Langford Sequences and Their
Variants: Partitions of
 $\{1, \dots, 2m + 2\} \setminus \{k, 2m + 1\}$,
 $\{1, \dots, 2m - 1, L\}$ and
 $\{1, \dots, 2m, L\} \setminus \{2\}$ into Differences
in $\{d, \dots, d + m - 1\}$

by

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Shai Joseph Mor

2007

Abstract

First investigated in the 1950's, Skolem sequences are partitions of an interval of positions $S = \{1, 2, \dots, 2m\}$ into pairs with differences in $D = \{1, 2, \dots, m\}$. Th. Skolem formulated the necessary conditions for their existence, constructed examples, applied them to the constructions of cyclic Steiner triple systems and paved the way for further research. A number of natural generalizations have been studied since. C.D. Langford considered partitions of $\{1, 2, \dots, 2m\}$ into pairs with differences in $\{d, \dots, d + m - 1\}$, where d is not fixed. Today these partitions bear his name.

We survey Langford sequences, their generalizations and applications. We also define new generalizations: *hooked extended Langford sequences*, *stretched Langford sequences* and *hooked stretched Langford sequences*. These are partitions of $\{1, \dots, 2m + 2\} \setminus \{k, 2m + 1\}$, $\{1, \dots, 2m - 1, L\}$, and $\{1, \dots, 2m, L\} \setminus \{2\}$, respectively, into pairs with differences in $\{d, d + 1, \dots, d + m - 1\}$. We give full necessary conditions for their existence and in the instance of extended sequences we provide solutions for all but a finite number of integers m for any choice of fixed d and all admissible k . For the stretched sequences we give two construction techniques which cover a large part of the parameter space.

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Chapter 1

Introduction

Skolem and Langford sequences were first defined about fifty years ago and have since been a productive and active research area. They are of interest in their own right and for their applications, the most notable of which is the construction of cyclic combinatorial block designs. In this thesis we begin by introducing the subject, notation and terminology. We derive the basic *parity* and *density* necessary conditions and proceed with a survey of the subject including some of the central constructions.

In Chapters 3 and 4 we present our new results. We define *hooked extended Langford sequences*, *stretched Langford sequences* and *hooked stretched Langford sequences*. These are partitions of $\{1, \dots, 2m+2\} \setminus \{k, 2m+1\}$, $\{1, \dots, 2m-1, L\}$, and $\{1, \dots, 2m, L\} \setminus \{2\}$, respectively, into pairs with differences in $\{d, d+1, \dots, d+m-1\}$. We give full necessary conditions for their existence and in the instance of extended sequences we provide solutions for all but a finite number of m for any choice of fixed d and all admissible k . For the stretched sequences we give two construction techniques which cover a large part of the parameter space.

Finally we end with a short chapter on the applications of Skolem and Langford sequences to the construction of combinatorial designs. We introduce the first and second Heffter problems of partitioning an interval into fixed size sets. Each of the Heffter problems is followed by a solution constructed from a Skolem sequence yielding a cyclic Steiner triple system. We also discuss constructions of class uniformly resolvable designs.

Chapter 2

Skolem and Langford Sequences and Their Variants

2.1 General necessary conditions

Throughout this thesis, if $\{a, a + 1, \dots, b\}$ is an interval of integers, we will use the notation $[a, b]$. Since we will often be concerned about integers of a particular parity in an interval, we use the conventions $[a, b]'' = [a, b] \cap 2\mathbb{Z}$ and $[a, b]' = [a, b] \cap (1 + 2\mathbb{Z})$. To save space we will often omit commas between the integers inside the sequences. Also for integers larger than 10 we denote $a = 10, b = 11, \dots, z = 35, A = 36, B = 37, \dots, Z = 71$.

We begin by defining a general notion of partitions into pairs with required differences:

Definition 2.1.1. *Let $P = \{p_1, \dots, p_{2n}\}$ be a set of $2n$ integers, and let $D = \{d_1, \dots, d_n\}$ be a multiset of positive integers. A **partition of P into differences in D** is a set $\Pi = \{(a_r, b_r) : 1 \leq r \leq n\}$ such that $\cup_{r=1}^n \{a_r, b_r\} = P$ and $b_r - a_r = d_r$ for each r . The elements of P are called **positions**, and the elements of D are called **differences**.*

Given P and D , the central question is whether there exists a partition of P into differences in D . Linek and Shalaby give necessary conditions for the existence of a partition into differences.

Theorem 2.1.2. *[39] Let $P = \{p_1, p_2, \dots, p_{2n}\}$ be a set of integers with $p_1 < p_2 < \dots < p_{2n}$, and let $D = \{d_1, d_2, \dots, d_n\}$ be a multiset of positive*

integers. If there exists a partition of P into differences in D , then

$$\sum_{i=1}^{2n} p_i \equiv \sum_{i=1}^n d_i \pmod{2} \quad (2.1)$$

$$\sum_{i=1}^{2n} (-1)^i p_i \leq \sum_{i=1}^n d_i \leq \sum_{i=1}^n (p_{i+n} - p_i) \quad (2.2)$$

Proof:

Let $\Pi = \{(a_r, b_r) : 1 \leq r \leq n\}$ be a partition of P into differences in D . Summing $b_r - a_r = d_r$ over $1 \leq r \leq n$ gives

$$\sum_r b_r - \sum_r a_r = \sum_r d_r,$$

which, when taken modulo 2, gives Congruence (2.1) since $\cup_{r=1}^n \{a_r, b_r\} = P$.

Because $a_r, b_r \in P$, we get the inequalities $\sum_r b_r \leq \sum_{r=n+1}^{2n} p_r$ and $\sum_r a_r \geq \sum_{r=1}^n p_r$. We take the difference of these inequalities,

$$\begin{aligned} \sum_{r=1}^n d_r &= \sum_{r=1}^n (b_r - a_r) \\ &= \sum_{r=1}^n b_r - \sum_{r=1}^n a_r \\ &\leq \sum_{r=n+1}^{2n} p_r - \sum_{r=1}^n p_r \\ &= \sum_{r=1}^n (p_{r+n} - p_r), \end{aligned}$$

yielding the upper bound of Inequality (2.2).

The lower bound of (2.2) is proved by induction on n . For $n = 1$ it is trivial, so let $n \geq 2$ and suppose the lower bound holds for $n - 1$.

Since $d_i > 0$ for each i , there is a pair $(p_1, p_k) \in \Pi$ with $p_k - p_1 = d_s$. If $p_k = p_2$, then $\Pi' = \Pi \setminus \{(p_1, p_k)\}$ is a partition of $P \setminus \{p_1, p_k\}$ into differences in $D \setminus \{d_s\}$, so by induction $\sum_{i=3}^{2n} (-1)^i p_i \leq \sum_{i \neq s} d_i$, and hence $\sum_{i=1}^{2n} (-1)^i p_i \leq \sum_{i=1}^n d_i$, as desired. If on the other hand $p_k > p_2$, then there is a pair $(p_2, p_\ell) \in \Pi$ with $p_\ell - p_2 = d_t$. If $p_\ell < p_k$, then let $d'_s = p_2 - p_1$

and $d'_t = p_k - p_\ell$. In this case $\Pi' = \Pi \setminus \{(p_1, p_k), (p_2, p_\ell)\} \cup \{(p_\ell, p_k)\}$ is a partition of $P \setminus \{p_1, p_2\}$ into differences in $D \setminus \{d_s, d_t\} \cup \{d'_t\}$, so by induction $\sum_{i=3}^{2n} (-1)^i p_i \leq d'_t + \sum_{i \neq s, t} d_i$, and so $\sum_{i=1}^{2n} (-1)^i p_i \leq d'_s + d'_t + \sum_{i \neq s, t} d_i$. However, $d'_s + d'_t < d_s + d_t$, hence $\sum_{i=1}^{2n} (-1)^i p_i < \sum_{i=1}^n d_i$, as desired. In the other case, $p_\ell > p_k$, we let $d'_s = p_2 - p_1$ and $d'_t = p_\ell - p_k$, and proceed similarly. ■

We call Congruence (2.1) the **parity condition** and Inequality (2.2) the **density condition**. The lower bound of Inequality (2.2) is seldom used. These conditions are not, unfortunately, sufficient in general.

Counterexample 2.1.3. *Let $D = \{1, 2, 6, 7\}$ and $P = [1, 8] = \{1, 2, \dots, 8\}$. All necessary conditions from Theorem 2.1.2 are satisfied, but there is no partition of P into differences in D . Indeed, any partition Π of $P = [1, 8]$ into differences in $D = \{1, 2, 6, 7\}$ must have $(1, 8) \in \Pi$, but then it is impossible to find two remaining elements from $[2, 7]$ with difference 6.*

However, G. Nordh conjectures Theorem 2.1.2 is sufficient under special circumstances when P is an interval.

Conjecture 2.1.4. *(G. Nordh, [48]) The following are necessary and sufficient for the existence of a partition of $P = [1, 2n]$ into differences in $D = \{d_1, d_2, \dots, d_n\}$ with $d_1 > d_2 > \dots > d_n$:*

1. *the number of even d_i is even,*
2. *the inequality $\sum_{i=1}^m d_i \leq m(2n - m)$ holds for each $1 \leq m \leq n$.*

This conjecture was checked by Nordh for up to twenty differences and it is also supported by known results on odd Langford sequences [34] and near Langford sequences [39]. This series of inequalities is stricter than Inequality (2.2). In the case of P being an interval these conditions are necessary.

When $P = [1, 2n]$, the parity condition is $\sum_{i=1}^n d_i \equiv n(2n + 1) \pmod{2}$; so if n is even, then the number of odd d_i is even, and if n is odd, the number of odd d_i is odd. In either case the number of even d_i is even.

The second condition in Nordh's conjecture says that there must be enough room in the set of positions to fit the m largest differences (compare Counterexample 2.1.3). That is, $d_1 \leq p_{2n} - p_1$, which is the $m = 1$ case

of the required inequalities. Similarly there must be enough room for the two largest differences:

$$\begin{aligned}
d_1 + d_2 &= (p_i - p_j) + (p_k - p_\ell) \\
&= (p_i + p_k) - (p_j + p_\ell) \\
&\leq p_{2n} + p_{2n-1} - (p_1 + p_2) \\
&= 2(2n - 2)
\end{aligned}$$

In general we have

$$\begin{aligned}
\sum_{i=1}^m d_i &\leq \left(\sum_{i=1}^m p_{2n-i+1} \right) - \left(\sum_{i=1}^m p_i \right) \\
&= m(2n - m),
\end{aligned} \tag{2.3}$$

for $1 \leq m \leq n$.

G. Nordh's conjecture motivates the definition of a perfect set. Simpson also implicitly refers to this concept by using the term "perfect Langford sequence" [59].

Definition 2.1.5. *A finite multiset $S \subset \mathbb{Z}$ is **perfect** if it contains an even number of even integers.*

Note also that a perfect set can only be partitioned into either two perfect sets or into two non-perfect sets, while a set that is not perfect can be partitioned only into one perfect set and one non-perfect set. Note also that when an integer interval is extended or shortened by exactly two elements (for instance $[a, b]$ extended to $[a - 2, b]$), then among these two is necessarily a single even difference, so the perfect/non-perfect status of the interval is flipped. We use both these facts in the proof of one of our main results in Section 3.6.

2.2 Skolem and Langford sequences

We start by defining a Skolem sequence.

Definition 2.2.1. *A Skolem sequence of order n is an integer sequence $S_n = s_1, s_2, \dots, s_{2n}$ that satisfies the following:*

1. for every $k \in \{1, 2, \dots, n\}$ there exist exactly two terms s_i, s_j such that $s_i = s_j = k$, and
2. for every $k \in \{1, 2, \dots, n\}$, if $s_i = s_j = k$, $i < j$, then $j - i = k$.

This is equivalent to a partition of $P = [1, 2n]$ into differences in $D = [1, n]$.

We prefer the notation $\mathcal{S}_n = s_1, s_2, \dots, s_{2n}$ because it is very convenient for the description of various concatenations that occur later. For a given n there are typically a multitude of different Skolem sequences of order n , but here the "=" means that the (specific) sequence on the right is *an example* of the type of sequence indicated on the left, not the unique example.

Example 2.2.2. *The sequence $\mathcal{S}_1 = 1, 1$ is the unique Skolem sequence of order 1. It corresponds to the partition of the set of positions $P = [1, 2]$ into differences in $D = \{1\}$; note that $\Pi = \{(1, 2)\}$.*

Example 2.2.3. *The sequence $\mathcal{S}_5 = 3, 4, 5, 3, 2, 4, 2, 5, 1, 1$ is a Skolem sequence of order 5. It corresponds to the partition of the set of positions $P = [1, 10]$ into differences in $D = [1, 5]$; note that $\Pi = \{(1, 4), (2, 6), (3, 8), (5, 7), (9, 10)\}$.*

Skolem, in his seminal 1957 paper, wrote about systems of pairings of integers [60]. The usage of a sequence notation is a modern development. Skolem was also interested in studying infinite partitions into differences. He defined a $m, +n$ system for $0 \leq m, n$ to be an infinite partition of the natural numbers such that the first difference is m and every subsequent pair has a difference increased by n . He constructed a $1, +1$ system by taking the first pair to be $(1, 2)$, then the n -th pair (a_n, b_n) is recursively built by letting a_n be the least integer that is not among the previous a_r and b_r , $1 \leq r \leq n - 1$. This is guaranteed to always work and never collide because this cover is built in order of successively increasing differences. Today these systems of "arithmetic" coverings that stretch out over all of the naturals are called **Beatty sequences** and for these we refer the reader to the work of Eggleton, Fraenkel and Simpson [18].

Skolem gave the necessary and sufficient conditions for the existence of the finite sequences named after him.

Theorem 2.2.4. [60] *For a Skolem sequence of order n (a $\mathcal{S}_n$) to exist it is necessary and sufficient that $n \equiv 0, 1 \pmod{4}$.*

Proof:

If a $\mathcal{S}_n$ exists, then there exists a partition of $P = [1, 2n]$ into differences in $D = [1, n]$. By the parity condition (2.1) of Theorem 2.1.2 we get $n(2n+1) \equiv n(n+1)/2 \pmod{2}$. However, if $n = 4k+2$ or $n = 4k+3$, then $n(2n+1)$ and $n(n+1)/2$ have opposite parity, hence $n = 4k$ or $4k+1$.

For sufficiency when $n \equiv 0 \pmod{4}$, let $n = 4m$. Π is composed from the following pairs:

1. all pairs $(4m+r, 8m-r)$ for $r = 0, 1, \dots, 2m-1$,
2. the pairs $(2m+1, 6m)$ and $(2m, 4m-1)$,
3. the pairs $(r, 4m-1-r)$ for $r = 1, 2, \dots, m-1$,
4. the pair $(m, m+1)$,
5. the pairs $(m+2+r, 3m-1-r)$ for $r = 0, 1, \dots, m-3$.

For $n \equiv 1 \pmod{4}$, let $n = 4m+1$ and Π is:

1. all pairs $(4m+2+r, 8m+2-r)$ for $r = 0, 1, \dots, 2m-1$,
2. the pairs $(2m+1, 6m+2)$ and $(2m+2, 4m+1)$,
3. the pairs $(r, 4m+1-r)$ for $r = 1, 2, \dots, m$,
4. the pair $(m+1, m+2)$,
5. the pairs $(m+2+r, 3m+1-r)$ for $r = 1, 2, \dots, m-2$.

■

Note that each line in Skolem's constructions either contains one or two pairs, or many pairs described by a parameter, r , ranging in value from 0 or 1 up to some linear function of the order m .

These are not the only solutions for each n . As we increase the order n , the number of Skolem sequences exhibits a combinatorial explosion, i.e., the number of different sequences as a function of the order grows very fast. Miller supplies an online, work-in-progress table where one can see the most recent numbers of distinct Skolem sequences of different orders [40].

Skolem also conjectured that for the remaining values of the order n , a slightly more relaxed partition exists, where one position does not occur in

a pair. This paved the way for a similar definition of a relaxed variant of $\mathcal{S}_n$ over the complementary range of n modulo 4.

Definition 2.2.5. *Given an integer sequence $s_1, s_2, \dots, s_\ell$, if $s_2 = 0$, then the position $p = 2$ is called a **left hook**, and if $s_{\ell-1} = 0$, then the position $p = \ell - 1$ is called a **right hook**.*

It is natural to speak of a sequence as being “hooked” if some position in the sequence is a hook.

Definition 2.2.6. *A **right-hooked Skolem sequence of order n** is an integer sequence $h\mathcal{S}_n = s_1, s_2, \dots, s_{2n+1}$ that satisfies the conditions (1) and (2) for a Skolem sequence (see Definition 2.2.1), and in addition satisfies (3) $s_{2n} = 0$.*

In the literature an underscore, “_”, or a star, “*”, is sometimes used instead of a zero to indicate the hook. Some authors refer to any zero-term (or the position that it occupies) as a hook. Also in the literature a right-hooked Skolem sequence of order n is simply called a hooked Skolem sequence, but we make the distinction here because later on we will be considering sequences with left hooks.

Remark 2.2.7. *A $h\mathcal{S}_n$ corresponds to a partition of $P = [1, 2n + 1] \setminus \{2n\}$ into differences in $D = [1, n]$. The correspondence is entirely similar to that between sequences and partitions given after Definition 2.2.1, provided that when the sequence is recovered from the partition, a zero is inserted at position $p = 2n$.*

Some examples help to make this definition concrete.

Example 2.2.8. *The sequence $h\mathcal{S}_3 = 1, 1, 2, 3, 2, 0, 3$ is a right-hooked Skolem sequence of order 3. It corresponds to the partition $\Pi = \{(1, 2), (3, 5), (4, 7)\}$ of the set of positions $P = [1, 7] \setminus \{6\}$ into differences in $D = [1, 3]$.*

Example 2.2.9. *The sequence $h\mathcal{S}_6 = 3, 1, 1, 3, 4, 5, 6, 2, 4, 2, 5, 0, 6$ is a right-hooked Skolem sequence of order 6. It corresponds to the partition $\Pi = \{(1, 4), (2, 3), (5, 9), (6, 11), (7, 13), (8, 10)\}$ of the set of positions $P = [1, 13] \setminus \{12\}$ into differences in $D = [1, 6]$.*

O’Keefe in 1958 established the existence of right-hooked Skolem sequences by constructing one for each order $n \equiv 2, 3 \pmod{4}$.

positions:	1	2	3	4	5	6	7	8	9
$\mathcal{S}_4(1)$	0	4	2	3	2	4	3	1	1
$\mathcal{S}_4(3)$	1	1	0	4	2	3	2	4	3
$\mathcal{S}_4(5)$	2	4	2	3	0	4	3	1	1
$\mathcal{S}_4(7)$	4	2	3	2	4	3	0	1	1
$\mathcal{S}_4(9)$	4	2	3	2	4	3	1	1	0

Table 2.1: k -extended Skolem sequences for $k \in \{1, 3, 5, 7, 9\}$.

Theorem 2.2.10. [50] *For a right-hooked Skolem sequence of order n ($h\mathcal{S}_n$) to exist it is necessary and sufficient that $n \equiv 2, 3 \pmod{4}$.*

The following is a generalization of a hooked sequence, as there is an independent variable introduced.

Definition 2.2.11. *A k -extended Skolem sequence of order n is an integer sequence $\mathcal{S}_n(k) = s_1, s_2, \dots, s_{2n+1}$ that satisfies conditions (1) and (2) for a Skolem sequence, and in addition satisfies (3) $s_k = 0$.*

Example 2.2.12. *Table 2.1 gives five examples of k -extended Skolem sequences of order four, $\mathcal{S}_4(k)$, for all values of $k \in \{1, 3, 5, 7, 9\}$. The given sequence $\mathcal{S}_4(3)$ corresponds to the partition $\Pi = \{(1, 2), (4, 8), (5, 7), (6, 9)\}$ of $P = [1, 9] \setminus \{3\}$ into differences in $D = [1, 4]$.*

A right-hooked Skolem sequence is then just a special case of a k -extended Skolem sequence. Indeed, a right-hooked Skolem sequence of order n is just a $2n$ -extended Skolem sequence of order n , and the notations $h\mathcal{S}_n$ and $\mathcal{S}_n(2n)$ are interchangeable. Baker in 1995 proved the sufficiency of the necessary conditions for the k -extended Skolem sequences [4]. The notation $(u, v) \equiv (a, b) \pmod{(4, 2)}$, used below, means that $u \equiv a \pmod{4}$ and $v \equiv b \pmod{2}$.

