1

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

National Library
of Canada

Acquisitions and
Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale
du Canada

Direction des acquisitions et
des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file *Votre référence*

Our file *Notre référence*

NOTICE

The quality of this microform is heavily dependent upon the quality of the original thesis submitted for microfilming. Every effort has been made to ensure the highest quality of reproduction possible.

If pages are missing, contact the university which granted the degree.

Some pages may have indistinct print especially if the original pages were typed with a poor typewriter ribbon or if the university sent us an inferior photocopy.

Reproduction in full or in part of this microform is governed by the Canadian Copyright Act, R.S.C. 1970, c. C-30, and subsequent amendments.

AVIS

La qualité de cette microforme dépend grandement de la qualité de la thèse soumise au microfilmage. Nous avons tout fait pour assurer une qualité supérieure de reproduction.

S'il manque des pages, veuillez communiquer avec l'université qui a conféré le grade.

La qualité d'impression de certaines pages peut laisser à désirer, surtout si les pages originales ont été dactylographiées à l'aide d'un ruban usé ou si l'université nous a fait parvenir une photocopie de qualité inférieure.

La reproduction, même partielle, de cette microforme est soumise à la Loi canadienne sur le droit d'auteur, SRC 1970, c. C-30, et ses amendements subséquents.

Canada

SERIES SOLUTIONS OF THE
LORENZ SYSTEM AND THE DUFFING EQUATION

by

Andrea Zypchen, B.Sc.

A thesis submitted to
the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics and Statistics

Carleton University

Ottawa, Canada

April, 1994

© Copyright, 1994

Andrea Zypchen

National Library
of Canada

Acquisitions and
Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale
du Canada

Direction des acquisitions et
des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

Your file Votre référence

Our file Notre référence

THE AUTHOR HAS GRANTED AN
IRREVOCABLE NON-EXCLUSIVE
LICENCE ALLOWING THE NATIONAL
LIBRARY OF CANADA TO
REPRODUCE, LOAN, DISTRIBUTE OR
SELL COPIES OF HIS/HER THESIS BY
ANY MEANS AND IN ANY FORM OR
FORMAT, MAKING THIS THESIS
AVAILABLE TO INTERESTED
PERSONS.

L'AUTEUR A ACCORDE UNE LICENCE
IRREVOCABLE ET NON EXCLUSIVE
PERMETTANT A LA BIBLIOTHEQUE
NATIONALE DU CANADA DE
REPRODUIRE, PRETER, DISTRIBUER
OU VENDRE DES COPIES DE SA
THESE DE QUELQUE MANIERE ET
SOUS QUELQUE FORME QUE CE SOIT
POUR METTRE DES EXEMPLAIRES DE
CETTE THESE A LA DISPOSITION DES
PERSONNE INTERESSEES.

THE AUTHOR RETAINS OWNERSHIP
OF THE COPYRIGHT IN HIS/HER
THESIS. NEITHER THE THESIS NOR
SUBSTANTIAL EXTRACTS FROM IT
MAY BE PRINTED OR OTHERWISE
REPRODUCED WITHOUT HIS/HER
PERMISSION.

L'AUTEUR CONSERVE LA PROPRIETE
DU DROIT D'AUTEUR QUI PROTEGE
SA THESE. NI LA THESE NI DES
EXTRAITS SUBSTANTIELS DE CELLE-
CI NE DOIVENT ETRE IMPRIMES OU
AUTREMENT REPRODUITS SANS SON
AUTORISATION.

ISBN 0-315-98640-9

Canada

Series Solutions of the Lorenz System and The Duffing Equation

Name _____

Dissertation Abstracts International is arranged by broad, general subject categories. Please select the one subject which most nearly describes the content of your dissertation. Enter the corresponding four-digit code in the spaces provided.

Mathematics

SUBJECT TERM

0465

SUBJECT CODE

U·M·I

Subject Categories

THE HUMANITIES AND SOCIAL SCIENCES

COMMUNICATIONS AND THE ARTS

Architecture 0729
Art History 0377
Cinema 0900
Dance 0378
Fine Arts 0357
Information Science 0723
Journalism 0391
Library Science 0399
Mass Communications 0708
Music 0413
Speech Communication 0459
Theater 0465

EDUCATION

General 0515
Administration 0514
Adult and Continuing 0516
Agricultural 0517
Art 0273
Bilingual and Multicultural 0282
Business 0688
Community College 0275
Curriculum and Instruction 0727
Early Childhood 0518
Elementary 0524
Finance 0277
Guidance and Counseling 0519
Health 0680
Higher 0745
History of 0520
Home Economics 0278
Industrial 0521
Language and Literature 0279
Mathematics 0280
Music 0522
Philosophy of 0998
Physical 0523

Psychology 0525
Reading 0535
Religious 0527
Sciences 0714
Secondary 0533
Social Sciences 0534
Sociology of 0340
Special 0529
Teacher Training 0530
Technology 0710
Tests and Measurements 0288
Vocational 0747

LANGUAGE, LITERATURE AND LINGUISTICS

Language 0679
General 0289
Ancient 0290
Linguistics 0291
Modern 0291
Literature 0401
General 0294
Classical 0295
Comparative 0297
Medieval 0298
Modern 0316
African 0591
American 0305
Asian 0352
Canadian (English) 0355
Canadian (French) 0593
English 0311
Germanic 0312
Latin American 0315
Middle Eastern 0313
Romance 0314
Slavic and East European

PHILOSOPHY, RELIGION AND THEOLOGY

Philosophy 0422
Religion 0318
General 0321
Biblical Studies 0319
Clergy 0320
History of 0322
Philosophy of 0469
Theology

SOCIAL SCIENCES

American Studies 0323
Anthropology 0324
Archaeology 0326
Cultural 0327
Physical 0310
Business Administration 0272
General 0770
Accounting 0454
Banking 0338
Management 0385
Marketing 0501
Canadian Studies 0503
Economics 0505
General 0508
Agricultural 0509
Commerce-Business 0510
Finance 0511
History 0358
Labor 0366
Theory 0351
Folklore 0578
Geography 0578
Gerontology 0578
History 0578
General

Ancient 0579
Medieval 0581
Modern 0582
Black 0328
African 0331
Asia, Australia and Oceania 0332
Canadian 0334
European 0335
Latin American 0336
Middle Eastern 0333
United States 0337
History of Science 0585
Law 0398
Political Science 0615
General 0616
International Law and Relations 0617
Public Administration 0814
Recreation 0452
Social Work 0626
Sociology 0627
General 0938
Criminology and Penology 0631
Demography 0628
Ethnic and Racial Studies 0629
Individual and Family Studies 0630
Industrial and Labor Relations 0700
Public and Social Welfare 0344
Social Structure and Development 0709
Theory and Methods 0999
Transportation 0453
Urban and Regional Planning 0453
Women's Studies

THE SCIENCES AND ENGINEERING

BIOLOGICAL SCIENCES

Agriculture 0473
General 0285
Agronomy 0475
Animal Culture and Nutrition 0476
Animal Pathology 0359
Food Science and Technology 0478
Forestry and Wildlife 0479
Plant Culture 0480
Plant Pathology 0817
Plant Physiology 0777
Range Management 0746
Wood Technology 0306
Biology 0287
General 0308
Anatomy 0309
Biostatistics 0370
Botany 0329
Cell 0353
Ecology 0369
Entomology 0793
Genetics 0410
Limnology 0307
Microbiology 0317
Molecular 0416
Neuroscience 0433
Oceanography 0821
Physiology 0778
Radiation 0472
Veterinary Science 0786
Zoology 0760
Biophysics 0425
General 0996
Medical

Geodesy 0370
Geology 0372
Geophysics 0373
Hydrology 0388
Mineralogy 0411
Paleobotany 0345
Paleoecology 0426
Paleontology 0418
Paleozoology 0985
Palaology 0427
Physical Geography 0368
Physical Oceanography 0415

HEALTH AND ENVIRONMENTAL SCIENCES

Environmental Sciences 0768
Health Sciences 0566
General 0300
Audiology 0992
Chemotherapy 0567
Dentistry 0350
Education 0769
Hospital Management 0758
Human Development 0982
Immunology 0564
Medicine and Surgery 0347
Mental Health 0569
Nursing 0570
Nutrition 0380
Obstetrics and Gynecology 0354
Occupational Health and Therapy 0381
Ophthalmology 0571
Pathology 0419
Pharmacology 0572
Pharmacy 0382
Physical Therapy 0573
Public Health 0574
Radiology 0575
Recreation

Speech Pathology 0460
Toxicology 0383
Home Economics 0386

PHYSICAL SCIENCES

Pure Sciences 0485
Chemistry 0749
General 0486
Agricultural 0487
Analytical 0488
Biochemistry 0738
Inorganic 0490
Nuclear 0491
Organic 0494
Pharmaceutical 0495
Physical 0754
Polymer 0405
Radiation 0605
Mathematics 0986
Physics 0606
General 0608
Acoustics 0748
Astronomy and Astrophysics 0607
Electronics and Electricity 0798
Elementary Particles and High Energy 0759
Fluid and Plasma 0609
Molecular 0610
Nuclear 0752
Optics 0756
Radiation 0611
Solid State 0463
Statistics 0346
Applied Sciences 0984
Applied Mechanics 0984
Computer Science

Engineering 0537
General 0538
Aerospace 0539
Agricultural 0540
Automotive 0541
Biomedical 0542
Chemical 0543
Civil 0544
Electronics and Electrical 0348
Heat and Thermodynamics 0545
Hydraulic 0546
Industrial 0547
Marine 0794
Materials Science 0548
Mechanical 0743
Metallurgy 0551
Mining 0552
Nuclear 0549
Packaging 0765
Petroleum 0554
Sanitary and Municipal 0790
System Science 0428
Geotechnology 0796
Operations Research 0795
Plastics Technology 0994
Textile Technology

PSYCHOLOGY

General 0621
Behavioral 0384
Clinical 0622
Developmental 0620
Experimental 0623
Industrial 0624
Personality 0625
Physiological 0989
Psychobiology 0349
Psychometrics 0632
Social 0451

The undersigned recommend to the Faculty of Graduate Studies
and Research acceptance of the thesis

"SERIES SOLUTIONS OF THE
LORENZ SYSTEM AND THE DUFFING EQUATION"

submitted by Andrea L. Zypchen, B.Sc.
in partial fulfilment of the requirements for
the degree of Master of Science

Thesis Supervisor

Chair, Department of Mathematics & Statistics

Carleton University

Abstract

Nonlinear differential equations with independent variables in the complex plane may have ordinary, Laurent or psi series solutions. For the Lorenz equations (3-dimensional and autonomous) and the Duffing equation (2-dimensional and non-autonomous) these series solutions are formulated, and constructive proofs are given regarding lower bounds on their radii of convergence.

The ordinary series solutions are plotted and compared to plots obtained by numerical integration.

For the psi series solutions of the Lorenz system it was calculated that the lower bound on the radii, when the logarithmic terms entered at the fourth term, was higher than when the logarithmic terms entered at the second term. The actual radius was observed to follow the same pattern. This is interesting in view of observations made by other researchers [10], that for a different system, the higher the order at which the logarithmic singularities enter, the smaller the chaotic region.

For the psi series solutions of the Duffing equation it was calculated that the lower bounds on the radii decreased as the strength of the forcing increased. This supports observations made by other researchers that as the motion became more chaotic, the singularity pattern condensed and the structures gradually collapsed towards the real axis [5, 7, 52, 55]

.

Acknowledgements

I would like to thank Dr. Sam Melkonian, Ken, Samantha, Nicholas and my parents.

Contents

Acceptance Sheet	ii
Abstract	iii
Acknowledgements	iv
Table of Contents	vi
List of Tables	viii
List of Figures	x
1 Introduction	1
1.1 The Study of Differential Equations in the Complex Domain	2
1.2 Results of the Past Decade	4
1.3 Series Solutions of Non-linear Differential Equations	7
1.4 Contents of the Thesis	9
2 The Lorenz System	11
2.1 Ordinary Series	11
2.1.1 Plots and Discussion	15
2.2 Laurent Series	17
2.3 Psi Series	25
2.3.1 Plots and Discussion	37
3 The Duffing Equation	39
3.1 Ordinary Series	39

3.1.1	Plots and Discussion	42
3.2	Laurent Series	43
3.3	Psi Series	43
4	Figures	48
5	Summary and Conclusions	71

List of Tables

- 1.1 Differential equations and the types of singularities of their solutions 3
- 2.1 $\mathcal{R}_{ib}$ and $\mathcal{R}_{act}$ as R varies, for both the $k_0 = 2$ and $k_0 = 4$ cases 38

List of Figures

4.1	$L(\alpha)$ versus α for the ordinary series solution of the Lorenz system . .	49
4.2	$\mathcal{R}_{est}$ versus R for the ordinary series solutions of the Lorenz system .	50
4.3	$\mathcal{R}_{est}$ versus time for the ordinary series solution for $x(t)$ of the Lorenz system	51
4.4	Ordinary series solution for $x(t)$ of the Lorenz system, truncated at 20 terms, versus time	52
4.5	Projection of the attractor on the z plane produced by the ordinary series solutions of the Lorenz system	53
4.6	$x(t)$ of the Lorenz system versus time, obtained by numerical integration	54
4.7	Projection of the attractor on the z plane for the Lorenz system, obtained by numerical integration	55
4.8	$x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.0001, 5, 5)$ (dashed)	56
4.9	$x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.001, 5, 5)$ (dashed)	57
4.10	$x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.01, 5, 5)$ (dashed)	58
4.11	$x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.1, 5, 5)$ (dashed)	59
4.12	$x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.2, 5, 5)$ (dashed)	60

4.13	$x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(-13.9, -15.8568, 32.1233)$ at $t = 10.9785$ (dashed) and $(5, 5, 5)$ at $t = 0$	61
4.14	Resummation of the double series along the lines $\alpha = k + 2l$	62
4.15	Solution of the inequality $\epsilon^{k_0 s} > 2K + Ks$	63
4.16	$L(\alpha)$ vs α for the psi series solution of the Lorenz system	64
4.17	The psi series for $x(t)$ of the Lorenz system versus the truncation point, $t = 0.9$, $t = 1.0$ and $t = 1.1$ (top to bottom)	65
4.18	The ordinary series for $x(t)$ of the Duffing equation, truncated at 25 terms, versus time	66
4.19	The ordinary series for $x(t)$ versus $x'(t)$ of the Duffing equation, truncated at 25 terms	67
4.20	$x(t)$ of the Duffing equation versus time, obtained by numerical integration	68
4.21	$x(t)$ versus $x'(t)$ of the Duffing equation, obtained by numerical integration	69
4.22	$x(t)$ versus t for the ordinary series of the Duffing equation, initial conditions $(0, 0)$ (solid) and $(0.01, 0, 0)$ (dashed)	70

Chapter 1

Introduction

Nonlinear phenomena abound in nature. Yet the differential equations which govern their dynamics rarely have an analytic solution. Numerical integration is often the only resort for estimating the system's trajectories.

Series solutions offer an alternative with several benefits. Firstly, the series *are* explicit expressions of the solutions. Secondly, if the independent variables are permitted to assume complex values, much additional information may be gained, including a strong indication of which values of the system's parameters will lead to integrability.

The remainder of the introduction covers the following topics. A brief account is given of how and why the study of differential equations in the complex domain began. Research of the past decade, on the relationship between a system's complex time singularity structure and its real time behaviour, is summarized. Finally, the various types of series solutions are described and the contents of the thesis outlined.

1.1 The Study of Differential Equations in the Complex Domain

In the second half of the nineteenth century, Cauchy began investigating the nature of differential equations in the complex plane. Instead of attempting to discover global solutions, valid for all initial conditions, researchers became interested in determining local solutions in the complex plane, which could then be analytically continued to form more general solutions. However, this method did not always succeed, since the ability to analytically continue solutions is very much related to the nature of their singularities.

Singularities (points at which a solution is not analytic) can be poles, branch points (algebraic or transcendental (e.g. logarithmic or irrational)), or essential. Each one of these can be fixed or movable. The location of a fixed singularity is determined by the coefficients of the differential equation. The location of a movable singularity is dependent upon the initial conditions. All singularities of linear differential equations are fixed, whereas nonlinear equations may possess either type. All singularities which are not poles are called critical points. Differential equations whose solutions admit various types of singularities are exhibited in Table 1.1.

The solution of a differential equation with fixed singularities can be analytically continued to produce a single-valued, general solution. On the other hand, if a differential equation possesses movable singularities which are not poles, analytic continuation of its solution may be difficult or impossible [39]. With this knowledge, many mathematicians became interested in classifying ordinary differential equations according to the types of singularities admitted by their solutions.

In 1884 Fuchs showed that the only equation of the form $\frac{dx}{dt} = F(x, t) = \frac{P(x, t)}{Q(x, t)}$, where P and Q are polynomial in x , with coefficients analytic in t , which is free from movable critical points, is the generalized Riccati equation

$$\frac{dx}{dt} = p_2(t)x^2 + p_1(t)x + p_0(t),$$

where $p_2(t)$, $p_1(t)$ and $p_0(t)$ are analytic functions [43].

In 1888 Painlevé proved that for first order equations of the form $G\left(\frac{dx}{dt}, x, t\right) = 0$,

<i>Differential Equation</i>	<i>Solution</i>	<i>Type of Singularity</i>
$w' + w^2 = 0$	$w(z) = (z - z_0)^{-1}$	movable pole
$w' - \lambda w^{1-\frac{1}{\lambda}} = 0$ $\lambda \in \mathbb{Q}(\text{rational})$	$w(z) = (z - z_0)^\lambda$	movable rational branch point
$w' - e^w = 0$	$w(z) = \ln(z - z_0)$	movable logarithmic branch point
$w' + w(\ln w)^2 = 0$	$w(z) = e^{\frac{1}{z-z_0}}$	movable essential singularity
$(z - c)w' = \lambda w$ $\lambda \in \mathbb{I}^-$	$w(z) = k(z - c)^\lambda$	fixed pole
$(z - c)w' = \lambda w$ $\lambda \in \mathbb{Q}$	$w(z) = k(z - c)^\lambda$	fixed rational branch point
$(z - c)w' = \lambda w$ $\lambda \in \bar{\mathbb{Q}}(\text{irrational})$	$w(z) = k(z - c)^\lambda$	fixed transcendental branch point
$w' = \frac{-w}{(z-c)^2}$	$w(z) = ke^{\frac{1}{z-c}}$	fixed essential singularity

Table 1.1: Differential equations and the types of singularities of their solutions

with G polynomial in $\frac{dx}{dt}$ and x , and analytic in t , the movable singularities are poles and/or algebraic branch points [43].

In 1888 Sophie Kowalevskaya won the Bordin Prize of the Paris Academy of Science for her research on the governing equations of a spinning top (sixth order). Motivated by the work of Fuchs, she determined for which values of the system's parameters, the solutions' were free from movable critical points. Of the four sets of parameters she found that met this condition, three sets gave equations which had known solutions. She successfully integrated the equations of motion for the fourth set of parameters [45, 46].

In 1897 Painlevé, aiming to answer a question posed by Picard, considered the singularities of all second-order ordinary differential equations (ODEs) of the form

$$\frac{d^2 w}{dz^2} = F(z, w, w'),$$

where F is rational in w' , algebraic in w and analytic in z . Painlevé, Fuchs, Gambier and Picard found fifty equations whose solutions were free from movable critical

points [32, 43]. Of these, forty-four were integrable by quadrature or linearization. The remaining six required new transcendental functions (PI - PVI) to be solved.

