step-by-step integration of the equations of motion is used wherein $r(t)$, $\dot{r}(t)$ and $\ddot{r}(t)$ are obtained in terms of their values at $t - \Delta t$: $r(t - \Delta t)$, $\dot{r}(t - \Delta t)$ and $\ddot{r}(t - \Delta t)$. The integration methods usually are based on an assumption regarding the variation of the acceleration during any time step (44).

In the frequency domain analysis, $R(t)$ is expressed as a sum of the harmonic components so that

$$R(t) = \sum_{\omega} \hat{R}(\omega) e^{-i\omega t} \quad (2.8)$$

where the summation is over those values of ω which are significant for the problem.

The response to the excitation given by $\hat{R}(\omega) e^{-i\omega t}$ will be of the form

$$r(t) = \hat{r}(\omega) e^{-i\omega t} \quad (2.9)$$

Substitution into the equations of motions Eqn. (2.6) gives

$$A(\omega) \hat{r}(\omega) = \hat{R}(\omega) \quad (2.10)$$

where

$$A(\omega) = [-\omega^2 M - i\omega C + K] \quad (2.11)$$

The system of equations (2.10) is a set of linear complex simultaneous equations which can be solved for $\hat{r}(\omega)$ by using standard equation's solving routines. The solution for $r(t)$ can then be obtained by Fourier synthesis of $\hat{r}(\omega)$ for all values of ω (45). The decomposition of the excitation into its harmonic components, the evaluation of the response to each of these components followed by the synthesis of the component response constitute the frequency domain analysis also called the Fourier transform analysis.

In general, a large computational effort is involved in the numerical evaluation of the Fourier transforms. However, the development of the Fast Fourier Transform algorithm (FFT) has made it possible to reduce the computation time by several orders of magnitudes, thus making the frequency domain analysis very efficient (46-49).

2.4 The Substructure Technique

The substructure method presents an alternative technique for determining the response of a structure including the soil-structure interaction effects. In the substructure method, soil and building are treated as two different substructure as shown in Fig. (2.2). The soil can then be analyzed by any method including the finite element method and the boundary element method. The soil substructure is analyzed first to obtain its response with respect to the connection degrees of freedom for a series of harmonic excitations. These responses are then used to assemble the foundation flexibility matrix. The inverse of this matrix, the stiffness matrix, is incorporated into the equations of motion of the structure. From this point onwards, the structure can be analyzed alone disregarding the presence of the foundation soil.

The substructure method has been used by J. Gutierrez for the earthquake analysis of soil-structure system (10). He achieved a significant reduction in the computational effort by separating the quasi-static displacements associated with the free field motion at the structure-soil interface and the remaining dynamic displacements; and expressing the latter as superposition of the first few modes of the structure on a fixed base. Chopra and Vaish later tested this technique and concluded that the approximation of taking only the first few modes on a rigid base in representing the structural response is not valid (1). Two reasons were given for this; first neglecting the higher modes for the equations of motion eliminates the direct contribution of the higher modes to the response of the superstructure and second the procedure also eliminates the indirect contribution due to the coupling between the higher modes and the base displacements causing inaccuracies in the evaluation of the latter.

Figure (2.2) Substructure model of The soil-structure system.

Chopra and Vaish introduced a new set of generalized shapes represented by the normalized modes of an associated soil-structure system. They argued that such modes approximate the deflections of the superstructure more accurately.

The equations of motion of the superstructure can be written as

$$M_c \ddot{r}_c^t + C_c \dot{r}_c^t + K_c r_c^t = R_c(t) \quad (2.12)$$

where r_c^t is the total displacement vector for the superstructure degrees of freedom including those at the soil-structure interface, and $R_c(t)$ is the vector of external forces in the structure. For an earthquake excitation, the only external forces are those exerted by the soil at the interface degrees of freedom.

The displacement vector r_c^t is now partitioned so that

$$\{r_c^t\} = \begin{Bmatrix} r_b^t \\ r_s^t \end{Bmatrix} \quad (2.13)$$

where r_b^t represents the displacement vector corresponding to the degrees of freedom at the base of the structure, and r_s^t represents the displacement vector of the superstructure degrees of freedom above the base.

Next, the total displacements are expressed as the sum of the quasi-static displacements due to free field motion at the interface degrees of freedom and the displacements relative to the former:

$$\begin{Bmatrix} r_b^t \\ r_s^t \end{Bmatrix} = \begin{Bmatrix} r_b \\ r_s \end{Bmatrix} + \begin{Bmatrix} r_{b0} \\ r_{s0} \end{Bmatrix} \quad (2.14)$$

Where r_b and r_s are the relative displacements and r_{b0} and r_{s0} are the quasi-static displacements due to free field motion. When the free field ground motion is identical at all interface degrees of freedom;

$$\begin{Bmatrix} r_{b0} \\ r_{s0} \end{Bmatrix} = a_g^u(t) 1^u + a_g^v(t) 1^v \quad (2.15)$$

in which the notations used are similar to those defined in Eqn. (2.3). Using Eqns. (2.13) through (2.15), Eqns. (2.12) can be rewritten in their partitioned form as (1)

$$\begin{aligned} & \begin{bmatrix} M_{bb} & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{r}_b \\ \ddot{r}_s \end{Bmatrix} + \begin{bmatrix} C_{bb} & C_b \\ C_b^T & C \end{bmatrix} \begin{Bmatrix} \dot{r}_b \\ \dot{r}_s \end{Bmatrix} + \begin{bmatrix} K_{bb} & K_b \\ K_b^T & K \end{bmatrix} \begin{Bmatrix} r_b \\ r_s \end{Bmatrix} \\ & = -a_g^u(t) E^u - a_g^v(t) E^v + \begin{Bmatrix} R_b \\ 0 \end{Bmatrix} \end{aligned} \quad (2.16)$$

In equations (2.16), M , C and K are respectively the mass, damping and stiffness matrices relative to the superstructure degrees of freedom above base, M_{bb} , C_{bb} , K_{bb} are the corresponding matrices for the interface degrees of freedom, and C_b , K_b are the coupling damping and stiffness matrices. Vectors E^u and E^v are similar to those defined in Eqn. (2.5), except that N now refers to the number of nodes in the superstructure including the base. Vector R_b represents the forces exerted by the soil on the structure at the connection degrees of freedom.

In deriving Eqn. (2.16), the free field motion is assumed to be identical at all base nodes. This assumption may not be appropriate in some cases such as the case of structures with long base or the case of embedded foundation. However, the extension of the above formulation to handle spatially varying ground motion is possible (10).

The interaction forces R_b are related to the base displacement r_b through the stiffness matrix of the foundation. The stiffness matrix of the foundation which can be determined from either the finite element method or the boundary element method is dependent on the excitation frequency. This means that the excitation as well as the responses must be expressed in terms of their harmonic components over the entire range of the excitation frequency,

By utilizing Eqns. (2.8) and (2.9), Eqn. (2.16) can be written as

$$\begin{aligned} & \left(-\omega^2 \begin{bmatrix} M_{bb} & 0 \\ 0 & M \end{bmatrix} - i\omega \begin{bmatrix} C_{bb} & C_b \\ C_b^T & C \end{bmatrix} + \begin{bmatrix} K_{bb} & K_b \\ K_b^T & K \end{bmatrix} \right) \begin{Bmatrix} \hat{r}_b \\ \hat{r}_s \end{Bmatrix} \\ & = -\hat{a}_g^u(\omega) E^u - \hat{a}_g^v(\omega) E^v + \begin{Bmatrix} \hat{R}_b(\omega) \\ 0 \end{Bmatrix} \end{aligned} \quad (2.17)$$

When the structure is not present, the displacements at the surface of the foundation are equal to the free field displacements and no external forces exist. The displacements of the foundation relative to the free field, r_f , are therefore caused by the forces that the superstructure exerts on the foundation. If these forces are denoted by R_f then

$$\hat{R}_f(\omega) = X_f(\omega) \hat{r}_f(\omega) \quad (2.18)$$

in which $X_f(\omega)$ is the frequency dependent stiffness matrix of the soil medium relative to the connection degrees of freedom. Compatibility and equilibrium require that

$$\hat{r}_b(\omega) = \hat{r}_f(\omega) \quad (2.19.a)$$

$$\hat{R}_b(\omega) = -\hat{R}_f(\omega) \quad (2.19.b)$$

substitution of Eqns. (2.16) and (2.17) into Eqn. (2.15) gives

$$\left(-\omega^2 \begin{bmatrix} M_{bb} & 0 \\ 0 & M \end{bmatrix} - i\omega \begin{bmatrix} C_{bb} & C_b \\ C_b^T & C \end{bmatrix} + \begin{bmatrix} K_{bb} + X_f(\omega) & K_b \\ K_b^T & K \end{bmatrix} \right) \begin{Bmatrix} \hat{r}_b(\omega) \\ \hat{r}_s(\omega) \end{Bmatrix} = -\hat{a}_g^u(\omega) E^u - \hat{a}_g^v(\omega) E^v \quad (2.20)$$

Once $X_f(\omega)$ is determined, Eqns. (2.20) can be solved for $\hat{r}(\omega)$, once for each frequency component, either directly or by employing appropriate Ritz shapes. The response in the time domain is then obtained by Fourier synthesizing the responses $\hat{r}(\omega)$.

Following the general idea presented by Vaish and Chopra (1), the mode shapes of an associated soil-structure system that will provide a good representation of the displacement of the structure founded on deformable soil are used as the generalized vectors. The associated soil-structure system is obtained from the real system by substituting a constant value for the foundation stiffness, X_f , in place of a frequency dependent value. The constant value can be chosen to be that corresponding to the static deformations, those with $\omega = 0.0$. Alternatively, one could use the real part of $X_f(\omega_0)$ where ω_0 is an arbitrarily selected small value of ω . The latter approach is used in this work.

The desired mode shapes are obtained by solving the following eigenvalue problem.

$$[\tilde{K} + \tilde{X}_f(\omega_0)] \Phi = \tilde{M} \Phi \lambda^2 \quad (2.21)$$

in which

$$\tilde{K} = \begin{bmatrix} K_{bb} & K_b \\ K_b^T & K \end{bmatrix} \quad (2.22.a)$$

$$\tilde{X}_f(\omega_0) = \begin{bmatrix} X_f(\omega_0) & 0 \\ 0 & 0 \end{bmatrix} \quad (2.22.b)$$

$$\tilde{M} = \begin{bmatrix} M_{bb} & 0 \\ 0 & M \end{bmatrix} \quad (2.22.c)$$

λ^2 is a diagonal matrix of the squared eigenvalues, Φ are the mode shapes normalized so that $\Phi^T \tilde{M} \Phi = I$, I being the identity matrix, and ω_0 is a specified frequency of excitation.

The structural displacements are now expressed in terms of the first m orthonormal modes

$$\begin{Bmatrix} r_b \\ r_s \end{Bmatrix} = \sum_{j=1}^m \phi_j Y_j(t) = \Phi Y(t) \quad (2.23)$$

which in the frequency domain becomes

$$\begin{Bmatrix} \hat{r}_b(\omega) \\ \hat{r}_s(\omega) \end{Bmatrix} = \sum_{j=1}^m \phi_j \hat{Y}_j(\omega) = \Phi \hat{Y}(\omega) \quad (2.24)$$

Substitution of Eqn. (2.24) into Eqn. (2.20), and premultiplication by Φ^T gives

$$\begin{aligned} & \left\{ -\omega^2 I - i\omega \Phi^T \tilde{C} \Phi + \Phi^T [\tilde{K} + \tilde{X}_f(\omega)] \Phi \right\} \dot{Y}(\omega) \\ & = -\Phi^T [\hat{a}_g^u(\omega) E^u + \hat{a}_g^v(\omega) E^v] \end{aligned} \quad (2.25)$$

in which

$$\tilde{X}_f(\omega) = \begin{bmatrix} X_f(\omega) & 0 \\ 0 & 0 \end{bmatrix} \quad (2.26)$$

Rather than specifying a damping matrix $\tilde{C}$ directly, it is the usual practice to assume that the matrix satisfies the following relationship

$$\Phi^T \tilde{C} \Phi = [2\zeta\lambda] \quad (2.27)$$

in which the matrix on the right hand side is a diagonal matrix with elements $2\zeta_i\lambda$, $i = 1, 2, \dots, m$ and ζ_i = damping in the i th mode.

Substitution of Eqns. (2.21) and (2.27) into Eqn. (2.25) gives

$$\begin{aligned} & \left\{ \omega^2 I - 2i\omega[\zeta\lambda] + \lambda^2 + \Phi^T [\tilde{X}_f(\omega) - \tilde{X}_f(\omega_0)] \Phi \right\} \dot{Y}(\omega) \\ & = -\Phi^T [\hat{a}_g^u(\omega) E^u + \hat{a}_g^v(\omega) E^v] \end{aligned} \quad (2.28)$$

Even with damping selected as in Eqn. (2.27), Eqns. (2.28) are coupled because the matrix $\Phi^T [\tilde{X}_f(\omega) - \tilde{X}_f(\omega_0)] \Phi$ is not diagonal except when $\omega = \omega_0$ in which case it is zero. However, the number of equations in (2.26) may be much smaller than those in Eq (2.1) giving a substantial reduction in the cost of computation.

2.5 Advantages of The Substructure Method

The substructure method has several advantages over the direct method both in terms of the accuracy and the efficiency. The substructure technique allows one to study the superstructure separately without repeating the analysis of the soil medium. This enables one to study the behaviour of the structure to any degree of accuracy required, and gives the choice of using any numerical technique for the analysis of the soil domain including the boundary element method. The flexibility of modelling the soil independently permits a more realistic representation of both the properties and the extent of the soil.

It should be noted that the direct-method can be used to analyse non-linear systems in which the substructure technique can not be used. Also, the introduction of infinite elements and consequent reduction in the size of the mesh required to represent the soil half plane has made the direct method equally effective in certain cases (9).

Chapter 3

FORMULATION OF THE BOUNDARY ELEMENT EQUATIONS

3.1 Introduction

The main objective of this chapter is to present the formulation of the boundary element equations for the solution of the problem of a semi-infinite soil medium under the effect of general harmonic stress state. The boundary element analysis constitutes a crucial step in the formulation of the substructure approach for obtaining the response of a soil-structure system to an arbitrary ground excitation using a frequency domain analysis.

As mentioned earlier, several assumptions have been made regarding the properties of the soil domain. These assumptions can be summarized as follows

- (1) A plane strain situation exists.
- (2) Soil medium consists of a homogeneous, linearly elastic half plane.
- (3) Material damping for soil is not considered but radiation damping is taken into account.

The solution of the steady state problem in linear elastodynamics by integral equations is possible in the Fourier transform space as the hyperbolic equations of motion become elliptic partial differential equations. The formulation of the boundary element equations relies on the existence of a fundamental solution for the Fourier transformed equations of motion corresponding to a dynamic concentrated force in the infinite domain. This fundamental solution in conjunction with the form of Betti's theorem appropriate to the transformed space yields a vector identity which corresponds to Somigliana's identity in elastostatic and Green's third identity in potential theory (33).

In the following presentation, the usual indicial notation of cartesian tensor analysis is used. The field equations are derived on the basis of small displacement theory for a linear, isotropic, homogeneous elastic material.

3.2 Equations of Motion

Let D denote an elastic body bounded by surface S as shown in Fig. (3.1). Associated with each differential surface element will be an outward pointing unit normal vector, n_i . If the displacement vector associated with each point $x \in D + S$ is given by

$$u_j = u_j(x, t) \quad (3.1)$$

then the equations of motion of the linear isotropic elastic body are

$$(c_1^2 - c_2^2)u_{k,jk} + c_2^2 u_{j,kk} + b_j = \ddot{u}_j \quad (3.2)$$

$$j = 1, 2 \quad k = 1, 2$$

with the propagation velocities of compression and shear waves in the unbounded material respectively given by

$$c_1^2 = \frac{\lambda + 2\mu}{\rho} \quad (3.3.a)$$

$$c_2^2 = \frac{\mu}{\rho} \quad (3.3.b)$$

In Eqns. (3.1) through (3.3),

u_j = Displacement vector at a point in D .

b_j = Body force vector at the point in D .

ρ = Mass density of the medium contained in the domain. and λ and μ are the Lamé's constants.