Theorem 2.2.13. [4] *The following are necessary and sufficient for the existence of a k -extended Skolem sequence of order n , $\mathcal{S}_n(k)$:*

1. $k \in [1, 2n + 1]$,
2. $(n, k) \equiv (0, 1), (1, 1), (2, 0), \text{ or } (3, 0) \pmod{(4, 2)}$.

Finally, these two variants can be combined.

Definition 2.2.14. A right-hooked k -extended Skolem sequence of order n is an integer sequence $h\mathcal{S}_n(k) = (s_1, s_2, \dots, s_{2n+2})$ that satisfies conditions (1) and (2) for a Skolem sequence, and in addition satisfies (3) $s_k = s_{2n+1} = 0$, $k \neq 2n + 1$.

Remark 2.2.15. A $h\mathcal{S}_n(k)$ corresponds to a partition of $P = [1, 2n + 2] \setminus \{k, 2n + 1\}$ into differences in $D = [1, n]$.

Example 2.2.16. The sequence $h\mathcal{S}_6(3) = 1, 1, 0, 4, 5, 6, 2, 4, 2, 5, 3, 6, 0, 3$ is a right-hooked 3-extended Skolem sequence of order 6. It corresponds to the partition $\Pi = \{(1, 2), (4, 8), (5, 10), (6, 12), (7, 9), (11, 14)\}$ of $P = [1, 14] \setminus \{3, 13\}$ into differences in $D = [1, 6]$.

Example 2.2.17. The sequence $h\mathcal{S}_6(5) = 3, 1, 1, 3, 0, 4, 5, 6, 2, 4, 2, 5, 0, 6$ is a right-hooked 5-extended Skolem sequence of order 6. It corresponds to the partition $\Pi = \{(1, 4), (2, 3), (6, 10), (7, 12), (8, 14), (9, 11)\}$ of $P = [1, 14] \setminus \{5, 13\}$ into differences in $D = [1, 6]$.

Linek and Jiang in 1999 established the existence of these sequences [37].

Theorem 2.2.18. [37] The following are necessary and sufficient for the existence of a right-hooked k -extended Skolem sequence of order n , $h\mathcal{S}_n(k)$:

1. $k \in [1, 2n + 2] \setminus \{2n + 1\}$,
2. $(n, k) \equiv (0, 0), (1, 0), (2, 1), \text{ or } (3, 1) \pmod{(4, 2)}$, and
3. $(n, k) \neq (1, 2)$.

Results for these sequences are summarized in Table 2.2 as adopted from the recent work of Linek and Shalaby [39].

Another natural generalization is to relax the shape of the difference set D and allow it to begin from an arbitrary number.

Definition 2.2.19. A Langford sequence of defect d and with m differences is an integer sequence $\mathcal{L}_d^m = s_1, s_2, \dots, s_{2m}$ that satisfies the following: (1) for every $k \in \{d, \dots, d + m - 1\}$ there exist exactly two terms s_i, s_j such that $s_i = s_j = k$, and (2) if $s_i = s_j = k$, $i < j$, then $j - i = k$.

Sequence	Positions	Diffs	Necessity	Sufficiency
$\mathcal{S}_n$	$[1, 2n]$	$[1, n]$	$n \equiv 0, 1 \pmod{4}$	[60]
$h\mathcal{S}_n$	$[1, 2n+1] \setminus \{2n\}$	$[1, n]$	$n \equiv 2, 3 \pmod{4}$	[50]
$\mathcal{S}_n(k)$	$[1, 2n+1] \setminus \{k\}$	$[1, n]$	$(n, k) \equiv (0, 1), (1, 1)$ $(2, 0), (3, 0) \pmod{4, 2}$	[4]
$h\mathcal{S}_n(k)$	$[1, 2n+2] \setminus \{2n+1, k\}$	$[1, n]$	$(n, k) \equiv (0, 0), (1, 0)$ $(2, 1), (3, 1) \pmod{4, 2}$ $(n, k) \neq (1, 2)$	[37]

Table 2.2: Skolem sequences and their variants.

In the literature the number of differences, m , is sometimes called the **length** of the Langford sequence, but we will not use this terminology because it can be confused with the length of the sequence. A Langford sequence $\mathcal{L}_d^m = s_1, s_2, \dots, s_{2m}$ is equivalent to a partition of $P = [1, 2m]$ into differences in $D = [d, d + m - 1]$.

Example 2.2.20. *The sequence $\mathcal{L}_2^4 = 3, 4, 5, 3, 2, 4, 2, 5$ is a Langford sequence of defect 2 and 4 differences. It corresponds to the partition $\Pi = \{(1, 4), (2, 6), (3, 8), (5, 7)\}$ of $P = [1, 8]$ into differences in $D = [2, 5]$.*

Example 2.2.21. *The sequence*

$$\mathcal{L}_5^9 = 10, 11, 12, 13, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}, \mathbf{5}, 10, \mathbf{6}, 11, \mathbf{7}, 12, \mathbf{8}, 13, \mathbf{9}$$

is a Langford sequence of defect 5 and 9 differences. It corresponds to the partition $\Pi = \{(1, 11), (2, 13), (3, 15), (4, 17), (5, 10), (6, 12), (7, 14), (8, 16), (9, 18)\}$ of $P = [1, 18]$ into differences in $D = [5, 13]$.

This example has a particularly nice structure. It is made of two parts, distinguished by boldface type and regular type. The two parts fit together like the jaws of a crocodile. These structures generalize to any defect $d \geq 1$; the corresponding partition is given in Table 2.3. Similar tables will be introduced in the next chapter, but there the corresponding sequences consist of four interlocking “jaws” rather than just two, with some sporadic differences appearing.

In 1958 C. Dudley Langford posed the problem that bears his name for $d = 2$ and Priddy solved the problem for all $m \equiv 0, 1 \pmod{4}$.

	a_r	b_r	$b_r - a_r$	
(1)	$\mathbf{d+j}$	$\mathbf{2d+2j}$	$\mathbf{d+j}$	$0 \leq j \leq d-1$
(2)	$1+j$	$2d+1+2j$	$2d+j$	$0 \leq j \leq d-2$

Table 2.3: A 1-parameter family of Langford sequences of type $\mathcal{L}_d^{2d-1}$.

Theorem 2.2.22. [32, 54] *For a Langford sequence of defect 2 and m differences (a $\mathcal{L}_2^m$) to exist it is necessary and sufficient that $m \equiv 0, 3 \pmod{4}$*

In 1976 Bermond et al. provided Langford sequences for some of the cases with $d > 2$ [7]. Simpson followed in 1983 with a paper that completely resolved all the remaining cases.

Theorem 2.2.23. [7, 59] *The following are necessary and sufficient for the existence of a Langford sequence of defect d and m differences, $\mathcal{L}_d^m$:*

1. $m \geq 2d - 1$,
2. $(m, d) \equiv (0, 0), (0, 1), (1, 1), \text{ or } (3, 0) \pmod{(4, 2)}$.

As with Skolem sequences it is also natural to speak of Langford sequences as being “hooked.”

Definition 2.2.24. *A right-hooked Langford sequence of defect d and m differences is an integer sequence $h\mathcal{L}_d^m = s_1, s_2, \dots, s_{2m+1}$ that satisfies the conditions (1) and (2) for a Langford sequence (see Definition 2.2.19), and in addition satisfies (3) $s_{2m} = 0$.*

Remark 2.2.25. *A right-hooked Langford sequence of defect d and m differences corresponds to a partition of $P = [1, 2m+1] \setminus \{2m\}$ into differences in $D = [d, d+m-1]$.*

Left hooks are defined correspondingly.

Example 2.2.26. *The sequence $h\mathcal{L}_2^2 = 2, 3, 2, 0, 3$ is a right-hooked Langford sequence of defect 2 and 2 differences. It corresponds to the partition $\Pi = \{(1, 3), (2, 5)\}$.*

Note that a Langford sequence of defect 1 is a Skolem sequence, so $\mathcal{L}_1^m = \mathcal{S}_m$ and $h\mathcal{L}_1^m = h\mathcal{S}_m$, and the same comment applies to the other variants of Langford sequences. Simpson completely resolved the existence problem for $h\mathcal{L}_d^m$ by working with (m, d) modulo $(4, 4)$ and providing tables for the corresponding partitions.

Theorem 2.2.27. [59] *The following are necessary and sufficient for the existence of a right-hooked Langford sequence of defect d and m differences, $h\mathcal{L}_d^m$:*

1. $m(m + 1 - 2d) + 2 \geq 0$,
2. $(m, d) \equiv (1, 0), (2, 0), (2, 1), \text{ or } (3, 1) \pmod{(4, 2)}$.

We can also generalize hooked Langford sequences to include arbitrary extensions.

Definition 2.2.28. *A k -extended Langford sequence of defect d and m differences is an integer sequence $\mathcal{L}_d^m(k) = s_1, s_2, \dots, s_{2m+1}$ that satisfies conditions (1) and (2) for a Langford sequence, and in addition satisfies (3) $s_k = 0$.*

Remark 2.2.29. *A k -extended Langford sequence of defect d and m differences corresponds to a partition of $P = [1, 2m + 1] \setminus \{k\}$ into differences in $D = [d, d + m - 1]$.*

Example 2.2.30. *The sequence $\mathcal{L}_4^6(4) = 9, 6, 8, 0, 4, 7, 5, 6, 4, 9, 8, 5, 7$ is a 4-extended Langford sequence of defect 4 and 6 differences. It corresponds to the partition $\Pi = \{(1, 10), (2, 8), (3, 11), (5, 9), (6, 13), (7, 12)\}$ of $P = [1, 13] \setminus \{4\}$ into differences in $D = [4, 9]$.*

Another example is $\mathcal{L}_4^6(6) = 9, 7, 4, 8, 6, 0, 4, 5, 7, 9, 6, 8, 5$; a 6-extended Langford sequence of defect 4 and 6 differences. It corresponds to the partition $\Pi = \{(1, 10), (2, 9), (3, 7), (4, 12), (5, 11), (8, 13)\}$ of $P = [1, 13] \setminus \{6\}$ into differences in $D = [4, 9]$.

Linek and Jiang were the first to approach the problem of constructing extended Langford sequences for small defects, namely $d = 2, 3$ [36]. Linek and Mor in 2004 completely solved the problem for $d = 4$, and for all but a finite number of m for all other d .

Theorem 2.2.31. [38] *The following are sufficient for the existence of a k -extended Langford sequence of defect d and m differences, $\mathcal{L}_d^m(k)$:*

1. $m \geq 2d - 1$ and $m \notin [2d + 2, 8d - 5]$,
2. $k \in [1, 2m + 1]$,
3. $(m, k) \equiv (0, 1), (1, d), (2, 0)$, or $(3, d + 1) \pmod{(4, 2)}$.

Definition 2.2.32. A right-hooked k -extended Langford sequence of defect d and m differences is an integer sequence $h\mathcal{L}_d^m(k) = s_1, s_2, \dots, s_{2m+2}$ that satisfies the conditions (1) and (2) for a Langford sequence, and in addition satisfies (3) $s_k = s_{2m+1} = 0, k \neq 2m + 1$.

Remark 2.2.33. A right-hooked k -extended Langford sequence of defect d and m differences corresponds to a partition of $P = [1, 2m + 2] \setminus \{k, 2m + 1\}$ into differences in $D = [d, d + m - 1]$.

Example 2.2.34. The sequence $h\mathcal{L}_3^3(3) = 5, 3, 0, 4, 3, 5, 0, 4$ is a right-hooked 3-extended Langford sequence of defect 3 and 3 differences. It corresponds to the partition $\Pi = \{(1, 6), (2, 5), (4, 8)\}$.

In 1999, Linek and Jiang settled the existence of right-hooked k -extended Langford sequences for defect 2.

Theorem 2.2.35. [36] The following are necessary and sufficient for the existence of a right-hooked k -extended Langford sequence of defect 2 and m differences, $h\mathcal{L}_2^m(k)$:

1. $(m, k) \equiv (0, 0), (1, 1), (2, 1)$, or $(3, 0) \pmod{(4, 2)}$,
2. $(m, k) \neq (2, 3), (3, 2)$.

Linek and Mor constructed right-hooked k -extended Langford sequences, $h\mathcal{L}_d^m(k)$, for $m = 2d, 2d \pm 1$ [38]. In Chapter 3, right-hooked k -extended Langford sequences are constructed for any given d , and almost all m for each d :

Theorem 3.6.7. [42] The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:

1. $m \geq 2d - 1$ and $m \notin [2d + 2, 8d + 4]$,
2. $k \in [1, 2m + 2] \setminus \{2m + 1\}$,

3. $(m, k) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}$.

In Chapter 4, we introduce a new generalization, a *stretched Langford sequence*.

Definition 2.2.36. For $L \geq 2m$, an *L -stretched Langford sequence* of defect d and m differences is an integer sequence $\mathcal{L}_d^m[L] = s_1, s_2, \dots, s_L$ that satisfies conditions (1) and (2) for a Langford sequence, and in addition satisfies (3) $s_{2m} = s_{2m+1} = \dots = s_{L-1} = 0$.

Remark 2.2.37. An L -stretched Langford sequence of defect d and m differences corresponds to a partition of $P = [1, 2m - 1] \cup \{L\}$ into differences in $D = [d, d + m - 1]$.

Example 2.2.38. The sequence $\mathcal{L}_2^3[8] = 2, 3, 2, 4, 3, 0, 0, 4$ is an 8-stretched Langford sequence of defect 2 and 3 differences. It corresponds to the partition $\Pi = \{(1, 3), (2, 5), (4, 8)\}$ of $P = [1, 5] \cup \{8\}$ into differences in $D = [2, 4]$.

Our main result on the existence of stretched Langford sequences is:

Theorem 4.1.7. Assume the necessary conditions of Lemma 4.1.1 are satisfied. Any condition below is then sufficient for the existence of an L -stretched Langford sequence of defect d and m differences, $\mathcal{L}_d^m[L]$:

1. $m = 2d - 2$ or $m = 1$;
2. $L = 2m, 2m + 1$, or $3m + d - 2$;
3. $L = 2m + 2$, $m \geq 2d - 1$, and $m \notin [2d + 2, 8d + 4]$;
4. $d = 1, 2, 3$, or 4 ;
5. $2m + 1 < L < 3m + d - 2$, $m \geq 2d$, and $m \notin [2d + 3, 2d + 5] \cup [2d + 9, 8d - 4]$;
6. $2m < L < 2m + d - 1$, $m \geq 2d + 2$, and $m \notin [2d + 5, 2d + 7] \cup [2d + 11, 8d + 4]$.

In the definition of the next variant, the hook is at the left rather than at the right to make room for the stretch.

Definition 2.2.39. For $L \geq 2m + 1$, a *left-hooked L -stretched Langford sequence* of defect d and m differences is an integer sequence $h\mathcal{L}_d^m[L] = s_1, s_2, \dots, s_L$ that satisfies conditions (1) and (2) for a Langford sequence, and in addition satisfies (3) $s_2 = s_{2m+1} = s_{2m+2} = \dots = s_{L-1} = 0$.

Remark 2.2.40. A left-hooked L -stretched Langford sequence of defect d and m differences corresponds to a partition of $P = ([1, 2m] \cup \{L\}) \setminus \{2\}$ into differences in $D = [d, d + m - 1]$.

Example 2.2.41. The sequence $h\mathcal{L}_3^2[7] = 3, 0, 4, 3, 0, 0, 4$ is a left-hooked 7-stretched Langford sequence of defect 3 and 2 differences. It corresponds to the partition $\Pi = \{(1, 4), (3, 7)\}$ of $P = [1, 4] \cup \{7\} \setminus \{2\}$ into differences in $D = [3, 4]$.

In Chapter 4, we construct left-hooked stretched Langford sequences for a very large portion of their parameter space:

Theorem 4.1.8. Assume the necessary conditions of Lemma 4.1.2 are satisfied. Any condition below is then sufficient for the existence of a left-hooked L -stretched Langford sequence of defect d and m differences, $h\mathcal{L}_d^m[L]$:

1. $m = 2d - 2$ or $m = 1$ or $(d, m, L) = (3, 2, 7)$ or $(3, 3, 9)$;
2. $L = 2m + 1$, or $3m + d - 1$;
3. $d = 1, 2$, or 3 ;
4. $2m + 1 < L < 3m + d - 1$, $m \geq 2d$, and $m \notin [2d + 3, 8d + 5]$;
5. $2m + 1 < L < 2m + d$, $m \geq 2d + 2$, and $m \notin [2d + 5, 8d + 13]$.

Table 2.4 is updated from the work of Linek and Shalaby with the addition of our recent results on stretched Langford sequences, and summarizes the current state of knowledge [39].

Sequence	Positions	Necessity	Sufficiency
$\mathcal{L}_d^m$	$[1, 2m]$	$(m, d) \equiv (0, 0), (0, 1), (1, 1), (3, 0) \pmod{(4, 2)}$.	[7], [59]
$h\mathcal{L}_d^m$	$[1, 2m + 1] \setminus \{2m\}$	$(m, d) \equiv (1, 0), (2, 0), (2, 1), (3, 1) \pmod{(4, 2)}$, $m(m + 1 - 2d) + 2 \geq 0$.	[59]
$\mathcal{L}_d^m(k)$	$[1, 2m + 1] \setminus \{k\}$	$(m, k) \equiv (0, 1), (1, d), (2, 0), (3, d + 1) \pmod{(4, 2)}$, $ k - (m + 1) \leq 1 + \frac{m}{2}(m - 2d + 3)$, $m \geq 2d - 1$, $\nexists \mathcal{L}_3^4(5)$.	$d = 2, 3$: [36] $d = 4$: [38] $m \notin [2d + 2, 8d - 5]$: [38]
$h\mathcal{L}_d^m(k)$	$[1, 2m + 2] \setminus \{k, 2m + 1\}$	$(m, k) \equiv (0, 0), (1, d + 1), (2, 1), (3, d) \pmod{(4, 2)}$, $\nexists h\mathcal{L}_2^2(3), h\mathcal{L}_2^3(2)$.	$m = 2d, 2d \pm 1$: [38] $d = 2$: [36] $m \geq 8d + 5, d \geq 1$, and more, see [42]
$\mathcal{L}_d^m[L]$	$\{1, \dots, 2m - 1, L\}$	$(m, L) \equiv (0, 0), (1, d + 1), (2, 1), (3, d) \pmod{(4, 2)}$.	$m \geq 8d - 3$ and $d \geq 1$, and more; see [43] [†]
$h\mathcal{L}_d^m[L]$	$\{1, \dots, 2m, L\} \setminus \{2\}$	$(m, L) \equiv (0, 0), (1, d + 1), (2, 1), (3, d) \pmod{(4, 2)}$.	$m \geq 8d + 6$ and $d \geq 1$, and more; see [43] [†]

[†]A partial result is given.

Table 2.4: Langford sequences and their variants.

Chapter 3

Partitions of $\{1, \dots, 2m + 2\} \setminus \{k, 2m + 1\}$ into Differences in $[d, d + m - 1]$: Right-hooked Extended Langford Sequences

In this chapter we focus on hooked extended Langford sequences. We obtain the general result by first constructing extended and hooked-extended Langford sequences for $m \in [2d + 5, 2d + 7]$.

3.1 Preliminaries, extended Langford sequences

Recall the definition of a perfect multiset:

Definition 2.1.5. *A finite multiset $S \subset \mathbb{Z}$ is **perfect** if it contains an even number of even integers.*

For example, $\emptyset, \{1\}, \{2, 3, 4\}$ and $\{2, 3, 5, 6\}$ are all perfect sets. In constructing hooked extended Langford sequences the parity of the extension k will depend on the perfect or non-perfect status of the set D .

Lemma 3.1.1. *(A) Let the set $[1, 2m+1] \setminus \{k\}$ be partitioned into the multiset of differences in $D = \{d_1, \dots, d_m\}$. If D is perfect, then k is odd; and if D is not perfect, then k is even.*

(B) Let the set $[1, 2m + 2] \setminus \{k, 2m + 1\}$ be partitioned into the multiset of differences in $D = \{d_1, \dots, d_m\}$. If D is perfect, then k is even; and if D is not perfect, then k is odd.

Proof:

(A) By the parity condition (2.1) of Theorem 2.1.2:

$$\begin{aligned} \sum_{i=1}^m d_i &\equiv \sum_{s \in [1, 2m+1] \setminus \{k\}} s \\ &\equiv (1 + 2 + \dots + (2m + 1)) - k \\ &\equiv (m + 1) - k \pmod{2}. \end{aligned}$$

If D is perfect then for even m the number of odd differences in D is even, and then the above reduces to $0 \equiv 1 - k \pmod{2}$ and k is odd. For odd m the number of odd differences in D is odd, and then the above reduces to $1 \equiv -k$ and k is odd. If D is not perfect a similar argument shows that k is even.