In 1983 Weiss, Tabor and Carnevale defined a Painlevé property for partial differential equations (PDEs), taking into account the fact that a function of several variables may be singular not at a point, but on a manifold [66].

Further classification of differential equations in terms of their movable singularities has been undertaken by Chazy [21], Garnier [33], Bureau [14, 15, 16, 17] (third-order equations), Ramani et al. [57] (two-dimensional quadratic systems), Hlavatý [40, 41, 42] and Cosgrove [23, 24, 25] (PDEs).

1.2 Results of the Past Decade

Over the past decade, many observations have been made, a few claims proven, and many conjectures put forth, on the relationship between a system's complex time singularity structure and real time behaviour, which are now summarized.

Ziglin [68] has proved that transversal intersections of separatrices in near-integrable Hamiltonian systems with two degrees of freedom imply the existence of infinitely sheeted solutions. This means that for this type of system, with the Painlevé property, there can be no transversal intersections of separatrices (considered a significant cause of chaotic behaviour). Bountis *et al.* [7] extended Ziglin's results to a particular type of non-Hamiltonian system.

There is no proof of a general connection between the Painlevé property and complete integrability of Hamiltonian systems. However, Adler and Moerbeke [1, 2] have studied the particular case of Hamiltonians where the constants of motion (I 's) are rational functions of the positions (p_i 's) and momenta (q_i 's), and the corresponding manifold is part of a torus ¹. In this case, the Hamilton-Jacobi procedure can be utilized and the system is deemed "algebraically integrable" ². Their theorem

¹Note that while the Liouville-Arnold theorem shows that when N I 's in involution exist the solutions lie on N dimensional real tori parametrized by the I 's, if t is allowed to be complex, then the manifold is no longer necessarily a torus.

²Algebraic integrability is defined by AM as being a property of those systems which possess a full set of rational integrals and whose motion is linearizable on a complex torus.

states that algebraic integrability implies that the system possesses the Painlevé property [1, 2]. The full converse of this theorem has yet to be proven, but no counterexamples have been found to date, and many examples have [55, 57].

Many researchers have experimented with various equations and the Painlevé property. The Painlevé technique has had unlimited success in determining parameter regions in which a system is integrable [20, 48, 51, 37, 56, 35, 34, 19, 28, 49, 8, 27, 59, 30, 5, 53, 65, 57, 18, 11, 52, 22, 54] (and references therein).

No ODE has been found that is integrable (has single-valued solutions), yet does not satisfy the Painlevé property [29]. The story is different for PDEs. The Dym-Kruskal equation ($u_t = u^3 u_{xxx}$), the nonlinear diffusion equation ($u_t = (u^{-2} u_x)_x$), the Wadati-Kohno-Ichikawa equation, and the Bruschi-Ragnisco hierarchy, among others, are solvable by Inverse Scattering (IS), but do not possess the Painlevé property [55]. The Dym-Kruskal equation, however, does possess the weak Painlevé property (it has rational branch points) and can be transformed into the KDV equation, which does have the Painlevé property [57].

What is known about the implications, regarding integrability, of an equation possessing the weak Painlevé property? Bountis *et al.* have found that those systems which satisfy the weak Painlevé property and are integrable have globally finitely sheeted solutions, whereas those which are not integrable have globally infinitely sheeted solutions [3, 4]. For two two-dimensional autonomous Hamiltonian systems possessing the weak Painlevé property, it was shown that an appropriate transformation of the dependent *and* independent variables resulted in a system with the *full* Painlevé property [57]. Sitnikov studied the equation of a periodically driven, one degree of freedom oscillator (non-autonomous). It possesses the weak Painlevé property, yet, when expanded about a fixed singularity, one of the compatibility conditions is not met and logarithmic terms must be introduced. For this case the system is known to behave chaotically [57, 58].

What role do fixed singularities play in integrability? Will a system with the full Painlevé property, but a *fixed* non-pole singularity be integrable? Up to second order, these singularities will disappear through linearization. For higher orders the question is unanswered [58]. Fixed singularities can cause non-integrability and

chaotic behaviour [47, 57, 58].

There are other reasons which indicate that the Painlevé property is an inadequate criterion for integrability. There are equations which satisfy the Painlevé property only after a suitable transformation [47]. Movable essential singularities in some cases are compatible with integrability [47]. There is a PDE that can be integrated by quadratures and may have *any* singularity structure ($u_{xt} + uu_{xx} + F(u_x) = 0$). Note that when an equation is integrable by quadratures, the singularity structure is irrelevant [13]. Logarithmic singularities are not incompatible with integrability (for example, the Lotka-Volterra system) [9].

So what complex singularity structure may signify integrability? The singularity distributions of several nonintegrable systems have been investigated. For systems with logarithmic branch points, the singularities are seen to cluster on infinitely many Riemann sheets in one-armed (ABC Flows [26]), two-armed (Lorenz [65], driven pendulum [53], Morse oscillator [54]) and four-armed (Duffing [30, 57]) structures. For systems with transcendental branch points (the Hénon-Heiles system [18, 20, 49] and Kuramoto's equation, self-similar spirals of singularities were found, and proven to form a natural boundary, beyond which the solution could not be analytically continued [18, 20, 49]. However, as the authors warn, there are cases, such as the third-order Chazy equation, where sufficiently nice natural boundaries can lead to integrability.

Interesting correlations between the complex time singularity structures of systems and their real time behaviour have been found. For the Lorenz system, the positions of singularities were found to correlate with the positions of the local maxima of one of the dependent variables. Also the amplitudes of the maxima were seen to correlate with the distances of the singularities from the real axis [65]. For Duffing's equation, it was found that as the motion became more chaotic, the singularity pattern condensed and the structures gradually collapsed towards the real axis [5, 7, 52, 55], and the same correlations between the local maxima and the singularity positions were found [7]. For the governing equations of anharmonic oscillators, it was also observed that the higher the order at which the logarithmic singularities enter, the smaller the chaotic region [10]. In addition, correlations have been found

between the timing of turbulent “bursts” and the real part of complex singularities of equations governing turbulence [31].

As demonstrated by the above results, there is clearly a deep connection between a system’s complex time singularity structure and its real time behaviour. It is quite apparent that the requirement that a system have only poles for movable singularities in order to be integrable is too restrictive, and ignores the possible consequences of essential singularities and fixed critical points. It is interesting to note that Painlevé defined a critical point as a point at which the solution was multivalued. He was interested in equations with *no* critical singularities (fixed or movable) [58].

As this body of research matures, various researchers have proposed new conjectures as to the types of singularity structures which will ensure integrability. T.C. Bountis *et al.* [6] have conjectured that the existence of infinitely sheeted solutions in complex time is a necessary condition for non-integrability and chaotic behaviour in real time. M.D. Kruskal conjectures that a more robust definition of integrability would require that the solutions’ possible values not be dense [47]. The validity of these conjectures remains to be tested, and many questions remain.

1.3 Series Solutions of Non-linear Differential Equations

Series solutions often provide a means of determining a system’s singularity structure, in particular by the following method.

We begin by assuming that the i^{th} component of the solution is of the form

$$u_i(t) = \sum_{\alpha=0}^{\infty} a_{\alpha}(t - t_0)^{\alpha-p}$$

where t_0 is the location of the movable singularity.

The possible values of p can then be calculated by substituting the series into the system, and balancing the dominant terms.

If p is a negative integer, or zero, t_0 is no longer a singular and an ordinary series solution will be obtained. For all systems, if an analytic solution exists, it is possible to obtain a balance by setting $p = 0$.

If p is irrational or complex, an appropriate psi series will form the general solution. For a further description of such series and proofs of convergence see the Master's thesis of M.A. Hemmi [36].

If p is a positive integer we then determine the recursion relation for the coefficients of the series, which will be of the form

$$B_{\alpha}a_{\alpha} = W_{\alpha}$$

where B_{α} is a square matrix and W_{α} is determined by the previous a_{α} 's. a_{α} will be arbitrary whenever B_{α} is singular, so long as a compatibility condition is met. The resonances are defined to be these particular values of α .

If p is rational, but not integer, it may be possible to make a change of variable so that for the differential equation in the new variable, p is a positive integer. If this transformation is possible, the analysis continues in the new variable, otherwise a psi series is necessary to form a general solution.

If one resonance is -1 and all others are positive integers, and all the compatibility conditions are met, a Laurent series solution will be valid, and the solution will have movable poles.

If not all of the compatibility conditions are met, a logarithmic psi series must be formed in order to regain the arbitrary coefficients lost by the incompatible resonances. This was discovered by noting that when an ordinary differential equation

$$y'' + p(x)y' + q(x)y = 0$$

has indicial roots r_1 and r_2 which differ by an integer, the Frobenius method states that one solution is

$$y_1(x) = \sum_{n=0}^{\infty} d_n x^{n+r_1}$$

and a second linearly independent solution is

$$y_2(x) = Ay_1 \log x + \sum_{n=0}^{\infty} e_n x^{n+r_2},$$

where A is a constant.

If the resonances do not fit this particular pattern an appropriate psi series will be necessary.

Two things must be noted. Since a general solution to an n^{th} order differential equation requires n arbitrary parameters, and the location of the singularity t_0 is arbitrary, these series form general solutions if and only if the number of resonances is equal to the order of the differential equation less one. Secondly, to form a solution the series must converge within a non-zero radius.

1.4 Contents of the Thesis

In this thesis we present the application of the previously described method to the Lorenz (third-order and autonomous) system

$$\begin{aligned}x' &= \sigma(y - x) \\y' &= -xz + Rx - y \\z' &= xy - Bz ,\end{aligned}\tag{1.1}$$

where x' , y' , z' denote derivatives with respect to time and σ , B and R are non-negative, real-valued parameters of the system, and the Duffing (second-order and nonautonomous) equation

$$x'' + kx' + px^3 = B_1 F(t) + B_0 ,\tag{1.2}$$

where k , p , B_1 and B_0 are real-valued parameters of the system.

A lower bound on the radius of convergence is estimated for each type of series (ordinary, Laurent, logarithmic psi), solutions are plotted and the truncation error is estimated. Although the psi series expansions for the solution near the singularities of the system have been thought to possibly serve as an explicit solution, to date their convergence is undetermined [3, 18, 30, 57].

All of the proofs which determine a bound on the radius of convergence of the series are similar and are based upon a convergence proof for logarithmic psi series solutions for Emden's equations (second-order and autonomous), carried out by Einar Hille [38]. A bound is first assumed to exist on the first $N - 1$ coefficients. It is then shown that for this same bound to hold for the N th and higher coefficients, N must be larger than a certain value, say N^* . If the bound is calculated based

on the first $N - 1$ coefficients, for N greater than or equal N^* , the bound will hold for all the coefficients. The ratio test is then applied to the bounding series, which indicates for what values of t the original series will converge.

The behaviour of the ordinary series solutions are compared to those obtained by numerical integration and factors affecting the bound are explored.

All calculations have been executed either in Mathematica or in C.

Chapter 2

The Lorenz System

The Lorenz equations (1.1) are the product of a search by a meteorologist, Edward Lorenz, for a three-dimensional system of ODEs which would demonstrate both the regular and the unpredictable behaviour characteristic of weather [61]. They are a truncated version of an infinite-dimensional system of equations modelling fluid convection in a two dimensional heat cell heated from below. The values of the parameters σ , B and R are proportional, respectively, to the Prandtl number, the Rayleigh number and some physical dimensions of the convection cell.

Numerous researchers have studied the dynamical behaviour of the Lorenz equations, most comprehensively Sparrow [61].

Painlevé analysis has been undertaken by Tabor [65, 60, 50].

2.1 Ordinary Series

About an ordinary point t_1 , power series solutions

$$x(t) = \sum_{\alpha=0}^{\infty} a_{\alpha} \tau^{\alpha} \text{ , } y(t) = \sum_{\alpha=0}^{\infty} b_{\alpha} \tau^{\alpha} \text{ and } z(t) = \sum_{\alpha=0}^{\infty} c_{\alpha} \tau^{\alpha} \text{ ,}$$

where $\tau = t - t_1$, will be valid within the radii of convergence of the series. Using the formula

$$\left(\sum_{\alpha=0}^{\infty} a_{\alpha} \tau^{\alpha-\alpha_1} \right) \left(\sum_{\alpha=0}^{\infty} b_{\alpha} \tau^{\alpha-\alpha_2} \right) = \sum_{\alpha=0}^{\infty} \left(\sum_{\beta=0}^{\alpha} a_{\beta} b_{\alpha-\beta} \right) \tau^{\alpha-(\alpha_1+\alpha_2)}$$

and substituting $x(t)$, $y(t)$ and $z(t)$ of this form into the Lorenz equations (1.1) yields

$$\sum_{\alpha=0}^{\infty} (a_{\alpha}(\alpha) - \sigma b_{\alpha-1} + \sigma a_{\alpha-1}) \tau^{\alpha-1} = 0, \quad (2.1)$$

$$\sum_{\alpha=0}^{\infty} \left(b_{\alpha}(\alpha) - Ra_{\alpha-1} + b_{\alpha-1} + \sum_{\beta=0}^{\alpha-1} a_{\beta} c_{\alpha-1-\beta} \right) \tau^{\alpha-1} = 0, \quad (2.2)$$

$$\sum_{\alpha=0}^{\infty} \left(c_{\alpha}(\alpha) - \sum_{\beta=0}^{\alpha-1} a_{\beta} b_{\alpha-1-\beta} + Bc_{\alpha-1} \right) \tau^{\alpha-1} = 0, \quad (2.3)$$

after shifting the indices of summation where necessary to obtain identical powers of τ . Note that termwise differentiation is valid since the uniform convergence of the sum of differentiated terms can be easily proven.

For these equations to be satisfied for all t it is necessary that the coefficient of each power of τ be zero. Thus we obtain the three recursion relations (RR) in matrix form

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \begin{pmatrix} a_{\alpha} \\ b_{\alpha} \\ c_{\alpha} \end{pmatrix} = \begin{pmatrix} \sigma b_{\alpha-1} - \sigma a_{\alpha-1} \\ Ra_{\alpha-1} - b_{\alpha-1} - \sum_{\beta=0}^{\alpha-1} a_{\beta} c_{\alpha-1-\beta} \\ \sum_{\beta=0}^{\alpha-1} a_{\beta} b_{\alpha-1-\beta} - Bc_{\alpha-1} \end{pmatrix},$$

or,

$$A_{\alpha} \mathbf{v}_{\alpha} = \mathbf{W}_{\alpha}.$$

Since A_{α} is singular only for $\alpha = 0$, a_0 , b_0 and c_0 are the three arbitrary coefficients necessary for these series to constitute a general solution.

Note that throughout the thesis, the norm of an n by n matrix A is defined as

$$\|A\| = \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |a_{ij}| \right).$$

In order to prove the existence of a lower bound on the radii of convergence of the three series, we first establish a bound on the coefficients of the series with the following proposition.

Proposition 1 *For all M , $0 < M < 1$, there exists K , $1 < K < \infty$, such that for all $\alpha \geq 1$*

$$\|\mathbf{v}_{\alpha}\| \leq MK^{\alpha}. \quad (2.4)$$

Proof.

The proposition is a consequence of the following

Claim 1 *If*

$$N \geq \max \left(2\sigma, \frac{R+1+|a_0|+|c_0|}{1-M}, \frac{B+|a_0|+|b_0|}{1-M} \right) \quad (2.5)$$

and if

$$\|\mathbf{v}_\alpha\| \leq MK^\alpha \quad (2.6)$$

for all $\alpha < N$, then

$$\|\mathbf{v}_\alpha\| \leq MK^\alpha \quad (2.7)$$

for $\alpha = N$.

The proposition will then follow by induction, with

$$K = \max_{1 \leq \alpha \leq N-1} L(\alpha), \quad (2.8)$$

where

$$L(\alpha) = \left(\frac{\|\mathbf{v}_\alpha\|}{M} \right)^{1/\alpha}.$$

We now prove the claim.

Given that (2.4) holds for $1 \leq \alpha \leq N-1$, for $\alpha = N$ we obtain

$$\begin{aligned} |a_\alpha| &\leq \frac{\sigma}{\alpha} (|b_{\alpha-1}| + |a_{\alpha-1}|) \\ &\leq \frac{2\sigma}{\alpha} MK^{\alpha-1}, \end{aligned}$$

$$\begin{aligned} |b_\alpha| &\leq \frac{1}{\alpha} \left(R|a_{\alpha-1}| + |b_{\alpha-1}| + \left(\sum_{\beta=1}^{\alpha-2} |a_\beta| |c_{\alpha-1-\beta}| \right) + |a_0| |c_{\alpha-1}| + |a_{\alpha-1}| |c_0| \right) \\ &\leq \frac{1}{\alpha} \left(RMK^{\alpha-1} + MK^{\alpha-1} + M^2 \left(\sum_{\beta=1}^{\alpha-2} K^\beta K^{\alpha-1-\beta} \right) + |a_0| MK^{\alpha-1} + |c_0| MK^{\alpha-1} \right) \\ &\leq \frac{1}{\alpha} \left((R+1+|a_0|+|c_0|) MK^{\alpha-1} + M^2 K^{\alpha-1} \left(\sum_{\beta=1}^{\alpha-2} 1 \right) \right) \\ &\leq \frac{1}{\alpha} (R+1+M\alpha+|a_0|+|c_0|) MK^{\alpha-1} \end{aligned}$$

and

$$\begin{aligned}
 |c_\alpha| &\leq \frac{1}{\alpha} \left(\left(\sum_{\beta=1}^{\alpha-2} |a_\beta| |b_{\alpha-1-\beta}| \right) + |a_0| |b_{\alpha-1}| + |a_{\alpha-1}| |b_0| + b |c_{\alpha-1}| \right) \\
 &\leq \frac{1}{\alpha} \left(M^2 \left(\sum_{\beta=1}^{\alpha-2} K^\beta K^{\alpha-1-\beta} \right) + |a_0| M K^{\alpha-1} + |b_0| M K^{\alpha-1} + B M K^{\alpha-1} \right) \\
 &\leq \frac{1}{\alpha} \left((B + |a_0| + |b_0|) M K^{\alpha-1} + M^2 K^{\alpha-1} \left(\sum_{\beta=1}^{\alpha-2} 1 \right) \right) \\
 &\leq \frac{1}{\alpha} (B + |a_0| + |b_0| + M\alpha) M K^\alpha.
 \end{aligned}$$

Since, for $\alpha \geq 1$, A_α is nonsingular, we obtain

$$\|v_\alpha\| \leq \|A_\alpha\| \|G_\alpha\| M K^\alpha = \frac{1}{\alpha} \|G_\alpha\| M K^\alpha,$$

where

$$G_\alpha = \begin{pmatrix} 2\sigma \\ R + 1 + |a_0| + |c_0| + M\alpha \\ B + |a_0| + |b_0| + M\alpha \end{pmatrix}. \quad (2.9)$$

In order for the bound to hold we must have, for $\alpha = N$,

$$\frac{\|G_\alpha\|}{\alpha} \leq 1,$$

or equivalently,

$$N \geq \max \left(2\sigma, \frac{R + 1 + |a_0| + |c_0|}{1 - M}, \frac{B + |a_0| + |b_0|}{1 - M} \right),$$

and the claim and thus the proposition are proved. $\square$

We are now prepared to state and prove the theorem regarding a lower bound on the radii of convergence of the three series.