Ignoring the body forces and assuming harmonic motion of the form

$$u_j = U_j e^{-i\omega t}, \quad (3.4)$$

equation (3.2) becomes

$$(c_1^2 - c_2^2)U_{k,jk} + c_2^2 U_{j,kk} + \omega^2 U_j = 0. \quad (3.5)$$

$$j = 1, 2 \quad k = 1, 2$$

Eqn. (3.5) is the frequency dependent equation of motion of the elastic medium. The stresses and displacements are assumed to satisfy the conditions prescribed on the boundary. Thus, if on part S_i of the surface the frequency dependent tractions are prescribed to be $(T_j)_0$ while on

Figure 3.1 Elastic Domain D .

the remaining surface S_u the frequency dependent displacements are specified to be $(U_j)_0$, the boundary conditions become

$$T_j = \sigma_{jk} n_k = (T_j)_0 \quad \text{on } S_t \quad (3.6.a)$$

$$U_j = (U_j)_0 \quad \text{on } S_u \quad (3.6.b)$$

where

T_j = The traction vector on the surface of the domain.

n_k = Unit vector along the outward normal to the surface.

σ_{jk} = The stress vector = $\hat{\sigma}_{jk} e^{-i\omega t}$.

3.3 Fundamental Singular Solution of Elastodynamics

The fundamental singular solution of elastodynamics is the response of the infinite medium to a unit concentrated harmonic force at a point P , or briefly, it is the displacement component $U_{jk}^*(P, Q)$ in the k direction at a point Q due to a unit concentrated force acting at point P in the direction j as shown in Fig. (3.1).

Following the work of Doyle (50), Kupradze (51), and Cruse and Rizzo (33,34), the fundamental solution can be obtained from the equations of motion of linear isotropic elastic medium. For a two dimensional space, it is given by

$$U_{jk}^* = \frac{i}{4G} \{ \psi \delta_{jk} - \chi r_{,k} r_{,j} \} \quad (3.9)$$

$$\begin{aligned} T_{jk}^* = \frac{i}{4} \left\{ \left(\frac{d\psi}{dr} - \frac{1}{r} \chi \right) \left(\delta_{kj} \frac{\partial r}{\partial n} + r_{,k} n_j \right) \right. \\ \left. - \frac{2}{r} \chi \left(n_k r_{,j} - 2 r_{,k} r_{,j} \frac{\partial r}{\partial n} \right) - 2 \frac{d\chi}{dr} r_{,k} r_{,j} \frac{\partial r}{\partial n} \right. \\ \left. + \left(\frac{c_1^2}{c_2^2} - 2 \right) \left(\frac{d\psi}{dr} - \frac{d\chi}{dr} - \frac{\chi}{r} \right) r_{,j} n_k \right\} \quad (3.10) \end{aligned}$$

where

$$\psi = H_0(k_2 r) + \frac{1}{k_2 r} \{ \gamma H_1(k_1 r) - H_1(k_2 r) \} \quad (3.11.a)$$

$$\chi = \gamma^2 H_2(k_1 r) - H_2(k_2 r) \quad (3.11.b)$$

For simplicity of computations the expression for T_{jk}^* shall be expanded. When the differentiations as indicated in Eqn. (3.10) are carried out the resulting form of T_{jk}^* is given by

$$T_{jk}^* = \frac{i}{4} \left\{ B_1 \left(\frac{\partial r}{\partial n} \delta_{jk} + \frac{\partial r}{\partial x_k} n_j \right) + B_2 \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial n} + B_3 \frac{\partial r}{\partial x_j} n_k \right\} \quad (3.12)$$

where

$$B_1 = -k_2 H_1(k_2 r) - \frac{2\chi}{r} \quad (3.13.a)$$

$$B_2 = 2k_1 \gamma^2 H_3(k_1 r) - 2k_2 H_3(k_2 r) \quad (3.13.b)$$

$$B_3 = -k_1 (1 - 2\gamma^2) H_1(k_1 r) - \frac{2\chi}{r} \quad (3.13.c)$$

$$\gamma = \frac{c_2}{c_1} \quad (3.13.d)$$

$$r(P, Q) = \sqrt{(X_P - X_Q)^2 + (Y_P - Y_Q)^2} \quad (3.13.e)$$

$$k_m = \frac{\omega}{c_m} \quad m = 1, 2 \quad (3.13.f)$$

$$n_m = \frac{\partial x_m}{\partial n} \quad m = 1, 2 \quad (3.13.g)$$

$$\frac{\partial r}{\partial n} = \frac{\partial r}{\partial x_m} n_m \quad m = 1, 2 \quad (3.13.h)$$

c_1 = Compression wave velocity.

c_2 = Shear wave velocity.

ω = Angular frequency of excitation.

G = Shear modulus of the medium.

δ_{lk} = Kronecker delta.

n = Outward pointing normal unit vector on the surface of the body.

$H_n(x)$ = Hankel function of the first kind of order n .

It should be noted that all derivatives of r and n are calculated at the field point Q . A detailed derivation is given in Appendix B.

3.4 Formulation of The Boundary Element Equations

The formulation of the boundary element equations relies on the dynamic Betti-Rayleigh reciprocal theorem, also known as Betti's third identity. Following Eringen and Suhubi (52), Betti's third identity can be stated as

$$\rho \int_D \left\{ \underline{u}^{(1)} \nabla \cdot \underline{u}^{(2)} - \underline{u}^{(2)} \cdot \nabla \underline{u}^{(1)} \right\} dD$$

$$= \int_S \left\{ \underline{u}^{(1)} \underline{t}^{(2)} - \underline{u}^{(2)} \underline{t}^{(1)} \right\} dS \quad (3.14)$$

where

D is an elastic domain with boundary S .

$\underline{u}^m, \underline{t}^m$ are the displacement vector in the domain D and the corresponding traction vector on a surface element with normal vector n corresponding to an elastodynamic state m .

$$\nabla^* = (c_1^2 - c_2^2) \nabla \nabla + c_2^2 \nabla^2 \quad (3.15)$$

and ∇ is the Del vector operator.

In Eqn. (3.14), let the elastodynamic state (1) correspond to the fundamental solution for concentrated force in the infinite space, namely the displacement and tractions for an applied force in the j direction and let the elastodynamic state (2) correspond to the set of displacements and tractions U_k and T_k in domain D and on surface S , containing the load point $P(x_i)$, and representing the unknowns in the problem to be solved. Substituting into Eqn. (3.14) the fundamental solution for state (1), and excluding the singular point one obtains (Appendix C).

$$C^P U_j(P) = - \int_S T_{jk}^*(P, Q) U_k(Q) dS + \int_S U_{jk}^*(P, Q) T_k(Q) dS \quad (3.16)$$

where

$C^P = 1$ when point P is inside region D , and

$C^P = .5$ when point P is on the smooth boundary (37).

Equation (3.16) is the boundary element equation. It has been derived assuming that domain D is finite. In case it is not finite as for example a soil half plane, it is necessary to consider the conditions at infinity so that a unique solution is insured and the possibility of energy source being located at infinity is ruled out. These conditions will be discussed later. For the solution of unknowns U_k and T_k , equation (3.16) is converted into a set of algebraic equations by discretizing the boundary into finite size elements. If there are N such elements, then the discretized form of the integral equation (3.16) is

$$C^P U_j(P) + \sum_{l=1}^N \int_{S_l} T_{jk}^*(P, Q) U_k(Q) dS = \sum_{l=1}^N \int_{S_l} U_{jk}^*(P, Q) T_k(Q) dS \quad (3.17)$$

The integration sign with subscript S_l indicates integration over a given element l . By locating point P in turn on each of the elements, N equations of the type (3.17) are obtained. The

simultaneous solution of these equations gives the unknowns U_k and T_k on the boundary. The numerical implementation of the equations and their solution are presented in the following section.

3.5 Numerical implementation of Boundary Element Equations

The numerical implementation of the boundary element equations (3.17) is characterized by the manner in which the boundary S is represented and by the assumptions that are made about the variation of the displacements and tractions over each element.

In general the smooth boundary is a plane curve and the N elements are assumed to be line segments. A local coordinate ξ which varies from -1 to +1 is introduced over the element as shown in Fig. (3.2). If m nodes are located on each element, and a suitable variation of U_k and T_k is assumed over the element, the values of $U_k(Q)$ and $T_k(Q)$ at some point Q on the element can be interpolated from the values at the element nodes as

$$U_k \{Q(\xi)\} = \sum_{n=1}^m M_n(\xi)(U_k)_n \quad (3.18.a)$$

$$T_k \{Q(\xi)\} = \sum_{n=1}^m M_n(\xi)(T_k)_n \quad (3.18.b)$$

where $(U_k)_n$ and $(T_k)_n$, $n = 1, 2, \dots, m$, are the values of U_k and T_k at the m nodes of the element, and $M_n(\xi)$, $n = 1, 2, \dots, m$, are the shape functions which have different forms according to the number of nodes over each element. The element of area $dS(Q)$ is also expressed in terms of ξ through the Jacobian of transformation from ξ to x as (37)

$$dS(Q) = J(\xi)d\xi \quad (3.19.a)$$

$$J(\xi) = \left| \frac{dx_i}{d\xi_i} \right| \quad (3.19.b)$$

Suppose that there are N elements on the boundary having a total of M nodes, and let the (c) node of element l be denoted by $d(l, c)$. The introduction of these notations and the shape functions into Eqn. (3.17) then leads to the following algebraic equation for the $(a)th$ node at which point P is located

$$\begin{aligned} .5U_j(P^a) + \sum_{l=1}^N \sum_{c=1}^m U_k \{d(l, c)\} \int_{S_l} T_{jk}^*(P^a, Q(\xi)) M_c(\xi) J(\xi) d\xi = \\ \sum_{l=1}^N \sum_{c=1}^m T_k \{d(l, c)\} \int_{S_l} U_{jk}^*(P^a, Q(\xi)) M_c(\xi) J(\xi) d\xi \end{aligned} \quad (3.20)$$

and

$$k, j = 1, 2$$

Two types of element are used in this study, namely, constant elements and linear elements. The constant element shown in Fig. (3.3) has one node at the centre and U_k and T_k are assumed to be constant over the element. Noting that $M_c(\xi) = 1.0$ for such an element, Eqn. (3.20) becomes

$$.5U_j(P) + \sum_{l=1}^N U_k(Q) \int_{S_l} T_{jk}^*(P, Q) dS = \sum_{l=1}^N T_k(Q) \int_{S_l} U_{jk}^*(P, Q) dS \quad (3.21)$$

The linear element has two nodes at the ends and a linear variation is assumed for both U_k and T_k as shown in Fig. (3.4). The shape functions become

$$M_A(\xi) = .5(1 - \xi) \quad (3.22.a)$$

$$M_B(\xi) = .5(1 + \xi) \quad (3.22.b)$$

Substitution of Eqn. (3.22) into Eqn. (3.20) gives

$$\begin{aligned} .5U_j(P^*) + \sum_{l=1}^N \sum_{c=1}^2 U_k \{d(l, c)\} \int_{S_l} T_{jk}^*(P^*, Q(\xi)) M_c(\xi) J(\xi) d\xi = \\ \sum_{l=1}^N \sum_{c=1}^2 t_k \{d(l, c)\} \int_{S_l} U_{jk}^*(P^*, Q(\xi)) M_c(\xi) J(\xi) d\xi \end{aligned} \quad (3.23)$$

and

$$k, j = 1, 2$$

The complete system of Eqns. (3.21) or (3.23) can be written in a compact form as

$$[H] \{U\} = [G] \{T\} \quad (3.24)$$

in which

$\{U\}$ = The vector containing the known and unknown displacement amplitudes at the nodes.

$\{T\}$ = The vector of corresponding tractions.

$$[H] = 0.5 [I] + [\tilde{H}]$$

$[\tilde{H}]$ = The matrix containing the terms $\int_{S_l} T_{jk}^* dS$.

$[I]$ = Identity matrix.

Figure 3.2 Local Coordinate system

Figure 3.3 Constant Element

Figure 3.4 Linear Element

$[G]$ = The matrix containing the terms $\int_S U_{jk}^* dS$.

The system of Eqns (3.24) contains n_1 unknown displacements and n_2 known displacements, the sum of $n_1 + n_2$ being equal to $2M$, where M is the number of nodes. Correspondingly there are n_1 known tractions and n_2 unknown tractions. If all the unknowns are transferred to the left side of Eqn. (3.24), the problem will be reduced to that of the solution of a set of algebraic equations of the form

$$[A] \{X\} = \{B\} \quad (3.25)$$

3.6 The Numerical Integration Scheme

The integrals in Eqn. (3.24) have to be evaluated numerically. A close examination of the integrals reveals that they depend on the distance $R(P, Q)$ between points P and Q . When the point P coincides with Q . The integrals become singular at P . Thus two different cases arise

Case 1

Point P not coincident with Q

The Gauss-Legendre n -point quadrature is used to evaluate the integrals. According to this formulae, the integral of a function $f(\xi)$ is approximated by

$$\int_{-1}^1 f(\xi) d\xi \approx \sum_{j=1}^n f(\xi_j) w_j \quad (3.26)$$

where w_j are the weight factors and ξ_j the abscissas for the n -point formulae. A four point formulae is used in this study since it is found to give good results.

Case 2

Point P coincident with point Q

As point P coincides with point Q , the radius vector $r(P, Q) = 0.0$, the traction tensor $T_{jk}^*(P, Q)$ and displacement tensor $U_{jk}^*(P, Q)$ become singular. It can be observed from Eqns. (3.9) through (3.13) that the singularities of T_{jk}^* and U_{jk}^* are of the order $O(1/r)$ and $O(\ln r)$ respectively, as explained in appendix C.

Constant Element Case

In the case of constant element, the shape function is unity and the integrands T_{jk}^* and U_{jk}^* remain singular. Since the singularity of U_{jk}^* is only logarithmic, its integrals exist and can be evaluated using a quadrature scheme with logarithmic weight function. On a close observation of the functions that make up U_{jk}^* , Eqns. (3.9) through (3.13), it will be noted that the only function that gives rise to the logarithmic singularity is the Bessel function of the second kind $Y_0(k_2 r)$. Thus functions other than $Y_0(k_2 r)$ are integrated numerically by the Gauss-Legendre scheme described above while use is made of the following scheme to evaluate the integral containing $Y_0(k_2 r)$ (27)

$$\int_0^1 f(\eta) \ln(\eta) d\eta = \sum_{j=1}^n f(\eta_j) b_j \quad (3.27)$$

Since the function $Y_0(k_2 r)$ is symmetric with respect to the singular point as shown in Fig. (3.5.a), Eqn. (3.27) can be used to evaluate the integral containing the term $Y_0(k_2 r)$ as

$$2 \int_0^1 \left\{ \frac{Y_0(k_2 r(\xi))}{\ln \xi} \right\} \ln \xi d\xi \approx \sum_{j=1}^n f(\xi_j) b_j \quad (3.28)$$

The integral containing T_{jk}^* for $j \neq k$ is antisymmetric with respect to the singular point as shown in Fig. (3.5.b). the singular point is proved to be equal to zero in appendix C. Once the singular point has been excluded, the integral over the left half of the element is equal to the integral over the right half of the element with opposite sign and the resultant becomes zero. For T_{jj} too, the integration leads to zero provided the point of singularity is excluded. Integration around the point of singularity gives a value of 0.5 on the smooth boundary, which has already been taken into account in the boundary element formulation. In conclusion, the integration of T_{jk}^* may be ignored for the element on which points P and Q coincide.

Linear Element Case

Referring to Fig. (3.6), point P may coincide with either point A or B . When point P coincides with node A , the shape function, $M_B(\xi)$ in Eqn. (3.22.b) is of order $O(r)$. Hence the product to be integrated for node B , namely, $T_{jk}^* M_B(\xi) J(\xi)$ and $U_{jk}^* M_B(\xi) J(\xi)$, become of order $O(r) \cdot O(\frac{1}{r})$ and $O(r)(\ln r)$ respectively. The above mentioned products tend to a finite limit at A , and so may be approximated by a polynomial over the element. The standard quadrature scheme, Eqn. (3.26), is therefore applied for integrating $T_{jk}^* M_B(\xi) J(\xi)$ and $U_{jk}^* M_B(\xi) J(\xi)$.