(B) Similarly,

$$\begin{aligned} \sum_{i=1}^m d_i &\equiv \sum_{s \in [1, 2m+2] \setminus \{k, 2m+1\}} s \\ &\equiv m - k \pmod{2}. \end{aligned}$$

Since $(m + 1) - k$ and $m - k$ have opposite parity, this case will be complementary to case (A). ■

A Langford sequence has $D = [d, d + m - 1]$ and

Lemma 3.1.2. For $m \geq 1$ the set $[d, d + m - 1]$ is perfect if and only if $(m, d) \equiv (0, 0), (0, 1), (1, 1)$, or $(3, 0) \pmod{(4, 2)}$.

Proof:

We write $m = 4t + r$, $0 \leq r \leq 3$, and partition the set $[d, d + m - 1]$ into $R_r = \{d, d + 1, \dots, d + r - 1\}$ and t intervals of length 4. Every interval of length 4 is perfect, regardless of whether it begins with an odd or an even integer. Since R_r is of size at most 3 we have that R_r is perfect if and only if R_r

has 0 or 2 even integers. These cases are exhausted by $R_0 = \emptyset$, $R_1 = \{2s+1\}$ and $R_3 = \{2t, 2t+1, 2t+2\}$, where s, t are integers. ■

Correspondingly, $D = [d, d+m-1]$ is not perfect if and only if $(m, d) \equiv (1, 0), (2, 0), (2, 1)$, or $(3, 1) \pmod{(4, 2)}$.

We now review the necessary conditions for the existence of a k -extended Langford sequence.

Theorem 3.1.3. [36], [38] *The following are necessary for the existence of a k -extended Langford sequence of defect d and m differences, $\mathcal{L}_d^m(k)$:*

1. $m \geq 2d - 3$,
2. $k \in [1, 2m+1]$,
3. $\frac{m}{2}(2d-1-m) + 1 \leq k \leq \frac{m}{2}(m-2d+5) + 1$,
4. $(m, k) \equiv (0, 1), (1, d), (2, 0)$, or $(3, d+1) \pmod{(4, 2)}$.

The necessary conditions for the existence of a right-hooked k -extended Langford sequence are somewhat similar.

Theorem 3.1.4. *The following are necessary for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m + 2/m \geq 2d - 3$,
2. $k \in [1, 2m+2] \setminus \{2m+1\}$,
3. $\frac{m}{2}(2d-1-m) \leq k \leq \frac{m}{2}(m-2d+5) + 2$,
4. $(m, k) \equiv (0, 0), (1, d+1), (2, 1)$, or $(3, d) \pmod{(4, 2)}$.

Proof:

A $h\mathcal{L}_d^m(k)$ is equivalent to a partition of $P = [1, 2m+2] \setminus \{k, 2m+1\}$ into differences in $D = [d, d+m-1]$. We have that

$$\begin{aligned} \sum_{i=1}^{2m} p_i &= \left(\sum_{i=1}^{2m+2} i \right) - k - (2m+1) \\ &= (m+1)(2m+3) - k - (2m+1) \\ &= (m+1)(2m+2) - k - m, \end{aligned}$$

$$\begin{aligned}\sum_{i=1}^m d_i &= \sum_{i=1}^m (d + i - 1) \\ &= md + (m - 1)m/2.\end{aligned}$$

The parity condition (2.1) says that $\sum_{i=1}^{2m} p_i \equiv \sum_{i=1}^m d_i \pmod{2}$, so plugging in the above we get $k \equiv md + (m + 1)m/2 \pmod{2}$, and the modular conditions on m and k follow.

Note that since $m > 0$ we have

$$m + 2/m \geq 2d - 3 \Leftrightarrow (m/2)(2d - 1 - m) \leq m + 1, \quad (3.1)$$

$$m + 2/m \geq 2d - 3 \Leftrightarrow m + 1 \leq (m/2)(m - 2d + 5) + 2. \quad (3.2)$$

If $k \geq m + 1$, the density condition (2.2) becomes

$$\begin{aligned}\sum_{i=1}^m d_i &= (m/2)(2d + m - 1) \\ &\leq [((m + 1) + \cdots + (2m + 2)) - k - (2m + 1)] - (1 + \cdots + m) \\ &= m^2 + 2m + 2 - k,\end{aligned}$$

hence $k \leq (m/2)(m - 2d + 5) + 2$. This implies that $m + 1 \leq (m/2)(m - 2d + 5) + 2$ which by (3.1) and (3.2) are equivalent to $m + 2/m \geq 2d - 3$ and $(m/2)(2d - 1 - m) \leq m + 1$. These combined yield $(m/2)(2d - 1 - m) \leq k \leq (m/2)(m - 2d + 5) + 2$.

Similarly, suppose that $k \leq m + 1$. Then the density condition (2.2) becomes

$$\begin{aligned}\sum_{i=1}^m d_i &= (m/2)(2d + m - 1) \\ &\leq [((m + 2) + \cdots + (2m + 2)) - (2m + 1)] - [(1 + \cdots + (m + 1)) - k] \\ &= m^2 + k,\end{aligned}$$

hence $k \geq (m/2)(2d - 1 - m)$. By (3.1) we have $m + 2/m \geq 2d - 3$, and then by (3.2) we have $m + 1 \leq (m/2)(m - 2d + 5)$. Then $(m/2)(2d - 1 - m) \leq k \leq (m/2)(m - 2d + 5) + 2$ follows. ■

If a $h\mathcal{L}_d^m(k)$ exists, then by Theorem 3.1.4 we have $m \geq 2d - 3 - (2/m) \geq 2d - 5$, so for small values of m we have

- If $m = 2d - 5$, then $(d, m, k) = (3, 1, 2)$,
- If $m = 2d - 4$, then $(d, m, k) = (3, 2, 3)$,
- If $m = 2d - 3$, then $2d - 3 \leq k \leq 2d - 1$,
- If $m = 2d - 2$, then $d - 1 \leq k \leq 3d - 1$.

The position of the k -extension is therefore affected by both parity and density when $m \leq 2d - 2$, while only parity affects the position of k when $m \geq 2d - 1$, as then the density condition is trivially satisfied for any $k \in [1, 2m + 2]$.

It is therefore convenient to write $D_i = [d, 3d - 2 + i]$ for $i \geq -4$. When $m = 2d - 5$, the only $h\mathcal{L}_d^m(k)$ with differences in D_{-4} is $h\mathcal{L}_3^1(2) = 3_3$. When $m = 2d - 4$, the only $h\mathcal{L}_d^m(k)$ with differences in D_{-3} is $h\mathcal{L}_3^2(3) = 34_3_4$. When $m = 2d - 3$, the differences are D_{-2} , and the parity consideration of Theorem 3.1.4 shows that $k = 2d - 3$, or $2d - 1$. A family of $h\mathcal{L}_d^{2d-3}(2d-1)$ for $d \geq 3$ is 3453_4_5, 567845_647_8, 789ab**567**_8**596**a_b, 9abcde**6789**_a**6b7c8d**_e, ..., and a family of $h\mathcal{L}_d^{2d-3}(2d-3)$ for $d \geq 2$ is **2_2**, 53_435_4, 756**8**_4576**4**_8, b9758**a**_6579b**86**_a, db97ce8**a**_679bd**86ca**_e, ... (nested bold evens are congruent mod 4 to the largest even difference, the remaining nested evens are congruent mod 4 to the next largest even difference). The case D_{-1} resists our methods, but $h\mathcal{L}_d^m(k)$ with differences D_0 , D_1 and D_2 are constructed by Linek and Mor [38].

The technique we have used to make our sequences involves *pivoting* a difference into a hook or the edge of the sequence to obtain a new sequence. For example, consider the following Langford sequence which has $d = 4$ and $m = 7$: 89a456748596a7. Any underlined symbol can be *pivoted*, that is, rewritten in position 1 so it replaces the null, “_”, thereby giving an extended Langford sequence with an extension at any one of the positions $k = 9, 11, 13$ or 15. Reversing these sequences gives extended Langford sequences with the remaining k -extensions $k = 1, 3, 5$ and 7. Pivoting depends on the placement of two “jaws” that fit together precisely: the 89a_____8_9_a and 45674_5_6_7. We use this method to produce almost all extended and hooked extended Langford sequences with D_6 , D_7 and D_8 . In each case we give tables with the necessary partitions of P into differences in D .

3.2 Difference set $D_6 = [d, 3d + 4], m = 2d + 5$

A $\mathcal{L}_d^{2d+5}(k)$ is equivalent to a partition of positions $P = [1, 4d + 11]$ into differences in $D_6 = [d, 3d + 4]$. Note that D_6 is not a perfect difference set, so k is necessarily even.

Lemma 3.2.1. *The following are sufficient for the existence of a k -extended Langford sequence of defect d and m differences, $\mathcal{L}_d^m(k)$:*

1. $m = 2d + 5$ and $d \geq 2$,
2. $k \in [1, 2m + 1]$,
3. $(m, k) \equiv (1, d)$ or $(3, d + 1) \pmod{(4, 2)}$.

Proof:

For even defects the union of the ranges of extensions in Table 3.1 is $[2d + 6, 4d + 10]''$. For odd defects the union of the ranges of extensions in Table 3.2 is $[2d + 6, 4d + 10]''$. By the symmetry of these sequences, the extension at position $4d + 10$ is provided by the extension at position 2. Reversing the sequences corresponding to Table 3.1 and Table 3.2, we obtain all extensions in $[1, 4d + 11]''$, for each parity of d . ■

We now construct right-hooked k -extended sequences for D_6 , $h\mathcal{L}_d^{2d+5}(k)$, $k \geq m + 2 = 2d + 7$.

Lemma 3.2.2. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $\mathcal{L}_d^m(k)$:*

1. $m = 2d + 5$ and $d \geq 2$,
2. $k \in [m + 2, 2m + 2] \setminus \{2m + 1\}$,
3. $(m, k) \equiv (1, d + 1)$ or $(3, d) \pmod{(4, 2)}$.

Proof:

For even defects the union of the ranges of extensions in Table 3.3 is $[2d + 7, 4d + 9]'$. For odd defects the union of the ranges of extensions in Table 3.4 is $[2d + 7, 4d + 9]'$. Therefore, on the position set, $[2d + 7, 4d + 9]$, for every d , we get every possible extension in the second half of P . ■

	a_i	b_i	$ b_i - a_i $	
(1)	1	$3d/2 + 5$	$3d/2 + 4$	–
(2)	2	2	0	–
(3)	3	$3d/2 + 6$	$3d/2 + 3$	–
(4)	$4 + j$	$2d + 9 + 2j$	$2d + 5 + j$	$0 \leq j \leq d - 1$
(5)	$d + 4 + j$	$2d + 6 + 2j$ <u>or</u> 2	$d + 2 + j$	$0 \leq j \leq d/2$
(6)	$3d/2 + 7 + j$	$3d + 12 + 2j$ <u>or</u> 2	$3d/2 + 5 + j$	$0 \leq j \leq d/2 - 2$
(7)	$2d + 7$	$4d + 11$	$2d + 4$	–
(8)	$3d + 8$	$4d + 9$	$d + 1$	–
(9)	$3d + 10$	$4d + 10$	d	–

	a_i	b_i	$ b_i - a_i $	
(1)	1	$d + 4$	$d + 3$	–
(2)	2	2	0	–
(3)	3	$d + 7$	$d + 4$	–
(4)	$4 + j$	$2d + 9 + 2j$	$2d + 5 + j$	$0 \leq j \leq d/2 - 2$
(5)	$d/2 + 3$	$2d + 8$	$3d/2 + 5$	–
(6)	$d/2 + 4 + j$	$3d + 9 + 2j$	$5d/2 + 5 + j$	$0 \leq j \leq d/2 - 1$
(7)	$d + 5$	$2d + 10$	$d + 5$	–
(8)	$d + 6$	$2d + 7$	$d + 1$	–
(9)	$d + 8 + j$	$2d + 14 + 2j$ <u>or</u> 2	$d + 6 + j$	$0 \leq j \leq d/2 - 2$
(10)	$3d/2 + 7$	$4d + 11$	$5d/2 + 4$	–
(11)	$3d/2 + 8 + j$	$3d + 14 + 2j$ <u>or</u> 2	$3d/2 + 6 + j$	$0 \leq j \leq d/2 - 2$
(12)	$2d + 12$	$3d + 12$	d	–
(13)	$3d + 7$	$4d + 9$	$d + 2$	–

Table 3.1: Extended Langford sequences: $m = 2d + 5$, defect is even, extensions ranges: $[2d + 6, 4d + 8]'' \setminus \{3d + 8, 3d + 10\}$ and $[2d + 14, 4d + 10]'' \setminus \{3d + 12\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	$(3d+9)/2$	$(3d+7)/2$	—
(2)	2	$d+3$	$d+1$	—
(3)	3	$(3d+11)/2$	$(3d+5)/2$	—
(4)	$4+j$	$2d+9+2j$	$2d+5+j$	$0 \leq j \leq d-2$
(5)	$d+4$	$4d+8$	$3d+4$	—
(6)	$d+5+j$	$2d+8+2j$	$d+3+j$	$0 \leq j \leq (d-3)/2$
(7)	$(3d+13)/2+j$	$3d+11+2j$	$(3d+9)/2+j$	$0 \leq j \leq (d-5)/2$
(8)	$2d+5$	$4d+9$	$2d+4$	—
(9)	$2d+6$	$2d+6$	0	—
(10)	$2d+7$	$4d+10$	$2d+3$	—
(11)	$3d+7$	$4d+7$	d	—
(12)	$3d+9$	$4d+11$	$d+2$	—

	a_i	b_i	$ b_i - a_i $	
(1)	1	$2d+4$	$2d+3$	—
(2)	2	2	0	—
(3)	3	$d+4$	$d+1$	—
(4)	$4+j$	$2d+9+2j$	$2d+5+j$	$0 \leq j \leq (d-5)/2$
(5)	$(d+5)/2$	$2d+5$	$(3d+5)/2$	—
(6)	$(d+7)/2$	$3d+7$	$(5d+7)/2$	—
(7)	$(d+9)/2+j$	$3d+10+2j$	$(5d+11)/2+j$	$0 \leq j \leq (d-3)/2$
(8)	$d+5+j$	$2d+8+2j$ <u>or</u> 2	$d+3+j$	$0 \leq j \leq (d-3)/2$
(9)	$(3d+9)/2$	$4d+9$	$(5d+9)/2$	—
(10)	$(3d+11)/2+j$	$3d+9+2j$ <u>or</u> 2	$(3d+7)/2+j$	$0 \leq j \leq (d-5)/2$
(11)	$2d+6$	$4d+8$	$2d+2$	—
(12)	$2d+7$	$4d+11$	$2d+4$	—
(13)	$3d+6$	$4d+6$	d	—
(14)	$3d+8$	$4d+10$	$d+2$	—

	a_i	b_i	$ b_i - a_i $	
(1)	1	$(3d+11)/2$	$(3d+9)/2$	—
(2)	2	2	0	—
(3)	3	$d+6$	$d+3$	—
(4)	$4+j$	$2d+9+2j$	$2d+5+j$	$0 \leq j \leq (d-1)/2$
(5)	$(d+9)/2$	$2d+8$	$(3d+7)/2$	—
(6)	$(d+11)/2$	$(3d+13)/2$	$d+1$	—
(7)	$(d+13)/2+j$	$3d+12+2j$	$(5d+11)/2+j$	$0 \leq j \leq (d-3)/2$
(8)	$d+7+j$	$2d+12+2j$ <u>or</u> 2	$d+5+j$	$0 \leq j \leq (d-5)/2$
(9)	$(3d+15)/2+j$	$3d+13+2j$ <u>or</u> 2	$(3d+11)/2+j$	$0 \leq j \leq (d-3)/2$
(10)	$2d+7$	$3d+11$	$d+4$	—
(11)	$2d+10$	$3d+10$	d	—
(12)	$3d+9$	$4d+11$	$d+2$	—

Table 3.2: Extended Langford sequences: $m = 2d + 5$, defect is odd, extensions ranges: $\{2d+6\}$, $[2d+8, 4d+4]'' \setminus \{3d+7\}$ and $[2d+12, 4d+10]'' \setminus \{3d+9, 3d+11\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	–
(2)	2	$3d/2 + 4$	$3d/2 + 2$	–
(3)	$3 + j$	$2d + 10 + 2j$	$2d + 7 + j$	$0 \leq j \leq d/2 - 2$
(4)	$d/2 + 2$	$2d + 5$	$3d/2 + 3$	–
(5)	$d/2 + 3$	$3d/2 + 3$	d	–
(6)	$d/2 + 4 + j$	$3d + 10 + 2j$	$5d/2 + 6 + j$	$0 \leq j \leq d/2 - 2$
(7)	$d + 3$	$3d + 8$	$2d + 5$	–
(8)	$d + 4 + j$	$2d + 7 + 2j$ <u>or</u> 1	$d + 3 + j$	$0 \leq j \leq d/2 - 2$
(9)	$3d/2 + 5 + j$	$3d + 9 + 2j$ <u>or</u> 1	$3d/2 + 4 + j$	$0 \leq j \leq d/2 - 2$
(10)	$2d + 4$	$4d + 10$	$2d + 6$	–
(11)	$2d + 6$	$4d + 9$	$2d + 3$	–
(12)	$2d + 8$	$4d + 12$	$2d + 4$	–
(13)	$3d + 5$	$4d + 7$	$d + 2$	–
(14)	$3d + 7$	$4d + 8$	$d + 1$	–
(15)	$4d + 11$	$4d + 11$	0	–

	a_i	b_i	$ b_i - a_i $	$0 \leq j \leq$
(1)	1	1	0	–
(2)	2	$3d/2 + 6$	$3d/2 + 4$	–
(3)	3	$d + 5$	$d + 2$	–
(4)	$4 + j$	$2d + 10 + 2j$	$2d + 6 + j$	$0 \leq j \leq d/2 - 1$
(5)	$d/2 + 4$	$2d + 9$	$3d/2 + 5$	–
(6)	$d/2 + 5$	$3d/2 + 5$	d	–
(7)	$d/2 + 6 + j$	$3d + 12 + 2j$	$5d/2 + 6 + j$	$0 \leq j \leq d/2 - 2$
(8)	$d + 6 + j$	$2d + 11 + 2j$ <u>or</u> 1	$d + 5 + j$	$0 \leq j \leq d/2 - 2$
(9)	$3d/2 + 7 + j$	$3d + 13 + 2j$ <u>or</u> 1	$3d/2 + 6 + j$	$0 \leq j \leq d/2 - 2$
(10)	$2d + 6$	$3d + 10$	$d + 4$	–
(11)	$2d + 7$	$4d + 12$	$2d + 5$	–
(12)	$2d + 8$	$3d + 11$	$d + 3$	–
(13)	$3d + 9$	$4d + 10$	$d + 1$	–
(14)	$4d + 11$	$4d + 11$	0	–

Table 3.3: Hooked extended Langford sequences: $m = 2d + 5$, defect is even, extensions ranges: $[2d + 7, 4d + 5]' \setminus \{3d + 5, 3d + 7\}$ and $[2d + 11, 4d + 9]' \setminus \{3d + 9, 3d + 11\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	—
(2)	2	$(3d+9)/2$	$(3d+5)/2$	—
(3)	$3+j$	$2d+10+2j$	$2d+7+j$	$0 \leq j \leq (d-3)/2$
(4)	$(d+5)/2$	$2d+6$	$(3d+7)/2$	—
(5)	$(d+7)/2$	$(3d+7)/2$	d	—
(6)	$(d+9)/2+j$	$3d+11+2j$	$(5d+13)/2+j$	$0 \leq j \leq (d-5)/2$
(7)	$d+3$	$3d+9$	$2d+6$	—
(8)	$d+4+j$	$2d+7+2j$ <u>or</u> 1	$d+3+j$	$0 \leq j \leq (d-3)/2$
(9)	$(3d+11)/2+j$	$3d+10+2j$ <u>or</u> 1	$(3d+9)/2+j$	$0 \leq j \leq (d-3)/2$
(10)	$2d+5$	$4d+10$	$2d+5$	—
(11)	$2d+8$	$4d+12$	$2d+4$	—
(12)	$3d+6$	$4d+8$	$d+2$	—
(13)	$3d+8$	$4d+9$	$d+1$	—
(14)	$4d+11$	$4d+11$	0	—

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	—
(2)	2	$(3d+13)/2$	$(3d+9)/2$	—
(3)	3	$d+5$	$d+2$	—
(4)	$4+j$	$2d+10+2j$	$2d+6+j$	$0 \leq j \leq (d-5)/2$
(5)	$(d+5)/2$	$(3d+11)/2$	$d+3$	—
(6)	$(d+7)/2$	$2d+9$	$(3d+11)/2$	—
(7)	$(d+9)/2+j$	$3d+9+2j$	$(5d+9)/2+j$	$0 \leq j \leq (d-1)/2$
(8)	$d+6+j$	$2d+11+2j$ <u>or</u> 1	$d+5+j$	$0 \leq j \leq (d-3)/2$
(9)	$(3d+15)/2+j$	$3d+14+2j$ <u>or</u> 1	$(3d+13)/2+j$	$0 \leq j \leq (d-5)/2$
(10)	$2d+6$	$3d+7$	$d+1$	—
(11)	$2d+7$	$4d+12$	$2d+5$	—
(12)	$2d+8$	$3d+12$	$d+4$	—
(13)	$3d+10$	$4d+10$	d	—
(14)	$4d+11$	$4d+11$	0	—

Table 3.4: Hooked extended Langford sequences: $m = 2d + 5$, defect is odd, extensions ranges: $[2d+7, 4d+7]' \setminus \{3d+6, 3d+8\}$ and $[2d+11, 4d+9]' \setminus \{3d+10, 3d+12\}$.