Theorem 1 *There exists $\mathcal{R}$, $0 < \mathcal{R} < \infty$, such that for $|\tau| < \mathcal{R}$, the series*

$$\sum_{\alpha=0}^{\infty} a_\alpha \tau^\alpha, \sum_{\alpha=0}^{\infty} b_\alpha \tau^\alpha \text{ and } \sum_{\alpha=0}^{\infty} c_\alpha \tau^\alpha$$

converge absolutely for all values of the system's parameters σ , B and R .

Proof. Using Proposition 1 the theorem follows easily from the comparison test.

Considering the series for $\alpha \geq 1$ we have

$$\sum_{\alpha=1}^{\infty} \mathbf{v}_{\alpha} \tau^{\alpha} \leq \left\| \sum_{\alpha=1}^{\infty} \mathbf{v}_{\alpha} \tau^{\alpha} \right\| \leq \sum_{\alpha=1}^{\infty} \|\mathbf{v}_{\alpha}\| |\tau|^{\alpha} \leq \sum_{\alpha=1}^{\infty} M K^{\alpha} |\tau|^{\alpha}.$$

Applying the ratio test to this last series gives

$$\lim_{\alpha \rightarrow \infty} \left(\frac{K^{\alpha+1}}{K^{\alpha}} |\tau| \right) = K |\tau|.$$

For the bounding series to converge we must have $K|\tau| < 1$ or $|\tau| < \frac{1}{K}$. Therefore the ordinary series converge for

$$\mathcal{R} \geq \frac{1}{K} = \mathcal{R}_{lb},$$

where $\mathcal{R}_{lb}$ denotes the estimated lower bound on the radii of convergence of the series. $\square$

2.1.1 Plots and Discussion

In this section, for specific parameter values, $\mathcal{R}_{lb}$ will be calculated and the solution plotted by repeatedly re-expanding the series about a point within the radius of convergence.

For the initial conditions $(x(0), y(0), z(0)) = (5, 5, 5)$ and the parameter values $\sigma = 10$, $B = \frac{8}{3}$ and $R = 28$, $\mathcal{R}_{lb}$ is entirely determined by $|b_1|$ since the maximum of $L(\alpha)$ occurs at $\alpha = 1$ (see Figure 4.1), and $|b_1|$ is the maximum of $|a_1|$, $|b_1|$ and $|c_1|$. Therefore, as $M \rightarrow 1$, $\mathcal{R}_{lb} \rightarrow \frac{1}{|b_1|} = 0.01$.

Note that $L(\alpha)$ appears to be converging to 5 which is approximately $\frac{1}{\mathcal{R}_{act}}$. This is what Hadamard's formula for the radius [44]

$$\mathcal{R} = \frac{1}{\limsup_{\alpha \rightarrow \infty} |a_{\alpha}|^{\frac{1}{\alpha}}}$$

predicts should occur as M approaches 1. This formula gives the radius as approximately 0.2.

A plot of $\mathcal{R}_b$ versus R , the system parameter usually varied in dynamical studies is shown in Figure 4.2. For all values of R the maximum of $L(\alpha)$ occurs at $\alpha = 1$ and K equals $|b_1|$. The value of $|b_1|$ is determined by R and the initial conditions. Therefore as R increases, the value of the estimated radius decreases.

A plot of $\mathcal{R}_b$ versus time (Figure 4.3) indicates that as the solution moves farther from the axis, the radius decreases. This is as expected, since $|a_0|$ is increasing, increasing $|b_1|$.

Plots of the trajectory defined by $x(t)$ and of a projection of the attractor on the z plane are shown in Figure 4.4 and Figure 4.5. Figure 4.6 and Figure 4.7 show the corresponding plots obtained by numerical integration for the same initial conditions and parameter values, which are identical to those obtained by the series solutions.

The series of plots, Figure 4.8, Figure 4.9, Figure 4.10, Figure 4.11 and Figure 4.12 illustrate how the point of divergence of the neighbouring solutions varies with the initial conditions. As the difference in initial conditions increases, the trajectories diverge sooner. It is interesting to note that although these plots are of chaotic motion, it is not due to real time singularities. However, as we shall see in the section on psi series, for these parameter values, logarithmic singularities exist in the complex domain.

Figure 4.13 shows the trajectory of $x(t)$ with the initial conditions $(x(0), y(0), z(0)) = (5, 5, 5)$, and a trajectory which has been perturbed slightly from this solution, at a time just prior to the divergence of the two trajectories from Figure 4.10. The "perturbed" trajectory diverges from the original trajectory (solid line) at a time greater than the previous point of divergence ($t \doteq 11$). This illustrates that divergence does not occur at a specific time, but rather that any two trajectories which begin "close" to each other, are bound to diverge after a sufficient amount of time has elapsed.

Although we can observe that the maximum of $L(\alpha)$ occurs at $\alpha = 1$, it does not appear possible to prove this. Therefore, the proof requires that K be calculated based on the first $N - 1$ coefficients, where N is determined from (2.5). Typical values of N fall within the range of 6 to 300. Note that since M can be chosen

between zero and one, the value of N can be decreased by decreasing M , thus reducing the amount of calculation required to determine K .

It is also interesting to note that since $L(\alpha)$ is a decreasing function, in order to better estimate the lower bound on the radii, the claim (2.4) could be made for α greater than an arbitrary value, say $\hat{\alpha}$. As $\hat{\alpha}$ tended to infinity, $L(\alpha)$ would tend to $\frac{1}{\mathcal{R}_{act}}$. However, this modification would greatly increase the value of N required to prove the claim.

Lastly, we estimate the truncation error. Given that

$$\sum_{\alpha=0}^{\infty} \mathbf{v}_{\alpha} \tau^{\alpha} \leq \left\| \sum_{\alpha=0}^{\infty} \mathbf{v}_{\alpha} \tau^{\alpha} \right\| \leq \sum_{\alpha=0}^{\infty} \|\mathbf{v}_{\alpha}\| |\tau|^{\alpha} \leq \sum_{\alpha=0}^{\infty} M K^{\alpha} |\tau|^{\alpha} = M \sum_{\alpha=0}^{\infty} (K|\tau|)^{\alpha}$$

and within $\mathcal{R}_{lb}$, $K|\tau| < 1$, the truncation error

$$\begin{aligned} E_n &= \frac{S_{\infty} - S_n}{S_{\infty}} \\ &\leq (K|\tau|)^{n+1}, \end{aligned}$$

where S_n is the n th partial sum of the geometric series.

The plots of the trajectory of $x(t)$ are insensitive to the point of truncation, as long as a minimum of eight terms are included. This corresponds to an accuracy of approximately four decimal places.

2.2 Laurent Series

In this section the Painlevé test is applied to the Lorenz system, and the parameter values for which the Laurent series is valid are determined.

We first determine the leading order behaviour close to a singularity by assuming the solutions are of the form

$$\begin{aligned} x(t) &= a_0 \tau^{\psi}, \\ y(t) &= b_0 \tau^{\psi}, \\ z(t) &= c_0 \tau^{\gamma}, \end{aligned} \tag{2.10}$$

where $\tau = (t - t_0)$, t_0 is the location of the arbitrary singularity, and $t \in \mathbb{C}$.

Substitution of (2.10) into (1.1) yields

$$a_0 \alpha \tau^{\psi-1} = \sigma (b_0 \tau^\beta - a_0 \tau^\psi) , \quad (2.11)$$

$$b_0 \beta \tau^{\beta-1} = -a_0 c_0 \tau^{\psi+\gamma} + R a_0 \tau^\psi - b_0 \tau^\beta , \quad (2.12)$$

$$c_0 \gamma \tau^{\gamma-1} = a_0 b_0 \tau^{\psi+\beta} - B c_0 \tau^\gamma . \quad (2.13)$$

Balancing the leading orders for equation (2.11) ($\tau^{\psi-1}$ and τ^β) yields

$$\psi - 1 = \beta ,$$

$$a_0 \psi = \sigma b_0 .$$

Balancing the leading orders for equation (2.12) ($\tau^{\psi+\gamma}$ and $\tau^{\beta-1}$) yields

$$\beta - 1 = \psi + \gamma ,$$

$$b_0 \beta = -a_0 c_0 .$$

Balancing the leading orders for equation (2.13) ($\tau^{\gamma-1}$ and $\tau^{\psi+\beta}$) yields

$$\gamma - 1 = \psi + \beta ,$$

$$c_0 \gamma = a_0 b_0 .$$

The solution of these six equations is $\psi = -1$, $\beta = -2$, $\gamma = 2$ and

$$a_0 = \pm 2i ,$$

$$b_0 = \mp \frac{2i}{\sigma} ,$$

$$c_0 = -\frac{2}{\sigma} ,$$

or a_0, b_0, c_0 equal to 0. In the latter case b_1 and c_1 are calculated to be zero and a_1, b_2 and c_2 are arbitrary. These are the initial conditions $x(0)$, $y(0)$ and $z(0)$, respectively, and t_0 is no longer a singularity. In other words, the Laurent series becomes an ordinary series, as in the last section.

Using this information, the solutions are assumed to be of the form

$$\begin{aligned} x(t) &= \sum_{\alpha=0}^{\infty} a_\alpha \tau^{\alpha-1} , \\ y(t) &= \sum_{\alpha=0}^{\infty} b_\alpha \tau^{\alpha-2} , \\ z(t) &= \sum_{\alpha=0}^{\infty} c_\alpha \tau^{\alpha-2} . \end{aligned} \quad (2.14)$$

Substitution of (2.14) into the Lorenz system (1.1) gives

$$\begin{aligned} \sum_{\alpha=0}^{\infty} a_{\alpha}(\alpha-1)\tau^{\alpha-2} &= \sigma \left(\sum_{\alpha=0}^{\infty} b_{\alpha}\tau^{\alpha-2} - \sum_{\alpha=0}^{\infty} a_{\alpha}\tau^{\alpha-1} \right), \\ \sum_{\alpha=0}^{\infty} b_{\alpha}(\alpha-2)\tau^{\alpha-3} &= R \sum_{\alpha=0}^{\infty} a_{\alpha}\tau^{\alpha-1} - \sum_{\alpha=0}^{\infty} b_{\alpha}\tau^{\alpha-2} - \left(\sum_{\alpha=0}^{\infty} a_{\alpha}\tau^{\alpha-1} \right) \left(\sum_{\alpha=0}^{\infty} c_{\alpha}\tau^{\alpha-2} \right), \\ \sum_{\alpha=0}^{\infty} c_{\alpha}(\alpha-2)\tau^{\alpha-3} &= \left(\sum_{\alpha=0}^{\infty} a_{\alpha}\tau^{\alpha-1} \right) \left(\sum_{\alpha=0}^{\infty} b_{\alpha}\tau^{\alpha-2} \right) - B \sum_{\alpha=0}^{\infty} c_{\alpha}\tau^{\alpha-2}. \end{aligned} \quad (2.15)$$

we obtain

$$\begin{aligned} \sum_{\alpha=0}^{\infty} (a_{\alpha}(\alpha-1) - \sigma b_{\alpha} + \sigma a_{\alpha-1}) \tau^{\alpha-2} &= 0, \\ \sum_{\alpha=0}^{\infty} \left(b_{\alpha}(\alpha-2) - R a_{\alpha-2} + b_{\alpha-1} + \sum_{\beta=1}^{\alpha-1} a_{\beta} c_{\alpha-\beta} + a_0 c_{\alpha} + c_0 a_{\alpha} \right) \tau^{\alpha-3} &= 0, \\ \sum_{\alpha=0}^{\infty} \left(c_{\alpha}(\alpha-2) - \sum_{\beta=1}^{\alpha-1} a_{\beta} b_{\alpha-\beta} - a_0 b_{\alpha} - b_0 a_{\alpha} + B c_{\alpha-1} \right) \tau^{\alpha-3} &= 0, \end{aligned}$$

after shifting the indices of summation where necessary, to obtain identical powers of τ .

For these equations to be satisfied for all τ it is necessary that the coefficient of each power of τ be zero. Thus we obtain the three recursion relations (RR)

$$\begin{aligned} a_{\alpha}(\alpha-1) - \sigma b_{\alpha} + \sigma a_{\alpha-1} &= 0, \\ b_{\alpha}(\alpha-2) - R a_{\alpha-2} + b_{\alpha-1} + \sum_{\beta=1}^{\alpha-1} a_{\beta} c_{\alpha-\beta} + a_0 c_{\alpha} + c_0 a_{\alpha} &= 0, \\ c_{\alpha}(\alpha-2) - \sum_{\beta=1}^{\alpha-1} a_{\beta} b_{\alpha-\beta} - a_0 b_{\alpha} - b_0 a_{\alpha} + B c_{\alpha-1} &= 0, \end{aligned}$$

where $a_{-1} = a_{-2} = b_{-1} = c_{-1} = 0$. Alternatively, in matrix form, we have

$$\begin{pmatrix} (\alpha-1) & -\sigma & 0 \\ -\frac{2}{\sigma} & (\alpha-2) & \pm 2i \\ \pm \frac{2i}{\sigma} & \mp 2i & (\alpha-2) \end{pmatrix} \begin{pmatrix} a_{\alpha} \\ b_{\alpha} \\ c_{\alpha} \end{pmatrix} = \begin{pmatrix} -\sigma a_{\alpha-1} \\ R a_{\alpha-2} - b_{\alpha-1} - \sum_{\beta=1}^{\alpha-1} a_{\beta} c_{\alpha-\beta} \\ \sum_{\beta=1}^{\alpha-1} a_{\beta} b_{\alpha-\beta} - B c_{\alpha-1} \end{pmatrix},$$

or,

$$A_{\alpha} \mathbf{v}_{\alpha} = W_{\alpha} = \begin{pmatrix} W_{\alpha 1} \\ W_{\alpha 2} \\ W_{\alpha 3} \end{pmatrix}.$$

Note that A_α is singular at $\alpha = 2$ and $\alpha = 4$. If the two sets of equations defined by $A_\alpha \mathbf{v}_\alpha = \mathbf{W}_\alpha$ are not inconsistent for these two values of α , there will be an infinite number of solutions for each system. This will provide us with the required further two arbitrary coefficients (in addition to the location of the moveable singularity, t_0).

The Freedholm Alternative Theorem states under which conditions a singular nonhomogeneous system of equations will be inconsistent or have infinitely many solutions. These conditions supply the compatibility conditions described in Section 1.3. The theorem states: Given the system $A_\alpha \mathbf{v}_\alpha = \mathbf{W}_\alpha$, where A_α is singular, there exist infinitely many solutions if and only if $\mathbf{V}_\alpha \cdot \mathbf{W}_\alpha = 0$, for all $\mathbf{V}_\alpha$ such that $A_\alpha^T \mathbf{V}_\alpha = 0$.

For $\alpha = 2$,

$$\mathbf{V}_2 = \begin{pmatrix} \mp 2i/\sigma \\ 0 \\ 1 \end{pmatrix} \text{ and}$$

$$\mathbf{W}_2 = \begin{pmatrix} -\sigma a_1 \\ -a_1 c_1 + R a_0 - b_1 \\ a_1 b_1 - B c_1 \end{pmatrix}.$$

Given that

$$\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \pm \frac{1}{3}(3\sigma - 1 - 2B) \\ \pm 2i \\ \frac{2}{3\sigma}(3\sigma + 1 - B) \end{pmatrix},$$

imposing $\mathbf{V}_2 \cdot \mathbf{W}_2 = 0$ yields the first compatibility condition, namely,

$$B(B - 1) = 6\sigma^2 - 2\sigma - B\sigma. \quad (2.16)$$

This concurs with the compatibility condition found by Tabor and Weiss [50]. When this condition is met the system $A_2 \mathbf{v}_2 = \mathbf{W}_2$ will not be inconsistent and will have infinitely many solutions. Thus for values of σ , B and R satisfying (2.16), one of a_2 , b_2 and c_2 can be made arbitrary.

For $\alpha = 4$,

$$V_4 = \begin{pmatrix} 0 \\ \pm i \\ 1 \end{pmatrix}$$

and

$$W_4 = \begin{pmatrix} -\sigma a_3 \\ -a_1 c_3 - a_2 c_2 - a_3 c_1 + R a_2 - b_3 \\ a_1 b_3 + a_2 b_2 + a_3 b_1 - B c_3 \end{pmatrix}.$$

Taking a_2 to be arbitrary, we have

$$b_2 = a_2/\sigma + a_1,$$

$$c_2 = \mp i a_2/\sigma \pm i(a_1 c_1 - R a_0 + b_1)/2$$

and

$$\begin{pmatrix} a_3 \\ b_3 \\ c_3 \end{pmatrix} = \begin{pmatrix} \pm \frac{-i-9iB-6iB^2+16iB^3-3i\sigma+24iB\sigma-102iB^2\sigma-54iBR\sigma+45i\sigma^2+171iB\sigma^2-81i\sigma^3}{108} \\ \pm \frac{-i-9iB-6iB^2+16iB^3-3i\sigma+24iB\sigma-102iB^2\sigma-54iBR\sigma+45i\sigma^2+171iB\sigma^2-81i\sigma^3}{54\sigma} \\ \frac{(1-B+3\sigma)(1\pm 36ia_2+4B+4B^2-6\sigma-12B\sigma+9\sigma^2)}{54\sigma} \end{pmatrix} + \begin{pmatrix} \frac{\frac{12}{2}(1-B-3\sigma)}{\sigma} \\ \frac{\frac{12}{2}(1-B+2\sigma)}{\sigma} \\ 0 \end{pmatrix}.$$

Imposing $V_4 \cdot W_4 = 0$ yields

$$a_3 c_1 + a_2 c_2 + a_1 c_3 - R a_2 + b_3 + i a_3 b_1 + i a_2 b_2 + i a_1 b_3 - i B c_3 = 0,$$

and upon the substitution of a_α , b_α and c_α , $\alpha = 1, 2, 3$, we obtain

$$\begin{aligned} & 54\sigma^4 - (144B - 99)\sigma^3 + (27BR + 129B^2 + 156B + 39)\sigma^2 - \\ & ((9B^2 + 45B)R + 47B^3 + 87B^2 - 27B + 1)\sigma + 6B^4 + 13B^3 - 9B^2 - 9B - 1 + \\ & a_2 (54i\sigma^2 + (144i - 171iB)\sigma + 45B^2 + 9B - 54) = 0. \end{aligned} \quad (2.17)$$

a_2 is arbitrary therefore the coefficient of a_2 must vanish, yielding the second compatibility condition

$$54\sigma^2 + (144 - 171B)\sigma + 45B^2 + 9B - 54 = 0. \quad (2.18)$$

The remainder of the terms in (2.17) must also vanish, yielding the third compatibility condition

$$\begin{aligned} 54\sigma^4 - (144B - 99)\sigma^3 + (27BR + 129B^2 + 156B + 39)\sigma^2 - \\ ((9B^2 + 45B)R + 47B^3 + 87B^2 - 27B + 1)\sigma + \\ 6B^4 + 13B^3 - 9B^2 - 9B - 1 = 0. \end{aligned} \quad (2.19)$$

The three compatibility conditions (2.16), (2.18), and (2.19) hold for the following values of σ , B and R

$$\begin{aligned} i) \quad & \sigma = 1/2 \quad B = 1 \quad R = 0, \\ ii) \quad & \sigma = 1 \quad B = 2 \quad R = 1/9, \\ iii) \quad & \sigma = 1/3 \quad B = 0 \quad R \text{ arbitrary}. \end{aligned} \quad (2.20)$$

For these three sets of parameter values the Lorenz system is completely integrable in terms of elliptic functions and Painlevé transcendents [50].

In order to prove the existence of a lower bound on the radii of convergence of the three series, we first establish a bound on the coefficients of the series with the following proposition, in exactly the same fashion as for the ordinary series.