Figure(3.5.a) $Y_0(k_2 r)$ Figure(3.5.b) $T_{jk}^*(C, C), j \neq k$

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As for node A , the shape function $M_A(\xi)$ tends to unity at A , the integrands $T_{jk}^* M_A(\xi) J(\xi)$ and $U_{jk}^* M_A(\xi) J(\xi)$ remain of order $O(1/r)$ and $O(\ln r)$ respectively, or in other words, remain singular. The singularity in $U_{jk}^* M_A(\xi) J(\xi)$ arises from $Y_0(k_2 r)$. The remaining functions are therefore evaluated ordinarily using Gauss-Legendre scheme. The Bessel function $Y_0(k_2 r)$ is evaluated using scheme (3.27). A new coordinate η shown in Fig. (3.6) is introduced and Eqn. (3.27) takes the form

$$\int_0^1 (\{Y_0(k_2 r(\eta)) M_A(\eta) J(\xi(\eta)) Z(\eta) \ln(\eta)\} / \ln(\eta)) d\eta \simeq \sum_{j=1}^n f(\eta_j) b_j \quad (3.29)$$

where $Z(\eta)$ is the Jacobian of transformation between ξ and η systems.

As described earlier in the case of constant element, the integral $T_{jk}^* M_A(\xi) J(\xi)$ is antisymmetric with respect to node A for both $j = k$ and $j \neq k$. Therefore, once the point of singularity is excluded, the resultant of integration over the two elements one to the right and the other to the left of node A is of negligible value since the two elements are usually of almost equal length.

The same situation exists when point P coincides with node B and the integration is very similar to that described above except that the singularities are at node B . The integral coefficients for node A are evaluated in the normal manner using the Gauss-Legendre scheme. The Bessel function $Y_0(k_2 r)$ is evaluated using the scheme (3.29) for node B , with the adjustment that the η coordinate is defined as in Fig. (3.7).

Figure 3.6 Point P at node A .

Figure 3.7 Point P at node B .

Chapter 4

VIBRATION OF A RIGID STRIP FOOTING

4.1 Introduction

The dynamic response of a rigid strip footing on an uniform elastic half space is a two dimensional plane strain problem which has been studied by several researchers (24,25,29,33,34) and for which analytical solutions are available. The problem can be equally well solved by the boundary element method and therefore provides a bench mark for the comparison of the boundary element results with the analytical results.

The prime objective of this chapter is to check out the accuracy of the formulation and the computer implementation against the available solutions. An extension of the work deals with the case of embedded footings and the effect of varying the value of Poisson's ratio on the dynamic behaviour of the footing.

4.2 Problem Definition

Fig. (4.1) shows the coordinate system and the basic dimensions of a massless rigid strip footing. The footing is subject to the action of harmonic motion along the three degrees of freedom V_1 , V_2 and V_3 defined in Fig. (4.1). The problem is a mixed boundary value problem with a prescribed stress condition ($\sigma_i, i = 1, 2$) on the free surface and a prescribed displacement condition under the footing. The problem is solved for the following two sets of boundary conditions along the contact surface between the footing and the half space.

4.2.1 Welded Contact Conditions

It will be noted that a vertical movement of the footing in Fig. (4.1) does not produce either a horizontal movement or a rocking movement, so that there is no coupling between the vertical and either horizontal or the rocking movement. However when the footing is bonded to the soil,

Figure 4.1 Single rigid strip footing bonded to a half space.

coupling exists between the horizontal and rocking movements. This is referred to as Welded contact condition. Under welded contact conditions, the compatibility between the displacements of the soil and the displacement of the footing requires that

$$u_2(x_1(i)) = \Delta V + \alpha x_1(i) \quad (4.1.a)$$

$$u_1(x_1(i)) = \Delta H \quad (4.1.b)$$

where i refers to a point under the footing, ΔV is the amplitude of the vertical displacement at the centre of the footing, α is the amplitude of the rocking angle, and ΔH is the amplitude of the horizontal displacement of the footing.

4.2.2 Relaxed Contact Conditions

In the case of relaxed contact, the boundary conditions on the contact area for vertical and rocking vibrations are

$$u_2(x_1(i)) = \Delta V + \alpha x_1(i) \quad (4.2.a)$$

$$P_1(i) = 0.0 \quad (4.2.b)$$

the corresponding relaxed conditions for horizontal vibrations are

$$u_1(x_1(i)) = \Delta H \quad (4.3.a)$$

$$P_2(i) = 0.0 \quad (4.3.b)$$

where, in the above equations, $P_1(i)$ and $P_2(i)$ are respectively the amplitudes of horizontal and vertical forces at point i .

4.2.3 Application of The BEM to Semi Infinite Domains

The boundary element Eqn. (3.21) has been formulated assuming that the domain D is finite. If the domain is not finite, as in the case of a half space, it is necessary to consider the behaviour of the solution at infinity carefully. There is no indication of this behaviour from the pure mathematical formulation of the problem, and it must be chosen in such a way that it agrees with the physical conditions of the problem. Green's formulae and similar formulae in the theory of elasticity, in particular Betti's formulae, serve as means to make this choice (33).

There are several ways to determine the asymptotic behaviour of vectors u_k and t_k . Eringen (52) and Kupradze (51) have proven, by decomposing the displacement vector $\underline{u}$ into its irrotational

part $\underline{u}^p$ and equivoluminal part $\underline{u}^s$, that the following two requirements result directly from the Sommerfeld radiation conditions of ordinary wave equations

$$\lim_{R \rightarrow \infty} R\{t^p - i\rho\omega c_1 \underline{u}^p\} = 0.0 \quad (4.4.a)$$

$$\lim_{R \rightarrow \infty} R\{t^s - i\rho\omega c_2 \underline{u}^s\} = 0.0 \quad (4.4.b)$$

One can arrive at the same conditions by equating the integration in Eqn. (3.21) over the infinitely distant parts of the boundary for any chosen accessible point of observation located on the plane part of the boundary to zero. In other words, the integration over the semi circle of infinite radius in Fig. (4.2) will be zero when the radiation conditions are satisfied. This implies that the integration of the boundary element equations can be restricted to the plane boundary of the half plane.

The existence and uniqueness of solutions based on these radiation conditions have been proven by Kupradze (51) and Doyle (50), and Doyle has indicated that these radiation conditions do admit to a physical interpretation which is consistent with the boundary conditions of the problem - they rule out the possibility of energy sources being located at infinity.

Although the integration is now restricted to the plane boundary of the half space, the plane boundary itself is of infinite extent which makes it impossible to perform a numerical solution over it. By carefully examining the radiation conditions, it becomes obvious that points distant from a given observation point P contribute very little to the integral in the BEM equations and the contribution decreases with the distance. It is thus possible to truncate the integration at a certain distance R on either side of the footing. It is expected that accuracy will improve with increasing values of R . Convergence is assessed by repeating the calculations with increasing values of R and where possible by comparing the results with published results. It should be noted that the published results with which comparison is made were themselves derived by mathematical techniques based on some simplifying assumptions and therefore represent only approximate solutions to the problem.

Constant elements have been used in the first stage of this study to examine the capability of such elements in producing results that compare well with published results. Linear elements have been tried subsequently.

A computer program has been developed using the proposed method to solve the problem of semi-infinite elastic domain undergoing steady state vibration for any set of boundary conditions using constant elements.

Figure 4.2 The half space Boundary.

4.3 Results and Discussions

4.3.1 Effect Of Truncating The Integral In The BEM Equations

In this section, the effect of the magnitude of R/b ratio, R being the distance on either side of the footing where the boundary element discretization is terminated and b the half width of footing, on the results of the analysis of the footing vibration is studied. The excitation is assumed to be in the vertical direction. In the analytical results presented here the following properties are used for the soil medium; Young's modulus = $2.016 \times 10^6 \text{ lb/ft}^2$ (96.5 MPa), Poisson's ratio = 0.25 and the mass density of the soil = $4.19 \text{ lb.s}^2/\text{ft}^4$ ($2.16 \text{ kN.s}^2/\text{m}^4$). The real and imaginary parts of the dynamic stiffness K_{22} are plotted against the frequency factor α_0 in Fig. (4.4) for values of $R/b = 0, 3$ and 5 . The corresponding boundary element meshes are shown in Fig. (4.3). The frequency factor α_0 and the dynamic stiffness K_{22} are defined as

$$\alpha_0 = \frac{\omega b}{c_2} \quad (4.5.a)$$

$$K_{22} = \frac{Q_2}{\pi G V_2} \quad (4.5.b)$$

where

ω The exciting frequency.

Q_2 The summation of the forces in the vertical direction under the footing.

c_2 The shear wave velocity in the soil medium.

G The modulus of rigidity of the soil.

The results for $R/b = 3$ and 5 are almost identical. It is therefore adequate to use the smaller value $R/b = 3$ which will give a smaller area to integrate and therefore a less expensive numerical integration. The remaining results presented in this part are based on a region of integration defined by $R/b = 3$.

4.3.2 Vertical Vibration

The equivalent dynamic stiffness for a relaxed contact condition is compared with the analytical results of Karasudhi (24) in Fig. (4.5) where excellent agreement is observed. For the welded contact condition, the dynamic compliance (the inverse of stiffness) has been computed in order to compare directly with the work of Luco and Westmann (29). This comparison is shown in Fig. (4.6) where the results are seen to agree closely.

Figure (4.3) Boundary Element Meshes.

Figure 4.4 Effect of truncation of the plane boundaries on the vertical stiffness of a single rigid strip footing, Poisson's ratio = 0.25.

Figure 4.5 Comparison of results for vertical excitation, relaxed conditions.

Figure 4.6 Comparison of results for vertical excitation, welded contact conditions.

A comparison between the results obtained for a relaxed and welded contact condition are shown in Fig. (4.7). It is observed that the difference between the two sets of results is small.

4.3.3 Horizontal Vibration

The horizontal stiffness for a relaxed contact condition is plotted in Fig. (4.8) along with the results given by Karasudhi. The agreement is good. It should be noted that relaxed condition here indicates the specification of zero normal stress at the contact surface between the foundation and the earth so that coupling with rocking or vertical vibration is precluded. The horizontal stiffness for welded condition is also shown on the same Figure. There is a maximum of 12 percent difference between the results for the two conditions. The horizontal compliance for welded condition is plotted in Fig. (4.9) along with the results of Luco and Westmann. Again, there is excellent agreement.

4.3.4 Rocking Vibration

The equivalent dynamic stiffness for the relaxed condition (zero shear stress) is plotted in Fig. (4.10) and compared with Karasudhi; again there is good agreement. The results for the welded condition are also shown on the same Figure. There is a maximum of 12 percent difference between the two sets of results. In Fig. (4.11) the rocking compliance for welded condition is compared to Luco and Westmann's results with excellent agreement.

4.3.5 Coupled Horizontal-Rocking Vibration

In this mode four compliances are involved. They are obtained from the inverse of the stiffness matrix. The four compliances are

C_{11} = Direct horizontal compliances.

C_{33} = Direct rocking compliances.

C_{13} = Coupling horizontal-rocking compliance.

C_{31} = Coupling rocking-horizontal compliance.

Numerical results indicated appreciable difference between C_{13} and C_{31} . Theoretically, the coupling compliances C_{13} and C_{31} should be equal by law of reciprocity. But due to numerical inaccuracies, differences did exist between the computed values. The maximum difference for the case of the mesh shown in Fig(4.1) is about 20 %. However, refining of the mesh near the edge of

Figure 4.7 Comparison between the stiffnesses under relaxed and welded contact conditions, vertical excitation case.

bf Figure 4.8 Comparison of the horizontal stiffnesses under relaxed and welded contact conditions.

Figure 4.9 Comparison of results for horizontal compliance, welded contact conditions

Figure 4.10 Comparison of results for rocking stiffness, relaxed and welded contact conditions.

Figure 4.11 Comparison of results for rocking compliance, welded contact conditions.

the rigid footing reduces the differences significantly although the differences do not entirely vanish. It is noted that Damisah (35) has reported similar differences between C_{13} and C_{31} despite the fact that he used higher order elements. The difference can be attributed to the strong singularity at the edge of the rigid footing. Better representation of the singularity at the edge by using a special singular elements at the edge may lead to better results.

In Fig. (4.12) shows the averages of the real and imaginary parts of C_{31} and C_{13} are compared with similar averages taken from Luco and Westmann (29), and the agreement is found to be good.

4.3.6 Conclusions Related to the response of a single footing

The results presented above give clear indication that the formulation of the problem as well as the computer work is correct. Also, it can be concluded that, the degree of accuracy which has been achieved using the constant element is quite satisfactory.

4.4 Effect of Variation of Poisson's Ratio

The analysis of a single footing under the action of different types of harmonic displacement, as previously presented, is repeated for three different values of Poisson's ratio: 0.1, 0.2 and 0.4.

The results illustrated in Figs. (4.13) to (4.18), indicate that with a few exceptions, the dynamic stiffness increases with increasing value of Poisson's ratio.

4.5 Effect of Embedment

This section investigates the effect of embedment in the soil medium on the dynamic behaviour of a single rigid footing subjected to a harmonic loading.

Fig. (4.19) depicts the significant dimensions of the model considered for the study. In Fig. (4.19), b represents the half width of the rigid massless footing and h is the depth below surface at which the footing is located. The ratio h/b is called the embedment ratio. The vertical sides of the pit are considered traction free. The footing has been analyzed for a frequency factor a_0 equal to 0.25 for six different values of h/b , namely 0.2, 0.4, 0.6, 0.8 and 1.0. For each case the boundary element mesh is shown in Fig. (4.20). The rigid strip footing located at the surface is considered as a special case with $h/b = 0.0$. Three different cases of loading are considered; vertical excitation, horizontal excitation and rocking excitation. In each case, a corresponding unit displacement is

Figure 4.12 Comparison of results for the coupling horizontal-rocking compliance.

Figure 4.13 Comparison of results for vertical stiffness, relaxed contact conditions.

Figure 4.14 Comparison of results for vertical stiffness, welded contact conditions.

Figure 4.15 Comparison of results for horizontal stiffness, relaxed contact conditions.

Figure 4.16 Comparison of results for horizontal stiffness, welded contact conditions.

Figure 4.17 Comparison of results for rocking stiffness, relaxed contact conditions.

Figure 4.18 Comparison of results for rocking stiffness, welded contact conditions.

Figure 4.19 The Embedded Single Footing Model

Figure (4.30) Boundary Element Meshes.

prescribed under the footing while the condition of zero traction is imposed on the free surface. The resulting tractions under the footing obtained from the analysis are summed up to obtain the corresponding stiffnesses.

In Figs. (4.21) and (4.22), the stress distribution under the footing is drawn for different h/b ratios for the case of vertical excitation and a relaxed contact condition. The relations between the different types of stiffnesses and the frequency factor are shown in Figs. (4.23) through (4.25).

Referring to these figures, it can be seen that:

- (1) For vertical excitation and relaxed condition, there is only a slight increase in the stresses under the footing as h/b increases.
- (2) The curve for the real part of the vertical stiffness is almost a horizontal straight line for both welded and relaxed conditions, while there is about 10% increase in the absolute magnitude of the imaginary part at $h/b = 1$. over that at $h/b = 0.0$.
- (3) Both the real and imaginary parts of the horizontal stiffness increase, but only slightly, as h/b increases.
- (4) There is only a slight change in the value of the rocking stiffness as h/b increases.
- (5) For all three set of excitation conditions, the difference between the stiffness's value corresponding to welded and relaxed conditions is small for all values of h/b .

It can be concluded that, for the above specified range of h/b and a_0 , the effect of embedment is not significant and need not be considered.

Figure (4.21) Stress Distribution under the footing for Excitation, Relaxed Contact Condition, the Real Part.

Figure (4.22) Stress Distribution under the footing for Vertical Excitation, Relaxed Contact Condition, The Imaginary Part.

Figure 4.23 Effect of embedment on the vertical stiffness of a single rigid strip footing.

Figure 4.24 Effect of embedment on horizontal stiffness of a single rigid strip footing.

Figure 4.25 Effect of embedment on rocking stiffness of a single rigid strip footing.

Chapter 5

THE COUPLING EFFECT OF TWO RIGID STRIP FOOTINGS

5.1 Introduction

Very often a structure is founded on several adjacent strip footings which may vary in dimensions. It is therefore of interest to study the effect of the presence of a neighbouring footing on the dynamic response of a given footing.

Macalden (54) studied the transmission of surface vibration from one rigid circular foundation to another. Warburton (55,56) has investigated this problem for the case of circular footings resting on the surface of an elastic half space. V.C. Liang (8) has studied the case of two parallel rigid strip footings of arbitrary width and separation. In Liang's model, the two footings were welded to the surface of a homogeneous, isotropic and viscoelastic layer. Another study available in a related area is by Gutierrez (10) who has studied the interaction of two structures founded on a layered soil underlain by bed rock using finite element model. This chapter investigates the dynamic interaction of two rigid footings shown in Fig. (5.1). The footing which is directly excited is called the active footing, while the footing excited by the ground motion which was transmitted from the active footing is called the passive footing. The active and passive footing are denoted by superscripts a and p respectively. In Fig. (5.1), d identifies the centre line distance between the two footings. The two footings are assumed to be of equal width, $2b$.