3.3 Difference set $D_7 = [d, 3d + 5], m = 2d + 6$.

A $\mathcal{L}_d^{2d+6}(k)$ is equivalent to a partition of positions $P = [1, 4d + 13]$ into differences in $D_7 = [d, 3d + 5]$. Note that D_7 is a perfect difference set when d is odd (in which case k is odd) and D_7 is not a perfect difference set when d is even (in which case k is even). With the following two lemmas we construct extended sequences $\mathcal{L}_d^{2d+6}(k)$ for even and odd defects, respectively.

Lemma 3.3.1. *The following are sufficient for the existence of a k -extended Langford sequence of even defect d and m differences, $\mathcal{L}_d^m(k)$:*

1. $m = 2d + 6$ and $d = 2s \geq 2$,
2. $k \in [1, 2m + 1]$,
3. $(m, k) \equiv (2, 0) \pmod{(4, 2)}$.

Proof:

The union of the ranges of extensions in Table 3.5 is $[2d + 8, 4d + 12]''$. Reversing the sequences in Table 3.5, we obtain all extensions in $[1, 4d + 13]''$. ■

Lemma 3.3.2. *The following are sufficient for the existence of a k -extended Langford sequence of odd defect d and m differences, $\mathcal{L}_d^m(k)$:*

1. $m = 2d + 6$ and $d = 2s + 1 \geq 3$,
2. $k \in [1, 2m + 1]$,
3. $(m, k) \equiv (0, 1) \pmod{(4, 2)}$.

Proof:

The union of the ranges of extensions in Table 3.6 is $[2d + 7, 4d + 13]'$. Reversing the sequences in Table 3.6, we obtain all extensions in $[1, 4d + 13]'$. ■

We now construct right-hooked k -extended sequences of the form $h\mathcal{L}_d^{2d+6}(k)$, $k \geq m + 2 = 2d + 8$. These sequences are equivalent to a partition of positions $P = [1, 4d + 14] \setminus \{k, 4d + 13\}$ into differences $D_7 = [d, 3d + 5]$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	$d + 3$	$d + 2$	–
(2)	2	2	0	–
(3)	3	$3d/2 + 6$	$3d/2 + 3$	–
(4)	$4 + j$	$2d + 11 + 2j$	$2d + 7 + j$	$0 \leq j \leq d/2 - 2$
(5)	$d/2 + 3$	$2d + 7$	$3d/2 + 4$	–
(6)	$d/2 + 4 + j$	$3d + 11 + 2j$	$5d/2 + 7 + j$	$0 \leq j \leq d/2 - 2$
(7)	$d + 4$	$3d + 9$	$2d + 5$	–
(8)	$d + 5 + j$	$2d + 8 + 2j$ <u>or</u> 2	$d + 3 + j$	$0 \leq j \leq d/2 - 1$
(9)	$3d/2 + 5$	$4d + 11$	$5d/2 + 6$	–
(10)	$3d/2 + 7 + j$	$3d + 12 + 2j$ <u>or</u> 2	$3d/2 + 5 + j$	$0 \leq j \leq d/2 - 2$
(11)	$2d + 6$	$4d + 12$	$2d + 6$	–
(12)	$2d + 9$	$4d + 13$	$2d + 4$	–
(13)	$3d + 8$	$4d + 9$	$d + 1$	–
(14)	$3d + 10$	$4d + 10$	d	–

	a_i	b_i	$ b_i - a_i $	
(1)	1	$d + 5$	$d + 4$	–
(2)	2	2	0	–
(3)	$3 + j$	$2d + 9 + 2j$	$2d + 6 + j$	$0 \leq j \leq d/2$
(4)	$d/2 + 4$	$3d/2 + 7$	$d + 3$	–
(5)	$d/2 + 5$	$2d + 10$	$3d/2 + 5$	–
(6)	$d/2 + 6 + j$	$3d + 13 + 2j$	$5d/2 + 7 + j$	$0 \leq j \leq d/2 - 2$
(7)	$d + 6$	$2d + 8$	$d + 2$	–
(8)	$d + 7 + j$	$2d + 12 + 2j$ <u>or</u> 2	$d + 5 + j$	$0 \leq j \leq d/2 - 1$
(9)	$3d/2 + 8 + j$	$3d + 14 + 2j$ <u>or</u> 2	$3d/2 + 6 + j$	$0 \leq j \leq d/2 - 1$
(10)	$3d + 11$	$4d + 11$	d	–
(11)	$3d + 12$	$4d + 13$	$d + 1$	–

Table 3.5: Extended Langford sequences: $m = 2d + 6$, defect is even, extensions ranges: $[2d + 8, 4d + 8]'' \setminus \{3d + 8, 3d + 10\}$ and $[2d + 12, 4d + 12]'' \setminus \{3d + 12\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	$d + 4$	$d + 3$	—
(2)	2	$d + 3$	$d + 1$	—
(3)	$3 + j$	$2d + 9 + 2j$	$2d + 6 + j$	$0 \leq j \leq (d - 3)/2$
(4)	$(d + 5)/2$	$(3d + 9)/2$	$d + 2$	—
(5)	$(d + 7)/2 + j$	$3d + 10 + 2j$	$(5d + 13)/2 + j$	$0 \leq j \leq (d - 3)/2$
(6)	$d + 5 + j$	$2d + 10 + 2j$	$d + 5 + j$	$0 \leq j \leq (d - 3)/2$
(7)	$(3d + 11)/2$	$4d + 11$	$(5d + 11)/2$	—
(8)	$(3d + 13)/2 + j$	$3d + 11 + 2j$	$(3d + 9)/2 + j$	$0 \leq j \leq (d - 1)/2$
(9)	$2d + 7$	$2d + 7$	0	—
(10)	$2d + 8$	$4d + 13$	$2d + 5$	—
(11)	$3d + 8$	$4d + 12$	$d + 4$	—
(12)	$3d + 9$	$4d + 9$	d	—

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	—
(2)	2	$d + 4$	$d + 2$	—
(3)	$3 + j$	$2d + 10 + 2j$	$2d + 7 + j$	$0 \leq j \leq (d - 3)/2$
(4)	$(d + 5)/2$	$2d + 8$	$(3d + 11)/2$	—
(5)	$(d + 7)/2$	$(3d + 13)/2$	$d + 3$	—
(6)	$(d + 9)/2 + j$	$3d + 11 + 2j$	$(5d + 13)/2 + j$	$0 \leq j \leq (d - 3)/2$
(7)	$d + 5 + j$	$2d + 9 + 2j$ <u>or</u> 1	$d + 4 + j$	$0 \leq j \leq (d + 1)/2$
(8)	$(3d + 15)/2 + j$	$3d + 14 + 2j$ <u>or</u> 1	$(3d + 13)/2 + j$	$0 \leq j \leq (d - 1)/2$
(9)	$3d + 9$	$4d + 10$	$d + 1$	—
(10)	$3d + 12$	$4d + 12$	d	—

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	—
(2)	2	$d + 4$	$d + 2$	—
(3)	$3 + j$	$2d + 10 + 2j$	$2d + 7 + j$	$0 \leq j \leq (d - 3)/2$
(4)	$(d + 5)/2$	$2d + 7$	$(3d + 9)/2$	—
(5)	$(d + 7)/2$	$(3d + 9)/2$	$d + 1$	—
(6)	$(d + 9)/2 + j$	$3d + 11 + 2j$	$(5d + 13)/2 + j$	$0 \leq j \leq (d - 3)/2$
(7)	$d + 5 + j$	$2d + 9 + 2j$ <u>or</u> 1	$d + 4 + j$	$0 \leq j \leq (d - 3)/2$
(8)	$(3d + 11)/2$	$3d + 9$	$(3d + 7)/2$	—
(9)	$(3d + 13)/2 + j$	$3d + 12 + 2j$ <u>or</u> 1	$(3d + 11)/2 + j$	$0 \leq j \leq (d - 3)/2$
(10)	$2d + 6$	$4d + 12$	$2d + 6$	—
(11)	$2d + 8$	$4d + 13$	$2d + 5$	—
(12)	$3d + 8$	$4d + 11$	$d + 3$	—
(13)	$3d + 10$	$4d + 10$	d	—

Table 3.6: Extended Langford sequences: $m = 2d + 6$, defect is odd, extensions ranges: $\{2d + 7\}$, $[2d + 9, 4d + 13]' \setminus \{3d + 12\}$ and $[2d + 9, 4d + 9]' \setminus \{3d + 8, 3d + 10\}$.

Lemma 3.3.3. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of even defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m = 2d + 6$ and $d = 2s \geq 2$,
2. $k \in [m + 2, 2m + 2] \setminus \{2m + 1\}$,
3. $(m, k) \equiv (2, 1) \pmod{(4, 2)}$.

Proof:

For even defects the union of the ranges of extensions in Table 3.7 is $[2d + 9, 4d + 11]'$. This includes every possible $k \geq m + 2$. Therefore, on the position set, $[2d + 9, 4d + 11]$, for each even d , we get every possible k on the second half of P . ■

Lemma 3.3.4. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of odd defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m = 2d + 6$ and $d = 2s - 1 \geq 1$,
2. $k \in [m + 2, 2m + 2] \setminus \{2m + 1\}$,
3. $(m, k) \equiv (0, 0) \pmod{(4, 2)}$.

Proof:

For even defects the union of the ranges of extensions in Table 3.8 is $[2d + 8, 4d + 14]''$. This includes all possible $k \geq m + 2$. Therefore, on the position set, $[2d + 8, 4d + 14]$, for each odd d , we get every possible extension on the second half of the position set. ■

3.4 Difference set $D_8 = [d, 3d + 6], m = 2d + 7$.

A $\mathcal{L}_d^{2d+7}(k)$ is equivalent to a partition of positions $P = [1, 4d + 15]$ into differences in $D_8 = [d, 3d + 6]$. Note that D_8 is always a perfect difference set, so necessarily k is odd.

Lemma 3.4.1. *The following are sufficient for the existence of a k -extended Langford sequence of defect d and m differences, $\mathcal{L}_d^m(k)$:*

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	–
(2)	2	$d + 4$	$d + 2$	–
(3)	$3 + j$	$2d + 10 + 2j$	$2d + 7 + j$	$0 \leq j \leq d/2 - 1$
(4)	$d/2 + 3$	$2d + 6$	$3d/2 + 3$	–
(5)	$d/2 + 4$	$3d/2 + 4$	d	–
(6)	$d/2 + 5 + j$	$3d + 12 + 2j$	$5d/2 + 7 + j$	$0 \leq j \leq d/2 - 2$
(7)	$d + 5 + j$	$2d + 9 + 2j$ <u>or</u> 1	$d + 4 + j$	$0 \leq j \leq d/2 - 2$
(8)	$3d/2 + 5 + j$	$3d + 9 + 2j$ <u>or</u> 1	$3d/2 + 4 + j$	$0 \leq j \leq d/2$
(9)	$2d + 7$	$4d + 12$	$2d + 5$	–
(10)	$2d + 8$	$4d + 14$	$2d + 6$	–
(11)	$3d + 7$	$4d + 10$	$d + 3$	–
(12)	$3d + 10$	$4d + 11$	$d + 1$	–
(13)	$4d + 13$	$4d + 13$	0	–

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	–
(2)	2	$d + 4$	$d + 2$	–
(3)	$3 + j$	$2d + 10 + 2j$	$2d + 7 + j$	$0 \leq j \leq d/2 - 1$
(4)	$d/2 + 3$	$2d + 7$	$3d/2 + 4$	–
(5)	$d/2 + 4$	$3d/2 + 5$	$d + 1$	–
(6)	$d/2 + 5 + j$	$3d + 12 + 2j$	$5d/2 + 7 + j$	$0 \leq j \leq d/2 - 2$
(7)	$d + 5 + j$	$2d + 9 + 2j$ <u>or</u> 1	$d + 4 + j$	$0 \leq j \leq d/2 - 1$
(8)	$3d/2 + 6 + j$	$3d + 11 + 2j$ <u>or</u> 1	$3d/2 + 5 + j$	$0 \leq j \leq d/2$
(9)	$2d + 8$	$4d + 14$	$2d + 6$	–
(10)	$3d + 9$	$4d + 12$	$d + 3$	–
(11)	$3d + 10$	$4d + 10$	d	–
(12)	$4d + 13$	$4d + 13$	0	–

Table 3.7: Hooked extended Langford sequences: $m = 2d + 6$, defect is even, extensions range: $[2d + 9, 4d + 11]' \setminus \{3d + 9\}$ and $[2d + 9, 4d + 9]' \setminus \{3d + 7\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	$d + 5$	$d + 4$	–
(2)	2	2	0	–
(3)	3	$d + 6$	$d + 3$	–
(4)	$4 + j$	$2d + 11 + 2j$	$2d + 7 + j$	$0 \leq j \leq (d - 1)/2$
(5)	$(d + 9)/2$	$2d + 9$	$(3d + 9)/2$	–
(6)	$(d + 11)/2$	$(3d + 13)/2$	$d + 1$	–
(7)	$(d + 13)/2 + j$	$3d + 14 + 2j$	$(5d + 15)/2 + j$	$0 \leq j \leq (d - 5)/2$
(8)	$d + 7 + j$	$2d + 12 + 2j$ <u>or</u> 2	$d + 5 + j$	$0 \leq j \leq (d - 3)/2$
(9)	$(3d + 15)/2 + j$	$3d + 13 + 2j$ <u>or</u> 2	$(3d + 11)/2 + j$	$0 \leq j \leq (d + 1)/2$
(10)	$2d + 10$	$3d + 12$	$d + 2$	–
(11)	$3d + 11$	$4d + 11$	d	–
(12)	$4d + 13$	$4d + 13$	0	–

	a_i	b_i	$ b_i - a_i $	
(1)	1	$(3d + 9)/2$	$(3d + 7)/2$	–
(2)	2	2	0	–
(3)	3	$d + 3$	d	–
(4)	$4 + j$	$2d + 11 + 2j$	$2d + 7 + j$	$0 \leq j \leq (d - 3)/2$
(5)	$(d + 7)/2$	$2d + 6$	$(3d + 5)/2$	–
(6)	$(d + 9)/2 + j$	$3d + 12 + 2j$	$(5d + 15)/2 + j$	$0 \leq j \leq (d - 5)/2$
(7)	$d + 4$	$3d + 10$	$2d + 6$	–
(8)	$d + 5 + j$	$2d + 8 + 2j$ <u>or</u> 2	$d + 3 + j$	$0 \leq j \leq (d - 3)/2$
(9)	$(3d + 11)/2$	$4d + 12$	$(5d + 13)/2$	–
(10)	$(3d + 13)/2 + j$	$3d + 11 + 2j$ <u>or</u> 2	$(3d + 9)/2 + j$	$0 \leq j \leq (d - 3)/2$
(11)	$2d + 7$	$4d + 11$	$2d + 4$	–
(12)	$2d + 9$	$4d + 14$	$2d + 5$	–
(13)	$3d + 7$	$4d + 9$	$d + 2$	–
(14)	$3d + 9$	$4d + 10$	$d + 1$	–
(15)	$4d + 13$	$4d + 13$	0	–

Table 3.8: Hooked extended Langford sequences: $m = 2d + 6$, defect is odd, extensions ranges: $[2d + 12, 4d + 14]'' \setminus \{3d + 11\}$ and $[2d + 8, 4d + 8]'' \setminus \{3d + 7, 3d + 9\}$.

1. $m = 2d + 7$ and $d \geq 2$,
2. $k \in [1, 2m + 1]$,
3. $(m, k) \equiv (1, d)$ or $(3, d + 1) \pmod{(4, 2)}$.

Proof:

For even defects the union of the ranges of extensions in Table 3.9 is $[2d + 9, 4d + 15]'$. For odd defects the union of the ranges of extensions in Table 3.10 is $[2d + 9, 4d + 13]'$. Reversing the sequences in Table 3.9 and Table 3.10, we obtain all extensions in $[1, 4d + 15]'$, for each parity of d . ■

We now construct right-hooked k -extended Langford sequences of the form $h\mathcal{L}_d^{2d+7}(k)$, $k \geq m + 2 = 2d + 9$. Such a sequence is equivalent to a partition of positions $P = [1, 4d + 16] \setminus \{k, 4d + 15\}$ into differences in $D_8 = [d, 3d + 6]$.

Lemma 3.4.2. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m = 2d + 7$ and $d \geq 2$,
2. $k \in [m + 2, 2m + 2] \setminus \{2m + 1\}$,
3. $(m, k) \equiv (1, d + 1)$ or $(3, d) \pmod{(4, 2)}$.

Proof:

For even defects the union of the ranges of extensions in Table 3.11 is $[2d + 10, 4d + 14]''$. This includes all possible $k \geq m + 2$. For odd defects the union of the ranges of extensions in Table 3.12 is $[2d + 10, 4d + 14]''$. This is all possible $k \geq m + 2$. ■

3.5 Combining the constructions

Previously published results and the constructions in Sections 3.2, 3.3 and 3.4 can be summarized in the following two theorems.