Proposition 2 *For all M , $0 < M < 1$, there exists K , $1 < K < \infty$ such that for all $\alpha \geq 5$*

$$\|v_\alpha\| \leq MK^\alpha. \quad (2.21)$$

Proof.

The proposition is a consequence of the following

Assertion 1 *If N is such that $Q(N) \geq 0$, where*

$$Q(\alpha) = \alpha^3(1-M) - \alpha^2 \left(5 + J + M \left(\frac{2}{\sigma} - 10 \right) \right) + \alpha \left(2 - J \left(\frac{2}{\sigma} - 1 \right) + M \left(\frac{10}{\sigma} \right) \right) + 8 \left(1 + \frac{J-9}{\sigma} \right)$$

where

$$J = \max \left(R + 1 + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |c_\beta|), B + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |b_\beta|) \right),$$

and if

$$\|\mathbf{v}_\alpha\| \leq MK^\alpha \quad (2.22)$$

for all $\alpha < N$, then

$$\|\mathbf{v}_\alpha\| \leq MK^\alpha . \quad (2.23)$$

for $\alpha = N$.

The proposition will then follow by induction if N is chosen such that $Q(\alpha) \geq 0$ for all $\alpha \geq N$, and

$$K = \max_{5 \leq \alpha \leq N-1} L(\alpha) , \quad (2.24)$$

where

$$L(\alpha) = \left(\frac{\|\mathbf{v}_\alpha\|}{M} \right)^{\frac{1}{\alpha}} .$$

We now prove the claim.

Given that (2.21) holds for $5 \leq \alpha \leq N-1$, for $\alpha = N$ we obtain

$$\begin{aligned} |W_{\alpha 1}| &\leq \sigma |a_{\alpha-1}| \\ &\leq \sigma MK^\alpha , \end{aligned}$$

$$\begin{aligned} |W_{\alpha 2}| &\leq R|a_{\alpha-2}| + |b_{\alpha-1}| + \sum_{\beta=5}^{\alpha-5} |a_\beta| |c_{\alpha-\beta}| + \\ &\quad \sum_{\beta=1}^4 |a_\beta| |c_{\alpha-\beta}| + \sum_{\beta=\alpha-4}^{\alpha-1} |a_\beta| |c_{\alpha-\beta}| \\ &\leq \left(R + 1 + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |c_\beta|) \right) MK^\alpha \end{aligned}$$

and

$$\begin{aligned} |W_{\alpha 3}| &\leq \sum_{\beta=5}^{\alpha-5} |a_\beta| |b_{\alpha-\beta}| + \sum_{\beta=1}^4 |a_\beta| |b_{\alpha-\beta}| + \\ &\quad \sum_{\beta=\alpha-4}^{\alpha-1} |a_\beta| |b_{\alpha-\beta}| + B|c_{\alpha-1}| \\ &\leq \left(B + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |b_\beta|) \right) MK^\alpha . \end{aligned}$$

Since for $\alpha \geq 5$ A_α is nonsingular,

$$\|v_\alpha\| = \|A_\alpha^{-1} \cdot W_\alpha\| \leq \|A_\alpha^{-1}\| \|G_\alpha\| M K^\alpha,$$

where

$$G_\alpha = \begin{pmatrix} \sigma \\ R + 1 + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |c_\beta|) \\ B + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |b_\beta|) \end{pmatrix} = \begin{pmatrix} G_{\alpha 1} \\ G_{\alpha 2} \\ G_{\alpha 3} \end{pmatrix}. \quad (2.25)$$

Given that

$$A_\alpha^{-1} = \frac{1}{(\alpha + 1)(\alpha - 2)(\alpha - 4)} \begin{pmatrix} \alpha(\alpha - 4) & \sigma(\alpha - 2) & -2i\sigma \\ \frac{2}{\sigma}(\alpha - 4) & (\alpha - 1)(\alpha - 2) & -2i(\alpha - 1) \\ \frac{-2i}{\sigma}(\alpha - 4) & 2i(\alpha - 2) & \alpha(\alpha - 3) \end{pmatrix},$$

for $\alpha \geq 5$,

$$\begin{aligned} \|A_\alpha^{-1}\| &= \frac{\max(\alpha^2 + \alpha(\sigma - 4), \alpha^2 + \alpha(\frac{2}{\sigma} - 1) - \frac{8}{\sigma}, \alpha^2 + \alpha(\frac{2}{\sigma} - 1) - \frac{8}{\sigma} - 4)}{(\alpha + 1)(\alpha - 2)(\alpha - 4)} \\ &= \frac{\max(\alpha^2 + \alpha(\sigma - 4), \alpha^2 + \alpha(\frac{2}{\sigma} - 1) - \frac{8}{\sigma})}{(\alpha + 1)(\alpha - 2)(\alpha - 4)} \\ &= \frac{\alpha^2 + \alpha(\frac{2}{\sigma} - 1) - \frac{8}{\sigma}}{(\alpha + 1)(\alpha - 2)(\alpha - 4)}, \end{aligned}$$

since $\alpha(\alpha - 4) < \alpha(\frac{2}{\sigma} - 1) - \frac{8}{\sigma}$ for all $\sigma < 3$.

It can also easily be shown that for all of the parameter values (2.20), $\|G_\alpha\|$ is equal to either $G_{\alpha 2}$ or $G_{\alpha 3}$. Therefore $\|G_\alpha\| = J + M(\alpha - 9)$ where

$$J = \max\left(R + 1 + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |c_\beta|), B + M(\alpha - 9) + \sum_{\beta=1}^4 (|a_\beta| + |b_\beta|)\right).$$

In order that (2.21) hold we must have

$$\|A_\alpha^{-1}\| \|G_\alpha\| \leq 1$$

for $\alpha = N$. This requires that $Q(\alpha) \geq 0$, where $Q(\alpha)$ is defined as in the assertion, and thus the assertion is proved. $\square$

We are now prepared to state and prove the theorem regarding a lower bound on the radii of convergence of the three series.

Theorem 2 *There exists $\mathcal{R}$, $0 < \mathcal{R} < \infty$, such that for $0 < \tau < \mathcal{R}$, the series*

$$\sum_{\alpha=0}^{\infty} a_{\alpha} \tau^{\alpha-1}, \quad \sum_{\alpha=0}^{\infty} b_{\alpha} \tau^{\alpha-2} \quad \text{and} \quad \sum_{\alpha=0}^{\infty} c_{\alpha} \tau^{\alpha-2}$$

converge absolutely for the values of the system's parameters given by (2.20).

Proof. Using Proposition 2 the theorem follows easily from the comparison test.

Considering the series for $\alpha \geq 5$ and using Proposition 2 we obtain the same radius of convergence as for the ordinary series

$$\mathcal{R} \geq \frac{1}{K} = \mathcal{R}_{lb},$$

where $\mathcal{R}_{lb}$ denotes the estimated lower bound on the radii of convergence of the series. □

2.3 Psi Series

In this section a series solution will be derived which will be valid if any of the three compatibility conditions (2.16), (2.18) and (2.19) are not met. As described in Section 1.3, in this case a general solution will be of the form

$$u(t) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} d_{kl} \tau^{k-p} (\tau^{k_0} \log \tau)^l, \quad (2.26)$$

where p is the integer obtained from the leading order analysis and k_0 is the smallest resonance for which the compatibility condition is not met.

For purposes of further analysis, the doubly infinite series is now converted into a single series with polynomial coefficients. Rather than summing over all k and l , Figure 4.14 illustrates that this is equivalent to summing over each line

$$\{(k, l) : k + k_0 l = \alpha\} \quad 0 \leq k, l < \infty,$$

for every integer α , $0 \leq \alpha < \infty$. Each line contains a finite number of points (k, l) with non-negative integer coordinates.

Letting $k + k_0 l = \alpha$, (2.26) becomes

$$u(t) = \sum_{\alpha=0}^{\infty} \tau^{\alpha-p} \sum_{k+k_0 l=\alpha} d_{kl} (\log \tau)^l .$$

Lastly, letting $z = \log(\tau)$ and $w_{\alpha}(z) = \sum_{k+k_0 l=\alpha} d_{kl} (\log \tau)^l$, the solution becomes

$$u(t) = \sum_{\alpha=0}^{\infty} \tau^{\alpha-p} w_{\alpha}(z) .$$

Note that since $l = \frac{\alpha-k}{k_0}$, the order of $w_{\alpha}(z)$ is $\lfloor \frac{\alpha}{k_0} \rfloor$, where $\lfloor T \rfloor$ denotes the greatest integer less than or equal to T .

Using this resummation technique for the psi series solutions of the Lorenz system, we form the series solutions

$$\begin{aligned} x(t) &= \sum_{\alpha=0}^{\infty} \tau^{\alpha-1} f_{\alpha}(z) , \\ y(t) &= \sum_{\alpha=0}^{\infty} \tau^{\alpha-2} g_{\alpha}(z) , \\ z(t) &= \sum_{\alpha=0}^{\infty} \tau^{\alpha-2} h_{\alpha}(z) , \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} f_{\alpha}(z) &= \sum_{k+k_0 l=\alpha} a_{kl} (\log \tau)^l , \\ g_{\alpha}(z) &= \sum_{k+k_0 l=\alpha} b_{kl} (\log \tau)^l , \\ h_{\alpha}(z) &= \sum_{k+k_0 l=\alpha} c_{kl} (\log \tau)^l . \end{aligned}$$

Differentiating these series with respect to time we obtain

$$\begin{aligned} x'(t) &= \sum_{\alpha=0}^{\infty} \tau^{\alpha-2} (f'_{\alpha}(z) + (\alpha-1)f_{\alpha}(z)) , \\ y'(t) &= \sum_{\alpha=0}^{\infty} \tau^{\alpha-3} (g'_{\alpha}(z) + (\alpha-2)g_{\alpha}(z)) , \\ z'(t) &= \sum_{\alpha=0}^{\infty} \tau^{\alpha-3} (h'_{\alpha}(z) + (\alpha-2)h_{\alpha}(z)) . \end{aligned} \tag{2.28}$$

since it can easily be shown that the sum of the derivatives converges uniformly, and thus it is valid to differentiate term by term. Briefly, this is proved by showing

that the sequence of partial sums is Cauchy, and in fact uniformly Cauchy, and thus uniformly convergent.

Substituting (2.27) and (2.28) into the Lorenz system (1.1) yields:

$$\begin{aligned} \sum_{\alpha=0}^{\infty} (f'_{\alpha}(z) + (\alpha-1)f_{\alpha}(z) + \sigma f_{\alpha-1}(z) - \sigma g_{\alpha}(z)) \tau^{\alpha-2} &= 0, \\ \sum_{\alpha=0}^{\infty} (g'_{\alpha}(z) + (\alpha-2)g_{\alpha}(z) - Rf_{\alpha-2}(z) + \\ &g_{\alpha-1} + \sum_{\beta=0}^{\alpha} f_{\alpha-\beta}(z)h_{\beta}(z)) \tau^{\alpha-3} = 0, \\ \sum_{\alpha=0}^{\infty} (h'_{\alpha}(z) + (\alpha-2)h_{\alpha}(z) + Bh_{\alpha-1}(z) - \\ &\sum_{\beta=0}^{\alpha} f_{\alpha-\beta}(z)g_{\beta}(z)) \tau^{\alpha-3} = 0, \end{aligned}$$

after shifting the indices of summation to obtain identical powers of τ in each series. For these equations to be satisfied for all t it is necessary that the coefficient of each power of τ be zero. Thus we obtain the following three recursion relations for $f_{\alpha}(z)$, $g_{\alpha}(z)$ and $h_{\alpha}(z)$:

$$\begin{aligned} f'_{\alpha}(z) + (\alpha-1)f_{\alpha}(z) + \sigma f_{\alpha-1}(z) - \sigma g_{\alpha}(z) &= 0, \\ g'_{\alpha}(z) + (\alpha-2)g_{\alpha}(z) - Rf_{\alpha-2}(z) + g_{\alpha-1} + \sum_{\beta=0}^{\alpha} f_{\alpha-\beta}(z)h_{\beta}(z) &= 0, \quad (2.29) \\ h'_{\alpha}(z) + (\alpha-2)h_{\alpha}(z) + Bh_{\alpha-1}(z) - \sum_{\beta=0}^{\alpha} f_{\alpha-\beta}(z)g_{\beta}(z) &= 0. \end{aligned}$$

Substituting $\alpha = 0$ into (2.29) yields

$$\begin{aligned} f_0 &= \pm 2i, \\ g_0 &= \mp 2i/\sigma, \\ h_0 &= -2/\sigma, \end{aligned} \quad (2.30)$$

or f_0 , g_0 , h_0 equal to 0. These are equivalent to the a_0 , b_0 and c_0 found for the Laurent series. In the case that f_0 , g_0 and h_0 are all zero, g_1 and h_1 are also zero and all the polynomials $f_{\alpha}(z)$, $g_{\alpha}(z)$ and $h_{\alpha}(z)$ become constant functions. In this case f_1 , g_2 and h_2 are arbitrary. These are the initial conditions $x(0)$, $y(0)$ and $z(0)$

respectively and t_0 is no longer a singularity. In other words, the psi series evolves as an ordinary series.

Substituting (2.30) into (2.29) yields

$$\begin{aligned}
 f'_\alpha(z) + (\alpha - 1)f_\alpha(z) + \sigma f_{\alpha-1}(z) - \sigma g_\alpha(z) &= 0, \\
 g'_\alpha(z) + (\alpha - 2)g_\alpha(z) - Rf_{\alpha-2}(z) + g_{\alpha-1} + \\
 + \sum_{\beta=1}^{\alpha-1} f_{\alpha-\beta}(z)h_\beta(z) \pm 2ih_\alpha(z) - (2/\sigma)f_\alpha(z) &= 0, \\
 h'_\alpha(z) + (\alpha - 2)h_\alpha(z) + Bh_{\alpha-1}(z) - \\
 \sum_{\beta=1}^{\alpha-1} f_{\alpha-\beta}(z)g_\beta(z) \pm (2i/\sigma)f_\alpha(z) \mp 2ig_\alpha(z) &= 0,
 \end{aligned} \tag{2.31}$$

or equivalently in matrix form,

$$\begin{aligned}
 \begin{pmatrix} f_\alpha(z) \\ g_\alpha(z) \\ h_\alpha(z) \end{pmatrix}' + \begin{pmatrix} (\alpha - 1) & -\sigma & 0 \\ -2/\sigma & (\alpha - 2) & \pm 2i \\ \pm \frac{2i}{\sigma} & \mp 2i & (\alpha - 2) \end{pmatrix} \begin{pmatrix} f_\alpha(z) \\ g_\alpha(z) \\ h_\alpha(z) \end{pmatrix} \\
 = \begin{pmatrix} -\sigma f_{\alpha-1}(z) \\ Rf_{\alpha-2}(z) - g_{\alpha-1}(z) - \sum_{\beta=1}^{\alpha-1} f_{\alpha-\beta}(z)h_\beta(z) \\ \sum_{\beta=1}^{\alpha-1} f_{\alpha-\beta}(z)g_\beta(z) - Bh_{\alpha-1}(z) \end{pmatrix},
 \end{aligned} \tag{2.32}$$

or

$$\mathbf{v}'_\alpha(z) + A_\alpha \mathbf{v}_\alpha(z) = \mathbf{V}_\alpha(z) = \begin{pmatrix} V_{\alpha 1} \\ V_{\alpha 2} \\ V_{\alpha 3} \end{pmatrix}. \tag{2.33}$$

In order to prove the existence of a lower bound on the radii of convergence of the three series, we first establish a bound on the coefficients of the series with the following proposition.

Proposition 3 *There exists K , $1 < K < \infty$, such that for all $z < 0$ and for all $\alpha \geq 5$,*

$$\|\mathbf{v}_\alpha(z)\| \leq \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha + 1}}, \tag{2.34}$$

where $k_0 = 2$ or $k_0 = 4$.

Proof.

The proposition is a consequence of the following

Assertion 2 *If N is such that the set of inequalities Q are satisfied for $\alpha = N$, where Q is defined by*

$$Q = \left\{ \begin{aligned} &\alpha \geq 4 + \frac{7\sqrt{6}}{2\sqrt{5}}P(\sigma)\sigma \\ &\left(\frac{4}{49P^2(\sigma)}\right)\alpha^2 - \left(\frac{4}{7P(\sigma)}\left(\frac{8}{7P(\sigma)} + \sqrt{\frac{3}{2}}J\right) - \pi^2\right)\alpha + \left(\left(\frac{8}{7P(\sigma)} + \sqrt{\frac{3}{2}}J\right)^2 - \pi^2\right) \geq 0 \\ &\left(\frac{4}{49P^2(\sigma)}\right)\alpha^2 - \left(\frac{4}{7P(\sigma)}\left(\frac{8}{7P(\sigma)} + \sqrt{\frac{3}{2}}JJ\right) - \pi^2\right)\alpha + \left(\left(\frac{8}{7P(\sigma)} + \sqrt{\frac{3}{2}}JJ\right)^2 - \pi^2\right) \geq 0 \end{aligned} \right\}, \quad (2.35)$$

where

$$J = R + 1 + \sum_{\beta=1}^{k_0-1} (|h_\beta(z)| + |f_\beta(z)|),$$

$$JJ = B + \sum_{j=1}^{k_0-1} (|f_\beta(z)| + |g_\beta(z)|)$$

and

$$P(\sigma) = \|P\|\|P^{-1}\| \quad (2.36)$$

$$= \begin{cases} \frac{7}{3}\left(\frac{1}{\sigma} + 1\right), & \sigma < \frac{7}{4} \\ \frac{4}{3}(\sigma + 1), & \frac{7}{4} < \sigma < 5 \\ \frac{8}{5}\sigma, & \sigma > 5 \end{cases},$$

and if

$$\|v_\alpha(z)\| \leq \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha + 1}} \quad (2.37)$$

for all $\alpha < N$, then

$$\|v_\alpha(z)\| \leq \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha + 1}} \quad (2.38)$$

for $\alpha = N$.

Given an N , since $f_\alpha(z)$, $g_\alpha(z)$, and $h_\alpha(z)$ are polynomials of order $\lfloor \frac{\alpha}{k_0} \rfloor$, it is possible to find a K such that (2.34) holds for $5 \leq \alpha \leq N - 1$. How such a K is calculated is shown in the next section.

The proposition will then follow by induction. if N is chosen such that the set of inequalities Q hold for all $\alpha \geq N$,

We now prove the claim.

We begin by solving (2.33) for $\mathbf{v}_\alpha$.