The results described here are presented as graphs between non-dimensional frequency factor a_0 and non-dimensional stiffness factors K_{ij}^a and K_{ij}^p where

$$a_0 = \frac{\omega b}{c_2} \quad (5.1)$$

$$K_{ij}^a = \frac{Q_i^a}{\pi G V_j^a} \quad (5.2)$$

$$K_{ij}^p = \frac{Q_i^p}{\pi G V_j^p} \quad (5.3)$$

and

Figure 5.1 The two rigid footings system

ω is the exciting frequency *rad/sec*.

b is the half width of each of the two footings.

G is the modulus of rigidity of the soil.

c_2 is the shear wave velocity in the soil medium.

K_{ij}^a is the non-dimensional stiffness of the active footing defined as the sum of the forces under the active footing in direction i due to a unit displacement of the active footing in direction j while all other degrees of freedom under the two footings are kept fixed.

K_{ij}^p is the non-dimensional stiffness of the passive footing defined as the sum of the forces under the footing in direction i due to a unit displacement of the active footing in the direction j while all other degrees of freedom under the two footing are kept fixed.

Q_i^a is the sum of the forces under the active footing in the direction i .

Q_i^p is the sum of the forces under the passive footing in direction i .

V_j^a is the displacement under the active footing in direction j .

i, j are subscripts that vary from 1 to 3.

$i = 1$ refers to the horizontal degree of freedom.

$i = 2$ refers to the vertical degree of freedom.

$i = 3$ refers to the rocking degree of freedom.

The main objectives of this chapter can be summarized as follows

- (1) To study the variation of different types of stiffnesses with the excitation frequency.
- (2) To study the effect of increasing the distance d between the two footings on the stiffnesses values.

The study is limited to the case of two rigid footings of identical dimensions resting on the surface of an elastic half space. Four d/b ratio cases 2., 4., 6. and 7.2 have been investigated in the practical range of frequency factor a_0 between 0.25 and 1.75. In addition, the case of a single footing is considered as a special case of two footings with d/b equal infinity. In the analytical results presented here the following properties are used for the soil medium; Young's modulus = $2.016 \times 10^6 \text{ lb/ft}^2$ (96.5 MPa), Poisson's ratio = 0.25 and the mass density of the soil = $4.19 \text{ lb.s}^2/\text{ft}^4$ ($2.16 \text{ kN.s}^2/\text{m}^4$). The Fig. (5.2) illustrates the different meshes used.

Figure (5.2) The boundary element meshes.

The problem is a mixed boundary value problem with a prescribed stress condition ($\sigma_i = 0.0, i = 1, 2$) on the free surface and a prescribed displacement condition under both footings. The problem is solved for two sets of boundary conditions along the contact surface between the footings and the half plane: welded contact condition and relaxed contact condition.

Welded Contact Conditions

This condition implies that there is a coupling between all the degrees of freedom.

Relaxed Contact Conditions

Under a relaxed contact condition, there is no coupling between any two of the degrees of freedom.

Again as for a single rigid strip footing case, a convergence test was carried out to determine the length R of the integration area. It was found that $R/L = 3$ ratio gives reasonably accurate results. Parameters R and L are defined in Fig. (5.1).

5.2 Results and Discussions

5.2.1 Response of Active Footing under Vertical Vibrations

Fig. (5.3) and Fig. (5.4) illustrate the response of the active rigid strip footing under the action of vertical excitation for relaxed contact condition and welded contact respectively.

By examining Fig. (5.3) it can be seen that

- (1) There is a significant difference between the values of K_{22}^* (both real and imaginary) for $d/b = 2$ and $d/b = \infty$. This difference in the case of the real part of K_{22}^* is almost 100 percent in the lower range of a_0 and decreases as a_0 increases until at $a_0 = 1.7$ where the two curves meet. The value of the real part of K_{22}^* for $d/b = 2.0$ is higher than that of $d/b = \infty$, $d/b = 4.0$, $d/b = 6.0$ and $d/b = 7.2$ in the lower range of excitation frequency.
- (2) For both the imaginary and real parts, as d/b increases the K_{22}^* curve becomes closer to that of the single footing case. It is noted that the maximum difference between the two sets of curves for $d/b = \infty$ and $d/b = 7.2$ is less than 10 percent.

Figure (5.3) Response of an active footing under vertical excitation, relaxed contact condition.

Figure 5.4 Response of an active footing under vertical excitation, welded contact condition.

- (3) By comparing the results in Figs. (5.3) and (5.4), it can be seen that, in general, the behaviour of the active footing under vertical excitation is similar for the two cases of relaxed and welded contact conditions.

5.2.2 Response of Active Footing under Horizontal Vibrations

Fig. (5.5) and (5.6) show the response of the active footing to a unit horizontal displacement under the relaxed and welded contact conditions respectively. It can be seen that

- (1) As for vertical excitation, the effect of the presence of a neighbouring footing on the stiffness of an active footing is very significant particularly when the distance between the two footing is small. This is readily seen by comparing the response values for the smaller values of d/b with those for a single footing. The K_{11}^* values for $d/b = 2$ and $d/b = 4$ are considerably higher than those for the other values of d/b .
- (2) As d/b increases, the real and imaginary parts of K_{11}^* values approach the values of K_{11}^* for $d/b = \infty$. A max difference of 15 percent exist between the values for $d/b = 7.2$ and $d/b = \infty$.

5.2.3 Response of Active Footing under Rocking Vibrations

Fig. (5.7) and (5.8) illustrate the relation between K_{33}^* and α_0 for different values of d/b for both relaxed and welded contact conditions.

By examining these curves, it can be seen that

- (1) The value of K_{33}^* is significantly affected by the presence of a neighbouring footing when the distance between the two is small.
- (2) As d/b ratio increases, the values of K_{33}^* approach the values corresponding to $d/b = \infty$.

5.2.4 Coupled Responses of the Active Footing

It was found that there is no coupling between the vertical stiffness of both the active and passive footings and any of the other stiffnesses.

The real and imaginary parts of the horizontal-rocking stiffness of the active footing, K_{31}^* for different values of d/b ratio are shown in Figs. (5.9) and (5.10). It can be seen that the presence of the passive footing has a significant effect on the value of K_{31}^* . Again, there are differences between values of K_{31}^* and K_{13}^* , but these differences are smaller than those in the case of single footing case.

Figure 5.5 Response of an active footing under horizontal excitation, relaxed contact condition.

Figure 5.6 Response of an active footing under horizontal excitation, welded contact condition.

Figure 5.7 Response of an active footing under rocking excitation, relaxed contact condition.

Figure 5.8 Response of an active footing under rocking excitation, welded contact condition.

Figure 5.9 Coupled horizontal-rocking response of an active footing, welded contact condition, the real part.

Figure 5.10 Coupled horizontal-rocking response of an active footing, welded contact condition, the imaginary part.

2

5.3 Response of Passive Footing

The relation between the real and imaginary parts of the vertical stiffness of passive footing and the frequency factor a_0 for different values of d/b are illustrated in Figs. (5.11) to (5.12). The other types of the stiffnesses exhibit the same type of behaviour. It can be seen that

- (1) There is a basic difference between the general shape of the curves for $d/b = 2$ and those for higher values of d/b . For $d/b = 2$, both real and imaginary curves are of the same sign through the entire range of a_0 under investigation and the magnitude of the real part increases as a_0 increases while the magnitude of the imaginary part decreases sharply for the range of a_0 between .25 and 1.25 then it starts to increase moderately.
- (2) The real part and the imaginary part of K_{22}^P for $d/b = 4$, $d/b = 6$ and $d/b = 7.2$ change sign very rapidly over the range of a_0 under study. This means that the dynamic forces generated on the passive footing due to the presence of a harmonic displacement on a neighbouring footing change rapidly between being tensile or compressive as the frequency of the applied displacement changes. The reason for this behaviour can be appreciated from Fig. (5.13) which illustrates the real displacement patterns for different values of frequency factor for the case of $d/b = 6$.

It is seen that the displacement of the passive footing depends on the apparent wave length of the displacement pattern at the surface. With changing frequency of excitation, the wave length and hence the displacement pattern changes. As a result, the passive footing has a tendency to move either upward or downwards and hence the restraining forces are compressive or tensile respectively.

The coupled stiffnesses, K_{31}^P and K_{13}^P , have the same general pattern of behaviour as the other stiffnesses of the passive footing.

5.4 Conclusion

Based on the limited investigation presented in this chapter, one can conclude that the presence of the passive footing has a significant effect on the behaviour of the active footing. This effect decreases as the distance between the two footings increases, except in cases of K_{13}^a and K_{13}^P . Further studies are needed before any conclusions can be reached.

Figure 5.11 Response of a passive footing under vertical excitation, relaxed contact conditions, the real part.

Figure 5.12 Response of a passive footing under vertical excitation, relaxed contact conditions, the imaginary part.

Figure 5.13 Displacement patterns for $d/b=6$

Chapter 6

DYNAMIC BEHAVIOUR OF SOIL MEDIUM

6.1 Introduction

As stated earlier, the substructure method of analysis requires the evaluation of $X_f(\omega)$, the frequency dependent stiffness matrix of the soil medium. This is best achieved by first calculating the flexibility matrix, $Z_f(\omega)$, of the soil medium corresponding to the connection degrees of freedom and then taking its inverse.

Matrix $Z_f(\omega)$ is a square matrix whose i, j th term represents the displacement at degree of freedom i due to a unit harmonic load, $e^{-i\omega t}$, imposed at degree of freedom j . For a surface foundation on a uniform, infinitely long soil layer of given characteristics, the i, j th term depends only on the distance d_{ij} , between the nodes corresponding to the i th and j th degrees of freedom. This implies that the deflection pattern of the soil surface is independent of the point at which the external force is applied and shifting the latter merely shifts the pattern by the same amount. The entire flexibility matrix can therefore be constructed from the results of two sets of analysis, one for a unit vertical load, and another for a unit horizontal load, applied say at node No. 0 as shown in Fig. (6.1).

The response of a semi-infinite or layered medium to a harmonic load applied at the surface is discussed in this chapter. The areas of discussion covered can be summarized as follows:

- (1) Study of the different parameters affecting the behaviour of soil medium under the action of unit vertical or horizontal concentrated loads.
- (2) Comparison of behaviour of a soil layer bounded by bed rock at sufficient depth and extended horizontally to a reasonable distance to that of a semi-infinite soil medium.
- (3) Presentation of a technique for assembling the flexibility matrix of the soil medium corresponding to the connection degrees of freedom. The assembled matrix is subsequently used to obtain the solution of the problem of rigid strip footing vibration.

(a) Vertical excitation

(b) Horizontal excitation

Figure (6.1) Analysis of soil medium for unit harmonic excitation.

6.2 The Boundary Element Analysis of a Soil Layer

Fig. (6.2) shows a typical soil layer of infinite extent in the horizontal direction as well as a soil half plane. The soil layer model is basically the same model used by some researchers for finite element analysis of the soil layer. The soil layer is analyzed for its response to the two load cases shown in Fig. (6.1).

... selecting a boundary element model to represent the soil region, the following three factors need to be considered

- (1) Selection of the size of the model which would accurately represent the infinite extent in the horizontal direction.
- (2) Selection of an appropriate size for the boundary elements in order to produce the correct response with the largest elements possible or, in other words, with the least number of elements possible.
- (3) Appropriate discretization of elements at and around the point of application of the load

Since only tractions can be specified over the boundary element, the only way to simulate a concentrated force is to apply a traction of a large magnitude over a relatively small element, which when integrated yields the proposed unit force as shown in Fig. (6.3).

A trial and error study was undertaken for a soil half plane to study the impact of the aforementioned factors on its behaviour. Various mesh lengths and element sizes were tried. In choosing the element length, at least ten elements were used to represent one full length of the shear wave. Previous studies (1) have shown that such a sub-division provides adequate representation. Also, constant elements were selected for this study. The assumption inherent in the selection of such element is that both the displacement and the traction are constant over the length of the element. For the purposes of the computer study, the value of mass density of the material was taken as $= 4.19 \text{ lb.s}^2/\text{ft}^4$ ($2.16 \text{ kN.s}^2/\text{m}^4$), Young's modulus $= 2.016 \times 10^6 \text{ lb/ft}^2$ (96.5 MPa), and Poisson's ratio $= 0.2$. Fig. (6.4) shows the five different meshes used. Meshes A, and B have the same length but different element lengths near the concentrated load, while meshes C, D, and E have larger length. All the meshes are analyzed for a normalized unit load which is equal in magnitude to the shear modulus for the soil medium. Fig. (6.5) to (6.8) illustrate the displacement pattern for two values of the excitation frequency: 2, and 50 rad/sec which are at the two ends of the frequency range of interest. By carefully examining these figures, it can be seen that

Figure (6.2.a) A typical soil layer.

Figure (6.2.b) Soil half space.

Figure (6.3.a) Constant Element Case.

Figure (6.3.b) Linear Element Case.

Figure (6.4) Boundary Element Meshes.

Figure (6.5) Displacement Pattern for central horizontal normalized unit load, frequency = 60 rad/sec.

Fig. (6.6) Displacement Pattern for central vertical normalized unit load, frequency = 50 rad/sec.

Fig. (8.7) Displacement Pattern for central vertical normalised unit load, frequency = 2 rad/sec.

Fig. (6.6) Displacement Pattern for central horizontal normalised unit load, frequency = 2 rad/sec.

- (1) Of the element sizes studied, all lead to approximately the same magnitude for the imaginary part of the response, even near the applied force.
- (2) Different mesh lengths studied produce approximately the same displacement pattern with the exception that the perturbations at the edge of a mesh are somewhat higher.
- (3) The size of the element has a significant effect on the magnitude of the real part of displacement near the applied load; the displacement magnitude increases as the length of the central element decreases. This behaviour may create a problem since the results do not show convergence at this point, which provides the diagonal terms in the flexibility matrix of the foundation. A procedure to establish a criteria for the selection of central element length will be discussed later.

Fig. (6.9) illustrates the vertical displacement pattern due to a unit normalized horizontal force at the central node along with the horizontal displacement pattern due to a unit normalized vertical force at the central node. The two patterns should be identical by the law of reciprocity, but, as can be seen from the figure, differences do exist between the two patterns. These differences apparently exist because of the inevitable inaccuracies in numerical computations and can be evened out by a process of averaging as discussed later in this chapter.

6.3 Linear Element

In many boundary element applications, elements which assumes linear variation of the displacement and tractions over the element are known to provide more accurate results when compared to constant elements. To examine the adequacy of constant elements in the representation of the problem under study, a comparison was made between the results obtained with two alternate models, one using constant elements and the other using linear elements.

Fig. (6.10) illustrates two meshes for the same soil domain; one using the constant boundary elements and the other the linear boundary elements. In Fig. (6.11) are shown the displacement patterns for the unit normalized vertical load case. It can be seen that there is no significant difference between the two sets of results. It was concluded that there was no significant advantage in using the linear element to analyze the problem under study.

Figure (6.9) Comparison of vertical displacement pattern due to horizontal load and horizontal displacement pattern due to vertical load.

Figure (6.10) Boundary Element Meshes used for Comparison between the Results for Constant Element Case and Linear Element Case.

Figure (0.11) Comparison of results for constant element case and linear element case.

6.4 Response of a Layered Soil Medium

Fig. (6.2.a) shows a region with a single layer of soil extended to infinity in the horizontal direction and bounded by bed rock at the bottom. The layer is analyzed under the action of either a unit normalized horizontal load or a unit normalized vertical load applied at the center.

For the purposes of the computer study, layers with different heights were analyzed. These heights were: 50, 150, 200 and 250 ft. The response of the layered medium were also compared with those of a half plane. The value of the mass density of soil material was $= 4.19 \text{ lb.s}^2/\text{ft}^4$ ($2.16 \text{ kN.s}^2/\text{m}^4$), Young's modulus $= 1.96 \times 10^8 \text{ lb/ft}^2$ ($9.381 \times 10^3 \text{ MPa}$), and Poisson's ratio $= 0.1667$.

Convergence tests were carried out to determine an appropriate mesh size and it was found that meshes shown in Figs. (6.12.a) and (6.12.b) were the smallest systems that would still yield good results. Studies were made with both a unit normalized horizontal and a unit normalized vertical load at the center. Constant elements were used for all cases.

The real parts of the horizontal and vertical deflections at central point and at 80 ft from the center line are plotted in Figs. (6.13) to (6.16) while the displacement patterns for different frequency values are plotted in Figs. (6.17) and (6.18). For the particular cases reported here the imaginary part is very small compared with the real part and is not shown here.