Theorem 3.5.1. *The following are sufficient for the existence of a k -extended Langford sequence of defect d and m differences, $\mathcal{L}_d^m(k)$:*

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	–
(2)	2	$3d/2 + 7$	$3d/2 + 5$	–
(3)	3	$d + 5$	$d + 2$	–
(4)	$4 + j$	$2d + 12 + 2j$	$2d + 8 + j$	$0 \leq j \leq d/2 - 1$
(5)	$d/2 + 4$	$2d + 10$	$3d/2 + 6$	–
(6)	$d/2 + 5$	$3d/2 + 6$	$d + 1$	–
(7)	$d/2 + 6 + j$	$3d + 14 + 2j$ <u>or</u> 1	$5d/2 + 8 + j$	$0 \leq j \leq d/2 - 2$
(8)	$d + 6 + j$	$2d + 11 + 2j$ <u>or</u> 1	$d + 5 + j$	$0 \leq j \leq d/2 - 1$
(9)	$3d/2 + 8 + j$	$3d + 15 + 2j$	$3d/2 + 7 + j$	$0 \leq j \leq d/2$
(10)	$2d + 9$	$3d + 13$	$d + 4$	–
(11)	$3d + 11$	$4d + 14$	$d + 3$	–
(12)	$3d + 12$	$4d + 12$	d	–

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	–
(2)	2	$d + 4$	$d + 2$	–
(3)	3	$d + 3$	d	–
(4)	$4 + j$	$2d + 12 + 2j$	$2d + 8 + j$	$0 \leq j \leq d/2 - 3$
(5)	$d/2 + 2$	$2d + 6$	$3d/2 + 4$	–
(6)	$d/2 + 3 + j$	$3d + 10 + 2j$	$5d/2 + 7 + j$	$0 \leq j \leq d/2 - 1$
(7)	$d + 5 + j$	$2d + 9 + 2j$ <u>or</u> 1	$d + 4 + j$	$0 \leq j \leq d/2 - 1$
(8)	$3d/2 + 5$	$4d + 11$	$5d/2 + 6$	–
(9)	$3d/2 + 6 + j$	$3d + 11 + 2j$ <u>or</u> 1	$3d/2 + 5 + j$	$0 \leq j \leq d/2 - 2$
(10)	$2d + 5$	$4d + 10$	$2d + 5$	–
(11)	$2d + 7$	$4d + 13$	$2d + 6$	–
(12)	$2d + 8$	$4d + 15$	$2d + 7$	–
(13)	$2d + 10$	$4d + 14$	$2d + 4$	–
(14)	$3d + 8$	$4d + 9$	$d + 1$	–
(15)	$3d + 9$	$4d + 12$	$d + 3$	–

Table 3.9: Extended Langford sequences: $m = 2d + 7$, defect is even, extensions ranges: $[2d + 11, 4d + 15]' \setminus \{3d + 11, 3d + 13\}$ and $[2d + 9, 4d + 7]' \setminus \{3d + 9\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	–
(2)	2	$2d + 9$	$2d + 7$	–
(3)	3	$(3d + 15)/2$	$(3d + 9)/2$	–
(4)	$4 + j$	$2d + 12 + 2j$	$2d + 8 + j$	$0 \leq j \leq (d - 3)/2$
(5)	$(d + 7)/2$	$2d + 10$	$(3d + 13)/2$	–
(6)	$(d + 9)/2$	$(3d + 11)/2$	$d + 1$	–
(7)	$(d + 11)/2 + j$	$3d + 13 + 2j$	$(5d + 15)/2 + j$	$0 \leq j \leq (d - 3)/2$
(8)	$d + 5$	$2d + 8$	$d + 3$	–
(9)	$d + 6 + j$	$2d + 11 + 2j$ <u>or</u> 1	$d + 5 + j$	$0 \leq j \leq (d - 3)/2$
(10)	$(3d + 13)/2$	$3d + 12$	$(3d + 11)/2$	–
(11)	$(3d + 17)/2 + j$	$3d + 16 + 2j$ <u>or</u> 1	$(3d + 15)/2 + j$	$0 \leq j \leq (d - 3)/2$
(12)	$3d + 10$	$4d + 12$	$d + 2$	–
(13)	$3d + 11$	$4d + 15$	$d + 4$	–
(14)	$3d + 14$	$4d + 14$	d	–

	a_i	b_i	$ b_i - a_i $	
(1)	1	1	0	–
(2)	2	$d + 4$	$d + 2$	–
(3)	3	$d + 3$	d	–
(4)	$4 + j$	$2d + 12 + 2j$	$2d + 8 + j$	$0 \leq j \leq (d - 5)/2$
(5)	$(d + 5)/2$	$2d + 6$	$(3d + 7)/2$	–
(6)	$(d + 7)/2 + j$	$3d + 11 + 2j$	$(5d + 15)/2 + j$	$0 \leq j \leq (d - 3)/2$
(7)	$d + 5 + j$	$2d + 9 + 2j$ <u>or</u> 1	$d + 4 + j$	$0 \leq j \leq (d - 3)/2$
(8)	$(3d + 9)/2$	$4d + 11$	$(5d + 13)/2$	–
(9)	$(3d + 11)/2 + j$	$3d + 10 + 2j$ <u>or</u> 1	$(3d + 9)/2 + j$	$0 \leq j \leq (d - 3)/2$
(10)	$2d + 5$	$4d + 10$	$2d + 5$	–
(11)	$2d + 7$	$4d + 13$	$2d + 6$	–
(12)	$2d + 8$	$4d + 15$	$2d + 7$	–
(13)	$2d + 10$	$4d + 14$	$2d + 4$	–
(14)	$3d + 8$	$4d + 9$	$d + 1$	–
(15)	$3d + 9$	$4d + 12$	$d + 3$	–

Table 3.10: Extended Langford sequences: $m = 2d + 7$, defect is odd, extensions ranges: $[2d + 11, 4d + 13]' \setminus \{3d + 10, 3d + 12, 3d + 14\}$ and $[2d + 9, 4d + 7]' \setminus \{3d + 8\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	$d + 4$	$d + 3$	–
(2)	2	2	0	–
(3)	3	$2d + 9$	$2d + 6$	–
(4)	4	$d + 5$	$d + 1$	–
(5)	$5 + j$	$2d + 13 + 2j$	$2d + 8 + j$	$0 \leq j \leq d/2 - 2$
(6)	$d/2 + 4$	$2d + 8$	$3d/2 + 4$	–
(7)	$d/2 + 5 + j$	$3d + 13 + 2j$	$5d/2 + 8 + j$	$0 \leq j \leq d/2 - 2$
(8)	$d + 6 + j$	$2d + 10 + 2j$ <u>or</u> 2	$d + 4 + j$	$0 \leq j \leq d/2 - 1$
(9)	$3d/2 + 6$	$4d + 13$	$5d/2 + 7$	–
(10)	$3d/2 + 7 + j$	$3d + 12 + 2j$ <u>or</u> 2	$3d/2 + 5 + j$	$0 \leq j \leq d/2 - 1$
(11)	$2d + 7$	$4d + 14$	$2d + 7$	–
(12)	$2d + 11$	$4d + 16$	$2d + 5$	–
(13)	$3d + 10$	$4d + 12$	$d + 2$	–
(14)	$3d + 11$	$4d + 11$	d	–
(15)	$4d + 15$	$4d + 15$	0	–

	a_i	b_i	$ b_i - a_i $	
(1)	1	$3d/2 + 8$	$3d/2 + 7$	–
(2)	2	2	0	–
(3)	3	$d + 6$	$d + 3$	–
(4)	$4 + j$	$2d + 11 + 2j$	$2d + 7 + j$	$0 \leq j \leq d/2 - 1$
(5)	$d/2 + 4$	$2d + 10$	$3d/2 + 6$	–
(6)	$d/2 + 5$	$3d + 12$	$5d/2 + 7$	–
(7)	$d/2 + 6$	$3d/2 + 7$	$d + 1$	–
(8)	$d/2 + 7 + j$	$3d + 15 + 2j$	$5d/2 + 8 + j$	$0 \leq j \leq d/2 - 2$
(9)	$d + 7 + j$	$2d + 12 + 2j$ <u>or</u> 2	$d + 5 + j$	$0 \leq j \leq d/2 - 1$
(10)	$3d/2 + 9$	$3d + 14$	$3d/2 + 5$	–
(11)	$3d/2 + 10 + j$	$3d + 18 + 2j$ <u>or</u> 2	$3d/2 + 8 + j$	$0 \leq j \leq d/2 - 2$
(12)	$2d + 9$	$3d + 13$	$d + 4$	–
(13)	$3d + 11$	$4d + 13$	$d + 2$	–
(14)	$3d + 16$	$4d + 16$	d	–
(15)	$4d + 15$	$4d + 15$	0	–

Table 3.11: Hooked extended Langford sequences: $m = 2d + 7$, defect is even, extensions ranges: $[2d + 10, 4d + 10]'' \setminus \{3d + 10\}$ and $[2d + 12, 4d + 14]'' \setminus \{3d + 12, 3d + 14, 3d + 16\}$.

	a_i	b_i	$ b_i - a_i $	
(1)	1	$d + 4$	$d + 3$	—
(2)	2	2	0	—
(3)	3	$2d + 9$	$2d + 6$	—
(4)	4	$d + 5$	$d + 1$	—
(5)	$5 + j$	$2d + 13 + 2j$	$2d + 8 + j$	$0 \leq j \leq (d - 3)/2$
(6)	$(d + 9)/2$	$2d + 8$	$(3d + 7)/2$	—
(7)	$(d + 11)/2 + j$	$3d + 14 + 2j$	$(5d + 17)/2 + j$	$0 \leq j \leq (d - 5)/2$
(8)	$d + 6 + j$	$2d + 10 + 2j$ <u>or</u> 2	$d + 4 + j$	$0 \leq j \leq (d - 3)/2$
(9)	$(3d + 11)/2$	$4d + 13$	$(5d + 15)/2$	—
(10)	$(3d + 13)/2 + j$	$3d + 11 + 2j$ <u>or</u> 2	$(3d + 9)/2 + j$	$0 \leq j \leq (d - 1)/2$
(11)	$2d + 7$	$4d + 14$	$2d + 7$	—
(12)	$2d + 11$	$4d + 16$	$2d + 5$	—
(13)	$3d + 9$	$4d + 11$	$d + 2$	—
(14)	$3d + 12$	$4d + 12$	d	—
(15)	$4d + 15$	$4d + 15$	0	—

	a_i	b_i	$ b_i - a_i $	$0 \leq j \leq$
(1)	1	$(3d + 13)/2$	$(3d + 11)/2$	—
(2)	2	2	0	—
(3)	3	$d + 6$	$d + 3$	—
(4)	$4 + j$	$2d + 11 + 2j$	$2d + 7 + j$	$0 \leq j \leq (d - 1)/2$
(5)	$(d + 9)/2$	$2d + 9$	$(3d + 9)/2$	—
(6)	$(d + 11)/2$	$(3d + 15)/2$	$d + 2$	—
(7)	$(d + 13)/2 + j$	$3d + 14 + 2j$	$(5d + 15)/2 + j$	$0 \leq j \leq (d - 3)/2$
(8)	$d + 7 + j$	$2d + 12 + 2j$ <u>or</u> 2	$d + 5 + j$	$0 \leq j \leq (d - 3)/2$
(9)	$(3d + 17)/2 + j$	$3d + 15 + 2j$ <u>or</u> 2	$(3d + 13)/2 + j$	$0 \leq j \leq (d - 1)/2$
(10)	$2d + 10$	$3d + 11$	$d + 1$	—
(11)	$3d + 12$	$4d + 16$	$d + 4$	—
(12)	$3d + 13$	$4d + 13$	d	—
(13)	$4d + 15$	$4d + 15$	0	—

Table 3.12: Hooked extended Langford sequences: $m = 2d + 7$, defect is odd, extensions ranges: $[2d + 10, 4d + 10]'' \setminus \{3d + 9\}$ and $[2d + 12, 4d + 14]'' \setminus \{3d + 11, 3d + 13\}$.

1. $m = 2d + 5, 2d + 6, \text{ or } 2d + 7,$
2. $k \in [1, 2m + 1],$
3. $(m, k) \equiv (0, 1), (1, d), (2, 0), \text{ or } (3, d + 1) \pmod{(4, 2)}.$

Proof:

If $d = 1$, then the result follows from Theorem 2.2.13. If $d = 2$, then the result follows from Theorem 4.1 of [36]. If $d \geq 3$, then Lemmas 3.2.1, 3.3.1, 3.3.2, and 3.4.1 complete the proof for the cases of difference sets D_6, D_7 with even defect, D_7 with odd defect, and D_8 , respectively. ■

Theorem 3.5.2. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m = 2d + 5, 2d + 6, \text{ or } 2d + 7,$
2. $k \in [m + 2, 2m + 2] \setminus \{2m + 1\},$
3. $(m, k) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}.$

Proof:

If $d = 1$, then the result follows from Theorem 2.2.18. If $d \geq 2$, then Lemmas 3.2.2, 3.3.3, 3.3.4, and 3.4.2 complete the proof for the cases of difference sets D_6, D_7 with even defect, D_7 with odd defect, and D_8 , respectively. ■

3.6 Main result: $h\mathcal{L}_d^m(k)$ for $m \geq 8d + 5$.

In this section we repeatedly use Theorem 2.2.23 and Theorem 2.2.27. For the proof of our main theorem we will also use the following results to construct smaller ingredient sequences.

Theorem 3.6.1. [38] *The following are sufficient for the existence of a k -extended Langford sequence of defect d and m differences, $\mathcal{L}_d^m(k)$:*

1. $m = 2d - 1, 2d, \text{ or } 2d + 1,$
2. $k \in [1, 2m + 1],$

3. $(m, k) \equiv (0, 1), (1, d), (2, 0), \text{ or } (3, d + 1) \pmod{(4, 2)}$.

Theorem 3.6.2. [38] *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m = 2d - 1, 2d, \text{ or } 2d + 1$,
2. $k \in [1, 2m + 2] \setminus \{2m + 1\}$,
3. $(m, k) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}$,
4. $(d, m, k) \notin \{(1, 1, 2), (2, 3, 2)\}$.

Note that in our statement of Theorem 3.6.2 the Langford sequence has position $2m + 1$ as a hook, whereas position 2 is the hook as stated originally in [38]. Furthermore, we alert the reader that the parity condition for the left-hooked k -extended Langford sequence as stated in [38] is incorrect.

The following four lemmas together will help us prove our main result regarding the existence of hooked k -extended Langford sequences.

Lemma 3.6.3. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m \geq 8d + 5$,
2. $k \in [1, 2m + 2]''$,
3. $[d, d' - 1]$ is a perfect set and $[d', d + m - 1]$ is a perfect set, where $m + d - 1 = (3d' - 2) + i, 0 \leq i \leq 2$.

Proof:

Let d' and i be as above, let $m' = m + d - d' = 2d' - 1 + i$, and let k' be any integer with $k' \in [1, 2m' + 2]'' = [1, 2(m + d - d') + 2]''$. We claim that for these values of d', m' and k' there exists a $h\mathcal{L}_{d'}^{m'}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.6.2 are satisfied.

Directly from the definitions of d', m' and k' we have that $m' = 2d' - 1, 2d'$, or $2d' + 1$, and that $k' \in [1, 2m' + 2] \setminus \{2m' + 1\}$. The number of differences in the set $[d', d + m - 1]$ is $m' = m + d - d'$, and this set is perfect, so by Lemma 3.1.2 one of the following congruences holds: $(m', d') \equiv (0, 0), (0, 1), (1, 1), \text{ or } (3, 0) \pmod{(4, 2)}$. There are now just three possibilities for (m', k') modulo $(4, 2)$ (recall that k' was defined to be an even integer):

- (a) If $m' \equiv 0 \pmod{4}$, then $(m', k') \equiv (0, 0) \pmod{(4, 2)}$.
- (b) If $(m', d') \equiv (1, 1) \pmod{(4, 2)}$, then $(m', k') \equiv (1, d' + 1) \pmod{(4, 2)}$.
- (c) If $(m', d') \equiv (3, 0) \pmod{(4, 2)}$, then $(m', k') \equiv (3, d') \pmod{(4, 2)}$.

Therefore one of the following congruences holds: $(m', k') \equiv (0, 0)$, $(1, d' + 1)$, or $(3, d') \pmod{(4, 2)}$. Finally, since $m \geq 8d + 5$, $d \geq 1$ and $0 \leq i \leq 2$, we have $(3d' - 2) + i = m + d - 1 \geq 13$, so $d' > 4$ and $(d', m', k') \notin \{(1, 1, 2), (2, 3, 2)\}$. It now follows from Theorem 3.6.2 that a $h\mathcal{L}_{d'}^{m'}(k')$ exists for all $k' \in [1, 2m' + 2]'' = [1, 2(m + d - d') + 2]''$.

Let $m'' = d' - d$. We now claim that there exists a $\mathcal{L}_d^{m''} = \mathcal{L}_d^{d'-d}$, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.23 are satisfied. Since $m \geq 8d + 5$, we have $3d' - 2 + i = m + d - 1 \geq 9d + 4$, so $d' \geq 3d + 2 - (i/3)$, where $0 \leq i \leq 2$ and d' is an integer. Hence

$$m'' = d' - d \geq 2d + 2. \quad (3.3)$$

The number of differences in the set $[d, d' - 1]$ is $m'' = d' - d$, so by (3.3) we have $m'' > 2d - 1$. Also $[d, d' - 1]$ is a perfect set, so by Lemma 3.1.2 we know that (m'', d) satisfies one of the congruences $(m'', d) \equiv (0, 0), (0, 1), (1, 1)$, or $(3, 0) \pmod{(4, 2)}$. It follows from Theorem 2.2.23 that a $\mathcal{L}_d^{m''} = \mathcal{L}_d^{d'-d}$ exists.

Having obtained the ingredient sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_d^{d'-d} h\mathcal{L}_{d'}^{m'+d-d'}(k'). \quad (3.4)$$

Since $k = k' + 2(d' - d)$, then as k' in $h\mathcal{L}_{d'}^{m'+d-d'}(k')$ is varied, all k -extensions for

$$k \in [2(d' - d) + 2, 2m + 2]'' \quad (3.5)$$

are achieved in $h\mathcal{L}_d^m(k)$.

In the last half of this proof we need slightly different ingredient sequences. This time for $k' \in [1, 2m' + 4]''$ we claim there exists a $\mathcal{L}_{d'-2}^{m'+2}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.5.1 are satisfied. Since $m' = 2d' - 1 + i$, $0 \leq i \leq 2$, we have $m' + 2 = 2(d' - 2) + (5 + i) = 2(d' - 2) + j$, $5 \leq j \leq 7$. Also, we deduced above that for even k' one of $(m', k') \equiv (0, 0)$, $(1, d' + 1)$, or $(3, d') \pmod{(4, 2)}$ holds, therefore one of the following congruences holds: $(m' + 2, k') \equiv (2, 0)$, $(3, (d' - 2) + 1)$, or $(1, d' - 2) \pmod{(4, 2)}$. By Theorem 3.5.1 a $\mathcal{L}_{d'-2}^{m'+2}(k') = \mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ exists for all $k' \in [1, 2m' + 4]'' = [1, 2(m + d - d') + 4]''$.

Next we claim that a $h\mathcal{L}_d^{m''-2}$ exists, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.27 are satisfied. By (3.3) we

have $m'' - 2 \geq 2d$ and therefore $(m'' - 2)((m'' - 2) + 1 - 2d) + 2 \geq 0$. Also we deduced above that $(m'', d) \equiv (0, 0), (0, 1), (1, 1), \text{ or } (3, 0) \pmod{(4, 2)}$, and therefore $(m'' - 2, d) \equiv (1, 0), (2, 0), (2, 1), \text{ or } (3, 1) \pmod{(4, 2)}$. It follows from Theorem 2.2.27 that a $h\mathcal{L}_d^{m''-2} = h\mathcal{L}_d^{d'-d-2}$ exists.

Having proved the existence of the new ingredients sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_{d'-2}^{m+d-d'+2}(k') h\mathcal{L}_d^{d'-d-2}. \quad (3.6)$$

Since $k = k'$, then as k' in $\mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ is varied, all k -extensions with

$$k \in [1, 2(m + d + d') + 4]'' \quad (3.7)$$

are achieved in $h\mathcal{L}_d^m(k)$.

To complete the proof it is enough to show that the interval in (3.5) overlaps with the interval in (3.7). This means showing that $2(d' - d) + 2 \leq 2(m + d - d') + 4$, which is equivalent to $4d' \leq 2m + 4d + 2$, which is, by the definition of $d' = 1/3(m + d + 1 - i)$, equivalent to $4m + 4d + 4(1 - i) \leq 6m + 12d + 6$. Clearly this last inequality holds by comparing individual summands. ■

Lemma 3.6.4. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m \geq 8d + 3$,
2. $k \in [1, 2m + 2]''$,
3. $[d, d' - 1]$ is not a perfect set and $[d', d + m - 1]$ is not a perfect set, where $m + d - 1 = (3d' - 2) + i, 0 \leq i \leq 2$.

Proof:

Let d' and i be as above, let $m' = m + d - d' = 2d' - 1 + i$, and let k' be any integer with $k' \in [1, 2m' + 2]'' = [1, 2(m + d - d') + 2]''$. We claim that for these values of d', m' and k' there exists a $\mathcal{L}_{d'}^{m'}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.6.1 are satisfied.

Directly from the definitions of d', m' and k' we have that $m' = 2d' - 1, 2d'$, or $2d' + 1$, and that $k' \in [1, 2m' + 2]$. The number of differences in the set $[d', d + m - 1]$ is $m' = m + d - d'$, and this set is not perfect, so by Lemma 3.1.2 one of the following congruences holds: $(m', d') \equiv (1, 0), (2, 0), (2, 1), \text{ or } (3, 1)$

(mod $(4, 2)$). There are now just three possibilities for (m', k') modulo $(4, 2)$ (recall that k' was defined to be an even integer):

- (a) If $(m', d') \equiv (1, 0) \pmod{(4, 2)}$, then $(m', k') \equiv (1, d') \pmod{(4, 2)}$.
- (b) If $m' \equiv 2 \pmod{4}$, then $(m', k') \equiv (2, 0) \pmod{(4, 2)}$.
- (c) If $(m', d') \equiv (3, 1) \pmod{(4, 2)}$, then $(m', k') \equiv (3, d' + 1) \pmod{(4, 2)}$.