Multiplying (2.33) through by the integrating factor $e^{A_\alpha z}$ yields

$$e^{A_\alpha z} \mathbf{v}_\alpha(z)' + A_\alpha e^{A_\alpha z} \mathbf{v}_\alpha(z) = \mathbf{V}_\alpha(z) e^{A_\alpha z} ,$$

i.e.,

$$(e^{A_\alpha z} \mathbf{v}_\alpha(z))' = e^{A_\alpha z} \mathbf{V}_\alpha(z) .$$

Integrating both sides from $-\infty$ to z , we obtain

$$e^{A_\alpha z} \mathbf{v}_\alpha(z) \Big|_{-\infty}^z = \int_{-\infty}^z e^{A_\alpha q} \mathbf{V}_\alpha(q) dq .$$

Since $\lim_{q \rightarrow -\infty} e^{A_\alpha q} \mathbf{v}_\alpha(z) = 0$, after multiplying through by $e^{-A_\alpha z}$, we obtain

$$\mathbf{v}_\alpha(z) = \int_{-\infty}^z e^{A_\alpha(q-z)} \mathbf{V}_\alpha(q) dq . \quad (2.39)$$

Noting that $e^{A_\alpha(q-z)} = P e^{D_\alpha(q-z)} P^{-1}$ (2.39) becomes

$$\mathbf{v}_\alpha(z) = \int_{-\infty}^z P e^{D_\alpha(q-z)} P^{-1} \mathbf{V}_\alpha(q) dq , \quad (2.40)$$

where P , the matrix of eigenvectors of A_α , is given by

$$P = \begin{pmatrix} \pm\sigma/2 & \mp\sigma & \pm\sigma/2 \\ \mp 1 & \mp 1 & \pm 3/2 \\ i & i & i \end{pmatrix}$$

and

$$D_\alpha = P^{-1} A_\alpha P = \begin{pmatrix} \alpha + 1 & 0 & 0 \\ 0 & \alpha - 2 & 0 \\ 0 & 0 & \alpha - 4 \end{pmatrix} .$$

To estimate the norm of $\mathbf{v}_\alpha(z)$ we must estimate each of the norms within the integral. Since $q \leq z \Rightarrow q - z \leq 0$, and $\alpha - 4 < \alpha - 2 < \alpha - 1$, we obtain

$$\begin{aligned} \|e^{D_\alpha(q-z)}\| &= \max(e^{(\alpha+1)(q-z)}, e^{(\alpha-2)(q-z)}, e^{(\alpha-4)(q-z)}) \\ &= e^{(\alpha-4)(q-z)} . \end{aligned} \quad (2.41)$$

Letting

$$\begin{aligned} P(\sigma) &= \|P\| \|P^{-1}\| \\ &= \begin{cases} \frac{7}{3}(\frac{1}{\sigma} + 1), & \sigma < \frac{7}{4} \\ \frac{4}{3}(\sigma + 1), & \frac{7}{4} < \sigma < 5 \\ \frac{8}{5}\sigma, & \sigma > 5 \end{cases} \end{aligned} \quad (2.42)$$

and substituting (2.41) and (2.42) into (2.40) to estimate $\|v_\alpha(z)\|$ we obtain

$$\|v_\alpha(z)\| \leq P(\sigma) \int_{-\infty}^z e^{(\alpha-4)(q-z)} \|V_\alpha(q)\| dq. \quad (2.43)$$

Given that (2.34) holds for $5 \leq \alpha \leq N-1$, for $\alpha = N$

$$\begin{aligned} |V_{\alpha 1}(z)| &= \sigma |f_{\alpha-1}(z)| \\ &\leq \sigma \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha}}, \end{aligned}$$

$$\begin{aligned} |V_{\alpha 2}(z)| &= R |f_{\alpha-2}(z)| + |g_{\alpha-1}(z)| + \sum_{\beta=k_0}^{\alpha-k_0} |h_\beta(z)| |f_{\alpha-\beta}(z)| \\ &\quad + \sum_{\beta=1}^{k_0-1} |h_\beta(z)| |f_{\alpha-\beta}(z)| + \sum_{\beta=\alpha-k_0+1}^{\alpha-1} |h_\beta(z)| |f_{\alpha-\beta}(z)|, \\ &\leq R \frac{(2K - Kz)^{\frac{\alpha-2}{k_0}}}{\sqrt{\alpha-1}} \\ &\quad + \left(1 + \sum_{\beta=1}^{k_0-1} (|h_\beta(z)| + |f_\beta(z)|) \right) \frac{(2K - Kz)^{\alpha-1/k_0}}{\sqrt{\alpha}} \\ &\quad + \sum_{\beta=2}^{\alpha-2} \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\beta+1} \sqrt{\alpha-\beta}}, \\ &\leq \left(\frac{R + 1 + \sum_{\beta=1}^{k_0-1} (|h_\beta(z)| + |f_\beta(z)|)}{\sqrt{\alpha-1}} + \pi \right) (2K - Kz)^{\alpha/k_0}, \end{aligned}$$

using the fact that

$$\lim_{\alpha \rightarrow \infty} \sum_{\beta=0}^{\alpha-1} \frac{1}{\sqrt{\beta+1}} \frac{1}{\sqrt{\alpha-\beta}} = \pi.$$

Similarly,

$$\begin{aligned}
 |V_{\alpha 3}(z)| &= B|h_{\alpha-1}(z)| + \sum_{\beta=k_0}^{\alpha-k_0} |g_{\beta}(z)||f_{\alpha-\beta}(z)| + \sum_{\beta=1}^{k_0-1} |g_{\beta}(z)||f_{\alpha-\beta}(z)| \\
 &\quad + \sum_{\beta=\alpha-k_0+1}^{\alpha-1} |g_{\beta}(z)||f_{\alpha-\beta}(z)|, \\
 &\leq \left(\frac{B + \sum_{\beta=1}^{k_0-1} (|g_{\beta}(z)| + |f_{\beta}(z)|)}{\sqrt{\alpha-1}} + \pi \right) (2K - Kz)^{\alpha/k_0}.
 \end{aligned}$$

Letting

$$G_{\alpha} = \begin{pmatrix} \frac{R+1+\sum_{\beta=1}^{k_0-1} (|h_{\beta}(z)| + |f_{\beta}(z)|)}{\sqrt{\alpha-1}} + \pi \\ \frac{B + \sum_{\beta=1}^{k_0-1} (|g_{\beta}(z)| + |f_{\beta}(z)|)}{\sqrt{\alpha-1}} + \pi \end{pmatrix} = \begin{pmatrix} G_{\alpha 1} \\ G_{\alpha 2} \\ G_{\alpha 3} \end{pmatrix} \quad (2.44)$$

we obtain

$$\|V_{\alpha}(z)\| \leq \|G_{\alpha}\| (2K - Kz)^{\alpha/k_0},$$

and (2.43) becomes

$$\|v_{\alpha}(z)\| \leq P(\sigma) \|G_{\alpha}\| \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\alpha/k_0} dq. \quad (2.45)$$

To estimate the R.H.S. we need the following lemma.

Lemma 1

$$\int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\alpha/k_0} dq \leq \frac{7(2K - Kz)^{\alpha/k_0}}{2(\alpha-4)} \quad (2.46)$$

for $\alpha > 4$.

Proof. Integrating the L.H.S. of (2.46) by parts yields

$$\begin{aligned}
 &\int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\alpha/k_0} dq \\
 &= \frac{e^{(\alpha-4)(q-z)}}{(\alpha-4)} (2K - Kz)^{\alpha/k_0} \Big|_{-\infty}^z - \frac{1}{(\alpha-4)} \int_{-\infty}^z e^{(\alpha-4)(q-z)} \frac{\alpha}{k_0} (-K) (2K - Kq)^{\frac{\alpha}{k_0}-1} dq \\
 &= \frac{(2K - Kz)^{\alpha/k_0}}{(\alpha-4)} - \frac{\frac{\alpha}{k_0} (-K)}{(\alpha-4)} \int_{-\infty}^z e^{(\alpha-4)(q-z)} \frac{\alpha}{k_0} (-K) (2K - Kq)^{\frac{\alpha}{k_0}-1} dq
 \end{aligned}$$

for $\alpha > 4$.

After $m + 1$ integrations by parts, we obtain

$$\begin{aligned}
 & \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\alpha/k_0} dq \\
 &= \frac{(2K - Kz)^{\alpha/k_0}}{(\alpha - 4)} - \frac{\frac{\alpha}{k_0}(-K)}{(\alpha - 4)^2} (2K - Kz)^{\frac{\alpha}{k_0}-1} \\
 &+ \frac{\frac{\alpha}{k_0}(\frac{\alpha}{k_0} - 1)(-K)^2}{(\alpha - 4)^3} (2K - Kz)^{\frac{\alpha}{k_0}-2} - \dots \\
 &+ \frac{(-1)^m \frac{\alpha}{k_0}(\frac{\alpha}{k_0} - 1) \dots (\frac{\alpha}{k_0} - m)(-K)^{m+1}}{(\alpha - 4)^{m+1}} \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\frac{\alpha}{k_0}-m-1} dq .
 \end{aligned}$$

To simplify we let

$$m = \lfloor \frac{\alpha}{k_0} \rfloor, \quad \gamma = \frac{\alpha}{k_0} \text{ and } X = (2K - Kz) .$$

Then

$$\begin{aligned}
 & \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\alpha/k_0} dq \\
 &= \frac{X^\gamma}{(\alpha - 4)} + \frac{\gamma K X^{\gamma-1}}{(\alpha - 4)^2} + \dots \\
 &+ \frac{\gamma(\gamma - 1) \dots (\gamma - m) K^{m+1}}{(\alpha - 4)^{m+1}} \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\frac{\alpha}{k_0}-m-1} dq .
 \end{aligned}$$

Let us estimate the last integral. Since $\gamma - m - 1 < 0$, for $q \leq z$

$$(2K - Kq)^{\gamma-m-1} \leq X^{\gamma-m-1}$$

and

$$\begin{aligned}
 & \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\frac{\alpha}{k_0}-m-1} dq \\
 &\leq X^{\gamma-m-1} \int_{-\infty}^z e^{(\alpha-4)(q-z)} dq , \\
 &= \frac{X^{\gamma-m-1}}{\alpha - 4} .
 \end{aligned}$$

Thus

$$\begin{aligned}
 & \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\alpha/k_0} dq \\
 &\leq \frac{X^\gamma}{(\alpha - 4)} + \frac{\gamma K X^{\gamma-1}}{(\alpha - 4)^2} + \dots
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\gamma(\gamma-1)\cdots(\gamma-m)K^{m+1}}{(\alpha-4)^{m+1}} K^{m+1} X^{\gamma-m-1} \\
& = \frac{X^\gamma}{(\alpha-4)} \left(1 + \frac{\gamma}{(\alpha-4)} \left(\frac{K}{X} \right) + \frac{\gamma(\gamma-1)}{(\alpha-4)^2} \left(\frac{K}{X} \right)^2 + \cdots \right. \\
& \quad \left. + \frac{\gamma(\gamma-1)\cdots(\gamma-m)}{(\alpha-4)^{m+1}} \left(\frac{K}{X} \right)^{m+1} \right).
\end{aligned}$$

Since for $z < 0$

$$\frac{K}{X} = \frac{K}{2K - Kz} = \frac{1}{2-z} \leq \frac{1}{2}$$

and for $j \geq 2$

$$\frac{\gamma-j}{\alpha-4} \leq \frac{1}{2}$$

we obtain

$$\begin{aligned}
& \int_{-\infty}^z e^{(\alpha-4)(q-z)} (2K - Kq)^{\alpha/k_0} dq \\
& \leq \frac{X^\gamma}{(\alpha-4)} \left(1 + \frac{\gamma}{2(\alpha-4)} \left(1 + \frac{\gamma-1}{2(\alpha-4)} \left(1 + \frac{1}{4} + \left(\frac{1}{4} \right)^2 + \cdots + \left(\frac{1}{4} \right)^{m-1} \right) \right) \right), \\
& \leq \frac{X^\gamma}{(\alpha-4)} \left(1 + \frac{\gamma}{2(\alpha-4)} \left(1 + \frac{\gamma-1}{2(\alpha-4)} \left(\frac{4}{3} \right) \right) \right), \\
& \leq \frac{7X^\gamma}{2(\alpha-4)},
\end{aligned}$$

which concludes the proof of Lemma 1. $\square$

We can now employ the estimate found in Lemma 1 in (2.45) to obtain

$$\|v_\alpha(z)\| \leq P(\sigma) \|G_\alpha\| \frac{7(2K - Kz)^{\alpha/k_0}}{2(\alpha-4)}. \quad (2.47)$$

Recall that we are attempting to show that

$$\|v_\alpha(z)\| < \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha+1}},$$

for $\alpha = N$. This requires that the inequality

$$P(\sigma) \|G_\alpha\| \frac{7}{2(\alpha-4)} \leq \frac{1}{\sqrt{\alpha+1}} \quad (2.48)$$

hold for $\alpha = N$.

This condition will be examined for the three cases $\|G_\alpha\| = G_{\alpha 1}$, $G_{\alpha 2}$ and $G_{\alpha 3}$.

When $\|G_\alpha\| = G_{\alpha 1}$, (2.48) becomes

$$P(\sigma) \frac{\sigma}{\sqrt{\alpha}} \frac{7}{2(\alpha - 4)} \leq \frac{1}{\sqrt{\alpha + 1}}.$$

Since $\frac{\sqrt{\alpha+1}}{\sqrt{\alpha}} \leq \sqrt{\frac{6}{5}}$, this is equivalent to

$$\alpha \geq 4 + \frac{7\sqrt{6}}{2\sqrt{5}} P(\sigma) \sigma \quad (2.49)$$

which can always be satisfied by sufficiently large N .

If we let

$$J = \left(R + 1 + \sum_{\beta=1}^{k_0-1} (|h_\beta(z)| + |f_\beta(z)|) \right)$$

if $\|G_\alpha\| = G_{\alpha 2}$, and let

$$J = \left(B + \sum_{\beta=1}^{k_0-1} (|f_\beta(z)| + |g_\beta(z)|) \right)$$

if $\|G_\alpha\| = G_{\alpha 3}$, for these two cases (2.48) becomes

$$P(\sigma) \left(\frac{J}{\sqrt{\alpha - 1}} + \pi \right) \frac{7}{2(\alpha - 4)} \leq \frac{1}{\sqrt{\alpha + 1}}.$$

Since $\frac{\sqrt{\alpha+1}}{\sqrt{\alpha-1}} \leq \sqrt{\frac{3}{2}}$, this is equivalent to

$$\pi \sqrt{\alpha + 1} \leq \frac{2}{7P(\sigma)} \alpha - \left(\frac{8}{7P(\sigma)} + \sqrt{\frac{3}{2}} J \right).$$

Squaring both sides and moving all terms to the R.H.S. gives the equivalent condition

$$\left(\frac{4}{49P^2(\sigma)} \right) \alpha^2 - \left(\frac{4}{7P(\sigma)} \left(\frac{8}{7P(\sigma)} + \sqrt{\frac{3}{2}} J \right) - \pi^2 \right) \alpha + \left(\left(\frac{8}{7P(\sigma)} + \sqrt{\frac{3}{2}} J \right)^2 - \pi^2 \right) \geq 0. \quad (2.50)$$

We can now see that if N satisfies Q (2.35) the assertion will be proved, and thus Proposition 3 as well. $\square$

We are now prepared to state and prove the theorem regarding a lower bound on the radii of convergence of the three series.

Theorem 3 *There exists $\mathcal{R}$, $0 < \mathcal{R} < \infty$, such that for $0 < \tau < \mathcal{R}$, the psi series*

$$\sum_{\alpha=0}^{\infty} f_{\alpha}(z)\tau^{\alpha-1}, \quad \sum_{\alpha=0}^{\infty} g_{\alpha}(z)\tau^{\alpha-2} \quad \text{and} \quad \sum_{\alpha=0}^{\infty} h_{\alpha}(z)\tau^{\alpha-2},$$

converge absolutely for all values of the system's parameters σ , B and R .

Using Proposition 3 the theorem follows from the comparison test.

Considering the series for $\alpha \geq 5$ and using the claim (2.34) we have

$$\sum_{\alpha=5}^{\infty} v_{\alpha}(z)\tau^{\alpha-a} \leq \sum_{\alpha=5}^{\infty} (2K - Kz)^{\alpha/k_0} \tau^{\alpha-a},$$

where $a = 1$ or $a = 2$.

Applying the ratio test to this series yields

$$\lim_{\alpha \rightarrow \infty} \left(\frac{\sqrt{\alpha+1}}{\sqrt{\alpha+2}} (2K - Kz)^{1/k_0} \tau \right) = \tau (2K - Kz)^{1/k_0}.$$

Therefore the three psi series converge iff

$$\tau (2K - Kz)^{1/k_0} < 1.$$

Substituting $\tau = e^z$, the condition for convergence becomes

$$K(2 - z) < e^{-k_0 z},$$

and letting $s = -z > 0$, becomes

$$K(2 + s) < e^{k_0 s}.$$

Let s_0 denote the point of intersection of the line $y = Ks + 2K$ and the curve $y = e^{k_0 s}$. From Figure 4.15, one can see that for all $s > s_0$, $K(2 + s) < e^{k_0 s}$. Thus for all $z < z_0$, where $z_0 = -s_0$, or equivalently for all τ , $0 < \tau < e^{z_0}$, the psi series will converge. This completes the proof of the theorem. $\square$

2.3.1 Plots and Discussion

In this section, for specific parameter values the estimated lower bound on the radii of convergence ($\mathcal{R}_{lb}$) will be calculated.

The number K must first be calculated.

Let

$$L_\alpha(z) = \left(\frac{\|\mathbf{v}_\alpha(z)\|}{(2-z)^{\alpha/k_0}} \right)^{k_0/\alpha} (\alpha+1)^{1/\alpha}.$$

Since $f_\alpha(z)$ is a polynomial in z of order $\lfloor \alpha/k_0 \rfloor$,

$$\begin{aligned} |f_\alpha(z)| &= \left| \sum_{n=0}^{\lfloor \alpha/k_0 \rfloor} a_n z^n \right| \\ &\leq \sum_{n=0}^{\lfloor \alpha/k_0 \rfloor} |a_n| |z|^n \\ &\leq \sum_{n=0}^{\lfloor \alpha/k_0 \rfloor} |a_n| (2-z)^n \\ &\leq \left(\sum_{n=0}^{\lfloor \alpha/k_0 \rfloor} |a_n| \right) (2-z)^{\alpha/k_0}. \end{aligned}$$

Thus

$$\|\mathbf{v}_\alpha\| \leq \left(\sum_{n=0}^{\lfloor \alpha/k_0 \rfloor} |d_n| \right) (2-z)^{\alpha/k_0},$$

where d_n denotes the coefficients belonging the series $f_\alpha(z)$, $g_\alpha(z)$ or $h_\alpha(z)$ for which the sum of the absolute values of the coefficients is the largest.

Substituting this estimate into the formula for $L_\alpha(z)$ we obtain

$$L_\alpha(z) \leq \left(\sum_{n=0}^{\lfloor \alpha/k_0 \rfloor} |d_n| \right)^{\alpha/k_0} (\alpha+1)^{1/\alpha}.$$

Therefore it suffices to have

$$K = \max_{k_0 \leq \alpha \leq N-1} L_\alpha. \quad (2.51)$$

The estimate $\mathcal{R}_{lb}$ is now computed. The arbitrary singularity is placed at $t = 0$ and the two arbitrary coefficients are given the value zero. The parameter values

Table 2.1: $\mathcal{R}_{lb}$ and $\mathcal{R}_{act}$ as R varies, for both the $k_0 = 2$ and $k_0 = 4$ cases

R	$k_0 = 2$			$k_0 = 4$		
	K	$\mathcal{R}_{lb}$	$\mathcal{R}_{act}$	K	$\mathcal{R}_{lb}$	$\mathcal{R}_{act}$
.50	6.93	.0400	.075 - .08	2.21	.1849	1.40-1.50
1.50	6.46	.0432	.075 - .08	3.85	.1011	0.80-0.90
14.00	34.86	.0063	.075 - .08	25.03	.0129	0.40-0.42
24.74	54.65	.0038	.090 - .10	39.51	.0079	0.35-0.37
100.00	165.75	.0011	.045 - .05	120.88	.0024	0.15-0.17

$\sigma = 10$, $B = \frac{8}{3}$, $R = 28$ are chosen as for the ordinary series. The derivation of the formula for $f_\alpha(z)$, $g_\alpha(z)$ and $h_\alpha(z)$ can be found in Appendix A.