It can be seen from Fig. (6.17) and (6.18) that at least 8-10 elements are representing a complete wave length. A careful examination of Figs. (6.13) and (6.14) shows that

- (1) There is a significant difference between the behaviour of the half plane case and that of layered medium. The difference is not only in the magnitude of displacement at each frequency, but also in the general shape of the curves. In Fig. (6.13) the real part of the vertical displacement for the half plane case decreases as the frequency increases, while the functions for a layered medium do not follow any such trend.
- (2) The vertical displacement curves for both $h = 200\text{ft}$ and 250 ft have the same general shape pattern with a peak value at frequency about 40 rad/sec with differences do not exceed about 15 percent. This might lead one to conclude that the system is converging to the true behaviour of half plane case, which is not true as seen from the half plane behaviour.
- (3) In Fig. (6.14), there is a significant difference in the behaviour of layer and half plane cases. The curves for layer height $= 150 \text{ ft}$, 200 ft and 250 ft do not show any convergence.
- (4) The central horizontal displacement value as shown in Fig. (6.15) shows similar trends for the cases of layer height $= 200 \text{ ft}$ and that of 250 ft height but they are both different from the

Detail at A

Figure (6.12.a) Boundary element mesh for half plane case.

Figure (6.12.b) Boundary element mesh for single layer case.

Figure(6.18) Vertical displacement pattern for horizontal load, half plane case.

Figure (6.14) Comparison of results for deflection at 80 ft from centre, vertical load case.

Figure (6.15) Comparison of results for central deflection, horizontal load case.

Figure (6.16) Comparison of results for deflection at 80 ft from centre, horizontal load case.

Figure(6.17) Horizontal displacement patterns for horizontal load, half-plane case.

Figure(6.18) Vertical displacement pattern for horizontal load, half plane case.

case of a half plane. The same observation can be made about the results presented in Fig. (6.16).

The behaviour of the soil layer indicated by the results presented above can be explained by considering resonance at the natural frequencies of a soil layer. Resonance phenomena does not appear in the case of a half space (8). However, in the case of a stratum over a rigid rock, waves are reflected at both boundary surfaces of the soil layer and as a result the layer possesses a set of natural frequencies, which are functions of the layer height and the elastic properties of the soil. Resonance occurs when the exiting harmonic motion has the same frequency as a natural frequency of the layer. If the medium is perfectly elastic, the amplitude of the resulting displacements becomes infinity at those frequencies; but if some amount of damping exists, the displacements are limited in magnitude due to the energy loss in the medium.

The resonant frequencies of the soil layer considered herein coincide with the natural frequencies of vibration of a prismatic beam having the same mass density as the soil and with length equal to the soil layer height (1,8). For horizontal vibrations, this equivalent beam has the same shear modulus, G , as the soil, but an infinite bending stiffness, i.e., it is a pure shear beam. For vertical vibrations, the equivalent beam has a modulus $(\lambda + 2G)$, and deforms only axially. The boundary conditions are in both cases the same as for the actual soil layer; one end fixed and the other free.

For horizontal vibrations, the resonant frequencies ω_n are (8)

$$\omega_n = \frac{2n-1}{2H} \pi c_2, \quad n = 1, 2, \dots \quad (6.1.a)$$

and for vertical vibrations, the resonant frequencies are

$$\omega_n = \frac{2n-1}{2H} \pi c_1, \quad n = 1, 2, \dots \quad (6.1.b)$$

where

H is the height of the layer.

The first shear beam frequencies of the layers are presented in Table (6.1). A closer examination of the results in Figs. (6.13) and (6.15) shows that the frequencies at which peaks occur in these curves approximately match the natural frequency values shown in Table (6.1).

6.5 Formulation of the Stiffness matrix of the foundation

As stated earlier, the stiffness matrix of the soil medium is best derived by calculating the

TABLE 6.1**First Shear Beam Frequency for the soil layers**

<i>H</i>	Vertical Vibration	Horizontal Vibration
50.	222.	140.55
150.	74.1	46.85
200.	55.57	35.15
250.	44.46	28.11

flexibility matrix corresponding to the connection degrees of freedom and taking its inverse.

Referring to Fig. (6.1), $C_{vv}(i, j)$ and $C_{hv}(i, j)$ are respectively the vertical and horizontal displacements at node i generated by a unit harmonic vertical load acting at node j , and $C_{vh}(i, j)$ and $C_{hh}(i, j)$ respectively the vertical and horizontal displacements at node i due to a unit harmonic horizontal load at node j . As discussed earlier, in order to evaluate these coefficients it is sufficient to obtain the displacements at equally spaced surface nodes generated by a concentrated vertical load and a concentrated horizontal load acting at just one node of the surface (e.g. node 0). The rules that generate all other terms of flexibility matrix are given by (57).

$$\left. \begin{aligned} C_{vv}(i, j) &= C_{vv}(j, i) = C_{vv}(|i - j|, 0) \\ C_{hh}(i, j) &= C_{hh}(j, i) = C_{hh}(|i - j|, 0) \\ C_{vh}(i, j) &= -C_{vh}(j, i) = \text{sign}(i - j)C_{vh}(|i - j|, 0) \\ C_{hv}(i, j) &= -C_{hv}(j, i) = \text{sign}(i - j)C_{hv}(|i - j|, 0) \end{aligned} \right\} \quad (6.2)$$

in which

$$\left. \begin{aligned} &1 && \text{for } i > j \\ \text{sign}(i - j) &= &0 && \text{for } i = j \\ &-1 && \text{for } i < j \end{aligned} \right\} \quad (6.3)$$

Theoretically speaking, the coupling terms C_{vh} and C_{hv} should be equal. However, differences did exist in their value because of numerical inaccuracies. It is observed that these differences are small, Fig. (6.11). Taking the average of $C_{vh}(i, j)$ and $C_{hv}(i, j)$ for these elements of the flexibility matrix leads to satisfactory results. The validity of this statement is confirmed by the solution of the bench mark problem of rigid strip footing for which classical solutions are available for comparison. Details of the strip footing analysis are presented below.

A full description of the rigid strip footing problem was presented in chapter 4 and will not repeated here. In chapter 4 the problem was analyzed by a direct application of the boundary element equations and the results were compared with the classical solutions available. This solution will be labelled here as 'rigid strip model'.

An alternative procedure of using the boundary element method to evaluate the response of the footing is to use the dynamic flexibility matrix of the soil layer developed in the preceding section. The first step in the method is to obtain the flexibility matrix of the soil corresponding to the boundary element nodes under the footing. The equations of equilibrium can then be stated as

$$[Z_f(\omega)] \{F\} = \{\delta\} \quad (6.4)$$

in which

$Z_f(\omega)$ = The $2N \times 2N$ matrix containing the flexibility coefficients,

$\{\delta\}$ = The prescribed displacements under the footing as given by Eqn. (4.1), and

$\{F\}$ = The interaction forces between the soil and the footing at the boundary element nodes.

Equation 6.3 can be solved once for a prescribed footing displacement $V_1 = 1$ and a second time for a prescribed displacement $V_2 = 1$. The resulting forces are then summed to obtain the dynamic stiffness. If desired, the rocking stiffness can be obtained by analyzing the footing for an imposed displacement $V_3 = 1$. For the purpose of reference, the results obtained by this method are labelled 'nodal point force model'.

In the analytical results presented here the following properties are used for the soil medium; Young's modulus = $2.016 \times 10^6 \text{ lb/ft}^2$ (96.5 MPa), Poisson's ratio = 0.25 and ρ , the mass density of the soil = $4.19 \text{ lb.s}^2/\text{ft}^4$ ($2.16 \text{ kN.s}^2/\text{m}^4$). The results are presented in terms of the non-dimensional frequency factor a_0 and the non-dimensional dynamic stiffness K_{ij} defined as follows:

$$a_0 = \frac{\omega b}{c_2} \quad (4.5.a)$$

$$K_{ij} = \frac{Q_i}{\pi G V_j} \quad (4.5.b)$$

in which, ω = the exciting frequency, Q_i = the sum of forces in the i direction under the footing, c_2 = the shear wave velocity in the soil medium, G = the modulus of rigidity of the soil, V_j = the imposed displacement along degree of freedom j , and b = the half width of the rigid footing.

The dynamic horizontal and vertical stiffnesses obtained from the two alternative methods of boundary element are plotted in Figs. (6.19) and (6.20) respectively along with the analytical results of Luco and Westmann (29). The results are seen to agree closely.

The results obtained by the flexibility matrix approach show some variation with the mesh used in the boundary element analysis of the two load cases mentioned earlier. This is particularly true of the value of $C_{vv}(1,1)$ or $C_{hh}(1,1)$ which are sensitive to the size of the central element. The rigid footing analysis can be used as a mean of determining the optimum size for this element. A trial and error technique is employed to determine the right size of the central element. The procedure is as follows:

- (1) A trial mesh is used to analyze the soil domain for the two point-load cases shown in Fig. (6.1). The results of the response calculation are used in evaluating $Z_f(\omega)$ for some reasonable values of ω .
- (2) The flexibility matrix is then assembled according to Eqns. (6.1) and (6.2) then Eqn. (6.3) is solved to get the dynamic stiffness coefficients.

Figure (6.19) Comparison of results for horizontal excitation for rigid strip footing case.

Figure (6.20) Comparison of results for vertical excitation for rigid footing case.

- (3) The dynamic stiffness coefficients are then compared with the known classical solutions.
- (4) If necessary the procedure is repeated with different size central element until good agreement is achieved.

It should be noted that the foregoing analysis will be required only once for a domain.

Another set of tests is also necessary to determine the length to which the mesh should extend beyond the base of the superstructure, it may be necessary to extend the soil mesh beyond the base range. But in some cases, where the structure has a long base, the value of the off-diagonal terms of the flexibility matrix, $C(i, j)$, may become very small compared with the diagonal terms as d_{ij} increases, and one might not need boundary element mesh under the whole base. This will lead to a banded flexibility matrix. Fig. (6.21) illustrates the system analyzed for a typical test case and the different meshes used. First, the mesh with $R/L = .25$ is analyzed for the two load cases shown in Fig. (6.1) for two values of the exciting frequency which are chosen so that one lies towards the lower end of the range of interest and the other towards the higher end. The results are then examined to decide whether there is need for boundary element mesh under the whole base or not based on the above mentioned condition. If the magnitude of the off diagonal terms corresponding to the nodes near the edge of the superstructure are small compared with the diagonal terms, a larger mesh is not required. If necessary, the other two meshes shown in Fig. (6.21) are analyzed and the resulting displacements at selected points are plotted as in Figs. (6.22) to (6.25). Careful examination of these curves will reveal the best mesh. For this particular case, the mesh with size equal to the base length appears to be adequate.

The stiffness matrix can be obtained by inverting the flexibility matrix.

Figure (6.21.a) Dimensions of a typical test system.

Figure (0.21.b) Boundary Element meshes.

Figure (6.22) Comparison of results for horizontal central deflection, horizontal load case.

Figure (6.23) Comparison of results for vertical central deflection, vertical load case.

Figure (6.24) Comparison of results for horizontal deflection at distance L from centre, horizontal load case.

Figure (6.25) Comparison of results for vertical deflection at distance L from centre, vertical load case.

Chapter 7

ANALYSIS OF EARTHQUAKE RESPONSE

7.1 Introduction

In the substructure technique used in this work, the soil is modelled by boundary elements while the superstructure is modelled by finite elements. The general formulation of the substructure technique as well as the finite element formulation of the superstructure were presented in Chapter 2. The boundary element formulation of the soil medium was presented in Chapter 3.

As discussed earlier, substructure analysis requires solution of the complex equations of the form given by Eqn. (2.18), once for each excitation frequency. This will involve a large volume of computation. However, as discussed earlier in Chapter 2, if appropriate Ritz shapes that will provide a good representation of the displacement of the structure founded on deformable soil can be found, it should be possible to reduce the size of the problem substantially.

This Chapter presents a comparative study of the results obtained by the above mentioned Ritz shapes procedure and those obtained by a direct approach. The Chapter also presents the results of a limited study of the effect of the ratio of the Young's modulus for the soil to that of the superstructure on the dynamic behaviour of the superstructure. Finally, a procedure to determine the response of a soil-structure system to a given arbitrary dynamic loading, such as for example an earthquake excitation, is outlined and two systems are solved to demonstrate the potential of the method.

7.2 Example of Substructure Analysis

To test the effectiveness of the Ritz vector suggested in Chapter 2, the soil-structure system shown in Fig. (7.1) is analysed for simple unit horizontal harmonic free field accelerations for a range of excitation frequencies, both by direct solution of Eqns. (2.18) and by the mode superposition method using 3 and 8 modes of the associated soil-structure system. The structure analysed has a Young's modulus $= 7.2 \times 10^8 \text{ lb/ft}^2$ ($3.44 \times 10^4 \text{ MPa}$), Poisson's ratio $= 0.17$ and a unit mass $= 4.81$

Figure (7.1) Soil-structure discrete model.

$lb.s^2/ft^4$ ($2.48kN.s^2/m^4$); the corresponding values for the foundation soil are $2.016 \times 10^6 lb/ft^2$ ($96.5MPa$), 0.25 and $4.19 lb.s^2/ft^4$ ($2.16kN.s^2/m^4$) respectively. The damping in the structure is assumed to be negligible.

Figure (7.1) also shows the finite element representation of the structure as well as the boundary element discretization of the surface of the half plane. It may be noted that the boundary element mesh need not match the finite element mesh at the interface, as long as each finite element node corresponds to a boundary element node.

The displacement responses of a selected node in the superstructure are shown in Figs. (7.2) and (7.3). It is seen that in all cases three modes suffice to give an excellent approximation to the results obtained by the direct method. The degree of accuracy can of course be improved by including a larger number of Ritz vectors.

7.3 Effect of Relative Stiffness of Soil

This section investigates the effect of the ratio, E_R , of the stiffness of the soil medium to the stiffness of the superstructure on the dynamic behaviour of the soil-structure system.

$$E_R = \frac{E_s}{E_{st}} \quad (7.1)$$

where

E_s Young's modulus of the soil medium.

E_{st} Young's modulus of the superstructure.

The soil structure system shown in Fig. (7.1) is analysed for a simple unit horizontal harmonic free field accelerations for a range of excitation frequencies. The Young's modulus of the superstructure is kept constant, while the Young's modulus of the soil is changed in such a way as to produce E_R values of .05, .1, .15, and .25. The superstructure on a rigid foundation is also analysed. Results are presented for selected degrees of freedom in Figs. (7.4) through (7.7). The first three natural frequencies of the system for different E_R values are presented in table (7.1). It can be seen that

- (1) The relative stiffness of soil, E_R , has a significant effect on the dynamic behaviour of soil-structure systems.
- (2) As would be expected, the value of the first natural frequency of the system decreases as E_R decreases.

Figure (7.2) Comparison of response at node 16, real part.

Figure (7.3) Comparison of response at node 16, imaginary part.

TABLE 7.1

Natural Frequencies of the System

E_R	1 st	2 nd	3 rd
.05	6.0	31.64	42.47
0.1	14.86	37.55	101.44
0.15	57.2	67.97	268.1
0.25	59.67	75.59	263.2
∞	62.86	215.13	243.3

P

Figure (7.4) Comparison of horizontal response at node 16, real part.

Figure (7.5) Comparison of horizontal response at node 16, imaginary part.

Figure (7.6) Comparison of vertical response at node 16, real part.

Figure (7.7) Comparison of vertical response at node 16, imaginary part.

(3) As the frequency value increases over 70 rad/sec, the displacements become very small.

7.4 Response to Arbitrary Horizontal Ground Motion

As stated earlier, the analysis of the response of a soil-structure system to given arbitrary loading, such as for example an earthquake excitation must be carried out in the frequency domain when the procedure outlined in this thesis is used.

For obtaining the response to an arbitrary excitation through the frequency domain analysis, the basic procedure is to express the excitation in terms of harmonic components, evaluate the response of the system to each component, and then superimpose the harmonic response to obtain the total response (58).

For numerical computations, frequency domain analysis relies on discrete Fourier Transformation which requires that the exciting functions be sampled at a small interval of time. The selection of the time step Δt is therefore an important step in the analysis. Once Δt has been selected, then knowing T_p , the duration of the exciting force which in this case is an earthquake; the number of time steps N_t , the frequency step $\Delta\omega$, and the maximum frequency included in the analysis, ω_{max} can be obtained by using the following equations

$$N_t = \frac{T_p}{\Delta t} \quad (7.2.a)$$

$$\Delta\omega = \frac{2\pi}{T_p} \quad (7.2.b)$$

$$\omega_{max} = \frac{N_t\pi}{T_p} \quad (7.2.c)$$

The considerations that govern the selection of Δt and T_p are fully discussed in the literature (59) and will not be repeated here.