Therefore one of the following congruences holds: $(m', k') \equiv (1, d')$, $(2, 0)$, or $(3, d' + 1) \pmod{(4, 2)}$. It now follows from Theorem 3.6.1 that a $\mathcal{L}_{d'}^{m'}(k')$ = $\mathcal{L}_{d'}^{m+d-d'}(k')$ exists for all $k' \in [1, 2m' + 1]'' = [1, 2(m + d - d') + 1]''$.

Let $m'' = d' - d$. We now claim that there exists a $h\mathcal{L}_d^{m''} = h\mathcal{L}_d^{d'-d}$, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.27 are satisfied. Since $m \geq 8d + 3$, we have $3d' - 2 + i = m + d - 1 \geq 9d + 2$, so $d' \geq 3d + 1 + (1 - i)/3$, where $0 \leq i \leq 2$ and d' is an integer. Hence

$$m'' = d' - d \geq 2d + 1. \quad (3.8)$$

The number of differences in the set $[d, d' - 1]$ is $m'' = d' - d$, so by (3.8) we have $m''(m'' + 1 - 2d) + 2 \geq 0$. Also $[d, d' - 1]$ is not a perfect set, so by Lemma 3.1.2 we know that (m'', d) satisfies one of the congruences $(m'', d) \equiv (1, 0)$, $(2, 0)$, $(2, 1)$, or $(3, 1) \pmod{(4, 2)}$. It follows from Theorem 2.2.27 that a $h\mathcal{L}_d^{m''} = h\mathcal{L}_d^{d'-d}$ exists.

Having obtained the ingredient sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_{d'}^{m+d-d'}(k') h\mathcal{L}_d^{d'-d}. \quad (3.9)$$

Since $k = k'$, then as k' in $\mathcal{L}_{d'}^{m+d-d'}(k')$ is varied, all k -extensions with

$$k \in [1, 2(m + d - d') + 1]'' \quad (3.10)$$

are achieved in $h\mathcal{L}_d^m(k)$.

In the last half of this proof we need slightly different ingredient sequences. This time for $k' \in [1, 2m' + 4]''$ we claim there exists a $h\mathcal{L}_{d'-2}^{m'+2}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.5.2 are satisfied. Since $m' = 2d' - 1 + i$, $0 \leq i \leq 2$, we have $m' + 2 = 2(d' - 2) + (5 + i) = 2(d' - 2) + j$, $5 \leq j \leq 7$. Also, we deduced above that for even k' one of $(m', k') \equiv (1, d')$, $(2, 0)$, or $(3, d' + 1) \pmod{(4, 2)}$ holds, therefore one of the following congruences holds: $(m' + 2, k') \equiv (3, d' - 2)$, $(0, 0)$, or $(1, (d' - 2) + 1) \pmod{(4, 2)}$. By Theorem 3.5.2 a $h\mathcal{L}_{d'-2}^{m'+2}(k') = h\mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ exists for all $k' \in [(m' + 2) + 2, 2(m' + 2) + 2]'' = [m + d - d' + 4, 2(m + d - d') + 6]''$.

Next we claim that a $\mathcal{L}_d^{m''-2}$ exists, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.23 are satisfied. By (3.8) we have $m'' - 2 \geq 2d - 1$. Also we deduced above that $(m'', d) \equiv (1, 0), (2, 0), (2, 1), \text{ or } (3, 1) \pmod{(4, 2)}$, and therefore $(m'' - 2, d) \equiv (0, 0), (0, 1), (1, 1), \text{ or } (3, 0) \pmod{(4, 2)}$. It follows from Theorem 2.2.23 that a $\mathcal{L}_d^{m''-2} = \mathcal{L}_d^{d'-d-2}$ exists.

Having proved the existence of the new ingredients sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_d^{d'-d-2} h\mathcal{L}_{d'-2}^{m+d-d'+2}(k'). \quad (3.11)$$

Since $k = k' + 2(d' - d - 2)$, then as k' in $h\mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ is varied, all k -extensions

$$k \in [m + d' - d, 2m + 2]'' \quad (3.12)$$

are achieved in $h\mathcal{L}_d^m(k)$.

To complete the proof it is enough to show that the interval in (3.10) overlaps with the interval in (3.12). This means showing that $m + d' - d \leq 2(m + d - d') + 1$, or equivalently showing $3d' \leq m + 3d + 1$, which holds since by definition of d' we have $m + 3d + 1 = 3d' + (2d + i)$. ■

Lemma 3.6.5. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m \geq 8d + 5$,
2. $k \in [1, 2m + 2]' \setminus \{2m + 1\}$,
3. $[d, d' - 1]$ is a perfect set and $[d', d + m - 1]$ is not a perfect set, where $m + d - 1 = (3d' - 2) + i, 0 \leq i \leq 2$.

Proof:

Let d' and i be as above, let $m' = m + d - d' = 2d' - 1 + i$, and let k' be any integer with $k' \in [1, 2m' + 2]' \setminus \{2m' + 1\} = [1, 2(m + d - d') + 2]' \setminus \{2(m + d - d') + 1\}$. We claim that for these values of d', m' and k' there exists a $h\mathcal{L}_{d'}^{m'}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.6.2 are satisfied.

Directly from the definitions of d', m' and k' we have that $m' = 2d' - 1, 2d'$, or $2d' + 1$, and that $k' \in [1, 2m' + 2] \setminus \{2m' + 1\}$. The number of differences

in the set $[d', d + m - 1]$ is $m' = m + d - d'$, and this set is not perfect, so by Lemma 3.1.2 one of the following congruences holds: $(m', d') \equiv (1, 0), (2, 0), (2, 1),$ or $(3, 1) \pmod{(4, 2)}$. There are now just three possibilities for (m', k') modulo $(4, 2)$ (recall that k' was defined to be an odd integer):

- (a) If $(m', d') \equiv (1, 0) \pmod{(4, 2)}$, then $(m', k') \equiv (1, d' + 1) \pmod{(4, 2)}$.
- (b) If $m' \equiv 2 \pmod{4}$, then $(m', k') \equiv (2, 1) \pmod{(4, 2)}$.
- (c) If $(m', d') \equiv (3, 1) \pmod{(4, 2)}$, then $(m', k') \equiv (3, d') \pmod{(4, 2)}$.

Therefore one of the following congruences holds: $(m', k') \equiv (1, d' + 1), (2, 1),$ or $(3, d') \pmod{(4, 2)}$. Finally, since $m \geq 8d + 5, d \geq 1$ and $0 \leq i \leq 2$, we have $(3d' - 2) + i = m + d - 1 \geq 13$, so $d' > 4$ and $(d', m', k') \notin \{(1, 1, 2), (2, 3, 2)\}$. It now follows from Theorem 3.6.2 that a $h\mathcal{L}_d^{m'}(k') = h\mathcal{L}_d^{m'+d-d'}(k')$ exists for all $k' \in [1, 2m' + 2]' \setminus \{2m' + 1\} = [1, 2(m + d - d') + 2]' \setminus \{2(m + d - d') + 1\}$.

Let $m'' = d' - d$. We now claim that there exists a $\mathcal{L}_d^{m''} = \mathcal{L}_d^{d'-d}$, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.23 are satisfied. Since $m \geq 8d + 5$, we have $3d' - 2 + i = m + d - 1 \geq 9d + 4$, so $d' \geq 3d + 2 - (i/3)$, where $0 \leq i \leq 2$ and d' is an integer. Hence

$$m'' = d' - d \geq 2d + 2. \quad (3.13)$$

The number of differences in the set $[d, d' - 1]$ is $m'' = d' - d$, so by (3.13) we have $m'' > 2d - 1$. Also $[d, d' - 1]$ is a perfect set, so by Lemma 3.1.2 we know that (m'', d) satisfies one of the congruences $(m'', d) \equiv (0, 0), (0, 1), (1, 1),$ or $(3, 0) \pmod{(4, 2)}$. It follows from Theorem 2.2.23 that a $\mathcal{L}_d^{m''} = \mathcal{L}_d^{d'-d}$ exists.

Having obtained the ingredient sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_d^{d'-d} h\mathcal{L}_d^{m+d-d'}(k'). \quad (3.14)$$

Since $k = k' + 2(d' - d)$, then as k' in $h\mathcal{L}_d^{m+d-d'}(k')$ is varied, all k -extensions with

$$k \in [2(d' - d) + 1, 2m + 2]' \setminus \{2m + 1\} \quad (3.15)$$

are achieved in $h\mathcal{L}_d^m(k)$.

In the last half of this proof we need slightly different ingredient sequences. This time for $k' \in [1, 2m' + 5]'$ we claim there exists a $\mathcal{L}_{d-2}^{m'+2}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.5.1 are

satisfied. Since $m' = 2d' - 1 + i, 0 \leq i \leq 2$, we have $m' + 2 = 2(d' - 2) + (5 + i) = 2(d' - 2) + j, 5 \leq j \leq 7$. Also, we deduced above that for odd k' one of $(m', k') \equiv (1, d' + 1), (2, 1)$, or $(3, d') \pmod{(4, 2)}$ holds, therefore one of the following congruences holds: $(m' + 2, k') \equiv (3, (d' - 2) + 1), (0, 1)$, or $(1, (d' - 2)) \pmod{(4, 2)}$. By Theorem 3.5.1 a $\mathcal{L}_{d'-2}^{m'+2}(k') = \mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ exists for all $k' \in [1, 2m' + 5]' = [1, 2(m + d - d') + 5]'$.

Next we claim that a $h\mathcal{L}_d^{m''-2}$ exists, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.27 are satisfied. By (3.13) we have $m'' - 2 \geq 2d$ and therefore $(m'' - 2)((m'' - 2) + 1 - 2d) + 2 \geq 0$. Also we deduced above that $(m'', d) \equiv (0, 0), (0, 1), (1, 1)$, or $(3, 0) \pmod{(4, 2)}$, and therefore $(m'' - 2, d) \equiv (1, 0), (2, 0), (2, 1)$, or $(3, 1) \pmod{(4, 2)}$. It follows from Theorem 2.2.27 that a $h\mathcal{L}_d^{m''-2} = h\mathcal{L}_d^{d'-d-2}$ exists.

Having proved the existence of the new ingredients sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_{d'-2}^{m+d-d'+2}(k') h\mathcal{L}_d^{d'-d-2}. \quad (3.16)$$

Since $k = k'$, then as k' in $\mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ is varied, all k -extensions with

$$k \in [1, 2(m + d + d') + 5]' \quad (3.17)$$

are achieved in $h\mathcal{L}_d^m(k)$.

To complete the proof it is enough to show that the interval in (3.15) overlaps with the interval in (3.17). This means showing that $2(d' - d) + 1 \leq 2(m + d - d') + 5$, which is equivalent to $2d' \leq m + 2d + 2$. However, by the definition of d' we have $m + 2d + 2 = (m + d - 1) + (d + 3) = (3d' - 2 + i) + (d + 3) = 3d' + 1 + i + d$, so the inequality follows. ■

Lemma 3.6.6. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m \geq 8d + 2$,
2. $k \in [1, 2m + 2]' \setminus \{2m + 1\}$,
3. $[d, d' - 1]$ is not a perfect set and $[d', d + m - 1]$ is a perfect set, where $m + d - 1 = (3d' - 2) + i, 0 \leq i \leq 2$.

Proof:

Let d' and i be as above, let $m' = m + d - d' = 2d' - 1 + i$, and let k' be any integer with $k' \in [1, 2m' + 1]' = [1, 2(m + d - d') + 1]'$. We claim that for these values of d', m' and k' there exists a $\mathcal{L}_{d'}^{m'}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.6.1 are satisfied.

Directly from the definitions of d', m' and k' we have that $m' = 2d' - 1, 2d'$, or $2d' + 1$, and that $k' \in [1, 2m' + 2]$. The number of differences in the set $[d', d + m - 1]$ is $m' = m + d - d'$, and this set is perfect, so by Lemma 3.1.2 one of the following congruences holds: $(m', d') \equiv (0, 0), (0, 1), (1, 1)$, or $(3, 0) \pmod{(4, 2)}$. There are now just three possibilities for (m', k') modulo $(4, 2)$ (recall that k' was defined to be an odd integer):

- (a) If $m' \equiv 0 \pmod{4}$, then $(m', k') \equiv (0, 1) \pmod{(4, 2)}$.
- (b) If $(m', d') \equiv (1, 1) \pmod{(4, 2)}$, then $(m', k') \equiv (1, d') \pmod{(4, 2)}$.
- (c) If $(m', d') \equiv (3, 0) \pmod{(4, 2)}$, then $(m', k') \equiv (3, d' + 1) \pmod{(4, 2)}$.

Therefore one of the following congruences holds: $(m', k') \equiv (0, 1), (1, d')$, or $(3, d' + 1) \pmod{(4, 2)}$. It now follows from Theorem 3.6.1 that a $\mathcal{L}_{d'}^{m'}(k')$ = $\mathcal{L}_{d'}^{m+d-d'}(k')$ exists for all $k' \in [1, 2m' + 1]' = [1, 2(m + d - d') + 1]'$.

Let $m'' = d' - d$. We now claim that there exists a $h\mathcal{L}_d^{m''} = h\mathcal{L}_d^{d'-d}$, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.27 are satisfied. Since $m \geq 8d + 2$, we have $3d' - 2 + i = m + d - 1 \geq 9d + 1$, so $d' \geq 3d + 1 - (i/3)$, where $0 \leq i \leq 2$ and d' is an integer. Hence

$$m'' = d' - d \geq 2d + 1. \quad (3.18)$$

The number of differences in the set $[d, d' - 1]$ is $m'' = d' - d$, so by (3.18) we have $m''(m'' + 1 - 2d) + 2 \geq 0$. Also $[d, d' - 1]$ is not a perfect set, so by Lemma 3.1.2 we know that (m'', d) satisfies one of the congruences $(m'', d) \equiv (1, 0), (2, 0), (2, 1)$, or $(3, 1) \pmod{(4, 2)}$. It follows from Theorem 2.2.27 that a $h\mathcal{L}_d^{m''} = h\mathcal{L}_d^{d'-d}$ exists.

Having obtained the ingredient sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_{d'}^{m+d-d'}(k') h\mathcal{L}_d^{d'-d}. \quad (3.19)$$

Since $k = k'$, then as k' in $\mathcal{L}_{d'}^{m+d-d'}(k')$ is varied, all k -extensions with

$$k \in [1, 2(m + d - d') + 1]' \quad (3.20)$$

are achieved in $h\mathcal{L}_d^m(k)$.

In the last half of this proof we need slightly different ingredient sequences. This time for $k' \in [1, 2m' + 6]' \setminus \{2m' + 5\}$ we claim there exists a $h\mathcal{L}_{d'-2}^{m'+2}(k')$, and we shall prove this by verifying that the sufficient conditions of Theorem 3.5.2 are satisfied. Since $m' = 2d' - 1 + i, 0 \leq i \leq 2$, we have $m' + 2 = 2(d' - 2) + (5 + i) = 2(d' - 2) + j, 5 \leq j \leq 7$. Also, we deduced above that for odd k' one of $(m', k') \equiv (0, 1), (1, d'), \text{ or } (3, d' + 1) \pmod{(4, 2)}$ holds, therefore one of the following congruences holds: $(m' + 2, k') \equiv (2, 1), (3, (d' - 2)), \text{ or } (1, (d' - 2) + 1) \pmod{(4, 2)}$. By Theorem 3.5.2 a $h\mathcal{L}_{d'-2}^{m'+2}(k') = h\mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ exists for all $k' \in [(m' + 2) + 2, 2(m' + 2) + 2]' \setminus \{2(m' + 2) + 1\} = [(m + d - d') + 4, 2(m + d - d') + 6]' \setminus \{2(m + d - d') + 5\}$.

Next we claim that a $\mathcal{L}_d^{m''-2}$ exists, and we shall prove this by verifying that the sufficient conditions of Theorem 2.2.23 are satisfied. By (3.18) we have $m'' - 2 \geq 2d - 1$. Also we deduced above that $(m'', d) \equiv (1, 0), (2, 0), (2, 1), \text{ or } (3, 1) \pmod{(4, 2)}$, and therefore $(m'' - 2, d) \equiv (0, 0), (0, 1), (1, 1), \text{ or } (3, 0) \pmod{(4, 2)}$. It follows from Theorem 2.2.23 that a $\mathcal{L}_d^{m''-2} = \mathcal{L}_d^{d'-d-2}$ exists.

Having proved the existence of the new ingredient sequences, we form the concatenation

$$h\mathcal{L}_d^m(k) = \mathcal{L}_d^{d'-d-2} h\mathcal{L}_{d'-2}^{m+d-d'+2}(k'). \quad (3.21)$$

Since $k = k' + 2(d' - d - 2)$, then as k' in $h\mathcal{L}_{d'-2}^{m+d-d'+2}(k')$ is varied, all k -extensions with

$$k \in [m + d' - d, 2m + 2]' \setminus \{2m + 1\} \quad (3.22)$$

are achieved in $h\mathcal{L}_d^m(k)$.

To complete the proof it is enough to show that the interval in (3.20) overlaps with the interval in (3.22). This means showing that $m + d' - d \leq 2(m + d - d') + 1$, which is equivalent to $3d' \leq m + 3d + 1$. However, by the definition of d' we have $m + 3d + 1 = 3d' + i + 2d$, so this last inequality clearly holds. ■

Now we use Lemmas 3.6.3 through 3.6.6 to prove the main result on constructing hooked extended Langford sequences.

Theorem 3.6.7. *The following are sufficient for the existence of a right-hooked k -extended Langford sequence of defect d and m differences, $h\mathcal{L}_d^m(k)$:*

1. $m \geq 2d - 1$ and $m \notin [2d + 2, 8d + 4]$,

2. $k \in [1, 2m + 2] \setminus \{2m + 1\}$,
3. $(m, k) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}$.
4. $(d, m, k) \notin \{(1, 1, 2), (2, 3, 2)\}$.

Proof:

If $2d - 1 \leq m \leq 2d + 1$, then the result is that of Theorem 3.6.2. So suppose that $m \geq 8d + 5$, and write $d + m - 1 = 3d' - 2 + i$ where $0 \leq i \leq 2$.

Consider first the case that k is even. In that case, since one of the congruences $(m, k) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}$ holds, it then follows that one of the congruences $(m, d) \equiv (0, 0), (0, 1), (1, 1), \text{ or } (3, 0)$ holds. By Lemma 3.1.2 this means that $[d, d + m - 1]$ is a perfect set. Since $[d, d + m - 1] = [d, d' - 1] \cup [d', m + d - 1]$, this means that $[d, d' - 1]$ and $[d', m + d - 1]$ are both perfect, or both not perfect, hence a $h\mathcal{L}_d^m(k)$ exists by (resp.) either Lemma 3.6.3 or Lemma 3.6.4.

Consider next the case that k is odd. In that case, since one of the congruences $(m, k) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}$ holds, it then follows that one of the congruences $(m, d) \equiv (1, 0), (2, 0), (2, 1), \text{ or } (3, 1)$ holds. By Lemma 3.1.2 this means that $[d, d + m - 1]$ is not a perfect set. Since $[d, d + m - 1] = [d, d' - 1] \cup [d', m + d - 1]$, this means that one of $[d, d' - 1]$ and $[d', m + d - 1]$ is perfect and the other is not perfect, hence a $h\mathcal{L}_d^m(k)$ exists by Lemma 3.6.5 or Lemma 3.6.6.

This completes the proof. ■

One can get a stronger intuition of the process carried out in the proof of Theorem 3.6.7 using Table 3.13. This example gives the two partitions and the specific perfect/non-perfect status case of the first partition. Columns pertaining to the first stage and second stage partitions are marked by ¹ and ², respectively. This example shows the first basic period of $d + m - 1$, from the initial value 46 to 58, given defect 5.