The maximum of $L(\alpha)$ occurs at $\alpha = 5$, as shown in Figure 4.16. $\mathcal{R}_{est}$ is approximately 0.02 and the actual radius of the series ($\mathcal{R}_{act}$) estimated by the ratio test, for these initial conditions is approximately 0.09.

Values of $\mathcal{R}_{lb}$ and $\mathcal{R}_{act}$ obtained for the $k_0 = 2$ and $k_0 = 4$ cases, as the system parameter R is varied, are compared in Table 2.1. $B = \frac{8}{3}$ and $\sigma = \frac{4}{3}$ are chosen to satisfy the compatibility condition for the resonance at 2, making $k_0 = 4$. The values $B = \frac{8}{3}$ and $\sigma = \frac{5}{3}$ do not satisfy any of the compatibility conditions, making $k_0 = 2$.

For the governing equations of anharmonic oscillators, researchers have found that the higher the order at which the logarithmic terms enter (i.e. the higher the k_0), the smaller the chaotic region is [10]. This analysis is interesting in view of their findings.

Lastly, the truncation error as a function of time is given by

$$E_n = \left((2K - K \log \tau)^{\frac{1}{k_0}} \tau \right)^{n+1}.$$

Chapter 3

The Duffing Equation

The foregoing analysis for the Lorenz equations is now repeated for the Duffing equation, of which there are many forms. Like the Lorenz equations it is a well studied system. Detailed dynamical studies have been completed by many researchers, most extensively Ueda [67]. The form of the equation used by Ueda is chosen for the ordinary series analysis. Painlevé analysis has been undertaken by many researchers [30]. The various forms of the equation give rise to different types of psi series. The parameter values which allow logarithmic singularities are chosen in order to parallel the Lorenz analysis.

3.1 Ordinary Series

First the recursion relation for the ordinary series is obtained. Substituting

$$x(t) = \sum_{\alpha=0}^{\infty} a_{\alpha} \tau^{\alpha} ,$$

where $\tau = (t - t_1)$ and t_1 is an ordinary point, and $F(t)$ as a power series about t_1 into Duffing's equation (1.2), we obtain

$$\sum_{\alpha=0}^{\infty} a_{\alpha}(\alpha)(\alpha - 1)\tau^{\alpha-2} + k \sum_{\alpha=0}^{\infty} a_{\alpha}(\alpha)\tau^{\alpha-1} + p \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} \sum_{\gamma=0}^{\beta} a_{\gamma} a_{\beta-\gamma} a_{\alpha-\beta} \tau^{\alpha} =$$

$$B_1 \sum_{\alpha=0}^{\infty} \frac{F^{(\alpha)}(t_1)}{\alpha!} \tau^{\alpha} + B_0 ,$$

where $F^{(\alpha)}(t_1)$ denotes the α 'th derivative of $F(t)$, evaluated at t_1 .

Equating coefficients of equal powers of τ gives the recursion relation

$$a_\alpha = \frac{1}{\alpha(\alpha-1)} \left[-k(\alpha-1)a_{\alpha-1} - p \left(\sum_{\beta=0}^{\alpha-2} \sum_{\gamma=0}^{\beta} a_\gamma a_{\beta-\gamma} a_{\alpha-\beta-2} \right) + B_1 \frac{F(t_1)^{(\alpha-2)}}{(\alpha-2)!} + B_0 \right].$$

In order to prove the existence of a lower bound on the radii of convergence of the series, we first establish a bound on the coefficients of the series with the following proposition.

Proposition 4 *For all M , $0 < M < 1$, there exists K , $\frac{1}{M} < K < \infty$, such that for all $\alpha \geq 1$*

$$|a_\alpha| \leq MK^\alpha. \quad (3.1)$$

Proof.

The proposition is a consequence of the following

Assertion 3 *If N is such for $\alpha = N$,*

$$\begin{aligned} & \alpha^2 \left(1 - \frac{M^2}{2} \right) - \alpha \left(1 + k - \frac{7M^2}{2} \right) \\ & - (6M^2 + 3p|a_0|M + 3p|a_0|^2 + AB_1 + B_0 - k) \geq 0, \end{aligned} \quad (3.2)$$

and if

$$|a_\alpha| \leq MK^\alpha, \quad (3.3)$$

for all $\alpha < N$, then

$$|a_\alpha| \leq MK^\alpha \quad (3.4)$$

for $\alpha = N$.

The proposition will then follow by induction if N is chosen such that the inequality (3.2) holds for all $\alpha \geq N$, and K is defined by

$$K = \max_{1 \leq \alpha \leq N-1} L(\alpha),$$

where

$$L(\alpha) = \left(\frac{|a_\alpha|}{M} \right)^{1/\alpha}.$$

We now prove the assertion.

Given that (3.1) holds for $1 \leq \alpha \leq N-1$, for $\alpha = N$ we obtain

$$\begin{aligned}
 |a_\alpha| &\leq \frac{1}{\alpha(\alpha-1)} \left[k(\alpha-1)|a_{\alpha-1}| + p \left(\sum_{\beta=0}^{\alpha-2} \sum_{\gamma=0}^{\beta} |a_\gamma| |a_{\beta-\gamma}| |a_{\alpha-\beta-2}| \right) + \right. \\
 &\quad \left. B_1 \frac{|F(t_1)^{(\alpha-2)}|}{(\alpha-2)!} + B_0 \right], \\
 &\leq \frac{1}{\alpha(\alpha-1)} \left(k(\alpha-1)MK^{\alpha-1} + M^3p \sum_{\beta=2}^{\alpha-3} \sum_{\gamma=1}^{\beta-1} K^\gamma K^{\beta-\gamma} K^{\alpha-\beta-2} \right. \\
 &\quad \left. + 3pM^2 \sum_{\beta=1}^{\alpha-3} |a_0| K^\beta K^{\alpha-\beta-2} + 3p|a_0|^2 MK^{\alpha-2} + AB_1 + B_0 \right), \\
 &\leq \frac{1}{\alpha(\alpha-1)} \left(k(\alpha-1) + M^2 \left(\frac{\alpha^2 - 7\alpha + 12}{2} \right) \right. \\
 &\quad \left. + 3p|a_0|M + 3p|a_0|^2 + AB_1 + B_0 \right) MK^\alpha,
 \end{aligned}$$

where A is the magnitude of the forcing term $F(t)$.

Therefore, for (3.1) to hold for $\alpha = N$, it is necessary that the coefficient of MK^α be less than or equal to one. Thus the following inequality must hold for $\alpha = N$:

$$\begin{aligned}
 &\alpha^2 \left(1 - \frac{M^2}{2} \right) - \alpha \left(1 + k - \frac{7M^2}{2} \right) \\
 &- \left(6M^2 + 3p|a_0|M + 3p|a_0|^2 + AB_1 + B_0 - k \right) \geq 0.
 \end{aligned} \tag{3.5}$$

Thus the assertion and proposition are proved. $\square$

We are now prepared to state and prove the theorem regarding a lower bound on the radius of convergence of the series.

Theorem 4 *There exists $\mathcal{R}$, $0 < \mathcal{R} < \infty$, such that for $0 < |\tau| < \mathcal{R}$, the series*

$$\sum_{\alpha=0}^{\infty} a_\alpha t^\alpha$$

converges absolutely for all values of the system's parameters, B_0 , B_1 and k .

Proof. Using Proposition 4 the theorem follows easily from the comparison test.

Considering the series for $\alpha \geq 1$ we have

$$\sum_{\alpha=1}^{\infty} a_{\alpha} \tau^{\alpha} \leq \left\| \sum_{\alpha=1}^{\infty} a_{\alpha} \tau^{\alpha} \right\| \leq \sum_{\alpha=1}^{\infty} |a_{\alpha}| |\tau|^{\alpha} \leq \sum_{\alpha=1}^{\infty} (K^{\alpha} + B) |\tau|^{\alpha}.$$

Applying the ratio test to this last series gives

$$\lim_{\alpha \rightarrow \infty} \left[\frac{(K^{\alpha+1} + B)}{(K^{\alpha} + B)} |\tau| \right] = K |\tau|,$$

and therefore

$$\mathcal{R} \geq \frac{1}{K} = \mathcal{R}_{lb},$$

where $\mathcal{R}_{lb}$ denotes the estimated lower bound on the radii of convergence of the series. □

3.1.1 Plots and Discussion

Choosing the values $p = 1$, $B_0 = 0.03$, $B_1 = 0.16$ and $k = 0.05$, to parallel the research done by Ueda [67], $\mathcal{R}_{lb}$ is calculated and the solution plotted by repeatedly re-expanding the series about a point within the radius of convergence.

The actual radius of the series ($\mathcal{R}_{act}$) for the initial conditions $(0,0)$, as determined by Hadamard's formula, is approximately 1. Since $|a_{\alpha}|$ is bounded by $B = 0.26$ for all α , the radius takes on its maximum possible value of one.

Plots of the trajectory defined by $x(t)$ and of the attractor are shown in Figure 4.18 and Figure 4.19. The initial conditions are $(x(0), x'(0)) = (0,0)$ and the series is truncated to twenty-five terms. Figure 4.20 and Figure 4.21 show the corresponding plots obtained by numerical integration for the same initial conditions and parameter values, which are identical to those obtained with the series solutions.

Figure 4.22 illustrates the chaotic nature of the Duffing equation. Two trajectories with initial conditions differing by 0.01 diverge after a finite time.

As B_1 , B_0 and k are increased, both $\mathcal{R}_{lb}$ and $\mathcal{R}_{act}$ are found to decrease. This supports the findings of other researchers that as the forcing is increased, the singularities condense [5, 7, 52, 55].

3.2 Laurent Series

Unless B_1 is zero, the Laurent series will not be valid for any parameter values, since the compatibility condition requires that the derivative of the forcing term, evaluated at the arbitrary singularity, take on a specific value, rendering the singularity no longer arbitrary.

3.3 Psi Series

In this section the psi series solution is formulated as for the Lorenz equations.

The system (1.2) is first converted into two first-order equations

$$x' = y, \quad (3.6)$$

$$y' = -ky - px^3 + B_1 F(t) + B_0. \quad (3.7)$$

Solutions $x(t)$ and $y(t)$ of the form

$$x(t) = \sum_{\alpha=0}^{\infty} \tau^{\alpha-1} f_{\alpha}(z),$$

$$y(t) = \sum_{\alpha=0}^{\infty} \tau^{\alpha-2} g_{\alpha}(z),$$

where

$$f_{\alpha}(z) = \sum_{k+k_0 l=\alpha} a_{kl} (\log \tau)^l,$$

$$g_{\alpha}(z) = \sum_{k+k_0 l=\alpha} b_{kl} (\log \tau)^l,$$

are substituted into (3.6) and (3.7) yielding the following recursion relations for $f_{\alpha}(z)$ and $g_{\alpha}(z)$

$$\begin{aligned} f'_{\alpha}(z) + (\alpha - 1)f_{\alpha}(z) - g_{\alpha}(z) &= 0, \\ g'_{\alpha}(z) + (\alpha - 2)g_{\alpha}(z) - B_1 \frac{F^{(\alpha-3)}(t_0)}{(\alpha - 3)!} - B_0 + kg_{\alpha-1}(z) \\ + p \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=0}^{\beta} f_{\gamma}(z) f_{\beta-\gamma}(z) f_{\alpha-\beta}(z) + p \sum_{\gamma=1}^{\alpha-1} f_{\gamma}(z) f_{\alpha-\gamma}(z) f_0 + 3pf_0(z)^2 f_{\alpha}(z) &= 0, \end{aligned}$$

after shifting the indices of summation and separating the terms involving $f_\alpha(z)$.

Alternatively, in matrix form we obtain

$$\begin{pmatrix} f_\alpha(z) \\ g_\alpha(z) \end{pmatrix}' + \begin{pmatrix} (\alpha-1) & -1 \\ 3pf_0^2 & (\alpha-2) \end{pmatrix} \begin{pmatrix} f_\alpha(z) \\ g_\alpha(z) \end{pmatrix} = \begin{pmatrix} 0 \\ B_1 \frac{F(t_0)(\alpha-3)}{(\alpha-3)!} + B_0 - kg_{\alpha-1}(z) - p \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=0}^{\beta} f_\gamma(z) f_{\beta-\gamma}(z) f_{\alpha-\beta}(z) + p \sum_{\gamma=1}^{\alpha-1} f_\gamma(z) f_{\alpha-\gamma}(z) f_0 \end{pmatrix}.$$

or

$$\mathbf{v}'_\alpha(z) + A_\alpha \mathbf{v}_\alpha(z) = \mathbf{V}_\alpha(z). \quad (3.8)$$

In order for this system to have valid logarithmic psi series solutions it must have integer resonances, which requires that the determinant of A_α , $\alpha^2 - 3\alpha + 2 + 3pf_0^2$, have integer roots. We choose the values $B_0 = 0$ and $pf_0^2 = -2$, which is satisfied for $p = \frac{1}{2}$ and $f_0 = \pm 2i$, giving resonances at -1 and at 4 .

The convergence proof parallels that for the Lorenz system, found in section 2.3.

In order to prove the existence of a lower bound on the radii of convergence of the series, we first establish a bound on the coefficients of the series with the following proposition.

Proposition 5 For all M , $0 < M < 1$, there exists K , $\frac{1}{\sqrt{5}} < K < \infty$, such that for all $z < 0$ and for all $\alpha \geq 5$,

$$\|\mathbf{v}_\alpha(z)\| < M \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha+1}}, \quad (3.9)$$

where $k_0 = 4$.

Proof. The proposition is a consequence of the following

Assertion 4 If $N \geq \left(\frac{14c}{1-28\pi M^2}\right)^2 + 9$ where

$$\begin{aligned} c = & B_1 A + k + 2 \sum_{\gamma=1}^4 |f_\gamma| \\ & + \sum_{\beta=5}^8 \sum_{\gamma=\beta-4}^4 |f_\gamma| |f_{\beta-\gamma}| + \sum_{\beta=\alpha-4}^{\alpha-1} \sum_{\gamma=\beta-4}^{\beta} |f_{\alpha-\beta}| |f_{\beta-\gamma}| \\ & + M(\pi + 2\pi \sum_{\gamma=0}^4 |f_\gamma| + \frac{3}{\sqrt{6}} \sum_{\gamma=0}^3 (4-\gamma) |f_\gamma|) - 2\sqrt{6}\pi M^2, \end{aligned}$$

and if

$$\|v_\alpha(z)\| < M \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha+1}} \quad (3.10)$$

for all $\alpha < N$, then

$$\|v_\alpha(z)\| < M \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha+1}} \quad (3.11)$$

for $\alpha = N$.

The proposition will then follow by induction.

The proof of the assertion follows that for the Lorenz psi series, with the following differences.

P , the matrix of eigenvectors of A_α , is

$$P = \begin{pmatrix} 1 & 1 \\ 2 & -3 \end{pmatrix},$$

$$D_\alpha = P^{-1} A_\alpha P = \begin{pmatrix} \alpha+1 & 0 \\ 0 & \alpha-4 \end{pmatrix},$$

$$P(\sigma) = \|P\| \|P^{-1}\| = 4$$

and

$$\begin{aligned} G_\alpha &= B_1 A + k + 2 \sum_{\gamma=1}^4 |f_\gamma| \\ &\quad + \sum_{\beta=5}^8 \sum_{\gamma=\beta-4}^4 |f_\gamma| |f_{\beta-\gamma}| + \sum_{\beta=\alpha-4}^{\alpha-1} \sum_{\gamma=\beta-4}^{\beta} |f_{\alpha-\beta}| |f_{\beta-\gamma}| \\ &\quad + M \left(\pi + 2\pi \sum_{\gamma=0}^4 |f_\gamma| + \frac{3}{\sqrt{6}} \sum_{\gamma=0}^3 (4-\gamma) |f_\gamma| \right) \\ &\quad + M^2 \left(2\pi \sqrt{\alpha-9} \right) - 2\sqrt{6}\pi M^2, \end{aligned}$$

if we restrict the value of K and M such that

$$K \geq \frac{1}{\sqrt[3]{5}},$$

which can always be satisfied by sufficiently large K .

The factor $\sqrt{\alpha - 9}$ appears due to the term arising from the double sum

$$p \sum_{\beta=1}^{\alpha-1} \sum_{\gamma=0}^{\beta} f_{\gamma}(z) f_{\beta-\gamma}(z) f_{\alpha-\beta}(z) \quad (3.12)$$

which, after substitution of the bound gives an expression including

$$\begin{aligned} \sum_{\beta=10}^{\alpha-5} \sum_{\gamma=5}^{\beta-5} \frac{M^3 K^{\alpha}}{\sqrt{\gamma+1} \sqrt{\beta-\gamma+1} \sqrt{\alpha-\beta+1}} &= M^3 K^{\alpha} \sum_{\beta=10}^{\alpha-5} \frac{1}{\sqrt{\alpha-\beta+1}} \sum_{\gamma=5}^{\beta-5} \frac{1}{\sqrt{\gamma+1} \sqrt{\beta-\gamma+1}} \\ &\leq M^3 K^{\alpha} \sum_{\beta=10}^{\alpha-5} \frac{1}{\sqrt{\alpha-\beta+1}} \sum_{\gamma=0}^{\beta} \frac{1}{\sqrt{\gamma+1} \sqrt{\beta-\gamma}} \\ &\leq \pi M^3 K^{\alpha} \sum_{\beta=10}^{\alpha-5} \frac{1}{\sqrt{\alpha-\beta+1}} \\ &= \pi M^3 K^{\alpha} \sum_{\beta=6}^{\alpha-9} \frac{1}{\sqrt{\beta}} \\ &\leq \pi M^3 K^{\alpha} \int_6^{\alpha-9} \frac{1}{\sqrt{\beta}} d\beta \\ &= 2\pi M^3 K^{\alpha} (\sqrt{\alpha-9} - \sqrt{6}) . \end{aligned}$$

Combining all the terms not involving α in the constant c we have

$$G_{\alpha} \leq M^2 2\pi \sqrt{\alpha-9} + c$$

Recall that we are attempting to show that

$$\|v_{\alpha}(z)\| < M \frac{(2K - Kz)^{\alpha/k_0}}{\sqrt{\alpha+1}} ,$$

for $\alpha = N$. This requires that the inequality

$$\frac{7P(\sigma)G_{\alpha}}{2(\alpha-4)} \leq \frac{1}{\sqrt{\alpha+1}} \quad (3.13)$$

be satisfied for $\alpha = N$.

Substituting G_{α} into (3.13) we obtain

$$\frac{14 (2\pi M^2 \sqrt{\alpha-9} + c)}{(\alpha-4)} \leq \frac{1}{\sqrt{\alpha+1}} .$$

Since $\frac{\alpha-4}{\sqrt{\alpha-1}} \geq \sqrt{\alpha-9}$, in order for this condition to hold for $\alpha = N$, we must have

$$N \geq \left(\frac{14c}{1-28\pi M^2} \right)^2 + 9.$$

The assertion and thus the proposition follow. $\square$

The comparison test can be applied as for the Lorenz psi series to obtain an estimate of the lower bound on the radii of convergence of the series.