Once the range of frequencies and the frequency step are determined, the complex frequency responses of the structure in terms of the generalized coordinates are obtained for each frequency for a unit harmonic free-field acceleration. The earthquake excitations are Fourier transformed, using the Fast Fourier algorithm (60). Thus

$$\tilde{a}_s^*(p\Delta\omega) = \sum_{k=0}^{N_t-1} a_s^*(t) e^{-\frac{2\pi i k p}{N_t}}, \quad s = u, v \quad (7.3)$$

where

u refers to the horizontal component.

v refers to the vertical component.

Only the horizontal component of the acceleration is considered in this investigation.

The generalized coordinate response in the frequency domain is then given by

$$\begin{aligned}\hat{Y}_n(\omega) &= \hat{Y}_n^u(\omega)\hat{a}_g^u(\omega) + \hat{Y}_n^v(\omega)\hat{a}_g^v(\omega) \\ n &= 1, 2, \dots, m\end{aligned}\quad (7.4)$$

The generalized coordinate response in the time domain is obtained by a Fourier synthesis

$$\begin{aligned}Y_n(k\Delta t) &= \frac{1}{N_t} \sum_{p=0}^{N_t-1} \hat{Y}_n(\omega) e^{\frac{2\pi i k p}{N_t}} \\ n &= 1, 2, \dots, m\end{aligned}\quad (7.5)$$

Finally, the displacement response for the structure is obtained by superimposing the responses in the m generalized coordinates

$$r(t) = \sum_{n=1}^m \phi_n Y_n(t) \quad (7.6)$$

Once the displacements of the structure are known, the stresses can be found using the finite element stress-displacement transformation matrices.

When using the discrete Fourier transformation (DFT), it is important to note that it is based on the assumption that both the excitation and the response are periodic of period T_p . This involves errors in obtaining the transient response of the system particularly when the damping in the system is small. The reason for these errors is simply that the system is not starting from rest due to the action of the excitations prior to the initial time. Two different techniques can be used to eliminate these errors. Both techniques are designed to ensure that the system starts from rest at the beginning of each period, that is, at time $0, +T_p, +2T_p$.

The first technique is known as a limiting process (58,61). In this technique, the excitation period is elongated by adding a grace band of zero excitations to the end, to ensure that the system will have a period of free vibration. Thus, when the system is damped and the period of the free vibration is long enough, the system comes to rest at the end of each period, which means that the calculated response is approximately the true transient response. It is clear from this discussion that this technique has two drawbacks. The addition of the zero excitation is computationally

inefficient because it requires more core memory and computation time to obtain the response. Also, the method is not applicable to undamped systems.

The second technique, which is used in this work, was proposed by Meek and Veletsos (34). The first step in that method is to evaluate the exact response of the system to periodic extension of the prescribed excitation. Next, a corrective periodic response is superimposed so as to bring the system to rest at the end of each excitation period. This ensures that a single period of the sum of the two periodic responses is identical to the response induced by one-time application of the sum of the prescribed excitation and the corrective excitation. By confining the corrective excitation to the last few time steps of the original excitation, the true transient response at all other time steps is obtained.

To apply this method to the problem being considered in this study which in effect consists of evaluating m single degrees of freedom systems, consider the k^{th} degree of freedom.

The periodic response given by Eqns. (7.3) and (7.4) can be written as

$$Y_k^P(m, \Delta t) = \frac{\Delta \omega}{2\pi} IDFT \left\{ \Delta t \cdot DFT \{a_g(m, \Delta t)\} \cdot \hat{Y}_k(n, \Delta \omega) \right\} \quad (7.7)$$

in which DFT means Discrete Fourier Transform and $IDFT$ is an inverse Discrete Fourier Transform. The corrective method is then applied as described below.

In order to ensure that the system starts from rest at the beginning of each period, the displacement and velocity at the last step should equal zero. The values of excitation function at the last two steps are therefore set equal to zero if they are not already zero; Fig. (7.8). Next, the periodic response $Y_k^P(m, \Delta t)$ is obtained as given by Eqn. (7.7).

Now, let it be assumed that a corrective excitation consisting of two impulses R_1 and R_2 placed at steps $N - 4$ and $N - 3$, as shown in Fig. (7.8), is applied to the system. Let this corrective excitation be called $x^*(t)$. The response of the system to this corrective excitation can be obtained by using eqn. (7.7) and replacing $a(t)$ by $x^*(t)$. Recognizing that the DFT of a unit impulse is a unit function (58), the periodic response of the system for a unit impulse at time zero becomes

$$Y_0^{*P}(m, \Delta t) = \frac{\Delta \omega}{2\pi} IDFT \left\{ \hat{Y}_k(n, \Delta \omega) \right\} \quad (7.8)$$

Using the time shifting property of DFT (60), the response at the last two steps due to unit impulse at time $(N - 4)\Delta t$ are

$$Y_1^{*P}((N - 2)\Delta t) = Y_0^{*P}(2\Delta t) \quad (7.9)$$

Figure (7.8) Steps of obtaining the response of the k -th generalized displacement.

$$Y_1^{*P}((N-1)\Delta t) = Y_0^{*P}(3\Delta t) \quad (7.10)$$

Similarly, for a unit impulse at time $(N-3)\Delta t$, the last two responses are

$$Y_2^{*P}((N-2)\Delta t) = Y_0^{*P}(\Delta t) \quad (7.11)$$

$$Y_2^{*P}((N-2)\Delta t) = Y_0^{*P}(2\Delta t) \quad (7.12)$$

The value of the corrective impulses R_1 and R_2 are now obtained from the following equations.

$$Y_k^P((N-2)\Delta t) + R_1 Y_0^{*P}(2\Delta t) + R_2 Y_0^{*P}(\Delta t) = 0 \quad (7.13)$$

$$Y_k^P((N-1)\Delta t) + R_1 Y_0^{*P}(3\Delta t) + R_2 Y_0^{*P}(2\Delta t) = 0 \quad (7.14)$$

The above equations are simply the conditions that the displacements at time $(N-2)\Delta t$ and $(N-1)\Delta t$ are set equal to zero, which ensures that the system starts from rest at the beginning of each period. The solution of Eqns. (7.13) and (7.14) yields the values of the R_1 and R_2 . Using these values and the function Y_0^{*P} , the response Y_k^{*P} of the system to the corrective excitation $x^*(t)$ is obtained from the following equation.

$$Y_k^{*P}(i\Delta t) = R_1 Y_0^{*P}(i_1\Delta t) + R_2 Y_0^{*P}(i_2\Delta t) \quad (7.15)$$

where

$$i_1 = i + 4 \quad (7.16.a)$$

and

$$i_2 = i + 3 \quad (7.16.b)$$

When the values of i_1 and i_2 are greater than $(N-1)$, they should be modified as follows

$$i_1 = i + 4 - N \quad (7.16.c)$$

and

$$i_2 = i + 3 - N \quad (7.16.d)$$

On superimposing the periodic responses $Y_k^P(t)$ and $Y_k^{*P}(t)$, the true transient response $Y_k(t)$ of the k^{th} system becomes Fig. (7.8)

$$Y_m(m.\Delta t) = Y_k^P(m.\Delta t) + Y_k^{*P}(m.\Delta t) \quad (7.17)$$

Having obtained the responses of the generalized displacements $Y_k(t); k = 1, 2, \dots, m$, the relative displacements of the structure nodal points can be evaluated at each time step using Eqn. (7.6). The stresses at each time step can then be obtained by using standard finite element techniques.

7.5 Procedure for the Earthquake Response of Soil-structure system

A computer program was developed to obtain the earthquake response of the soil-structure system on the basis of the analytical treatment presented in this and the previous Chapters. The program considered the case of horizontal ground motion only. However, it can be modified by simple changes to take into account a vertical ground motion. The program was limited to include three mode shapes in the analysis.

The main steps in obtaining the earthquake response are

- (1) The soil-structure system is analyzed to obtain the generalized mode shapes.
- (2) For each frequency component of excitation
 - (a) The response of the system to the two unit harmonic central loads; a horizontal load and a vertical one are obtained through the boundary element analyses.
 - (b) The flexibility matrix corresponding to the connection degrees of freedom is then assembled and inverted to get the stiffness matrix, $X_f(\omega)$.
 - (c) The complex frequency response functions for the three generalized displacements of the associated soil-structure system are obtained by solving the equations of motion (2.23).
- (3) The frequency response $\hat{Y}_k(m, \Delta\omega); k = 1, 2, 3$ is obtained as described earlier for the range of frequency steps from $m = 0$ to $m = N/2$. the fact that the *IDFT* of the complex frequency response is a real function requires that the real part of $\hat{Y}_k(\omega)$ be an even function and the imaginary part be an odd function (58). This is done by folding the first $N/2$ real values of $\hat{Y}_k(\omega)$ about the $N/2$ step, and folding and flipping the first $N/2$ imaginary values of $\hat{Y}_k(\omega)$ about the $N/2$ step. In fact the second $N/2$ values of $\hat{Y}_k(\omega)$ are the frequency response for the range of negative frequencies since the frequency range extends periodically from $(-N\Delta\omega/2)$ to $(+N\Delta\omega/2)$.
- (4) For each mode k , ($k = 1, 2, 3$) the following steps are repeated
 - (a) The periodic response Y_k^P as given by Eqn. (7.7) is obtained using the *FFT* algorithm.

- (b) The periodic response Y_0^P as given by Eqn. (7.8) is obtained using the *FFT* algorithm.
 - (c) Eqns. (7.13) and (7.14) are used to obtain the corrective impulses R_1 and R_2 .
 - (d) The corrective periodic response Y_k^{*P} is computed using Eqn. (7.15).
 - (e) The two periodic responses Y_k^P and Y_k^{*P} as given by Eqn. (7.18) are superimposed to obtain the true transient response Y_k .
- (5) For each time step, the relative displacements of the structure nodal points are calculated (Eqn. (7.8)).

Two soil-structure problems were solved by the frequency domain analysis using the program described in this section and the results are presented in the next section.

7.6 Analytical Results for the Soil-Structure Problem

This section presents the results of analytical studies carried out using the frequency domain analysis procedure described earlier. Two example problems are solved and the results are presented.

Before attempting the solution of a practical problem, a problem of smaller size was solved to study the effect of soil flexibility. The soil-structure system shown in Fig. (7.1), was considered. The damping effects for the superstructure are considered. The damping ratio is taken as .02 for each mode.

The system is analysed twice, the first time taking the flexibility of the soil into account, and the second time considering the foundation medium to be rigid. Response analysis is carried out for the first ten seconds of the North-South horizontal ground acceleration of El Centro earthquake (1940). The time step is chosen to be 0.01 sec. Correspondingly, the frequency step, $\Delta\omega$, becomes .314 rad/sec. Fig. (7.9) shows the earthquake acceleration record for the period considered in the analysis, while Figs. (7.10.a) and (7.10.b) are the real and the imaginary parts of the *DFT* of that acceleration record. The horizontal and vertical relative displacement of a selected node of the superstructure are shown in Figs. (7.11) through (7.14). It can be seen that the relative response of the structure founded on flexible soil has a higher value of the peak displacement than does the response of the structure founded on a rigid base. This brings out the importance of including interaction in the earthquake analysis of such a structure.

Figure (7.9) Acceleration record - El Centro, California, Earthquake, May 1940.

Figure (7.10.a) Discrete Fourier transform of El Centro earthquake acceleration history, real part.

Figure (7.10.b) Discrete Fourier transform of El Centro earthquake acceleration history, imaginary part.

Figure (7.11) Horizontal displacement at node 10, flex. foundation case.

Figure (7.12) Vertical displacement at node 16, flex. foundation case.

Figure (7.13) Horizontal displacement at node 16, rigid foundation case.

Figure (7.14) Vertical displacement at node 10, rigid foundation case.

These results are in general agreement with the results obtained by other researchers (1) with the exception that the differences between the peak displacements for the flexible and rigid foundation cases are higher in the present study. This can be explained in light of the fact that the relative stiffness of the soil in the present study, although realistic for a dense gravel soil, is very low as compared to that used in other studies (1,14).

Figure (7.14.b) illustrates the history of horizontal stress at the upper right corner of the structure (element No 18). Table (7.2) gives the horizontal, vertical and shear stresses in the same element. It can be seen that, the stresses in the case of a structure resting on flexible foundation are smaller than the corresponding stresses in the case of structure resting on rigid foundation. This can be attributed to the fact that more energy will be lost through radiation in the flexible foundation than that lost in the rigid foundation.

In the second problem of soil-structure interaction, a dam of height 300 ft and a vertical upstream face was considered. The excitation was again the first 10 seconds of El Centro earthquake. The finite element model of the dam as well as the boundary element model of the soil are shown in Fig. (7.15)

The dam analysed has a Young's modulus $= 7.2 \times 10^9 \text{ lb/ft}^2$ ($3.44 \times 10^4 \text{ MPa}$), Poisson's ratio $= 0.17$ and a unit mass $= 4.81 \text{ lb.s}^2/\text{ft}^4$ ($2.48 \text{ kN.s}^2/\text{m}^4$); the corresponding values for the foundation soil are $3.6 \times 10^7 \text{ lb/ft}^2$ (1723.2 MPa), 0.25 and $4.19 \text{ lb.s}^2/\text{ft}^4$ ($2.16 \text{ kN.s}^2/\text{m}^4$) respectively. The damping in the dam is assumed to be 5 % of the critical in the first three modes.

Figs. (7.16) through (7.18) illustrate the mode shapes and the natural frequencies of the system.

The complex displacements were calculated for every second frequency step. Figs. (7.19) through (7.24) show these complex frequency responses. Figs. (7.25) through (7.27) show the time series responses of the three generalized displacements. Figs. (7.28) and (7.29) are the horizontal and vertical displacement histories of the crest. The first 996 steps in these Figures represent the true response; the last four steps are not correct because of the manner in which the frequency analysis was formulated and should be disregarded.

7.7 Effect of the Time Step and Total Time on the Response analysis

To study the effect of time step, Δt , on the present analysis, the problem shown in Fig. (7.1) was solved again with a time step of 0.02 seconds instead of 0.01 seconds. The horizontal and

Figure (7.14.b) Horizontal Stress in element No 18.