3.7 Final remarks on hooked extended Langford sequences.

The many tables in this chapter were constructed to contain two jaws with pivots. From empirical evidence it seemed that having another two jaws

ser#	d'	$d + m - 1$	Simpson's sequence ¹	D_i^1	Simpson's sequence ²	D_{i+6}^2	case ¹
(1)	17	50		hD_1		D_7	P, NP
(2)		51		hD_2		D_8	P, NP
(3)	18	52	$\mathcal{L}_5^{13}$	hD_0	$h\mathcal{L}_5^{11}$	D_6	P, P
(4)		53		hD_1		D_7	P, P
(5)		54		hD_2		D_8	P, NP
(6)	19	55	$h\mathcal{L}_5^{14}$	D_0	$\mathcal{L}_5^{12}$	hD_6	NP, P
(7)		56		D_1		hD_7	NP, NP
(8)		57		D_2		hD_8	NP, NP
(9)	20	58	$h\mathcal{L}_5^{15}$	D_0	$\mathcal{L}_5^{13}$	hD_6	NP, NP
(10)		59		D_1		hD_7	NP, P
(11)		60		D_2		hD_8	NP, P
(12)	21	61	$h\mathcal{L}_5^{16}$	hD_0	$\mathcal{L}_5^{14}$	D_6	P, P
(13)		62		hD_1		D_7	P, NP
(14)		63		hD_2		D_8	P, NP

Table 3.13: A demonstration how the proof of Theorem 3.6.7 works for defect 5 and $m \in [46, 58]$. The first partition $[d, d' - 1] \cup [d', d + m - 1]$ ¹ is followed by the second partition $[d, d' - 3] \cup [d' - 2, d + m - 1]$ ² followed by the perfect(P) and/or non-perfect(NP) status for the first partition.

containing the majority of the large differences worked well. Moreover, these two new jaws could reciprocate with the two pivot jaws to completely cover large segments of the set of positions. The formalization of each of these tables was done by a computer program receiving as input all possible pairs of two sequences with consecutive parameters and checking if these two define a viable infinite family.

To complete the general existence problem for hooked k -extended Langford sequences, it is therefore enough to give constructions of hooked extended Langford sequences with $m = 2d - 2$ and the “No Man’s Land:” $m \in [2d + 2, 8d + 4]$. For small values of d this may be done directly, as for $d = 4$ in the case of extended Langford sequences [38]. We hope that the techniques used here can be applied to other problems and may inspire a complete solution of the problem of existence of the extended and hooked extended Langford sequence.

Chapter 4

Partitions of $[1, 2m - 1] \cup \{L\}$ into Differences in $[d, d + m - 1]$: Stretched Langford Sequences

In this chapter we consider the problems of constructing stretched and hooked stretched Langford sequences. Hooked sequences are often used in constructions by joining two sequences at the hook:

$$83743564875\mathbf{26211}$$

Extended Langford sequences can be thought of as containing a generalization of a hook; stretched sequences, with a “wide hook”, are the complementary object that could be used in such constructions:

$$67845964758\mathbf{1139232} \tag{4.1}$$

We will use constructive results about extended and hooked-extended Langford sequences from [38, 42] to build stretched and hooked stretched Langford sequences. In each case the construction method will be similar: for a given stretch of hook we will use an extended sequence with an appropriate k and insert a new difference, letting $s_k = (L - k)$ and $s_L = (L - k)$. In Chapter 5.3 we give an application of stretched Langford sequences to the construction of class uniformly resolvable designs.

4.1 Stretched Langford sequences: definitions, necessary and sufficient conditions

We recall the definitions of an L -stretched Langford sequence and a left-hooked L -stretched Langford sequence.

Definition 2.2.36. For $L \geq 2m$, an L -stretched Langford sequence of defect d and m differences is an integer sequence $\mathcal{L}_d^m[L] = s_1, s_2, \dots, s_L$ that satisfies conditions (1) and (2) for a Langford sequence, and in addition satisfies (3) $s_{2m} = s_{2m+1} = \dots = s_{L-1} = 0$.

Definition 2.2.39. For $L \geq 2m + 1$, a left-hooked L -stretched Langford sequence of defect d and m differences is an integer sequence $h\mathcal{L}_d^m[L] = s_1, s_2, \dots, s_L$ that satisfies conditions (1) and (2) for a Langford sequence, and in addition satisfies (3) $s_2 = s_{2m+1} = s_{2m+2} = \dots = s_{L-1} = 0$.

By the remarks following these definitions in Chapter 2, an L -stretched Langford sequence of defect d and m differences is equivalent to a partition of the position set $P = [1, 2m-1] \cup \{L\}$ into differences in $D = [d, d+m-1]$, and a left-hooked L -stretched Langford sequence of defect d and m differences is equivalent to a partition of the position set $P = [1, 2m] \cup \{L\} \setminus \{2\}$ into differences in $D = [d, d+m-1]$. We first establish the necessary conditions for these sequences to exist.

Lemma 4.1.1. *The following are necessary for the existence of an L -stretched Langford sequence of defect d and m differences, $\mathcal{L}_d^m[L]$:*

1. $m \geq 2d - 2$ or $m = 1$, and
 - (a) If $m = 2d - 2$, then $L = 5d - 5$,
 - (b) If $m = 2d - 1$, then $4d - 2 \leq L \leq 6d - 4$,
 - (c) If $m \geq 2d$, then $2m \leq L \leq 3m + d - 2$.
2. $(m, L) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}$.

Proof:

If a $\mathcal{L}_d^m[L]$ exists, then there is a partition of $P = [1, 2m-1] \cup \{L\}$ into

differences in $D = [d, d + m - 1]$. In the notation of Theorem 2.1.2 we have

$$\sum_{i=1}^m d_i = \sum_{i=1}^m (d + i - 1) = md + m(m - 1)/2, \quad (4.2)$$

$$\sum_{i=1}^{2m} p_i = \left(\sum_{i=1}^{2m-1} i \right) + L = m(2m - 1) + L. \quad (4.3)$$

Parity condition (2.1) is $\sum d_i \equiv \sum p_i \pmod{2}$ so by the above and some algebra $L \equiv md + m(m + 1)/2 \pmod{2}$; the congruences on (m, L) follow.

We have $L \geq 2m$ from the definition of an L -stretched Langford sequence, and clearly L is at most the largest difference plus the second-largest position, so $L \leq (2m - 1) + (d + m - 1) = 3m + d - 2$. Density condition (2.2) gives

$$\begin{aligned} \sum_{i=1}^m d_i &= md + m(m - 1)/2 \\ &\leq ((m + 1) + \cdots + (2m - 1) + L) - (1 + \cdots + m) \\ &= m^2 - 2m + L, \end{aligned} \quad (4.4)$$

and isolating L gives $m(2d + 3 - m)/2 \leq L$. Putting these together gives

$$\max\{2m, m(2d + 3 - m)/2\} \leq L \leq 3m + d - 2, \quad (4.5)$$

from which follow (1c), the lower bound of (1b) and the lower bound of (1a), i.e., $5d - 5 \leq L$. It now suffices to prove (1) and the upper bounds in (1a), (1b) under the assumptions $m \leq 2d - 1$ and $m > 1$.

Some difference $\delta \in D = [d, d + m - 1]$ is in positions $L, L - \delta$, so there exists a partition of $P' = [1, 2m - 1] \setminus \{L - \delta\}$ into differences in $D' = [d, d + m - 1] \setminus \{\delta\}$. The sum of the differences in D' is

$$\sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i = m(m + 2d - 1)/2 - \delta,$$

which is bounded below by

$$m(m + 2d - 1)/2 - (m + d - 1).$$

By the density condition (2.2), this must be less than the sum of the $m - 1$ largest positions minus the sum of the $m - 1$ smallest positions in P' . What

this sum is exactly depends on where $L - \delta$ is in P' but it achieves its maximum, $m(m - 1)$, when $L - \delta = m$. Putting these together we get

$$\frac{m(m + 2d - 1)}{2} - (m + d - 1) \leq m(m - 1).$$

This reduces to

$$0 \leq (m - (2d - 2))(m - 1),$$

from which we can conclude that $m \geq 2d - 2$ because we can assume that $m > 1$.

To establish the upper bounds in (1a) and (1b) we consider two cases according to the position of $L - \delta$ with respect to $m - 1$. If $L - \delta \leq m - 1$ then

$$L \leq m - 1 + \delta \leq 2m + d - 2.$$

When $m = 2d - 2$, this contradicts our previous lower bound, and when $m = 2d - 1$ we get $L \leq 5d - 4 < 6d - 4$.

When $L - \delta \geq m$ the density condition (2.2) for P' and D' is

$$\begin{aligned} \sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i &= m(m + 2d - 1)/2 - \delta \\ &\leq (m + \dots + (2m - 1) - (L - \delta)) - (1 + \dots + (m - 1)) \\ &= m^2 - L + \delta. \end{aligned} \tag{4.6}$$

From (4.5) we have $m(2d - m + 3)/2 \leq L$, so isolating 2δ in (4.6) gives

$$m(2d + 1 - m) \leq L + m(2d - 1 - m)/2 \leq 2\delta \leq 2(m + d - 1). \tag{4.7}$$

Finally from (4.7) we get $L \leq 2(m + d - 1) - m(2d - 1 - m)/2$, from which the upper bounds of (1a) and (1b) follow. ■

In a similar vein we establish the necessary conditions for left-hooked stretched Langford sequences.

Lemma 4.1.2. *The following are necessary for the existence of a left-hooked L -stretched Langford sequence of defect d and m differences, $h\mathcal{L}_d^m[L]$:*

1. $m \geq 2d - 2$ or $m = 1$ or $(d, m, L) = (3, 2, 7), (3, 3, 9)$, and
 - (a) If $m = 2d - 2$, then $L = 5d - 5$ or $5d - 3$,

- (b) If $m = 2d - 1$, then $4d - 1 \leq L \leq 6d - 2$,
- (c) If $m \geq 2d$, then $2m + 1 \leq L \leq 3m + d - 1$;
- 2. $(m, L) \equiv (0, 0), (1, d + 1), (2, 1), \text{ or } (3, d) \pmod{(4, 2)}$,
- 3. $(d, m, L) \notin \{(1, 1, 4), (2, 3, 8)\}$.

Proof:

The congruence on (m, L) is the same as in Lemma 4.1.1 since in the present case $P = [1, 2m - 1] \cup \{L\}$ is altered by changing position 2 to position $2m$, and each is even. Due to this alteration, the sum of the m largest positions goes up by $m - 1$, and the sum of the m smallest positions also goes up by $m - 1$, so the bound $m(2d + 3 - m)/2 \leq L$ is unchanged; but the upper bound on L is one larger as the second largest position is one larger. The analog to (4.5) is

$$\max\{2m + 1, m(2d + 3 - m)/2\} \leq L \leq 3m + d - 1. \quad (4.8)$$

Now (1c) and the lower bounds in (1a),(1b) follow from (4.8). When $m \leq 3$, Equation 4.8 gives $m \leq 4$. If $d = 4$ and $(m = 2 \text{ or } m = 3)$ it is easy to verify that $L = 9$ or 12 respectively and that these left-hooked L -stretched Langford sequences do not exist. Thus the only $h\mathcal{L}_d^m[L]$ with $m \leq 3$ are:

$$\begin{aligned} h\mathcal{L}_d^1[d + 1] &= d, 0^{d-1}, d \quad (d > 1); \\ h\mathcal{L}_1^2[5] &= 2, 0, 2, 1, 1; \\ h\mathcal{L}_2^2[5] &= 3, 0, 2, 3, 2; \\ h\mathcal{L}_2^2[7] &= 2, 0, 2, 3, 0, 0, 3; \\ h\mathcal{L}_3^2[7] &= 3, 0, 4, 3, 0, 0, 4; \\ h\mathcal{L}_1^3[7] &= 3, 0, 2, 3, 2, 1, 1 \\ &\text{(or } h\mathcal{L}_1^3[7] = 2, 0, 2, 3, 1, 1, 3); \\ h\mathcal{L}_1^3[9] &= 2, 0, 2, 1, 1, 3, 0, 0, 3; \\ h\mathcal{L}_2^3[10] &= 3, 0, 2, 3, 2, 4, 0, 0, 0, 4; \\ h\mathcal{L}_3^3[9] &= 4, 0, 3, 4, 5, 3, 0, 0, 5. \end{aligned}$$

The $h\mathcal{L}_2^3[8] = h\mathcal{L}_2^3(7)$ and $h\mathcal{L}_3^3[7] = h\mathcal{L}_3^3$ do not exist because the necessary conditions for the equivalent non-stretched sequences are not satisfied. So if $m \leq 3$, then all necessary conditions are satisfied, so we may take $4 \leq m \leq 2d - 1$ in what follows.

Some difference $\delta \in D = [d, d + m - 1]$ is in positions $L, L - \delta$, so there exists a partition of $P' = [1, 2m] \setminus \{2, L - \delta\}$ into differences in $D' =$

$[d, d + m - 1] \setminus \{\delta\}$. The sum of the differences in D' is

$$\sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i = m(m + 2d - 1)/2 - \delta,$$

which is bounded below by

$$m(m + 2d - 1)/2 - (m + d - 1).$$

By the density condition (2.2), this must be less than the sum of the $m - 1$ largest positions minus the sum of the $m - 1$ smallest positions. What this sum is exactly depends on where $L - \delta$ is in P' but it achieves its maximum, $m(m - 1) + 1$, when $L - \delta = m + 1$. Putting these together we get

$$\frac{m(m + 2d - 1)}{2} - (m + d - 1) \leq m(m - 1) + 1.$$

This reduces to

$$-2 \leq (m - (2d - 2))(m - 1),$$

from which we can conclude that $m \geq 2d - 2$ because we can assume that $m \geq 4$.

To establish the upper bounds in (1a) and (1b) we consider two cases: the position of $L - \delta$ with respect to m . If $L - \delta \leq m$, then

$$L \leq m + \delta \leq 2m + d - 1.$$

When $m = 2d - 2$, this reduces to $L \leq 5d - 5 < 5d - 3$, and when $m = 2d - 1$, we get $L \leq 5d - 3 < 6d - 2$.

When $L - \delta \geq m + 1$, the density condition (2.2) for P' and D' is

$$\begin{aligned} \sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i &= m(m + 2d - 1)/2 - \delta \\ &\leq [(m + 1) + \dots + 2m - (L - \delta)] - (1 + \dots + (m - 2)) \\ &= m^2 + 2 - L + \delta. \end{aligned} \tag{4.9}$$

From (4.8) we have $m(m - 2d + 3)/2 \leq L$, so isolating 2δ in (4.9) gives

$$\begin{aligned} m(2d + 1 - m) - 2 &\leq L + m(2d - 1 - m)/2 - 2 \\ &\leq 2\delta \leq 2(m + d - 1). \end{aligned} \tag{4.10}$$

From Equation (4.10) we get $L \leq 2(m + d - 1) - m(2d - 1 - m)/2 + 2$, from which the upper bounds of (1a) and (1b) follow. Finally, when $m = 2d - 2$ the congruence on (m, L) shows that $L \neq 5d - 4$. ■

We now develop constructions for stretched Langford sequences. While a $h\mathcal{L}_d^m(k)$ is right-hooked, it is nevertheless more convenient to state the next result for a left-hooked version of the extended Langford sequence.

Lemma 4.1.3 (Adding difference $d+m-1$.) *If there exists a $\mathcal{L}_d^{m-1}(k)$ with $k \geq m - d + 1$, then there exists a $\mathcal{L}_d^m[k+d+m-1]$. If there exists a left-hooked $h\mathcal{L}_d^{m-1}(k)$ with $k \geq m - d + 2$, then there exists a $h\mathcal{L}_d^m[k+d+m-1]$.*

Proof:

To each sequence add difference $d+m-1$ at positions k and $k+d+m-1$. ■

Example 4.1.4. *For $d = 2, m = 4$, the possible L are 8, 10 and 12. When $L = 8$, this is simply a Langford sequence:*

$$5, 2, 4, 2, 3, 5, 4, 3.$$

For $L = 10$ we start with an extended Langford sequence 2, 4, 2, 3, 0, 4, 3 and add the difference 5 at positions 5 and 10 to get

$$2, 4, 2, 3, 5, 4, 3, 0, 0, 5.$$

(Note that this is also a $h\mathcal{L}_2^5(8)$.) For $L = 12$ we start with the Langford sequence 4, 2, 3, 2, 4, 3 and add the difference 6 at positions 7 and 12 to get

$$4, 2, 3, 2, 4, 3, 5, 0, 0, 0, 0, 5$$

Example 4.1.5. *To construct a partition of $P = [1, 10] \cup \{15\} \setminus \{2\}$ into $D = [2, 6]$ start with 5, 0, 2, 4, 2, 5, 3, 4, 0, 3 and add the difference 6 at positions 9 and 15 to get*

$$5, 0, 2, 4, 2, 5, 3, 4, 6, 3, 0, 0, 0, 0, 6.$$

Lemma 4.1.6 (Adding difference d .) *If there exists a $\mathcal{L}_{d+1}^{m-1}(k)$ with $k \geq 2m - d$, then there exists a $\mathcal{L}_d^m[k+d]$. If there exists a left-hooked $h\mathcal{L}_{d+1}^{m-1}(k)$ with $k \geq 2m - d + 1$, then there exists a $h\mathcal{L}_d^m[k+d]$.*

Proof:

To each sequence add difference d at positions k and $k + d$. ■

From these lemmas and the results known about extended Langford sequences, namely Theorem 2.2.13 [4], Theorem 2.2.18 [37], Theorem 2.2.31 [36, 38] and Theorem 3.6.7, we can derive some sufficient conditions for stretched and left-hooked stretched Langford sequences to exist.

Theorem 4.1.7. *Assume the necessary conditions of Lemma 4.1.1 are satisfied. Any condition below is then sufficient for the existence of an L -stretched Langford sequence of defect d and m differences, $\mathcal{L}_d^m[L]$:*

1. $m = 2d - 2$ or $m = 1$;
2. $L = 2m, 2m + 1$, or $3m + d - 2$;
3. $L = 2m + 2$, $m \geq 2d - 1$ and $m \notin [2d + 2, 8d + 4]$;
4. $d = 1, 2, 3$, or 4 ;
5. $2m + 1 < L < 3m + d - 2$, $m \geq 2d$, and $m \notin [2d + 3, 2d + 5] \cup [2d + 9, 8d - 4]$;
6. $2m < L < 2m + d - 1$, $m \geq 2d + 2$, and $m \notin [2d + 5, 2d + 7] \cup [2d + 11, 8d + 4]$.

Proof:

1. If $m = 2d - 2$, then $L = 5d - 5$ by Lemma 4.1.1(1), and a $\mathcal{L}_d^{2d-2}[5d - 5]$ comes from a $\mathcal{L}_d^{2d-3}(2d - 2)$ (given in [38], p. 423, first paragraph) and Lemma 4.1.3. Case $m = 1$ is trivial.

2. We have $\mathcal{L}_d^m[2m] = \mathcal{L}_d^m$ and $\mathcal{L}_d^m[2m + 1] = h\mathcal{L}_d^m$, and these exist by (resp.) Theorem 2.2.23 and Theorem 2.2.27. If $L = 3m + d - 2$, then take a $\mathcal{L}_d^{m-1}$ and add the difference $m + d - 1$ at positions $2m - 1$ and $3m + d - 2$.

3. Reversing a $\mathcal{L}_d^m[2m + 2]$ yields a $h\mathcal{L}_d^m(3)$ whose current existence conditions are described by Theorem 3.6.7.

4. For $m \geq 2d$ or $d = 1, 2$ apply Lemma 4.1.3 to the extended Langford sequences in Theorem 2.2.13 and Theorem 2.2.31. By the above cases and Theorem 3.6.2, it only remains to give sequences for $d = 3, 4$; $m = 2d - 1$ and $2m + 2 = 4d \leq L \leq 6d - 4$:

$$\mathcal{L}_3^5[12] = h\mathcal{L}_3^5(10) \text{ (Theorem 3.6.2);}$$

$$\begin{aligned}
\mathcal{L}_3^5[14] &= 4, 6, 3, 5, 4, 3, 7, 6, 5, 0, 0, 0, 0, 7; \\
\mathcal{L}_4^7[16] &= 6, 7, 8, 9, 5, a, 6, 4, 7, 5, 8, 4, 9, 0, 0, a; \\
\mathcal{L}_4^7[18] &= 9, 4, 8, 5, 7, 4, 6, a, 5, 9, 8, 7, 6, 0, 0, 0, a; \\
\mathcal{L}_4^7[20] &= 7, 5, 8, 9, 4, 6, 5, 7, 4, a, 8, 6, 9, 0, 0, 0, 0, a.
\end{aligned}$$

5. By Theorem 2.2.31, Theorem 3.5.1 and Theorem 3.6.7, a $\mathcal{L}_d^m(k)$ exists whenever the necessary conditions are satisfied and $m \in [2d - 1, 2d + 1] \cup [2d + 5, 2d + 7] \cup [8d - 4, \infty)$. Add difference $d + m$ to these extended sequences as in Lemma 4.1.3 to get $\mathcal{L}_d^{m+1}[L]$ for all admissible L , and with $m + 1 \in [2d, 2d + 2] \cup [2d + 6, 2d + 8] \cup [8d - 3, \infty)$. Now replace $m + 1$ by m , and take the set-complement.