Chapter 4

Figures

Figure 4.1: $L(\alpha)$ versus α for the ordinary series solution of the Lorenz system

Figure 4.2: R_{est} versus R for the ordinary series solutions of the Lorenz system

Figure 4.3: $\mathcal{R}_{est}$ versus time for the ordinary series solution for $x(t)$ of the Lorenz system

Figure 4.4: Ordinary series solution for $x(t)$ of the Lorenz system, truncated at 20 terms, versus time

Figure 4.5: Projection of the attractor on the z plane produced by the ordinary series solutions of the Lorenz system

Figure 4.6: $x(t)$ of the Lorenz system versus time, obtained by numerical integration

Figure 4.7: Projection of the attractor on the z plane for the Lorenz system, obtained by numerical integration

Figure 4.8: $x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.0001, 5, 5)$ (dashed)

Figure 4.9: $x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5.5, 5, 5)$ (solid) and $(5.001, 5, 5)$ (dashed)

Figure 4.10: $x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.01, 5, 5)$ (dashed)

Figure 4.11: $x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.1, 5, 5)$ (dashed)

Figure 4.12: $x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(5, 5, 5)$ (solid) and $(5.2, 5, 5)$ (dashed)

Figure 4.13: $x(t)$ versus time for the ordinary series of the Lorenz system, initial conditions $(-13.9, -15.8568, 32.1233)$ at $t = 10.9785$ (dashed) and $(5, 5, 5)$ at $t = 0$

Figure 4.14: Resummation of the double series along the lines $\alpha = k + 2l$

Figure 4.15: Solution of the inequality $e^{k_0 s} > 2K + Ks$

CHAPTER 4. FIGURES

Figure 4.16: $L(\alpha)$ vs α for the psi series solution of the Lorenz system

Figure 4.17: The psi series for $x(t)$ of the Lorenz system versus the truncation point, $t = 0.9$, $t = 1.0$ and $t = 1.1$ (top to bottom)

Figure 4.18: The ordinary series for $x(t)$ of the Duffing equation, truncated at 25 terms, versus time

Figure 4.19: The ordinary series for $x(t)$ versus $x'(t)$ of the Duffing equation, truncated at 25 terms

Figure 4.20: $x(t)$ of the Duffing equation versus time, obtained by numerical integration

Figure 4.21: $x(t)$ versus $x'(t)$ of the Duffing equation, obtained by numerical integration

Figure 4.22: $x(t)$ versus t for the ordinary series of the Duffing equation, initial conditions $(0,0)$ (solid) and $(0.01,0,0)$ (dashed)

Chapter 5

Summary and Conclusions

In this thesis we have formed series solutions about non-singular points, poles and logarithmic branch points, for the Lorenz system and the Duffing equation. For each series it was proved that a lower bound on the radius of convergence exists. Previously the convergence of the psi series for these systems was unknown. The bound was calculated for various values of the system's parameters, and found to be generally within an order of magnitude of the actual radii of convergence of the series.

For both systems, the ordinary series were continued by re-expanding the series within their radii of convergence. These solutions were shown to provide plots of the trajectories identical to those obtained by numerical integration.

Using the ordinary series solutions for the Lorenz system, it was observed that as the difference in initial conditions between two neighbouring trajectories increased, their point of divergence decreased.

For the psi series solutions for the Lorenz system it was calculated that the lower bound on the radii, when the logarithmic terms entered at the fourth term, was higher than when the logarithmic terms entered at the second term. The actual radius was observed to follow the same pattern. This is interesting in view of the observations made by other researchers, for the governing equations of anharmonic oscillators [10], that the higher the order at which the logarithmic singularities enter, the smaller the chaotic region.

For the psi series solutions for the Duffing equation it was calculated that the

lower bound on the radii decreased as the strength of the forcing increased. This supports observations made by other researchers [5, 7, 52, 55] that as the motion became more chaotic, the singularity pattern condensed and the structures gradually collapsed towards the real axis [5, 7, 52, 55].

At the conclusion of the thesis several questions naturally suggest themselves.

Firstly, after applying this method of proof to these two systems several observations can be made. For ordinary differential equations of the form

$$u^{(n)} = L(t, u^1, \dots) + P(u) ,$$

where P is a polynomial function and L is a linear function, one can see that if P is at most of third order, this method of proof will be successful.

When P is permitted to be of the fourth order, it is not obvious how to modify the proof so that it will succeed. The difficulty occurs in the condition on N . The term similar to (3.12) will be a triple sum rather than a double sum. G_α will be of the form $M^3(\alpha - C)^{\frac{3}{2}} + \text{constants}$, where C is some positive integer, due to the double integration of $\frac{1}{\sqrt{\alpha}}$.

The inequality that N must satisfy for the induction process to succeed will appear as

$$M^3(\alpha - C)^{\frac{3}{2}} \leq (\alpha - C)^{\frac{1}{2}}$$

or

$$\alpha \leq \frac{1}{M^3} + C ,$$

which makes the proof unsuccessful.

Secondly, an estimate of an upper bound on the radii of convergence would be desirable. However, for all the series, the coefficients appear to show no pattern in their signs, and thus how an upper bound could be estimated is unclear. Do the sequence of signs give us any information regarding the integrability of the system?

Lastly, T.C. Bountis et. al [3, 4] have found that those systems which satisfy the weak Painlevé property and are integrable have finitely sheeted solutions, whereas those which are not integrable have infinitely sheeted solutions. How do the radii of convergence of their psi series solutions compare?

To conclude, series solutions of nonlinear differential equations in the complex plane may possibly lead to insight into why certain singularity structures lead to integrability, and others do not.

Appendix A Derivation of the formulae for $f_\alpha(z)$, $g_\alpha(z)$ and $h_\alpha(z)$ for the Psi Series of the Lorenz equations

From (2.40) we have

$$\mathbf{v}_\alpha(z) = \int_{-\infty}^z P e^{D_\alpha(q-z)} P^{-1} \mathbf{V}_\alpha(q) dq, \quad (5.1)$$

where

$$\mathbf{v}_\alpha(z) = \begin{pmatrix} f_\alpha(z) \\ g_\alpha(z) \\ h_\alpha(z) \end{pmatrix},$$

$$\mathbf{V}_\alpha(z) = \begin{pmatrix} -\sigma f_{\alpha-1}(z) \\ Rf_{\alpha-2}(z) - g_{\alpha-1}(z) - \sum_{\beta=1}^{\alpha-1} f_{\alpha-\beta}(z)h_\beta(z) \\ \sum_{\beta=1}^{\alpha-1} f_{\alpha-\beta}(z)g_\beta(z) - Bh_{\alpha-1}(z) \end{pmatrix},$$

P , the matrix of eigenvectors of A_α , is given by

$$P = \begin{pmatrix} \pm\sigma/2 & \mp\sigma & \pm\sigma/2 \\ \mp 1 & \mp 1 & \pm 3/2 \\ i & i & i \end{pmatrix}$$

and

$$D_\alpha = P^{-1} A_\alpha P = \begin{pmatrix} \alpha + 1 & 0 & 0 \\ 0 & \alpha - 2 & 0 \\ 0 & 0 & \alpha - 4 \end{pmatrix}.$$

After multiplying the four matrices under the integral sign together, we obtain

$$\mathbf{v}_\alpha(z) = \int_{-\infty}^z T \begin{pmatrix} e^{(\alpha+1)(q-z)} \\ e^{(\alpha-2)(q-z)} \\ e^{(\alpha-4)(q-z)} \end{pmatrix} dq, \quad (5.2)$$

where

$$T = \begin{pmatrix} \frac{1}{3} \mathbf{V}_{1\alpha}(z) - \frac{2}{5} \mathbf{V}_{2\alpha}(z) - \frac{2\sigma}{15} \mathbf{V}_{3\alpha}(z) & \frac{2}{3} \mathbf{V}_{1\alpha}(z) + \frac{1}{3} \mathbf{V}_{3\alpha}(z) & \frac{2}{5} \mathbf{V}_{2\alpha}(z) - \frac{1}{5} \mathbf{V}_{3\alpha}(z) \\ -\frac{2}{3\sigma} \mathbf{V}_{1\alpha}(z) + \frac{2}{5} \mathbf{V}_{2\alpha}(z) + \frac{4\sigma}{15} \mathbf{V}_{3\alpha}(z) & \frac{2}{3\sigma} \mathbf{V}_{1\alpha}(z) + \frac{1}{3} \mathbf{V}_{3\alpha}(z) & \frac{3}{5} \mathbf{V}_{2\alpha}(z) - \frac{3}{5} \mathbf{V}_{3\alpha}(z) \\ \frac{2i}{3\sigma} \mathbf{V}_{1\alpha}(z) - \frac{2i}{5} \mathbf{V}_{2\alpha}(z) + \frac{4i}{15} \mathbf{V}_{3\alpha}(z) & -\frac{2i}{3\sigma} \mathbf{V}_{1\alpha}(z) + \frac{1}{3} \mathbf{V}_{3\alpha}(z) & \frac{2i}{5} \mathbf{V}_{2\alpha}(z) + \frac{2i}{5} \mathbf{V}_{3\alpha}(z) \end{pmatrix}$$

Letting t denote an entry in T , we must now find an expression for

$$\int_{-\infty}^z e^{\lambda(q-z)} t(q) dq, \quad (5.3)$$

where λ equals $\alpha + 1$, $\alpha - 2$ or $\alpha - 4$.

Note that each entry in T is a function of $f_\alpha(z)$, $g_\alpha(z)$ and $h_\alpha(z)$ and therefore is a polynomial in z of order less than or equal to $\lfloor \frac{\alpha}{2} \rfloor$.

Integrating (5.3) by parts yields the following, if $e^{\lambda(q-z)}$ evaluated at $-\infty$ vanishes (which is true for $\alpha > 4$)

$$\int_{-\infty}^z e^{\lambda(q-z)} t(q) dq = \frac{1}{\lambda} t(z) - \frac{1}{\lambda} \int_{-\infty}^z e^{\lambda(q-z)} t'(q) dq.$$

After n integration by parts we obtain

$$\int_{-\infty}^z e^{\lambda(q-z)} t(q) dq = \sum_{k=0}^{n-1} \frac{(-1)^k t^{(k)}(z)}{\lambda^{k+1}} + \frac{(-1)^n}{\lambda^n} \int_{-\infty}^z e^{\lambda(q-z)} t^{(n)}(q) dq.$$

If we let $n - 1 = \lfloor \frac{\alpha}{2} \rfloor$, then $t^{(n)} = 0$, and

$$\int_{-\infty}^z e^{\lambda(q-z)} t(q) dq = \sum_{k=0}^{\lfloor \frac{\alpha}{2} \rfloor} \frac{(-1)^k t^{(k)}(z)}{\lambda^{k+1}}.$$

Therefore, for $\alpha > 4$, we obtain the following equations for $f_\alpha(z)$, $g_\alpha(z)$ and $h_\alpha(z)$:

$$\begin{aligned} f_\alpha(z) &= \sum_{k=0}^{\lfloor \frac{\alpha}{2} \rfloor} (-1)^k \left(\frac{t_{11}^{(k)}(z)}{(\alpha+1)^{(k+1)}} + \frac{t_{12}^{(k)}(z)}{(\alpha-2)^{(k+1)}} + \frac{t_{13}^{(k)}(z)}{(\alpha-4)^{(k+1)}} \right), \\ g_\alpha(z) &= \sum_{k=0}^{\lfloor \frac{\alpha}{2} \rfloor} (-1)^k \left(\frac{t_{21}^{(k)}(z)}{(\alpha+1)^{(k+1)}} + \frac{t_{22}^{(k)}(z)}{(\alpha-2)^{(k+1)}} + \frac{t_{23}^{(k)}(z)}{(\alpha-4)^{(k+1)}} \right), \\ h_\alpha(z) &= \sum_{k=0}^{\lfloor \frac{\alpha}{2} \rfloor} (-1)^k \left(\frac{t_{31}^{(k)}(z)}{(\alpha+1)^{(k+1)}} + \frac{t_{32}^{(k)}(z)}{(\alpha-2)^{(k+1)}} + \frac{t_{33}^{(k)}(z)}{(\alpha-4)^{(k+1)}} \right). \end{aligned}$$

$f_\alpha(z)$, $g_\alpha(z)$ and $h_\alpha(z)$, $1 \leq \alpha \leq 4$ must be calculated by hand, as follows, taking the top value for f_0 and g_0 from (2.30).

$$\begin{pmatrix} f_0 \\ g_0 \\ h_0 \end{pmatrix} = \begin{pmatrix} 2i \\ -2i/\sigma \\ -2/\sigma \end{pmatrix},$$

$$\begin{pmatrix} f_1 \\ g_1 \\ h_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}(3\sigma - 1 - 2B) \\ 2i \\ \frac{2}{3\sigma}(1 - B + 3\sigma) \end{pmatrix},$$

$$\begin{pmatrix} f_2(z) \\ g_2(z) \\ h_2(z) \end{pmatrix} = \begin{pmatrix} i \frac{3\sigma^2 + \sigma(-9R + 9a_h + 11 - 8B) + 3B^2 - 2B - 1}{3} \\ i \frac{\sigma(-9R + 9a_h + 12 - 12B) + 5B^2 - 4B - 1}{9\sigma} \\ a_h \end{pmatrix} + \begin{pmatrix} \frac{2i}{9}(B^2 - B + (B + 2)\sigma - 6\sigma^2) \\ \frac{2i}{9\sigma}(B^2 - B + (B + 2)\sigma - 6\sigma^2) \\ \frac{2}{9\sigma}(B^2 - B + (B + 2)\sigma - 6\sigma^2) \end{pmatrix} z,$$

where a_h is arbitrary,

$$\begin{pmatrix} f_3(z) \\ g_3(z) \\ h_3(z) \end{pmatrix} = \begin{pmatrix} c_f \\ c_g \\ c_h \end{pmatrix} + \begin{pmatrix} d_f \\ d_g \\ d_h \end{pmatrix} z,$$

where

$$\begin{aligned} c_f = & -\left(2i + 6iB - 18iB^2 + 10iB^3 + (-13i - 18ia_h + B(101i + 18ia_h) \right. \\ & \left. + B - 73iB^2 - 15iB^3 + (18i - 18iB)R\right)\sigma + (-169i - 90ia_h + (62i + 36ia_h)B + \\ & 107iB^2 + (90i + 18iB)R)\sigma^2 + (255i + 216ia_h - 309iB - 216iR) \\ & \left.\sigma^3 + 297i\sigma^4\right)/108\sigma, \end{aligned}$$

$$\begin{aligned} c_g = & -\left(2i + 6iB - 18iB^2 + 10iB^3 + (-13i - 18ia_h + (107i + 18ia_h) \right. \\ & \left. B - 85iB^2 - 9iB^3 + (18i - 18iB)R\right)\sigma + (-175i - 90ia_h + (62i + 36ia_h)B + \\ & 113iB^2 + (90i + 18iB)R)\sigma^2 + (261i + 162ia_h - 279iB - 162iR) \\ & \left.\sigma^3 + 171i\sigma^4\right)/(54\sigma^2), \end{aligned}$$

$$\begin{aligned} c_h = & -\left(-2 - 6B + 18B^2 - 10B^3 + (15 + 18a_h + (-93 - 18a_h)B + 87B^2 - 9B^3 + \right. \\ & \left. (-18 + 18B)R\right)\sigma + (123 + 72a_h + (-102 - 18a_h)B - 21B^2 + \\ & (-72 + 18B)R)\sigma^2 + (-63 + 54a_h + 9B - 54R) \\ & \left.\sigma^3 + 135\sigma^4\right)/54\sigma^2, \end{aligned}$$

$$d_f = \frac{-iB + 2iB^2 - iB^3 + (2i + 2iB - 4iB^2)\sigma + (-12i + 3iB)\sigma^2 + 18i\sigma^3}{9},$$

$$d_g = \frac{-2iB + 4iB^2 - 2iB^3 + (4i + 2iB - 6iB^2)\sigma + (-20i + 8iB)\sigma^2 + 24i\sigma^3}{9\sigma} \text{ and}$$

$$d_h = \frac{4B - 8B^2 + 4B^3 + (-8 + 16B - 8B^2)\sigma - 36B\sigma^2 + 72\sigma^3}{27\sigma},$$

$$\begin{pmatrix} f_A(z) \\ g_A(z) \\ h_A(z) \end{pmatrix} = \begin{pmatrix} k_f \\ k_g \\ k_h \end{pmatrix} + \begin{pmatrix} l_f \\ l_g \\ l_h \end{pmatrix} z + \begin{pmatrix} m_f \\ m_g \\ m_h \end{pmatrix} z^2,$$

where

$$k_f = \begin{aligned} & (-2i - 16iB - 12iB^2 + 80iB^3 - 50iB^4 + (29i + 18ia_h + (-18 + 12i + 72ia_h)B \\ & (18 - 562i - 90ia_h)B^2 + 540iB^3 - 19iB^4 + (-18i - 72iB + 90iB^2)R) \\ & \sigma + (36 - 36ia_h + (1588i + 630ia_h)B + (18 - 1376i - 270ia_h)B^2 - \\ & 304iB^3 + i(92 - 324KH) + (36i - 684iB + 162iB^2)R)\sigma^2 + \\ & (-72 - 1906i - 1386ia_h - 162ia_h^2 + (18 + 728i + 522ia_h)B + 1034iB^2 + \\ & (1386i + 324ia_h - 36iB)R - 162iR^2)\sigma^3 + (-108 + 1482i + 1728ia_h - 2256iB - \\ & 1728iR)\sigma^4 + 3015i\sigma^5) / 648\sigma, \end{aligned}$$

$$k_g = \begin{aligned} & 90i - 240iB - 2340iB^2 + 5040iB^3 - 2550iB^4 + (75i - 810ia_h + \\ & (-990 + 9808i + 5400ia_h)B + (990 - 35522i - 4590ia_h)B^2 + 26200iB^3 \\ & - 561iB^4 + (810i - 5400iB + 4590iB^2)R)\sigma + (1980 - 8640ia_h + \\ & (144 + 85094i + 35910ia_h)B + (846 - 58426i - 15930ia_h)B^2 - \\ & 17222iB^3 + i(-9446 - 24300k_h) + (8640i - 34416iB + 8766iB^2)R \\ & \sigma^2 + (-4248 - 73640i - 60210ia_h - 12150ia_h^2 + \\ & (846 + 7372i + 22950ia_h)B + 61228iB^2 + (59922i + 24300ia_h + \\ & 2016iB)R - 12150iR^2)\sigma^3 + (-5076 + 90300i + 90720ia_h - \\ & 131106iB - 89856iR)\sigma^4 + 159687i\sigma^5 / 16200\sigma^2, \end{aligned}$$

$$l_f = -(-30i - 120iB + 180iB^2 + 120iB^3 - 150iB^4 + (225i + 270ia_h +$$

$$\begin{aligned}
& (-45 - 1226i) B + (45 - 491i - 270ia_h) B^2 + 1600i B^3 - 108i B^4 + \\
& \left(-270i + 270i B^2 \right) R \sigma + (90 + 2287i + 1080ia_h + (-18 + 2357i + 1080ia_h) B + \\
& (63 - 4228i - 540ia_h) B^2 - 416i B^3 + (-1080i - 1773i B + 423i B^2) R) \sigma^2 + \\
& \left(-144 - 5495i - 3780ia_h + (63 + 4741i + 1350ia_h) B + 34i B^2 + (3816i - 927i B) \right. \\
& \left. \sigma^3 + (-378 - 2175i + 810ia_h + 1032i B - 918i R) \sigma^4 + 2736i \sigma^5 \right) / 2025\sigma,
\end{aligned}$$