TABLE 7.2

Stresses in element (18). of the structure Model Fig. 7.1

Units lbs/ft^2

time	flex foundation			rigid foundation		
(sec.)	σ_{XX}	σ_{YY}	σ_{XY}	σ_{XX}	σ_{YY}	σ_{XY}
0	$-.9 \times 10^1$	$.104 \times 10^3$	$-.25 \times 10^2$	$.24 \times 10^2$	$-.27 \times 10^3$	$.86 \times 10^2$
0.5	$.4 \times 10^1$	$.202 \times 10^3$	$-.72 \times 10^2$	$-.157$	$-.133 \times 10^1$	$-.25 \times 10^1$
1.0	$-.605 \times 10^1$	$-.330 \times 10^1$	$.29 \times 10^1$	$-.490 \times 10^2$	$.550 \times 10^3$	$-.175 \times 10^3$
1.5	$-.23 \times 10^2$	$.11 \times 10^3$	$-.13 \times 10^2$	$.101 \times 10^2$	$-.12 \times 10^3$	$.33 \times 10^2$
2.0	$-.33 \times 10^2$	$-.84 \times 10^3$	$.318 \times 10^3$	$-.150 \times 10^3$	$.164 \times 10^4$	$-.54 \times 10^3$
2.5	$-.39 \times 10^2$	$.187 \times 10^3$	$-.620 \times 10^2$	$-.480 \times 10^2$	$.568 \times 10^3$	$-.158 \times 10^3$
3.0	$.35 \times 10^2$	$.147 \times 10^2$	$-.430 \times 10^2$	$-.210 \times 10^2$	$.220 \times 10^3$	$-.820 \times 10^2$
3.5	$.29 \times 10^2$	$.426 \times 10^3$	$-.175 \times 10^3$	$-.550 \times 10^2$	$.650 \times 10^3$	$-.180 \times 10^3$
4.0	$.52 \times 10^2$	$.240 \times 10^3$	$.136 \times 10^3$	$-.160 \times 10^3$	$.188 \times 10^4$	$-.559 \times 10^3$
4.5	$.15 \times 10^2$	$.370 \times 10^3$	$.140 \times 10^3$	$-.160 \times 10^3$	$.187 \times 10^4$	$-.530 \times 10^3$
5.0	$.36 \times 10^2$	$.365 \times 10^3$	$-.16 \times 10^3$	$-.115 \times 10^2$	$-.165 \times 10^3$	$-.22 \times 10^2$
5.5	$.016 \times 10^2$	$-.20 \times 10^3$	$.493 \times 10^2$	$-.660 \times 10^2$	$-.750 \times 10^3$	$-.23 \times 10^3$
6.0	$.330 \times 10^1$	$.41 \times 10^2$	$-.173 \times 10^2$	$.17 \times 10^2$	$-.196 \times 10^3$	$.62 \times 10^2$
6.5	$.73 \times 10^1$	$.12 \times 10^3$	$-.5 \times 10^2$	$-.325 \times 10^2$	$.38 \times 10^3$	$-.1073 \times 10^3$
7.0	$-.62 \times 10^1$	$.39 \times 10^2$	$-.65 \times 10^1$	$-.150 \times 10^2$	$.188 \times 10^3$	$-.475 \times 10^2$
7.5	$-.236 \times 10^2$	$-.237 \times 10^3$	$.105 \times 10^3$	$.166 \times 10^2$	$-.19 \times 10^2$	$.58 \times 10^2$
8.0	$.283 \times 10^1$	$-.13 \times 10^3$	$.41 \times 10^2$	$.54 \times 10^2$	$-.61 \times 10^3$	$.194 \times 10^3$
8.5	$.318 \times 10^2$	$.27 \times 10^2$	$-.125 \times 10^3$	$.77 \times 10^2$	$-.876 \times 10^3$	$.27 \times 10^3$
9.0	$.244 \times 10^2$	$.540 \times 10^3$	$-.21 \times 10^3$	$.21 \times 10^3$	$-.24 \times 10^4$	$.73 \times 10^3$
9.5	$-.256 \times 10^2$	$.15 \times 10^3$	$-.23 \times 10^2$	$.36 \times 10^3$	$-.417 \times 10^4$	$.122 \times 10^4$
10.0	0.0	0.0	0.0	0.0	0.0	0.0

Figure 7.15.a) Finite element model of the dam.

Figure (7.15.b) Boundary element model of the soil half plane.

1st mode
Frequency=9.517 rad/sec

Figure (7.16) First mode shape of the dam.

2nd mode
Frequency=10.19 rad/sec

Figure (7.17) Second mode shape of the dam.

3rd mode
Frequency=50.64 rad/sec

Figure (7.18) Third mode shape of the dam.

Figure (7.19) Complex frequency response of 1st generalized displacement, real part.

Figure (7.20) Complex frequency response of 1st generalized displacement, imaginary part.

Figure(7.21) Complex frequency response of ind generalized displacement,real part.

Figure (7.22) Complex frequency response of 2nd generalized displacement, imaginary part.

Figure (7.23) Complex frequency response of 3rd generalized displacement, real part.

Figure (7.24) Complex frequency response of 3rd generalized displacement, imaginary part.

Figure (7.25) Time response of first generalized displacement.

Figure (7.26) Time response of second generalized displacement.

Figure (7.27) Time response of third generalized displacement.

Figure (7.28) Horizontal displacement of crest.

Figure (7.29) Vertical displacement of crest.

Figure (7.30) Effect of the time step on the horizontal displacement at crest

- (3) The extension of this work to a three dimensional problem may be considered. Three dimensional finite elements and boundary elements are available in references (35-37,43), and can be used in the formulation of the problem.
- (4) The frequency domain analysis method proposed here can be used to carry out an in-depth parametric study to obtain the effect of all variables affecting the earthquake response of an emptied reservoir dam-soil system.

vertical displacement histories of the crest are plotted in Figs. (7.30) and (7.31) respectively. It can be seen that there is no significant difference between the two sets of results for 0.01 and .02 second time step.

In order to study the effect of the total time, the same problem is considered again with total time, T_p , of 5 seconds and the time step, Δt , equal to 0.01 second. The results are presented in Figs. (7.32) and (7.33) and compared with those for the case of T_p equal to 10 seconds. It can be seen that the choice of T_p does affect the response analysis. That brings about the importance of choosing the proper duration time for a particular earthquake so that the resulting response would be realistic. Several trial runs may be needed to ensure that the results converge with increasing T_p .

Figure (7.32) Effect of total time on the horizontal response at crest

Figure (7.33) Effect of total time on the vertical response at crest

Chapter 8

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE WORK

8.1 Summary

Research into the response of structures to the dynamic loading caused by an earthquake has received the attention of many investigators during the past few years because of the damage caused to many important structures, such as dams, tall buildings and bridges by recent earthquakes. Studies have shown that in many cases the soil-structure interaction has a significant effect on the response and must be accounted for. Due to the complex nature of the problem of earthquake response of a soil-structure system, a numerical procedure must be used to solve the problem. The primary objective of this study has been to develop an accurate and efficient numerical procedure to solve the problem. The suggested procedure employs a substructuring technique, in which the structure is modelled as an assemblage of finite elements while the soil half space is modelled by boundary elements.

The following assumptions are made to simplify the problem.

- (1) The study is limited to the two dimensional case, the motion in any vertical plane perpendicular to the axis of the structure being similar.
- (2) A plane strain situation exists.
- (3) The structure and soil materials are homogeneous and elastic.
- (4) Material damping in the soil material is not considered, but the radiation damping is taken into account.
- (5) The damping ratio corresponding to each mode of the superstructure is independently selected.

The substructure technique is used in this work. In this technique the soil is treated as one substructure and the superstructure as the other. The stiffness matrix of the soil substructure corresponding to the connection degrees of freedom between the superstructure and the soil is obtained first. This is then combined with a finite element representation of the superstructure

and the given free field excitation to obtain the equations of motion of the combined system. To reduce the size of the resulting problem, the displacement vector is expressed as a sum of a few appropriately chosen Ritz shapes weighted by unknown generalized coordinates. The most appropriate Ritz shapes for this purpose are the first few mode shapes of an associated soil-structure system.

The substructure analysis is performed in the frequency domain using the Fast Fourier Transform algorithm. The frequency domain analysis is based on a resolution of the free field excitation into its harmonic components, and the solution of the equations of motion once for each such harmonic component followed by a Fourier synthesis of the resulting responses.

The formulation of the problem of a soil-structure system under the action of an earthquake excitation is reviewed first. This is followed by a discussion of the manner in which the substructure technique is applied to the vibrations problems of soil-structure systems.

A boundary element formulation of the soil medium is presented next. The problem of vibration of a rigid strip footing is then analysed mainly with a view to benchmark the boundary element formulations as well as the computer implementation by comparing the results with the existing classical solutions of rigid strip footings. The comparisons are found to be satisfactory. A study is then made of the coupling effects between two neighbouring rigid strip footings under steady state vibrations.

The substructure method used in the analysis requires that a dynamic stiffness matrix corresponding to the connection degrees of freedom at the base of the superstructure be obtained for the soil for each component frequency of excitation. This is accomplished by first obtaining the dynamic flexibility matrix of the soil medium and then inverting it. The frequency dependent stiffness matrix of the soil medium is combined with the finite element model of the superstructure to obtain the equations of motion.

It is shown that the frequency dependent flexibility matrix of the soil region with a horizontal ground surface can be assembled from the analyses of two separate harmonic load cases: one horizontal and the other vertical, both applied at the central point of the soil boundary element discretization.

The different factors affecting the choice of the boundary element mesh are investigated first. A procedure to assemble the flexibility matrix is presented next. The assembled matrix is used to analyze the problem of a rigid strip footing and close agreement with the classical solutions is observed.

A brief review of the finite element formulation is presented next. Since a direct solution of the equations of motion for each harmonic component of excitation would involve a large volume of computations, a set of generalized Ritz shapes is introduced to permit considerable reduction in the size of the problem. The Ritz shapes chosen are the first three modes of an associated soil-structure system. A comparison between the direct method of solving the equations of motion and the modal analyses employing three and eight modes of the above mentioned soil-structure system proves the effectiveness of the modal technique and shows that even three modes give adequate accuracy.

A study on the effect on response of the ratio of soil modulus of elasticity to the structure modulus of elasticity follows. As would be anticipated, that ratio has a significant effect on the dynamic behaviour of the soil-structure system.

A procedure for obtaining the response of the soil-structure system through a frequency domain analysis is then developed. The method can be summarized as follows:

- (1) The soil-structure is analyzed to obtain the generalized mode shapes.
- (2) For each frequency component of excitation
 - (a) The boundary element program is used to obtain the responses of the soil substructure to the two unit harmonic central loads; a horizontal load and a vertical load.
 - (b) The flexibility matrix corresponding to the connection degrees of freedom is then assembled and inverted to get the stiffness matrix.
 - (c) The complex frequency response functions for the three generalized displacements of the associated soil-structure system are obtained by solving the three coupled equations of motion.
- (3) Having obtained the complex frequency response functions of the generalized displacements, the modal responses to an arbitrary ground motion are obtained by a Fourier synthesis. The effectiveness of the method of the analysis derives from the highly efficient fast Fourier transform algorithms which cut down the computation time by several orders of magnitude.
- (4) Having obtained the time history of generalized displacements, the displacements and stresses for the structure are obtained by standard transformations.

A computer program was developed to obtain the response of the dam reservoir system using the proposed frequency domain analysis.

Two soil-structure problems are solved and presented to demonstrate the effectiveness of the proposed method.

8.2 Conclusions

As a result of the present work, the following major conclusions are drawn:

- (1) The boundary element method can be employed effectively to obtain the response of a semi-infinite soil domain. The BEM provides a simple alternative to the existing techniques such as the finite elements or the direct mathematical formulation of the soil medium treated as a viscoelastic half plane. In the boundary element analysis, it is necessary only to discretize the plane surface of the soil medium and the boundary element representation can be truncated at a reasonable distance from the region of interest.
- (2) The approximation of the response of a semi-infinite soil domain by a soil layer underlain by bed rock is not adequate.
- (3) The success of the substructure analysis ultimately depends on the evaluation of the frequency dependent stiffness matrix of the unbounded soil medium. To achieve this, the soil must be discretized in some manner and the soil model analysed for the effect of a unit harmonic load at each connection degree of freedom. It is demonstrated in this work, that a boundary element representation proves to be very effective in deriving the stiffness matrix of the soil. Further, when the structure-soil interface consists of a plane surface, the stiffness matrix can be assembled from just two sets of analysis, one for a unit harmonic horizontal load and another from a unit harmonic vertical load.

8.3 Recommendations For Future Work

The following recommendations are made for future work

- (1) The present computer program can be modified by simple changes to include the vertical ground motion.
- (2) The extension of this work to include the water reservoir in the problem of soil-dam-reservoir interaction is recommended. The boundary element formulation of the infinite reservoir is available in references (61), and it can be incorporated in the present formulation.

- (3) The extension of this work to a three dimensional problem may be considered. Three dimensional finite elements and boundary elements are available in references (35-37,43), and can be used in the formulation of the problem.
- (4) The frequency domain analysis method proposed here can be used to carry out an in-depth parametric study to obtain the effect of all variables affecting the earthquake response of an emptied reservoir dam-soil system.

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Appendix A

Finite Element Characteristics

The triangular element, Fig. A-1, selected for this study, has two translation components of displacements at each node. If the displacements u and v at any point are assumed to be linear, they may be expressed in terms of the six nodal displacements, $\{\delta^e\}$, through the shape functions $[N_e]$ as

$$\{u\} = [N_e] \{\delta^e\} \quad (A-1)$$

The shape functions are

$$[N_e] = [IN_i \quad IN_j \quad IN_m] \quad (A-2)$$

where I is a 2x2 identity matrix and N_i , N_j and N_m are given by

$$N_i = \frac{1}{2\Delta} (a_i + b_i x + c_i y) \quad (A-3)$$

where i, j and m represent the three nodes of the triangular finite element with coordinates (x_i, y_i) , (x_j, y_j) and (x_m, y_m) referred to the centroidal axes x and y . Δ is the area of the triangular element given by

$$\Delta = 0.5ABS (a_i + a_j + a_m) \quad (A-4)$$

and

$$a_i = x_j y_m - x_m y_j \quad ; \quad b_i = y_j - y_m \quad ; \quad c_i = x_m - x_j$$

$$a_j = x_m y_i - x_i y_m \quad ; \quad b_j = y_m - y_i \quad ; \quad c_j = x_i - x_m$$

$$a_m = x_i y_j - x_j y_i \quad ; \quad b_m = y_i - y_j \quad ; \quad c_m = x_j - x_i$$

Strains and Stresses

For a problem in two dimensional space, the strain component at any point may be expressed as (36)

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{Bmatrix} \quad (A-5)$$

Fig.(A-1) A Constant Strain Traingular Finite Element.

The resulting relationship may be expressed as

$$\{\epsilon\} = [B] \{\delta^e\} \quad (A-7)$$

Hence matrix $[B]$ is given by

$$[B] = \frac{1}{2\Delta} \begin{bmatrix} b_i & 0 & b_j & 0 & b_m & 0 \\ 0 & c_i & 0 & c_j & 0 & c_m \\ c_i & b_i & c_j & b_j & c_m & b_m \end{bmatrix} \quad (A-6)$$

The relationship between stresses and strains take the form

$$\{\sigma\} = [D] \{\epsilon\} \quad (A-7)$$

and for the plane strain case considered here,

$$[D] = \frac{E(1-\nu)}{(1-\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \quad (A-8)$$

In the above equations,

E = modulus of elasticity

and

ν = Poisson's ratio.

Stiffness matrix

Once matrices $[B]$ and $[D]$ have been established, the stiffness matrix $[K]$ may be obtained by

$$K = \int_{vol} B^T D B dV \quad (A-9)$$

Mass Matrix

The mass matrix is approximated by 'lumping' the total mass of an element at its nodes. In that case, the mass matrix becomes diagonal resulting in a significant reduction in the computational efforts. The lumped mass matrix is given by:

$$[M^e] = \frac{\rho_s \Delta}{3} [I] \quad (A-10)$$

where

ρ_s the mass density of the structure.

APPENDIX B

Simplification of T_{jk}^*

The fundamental solution of the equations of motion of the elastic domain is given by

$$U_{jk}^* = \frac{1}{4G} \{ \psi \delta_{jk} - \chi r_{,k} r_{,j} \} \quad (B-1)$$

$$\begin{aligned} T_{jk}^* = & \frac{i}{4} \left\{ \left(\frac{d\psi}{dr} - \frac{1}{r} \chi \right) \left(\delta_{kj} \frac{\partial r}{\partial n} + r_{,k} n_j \right) \right. \\ & - \frac{2}{r} \chi \left(n_{k,r,j} - 2 r_{,k} r_{,j} \frac{\partial r}{\partial n} \right) - 2 \frac{d\chi}{dr} r_{,k} r_{,j} \frac{\partial r}{\partial n} \\ & \left. + \left(\frac{c_1^2}{c_2^2} - 2 \right) \left(\frac{d\psi}{dr} - \frac{d\chi}{dr} - \frac{\chi}{r} \right) r_{,j} n_k \right\} \quad (B-2) \end{aligned}$$

where

$$\psi = H_0(k_2 r) + \frac{1}{k_2 r} \{ \gamma H_1(k_1 r) - H_1(k_2 r) \} \quad (B-3)$$

$$\chi = \gamma^2 H_2(k_1 r) - H_2(k_2 r) \quad (B-4)$$

Note that (37)

$$\frac{d}{dz} H_0(z) = -H_1(z) \quad (B-5)$$

$$\frac{d}{dz} H_1(z) = \frac{1}{z} H_1(z) - H_2(z) \quad (B-6)$$

$$\frac{d}{dz} H_2(z) = H_1(z) - \frac{2}{z} H_2(z) \quad (B-7)$$

By employing Eqns. (B-5) to (B-7), and after some manipulation, the following expressions can be obtained.

$$\frac{d\psi}{dr} = -k_2 H_1(k_2 r) - \frac{\chi}{r} \quad (B-8)$$

$$\frac{d\chi}{dr} = k_2 [\gamma^3 H_1(k_1 r) - H_1(k_2 r)] - \frac{2\chi}{r} \quad (B-9)$$

and

$$\frac{d\psi}{dr} - \frac{d\chi}{dr} - \frac{\chi}{r} = -k_2 \gamma^3 H_1(k_1 r) \quad (B-10)$$

also

$$\frac{d\psi}{dr} - \frac{\chi}{r} = -k_2 H_1(k_2 r) - \frac{2\chi}{r} \quad (B-11)$$

and

$$-2 \frac{d\chi}{dr} \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial n} + \frac{4\chi}{r} \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial n} =$$

$$2 \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial n} \left[\frac{4\chi}{r} - k_2 \{ \gamma^3 H_1(k_1 r) - H_1(k_2 r) \} \right] \quad (B-12)$$

Substitution of Eqns. (B-8) to (B-12) in Eqn. (B-2) and regrouping leads to the following expression

$$T_{jk}^* = \frac{1}{4} \left\{ B_1 \left(\frac{\partial r}{\partial n} \delta_{lk} + \frac{\partial r}{\partial x_k} n_j \right) + B_2 \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \frac{\partial r}{\partial n} + B_3 \frac{\partial r}{\partial x_j} n_k \right\} \quad (B-12)$$

where

$$B_1 = -k_2 H_1(k_2 r) - \frac{2\chi}{r} \quad (B-13)$$

$$B_2 = 2k_1 \gamma^2 H_3(k_1 r) - 2k_2 H_3(k_2 r) \quad (B-14)$$

$$B_3 = -k_1 (1 - 2\gamma^2) H_1(k_1 r) - \frac{2\chi}{r} \quad (B-15)$$

$$\gamma = \frac{c_2}{c_1} \quad (B-16)$$

$$r(P, Q) = \sqrt{(X_P - X_Q)^2 + (Y_P - Y_Q)^2} \quad (B-17)$$

$$k_m = \frac{\omega}{c_m} \quad m = 1, 2 \quad (B-18)$$

$$n_m = \frac{\partial x_m}{\partial n} \quad m = 1, 2 \quad (B-19)$$

$$\frac{\partial r}{\partial n} = \frac{\partial r}{\partial x_m} n_m \quad m = 1, 2 \quad (B-20)$$

c_1 = Compression wave velocity.

c_2 = Shear wave velocity.