6. Add difference $d - 1$ to the extended sequences in Case 5 according to Lemma 4.1.6 to get $\mathcal{L}_d^{m'}[L]$ for $d' = d - 1$, $m' = m + 1$, $L \leq 2m' + d' - 1$ and with $m' = m + 1 \in [2d' + 2, 2d' + 4] \cup [2d' + 8, 2d' + 10] \cup [8d' + 5, \infty)$. Replace m', d' by m, d , and take the set-complement to get the result. ■

We have a similar result for left-hooked L -stretched Langford sequences.

Theorem 4.1.8. *Assume the necessary conditions of Lemma 4.1.2 are satisfied. Any condition below is then sufficient for the existence of a left-hooked L -stretched Langford sequence of defect d and m differences, $h\mathcal{L}_d^m[L]$:*

1. $m = 2d - 2$ or $m = 1$ or $(d, m, L) = (3, 2, 7), (3, 3, 9)$;
2. $L = 2m + 1$ or $3m + d - 1$;
3. $d = 1, 2, 3$;
4. $2m + 1 < L < 3m + d - 1, m \geq 2d$ and $m \notin [2d + 3, 8d + 5]$;
5. $2m + 1 < L < 2m + d, m \geq 2d + 2$ and $m \notin [2d + 5, 8d + 13]$.

Proof:

1. Sequences $h\mathcal{L}_d^{2d-3}(2d-3)$, $d \geq 2$, and $h\mathcal{L}_d^{2d-3}(2d-1)$, $d \geq 3$, are given in the next to last paragraph of Section 3.1. Reverse these and apply Lemma 4.1.3 to get (resp.) $h\mathcal{L}_d^{2d-2}[5d-3]$, $d \geq 2$, and $h\mathcal{L}_d^{2d-2}[5d-5]$, $d \geq 3$. If $d = 2$, then $h\mathcal{L}_d^{2d-2}[5d-5] = h\mathcal{L}_2^2[5]$ is a reverse $\mathcal{L}_2^2(2)$ (Theorem 2.2.23). Case $m = 1$ is trivial and the two sporadic sequences are in the proof of Theorem 4.1.2.

2. For $L = 2m + 1$ reverse a $h\mathcal{L}_d^m$ (Theorem 2.2.23). For $L = 3m + d - 1$ reverse a $h\mathcal{L}_d^{m-1}$ and add the difference $m + d - 1$ at positions $2m$ and $3m + d - 1$.

3. By the above and the construction of all $h\mathcal{L}_d^m[L]$ with $m \leq 3$ in the proof of Theorem 4.1.2, it remains to give $h\mathcal{L}_d^m[L]$ for

(1b)' $m = 2d - 1 \geq 5$, $L \in [4d, 6d - 2]$,

(1c)' $m \geq 2d \geq 4$, $L \in [2m + 2, 3m + d - 2]$,

L of correct parity and (d, m, L) not in the list of exceptions. For $d = 1$ apply Lemma 4.1.3 to the $h\mathcal{S}_n(k)$ of Theorem 2.2.18 to get all required $h\mathcal{L}_1^m[L]$. For $d = 2$ apply Lemma 4.1.3 to the $h\mathcal{L}_2^m(k)$ of Theorem 2.2.35 to get all required $h\mathcal{L}_2^m[L]$ except $h\mathcal{L}_2^4[12] = 5, 0, 2, 3, 2, 5, 3, 4, 0, 0, 4$. For $d = 3$ apply Lemma 4.1.3 to the $h\mathcal{L}_d^m(k)$ ($d = 3$) of Theorem 3.6.7 to get all required $h\mathcal{L}_3^m[L]$ except $h\mathcal{L}_3^5[12] = 5, 0, 6, 4, 7, 5, 3, 4, 6, 3, 0, 7$; $h\mathcal{L}_3^5[14] = 4, 0, 5, 6, 4, 3, 7, 5, 3, 6, 0, 0, 0, 7$; $h\mathcal{L}_3^5[16] = 6, 0, 3, 4, 5, 3, 6, 4, 7, 5, 0, 0, 0, 0, 7$.

4. By Theorem 3.6.2 and Theorem 3.6.7 a $h\mathcal{L}_d^m(k)$ exists for all admissible k whenever $m \in [2d - 1, 2d + 1] \cup [8d + 5, \infty)$. Reverse the sequences and add difference $d + m$ as in Lemma 4.1.3 to get $h\mathcal{L}_d^{m+1}[L]$ for all admissible L and $m + 1 \in [2d, 2d + 2] \cup [8d + 6, \infty)$. Now replace $m + 1$ by m and take set complements.

5. Proceed as in the preceding case, but apply Lemma 4.1.6 instead. ■

4.2 Final remarks on stretched Langford sequences

We construct $\mathcal{L}_d^m[L]$ sequences primarily from $\mathcal{L}_d^{m-1}(k)$ or $\mathcal{L}_{d+1}^{m-1}(k)$ sequences, with similar constructions in the hooked case. Our central Theorems 4.1.7 and 4.1.8 make it plain that once the necessary conditions for (hooked) extended Langford sequences are shown to be sufficient, then the existence of (hooked) stretched Langford sequences will be settled. *A priori* this strong connection is not obvious, as position L in a (hooked) stretched Langford sequence need not be occupied by the largest or smallest difference, but it becomes evident after Lemmas 4.1.1 and 4.1.2 are carefully worked out.

Given the necessary conditions formulated in Lemmas 4.1.1 and 4.1.2, we can restate the global behavior of the central Theorems 4.1.7 and 4.1.8 in three ways:

1. For a given d and m , where m is sufficiently large with respect to d , every possible L of the correct parity and in the correct range can be achieved.
2. For a given d and $L \geq 16d$, then almost every $8d \leq m \leq L/2$ in the correct classes modulo 4 can be achieved.
3. For a given m and $L \geq 2m$ sufficiently close to $2m$, almost every d of the correct parity in the range $L - 3m \leq d \leq m/2$ can be achieved.

The primary motivation for the definition and construction of stretched Langford sequences is the construction of partitions of an interval into collections of sets that contain given differences a required number of times. For example using several Langford sequences to construct a longer Langford sequence. Two hooked sequences can be “hooked” together to form a longer sequence that is still a partition of the longer interval into the multi-set union of the differences from the two sequences. The “wide hook” is complementary to the extensions in an extended Langford sequence for this same application. Here we think of the extension as a “deep” hook. For example, we can “hook” together an extended Langford sequence with $d = 1$, $m = 3$ and $k = 4$, and a stretched Langford sequence with $d = 4$, $m = 6$ and $L = 15$ as shown in Equation 4.1 to construct a Skolem sequence with nine differences.

Chapter 5

Applications of Skolem-Langford Sequences.

In this chapter we survey some select applications of Skolem and Langford sequences in the construction of designs. We demonstrate the constructions of triple systems and class uniformly resolvable designs. For the reader who wants to see a wider spectrum of such applications or to learn more about the topic we supply a brief discussion and references for other applications in Section 5.4.

5.1 Definitions of designs

Definition 5.1.1. *A Steiner triple system, $STS_\lambda(v)$, is a pair $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a v -set and $\mathcal{B}$ is a multiset of 3-subsets of $\mathcal{X}$, called **blocks**, such that any pair from $\mathcal{X}$ is contained in exactly λ members of $\mathcal{B}$. When $\lambda = 1$, it is often omitted.*

Steiner triple systems are examples of combinatorial designs [58]. The necessary conditions for the existence of a $STS_\lambda(v)$ are that $\lambda v(v-1) \equiv 0 \pmod{6}$ and $\lambda(v-1) \equiv 0 \pmod{2}$; these are known to be sufficient [58].

Definition 5.1.2. *A permutation of set S is a bijection from S to itself.*

Definition 5.1.3. *An automorphism of a design $(\mathcal{X}, \mathcal{B})$ is a permutation of the set $\mathcal{X}$ such that the image of every block is a block. The automorphism group of a design is the set of all its automorphisms.*

Langford sequences are most useful for constructing designs that have non-trivial automorphism groups.

Definition 5.1.4. *A design is **cyclic** if it has an automorphism of the form $(0, 1, \dots, v-1)$, where $\mathcal{X} = \{0, 1, \dots, v-1\}$. A design is **1-rotational** if it has an automorphism of the form $(\infty)(0, 1, \dots, v-2)$, where $\mathcal{X} = \{\infty, 0, 1, \dots, v-2\}$.*

In these cases the point set is often identified with $\mathbb{Z}_v$ or $\mathbb{Z}_{v-1} \cup \{\infty\}$. When a design has such an automorphism group we only need to present one member from each orbit of the action on the blocks. Thus such designs are easier to find and shorter to present.

Definition 5.1.5. *Let $D_1, D_2, \dots, D_t$ be sets of size 3 in $\mathbb{Z}_v$ such that the differences arising from all pairs of points in the D_i give each non-zero element of $\mathbb{Z}_v$ exactly λ times. Then these sets are said to form a $(v, 3, \lambda)$ **difference system** in $\mathbb{Z}_v$.*

Consider the sets $D_1 = \{1, 2, 5\}$ and $D_2 = \{1, 3, 9\}$ in $\mathbb{Z}_{13}$. The differences arising from D_1 are 1, 12, 4, 9, 3 and 10 and those arising from D_2 are 2, 11, 8, 5, 6 and 7. So between them, D_1 and D_2 give each non-zero element of $\mathbb{Z}_{13}$ as a difference exactly once.

Theorem 5.1.6. [58] *Let $D_1, D_2, \dots, D_t$ form a $(v, 3, \lambda)$ difference system in $\mathbb{Z}_v$. Then the sets $D_i + j, 1 \leq i \leq t, 0 \leq j \leq v-1$, are the blocks of a $STS_\lambda(v)$.*

In our example the $STS(13)$ is

$\{1, 2, 5\},$	$\{2, 3, 6\},$	$\{3, 4, 7\},$	$\{4, 5, 8\},$	$\{5, 6, 9\},$	$\{6, 7, 10\},$	$\{7, 8, 11\},$	$\{8, 9, 12\},$
$\{9, 10, 0\},$	$\{10, 11, 1\},$	$\{11, 12, 2\},$	$\{12, 0, 3\},$	$\{0, 1, 4\},$	$\{1, 3, 9\},$	$\{2, 4, 10\},$	$\{3, 5, 11\},$
$\{4, 6, 12\},$	$\{5, 7, 0\},$	$\{6, 8, 1\},$	$\{7, 9, 2\},$	$\{8, 10, 3\},$	$\{9, 11, 4\},$	$\{10, 12, 5\},$	$\{11, 0, 6\},$
$\{12, 1, 7\},$	$\{0, 2, 8\},$						

5.2 Heffter problems and cyclic triple systems

In the 1890s, Heffter and Netto studied the problem of constructing a cyclic $STS(v)$, $v \equiv 1 \pmod{6}$, [26, 46]. They were looking for a $(v, 3, 1)$ difference system. In particular they observed that some of these systems can be constructed in a particularly attractive way for some values of v .

Problem 5.2.1 (Heffter's first difference problem). [26, 46] Partition $\{1, \dots, 3n\}$ into n triples (a, b, c) such that $a + b \pm c \equiv 0 \pmod{6n+1}$. Then the set of all triples $\{0, a, a+b\}$ is a $(6n+1, 3, 1)$ -difference system.

Skolem used his sequences to solve Heffter's problem for $n \equiv 0, 1 \pmod{4}$ and suggested that hooked sequences could be used to solve the other cases of n .

Construction 5.2.2 (A cyclic Steiner triple system $\text{STS}(6n+1)$). [60] From a Skolem sequence or a right-hooked Skolem sequence of order n , we get n disjoint pairs (a_r, b_r) such that $b_r - a_r = r$, for $1 \leq r \leq n$. Then the set of all triples $\{r, a_r + n, b_r + n\}$, for $1 \leq r \leq n$, is a solution to Heffter's first difference problem. The collection of all 3-sets $\{s, r+s, a_r + n + r + s\}$, for $1 \leq r \leq n$ and $0 \leq s < 6n+1$, form a $\text{STS}(6n+1)$.

For example, take the Skolem sequence $S_4 = 42324311$. The resulting triples from Skolem's construction are $\{0, 4, 9\}, \{0, 2, 8\}, \{0, 3, 10\}, \{0, 1, 12\}$ which is a $(25, 3, 1)$ -difference system and yields a $\text{STS}(25)$. The utility of a hooked Skolem sequence to construct an $\text{STS}(v)$ when a Skolem sequence does not exist, namely for $n \equiv 2, 3 \pmod{4}$, takes advantage of the fact that in $\mathbb{Z}_v$, v odd, the distance from $(v+1)/2$ to 0 is the same as from $(v-1)/2$ to 0. For example, the right-hooked Skolem sequence 11232_3 yields the triples $\{\{0, 1, 5\}, \{0, 2, 8\}, \{0, 3, 10\}\}$ which is a $(19, 3, 1)$ -difference system. Heffter himself gave the sets $\{0, 2i-1, 4i-1\}, \{0, 2m+2i-1, 4i-3\}, 1 \leq i \leq m$, as a solution to his problem [26].

If $v \equiv 3 \pmod{6}$, say $v = 6n+3$, the number of triples in an $\text{STS}(v)$ is $v(n+1)/3$ which is not a multiple of v , so we cannot obtain a cyclic design from a difference system as in the previous case. However the method can be adapted. Peltesohn in 1938 was the first to prove that a cyclic Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \pmod{6}$ and $v \neq 9$ [51].

Problem 5.2.3 (Heffter's second difference problem). [26, 46] Partition $\{1, \dots, 3n+1\} \setminus \{2n+1\}$ into n triples (a, b, c) such that $a + b \pm c \equiv 0 \pmod{6n+3}$. Then the set of all triples $\{0, a, a+b\}$, together with the 'short block' $\{0, 2n+1, 4n+2\}$, is a $(6n+3, 3, 1)$ -difference system.

In 1966 Rosa solved Heffter's second problem, again using Skolem-type sequences:

Construction 5.2.4 (A cyclic Steiner triple system $\text{STS}(6n+3)$, $n \neq 1$). [55] From an $(n+1)$ -extended Skolem sequence or a right-hooked

$(n+1)$ -extended Skolem sequence of order n , we get n disjoint pairs (a_r, b_r) such that $b_r - a_r = r$, for $1 \leq r \leq n$. The triples $(r, a_r + n, b_r + n)$, for $1 \leq r \leq n$, are a solution to Heffter's second difference problem. The collection of all 3-sets $\{s, r + s, a_r + n + r + s\}$ for $1 \leq r \leq n$ and $0 \leq s < 6n + 3$, and $\{s, 2n + 1 + s, 4n + 2 + s\}$ for $0 \leq s \leq 2n$ form a STS($6n + 3$).

Here we use the fact that v is divisible by 3. We make the 'short block' only contain differences $\pm v/3$. For example, the 5-extended Skolem sequence $S_4(5) = 2423.4311$ gives triples $\{0, 2, 7\}$, $\{0, 4, 10\}$, $\{0, 3, 11\}$, $\{0, 1, 13\}$, $\{0, 9, 18\}$ in $\mathbb{Z}_{27}$. The hook plays the same role it did in the solution to Heffter's first difference problem. For example the right-hooked 3-extended Skolem sequence $hS_2(3) = 11_2_2$ gives the triples $\{0, 1, 4\}$, $\{0, 2, 8\}$ and $\{0, 5, 10\}$ in $\mathbb{Z}_{15}$.

5.3 Class uniformly resolvable designs

Let $\mathcal{K}$ be a set of positive integers. A **pairwise balanced design**, denoted $PBD(v, \mathcal{K}, 1)$ is a pair $(\mathcal{V}, \mathcal{B})$ where $|\mathcal{V}| = v$, and $\mathcal{B}$ is a collection of subsets of $\mathcal{V}$ called blocks. Each subset has size $k \in \mathcal{K}$ and each pair of points of $\mathcal{V}$ occurs exactly one time in the blocks.

A **parallel class** (or **resolution class**) of a design is a subset of the blocks that partition the point set $\mathcal{V}$. If the blocks can be partitioned into parallel classes, the design is called **resolvable**, and we write it as $RPBD(\mathcal{V}, \mathcal{K}, 1)$. If every parallel class of a resolvable pairwise balanced design has the same number of blocks of each size, we say that it is a **class-uniformly resolvable design**, or CURD.

Here we give a brief reference history on the research done on the existence of CURDs. Initially, Lamken et al. in 1991 introduced class-uniformly resolvable designs (CURDs) [33]. Shortly thereafter Wevrick and Vanstone in 1996 developed a classification scheme based on the ratio $a : b$ of the number of pairs to the number of triples in a CURD [63]. Danziger and Stevens settle the necessary and sufficient conditions for CURDs when $(a, b) = (3, 1)$ [16]. Keranen, Rees and Ling in 2003 constructed all possible CURDs when the ratio of the number of blocks of size 2 to the number of blocks of size 3 is $3m : 1$ for all $m \geq 1$ [30]. To get that result they made use of Skolem and Langford sequences to build CURDs for all such ratios.

We present an application of a stretched Langford sequence to build a particular CURD.

A $CURD(76, 57, 26, 8)$ is a pairwise balanced resolvable design with blocks of size 2 and 3 such that every resolution class has 26 blocks of size 2 and 8 blocks of size 3, i.e., of type $2^{26}3^8$. Using Theorem 3.4 from [16], we could build a $CURD(76, 57, 26, 8)$ if we had a CURD of type $1^{12}2^63^2$ on $v = 19$ points and with replication number $\lambda = 4$ with the additional property that the blocks of size three are precisely multiple copies of the same $\lambda = 1$ packing on 19 points. This CURD has 57 resolution classes and so could conceivably be built cyclically from three base classes.

Since the triples must be a $\lambda = 1$ packing replicated we could ask that each base resolution class contain the triples $\{0, 1, 9\}$ and $\{2, 5, 7\}$, which develops into a packing on 19 points. If we also add pairs $\{4, 10\}$ and $\{6, 18\}$, we have used each difference in $\{1, 2, 3, 5, 6, 7, 8, 9\}$ three times each and it remains to place the difference 4 four times and all other differences once more across the three base resolution classes. Since the points untouched in each class so far are an interval with two isolated points, we can fill one of the base classes with a stretched Langford sequence with parameters $d = 1, m = 4$ and $L = 10$.

Only two more classes remain to be completed, rather than three! A quick search reveals that the other two base classes can be completed with

$$\begin{aligned} &\{3, 12\}, \{8, 14\}, \{11, 16\}, \{13, 17\} \\ &\{3, 11\}, \{8, 15\}, \{12, 16\}, \{13, 17\} \end{aligned}$$

The use of the stretched Langford sequence reduced the search space dramatically. For larger designs the application should be just as useful.

5.4 Other applications and survey listing

Naturally, Chapter 5 could not cover all of the many applications but only a couple. We cite here additional references of the many other applications of Langford sequences.

There is a great deal of work on the application to constructing various designs with prescribed automorphisms, including bicyclic triple systems, 1-rotational even cycle systems, 2-fold triple systems, balanced sampling plans, Mendelsohn triples systems, etc., that are cyclic, 1-rotational, or (f, k) -rotational [6], [11] [12], [13], [14], [20] [28], [15], [24], [25], [27], [29], [41] [62], [64], [65].

Langford sequences have also been used to construct more graph theoretic objects such as 1-factorizations, m -cycle systems, graceful labellings, and colorings of Steiner quadruple systems [1], [10], [23], [35], [45], [44], [49], [52], [53].

There are a host of applications which do not fit so neatly into a single research area. These include additive permutations, information theory, radio astronomy, and ballistic missile guidance codes [2], [8], [17], [22]. Though this list of references is short, the author feels strongly that there are many other applications of this study to real world problems.

Chapter 6

Conclusion

The construction and application of Langford sequences and their generalizations is a very wide research topic and we have only been able to sample and present a small subset of it. There are many open problems and conjectures remaining to be solved. We hope that this thesis provides useful extensions (pun intended) of the definitions of Langford sequences, new construction methods and a brief but thorough survey of the body of knowledge. We hope it will be a point of departure for future research and researchers. The central open questions arising from the work done here are as follows.

1. Are there hooked k -extended Langford sequences for the “No Man’s Land” range of m for which our method does not provide a construction?
2. Are there L -stretched Langford sequences for the values our methods do not construct?

We suspect the answers are positive and offer two conjectures.

Conjecture 6.0.1. *The necessary conditions for the existence of hooked k -extended Langford sequences are also sufficient, with the exception of a small number of values of (d, m, k) .*

Conjecture 6.0.2. *The necessary conditions for the existence of stretched and hooked stretched Langford sequences are also sufficient, with the exception of a small number of values of (d, m, L) .*

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