$$\begin{aligned}
l_g = & - \left(-90i - 360i B + 540i B^2 + 360i B^3 - 450i B^4 + (675i + 810ia_h + \right. \\
& (-135 - 3848i) B + (135 - 1243i - 810ia_h) B^2 + 4850i B^3 - 434i B^4 + \\
& \left. (-810i + 810i B^2) R \right) \sigma + (270 + 7201i + 3240ia_h + (36 + 7636i + 3240ia_h) B + \\
& (99 - 14144i - 1620ia_h) B^2 - 693i B^3 + (-3240i - 5229i B + 1179i B^2) R) \sigma^2 + \\
& \left(-612 - 18535i - 11340ia_h + (99 + 14843i + 4050ia_h) B + 1982i B^2 + \right. \\
& (11268i - 2871i B) R) \sigma^3 + (-594 - 2325i + 2430ia_h - 339i B - \\
& \left. 2214i R) \sigma^4 + 4878i \sigma^5 \right) / 2025\sigma^2,
\end{aligned}$$

$$\begin{aligned}
l_h = & - \left(60 + 240 B - 360 B^2 - 240 B^3 + 300 B^4 + (-450 - 540a_h + (2527 + 135i) B + \right. \\
& (1082 - 135i + 540a_h) B^2 - 3625 B^3 + 466 B^4 + (540 - 540 B^2) R) \sigma + \\
& \left(-4724 - 270i - 2160a_h + (-5864 - 36i - 2160a_h) B + (10581 - 99i + 1080a_h) B^2 \right. \\
& + 207 B^3 + (2160 + 3546 B - 846 B^2) R) \sigma^2 + (12890 + 612i + 7560a_h + \\
& (-7957 - 99i - 2700a_h) B - 3943 B^2 + (-7632 + 1854 B) R) \sigma^3 + \\
& \left. (-3000 + 594i - 1620a_h + 1686 B + 1836 R) \sigma^4 + 3528 \sigma^5 \right) / 2025\sigma^2,
\end{aligned}$$

$$\begin{aligned}
m_f = & - \left(-17i B + 23i B^2 + 5i B^3 - 11i B^4 + (34i + (9 + 34i) B + \right. \\
& (-9 - 101i) B^2 + 33i B^3 + (9i B - 9i B^2) R) \sigma + (-18 - 160i + (-9 + 107i) B - \\
& 98i B^2 + (-18i - 9i B) R) \sigma^2 + (54 + 150i - 276i B + 54i R) \sigma^3 + 72i \sigma^4 \Big) / 405\sigma.
\end{aligned}$$

$$\begin{aligned}
m_g = & - \left(-17i B + 23i B^2 + 5i B^3 - 11i B^4 + (34i + (9 + 34i) B + \right. \\
& (-9 - 101i) B^2 + 33i B^3 + (9i B - 9i B^2) R) \sigma + (-18 - 160i + (-9 + 107i) B - \\
& 98i B^2 + (-18i - 9i B) R) \sigma^2 + (54 + 150i - 276i B + 54i R) \sigma^3 + 72i \sigma^4 \Big) / 135\sigma.
\end{aligned}$$

$$m_h = - \left(34 B - 36 B^2 - 30 B^3 + 32 B^4 + (-68 + (-108 + 18i) B + (222 - 18i) B^2 - 46 B^3 + (-18 B + 18 B^2) R) \sigma + (360 - 36i + (-54 - 18i) B - 306 B^2 + (36 + 18 B) R) \sigma^2 + (-540 + 108i + 432 B - 108 R) \sigma^3 + 216 \sigma^4 \right) / 405 \sigma .$$

If $\log^4(\tau)$ is required, fgh , $\alpha = 0, 1, 2, 3$, and 4 are

$$\begin{aligned} f_0 &= 2i , \\ f_1 &= \frac{i}{3} (-1 - 2B + 3\sigma) , \\ g_0 &= \frac{-2i}{\sigma} , \\ g_1 &= 2i , \\ h_0 &= \frac{-2}{\sigma} , \\ h_1 &= \frac{2(1 - B + 3\sigma)}{3\sigma} , \\ f_2 &= i h_2 \sigma - \frac{\sigma(R f_0 - g_1 - f_1 h_1)}{2} , \\ g_2 &= \frac{\sigma f_1 + f(2)}{\sigma} , \\ f_3 &= \frac{-3\sigma f(2)}{4} + \frac{i}{2} \sigma (f(2) g_1 + f_1 g(2) - B h(2)) - \frac{\sigma(R f_1 - g_2 - f_2 h_1 - f_1 h_2)}{4} , \\ g_3 &= \frac{-f_2}{2} + i (f_2 g_1 + f_1 g_2 - B h_2) - \frac{R f_1 - g_2 - f_2 h_1 - f_1 h_2}{2} , \\ h_3 &= \frac{i}{2} f_2 - \frac{i}{2} (R f_1 - g_2 - f_2 h_1 - f_1 h_2) , \\ \begin{pmatrix} f_4(z) \\ g_4(z) \\ h_4(z) \end{pmatrix} &= \begin{pmatrix} a_f \\ a_g \\ a_h \end{pmatrix} + \begin{pmatrix} b_f \\ b_g \\ b_h \end{pmatrix} z , \end{aligned}$$

where

$$\begin{aligned} b_h &= 2i R f_1 + 2i B R f_1 - 4 R f_1^2 + 2i f_2 - 2i B f_2 + 4i R f_2 - \\ &R \sigma f_1 g_1 + 4 f_2 g_1 - 3 \sigma f_2 g_1 + 4i f_1 f_2 g_1 + 2i \sigma f_2 g_1^2 - \\ &2i g_2 - 2i B g_2 + 8 f_1 g_2 + 4i f_1^2 g_2 + 4 f_2 g_2 + \sigma g_1 g_2 + \end{aligned}$$

$$\begin{aligned}
& 2i\sigma f_1 g_1 g_2 + iR\sigma f_1 h_1 - 2i f_2 h_1 - 2iB f_2 h_1 + \\
& 3i\sigma f_2 h_1 + 4f_1 f_2 h_1 + 3\sigma f_2 g_1 h_1 - i\sigma g_2 h_1 + \\
& 2\sigma f_1 g_2 h_1 - i\sigma f_2 h_1^2 - 4B h_2 - 2i f_1 h_2 - 6iB f_1 h_2 + \\
& 4f_1^2 h_2 - 4i f_2 h_2 - 2iB\sigma g_1 h_2 + \sigma f_1 g_1 h_2 - \\
& 2B\sigma h_1 h_2 - i\sigma f_1 h_1 h_2/10 ,
\end{aligned}$$

$$b_f = \frac{-i}{2} b_h \sigma ,$$

$$b_g = \frac{-3i}{2} b_h ,$$

$$\begin{aligned}
a_f = & -4b_f - 4ia_h\sigma + R\sigma^2 f_1 + 3\sigma^2 f_2 - 2i\sigma^2 f_2 g_1 - \sigma^2 g_2 - \\
& 2i\sigma^2 f_1 g_2 - \sigma^2 f_2 h_1 + 2iB\sigma^2 h_2 - \sigma^2 f_1 h_2/8 ,
\end{aligned}$$

$$\begin{aligned}
a_g = & \frac{\frac{-3i}{2} a_h - b_f}{2\sigma + \frac{R\sigma f_1}{8} + \frac{3\sigma f_2}{8} - \frac{i}{4}\sigma f_2 g_1 - \frac{\sigma g_2}{8} -} \\
& \frac{\frac{i}{4}\sigma f_1 g_2 - \frac{\sigma f_2 h_1}{8} + \frac{i}{4}B\sigma h_2 - \frac{\sigma f_1 h_2}{8}}{.}
\end{aligned}$$

Bibliography

- [1] M. Adler, P. Van Moerbeke: *Completely integrable systems, Euclidean Lie algebras, and curves*, Adv. Math. **38**, 267 (1980).
- [2] M. Adler, P. Van Moerbeke: *Linearization of systems, Jacobi varieties and representation theory*, Adv. Math. **38**, 318 (1980).
- [3] T.C. Bountis: *On nonintegrable systems with square root singularities in complex time*, Phys. Lett. A **159**, 1 (1991).
- [4] T.C. Bountis: *Nonintegrable systems with algebraic singularities in complex time*, J. Phys. A **24**, 3217 (1991).
- [5] T.C. Bountis, M. Bier, Y. Papageorgiou: *A Singularity Analysis Approach to the Solutions of Duffing's Equation* in Symmetries and Singularity Structures, ed. by M. Lakshmanan and M. Daniel, 112 (1990).
- [6] T.C. Bountis, L.B. Drossos, M. Lakshmanan, S. Parthasarathy: *On the Nonintegrability of a Family of Duffing-van der Pol Oscillators*, J. Phys. A **26**, 6927 (1993).
- [7] T.C. Bountis, V. Papageorgiou, M. Bier: *On the Singularity Analysis of Intersecting Separatrices in Near-Integrable Dynamical Systems*, Physica **24D**, 292 (1987).
- [8] T.C. Bountis, A. Ramani, B. Grammaticos, B. Dorrizo: *On the Complete and Partial Integrability of Non-Hamiltonian systems*, Physica **128A**, 268 (1984).

- [9] T.C. Bountis, H. Segur: *Logarithmic Singularities and Chaotic Behaviour in Hamiltonian Systems* in *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems* ed. M. Tabor and Yvain M. Treve, 279 (1981).
- [10] T.C. Bountis and H. Segur: *A Singularity Analysis of Integrability and Chaos in Dynamical Systems* in *Singularities and Dynamical Systems*, ed S. Pnevmatikos (1984).
- [11] T.C. Bountis, H. Segur, F. Vivaldi: *Integrable Hamiltonian Systems and the Painlevé Property*, *Phys. Rev. A* **25**, 1257 (1982).
- [12] B. Boyce, D. DiPrima: *Differential Equations*
- [13] F. Calogero: *A Solvable Nonlinear Wave Equation*, *Studies in Applied Mathematics* **70**, 189 (1984).
- [14] F. Bureau: *Differential Equations with Fixed Critical Points*, *Ann. Math. Pura Appl. (4)* **64**, 229 (1964).
- [15] F. Bureau: *Differential Equations with Fixed Critical Points*, *Ann. Math. Pura Appl. (4)* **66**, 1 (1964).
- [16] F. Bureau: *Integration of some Nonlinear Systems of ODEs*, *Ann. Math. Pura Appl. (4)* **94**, 344 (1972).
- [17] F. Bureau: *Sur des Systèmes différentiels du Troisième Ordre et les équations Différentielles associées*, *Acad. Roy. Belg. Bull. Cl. Sci. (5)* **73**, 335 (1987).
- [18] Y.F. Chang, J.M. Greene, M. Tabor, J. Weiss: *The Analytic Structure of Dynamical Systems and Self-Similar Natural Boundaries*, *Physica* **8D**, 183 (1983).
- [19] Y.F. Chang, M. Tabor, J. Weiss, G. Corliss: *On the Analytic Structure of the Henon-Heiles System*, *Phys. Lett. A* **85**, 211 (1981).
- [20] Y.F. Chang, M. Tabor, J. Weiss: *Analytic Structure of the Henon-Heiles Hamiltonian in Integrable and Nonintegrable Regimes*, *J. Math. Phys.* **23**, 531 (1982).

- [21] J. Chazy: *Sur les Équations Différentielles du Troisième ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes*, Acta Math. **34**, 317 (1911).
- [22] P.A. Clarkson, J.B. McCleod, P.J. Olver, A. Ramani: *Integrability of Klein-Gordon Equations* SIAM J. Math. Anal. **17**, 798 (1986).
- [23] C.M. Cosgrove: *Painlevé Classification of all Semilinear Partial Differential Equations of the Second Order. I Hyperbolic equations in two independent variables. II. Parabolic and Higher Dimensional Equations*, Studies in Applied Mathematics **89**, 1 (1993).
- [24] C. M. Cosgrove: *All binomial-type Painlevé equations of the Second Order and degree Three or Higher*, Studies in Applied Mathematics **90**, 119 (1993).
- [25] C.M. Cosgrove, G. Scoufis: *Painlevé Classification of a Class of Differential Equations of the Second Order and Second Degree*, Studies in Applied Mathematics **88**, 25 (1993).
- [26] T. Dombre, U. Frisch, J.M. Greene, M. Henon, A. Mehr, A.M. Soward: *Chaotic Streamlines in the ABC Flows*, J. Fluid Mech. **167**, 353 (1986).
- [27] B. Dorizzi, B. Grammaticos: *Explicit Integrability for Hamiltonian Systems and the Painlevé Conjecture*, J. Math. Phys. **25**(3), 481 (1984).
- [28] B. Dorizzi, B. Grammaticos, A. Ramani, P. Winternitz: *Integrable Hamiltonian Systems with Velocity-dependent Potentials*, J. Math. Phys. **26**, 3070 (1985).
- [29] H. Flaschka, A.C. Newell, M. Tabor: *Integrability*, in What is Integrability, ed. by V.E. Zakharov, 73 (1991).
- [30] J.D. Fournier, G. Levine, M. Tabor: *Singularity Clustering in the Duffing Oscillator*, J. Phys. A **21**, 33 (1988).
- [31] U. Frisch, R. Morf: *Intermittency in Nonlinear Dynamics and Singularities at Complex Times*, Phys. Rev. A **23**, 2673 (1981).

- [32] B. Gambier: *Sur les Équations Différentielles du Second Ordre et du Premier degré dont l'intégrale Générale est à points critiques fixes*, Acta Math. **33**, 1 (1910).
- [33] R. Garnier: *Sur des Équations différentielles du Troisième ordre dont l'intégrale Générale est Uniforme et sur une Classe d'équations nouvelles d'ordre Supérieur dont l'intégrale Générale est Uniforme*, Ann. Sci. Ecole Norm. Sup. **29**, 1 (1912).
- [34] B. Grammaticos, B. Dorrizo, A. Ramani: *Integrability of Hamiltonians with third and fourth degree Polynomial Potentials*, J. Math. Phys. **24**, 2289 (1984).
- [35] B. Grammaticos, B. Dorizzi, A. Ramani: *Hamiltonians with high-order Integrals and the Weak Painlevé Concept*, J. Math. Phys. **25**, 3470 (1984).
- [36] M.A. Hemmi: *Series Solutions of Nonlinear Ordinary Differential Equations*, Masters Thesis at Carleton University, (1994).
- [37] J. Hietarinta, B. Dorizzi, B. Grammaticos: *Complete Painlevé Analysis for Coupled Quartic Oscillators in N Dimensions*, Phys. Lett. A **108**, 55 (1985).
- [38] E. Hille: *A Note on Quadratic Systems*, Proc. RSE(A) **72**, 1 (1972).
- [39] E. Hille: *Differential Equations in the Complex Plane*. (1976).
- [40] L. Hlavatý: *On the Painlevé Classification of Partial Differential Equations*, J. Math. Phys **31**, 605 (1990).
- [41] L. Hlavatý: *On the Painlevé Classification of Partial Differential Equations II. Rational Equations with One Pole*, Czechoslovak Journal of Physics **42**, 765 (1992).
- [42] L. Hlavatý: *On the Painlevé Classification of Partial Differential Equations III. Rational Equations with Several Poles*, J. Math. Phys. **33**, 888 (1992).
- [43] E. Ince: *Ordinary Differential Equations*, (1926).

2 of/de 2

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

2 of/de 2

PM-1 3½"x4" PHOTOGRAPHIC MICROCOPY TARGET
NBS 1010a ANSI/ISO #2 EQUIVALENT

PRECISIONSM RESOLUTION TARGETS

- [44] K. Knopp: *Theory and Application of Infinite Series*, Blackie and Son Ltd: Glasgow (1951).
- [45] S.V. Kowalevskaya: *Sur le Problème de la Rotation d'un Corps Solide Autour d'un Point Fixe*, Acta Math. **12**, 177 (1889).
- [46] S.V. Kowalevskaya: *Sur une Propriété du Système d'équations Différentielles qui Définit la Rotation d'un Corps Solide Autour d'un Point Fixe*, Acta Math., **14**, 81 (1889).
- [47] M.D. Kruskal, P.A. Clarkson: *Painlevé-Kovalevskaya and Poly-Painlevé Tests*, Studies in Applied Math. **86**, 87 (1992).
- [48] M. Lakshmanan, R. Sahadevan: *Painlevé Property of Coupled Anharmonic Oscillators with N degrees of Freedom*, Phys. Lett. A **101**, 189 (1984).
- [49] M. Lakshmanan, R. Sahadevan: *Painlevé Analysis, Lie Symmetries, and Integrability of coupled nonlinear oscillators of polynomial type*, Phys. Rep. **224**, 3 (1993).
- [50] G. Levine, M. Tabor: *Integrating the Non-integrable: Analytic Structure of the Lorenz System Revisited*, Physica **33D** 189, (1988).
- [51] C.R. Menyuk, H.H. Chen, Y.C. Lee: *Restricted Multiple Three-wave Interactions*, Phys. Rev. A **27**, 1597 (1983).
- [52] S. Parthasarathy: *Singularity Structures and Chaotic Dynamics of the Parametrically Driven Duffing Oscillator* in Symmetries and Singularity Structures, ed. by M. Lakshmanan and M. Daniel, 104 (1990).
- [53] S. Parthasarathy, M. Lakshmanan: *On the Analytic Structure of the Driven Pendulum*, J. Phys. A **23** L1223 (1990).
- [54] S. Parthasarathy, M. Lakshmanan: *Analytic Structure of the Damped Driven Morse Oscillator*, Phys. Lett. A **157**, 365 (1991).

- [55] S. Puri: *Painlevé Property in Hamiltonian and Non-Hamiltonian Systems* in Symmetries and Singularity Structures, ed. by M. Lakshmanan and M. Daniel, 92 (1990).
- [56] A. Ramani, B. Dorizzi, B. Grammaticos, T.C. Bountis: *Integrability and the Painlevé Property for Low-dimensional Systems*, J. Math. Phys. **25**, 878 (1984).
- [57] A. Ramani, B. Grammaticos, T.C. Bountis: *The Painlevé Property and Singularity Analysis of Integrable and Non-integrable Systems*, Phys.Rep.**180**.159 (1989).
- [58] A.F. Ranada, A. Ramani, B. Dorizzi, B. Grammaticos: *The Weak Painlevé property as a Criterion for the Integrability of Dynamical Systems*, J. Math. Phys. **26** 708, (1985).
- [59] R. Sahadevan and M. Lakshmanan: *Invariance and Integrability: Hénon-Heiles and two coupled quartic anharmonic oscillator systems*, J. Phys.A **19**, 949 (1986).
- [60] T. Sen, M. Tabor: *Lie Symmetries of the Lorenz Model*, Physica **44D**,313 (1990).
- [61] C. Sparrow : *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors* (1982).
- [62] M. Tabor: *Modern Dynamics and Classical Analysis*, Nature **310**,277 (1984).
- [63] M. Tabor: *Chaos and Integrability in Nonlinear Dynamics* (1989).
- [64] M. Tabor: *Analytic Structure of Dynamical Systems*, Pramana - J. Phys. **2**, 315 (1989).
- [65] M. Tabor, J. Weiss: *Analytic Structure of the Lorenz System*, Phys. Rev. A **24**, 2157 (1981).
- [66] J. Weiss, M. Tabor, G. Carnevale: *The Painlevé Property for Partial Differential Equations*, J. Math. Phys. **24**(3),522 (1983).

- [67] Y. Ueda: *Explosion of Strange Attractors Exhibited by Duffing's Equation*, Ann. N.Y. Acad. Sci. **357** (1980).
- [68] S.L. Ziglin: *Splitting of Separatrices. Branching of Solutions and Nonexistence of an Integral in the Dynamics of a Solid Body*, Trans. Moscow Math. Soc., 283 (1982).

END

0 1-0 6-9 5

FIN