ω = Angular frequency of excitation.

G = Shear modulus of the medium.

δ_{jk} = Kronecker delta.

n = Outward pointing normal unit vector on the surface of the body.

$H_n(x)$ = Hankel function of the first kind of order n .

Appendix C

Formulation of The Boundary Element Equations

The formulation of the boundary element equations relies on the dynamic Betti-Rayleigh reciprocal theorem, this theorem is also known as Betti's third identity. Following Erigen and Suhbi (25), Betti's third identity can be stated as

$$\rho \int_D \left\{ \underline{u}^{(1)} \nabla \cdot \underline{u}^{(2)} - \underline{u}^{(2)} \cdot \nabla \underline{u}^{(1)} \right\} dD \\ - \int_S \left\{ \underline{u}^{(1)} \underline{t}^{(2)} - \underline{u}^{(2)} \underline{t}^{(1)} \right\} dS \quad (3.14)$$

where

D is an elastic domain with boundary S .

$\underline{u}^m, \underline{t}^m$ are the displacement vector in the domain D and the corresponding traction vector on a surface element with normal vector $\underline{n}$ corresponding to an elastodynamic state m .

$$\nabla \cdot = (c_1^2 - c_2^2) \nabla \nabla + c_2^2 \nabla^2 \quad (3.15)$$

∇ is the Del vector operator.

In Eqn. (3.14), let the elastodynamic state (1) correspond to the fundamental solution for concentrated force in the infinite space, namely the displacement and tractions for an applied force in the j direction and let the elastodynamic state (2) correspond to the set of displacements and tractions U_k and T_k in domain D and on surface S , containing the load point $P(x_i)$, and representing the unknowns in the problem to be solved. Substituting into Eqn. (3.14) the fundamental solution for state (1), and neglecting the body forces one obtains

$$- \int_S T_{jk}^*(P, Q) U_k dS(Q) + \int_S T_k U_{jk}^*(P, Q) dS = 0.0 \quad (C-1)$$

To exclude the singular point P , as shown in Fig. (C-1), the basic procedure is to evaluate the integrals over the domain $D - D^*$ and take the limiting values as $\epsilon \rightarrow 0$. Eqn. (C-1) can be written as

$$- \int_{S+S^*} T_{jk}^*(P, Q) U_k dS(Q) + \int_{S+S^*} T_k U_{jk}^*(P, Q) dS = 0.0 \quad (C-2)$$

Fig. (C-1) Domain $D - D^*$

The left hand side of the integral can be written as

$$-\int_S T_{jk}^* U_k dS + \int_S T_k U_{jk}^* dS - \int_{S^*} T_{jk}^* U_k dS + \int_{S^*} T_k U_{jk}^* dS \quad (C-3)$$

Let

$$\int_{S^*} T_{jk}^* U_k dS = I_1 \quad (B-4.a)$$

and

$$\int_{S^*} T_k U_{jk}^* dS = I_2 \quad (B-4.b)$$

Since U_k is regular, it tends to U_k^P as $\epsilon \rightarrow 0$, then

$$\lim_{\epsilon \rightarrow 0} I_1 = U_k^P \lim_{\epsilon \rightarrow 0} \int_{S^*} T_{jk}^* dS \quad (C-5)$$

similarly,

$$\lim_{\epsilon \rightarrow 0} I_2 = T_k^P \lim_{\epsilon \rightarrow 0} \int_{S^*} U_{jk}^* dS \quad (C-6)$$

In the limits as $z \rightarrow 0$, the following relations hold

$$J_0(z) = 1 \quad (C-6)$$

$$J_1(z) = 0 \quad (C-7)$$

$$J_2(z) = 0 \quad (C-8)$$

$$Y_0(z) = \frac{2}{\pi} \ln(z) \quad (C-9)$$

$$Y_1(z) = -\frac{2}{\pi z} \quad (C-10)$$

$$Y_2(z) = -\frac{4}{\pi z^2} \quad (C-11)$$

$$H_0(z) = \frac{2i}{\pi} \ln(z) + 1 \quad (C-12)$$

$$H_1(z) = -\frac{2i}{\pi z} \quad (C-13)$$

$$H_2(z) = -\frac{4i}{\pi z^2} \quad (C-14)$$

On the surface S^*

$$r = \epsilon, \quad dS = \epsilon d\theta, \quad dn = -dr,$$

$$\frac{\partial r}{\partial x_1} = \cos \theta, \quad \frac{\partial r}{\partial x_2} = \sin \theta,$$

$$n_1 = -\cos \theta, \quad n_2 = -\sin \theta$$

Substituting relationships (C-6) through (C-14) in (B-4), one obtains

$$\lim_{r \rightarrow 0} \chi = -\frac{4ic_2^2 c_1^2}{c_1^2 \pi \omega^2 r^2} + \frac{4ic_2^2}{\pi \omega^2 r^2} = 0.0 \quad (C-15)$$

Similarly, substitution into (B-3) gives

$$\lim_{r \rightarrow 0} \psi = \frac{2i}{\pi} \ln(k_2 r) + 1 \quad (C-16)$$

thus

$$\lim_{r \rightarrow 0} \int_{S^+} U_{jk}^* dS = \frac{i}{4G} \int_{S^+} \left\{ 1 + \frac{2i}{\pi} \ln(k_2 r) \right\} \delta_{jk} dS = 0 \quad (C-17)$$

For small values of r ($r = \epsilon$).

$$B_1 = \frac{2i}{\pi \epsilon} \quad (C-18)$$

$$B_2 = \frac{4ic_2^2}{\pi \epsilon c_1^2} - \frac{4i}{\pi \epsilon} \quad (C-19)$$

$$B_3 = \frac{2i}{\pi \epsilon} - \frac{4ic_2^2}{\pi \epsilon c_1^2} \quad (C-20)$$

By careful manipulation, it can be proved that for small values of r ($r = \epsilon$)

$$T_{11}^* = T_{22}^* = \frac{1}{2\pi \epsilon} \quad (C-21)$$

$$\int_{S^+} T_{11}^* dS = \int_{S^+} T_{22}^* dS = \frac{1}{2\pi \epsilon} \int_{\epsilon^+} \epsilon d\theta = 1 \quad (C-22)$$

also

$$\int_{S^+} T_{21}^* dS = \int_{S^+} T_{12}^* dS = 0 \quad (C-23)$$

Finally

$$I_1 = \delta_{jk} \quad (C-24)$$

$$I_2 = 0 \quad (C-25)$$

When point P is on the surface, the limits of integration are 0 and π and the integrals become

$$I_1 = 0.5\delta_{jk} \quad (C-26)$$

$$I_2 = 0 \quad (C-27)$$

Thus, the boundary integral equation become

$$C^P U_k(P) + \sum_{n=1}^N \int_{S_n} T_{jk}^*(P, Q) U_k dS = \sum_{n=1}^N \int_{S_n} U_{jk}^*(P, Q) T_k dS \quad (3.17)$$

$C^P = 1$ when point P is inside region D , and

$C^P = .5$ when point P is on the smooth boundary (26).

It is clear from Eqns. (B-1), (C-3) and (C-16) that U_{jj}^* is of order $O(\ln(r))$ for small values of r . Also, from Eqn. (C-21), it is seen that T_{jj}^* is of order $1/r$.

Appendix D

NOTATION

The following symbols are used in this thesis:

α_0 = frequency factor.

$\ddot{a}_g^x(t)$ = ground acceleration in the horizontal direction.

$\ddot{a}_g^x(\omega)$ = Fourier transform of ground acceleration in the horizontal direction.

$\ddot{a}_g^y(t)$ = ground acceleration in the vertical direction.

$\ddot{a}_g^y(\omega)$ = Fourier transform of ground acceleration in the vertical direction.

b = half width of the rigid strip footing.

b_j = body forces vector at a point in domain D .

BEM = boundary element method.

C = damping matrix of the structure.

C_{bb} = damping matrix for the structural degrees of freedom at the base.

C_c = damping matrix for the superstructure degrees of freedom.

c_1 = compression wave velocity.

c_2 = shear wave velocity.

$C_{vv}(i, j)$ = the vertical displacements at node i generated by a unit harmonic vertical load acting at node j .

$C_{hv}(i, j)$ = horizontal displacement at node i generated by a unit harmonic vertical load acting at node j .

$C_{vh}(i, j)$ = the vertical displacement at node i due to a unit harmonic horizontal load at node j .

$C_{hh}(i, j)$ = the horizontal displacement at node i due to a unit harmonic horizontal load at node j .

DFT = discrete Fourier transform.

E = modulus of elasticity.

$$\{E^u\}^T = \{M_1 \ 0 \ M_2 \ 0 \dots M_N \ 0\}$$

$$\{E^v\}^T = \{0 \ M_1 \ 0 \ M_2 \dots 0 \ M_N\}$$

$F(t)$ = nodal forces in the finite element mesh.

FEM = finite element method.

FFT = Fast Fourier transform.

G = modulus of rigidity.

$H_n(z)$ = Hankel function of the first kind of order n .

I = identity matrix.

IDFT = inverse discrete Fourier transform.

J_n = Bessel function of the first kind.

$J(\zeta)$ = Jacobian of transformation from ζ coordinates to z coordinates.

K = stiffness matrix for the structural degrees of freedom above the base.

K_{bb} = stiffness matrix for the base degrees of freedom.

K_e = stiffness matrix for the superstructure degrees of freedom.

K_{ij} = the non-dimensional stiffness of the rigid strip footing defined as the sum of the forces under the footing in direction i due to a unit displacement in direction j while all other degrees of freedom are kept fixed.

K_{ij}^a = the non-dimensional stiffness of the active footing defined as the sum of the forces under the active footing in direction i due to a unit displacement of the active footing in direction j while all other degrees of freedom under the two footings are kept fixed.

K_{ij}^p = the non-dimensional stiffness of the passive footing defined as the sum of the forces under the footing in direction i due to a unit displacement of the active footing in the direction j while all other degrees of freedom under the two footings are kept fixed.

M = mass matrix for the structural degrees of freedom above the base.

M_{bb} = mass matrix for the structural degrees of freedom of the base.

M_e = mass matrix for the superstructure degrees of freedom.

m = number of generalised mode shapes used to represent the structural response.

$M_n(\zeta)$ = the shape function.

n_k = unit vector along the outward normal to the surface S .

N_t = number of time steps in Fourier synthesis.

$P_1(i)$ = the amplitude of the horizontal force at point i under the footing.

$P_2(i)$ = the amplitude of vertical force at point i under the footing.

Q_1 = the summation of the forces in the horizontal direction under the footing.

Q_2 = the summation of the forces in the vertical direction under the footing.

$r(t)$ = vector of structural displacements relative to the free-field displacements.

$\hat{r}(\omega)$ = Fourier transform of $r(t)$.

$\dot{r}(t)$ = vector of structural velocities relative to the free-field velocity.

$\ddot{r}(t)$ = vector of structural acceleration relative to the free-field acceleration.

$\hat{\ddot{r}}(\omega)$ = Fourier transform of $\ddot{r}(t)$.

r_c^t = vector of total displacement of the superstructure.

$\dot{r}_c^t$ = vector of total velocity of the superstructure.

$\ddot{r}_c^t$ = vector of total acceleration of the superstructure.

$\ddot{r}^t$ = vector of total acceleration of the entire system.

$r_b(t)$ = vector of the structural displacements for the base degrees of freedom relative to the free-field displacements.

$\hat{r}_b(\omega)$ = Fourier transform of $r_b(t)$.

r_{b0} = vector of the quasi-static displacement due to the free field motion at the base degrees of freedom.

r_b^t = vector of the total displacement of the structural D.O.F above the base.

$r_f(t)$ = vector of displacements for the soil medium at the base degrees of freedom relative to the free-field displacements.

$\hat{r}_f(\omega)$ = Fourier transform of $r_f(t)$.

$r_s(t)$ = vector of the structural displacements for the non-base degrees of freedom relative to the free-field displacements.

$\hat{r}_s(\omega)$ = Fourier transform of $r_s(t)$.

$r_s^t(t)$ = vector of the total displacement of the superstructure degrees of freedom above the base.

r_{s0} = vector of the quasi-static displacement of the structural degrees of freedom above the base due to the free field motion.

R_1 = corrective impulse at step $N - 4$.

R_2 = corrective impulse at step $N - 3$.

$R_b(t)$ = vector of the interaction forces for the base degrees of freedom.

$\hat{R}_b(\omega)$ = Fourier transform of $R_b(t)$.

$R_e^t(t)$ = vector of external forces in the structure.

$R_f(t)$ = vector of interaction forces at the soil for the base degrees of freedom.

S = surface of the domain D .

$T_j(Q)$ = the traction vector on the surface S of the domain D at point Q .

T_{jk}^* = the traction component in the direction k at point Q due to a unit concentrated force acting at point P in the direction j .

$\underline{t}^m$ = the traction vector on a surface element with normal vector n corresponding to an elastodynamic state m .

T_p = duration time of the earthquake.

$U_j(Q)$ = the displacement vector on the surface S of the domain D at point Q .

U_{jk}^* = the displacement component in the direction k at point Q due to a unit concentrated force acting at point P in the direction j .

$\underline{u}^m$ = the displacement vector on a surface element with normal vector n corresponding to an elastodynamic state m .

V_1 = amplitude of the horizontal displacement of a rigid strip footing.

V_2 = amplitude of the vertical displacement of a rigid strip footing.

V_3 = amplitude of the rocking displacement of a rigid strip footing.

V_1^a = amplitude of the horizontal displacement of the active footing.

V_2^a = amplitude of the vertical displacement of the active footing.

V_3^a = amplitude of the rocking displacement of the active footing.

$x^*(t)$ = corrective excitation history.

$X_f(\omega)$ = foundation dynamic stiffness matrix.

$Y_n(x)$ = Bessel function of the second kind of order n .

$Y(t)$ = vector of the generalized coordinates of the structure.

$\hat{Y}(\omega)$ = Fourier transform of $Y(t)$.

$Y_n(t)$ = response in the n^{th} generalised mode.

$\hat{Y}_n(\omega)$ = Fourier transform of $Y_n(t)$.

$Y_n^P(t)$ = periodic response in the n^{th} generalized mode.

$Z_f(\omega)$ = flexibility matrix of the foundation.

α = the amplitude of the rocking angle.

δ_{jk} = Kroncker delta.

$\Delta\omega$ = frequency step.

Δt = time step.

ρ = mass density of the medium.

ω = the exciting frequency.

ω_{max} = the maximum frequency.

ζ_m = damping ratio in the m^{th} mode of the system.

ϕ_n = vector of mode shape n .

Φ = matrix of normalized mode shapes.

λ^2 = diagonal matrix of the squared eigenvalues.

END

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